

# Order preserving and order reversing operators on the class of convex functions in Banach spaces

Alfredo N. Iusem\*    Daniel Reem†    Benar F. Svaiter‡

November 28, 2012

## Abstract

A remarkable recent result by S. Artstein-Avidan and V. Milman states that, up to precomposition with affine operators, addition of affine functionals, and multiplication by positive scalars, the only *fully order preserving* mapping acting on the class of lower semicontinuous proper convex functions defined on  $\mathbb{R}^n$  is the identity operator, and the only *fully order reversing* one acting on the same set is the Fenchel conjugation. Here *fully order preserving (reversing) mappings* are understood to be those which preserve (reverse) the pointwise order among convex functions, are invertible, and such that their inverses also preserve (reverse) such order. In this paper we establish a suitable extension of these results to order preserving and order reversing operators acting on the class of lower semicontinuous proper convex functions defined on arbitrary Banach spaces.

**Key words:** order preserving operators, order reversing operators, Fenchel conjugation, convex functions, lower semicontinuous functions, Banach space, involution.

**Mathematical Subject Classification (2010):** 46N10, 46B10.

---

\*Instituto de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina 110, Rio de Janeiro, RJ, CEP 22460-320, Brazil, [iusp@impa.br](mailto:iusp@impa.br). The work of this author was partially supported by CNPq grant no. 301280/86.

†Instituto de Matemática Pura e Aplicada, (IMPA), Estrada Dona Castorina 110, Rio de Janeiro, RJ, CEP 22460-320, Brazil, [dream@impa.br](mailto:dream@impa.br).

‡Instituto de Matemática Pura e Aplicada, (IMPA), Estrada Dona Castorina 110, Rio de Janeiro, RJ, CEP 22460-320, Brazil, [benar@impa.br](mailto:benar@impa.br). The work of this author was partially supported by CNPq grants no. 474944/2010-7, 303583/2008-8 and FAPERJ grant E-26/110.821/2008.

# 1 Introduction

In their recent paper [3], S. Artstein-Avidan and V. Milman established a remarkable result on certain mappings acting on the class of convex functions defined on  $\mathbb{R}^n$ . More precisely, let  $\mathcal{C}(\mathbb{R}^n)$  be the set of lower semicontinuous proper convex functions  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ . For  $f, g \in \mathcal{C}(\mathbb{R}^n)$ , we write  $f \leq g$  if and only if  $f(x) \leq g(x)$  for all  $x \in \mathbb{R}^n$ . An operator  $T : \mathcal{C}(\mathbb{R}^n) \rightarrow \mathcal{C}(\mathbb{R}^n)$  is said to be *order preserving* if  $T(f) \leq T(g)$  whenever  $f \leq g$ , and *order reversing* if  $T(f) \geq T(g)$  whenever  $f \leq g$ . We will say that  $T$  is fully order preserving (reversing) if, in addition,  $T$  is invertible and its inverse  $T^{-1}$  is also order preserving (reversing). The main results in [3] state that, up to pre-composition with affine operators, addition of affine functionals, and multiplication by positive scalars, the only fully order preserving mapping acting on  $\mathcal{C}(\mathbb{R}^n)$  is the identity operator, and the only fully order reversing one acting on the same set is the Fenchel conjugation, meaning the operator  $T$  given by  $T(f) = f^*$  for all  $f \in \mathcal{C}(\mathbb{R}^n)$ , where  $f^*$  denotes the Fenchel conjugate of  $f$ , defined as  $f^*(u) = \sup_{x \in \mathbb{R}^n} \{\langle u, x \rangle - f(x)\}$  for all  $u \in \mathbb{R}^n$ . We mention, parenthetically, that the Fenchel conjugation is called “Legendre transform” in [3]. We think that our notation follows the usual practice in the convex analysis literature.

These results from [3] were the starting point of several interesting developments, consisting mainly of replacing  $\mathcal{C}(X)$  by other sets. Along this line, we mention for instance the characterizations of fully order reversing mappings acting on:

- i) the class of  $s$ -concave functions (see [2]),
- ii) the family of convex functions vanishing at 0 (see [4]),
- iii) closed and convex cones, with the order given by the set inclusion, and the polarity mapping playing the role of the Fenchel conjugation (see [8]),
- iv) closed and convex sets (see [9]),
- v) ellipsoids (see [5]).

All these papers deal exclusively with the finite dimensional case. In [11], it is proved that the composition of two order reversing bijections acting on the class of convex functions defined on certain locally convex topological vector spaces is the identity if and only in the same happens with their restrictions to the subset of affine functions, but no characterization of the kind established in [3] is given. To our knowledge, the extension of the above described results in [3] to infinite dimensional spaces remains as an open issue. Addressing it is the main goal of this paper. More precisely, we will generalize the analysis of order preserving and order reversing operators developed by Milman and Artstein-Avidan to the setting of arbitrary Banach spaces. As a corollary, we also obtain a characterization of order preserving involutions (see Corollary 6).

We mention now the main differences between [3] and our work.

When dealing, as we do, with arbitrary Banach spaces, the analysis becomes substantially more involved, demanding different tools. This situation occurs at several points in the analysis. For

instance, a key element of the proof consists of establishing the fact that the restriction of the operators of interest to the space of affine functions is itself affine (see Propositions 5 and 7 for a precise statement). In the infinite dimensional case, we need to show that such restriction is also continuous (cf. Propositions 8 and 9), which turns out to be rather nontrivial. Of course, such continuity is a moot issue in a finite dimensional setting.

Other specific features of our infinite dimensional analysis are the use of the Fenchel biconjugate  $f^{**}$  and its properties (cf. the proof of Theorem 1), and the fact that, in the nonreflexive case, the order reversing operators under consideration send lower semicontinuous functions with respect to the strong topology in  $X$  to lower semicontinuous ones with respect to the weak\* topology in  $X^*$  (see Theorem 2).

There is also another important difference between our approach and the one in [3]. As Milman and Artstein-Avidan rightly point out in [3], one can establish the characterization for the order preserving case and obtain the one for the order reversing case as an easy consequence, or make the full analysis for order reversing operators and then get the result for order preserving ones as a corollary. In [3], they chose the second option, while we follow in this paper the first one. As a consequence, excepting for some basic results (e.g. Proposition 1), or the use of some classical tools (like Lemma 1), which appear in [3] as well as in our work, both prooflines are basically different. For instance, the role played by the  $\delta$ -functions (i.e., the indicator functions of singletons), in the analysis of the order reversing case in [3], is taken by the affine functions in the order preserving one, developed here (e.g., our Propositions 3, 4, 5, 6 and 7). We also obtain, as a by-product of our analysis, some results which are possibly interesting on their own, like two characterizations of affine functions in general Banach spaces (see Proposition 2 and Lemma 2), which are fully unrelated to the contents of [3].

## 2 The order preserving case

We start by introducing some quite standard notation. Let  $X$  be a real Banach space. We denote by  $X^*$  and  $X^{**}$  its topological dual and bidual, respectively. We define as  $\mathcal{C}(X)$  the set of lower semicontinuous proper convex functions  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , where the lower semicontinuity is understood to hold with respect to the strong (or norm) topology in  $X$ . For  $f \in \mathcal{C}(X)$ ,  $f^* : X^* \rightarrow \mathbb{R} \cup \{+\infty\}$  will denote its Fenchel conjugate, defined as  $f^*(u) = \sup_{x \in X} \{\langle u, x \rangle - f(x)\}$  for all  $u \in X^*$ , where  $\langle \cdot, \cdot \rangle$  stands for the duality coupling, i.e.  $\langle u, x \rangle = u(x)$  for all  $(u, x) \in X^* \times X$ . As usual,  $f^{**} : X^{**} \rightarrow \mathbb{R} \cup \{+\infty\}$  will indicate the biconjugate of  $f$ , i.e.  $f^{**} = (f^*)^*$ . Given a linear and continuous operator  $C : X^* \rightarrow X^*$ ,  $C^* : X^{**} \rightarrow X^{**}$  will denote the adjoint of  $C$ , defined by the equation  $\langle C^*a, u \rangle = \langle a, Cu \rangle$  for all  $(a, u) \in X^{**} \times X^*$ . We consider the pointwise order in  $\mathcal{C}(X)$ , i.e., given  $f, g \in \mathcal{C}(X)$ , we write  $f \leq g$  whenever  $f(x) \leq g(x)$  for all  $x \in X$ . We denote as  $\mathbb{R}_{++}$  the set of strictly positive real numbers.

The next definition introduces the family of operators on  $\mathcal{C}(X)$  whose characterization is the main goal of this paper.

**Definition 1.** i) An operator  $T : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  is order preserving whenever  $f \leq g$  implies  $T(f) \leq T(g)$ .

ii) An operator  $T : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  is fully order preserving whenever  $f \leq g$  iff  $T(f) \leq T(g)$ , and additionally  $T$  is onto.

iii) We define as  $\mathcal{B}$  the family of fully order preserving operators  $T : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$ .

We will prove that  $T$  belongs to  $\mathcal{B}$  if and only if there exist  $c \in X$ ,  $w \in X^*$ ,  $\beta \in \mathbb{R}$ ,  $\tau \in \mathbb{R}_{++}$  and a continuous automorphism  $E$  of  $X$  such that

$$T(f)(x) = \tau f(Ex + c) + \langle w, x \rangle + \beta,$$

for all  $f \in \mathcal{C}(X)$  and all  $x \in X$ . We begin with an elementary property of the operators in  $\mathcal{B}$ .

**Proposition 1.** If  $T$  belongs to  $\mathcal{B}$  then,

i)  $T$  is one-to-one,

ii) Given a family  $\{f_i\}_{i \in I} \subset \mathcal{C}(X)$  such that  $\sup_{i \in I} f_i$  belongs to  $\mathcal{C}(X)$  (i.e., the supremum is not identically equal to  $+\infty$ ), it holds that  $T(\sup_{i \in I} f_i) = \sup_{i \in I} T(f_i)$ .

*Proof.* i) If  $T(f) = T(g)$  then  $T(f) \leq T(g)$  and  $T(g) \leq T(f)$ , so that, since  $T$  is fully order preserving,  $f \leq g$  and  $g \leq f$ , implying that  $f = g$ .

ii) Let  $\hat{f} = \sup_{i \in I} f_i$ . Since  $f_i \leq \hat{f}$  for all  $i \in I$  and  $T$  is order preserving, we get that  $T(f_i) \leq T(\hat{f})$  for all  $i \in I$ , so that

$$\sup_{i \in I} T(f_i) \leq T(\hat{f}) = T\left(\sup_{i \in I} f_i\right). \quad (1)$$

In view of (1),  $\sup_{i \in I} T(f_i)$  belongs to  $\mathcal{C}(X)$ . Since  $T$  is onto, there exists  $g \in \mathcal{C}(X)$  such that  $\sup_{i \in I} T(f_i) = T(g)$ . Observe that  $T(f_i) \leq \sup_{i \in I} T(f_i) = T(g)$ , so that, since  $T$  is fully order preserving,  $f_i \leq g$  for all  $i \in I$ , and hence  $\hat{f} = \sup_{i \in I} f_i \leq g$ . Since  $T$  is order preserving,

$$T\left(\sup_{i \in I} f_i\right) = T(\hat{f}) \leq T(g) = \sup_{i \in I} T(f_i). \quad (2)$$

The result follows from (1) and (2). □

An important consequence of Proposition 1(ii) is that an operator in  $\mathcal{B}$  is fully determined by its action on the family of affine functions on  $X$ , because any element of  $\mathcal{C}(X)$  is indeed a

supremum of affine functions (see, e.g., [10], p. 90). Hence, we will analyze next the behavior of the restrictions of operators in  $\mathcal{B}$  to the family of affine functions.

Define  $\mathcal{A} \subset \mathcal{C}(X)$  as  $\mathcal{A} = \{h : X \rightarrow \mathbb{R} \text{ such that } h \text{ is affine and continuous}\}$ . For  $u \in X^*$ ,  $\alpha \in \mathbb{R}$ , define  $h_{u,\alpha} \in \mathcal{A}$  as  $h_{u,\alpha}(x) = \langle u, x \rangle + \alpha$ . Clearly, every  $h \in \mathcal{A}$  is of the form  $h = h_{u,\alpha}$  for some  $(u, \alpha) \in X^* \times \mathbb{R}$ . For  $f \in \mathcal{C}(X)$ , define  $A(f) \subset \mathcal{A}$  as  $A(f) = \{h \in \mathcal{A} : h \leq f\}$ .

We continue with an elementary characterization of affine functions.

**Proposition 2.** *For all  $f \in \mathcal{C}(X)$ ,  $f$  belongs to  $\mathcal{A}$  if and only if there exists a unique  $u \in X^*$  such that  $h_{u,\alpha} \in A(f)$  for some  $\alpha \in \mathbb{R}$ .*

*Proof.* We start with the “only if” statement. If  $f \in \mathcal{A}$  then  $f = h_{u',\alpha'}$  for some  $(u', \alpha') \in X^* \times \mathbb{R}$ . Take an arbitrary  $h_{u,\alpha} \in A(f)$ , so that  $\langle u, x \rangle + \alpha \leq \langle u', x \rangle + \alpha'$  for all  $x \in X$ , or equivalently

$$\langle u - u', x \rangle \leq \alpha' - \alpha \quad (3)$$

for all  $x \in X$ . We claim that  $u - u' = 0$ . Otherwise, there exists  $z \in X$  such that  $\langle u - u', z \rangle \neq 0$ , and WLOG we may assume that  $\langle u - u', z \rangle > 0$  (otherwise, substitute  $-z$  for  $z$ ). Take now  $x = tz$  in (3), with  $t \in \mathbb{R}_{++}$ , getting  $\langle u - u', z \rangle \leq (\alpha' - \alpha)/t$ . Taking limits with  $t \rightarrow \infty$ , we obtain  $\langle u - u', z \rangle \leq 0$ , a contradiction, and henceforth  $u = u'$ .

We prove now the “if” statement. A well known result in convex analysis (see, e.g., [10], p. 90), establishes that  $f = \sup_{h \in A(f)} h$  for all  $f \in \mathcal{C}(X)$ , so that  $A(f) \neq \emptyset$  for all  $f \in \mathcal{C}(X)$ . If there exists a unique  $u \in X^*$  such that  $h_{u,\alpha}$  belongs to  $A(f)$ , then  $A(f) = \{h_{u,\alpha_i}\}_{i \in I}$  for some set  $I \neq \emptyset$ , and hence,

$$f(x) = \sup_{h \in A(f)} h(x) = \sup_{i \in I} h_{u,\alpha_i}(x) = \sup_{i \in I} \{\langle u, x \rangle + \alpha_i\} = \langle u, x \rangle + \sup_{i \in I} \{\alpha_i\} \quad (4)$$

for all  $x \in X$ . Since  $f$  is proper,  $\sup_{i \in I} \{\alpha_i\} < \infty$ . Taking  $\alpha = \sup_{i \in I} \{\alpha_i\}$ , we conclude from (4) that  $f(x) = \langle u, x \rangle + \alpha$ , i.e.,  $f$  belongs to  $\mathcal{A}$ .  $\square$

**Corollary 1.** *If  $f = h_{u,\alpha} \in \mathcal{A}$  then  $A(f) = \{h_{u,\delta} : \delta \leq \alpha\}$ .*

*Proof.* Immediate from Proposition 2.  $\square$

**Corollary 2.** *Consider  $h_{u,\alpha} \in \mathcal{A}$ ,  $f \in \mathcal{C}(X)$ . If  $f \leq h_{u,\alpha}$ , then  $f = h_{u,\delta}$  with  $\delta \leq \alpha$ .*

*Proof.* Take  $h_{u',\alpha'} \in A(f)$ . Then  $h_{u',\alpha'} \leq f \leq h_{u,\alpha}$ , i.e.  $h_{u',\alpha'} \in A(h_{u,\alpha})$ . By Corollary 1,  $u' = u$ , i.e., there exists a unique  $u$  such that  $h_{u,\alpha'} \in A(f)$  for some  $\alpha' \in \mathbb{R}$ . By Proposition 2,  $f$  is affine, and the result follows from Corollary 1.  $\square$

Next, we prove that operators in  $\mathcal{B}$  map affine functions to affine functions.

**Proposition 3.** *If  $T$  belongs to  $\mathcal{B}$  then*

*i)  $T(h) \in \mathcal{A}$  for all  $h \in \mathcal{A}$ ,*

ii) if  $T(f) \in \mathcal{A}$  for some  $f \in \mathcal{C}(X)$  then  $f \in \mathcal{A}$ .

*Proof.* i) Let  $h = h_{u,\alpha}$ . We will use Proposition 2 in order to establish affinity of  $T(h)$ . Take  $h_{z,\delta}, h_{z',\delta'} \in A(T(h))$ , i.e.  $h_{z,\delta} \leq T(h), h_{z',\delta'} \leq T(h)$ . Since  $T$  is onto, there exist  $g, g' \in \mathcal{C}(X)$  such that  $h_{z,\delta} = T(g), h_{z',\delta'} = T(g')$ , so that  $T(g) \leq T(h), T(g') \leq T(h)$ . Since  $T$  is fully order preserving, we get  $g \leq h, g' \leq h$ . By Corollary 2, there exist  $\eta, \eta' \in \mathbb{R}$  such that  $g = h_{u,\eta}, g' = h_{u,\eta'}$ . Assume, without loss of generality, that  $\eta \leq \eta'$  so that  $g \leq g'$ , and hence, since  $T$  is order preserving,

$$h_{z,\delta} = T(g) \leq T(g') = h_{z',\delta'}. \quad (5)$$

In view of Corollary 2, we get from (5) that  $z = z'$ . We have proved that there exists a unique  $z \in X^*$  such that  $h_{z,\delta} \in A(T(h))$  for some  $\delta \in \mathbb{R}$ , and hence  $T(h)$  is affine by Proposition 2.

ii) Note that whenever  $T \in \mathcal{B}$ , its inverse, the operator  $T^{-1}$ , also belongs to  $\mathcal{B}$ . It follows that  $T^{-1}$  also maps affine functions to affine functions, which establishes the result.  $\square$

We have seen that operators  $T \in \mathcal{B}$  map  $\mathcal{A}$  to  $\mathcal{A}$ . For  $T \in \mathcal{B}$ , we denote as  $\widehat{T} : \mathcal{A} \rightarrow \mathcal{A}$  the restriction of  $T$  to  $\mathcal{A}$ . Note that  $\widehat{T}$  inherits from  $T$  the properties of being onto and fully order preserving in  $\mathcal{A}$ . However, if we start with an operator in  $\mathcal{A}$  with these properties, and we “lift it up” to  $\mathcal{C}(X)$ , the resulting operator may not belong to  $\mathcal{B}$ . This is due to the fact that the order preserving property is rather weak in  $\mathcal{A}$  because, as we have seen, a pair of affine functions is ordered only when they have the same linear part. More precisely, consider an operator  $\widehat{R} : \mathcal{A} \rightarrow \mathcal{A}$  which is onto and fully order preserving. We can extend it to  $\mathcal{C}(X)$  in a natural way, defining  $R : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  as  $R(f) = \sup_{h \in A(f)} \widehat{R}(h)$ . Since  $f \leq g$  implies  $A(f) \subset A(g)$ , it is easy to prove that  $R$  is order preserving, but it might fail to be fully order preserving, as the following example shows:

Take  $X = \mathbb{R}$  and define  $\widehat{R} : \mathcal{A} \rightarrow \mathcal{A}$  as

$$\widehat{R}(h_{u,\alpha}) = \begin{cases} h_{u,\alpha} & \text{if } u \in (-1, 1) \\ h_{-u,\alpha} & \text{otherwise.} \end{cases}$$

It is immediate that  $\widehat{R}$  is onto and fully order preserving; in fact it is an involution, meaning that  $\widehat{R}(\widehat{R}(h)) = h$  for all  $h \in \mathcal{A}$ . Consider its extension  $R : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  as defined above, and take  $f_1(x) = 1/2|x|$ ,  $f_2(x) = \max\{x, 0\}$ . It is easy to check that  $R(f_1) = f_1 = (1/2)|x|$ ,  $R(f_2) = |x|$ , so that  $R(f_1) \leq R(f_2)$ , but it is not true that  $f_1 \leq f_2$  (the inequality fails for  $x < 0$ ). In fact any operator  $\widehat{R} : \mathcal{A} \rightarrow \mathcal{A}$  of the form  $\widehat{R}(h_{u,\alpha}) = h_{\psi(u),\varphi(\alpha)}$  where  $\psi : X^* \rightarrow X^*$  is a bijection and  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is increasing and onto, turns out to be onto and fully order preserving in  $\mathcal{A}$ , while operators  $\widehat{T} : \mathcal{A} \rightarrow \mathcal{A}$  which are the restrictions of operators  $T$  in  $\mathcal{B}$  have a much more specific form; we will prove next that  $\widehat{T} : \mathcal{A} \rightarrow \mathcal{A}$  is indeed affine for all  $T \in \mathcal{B}$ , with the following meaning: we identify  $\mathcal{A}$  with  $X^* \times \mathbb{R}$ , associating  $h_{u,\alpha}$  to the pair  $(u, \alpha)$ , and hence we write  $\widehat{T}(u, \alpha)$  instead of  $\widehat{T}(h_{u,\alpha})$ , and consider affinity of  $\widehat{T}$  as an operator acting on  $X^* \times \mathbb{R}$ .

In view of Proposition 3, with this notation, for  $T \in \mathcal{B}$ ,  $\widehat{T}$  maps  $X^* \times \mathbb{R}$  to  $X^* \times \mathbb{R}$ , and so we will write

$$\widehat{T}(u, \alpha) = (y(u, \alpha), \gamma(u, \alpha)), \quad (6)$$

with  $y : X^* \times \mathbb{R} \rightarrow X^*$ ,  $\gamma : X^* \times \mathbb{R} \rightarrow \mathbb{R}$ , i.e.  $y(u, \alpha)$  is the linear part of  $T(u, \alpha)$  and  $\gamma(u, \alpha)$  is its additive constant, so that  $T(h_{u,\alpha})(x) = \langle y(u, \alpha), x \rangle + \gamma(u, \alpha)$ .

We must prove that both  $y$  and  $\gamma$  are affine. We start with  $y$ , establishing some of its properties in the following two propositions.

**Proposition 4.** *If  $T \in \mathcal{B}$  then  $y(\cdot, \cdot)$ , defined by (6), depends only upon its first argument.*

*Proof.* Consider arbitrary pairs  $(u, \alpha), (u, \delta) \in X^* \times \mathbb{R}$ , and assume WLOG that  $\alpha \leq \delta$ , so that  $h_{u,\alpha} \leq h_{u,\delta}$ , and, since  $T$  is order preserving,

$$h_{y(u,\alpha),\gamma(u,\alpha)} = T(h_{u,\alpha}) \leq T(h_{u,\delta}) = h_{y(u,\delta),\gamma(u,\delta)}. \quad (7)$$

It follows from (7) and Corollary 2 that  $y(u, \alpha) = y(u, \delta)$  and hence  $y$  does not depend upon its second argument.  $\square$

In view of Proposition 4, we will write in the sequel  $y : X^* \rightarrow X^*$ , with  $y(u) = y(u, \alpha)$  for an arbitrary  $\alpha \in \mathbb{R}$ , and also

$$\widehat{T}(u, \alpha) = (y(u), \gamma(u, \alpha)). \quad (8)$$

We recall that  $\phi : X^* \rightarrow \mathbb{R}$  is *quasiconvex* if  $\phi(su_1 + (1-s)u_2) \leq \max\{\phi(u_1), \phi(u_2)\}$  for all  $u_1, u_2 \in X^*$  and all  $s \in [0, 1]$ .

**Proposition 5.** *Assume that  $T$  belongs to  $\mathcal{B}$ . Then,*

*i)  $y : X^* \rightarrow X^*$ , defined by (8), is one-to-one and onto,*

*ii) both  $\langle y(\cdot), x \rangle : X^* \rightarrow \mathbb{R}$  and  $\langle y^{-1}(\cdot), x \rangle : X^* \rightarrow \mathbb{R}$  are quasiconvex for all  $x \in X$ .*

*Proof.* i) Suppose that  $y(u_1) = y(u_2)$  for some  $u_1, u_2 \in X^*$ . Assume WLOG that  $\gamma(u_1, 0) \leq \gamma(u_2, 0)$ . Then  $T(h_{u_1,0}) \leq T(h_{u_2,0})$ , implying, since  $T$  is fully order preserving, that  $h_{u_1,0} \leq h_{u_2,0}$ , and hence  $u_1 = u_2$  by Corollary 2, so that  $y$  is one-to-one. Surjectivity of  $y$  follows from surjectivity of  $\widehat{T}$ , which is a consequence of Proposition 3(ii).

ii) Take  $u_1, u_2 \in X^*, s \in [0, 1]$ . Note that

$$h_{su_1+(1-s)u_2,0} = sh_{u_1,0} + (1-s)h_{u_2,0} \leq \max\{h_{u_1,0}, h_{u_2,0}\}.$$

Hence,

$$T(h_{su_1+(1-s)u_2,0}) \leq T(\max\{h_{u_1,0}, h_{u_2,0}\}) = \max\{T(h_{u_1,0}), T(h_{u_2,0})\}, \quad (9)$$

using the order preserving property of  $T$  in the inequality and Proposition 1(ii) in the equality. Evaluating the leftmost and rightmost expressions in (9) at a point of the form  $tx$  with  $x \in X, t \in \mathbb{R}_{++}$ , we get

$$\langle y(su_1 + (1-s)u_2), tx \rangle + \gamma(su_1 + (1-s)u_2), 0 \leq \max\{\langle y(u_1), tx \rangle + \gamma(u_1, 0), \langle y(u_2), tx \rangle + \gamma(u_2, 0)\}. \quad (10)$$

Dividing both sides of (10) by  $t$  and taking limits with  $t \rightarrow \infty$ , we get

$$\langle y(su_1 + (1-s)u_2), x \rangle \leq \max\{\langle y(u_1), x \rangle, \langle y(u_2), x \rangle\},$$

establishing quasiconvexity of  $\langle y(\cdot), x \rangle$  for all  $x \in X$ . Quasiconvexity of  $\langle y^{-1}(\cdot), x \rangle$  follows using the same argument with  $\widehat{T}^{-1}$  instead of  $\widehat{T}$ .  $\square$

Next we recall a well known result on finite dimensional affine geometry.

**Lemma 1.** *Let  $V, V'$  be finite dimensional real vector spaces, with  $\dim(V) \geq 2$ . If  $Q : V \rightarrow V'$  is one-to-one and maps segments to segments, then  $Q$  is affine.*

*Proof.* The result appears, e.g., in [1]. A short proof can be found in Remark 6 of [3].  $\square$

The next corollary extends this result to our infinite dimensional setting.

**Corollary 3.** *Let  $Z, Z'$  be arbitrary real vector spaces with  $\dim(Z) \geq 2$ . If  $Q : Z \rightarrow Z'$  is one-to-one and maps segments to segments, then  $Q$  is affine.*

*Proof.* Take vectors  $z_1, z_2 \in Z$  and  $s \in [0, 1]$ . We must prove that

$$Q(sz_1 + (1-s)z_2) = sQ(z_1) + (1-s)Q(z_2). \quad (11)$$

We may suppose WLOG that  $Q(0) = 0$  (otherwise, replace  $Q$  by  $Q'; Z \rightarrow Z'$ , defined as  $Q'(z) = Q(z) - Q(0)$ , which also maps segments to segments). Assume first that  $z_1, z_2$  are linearly independent. Let  $V \subset Z$  be the two-dimensional subspace spanned by  $z_1, z_2$ , and  $V' \subset Z'$  the subspace spanned by  $Q(z_1), Q(z_2)$ . Since  $Q$  maps segments to segments, and henceforth lines to lines, the line through  $z_1$  must be mapped to the line through  $Q(z_1)$  in  $Z'$ , and the line through  $z_2$  to the line through  $Q(z_2)$  in  $Z'$ . Since these two lines in  $Z'$  are indeed contained in  $V'$ , and every point in  $V$  belongs to some segment with extremes in the lines through  $z_1$  and  $z_2$ , we conclude that the image of the restriction  $Q|_V$  of  $Q$  to  $V$  is contained in  $V'$ . Hence, since  $V$  and  $V'$  are finite dimensional, and  $\dim(V) = 2$ , we may apply Lemma 1 to  $Q|_V : V \rightarrow V'$ , concluding that  $Q|_V$  is affine, and hence (11) holds. Assume now that  $z_1, z_2$  are linearly dependent, i.e. colinear. Since (11) holds trivially when both  $z_1$  and  $z_2$  vanish, assume WLOG that  $z_1 \neq 0$ , and replace  $z_2$  by  $\tilde{z}_2 = z_2 + z_3$ , with  $z_3$  linearly independent of  $z_1, z_2$ . Now apply the same argument as above to the two dimensional subspace  $\tilde{V}$  spanned by  $z_1, \tilde{z}_2$ , concluding that the restriction of  $Q$  to  $\tilde{V}$  is affine (henceforth continuous), and that (11) holds with  $\tilde{z}_2 = z_2 + z_3$  substituting for  $z_2$ . Take limits then in (11) with  $z_3 \rightarrow 0$  in  $\tilde{V}$ , and conclude from the continuity of  $Q$  in  $\tilde{V}$  that (11) holds with  $z_1, z_2$  also in the colinear case. Since  $z_1$  and  $z_2$  are arbitrary points in  $Z$ ,  $Q$  itself is affine.  $\square$



We continue with another result, possibly of some interest on its own.

**Lemma 2.** *If a mapping  $M : X^* \rightarrow X^*$  satisfies:*

*i)  $M$  is one-to-one and onto,*

*ii) both  $\langle M(\cdot), x \rangle : X^* \rightarrow \mathbb{R}$  and  $\langle M^{-1}(\cdot), x \rangle : X^* \rightarrow \mathbb{R}$  are quasiconvex for all  $x \in X$ ,*

*then  $M$  is affine.*

*Proof.* Take  $u_1, u_2 \in X^*$ ,  $s \in [0, 1]$ . We show next that  $M(su_1 + (1-s)u_2)$  belongs to the segment between  $M(u_1)$  and  $M(u_2)$ . Define  $\bar{u} = su_1 + (1-s)u_2$ . By assumption (ii),

$$\langle M(\bar{u}), x \rangle \leq \max\{\langle M(u_1), x \rangle, \langle M(u_2), x \rangle\}. \quad (12)$$

Note that if  $M(u_1) = M(u_2)$  then  $u_1 = u_2 = \bar{u}$  because  $M$  is one-to-one, and the result holds trivially. Assume that  $M(u_1) \neq M(u_2)$ , and consider the halfspaces  $U = \{x \in X : \langle M(u_1) - M(u_2), x \rangle \geq 0\}$ ,  $W = \{x \in X : \langle M(u_1) - M(\bar{u}), x \rangle \geq 0\}$ . Note that if  $x \in U$  then the maximum in the right hand side of (12) is attained in the first argument, in which case  $\langle M(\bar{u}), x \rangle \leq \langle M(u_1), x \rangle$ , i.e.,  $x$  belongs to  $W$ . We have shown that  $U \subset W$ . It is an easy consequence of the Convex Separation Theorem that  $M(u_1) - M(u_2)$  and  $M(u_1) - M(\bar{u})$  belong to the same halfline, i.e., there exists  $\sigma > 0$  such that  $M(u_1) - M(\bar{u}) = \sigma(M(u_1) - M(u_2))$ , or equivalently,

$$M(su_1 + (1-s)u_2) = \sigma M(u_2) + (1-\sigma)M(u_1). \quad (13)$$

Reversing the roles of  $u_1, u_2$ , we conclude, with the same argument, that there exists  $\sigma' > 0$  such that

$$M(su_1 + (1-s)u_2) = (1-\sigma')M(u_2) + \sigma'M(u_1),$$

implying that  $\sigma = 1 - \sigma'$  and therefore  $\sigma \in [0, 1]$ . Thus, (13) shows that  $M(su_1 + (1-s)u_2)$  belongs to the segment between  $M(u_1)$  and  $M(u_2)$ , and therefore  $M$  maps all points in the segment between  $u_1$  and  $u_2$  to points between  $M(u_1)$  and  $M(u_2)$ . The fact that the image of the first segment fills the second one results from the same argument applied to  $M^{-1}$ , which enjoys, by assumption, the same quasiconvexity property as  $M$ . We have shown that  $M$  maps segments to segments, and then affinity of  $M$  follows from Corollary 3.  $\square$

**Corollary 4.** *If  $T \in \mathcal{B}$  then  $y : X^* \rightarrow X^*$ , defined by (8), is affine.*

*Proof.* The result follows from Proposition 5 and Lemma 2.  $\square$

We establish next affinity of  $\gamma(\cdot, \cdot)$ . We start with an elementary property of  $\gamma(u, \cdot)$ .

**Proposition 6.** *If  $T$  belongs to  $\mathcal{B}$  then  $\gamma(u, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$  is strictly increasing, one-to-one and onto for all  $u \in X^*$ .*

*Proof.* Since  $T$  is a bijection, we get from Proposition 3(ii) that  $\widehat{T}$  is also a bijection, and hence the same holds for  $\gamma(u, \cdot)$ . If  $\alpha \leq \delta$  then  $h_{u, \alpha} \leq h_{u, \delta}$ , which implies, since  $T$  is order preserving,  $h_{y(u), \gamma(u, \alpha)} \leq h_{y(u), \gamma(u, \delta)}$ , so that, in view of Corollary 2,  $\gamma(u, \alpha) \leq \gamma(u, \delta)$ . Since  $\gamma(u, \cdot)$  is one-to-one, we conclude that it is strictly increasing.  $\square$

Next we establish affinity of  $\gamma(\cdot, \cdot)$ .

**Proposition 7.** *If  $T \in \mathcal{B}$  then  $\gamma : X^* \times \mathbb{R} \rightarrow \mathbb{R}$ , defined by (8), is affine.*

*Proof.* We start by proving that  $\gamma$  is convex. Take  $u_1, u_2 \in X^*$ ,  $\alpha_1, \alpha_2 \in \mathbb{R}$  and  $s \in [0, 1]$ . We consider first the case in which  $u_1 \neq u_2$ . Define  $\bar{u} = su_1 + (1-s)u_2, \bar{\alpha} = s\alpha_1 + (1-s)\alpha_2$ . Observe that

$$sh_{u_1, \alpha_1} + (1-s)h_{u_2, \alpha_2} \leq \max\{h_{u_1, \alpha_1}, h_{u_2, \alpha_2}\}. \quad (14)$$

As in the proof of Proposition 5(i), we get from (14)

$$T(h_{\bar{u}, \bar{\alpha}}) \leq \max\{T(h_{u_1, \alpha_1}), T(h_{u_2, \alpha_2})\},$$

so that, for all  $x \in X$ ,

$$\begin{aligned} \langle y(su_1 + (1-s)u_2), x \rangle + \gamma(\bar{u}, \bar{\alpha}) &= s\langle y(u_1), x \rangle + (1-s)\langle y(u_2), x \rangle + \gamma(\bar{u}, \bar{\alpha}) \leq \\ &\max\{\langle y(u_1), x \rangle + \gamma(u_1, \alpha_1), \langle y(u_2), x \rangle + \gamma(u_2, \alpha_2)\}, \end{aligned} \quad (15)$$

using Corollary 4 in the equality. Since  $u_1 \neq u_2$ , and  $y$  is one-to-one by Proposition 5(i), we have that  $y(u_1) \neq y(u_2)$ , and hence there exists  $\bar{x} \in X$  such that

$$\langle y(u_1), \bar{x} \rangle + \gamma(u_1, \alpha_1) = \langle y(u_2), \bar{x} \rangle + \gamma(u_2, \alpha_2),$$

so that

$$\langle y(u_1) - y(u_2), \bar{x} \rangle = \gamma(u_2, \alpha_2) - \gamma(u_1, \alpha_1). \quad (16)$$

Replacing (16) in (15), and noting that for  $x = \bar{x}$  there is a tie between both arguments for the maximum in the rightmost expression of (15), we obtain, after some elementary algebra,

$$\langle y(u_2), \bar{x} \rangle + s\gamma(u_2, \alpha_2) - s\gamma(u_1, \alpha_1) + \gamma(\bar{u}, \bar{\alpha}) \leq \langle y(u_2), \bar{x} \rangle + \gamma(u_2, \alpha_2). \quad (17)$$

Rearranging terms in (17) and using the definitions of  $\bar{u}, \bar{\alpha}$ , we get

$$\gamma(s(u_1, \alpha_1) + (1-s)(u_2, \alpha_2)) \leq s\gamma(u_1, \alpha_1) + (1-s)\gamma(u_2, \alpha_2), \quad (18)$$

establishing joint convexity of  $\gamma$  in its two arguments. In order to obtain affinity of  $\gamma$ , it suffices to prove that (18) holds indeed with equality. Suppose that, on the contrary, there exist  $u_1, u_2 \in X^*$ ,  $u_1 \neq u_2$ ,  $\alpha_1, \alpha_2 \in \mathbb{R}$  and  $\bar{s} \in [0, 1]$  such that (18) holds with strict inequality, and take  $\theta \in \mathbb{R}$  such that

$$\gamma(\bar{s}(u_1, \alpha_1) + (1-\bar{s})(u_2, \alpha_2)) < \theta < \bar{s}\gamma(u_1, \alpha_1) + (1-\bar{s})\gamma(u_2, \alpha_2). \quad (19)$$

For the sake of a lighter notation, define  $\bar{\gamma} = \gamma(\bar{s}(u_1, \alpha_1) + (1 - \bar{s})(u_2, \alpha_2))$ ,  $\gamma_1 = \gamma(u_1, \alpha_1)$ ,  $\gamma_2 = \gamma(u_2, \alpha_2)$ . Since  $\gamma(\bar{u}, \cdot)$  is onto by Proposition 6, there exists  $\eta \in \mathbb{R}$  such that  $\theta = \gamma(\bar{u}, \eta)$ , and hence it follows from (19) that

$$\begin{aligned} h_{y(\bar{u}), \bar{\gamma}} &< h_{y(\bar{u}), \gamma(\bar{u}, \eta)} < h_{y(\bar{u}), \bar{s}\gamma_1 + (1 - \bar{s})\gamma_2} \leq \max \{h_{y(u_1), \gamma_1}, h_{y(u_2), \gamma_2}\} = \\ &\max \{\widehat{T}(u_1, \alpha_1), \widehat{T}(u_2, \alpha_2)\} = T(\max \{h_{u_1, \alpha_1}, h_{u_2, \alpha_2}\}). \end{aligned} \quad (20)$$

Since  $h_{y(\bar{u}), \bar{\gamma}} = T(h_{\bar{u}, \bar{\alpha}})$  and  $h_{y(\bar{u}), \gamma(\bar{u}, \eta)} = T(h_{\bar{u}, \eta})$ , we can rewrite (20) as

$$T(h_{\bar{u}, \bar{\alpha}}) < T(h_{\bar{u}, \eta}) \leq T(\max \{h_{u_1, \alpha_1}, h_{u_2, \alpha_2}\}),$$

which gives, since  $T$  is fully order preserving and one-to-one,

$$h_{\bar{u}, \bar{\alpha}} < h_{\bar{u}, \eta} \leq \max \{h_{u_1, \alpha_1}, h_{u_2, \alpha_2}\},$$

and therefore, for all  $x \in X$ ,

$$\langle \bar{u}, x \rangle + \bar{\alpha} < \langle \bar{u}, x \rangle + \eta \leq \max \{\langle u_1, x \rangle + \alpha_1, \langle u_2, x \rangle + \alpha_2\}. \quad (21)$$

Since  $u_1 \neq u_2$ , there exists  $\bar{x}$  such that  $\langle u_1 - u_2, \bar{x} \rangle = \alpha_2 - \alpha_1$ , in which case

$$\langle \bar{u}, \bar{x} \rangle + \bar{\alpha} = \langle u_1, \bar{x} \rangle + \alpha_1 = \langle u_2, \bar{x} \rangle + \alpha_2,$$

which contradicts (21), and establishes that

$$\gamma(s(u_1, \alpha_1) + (1 - s)(u_2, \alpha_2)) = s\gamma(u_1, \alpha_1) + (1 - s)\gamma(u_2, \alpha_2), \quad (22)$$

completing the proof of affinity of  $\gamma$  when  $u_1 \neq u_2$ . Now we deal with the remaining case, namely  $u_1 = u_2$ , and so we proceed to establish affinity of  $\gamma(u, \cdot)$  for a fixed  $u$ . First note that we have indeed established affinity of  $\gamma(\cdot, \alpha)$  for fixed  $\alpha$ , because for a fixed second argument (i.e.,  $\alpha_1 = \alpha_2$ ), (22) holds trivially when  $u_1 = u_2$ . Hence, for all  $\alpha \in \mathbb{R}$  the restriction of  $\gamma(\cdot, \alpha)$  to any finite dimensional subspace of  $X^*$ , being affine, is continuous. Take now  $u, v \in X^*$  with  $v \neq 0$ , so that  $u \neq u + v$ , and observe that (22) holds indeed with  $u_1 = u + v$ ,  $u_2 = u$ , and hence

$$\begin{aligned} \gamma(u + sv, \bar{\alpha}) &= \gamma(u + sv, s\alpha_1 + (1 - s)\alpha_2) = \\ &\gamma(s(u + v, \alpha_1) + (1 - s)(u, \alpha_2)) = s\gamma(u + v, \alpha_1) + (1 - s)\gamma(u, \alpha_2). \end{aligned} \quad (23)$$

We look now at the restrictions of  $\gamma(\cdot, \bar{\alpha})$  and  $\gamma(\cdot, \alpha_1)$  to the two dimensional subspace of  $X$  spanned by  $u$  and  $v$ . Since such restrictions are continuous, as already noted, after taking limits with  $v \rightarrow 0$  in (23), we obtain

$$\gamma(u, s\alpha_1 + (1 - s)\alpha_2) = s\gamma(u, \alpha_1) + (1 - s)\gamma(u, \alpha_2),$$

proving that  $\gamma(u, \cdot)$  is affine for all  $u \in X^*$ , and completing the proof of affinity of  $\gamma(\cdot, \cdot)$ .  $\square$

In the finite dimensional case, Proposition 7 would suffice for obtaining an explicit form of  $\widehat{T}$ , but in our setting we still have to prove continuity of  $\widehat{T}$ , which is not immediate from its affinity. For proving continuity, we will need the following elementary result.

**Proposition 8.** *Take  $T \in \mathcal{B}$ . If  $f \in \mathcal{C}(X)$  is finite everywhere, then  $T(f)$  is also finite everywhere.*

*Proof.* Take  $T \in \mathcal{B}$  and a finite everywhere  $f \in \mathcal{C}(X)$ . Suppose that  $T(f)(x_0) = \infty$  for some  $x_0 \in X$  and define  $g_0 : X \rightarrow \mathbb{R} \cup \{\infty\}$  as

$$g_0(x) = \begin{cases} 0 & \text{if } x = x_0 \\ \infty & \text{otherwise.} \end{cases}$$

Since  $T$  is onto, there exists  $f_0 \in \mathcal{C}(X)$  such that  $T(f_0) = g_0$ . Since  $f$  is finite everywhere,  $\tilde{f} = \max\{f, f_0\}$  belongs to  $\mathcal{C}(X)$ . Therefore, using Proposition 1(ii),

$$T(\tilde{f}) = \max\{T(f), T(f_0)\} = \max\{T(f), g_0\},$$

so that  $T(\tilde{f})(x) = \infty$  for all  $x \in X$ . Hence, we have that  $\tilde{f} \in \mathcal{C}(X)$  and  $T(\tilde{f}) \notin \mathcal{C}(X)$ , in contradiction with our assumptions on  $T$ .  $\square$

**Proposition 9.** *If  $T \in \mathcal{B}$  then both  $y : X^* \rightarrow X^*$  and  $\gamma : X^* \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous.*

*Proof.* First we prove that  $\gamma$  is continuous. Let  $g = \|\cdot\| \in \mathcal{C}(X)$  and define  $\bar{g} = T(g)$ . Let  $B$  be the unit ball in  $X^*$ , i.e.  $B = \{u \in X^* : \|u\| \leq 1\}$ . Since  $\|x\| = \sup_{u \in B} \{h_{u,0}(x) = \langle u, x \rangle\}$ , we have

$$\bar{g}(x) = T(g)(x) = \sup_{u \in B} \{(T(h_{u,0}))(x)\} = \sup_{u \in B} \{h_{\widehat{T}(u,0)}(x)\} = \sup_{u \in B} \{\langle y(u), x \rangle + \gamma(u, 0)\}. \quad (24)$$

Therefore,

$$\bar{g}(0) = \sup_{u \in B} \{\gamma(u, 0)\}. \quad (25)$$

Since  $\gamma : X^* \times \mathbb{R} \rightarrow \mathbb{R}$  is affine by Proposition 7, it can be written as  $\gamma(u, \alpha) = \tilde{\gamma}(u) + \hat{\gamma}(\alpha) + \mu$  where  $\mu = \gamma(0, 0) \in \mathbb{R}$  and  $\tilde{\gamma} : X^* \rightarrow \mathbb{R}, \hat{\gamma} : \mathbb{R} \rightarrow \mathbb{R}$  are linear functionals. Using this representation of  $\gamma$  and (25), we conclude that  $\sup_{u \in B} \{\tilde{\gamma}(u)\} + \mu = \bar{g}(0) < \infty$ , where the inequality follows from Proposition 8 and the definition of  $g$ . Therefore

$$|\tilde{\gamma}(u)| = \max\{\tilde{\gamma}(u), \tilde{\gamma}(-u)\} \leq \bar{g}(0) - \mu \in \mathbb{R},$$

for all  $u \in B$ , and hence  $\tilde{\gamma}$  is a bounded linear functional with  $\|\tilde{\gamma}\| \leq \bar{g}(0) - \mu$ , so that  $\tilde{\gamma}$  is continuous. Continuity of  $\hat{\gamma}$  follows trivially from its linearity and the fact that its domain is finite dimensional. Hence  $\gamma$  is continuous.

For proving continuity of  $y$ , observe that  $y$  can be written as  $y(u) = Du + w$  where  $D : X^* \rightarrow X^*$  is a linear operator and  $w = y(0) \in X^*$ . Combining this representation of  $y$  with (24) we have

$$\sup_{u \in B} \{\langle Du + w, x \rangle + \langle \tilde{\gamma}, u \rangle + \mu\} = \bar{g}(x) < \infty \quad (26)$$

for all  $x \in X$ , where the inequality follows from Proposition 8 and the definition of  $g$ . Take  $u \in B$  and conclude from (26) that

$$\langle Du, x \rangle \leq \bar{g}(x) - \langle w, x \rangle - \tilde{\gamma}(u) - \mu \leq \bar{g}(x) - \langle w, x \rangle + \|\tilde{\gamma}\| \|u\| - \mu \leq \bar{g}(x) - \langle w, x \rangle + \|\tilde{\gamma}\| - \mu.$$

Therefore

$$\sup_{u \in B} \{|\langle Du, x \rangle|\} = \sup_{u \in B} \{\langle Du, x \rangle\} < \infty$$

for all  $x \in X$ . This means that the family of bounded linear operators  $\{Du \mid u \in B\}$  is pointwise bounded. Applying Banach-Steinhaus uniform boundedness principle (see, e.g., [7], p. 32), we get that this family is uniformly bounded, i.e., there exists  $\nu < \infty$  such that  $\|Du\| \leq \nu$  for all  $u \in B$ . It follows that  $\|D\| \leq \nu$ , and therefore  $D$  is bounded and linear, hence continuous. We conclude that the operator  $y$  defined as  $y(u) = Du + w$  is continuous, completing the proof.  $\square$

Now we present a more explicit formula for the operator  $\widehat{T}$ .

**Corollary 5.** *If  $T \in \mathcal{B}$ , then there exist  $d \in X^{**}$ ,  $w \in X^*$ ,  $\beta \in \mathbb{R}$ ,  $\tau \in \mathbb{R}_{++}$  and a continuous automorphism  $D$  of  $X^*$  such that  $\widehat{T}(u, \alpha) = (Du + w, \langle d, u \rangle + \tau\alpha + \beta)$ .*

*Proof.* Being affine and continuous by virtue of Corollary 4 and Propositions 7 and 9 we have that  $y(u) = Du + w$ ,  $\gamma(u, \alpha) = \langle d, u \rangle + \tau\alpha + \beta$ , with  $D, d, w, \tau$  and  $\beta$  as in the statement of the corollary. The facts that  $D$  is an automorphism of  $X^*$  and that  $\tau$  is positive follow from Propositions 5(i) and 6, which establish that  $y$  is a bijection and that  $\gamma(u, \cdot)$  is increasing, respectively.  $\square$

We recall now two basic properties of the Fenchel biconjugate, which will be needed in the proof of our main result.

**Proposition 10.** *i) For all  $f \in \mathcal{C}(X)$ ,  $f|_X^{**} = f$ , where  $f|_X^{**}$  denotes the restriction of  $f^{**}$  to  $X \subset X^{**}$ .*

*ii) For all  $f \in \mathcal{C}(X)$  and all  $a \in X^{**}$ ,  $f^{**}(a) = \sup_{h_{u,\alpha} \in A(f)} \{\langle a, u \rangle + \alpha\}$ .*

*Proof.* A proof of item (i), sometimes called Fenchel-Moreau Theorem, can be found, for instance, in Theorem 1.11 of [7].

We proceed to prove item (ii). Observe that, by definition of  $f^{**}$ ,  $f^{**}(a) = \sup_{u \in X^*} \{\langle a, u \rangle - f^*(u)\}$ . Also, by definition of  $f^*$ ,  $f^*(u) = \sup_{x \in X} \{\langle u, x \rangle - f(x)\}$ , so that  $h_{u, -f^*(u)}(x) = \langle u, x \rangle - f^*(u) \leq f(x)$  for all  $x \in X$ , and therefore  $h_{u, -f^*(u)}$  belongs to  $A(f)$  for all  $u \in X^*$  (we may assume

$f^*(u) < +\infty$  when taking the supremum over  $u \in X^*$ , in the formula of  $f^{**}(a)$ ). Next, we prove that if  $h_{u,\alpha}$  belongs to  $A(f)$  then  $\alpha \leq -f^*(u)$ . Indeed, if  $h_{u,\alpha} \in A(f)$  then, for all  $x \in X$ ,

$$\alpha \leq \inf_{x \in X} \{f(x) - \langle u, x \rangle\} = \inf_{x \in X} \{-(\langle u, x \rangle - f(x))\} = -\sup_{x \in X} \{\langle u, x \rangle - f(x)\} = -f^*(u). \quad (27)$$

In view of (27) and the fact that  $h_{u,-f^*(u)} \in A(f)$  for all  $u \in X^*$ , we have

$$\sup_{h_{u,\alpha} \in A(f)} \{\langle a, u \rangle + \alpha\} = \sup_{h_{u,-f^*(u)} \in A(f)} \{\langle a, u \rangle - f^*(u)\} = \sup_{u \in X^*} \{\langle a, u \rangle - f^*(u)\} = f^{**}(a).$$

□

We mention, parenthetically, that the Fenchel-Moreau Theorem can be easily deduced from Proposition 10(ii). Next we present our first main result, characterizing the operators  $T \in \mathcal{B}$

**Theorem 1.** *An operator  $T : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  is fully order preserving if and only if there exist  $c \in X$ ,  $w \in X^*$ ,  $\beta \in \mathbb{R}$ ,  $\tau \in \mathbb{R}_{++}$  and a continuous automorphism  $E$  of  $X$  such that*

$$T(f)(x) = \tau f(Ex + c) + \langle w, x \rangle + \beta, \quad (28)$$

for all  $f \in \mathcal{C}(X)$  and all  $x \in X$ .

*Proof.* We start with the “only if” statement. As already mentioned, a basic convex analysis result establishes that  $f = \sup_{h \in A(f)} h$ . In view of Proposition 1(ii), we have

$$T(f) = \sup_{h \in A(f)} \widehat{T}(h) = \sup_{h_{u,\alpha} \in A(f)} \widehat{T}(h_{u,\alpha}) = \sup_{h_{u,\alpha} \in A(f)} h_{y(u),\gamma(u,\alpha)},$$

meaning that

$$T(f)(x) = \sup_{h_{u,\alpha} \in A(f)} \{\langle y(u), x \rangle + \gamma(u, \alpha)\} = \sup_{h_{u,\alpha} \in A(f)} \{\langle Du + w, x \rangle + \langle d, u \rangle + \tau\alpha + \beta\}, \quad (29)$$

using Corollary 5 in the last equality. Define now  $C \in \text{Aut}(X^*)$ ,  $c \in X^{**}$  as  $C = \tau^{-1}D$ ,  $c = \tau^{-1}d$ . Continuing from (29),

$$T(f)(x) = \sup_{h_{u,\alpha} \in A(f)} \{\langle Du, x \rangle + \langle d, u \rangle + \tau\alpha\} + \langle w, x \rangle + \beta = \tau \left[ \sup_{h_{u,\alpha} \in A(f)} \{\langle Cu, x \rangle + \langle c, u \rangle + \alpha\} \right] + \langle w, x \rangle + \beta = \tau \left[ \sup_{h_{u,\alpha} \in A(f)} \{\langle C^*x + c, u \rangle + \alpha\} \right] + \langle w, x \rangle + \beta. \quad (30)$$

Note that  $C^*$  is an operator in  $X^{**}$ , so that the expression  $C^*x$  in (30) must be understood through the natural immersion of  $X$  in  $X^{**}$  (i.e., from now on we consider  $X$  as a subspace of  $X^{**}$ ). Observe that, by Proposition 10(ii),

$$\sup_{h_{u,\alpha} \in A(f)} \{\langle a, u \rangle + \alpha\} = f^{**}(a). \quad (31)$$

Replacing (31) in (30), with  $a = C^*x + c \in X^{**}$ , we get

$$T(f)(x) = \tau f^{**}(C^*x + c) + \langle w, x \rangle + \beta. \quad (32)$$

Define  $Y \subset X^{**}$  as  $Y = \{C^*x + c : x \in X\}$ . We claim that  $Y = X$ . Assume first that there exists  $\tilde{x} \in X \setminus Y$ , and define  $f_1 : X \rightarrow \mathbb{R} \cup \{+\infty\}$  as the indicator function of  $\{\tilde{x}\}$ , i.e.  $f_1(x) = 0$  if  $x = \tilde{x}$ ,  $f_1(x) = +\infty$  otherwise. It is easy to check that  $f_1^{**} : X^{**} \rightarrow \mathbb{R} \cup \{+\infty\}$  is still the indicator function of  $\{\tilde{x}\}$ , seen now as a subset of  $X^{**}$ . We look now at  $T(f_1)$ . Since  $\tilde{x} \notin Y$ , we have  $C^*x + c \neq \tilde{x}$  for all  $x \in X$ , so that  $f_1^{**}(C^*x + c) = +\infty$  for all  $x \in X$ . In view of (32),  $T(f_1)(x) = +\infty$  for all  $x \in X$ , implying that  $T(f_1)$  is not proper, and so  $T(f_1) \notin \mathcal{C}(X)$ , contradicting our assumptions on  $T$ . We have shown that  $X \subset Y$ . Suppose now that there exists  $\tilde{x} \in Y \setminus X$ . Since  $\tilde{x}$  belongs to  $Y$ , there exists  $x' \in X$  such that  $C^*x' + c = \tilde{x}$ . Define  $f_2 : X \rightarrow \mathbb{R} \cup \{+\infty\}$  as the indicator function of  $\{x'\}$ , i.e.  $f_2(x) = 0$  if  $x = x'$ ,  $f_2(x) = +\infty$  otherwise. Since  $T$  is onto, there exists  $f \in \mathcal{C}(X)$  such that  $T(f) = f_2$ . Let  $\xi = -\tau^{-1}(\langle w, x' \rangle + \beta)$ . In view of (31) we have  $f^{**}(C^*x + c) = \xi$  if  $x = x'$ ,  $f^{**}(C^*x + c) = +\infty$  otherwise. So, there exists a unique point in  $Y$ , namely  $C^*x' + c = \tilde{x}$ , where  $f^{**}$  takes a finite value, or equivalently,  $f^{**}(z) = +\infty$  for all  $z \in Y \setminus \{\tilde{x}\}$ . Since we already know that  $X \subset Y$  and  $\tilde{x} \notin X$ , so that  $X \subset Y \setminus \{\tilde{x}\}$ , we conclude that  $f^{**}(z) = +\infty$  for all  $z \in X$ , and hence, in view of Proposition 10(i),  $f(x) = +\infty$  for all  $x \in X$ , so that  $f$  is not proper, implying that  $f \notin \mathcal{C}(X)$ , a contradiction. We have completed the proof of the equality between  $X$  and  $Y$ , establishing the claim. Now, since  $C^*x + c \in X$  for all  $x \in X$ , as we have just proved, we apply Proposition 10(i), and rewrite (32) as:

$$T(f)(x) = \tau f(C^*x + c) + \langle w, x \rangle + \beta. \quad (33)$$

We observe now that, since  $C^*x + c \in X$  for all  $x \in X$ , taking  $x = 0$ , we get  $c \in X$ . As a consequence, the restriction of  $C^*$  to  $X$  is an automorphism of  $X$ . At this point, that fact that this automorphism is the restriction to  $X$  of the adjoint of an automorphism  $C$  of  $X^*$ , is not significant any more, since any automorphism  $E$  of  $X$  is of the restriction to  $X$  of some  $C^* : X^{**} \rightarrow X^{**}$ : it suffices to define  $C : X^* \rightarrow X^*$  as  $C = E^*$ . Substituting  $E$  for  $C^*$  in (33), we recover (28), completing the proof of the “only if” statement.

Now we prove the “if” statement, i.e., we must show that operators  $T$  of the form given by (28) belong to  $\mathcal{B}$ . First, it is immediate that such operators map  $\mathcal{C}(X)$  to  $\mathcal{C}(X)$ . Now, we will denote as  $T[E, c, w, \tau, \beta]$  the operator  $T$  defined by (28). Clearly,  $T[E, c, w, \tau, \beta]$  is order preserving. In order to complete the proof, we must show that  $T[E, c, w, \tau, \beta]$  is fully order preserving and onto. It suffices to show that  $T[E, c, w, \tau, \beta]$  is invertible, and that its inverse is also order preserving. We claim that

$$T[E, c, w, \tau, \beta]^{-1} = T[\bar{E}, \bar{c}, \bar{w}, \bar{\tau}, \bar{\beta}], \quad (34)$$

with

$$\bar{E} = E^{-1}, \quad \bar{c} = -E^{-1}c, \quad \bar{w} = -\tau^{-1}(E^{-1})^*w, \quad \bar{\tau} = \tau^{-1}, \quad \bar{\beta} = \tau^{-1}(\langle E^{-1}c, w \rangle - \beta). \quad (35)$$

Note that  $E^{-1}$  exists because  $E$  is a continuous automorphism of  $X$ , and is continuous by virtue of the Closed Graph Theorem (see, e.g., Corollary 2.7 in [7]). For the sake of a lighter notation, define  $T = T[E, c, w, \tau, \beta]$ ,  $\bar{T} = T[\bar{E}, \bar{c}, \bar{w}, \bar{\tau}, \bar{\beta}]$ . Observe that, for all  $f \in \mathcal{C}(X)$ , in view of the definitions of  $T$  and  $\bar{T}$ ,

$$\begin{aligned}\bar{T}(T(f)) &= \bar{T}(\tau f(E(\cdot) + c) + \langle w, \cdot \rangle + \beta) = \\ &= \bar{\tau}[\tau f(E(\bar{E}(\cdot) + \bar{c}) + c) + \langle w, \bar{E}(\cdot) + \bar{c} \rangle + \beta] + \langle \bar{w}, \cdot \rangle + \bar{\beta} = f(\cdot),\end{aligned}$$

using the definitions of  $\bar{E}, \bar{c}, \bar{w}, \bar{\tau}$  and  $\bar{\beta}$ . It follows that  $\bar{T}(T(f)) = f$  for all  $f \in \mathcal{C}(X)$ . A similar computation shows that  $T(\bar{T}(f)) = f$  for all  $f \in \mathcal{C}(X)$ , and hence  $\bar{T} = T^{-1}$  so that  $T$  is onto. Since  $\bar{T} = T[\bar{C}, \bar{c}, \bar{w}, \bar{\tau}, \bar{\beta}]$  is also of the form given by (28), we conclude that  $T^{-1} = \bar{T}$  is order preserving, showing that  $T$  is fully order preserving, and completing the proof.  $\square$

We remark that in the reflexive case, the fact that  $X = Y$  follows immediately from the equality between  $X$  and  $X^{**}$ .

The result in Theorem 1 can be rephrased as saying that the identity operator is the only fully order preserving operator in  $\mathcal{C}(X)$ , up to addition of affine functionals, pre-composition with affine operators, and multiplication by positive scalars, thus extending to Banach spaces the result established in [3] for the finite dimensional case.

It is also worthwhile to consider another set of operators on  $\mathcal{C}(X)$ , namely the one consisting of *order preserving involutions*, i.e., of those order preserving operators  $T : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  which are *involutions*, namely operators  $T$  such that  $T(T(f)) = f$  for all  $f \in \mathcal{C}(X)$ . Since for such a  $T$  we have  $T^{-1} = T$ , it follows that  $T^{-1}$  is also order preserving, and hence  $T$  is fully order preserving, so that such set is indeed a subset of  $\mathcal{B}$ . The characterization of order preserving involutions in  $\mathcal{C}(X)$  is given in the following corollary, whose finite dimensional version can be found in Theorem 1 of [3].

**Corollary 6.** *An operator  $T : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  is an order preserving involution if and only if there exist  $c \in X$ ,  $w \in X^*$  and a continuous automorphism  $E$  of  $X$ , satisfying  $E^2 = I_X, c \in \text{Ker}(E + I_X), w \in \text{Ker}(E^* + I_{X^*})$  such that*

$$T(f)(x) = f(Ex + c) + \langle w, x \rangle - \frac{1}{2}\langle c, w \rangle \quad (36)$$

for all  $x \in X$ , where  $I_X$  denotes the identity operator in  $X$ .

*Proof.* It is immediate that  $T$  is an order preserving involution if and only if  $T \in \mathcal{B}$  and  $T = T^{-1}$ . In view of the formula for  $T^{-1}$  given by (34), an operator  $T \in \mathcal{B}$  satisfies  $T = T^{-1}$  if and only if  $T[E, c, w, \tau, \beta] = T[\bar{E}, \bar{c}, \bar{w}, \bar{\tau}, \bar{\beta}]$ , with  $\bar{E}, \bar{c}, \bar{w}, \bar{\tau}, \bar{\beta}$  given by (35), which occurs if and only if  $\bar{E} = E, \bar{c} = c, \bar{w} = w, \bar{\tau} = \tau$  and  $\bar{\beta} = \beta$ , as can be seen after some elementary algebra. It is easy to check, using (35), that these equalities are equivalent to  $E^2 = I_X, c \in \text{Ker}(E + I_X), w \in \text{Ker}(E^* + I_{X^*}), \tau = 1$  and  $\beta = -\langle w, c \rangle/2$ , in which case (28) reduces to (36), so that the result is just a consequence of Theorem 1.  $\square$



### 3 The order reversing case

In this section we will characterize fully order reversing operators, but before defining them formally, we must discuss the appropriate co-domains for such operators. In [3], the prototypical fully order reversing operator turns out to be the Fenchel conjugation, and it is reasonable to expect that it will play a similar role in the infinite dimensional case. Of course, the operator which sends  $f \in \mathcal{C}(X)$  to its Fenchel conjugate  $f^* : X^* \rightarrow \mathbb{R} \cup \{+\infty\}$  is well defined in any Banach space; however its co-domain is not  $\mathcal{C}(X)$ , but rather  $\mathcal{C}(X^*)$ . There is an additional complication. As we have seen in the order preserving case, the fact that we are dealing with surjective operators, whose inverses enjoy a similar order preserving property, is rather essential for the characterization. It turns out to be the case that in nonreflexive Banach spaces the Fenchel conjugation is not onto, generally speaking. In fact, if we take  $c \in X^{**} \setminus X$ , it is easy to check that the linear functional  $\langle c, \cdot \rangle : X^* \rightarrow \mathbb{R}$  is not the Fenchel conjugate of any function in  $\mathcal{C}(X)$  (the natural candidate would be the indicator function of the singleton  $\{c\}$ , but it does not belong to  $\mathcal{C}(X)$ , because  $c \notin X$ ). One way to overcome this obstacle is to reduce somewhat the co-domain of the operators of interest. It is well known that convex functions are lower semicontinuous in the strong topology if and only if they are lower semicontinuous in the weak one. On the other hand, if we consider a dual space  $X^*$  with the weak\* topology, the class of lower semicontinuous convex functions with respect to this topology is strictly smaller; in fact a linear functional  $\langle c, \cdot \rangle$  with  $c \in X^{**} \setminus X$  is not weak\* lower semicontinuous. It is well known that the topological dual of  $X^*$  with the weak\* topology is precisely  $X$ , so that the topological bidual of  $X^*$  with this topology is  $X^*$ . Also, the biconjugate of a weak\* lower semicontinuous proper convex  $g : X^* \rightarrow \mathbb{R} \cup \{+\infty\}$  coincides with  $g$ , and hence the image of  $\mathcal{C}(X)$  through the Fenchel conjugation coincides with the class of weak\* lower semicontinuous proper convex functions defined on  $X^*$ .

Thus, we define  $\mathcal{C}_{w^*}(X^*)$  as the set of weak\* lower semicontinuous proper convex functions  $g : X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ . We give next the formal definition of order reversing operators.

**Definition 2.** *i) An operator  $S : \mathcal{C}(X) \rightarrow \mathcal{C}_{w^*}(X^*)$  is order reversing whenever  $f \leq g$  implies  $S(f) \geq S(g)$ .*

*ii) An operator  $S : \mathcal{C}(X) \rightarrow \mathcal{C}_{w^*}(X^*)$  is fully order reversing whenever  $f \leq g$  iff  $S(f) \geq S(g)$ , and additionally  $S$  is onto.*

We will characterize fully order reversing operators with this definition, and the result will be an easy consequence of Theorem 1. The issue of characterizing fully order reversing operators  $S : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  (or  $S : \mathcal{C}(X) \rightarrow \mathcal{C}(X^*)$  in the nonreflexive case), are left as open problems, deserving future research.

We start with an elementary result on the conjugate of compositions with affine operators and additions of affine functionals.

**Proposition 11.** Consider  $f \in \mathcal{C}(X)$ . Let  $E$  be a continuous automorphism of  $X$ , and take  $c \in X, w \in X^*, \beta \in \mathbb{R}, \tau \in \mathbb{R}_{++}$ . Define  $g \in \mathcal{C}(X)$  as  $g(x) = \tau f(Ex + c) + \langle w, x \rangle + \beta$ . Then

$$g^*(u) = \tau f^*(Hu + v) + \langle u, y \rangle + \rho$$

for all  $u \in X^*$ , where

$$H = \tau^{-1}(E^{-1})^*, \quad v = -\tau^{-1}(E^{-1})^*w, \quad y = -\tau^{-1}E^{-1}c \in X, \quad \rho = \tau^{-1}(\langle w, E^{-1}c \rangle - \beta).$$

*Proof.* Elementary. □

Now we present our characterization of fully order reversing operators, which extends to Banach spaces the similar finite dimensional result in [3].

**Theorem 2.** An operator  $S : \mathcal{C}(X) \rightarrow \mathcal{C}_{w^*}(X^*)$  is fully order reversing if and only if there exist  $v \in X^*, y \in X, \rho \in \mathbb{R}, \tau \in \mathbb{R}_{++}$  and a continuous automorphism  $H$  of  $X^*$  such that

$$S(f)(u) = \tau f^*(Hu + v) + \langle u, y \rangle + \rho,$$

for all  $f \in \mathcal{C}(X)$  and all  $u \in X^*$ .

*Proof.* Define  $F : \mathcal{C}(X) \rightarrow \mathcal{C}(X^*)$  as  $F(f) = f^*$ , i.e.  $F$  is the Fenchel conjugation. It is immediate from its definition that  $F$  is order reversing. Note that  $f^*$  is convex, proper, and also lower semicontinuous in the weak\* topology for all  $f \in \mathcal{C}(X)$ , being the supremum of affine functions which are continuous in this topology. Hence,  $F(\mathcal{C}(X)) \subset \mathcal{C}_{w^*}(X^*)$ . We recall also that the topological dual of  $X^*$  with the weak\* topology is precisely  $X$  with the strong topology (see, e.g., Proposition 2.3(ii) in [6]). We observe that  $F$  is invertible; in fact, its inverse  $F^{-1} : \mathcal{C}_{w^*}(X^*) \rightarrow \mathcal{C}(X)$  is given by

$$F^{-1}(g) = \sup_{u \in X^*} \{\langle u, \cdot \rangle - g(u)\} \tag{37}$$

for all  $g \in \mathcal{C}_{w^*}(X^*)$ . We conclude from (37) that  $F(\mathcal{C}(X)) = \mathcal{C}_{w^*}(X^*)$ . It follows also from (37) that  $F^{-1}$  is order reversing, and, since  $F$  is also order reversing, we obtain from the characterization of the image of  $F$  that  $F^{-1}$  is fully order reversing.

Consider now the operator  $F^{-1} \circ S : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$ . Being the composition of two fully order reversing operators, it is fully order preserving. It follows from Theorem 1 that there exist  $c \in X, w \in X^*, \beta \in \mathbb{R}, \tau \in \mathbb{R}_{++}$  and a continuous automorphism  $E$  of  $X$ , such that

$$F^{-1} \circ S = T[E, c, w, \tau, \beta], \tag{38}$$

where  $T[E, c, w, \tau, \beta] \in \mathcal{B}$ , as in the proof of Theorem 1, is defined as

$$T[E, c, w, \tau, \beta](f)(x) = \tau f(Ex + c) + \langle w, x \rangle + \beta,$$

for all  $f \in \mathcal{C}(X)$  and all  $x \in X$ . It follows from (38) that  $S = F \circ T[E, c, w, \tau, \beta]$ , so that  $S(f) = g^*$ , where  $g \in \mathcal{C}(X)$  is defined as  $g(x) = \tau f(Ex + c) + \langle w, x \rangle + \beta$ . The “only if” statement follows then from Proposition 11. The “if” statement is a consequence of the facts that the Fenchel conjugation is a fully order reversing operator and the affine operator  $L(u) = Hu + c$  is invertible. □

We remark that when  $X$  is reflexive the weak and the weak\* topologies in  $X^*$  coincide, so that  $\mathcal{C}(X^*) = \mathcal{C}_{w^*}(X^*)$ , and hence in this case Theorem 2 characterizes the set of fully order reversing operators  $S : \mathcal{C}(X) \rightarrow \mathcal{C}(X^*)$ . In fact, it is easy to check that reflexivity of  $X$  is equivalent to surjectivity of the Fenchel conjugation as a map from  $\mathcal{C}(X)$  to  $\mathcal{C}(X^*)$ .

As in the case of fully order preserving operators, the result of Theorem 2 can be rephrased as saying that the Fenchel conjugation is the only fully order reversing operator from  $\mathcal{C}(X)$  to  $\mathcal{C}_{w^*}(X^*)$ , up to addition of affine functionals, pre-composition with affine operators, and multiplication by positive scalars.

## References

- [1] Artin, E. *Geometric Algebra*. Interscience, New York (1957).
- [2] Artstein-Avidan, S., Milman, V. The concept of duality for measure projections of convex bodies. *Journal of Functional Analysis* **254** (2008), 2648-2666.
- [3] Artstein-Avidan, S., Milman, V. The concept of duality in convex analysis, and the characterization of the Legendre transform. *Annals of Mathematics* **169** (2009), 661-674.
- [4] Artstein-Avidan, S., Milman, V. Hidden structures in the class of convex functions and a new duality transform. *Journal of the European Mathematical Society* **13** (2011), 975-1004.
- [5] Artstein-Avidan, S., Slomka, B.A. Order isomorphisms in cones and a characterization of duality for ellipsoids. *Selecta Mathematica* **18** (2012), 391-415.
- [6] Barbu, V., Precupanu, T. *Convexity and optimization in Banach spaces*. Editura Academiei, Bucarest (1978).
- [7] Brezis, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, Berlin (2011).
- [8] Schneider, R. The endomorphisms of the lattice of closed convex cones. *Beiträge zur Algebra und Geometrie* **49** (2008) 541-547.
- [9] Slomka, B.A. On duality and endomorphisms of lattices of closed convex sets. *Advances in Geometry* **11** (2011), 225-239.
- [10] Van Tiel, J. *Convex Analysis, an Introductory Text*. John Wiley, New York (1984).
- [11] Wright, S.E. Invertibility of order-reversing transforms on convex functions. *Journal of Convex Analysis* **17** (2010), 103-110.