

ON CONVEX CONES WITH INFINITELY MANY CRITICAL ANGLES

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Abstract. This note deals with some cardinality issues concerning the set of critical angles of a convex cone $K \subset \mathbb{R}^d$. Such set is referred to as the angular spectrum of the cone. In a recent work of ours, it has been shown that the angular spectrum of a polyhedral cone is necessarily finite and that its cardinality can grow at most polynomially with respect to the number of generators. In this note we explore the case of non-polyhedral cones. More specifically, we construct a cone whose angular spectrum is infinite (but possibly countable), and, what is harder to achieve, we construct a cone with noncountable angular spectrum. The construction procedure is highly technical in both cases, but the obtained results are useful for better understanding why some convex cones exhibit such a complicated angular structure.

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1 Introduction

The discussion takes place in the context of an Euclidean space \mathbb{R}^d equipped with the usual inner product $\langle u, v \rangle = u^T v$ and the associated norm $\|\cdot\|$. The symbol S_d refers to the corresponding unit sphere. The dimension d is assumed to be greater or equal than 2.

Consider an arbitrary closed convex cone $K \subset \mathbb{R}^d$. The problem of finding a pair of unit vectors achieving the maximal angle

$$\theta_{\max}(K) = \sup_{u, v \in K \cap S_d} \arccos \langle u, v \rangle \quad (1)$$

arises in different areas of mathematics. By way of example, we mention that $\theta_{\max}(K)$ serves to measure the degree of pointedness of K (cf. [2]), as well as the efficiency of certain interior point methods for solving feasibility systems with inequalities described by K (cf. [5]).

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As shown in [3], a necessary condition for a pair $(u, v) \in \mathbb{R}^d \times \mathbb{R}^d$ to solve the maximization problem (1) is that

$$u, v \in K \cap S_d, \quad v - \langle u, v \rangle u \in K^+, \quad u - \langle u, v \rangle v \in K^+, \quad (2)$$

where $K^+ = \{y \in \mathbb{R}^d : \langle y, x \rangle \geq 0 \forall x \in K\}$ denotes the dual cone of K . The criticality (or stationarity) conditions stated in (2) can be easily derived by working out the equivalent minimization problem³

$$\cos[\theta_{\max}(K)] = \inf_{u, v \in K \cap S_d} \langle u, v \rangle. \quad (3)$$

Definition 1. Let $K \subset \mathbb{R}^d$ be a closed convex cone. By a *critical pair* of K one understands any pair (u, v) of vectors satisfying (2). The angle formed by a critical pair is called a *critical angle*. The set

$$\Omega(K) = \{\arccos \langle u, v \rangle : (u, v) \text{ is a critical pair of } K\}$$

is called the *angular spectrum* of K .

Remark. Sometimes it is convenient to drop from $\Omega(K)$ the critical angles which are improper, that is to say, the critical angles formed by vectors u and v such that $|\langle u, v \rangle| = 1$. A cone admits at most two improper critical angles, so this point will not affect our discussion.

The angular spectrum is a set which describes somehow the angular structure of the cone. Not only the maximal angle enters into the picture, but also a full bunch of additional angles having a special meaning. For polyhedral (or finitely generated) cones the situation is fairly well understood. Such type of cones has always a finite number of critical angles ([3]). Upper bounds for the cardinality of their angular spectra are proposed in [4].

In this note we focuss the attention on non-polyhedral cones. We will exhibit a cone whose angular spectrum is infinite (but possibly countable), and, what is harder to achieve, we will construct a cone with noncountable angular spectrum.

2 Two preliminary lemmas on cones with infinite generators

We start with two results on cones with a possibly infinite number of generators. In the sequel \mathbb{N} refers to the set $\{1, 2, \dots\}$ of positive integers. For $G \subset S_d$, the symbol K_G will denote the cone

³As pointed out by one of the referees, the criticality conditions stated in (2) can also be derived by using a direct geometric argument. Assuming $u \neq -v$, denote by P the $(d-2)$ -dimensional subspace orthogonal to the vectors u, v and by Γ_1, Γ_2 the hyperplanes containing $P \cup \{u\}$ and $P \cup \{v\}$, respectively. The fact that u and v are unit vectors in K achieving the maximal angle (1) implies that Γ_1 and Γ_2 are support hyperplanes of K . Observe now that the vectors $v - \langle u, v \rangle u$ and $u - \langle u, v \rangle v$ are orthogonal to Γ_1 and Γ_2 , respectively.

generated by G , i.e.,

$$K_G = \left\{ \sum_{i=1}^m \alpha_i g^i : m \in \mathbb{N}, g^i \in G \ (1 \leq i \leq m), \alpha \in \mathbb{R}_+^m \right\}.$$

Lemma 1. *If G is closed and there exists $z \in S_d$ such that $\langle z, g \rangle > 0$ for all $g \in G$, then K_G is closed.*

Proof. Take a sequence $\{x^k\}_{k \in \mathbb{N}} \subset K_G$ converging to some $\bar{x} \in \mathbb{R}^d$. We will prove that \bar{x} belongs to K_G . By the conical version of Caratheodory's Theorem (cf. [9, Corollary 17.1.2]), each x^k can be written as a positive combination of up to d elements of G , i.e., $x^k = \sum_{j=1}^d \alpha_{kj} g^{kj}$ with $\alpha_{kj} \geq 0$, $g^{kj} \in G$. Clearly $\{g^{kj}\}_{k \in \mathbb{N}}$ is bounded for all $j \in \{1, \dots, d\}$. We claim that $\{\alpha_{kj}\}_{k \in \mathbb{N}}$ is also bounded for all $j \in \{1, \dots, d\}$. Note that

$$\langle z, x^k \rangle = \sum_{i=1}^d \alpha_{ki} \langle z, g^{ki} \rangle.$$

Since G is compact (because it is bounded and closed), and $\langle z, g \rangle > 0$ for all $g \in G$, there exists $\nu > 0$ such that $\langle z, g \rangle \geq \nu$ for all $g \in G$. Thus,

$$\|x^k\| \geq \langle z, x^k \rangle \geq \nu \sum_{i=1}^d \alpha_{ki} \geq \nu \alpha_{kj},$$

i.e., $\alpha_{kj} \in [0, \|x^k\|/\nu]$. Since $\{x^k\}$ is bounded, because it is convergent, the claim holds. Thus we can refine $\{x^k\}$ to $\{x^{\ell_k}\}$ so that $\{g^{\ell_k, j}\}_{k \in \mathbb{N}}$ converges for all $j \in \{1, \dots, d\}$, say to \bar{g}^j , and $\{\alpha_{\ell_k, j}\}_{k \in \mathbb{N}}$ converges for all $j \in \{1, \dots, d\}$, say to $\bar{\alpha}_j$. It follows that

$$\bar{x} = \lim_{k \rightarrow \infty} x^k = \lim_{k \rightarrow \infty} x^{\ell_k} = \lim_{k \rightarrow \infty} \sum_{j=1}^d \alpha_{\ell_k, j} g^{\ell_k, j} = \sum_{j=1}^d \bar{\alpha}_j \bar{g}^j.$$

Since G is closed, \bar{g}^j belongs to G for all $j \in \{1, \dots, d\}$ and \bar{x} turns out to be a nonnegative combination of elements of G , so that it belongs to K_G . \square

The above result is most probably known, for instance a variant of Lemma 1 is mentioned without proof in an old reference by Wets [10]. The hypothesis on the existence of z amounts to saying that the dual cone of K_G has nonempty interior, or equivalently that K_G is pointed in the sense that $K_G \cap -K_G = \{0\}$. The way it is formulated in the statement of the lemma is more appropriate for the sequel.

We mention in passing that the dual cone of K_G admits the characterization

$$[K_G]^+ = \{y \in \mathbb{R}^d : \langle y, g \rangle \geq 0 \ \forall g \in G\}.$$

This observation leads to the following lemma whose proof is immediate and therefore omitted.

Lemma 2. *The unit vectors $u, v \in K_G$ form a critical pair of K_G if and only if, for all $g \in G$,*

$$\begin{aligned}\langle v, g \rangle &\geq \langle u, v \rangle \langle u, g \rangle, \\ \langle u, g \rangle &\geq \langle u, v \rangle \langle v, g \rangle.\end{aligned}$$

3 A general construction procedure

In order to construct a cone K_G such that $\Omega(K_G)$ is infinite, we start by considering the revolution cone

$$\widehat{K} = \left\{ (x, t) \in \mathbb{R}^{d-1} \times \mathbb{R} : \sqrt{3}\|x\| \leq t \right\}.$$

Clearly, all pairs $(\frac{1}{2}(x, \sqrt{3}), \frac{1}{2}(-x, \sqrt{3}))$ with $x \in S_{d-1}$ are critical for \widehat{K} , and for all these pairs the critical angle is $\pi/3$, so that

$$\Omega(\widehat{K}) = \left\{ \left\langle \frac{1}{2}(x, \sqrt{3}), \frac{1}{2}(-x, \sqrt{3}) \right\rangle \right\} = \left\{ \frac{1}{2} \right\}.$$

We now select a closed subset $Q \subset S_{d-1}$ and a continuous function $\gamma : Q \rightarrow [1, 3/2]$. First we perturb each $x \in Q$, multiplying it by the scalar $\gamma(x)$, and then we construct perturbed pairs $(u(x), v(x))$ in \mathbb{R}^d , duly normalized, as

$$\begin{cases} u(x) = \frac{1}{\sqrt{3+\gamma^2(x)}} (\gamma(x)x, \sqrt{3}), \\ v(x) = \frac{1}{\sqrt{3+\gamma^2(x)}} (-\gamma(x)x, \sqrt{3}). \end{cases} \quad (4)$$

We take next the set

$$G = \{u(x) : x \in Q\} \cup \{v(x) : x \in Q\} \subset S_d, \quad (5)$$

and consider the cone K_G generated by G . The idea is that Q and γ should be selected in such a way that all pairs $(u(x), v(x))$ with $x \in Q$ are critical for K_G , and that angles corresponding to different pairs are different.

We identify next some properties of Q and γ that ensure these intended goals.

Theorem 1. *Take a closed subset $Q \subset S_{d-1}$ such that $\langle x, y \rangle \geq 0$ for all $x, y \in Q$, and a continuous and one-to-one function $\gamma : Q \rightarrow [1, 3/2]$. Consider the set G given by (5). Assume that*

$$\langle x, y \rangle \leq \frac{\gamma(x)}{\gamma(y)} \quad (6)$$

for all $x, y \in Q$. Then,

i) K_G is closed,

ii) All pairs $(u(x), v(x))$, as defined by (4), are critical for K_G ,

iii) $\text{card}[\Omega(K_G)] \geq \text{card}(Q)$.

Proof. i) Since Q is compact and the functions $u(\cdot), v(\cdot)$ defined by (4) are continuous, it follows that $\{u(x) : x \in Q\}$ and $\{v(x) : x \in Q\}$ are both compact. Hence G is closed. Take now the vector $z = (0, \dots, 0, 1) \in \mathbb{R}^d$ and notice that $\langle z, u(x) \rangle = \langle z, v(x) \rangle = \sqrt{3/(3 + \gamma^2(x))} > 0$ for all $x \in Q$. Thus, we are within the assumptions of Lemma 1, which ensures that K_G is closed.

ii) In view of Lemma 2, it suffices to check that, for all $x, y \in Q$,

$$\langle v(x), u(y) \rangle \geq \langle u(x), v(x) \rangle \langle u(x), u(y) \rangle, \quad (7)$$

$$\langle u(x), u(y) \rangle \geq \langle u(x), v(x) \rangle \langle v(x), u(y) \rangle, \quad (8)$$

$$\langle v(x), v(y) \rangle \geq \langle u(x), v(x) \rangle \langle u(x), v(y) \rangle, \quad (9)$$

$$\langle u(x), v(y) \rangle \geq \langle u(x), v(x) \rangle \langle v(x), v(y) \rangle. \quad (10)$$

Using (4) we get that, for all $x, y \in Q$,

$$\langle u(x), v(x) \rangle = \frac{3 - \gamma^2(x)}{3 + \gamma^2(x)}, \quad (11)$$

$$\langle u(x), u(y) \rangle = \langle v(x), v(y) \rangle = \frac{3 + \gamma(x)\gamma(y)\langle x, y \rangle}{\sqrt{3 + \gamma^2(x)}\sqrt{3 + \gamma^2(y)}}, \quad (12)$$

$$\langle u(x), v(y) \rangle = \langle v(x), u(y) \rangle = \frac{3 - \gamma(x)\gamma(y)\langle x, y \rangle}{\sqrt{3 + \gamma^2(x)}\sqrt{3 + \gamma^2(y)}}. \quad (13)$$

Using (11)-(13), we get that (8) and (9) become

$$3 + \gamma(x)\gamma(y)\langle x, y \rangle \geq \left[\frac{3 - \gamma^2(x)}{3 + \gamma^2(x)} \right] (3 - \gamma(x)\gamma(y)\langle x, y \rangle), \quad (14)$$

and (7) and (10) become

$$3 - \gamma(x)\gamma(y)\langle x, y \rangle \geq \left[\frac{3 - \gamma^2(x)}{3 + \gamma^2(x)} \right] (3 + \gamma(x)\gamma(y)\langle x, y \rangle). \quad (15)$$

For (14), note that, since $1 \leq \gamma(x) \leq 3/2$ for all $x \in Q$, and $0 \leq \langle x, y \rangle \leq 1$ for all $x, y \in Q$, the left-hand side is not smaller than 3, while both factors in the right-hand side are positive, the first one being not bigger than 1 and the second one not bigger than 3. It follows that the inequality holds. We look now at (15). A simple algebraic manipulation shows that it is equivalent to

$$\gamma^2(x) - \gamma(x)\gamma(y)\langle x, y \rangle \geq -\gamma^2(x) + \gamma(x)\gamma(y)\langle x, y \rangle,$$

which is itself equivalent to assumption (6).

iii) In view of (ii) and (11),

$$\Omega(K_G) \supset \left\{ \arccos \left[\frac{3 - \gamma^2(x)}{3 + \gamma^2(x)} \right] : x \in Q \right\}. \quad (16)$$

Note that $\tau \in]0, \infty[\mapsto \psi(\tau) := \frac{3-\tau}{3+\tau} = 1 - \frac{2}{1+(3/\tau)}$ is decreasing. Since γ is one-to-one and $\gamma(x) > 0$ for all $x \in Q$, we get that $\hat{\psi} : Q \rightarrow \mathbb{R}$, defined as $\hat{\psi}(x) = \psi(\gamma(x)) = \arccos[(3 - \gamma^2(x))/(3 + \gamma^2(x))]$, is one-to-one. The result follows from (16). \square

We mention that the specific angle of the initial revolution cone \widehat{K} , namely $\pi/3$, as well as the specific bounds on the image of γ , namely 1 and $3/2$, are inessential. The chosen values make computations easier because they ensure that the perturbed critical angles will be larger than $\pi/3$ and still acute, i.e., lower the $\pi/2$ (in fact we have that $1/7 \leq \langle u(x), v(x) \rangle \leq 1/2$ for all $x \in Q$), but we could have started with other revolution cones and others bounds for γ .

4 A cone with countable angular spectrum

Next we construct a pair Q, γ where Q is infinite and countable. Take $d \geq 3$ and consider a convergent sequence $\{x^k\}_{k \in \mathbb{N}} \subset S_{d-1}$ such that

$$0 \leq \langle x^j, x^k \rangle < 1 \quad \text{for all } j, k \in \mathbb{N}, j \neq k \quad (17)$$

$$\langle x^\ell, x^j \rangle \leq \langle x^k, x^j \rangle \quad \text{whenever } j \leq k \leq \ell. \quad (18)$$

An example of such a sequence is obtained by taking $x^k = (\cos k^{-1}, \sin k^{-1}, 0, \dots, 0)$ for all k , in which case $\langle x^j, x^k \rangle = \cos(j^{-1} - k^{-1})$. Let now $x^\infty = \lim_{k \rightarrow \infty} x^k$. Define $Q = \{x^k\}_{k \in \mathbb{N}} \cup \{x^\infty\}$, i.e., Q consists of the sequence $\{x^k\}_{k \in \mathbb{N}}$ together with its limit x^∞ . Clearly, Q is closed and satisfies the assumptions of Theorem 1. Regarding γ , let us write

$$\gamma(x^k) = \gamma_k \quad \forall k \in \mathbb{N} \cup \{\infty\}. \quad (19)$$

In view of the definition of Q , continuity of γ will be ensured if $\gamma_\infty = \lim_{k \rightarrow \infty} \gamma_k$. On the other hand, assumption (6) is much harder to achieve. We construct next a sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ leading to

a function γ which satisfies this assumption. For all $k \in \mathbb{N}$, let $\eta_k = \langle x^k, x^{k+1} \rangle$. Consider $\{\gamma_k\}_{k \in \mathbb{N}}$ defined recursively by

$$\gamma_1 \in]1, 3/2], \quad \gamma_{k+1} \in]\max\{1, \delta_k\}, \gamma_k[\quad \text{for } k = 1, 2, \dots, \quad (20)$$

where $\delta_k = \max_{1 \leq j \leq k} \{\gamma_j \eta_j\}$. Finally, define

$$\gamma_\infty = \lim_{k \rightarrow \infty} \gamma_k. \quad (21)$$

Next we prove that the sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ is well defined and that it satisfies the required properties.

Proposition 1. *One has:*

- i) *The choice (20) is feasible, i.e., $\max\{1, \delta_k\} < \gamma_k$ for all $k \in \mathbb{N}$.*
- ii) *$\{\gamma_k\}_{k \in \mathbb{N}}$ is decreasing and convergent.*
- iii) *$\langle x^k, x^j \rangle \leq \gamma_k / \gamma_j$ for all $j, k \in \mathbb{N} \cup \{\infty\}$.*

Proof. i) By induction. For $k = 1$ we need $\max\{1, \gamma_1 \eta_1\} < \gamma_1$, which holds because $1 < \gamma_1$ and $0 \leq \eta_1 < 1$. Suppose now that $\max\{1, \delta_j\} < \gamma_j$ for $j = 1, 2, \dots, k-1$ and that γ_j has been chosen according to (20) for $j = 2, \dots, k$. We must prove that $\max\{1, \delta_k\} < \gamma_k$. Note that $\delta_k = \max\{\delta_{k-1}, \gamma_k \eta_k\}$, so that $\max\{1, \delta_k\} = \max\{1, \delta_{k-1}, \gamma_k \eta_k\}$, and hence it suffices to prove that $\max\{1, \delta_{k-1}\} < \gamma_k$ and that $\gamma_k \eta_k < \gamma_k$. The first of these inequalities is a consequence of the inductive hypothesis, while the second one follows from (17) and the fact that γ_k is positive, ensured by (20).

ii) By construction, $\{\gamma_k\}_{k \in \mathbb{N}}$ is decreasing and bounded from below by 1, hence convergent.

iii) Let $j, k \in \mathbb{N}$. Since $\langle x^j, x^k \rangle \leq 1 \leq \gamma_k / \gamma_j$ for $j \geq k$, it suffices thus to deal with the case when $1 \leq j < k$. Note that

$$\gamma_j \langle x^k, x^j \rangle \leq \gamma_j \langle x^{j+1}, x^j \rangle = \gamma_j \eta_j \leq \max_{1 \leq \ell \leq k} \{\gamma_\ell \eta_\ell\} = \delta_k < \gamma_k, \quad (22)$$

using (18) in the first inequality and (20) in the last one. The case $j \in \mathbb{N}, k = \infty$ follows from (22) by letting $k \rightarrow \infty$. Letting then $j \rightarrow \infty$ one takes care of the remaining case $k = \infty, j = \infty$. \square

Remark. If a more specific definition of the γ_k 's is wanted, one can take e.g. the midpoint in each interval, i.e., $\gamma_1 = 5/4$, $\gamma_{k+1} = (1/2)[\gamma_k + \max\{1, \delta_k\}]$.

Corollary 1. *Consider the cone K_G generated by the set G defined by (4)–(5), with $Q = \{x^k\}_{k \in \mathbb{N}} \cup \{x^\infty\}$ according to (17)–(18), and γ as defined by (19)–(21). Then, K_G is closed, all pairs (u^k, v^k) defined as*

$$u^k = \frac{1}{\sqrt{3 + \gamma_k^2}} \left(\gamma_k x^k, \sqrt{3} \right), \quad v^k = \frac{1}{\sqrt{3 + \gamma_k^2}} \left(-\gamma_k x^k, \sqrt{3} \right) \quad (k \in \mathbb{N} \cup \{\infty\})$$

are critical for K_G , and $\Omega(K_G)$ is infinite.

Proof. We invoke Theorem 1. The set Q is closed because it consists of a converging sequence with its limit. Notice that $\langle x, y \rangle \geq 0$ for all $x, y \in Q$ because the first inequality in (17) extends to the case in which $k = \infty$ or $j = \infty$. The function γ is continuous by (21), one-to-one because $\gamma_1 > \gamma_2 > \dots > \gamma_\infty$, and condition (6) holds by Proposition 1(iii). The fact that Q is infinite follows from the second inequality in (17), which implies that $x^j \neq x^k$ for all $j \neq k$. Then the three statements of the corollary are consequences of items (i), (ii) and (iii) of Theorem 1, respectively. \square

5 A cone with noncountable angular spectrum

Given two different critical angles, say θ_0 and θ_1 , it is very tempting trying to construct a continuous curve $\{(u(t), v(t)) : t \in [0, 1]\}$ of critical pairs such that

$$\arccos\langle u(0), v(0) \rangle = \theta_0 \quad \text{and} \quad \arccos\langle u(1), v(1) \rangle = \theta_1.$$

If this were possible, then the whole interval $[\theta_0, \theta_1]$ would be contained in the angular spectrum of the cone. The following proposition prevents us however from being too optimistic.

Proposition 2. *Consider a closed convex cone $K \subset \mathbb{R}^d$ and an absolutely continuous curve $\mathcal{C} = \{(u(t), v(t)) : t \in [0, 1]\}$ formed by critical pairs of K . Then,*

$$\begin{cases} \langle u(t), v'(t) \rangle = 0 \\ \langle u'(t), v(t) \rangle = 0 \end{cases} \quad (23)$$

almost everywhere on $[0, 1]$, and, in particular,

$$t \in [0, 1] \mapsto \psi(t) = \arccos\langle u(t), v(t) \rangle \text{ is a constant function.} \quad (24)$$

Proof. Consider a point t in $]0, 1[$ at which $u(\cdot)$ and $v(\cdot)$ are differentiable. The set of such points has full Lebesgue measure in $[0, 1]$. For notational convenience, let us introduce the function $\lambda(\cdot) = \langle u(\cdot), v(\cdot) \rangle$. Since \mathcal{C} is formed by critical pairs of K , one can write

$$\langle u(t) - \lambda(t)v(t), v(t + \varepsilon h) \rangle \geq 0$$

for $h = \pm 1$ and $\varepsilon > 0$ small enough. By subtracting $\langle u(t) - \lambda(t)v(t), v(t) \rangle = 0$, dividing by ε and letting $\varepsilon \rightarrow 0$, one ends up with

$$\langle u(t) - \lambda(t)v(t), v'(t) \rangle = 0.$$

But $\langle v(t), v'(t) \rangle = 0$ because $v(\cdot)$ is contained in the unit sphere. In this way one gets the first orthogonality condition stated in (23). The second one is obtained by permuting the roles of $u(\cdot)$ and $v(\cdot)$. Since λ is absolutely continuous, one can write

$$\lambda(t) = \lambda(0) + \int_0^t \lambda'(\tau) d\tau.$$

But $\lambda'(\cdot) = \langle u(\cdot), v'(\cdot) \rangle + \langle u'(\cdot), v(\cdot) \rangle$ vanishes almost everywhere on $[0, 1]$, so the constancy condition (24) follows from the constancy of λ . \square

In view of the above proposition, it is not a wise idea to work with a set Q of the form $Q = \{(\cos t, \sin t, 0, \dots, 0) : t \in [a, b]\}$. Instead of an interval $[a, b]$, we must try our chance with a noncountable set having a more complicated structure. What we propose in fact is to look at

$$Q = \{(\cos t, \sin t) : t \in T\} \subset S_{3-1}, \quad (25)$$

$$T = \text{Cantor ternary set}. \quad (26)$$

We mention that we have fixed $d = 3$ with the only purpose of simplifying the notation; we could as well work with any dimension d and $Q = \{(\cos t, \sin t, 0, \dots, 0) : t \in T\} \subset S_{d-1}$.

Cantor's ternary set is known well enough (cf. [1], [11]) so we don't need to burden the presentation with all the details. What is strictly needed for our exposition is recalling that T can be represented in the form

$$T = [0, 1] \setminus \bigcup_{m=1}^{\infty}]\alpha_m, \beta_m[, \quad (27)$$

where the open intervals $] \alpha_m, \beta_m [$ are pairwise disjoint, and for all $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $\beta_m - \alpha_m = 3^{-k}$.

So, T is closed because its complement is open. That T is noncountable can be better seen if one uses the ternary expansion of the elements of the interval $[0, 1]$. We admit it as a fact that $\text{card}(T) = 2^{\mathbb{N}}$. Also known is that $\bigcup_{m=1}^{\infty}] \alpha_m, \beta_m [$ is dense in $[0, 1]$, so that T is nowhere dense, and hence totally disconnected (i.e., all its connected components are singletons).

Having already chosen Q , we must now select an appropriate γ . To do this we introduce first an auxiliary function $\sigma : [0, 1] \rightarrow \mathbb{R}$ in the following way:

$$\sigma(t) = 0 \quad \text{if } t \in T, \quad (28)$$

and on each interval $] \alpha_m, \beta_m [$, σ is defined as

$$\sigma(t) = \begin{cases} (\beta_m - \alpha_m)(t - \alpha_m) & \text{if } \alpha_m < t \leq (\alpha_m + \beta_m)/2 \\ (\beta_m - \alpha_m)(\beta_m - t) & \text{if } (\alpha_m + \beta_m)/2 < t < \beta_m. \end{cases} \quad (29)$$

By construction, σ vanishes over T and is positive on any interval of the form $] \alpha_m, \beta_m [$. One can check that σ is continuous on $[0, 1]$, so one can define $\gamma : Q \rightarrow \mathbb{R}$ by means of the integral

$$\gamma(x(t)) = 1 + \int_0^t \sigma(\tau) d\tau, \quad (30)$$

where we use the notation $x(t) = (\cos t, \sin t) \in S_2$ for $t \in T$. We must check that this choice of Q and γ satisfies the assumptions of Theorem 1. The critical item is (6), namely

$$\langle x(s), x(t) \rangle \leq \gamma(x(s))/\gamma(x(t)) \quad \forall s, t \in T, \quad (31)$$

for which we need two technical results.

Lemma 3. *For all $\tau \in [-1, 1]$, $\cos \tau \leq 1 - \tau^2/4$.*

Proof. The Maclaurin series expansion of $\cos(\cdot)$ gives readily the inequality

$$\cos \tau \leq 1 - \frac{\tau^2}{2} + \frac{\tau^4}{24}$$

for any $\tau \in \mathbb{R}$. For proving the result it suffices to show that

$$-\frac{\tau^2}{2} + \frac{\tau^4}{24} \leq -\frac{\tau^2}{4},$$

but this turns out to be equivalent to $\tau^2 \leq 6$, which certainly holds when $\tau \in [-1, 1]$. \square

Lemma 4. *Consider σ as defined by (28)–(29). Then,*

$$i) \int_{\alpha_m}^{\beta_m} \sigma(t) dt = (\beta_m - \alpha_m)^3/4,$$

$$ii) \int_0^1 \sigma(t) dt = 1/100,$$

$$iii) \int_{\alpha}^{\beta} \sigma(t) dt \leq (\beta - \alpha)^3/4 \text{ for all } \alpha, \beta \in T \text{ with } \alpha < \beta.$$

Proof. From the definition of σ over $[\alpha_m, \beta_m]$, one sees that the integral (i) corresponds to the area of a triangle with basis $\beta_m - \alpha_m$ and height $(\beta_m - \alpha_m)^2/2$. For computing the integral (ii), we write

$$\int_0^1 \sigma(t) dt = \sum_{m=1}^{\infty} \int_{\alpha_m}^{\beta_m} \sigma(t) dt = \sum_{m=1}^{\infty} \frac{(\beta_m - \alpha_m)^3}{4}.$$

As mentioned before, $\beta_m - \alpha_m = 3^{-k}$ for some $k \in \mathbb{N}$. From the very construction of the Cantor ternary set, one knows that there are exactly 2^k intervals of length $3^{-(k+1)}$. Thus,

$$\int_0^1 \sigma(t) dt = \frac{1}{4} \sum_{k=0}^{\infty} 2^k [3^{-(k+1)}]^3 = \frac{1}{108} \sum_{k=0}^{\infty} \left(\frac{2}{27}\right)^k = \frac{1}{108} \left[\frac{1}{1 - \frac{2}{27}} \right] = \frac{1}{100}.$$

As far as (iii) is concerned, consider the function $\rho : [\alpha, \beta] \rightarrow \mathbb{R}$ defined as

$$\rho(t) = \begin{cases} (\beta - \alpha)(t - \alpha) & \text{if } \alpha < t \leq (\alpha + \beta)/2 \\ (\beta - \alpha)(\beta - t) & \text{if } (\alpha + \beta)/2 < t < \beta. \end{cases} \quad (32)$$

Since $\int_{\alpha}^{\beta} \rho(t) dt = (\beta - \alpha)^3/4$ by the same argument as in (i), it suffices to show that $\sigma(t) \leq \rho(t)$ for all $t \in (\alpha, \beta)$. This is the case if $t \in T$, in view of (28). Otherwise $t \in]\alpha_m, \beta_m[$ for some m , and since $\alpha, \beta \notin]\alpha_m, \beta_m[$, because $\alpha, \beta \in T$, we have

$$\alpha \leq \alpha_m < \beta_m \leq \beta. \quad (33)$$

We emphasize that it is in (33) where we use in an essential way the fact that $\alpha, \beta \in T$. Now we must consider, in view of (29) and (32), four cases

- (a) $t \leq (\alpha + \beta)/2, t \leq (\alpha_m + \beta_m)/2,$
- (b) $t \leq (\alpha + \beta)/2, t > (\alpha_m + \beta_m)/2,$
- (c) $t > (\alpha + \beta)/2, t \leq (\alpha_m + \beta_m)/2,$
- (d) $t > (\alpha + \beta)/2, t > (\alpha_m + \beta_m)/2.$

Note that, by virtue of (33), $\beta_m - \alpha_m \leq \beta - \alpha$, $\beta_m - t \leq \beta - t$, and $t - \alpha_m \leq t - \alpha$, from which we get the result in cases (a) and (d). For case (c), we have $t \leq (\alpha_m + \beta_m)/2 \leq (\alpha_m + \beta)/2$, which implies $t - \alpha_m \leq \beta - t$, and therefore $\sigma(t) \leq \rho(t)$. For case (b), we have $t > (\alpha_m + \beta_m)/2 \geq (\alpha + \beta_m)/2$, which implies $\beta_m - t \leq t - \alpha$, and we get again $\sigma(t) \leq \rho(t)$. \square

Remark. Item (iii) of Lemma 4 is the critical point in the analysis and it is here that the set T plays its essential role. We mention that T is maximal with respect to the inequality (33). If α does not belong to T , there exists some $\beta \in T$ for which the inequality fails, namely $\beta = \beta_m$ if $\alpha \in (\alpha_m, \beta_m)$, as a simple geometrical argument shows, and the same situation occurs if $\beta \notin T$. Thus, there exists no $T' \subset [0, 1]$, strictly bigger than T , such that $\int_{\alpha}^{\beta} \sigma(t) dt \leq (\beta - \alpha)^3/4$ for all $\alpha, \beta \in T'$.

Now everything is prepared to state the main result of this section.

Theorem 2. *Consider the cone K_G , generated by G defined in (4)–(5), with Q and T given by (25)–(26). Let γ defined as in (30), with σ given by (28)–(29). Then K_G is closed, for every $t \in T$ the pair $(u(t), v(t))$ defined as*

$$\begin{aligned} u(t) &= \frac{1}{\sqrt{3 + \gamma^2(x(t))}} \left(\gamma(x(t))x(t), \sqrt{3} \right), \\ v(t) &= \frac{1}{\sqrt{3 + \gamma^2(x(t))}} \left(-\gamma(x(t))x(t), \sqrt{3} \right) \end{aligned}$$

with $x(t) = (\cos t, \sin t)$, is critical for K_G , and the angular spectrum $\Omega(K_G)$ contains the set

$$\Theta = \left\{ \arccos \left[\frac{3 - \gamma^2(x(t))}{3 + \gamma^2(x(t))} \right] : t \in T \right\},$$

which is noncountable.

Proof. We must check the assumptions of Theorem 1. The set Q is closed by compactness of T and continuity of the trigonometric functions. For arbitrary $s, t \in T$, the inner product

$$\langle x(s), x(t) \rangle = \cos s \cos t + \sin s \sin t = \cos(t - s) \quad (34)$$

is nonnegative because $|t - s| \leq 1$. The continuity of γ follows directly from its definition (30). On the other hand,

$$\gamma(x(t)) - \gamma(x(s)) = \int_s^t \sigma(\tau) d\tau$$

is positive if $s < t$. This proves not only that $t \in T \mapsto \gamma(x(t))$ is increasing, but also that $\gamma : Q \rightarrow \mathbb{R}$ is one-to-one. Also $\gamma(x(t)) \geq 1$ by (30) and nonnegativity of σ , and

$$\gamma(x(t)) \leq \gamma(x(1)) = 1 + \int_0^1 \sigma(\tau) d\tau = \frac{101}{100} < \frac{3}{2},$$

by Lemma 4(ii). It remains to check the hypothesis (6), which here takes the form (31). Since $\gamma(x(\cdot))$ is increasing and $x(s), x(t)$ are unit vectors, the result holds if $s \geq t$. Assume that $0 \leq s < t$. By (34) and Lemma 3,

$$\langle x(s), x(t) \rangle = \cos(t - s) \leq 1 - (t - s)^2/4. \quad (35)$$

On the other hand, using (30) and Lemma 4(iii),

$$\frac{\gamma(x(s))}{\gamma(x(t))} = \frac{1 + \int_0^s \sigma(\tau) d\tau}{1 + \int_0^t \sigma(\tau) d\tau} = 1 - \frac{\int_s^t \sigma(\tau) d\tau}{1 + \int_0^t \sigma(\tau) d\tau} \geq 1 - \int_s^t \sigma(\tau) d\tau \geq 1 - \frac{(t - s)^3}{4}. \quad (36)$$

In view of (35) and (36), it suffices to check that

$$1 - \frac{(t - s)^2}{4} \leq 1 - \frac{(t - s)^3}{4},$$

equivalent to $t - s \leq 1$, which holds because $0 \leq s < t \leq 1$. We have checked all the assumptions of Theorem 1, and the conclusion follows directly from it. \square

6 Final remarks

The result presented in Theorem 2 is quite striking but it doesn't close entirely the analysis of angular spectra. Note that the function

$$t \mapsto f(t) = \arccos \left[\frac{3 - \gamma^2(x(t))}{3 + \gamma^2(x(t))} \right]$$

is an homeomorphism between the sets T and Θ . This means that Θ inherits several of the topological properties of T like for instance being closed, nowhere dense, etc.

In principle, the set $\Omega(K_G)$ could be larger than Θ , and, in particular, it could contain an interval of positive length. This is not however the case. It is possible to prove that $\Omega(K_G)$ is indeed equal to Θ , but the analysis is too long and tedious, and goes beyond our purpose, namely constructing a cone with noncountable angular spectrum.

This fact, nevertheless, gives rise to the following conjecture: the angular spectrum of any closed and convex cone in \mathbb{R}^d is nowhere dense, and hence totally disconnected. Note that the set of critical pairs is closed, as follows directly from (2), and hence, since it is contained in $S_d \times S_d$, it is compact. It follows that the angular spectrum is always closed, and so it would be enough to prove that its complement is dense. This is certainly the case when the spectrum is either finite or countable.

Our second remark concerns the localization of critical angles. Since f is increasing, the smallest element of Θ is $f(0) = 0$, while the largest one is

$$f(1) = \arccos \left[\frac{3 - (101/100)^2}{3 + (101/100)^2} \right] \approx 0.336\pi$$

This means that we have constructed a cone K_G which exhibits noncountable many critical angles scattered between 0 and 60.495 degrees (approximately). By applying a general duality result established in [4], we deduce that $[K_G]^+$ exhibits noncountable many critical angles scattered between $180 - 60.495 = 119.505$ and $180 - 0 = 180$ degrees. This time the concentration of critical angles occurs in the region of obtuse angles.

We close this note with a last remark. Although the results presented in this work are rather technical and fall in a narrow area of research, it is possible to reformulate them in a broader context. A very interesting challenge in applied linear algebra is that of solving an eigenvalue problem described by linear complementarity conditions: given a matrix A of size $n \times n$ and a closed convex cone $P \subset \mathbb{R}^n$, one must find all the real numbers λ such that the system

$$x \in P, \quad Ax - \lambda x \in P^+, \quad \langle x, Ax - \lambda x \rangle = 0 \tag{37}$$

admits a nonzero solution $x \in \mathbb{R}^n$. The spectrum (or set of eigenvalues) of A relative to P is by definition the collection of all such λ 's. The eigenvalue problem (37) arises in many different areas and has been studied in [6, 7, 8], just to mention a few references. It is not difficult to see that if θ is a critical angle of $K \subset \mathbb{R}^d$, then $\cos \theta$ is an eigenvalue of the $2d \times 2d$ symmetric matrix

$$A = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

relative to the cone $P = K \times K \subset \mathbb{R}^{2d}$. By choosing K_G as in Corollary 1 one shows that, relatively to a nonpolyhedral cone, the number of eigenvalues of a symmetric matrix may be infinite (but

possibly countable). The cone K_G of Theorem 2 leads to an example of a noncountable set of eigenvalues for a symmetric matrix. It was already known that a revolution cone may produce a set of eigenvalues which contains an interval of positive length but this can occur only for matrices which are not symmetric!

Of course, it is also possible to immerse our work in the general setting of a nonconvex constrained optimization problem. The λ 's of the previous paragraph are interpreted now as Lagrange multipliers associated to a nonlinear equality constraint. However, by lifting our original problem to such an abstract level, we lose touch with our main motivation, namely, the analysis of the angular geometry of a cone.

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