# A NEUMANN-NEUMANN METHOD FOR DG DISCRETIZATION OF ELLIPTIC PROBLEMS

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**Abstract.** A discontinuous Galerkin (DG) discretization of Dirichlet problem for second order elliptic equations with discontinuous coefficients in the 2-D is considered. For this discretization, a Neumann-Neumann (N-N) algorithm is designed and analyzed as an additive Schwarz method (ASM). The coarse spaces is defined usinga special partition of unity. The method is almost optimal under the natural assumption on the triangulation. Its rate of convergence is independent of jumps of coefficients. The method is well suited for parallel computations.

**Key words.** interior penalty discretization, discontinuous Galerkin method, elliptic problems with discontinuous coefficients, finite element method, Neumann-Neumann algorithms, Schwarz methods, preconditioners

**AMS subject classifications.** 65F10, 65N20, 65N30

**1. Introduction.** In this paper, discontinuous Galerkin approximation of elliptic problems with discontinuous coefficients is considered. The problem is considered in a polygonal region $\Omega$ which is a union of disjoint polygonal subregions $\Omega_i$. The discontinuities of the coefficients occur across $\partial\Omega_i$. The problem is approximated by a conforming finite element method (FEM) on matching triangulation in each $\Omega_i$ and nonmatching one across $\partial\Omega_i$. This kind of triangulation and composite discretization are motivated first of all by the regularity of the solution of the problem being discussed. Discrete problems are formulated using DG methods, symmetric and with interior penalty terms on the $\partial\Omega_i$; see [1, 2, 4]. A goal of this paper is to design and analyze Neumann-Neumann (N-N) algorithms for the resulting discrete problem; see [6, 9] and also [10]. The first step, the problem is reduced to the Schur complement problem with respect to unknowns on $\partial\Omega_i$, for $i = 1, \ldots, N$. For that discrete harmonic functions defined in a special way are used. The method is designed and analyzed for the Schur complement problem using the general theory of ASMs; see [6, 10]. The local problems are defined on $\Omega_i$ and faces of $\partial\Omega_j$ which are common to $\Omega_i$. The coarse space is defined using a special partitioning of unity with respect to $\Omega_i$ and introducing master and slave sides of substructures. A side $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$ is master when $\rho_i \geq \rho_j$, otherwise it is slave, so if $F_{ij} \subset \partial\Omega_i$ is master then $F_{ji} \subset \partial\Omega_j$, $F_{ij} = F_{ji}$, is slave. The $h_i-$ and $h_j-$ triangulations on $F_{ij}$ and $F_{ji}$, respectively, are built in a way that $h_i \geq h_j$ if $\rho_i \geq \rho_j$ where $h_i$ and $h_j$ are the parameters of these triangulations. It is proved that the algorithm is almost optimal and its rate of convergence is independent of $h_i$ and $h_j$, the number of subdomains $\Omega_i$ and the jumps of coefficients. The algorithm is well suited for parallel computations and it can be straitforwardly extended to the problems in $3 - D$ cases.

DG methods are becoming more and more popular for approximation of PDEs; see [1, 2] and literature therein. There are also several papers devoted to algorithms for solving the resulting discrete problem, in particular domain decomposition methods. We first mention [7] and [8] where composite discretization similar to those

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discussed in this paper are considered. In these papers overlapping Schwarz and Neumann-Dirichlet methods were proposed and analyzed for DG discretization of elliptic problems with continuous coefficients. In [4] for the considered discrete problem, a multilevel ASM is designed and analized but it is not optimal. In [3] a two-level ASM is proposed and analyzed for DG discretization of fourth order problems. For our knowledge N-N algorithms for DG discretization of elliptic problems with continuous and discontinuous coefficients have not been analyzed in literature.

The paper is organized as follows. In Section 2 the differential problem and its DG discretization are formulated. In Section 3 the Schur complement problem is derived using descrite harmonic function in a special way. Sections 4 and 5 are devoted to designing a N-N algorithm while Section 6 is devoted to the proof of the main result, Theorem 5.2. In Section 7 auxiliary results are proved for the used coarse space.

## 2. Differential and discrete problems.

**2.1. Differential problem.** Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \quad v \in H_0^1(\Omega) \tag{2.1}$$

where

$$a(u, v) = \sum_{i=1}^{N} \int_{\Omega_i} \rho_i \nabla u \nabla v dx, \quad f(v) = \int_{\Omega} f v dx.$$

We assume that $\bar{\Omega} = \cup_{i=1}^{N} \bar{\Omega}_i$ and the substructures $\Omega_i$ are disjoint shaped regular polygonal subregions of diameter $H_i$ and form a geometrical conforming partition of $\Omega$, i.e. $\forall i \neq j$ the intersection $\partial\Omega_i \cap \partial\Omega_j$ is empty or is a common vertex or face of $\partial\Omega_i$ and $\partial\Omega_j$. We assume that $f \in L^2(\Omega)$ and the coefficients $\rho_i$ are constants larger than a positive constant $\rho_0$ what guarantee that the problem is well posed in $H_0^1(\Omega)$.

**2.2. Discrete problem.** Let us introduce the shape regular triangulation in each $\Omega_i$ with triangular elements and $h_i$ as the mesh parameter. The resulting triangulation on $\Omega$ is in general nonmatching across $\partial\Omega_i$. Let $X_i(\Omega_i)$ be a finite element (FE) space of piecewise linear continuous functions $\Omega_i$. Note that we do not assume that functions in $X_i(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$; see Remark 5.4 for others variations. Define

$$X_h(\Omega) = X_1(\Omega_1) \times \cdots \times X_N(\Omega_N).$$

A discrete problem obtained by DG method, see [1, 2, 4], is of the form:

Find $u_h^* \in X_h(\Omega)$ such that

$$a_h(u_h^*, v) = f(v), \quad v \in X_h(\Omega) \tag{2.2}$$

where

$$a_h(u, v) \equiv \sum_{i=1}^{N} a_i(u, v) + s_a(u, v) + s_p(u, v). \tag{2.3}$$

Here for $u = \{u_i\}_{i=1}^N \in X_h(\Omega)$ and $v = \{v_i\}_{i=1}^N \in X_h(\Omega)$

$$a_i(u, v) \equiv \int_{\Omega_i} \rho_i \nabla u_i \nabla v_i dx, \tag{2.4}$$

$$s_a(u,v) \equiv \sum_{i=1}^{N} \sum_{F_{ij}\subset\partial\Omega_i} \frac{1}{l_{ij}} \int_{F_{ij}} \frac{\rho_{ij}}{2}\left(\frac{\partial u_i}{\partial n} + \frac{\partial u_j}{\partial n}\right)(v_j - v_i)ds \tag{2.5}$$

where $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$ is the common face of $\partial\Omega_i$ and $\partial\Omega_j$ and let $l_{ij} = 2$. We also include $F_{i0} = \partial\Omega_i \cap \partial\Omega$ whenever it has positive measure and let $l_{i0} = 1$. The index $l_{ij}$ says how many subdomains shares $F_{ij}$. The $\frac{\partial}{\partial n}$ denotes the normal outward derivative on $\partial\Omega_i$ and $\rho_{ij} = 2\rho_i\rho_j/(\rho_i + \rho_j)$ the harmonic average of $\rho_i$ and $\rho_j$ when $j \neq 0$ and $\rho_{i0} = \rho_i$. To make the notation even more compact, when $j = 0$ we take $u_j = 0$ and $v_j = 0$, and $\frac{\partial u_j}{\partial n} = \frac{\partial u_i}{\partial n}$ and $\frac{\partial v_j}{\partial n} = \frac{\partial v_i}{\partial n}$. We note that when $\rho_{ij}$ is given by harmonic average, $\min\{\rho_i, \rho_j\} \le \rho_{ij} \le \max\{\rho_i, \rho_j\}$, and also $\rho_{ij} \le 2\rho_i$ and $\rho_{ij} \le 2\rho_j$

The bilinear form $s_a(.,.)$ also can be written as

$$s_a(u,v) \equiv \sum_{i=1}^{N} \frac{\rho_{ij}}{l_{ij}} \sum_{F_{ij}\subset\partial\Omega_i} \left\{\int_{F_{ij}} \frac{\partial u_i}{\partial n}(v_j - v_i)ds + \int_{F_{ij}} \frac{\partial v_i}{\partial n}(u_j - u_i)ds\right\} \tag{2.6}$$

where the term $\frac{1}{4}(\frac{\partial u_j}{\partial n}, u_j - u_i)_{L^2(F_{ij})}$ on $F_{ij} \subset \partial\Omega_j$ has been added ($F_{ji} = F_{ij}$) to $\frac{1}{4}(-\frac{\partial u_j}{\partial n}, v_i - v_j)_{L^2(F_{ji})}$ and replacing $\frac{\partial}{\partial n}$ to $\partial\Omega_i$ by $-\frac{\partial}{\partial n}$ to $\partial\Omega_j$. In a similar way we proceed with the term $\frac{1}{4}(\frac{\partial v_j}{\partial n}, u_j - u_i)_{L^2(F_{ji})}$. The penalty term is given as

$$s_p(u,v) \equiv \sum_{i=1}^{N} \sum_{F_{ij}\subset\partial\Omega_i} \int_{F_{ij}} \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_i}(u_j - u_i)(v_j - v_i)ds \tag{2.7}$$

where $\delta$ is a penalty positive parameter. It is known that there is $\delta_0 = O(1) > 0$ such that for $\delta \ge \delta_0$ the problem (2.2) has a unique solution. An error bound of the method is optimal for $\rho_i = 1$, see [1, 2], but is not for discontinuous coefficients, see [4]. In the later case the error is $O(h^{3/2})$ only in the $H^1-$ broken norm if the solution of (2.1) $u^* \in H^{3/2+\epsilon}(\Omega)$, with $\epsilon > 0$. On the other hand we cannot expect more regularity of $u^*$ in the case of discontinuous coefficients in a general case. In the discretization of $s_a(.,.)$ and $s_p(.,.)$, see (2.6) and (2.7), we use the harmonic average of $\rho_i$ and $\rho_j$, $\rho_{ij} = 2\rho_i\rho_j/(\rho_i + \rho_j)$ instead of $\rho_i$ and $\rho_j$. In the case of jump on the coefficients across interfaces, it is a natural way of using it.

We introduce the so-called broken norm in $X_h(\Omega)$ with weights given by $\rho_i$ and $\rho_{ij}$. For $u = \{u_i\} \in X_h(\Omega)$ define

$$\| u \|_{1,h}^2 \equiv \sum_{i=1}^{N} \{\rho_i \| \nabla u_i \|_{L^2(\Omega_i)}^2 + \sum_{F_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_i} \int_{F_{ij}} (u_i - u_j)^2 ds\}. \tag{2.8}$$

LEMMA 2.1. *There exists $\delta_0 > 0$ such that for $\delta \ge \delta_0$ and $u \in X_h(\Omega)$*

$$\gamma \| u \|_{1,h}^2 \le a_h(u,u) \le M \| u \|_{1,h}^2 \tag{2.9}$$

*where $\gamma$ and $M$ are positive constants independent of the $\rho_i$, $h_i$ and $H_i$.*

For the proof see, for example [4].

**3. Schur complement problem.** In this section we derive a Schur complement problem for the problem (2.2). We first introduce some auxiliary notations.

Let $u = \{u_i\} \in X_h(\Omega)$ be given. We can represent $u_i$ as

$$u_i = \mathcal{H}_i u_i + P_i u_i \tag{3.1}$$

where $\mathcal{H}_i u_i$ is the discrete harmonic part of $u_i$ in the sense of $a_i(.,.)$, see (2.4), i.e.

$$a_i(\mathcal{H}_i u_i, v_i) = 0 \qquad v_i \in \overset{o}{X}_i(\Omega_i) \tag{3.2}$$

$$\mathcal{H}_i u_i = u_i \quad \text{on} \quad \partial\Omega_i, \tag{3.3}$$

while $P_i u_i$ is the projection of $u_i$ on $\overset{o}{X}_i(\Omega_i)$ in the sense of $a_i(.,.)$, i.e.

$$a_i(P_i u_i, v_i) = a_i(u_i, v_i), \quad v_i \in \overset{o}{X}_i(\Omega_i). \tag{3.4}$$

Here $\overset{o}{X}_i(\Omega_i)$ is a subspace of $X_i(\Omega_i)$ of functions which vanish on $\partial\Omega_i$, and $\mathcal{H}_i u_i$ is the classical discrete harmonic part of $u_i$. Let us denote $\overset{o}{X}_h(\Omega) \equiv \{\overset{o}{X}_i(\Omega_i)\}_{i=1}^N$ to be a subspace of $X_h(\Omega)$ and consider the global projections $\mathcal{H}u \equiv \{\mathcal{H}_i u_i\}_{i=1}^N$ and $Pu \equiv \{P_i u_i\}_{i=1}^N : X_h(\Omega) \rightarrow \overset{o}{X}_h(\Omega)$ in the sense of $\sum_{i=1}^N a_i(.,.)$. A function $u \in X_h(\Omega)$ can therefore be decomposed as

$$u = \mathcal{H}u + Pu. \tag{3.5}$$

The function $u \in X_h(\Omega)$ can also be represented as

$$u = \hat{\mathcal{H}}u + \hat{P}u \tag{3.6}$$

where $\hat{P}u = \{\hat{P}_i u_i\}_{i=1}^N : X_h(\Omega) \rightarrow \overset{o}{X}_h(\Omega)$ is the projection in the sense of $a_h(.,.)$, the original bilinear form of (2.2), see (2.3). Since $\hat{P}_i u_i \in \overset{o}{X}_i(\Omega_i)$ and $v_i \in \overset{o}{X}_i(\Omega_i)$, we have

$$a_i(\hat{P}_i u, v_i) = a_h(u, v_i).$$

The discrete solution of (2.2) can be decomposed as $u_h^* = \hat{\mathcal{H}}u_h^* + \hat{P}u_h^*$. To find $\hat{P}u_h^*$ we need to solve the following set of usual discrete Dirichlet problems:

Find $\hat{P}_i u_h^* \in \overset{o}{X}_i(\Omega)$ such that

$$a_i(\hat{P}_i u_h^*, v_i) = f(v_i), \qquad v_i \in \overset{o}{X}_i(\Omega_i) \tag{3.7}$$

for $i = 1, \cdots, N$. Note that these problems are local and independent, so they can be solved in parallel. This is a precomputational step.

We now formulate the problem for $\hat{\mathcal{H}}u_h^*$. Let $\hat{\mathcal{H}}_i u$ be the discrete harmonic part of $u$ in the sense of $\hat{a}_i(.,.)$, where $\hat{\mathcal{H}}_i u \in X_i(\Omega_i)$ is the solution of

$$\hat{a}_i(\hat{\mathcal{H}}_i u, v_i) = 0 \qquad v_i \in \overset{o}{X}_i(\Omega_i), \tag{3.8}$$

$$u_i \quad \text{on} \quad \partial\Omega_i \quad \text{and} \quad u_j \quad \text{on} \quad F_{ji} \subset \partial\Omega_j \quad \text{are given} \tag{3.9}$$

where $u_j$ are given on $F_{ji} = \partial\Omega_i \cap \partial\Omega_j$ and

$$\hat{a}_i(u_i, v_i) \equiv \rho_i(\nabla u_i, \nabla v_i)_{L^2(\Omega_i)} + \sum_{F_{ij} \subset \partial\Omega_i} \frac{\rho_{ij}}{l_{ij}} (\frac{\partial v_i}{\partial n}, u_j - u_i)_{L^2(F_{ij})}. \tag{3.10}$$

Note that (3.8) - (3.9) has a unique solution. To see this, let us rewrite (3.8) in the form

$$\rho_i(\nabla \hat{\mathcal{H}}_i u, \nabla \varphi_i^k)_{L^2(\Omega_i)} = -\sum_{F_{ij}\subset\partial\Omega_i} \frac{\rho_{ij}}{l_{ij}}(\frac{\partial \varphi_i^k}{\partial n}, u_j - u_i)_{L^2(F_{ij})} \tag{3.11}$$

where $\varphi_i^k$ are nodal basis functions of $\overset{o}{X}_i\ (\Omega_i)$ associated with interior nodal points $x_k$ of the $h_i$-triangulation of $\Omega_i$. Note that $\frac{\partial\varphi_i^k}{\partial n}$ does not vanish on $\partial\Omega_i$ when $x_k$ is an interior node close to $\partial\Omega_i$. We see that $\hat{\mathcal{H}}_i u$ is a special extension into $\Omega_i$ of $u$ given on $\partial\Omega_i$ and on the $F_{ji}$ and therefore it depends on the values of $u_j$ given on $F_{ji} = \partial\Omega_i \cap \partial\Omega_j$, and on $F_{0i}$ we already had assumed $u_j = 0$ for $j = 0$. Note that $\hat{\mathcal{H}}_i u$ is the discrete harmonic except on nodal points close to $\partial\Omega_i$. We will call sometimes $\hat{\mathcal{H}}_i u$ as discrete harmonic in special sense, i.e. in the sense of $\hat{a}_i(.,.)$ or $\hat{\mathcal{H}}_i$. We set that $\hat{\mathcal{H}}u = \{\hat{\mathcal{H}}_i u\}_{i=1}^N \in X_h(\Omega)$.

Note that (3.8) is obtained from

$$a_h(\hat{\mathcal{H}}u, v) = 0 \tag{3.12}$$

for $u \in X_h(\Omega)$ and when taking $v = \{v_i\}_{i=1}^N \in \overset{o}{X}_h(\Omega)$. It is easy to see that $\hat{\mathcal{H}}u = \{\hat{\mathcal{H}}_i u\}_{i=1}^N$ and $\hat{P}u = \{\hat{P}_i u_i\}_{i=1}^N$ are orthogonal in the sense of $a_h(.,.)$, i.e.

$$a_h(\hat{\mathcal{H}}u, \hat{P}v) = 0, \qquad u, v \in X^h(\Omega). \tag{3.13}$$

In addition,

$$\mathcal{H}\hat{\mathcal{H}}u = \mathcal{H}u, \qquad \hat{\mathcal{H}}\mathcal{H}u = \hat{\mathcal{H}}u \tag{3.14}$$

since $\hat{\mathcal{H}}u$ and $\mathcal{H}u$ do not change the values of $u$ on all the nodes on boundaries of the subdomais $\Omega_i$ also denoted by

$$\Gamma = (\cup_i \partial\Omega_{ih_i}). \tag{3.15}$$

We note that definition of $\Gamma$ includes the nodes on both side of $\cup_i\partial\Omega_i$.

We are now in the position to derive a Schur complement problem for (2.2). Let us apply the decomposition (3.6) into (2.2). We get

$$a_h(\hat{\mathcal{H}}u_h^* + \hat{P}u_h^*, \hat{\mathcal{H}}v_h + \hat{P}v_h) = f(\hat{\mathcal{H}}v_h + \hat{P}v_h)$$

or

$$a_h(\hat{\mathcal{H}}u_h^*, \hat{\mathcal{H}}v_h) + 2a_h(\hat{\mathcal{H}}u_h^*, \hat{P}v_h) + a_h(\hat{P}u_h^*, \hat{P}v_h) = f(\hat{\mathcal{H}}v_h) + f(\hat{P}v_h).$$

Using (3.7) and (3.12) we have

$$a_h(\hat{\mathcal{H}}u_h^*, \hat{\mathcal{H}}v_h) = f(\hat{\mathcal{H}}v_h), \qquad v_h \in X_h(\Omega). \tag{3.16}$$

This problem is the Schur complement problem for (2.2). We denote the space $V_h(\Gamma)$ or in short notation $V$ as the set of all functions $v_h$ in $X_h(\Omega)$ such $\hat{P}v_h = 0$, i.e. the space of discrete harmonic functions in the sense of the $\hat{\mathcal{H}}_i$. We rewrite the Schur complement problem as:

Find $u_h^* \in V_h(\Gamma)$ such that

$$s(u_h^*, v_h) = g(v_h), \qquad v_h \in V_h(\Gamma) \tag{3.17}$$

where here and below $u_h^* \equiv \hat{\mathcal{H}} u_h^*$, and

$$s(u_h, v_h) = a_h(\hat{\mathcal{H}} u_h, \hat{\mathcal{H}} v_h), \qquad g(v_h) = f(\hat{\mathcal{H}} v_h). \tag{3.18}$$

This problem has a unique solution.

**4. Technical tools.** A goal is to design and analyze a Neumann-Neumann (N-N) method for solving (3.17). This will be done in next section. We now introduce some notations and facts used for that. Let $u = \{u_i\}_{i=1}^N \in X_h(\Omega)$ and $v = \{v_i\}_{i=1}^N \in X_h(\Omega)$. Let

$$d_i(u_i, v_i) \equiv \rho_i (\nabla u_i, \nabla v_i)_{L^2(\Omega_i)} + \sum_{F_{ij} \subset \partial\Omega_i} \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_i} (u_j - u_i, v_j - v_i)_{L^2(F_{ij})} \tag{4.1}$$

and

$$d_h(u, v) = \sum_{i=1}^N d_i(u, v). \tag{4.2}$$

Note that for $u, v \in \overset{o}{X}_h(\Omega)$

$$d_i(u, v) = a_i(u, v) = \rho_i (\nabla u_i, \nabla v_i)_{L^2(\Omega_i)} \tag{4.3}$$

and for $u \in X_h(\Omega)$

$$C_0 d_h(u, u) \le a_h(u, u) \le C_1 d_h(u, u) \tag{4.4}$$

in view of Lemma 2.1, where $C_0$ and $C_1$ are positive constants independent of $h_i$, $H_i$ and $\rho_i$. The next lemma shows the equivalence between discrete harmonic functions in the sense $\mathcal{H}$ and in the sense $\hat{\mathcal{H}}$, and therefore we can take advantage of all the discrete Sobolev results known for $\mathcal{H}$ discrete harmonic extensions.

LEMMA 4.1. *For $u \in X_h(\Omega)$*

$$d_h(\mathcal{H}u, \mathcal{H}u) \le d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) \le C d_h(\mathcal{H}u, \mathcal{H}u) \tag{4.5}$$

*where $\mathcal{H}u = \{\mathcal{H}_i u_i\}_{i=1}^N$ and $\hat{\mathcal{H}}u = \{\hat{\mathcal{H}}_i u\}_{i=1}^N$ are defined by (3.2) - (3.3) and (3.8) - (3.9) respectively, where $C$ is a positive constant independent of $h_i$, $u$, $\rho_i$ and $H_i$.*

*Proof.* We note that $P$ and $\mathcal{H}$ are projections in the sense of $d_h(.,.)$ while $\hat{P}$ and $\hat{\mathcal{H}}$ are projections in the sense of $a_h(.,.)$. Therefore, the LHS of (4.5) follows from properties of minimum energy of discrete harmonic extensions in the $d_h(.,.)$ sense. To prove RHS of (4.5) note that

$$d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) = d_h(\hat{\mathcal{H}}u, \mathcal{H}\hat{\mathcal{H}}u + P\hat{\mathcal{H}}u) = d_h(\hat{\mathcal{H}}u, \mathcal{H}u)) + d_h(\hat{\mathcal{H}}u, P\hat{\mathcal{H}}u) \tag{4.6}$$

in viev of (3.14). The first term is estimated as

$$d_h(\hat{\mathcal{H}}u, \mathcal{H}u) \le \varepsilon d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) + \frac{1}{4\varepsilon} d_h(\mathcal{H}u, \mathcal{H}u), \tag{4.7}$$

with arbitrary $\varepsilon > 0$. To estimate the second term of RHS of (4.6) note that for $v \equiv P\hat{\mathcal{H}}u \in \overset{o}{X}(\Omega)$ and using (3.11), we get

$$
\begin{aligned}
(4.8) \qquad d_h(\hat{\mathcal{H}}u, v) &= \sum_{i=1}^{N} \rho_i (\nabla \hat{\mathcal{H}}_i u_i, \nabla v_i)_{L^2(\Omega_i)} \\
&= -\sum_{i=1}^{N} \sum_{F_{ij}\subset\partial\Omega_i} \frac{\rho_{ij}}{l_{ij}} (\frac{\partial v_i}{\partial n}, u_j - u_i)_{L^2(F_{ij})}.
\end{aligned}
$$

The terms of RHS of (4.8) are estimated as

$$
\begin{aligned}
&|\rho_{ij}(\frac{\partial v_i}{\partial n}, u_j - u_i)_{L^2(F_{ij})}| \leq \ \rho_{ij} \parallel \frac{\partial v_i}{\partial n} \parallel_{L^2(F_{ij})} \parallel u_i - u_j \parallel_{L^2(F_{ij})} \\
&\leq C \frac{\rho_{ij}}{h_i^{1/2}} \parallel \nabla v_i \parallel_{L^2(\Omega_i)} \parallel u_i - u_j \parallel_{L^2(F_{ij})} \\
&\leq C\{\varepsilon \rho_{ij} \parallel \nabla v_i \parallel^2_{L^2(\Omega_i)} + \frac{\rho_{ij}}{4\varepsilon h_i} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})}\} \\
&\leq C\{2\varepsilon \rho_i \parallel \nabla v_i \parallel^2_{L^2(\Omega_i)} + \frac{\rho_{ij}}{4\varepsilon h_i} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})}\},
\end{aligned}
$$

where we have used that $\rho_{ij} \leq 2\rho_i$. Substituting this into (4.8), we get

$$
d_h(\hat{\mathcal{H}}u, v) \leq C \sum_{i=1}^{N} \{2\varepsilon \rho_i \parallel \nabla P_i \hat{\mathcal{H}}_i u_i \parallel^2_{L^2(\Omega_i)} + \frac{\rho_{ij}}{4 h_i \varepsilon} \sum_{F_{ij}\subset\partial\Omega_i} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})}\},
$$

(4.9)
and using

$$
\parallel \nabla P_i \hat{\mathcal{H}}_i u_i \parallel_{L^2(\Omega_i)} \leq \parallel \nabla \hat{\mathcal{H}}_i u_i \parallel_{L^2(\Omega_i)},
$$

we obtain

$$
(4.10) \qquad d_h(\hat{\mathcal{H}}u, v) \leq C\{\varepsilon d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) + \frac{1}{4\varepsilon} d_h(\mathcal{H}u, \mathcal{H}u)\},
$$

and then

$$
d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) \leq C\{\varepsilon d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) + \frac{1}{4\varepsilon} d_h(\mathcal{H}u, \mathcal{H}u)\}.
$$

Choosing now $\varepsilon$ sufficiently small, the RHS of (4.5) follows. ◻

**5. Neumann-Neumann method.** We design and analyze Neumann-Neumann (N-N) methods for solving (3.17); see [6, 9]. For that we follow the general framework of ASM; see [10], and stated in the next lemma. The operators $I_i$ and $T$, the bilinear forms $b_i$ and the spaces $V_i$ are defined on the next subsections while the space $V$ and bilinear form $a_h$ are the same as above.

Lemma 5.1. *Suppose the following three assumptions hold:*

*i) There exists a constant $C_0$ such that for all $u \in V$ there exists a decomposition $u = \sum_{i=0}^{N} I_i u_i$, $u_i \in V_i$ with*

$$
\sum_{i=0}^{N} b_i(u_i, u_i) \leq C_0^2 a_h(u, u).
$$

*ii) There exist constants $\epsilon_{i,j}, i,j = 1,\ldots,N$ such that*

$$a_h(I_iu_i, I_ju_j) \le \epsilon_{i,j} a_h(I_iu_i, I_iu_j)^{1/2} a_h(I_ju_j, I_ju_j)^{1/2}, \quad \forall u_i \in V_i \quad \forall u_j \in V_j.$$

*iii) There exists a constant $\omega$ such that*

$$a_h(I_iu, I_iu) \le \omega b_i(u,u) \quad \forall u \in V_i, \quad i = 0,\ldots,N.$$

*Then, $T$ is invertible and*

$$C_0^2 a_h(u,u) \le a_h(Tu,u) \le (\rho(\epsilon)+1)\omega a_h(u,u), \quad \forall u \in V.$$

*Here, $\rho(\epsilon)$ is the spectral radius of the matrix $\epsilon = \{\epsilon\}_{i,j=1}^N$.*

**5.1. Local spaces $V_i$.** Let us denote $V_i(\Gamma_i)$, in short notation $V_i$, as the vector space defined by the nodal values on $\partial\Omega_i$ and by nodal values on the neighboring faces of $\Omega_i$, i.e. on $F_{ji} \subset \partial\Omega_j$, where $F_{ij} = F_{ji} = \partial\Omega_i \cap \partial\Omega_j$. We denote such nodes by $\Gamma_i$. We note that we **do include** the nodal values of $\partial F_{ji}$ (which are vertices of $\Omega_j$) as degrees of freedom of $V_i$. We denote by $u \in V_i$, if $u = \{u_l^{(i)}\}_{l\in\#(i)}$, where $\#(i)$ is the index set composed of $i$ and the $j$ indices where $F_{ij}$ is a face of $\partial\Omega_i$, where the function $u_i^{(i)}$ is $u$ restricted to $\partial\Omega_i$ and the function $u_j^{(i)}$ is $u$ restricted to $F_{ji}$. To simplify notation we also use $u = \{u_l\} \in V_i$ to refer to a function defined on $\Gamma_i$, and $u = \{u_i\} \in V$ to refer to a function defined on all $\Gamma$.

Let us define the regular zero extension operator $\tilde{I}_i : V_i \to V$ as follows: Given $u \in V_i$, let $\tilde{I}_i u$ be equal to $u$ on nodes $\Gamma_i$ and zero on $\Gamma\backslash\Gamma_i$. Then we associate with each $\Omega_k,\ k = 1,\cdots,N$, the discrete harmonic function $u_k$ inside each $\Omega_k$ in the sense of $\hat{\mathcal{H}}_k$, see (3.8) and (3.9).

**5.2. Master and slave sides.** We first classify faces $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$, common to $\Omega_i$ and $\Omega_j$, as master and slave. The face $F_{ij}$ is master if $h_i \ge h_j$ and denoted by $\gamma_{ij}$ and slave if $h_i < h_j$ and denoted by $\delta_{ij}$. Here $h_i$ and $h_j$ are parameters of the $h_i$- and $h_j$- triangulation of $F_{ij} \subset \partial\Omega_i$ and $F_{ji} \subset \partial\Omega_j, F_{ij} = F_{ji}$, respectively. We consider in the analysis $F_{i0}$ as the master side.

**ASSUMPTION M: $h_i \ge h_j$ if and only if $\rho_i \ge \rho_j$**

**5.3. Weighted prolongation operators $I_i$.** We associate with each $\Omega_i,\ i = 1,\cdots,N$, the weighting diagonal matrices $D^{(i)} = \{D_l^{(i)}\}_{l\in\#(i)}$ on $\Gamma_i$ as follows:

- On $\partial\Omega_i$ $(l = i)$

$$D_i^{(i)} = \begin{cases} 1 & x \in \gamma_{ijh_i} \subset \partial\Omega_i, \\ 0 & x \in \delta_{ijh_i} \subset \partial\Omega_i, \\ 1 & x \in \nu_i \subset \partial\Omega_i \end{cases} \tag{5.1}$$

  where $\gamma_{ijh_i}$ and $\delta_{ijh_i}$ are the sets of nodal points of $\gamma_{ij} \subset \partial\Omega_i$ and $\delta_{ij} \subset \partial\Omega_i$, respectively; $\nu_i$ is the set of vertices of $\Omega_i$.

- On $\partial\Omega_j$ $(l = j,\ F_{ij} = F_{ji} = \partial\Omega_i \cap \partial\Omega_j)$

$$D_j^{(i)} = \begin{cases} 0, & x \in \gamma_{jih_j} \subset \partial\Omega_j,\ \gamma_{ji} = \delta_{ij},\ \delta_{ij} \subset \partial\Omega_i, \\ 1, & x \in \delta_{jih_j} \subset \partial\Omega_i, \\ 0, & x \in (\partial\gamma_{jih_j} \cup \partial\delta_{jih_j}) \end{cases} \tag{5.2}$$

  where for $F_{i0} \in \partial\Omega$ we set $D_j^{(i)} = 0$.

The extension operators $I_i : V_i \to V$, $i = 1, \ldots, N$ are defined as

$$I_i = \tilde{I}_i D^{(i)}. \tag{5.3}$$

**5.4. Coarse space and subspace decomposition.** Note that

$$\sum_{i=1}^{N} I_i \tilde{I}_i^T = I_\Gamma \tag{5.4}$$

is a partition of unity on every node of $\Gamma$, where $I_\Gamma$ is the identity operator on $\Gamma$. The coarse space $V_0 \subset V$ and its coarse basis functions $\Theta^{(i)}$ are introduced as follows:

$$V_0 = Span\{\Theta^{(i)}\}_{i=1}^{N}, \tag{5.5}$$

where $\Theta^{(i)} = I_i \Phi^{(i)}$ where $\Phi^{(i)} \in V_i$ is defined as follows: If the substructure $\Omega_i$ does not share a face with the boundary of $\Omega$, i.e. when $\partial\Omega_i \cap \partial\Omega$ has measure zero, then we define $\Phi^{(i)}$ to be equal to one at every node of $\Gamma_i$. Such substructure we denote by $N_I$ substructure. If not, $\Omega_i$ is a $N_B$-substructure and we define $\Phi^{(i)}$ equal to one on the faces $F_{ij}$ and $F_{ji}$ that do not touch $\partial\Omega$, the linear function decreasing from one to zero on the faces $F_{ij}$ and $F_{ji}$ touching $\partial\Omega$, and equal to zero on $F_{i0}$. Because of the linearity, the function $\Phi^{(i)}$ matches across $F_{ij}$ and $F_{ji}$.

Let us denote $I_0 = I_\Gamma$, i.e. the identity operator on $\Gamma$. Hence, $V$ can be decomposed as

$$V = \sum_{i=0}^{N} I_i V_i. \tag{5.6}$$

We now define bilinear form $b_0$ as

$$b_0(u, v) = (1 + \log \frac{H}{h})^{-1} d_h(\mathcal{H}u, \mathcal{H}v), \quad u, v \in V_0. \tag{5.7}$$

REMARK 5.1. *Other choices of coarse problems can be considered. We can replace the bilinear form (5.7) to*

$$b_0(u, v) = (1 + \log \frac{H}{h})^{-1} a_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}v), \quad u, v \in V_0 \tag{5.8}$$

*and the analysis will follow straightforwardly from the analysis for (5.7) and using (4.4) and (4.5).*

*In the case when $\Omega_i$ are triangles, we can replace the space $V_0$ by conforming continuous piecewise linear functions on the coarse triangulation associated to $\Gamma$, i.e. the functions are linear and matching on $F_{ij}$ and $F_{ji}$ and continuous at the vertices of the substructures, and vanishing on $\partial\Omega$. Theorem 5.2, see below, is valid for this variant of method when the coefficient $\rho_i$ are quasimonotonic, see [5]. In the proof $u_0 \in V_0$ is defined by values at common vertices $x_k$ of the substructures which are equal to an algebraic average values of $u_i$ over faces of $\Omega_i$ with $x_k$ as common vertex.*

**5.5. Local bilinear forms.** For $i = 1, \cdots, N$, and for $u = \{u_l\} \in V_i$ and $v = \{v_l\} \in V_i$ define

$$\begin{aligned} b_i(u, v) = &\int_{\Omega_i} \rho_i \nabla \mathcal{H}_i u_i \nabla \mathcal{H}_i v_i dx + \frac{\rho_i}{H_i^2} \int_{\Omega_i} (\mathcal{H}_i u_i)(\mathcal{H}_i v_i) dx + \\ &+ \sum_{F_{ij} \subset \partial\Omega_i} \delta \frac{\rho_{ij}}{h_{ij}} \int_{F_{ij}} (u_j - u_i)(v_j - v_i) ds, \end{aligned} \tag{5.9}$$

where $h_{ij} = 2h_ih_j/(h_i+h_j)$ is the harmonic average of $h_i$ and $h_j$. For a face $F_{i0}$, we let $h_{ij} = h_i$, and again $\rho_{ij} = \rho_i$ and $v_j = u_j = 0$, $v_i - v_j = v_i$ and $u_i - u_j = 0$. Note that $b_i(.,.)$ differs from $d_i(.,.)$ by the $L_2(\Omega_i)$ term and also by the factor multiplying the penalty term, where here we add the factors from neighboring subdomains, see (4.1). The addition of the $L_2(\Omega_i)$ term makes $b_i(u,u)$ a norm also in the case where $\partial\Omega_i$ does not touch the Dirichlet boundary of the original domain $\partial\Omega$, and as a consequence the local problems will be uniquely solvable.

Remark 5.2. *In this paper we also consider the case where*

$$b_i(u,v) = a_i(\mathcal{H}_iu_i, \mathcal{H}_iv_i) + \sum_{F_{ij}\subset\partial\Omega_i} \delta\frac{\rho_{ij}}{h_{ij}}\int_{F_{ij}}(u_j-u_i)(v_j-v_i)ds. \tag{5.10}$$

*To fix the solvability issue of the local problems, where the constant functions might be in the kernel, we replace $V_i$ by the space of functions in $V_i$ with zero average on $\partial\Omega_i$ or in $\Omega_i$. The analysis developed here includes also this case; see Remark 6.1.*

Remark 5.3. *Like in Remark 5.2, a natural question to ask is if we can replace the bilinear form (5.10) to*

$$\begin{aligned} b_i(u,v) &= a_i(\hat{\mathcal{H}}_iu_i, \hat{\mathcal{H}}_iv_i) + \\ &+ \frac{\rho_{ij}}{l_{ij}}\sum_{F_{ij}\subset\partial\Omega_i}\{\int_{F_{ij}}\frac{\partial\hat{\mathcal{H}}_iu_i}{\partial n}(v_j-v_i)ds + \int_{F_{ij}}\frac{\partial\hat{\mathcal{H}}_iv_i}{\partial n}(u_j-u_i)ds\} + \\ &+ \sum_{F_{ij}\subset\partial\Omega_i}\int_{F_{ij}}\frac{\delta}{l_{ij}}\frac{\rho_{ij}}{h_i}(u_j-u_i)(v_j-v_i)ds \end{aligned} \tag{5.11}$$

*and define a version of balancing domain decomposition as in ([9]). The answer is no because we cannot estimate (6.8) with $C$ independent of the ratio $h_i/h_j$. In addition, we cannot replace the last term of (5.11) to*

$$\sum_{F_{ij}\subset\partial\Omega_i}\int_{F_{ij}}\frac{\delta}{2}\frac{\rho_{ij}}{h_{ij}}(u_j-u_i)(v_j-v_i)ds$$

*since the associated bilinear form might not be positive definite whenever $h_i < h_j$, except if we take $\delta \geq 2\delta_0$, where $\delta_0$ is given on Lemma 2.1, since we can use that $h_{ij} \geq h_i$.*

**5.6. Projection-like operators.** For $i = 0, \cdots N$, let $\tilde{T}_i : V \to V_i$ be defined as

$$b_i(\tilde{T}_iu, v) = a_h(u, I_iv), \quad v \in V_i, \tag{5.12}$$

and let $T_i = I_i\tilde{T}_i$. .

**5.7. Preconditioner and main theorem.** Find $u_h^* \in V$ such that

$$Tu_h^* = g_h \tag{5.13}$$

where

$$T = \sum_{i=0}^{N} T_i \tag{5.14}$$

and

$$g_h = \sum_{i=0}^{N} g_i \quad g_i = T_i u_h^*,$$

and $u_h^*$ is the solution of (3.17).

THEOREM 5.2. *Assume that Assumption M holds. Then there exist positive constants $C_0$ and $C_1$ independent of $h_i, H_i$ and the jumps of $\rho_i$ such that*

$$C_0 a_h(u,u) \le a_h(Tu,u) \le C_1(1+\log\frac{H}{h})^2 a_h(u,u) \qquad \forall u \in V. \tag{5.15}$$

*Here* $\log(H/h) = \max_i \ \log(H_i/h_i)$.

REMARK 5.4.

*Two other possible discretizations rather the one defined in Section 2 can be considered: In the first discretization, for the case where $\Omega_i$ is a $N_B$ substructure, we modify the space $X_i(\Omega_i)$ as the discrete functions vanishing on $F_{i0}$. In the second discretization, for the case $\Omega_i$ is such that $\partial\Omega_i \cap \partial\Omega \neq$ we modify the space $X_i(\Omega_i)$ as the discrete functions vanishing on $\partial\Omega$. The essential difference of these two discretizations is that the first discretization does not assume that the $X_i(\Omega_i)$ should vanish on $\partial\Omega_i \cap \partial\Omega$ whenever $\Omega_i$ touches $\partial\Omega$ at only a vertex. In the second discretization all nodes of $\partial\Omega_{ih_i} \cap \partial\Omega$ are not degrees of freedom of the problem, while in the first discretization, all nodes but those that $\partial\Omega_{ih_i}$ touches $\partial\Omega$ at only a vertex are not degrees of freedom. In both cases, the boundary terms and penalty terms on $F_{i0}$, see (2.6) and (2.7), do not exist and are not required.*

*Neumann-Neumann methods can also be developed for those cases. For the first discretization no changes are required for the coarse problem, and for the local problems associated to $N_I$ substructures. For the local problems on $N_B$ substructurez $\Omega_i$ we simply eliminate all the degrees of freedom associated to the nodes on $\partial\Omega_{ih_i} \cap \partial\Omega_i$ and Theorem 5.2 will hold with a similar proof. For the second discretization more changes are required to design the preconditioner. For $N_I$ substructures that touch $\partial\Omega$ at just one vertex, we modify $V_0$ considering $\Phi_{(i)}$ to be linear in the coarse trinagulation on $\Gamma_i$ and vanishing at that vertex, i.e. like what was done for $N_B$ substructures. The space $V$ and the nodes $\Gamma$ now do not have any degrees of freedom on $\partial\Omega$, and the spaces $V_i$ are defined as the space $V$ restricted to $\Gamma_i$, where now the $\Gamma_i$ do not include nodes on $\partial\Omega$. The definition of the $D^{(i)}$ also do not have any entrance associated to nodes on $\partial\Omega$. When proving Theorem 5.2, a technical problem will arise: How to bound $\bar{u}_i$ in (6.26) for the case that $\Omega_i$ touches $\partial\Omega$ at only one vertex? There are two possibilitites for the analysis: If there exists a $N_B$ substructure $\Omega_j$ where $\rho_i \le \rho_j$ and with a face $F_{ij}$ in common, then $\bar{u}_i$ can be estimated from the energy norm on $\Omega_i$ and on $\Omega_j$; in this case Theorem 5.2 holds. If not, a log factor in the estimation of $\bar{u}_i$ is obtained and therefore, Theorem 5.2 will hold with three-logs.*

**6. Proof of Theorem 5.2.** By the general theorem of ASMs we need to check the three key assumptions of Lemma 5.1.
Assumption(ii). We need to prove that

$$a_h(u,u) \le \omega b_0(u,u), \quad u \in V_0 \tag{6.1}$$

and for $i = 1, \cdots, N$

$$a_h(I_i u, I_i u) \le \omega b_i(u,u), \ \ u \in V_i \tag{6.2}$$

with $\omega \leq C(1+\log\frac{H}{h})^2$ where $C$ is a positive constant independent of $h_i$, $H_i$ and $\rho_i$.

By Lemma 2.1 and Lemma 4.1

$$a_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) \leq C d_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) \leq C d_h(\mathcal{H}u, \mathcal{H}u). \tag{6.3}$$

where $d_h(.,.)$ is defined by (4.2). The proofs of (6.1) and (6.2) then reduce to $d_h(\mathcal{H}u, \mathcal{H}u)$ instead of $a_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u)$.

The proof of (6.1) follows from the definition of $b_0$, see (5.7), where

$$a_h(\hat{\mathcal{H}}u, \hat{\mathcal{H}}u) \leq C d_h(\mathcal{H}u, \mathcal{H}u) = C(1+\log\frac{H}{h}) b_0(\mathcal{H}u, \mathcal{H}u) \tag{6.4}$$

with $\omega \leq C(1+\log\frac{H}{h})$.

We now prove (6.2). In order to simplify notations, all the functions are considered as harmonic extensions in the $\mathcal{H}$ sense. Hence, we denote $\mathcal{H}I_i u$ by $I_i u$ and let $u = \{u_l\}_{l\in\#(i)} \in V_i$. Using (4.1), (4.2), (5.3) and (5.9) we have

$$d_h(I_i u, I_i u) = d_i(D^{(i)}u, D^{(i)}u) + \sum_j d_j(D^{(i)}u, D^{(i)}u) \tag{6.5}$$

where the sum is taken over $\Omega_j$ with common faces to $\Omega_i$. We now estimate the two RHS terms of (6.5) as follows:

$$\begin{aligned} d_i(D^{(i)}u, D^{(i)}u) &= \int_{\Omega_i} \rho_i |\nabla D_i^{(i)} u_i|^2 dx + \\ &+ \sum_{F_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_i} \int_{F_{ij}} (D_i^{(i)} u_i - D_j^{(i)} u_j)^2 dx. \end{aligned} \tag{6.6}$$

We now estimate the first term of (6.6). We have

$$\rho_i \parallel \nabla D_i^{(i)} u_i \parallel^2_{L^2(\Omega_i)} \leq 2\rho_i \{\parallel \nabla (D_i^{(i)} u_i - u_i) \parallel^2_{L^2(\Omega_i)} + \parallel \nabla u_i \parallel^2_{L^2(\Omega_i)}\}$$

and

$$\rho_i \parallel \nabla (D_i^{(i)} u_i - u_i) \parallel^2_{L^2(\Omega_i)} \leq C \sum_{\delta_{ij}\subset\partial\Omega_i} \rho_i \parallel \tilde{u}_i \parallel^2_{H_{00}^{1/2}(\delta_{ij})},$$

where $\tilde{u}_i = u_i$ at the interior nodal points of $\delta_{ij}$ and $\tilde{u}_i = 0$ on $\partial\delta_{ij}$. It can be proved, see for example [10], that

$$\rho_i \parallel \tilde{u}_i \parallel^2_{H_{00}^{1/2}(\delta_{ij})} \leq C(1+\log\frac{H_i}{h_i})^2 \rho_i \parallel u_i \parallel^2_{H^1(\Omega_i)}, \tag{6.7}$$

where we have denoted

$$\parallel u_i \parallel^2_{H^1(\Omega_i)} = \parallel \nabla u_i \parallel^2_{L^2(\Omega_i)} + \frac{1}{H_i^2} \parallel u_i \parallel^2_{L^2(\Omega_i)}.$$

Remark 6.1. *In the case we use the approach described in Remark 5.2, we use the fact that $u_i$ has average zero on $\partial\Omega_i$ and then use Friedrich's inequality to obtain semi-norm on the RHS of (6.7). See also (6.8) below and after (6.10).*

We now estimate the second term of (6.6) and (6.10). Note that for $F_{i0}$, i.e. for faces on $\partial\omega$, the estimates of the terms corresponding to $F_{i0}$ follow straightfowardly. On a slave face $F_{ij}$ of $\partial\Omega_i$, i.e. where $h_i < h_j$ and $\rho_i < \rho_j$, or on $F_{i0}$, we have

$$\| D_i^{(i)} u_i - D_j^{(i)} u_j \|^2_{L^2(F_{ij})} \leq C h_i \max_{F_{ij}} |u_i|^2$$

hence,

$$\frac{\rho_{ij}}{h_{ij}} \| D_i^{(i)} u_i - D_j^{(i)} u_j \|^2_{L^2(F_{ij})} \leq C \rho_i \max_{F_{ij}} |u_i|^2 \leq C(1 + \log \frac{H_i}{h_i}) \rho_i \| u_i \|^2_{H^1(\Omega_i)},$$

where we have used that $\rho_{ij} \leq 2\rho_i$, and since $h_i < h_j$ we also have $h_{ij} > h_i$.

On a master side $F_{ij}$ of $\partial\Omega_i$, i.e. where $h_i \geq h_j$ and $\rho_i \geq \rho_j$, we have

$$\| D_i^{(i)} u_i - D_j^{(i)} u_j \|_{L^2(F_{ij})} \leq \| u_i - u_j \|_{L^2(F_{ij})} + \| u_j(0)\varphi_j^0 + u_j(H)\varphi_j^H \|_{L^2(F_{ij})},$$

where $\varphi_j^0$ and $\varphi_j^H$ are the nodal basis functions on $\partial\Omega_j$ associated to the endpoints of the face $F_{ji} \equiv (0, H)$. Using a triangular inequality we have

$$\| u_j(0)\varphi_j^0 \|_{L^2(F_{ij})} \leq C \| u_j \|_{L^2(0,h_j)} \leq C(\| u_i \|_{L^2(0,h_j)} + \| u_i - u_j \|_{L^2(F_{ij})})$$

and

$$\| u_i \|^2_{L^2(0,h_j)} \leq C \max_{F_{ij}} |u_i|^2 h_j \leq C h_j (1 + \log \frac{H_i}{h_i}) \| u_i \|^2_{H^1(\Omega_i)} .$$

Using similar arguments for bounding $\| u_j(H)\varphi_j^H \|_{L^2(F_{ij})}$, and using that $\rho_{ij} \leq 2\rho_i$, and $h_i \geq h_j$ which implies $h_{ij} \geq h_j$, we obtain

$$\frac{\rho_{ij}}{h_{ij}} \| D_i^{(i)} u_i - D_j^{(i)} u_j \|^2_{L^2(F_{ij})} \leq C(1 + \log \frac{H_i}{h_i}) b_i(u, u), \tag{6.8}$$

and the estimate

$$d_i(I_i u, I_i u) \leq C(1 + \log \frac{H_i}{h_i})^2 b_i(u, u) \tag{6.9}$$

follows.

We now estimate the second term of (6.6) $d_j(D^{(i)}u, D^{(i)}u)$ by $b_i(u, u)$. For $u = \{u_l\} \in V_i$ we have

$$d_j(D^{(i)}u, D^{(i)}u) = \rho_j \| \nabla D_j^{(i)} u_j \|^2_{L^2(\Omega_j)} + \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_j} \int_{F_{ij}} (D_i^{(i)} u_i - D_j^{(i)} u_j)^2 dx. \tag{6.10}$$

We need to estimate the first term only since the second term has been already estimated, see (6.8). If $F_{ij}$ is a slave side of $\partial\Omega_i$ then $D_j^{(i)}$ vanishes, and so vanishes $\| \nabla D_j^{(i)} u_j \|^2_{L^2(\Omega_j)}$. We now estimate the case where $F_{ij}$ is a master side of $\partial\Omega_i$ and it is not equal to $F_{i0}$. On $F_{ji}$ we decompose $u_j = w_j + u_j(0)\varphi_j^0 + u_j(H)\varphi_j^H$, where $w_j = D_j^{(i)} u_j$. We have

$$\| \nabla w_j \|^2_{L^2(\Omega_j)} \leq C \| w_j \|^2_{H_{00}^{1/2}(F_{ji})} = C\{ |w_j|^2_{H^{1/2}(F_{ji})} + \int_{F_{ji}} \frac{w_j^2}{dist(s, \partial F_{ji})} ds \}. \tag{6.11}$$

We estimate the first term of RHS of (6.11). Let $Q_j$ be the $L_2$- projection on the $h_j$-triangulation of $F_{ji}$. Using this we have

$$(6.12)\qquad \begin{aligned} |w_j|^2_{H^{1/2}(F_{ji})} &\leq 2\{|w_j - Q_j u_i|^2_{H^{1/2}(F_{ij})} + |Q_j u_i|^2_{H^{1/2}(F_{ij})}\} \\ &\leq C\{\frac{1}{h_j} \parallel w_j - u_i \parallel^2_{L^2(F_{ij})} + \parallel \nabla u_i \parallel^2_{L^2(\Omega_i)}\} \end{aligned}$$

and

$$(6.13) \parallel w_j - u_i \parallel^2_{L^2(F_{ij})} \leq 2 \parallel u_j - u_i \parallel^2_{L^2(F_{ij})} + 2 \parallel u_j(0)\varphi_j^0 + u_j(H)\varphi_j^H \parallel^2_{L^2(F_{ij})}$$

where the second term of the RHS of (6.13) can be bounded as before and using the fact that $\rho_j \leq \rho_i$.

It remains to estimate the second term of (6.11). We have

$$(6.14)\qquad \int_{F_{ji}} \frac{w_j^2}{dist(s,\partial F_{ji})} ds \leq C\{\int_0^{H/2} \frac{w_j^2}{s} ds + \int_{H/2}^{H} \frac{w_j^2}{(H-s)}) ds\}.$$

Let us estimate the first term of RHS of (6.14). We have

$$\begin{aligned} \int_0^{H/2} \frac{w_j^2}{s} ds &= \int_0^{h_j} \frac{w_j^2}{s} ds + \int_{h_j}^{H/2} \frac{u_j^2}{s} ds \\ &\leq C\{u_j^2(h_j) + \int_{h_j}^{H/2} \frac{u_i^2 - u_j^2}{s} ds + \int_{h_j}^{H/2} \frac{u_i^2}{s} ds\} \\ &\leq C\{u_j^2(h_j) + \frac{1}{h_j} \parallel u_i - u_j \parallel^2_{L^2(F_{ji})} + (1 + \log \frac{H_j}{h_j}) \max_{F_{ij}} |u_i|^2\} \\ &\leq C\{\frac{1}{h_j} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})} + (1 + \log \frac{H_i}{h_i})(1 + \log \frac{H_j}{h_j}) \parallel u_i \parallel^2)_{H^1(\Omega_i)}\}. \end{aligned}$$

The second term of (6.14) is estimated similarly. Substituting these estimates to (6.14) we get

$$(6.15)\qquad \begin{aligned} \int_{F_{ji}} \frac{u_j^2}{dist(s,\delta F_{ji})} ds &\leq C\{(1 + \log \frac{H}{h})^2 (\parallel \nabla u_i \parallel^2_{L^2(\Omega_i)} + \\ &+ \frac{1}{H_i^2} \parallel u_i \parallel^2_{L^2(\Omega_i)}) + \frac{1}{h_j} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})}\}. \end{aligned}$$

In turn, substituting (6.12) and (6.15) into (6.11), and the resulting estimate and (6.8) into (6.10) we get

$$(6.16)\qquad d_j(D^{(i)}u, D^{(i)}u) \leq C(1 + \log \frac{H}{h})^2 b_i(u,u).$$

Using (6.9) and (6.16) into (6.2), we get

$$d_h(u,u) \leq C(1 + \log \frac{H}{h})^2 b_i(u,u).$$

The proof of Assumption(ii) is complete.

Assumption(iii) We need to prove that

$$a_h(I_iu^{(i)}, I_ju^{(j)}) \le C\varepsilon_{ij}a_h^{1/2}(I_iu^{(i)}, I_iu^{(i)})\; a_h^{1/2}(I_ju^{(j)}, I_ju^{(j)}) \tag{6.17}$$

for $u^{(i)} \in V_i$ and $u^{(j)} \in V_j$, $\quad i,j = 1,\cdots,N$, and the spectral radius of $\varepsilon = \{\varepsilon_{ij}\}_{i,j=1}^N$, $\varrho(\varepsilon)$, is bounded. In our case $\varrho(\varepsilon) \le C$ with constant independent of $h_i$ and $H_i$. This follows from the fact that $u^{(i)}$ and $u^{(j)}$ are different from zero on $\Omega_i$ and $\Omega_j$ and their neighbor substructures.

Assumption(i) By Lemma 2.1 and Lemma 4.1, we need to prove that for $u = \{u_i\}_{i=1}^N \in V$ there exist $v^{(0)} \in V_0$ and $v^{(i)} \in V_i$ such that

$$v^{(0)} + \sum_{i=1}^N I_iv^{(i)} = u \tag{6.18}$$

and

$$b_0(v^{(0)}, v^{(0)}) + \sum_{i=1}^N b_i(v^{(i)}, v^{(i)}) \le Cd_h(u,u) \tag{6.19}$$

where $C$ is independent of $h_i$ and $H_i$.

We first set

$$v^{(0)} = \sum_{i=1}^N \bar{u}_i\Theta^{(i)}, \quad \bar{u}_i = \frac{1}{|\Omega_i|}\int_{\Omega_i} u_i ds \tag{6.20}$$

where $u = \{u_i\}_{i=1}^N \in V$. We note that another possibility would be to define $\bar{u}_i$ as the average of $u_i$ on $\partial\Omega_i$ or a face of it. The $v^{(0)}$ also can be represented, see (5.5), as

$$v^{(0)} = \sum_{i=1}^N I_i\bar{u}_i\Phi^{(i)}.$$

Using the partition of unity (5.4) we compute

$$w \equiv u - v^{(0)} = \sum_{i=1}^N I_i(\tilde{I}_i^T u - \bar{u}_i\Phi^{(i)}), \tag{6.21}$$

and define

$$v^{(i)} \equiv \tilde{I}_i^T u - \bar{u}_i\Phi^{(i)},$$

i.e. $v^{(i)} = \{v_l^{(i)}\}_{l\in\#(i)} \in V_i$ is defined as $v_i^{(i)} = u_i - \bar{u}_i\Phi_i^{(i)}$ on $\partial\Omega_i$, $v_j^{(i)} = u_j - \bar{u}_i\Phi_j^{(i)}$ on neighboring faces $F_{ji}$ including also the nodes on $\partial F_{ji}$.

By Lemma 7.1, see below, we have

$$b_0(v^{(0)}, v^{(0)}) = (1 + \log\frac{H}{h})^{-1}d_h(v^{(0)}, v^{(0)}) \le Cd_h(u,u). \tag{6.22}$$

It remains to estimate $b_i(v^{(i)}, v^{(i)})$ for $i = 1,\cdots,N$. We have

$$b_i(v^{(i)}, v^{(i)}) \le C\{\rho_i \parallel \nabla\mathcal{H}_i(u_i^{(i)} - \bar{u}_i\Phi_i^{(i)}) \parallel_{L^2(\Omega_i)}^2 + \tag{6.23}$$

$$+\frac{\rho_i}{H_i^2} \parallel \mathcal{H}_i(u_i^{(i)} - \bar{u}_i\Phi_i^{(i)}) \parallel_{L^2(\Omega_i)}^2 + \sum_{F_{ij}\subset\partial\Omega_i} \delta\frac{\rho_{ij}}{h_{ij}} \parallel u_i^{(i)} - u_j^{(i)} \parallel_{L^2(F_{ij})}^2\},$$

where we note that we have used in the last term of the LHS of (6.23) that $\Phi_i^{(i)} = \Phi_j^{(i)}$ on the faces $F_{ij}$ and $F_{ji}$ and $\Phi_i^{(i)}$ vanishes on faces of $\partial\Omega_i \cap \partial\Omega$. If $\Omega_i$ is a $N_I$ substructure, i.e. it does not share a face with $\partial\Omega$, the function $\Phi_i^{(i)}$ is the constant equal to one and using Poincaré's inequality we obtain

$$\| \mathcal{H}_i(u_i - \bar{u}_i\Phi_i^{(i)}) \|_{L^2(\Omega_i)}^2 \le CH_i^2 \| \nabla\mathcal{H}_i u_i \|_{L^2(\Omega_i)}^2 . \tag{6.24}$$

When $\Omega_i$ is a $N_B$ substructure, then

$$\begin{aligned} \| \nabla\mathcal{H}_i(u_i - \bar{u}_i\Phi_i^{(i)}) \|_{L^2(\Omega_i)}^2 \le 2 \| \nabla\mathcal{H}_i(u_i - \bar{u}_i) \|_{L^2(\Omega_i)}^2 + \\ + 2\bar{u}_i^2 \| \nabla\mathcal{H}_i(1 - \Phi_i^{(i)}) \|_{L^2(\Omega_i)}^2 . \end{aligned} \tag{6.25}$$

To estimate the second term of (6.25), we use that $1 - \Phi_i^{(i)}$ is linear on the faces of $\partial\Omega_i$ and vanishes in one of them and minimum energy arguments to have

$$\| \mathcal{H}_i(1 - \Phi_i^{(i)}) \|_{L^2(\Omega_i)}^2 \le C.$$

To bound $\bar{u}_i$, we consider $\bar{u}_{i0}$ the average of $u_i$ on $F_{i0}$ and we use a Poincaré inequality to obtain

$$\bar{u}_i^2 \le C \| \nabla\mathcal{H}_i u_i \|_{L^2(\Omega_i)}^2 + 2\bar{u}_{i0}^2, \tag{6.26}$$

and using that

$$\bar{u}_{i0}^2 \le \frac{C}{H_i} \|u_i\|_{L^2(F_{i0})}^2$$

and $H_i \ge h_i$ we obtain

$$\| \nabla\mathcal{H}_i(u_i - \bar{u}_i\Phi_i^{(i)}) \|_{L^2(\Omega_i)}^2 \le C\{\| \nabla\mathcal{H}_i u_i \|_{L^2(\Omega_i)}^2 + \frac{1}{h_i}\|u_i\|_{L^2(F_{i0})}^2\}, \tag{6.27}$$

and then use Poincaré inequality to bound the first term of the RHS of (6.23). An estimate of the last term of (6.23) is obvious and the proof of Theorem 5.2 is complete.

**7. Auxiliary lemma.** Let $u_0 \in V_0$ be defined for $u = \{u_i\}_{i=1}^N \in V$ as

$$u_0 = \sum_{i=1}^N \bar{u}_i\Theta^{(i)}, \quad \bar{u}_i \equiv \frac{1}{|\Omega_i|}\int_{\Omega_i} u_i ds. \tag{7.1}$$

LEMMA 7.1. *Assume that Assumption M holds. Then for $u_0$ defined by (7.1) holds*

$$d_h(u_0, u_0) \le C(1 + \log\frac{H}{h})d_h(u, u) \tag{7.2}$$

*where $C$ is independent of $h_i$, $H_i$ and the jumps of $\rho_i$*

Proof By Lemma 4.1 the estimate (7.2) is enough to prove for $\mathcal{H}u_0 = \{\mathcal{H}_i u_i^0\}_{i=1}^N$. Let us below denote $\mathcal{H}u_0$ by $u_0$. We have

$$d_h(u_0, u_0) = \sum_{i=1}^N\{\rho_i \| \nabla u_i^0 \|_{L^2(\Omega_i)}^2 + \sum_{F_{ij}\subset\partial\Omega_i} \delta\frac{\rho_{ij}}{h_{ij}} \| u_i^0 - u_j^0 \|_{L^2(F_{ij})}^2\}. \tag{7.3}$$

We estimate the first term. Note that on $\partial\Omega_i$

$$u_0 = \bar{u}_i\Theta_i^{(i)} + \sum_{\delta_{ij}\subset\partial\Omega_i} \bar{u}_j\Theta_i^{(j)} \tag{7.4}$$

and

$$\bar{u}_i\Phi_i^{(i)} = \bar{u}_i\Theta_i^{(i)} + \sum_{\delta_{ij}\subset\partial\Omega_i} \bar{u}_i\Theta_i^{(j)}. \tag{7.5}$$

Note that

$$\begin{aligned} \| \nabla u_i^0 \|^2_{L^2(\Omega_i)} &= \| \nabla(u_i^0 - \bar{u}_i) \|^2_{L^2(\Omega_i)} \leq \\ &\leq 2 \| \nabla(u_i^0 - \Phi_i^{(i)}\bar{u}_i) \|^2_{L^2(\Omega_i)} + 2\bar{u}_i^2 \| \nabla(1-\Phi_i^{(i)}) \|^2_{L^2(\Omega_i)} . \end{aligned} \tag{7.6}$$

When $\Omega_i$ is a $N_I$ substructure, the second term of (7.6) vanishes, otherwise we can use similar arguments as in (6.27) and show that

$$\bar{u}_i^2 \| \nabla(1-\mathcal{H}_i\Phi_i^{(i)}) \|^2_{L^2(\Omega_i)} \leq C\{\| \nabla\mathcal{H}_i u_i \|^2_{L^2(\Omega_i)} + \frac{1}{H_i}\|u_i\|^2_{L^2(F_{i0})}\}.$$

To bound the first term of (7.6), we use (7.4) and (7.5) to have

$$\begin{aligned} \| \nabla(u_i^0 - \Phi_i^{(i)}\bar{u}_i) \|^2_{L^2(\Omega_i)} &\leq C \sum_{\delta_{ij}\subset\partial\Omega_i} (\bar{u}_i - \bar{u}_j)^2 \| \Theta_i^{(j)} \|^2_{H_{00}^{1/2}(\delta_{ij})} \leq \\ &\leq C(\bar{u}_i - \bar{u}_j)^2 (1 + \log\frac{H_i}{h_i}) \end{aligned} \tag{7.7}$$

since

$$\| \Theta_i^{(j)} \|^2_{H_{00}^{1/2}(\delta_{ij})} \leq C(1+\log\frac{H_i}{h_i}).$$

Let for $F_{ij} = F_{ji}, F_{ij} \subset \partial\Omega_i, F_{ji} \subset \partial\Omega_j$

$$\bar{u}_{iF_{ij}} = \frac{1}{|F_{ij}|}\int_{F_{ij}} u_i ds, \qquad \bar{u}_{jF_{ji}} = \frac{1}{|F_{ji}|}\int_{F_{ji}} u_j ds.$$

Using this we get

$$\begin{aligned} (\bar{u}_i - \bar{u}_j)^2 &\leq C\{(\bar{u}_i - \bar{u}_{iF_{ij}})^2 + (\bar{u}_{iF_{ij}} - \bar{u}_{jF_{ji}})^2 + (\bar{u}_{jF_{ji}} - \bar{u}_j)^2 \leq \\ &\leq C\{\| \nabla u_i \|^2_{L^2(\Omega_i)} + \frac{1}{h_{ij}} \| u_i - u_j \|^2_{L^2(F_{ij})} + \| \nabla u_j \|^2_{L^2(\Omega_j)}\}. \end{aligned} \tag{7.8}$$

We have used Poincare' inequality and that $H_i \geq h_{ij}$

$$(\bar{u}_{iF_{ij}} - \bar{u}_{jF_{ji}})^2 \leq C\frac{1}{H_i^2}(u_i - u_j, 1)^2_{L^2(F_{ij})} \leq C\frac{1}{H_i}\|u_i - u_j\|^2_{L^2(F_{ij})}.$$

Substituting (7.8) into (7.7), and using that on $\delta_{ij}$ we have $\rho_i \leq \rho_j$, and we obtain

$$\begin{aligned} \rho_i \| \nabla u_i^0 \|^2_{L^2(\Omega_i)} \leq C(1+\log\frac{H_i}{h_i})\{&\rho_i \| \nabla u_i \|^2_{L^2(\Omega_i)} + \rho_j \| \nabla u_j \|^2_{L^2(\Omega_j)} + \\ &+ \frac{\rho_{ij}}{h_{ij}} \| u_i - u_j \|^2_{L^2(F_{ij})}\}. \end{aligned} \tag{7.9}$$

We now estimate the second term of (7.3). Let $F_{ij} = \gamma_{ij}$, i.e. $F_{ij}$ be the master side. Note that for $F_{i0}$ the estimate is obvious. For the remaining we have

$$u_i^0 - u_j^0 = \bar{u}_i \Theta_i^{(i)} - (\bar{u}_j \Theta_j^{(j)} + \bar{u}_i \Theta_j^{(i)}) = (\bar{u}_i - \bar{u}_j)\Theta_j^{(j)}, \tag{7.10}$$

therefore

$$\frac{1}{h_{ij}} \parallel u_i^0 - u_j^0 \parallel^2_{L^2(F_{ij})} = \frac{1}{h_{ij}} (\bar{u}_i - \bar{u}_j)^2 \parallel \Theta_j^{(j)} \parallel^2_{L^2(F_{ij})} \leq C(\bar{u}_i - \bar{u}_j)^2$$

since

$$\parallel \Theta_j^{(j)} \parallel^2_{L^2(F_{ij})} \leq h_j \tag{7.11}$$

and $h_j \leq h_i$ and $h_{ij} \geq h_j$. Using (7.8) and that $\rho_{ij} \leq \rho_i$ and $\rho_{ij} \leq 2\rho_j$, we get

$$\begin{aligned} \frac{\rho_{ij}}{h_{ij}} \parallel u_i^0 - u_j^0 \parallel^2_{L^2(F_{ij})} &\leq C\{\rho_i \parallel \nabla u_i \parallel^2_{L^2(\Omega_i)} + \rho_j \parallel \nabla u_j \parallel^2_{L^2(\Omega_j)} + \\ &+ \frac{\rho_{ij}}{h_{ij}} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})}\}. \end{aligned} \tag{7.12}$$

Let $F_{ij} = \delta_{ij}$, i.e $F_{ij}$ be the slave side. In this case (7.10) reduces to

$$u_i^0 - u_j^0 = (\bar{u}_i - \bar{u}_j)\Theta_i^{(i)}$$

therefore we get

$$\begin{aligned} \frac{\rho_{ij}}{h_{ij}} \parallel u_i^0 - u_j^0 \parallel^2_{L^2(F_{ij})} &= \frac{\rho_{ij}}{h_{ij}} (\bar{u}_i - \bar{u}_j)^2 \parallel \Theta_i^{(i)} \parallel^2_{L^2(F_{ij})} \leq \\ &\leq C\rho_{ij}(\bar{u}_i - \bar{u}_j)^2 \leq C\{\rho_i \parallel \nabla u_i \parallel^2_{L^2(\Omega_i)} + \rho_j \parallel \nabla u_j \parallel^2_{L^2(\Omega_j)} + \\ &+ \frac{\rho_{ij}}{h_{ij}} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})}\} \end{aligned} \tag{7.13}$$

in view of (7.11) for $\delta_{ij} \subset \partial\Omega_i$ and (7.8).

Substituting (7.9), (7.13) into (7.3) we get (7.2). The proof of Lemma 7.1 is complete.

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