THE EDDY VISCOSITY FOR GRAVITY WAVES PROPAGATING OVER TURBULENT SURFACES

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Abstract. This paper analyses (one-dimensional) nonlinear wave propagation over a disordered fluid body having a small viscosity. The lower boundary is disordered and modelled by a random process. As a pulse shaped nonlinear wave propagates over this turbulent boundary, the velocity and wave elevation are viewed as random fields. Starting from first principles the eddy viscosity is characterized and shown to depend on different scales. This is captured as the leading order pseudo-differential operator resulting from the asymptotic analysis of stochastic differential equations. A discussion is provided showing that mean-field theory would have not captured the correct attenuation rate for the large scale object. Numerical results are provided illustrating the accuracy of the eddy viscosity expression.

Key words. Nonlinear waves, inhomogeneous media, viscous shock.

1. Introduction. In this paper we address the (one-dimensional) propagation of nonlinear (free surface) waves over a disordered fluid body having a small viscosity. The lower boundary is a disordered surface modelled by a random process. As a pulse shaped nonlinear wave propagates over this turbulent boundary, the velocity and wave elevation are viewed as random fields. We assume that the typical amplitude of the fluctuations of the (lower) turbulent boundary is small compared to the average thickness of the fluid layer, and that the propagation distance is large. We carry out an asymptotic analysis based on these assumptions. Our goal in this study is to provide a tractable mathematical model that can accurately predict properties of large-scale motion (the wave) in turbulent flows. Keeping this in mind, and starting from first principles, we characterize the flow's *eddy viscosity* due to the presence of a turbulent boundary. This is done by applying an asymptotic stochastic analysis to the Lagrangian formulation of the problem. Namely we start with a one-dimensional, viscous shallow water system which is transformed into a Lagrangian frame by using the Riemann invariants of the underlying inviscid, constant coefficient system. Applying a limit theorem for stochastic differential equations we characterize the flow along the wavefront by a viscous Burgers equation, where its *effective viscosity* automatically incorporates the turbulent (eddy) viscosity. Hence from first principles we construct effective equations that accurately capture, along the large-scale, the effect of the unpredictable fine-scale turbulent features of the flow, for this specific class of problems. A preliminary communication reporting on the eddy viscosity has been published in the Physical Review Letters [14].

It is important to note that we are not performing a mean-field theory. Our asymptotic analysis provides the leading order stochastic dynamics through which a Burgers-type equation can be characterized. Actually the Burgers' diffusive-like term is more general than an effective viscosity, but it reduces to such a term when the turbulent surface's correlation length is much smaller than the pulse width. The general form of the diffusive like term is described through a pseudo-differential operator.

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It is clearly seen through the Fourier representation of this operator that the eddy viscosity, in the more general setting, is scale dependent. This is along the direction pointed out by some recent work such as for example Germano *et al.* [15] and Hughes *et al.* [16].

We also remark that no a priori hypothesis is made on the turbulent viscosity, as for example a stress-strain relation to be satisfied. A nice discussion on all these modelling issues is given in Pope's book [28]. In particular in chapter 10 Pope describes *turbulent-viscosity models* and their underlying hypothesis, while in chapter 12 *PDF (probability density function) methods* are introduced. The turbulent-viscosity models provide strategies for defining, or estimating, the associated eddy viscosity of the flow. We point out that in our study the eddy viscosity arises from the scaling adopted in the analysis of the randomly driven pulse propagation. Moreover, through the limit theorem provided by the probabilistic modelling adopted, we identify a stochastic partial differential equation (SPDE) governing the dynamics. This is conceptually different from the PDF methods in [28] where stochastic differential equations (SDEs), say the Langevin model, are used to model random features of the velocity field through the momentum balance equations. Nevertheless we found it interesting to report how a SDE arises (also) from first principles in this simpler one-dimensional model.

The fact that our flow is one-dimensional is a simplification that, as mentioned above (c.f. also Pope, section, 1.2 [28]), allows for a theoretical study of "tractable mathematical models that can accurately predict properties of turbulent flows". As mentioned in the same section "The large-scale motions are strongly influenced by the geometry of the flow (i.e., by boundary conditions), and they control the transport and mixing. The behaviour of the small-scale motions, on the other hand, is determined almost entirely by the rate at which they receive energy from the large-scales, and by the viscosity. Hence these small-scale motions have a universal character, independent of the flow geometry". This is the spirit of our analysis. The large-scale motion, represented by the pulse shaped wave is strongly influenced by the geometry of the turbulent boundary. Starting from the governing fluid equations we predict a universal character (namely the eddy viscosity) which is independent of the particular small scale features of the turbulent boundary's geometry. We hope that our findings in this particular wave scenario will stimulate further connections and further research from experts in the field.

We give a brief background on the mathematical theory that has been developed and which is along the lines of our study. The propagation of a linear pulse through a random medium has been extensively studied (see for instance the review [3]). In particular the O'Doherty-Anstey theory predicts that if the pulse is observed in a Lagrangian frame that moves with a random velocity, then the pulse appears to retain its shape up to a slow spreading and attenuation [26]. A rather convincing heuristic explanation of this phenomenon is given in [7]. The mathematical treatment of this issue is addressed in [7, 6, 8, 18, 19, 4] and migrated to shallow water waves in [25]. An extension to dispersive water waves is provided in [11, 13, 21, 22]. We have recently extended this theory to inviscid nonlinear waves in [12].

The paper is organized as follows. In Section 2 we introduce the nonlinear shallow water wave model with a random depth together with the corresponding Riemann invariants. In Section 3 we derive the effective viscous Burgers equation governing the evolution of the front pulse. We discuss the properties of this effective equation in Section 4 and identify the eddy viscosity term.

2. Shallow water waves with random depth. We develop an asymptotic probabilistic theory for the viscous shallow water equations in the regime of long waves propagating over a rapidly varying disordered (random) topography. This choice is based on our long experience in this problem [2, 11, 12, 13, 14, 21, 22, 23, 24, 25]. Nevertheless we believe it also applies to other convection-diffusion problems, where waves interact with a turbulent surface (or layer) under time scales such that the wave feels this surface as frozen in time.

The dimensionless shallow water equations are given by [9]

$$\frac{\partial \eta}{\partial t} + \frac{\partial (1+\varepsilon h + \alpha \eta) u}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial \eta}{\partial x} + \alpha u \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2}, \tag{2.2}$$

where η is the free surface elevation, u is the horizontal velocity component and the reference shallow water speed is one. The fluid body is given by $H(x,t) = 1+\varepsilon h(x) + \alpha\eta(x,t)$. The parameter α is the ratio of the typical wave amplitude over the mean depth. It governs the strength of the nonlinearity. The viscosity is given by μ. These two parameters are assumed to be small. The parameter ε is the order of magnitude of the fluctuations of the depth, which are described by the stationary zero-mean random process $h(x)$. The parameter ε is assumed to be small. We shall see that a suitable scaling between ε, α, and μ to exhibit the interplay between these effects is

$$\alpha = \varepsilon^2 \alpha_0, \qquad \mu = \varepsilon^2 \mu_0, \tag{2.3}$$

where α_0 (resp. μ_0) is the normalized nonlinear (resp. viscosity) parameter which is a nonnegative number of order 1. As will be shown below, under this scaling (regime) the eddy viscosity is fully characterized from first principles. Nonlinearity is not as weak as indicated by the scaling (2.3), because we deal with propagation distances and times of order ε^{-2}. As a consequence the cumulative nonlinear effects are of order one and the solution is considered all the way up to the shock formation time. In fact scaling (2.3) has been chosen so that the *nonlinear, viscosity and random effects* become of order 1 for propagation distances of order ε^{-2}. Another problem of interest is to investigate other regimes where a similar analysis can be performed.

The random process h is assumed to be stationary, to possess derivatives, to satisfy the moment conditions $\mathbb{E}[h(0)] = 0$, $\mathbb{E}[h(0)^2] < \infty$, and $\mathbb{E}[(\partial_x h(0))^2] < \infty$. The autocorrelation function

$$\phi_0(x) = \mathbb{E}[h(y)h(y+x)] \tag{2.4}$$

is also assumed to decay fast enough so that it belongs to $L^{1/2}$, i.e. ϕ_0 decays at infinity fast enough to ensure the convergence of the integral $\int_{-\infty}^{\infty} |\phi_0(x)|^{1/2} dx$. In particular this implies ergodicity for the process h. We define the correlation length of the medium as

$$l_c = \frac{\int_0^\infty |\phi_0(x)| dx}{\phi_0(0)}. \tag{2.5}$$

It represents the typical variation length scale of the random surface, namely the topography.

In the sequel we will rewrite our system in a way that a Lagrangian formulation follows directly. Then an asymptotic probabilistic theory (namely a limit theorem) for ordinary differential equations (ODEs) is applied, characterizing the eddy viscosity and the underlying stochastic dynamical model.

We start by introducing the "deterministic" local propagation speed

$$c = \sqrt{1 + \alpha\eta}, \tag{2.6}$$

which does not include the term εh, but it is nevertheless random through the term $\alpha\eta$. We can reformulate the above equations in terms of c and u to obtain

$$\frac{\partial c}{\partial t} + \frac{\alpha}{2} c \frac{\partial u}{\partial x} + \alpha u \frac{\partial c}{\partial x} + \frac{\alpha\varepsilon}{2c}\frac{\partial hu}{\partial x} = 0, \tag{2.7}$$

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \frac{2c}{\alpha}\frac{\partial c}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2}. \tag{2.8}$$

We define the Riemann invariants

$$A(x,t) = \frac{\alpha u - 2c + 2}{\alpha}, \qquad B(x,t) = \frac{\alpha u + 2c - 2}{\alpha}. \tag{2.9}$$

If the viscosity parameter is vanishing ($\mu = 0$) and the bottom is flat ($\varepsilon = 0$), then we get back the standard left- and right-going modes (A and B, respectively) of the hyperbolic system. In presence of nonlinearity, viscosity, and randomness the Riemann invariants satisfy

$$\frac{\partial A}{\partial t} + (-1 + \alpha\frac{3A+B}{4})\frac{\partial A}{\partial x} = \frac{\varepsilon}{2}\frac{\partial h(A+B)}{\partial x}\frac{1}{1+\alpha(B-A)/4} + \frac{\mu}{2}\frac{\partial^2(A+B)}{\partial x^2}, \tag{2.10}$$

$$\frac{\partial B}{\partial t} + (1 + \alpha\frac{A+3B}{4})\frac{\partial B}{\partial x} = -\frac{\varepsilon}{2}\frac{\partial h(A+B)}{\partial x}\frac{1}{1+\alpha(B-A)/4} + \frac{\mu}{2}\frac{\partial^2(A+B)}{\partial x^2}. \tag{2.11}$$

Note that in absence of random perturbations $\varepsilon = 0$ and viscosity $\mu = 0$, the two Riemann invariants are constant along different characteristics. Taking into account the fact that $\alpha = \alpha_0\varepsilon^2$ and $\mu = \varepsilon^2\mu_0$, we can rewrite the evolution equations for A and B in the following way, up to terms of order ε^3:

$$\begin{aligned}
\frac{\partial}{\partial x}\begin{pmatrix} A \\ B \end{pmatrix} &= Q(x)\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix} - \varepsilon\frac{h'}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \\
&+\varepsilon^2\frac{\alpha_0}{4}\begin{pmatrix} 3A+B & 0 \\ 0 & A+3B \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix} \\
&+\varepsilon^2\frac{\mu_0}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\frac{\partial^2}{\partial t^2}\begin{pmatrix} A \\ B \end{pmatrix} + O(\varepsilon^3),
\end{aligned} \tag{2.12}$$

where h' stands for the spatial derivative of h and

$$Q(x) = \frac{1}{1+\varepsilon h}\begin{pmatrix} 1+\varepsilon\frac{h}{2} & \varepsilon\frac{h}{2} \\ -\varepsilon\frac{h}{2} & -1-\varepsilon\frac{h}{2} \end{pmatrix}. \tag{2.13}$$

The system is completed by the initial condition corresponding to a right-going wave incoming from the homogeneous half-space $x < 0$

$$A(x,t) = 0, \quad B(x,t) = f(t-x), \quad t < 0, \tag{2.14}$$

where the function f is compactly supported in $(0, \infty)$.

3. Derivation of the effective equation for the front pulse. In this section we perform a series of transformations to rewrite the evolution equations of the modes by centering along the characteristic of the right-going mode. We shall then obtain a upper-triangular system that can be integrated more easily. In a second step we shall apply a limit theorem to this system to establish an effective nonlinear equation for the front pulse.

The random topography affects the propagation of the Riemann invariants by perturbing their characteristics, so that the matrix Q in Eq. (2.12) is not the identity matrix. Two main effects can be distinguished: the diagonal terms describe random corrections to the local speed, while the off-diagonal parts describe random coupling. Our first goal is to center the propagation equations along the randomly perturbed characteristics. This can be done by computing the eigenvalues and eigenvectors of the matrix Q. The eigenvalues of the matrix $Q(x)$ are $\pm\gamma(x)$ with $\gamma(x) = (1+\varepsilon h)^{-1/2}$. The two eigenvectors form a basis so we introduce the matrix U

$$U = \frac{1}{2} \begin{pmatrix} (1+\varepsilon h)^{1/4} + (1+\varepsilon h)^{-1/4} & (1+\varepsilon h)^{1/4} - (1+\varepsilon h)^{-1/4} \\ (1+\varepsilon h)^{1/4} - (1+\varepsilon h)^{-1/4} & (1+\varepsilon h)^{1/4} + (1+\varepsilon h)^{-1/4} \end{pmatrix} \tag{3.1}$$

which is such that

$$UQU^{-1} = \gamma(x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The propagation equation in this frame exhibits a propagation matrix that is diagonal with x-dependent entries. We push the simplification forward by considering a new spatial variable which is related to the travel time along the characteristics:

$$z(x) = \int_0^x \gamma(x')dx'. \tag{3.2}$$

We now introduce the modified modes

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}(z,t) = U \begin{pmatrix} A \\ B \end{pmatrix}(x(z),t) \exp\left(\varepsilon \frac{h(x(z))}{2}\right). \tag{3.3}$$

Note that $\frac{h'(x(z))}{\gamma(x(z))} = \frac{d}{dz}h(x(z))$. We still denote this quantity by h'. Finally we consider the reference frame that moves with the right-going mode B_1

$$\tau = t - z, \tag{3.4}$$

so that the equation for (A_1, B_1) reads

$$\begin{aligned} \frac{\partial}{\partial z}\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} &= \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \frac{\partial}{\partial \tau} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} - \varepsilon \frac{h'}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \\ &\quad + \varepsilon^2 \frac{\alpha_0}{4} \begin{pmatrix} 3A_1 + B_1 & 0 \\ 0 & A_1 + 3B_1 \end{pmatrix} \frac{\partial}{\partial \tau} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \\ &\quad + \varepsilon^2 \frac{\mu_0}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{\partial^2}{\partial \tau^2} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} + O(\varepsilon^3). \end{aligned} \tag{3.5}$$

Note that the random medium introduces a coupling between the two modes through the term proportional to h' as a consequence of multiple scattering. But these two modes also exchange energy through the nonlinearity and viscosity terms. This equation is very similar to Eq. (2.12), but the propagation matrix $Q(x)$ has been simplified into a matrix with a single non-vanishing (constant) entry. As mentioned earlier, this is the upper-triangular system that can be more easily integrated in an ODE like fashion, along the deterministic characteristics. This will give rise to an ODE, randomly forced by the turbulent surface. The equation for B_1 can be integrated as

$$B_1(z,\tau) = \int_0^z S_B(y,\tau)dy + f(\tau), \tag{3.6}$$

$$\begin{aligned} S_B(y,\tau) = &-\varepsilon\frac{h'(y)}{4}A_1(y,\tau) + \varepsilon^2\frac{\alpha_0}{4}(A_1+3B_1)\frac{\partial B_1}{\partial\tau}(y,\tau) \\ &+\varepsilon^2\frac{\mu_0}{2}\frac{\partial^2(A_1+B_1)}{\partial\tau^2} + O(\varepsilon^3). \end{aligned} \tag{3.7}$$

In this paper we consider large propagation distance z of order ε^{-2}. We shall show that A_1 is of order ε, so that S_B is of order ε^2, and the integral in Eq. (3.6) will turn out to be of order 1.

The equation for A_1 can be integrated as

$$A_1(z,\tau) = -\frac{1}{2}\int_{-\infty}^{\tau} S_A(z+\frac{\tau-s}{2}, s)ds, \tag{3.8}$$

$$\begin{aligned} S_A(z,s) = &-\varepsilon\frac{h'(z)}{4}B_1(z,s) + \varepsilon^2\frac{\alpha_0}{4}(3A_1+B_1)\frac{\partial A_1}{\partial\tau}(z,s) \\ &+\varepsilon^2\frac{\mu_0}{2}\frac{\partial^2(A_1+B_1)}{\partial\tau^2} + O(\varepsilon^3). \end{aligned} \tag{3.9}$$

The integral in Eq. (3.8) seems to have an infinite support $(-\infty,\tau)$. However, we are interested in the front pulse which means that we only consider local times τ lying in some interval $[-T,T]$ with a fixed T of order 1. On the other hand, the contribution of the negative axis can be bounded as we now discuss. Two cases can be distinguished.

Inviscid case $\mu_0 = 0$. The initial conditions (2.14) impose that A_1 and B_1 are zero for $\tau < 0$ and $z = 0$. The transport equations (3.5) then show that A_1 and B_1 are zero for $\tau < 0$ whatever $z \geq 0$. Thus the integral with respect to s in Eq. (3.8) actually goes from 0 to τ. Furthermore Eq. (3.9) shows that S_A is of order ε. This allows us to claim that

$$\sup_{z\in[0,L/\varepsilon^2],\tau\in[-T,T]} |A_1(z,\tau)| \leq K\varepsilon \text{ and } \sup_{z\in[0,L/\varepsilon^2],\tau\in[-T,T]} |\frac{\partial A_1}{\partial\tau}(z,\tau)| \leq K\varepsilon. \tag{3.10}$$

Viscous case $\mu_0 > 0$. The viscous term in Eq. (3.5) is responsible for an instantaneous pulse spreading. We cannot claim anymore that the integral with respect to s in Eq. (3.8) starts from 0. However we can a priori control the contribution of the negative axis by using estimates of the heat kernel. The viscosity parameter is of order ε^2, so for propagation distances of order ε^{-2} the diffusion induced by the viscosity into the negative axis $\tau < 0$ is at most of order 1, and we get the following estimate for $\tau \leq 0$:

$$\sup_{z\in[0,L/\varepsilon^2]} (A_1^2 + B_1^2)(z,\tau) \leq C_L \exp\left(-\frac{2\tau^2}{\mu_0 L}\right).$$

Substituting this estimate into Eqs. (3.8-3.9) establishes that the inequalities (3.10) still hold true.

We can now substitute the integral representation of A_1 into the one of B_1

$$B_1(z,\tau) = f(\tau) - \frac{\varepsilon^2}{32}\int_0^z h'(y)\int_{-\infty}^{\tau} h'(y+\frac{\tau-s}{2})B_1(y+\frac{\tau-s}{2},s)dsdy$$
$$+\varepsilon^2\frac{3\alpha_0}{4}\int_0^z B_1\frac{\partial B_1}{\partial\tau}(y,\tau)dy + \varepsilon^2\frac{\mu_0}{2}\int_0^z \frac{\partial^2 B_1}{\partial\tau^2}(y,\tau)dy + O(\varepsilon^3(1+z)). \tag{3.11}$$

Note that we have eliminated the terms $\varepsilon^2 A_1\partial_\tau B_1$, $\varepsilon^2 A_1\partial_\tau A_1$, $\varepsilon^2 B_1\partial_\tau A_1$, as they are of order ε^3 and are negligible for propagation distance of order ε^{-2}. We introduce the re-scaled right-going mode

$$B_1^\varepsilon(z,\tau) = B_1(\frac{z}{\varepsilon^2},\tau)$$

which satisfies

$$B_1^\varepsilon(z,\tau) = f(\tau) - \frac{1}{32}\int_0^z h'(\frac{y}{\varepsilon^2})\int_{-\infty}^{\tau} h'(\frac{y}{\varepsilon^2}+\frac{\tau-s}{2})B_1^\varepsilon(y+\varepsilon^2\frac{\tau-s}{2},s)dsdy$$
$$+\frac{3\alpha_0}{4}\int_0^z B_1^\varepsilon\frac{\partial B_1^\varepsilon}{\partial\tau}(y,\tau)dy + \frac{\mu_0}{2}\int_0^z \frac{\partial^2 B_1^\varepsilon}{\partial\tau^2}(y,\tau)dy + O(\varepsilon). \tag{3.12}$$

In a formal way, we can write this equation in the form

$$B_1^\varepsilon(z) = f + \int_0^z F(\frac{y}{\varepsilon^2})B_1^\varepsilon(y)dy + \int_0^z G(B_1^\varepsilon(y))dy, \tag{3.13}$$

where $F(y)$ is a linear random operator acting on functions $b(\tau)$ as

$$[F(y)b](\tau) = -\frac{1}{32}h'(y)\int_{-\infty}^{\tau} h'(y+\frac{\tau-s}{2})b(s)ds.$$

$F(y)$ is random and it possesses nice ergodic properties inherited through h'. Thus an averaging over the fast-varying component of Eq. (3.13) can be applied. The averaging principle is valid in the deterministic case under very general assumptions if the limit $\tilde{F}$ of the average value of F over the interval $[0,y_0]$ exists when y_0 goes to infinity [5]. It turns out that this result is also valid in the random case where $y_0^{-1}\int_0^{y_0} F(y)dy$ converges to $\tilde{F}$ in probability only. The rigorous way makes use of an extended version of Khasminskii's limit theorem for randomly forced ODEs[17]. We show in the Appendix that, in the limit $\varepsilon\to 0$, we get that B_1^ε converges to $\tilde{B}_1$ solution of

$$\tilde{B}_1(z) = f + \int_0^z \tilde{F}\tilde{B}_1(y)dy + \int_0^z G(\tilde{B}_1(y))dy,$$

where $\tilde{F} = \mathbb{E}[F(y)]$, that is to say

$$[\tilde{F}b](\tau) = -\frac{1}{32}\int_{-\infty}^{\tau}\mathbb{E}[h'(y)h'(y+\frac{\tau-s}{2})]b(s)ds.$$

This is the precise point where the stochastic modeling comes into the play by interpreting the disordered boundary component as a microscale random process. The integral equation satisfied by the limiting pulse front $\tilde{B}_1$ reads explicitly as

$$\tilde{B}_1(z,\tau) = f(\tau) - \frac{1}{16}\int_0^z \Lambda\tilde{B}_1(y,\tau)dy + \frac{3\alpha_0}{4}\int_0^z \tilde{B}_1\frac{\partial\tilde{B}_1}{\partial\tau}(y,\tau)dy + \frac{\mu_0}{2}\int_0^z \frac{\partial^2\tilde{B}_1}{\partial\tau^2}(y,\tau)dy, \tag{3.14}$$

where the linear operator Λ is a convolution operator

$$\Lambda B(\tau) = \frac{1}{2}\int_0^\infty \phi_1(\frac{s}{2})B(\tau - s)ds = \left[\frac{1}{2}\phi_1\left(\frac{\cdot}{2}\right)\mathbf{1}_{[0,\infty)}(\cdot)\right] * B(\tau), \tag{3.15}$$

$\phi_1(y) = \mathbb{E}[h'(z)h'(z+y)]$, and $*$ is the standard convolution product. In the Fourier domain

$$\int_{-\infty}^\infty \Lambda B(\tau)e^{i\omega\tau}d\tau = b_1(2\omega)\int_{-\infty}^\infty B(\tau)e^{i\omega\tau}d\tau, \tag{3.16}$$

where b_1 is the Fourier transform of the positive lag part of the autocorrelation function of the random stationary process h'

$$b_1(\omega) = \int_0^\infty \phi_1(\tau)e^{i\omega\tau}d\tau. \tag{3.17}$$

We will now show how to express b_1 in terms of the autocorrelation function of the random stationary process h. Let us denote

$$b_0(\omega) = \int_0^\infty \phi_0(y)e^{i\omega y}dy, \tag{3.18}$$

where ϕ_0 is the autocorrelation function of h. We will show that

$$\phi_1(x) = -\phi_0''(x), \tag{3.19}$$

$$b_1(\omega) = -i\omega\phi_0(0) + \omega^2 b_0(\omega), \tag{3.20}$$

which establishes the desired relationship. On the one hand, we have $\partial_y^2\phi_0(y) = \mathbb{E}[h(z)h''(z+y)]$. On the other hand, ϕ_0 is independent of z by stationarity of the random process h, so $0 = \partial_z\partial_y\phi_0(y) = \mathbb{E}[h(z)h''(z+y)] + \mathbb{E}[h'(z)h'(z+y)]$. As a result we obtain (3.19). Furthermore, by integrating by parts, we get

$$b_1(\omega) = -\int_0^\infty \phi_0''(y)e^{i\omega y}dy = \left[\phi_0'(y)e^{i\omega y}\right]_0^\infty + i\omega\int_0^\infty \phi_0''(y)e^{i\omega y}dy.$$

Since ϕ_0 is even and differentiable, we have $\phi_0'(0) = 0$. As a result the first term of the right-hand side vanishes. Integrating once again by parts

$$b_1(\omega) = i\omega\left[\phi_0(y)e^{i\omega y}\right]_0^\infty + \omega^2\int_0^\infty \phi_0(y)e^{i\omega y}dy,$$

which yields (3.20).

The physical interpretation of results will be readily available below and in the next section. Accordingly Λ can be decomposed into the sum of a transport operator corresponding to the term $-i\omega\phi_0(0)$ in Eq. (3.20), and a pseudo-differential operator corresponding to $\omega^2 b_0(\omega)$. In terms of the true mode B, we have to take care of the change of variable $x \mapsto z(x)$. In the macroscopic scales

$$z(\frac{x}{\varepsilon^2}) = \frac{x}{\varepsilon^2} - \frac{\varepsilon}{2}\int_0^{\frac{x}{\varepsilon^2}} h(x')dx' + \frac{3\varepsilon^2}{8}\int_0^{\frac{x}{\varepsilon^2}} h(x')^2dx' + O(\varepsilon). \tag{3.21}$$

Applying the central limit theorem for the second term of the r.h.s., and the law of large numbers for the third term, we get the convergence result

$$z(\frac{x}{\varepsilon^2}) - \frac{x}{\varepsilon^2} \overset{\varepsilon\to 0}{\longrightarrow} \frac{1}{\sqrt{2}}\sqrt{b_0(0)}W_x + \frac{3}{8}\phi_0(0)x, \tag{3.22}$$

where W_x is a standard Brownian motion.

We can then **summarize** the calculations above by stating the following proposition.

PROPOSITION 3.1. *The front pulse $B^\varepsilon(x,\tau) := B(x/\varepsilon^2, \tau + x/\varepsilon^2)$ converges to $\tilde{B}$ given by*

$$\tilde{B}(x,\tau) = \tilde{B}_0\left(x, \tau - \frac{\sqrt{b_0(0)}}{\sqrt{2}}W_x - \frac{\phi_0(0)}{2}x\right). \tag{3.23}$$

$\tilde{B}_0$ is the solution to the deterministic equation

$$\frac{\partial \tilde{B}_0}{\partial x} = \mathcal{L}\tilde{B}_0 + \frac{3\alpha_0}{4}\tilde{B}_0\frac{\partial \tilde{B}_0}{\partial \tau}, \tag{3.24}$$

$$\tilde{B}_0(0,\tau) = f(\tau), \tag{3.25}$$

where the operator $\mathcal{L}$ can be written explicitly in the Fourier domain as

$$\int_{-\infty}^{\infty} \mathcal{L}B(\tau)e^{i\omega\tau}d\tau = -\left(\frac{\mu_0\omega^2}{2} + \frac{b_0(2\omega)\omega^2}{4}\right)\int_{-\infty}^{\infty} B(\tau)e^{i\omega\tau}d\tau.$$

$\mathcal{L}$ results from the action of the kinematic viscosity μ_0 and the effective pseudo-viscosity originating from the random forcing. The physical description for this operator is given in detail in the next section.

4. Characterization of the eddy viscosity. In this section we analyze the main properties of the effective equation for the front pulse. The important function affecting the dynamics is the Fourier transform $b_0(\omega)$ of the positive lag part of the autocorrelation function of the random fluctuations of the bottom. We have proved that $B(x/\varepsilon^2, x/\varepsilon^2 + \tau)$ converges to $\tilde{B}$ given by (3.23-3.24). The Brownian motion W_x represents the random time shift imposed by the random propagation speed. The effect of the random speed is described through a simple example in the next section. $\mathcal{L}$ is a pseudo-differential operator that models the deterministic pulse deformation. Note that the effective equation for the front pulse depends on the randomness (through the function b_0), on the kinematic viscosity (through μ_0), and on nonlinearity (through the parameter α_0).

The first qualitative property satisfied by the pseudo-differential operator $\mathcal{L}$ is that it preserves the hyperbolic nature of the original equation. Indeed, in the time domain, we can write

$$\mathcal{L}B(\tau) = \left[\frac{1}{8}\phi_0\left(\frac{\tau}{2}\right)\mathbf{1}_{[0,\infty)}(\tau)\right] * \left[\frac{\partial^2 B}{\partial \tau^2}(\tau)\right] = \frac{1}{8}\int_0^\infty \phi_0\left(\frac{s}{2}\right)\frac{\partial^2 B}{\partial \tau^2}(\tau - s)ds$$

The indicator function $\mathbf{1}_{[0,\infty)}$ is essential to interpret correctly the convolution. If τ_0 is a time such that B is vanishing for $\tau < \tau_0$, then $\mathcal{L}B$ is also vanishing for $\tau < \tau_0$. This means that the effective viscosity (to be explicitly highlighted in the following) cannot diffuse the wave in the forward direction (ahead the front), but only in the backward direction (behind the front). This in turn implies that the reduction of the pseudo-differential operator $\mathcal{L}$ to a second-order diffusion operator should be handled with precaution.

The pseudo-spectral operator $\mathcal{L}$ can be divided into two parts

$$\mathcal{L} = \mathcal{L}_r + \mathcal{L}_i, \tag{4.1}$$

$$\int_{-\infty}^{\infty} \mathcal{L}_r B(\tau) e^{i\omega\tau} d\tau = -\frac{[b_r(2\omega)+2\mu_0]\omega^2}{4} \int_{-\infty}^{\infty} B(\tau) e^{i\omega\tau} d\tau, \tag{4.2}$$

$$\int_{-\infty}^{\infty} \mathcal{L}_i B(\tau) e^{i\omega\tau} d\tau = -\frac{i b_i(2\omega)\omega^2}{4} \int_{-\infty}^{\infty} B(\tau) e^{i\omega\tau} d\tau, \tag{4.3}$$

where b_r and b_i are respectively the real and imaginary part of b_0

$$b_r(\omega) = \int_0^\infty \mathbb{E}[h(0)h(x)] \cos(\omega x) dx, \qquad b_i(\omega) = \int_0^\infty \mathbb{E}[h(0)h(x)] \sin(\omega x) dx.$$

By the Wiener-Khintchine theorem [20], b_r is proportional to the power spectral density of the random stationary process h. As a result, b_r is nonnegative which shows that $\mathcal{L}_r$ can be interpreted as an effective diffusion operator. More precisely, for small frequencies, $\mathcal{L}_r$ behaves like a second-order diffusion. Indeed, if $\omega l_c \ll 1$, then $b_r(\omega) \simeq \mu_1$ where $\mu_1 := \int_0^\infty \phi_0(x)dx$, and

$$\mathcal{L}_r \simeq \frac{\mu_1 + 2\mu_0}{4} \frac{\partial^2}{\partial \tau^2}.$$

On the other hand b_r decays to zero for high-frequencies, so that $\mathcal{L}_r$ can be reduced to the kinematic viscosity term. The proof of the decay of b_r is based on Fourier theory: ϕ_0 is assumed to belong to $L^{1/2}$ and it is bounded by the variance $\phi_0(0)$, so it belongs to L^1. As a result the Fourier theory ensures that $b_r(\omega)$ converges to 0 as ω goes to infinity. Actually the decay can be estimated more precisely. Indeed we have assumed that $\mathbb{E}[h'^2(0)] < \infty$, which is equal to $-\phi_0''(0)$. By use of the inverse Fourier transform this shows that $\int \omega^2 b_r(\omega) d\omega < \infty$. Thus $\omega^2 b_r(\omega)$ should decay fast enough as ω goes to infinity to ensure the convergence of this integral. This property that the limiting behavior of the eddy viscosity is similar to the kinematic viscosity is due to our scaling choices. There might be other regimes where the turbulent component does not vanish.

$\mathcal{L}_i$ is an effective dispersion operator, since it preserves the energy. It behaves like a third-order dispersion for small frequencies. Indeed, if $\omega l_c \ll 1$, then $b_i(\omega) \simeq \omega\beta_1$ where $\beta_1 := \int_0^\infty x\phi_0(x)dx$, and

$$\mathcal{L}_i \simeq -\frac{\beta_1}{2} \frac{\partial^3}{\partial \tau^3}.$$

It is interesting to determine which operator, $\mathcal{L}_r$ or $\mathcal{L}_i$, is the most important one. By scaling arguments, we get that $\omega^3\beta_1$ is of the order of $(\omega l_c)\mu_0 l_c^2$ which is smaller than $\mu_0\omega^2$ if $\omega l_c \ll 1$. As a result, the effective dispersion for small-frequencies is usually smaller than the effective diffusion. Furthermore, we usually have $\beta_1 > 0$. This is the case, for instance, for a Gaussian autocorrelation function $\phi_0(x) = \exp(-x^2/x_c^2)$: we then have $l_c = x_c\sqrt{\pi}/2$, $\mu_0 = l_c$, and $\beta_1 = x_c^2/2 = 2l_c^2/\pi$. The fact that $\beta_1 > 0$ shows that the dispersion is reduced compared to the original one: the third-order dispersion coefficient, that is equal to $\beta_0/6$ in absence of randomness, takes the value $\beta_0/6 - \beta_1/2$ in presence of random topography. Note, however, that special configurations can be encountered that do not belong to the general case described above. One interesting case deserves an aparte. Let us consider for a while that the process m is the derivative of a smooth stationary zero-mean random process ν, such as a Gaussian random process with Gaussian autocorrelation function. We then have $\phi_0(u) = -\partial_u^2\mathbb{E}[\nu(0)\nu(u)]$, and $\mu_0 = 0$ while $\beta_1 = -\mathbb{E}[\nu(0)^2] < 0$. This shows that, in this very particular case, the dominant operator is the dispersion operator, and it enhances the original dispersion.

Finally, similarly as b_r, b_i decays to zero for high-frequencies, so that $\mathcal{L}_i$ has no effect on the high-frequency components.

Let us address the case where the power spectral density of the process h can be considered as constant over the spectral range of f: $b_0(\omega) \equiv \mu_1$. This arises if the typical wavelength of the pulse f is larger than the correlation radius of the medium. In such a case the early steps of the effective evolution equation is that of the viscous Burgers equation

$$\frac{\partial \tilde{B}_0}{\partial x} = \mu \frac{\partial^2 \tilde{B}_0}{\partial \tau^2} + \frac{3\alpha_0}{4} \tilde{B}_0 \frac{\partial \tilde{B}_0}{\partial \tau}, \tag{4.4}$$

where $\mu = \mu_0/2 + \mu_1/4$. In this case we have an eddy viscosity [28] which arises from the combination of the kinematic viscosity together with the apparent viscosity. To leading order both components of the eddy viscosity removes energy from the coherent wavefront in a diffusive like manner. We can think of two Gaussian filters: one due to the physical viscosity (say G_0) and the other due to the apparent viscosity (say G_1). The energy filtered by G_0 is lost forever and can not be recovered. As discussed in a recent letter [14] the energy filtered through G_1 corresponds to a conversion of coherent energy transported by the front pulse into incoherent energy contained in the small incoherent wave fluctuations following the front pulse. The energy filtered by G_1 can be recovered along the coherent wavefront by a time-reversal recompression using a time-reversal mirror [14]. This surprising result shows that, although the kinematic and apparent viscosity appear with the same form in Eq. (4.4), they are of very different nature.

However Eq. (4.4) may eventually fail describing the wavefront propagation. Indeed new frequencies are generated by the nonlinear term, that may fall in the tail of the function b_0, and one must consider the complete equation with the pseudo-differential operator $\mathcal{L}$. Note that we cannot consider the true white noise case, because a white noise does not fulfill the smoothness requirement that is necessary to ensure the convergence result. We have skipped some technical details, but the assumptions $\mathbb{E}[h(0)^2] < \infty$ and $\mathbb{E}[h'(0)^2] < \infty$ are important for the proof of the convergence result and they are not fulfilled by the white noise whose variance is infinity. Nevertheless, qualitatively speaking, white noise disorder would affect the entire spectrum of the pulse as opposed to the case discussed here.

In conclusion, regarding the material presented in this section. The important difference for the viscous case is that a shock can no longer form. Hence fluctuations enhance diffusivity rather than postpone the shock formation time as in [12]. There are also technical differences. In particular the a priori estimates [12] were based on hyperbolic arguments (i.e. in the inviscid case no wave can go beyond the front). Here, in the viscous case, they are based on estimates of the heat kernel.

5. Comparisons with other approaches in the linear regime. The derivation of the effective equation for the front pulse is based on an integral representation of the front pulse and the application of the Khaminskii's limit theorem. Our approach gives more precise results than a mean-field theory, as we discuss now in the linear framework. Indeed, if $\alpha_0 = 0$, then the effective dynamics described in Proposition 3.1 can be rewritten in terms of a stochastic partial differential equation (SPDE) for the front pulse that takes into account the deterministic pulse spreading as well

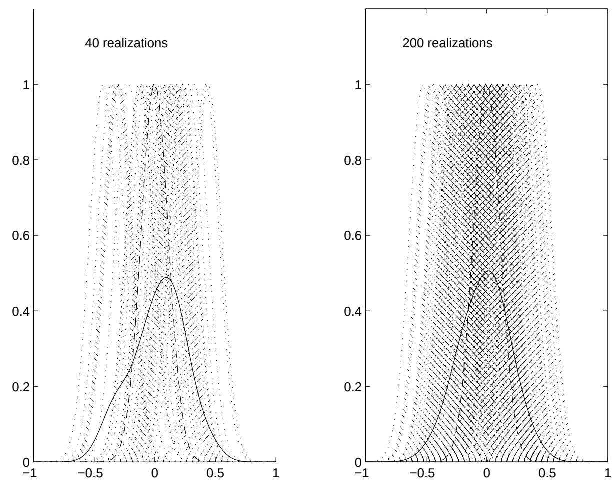

FIG. 5.1. *Averaging over solutions: in dashed line a typical profile centered at the origin; in dotted lines realizations over the solution space; in solid line the mean field profile.*

as the random time shift:

$$d\tilde{B} = \mathcal{L}\tilde{B}dx - \frac{\sqrt{b_0(0)}}{\sqrt{2}}\frac{\partial \tilde{B}}{\partial \tau} \circ dW_x - \frac{\phi_0(0)}{2}\frac{\partial \tilde{B}}{\partial \tau}dx. \tag{5.1}$$

Mathematically speaking the stochastic integral is a Stratonovich integral [1]. In the standard Ito form, this SPDE reads

$$d\tilde{B} = \mathcal{L}\tilde{B}dx + \frac{b_0(0)}{4}\frac{\partial^2 \tilde{B}}{\partial \tau^2}dx - \frac{\sqrt{b_0(0)}}{\sqrt{2}}\frac{\partial \tilde{B}}{\partial \tau}dW_x - \frac{\phi_0(0)}{2}\frac{\partial \tilde{B}}{\partial \tau}dx. \tag{5.2}$$

Consider now the mean field $\tilde{B}_{\rm mf}(x,\tau) = \mathbb{E}[\tilde{B}(x,\tau)]$. Taking the expectation of the SPDE (5.2) establishes the effective equation for the mean field

$$\frac{\partial \tilde{B}_{\rm mf}}{\partial x} = \mathcal{L}_{\rm mf}\tilde{B}_{\rm mf} - \frac{\phi_0(0)}{2}\frac{\partial \tilde{B}_{\rm mf}}{\partial \tau}, \tag{5.3}$$

where the effective viscosity is described by the pseudo-differential operator

$$\mathcal{L}_{\rm mf} = \mathcal{L} + \frac{b_0(0)}{4}\frac{\partial^2}{\partial \tau^2}.$$

Accordingly, a correct mean field approach would lead to the effective equation (5.3) which clearly overestimates the effective viscosity, because the random time shift is averaged out and leads to enhanced apparent diffusion and attenuation.

A very simple example of this fact is shown through figure 5.1. Consider a transport model in which we have travelling waves (say Gaussian wave profiles) propagating with a random speed. Let the speed have a Gaussian distribution. In figure 5.1 (at the left) we present 40 realizations and (to the right) 200 realizations of the solution space. Note that the mean field (average) profile is converging to a highly attenuated Gaussian profile. In this simple example there is no attenuation whatsoever.

The effective equation for the front pulse that we have obtained is in agreement with previous results obtained by several authors in absence of nonlinearity and viscosity [7, 6, 8, 27, 25]. In this linear framework, the usual approach consists in taking a Fourier transform with respect to time, which reduces the partial differential equation (PDE) to a set of ordinary differential equations driven by random coefficients. The solution to each equation can be split into a right- and left-going mode. The joint statistical distribution of the right-going modes for all frequencies can be obtained by use of an approximation-diffusion theorem. The application of an inverse Fourier transform gives an integral representation of the front pulse. The result obtained by this approach exhibits a random time shift and a deterministic spreading of the front pulse, which is in agreement with the result of Proposition 3.1 in the case $\alpha_0 = \mu_0 = 0$. The Fourier approach can still be applied in presence of viscosity, but it turns out to be more tricky in presence of nonlinearity. Our approach makes use of a more complicated approach from the PDE point of view, but it allows us to get the effective dynamics of the front pulse in presence of a weak nonlinearity.

6. Numerical experiments. In this section we present the results of full numerical simulations of the shallow water equations

$$\frac{\partial \eta}{\partial t} + \frac{\partial (1+h+\eta)u}{\partial x} = 0, \tag{6.1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial \eta}{\partial x} + u\frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2}, \tag{6.2}$$

with a random topography $h(x)$. The initial conditions for the wave elevation and velocity field are $\eta(t, x = 0) = f(t)$ and $u(t, x = 0) = f(t)$. These initial conditions for a small-amplitude f give rise to an almost pure right-going wave given by $A(t, x = 0) \simeq 0$ and $B(t, x = 0) \simeq 2f(t)$. In the following we are interested in the transmitted front pulse for the wave elevation $\eta \simeq B/2$.

The random bottom $h(x)$ is modeled as the realization of a stepwise constant process which takes uniformly distributed random values between $-\sigma$ and σ over elementary intervals with length l_c. In the case where l_c is smaller than the typical wavelength of the input pulse, the white noise approximation can be applied and then

$$b_0(\omega) \equiv \frac{l_c \sigma^2}{6}.$$

Note that the stepwise constant model used for the turbulent surface in our simulations is beyond the class of models that satisfy the hypothesis set for the derivation of the results. We made this choice because we believe that the validity of the main statements does not depend on the technical assumptions used to prove the results in this paper.

The numerical simulations are performed with the semi-Lagrangian finite difference scheme used in [12]. We carry out numerical experiments which illustrate that the solution of the shallow water equations (2.1-2.2) is accurately described by the effective equation (3.24) in the regime of small parameters considered in this paper.

We first consider an initial Gaussian pulse and show the pulse shaping in presence of random topography and very weak nonlinearity (figure 6.1). In the experiments the kinematic viscosity is vanishing, so we can observe the effect of the eddy viscosity. The theoretical prediction, based on the ODA theory, is found to be in very good agreement with the results of the numerical simulations.

Second we consider an initial Gaussian pulse in presence of random topography and nonlinearity. We compare in Figure 6.2a the theoretical prediction given by the solution to the effective Burgers equation (4.4) with the numerical simulation, which shows good quantitative agreement. To enhance the role of the eddy viscosity, we

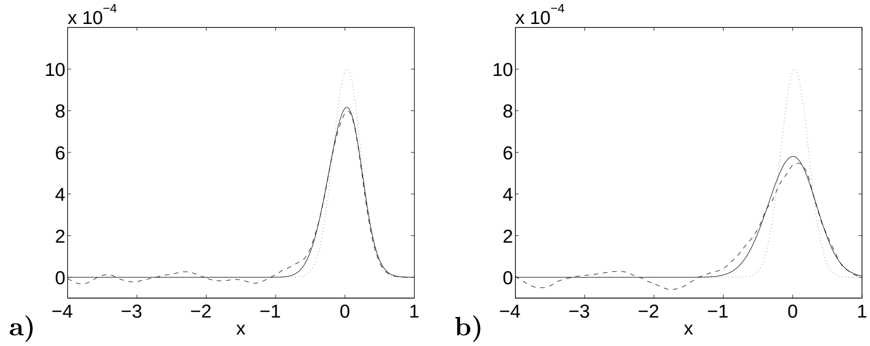

Fig. 6.1. *Transmitted pulse shape in presence of stochastic forcing. The initial pulse is a Gaussian $f(t) = \alpha \exp(-t^2/0.08)$ (dotted line). Here $\mu = 0$, $\alpha = 0.001$, $l_c = 0.1$, $L = 59$, $\sigma = 0.2$ (picture a), and $\sigma = 0.4$ (picture b). The dashed lines represent the numerical solutions shifted for better comparison. The solid lines represent the theoretical front pulses.*

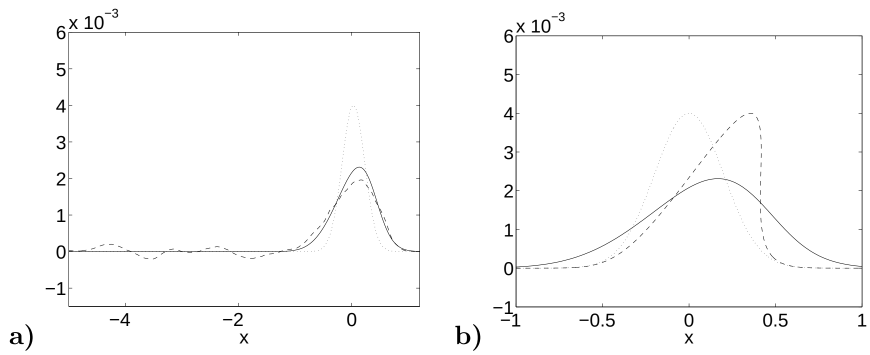

Fig. 6.2. *Picture a: Transmitted pulse shape in presence of stochastic forcing. The initial pulse is a Gaussian $f(t) = \alpha \exp(-t^2/0.08)$ (dotted line). Here $\mu = 0$, $\alpha = 0.004$, $l_c = 0.1$, $L = 59$, and $\sigma = 0.4$. The dashed line represents the numerical solution and the solid line represents the theoretical solution. Picture b: Theoretical transmitted pulse shape in presence (solid line) or in absence (dashed line) of stochastic forcing, where a shock has formed.*

report in figure 6.2b the theoretical pulse shapes in presence or in absence of the random topography. Note that even though α seems to be small, we have chosen the nonlinear parameter and the propagation distance so that the experiment corresponds precisely to the shock propagation distance. The regularizing effect of the noise-induced viscosity is obvious. Note that the theoretical solution to the inviscid Burgers equation is computed by the characteristic method, while the theoretical solution to the viscous Burgers equation (4.4) is computed by the Cole-Hopf transformation [9].

Finally, we consider a different initial profile, namely the derivative of a Gaussian which has a broader spectrum, and plot the result in figure 6.3. This experiment is carried out to confirm that the results can be applied to any type of initial conditions, and not only the usual Gaussian profiles.

7. Conclusion. In this paper we have addressed the propagation of nonlinear water waves over a disordered one-dimensional topography. In the presence of properly scaled stochastic forcing the solution is regularized leading to a viscous shock profile which depends on the degree of nonlinearity and on the power spectral density of the random fluctuations of the bottom. We have actually shown that the transmitted

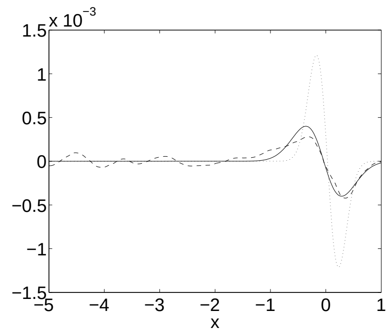

FIG. 6.3. *Transmitted pulse shape in presence of stochastic forcing. The initial pulse is the derivative of a Gaussian* $f(t) = -10\alpha t \exp(-t^2/0.08)$ *(dotted line). Here* $\alpha = 0.004$*,* $l_c = 0.1$*,* $\sigma = 0.4$*, and* $L = 59$*.*

wave is governed by an effective viscous Burgers equation. In deriving this effective equation we were able to characterize, from first principles, the eddy viscosity due to the interaction of a long wave with a turbulent surface. We hope that the results obtained in the specific case of this paper stimulate further investigations.

Appendix A. Technical comments in applying Khasminskii's theorem. Khasminskii's limit theorem is for ordinary differential equations. Typically Fluid Dynamics equations are partial differential equations. In absence of nonlinearity, an efficient and rapid proof consists in taking the Fourier transform in time, which reduces the system to an uncoupled system of ODEs, and the direct application of Khasminskii's theorem gives the desired result. In presence of nonlinearity, however, this strategy cannot be applied, but a proof can be obtained by the introduction of a corrector. This proof requires the technical assumption that the random function h' is bounded and strongly mixing

$$\sup_{A\in\mathcal{F}_{-\infty}^{y}, B\in\mathcal{F}_{y+\tau}^{\infty}} |\mathbb{P}(B|A) - \mathbb{P}(B)| \leq \Phi(\tau)$$

where $\mathcal{F}_y^z$ is the sigma-algebra (or information) generated by $\{h'(\tau), y \leq \tau \leq z\}$, and the mixing function Φ decays fast enough so that $\int \Phi(\tau)^{1/2} d\tau < \infty$.

We denote, for any bounded function $b(\tau)$,

$$[F(y)b](\tau) = \int_{-\infty}^{\tau} h'(y)h'(y + \frac{\tau - s}{3})b(s)ds,$$
$$[\tilde{F}b](\tau) = \int_{-\infty}^{\tau} \phi_1(\frac{\tau - s}{2})b(s)ds,$$

where $\phi_1(\tau) = \mathbb{E}[h'(y)h'(y + \tau)]$.

First step: If $y \leq \xi$ and $\tau \geq 0$, then

$$|\mathbb{E}\left[h'(\xi)h'(\xi + \tau) - \phi_1(\tau)|\mathcal{F}_{-\infty}^{y}\right]| \leq 2\|h'\|_\infty^2 \Phi^{1/2}(\tau)\Phi^{1/2}(\xi - y).$$

This mixing Lemma is essentially proved in [10]. For consistency, we give it here. On the one hand, $h'(\xi)h'(\xi + \tau)$ is $\mathcal{F}_\xi^\infty$-adapted, so the mixing hypothesis implies

$$|\mathbb{E}\left[h'(\xi)h'(\xi + \tau) - \phi_1(\tau)|\mathcal{F}_{-\infty}^{y}\right]| \leq 2\|h'\|_\infty^2 \Phi(\xi - y).$$

On the other hand, we have

$$|\mathbb{E}[h'(\xi+\tau)|\mathcal{F}^{\xi}_{-\infty}]| \le \|h'\|_\infty \Phi(\tau)$$

and

$$\phi_1(\tau) \le \|h'\|^2_\infty \Phi(\tau)$$

so that

$$|\mathbb{E}\left[h'(\xi)h'(\xi+\tau) - \phi_1(\tau)|\mathcal{F}^{y}_{-\infty}\right]| \le 2\|h'\|^2_\infty \Phi(\tau).$$

Combining the two estimates gives the result.

Second step: For $y_0 < y$, we define the corrector

$$[F_1(y_0,y)b](\tau) = -\int_{-\infty}^{\tau} ds \int_y^{L/\varepsilon^2} d\xi \mathbb{E}\left[h'(\xi)h'(\xi+\frac{\tau-s}{2}) - \phi_1(\frac{\tau-s}{2})|\mathcal{F}^{y_0}_{-\infty}\right] b(s).$$

Then $[F_1(y_0,y)b](\tau)$ is bounded uniformly in $y_0, y \in [0, L/\varepsilon^2]$ and τ and it satisfies

$$\mathbb{E}\left[[F(\frac{y_0}{\varepsilon^2},\frac{y}{\varepsilon^2})b](\tau)|\mathcal{F}^{y_0/\varepsilon^2}_{-\infty}\right] = [\tilde{F}b](\tau) + \varepsilon^2 \frac{\partial}{\partial y}[F_1(\frac{y_0}{\varepsilon^2},\frac{y}{\varepsilon^2})b](\tau).$$

The bounded properties of the corrector follows from the mixing lemma:

$$\begin{aligned}
|[F_1(y_0,y)b](\tau)| &\le 2\int_{-\infty}^{\tau} ds \int_y^{L/\varepsilon^2} d\xi \|h'\|^2_\infty \Phi^{1/2}(\frac{\tau-s}{2})\Phi^{1/2}(\xi-y_0) \\
&\le 2\|h'\|^2_\infty \int_0^\infty \Phi^{1/2}(s)ds \int_{y-y_0}^\infty \Phi^{1/2}(\xi)d\xi \le 2\|h'\|^2_\infty \left[\int_0^\infty \Phi^{1/2}(s)ds\right]^2.
\end{aligned}$$

Third step: Let

$$X^\varepsilon(z,\tau) = \int_0^z [F(\frac{y}{\varepsilon^2})b - \tilde{F}b](\tau)dy. \tag{A.1}$$

Then, uniformly in $z \in [0,L]$ and τ,

$$\mathbb{E}[X^\varepsilon(z,\tau)^2] \le C\varepsilon^2. \tag{A.2}$$

We first carry out the following calculation

$$\begin{aligned}
\mathbb{E}[X^\varepsilon(z,\tau)^2] &= 2\int_0^z dy \int_0^y dy_0 \mathbb{E}\left[[F(\frac{y}{\varepsilon^2})b - \tilde{F}b](\tau)[F(\frac{y_0}{\varepsilon^2})b - \tilde{F}b](\tau)\right] \\
&= 2\int_0^z dy_0 \int_{y_0}^z dy \mathbb{E}\left[\mathbb{E}\left[[F(\frac{y}{\varepsilon^2})b - \tilde{F}b](\tau)|\mathcal{F}^{y_0/\varepsilon^2}_{-\infty}\right][F(\frac{y_0}{\varepsilon^2})b - \tilde{F}b](\tau)\right] \\
&= 2\varepsilon^2 \int_0^z dy_0 \mathbb{E}\left[\left([F_1(\frac{y_0}{\varepsilon^2},\frac{z}{\varepsilon^2})b](\tau) - [F_1(\frac{y_0}{\varepsilon^2},\frac{y_0}{\varepsilon^2})b](\tau)\right)[F(\frac{y_0}{\varepsilon^2})b - \tilde{F}b](\tau)\right].
\end{aligned}$$

Using the fact that F_1 is bounded we get the desired result. This proof can be readily extended to the case where b is a function of τ and y independent of ε.

Step 4: We can now consider the convergence of B_1^ε defined by (3.13). We write that

$$B_1^\varepsilon - \tilde{B}_1 = \int_0^z F(\frac{y}{\varepsilon^2})[B_1^\varepsilon - \tilde{B}_1]dy + \int_0^z [F(\frac{y}{\varepsilon^2}) - \tilde{F}]\tilde{B}_1 dy + \int_0^z G(B_1^\varepsilon) - G(\tilde{B}_1)dy.$$

The second term of the r.h.s. is of the form (A.1), so that it can be bounded as in (A.2). We finally use the fact that G is locally Lipschitz and Gronwall Lemma to get the final convergence result.

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