LINEAR AND MULTIPLICATIVE 2-FORMS

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Abstract. We study the relationship between multiplicative 2-forms on Lie groupoids and linear 2-forms on Lie algebroids, which leads to a new approach to the infinitesimal description of multiplicative 2-forms and to the integration of twisted Dirac manifolds.

Contents

1. Introduction

The main purpose of this paper is to offer an alternate viewpoint to the study of multiplicative 2-forms on Lie groupoids and their infinitesimal counterparts carried out in [3]. This study turns out to be closely related to topics such as equivariant cohomology and generalized moment maps theories, see e.g. [2, 3, 19]. A particularly important case is that of symplectic multiplicative 2-forms (i.e., *symplectic groupoids*), whose infinitesimal counterparts are Poisson structures [6]. As shown in [3], infinitesimal versions of more general multiplicative 2-forms include twisted Dirac structures in the sense of [17].

Let $\mathcal{G}$ be a Lie groupoid over M, with source and target maps $\mathsf{s}, \mathsf{t} : \mathcal{G} \longrightarrow M$, and multiplication $m : \mathcal{G}^{(2)} \longrightarrow \mathcal{G}$. Let A be the Lie algebroid of $\mathcal{G}$, with Lie bracket $[\cdot,\cdot]$ on $\Gamma(A)$ and anchor $\rho : A \longrightarrow TM$. A 2-form $\omega \in \Omega^2(\mathcal{G})$ is called *multiplicative* if

$$m^*\omega = p_1^*\omega + p_2^*\omega,$$

where $p_1, p_2 : \mathcal{G}^{(2)} \longrightarrow \mathcal{G}$ are the natural projections. Given $\phi \in \Omega^3(M)$ closed, we say that ω is *relatively ϕ-closed* if $d\omega = \mathsf{s}^*\phi - \mathsf{t}^*\phi$. The main result in [3] asserts that, if $\mathcal{G}$ is s-simply-connected, then there exists a one-to-one correspondence between multiplicative 2-forms $\omega \in \Omega^2(\mathcal{G})$ and vector bundle maps $\sigma : A \longrightarrow T^*M$ satisfying

$$\langle \sigma(u), \rho(v) \rangle = -\langle \sigma(v), \rho(u) \rangle$$
$$\sigma([u,v]) = \mathcal{L}_{\rho(u)}\sigma(v) - i_{\rho(v)} d\sigma(u) + i_{\rho(v)} i_{\rho(u)}\phi,$$

for all $u, v \in \Gamma(A)$. We refer to such maps σ as *IM 2-forms* relative to ϕ (*IM* stands for *infinitesimal multiplicative*). If $L \subset TM \oplus T^*M$ is a ϕ-twisted Dirac structure, then the projection $L \longrightarrow T^*M$ is naturally an IM 2-form, so the correspondence above includes the integration of twisted Dirac structures as a special case.

The IM 2-form associated with a multiplicative 2-form $\omega \in \Omega^2(\mathcal{G})$ is simply

$$\sigma(u) = i_u\omega|_{TM}, \quad u \in A, \tag{1.1}$$

where A and TM are naturally viewed as subbundles of $T\mathcal{G}|_M$. The construction of ω from a given $\sigma : A \longrightarrow T^*M$ in [3, Sec. 5] relies on the identification of $\mathcal{G}$ with A-homotopy classes of A-paths (in the sense of [9], c.f. [16]), in such a way that ω is obtained by a variation of the infinite dimensional reduction procedure of [9]. A different, more general, viewpoint to this problem has been recently studied in [1], where this correspondence is seen as part of a general Van Est isomorphism.

In this paper, we avoid the use of path spaces by noticing that the construction of a multiplicative $\omega \in \Omega^2(\mathcal{G})$ out of an IM 2-form σ can be phrased as the integration of a suitable Lie algebroid morphism, similar in spirit to the approach of Mackenzie and Xu [13, 14] to the problem of integrating Lie bialgebroids to Poisson groupoids, which served as our main source of inspiration.

We notice that any multiplicative 2-form $\omega \in \Omega^2(\mathcal{G})$ naturally induces a 2-form $\Lambda \in \Omega^2(A)$ on the total space of A, which is *linear* in a suitable sense. We show that, when ω is relatively ϕ-closed, the 2-form Λ is totally determined by the map σ (1.1) and ϕ via the formula

$$\Lambda = -(\sigma^*\omega_{can} + \rho^*\tau(\phi)), \tag{1.2}$$

where ω_{can} is the canonical symplectic form on T^*M, and $\tau(\phi) \in \Omega^2(TM)$ is the 2-form defined, at each point $X \in TM$, by $\tau(\phi)|_X = p_M^*(i_X\phi)$, where $p_M : TM \longrightarrow M$ denotes the natural projection.

As a key step to reconstruct multiplicative 2-forms from infinitesimal data, consider an arbitrary Lie algebroid $A \longrightarrow M$, along with a vector bundle map $\sigma : A \longrightarrow T^*M$ and a closed $\phi \in \Omega^3(M)$. Let us use σ and ϕ to define $\Lambda \in \Omega^2(A)$ by (1.2). Our main observation is that the bundle map

$$\Lambda^\sharp : TA \longrightarrow T^*A, \;\; U \mapsto i_U\Lambda$$

is a morphism between tangent and cotangent Lie algebroids (see [13]) if and only if σ is an IM 2-form relative to ϕ. This result can be immediately applied to the integration of IM 2-forms: the morphism of groupoids $T\mathcal{G} \longrightarrow T^*\mathcal{G}$ obtained by integrating the morphism $\Lambda^\sharp : TA \longrightarrow T^*A$ determines the desired multiplicative 2-form. Our approach to multiplicative 2-forms can be naturally extended in different directions, e.g. to forms of higher degree or forms with no prescription on their exterior derivatives, as recently done in [1] from a different perspective. These extensions and a comparison with [1] will be discussed in a separate paper.

The paper is organized as follows. In Section 2 we briefly recall the definitions and main properties of tangent and cotangent Lie algebroids and groupoids. In Section 3, we discuss the construction of linear 2-forms on Lie algebroids associated with multiplicative 2-forms on Lie groupoids. In Section 4, we relate IM 2-forms with linear 2-forms defining algebroid morphisms $TA \longrightarrow T^*A$, and apply our results to integration of IM 2-forms.

1.1. **Notations and conventions.** For a Lie groupoid $\mathcal{G}$ over M, its source and target maps are denoted by s, t. Composable pairs $(g,h) \in \mathcal{G}^{(2)} = \mathcal{G} \times_M \mathcal{G}$ are such that $\mathsf{s}(g) = \mathsf{t}(h)$, and the multiplication map is denoted by $m : \mathcal{G}^{(2)} \to \mathcal{G}$, $m(g,h) = gh$. Its Lie algebroid is $A\mathcal{G} = \ker(Ts)|_M$, with anchor $T\mathsf{t}|_A : A \longrightarrow TM$, and bracket induced by right-invariant vector fields. For a general Lie algebroid $A \longrightarrow M$, we denote its anchor by ρ_A and bracket by $[\cdot,\cdot]_A$ (or simply ρ and $[\cdot,\cdot]$ if there is no risk of confusion). Given vector bundles $A \to M$ and $B \to M$, vector bundle maps $A \to B$ in this paper are assumed to cover the identity map (unless otherwise stated). Einstein's summation convention is consistently used throughout the paper.

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2. Tangent and cotangent structures

In this section, we briefly recall tangent and cotangent algebroids and groupoids, following [12, 13], where readers can find more details.

2.1. **Tangent and cotangent Lie groupoids.** Let $\mathcal{G}$ be a Lie groupoid over M, with Lie algebroid $A\mathcal{G}$ (if there is no risk of confusion, we may denote $A\mathcal{G}$ simply by A). The tangent bundle $T\mathcal{G}$ has a natural Lie groupoid structure over TM, with source (resp. target) map given by $T\mathsf{s} : T\mathcal{G} \longrightarrow TM$ (resp. $T\mathsf{t} : T\mathcal{G} \longrightarrow TM$). The multiplication on $T\mathcal{G}$ is defined by $Tm : T\mathcal{G}^{(2)} = (T\mathcal{G})^{(2)} \longrightarrow T\mathcal{G}$. We refer to this groupoid as the **tangent groupoid** of $\mathcal{G}$.

The cotangent bundle $T^*\mathcal{G}$ has a Lie groupoid structure over A^*, known as the **cotangent groupoid** of $\mathcal{G}$. The source and target maps are given by

$$\tilde{\mathsf{s}}(\alpha_g)u = \alpha_g(Tl_g(u - T\mathsf{t}(u))), \qquad \tilde{\mathsf{t}}(\beta_g)v = \beta_g(Tr_g(v))$$

where $\alpha_g, \beta_g \in T^*_g\mathcal{G}$, $u \in A_{s(g)}$, and $v \in A_{t(g)}$. Here $l_g : \mathsf{t}^{-1}(\mathsf{s}(g)) \longrightarrow \mathsf{t}^{-1}(\mathsf{t}(g))$ and $r_g : \mathsf{s}^{-1}(\mathsf{t}(g)) \longrightarrow \mathsf{s}^{-1}(\mathsf{s}(g))$ denote the left and right multiplications by $g \in \mathcal{G}$, respectively. The multiplication on $T^*\mathcal{G}$, denoted by $\circ$, is defined by

$$\alpha_g \circ \beta_h(Tm(X_g, Y_h)) = \alpha_g(X_g) + \beta_h(Y_h), \tag{2.3}$$

for $(X_g, Y_h) \in T_{(g,h)}\mathcal{G}^{(2)}$.

2.2. **Tangent double vector bundles and duals.** Let $q_A : A \longrightarrow M$ be a vector bundle. There is a natural *double vector bundle* [12, 15] associated with it, referred to as the **tangent double vector bundle** of A, and defined by the following diagram:

(2.4)
$$\begin{array}{ccc} TA & \xrightarrow{Tq_A} & TM \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p_M} \\ A & \xrightarrow[q_A]{} & M \end{array}$$

Here the vertical arrows are the usual tangent bundle structures. Similarly, one can consider the tangent double vector bundle of $q_{A^*} : A^* \longrightarrow M$, which defines a double vector bundle TA^*:

(2.5)
$$\begin{array}{ccc} TA^* & \xrightarrow{Tq_{A^*}} & TM \\ {\scriptstyle p_{A^*}}\downarrow & & \downarrow{\scriptstyle p_M} \\ A^* & \xrightarrow[q_{A^*}]{} & M \end{array}$$

It will be useful to consider coordinates on these bundles. If (x^j), $j = 1, \dots, \dim(M)$, are local coordinates on M and $\{e_d\}$, $d = 1, \dots, \mathrm{rank}(A)$, is a basis of local sections of A, we write the corresponding coordinates on A as (x^j, u^d) and tangent coordinates on TA as $(x^j, u^d, \dot{x}^j, \dot{u}^d)$. For each $x = (x^j)$, note that (u^d) specifies a point in A_x, $(\dot{x}^j)$ gives a point in T_xM, whereas $(\dot{u}^d)$ determines a point on a second copy of A_x, tangent to the fibres of $A \longrightarrow M$, known as the *core* of TA (defined by $\ker(p_A) \cap \ker(Tq_A)$, see [12, 15]). Similarly, we have local coordinates (x^j, ξ_d) on A^* (relative to the basis $\{e^d\}$, dual to $\{e_d\}$), and tangent coordinates $(x^j, \xi_d, \dot{x}^j, \dot{\xi}_d)$, where now the coordinates $(\dot{\xi}_d)$ represent the core directions.

Let $T^\bullet A \longrightarrow TM$ be the vector bundle defined by dualizing the fibres of $Tq_A : TA \longrightarrow TM$, $(x^j, u^d, \dot{x}^j, \dot{u}^d) \mapsto (x^j, \dot{x}^j)$. This fits into the double vector bundle

(2.6)
$$\begin{array}{ccc} T^\bullet A & \longrightarrow & TM \\ \downarrow & & \downarrow{\scriptstyle p_M} \\ A^* & \xrightarrow[q_{A^*}]{} & M \end{array}$$

Here the vertical map $T^\bullet A \to A^*$ is defined by $(x^j, \zeta_d, \dot{x}^j, \eta_d) \mapsto (x^j, \eta_d)$, where $T^\bullet A$ is locally written as $(x^j, \zeta_d, \dot{x}^j, \eta_d)$, with (ζ_d) dual to (u^d), and (η_d) dual to $(\dot{u}^d)$.

The double vector bundles (2.5) and (2.6) turn out to be isomorphic: as shown in [13, Prop. 5.3], by applying the tangent functor to the natural pairing $A^* \times_M A \longrightarrow \mathbb{R}$ (followed by the fibre projection $T\mathbb{R} \to \mathbb{R}$) one obtains a nondegenerate pairing $TA^* \times_{TM} TA \longrightarrow \mathbb{R}$, which induces an isomorphism of double vector bundles

(2.7)
$$I : TA^* \longrightarrow T^\bullet A.$$

Locally, this identification amounts to the flip

$$(x^j, \xi_d, \dot{x}^j, \dot{\xi}_d) \mapsto (x^j, \dot{\xi}_d, \dot{x}^j, \xi_d).$$

The cotangent bundle T^*A can be locally written in coordinates (x^j, u^d, p_j, ζ_d), where (p_j) determines a point in T^*_xM and ζ_d in A^*_x (dual to the direction tangent to the fibres $A \longrightarrow M$). If $c_A : T^*A \longrightarrow A$, $c_A(x^j, u^d, p_j, \zeta_d) = (x^j, u^d)$ denotes the natural projection, we see that T^*A fits into the following double vector bundle:

(2.8)
$$\begin{array}{ccc} T^*A & \xrightarrow{\;r\;} & A^* \\ {\scriptstyle c_A}\downarrow & & \downarrow{\scriptstyle q_{A^*}} \\ A & \xrightarrow[q_A]{} & M \end{array}$$

where the bundle projection $r : T^*A \longrightarrow A^*$ is given locally by $r(x^j, u^d, p_j, \zeta_d) = (x^j, \zeta_d)$. The same construction can be applied to the vector bundle $A^* \longrightarrow M$, yielding a double vector bundle structure for T^*A^*. These double vector bundles can be identified by a Legendre type transform [13, Thm. 5.5] (c.f. [18]):

(2.9)
$$R : T^*A^* \longrightarrow T^*A,$$

given locally by $(x^j, \xi_d, p_j, u^d) \mapsto (x^j, u^d, -p_j, \xi_d)$.

There are two other identifications involving tangent and cotangent double vector bundles that we need to recall. For an arbitrary manifold M, we first have the *canonical involution*

(2.10)
$$\begin{array}{ccc} TTM & \xrightarrow{\;J_M\;} & TTM \\ {\scriptstyle p_{TM}}\downarrow & & \downarrow{\scriptstyle Tp_M} \\ TM & \xrightarrow[\mathrm{Id}]{} & TM \end{array}$$

which is an isomorphism of double vector bundles (restricting to the identity on side bundles and cores). Writing local coordinates $(x^j, \dot{x}^j)$ for TM, and tangent coordinates $(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j)$ for $T(TM)$, J_M is given by

$$J_M(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j) = (x^j, \delta x^j, \dot{x}^j, \delta \dot{x}^j).$$

There is also an isomorphism of double vector bundles (also restricting to the identity on side bundles and cores),

(2.11)
$$\Theta_M : TT^*M \longrightarrow T^*TM,$$

defined in local coordinates by

$$\Theta_M(x^j, p_j, \dot{x}^j, \dot{p}_j) = (x^j, \dot{x}^j, \dot{p}_j, p_j).$$

Here (x^j, p_j) are cotangent coordinates on T^*M. Equivalently, $\Theta_M = J_M^* \circ I_M$, where $J_M^* : T^\bullet TM \longrightarrow T^*TM$ is the dual of (2.10), and

(2.12)
$$I_M : TT^*M \longrightarrow T^\bullet TM$$

is as in (2.7) (with $A = TM$).

2.3. **Tangent and cotangent Lie algebroids.** Suppose that the vector bundle $A \longrightarrow M$ carries a Lie algebroid structure, which can be equivalently described by a fibrewise linear Poisson structure on A^* (see e.g. [4, Sec. 16.5]). Since any Poisson structure on a manifold defines a Lie algebroid structure on its cotangent bundle (see e.g. [4, Sec. 17.3]), we obtain a Lie algebroid structure on T^*A^*; it follows that TA^* inherits a Poisson structure, which turns out to be linear with respect to *both* vector bundle structures on TA^* (2.5). Hence the vector bundle $T^\bullet A^* \longrightarrow TM$, dual to $TA^* \longrightarrow TM$, is a Lie algebroid. Using the identification $T^\bullet A^* \cong TA$ as in (2.7), we obtain a Lie algebroid structure on $TA \longrightarrow TM$, referred to as the **tangent Lie algebroid** of A.

To describe this algebroid structure more explicitly, we recall that any section $u \in \Gamma(A)$ gives rise to two types of sections on TA: the first one is just $Tu : TM \to TA$, and the second one, denoted by $\widehat{u}$, identifies u at each point with a core element in TA; locally, using coordinates (x^j, u^d) for A and $(x^j, u^d, \dot{x}^j, \dot{u}^d)$ for TA, $\widehat{u} : TM \to TA$ is defined by

$$\widehat{u}(x^j, \dot{x}^j) = (x^j, 0, \dot{x}^j, u^d(x)). \tag{2.13}$$

These two types of sections generate the space of sections of $TA \longrightarrow TM$. The Lie algebroid structure on TA is completely described in terms of these sections by the relations [13]:

$$[\widehat{u}, \widehat{v}]_{TA} = 0, \quad [Tu, \widehat{v}]_{TA} = \widehat{[u, v]_A}, \quad [Tu, Tv]_{TA} = T[u, v]_A, \tag{2.14}$$

for $u, v \in \Gamma(A)$; the anchor map is $\rho_{TA} = J_M \circ T\rho_A$, where $J_M : T(TM) \longrightarrow T(TM)$ is as in (2.10).

On the other hand, since $T^*A^* \to A^*$ is a Lie algebroid (defined by the linear Poisson structure on A^*), one can induce a Lie algebroid structure on $r : T^*A \to A^*$ using the identification (2.9). This is known as the **cotangent Lie algebroid** of A. Explicit formulas for its bracket and anchor will be recalled in Section 4.3.

Suppose that $A = A\mathcal{G}$ is the Lie algebroid of a Lie groupoid $\mathcal{G}$, and consider the natural inclusion $i_{A\mathcal{G}} : A\mathcal{G} \longrightarrow T\mathcal{G}$, which is a bundle map over the identity section $M \hookrightarrow \mathcal{G}$. Then the canonical involution $J_{\mathcal{G}} : T(T\mathcal{G}) \longrightarrow T(T\mathcal{G})$ (2.10) restricts to a Lie algebroid isomorphism

$$j_{\mathcal{G}} : T(A\mathcal{G}) \longrightarrow A(T\mathcal{G}). \tag{2.15}$$

In other words, we have a commutative diagram

$$\begin{array}{ccc} T(A\mathcal{G}) & \xrightarrow{\;j_{\mathcal{G}}\;} & A(T\mathcal{G}) \\ {\scriptstyle Ti_{A\mathcal{G}}}\downarrow & & \downarrow{\scriptstyle i_{A(T\mathcal{G})}} \\ T(T\mathcal{G}) & \xrightarrow[\;J_{\mathcal{G}}\;]{} & T(T\mathcal{G}) \end{array} \tag{2.16}$$

The canonical pairing $T^*\mathcal{G} \times_{\mathcal{G}} T\mathcal{G} \longrightarrow \mathbb{R}$ is a morphism of groupoids, and applying the Lie functor one obtains a nondegenerate pairing $A(T^*\mathcal{G}) \times_{A\mathcal{G}} A(T\mathcal{G}) \longrightarrow \mathbb{R}$, explicitly given by

$$\langle U, V \rangle = \big\langle I_{\mathcal{G}}(i_{A(T^*\mathcal{G})}(U)), i_{A(T\mathcal{G})}(V) \big\rangle,$$

where $U \in A(T^*\mathcal{G})$, $V \in A(T\mathcal{G})$, and $I_{\mathcal{G}}$ is as in (2.12). This induces an isomorphism $A(T^*\mathcal{G}) \longrightarrow A^\bullet(T\mathcal{G})$, where $A^\bullet(T\mathcal{G})$ is obtained by dualizing the fibres of $A(T\mathcal{G}) \longrightarrow A(\mathcal{G})$, and the composition of this map with $j_{\mathcal{G}}^* : A^\bullet(T\mathcal{G}) \longrightarrow T^*(A\mathcal{G})$ defines a Lie algebroid isomorphism

$$\theta_{\mathcal{G}} : A(T^*\mathcal{G}) \longrightarrow T^*(A\mathcal{G}). \tag{2.17}$$

Alternatively, one can check that $\theta_{\mathcal{G}} = (Ti_{A\mathcal{G}})^* \circ \Theta_{\mathcal{G}} \circ i_{A(T^*\mathcal{G})}$, where $(Ti_{A\mathcal{G}})^* : i_{A\mathcal{G}}^* T^*(T\mathcal{G}) \longrightarrow T^*(A\mathcal{G})$ is dual to the tangent map $Ti_{A\mathcal{G}} : T(A\mathcal{G}) \longrightarrow i_{A\mathcal{G}}^* T(T\mathcal{G})$.

3. Tangent lifts and the Lie functor

We now discuss how multiplicative forms on Lie groupoids relate to differential forms on Lie algebroids. As a first step, we need to recall a natural operation that lifts differential forms on a manifold to its tangent bundle,

$$\Omega^k(M) \longrightarrow \Omega^k(TM), \;\; \alpha \mapsto \alpha_T, \tag{3.18}$$

known as the **tangent (or complete) lift**, see [10, 20].

3.1. **Tangent lifts of differential forms.** The properties of tangent lifts recalled in this subsection can be found (often in more generality) in [10]; we included the proofs of some key facts for the sake of completeness.

Given the tangent bundle $p_M : TM \longrightarrow M$, $(x^j, \dot{x}^j) \mapsto (x^j)$, consider the two vector bundle structures associated with $T(TM)$:

$$\begin{array}{ccc} T(TM) & \xrightarrow{Tp_M} & TM \\ {\scriptstyle p_{TM}}\downarrow & & \\ TM, & & \end{array} \tag{3.19}$$

where $p_{TM}(x^j, \dot{x}^j, \delta x^j, \delta\dot{x}^j) = (x^j, \dot{x}^j)$ and $Tp_M(x^j, \dot{x}^j, \delta x^j, \delta\dot{x}^j) = (x^j, \delta x^j)$. We use the notation

$$T(TM) \times_{Tp_M} T(TM), \quad T(TM) \times_{p_{TM}} T(TM),$$

to specify the vector bundle structure used for fibre products over TM; more general k-fold fibre products over TM are denoted by

$$\prod_{Tp_M}^{k} T(TM), \quad \prod_{p_{TM}}^{k} T(TM).$$

Using the involution (2.10), given by $J_M(x^j, \dot{x}^j, \delta x^j, \delta\dot{x}^j) = (x^j, \delta x^j, \dot{x}^j, \delta\dot{x}^j)$ in local coordinates, we obtain a natural isomorphism

$$J_M^{(k)} : \prod_{p_{TM}}^{k} T(TM) \longrightarrow \prod_{Tp_M}^{k} T(TM). \tag{3.20}$$

Given a k-form $\alpha \in \Omega^k(M)$, $k \geq 1$, consider the bundle map

$$\alpha^\sharp : \prod_{p_M}^{k-1} TM \longrightarrow T^*M, \;\; \alpha^\sharp(X_1, \ldots, X_{k-1}) = i_{X_{k-1}} \ldots i_{X_1}\alpha. \tag{3.21}$$

(For $k = 1$, $\alpha^\sharp : M \longrightarrow T^*M$ is just α viewed as a section of T^*M.) Using the natural identification $T(\prod_{p_M}^k TM) = \prod_{Tp_M}^k T(TM)$, we consider the tangent map

$$T\alpha^\sharp : \prod_{Tp_M}^{k-1} T(TM) \longrightarrow T(T^*M).$$

The **tangent (or complete) lift** of a k-form on M is defined as follows (c.f. [20]):

- If $f \in \Omega^0(M) = C^\infty(M)$, then $f_T \in C^\infty(TM)$ is the fibrewise linear function on TM defined by df,

$$f_T(X) = (df)_{p_M(X)}(X), \;\; X \in TM.$$

- If $\alpha \in \Omega^k(M)$, $k \geq 1$, we define
$$(\alpha_T)^\sharp : \prod_{p_{TM}}^{k-1} T(TM) \longrightarrow T^*(TM), \quad (\alpha_T)^\sharp := \Theta_M \circ T\alpha^\sharp \circ J_M^{(k-1)},$$
and then $\alpha_T \in \Omega^k(TM)$ is given by
$$\alpha_T(U_1, \ldots, U_k) := \left\langle \alpha_T^\sharp(U_1, \ldots, U_{k-1}), U_k \right\rangle.$$

One can directly verify that α_T is multilinear. The fact that it is indeed a k-form on TM follows from the next lemma (c.f. [10, 20]).

Lemma 3.1. *The following holds:*

(*i*) *For $f \in C^\infty(M)$, $df_T = (df)_T$.*

(*ii*) *For $f \in C^\infty(M)$, $\alpha \in \Omega^k(M)$,*
$$(f\alpha)_T = f_T \alpha^\vee + f^\vee \alpha_T,$$
where $\beta^\vee = p_M^* \beta$ for any $\beta \in \Omega^l(M)$.

(*iii*) *For $k \geq 2$, the tangent lift $(dx^{i_1} \wedge \ldots \wedge dx^{i_k})_T$ equals*
$$\sum_{m=1}^{k} (dx^{i_1})^\vee \wedge \ldots \wedge (dx^{i_{m-1}})^\vee \wedge (dx^{i_m})_T \wedge (dx^{i_{m+1}})^\vee \wedge \ldots \wedge (dx^{i_k})^\vee.$$
(Whenever there is no risk of confusion, we write $(dx^j)^\vee$ simply as dx^j.)

Proof. To verify (*i*), let us consider $X \in TM$ and $U \in T_X(TM)$. In local coordinates, we write $X = (x^j, \dot{x}^j)$ and $U = (x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j)$. Then $f_T(X) = \frac{\partial f}{\partial x^i} \dot{x}^i$, and

$$d(f_T)_X(U) = \frac{\partial^2 f}{\partial x^j \partial x^i} \dot{x}^i \delta x^j + \frac{\partial f}{\partial x^j} \delta \dot{x}^j. \tag{3.22}$$

On the other hand, we may view df as a section
$$(df)^\sharp : M \longrightarrow T^*M, \quad x = (x^j) \mapsto (x^j, \frac{\partial f}{\partial x^j}).$$
Hence $T(df)^\sharp : TM \longrightarrow T(T^*M)$ is given by
$$T(df)^\sharp(x^j, \dot{x}^j) = (x^j, \frac{\partial f}{\partial x^j}, \dot{x}^j, \frac{\partial^2 f}{\partial x^i \partial x^j} \dot{x}^i),$$
and, as a consequence,
$$(df)_T(x^j, \dot{x}^j) = \Theta_M(T(df)^\sharp(x^j, \dot{x}^j)) = (x^j, \dot{x}^j, \frac{\partial^2 f}{\partial x^i \partial x^j} \dot{x}^i, \frac{\partial f}{\partial x^j}).$$
It immediately follows that $((df)_T)_X(U)$ agrees with (3.22).

Let us show that (*ii*) holds for $k > 1$ (the cases $k = 0, 1$ are simpler). One can directly check that $(f\alpha)^\sharp = f\alpha^\sharp$ and
$$T(f\alpha)^\sharp(U_1, \ldots, U_{k-1}) = \alpha^\sharp(X_1, \ldots, X_{k-1})(df)_x(Y) + f(x) T\alpha^\sharp(U_1, \ldots, U_{k-1}),$$
where $X_i = p_{TM}(U_i) \in T_xM$, and $Y = (p_M)_*(U_1) = \ldots = (p_M)_*(U_{k-1})$. In the last formula, addition and multiplication by scalars are with respect to the vector bundle structure $T(T^*M) \longrightarrow TM$ (in the fibre over $Y \in T_xM$), and $\alpha^\sharp(X_1, \ldots, X_{k-1}) \in T_x^*M$ is viewed inside $T(T^*M)$ as the core (i.e., tangent to T^*M-fibres). Since

$\Theta_M : T(T^*M) \longrightarrow T^*(TM)$ is a double vector bundle isomorphism restricting to the identity on side bundle and cores, we have

$$\Theta_M T(f\alpha)^\sharp(U_1,\ldots,U_{k-1}) = \alpha^\sharp(X_1,\ldots,X_{k-1})(df)_x(Y) + f(x)\Theta_M T\alpha^\sharp(U_1,\ldots,U_{k-1}), \tag{3.23}$$

where now the addition and scalar multiplication operations are relative to the vector bundle $T^*(TM) \longrightarrow TM$, and $\alpha^\sharp(X_1,\ldots,X_{k-1})$ belongs to the core fibre in $T^*(TM)$ (i.e., cotangent to M). Writing $(U_1,\ldots,U_{k-1}) = J_M^{(k-1)}(V_1,\ldots,V_{k-1})$, then $X_i = (p_M)_*(V_i)$ and $Y = p_{TM}(V_i)$, so (3.23) yields

$$(f\alpha)_T^\sharp = (f_T\alpha^\vee + f^\vee\alpha_T)^\sharp.$$

Let us now prove (iii). Note that

$$(dx^{i_1}\wedge\ldots\wedge dx^{i_k})^\sharp(X_1,\ldots,X_{k-1}) = \sum_{\sigma\in S_k}(-1)^\sigma \dot{x}_1^{i_{\sigma(1)}}\ldots\dot{x}_{k-1}^{i_{\sigma(k-1)}}dx^{i_{\sigma(k)}},$$

where $X_l = (x^j, \dot{x}_l^j) \in T_xM$. Then $\langle\Theta_M(T(dx^{i_1}\wedge\ldots\wedge dx^{i_k})^\sharp(U_1,\ldots,U_{k-1})), V_k\rangle$ equals

$$\sum_{\sigma\in S_k}(-1)^\sigma\sum_{n=1}^{k}\dot{x}_1^{i_{\sigma(1)}}\ldots\dot{x}_{n-1}^{i_{\sigma(n-1)}}(\delta\dot{x})_n^{i_{\sigma(n)}}\dot{x}_{n+1}^{i_{\sigma(n+1)}}\ldots\dot{x}_k^{i_{\sigma(k)}}, \tag{3.24}$$

where $U_l = (x^j, \dot{x}_l^j, (\delta x)^j, (\delta\dot{x})_l^j)$ and $V_k = (x^j, (\delta x)^j, \dot{x}_k^j, (\delta\dot{x})_k^j)$. Since $(dx^j)_T = d\dot{x}^j$ (by (i)), one checks that

$$\sum_{n=1}^{k}(dx^{i_1})^\vee\wedge\ldots\wedge(dx^{i_{n-1}})^\vee\wedge(dx^{i_n})_T\wedge(dx^{i_{n+1}})^\vee\wedge\ldots\wedge(dx^{i_k})^\vee(V_1,\ldots,V_k),$$

where $V_l = (x^j, (\delta x)^j, \dot{x}_l^j, (\delta\dot{x})_l^j)$ (so that $J_M(V_l) = U_l$), equals

$$\sum_{\sigma\in S_k}(-1)^\sigma\sum_{n=1}^{k}\dot{x}_{\sigma(1)}^{i_1}\ldots\dot{x}_{\sigma(n-1)}^{i_{n-1}}(\delta\dot{x})_{\sigma(n)}^{i_n}\dot{x}_{\sigma(n+1)}^{i_{n+1}}\ldots\dot{x}_{\sigma(k)}^{i_k},$$

which agrees with (3.24) after reshuffling indices.

□

Let us now consider the operation

$$\tau : \Omega^k(M) \longrightarrow \Omega^{k-1}(TM), \quad \tau(\alpha)_X = p_M^*(i_X\alpha), \tag{3.25}$$

where $X \in TM$ and $k \geq 1$. In other words, given $U_1,\ldots,U_{k-1} \in T_X(TM)$,

$$\tau(\alpha)_X(U_1,\ldots,U_{k-1}) = \alpha(X, (p_M)_*(U_1),\ldots,(p_M)_*(U_{k-1})).$$

In coordinates, writing $\alpha = \frac{1}{k!}\alpha_{i_1\ldots i_k}(x)dx^{i_1}\wedge\ldots\wedge dx^{i_k}$ (with $\alpha_{i_1\ldots i_k}$ totally anti-symmetric), we have

$$\tau(\alpha)_X = \frac{1}{(k-1)!}\alpha_{i_1\ldots i_k}(x)X^{i_1}dx^{i_2}\wedge\ldots\wedge dx^{i_k}.$$

Example 3.2. *Consider the map* $\omega^\sharp : TM \longrightarrow T^*M$*,* $\omega^\sharp(X) = i_X\omega$*, associated with a 2-form* $\omega \in \Omega^2(M)$*. A direct computation shows that*

$$\tau(\omega) = (\omega^\sharp)^*\theta_{can},$$

where $\theta_{can} \in \Omega^1(T^*M)$ *is the canonical 1-form,* $\theta_{can} = p_i dx^i$*.*

The tangent lift operation is defined by the following Cartan-like formula (c.f. [10]).

Proposition 3.3. *For* $\alpha \in \Omega^k(M)$*, its tangent lift is given by the formula*

$$\alpha_T = d\tau(\alpha) + \tau(d\alpha). \tag{3.26}$$

Proof. It suffices to check (3.26) locally, so we replace M by a neighborhood with coordinates (x^j), so that TM has coordinates $(x^j, \dot{x}^j)$. Let us consider the vector field V on TM defined by

$$V_X := \dot{x}^j \frac{\partial}{\partial x^j} \in T_X(TM),$$

where $X = (x^j, \dot{x}^j) \in TM$. This vector field has the property that $Tp_M(V_X) = X$. One can directly check that

$$f_T = \mathcal{L}_V(p_M^* f), \quad \text{and} \quad (dx^j)_T = d\dot{x}^j = \mathcal{L}_V(p_M^* dx^j), \tag{3.27}$$

where $f \in C^\infty(M)$. From the definition of τ, it immediately follows that

$$\tau(\beta) = i_V p_M^* \beta, \quad \beta \in \Omega^k(M). \tag{3.28}$$

Given $\alpha = \frac{1}{k!}\alpha_{i_1\ldots i_k}(x) dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, using Lemma 3.1 we obtain

$$\begin{aligned}\alpha_T =& \frac{1}{k!}(\alpha_{i_1\ldots i_k})_T\, p_M^*(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) + \frac{1}{k!} p_M^* \alpha_{i_1\ldots i_k} (dx^{i_1} \wedge \ldots \wedge dx^{i_k})_T \\ =& \frac{1}{k!}(\alpha_{i_1\ldots i_k})_T\, p_M^*(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) + \\ & \frac{1}{k!} p_M^* \alpha_{i_1\ldots i_k} \sum_{n=1}^{k} dx^{i_1} \wedge \ldots \wedge (dx^{i_n})_T \wedge \cdots \wedge dx^{i_k}.\end{aligned}$$

It then follows from (3.27) that $\alpha_T = \mathcal{L}_V p_M^* \alpha$. Using (3.28) and Cartan's formula, we have

$$\alpha_T = d(i_V p_M^* \alpha) + i_V p_M^* d\alpha = d\tau(\alpha) + \tau(d\alpha).$$

□

Example 3.4. *From Example 3.2, it follows that if* $\omega \in \Omega^2(M)$*, then*

$$\omega_T = -(\omega^\sharp)^*\omega_{can} + \tau(d\omega).$$

Here $\omega_{can} = -d\theta_{can} = dx^i \wedge dp_i$ *is the canonical symplectic form on* T^*M*. (For the tangent lift of closed 2-forms, see also* [8, Sec. 3]*).*

An immediate consequence of (3.26) is the fact that tangent lifts and exterior derivatives commute.

Corollary 3.5. *For* $\alpha \in \Omega^k(M)$*,* $d(\alpha_T) = (d\alpha)_T$*.*

3.2. **Lie functor on multiplicative differential forms.** Let $\mathcal{G}$ be a Lie groupoid over M, $A = A\mathcal{G}$ its Lie algebroid, and let $\alpha \in \Omega^k(\mathcal{G})$. We can define an induced k-form on A by pulling back the tangent lift $\alpha_T \in \Omega^k(T\mathcal{G})$ via the inclusion $i_A : A \longrightarrow T\mathcal{G}$. This section discusses this operation when α is multiplicative.

Recall that a k-form $\alpha \in \Omega^k(\mathcal{G})$ is *multiplicative* if

$$m^*\alpha = p_1^*\alpha + p_2^*\alpha, \tag{3.29}$$

where $p_1, p_2 : \mathcal{G}^{(2)} \to \mathcal{G}$ are the natural projections, and m is the groupoid multiplication. We denote the associated k-form on A by

$$\mathrm{Lie}(\alpha) := i_A^*\alpha_T. \tag{3.30}$$

Note that it follows from Corollary 3.5 that

$$d\mathrm{Lie}(\alpha) = \mathrm{Lie}(d\alpha). \tag{3.31}$$

In order to explain in which sense $\mathrm{Lie}(\alpha)$ is the infinitesimal counterpart of α, we will need a known alternative characterization of multiplicative forms.

The tangent groupoid structure on the tangent bundle $p_{\mathcal{G}} : T\mathcal{G} \longrightarrow \mathcal{G}$ over TM induces a groupoid structure on the direct sum

$$\prod_{p_{\mathcal{G}}}^{n} T\mathcal{G} = T\mathcal{G} \oplus \ldots \oplus T\mathcal{G}$$

over the base $\prod_{p_M}^n TM = TM \oplus \ldots \oplus TM$ in a canonical way.

Lemma 3.6. *A k-form $\alpha \in \Omega^k(\mathcal{G})$ ($k \geq 1$) is multiplicative if and only if the bundle map $\alpha^\sharp : \prod_{p_{\mathcal{G}}}^{k-1} T\mathcal{G} \longrightarrow T^*\mathcal{G}$ (see (3.21)) is a groupoid morphism.*

Proof. Let us consider the following identities, obtained by differentiating basic identities on any Lie groupoid (see [3, Lem. 3.1]):

$$(Tm)_{(\mathsf{t}(g),g)}(T\mathsf{t}(X), X) = X = (Tm)_{(g,\mathsf{s}(g))}(X, T\mathsf{s}(X)), \ \ \forall X \in T_g\mathcal{G}, \tag{3.32}$$

$$(Tr_g)_{\mathsf{t}(g)}(u) = (Tm)_{(\mathsf{t}(g),g)}(u, 0), \ \ (Tl_g)_{\mathsf{s}(g)}(v) = (Tm)_{(g,\mathsf{s}(g))}(0, v) \tag{3.33}$$

where $u \in A_{\mathsf{t}(g)} = \mathrm{Ker}(Ts)|_{\mathsf{t}(g)}$ and $v \in \mathrm{Ker}(Tt)|_{\mathsf{s}(g)}$. Using the first identities in (3.32) and (3.33), we see that if α is multiplicative, then by (3.29) we have

$$\alpha(T\mathsf{t}(X_1), \ldots, T\mathsf{t}(X_{k-1}), u) = \alpha(X_1, \ldots, X_{k-1}, Tr_g(u)),$$

where $X_i \in T_g\mathcal{G}, u \in A_{\mathsf{t}(g)}$. This is precisely the compatibility of $\alpha^\sharp$ with the target maps on $\prod_{p_{\mathcal{G}}}^{k-1} T\mathcal{G}$ and $T^*\mathcal{G}$. Similarly, note that (3.32) and (3.29) imply that, if $Z_1, \ldots, Z_k \in TM$, then $\alpha(Z_1, \ldots, Z_k) = 0$. Using this fact, along with (3.29) and the second identities in (3.32) and (3.33), we obtain the compatibility between $\alpha^\sharp$ and source maps:

$$\alpha(T\mathsf{s}(X_1), \ldots, T\mathsf{s}(X_{k-1}), u) = \alpha(X_1, \ldots, X_{k-1}, Tl_g(u - T\mathsf{t}(u))),$$

where $X_i \in T_g\mathcal{G}, u \in A_{\mathsf{t}(g)}$.

Assuming that $\alpha^\sharp$ is compatible with source and target maps, we see that it is a groupoid morphism if and only if

$$\alpha^\sharp(Tm(X_1, Y_1), \ldots, Tm(X_{k-1}, Y_{k-1})) = \alpha^\sharp(X_1, \ldots, X_{k-1}) \circ \alpha^\sharp(Y_1, \ldots, Y_{k-1}).$$

By evaluating each side of the last equation on $Tm(X_k, Y_k)$, we see that this condition is equivalent to

$$\alpha(Tm(X_1, Y_1), \dots, Tm(X_k, Y_k)) = \alpha(X_1, \dots, X_k) + \alpha(Y_1, \dots, Y_k),$$

which is precisely the multiplicativity condition (3.29). □

Given a groupoid morphism $\psi : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$, we denote the associated morphism of Lie algebroids (given by the restriction of $T\psi : T\mathcal{G}_1 \longrightarrow T\mathcal{G}_2$ to $A\mathcal{G}_1 \subset T\mathcal{G}_1$) by

$$\mathrm{Lie}(\psi) : A\mathcal{G}_1 \longrightarrow A\mathcal{G}_2.$$

The natural projection $p_{\mathcal{G}} : T\mathcal{G} \to \mathcal{G}$ is a groupoid morphism, and one can directly verify that there is a canonical identification

$$A(\prod_{p_{\mathcal{G}}}^{k-1} T\mathcal{G}) = \prod_{\mathrm{Lie}(p_{\mathcal{G}})}^{k-1} A(T\mathcal{G}).$$

Using this identification we get, for any given multiplicative k-form $\alpha \in \Omega^k(\mathcal{G})$, a Lie algebroid morphism

$$\mathrm{Lie}(\alpha^\sharp) : \prod_{\mathrm{Lie}(p_{\mathcal{G}})}^{k-1} A(T\mathcal{G}) \longrightarrow A(T^*\mathcal{G}). \tag{3.34}$$

The isomorphism $j_{\mathcal{G}} : T(A\mathcal{G}) \xrightarrow{\sim} A(T\mathcal{G})$, see (2.15), induces an identification

$$j_{\mathcal{G}}^{(k)} : \prod_{p_A}^{k} T(A\mathcal{G}) \longrightarrow \prod_{\mathrm{Lie}(p_{\mathcal{G}})}^{k} A(T\mathcal{G}). \tag{3.35}$$

Recall the isomorphism $\theta_{\mathcal{G}} : A(T^*\mathcal{G}) \to T^*(A\mathcal{G})$ defined in (2.17).

Proposition 3.7. *For a multiplicative k-form $\alpha \in \Omega^k(\mathcal{G})$,* $\mathrm{Lie}(\alpha)$ *and* $\mathrm{Lie}(\alpha^\sharp)$ *are related by*

$$\mathrm{Lie}(\alpha)^\sharp = \theta_{\mathcal{G}} \circ \mathrm{Lie}(\alpha^\sharp) \circ j_{\mathcal{G}}^{(k-1)} : \prod_{p_A}^{k-1} T(A\mathcal{G}) \longrightarrow T^*(A\mathcal{G}).$$

Proof. Recall that $\theta_{\mathcal{G}} = (Ti_{A\mathcal{G}})^* \circ \Theta_{\mathcal{G}} \circ i_{A(T^*\mathcal{G})}$ and $J_{\mathcal{G}} \circ Ti_{A\mathcal{G}} = i_{A(T\mathcal{G})} \circ j_{\mathcal{G}}$. This last identity immediately implies that

$$J_{\mathcal{G}}^{(k)} \circ (\prod^{k} Ti_{A\mathcal{G}}) = (\prod^{k} i_{A(T\mathcal{G})}) \circ j_{\mathcal{G}}^{(k)}.$$

Since $i_{A(T^*\mathcal{G})} \circ \mathrm{Lie}(\alpha^\sharp) = T\alpha^\sharp \circ \prod^{k-1} i_{A(T\mathcal{G})}$, it follows that

$$\begin{aligned}\theta_{\mathcal{G}} \circ \mathrm{Lie}(\alpha^\sharp) \circ j_{\mathcal{G}}^{(k-1)} &= (Ti_{A\mathcal{G}})^* \circ \Theta_{\mathcal{G}} \circ T\alpha^\sharp \circ \prod^{k-1} i_{A(T\mathcal{G})} \circ j_{\mathcal{G}}^{(k-1)} \\ &= (Ti_{A\mathcal{G}})^* \circ \alpha_T^\sharp \circ (\prod^{k-1} Ti_{A\mathcal{G}}),\end{aligned}$$

and this last term is $(i_A^* \alpha_T)^\sharp = (\mathrm{Lie}(\alpha))^\sharp$. □

Corollary 3.8. *If $\alpha \in \Omega^k(\mathcal{G})$ is multiplicative and $\mathcal{G}$ is* s*-connected, then $\alpha = 0$ if and only if* $\mathrm{Lie}(\alpha) = 0$.

Proof. If $\mathcal{G}$ is s-connected, then $\prod_{p_{\mathcal{G}}}^{k-1} T\mathcal{G}$ also has connected source-fibres. We now use the fact that if two groupoid morphisms $\mathcal{G}_1 \longrightarrow \mathcal{G}_2$ induce the same Lie algebroid morphism and $\mathcal{G}_1$ has source-connected fibres, then they must coincide. Hence $\alpha^\sharp = 0$ if and only if $\mathrm{Lie}(\alpha^\sharp) = 0$. The conclusion now follows since $\alpha = 0$ (resp. $\mathrm{Lie}(\alpha) = 0$) is equivalent to $\alpha^\sharp = 0$ (resp. $\mathrm{Lie}(\alpha)^\sharp = 0$), and $\mathrm{Lie}(\alpha)^\sharp = 0$ if only if $\mathrm{Lie}(\alpha^\sharp) = 0$ by Prop. 3.7. □

4. Multiplicative 2-forms and their infinitesimal counterparts

4.1. **Linear 2-forms on vector bundles.** Let $q : A \longrightarrow M$ be a vector bundle, and consider the double vector bundles TA and T^*A, as in Section 2.2. A 2-form $\Lambda \in \Omega^2(A)$ is called **linear** if

$$\Lambda^\sharp : TA \longrightarrow T^*A$$

is a morphism of double vector bundles (c.f. [11, Sec. 7.3]). In particular, there is a vector bundle map $\lambda : TM \to A^*$ (over the identity) making the following diagram commute:

$$\begin{array}{ccc} TA & \xrightarrow{\Lambda^\sharp} & T^*A \\ {\scriptstyle Tq}\downarrow & & \downarrow{\scriptstyle r} \\ TM & \xrightarrow[\lambda]{} & A^* \end{array} \quad . \tag{4.36}$$

In this case we say that Λ **covers** λ.

Remark 4.1. *The fact that a bivector field π on a vector bundle A is linear is equivalent* [11, 13] *to the bundle map $\pi^\sharp : T^*A \to TA$ being a morphism of double vector bundles. Hence linear 2-forms are just their dual analogues.*

It is simple to check from the definition that a linear 2-form has a local expression of the form:

$$\begin{aligned} \Lambda &= \frac{1}{2}\Lambda_{ij}(x,u)dx^i \wedge dx^j + \Lambda_{jd}(x,u)dx^j \wedge du^d \\ &= \frac{1}{2}\Lambda_{ij,d}(x)u^d dx^i \wedge dx^j + \lambda_{jd}(x)dx^j \wedge du^d. \end{aligned} \tag{4.37}$$

where $(x,u) = (x^j, u^d)$ are local coordinates in A (relative to a local basis $\{e_d\}$), and $\lambda_{jd} = \langle \lambda(\frac{\partial}{\partial x^j}), e_d \rangle$.

Example 4.2. *The canonical symplectic form $\omega_{can} = dx^j \wedge dp^j$ on the cotangent bundle T^*M is linear. Any vector bundle map $\sigma : A \longrightarrow T^*M$, locally written as $\sigma(e_d) = \sigma_{jd}dx^j$, defines a linear 2-form on A by pullback,*

$$\sigma^*\omega_{can} = u^d \frac{\partial \sigma_{id}}{\partial x^k} dx^i \wedge dx^k + \sigma_{id}dx^i \wedge du^d,$$

covering the map $\lambda = \sigma^t : TM \longrightarrow A^*$.

From the local expression (4.37), one can directly verify that Example 4.2 completely characterizes linear closed 2-forms:

Proposition 4.3. *A linear 2-form $\Lambda \in \Omega^2(A)$ is closed if and only if it is of the form*

$$\Lambda = (\lambda^t)^*\omega_{can},$$

where $\lambda^t : A \to T^*M$ is the fibrewise transpose of λ.

A proof of this result can be found in [11, Sec. 7.3].

Example 4.4. *If $\omega \in \Omega^2(M)$, then its tangent lift $\omega_T \in \Omega^2(TM)$ is linear and covers the map $\lambda = \omega^\sharp : TM \to T^*M$. If ω is closed, then so is ω_T (it is in fact exact, by Prop. 3.3). It follows from Prop. 4.3 and the fact that $(\omega^\sharp)^t = -\omega^\sharp$ that*

$$\omega_T = -(\omega^\sharp)^*\omega_{can},$$

in agreement with Example 3.4.

Example 4.5. *Let $\phi \in \Omega^3(M)$ be a 3-form on M. Then the 2-form $\tau(\phi)$ on TM is linear; it covers the bundle map $\lambda : A \to T^*M$ that is zero on each fibre.*

4.2. **Linear 2-forms on Lie algebroids.** Let $A \longrightarrow M$ be a Lie algebroid. We will discuss two natural ways to obtain linear 2-forms on A.

First, given any 3-form $\phi \in \Omega^3(M)$, we can use the anchor $\rho : A \longrightarrow TM$ to pull-back the linear 2-form $\tau(\phi)$ to A. The resulting 2-form

$$\rho^*(\tau(\phi)) \in \Omega^2(A)$$

is linear, covering the map $\lambda : TM \longrightarrow A^*$ which is zero on each fibre.

On the other hand, if $A = A\mathcal{G}$ is the Lie algebroid of a Lie groupoid $\mathcal{G}$, then one obtains linear 2-forms on A as infinitesimal versions of multiplicative 2-forms on $\mathcal{G}$:

Proposition 4.6. *Let $\omega \in \Omega^2(\mathcal{G})$ be a multiplicative 2-form, and let $\lambda : TM \longrightarrow A^*$ be the defined by $\lambda(X)(u) = \omega(X, u)$, for $X \in TM$ and $u \in A$. Then*

(1) *$\Lambda = \mathrm{Lie}(\omega) \in \Omega^2(A)$ is linear and covers λ.*
(2) *Given $\phi \in \Omega^3(M)$ closed and if $\mathcal{G}$ is s-connected, then $d\omega = \mathsf{s}^*\phi - \mathsf{t}^*\phi$ if and only if*
$$\Lambda = (\lambda^t)^*\omega_{can} - \rho^*(\tau(\phi)).$$

Proof. Let us prove (1). Note that $\mathrm{Lie}(\omega) = i_A^*\omega_T$ is linear since $\omega_T \in \Omega^2(T\mathcal{G})$ is linear, and the pull back of a linear 2-form to a vector subbundle is again linear.

From Lemma 3.6, we know that $\omega^\sharp : T\mathcal{G} \longrightarrow T^*\mathcal{G}$ is a groupoid morphism, which restricts to the map $\lambda : TM \longrightarrow A^*$ on identity sections. As a result, $\mathrm{Lie}(\omega^\sharp)$ fits into the following commutative diagram:

$$\begin{array}{ccc} A(T\mathcal{G}) & \xrightarrow{\mathrm{Lie}(\omega^\sharp)} & A(T^*\mathcal{G}) \\ \downarrow & & \downarrow \\ TM & \xrightarrow[\lambda]{} & A^*, \end{array}$$

and it follows from Prop. 3.7 that $\Lambda = \mathrm{Lie}(\omega)$ covers λ.

For part (2), note that

$$\mathrm{Lie}(\mathsf{s}^*\phi - \mathsf{t}^*\phi) = i_A^*(\mathsf{s}^*\phi)_T - i_A^*(\mathsf{t}^*\phi)_T.$$

From (3.26) and the fact that $d\phi = 0$, we see that $(\mathsf{s}^*\phi)_T = d\tau(\mathsf{s}^*\phi)$ and $(\mathsf{t}^*\phi)_T = d\tau(\mathsf{t}^*\phi)$. A simple computation shows that $\tau(\mathsf{s}^*\phi) = (T\mathsf{s})^*\tau(\phi)$ and $\tau(\mathsf{t}^*\phi) = (T\mathsf{t})^*\tau(\phi)$. Hence $\mathrm{Lie}(\mathsf{s}^*\phi - \mathsf{t}^*\phi) = d(i_A^*(T\mathsf{s})^*\tau(\phi) - i_A^*(T\mathsf{t})^*\tau(\phi))$. Since $T\mathsf{s} \circ i_A = 0$ (A is tangent to s-fibres) and $T\mathsf{t} \circ i_A = \rho$, we obtain $\mathrm{Lie}(\mathsf{s}^*\phi - \mathsf{t}^*\phi) = -d\rho^*\tau(\phi)$. By Corollary 3.8, we know that

$$d\omega - (\mathsf{s}^*\phi - \mathsf{t}^*\phi) = 0 \iff \mathrm{Lie}(d\omega - (\mathsf{s}^*\phi - \mathsf{t}^*\phi)) = 0.$$

But Lie$(d\omega - (\mathsf{s}^*\phi - \mathsf{t}^*\phi)) = d(\Lambda + \rho^*\tau(\phi))$. Since the linear 2-form $\Lambda + \rho^*\tau(\phi)$ covers λ, it follows from Prop. 4.3 that

$$d(\Lambda + \rho^*\tau(\phi)) = 0 \iff \Lambda + \rho^*\tau(\phi) = (\lambda^t)^*\omega_{can},$$

as desired. □

To make the connection between this paper and the results in [3] more transparent, it will be convenient to consider the map $\sigma_\omega : A \longrightarrow T^*M$ induced by $\omega \in \Omega^2(\mathcal{G})$ via

$$\sigma_\omega(u)(X) = \omega(u, X), \quad u \in A, X \in TM. \tag{4.38}$$

In the notation of Prop. 4.6, we have $\sigma_\omega = -\lambda^t$, so under the assumptions in part (2), $\Lambda = \mathrm{Lie}(\omega)$ and σ_ω are related by

$$\Lambda = -(\sigma_\omega^*\omega_{can} + \rho^*\tau(\phi)), \tag{4.39}$$

in such a way that Λ covers $-\sigma_\omega^t : TM \longrightarrow A^*$.

4.3. IM 2-forms and Lie algebroid morphisms.

This subsection presents to key step to integrate IM 2-forms.

Let $A \longrightarrow M$ be a Lie algebroid, with bracket $[\cdot,\cdot]$ and anchor ρ. Let $\sigma : A \longrightarrow T^*M$ be a vector bundle map (over the identity) and $\phi \in \Omega^3(M)$ a closed 3-form. Motivated by (4.39), let us consider the linear 2-form $\Lambda \in \Omega^2(A)$ defined by

$$\Lambda = -(\sigma^*\omega_{can} + \rho^*\tau(\phi)), \tag{4.40}$$

covering $-\sigma^t : TM \longrightarrow A^*$. The following result describes when such 2-form induces a morphism between tangent and cotangent algebroid structures.

Theorem 4.7. *Let $\Lambda \in \Omega^2(A)$ be as in* (4.40). *The following are equivalent:*

(i) *The map $\Lambda^\sharp : TA \longrightarrow T^*A$ is a Lie algebroid morphism.*
(ii) *The map $\sigma : A \longrightarrow T^*M$ satisfies*

$$\langle\sigma(u), \rho(v)\rangle = -\langle\sigma(v), \rho(u)\rangle \tag{4.41}$$

$$\sigma([u, v]) = \mathcal{L}_{\rho(u)}\sigma(v) - i_{\rho(v)}d\sigma(u) + i_{\rho(v)}i_{\rho(u)}\phi, \tag{4.42}$$

for all $u, v \in \Gamma(A)$.

Vector bundle maps $\sigma : A \longrightarrow T^*M$ satisfying conditions (4.41) and (4.42) were introduced in [3] and are referred to as **IM 2-forms** on A (relative to ϕ). We also recall that a **morphism** between Lie algebroids $A \longrightarrow M$ and $B \longrightarrow N$ (see e.g. [12]) is a vector bundle map $\Psi : A \longrightarrow B$, covering $\psi : M \longrightarrow N$, which is compatible with anchors, meaning that

$$\rho_B \circ \Psi = T\psi \circ \rho_A,$$

and compatible with brackets in the following sense. Consider the pull-back bundle $\psi^*B \longrightarrow M$, and let us keep denoting by Ψ the induced map $\Gamma(A) \longrightarrow \Gamma(\psi^*B)$ at the level of sections. Given sections $u, v \in \Gamma(A)$ such that $\Psi(u) = f_j\psi^*u_j$ and $\Psi(v) = g_i\psi^*v_i$, where $f_j, g_i \in C^\infty(M)$ and $u_j, v_i \in \Gamma(B)$, the following condition should be valid:

$$\Psi([u, v]_A) = f_jg_i\psi^*[u_j, v_i]_B + \mathcal{L}_{\rho_A(u)}g_i\psi^*v_i - \mathcal{L}_{\rho_A(v)}f_j\psi^*u_j. \tag{4.43}$$

We will need explicit local formulas for the tangent and cotangent Lie algebroids. For a basis of local sections $\{e_d\}$ of A, we denote the corresponding Lie algebroid structure functions by ρ_a^j and C_{ab}^c,

$$\rho_A(e_a) = \rho_a^j \frac{\partial}{\partial x^j}, \quad [e_a, e_b] = C_{ab}^c e_c.$$

Recall from Section 2.3 that any section $u : M \longrightarrow A$ defines two types of sections of $TA \longrightarrow TM$, denoted by Tu and $\widehat{u}$. From (2.14), the tangent Lie algebroid structure can be written as follows:

$$(4.44) \qquad [\widehat{e}_a, \widehat{e}_b]_{TA} = 0, \;\; [Te_a, \widehat{e}_b]_{TA} = C_{ab}^c \widehat{e}_c, \;\; [Te_a, Te_b]_{TA} = C_{ab}^c Te_c + dC_{ab}^c \widehat{e}_c,$$

$$(4.45) \qquad \rho_{TA}(Te_a) = \rho_a^j \frac{\partial}{\partial x^j} + d\rho_a^j \frac{\partial}{\partial \dot{x}^j}, \quad \rho_{TA}(\widehat{e}_a) = \rho_a^j \frac{\partial}{\partial \dot{x}^j}.$$

To describe the Lie algebroid structure on $T^*A \longrightarrow A^*$ explicitly, we also consider two types of sections that generate the space of sections of T^*A over A^*. The first type is induced from a section $u \in \Gamma(A)$, and denoted by u^L. In local coordinates (x^j, ξ_d) on A^* (relative to the basis of local sections $\{e^d\}$ of A^*, dual to $\{e_d\}$), it is given by

$$(4.46) \qquad u^L(x^j, \xi_d) = (x^j, u^d(x), 0, \xi_d),$$

where T^*A is written locally in coordinates (x^j, u^d, p_j, ζ_d) as in Section 2.2. The second type are *core sections*: locally, for each $\alpha = \alpha_j dx^j \in \Gamma(T^*M)$, we define the section $\widehat{\alpha}$ of $T^*A \longrightarrow A^*$ by

$$(4.47) \qquad \widehat{\alpha}(x^j, \xi_d) = (x^j, 0, \alpha_j(x), \xi_d).$$

The cotangent Lie algebroid is defined by the relations:

$$(4.48) \quad [\widehat{dx^i}, \widehat{dx^j}]_{T^*A} = 0, \;\; [e_a^L, \widehat{dx^j}]_{T^*A} = \widehat{d\rho_a^j}, \;\; [e_a^L, e_b^L]_{T^*A}|_{(x,\xi)} = -\widehat{dC_{ab}^c}\xi_c + C_{ab}^c e_c^L,$$

$$(4.49) \quad \rho_{T^*A}(\widehat{dx^i}) = \rho_a^i \frac{\partial}{\partial \xi_a}, \quad \rho_{T^*A}(e_a^L)|_{(x,\xi)} = \rho_a^i \frac{\partial}{\partial x^i} + C_{ab}^c \xi_c \frac{\partial}{\partial \xi_b}.$$

We now turn to the proof of Theorem 4.7.

Proof. We work locally, so we assume that M has coordinates (x^j). Then A has coordinates (x^j, u^d) (relative to a basis of local sections $\{e_d\}$), TA has tangent coordinates $(x^j, u^d, \dot{x}^j, \dot{u}^d)$, while induced coordinates on T^*A are denoted by (x^j, u^d, p_j, ζ_d). Similarly, A^* has dual coordinates (x^j, ξ_d), inducing coordinates $(x^j, \xi_d, \dot{x}^j, \dot{\xi}_d)$ on TA^*.

We start by discussing when $\Lambda^\sharp$ is compatible with the anchors, i.e.,

$$(4.50) \qquad T(-\sigma^t) \circ \rho_{TA} = \rho_{T^*A} \circ \Lambda^\sharp.$$

Let us consider local expressions of the relevant maps. We write $\sigma : A \longrightarrow T^*M$ and $\sigma^t : TM \longrightarrow A^*$ locally as

$$\sigma(x^j, u^d) = (x^j, u^d \sigma_{jd}(x)), \quad \sigma^t(x^j, \dot{x}^j) = (x^j, \dot{x}^j \sigma_{jd}(x)).$$

Denoting coordinates on TM by $(x^j, \dot{x}^j)$, and on $T(TM)$ by $(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j)$, we get

$$T(-\sigma^t)(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j) = (x^j, -\dot{x}^l \sigma_{ld}, \delta x^j, -\dot{x}^l \frac{\partial \sigma_{ld}}{\partial x^k} \delta x^k - \sigma_{ld} \delta \dot{x}^l) \in TA^*.$$

One can directly verify that the map $\Lambda^\sharp$ can be locally written as follows:

$$\Lambda^\sharp(x^j, u^d, \dot{x}^j, \dot{u}^d) = (x^j, u^d, p_j, \zeta_d), \tag{4.51}$$

where

$$p_j = \dot{x}^l u^d \left(\frac{\partial \sigma_{jd}}{\partial x^l} - \frac{\partial \sigma_{ld}}{\partial x^j} \right) + \dot{u}^d \sigma_{jd} - \phi_{ijk} u^d \rho^k_d \dot{x}^i, \quad \zeta_d = -\dot{x}^l \sigma_{ld}.$$

The space of sections of $TA \longrightarrow TM$ is generated by sections of types Te_a and $\widehat{e}_b$. We have

$$\Lambda^\sharp(Te_a|_{(x,\dot{x})}) = \left(x^j, \delta_{ad}, \dot{x}^l \left(\frac{\partial \sigma_{ja}}{\partial x^l} - \frac{\partial \sigma_{la}}{\partial x^j} \right) - \phi_{ijk} \rho^k_a \dot{x}^i, -\dot{x}^l \sigma_{ld} \right) \tag{4.52}$$

$$\Lambda^\sharp(\widehat{e}_b|_{(x,\dot{x})}) = (x^j, 0, \sigma_{jb}, -\dot{x}^l \sigma_{ld}) \tag{4.53}$$

Using (4.45) and (4.49), one can directly check that

$$T(-\sigma^t)(\rho_{TA}(\widehat{e}_b|_{(x,\dot{x})})) = (x^j, -\dot{x}^l \sigma_{ld}, 0, -\sigma_{ld}\rho^l_b) \in (-\sigma^t)^* TA^*.$$

On the other hand, using the local expression (4.51), we have

$$\rho_{T^*A}(\Lambda^\sharp(\widehat{e}_b|_{(x,\dot{x})})) = (x^j, -\dot{x}^l \sigma_{ld}, 0, \rho^l_d \sigma_{lb})$$

It follows that the compatibility (4.50) for core sections amounts to

$$\langle \rho(e_b), \sigma(e_d) \rangle = -\langle \rho(e_d), \sigma(e_b)) \rangle,$$

which is equivalent to (4.41).

For sections of type Te_b, again using (4.45) and (4.49), we get

$$T(-\sigma^t)(\rho_{TA}(Te_b|_{(x,\dot{x})})) = (x^j, -\dot{x}^l \sigma_{ld}, \rho^j_b, \zeta_d) \in (-\sigma^t)^* TA^*,$$

where

$$\zeta_d = -\dot{x}^l \left(\frac{\partial \sigma_{ld}}{\partial x^k} \rho^k_b + \sigma_{id} \frac{\partial \rho^i_b}{\partial x^l} \right) = -\left\langle \mathcal{L}_{\rho(e_b)} \sigma(e_d), \dot{x} \right\rangle \tag{4.54}$$

Similarly, we compute

$$\rho_{T^*A}(\Lambda^\sharp(Te_b|_{(x,\dot{x})})) = (x^j, -\dot{x}^l \sigma_{ld}, \rho^j_b, \zeta'_d),$$

where

$$\begin{aligned} \zeta'_d &= \dot{x}^l \rho^k_d \left(\frac{\partial \sigma_{kb}}{\partial x^l} - \frac{\partial \sigma_{lb}}{\partial x^k} \right) - \phi_{ijk} \rho^k_b \dot{x}^i \rho^j_d + C^c_{db} \dot{x}^l \sigma_{lc} \\ &= \left\langle -i_{\rho(e_d)}(d\sigma(e_b)) + i_{\rho(e_d)} i_{\rho(e_b)} \phi + \sigma([e_d, e_b]), \dot{x} \right\rangle. \end{aligned} \tag{4.55}$$

Comparing (4.54) and (4.55), it follows that the compatibility (4.50) for sections of the type Te_b is verified if and only if (4.42) holds.

Let us now check the bracket preserving condition (4.43), that in our case reads

$$\begin{aligned} \Lambda^\sharp([U, V]_{TA}|_{(x,\dot{x})}) =& f_j g_i [U_j, V_i]_{T^*A}|_{-\sigma^t(x,\dot{x})} + \mathcal{L}_{\rho_{TA}(U)} g_i V_i|_{-\sigma^t(x,\dot{x})} \\ &- \mathcal{L}_{\rho_{TA}(V)} f_j U_j|_{-\sigma^t(x.\dot{x})}, \end{aligned} \tag{4.56}$$

where $U, V \in \Gamma(TA)$, and $f_j, g_i \in C^\infty(TM)$, $U_j, V_i \in \Gamma(T^*A)$ are such that $\Lambda^\sharp(U) = f_j(-\sigma^t)^* U_j$ and $\Lambda^\sharp(V) = g_i(-\sigma^t)^* V_i$.

From (4.52), (4.53), we can write

$$\Lambda^\sharp(Te_a|_{(x,\dot{x})}) = e^L_a|_{-\sigma^t(x,\dot{x})} + f^a_j \widehat{dx^j}|_{-\sigma^t(x,\dot{x})} \tag{4.57}$$

$$\Lambda^\sharp(\widehat{e}_a|_{(x,\dot{x})}) = \widehat{\sigma(e_a)}|_{-\sigma^t(x,\dot{x})} = g^a_i \widehat{dx^i}|_{-\sigma^t(x,\dot{x})}, \tag{4.58}$$

where

$$f_j^a = \dot{x}^l \left(\frac{\partial \sigma_{ja}}{\partial x^l} - \frac{\partial \sigma_{la}}{\partial x^j} \right) - \phi_{ijk} \rho_a^k \dot{x}^i, \quad g_i^a = \sigma_{ia}, \tag{4.59}$$

so we can express the images in terms of sections of types (4.46) and (4.47) on T^*A. It will be useful to note that the functions $f_j^a = f_j^a(x, \dot{x})$ satisfy

$$f_j^a dx^j = i_{\dot{x}} d\sigma(e_a) - i_{\dot{x}} i_{\rho(e_a)} \phi, \tag{4.60}$$

viewed as an equality of *horizontal* 1-forms on TM, i.e. 1-forms of type $\alpha_j(x, \dot{x}) dx^j$ (in this formula, $\dot{x}$ is seen as the vector field $\dot{x}^l \frac{\partial}{\partial x^l}$ on TM). In fact, locally, there is an identification of the space of horizontal 1-forms on TM with a subspace of sections of $(-\sigma^t)^* T^* A$ via

$$\Omega^1_{hor}(TM) \longrightarrow \Gamma((-\sigma^t)^* T^* A), \quad \alpha_j(x, \dot{x}) dx^j \mapsto \alpha_j(x, \dot{x}) \widehat{dx^j}|_{-\sigma^t(x, \dot{x})}. \tag{4.61}$$

In the remainder of this section, we will use this identification to view horizontal 1-forms on TM as sections on the bundle $(-\sigma^t)^* T^* A$. In particular, in order to simply our notation, we will write $\widehat{dx^j}|_{-\sigma^t(x,\dot{x})}$ just as dx^j.

Since it suffices to verify condition (4.56) for sections of types Te_a (linear) and $\widehat{e}_a$ (core), we have three cases to analyze.

Core-core sections

If $U = \widehat{e}_a$ and $V = \widehat{e}_b$ are core sections, then by (4.44) we know that $[\widehat{e}_a, \widehat{e}_b]_{TA} = 0$, so the l.h.s. of (4.56) vanishes. On the other hand, from (4.45), the Lie derivatives on the r.h.s. of (4.56) are only with respect to the variable $\dot{x}$. Since the functions g_j in (4.58) do not depend on $\dot{x}$ and $[\widehat{dx^i}, \widehat{dx^j}]_{T^*A} = 0$, it follows that, for a pair of core sections, the r.h.s. of (4.56) vanishes as well.

Core-linear sections

Let us consider (4.56) when $U = Te_a$ and $V = \widehat{e}_b$. Since $[Te_a, \widehat{e}_b]_{TA} = C_{ab}^c \widehat{e}_c$, it follows from (4.58) that the l.h.s. of (4.56) is

$$\Lambda^\sharp([Te_a, \widehat{e}_b]_{TA}) = \sigma([e_a, e_b]).$$

Using the bracket relations (4.48), one directly sees that the first term on the r.h.s. of (4.56) is just $\sigma_{ib} d\rho_a^i$. For the second term, we have

$$\begin{aligned}(\mathcal{L}_{\rho_{TA}(Te_a)} \sigma_{ib}) dx^i &= \mathcal{L}_{\rho_a^l \frac{\partial}{\partial x^l}} (\sigma_{ib} dx^i) - \sigma_{ib} \mathcal{L}_{\rho_a^l \frac{\partial}{\partial x^l}} dx^i \\ &= \mathcal{L}_{\rho(e_a)} \sigma(e_b) - \sigma_{ib} d\rho_a^i.\end{aligned}$$

The third term on the r.h.s. of (4.56) is given by

$$\begin{aligned}(\mathcal{L}_{\rho_{TA}(\widehat{e}_b)} f_j^a) dx^j &= \left(\rho_b^l \left(\frac{\partial \sigma_{ja}}{\partial x^l} - \frac{\partial \sigma_{la}}{\partial x^j} \right) - \phi_{ijk} \rho_a^k \rho_b^i \right) dx^j \\ &= i_{\rho(e_b)} d\sigma(e_a) - i_{\rho(e_b)} i_{\rho(e_a)} \phi.\end{aligned}$$

As a result, in this case, (4.56) is equivalent to

$$\sigma([e_a, e_b]) = \mathcal{L}_{\rho(e_a)} \sigma(e_b) - i_{\rho(e_b)} d\sigma(e_a) + i_{\rho(e_b)} i_{\rho(e_a)} \phi,$$

which agrees with condition (4.42).

Linear-linear sections

We finally consider (4.56) when $U = Te_a$ and $V = Te_b$. From (4.44), (4.57) and (4.58), and using (4.60), we see that the l.h.s. of (4.56) is

$$\begin{aligned}\Lambda^\sharp([Te_a, Te_b]_{TA}) &= C^c_{ab} e^L_c|_{-\sigma^t(x,\dot{x})} + C^c_{ab} f^c_j dx^j + dC^c_{ab}(\dot{x})\sigma(e_c)\\ &= [e_a, e_b]^L|_{-\sigma^t(x,\dot{x})} + C^c_{ab}(i_{\dot{x}} d\sigma(e_c) - i_{\dot{x}} i_{\rho(e_c)}\phi) + dC^c_{ab}(\dot{x})\sigma(e_c)\\ &= [e_a, e_b]^L|_{-\sigma^t(x,\dot{x})} + i_{\dot{x}} d\sigma([e_a, e_b]) + dC^c_{ab}\langle\sigma(e_c), \dot{x}\rangle - i_{\dot{x}} i_{[\rho(e_a),\rho(e_b)]}\phi.\end{aligned}$$

As in (4.60), we abuse notation and use $\dot{x}$ also to represent the vector field $\dot{x}^l \frac{\partial}{\partial x^l}$.

By (4.57), we can write $\Lambda^\sharp(Te_a) = e^L_a + f^a_j dx^j$ and $\Lambda^\sharp(Te_b) = e^L_b + f^b_i dx^i$. Using (4.48), we see that the first term on the r.h.s. of (4.56) is

$$[e_a, e_b]^L|_{-\sigma^t(x,\dot{x})} + dC^c_{ab}\big\langle\sigma^t(\dot{x}), e_c\big\rangle - f^a_j d\rho^j_b + f^b_i d\rho^i_a.$$

Note that the second and third terms on the r.h.s. of (4.56) are $\mathcal{L}_{\rho_{TA}(Te_a)} f^b_i dx^i$ and $\mathcal{L}_{\rho_{TA}(Te_b)} f^a_j dx^j$, respectively. Let us find a more explicit expression for the latter (the former is clearly completely analogous). Since

$$\mathcal{L}_{\rho_{TA}(Te_b)} f^a_j dx^j = \mathcal{L}_{\rho_{TA}(Te_b)}(f^a_j dx^j) - f^a_j \mathcal{L}_{\rho_{TA}(Te_b)} dx^j,$$

it follows that

$$\mathcal{L}_{\rho_{TA}(Te_b)} f^a_j dx^j = \mathcal{L}_{\rho_{TA}(Te_b)} i_{\dot{x}} d\sigma(e_a) - \mathcal{L}_{\rho_{TA}(Te_b)} i_{\dot{x}} i_{\rho(e_a)}\phi - f^a_j d\rho^j_b.$$

Let us consider the (local) vector fields on TM given by $V_b = d\rho^l_b(\dot{x})\frac{\partial}{\partial x^l}$ and $V'_b = d\rho^l_b(\dot{x})\frac{\partial}{\partial \dot{x}^l}$, so that $\rho_{TA}(Te_b) = \rho(e_b) + V'_b$. It is simple to check that $[\rho(e_a), \dot{x}] = -V_b$ and $\mathcal{L}_{V'_b} i_{\dot{x}}\alpha = i_{V_b}\alpha$ for any $\alpha = \frac{1}{2}\alpha_{ij}(x)dx^i \wedge dx^j$. Using Cartan calculus, we find

$$\begin{aligned}\mathcal{L}_{\rho_{TA}(Te_b)} i_{\dot{x}} d\sigma(e_a) &= \mathcal{L}_{\rho(e_b)} i_{\dot{x}} d\sigma(e_a) + \mathcal{L}_{V'_b} i_{\dot{x}} d\sigma(e_a)\\ &= -i_{V_b} d\sigma(e_a) + i_{\dot{x}}\mathcal{L}_{\rho(e_b)} d\sigma(e_a) + i_{V_b} d\sigma(e_a)\\ &= i_{\dot{x}} d i_{\rho(e_b)} d\sigma(e_a).\end{aligned}$$

Similarly,

$$\mathcal{L}_{\rho_{TA}(Te_b)} i_{\dot{x}} i_{\rho(e_a)}\phi = i_{\dot{x}}\mathcal{L}_{\rho(e_b)} i_{\rho(e_a)}\phi.$$

As a result, we obtain

$$\mathcal{L}_{\rho_{TA}(Te_b)} f^a_j dx^j = i_{\dot{x}} d i_{\rho(e_b)} d\sigma(e_a) - i_{\dot{x}}\mathcal{L}_{\rho(e_b)} i_{\rho(e_a)}\phi - f^a_j d\rho^j_b.$$

Analogously, we have

$$\mathcal{L}_{\rho_{TA}(Te_a)} f^b_i dx^i = i_{\dot{x}} d i_{\rho(e_a)} d\sigma(e_b) - i_{\dot{x}}\mathcal{L}_{\rho(e_a)} i_{\rho(e_b)}\phi - f^b_i d\rho^i_a.$$

Hence (4.56) amounts to the identity

$$\begin{aligned}i_{\dot{x}} d\sigma([e_a, e_b]) - i_{\dot{x}} i_{[\rho(e_a),\rho(e_b)]}\phi =& i_{\dot{x}} d i_{\rho(e_a)} d\sigma(e_b) - i_{\dot{x}}\mathcal{L}_{\rho(e_a)} i_{\rho(e_b)}\phi\\ &- i_{\dot{x}} d i_{\rho(e_b)} d\sigma(e_a) + i_{\dot{x}}\mathcal{L}_{\rho(e_b)} i_{\rho(e_a)}\phi\end{aligned} \tag{4.62}$$

By basic Cartan calculus of forms, we have the identity

$$i_{[\rho(e_a),\rho(e_b)]}\phi - \mathcal{L}_{\rho(e_a)} i_{\rho(e_b)}\phi + \mathcal{L}_{\rho(e_b)} i_{\rho(e_a)}\phi = d i_{\rho(e_b)} i_{\rho(e_a)}\phi.$$

It follows that (4.62) is equivalent to

$$\begin{aligned}d\sigma([e_a, e_b]) &= d(i_{\rho(e_a)} d\sigma(e_b) - i_{\rho(e_b)} d\sigma(e_a) + i_{\rho(e_b)} i_{\rho(e_a)}\phi)\\ &= d(\mathcal{L}_{\rho(e_a)}\sigma(e_b) - d i_{\rho(e_a)}\sigma(e_b) - i_{\rho(e_b)} d\sigma(e_a) + i_{\rho(e_b)} i_{\rho(e_a)}\phi)\\ &= d(\mathcal{L}_{\rho(e_a)}\sigma(e_b) - i_{\rho(e_b)} d\sigma(e_a) + i_{\rho(e_b)} i_{\rho(e_a)}\phi)\end{aligned}$$

which holds by (4.42). $\square$

4.4. **Applications to integration.** In this section we present an alternative proof to the main result in [3], which describes IM 2-forms as infinitesimal versions of multiplicative 2-forms ([3, Thm. 2.5]).

Let $\mathcal{G}$ be a Lie groupoid over M, with Lie algebroid A. Let us denote the space of multiplicative 2-forms on $\mathcal{G}$ by $\Omega^2_{mult}(\mathcal{G})$, and the space of linear 2-forms on A by $\Omega^2_{lin}(A)$. We also consider the subspace $\Omega^2_{alg}(A) \subset \Omega^2_{lin}(A)$ of linear 2-forms Λ for which $\Lambda^\sharp : TA \longrightarrow T^*A$ is a Lie algebroid morphism.

A direct consequence of Prop. 3.7 is that $\mathrm{Lie}(\alpha)^\sharp$ is a Lie algebroid morphism for any multiplicative k-form α on $\mathcal{G}$. Using Prop. 4.6, part (1), we conclude that the Lie functor on multiplicative forms gives rise to a well-defined map

$$\mathrm{Lie} : \Omega^2_{mult}(\mathcal{G}) \longrightarrow \Omega^2_{alg}(A), \;\; \omega \mapsto \Lambda = \mathrm{Lie}(\omega). \tag{4.63}$$

Proposition 4.8. *If $\mathcal{G}$ is s-simply-connected, then* (4.63) *is a bijection.*

Proof. We will show that (4.63) has an inverse map. If $\Lambda \in \Omega^2_{alg}(A)$, then $\Lambda^\sharp : TA \longrightarrow T^*A$ is a morphism of algebroids. So

$$\theta_{\mathcal{G}}^{-1} \circ \Lambda^\sharp \circ j_{\mathcal{G}}^{-1} : A(T\mathcal{G}) \longrightarrow A(T^*\mathcal{G})$$

is a Lie algebroid morphism. Since $\mathcal{G}$ is s-simply-connected, so is $T\mathcal{G}$. By Lie's second theorem for algebroids (see e.g. [12]), there exists a unique Lie groupoid morphism $\omega^\sharp : T\mathcal{G} \longrightarrow T^*\mathcal{G}$ with $\mathrm{Lie}(\omega^\sharp) = \theta_{\mathcal{G}}^{-1} \circ \Lambda^\sharp \circ j_{\mathcal{G}}^{-1}$, or $\Lambda^\sharp = \theta_{\mathcal{G}} \circ \mathrm{Lie}(\omega^\sharp) \circ j_{\mathcal{G}}$.

It remains to check that $\omega^\sharp$ is indeed the bundle map associated with a 2-form $\omega \in \Omega^2(\mathcal{G})$, i.e., that it is a vector bundle map (covering the identity) with respect to the bundle structures $T\mathcal{G} \longrightarrow \mathcal{G}$ and $T^*\mathcal{G} \longrightarrow \mathcal{G}$, and that $(\omega^\sharp)^* = -\omega^\sharp$. A proof of this fact can be given just as in [14]: the key point is that the bundle projections $p_{\mathcal{G}} : T\mathcal{G} \longrightarrow \mathcal{G}$, $c_{\mathcal{G}} : T^*\mathcal{G} \longrightarrow \mathcal{G}$, the vector bundle sums $T\mathcal{G} \times_{p_{\mathcal{G}}} T\mathcal{G} \longrightarrow T\mathcal{G}$, $T^*\mathcal{G} \times_{c_{\mathcal{G}}} T^*\mathcal{G} \longrightarrow T^*\mathcal{G}$, and scalar multiplications $T\mathcal{G} \times \mathbb{R} \longrightarrow T\mathcal{G}$, $T^*\mathcal{G} \times \mathbb{R} \longrightarrow T^*\mathcal{G}$, as well as the natural pairing $T\mathcal{G} \times_{(p_{\mathcal{G}}, c_{\mathcal{G}})} T^*\mathcal{G} \longrightarrow \mathbb{R}$ are all groupoid morphisms. The corresponding maps for Lie algebroids (after the identifications (2.15) and (2.17)) are precisely the vector bundle structure maps and pairing for $p_A : TA \longrightarrow A$ and $c_A : T^*A \longrightarrow A$, see e.g. [13]. For example, to check that $c_{\mathcal{G}} \circ \omega^\sharp = p_{\mathcal{G}}$, it suffices to verify this condition (by connectivity of source-fibres) at the level of algebroids. But then we have

$$\mathrm{Lie}(c_{\mathcal{G}} \circ \omega^\sharp) = c_A \circ \Lambda^\sharp = p_A = \mathrm{Lie}(p_{\mathcal{G}}).$$

The other properties of $\omega^\sharp$ are derived from those of $\Lambda^\sharp$ similarly, as in [14, Thm. 4.1]. $\square$

Corollary 4.9 ([3])**.** *If $\mathcal{G}$ is s-simply-connected and $\phi \in \Omega^3(M)$ is closed, there is a one-to-one correspondence between multiplicative 2-forms on $\mathcal{G}$ satisfying $d\omega = \mathsf{s}^*\phi - \mathsf{t}^*\phi$ and IM 2-forms $\sigma : A \longrightarrow T^*M$ relative to ϕ.*

Proof. We know that $\mathrm{Lie} : \Omega^2_{mult}(\mathcal{G}) \longrightarrow \Omega^2_{alg}(A)$, $\omega \mapsto \Lambda = \mathrm{Lie}(\omega)$ is a bijection, and by Prop. 4.6, part (2), $d\omega = \mathsf{s}^*\phi - \mathsf{t}^*\phi$ if and only if $\Lambda = -(\sigma_\omega^*\omega_{can} + \rho^*\tau(\phi))$. The conclusion now follows from Theorem 4.7. $\square$

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