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Normality and Modulability Indices. Part I: Convex Cones in Normed Spaces

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Abstract. This paper proposes and compares several ways of measuring the degree of normality of a convex cone contained in a normed space. The dual concept of modulability is also considered. Other notions like solidity and sharpness are also analyzed from a quantitative point of view.

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1 Introduction

In recent years we have devoted a great deal of effort into describing the angular structure of closed convex cones in finite dimensional vector spaces [21, 22, 24]. We have addressed also the issue of measuring the degree of solidity and the degree of pointedness of a closed convex cone [19, 20, 23]. Finite dimensionality of the underlying vector space was a crucial hypothesis.

The situation is more complex in an infinite dimensional setting. Most of our results simply don't extend in a trivial manner to the context of a Hilbert space. The discussion is even more involved if one works in a general normed space, say $(X, \|\cdot\|)$. The intrinsic geometry of the closed unit ball

$$B_X = \{x \in X : \|x\| \leq 1\}$$

has an important impact on the way we measure and perceive properties like pointedness, solidity, reproducibility, normality, and so on.

To proceed further with the exposition we need to lay down some notation and explain the basic terminology that is being employed. The unit sphere in X is indicated with the symbol S_X. The main object of our attention is the set

$$\Xi(X) \equiv \text{nontrivial closed convex cones in } X$$

which we equip with the truncated Pompeiu-Hausdorff metric [37]

$$\varrho(K_1, K_2) = \text{haus}(K_1 \cap B_X, K_2 \cap B_X).$$

Here

$$\text{haus}(C_1, C_2) = \max\left\{\sup_{z \in C_1} \text{dist}[z, C_2], \sup_{z \in C_2} \text{dist}[z, C_1]\right\} \tag{1}$$

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stands for the classical Pompeiu-Hausdorff distance between two bounded closed nonempty sets C_1, C_2, and $\mathrm{dist}[x, C]$ refers to the distance from x to the set C. By a convex cone we understand a nonempty set K satisfying $K + K \subset K$ and $\mathbb{R}_+ K \subset K$. Saying that a convex cone K is nontrivial simply means that K is different from $\{0\}$ and different from the whole space X.

We are concerned also with duality issues. The topological dual space X^* is equipped with the norm

$$\|y\|_* = \sup_{\|x\| \leq 1} \langle y, x \rangle,$$

where the bilinear form $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$ stands for the duality product between X and X^*. The notation B_{X^*} refers to the closed unit ball in X^*.

Table 1 indicates the main properties of convex cones that we want to explore: pointedness, normality, sharpness, reproducibility, modulability, and solidity. The emphasis of our work lies on the quantitative aspect, that is to say, we introduce and study various coefficients that measure to which extent a certain property is present in a given convex cone.

"Primal" concept	coefficient(s)	"Dual" concept	coefficient(s)
pointedness	-	almost reproducibility	-
normality	$\sigma,\ \beta,\ \nu,\ \rho_{\rm nor}$	modulability	$\mu,\ \rho_{\rm mod}$
sharpness	$\tau,\ \rho_{\rm sh}$	solidity	$\varphi,\ \rho_{\rm sol}$

Table 1. Six properties for convex cones in normed spaces

Our research program is too vast to be treated in a single paper. An important portion is left for the Part II of our work [25], specially the results that are valid only in a Hilbert space setting.

2 Beyond Reproducibility

2.1 From Reproducibility to Modulability

For closed convex cones in Banach spaces there is no difference between reproducibility and modulability. In order to better explain the motivation behind the introduction of the concept of modulability, we lift the discussion to the more abstract setting of a general normed space.

Recall that a convex cone K in a normed space $(X, \|\cdot\|)$ is said to be *reproducing* (or generating) if the linear subspace

$$\mathrm{span}(K) = K - K$$

spanned by K is the whole space X. Reproducibility is a purely algebraic concept, the norm $\|\cdot\|$ playing no role in it. Sometimes it is helpful to view K as the set of "nonnegative" elements of the space X. What reproducibility says is that every vector x in X can be decomposed in the form

$$x = u - v \quad \text{with} \quad u, v \in K,$$

i.e., as difference of two nonnegative elements. Of course, such a decomposition of x is not unique.

Defining and computing a "best" decomposition is a fundamental problem of the theory of convex cones. We will not elaborate here on this classical issue (cf. [7, 15]). Suffice it to say that a convenient decomposition of x is one for which $\|u\|$ and $\|v\|$ are are not too large while compared to $\|x\|$. It is natural to ask whether it is possible to choose $u, v \in K$ so that the ℓ^2-norm

$$\|(u, v)\| = \sqrt{\|u\|^2 + \|v\|^2} \tag{2}$$

doesn't exceed a certain multiple of $\|x\|$. Such multiple should depend on the cone K but not on the particular x that we want to decompose.

Definition 1. *A convex cone K in a normed space X is said to be modulable if there is a constant $\gamma > 0$ such that*

$$\left\{ \begin{array}{l} \textit{any } x \in X \textit{ is expressible in the form } x = u - v \\ \textit{with } u, v \in K \textit{ satisfying } \gamma \, \|(u, v)\| \leq \|x\|. \end{array} \right. \tag{3}$$

Such a scalar γ is called a modulability constant for K.

The choice of the ℓ^2-norm in the product space $X \times X$ is not essential. We could have used instead an equivalent norm, for instance the ℓ^∞-norm

$$\|(u, v)\|_\infty = \max\{\|u\|, \|v\|\}. \tag{4}$$

Needless to say, a modulable convex cone is necessarily reproducing. What is more striking is the following converse result.

Theorem 1. *If K is a reproducing closed convex cone K in a Banach space X, then K is modulable.*

Theorem 1 is mentioned without proof in the book by Krasnosel'ski and Zabreiko [29, Section 33.1]. A more elaborate formulation and a proof of this result can be found in the book by Kusraev and Kutateladze [31, Section 3.1]. Most soviet authors refer to modulability as the "non-oblateness property", but the later terminology has the inconvenience of using a negative prefix. For the same reason we are not using the expression "non-flattening property" adopted by a few authors (cf. [26]). Theorem 1 can be found also in Ando [1] and in the classical book by Peressini [35]. The later author refers to a modulable convex cone as being a "strict b-cone".

Theorem 1 is no longer true if the normed space X fails to be complete. In Section 2.3 we will present an interesting example of a closed convex cone which is reproducing but not modulable. Of course, such a cone lives in a non-complete normed space. For the time being we ask the reader to keep always in mind the following sentence:

While dealing with closed convex cones in Banach spaces,
modulability and reproducibility are the same concept.

The next proposition sheds additional light on modulability. The notation $\mathcal{N}_X(z)$ stands for the filter of neighborhoods of a point $z \in X$.

Proposition 1. *For a convex cone K in a normed space X, the following conditions are equivalent:*

(a) K *is modulable.*

(b) $0 \in \operatorname{int}\{u - v : \ u, v \in K, \ \|(u, v)\| \leq 1\}$.

(c) $0 \in \operatorname{int}[K \cap B_X - K \cap B_X]$.

(d) $0 \in \operatorname{int}[K \cap V - K \cap V]$ *for all* $V \in \mathcal{N}_X(0)$.

(e) $0 \in \operatorname{int}[K \cap V_1 - K \cap V_2]$ *for all* $V_1, V_2 \in \mathcal{N}_X(0)$.

Furthermore, a scalar $\gamma > 0$ *is a modulability constant for* K *if and only if*

$$\gamma B_X \subset \{u - v: \; u, v \in K, \; \|(u,v)\| \leq 1\}. \tag{5}$$

Proof. Some portions of this theorem are probably known. We divide the proof in several parts:
$(e) \Rightarrow (d) \Rightarrow (c)$. Take both neighborhoods V_1, V_2 equal to V, and then choose V as the unit ball B_X.
$(c) \Rightarrow (e)$. Let $V_1, V_2 \in \mathcal{N}_X(0)$. In view of (c), there are positive numbers r and $\varepsilon_1, \varepsilon_2$ such that

$$\begin{aligned} rB_X &\subset K \cap B_X - K \cap B_X \\ \varepsilon_i B_X &\subset V_i \; \text{ for } i = 1, 2. \end{aligned}$$

By letting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, one gets

$$\begin{aligned} \varepsilon r B_X &\subset \varepsilon(K \cap B_X - K \cap B_X) \\ &= K \cap \varepsilon B_X - K \cap \varepsilon B_X \\ &\subset K \cap V_1 - K \cap V_2, \end{aligned}$$

showing in this way that $K \cap V_1 - K \cap V_2$ contains 0 in its interior.
$(b) \Leftrightarrow (c)$. This is due to the fact that the ℓ^2-norm (2) is equivalent to the ℓ^∞-norm (4).
$(a) \Rightarrow (b)$. Let $\gamma > 0$ be a modulability constant for K. We shall prove that the inclusion (5) holds true. Take $x \in \gamma B_X$. By (3), there is a decomposition $(u, v) \in K \times K$ of x satisfying $\gamma\|(u,v)\| \leq \|x\|$. Hence, $x = u - v$ with $u, v \in K$ such that $\|(u,v)\| \leq 1$. So, x belongs to the right-hand side of (5) as desired.
$(b) \Rightarrow (a)$. Let $\gamma > 0$ be as in (5). We shall prove that γ serves as modulability constant for K. For any nonzero vector $x \in X$ one can write

$$\gamma \|x\|^{-1} x = u' - v'$$

with $u', v' \in K$ such that $\|(u', v')\| \leq 1$. This means that x is expressible in the form

$$x = \underbrace{(\|x\|/\gamma)u'}_{u} - \underbrace{(\|x\|/\gamma)v'}_{v}$$

with $(u, v) \in K \times K$ satisfying $\gamma\|(u,v)\| = \|x\| \, \|(u', v')\| \leq \|x\|$. □

2.1.1 The Use of Absolutely Convex Hulls

In the sequel we use the notation $\operatorname{aco}(C)$ to indicate the *absolutely convex hull* of a subset C of X. By construction,

$$\operatorname{aco}(C) = \operatorname{co}[C \cup -C]$$

corresponds to the smallest symmetric convex set containing C.

Theorem 2. *A convex cone* K *in a normed space* X *is modulable if and only if the set*

$$K^\bullet = \operatorname{aco}[K \cap B_X] \tag{6}$$

is a neighborhood of the origin.

Proof. This result is in the same spirit as Proposition 1. For putting everything in the right perspective we start by mentioning that any ℓ^p-norm

$$\|(u,v)\|_p = [\|u\|^p + \|v\|^p]^p \qquad (1 \le p < \infty)$$

is equivalent to the ℓ^2-norm (2), so the condition (b) in Proposition 1 amounts to saying that

$$V_p(K) = \{u - v : \ u, v \in K, \ \|(u,v)\|_p \le 1\}$$

is a neighborhood of 0. It turns out that $V_p(K)$ can be written as function of the set $K \cap B_X$. The particular choice $p = \infty$ yields of course

$$V_\infty(K) = K \cap B_X - K \cap B_X,$$

an expression already encountered in Proposition 1(c). The case $p \in [1, \infty[$ can also be worked out, but this time one gets a more involved expression, namely

$$V_p(K) = \bigcup_{\substack{\alpha^p + \beta^p \le 1 \\ \alpha, \beta \ge 0}} \{\alpha(K \cap B_X) - \beta(K \cap B_X)\}.$$

The case $p = 1$ is of special relevance because

$$V_1(K) = \bigcup_{\substack{\alpha + \beta \le 1 \\ \alpha, \beta \ge 0}} \{\alpha(K \cap B_X) - \beta(K \cap B_X)\}$$

corresponds exactly to the absolutely convex hull of $K \cap B_X$. This completes the proof of the theorem. $\square$

Before continuing our discussion on modulability, we pause for a moment and say a few words on the absolutely convex hull of a set like $K \cap B_X$. First of all, observe that (6) can be written in the equivalent form

$$K^\bullet = \mathrm{co}\,[(K \cup -K) \cap B_X]\,.$$

Such representation of $K^\bullet$ facilitates sometimes the computation of this set. The following lemma shows that $K \cap B_X$ and $K \cap S_X$ have the same absolutely convex hull. Such result is probably known, but we record it for the sake of completeness.

Lemma 1. *For a nontrivial convex cone K in a normed space X, one has*

$$K^\bullet = \mathrm{aco}\,[K \cap S_X] = \mathrm{co}\,[(K \cup -K) \cap S_X]\,. \tag{7}$$

Proof. The second equality in (7) is clear, so we concentrate on the first one. It is enough to prove that

$$K \cap B_X \subset \mathrm{aco}\,[K \cap S_X]$$

because a simple logic argument leads then to the desired conclusion. Take x in $K \cap B_X$. Notice that $0 \in \mathrm{aco}\,[K \cap S_X]$, so there is no loss of generality in assuming that $x \neq 0$. In such a case, one can write

$$x = \alpha u + (1 - \alpha)(-v)$$

with $u = v = \|x\|^{-1}x$ belonging to $K \cap S_X$ and $\alpha = (1 + \|x\|)/2$ belonging to $]0, 1]$. This shows that $x \in \mathrm{aco}\,[K \cap S_X]$. $\square$

We mention in passing that $K^\bullet$ behaves in a Lipschitz-continuous manner with respect to changes in the argument K. In the next lemma we use the classical Pompeiu-Hausdorff distance. The term $\mathrm{haus}(C_1, C_2)$ introduced in (1) is finite and well defined as long as the sets C_1, C_2 are nonempty and bounded. However, the function $\mathrm{haus}(\cdot,\cdot)$ is a true metric only if the sets C_1, C_2 are further required to be closed.

Lemma 2. *Let K_1, K_2 be nontrivial closed convex cones in a normed space X. Then,*

$$\mathrm{haus}(K_1^\bullet, K_2^\bullet) \leq \varrho(K_1, K_2).$$

Proof. Take any $x \in K_1^\bullet$ and write it in the form $x = \alpha u - (1-\alpha)v$ with $\alpha \in [0,1]$ and $u, v \in K_1 \cap B_X$. For each $\varepsilon > 0$ one can find a pair $u_\varepsilon, v_\varepsilon \in K_2 \cap B_X$ such that

$$\begin{aligned} \|u - u_\varepsilon\| &\leq \mathrm{dist}[u, K_2 \cap B_X] + \varepsilon, && (8)\\ \|v - v_\varepsilon\| &\leq \mathrm{dist}[v, K_2 \cap B_X] + \varepsilon. && (9) \end{aligned}$$

Observe that $x_\varepsilon = \alpha u_\varepsilon - (1-\alpha)v_\varepsilon$ belongs to $K_2^\bullet$ and

$$\|x - x_\varepsilon\| = \|\alpha(u - u_\varepsilon) + (1-\alpha)(v_\varepsilon - v)\| \leq \alpha\,\|u - u_\varepsilon\| + (1-\alpha)\,\|v_\varepsilon - v\|\,.$$

Given (8)-(9), one obtains

$$\mathrm{dist}[x, K_2^\bullet] \leq \|x - x_\varepsilon\| \leq \left\{ \sup_{z \in K_1 \cap B_X} \mathrm{dist}[z, K_2 \cap B_X] \right\} + \varepsilon.$$

By letting first $\varepsilon \to 0$ and taking then the supremum with respect to $x \in K_1^\bullet$, one arrives at

$$\sup_{x \in K_1^\bullet} \mathrm{dist}[x, K_2^\bullet] \leq \sup_{z \in K_1 \cap S_X} \mathrm{dist}[z, K_2 \cap B_X].$$

This is half of the proof. The other half is obtained by exchanging the roles of K_1 and K_2. $\square$

2.2 Measuring the Degree of Modulability of a Convex Cone

Theorem 2 suggests introducing the number

$$\mu(K) = \sup\{r \geq 0 : \; rB_X \subset K^\bullet\} \tag{10}$$

as a tool for measuring the degree of modulability of K. Clearly one has

$$0 \leq \mu(K) \leq 1$$

for every nontrivial convex cone K in any normed space X. Let us examine more carefully the definition of the function μ and see what the term (10) is actually telling us about the structure of K. For warming up nothing is better than considering a simple example in a finite dimensional context.

Example 1. By way of illustration we work out the case of an elliptic cone

$$\mathcal{E}(A) = \{(\xi, t) \in X : \; \sqrt{\xi^T A \xi} \leq t\}$$

in the Euclidean space $X = \mathbb{R}^n \times \mathbb{R}$. Here A denotes a positive definite symmetric matrix of size $n \times n$. In order to form the convex hull of the set $[\mathcal{E}(A) \cup -\mathcal{E}(A)] \cap B_X$ we draw a segment joining the points (ξ, t) and $(\xi, -t)$. We do this for all (ξ, t) such that

$$\sqrt{\xi^T A \xi} = t, \tag{11}$$

$$\|\xi\|^2 + t^2 = 1. \tag{12}$$

A geometric argument shows that the largest ball rB_X contained in $[\mathcal{E}(A) \cup -\mathcal{E}(A)] \cap B_X$ has a positive radius r given by

$$r = \inf_{\xi, t} \|(0,0) - (\xi, 0)\|,$$

where the infimum is taken with respect to $(\xi, t) \in X$ satisfying the constraints (11)-(12). By getting rid of the variable t one arrives at

$$r^2 = \inf_{\xi^T(I+A)\xi = 1} \|\xi\|^2 = \left[\sup_{\xi \neq 0} \frac{\xi^T(I+A)\xi}{\|\xi\|^2}\right]^{-1}.$$

One has proven in this way that

$$\mu(\mathcal{E}(A)) = \frac{1}{\sqrt{1 + \lambda_{\max}(A)}} \tag{13}$$

with $\lambda_{\max}(A)$ denoting the largest eigenvalue of A. For the Lorentz or "ice-cream" cone

$$\mathcal{L} = \{(\xi, t) \in X : \ \|\xi\| \leq t\}$$

one gets in particular $\mu(\mathcal{L}) = \sqrt{2}/2$.

It is too early to draw a general conclusion from Example 1, but formula (13) strongly suggests that μ has something to do with the concept of radius of solidity introduced and studied in [19, Section 4]. We will come back to this point in due course.

Needless to say, computing $\mu(K)$ for a given convex cone K in an arbitrary normed space X is not always as easy as in Example 1. When it comes to practical computations, perhaps the simplest way of estimating $\mu(K)$ is by solving first the problem which consists in finding the least ℓ^1-norm element in

$$D_K(x) = \{(u, v) \in X \times X : \ u, v \in K, u - v = x\},$$

the set of all decompositions of a given $x \in X$. The details are explained in the next proposition.

Proposition 2. *A convex cone K in a normed space X is modulable if and only if*

$$\zeta(K) = \sup_{\|x\| \leq 1} \inf_{(u,v) \in D_K(x)} \|(u, v)\|_1 \tag{14}$$

is a finite number. Furthermore, one has the relation

$$\mu(K) = \frac{1}{\zeta(K)} \tag{15}$$

with the usual convention $1/\infty = 0$ being in force.

Proof. A quick sketch of the proof will do. The term (14) corresponds to the smallest real $\zeta > 0$ such that

$$\inf_{(u,v)\in D_K(x)} \|(u,v)\|_1 \;\le\; \zeta\|x\| \qquad \forall x \in X.$$

The reciprocal $1/\zeta(K)$ is then the largest constant $\gamma > 0$ such that

$$\left\{ \begin{array}{l} \text{any } x \in X \text{ is expressible in the form } x = u - v \\ \text{with } u, v \in K \text{ satisfying } \gamma\|(u,v)\|_1 \le \|x\|. \end{array} \right. \tag{16}$$

If one looks back again at the proof of Proposition 1, one sees that (16) is equivalent to

$$\gamma B_X \subset \{u - v : \; u, v \in K, \; \|(u,v)\|_1 \le 1\}.$$

The link between $\zeta(K)$ and $\mu(K)$ is now clear. □

We have used the ℓ^1-norm in the definition of $\zeta(K)$ because in such a way one gets a direct and simple relation with the coefficient $\mu(K)$. Let us illustrate the use of formula (15) with the help of an illuminating example.

Example 2. Consider the vector space $\mathcal{B}([a,b],\mathbb{R})$ of bounded functions $x : [a,b] \to \mathbb{R}$ equipped with the uniform (or Chebyshev) norm $\|x\| = \sup_{a\le t\le b} |x(t)|$. The set

$$K = \{u \in \mathcal{B}([a,b],\mathbb{R}) : u(t) \ge 0 \; \forall t \in [a,b]\}$$

is a reproducing closed convex in the Banach space $(\mathcal{B}([a,b],\mathbb{R}), \|\cdot\|)$. Hence, it is modulable. In order to evaluate $\mu(K)$ we proceed as follows. First, observe that any $x \in \mathcal{B}([a,b],\mathbb{R})$ can be decomposed as difference

$$x(t) = \underbrace{\max\{0, x(t)\}}_{x_+(t)} - \underbrace{\max\{0, -x(t)\}}_{x_-(t)} \qquad \forall t \in [a,b]$$

of two functions $x_+, x_- \in K$ such that $\|x_+\| \le \|x\|$, $\|x_-\| \le \|x\|$. Hence,

$$\inf_{(u,v)\in D_K(x)} \|(u,v)\|_1 \;\le \|(x_+, x_-)\|_1 \le \|x_+\| + \|x_-\| \le 2\|x\|.$$

By taking the supremum with respect to $x \in B_X$ one gets the estimate $\zeta(K) \le 2$. We now show that this estimate is optimal. Consider any function $\hat{x} : [a,b] \to \mathbb{R}$ such that

$$\inf_{a\le t\le b} \hat{x}(t) = -1 \quad \text{and} \quad \sup_{a\le t\le b} \hat{x}(t) = 1. \tag{17}$$

Such a function $\hat{x}$ is clearly in S_X. We claim that

$$\|(u,v)\|_1 \ge 2 \qquad \forall (u,v) \in D(\hat{x}).$$

Although it is not necessary, for shortening the proof we will ask the extrema in (17) to be attained. Suppose that $\hat{x}$ attains its infimum at $t_* \in [a,b]$ and its supremum at $t^* \in [a,b]$. If $(u,v) \in D_K(\hat{x})$, then one has in particular

$$\begin{aligned} u(t^*) - v(t^*) = \hat{x}(t^*) &\;=\; 1, \\ u(t_*) - v(t_*) = \hat{x}(t_*) &\;=\; -1. \end{aligned}$$

Given that u, v are nonnegative functions, one gets $u(t^*) \ge 1$ and $v(t_*) \ge 1$. Hence, $\|u\| \ge 1, \|v\| \ge 1$, and the proof of our claim is complete. In conclusion, $\zeta(K) = 2$ and formula (15) yields $\mu(K) = 1/2$.

As explained in the next proposition, evaluating $\mu(K)$ is also a matter of estimating the least-norm element in the boundary of $K^\bullet$.

Proposition 3. *Let K be a convex cone in a normed space X. Then,*

$$\mu(K) = \inf_{x \in \mathrm{bd}(K^\bullet)} \|x\|. \tag{18}$$

Proof. Suppose that $K^\bullet$ contains the origin in its interior, otherwise both sides in (18) are equal to 0. The next reasoning applies to any bounded convex set C containing the origin in its interior, but, of course, we have the particular case $C = K^\bullet$ in mind. First of all, we claim that

$$\inf_{x \in \mathrm{bd}(C)} \|x\| = \mathrm{dist}[0, X \backslash C]. \tag{19}$$

Since C and its complement $X\backslash C$ have the same boundary, it follows that

$$\inf_{x \in \mathrm{bd}(C)} \|x\| = \inf_{x \in \mathrm{bd}(X \backslash C)} \|x\| = \inf_{\substack{x \in \mathrm{cl}(X \backslash C) \\ x \notin \mathrm{int}(X \backslash C)}} \|x\|.$$

But the constraint $x \notin \mathrm{int}(X\backslash C)$ in the last infimum is superfluous because the norm of a point in the interior of $X\backslash C$ can always be reduced a bit further. Hence,

$$\inf_{x \in \mathrm{bd}(C)} \|x\| = \inf_{x \in \mathrm{cl}(X \backslash C)} \|x\| = \mathrm{dist}[0, \mathrm{cl}(X \backslash C)] = \mathrm{dist}[0, X \backslash C].$$

This takes care of our claim. Now, since the implications

$$\mathrm{dist}[0, X\backslash C] > r \quad \Longrightarrow \quad rB_X \subset C \quad \Longrightarrow \quad \mathrm{dist}[0, X \backslash C] \geq r$$

holds for any scalar $r \geq 0$, one readily gets

$$\sup\{r \geq 0 : rB_X \subset C\} = \mathrm{dist}[0, X\backslash C].$$

This and (19) yield the announced formula. □

In the next proposition we characterize the coefficient $\mu(K)$ in terms of the support function of $K^\bullet$. Recall that the support function Ψ_C^* of a nonempty set $C \subset X$ is defined as

$$y \in X^* \mapsto \Psi_C^*(y) = \sup_{x \in C} \langle y, x \rangle.$$

The representation formulas stated in Proposition 4 require K to satisfy a certain "qualification condition". Checking this technical hypothesis is sometimes a bit bothersome, but unfortunately this is something not to be neglegted.

Proposition 4. *Let K be nontrivial convex cone in a normed space X. Suppose that K is "qualified" in the sense that*

$$K^\bullet \text{ and } \mathrm{cl}(K^\bullet) \text{ have the same interior.} \tag{20}$$

Then, one can write

$$\mu(K) = \max\{r \geq 0 : \ rB_X \subset \mathrm{cl}(K^\bullet)\} \tag{21}$$

and also

$$\begin{aligned} \mu(K) &= \inf_{\|y\|_*=1} \max\{\Psi^*_{K\cap B_X}(y), \Psi^*_{-K\cap B_X}(y)\} && (22) \\ &= \inf_{\|y\|_*=1} \max\{\Psi^*_{K\cap S_X}(y), \Psi^*_{-K\cap S_X}(y)\}. && (23) \end{aligned}$$

Proof. If $K^\bullet$ and $\mathrm{cl}(K^\bullet)$ have the same interior, then both sets have also the same boundary. In such a case, formula (18) can be written in the form

$$\mu(K) = \inf_{x\in \mathrm{bd}[\mathrm{cl}(K^\bullet)]} \|x\|.$$

If one applies the proof technique of Proposition 3 to the set $C = \mathrm{cl}(K^\bullet)$, one gets

$$\inf_{x\in \mathrm{bd}[\mathrm{cl}(K^\bullet)]} \|x\| = \sup\{r \geq 0 : \; rB_X \subset \mathrm{cl}(K^\bullet)\}. \tag{24}$$

Since the interval $\{r \geq 0 : \; rB_X \subset K^\bullet\}$ is compact, the supremum in (24) is attained. This completes the proof of (21). We now take care of formula (22). For any $r \geq 0$ and any bounded closed convex set $C \subset X$ containing the origin $0 \in X$, one has

$$\begin{aligned} rB_X \subset C &\iff \Psi^*_{rB_X}(y) \leq \Psi^*_C(y) \quad \forall y \in X^* \\ &\iff r\,\|y\|_* \leq \Psi^*_C(y) \quad \forall y \in X^* \\ &\iff r \leq \inf_{\|y\|_*=1} \Psi^*_C(y). \end{aligned}$$

In view of (21), one gets

$$\mu(K) = \inf_{\|y\|_*=1} \Psi^*_{\mathrm{cl}(K^\bullet)}(y).$$

But standard calculus rules on support functions yield

$$\Psi^*_{\mathrm{cl}(K^\bullet)}(y) = \Psi^*_{K^\bullet}(y) = \max\{\Psi^*_{K\cap B_X}(y), \Psi^*_{-K\cap B_X}(y)\},$$

completing in this way the proof of (22). Formula (23) is proven analogously but now one uses the representation (7) of $K^\bullet$. $\square$

Remark 1. There are two easy ways of ensuring the qualification hypothesis (20). The first way is asking K to be modulable. Indeed, the modulability of K implies that $K^\bullet$ has nonempty interior, and this in turn implies (20). The second way of ensuring the qualification hypothesis is asking $K^\bullet$ to be closed. This happens, for instance, if K is a closed convex cone in a reflexive Banach space. Indeed, it is easy to check that in a reflexive Banach space the convex hull of the union of two bounded closed convex sets is convex and weakly closed, hence closed.

Corollary 1. *Let K be a qualified nontrivial convex cone in a normed space X. Then,*

$$\mu(\mathrm{cl}(K)) = \mu(K).$$

In particular, $\mathrm{cl}(K)$ *is modulable if and only if K is modulable.*

Proof. It follows from the representation formula (23) and the fact that

$$\Psi^*_{S_X \cap P} = \Psi^*_{\mathrm{cl}(S_X \cap P)} = \Psi^*_{S_X \cap \mathrm{cl}(P)}$$

for any convex cone $P \subset X$. $\square$

The qualification assumption is essential in Corollary 1. The following example shows that, in general, the concept of modulability is not blind with respect to topological closure.

Example 3. Let $\ell_2(\mathbb{R})$ denote the Hilbert space of real sequences $\{x_k\}_{k\geq 1}$ such that $\sum_{k=1}^{\infty} x_k^2 < \infty$. In this space consider the convex cone K given by

$$x \in K \quad \Leftrightarrow \quad \exists n \geq 1 \text{ such that } x_1 \geq 0, \ldots, x_n \geq 0 \text{ and } x_k = 0 \text{ for all } k \geq n+1.$$

Its closure

$$\mathrm{cl}(K) = \{x \in \ell_2(\mathbb{R}) : x_k \geq 0 \ \forall k \geq 1\}$$

is clearly modulable, but K itself is not.

2.3 Reproducibility without Modulability

As promised before, we now display a nice example of a closed convex cone which is reproducing but not modulable.

Example 4. Denote by $\mathcal{BV}([a,b],\mathbb{R})$ the vector space of functions $x : [a,b] \to \mathbb{R}$ of bounded variation. This space is not complete while equipped with the uniform norm $\|x\| = \sup_{a\leq t\leq b} |x(t)|$. According to a classical result in analysis, functions of bounded variation on a compact interval are exactly those which can be written as difference of two nondecreasing functions on that interval. It follows that the closed convex cone

$$K = \{x \in \mathcal{BV}([a,b],\mathbb{R}) : x \text{ is nondecreasing}\} \tag{25}$$

is reproducing. We claim that (25) is not modulable. We shall construct a sequence $\{x_k\}_{k\geq 1}$ of unit vectors in $\mathcal{BV}([a,b],\mathbb{R})$ such that

$$\inf_{(u,v)\in D_K(x_k)} \|(u,v)\|_1 \ \to \infty \quad \text{as} \quad k \to \infty. \tag{26}$$

In view of Proposition 2, the existence of such sequence would imply the non-modulability of K. For notational simplicity we work out only the particular case $a = 0, b = 1$. For each $k \geq 1$, consider the function

$$t \in [0,1] \mapsto x_k(t) = \cos(2k\pi t).$$

Observe that the x_k's are of bounded variation and have unit length with respect to the uniform norm. Checking (26) is quite cumbersome but it can be done with a bit of patience. The key observation is that the trigonometric function $x_k(\cdot)$ oscillates more and more as k increases. Consider a given k and an arbitrary pair $u, v : [0,1] \to \mathbb{R}$ of nondecreasing functions such that

$$u(t) - v(t) = x_k(t) \quad \forall t \in [0,1].$$

Let $t_i = i/(2k)$, with $i \in \{1, 2, \ldots, 2k-1\}$, be the points at which $x_k(\cdot)$ changes the type of monotonicity. On the interval $[0, t_1]$ the function $x_k(\cdot)$ is decreasing and

$$v(t_1) = u(t_1) - x_k(t_1) = u(t_1) + 1 \geq u(0) + 1.$$

On $[t_1, t_2]$ the function $x_k(\cdot)$ is increasing and

$$u(t_2) = v(t_2) + x_k(t_2) = v(t_2) + 1 \geq v(t_1) + 1 \geq u(0) + 2.$$

By repeating the same argument one gets

$$\begin{aligned} v(t_3) &\geq u(0) + 3, \\ u(t_4) &\geq u(0) + 4, \end{aligned}$$

and so on. One ends up with $u(1) \geq u(0) + 2k$. This inequality yields

$$\|(u, v)\|_1 \geq \|u\| \geq k.$$

Since the pair (u, v) was an arbitrary decomposition of x_k, we conclude that (26) holds.

2.4 Properties of $\mu(\cdot)$ as Function on $\Xi(X)$

2.4.1 Nonexpansiveness

The classical Pompeiu-Hausdorff distance (1) admits a support function characterization when it is applied to convex sets.

Lemma 3. *If C_1, C_2 are bounded closed convex nonempty sets in a normed space X, then*

$$\operatorname{haus}(C_1, C_2) = \sup_{\|y\|_* = 1} |\Psi^*_{C_1}(y) - \Psi^*_{C_2}(y)|.$$

Proof. See for instance [4, Corollary 3.2.8] or [8, Theorem 2.18]. □

The next proposition is obtained straightforwardly by combining Lemma 3 and Proposition 4.

Proposition 5. *Let K_1, K_2 be nontrivial closed convex cones in a normed space X. The inequality*

$$|\mu(K_1) - \mu(K_2)| \leq \varrho(K_1, K_2) \tag{27}$$

holds in case K_1, K_2 are both qualified (or in case both are non-modulable).

Proof. If K_1, K_2 are both non-modulable, then $\mu(K_1) = \mu(K_2) = 0$ and (27) holds trivially. If K_1, K_2 are both qualified, then the proof of (27) relies on the representation formula (22). Lemma 3 yields

$$\Psi^*_{K_1 \cap B_X}(y) \leq \Psi^*_{K_2 \cap B_X}(y) + \|y\|_* \underbrace{\operatorname{haus}(K_1 \cap B_X, K_2 \cap B_X)}_{\varrho(K_1, K_2)}$$

for every $y \in X^*$. Similarly,

$$\begin{aligned} \Psi^*_{-K_1 \cap B_X}(y) &\leq \Psi^*_{-K_2 \cap B_X}(y) + \|y\|_* \, \varrho(-K_1, -K_2) \\ &= \Psi^*_{-K_2 \cap B_X}(y) + \|y\|_* \, \varrho(K_1, K_2). \end{aligned}$$

One gets in this way

$$\max\{\Psi^*_{K_1 \cap B_X}(y), \Psi^*_{-K_1 \cap B_X}(y)\} \leq \max\{\Psi^*_{K_2 \cap B_X}(y), \Psi^*_{-K_2 \cap B_X}(y)\} + \|y\|_* \, \varrho(K_1, K_2).$$

By passing to the infimum with respect to $y \in S_{X^*}$, one arrives at

$$\mu(K_1) \leq \mu(K_2) + \varrho(K_1, K_2).$$

For completing the proof it suffices now to exchange the roles of K_1 and K_2. □

Corollary 2. *Let X be a reflexive Banach space. Then,*

$$|\mu(K_1) - \mu(K_2)| \leq \varrho(K_1, K_2) \qquad \forall K_1, K_2 \in \Xi(X),$$

i.e., $\mu : (\Xi(X), \varrho) \to \mathbb{R}$ is a nonexpansive function.

It is not clear whether Corollary 2 remains true if X is not a reflexive Banach space. In any case, constructing a counterexample is not a trivial matter. We mention that (27) is valid in a general normed space for many configurations concerning the pair $K_1, K_2 \in \Xi(X)$. If there is a trouble at all with (27), then one cone must be modulable and the other cone must be non-modulable and not-qualified. We skip this technical point and go on with the discussion of other properties concerning the modulability coefficient.

2.4.2 Other Properties

We need to introduce a particular class of normed spaces. We don't know if this class has been considered already in the literature.

Definition 2. *A normed space X is gentle if a closed set M satisfying*

$$\operatorname{int}(B_X) \subset \operatorname{co}(M) \subset B_X \tag{28}$$

contains necessarily the unit sphere S_X.

The above definition is a bit technical, so it is helpful to recall the known concept of dentability. One says that $z \in C$ is a *denting point* of C if for all $\varepsilon > 0$ the closed convex hull of

$$\{x \in C : \|x - z\| \geq \varepsilon\}$$

leaves z aside. Several equivalent characterizations of dentability can be found in [5] and [33].

Proposition 6. *Let X be a vector space equipped with a norm such that every unit vector of X is a denting point of B_X. Then, X is gentle.*

Proof. Let M be a closed set satisfying (28). In particular, one has $\operatorname{clco}(M) = B_X$, where the notation $\operatorname{clco}(M)$ refers to the closure of the convex hull of M. We must prove that $S_X \subset M$. Suppose on the contrary that $z \notin M$ for some unit vector $z \in X$. Since M is closed, we can find a small $\varepsilon > 0$ such that

$$\{x \in B_X : \|x - z\| \geq \varepsilon\} \supset M.$$

Hence,

$$\operatorname{clco}\{x \in B_X : \|x - z\| \geq \varepsilon\} \supset B_X. \tag{29}$$

Notice that $z \in B_X$, but z doesn't belong to the set on the left-hand side of (29) because z is a denting point of B_X. This contradiction confirms that $S_X \subset M$. □

Corollary 3. *Any locally uniformly rotund Banach space is gentle. In particular, any Hilbert space is gentle.*

Proof. That a Banach space, say X, is locally uniformly rotund means that

$$\|x_n - x\| \to 0 \quad \text{whenever} \quad x, x_n \in X \;\text{ and }\; 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 \to 0.$$

It is known (cf. [5, 32, 40]) that in a locally uniformly rotund Banach space every unit vector is a denting point of the closed unit ball. □

Remark 2. Consider the space $\ell_\infty(\mathbb{R})$ of bounded real sequences $\{x_k\}_{k\geq 1}$ equipped with its usual norm $\|x\| = \sup_{k\geq 1} |x_k|$. This is a typical example of normed space that is not gentle. To see this we suggest examining the closed set

$$M = \{x \in \ell_\infty(\mathbb{R}) : \|x\| \leq 1,\ x_1^2 + (x_2 - 1)^2 \geq 1/9\}.$$

Notice that M doesn't contain the unit sphere because it leaves the unit vector $(0, 1, 0, 0, \dots)$ aside. However, the convex hull of M contains the open unit ball. Indeed, any $x \in \ell_\infty(\mathbb{R})$ of length less than 1 can be written in the form

$$x = \alpha \underbrace{(1, x_2, x_3, \dots)}_{\text{in } M} + \beta \underbrace{(-1, x_2, x_3, \dots)}_{\text{in } M},$$

where $\alpha = (1 + x_1)/2$ and $\beta = (1 - x_1)/2$ are nonnegative coefficients adding up to 1. Another example of normed space that is not gentle is the space $\ell_1(\mathbb{R})$ of absolutely summable real sequences $\{x_k\}_{k\geq 1}$ equipped with the norm $\|x\| = \sum_{k\geq 1} |x_k|$. As set M one takes this time

$$M = \left\{ x \in \ell_1(\mathbb{R}) : \|x\| \leq 1,\ (x_1 - (1/2))^2 + (x_2 - (1/2))^2 \geq 1/9 \right\}.$$

M doesn't contain the unit sphere because it leaves the unit vector $(1/2, 1/2, 0, 0, \dots)$ aside. Let $x \in \ell_1(\mathbb{R})$ be a vector of length less than 1. If $x_1 = 0$ or $x_2 = 0$, then x is already in M, otherwise we write x as convex combination

$$x = \frac{|x_1|}{|x_1| + |x_2|} \left(\frac{|x_1| + |x_2|}{|x_1|} x_1, 0, x_3, x_4, \dots \right) + \frac{|x_2|}{|x_1| + |x_2|} \left(0, \frac{|x_1| + |x_2|}{|x_2|} x_2, x_3, x_4, \dots \right)$$

of two vectors from M. In short, $\mathrm{co}(M)$ contains the open unit ball.

We now come back to the main stream of our exposition. Gentleness of X is an essential assumption for the validity of the property (e) in Theorem 3.

Theorem 3. *Let X be a normed space. Then, the function $\mu : \Xi(X) \to \mathbb{R}$ enjoys the following properties:*

(a) $K_1 \subset K_2$ implies $\mu(K_1) \leq \mu(K_2)$.

(b) $\mu(T(K)) = \mu(K)$ for all $K \in \Xi(X)$ and all invertible linear isometry $T : X \to X$.

(c) $\mu(K) = 0$ if and only if K is not modulable.

(d) $K \cup -K = X$ implies $\mu(K) = 1$.

If the normed space X is gentle, then one can add the next property to the list:

(e) $\mu(K) = 1$ implies $K \cup -K = X$.

Proof. (a) $K_1 \subset K_2$ implies $K_1^\bullet \subset K_2^\bullet$, so the monotonicity of $\mu(\cdot)$ is obvious.
(b) Take $K \in \Xi(X)$ and an invertible linear isometry $T : X \to X$. Recall that a linear map $T : X \to X$ is called an isometry if $\|Tx\| = \|x\|$ for all $x \in X$. To start with, observe that $T(K)$ belongs to $\Xi(X)$. Since T is assumed to be invertible, one can write

$$\begin{aligned} D_{T(K)}(\tilde{x}) &= \{(\tilde{u}, \tilde{v}) \in X \times X :\ \tilde{u}, \tilde{v} \in T(K), \tilde{u} - \tilde{v} = \tilde{x}\} \\ &= \{(Tu, Tv) \in X \times X :\ (u, v) \in D_K(x)\} \end{aligned}$$

with $x = T^{-1}(\tilde{x})$. Hence,

$$\zeta(T(K)) = \sup_{\|\tilde{x}\|\leq 1} \inf_{(\tilde{u},\tilde{v})\in D_{T(K)}(\tilde{x})} \|(\tilde{u},\tilde{v})\|_1 = \sup_{\substack{x\in X\\ \|Tx\|\leq 1}} \inf_{(u,v)\in D_K(x)} \|(Tu,Tv)\|_1 \ = \ \zeta(K).$$

It suffices now to apply Proposition 2.
(c) It follows from Theorem 2 and the very definition of $\mu(\cdot)$.
(d) If $K \cup -K = X$, then $K^\bullet = B_X$. The later equality implies that $\mu(K) = 1$.
(e) Suppose that the normed space X is gentle. Take any $K \in \Xi(X)$ such that $\mu(K) = 1$. Notice that the inclusion $rB_X \subset \mathrm{co}\,[(K \cup -K) \cap B_X]$ holds for any $r \in]0,1[$. Hence,

$$\mathrm{int}(B_X) \subset \mathrm{co}\,[(K \cup -K) \cap B_X]\,.$$

From here and the fact that $(K \cup -K) \cap B_X$ is a closed set contained in B_X, we deduce that

$$S_X \subset (K \cup -K) \cap B_X.$$

Due to a simple homogeneity argument, the later inclusion implies that $K \cup -K = X$. □

The next example shows that the property (e) in Theorem 3 may fail if X is not gentle.

Example 5. In the space $X = \ell_\infty(\mathbb{R})$ equipped with its usual norm, consider the closed convex cone

$$K = \{x \in \ell_\infty(\mathbb{R}) : |x_k| \leq x_1 \ \forall k \geq 2\}.$$

Clearly $K \cap S_X = \{x \in \ell_\infty(\mathbb{R}) : x_1 = 1,\ |x_k| \leq 1 \ \forall k \geq 2\}$. Any $x \in B_X$ can be represented in the form

$$x = \alpha\, \underbrace{(1, x_2, x_3, \ldots)}_{\text{in } K\cap\in S_X} + \beta\, \underbrace{(-1, x_2, x_3, \ldots)}_{\text{in } -K\cap\in S_X}$$

with $\alpha = (1 + x_1)/2$ and $\beta = (1 - x_1)/2$ being nonnegative coefficients adding up to 1. This proves the inclusion $B_X \subset K^\bullet$ and yields $\mu(K) = 1$. On the other hand, $K \cup -K \neq \ell_\infty(\mathbb{R})$ because the bounded sequence $(0, 1, 1, \ldots)$ is neither in K nor in $-K$.

2.5 The Radius of Modulability

For closed convex cones in reflexive Banach spaces there are also other ways of quantifying modulability. As an alternative to the coefficient $\mu(K)$ one might consider

$$\rho_{\mathrm{mod}}(K) = \inf_{\substack{Q\in\Xi(X)\\ Q \text{ not modulable}}} \varrho(K, Q), \tag{30}$$

a number called the *radius of modulability* of K. The interpretation of the above minimization problem is clear: we are looking for the non-modulable element of $\Xi(X)$ lying at shortest distance from K.

What motivates the use of (30) as tool for quantifying modulability is the following topological result.

Proposition 7. *Let X be a reflexive Banach space. Then,*

$$\mathrm{Mod}(X) = \{K \in \Xi(X) : K \text{ is modulable}\}$$

is an open set in the metric space $(\Xi(X), \varrho)$.

Proof. Combine Corollary 2 and Theorem 3(c). □

The link between the functions $\mu(\cdot)$ and $\rho_{\text{mod}}(\cdot)$ is explained in the next corollary.

Corollary 4. *Let X be a reflexive Banach space. Then,*

(a) $\rho_{\text{mod}} : (\Xi(X), \varrho) \to \mathbb{R}$ is the largest nonexpansive map that vanishes exactly over the non-modulable elements of $\Xi(X)$.

(b) $\mu(K) \leq \rho_{\text{mod}}(K)$ for all $K \in \Xi(X)$.

Proof. Notice that $\rho_{\text{mod}}(\cdot)$ is the distance function to a closed set, namely $\Xi(X) \backslash \text{Mod}(X)$. This fact yields immediately the following two properties:

$$|\rho_{\text{mod}}(K_1) - \rho_{\text{mod}}(K_2)| \leq \varrho(K_1, K_2) \ \ \forall K_1, K_2 \in \Xi(X), \tag{31}$$

$$\rho_{\text{mod}}(K) = 0 \ \text{ iff } \ K \in \Xi(X) \ \text{ is not modulable.} \tag{32}$$

Now, let $\rho : \Xi(X) \to \mathbb{R}$ be another function satisfying the properties (31)-(32). For any $K \in \Xi(X)$, one has

$$\rho(K) \leq \rho(Q) + \varrho(K, Q) \quad \forall Q \in \Xi(X).$$

By taking the infimum with respect to Q in $\Xi(X) \backslash \text{Mod}(X)$ one arrives at $\rho(K) \leq \rho_{\text{mod}}(K)$. This proves the pointwise maximality of $\rho_{\text{mod}}(\cdot)$. The part (b) follows from (a) and Corollary 2. □

3 Solidity

3.1 Modulability versus Solidity

According to a famous result often attributed to M. A. Krasnosel'skii [28], every solid closed convex cone in a Banach space is reproducing. Also observed by Krasnosel'skii is the fact that in an infinite dimensional context, reproducibility doesn't imply solidity.

In this section we elaborate a bit more on this theme. More specifically, we compare the expression $\mu(K)$ and the Frobenius solidity coefficient

$$\varphi(K) = \sup\{r : \ \|z\| = 1, \ r \geq 0, \ z + rB_X \subset K\}. \tag{33}$$

The expression (33) has been extensively studied and used by numerous authors [9, 10, 11, 12, 20], specially in a finite dimensional setting. In this paper we place ourselves in the context of an arbitrary normed space. Directly from its definition, one can see that Frobenius solidity coefficient satisfies

$$0 \leq \varphi(K) \leq 1$$

for every nontrivial convex cone K in any normed space X.

Recall that a convex cone K in a normed space X is said to be *solid* if $\text{int}(K)$ is nonempty. The motivation behind the introduction of (33) is the fact that for a nontrivial convex cone $K \subset X$ one has

$$K \text{ is solid} \quad \Longleftrightarrow \quad \varphi(K) > 0.$$

The next proposition provides a lower bound for $\mu(K)$ in terms of the coefficient $\varphi(K)$.

Proposition 8. *Let K be a nontrivial convex cone in a normed space X. Then,*

$$\frac{\varphi(K)}{1+\varphi(K)} \;\leq\; \mu(K). \tag{34}$$

Proof. If K is not solid, then $\varphi(K)=0$ and (34) holds trivially. Suppose then that K is solid, i.e., one can find a vector $z \in X$ and a positive scalar r such that $z + rB_X \subset K$. There is no loss of generality in taking z of unit length. Observe that any nonzero vector $x \in X$ can be decomposed as difference

$$x = \underbrace{\frac{\|x\|}{2r}\left[z + \frac{r}{\|x\|}x\right]}_{u} \;-\; \underbrace{\frac{\|x\|}{2r}\left[z + \frac{r}{\|x\|}(-x)\right]}_{v}$$

of two vectors u, v lying in K. Incidentally, this proves that every solid convex cone in a normed space is reproducing. For obtaining (34) we estimate the ℓ^1-norm of the decomposition (u,v). By using the triangle inequality on $(X, \|\cdot\|)$ one gets

$$\begin{aligned}
\|u\| + \|v\| &= \frac{\|x\|}{2r}\left\{\left\|z + \frac{r}{\|x\|}x\right\| + \left\|z - \frac{r}{\|x\|}x\right\|\right\} \\
&\leq \frac{\|x\|}{2r}\left\{(1+r)+(1+r)\right\} \\
&= \left(1 + \frac{1}{r}\right)\|x\|.
\end{aligned}$$

We have shown in this way that $\zeta(K) \leq 1 + (1/r)$. We now take r as large as possible. By letting $r \to \varphi(K)$ one arrives at

$$\zeta(K) \leq 1 + \frac{1}{\varphi(K)}\;.$$

Proposition 2 does the rest of the job. □

Is the lower bound (34) optimal or, on the contrary, is there room for improvement? A first answer is this: if we don't have any additional information on the structure of the normed space X, then the lower bound (34) is the best one can get.

Example 6. Consider the normed space X and the convex cone K introduced in Example 2. Consider the vector $z \in X$ defined by $z(t) = 1$ for all $t \in [a,b]$. This vector has unit length and $z + B_X$ is contained in K. Hence $\varphi(K) = 1$. On the other hand, we know already that $\mu(K) = 1/2$. So, for this example, the relation (34) is in fact an equality.

And what happens if the structure of X is somewhat special? Imagine, for instance, that the norm of X derives from an inner product. Is this information of any use? Before answering this question, we start by introducing the following technical definition.

Definition 3. *A normed space X is called polite if for all $z \in S_X$, $r \in]0,1[$ and $x \in rS_X$ there are scalars $\gamma > 0$ and $\alpha > 0$ such that*

$$\|x + \gamma z\| = 1, \tag{35}$$

$$\|\alpha(x + \gamma z) - z\| = r. \tag{36}$$

Regardless of whether X is polite or not, a scalar $\gamma > 0$ satisfying (35) always exists. However, the condition (36) is harder to achieve because it forces the ball B_X to posses some kind of "rotundity"

Lemma 4. *Suppose that X is a pre-Hilbert space, i.e., the norm of X derives from an inner product. Then, X is polite.*

Proof. Let $\langle \cdot, \cdot \rangle$ denote the inner product yielding the norm of X. We take γ as the positive root of the quadratic function

$$\begin{aligned} t \in \mathbb{R} \mapsto \phi(t) &= \|x + tz\|^2 - 1 \\ &= t^2 + 2t\langle x, z\rangle + r^2 - 1. \end{aligned}$$

One gets of course $\gamma = \beta - \langle x, z\rangle$ with $\beta = \sqrt{1 - r^2 + \langle x, z\rangle^2}$. We now look for the roots of the quadratic function

$$\begin{aligned} t \in \mathbb{R} \mapsto \varphi(t) &= \|t(x + \gamma z) - z\|^2 - r^2 \\ &= t^2 - 2t\langle x + \gamma z, z\rangle + 1 - r^2 \\ &= t^2 - 2\beta t + 1 - r^2. \end{aligned}$$

Both roots $\alpha = \beta \pm \langle x, z\rangle$ are positive and solve the equation (36). □

We are ready to state:

Theorem 4. *Let K be a nontrivial convex cone in a polite normed space X. Then,*

$$\varphi(K) \leq \mu(K). \tag{37}$$

Proof. Suppose that $\varphi(K) > 0$, otherwise the result is trivial. Consider a unit vector $z \in X$ and a scalar $\bar{r} \in]0, 1]$ such that $z + \bar{r}B_X \subset K$. Pick up $r < \bar{r}$ and a vector $x \in rS_X$. All we need to do is proving that

$$x \in \mathrm{co}[(K \cap B_X) \cup (-K \cap B_X)]. \tag{38}$$

We start by writing

$$x = \frac{\delta}{\gamma + \delta}(x + \gamma z) + \frac{\gamma}{\gamma + \delta}(x - \delta z),$$

i.e., we express x as a convex combination of $x + \gamma z$ and $x - \delta z$. We choose $\gamma > 0$ and $\delta > 0$ so that

$$\|x + \gamma z\| = 1, \quad \|x - \delta z\| = 1.$$

In order to complete the proof of (38) we must check that

$$x + \gamma z \in K \quad \text{and} \quad x - \delta z \in -K. \tag{39}$$

It is here where the politeness assumption enters into the picture. The politeness of the normed space X ensures the existence of a scalar $\alpha > 0$ such that

$$\alpha(x + \gamma z) \in z + rS_X.$$

But $z + rS_X \subset z + rB_X \subset z + \bar{r}B_X$. Hence, $\alpha(x + \gamma z) \in K$. By dividing by α one gets the first condition in (39). We apply the politeness assumption again but this time with respect to $-x \in rS_X$. We deduce the existence of a scalar $\alpha' > 0$ such that

$$\alpha'(-x + \delta z) \in z + rS_X.$$

A similar argument as before yields $-x + \delta z \in K$, that is, the second condition in (39). □

Remark 3. The space $X = \mathcal{B}([a,b],\mathbb{R})$ equipped with the uniform norm $\|x\| = \sup_{a\le t\le b}|x(t)|$ is not polite. If this space were polite, then we should have obtained the estimate (37) for the convex cone of nonnegative functions, contradicting what we have learned from Example 6.

3.2 The Set of Solid Cones is Open

That solid cones form an open set in the metric space $(\Xi(X),\varrho)$ was established in [18, Corollary 5.2], but this was done only in a finite dimensional context. We now extend such a result to a general normed space X by using a proof which is more elaborate and entirely different.

We start by introducing the gap distance

$$\begin{aligned}\delta(K_1,K_2) &= \max\left\{\sup_{x\in K_1\cap B_X}\mathrm{dist}[x,K_2],\ \sup_{x\in K_2\cap B_X}\mathrm{dist}[x,K_1]\right\}\\ &= \max\left\{\sup_{x\in K_1\cap S_X}\mathrm{dist}[x,K_2],\ \sup_{x\in K_2\cap S_X}\mathrm{dist}[x,K_1]\right\}\end{aligned}\tag{40}$$

between two elements K_1, K_2 in $\Xi(X)$. The function $\delta(\cdot,\cdot)$ satisfies all the axioms of a metric except for the triangular inequality (cf. [6]). Anyway, it is good to know that ϱ and δ induce the same topology on $\Xi(X)$. Not only that, ϱ and δ are equivalent[3] in the sense described below.

Lemma 5. *Let X be a normed space X. Then,*

$$\delta(K_1,K_2)\le\varrho(K_1,K_2)\le 2\delta(K_1,K_2)$$

for all $K_1,K_2\in\Xi(X)$.

Proof. The first inequality is obvious. The second one is a particular case of a more general result by Attouch and Wets [2, Proposition 1.4] on truncated distances between convex sets. We give an independent (and shorter) proof of the second inequality in order to see why the coefficient 2 is showing up. We claim that

$$\mathrm{dist}[x,K\cap B_X]\le 2\,\mathrm{dist}[x,K]\quad\forall x\in B_X.\tag{41}$$

Let $x\in B_X$. For any $\varepsilon>0$, there is a point $x_\varepsilon\in K$ satisfying

$$\|x-x_\varepsilon\|\le\mathrm{dist}[x,K]+\varepsilon.\tag{42}$$

If such x_ε can be found in the unit ball B_X, then one gets not just (41), but also the sharper estimate $\mathrm{dist}[x,K\cap B_X]\le\mathrm{dist}[x,K]$ and, a posteriori, the equality $\mathrm{dist}[x,K\cap B_X]=\mathrm{dist}[x,K]$. By-the-way, this special situation is occurring in a Hilbert space setting because the projection on a closed convex cone is a norm-reducing operation. If x_ε is not in B_X, then the normalized vector $\hat{x}_\varepsilon = x_\varepsilon/\|x_\varepsilon\|$ belongs to $K\cap B_X$ and

$$\mathrm{dist}[x,K\cap B_X]\le\|x-\hat{x}_\varepsilon\|\le\|x-x_\varepsilon\|+\|x_\varepsilon-\hat{x}_\varepsilon\|.\tag{43}$$

But

$$\|x_\varepsilon-\hat{x}_\varepsilon\| = \left(1-\frac{1}{\|x_\varepsilon\|}\right)\|x_\varepsilon\| = \|x_\varepsilon\|-1\le\|x_\varepsilon\|-\|x\|\le\|x_\varepsilon-x\|.\tag{44}$$

Now it is a matter of combining (42), (43), (44), and letting then $\varepsilon\to 0$. ☐

[3] The coefficient 2 appearing in Lemma 5 is not necessarily the best possible constant. Finding the best constant would require a deeper analysis of the geometry of the normed space X. If X is a Hilbert space, then ϱ and δ are not just equivalent, but they are in fact identical.

We continue with two technical lemmas.

Lemma 6. *Let Q be a nonempty convex set in a normed space X. If $u \in Q$, then*

$$\mathrm{dist}[u + \lambda(y - u), Q] \geq \lambda \, \mathrm{dist}[y, Q]$$

for all $y \in X$ and all $\lambda \geq 1$.

Proof. Ab absurdo, suppose that $\mathrm{dist}[v, Q] < \lambda\theta$ with $v = u + \lambda(y - u)$ and $\theta = \mathrm{dist}[y, Q]$. In such a case, there exists $w \in Q$ such that $\|v - w\| < \lambda\theta$. Observe that the vector

$$z = \frac{1}{\lambda}\, w + \left(1 - \frac{1}{\lambda}\right) u$$

belongs to Q because it is a convex combination of two points lying in Q. Note also that

$$\|z - y\| = \left\| y - \frac{1}{\lambda} w - \left(1 - \frac{1}{\lambda}\right) u \right\| = \frac{1}{\lambda}\, \|u + \lambda(y - u) - w\| = \frac{1}{\lambda}\, \|v - w\| < \theta,$$

in contradiction with the definition of θ. $\square$

Lemma 7. *Let K and Q be nontrivial closed convex cones in a normed space X. Let r be a positive scalar and $x \in X$ a unit vector such that $x \notin \mathrm{int}(Q)$ and $x + rB_X \subset K$. Then,*

$$\delta(K, Q) \geq \frac{r}{1 + r}\,.$$

Proof. Since $x \notin \mathrm{int}(Q)$, there exists a sequence $\{x^n\}_{n \geq 1}$ such that $\|x^n - x\| \leq 1/n$ and $x^n \notin Q$ for all $n \in \mathbb{N}$. Let $\theta_n = \mathrm{dist}[x^n, Q]$. Note that $\theta_n > 0$ for all n because Q is closed. For each $k \in \mathbb{N}$, pick up $u^{k,n} \in Q$ such that

$$\left\|x^n - u^{k,n}\right\| \leq \theta_n + \frac{1}{k}\,. \tag{45}$$

We remark, parenthetically, that we do not assume the existence of a vector in Q which realizes the distance from x^n to Q. For $n, k \in \mathbb{N}$ with $n > 1/r$, define

$$\begin{aligned} \lambda_{n,k} &= 1 + \frac{r - \frac{1}{n}}{\theta_n + \frac{1}{k}}\,, \\ v^{n,k} &= u^{n,k} + \lambda_{n,k}(x^n - u^{n,k}). \end{aligned}$$

A direct application of Lemma 6 yields

$$\mathrm{dist}[v^{n,k}, Q] \geq \left[1 + \frac{r - \frac{1}{n}}{\theta_n + \frac{1}{k}}\right] \theta_n. \tag{46}$$

Note that

$$\left\|v^{n,k} - x\right\| \leq \left\|v^{n,k} - x^n\right\| + \|x^n - x\| \leq (\lambda_{n,k} - 1) \left\|x^n - u^{n,k}\right\| + \frac{1}{n}\,.$$

Given (45) and the definition of $\lambda_{n,k}$, one gets

$$\left\|v^{n,k} - x\right\| \leq (\lambda_{n,k} - 1) \left(\theta_n + \frac{1}{k}\right) + \frac{1}{n} = r,$$

that is to say, $v^{n,k} \in x + rB_X \subset K$. On the other hand, $\mathrm{dist}[v^{n,k}, Q] > 0$ because the rightmost expression in (46) is positive. Since Q is a cone, we conclude that $v^{n,k} \neq 0$. Notice that

$$\bar{v}^{n,k} = \frac{v^{n,k}}{\|v^{n,k}\|} \in K \cap S_X,$$

$$\mathrm{dist}[\bar{v}^{n,k}, Q] = \frac{1}{\|v^{n,k}\|}\,\mathrm{dist}[v^{n,k}, Q] \geq \left[1 + \frac{r - \frac{1}{n}}{\theta_n + \frac{1}{k}}\right] \frac{\theta_n}{\|v^{n,k}\|}\,,$$

and

$$\left\|v^{n,k}\right\| \leq \left\|v^{n,k} - x\right\| + \|x\| \leq r + 1.$$

By combining these three conditions and the characterization (40) of δ, one ends up with

$$\delta(K,Q) \geq \mathrm{dist}[\bar{v}^{n,k}, Q] \geq \left[1 + \frac{r - \frac{1}{n}}{\theta_n + \frac{1}{k}}\right] \frac{\theta_n}{r+1}$$

for all $k, n \in \mathbb{N}$ such that $n > 1/r$. By letting first $k \to \infty$, one gets

$$\delta(K,Q) \geq \left[1 + \frac{r - \frac{1}{n}}{\theta_n}\right] \frac{\theta_n}{r+1} \; = \; \frac{\theta_n + r - \frac{1}{n}}{r+1} \; \geq \; \frac{r - \frac{1}{n}}{r+1}\,.$$

By letting now $n \to \infty$ one arrives at the desired conclusion. ☐

We now are ready to establish a robustness result for the concept of solidity. Recall that $\varphi(K)$ stands for the Frobenius solidity coefficient of K.

Theorem 5. *Let K and Q be nontrivial closed convex cones in a normed space X. Suppose that K is solid. If*

$$\delta(K,Q) < \frac{\varphi(K)}{1 + \varphi(K)}\,, \tag{47}$$

then Q is solid as well.

Proof. Pick up $r > 0$ and $x \in S_X$ such that $x + rB_X \subset K$. Such a pair (x, r) exists because K is assumed to be solid. Since the inequality (47) is strict, one can take r close enough to $\varphi(K)$ so that

$$\delta(K,Q) < \frac{r}{1+r}\,. \tag{48}$$

We must prove that $\mathrm{int}(Q)$ is nonempty. In fact, $\mathrm{int}(Q)$ contains the vector x because otherwise Lemma 7 would contradict (48). ☐

Two important conclusions can be drawn from Theorem 5.

Corollary 5. *Let X be a normed space. Then,*

$$\mathrm{Sol}(X) = \{K \in \Xi(X) : K \textit{ is solid}\}$$

is an open set in the metric space $(\Xi(X), \varrho)$.

Proof. Each K belonging to $\mathrm{Sol}(X)$ is the center of a "gap" ball

$$\left\{Q \in \Xi(X) : \delta(K,Q) < \frac{\varphi(K)}{1+\varphi(K)}\right\}$$

that is fully contained in $\mathrm{Sol}(X)$. In view of Lemma 5, the ball

$$\left\{Q \in \Xi(X) : \varrho(K,Q) < \frac{\varphi(K)}{1+\varphi(K)}\right\}$$

is also contained in $\mathrm{Sol}(X)$. □

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Corollary 6. *Let X be a normed space. Define the radius of solidity of $K \in \Xi(X)$ as the number*

$$\rho_{\mathrm{sol}}(K) = \inf_{\substack{Q\in\Xi(X)\\ Q \text{ not solid}}} \varrho(K,Q).$$

Then, $\rho_{\mathrm{sol}} : (\Xi(X),\varrho) \to \mathbb{R}$ is the largest nonexpansive map that vanishes exactly over the nonsolid elements of $\Xi(X)$. Furthermore,

$$\frac{\varphi(K)}{1+\varphi(K)} \leq \rho_{\mathrm{sol}}(K) \leq \rho_{\mathrm{mod}}(K) \qquad \forall K \in \Xi(X). \tag{49}$$

Proof. The first part can be proven as in Corollary 4. The first inequality in (49) is contained implicitly in the proof of Corollary 5, while the second one is a consequence of the inclusion $\mathrm{Sol}(X) \subset \mathrm{Mod}(X)$. □

4 Beyond Pointedness

4.1 From Pointedness to Normality

A convex cone K in a normed space is said to be *pointed* if K doesn't contain a line, i.e., $K \cap -K = \{0\}$. Pointedness is fundamental concept of the theory of convex cones.

When one works in an infinite dimensional context, pointedness needs sometimes to be changed by a stronger assumption. Two alternative concepts emerge as natural substitutes: normality and sharpness. The precise definition of normality slightly differs from one author to another. The definition that we adopt reads as follows.

Definition 4. *A convex cone K in a normed space X is called normal if there is a constant $\beta > 0$ such that*

$$\beta\,(\|u\| + \|v\|) \leq \|u+v\| \quad \text{for all } u,v \in K.$$

One refers to β as a normality constant for K. The term "abnormal" is used to indicate the absence of normality.

Normality is a useful assumption precluding pathological situations. A normal convex cone in an infinite dimensional normed space is not just pointed, but it is a bit more than that. For the sake of completeness we state below several equivalent characterization of normality. Recall that a convex cone K induces in the underlying space a pre-order (i.e., a reflexive and transitive relation) by writing $x_1 \leq_K x_2$ whenever $x_2 - x_1 \in K$.

Proposition 9. *For a convex cone K in a normed space X, the following statements are equivalent:*

(a) K *is normal.*

(b) $(B_X + K) \cap (B_X - K)$ *is a bounded set.*

(c) $(V + K) \cap (V - K)$ *is bounded for all bounded* $V \subset X$.

(d) $(V_1 + K) \cap (V_2 - K)$ *is bounded for all bounded* $V_1, V_2 \subset X$.

(e) there is a scalar $\alpha > 0$ *such that* $x_1 \leq_K x \leq_K x_2$ *implies* $\|x\| \leq \alpha \max\{\|x_1\|, \|x_2\|\}$.

(f) there is a scalar $\gamma > 0$ *such that* $0 \leq_K u \leq_K v$ *implies* $\gamma\|u\| \leq \|v\|$.

Proof. See [1, Lemma 2], [38, Chapter 5.3], [35, Chapter 2.1], and [17, Chapter 1]. $\square$

4.2 Measuring the Degree of Normality of a Convex Cone

According to Definition 4, a convex cone K in a normed space X is normal if and only if the coefficient

$$\beta(K) = \inf_{\substack{u,v \in K \\ (u,v) \neq (0,0)}} \frac{\|u + v\|}{\|u\| + \|v\|} \tag{50}$$

is different from 0. The term $\beta(K)$ is a natural candidate for measuring the degree of normality of K, but there are also other possibilities. As alternative to (50) we propose considering the expression

$$\nu(K) = \sup\{r \geq 0 : \; rK_\bullet \subset B_X\} \tag{51}$$

with $K_\bullet = (B_X + K) \cap (B_X - K)$. Needless to say, definition (51) is directly inspired by Proposition 9(b). Notice incidentally that

$$0 \leq \nu(K) \leq 1$$

for every nontrivial convex cone K in any normed space X.

The following two examples are given for the sake of comparison.

Example 7. Consider the vector space $\mathcal{C}([a, b], \mathbb{R})$ of continuous functions $x : [a, b] \to \mathbb{R}$ equipped with the uniform norm $\|x\| = \max_{a \leq t \leq b} |x(t)|$. The closed convex cone

$$K = \{u \in \mathcal{C}([a, b], \mathbb{R}) : u(t) \geq 0 \; \forall t \in [a, b]\}$$

is normal. In fact, we claim that $\beta(K) = 1/2$. To see this, take any pair of vectors $u, v \in K$ with $(u, v) \neq (0, 0)$. Let $t_1, t_2 \in [a, b]$ be such that $u(t_1) = \|u\|$ and $v(t_2) = \|v\|$. Then,

$$\|u + v\| \geq u(t_1) + v(t_1) \geq \|u\|,$$

$$\|u + v\| \geq u(t_2) + v(t_2) \geq \|v\|.$$

One gets $\|u + v\| \geq (1/2)(\|u\| + \|v\|)$. Thus, $\beta(K) \geq 1/2$. Finally, observe that the bound $1/2$ is attained by choosing u, v in a suitable way, for instance

$$u(t) = \frac{t - a}{b - a}\,, \quad v(t) = \frac{b - t}{b - a}\,.$$

Computing the coefficient $\nu(K)$ is also easy. We claim that $\nu(K) = 1$. For obtaining this estimate, we shall prove the inclusion

$$(B_X + K) \cap (B_X - K) \subset B_X.$$

If $x \in (B_X + K) \cap (B_X - K)$, then it is possible to write

$$x(t) = w(t) + u(t), \tag{52}$$
$$x(t) = z(t) - v(t), \tag{53}$$

with $u, v \in K$ and $w, z \in B_X$. It follows that $-1 \le w(t) \le x(t) \le z(t) \le 1$, from where one gets $x \in B_X$.

Example 8. We now equip the vector space $\mathcal{C}([a,b],\mathbb{R})$ with the L^1- norm $\|x\| = \int_a^b |x(t)|dt$. We consider the same cone K as in Example 7. This time one gets $\beta(K) = 1$ because

$$\|u + v\| = \|u\| + \|v\| \qquad \forall u, v \in K.$$

The computation of $\nu(K)$ is a bit harder. We claim that $\nu(K) = 1/2$. First we show that

$$(B_X + K) \cap (B_X - K) \subset 2B_X. \tag{54}$$

We take a vector $x \in (B_X + K) \cap (B_X - K)$ and decompose it as in (52)-(53). Let

$$T_1 = \{t \in [a,b] : x(t) \ge 0\}, \;\; T_2 = [a,b] \backslash T_1.$$

Observe that

$$\begin{aligned} |x(t)| = x(t) &\le z(t) = |z(t)| \qquad \forall t \in T_1, \\ |x(t)| = -x(t) &\le -w(t) = |w(t)| \qquad \forall t \in T_2. \end{aligned}$$

Hence,

$$\|x\| = \int_{T_1} |x(t)|dt + \int_{T_2} |x(t)|dt \le \int_{T_1} |z(t)|dt + \int_{T_2} |w(t)|dt.$$

This shows that $\|x\| \le \|z\| + \|w\| \le 2$ and completes the proof of (54). We now show that the coefficient 2 on the right-hand side of (54) is the smallest possible. Consider an arbitrary $c \in]a,b[$. We pick up a pair $z, w : [a,b] \to \mathbb{R}$ of continuous functions such that

$$\begin{array}{ccc} z(t) > 0 \quad \forall t \in [a,c[& , & z(t) = 0 \quad \forall t \in [c,b], \\ w(t) < 0 \quad \forall t \in]c,b] & , & w(t) = 0 \quad \forall t \in [a,c], \\ \displaystyle\int_a^c z(t)dt = 1 & , & \displaystyle\int_c^b w(t)dt = -1. \end{array}$$

Now, we take $x = z + w$. Since $z, w \in B_X$, $z \in K$, $w \in -K$, one has $x \in (B_X + K) \cap (B_X - K)$ and

$$\|x\| = \int_a^b |x(t)|\,dt = \int_a^c z(t)\,dt - \int_c^b w(t)\,dt = 2.$$

Remark 4. Two important lessons can be drawn from Examples 7 and 8. On the one hand side, the values of $\beta(K)$ and $\nu(K)$ depend not just on K but also on the intrinsic geometry of the unit ball B_X, i.e., the choice of the norm $\|\cdot\|$ plays an important role in the way one measures the degree of normality of a convex cone. On the other hand, $\beta(K) < \nu(K)$ in Example 7 and $\beta(K) > \nu(K)$ in Example 8, so one cannot always predict which one of these coefficients will be larger.

Sometimes it is convenient to represent $\nu(K)$ in a slightly different form. The following lemma will be useful in the sequel.

Lemma 8. *Let K be a convex cone in a normed space X. Then,*

$$\begin{aligned} \nu(K) &= \max\{r \geq 0 : \, r\, K_\natural \subset B_X\} && (55) \\ &= \sup_{\|y\|_*=1} \frac{1}{\Psi^*_{K_\natural}(y)} && (56) \\ &= \left[\inf_{\|y\|_*=1} \Psi^*_{K_\natural}(y)\right]^{-1} \end{aligned}$$

with $K_\natural = \mathrm{cl}(B_X + K) \cap \mathrm{cl}(B_X - K)$ being a closed convex set containing the ball B_X.

Proof. We claim that the equivalence

$$rK_\natural \subset B_X \quad \Longleftrightarrow \quad r\, K_\bullet \subset B_X$$

holds for any $r \geq 0$. The implication $\Longrightarrow$ is obvious because $K_\bullet \subset K_\natural$. For proving the reverse implication, suppose that $r\, K_\bullet \subset B_X$ and take $x \in rK_\natural$. For any $\varepsilon > 0$, one can write

$$\begin{aligned} x &\in r[B_X + K + \varepsilon B_X], \\ x &\in r[B_X - K + \varepsilon B_X]. \end{aligned}$$

Hence, $x \in r(1+\varepsilon)K_\bullet$. This yields in turn $x \in (1+\varepsilon)B_X$. By passing to the intersection with respect to $\varepsilon > 0$, one ends up with $x \in B_X$. Observe that the maximum in (55) is attained because $\{r \geq 0 : \, r\, K_\natural \subset B_X\}$ is a compact interval. Formula (56) is obtained from (55) and the fact that

$$\begin{aligned} rK_\natural \subset B_X \quad &\Longleftrightarrow \quad r\, \Psi^*_{K_\natural}(y) \leq \|y\|_* \quad \forall y \in X^* \\ &\Longleftrightarrow \quad r \leq \inf_{\|y\|_*=1} \frac{1}{\Psi^*_{K_\natural}(y)}\, . \end{aligned}$$

Division by $\Psi^*_{K_\natural}(y)$ causes no troubles because the support function $\Psi^*_{K_\natural}(\cdot)$ never vanishes over the unit sphere S_{X^*} (recall that $K_\natural$ contains the ball B_X). ☐

Remark 5. The closure operation appearing in the definition of $K_\natural$ is superfluous when X is a reflexive Banach space and the convex cone K is closed. In such a particular setting, $K_\natural = K_\bullet$ and the first part of Lemma 8 reduces to saying that the supremum in (51) is attained.

Corollary 7. *Let K be a convex cone in a normed space X. Then,*

$$\nu(\mathrm{cl}(K)) = \nu(K).$$

In particular, $\mathrm{cl}(K)$ *is normal if and only if K is normal.*

Proof. Combine the representation formula (55) and the general equality $[\mathrm{cl}(K)]_\natural = K_\natural$. The second part of the corollary is known [38, Section 5.3.1]. ☐

4.3 Duality Between Normality and Modulability

The link between normality and modulability is well understood. The situation is summarized in the next proposition. As usual, the notation

$$K^+ = \{y \in X^* : \langle y, x\rangle \geq 0 \;\forall x \in K\}$$

stands for the dual cone of K.

Proposition 10. *For a closed convex cone K in a reflexive Banach*[4] *space X, one has:*

(a) K is modulable if and only if K^+ is normal.

(b) K is normal if and only if K^+ is modulable.

It is not simple to single out the first historically documented evidence for this nice result. Anyway, an appropriate reference for Proposition 10 is Grosberg and Krein [15]; see also Ando [1], Kist [27], Krein [30], Schaefer [38, Chapter 5.3], and Weston [42]. A quantitative version of Proposition 10 reads as follows:

Theorem 6. *For a closed convex cone K in a reflexive Banach space X, one has*

$$\mu(K) = \nu(K^+) \quad \text{and} \quad \nu(K) = \mu(K^+). \tag{57}$$

Proof. The proof of this theorem relies on the use of the polarity operator

$$C \mapsto \mathrm{pol}(C) = \{y \in X^* : \Psi_C^*(y) \leq 1\}.$$

Let us prove first the inequality

$$\mu(K) \leq \nu(K^+). \tag{58}$$

Suppose that K is modulable, otherwise we are done. Consider a positive r such that

$$rB_X \subset K^\bullet. \tag{59}$$

This inclusion is reversed by taking the polar set on each side, i.e.,

$$\mathrm{pol}(K^\bullet) \subset \mathrm{pol}(rB_X).$$

But $\mathrm{pol}(rB_X) = (1/r)B_{X^*}$. On the other hand, by applying standard calculus rules for computing polar sets in a reflexive Banach space, one gets

$$\begin{aligned}
\mathrm{pol}(K^\bullet) &= \mathrm{pol}(\mathrm{co}\,[(K \cap B_X) \cup (-K \cap B_X)]) \\
&= \mathrm{pol}(K \cap B_X) \cap \mathrm{pol}(-K \cap B_X) \\
&= (B_{X^*} - K^+) \cap (B_{X^*} + K^+) \\
&= (K^+)_\bullet.
\end{aligned}$$

In short, starting from (59) one arrives at $r(K^+)_\bullet \subset B_{X^*}$. Besides trivial details that we are omitting, this is in essence the proof of (58). The inequality

$$\nu(K) \leq \mu(K^+)$$

[4] Whenever dualization is concerned, we automatically assume that X is a reflexive Banach space. This simplificatory assumption is not always needed, but it greatly helps the smooth flow of the presentation.

can be proven by following a similar pattern. Assume that K is normal, otherwise there is nothing to prove. We start with the relation $r\, K_\bullet \subset B_X$, we divide on both sides by r, and then we pass to the polars. The key observation now is that

$$\begin{aligned}\mathrm{pol}(K_\bullet) &= \mathrm{pol}[(B_X + K) \cap (B_X - K)] \\ &= \mathrm{co}\,[\mathrm{pol}(B_X + K) \cup \mathrm{pol}(B_X - K)] \\ &= \mathrm{co}\left[(-K^+ \cap B_{X^*}) \cup (K^+ \cap B_{X^*})\right] \\ &= (K^+)^\bullet.\end{aligned}$$

Finally, that X is a reflexive Banach space implies that the dual cone of K^+ can be identified with K (cf. [3, Theorem 2.4.3]). So, one obtains

$$\nu(K^+) \leq \mu((K^+)^+) = \mu(K),$$
$$\mu(K^+) \leq \nu((K^+)^+) = \nu(K),$$

completing in this way the proof of (57). □

Theorem 6 is not entirely new. In fact, a similar result has been established by Ng [34], see also [14, Lemma 2.1]. We have included the proof of Theorem 6 only for the sake of completeness.

5 Normality versus Sharpness

What does sharpness mean? We reserve this term to a property that can be seen as dual to solidity.

Definition 5. *A convex cone K in a normed space X is said to be sharp if there is a nonzero vector $y \in X^*$ such that $\|x\| \leq \langle y, x\rangle$ for all $x \in K$.*

This notion of sharpness can be found in numerous references but sometimes under a different name, see for instance [17] and [29]. It is clear that sharpness implies normality but the reverse implication is not true.

Proposition 11. *For a nontrivial closed convex cone K in a reflexive Banach space X, one has:*

(a) K is sharp if and only if K^+ is solid.

(b) K is solid if and only if K^+ is sharp.

The above duality result is formulated in a slightly different wording by Han [16, Theorem 2.4]. Reflexivity of the Banach space X is required to make sure that the dual of K^+ can be identified with K. Reflexivity is an essential assumption for the "if" part of Proposition 11(b). Indeed, Qiu [36] constructed an example of a non-solid closed convex cone K in a non-reflexive Banach space X whose dual K^+ is sharp.

In what follows we refer to the set

$$\Phi(c, y) = \{x \in X : \ c\, \|x\| \leq \langle y, x\rangle\} \tag{60}$$

as the revolution-like cone with parameters $c \in \mathbb{R}_+$ and $y \in S_{X^*}$. If X is a pre-Hilbert space, then

$$\mathrm{rev}(\theta, y) = \Phi(\cos\theta, y)$$

is a genuine revolution cone: the ray $\mathbb{R}_+ y$ corresponds to the axis of revolution and $\theta \in [0, \pi/2]$ is the angle of revolution (or half-aperture angle according to Goffin's terminology [13]).

The closed convex cone (60) is sharp if and only if the parameter c is different from 0. In fact, one has the following result.

Proposition 12. *For a convex cone K in a normed space X, the following conditions are equivalent:*

(a) K is sharp.

(b) there are $c > 0$ and $y \in S_{X^*}$ such that $K \subset \Phi(c, y)$.

Proof. It is straightforward. □

Inspired by Proposition 12(b), it is natural to introduce

$$\tau(K) = \sup_{\substack{(c,y)\in\mathbb{R}_+\times S_{X^*} \\ K\subset\Phi(c,y)}} c \tag{61}$$

and consider this coefficient as a tool for measuring the degree of sharpness of K. The term (61) can be represented in manifold ways. One clearly has

$$\begin{aligned} \tau(K) &= \sup_{\|y\|_*=1} \sup_{\substack{c\geq 0 \\ K\subset\Phi(c,y)}} c \\ &= \sup_{\|y\|_*=1} \inf_{x\in K\cap S_X} \langle y, x\rangle \\ &= \sup_{\|y\|_*=1} \inf_{x\in \mathrm{clco}(K\cap S_X)} \langle y, x\rangle. \end{aligned} \tag{62}$$

The next theorem is obtained by trying to exchange the order of the supremum and the infimum in the expression (62).

Theorem 7. *Let K be a nontrivial closed convex cone in a reflexive Banach space X. Then,*

$$\tau(K) = \mathrm{dist}[0, \mathrm{co}(K\cap S_X)]. \tag{63}$$

In particular, K is sharp if and only if the closed convex hull of $K \cap S_X$ doesn't contain the origin.

Proof. By homogeneity, one can write (62) in the equivalent form

$$\tau(K) = \sup_{y\in B_{X^*}} \inf_{x\in \mathrm{clco}(K\cap S_X)} \langle y, x\rangle.$$

For exchanging the order of the supremum and the infimum we invoke the following three facts: firstly, B_{X^*} is a weakly compact convex set in X^*; secondly, $\mathrm{clco}(K \cap S_X)$ is a closed convex set in X; and, thirdly, the bilinear form $\langle y, x\rangle$ is continuous with respect to the variable $x \in X$, and weakly continuous with respect to the variable $y \in X^*$. Under these circumstances it is possible to apply Sion's minimax theorem [39] and get

$$\tau(K) = \inf_{x\in \mathrm{clco}(K\cap S_X)} \sup_{y\in B_{X^*}} \langle y, x\rangle = \inf_{x\in \mathrm{clco}(K\cap S_X)} \|x\|.$$

For obtaining (63) it suffices to observe that the closure operation can be dropped in the above line. □

It is worthwhile to note that $\tau(K)$ behaves in a Lipschitz-continuous manner with respect to perturbations in the argument K. Indeed, one has:

Proposition 13. *Let X be a normed space. Then,*

$$|\tau(K_1) - \tau(K_2)| \leq \vartheta(K_1, K_2) \qquad \forall K_1, K_2 \in \Xi(X) \tag{64}$$

with $\vartheta(K_1, K_2)$ denoting the Pompeiu-Hausdorff distance between the traces $K_1 \cap S_X$ and $K_2 \cap S_X$.

Proof. By combining Lemma 3 and formula (62), one readily gets

$$|\tau(K_1) - \tau(K_2)| \leq \mathrm{haus}(\mathrm{clco}(K_1 \cap S_X), \mathrm{clco}(K_2 \cap S_X))$$

for all $K_1, K_2 \in \Xi(X)$. For arriving at (64) we just need now to exploit the general inequality

$$\mathrm{haus}(\mathrm{clco}(C_1), \mathrm{clco}(C_2)) \leq \mathrm{haus}(C_1, C_2), \tag{65}$$

which holds for any pair C_1, C_2 of nonempty closed bounded sets in a general normed space X. We omit the proof of (65) because this inequality can be found in the specialized literature concerning the Pompeiu-Hausdorff metric. □

With the help of Proposition 13 one gets the following topological result.

Proposition 14. *Let X be a normed space. Then,*

$$\mathrm{Sh}(X) = \{K \in \Xi(X) : K \textit{ is sharp}\}$$

is an open set in the metric space $(\Xi(X), \varrho)$.

Proof. By proceeding as in Lemma 5, one can show that ϑ is majorized by 2δ, with δ as in (40). Hence,

$$|\tau(K_1) - \tau(K_2)| \leq 2\,\delta(K_1, K_2) \qquad \forall K_1, K_2 \in \Xi(X). \tag{66}$$

Since δ is majorized by ϱ, it follows that $\tau(\cdot)$ is Lipschitz continuous as function on the metric space $(\Xi(X), \varrho)$. This proves the announced result. □

Corollary 8. *Let X be a normed space. Define the radius of sharpness of $K \in \Xi(X)$ as the number*

$$\rho_{\mathrm{sh}}(K) = \inf_{\substack{Q \in \Xi(X) \\ Q \textit{ not sharp}}} \varrho(K, Q).$$

Then, $\rho_{\mathrm{sh}} : (\Xi(X), \varrho) \to \mathbb{R}$ is the largest nonexpansive map that vanishes exactly over the nonsharp elements of $\Xi(X)$. In particular, $\tau(K) \leq 2\,\rho_{\mathrm{sh}}(K)$ for all $K \in \Xi(X)$.

The next theorem corresponds to a quantitative version of Proposition 11.

Theorem 8. *For a nontrivial closed convex cone K in a reflexive Banach space X, one has*

$$\tau(K) = \varphi(K^+) \quad \textit{and} \quad \varphi(K) = \tau(K^+). \tag{67}$$

Proof. Since K is a closed convex cone in a reflexive Banach space, for all $(c, y) \in \mathbb{R}_+ \times S_{X^*}$, one has

$$\begin{aligned} K \subset \Phi(c, y) \quad &\Leftrightarrow \quad c\,\|x\| \leq \langle y, x\rangle \;\; \forall x \in K \\ &\Leftrightarrow \quad y + cB_{X^*} \subset K^+. \end{aligned}$$

This yields the first relation in (67). The second relation is obtained by a simple duality argument. □

Several conclusions can be drawn from Theorem 8. We just mention three of them.

Corollary 9. *Let X be a reflexive Banach space. Then,*

$$|\varphi(K_1) - \varphi(K_2)| \leq 2\,\delta(K_1, K_2) \qquad \forall K_1, K_2 \in \Xi(X) \tag{68}$$

Proof. Take K_1, K_2 in $\Xi(X)$. Similarly as in (66), one can write

$$|\tau(K_1^+) - \tau(K_2^+)| \leq 2\,\delta_*(K_1^+, K_2^+)$$

with $\delta_*(\cdot,\cdot)$ measuring gap distances between elements of $\Xi(X^*)$. It suffices then to combine (67) and the Walkup-Wets Isometry Theorem [41] which asserts that $\delta_*(K_1^+, K_2^+) = \delta(K_1, K_2)$. □

Corollary 10. *Let K be a nontrivial closed convex cone in a reflexive Banach space X. Then,*

$$\frac{\tau(K)}{1+\tau(K)} \;\leq\; \nu(K).$$

Proof. Combine (67) with Proposition 8 and Theorem 6. □

Corollary 11. *Let K be a nontrivial closed convex cone in a reflexive Banach space X whose dual is polite. Then,*

$$\tau(K) \;\leq\; \nu(K).$$

Proof. Combine (67) with Theorems 4 and 6. □

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