RAMIFIED PULL-BACK COMPONENTS OF THE SPACE OF CODIMENSION ONE FOLIATIONS

AUTHOR: WANDERSON COSTA E SILVA
IMPA

MARCH, 2013

ADVISOR: ALCIDES LINS NETO

Agradecimentos

Agradeço ao meu mestre Alcides Lins Neto, meu orientador, por ter me passado um problema de tese interessante, por tudo que me ensinou, pelas valiosas conversas e inúmeras sugestões, e sobretudo pela paciência e generosidade que teve comigo. Gostaria de dizer que foi um prazer ter sido seu aluno. E o senhor é sem dúvida uma das maiores referências profissionais que tenho.

Ao IMPA pela excelente infra-estrutura, um excelente ambiente de pesquisa e um corpo de funcionários de dar inveja a qualquer instituição do mundo!

Agradeço aos membros da banca, Alcides Lins Neto, Paulo Sad(com quem aprendi a desenhar), César Camacho, Israel Vaisencher, Thiago Fassarella e Carolina Araújo pela leitura atenciosa, pelas inúmeras sugestões e correções que permitiram melhorar significativamente o texto desta tese. Gostaria de agradecer a todos os colegas com quem discuti pontos desta tese, em especial Thiago Fassarella, Maurício Corrêa Júnior(Tonesco), Arturo Fernandes Perez.

Gostaria de agradecer também aos professores Jacob Palis Júnior(a pessoa que me incentivou a estudar folheações e seus conselhos foram decisivos na escolha da área), Marcelo Viana(por toda a ajuda de sempre, pela confiança e pela seriedade e profissionalismo), Gugu, Henrique Bursztyn, Jorge Zubelli, Fernando Codá, Karl Otto Stöhr, Amílcar Pacheco e Raquel, Bruno Scárdua. Aos funcionários do IMPA como um todo. Em especial, à turma da biblioteca, Carol, Cecília, Fábio, Geysa, e Carla. Um agradecimento especial vai também ao professor César Camacho por toda a sua ajuda nos momentos difíceis. Sua generosidade sempre seguirá comigo! Ao pessoal da sala 210 (especialmente Guilherme Brondi e Rogerinho), agradeço ao Paulinho, Cássia Pessanha, Suely, Jurandira, Ana Paula, Juliana Bressan e Pedro Faro.

Aos professores da UFMG, Mário Jorge, Sylvie, Sônia Carvalho, Carlos Moreira, Armando Neves, Cica, Francisco Dutenhefner, Elmo Salomão, Agostinho, Zé Guilherme, Wagner Nunes, Waguinho, Gabriel Pelegatti, Márcia Fusaro, Rogério Santos Mol (a quem devo o gosto pela Geometria Complexa) e a turma do Xerox(Antônio Carlos, Miguel e Valdo).

À Marizina, Vilma, Lívia, Sander Bôsco e Edmilson pelos excelentes anos passados no CCDA.

Aos meus amigos Marlon López, Michael Deutsch pela ajuda na revisão da tese. Em especial ao Marlon López por toda a ajuda com as figuras e sobretudo com a preparação na apresentação da defesa. Sua ajuda foi imprescindível. Aos amigos Maurício Collares e Victor Duarte pela ajuda nas partes computacionais. Ao meu grande amigo Jean pela ajuda nos momentos difíceis e pela amizade.

Ao Érico Goulart pela amizade e por compartilharmos o sonho da carreira ciêntífica desde a adolescência. Aos meus amigos com quem dividi moradia no Rio, Augusto Quadros, Marcelo Hilário, Luca Merténs, Stefan Weiss, Dan jane, Zulu, Marcos César, Luiz Paulo, Elias Ramos, Dona Celisete, Márcia, Tuti.

Eu sei que fiz muitos amigos IMPA vou me lidar a colocar o nome de alguns aqui: Ivaldo Nunes, Vanessa Ramos, José Eduardo(Severino), Adriana Neumann, Yuri Lima, Arturo, Nivaldo Medeiros, Heudson Mirandola, Fernando (Nando), Walmir, Ieda Polessa, Fernando Codá, Maurício Vilches, Sérgio, Rodrigo Salomão, Fábio Simas, Carlos Morales, Andreia Nascimento, Isabel Cherques, Fátima Russo, Carlos Moraes (Pará), LG, Sobrinho, Didier, Felipe Linares, Diego Nehab, Reymundo Heluani, Eduardo Esteves, Alfredo Iusem, Benar Fux Svaiter, Diogo Gobira, Fernando Ferraz, Leandro Loriato, Weliton, Cássio Félix, Camila Veneo, Luiza Roris, Teresa Amorelli, Tertuliano, Emílio, Júlio Daniel, Fernando Carneiro, Joyce, Flávia Furtado, Fernanda Lopes, Pablo Guarino, Flávio Rocha, Pedro Rizzo, Alejandro Simarra, Ruben Lizarbe, Raphael Constant, Liliana, Evilson, Rudy, Luiz Carlos, Josenildo, Kênia, família Petrúcio(os 3 irmãos).

Enfim a todos os meus amigos que fiz no IMPA. Aos meus amigos de Ouro Branco especialmente muito especialmente ao Cristiano(Tibém), Domingos, Vérton, Cristiano do Liu, Romero, Tcheco, Théa Beatriz e Fermando Rômulo Costa, Clarice, Celmar, Gilvan, Flávio Ferreira(e toda a família). Agradecimento singelo para Maíra Frid pelos bons momentos de convivência e também por tudo que me ensinou! Aos meus irmãos Édson, Fabiana e Vanessa(in memorian) e ao Tio Vavá pela torcida. Dedico também à memória dos Professors Carlos Isnard, Paulo Sabini, Flaviano Bahia, Marco Brunella e do colega José Leandro Pinheiro(O Tiú).

Esses são os agradecimentos que faria neste momento, certamente seriam outros em momentos diferentes. Não se excluem, se somam. Por fim gostaria gostaria de agradecer ao governo brasileiro através dos orgãos CNPq e CAPES pelo auxílio financeiro.

Um Meio ou uma Desculpa

"Não conheço ninguém que conseguiu realizar seu sonho, sem sacrificar feriados e domingos pelo menos uma centena de vezes. Da mesma forma, se você quiser construir uma relação amiga com seus filhos, terá que se dedicar a isso, superar o cansaço, arrumar tempo para ficar com eles, deixar de lado o orgulho e o comodismo. Se quiser um casamento gratificante, terá que investir tempo, energia e sentimentos nesse objetivo. O sucesso é construído à noite! Durante o dia você faz o que todos fazem. Mas, para obter um resultado diferente da maioria, você tem que ser especial. Se fizer igual a todo mundo, obterá os mesmos resultados. Não se compare à maioria, pois, infelizmente ela não é modelo de sucesso. Se você quiser atingir uma meta especial, terá que estudar no horário em que os outros estão tomando chope com batatas fritas. Terá que planejar, enquanto os outros permanecem à frente da televisão. Terá que trabalhar enquanto os outros tomam sol à beira da piscina. A realização de um sonho depende de dedicação, há muita gente que espera que o sonho se realize por mágica, mas toda mágica é ilusão, e a ilusão não tira ninguém de onde está, em verdade a ilusão é combustível dos perdedores pois... Quem quer fazer alguma coisa, encontra um MEIO. Quem não quer fazer nada, encontra uma DESCULPA." *Roberto Shinyashiki*

”No que diz respeito ao empenho, ao compromisso, ao esforço, à dedicação, não existe meio termo. Ou você faz uma coisa bem feita ou não faz.”

Ayrton Senna

Dedico este trabalho aos meus pais, José Paulino da Silva e Salete da Costa Silva(in memorian), que são, sem sombra de dúvida, as minhas maiores referências.

Abstract. Let $\mathbb{F}\mathrm{ol}\,(k;n)$ be the space of codimension one holomorphic foliations of degree k on $\mathbb{P}^n$. In this work we prove that, if $n \geq 3$, then the set of foliations $\mathcal{F}$ of $\mathbb{P}^n$ which can be written as $\mathcal{F} = f^*(\mathcal{G})$, where $\mathcal{G}$ is a foliation in $\mathbb{P}^2$ of degree $d \geq 2$ with three invariant lines in general position and $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$, $deg\,(f) = \nu \geq 2$, $f = \left(F_0^{\alpha} : F_1^{\beta} : F_2^{\gamma}\right)$, $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ such that $1 \leq \alpha < \beta < \gamma$ and g.c.d $(\beta.\gamma, \alpha.\gamma, \alpha.\beta) = 1$, is an irreducible component of the following space:

$$\mathbb{F}\mathrm{ol}\left(\left(\nu\left[(d-1)+\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right]-2, n\right)\right).$$

Contents

1. Notations

(1) $\Pi_n : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n -$ The canonical projection which defines $\mathbb{P}^n$. If $p \in \mathbb{C}^{n+1} \setminus \{0\}$ then $\Pi_n(p) = [p]=$ line in $\mathbb{C}^{n+1}$ joining 0 to p.

(2) $deg\,(f) -$ the algebraic degree of f.

(3) $Il_3\,(d,2) -$ Subset of $\mathbb{F}\mathrm{ol}\,(d,2)$ corresponding to the foliations with three invariant straight lines in general position.

(4) $BRM\,(n,\nu,\alpha,\beta,\gamma) = \{f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2\}$ of degree ν which are given by $f = \left(F_0^{\alpha} : F_1^{\beta} : F_2^{\gamma}\right)$, where $deg\,(F_0)\,.\alpha = deg\,(F_1)\,.\beta = deg\,(F_2)\,.\gamma = \nu, \nu \geq 2, (\alpha,\beta,\gamma) \in \mathbb{N}^3$ such that $1 \leq \alpha < \beta < \gamma$ and g.c.d $(\beta.\gamma, \alpha.\gamma, \alpha.\beta) = 1$. (Branched Rational Mappings)

(5) $Gen\,(n,\nu,\alpha,\beta,\gamma) -$ The Zariski open subset corresponding to those $f \in BRM\,(n,\nu,\alpha,\beta,\gamma)$ such that for all $p \in \tilde{f}^{-1}\,(0) \setminus \{0\}$ we have $dF_0\,(p) \wedge dF_1\,(p) \wedge dF_2\,(p) \neq 0$, here we are denoting by $\tilde{f}$ the lifting of f.

(6) $Sing\,(\mathcal{F}) -$ Singular set of $\mathcal{F}$.

(7) $I\,(f) -$ The indeterminacy set of f.

(8) $Z\,(F_0,...,F_m) -$ Common zero set of the polynomials $F_0,...,F_m$.

(9) $P\,(f) -$ Critical set of $f : \mathbb{P}^n \to \mathbb{P}^2$.

(10) $V\,(f) -$ The set $V\,(f) = \mathbb{P}^n \backslash I\,(f)$.

(11) $C\,(f) -$ The image of the critical set of $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$, $C\,(f) = f\,(P\,(f))$.

(12) $\mathcal{H}\,(d,2) =$ Subset of $\mathbb{F}\mathrm{ol}\,(d,2)$ corresponding to the foliations such that all their singularities are of Hyperbolic-type.

(13) Submanifold $-$ A smooth complex subvariety in $\mathbb{P}^n$.

(14) $\mathbb{P}^2_{[r,s,t]} -$ the weighted projective plane with weights (r,s,t).

2. Introduction

Let $\mathcal{F}$ be a holomorphic singular foliation on $\mathbb{P}^n$ of codimension 1, $\Pi_n : \mathbb{C}^{n+1}\backslash \{0\} \to \mathbb{P}^n$ be the natural projection and $\mathcal{F}^* = \Pi_n^* (\mathcal{F})$. It is known that $\mathcal{F}^*$ can be defined by an integrable $1-$form $\Omega = \sum_{j=0}^n A_j dz_j$, where the $A_j's$ are homogeneous polynomials of the same degree, satisfying the so called Euler condition:

$$\sum_{j=0}^{n} z_j A_j \equiv 0, \tag{1}$$

and codim$(S(\Omega)) \geq 2$, where $S(\Omega)$ is the singular set of Ω, $S(\Omega) = \{A_0 = ... = A_n = 0\}$.

Let us give an idea of the proof of this fact. Since $H^1(\mathbb{C}^{n+1}\backslash \{0\}, \mathcal{O}^*) = 0$ and $\mathcal{F}^*$ can be defined locally by an integrable 1-forms, it follows that $\mathcal{F}^*$ can be defined by an integrable holomorphic 1-form η in $\mathbb{C}^{n+1}\backslash \{0\}$. Hartog's Theorem implies that η can be extended to $\mathbb{C}^{n+1}$, so that we can consider its Taylor series $\eta = \eta_k + \eta_{k+1} + ...$, where the coefficients of η_j are homogeneous polynomials of degree j and η_k is integrable. On the other hand, since the fibers of Π_n are contained in the leaves of $\mathcal{F}^*$, we get that $\mathcal{F}^*$ is invariant by the homotheties of $\mathbb{C}^{n+1}$. This implies that $\mathcal{F}^*$ can be represented by $\Omega = \eta_k$ and that the coefficients of Ω satisfy (1). The singular set of $\mathcal{F}$, $S(\mathcal{F})$, is $\Pi_n(S(\Omega)) = \Pi_n(S(\mathcal{F}^*))$. Recall that the integrability condition is given by

$$\Omega \wedge d\Omega = 0. \tag{2}$$

The leaves of $\mathcal{F}$ are of the form $\Pi_n(L)$, where L is a leaf of $\mathcal{F}^*$, that is, a codimension-1 solution of the differential equation $\Omega = 0$. The form Ω will be called a homogeneous expression of $\mathcal{F}$. The degree of $\mathcal{F}$ is, by definition, the number of tangencies (counted with multiplicities) of a generic linearly embedded $\mathbb{P}^1$ with $\mathcal{F}$. If we denote it by $deg(\mathcal{F})$ then $deg(\mathcal{F}) = d - 1$, where $d = deg(A_0) = ... = deg(A_n)$. We will denote the space of foliations of a fixed degree, say k, in $\mathbb{P}^n$ by $\mathbb{F}\text{ol}(k, n)$. If we consider relation (2), and the fact that $S(\mathcal{F})$ has codimension ≥ 2, then we see that $\mathbb{F}\text{ol}(k, n)$ can be identified with a Zariski's open set in the variety obtained by projectivizing the space of forms Ω which satisfy (1) and (2). It is in fact an intersection of quadrics. Hence we have the following:

Problem: ***Describe and classify the irreducible components of*** $\mathbb{F}\text{ol}(k; n)$ $k \geq 3$ ***on*** $\mathbb{P}^n$***,*** $n \geq 3$.

Before stating the precise results, let us describe some known results and examples. We recall that this list is not complete.

(1) *Example 0.* In the case $n = 2$ condition (1) implies condition (2), so that $\mathbb{F}\text{ol}(k, 2)$ is a Zariski open set of a linear variety.

(2) *Example 1.* A foliation of degree 0 in $\mathbb{P}^n$ has meromorphic first integral of the form f/g, where $f, g : \mathbb{C}^{n+1} \to \mathbb{C}$ are homogeneous polynomials of degree 1. The form Ω, which describes the foliation, is in this case $\Omega = fdg - gdf$. As a consequence, all foliations of degree 0 are linearly equivalent.

(3) *Example* 2. A foliation of degree 1 in $\mathbb{P}^n$ can be described by a 1-form in $\mathbb{C}^{n+1}$ which has an integrating factor. We say that a meromorphic function ϕ is an integrating factor of Ω, if the form Ω/ϕ is closed. In the case of degree 1, the form Ω can be of three types:

(a) $\Omega = f_1 f_2 f_3 \sum_{j=1}^{3} \lambda_j \frac{df_j}{f_j}$, where f_1, f_2 and f_3 are homogeneous polynomials of degree 1 and $\lambda_1 + \lambda_2 + \lambda_3 = 0$. In this case the integrating factor is $\phi = f_1 f_2 f_3$.

(b) $\Omega = f_2^2 df_1 - f_1 (f_2 + f_3) df_2 + f_1 f_2 df_3$, where f_1, f_2 and f_3 are homogeneous polynomials of degree 1. The integrating factor is $\phi = f_1 f_2^2$. As is shown in [C.Ln] this case can be considered as a limit case of (3a).

(c) $\Omega = f_2 df_1 - 2 f_1 df_2$, where $deg(f_1) = 2$ and $deg(f_2) = 1$. The proof of the above result can be found in [Jou]. As a consequence, $\mathbb{F}\text{ol}(1, n)$, for $n \geq 3$, has two irreducible components, where a generic point of the first (resp. second) component is given by a form of the type (3a) (resp.(3c)).

Next we will describe some known irreducible components of $\mathbb{F}\text{ol}\,(k, n)\,,\ n \geq 3$.

(4) *Example* 3. Linear Pull-back foliations. Let $\mathcal{F}$ be a foliation of degree k in $\mathbb{P}^2$ and $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2, n \geq 3$, be a linear submersion $\tilde{f} : \mathbb{C}^{n+1} \to \mathbb{C}^3$. Then $\mathcal{F}^* = \tilde{f}^* (\mathcal{F})$ is a foliation in $\mathbb{F}\text{ol}\,(k, n)$. We denote the space of foliations of this type by $\mathcal{PBL}\,(k; n)$. It can be proved, by using the techniques developed in [Ca.Ln], that $\mathcal{PBL}\,(k; n)$ is an irreducible component of $\mathbb{F}\text{ol}(k, n)$ for all $k \geq 2$ and $n \geq 3$.

(5) *Example* 4. Rational Components. Given $l, m \in \mathbb{N}$ such that $l + m - 2 = k$, let f and g be homogeneous polynomials in $\mathbb{C}^{n+1}$ such that:

(a) $deg\,(f) = l, deg\,(g) = m$, and $l/m = r/s$, where $g.c.d\,(r, s) = 1$.

(b) If $z \in \{f = g = 0\} - \{0\}$, then $df\,(z) \wedge dg\,(z) \neq 0$.

(c) The hypersurface $\pi\,(f = 0) \subset \mathbb{P}^n$ is smooth.
Let $\mathcal{F}$ be the foliation in $\mathbb{P}^n$, $n \geq 3$, whose leaves are the level surfaces of $\phi = f^r/g^s$, considered as a meromorphic function on $\mathbb{P}^n$. In homogeneous coordinates $\mathcal{F}$ can be defined by the form $\Omega = rgdf - sfdg$, where $\text{codim}(S\,(\Omega)) \geq 2$,. Therefore $\mathcal{F} \in \mathbb{F}\text{ol}\,(k, n)$ for $k = l + m - 2$. We denote by $\mathcal{R}\,(l, m; n)$ the set of foliations in $\mathbb{F}\text{ol}\,(k, n)$ of this type. The following result is a direct consequence of [G.Ln] and [C.Ln1].

Theorem 2.1. $\overline{\mathcal{R}\,(l, m; n)}$ *is a irreducible component of* $\mathbb{F}\text{ol}\,(k, n)$, *if* $n \geq 3$.

In the case $k = 2$ we have two possibilities: $l = m = 2$ and $l = 1, m = 3$.

(6) *Example* 5. Logarithmic components. Let $f_1, ..., f_m$ be homogeneous polynomials in $\mathbb{C}^{n+1}$, where $m \geq 3$, and $\lambda_1, ..., \lambda_m \in \mathbb{C}^*$ be such that $\sum_{j=1}^{m} \lambda_j d_j = 0$ where $d_j = deg(f_j)$. The form

$$\Omega = f_1 ... f_m \sum_{j=1}^{m} \lambda_j \frac{df_j}{f_j},$$

is integrable. The condition $\sum_{j=1}^{m} \lambda_j d_j = 0$ implies that Ω satisfies (1), so that it defines a foliation $\mathcal{F} = \mathcal{F}(\Omega) \in \mathbb{P}^n$. A 1-form as in (3) will be called a logarithmic form, and the foliation it induces in $\mathbb{P}^n$ a logarithmic foliation.

When $f_1, ..., f_m$ are irreducibles and relatively prime we will denote $\mathcal{F}$ by

$$\mathcal{F}(f_1, ..., f_m, \lambda_1, ..., \lambda_m).$$

The set of the foliations of this type will be denoted by $L(d_1, ..., d_m) \subset \mathbb{F}\mathrm{ol}(k, n)$, where $k = d_1 + ... + d_m - 2$.

The following is a known result:

Theorem 2.2. *If* $n \geq 3, m \geq 3$ *and* $k = d_1 + ... + d_m - 2$*, then* $\overline{L(d_1, ..., d_m)}$ *is an irreducible component of* $\mathbb{F}\mathrm{ol}(k, n)$.

The proof of this theorem is done in [C.A].

In the case $k = 2$ we have two possibilities: $m = 4, d_1 = d_2 = d_3 = d_4 = 1$ and $m = 3, d_1 = d_2 = 1, d_3 = 2$. We have so far described five types of irreducible components of $\mathbb{F}\mathrm{ol}(k, n); n \geq 3$.

To obtain a satisfactory description of $\mathbb{F}\mathrm{ol}(d; n)$ (for example, to talk about deformations) it would be reasonable to know the decomposition of $\mathbb{F}\mathrm{ol}(d; n)$ in irreducible components.

In the paper [C.Ln], the authors have proved that the space of holomorphic codimension one foliations of degree 2 on $\mathbb{P}^n, n \geq 3$, has six irreducible components, which can be described by geometric and dynamic properties of a generic element. Five of these components are

$$\overline{\mathcal{R}(2, 2; n)}, \overline{\mathcal{R}(1, 3; n)}, \overline{\mathcal{L}(1, 1, 1, 1; n)}, \overline{\mathcal{L}(1, 1, 2; n)}; \overline{\mathcal{PBL}(2; n)}.$$

The sixth component is called exceptional, in particular, because two generic elements in it are equivalent by an automorphism of $\mathbb{P}^n$. We refer the curious reader to [C.Ln], [Ln] for a detailed discussion and for the explanation about the notation.

A consequence of their classification is that we have two possibilities for a degree two foliation $\mathcal{F}$ on $\mathbb{P}^n, n \geq 3$: either $\mathcal{F}$ is defined by a meromorphic closed 1-form on $\mathbb{P}^n$, or $\mathcal{F} = f^*(\mathcal{G})$, where $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a linear map and $\mathcal{G}$ is a degree two foliation of $\mathbb{P}^2$. A foliation defined by a meromorphic closed 1-form admits a special transverse projective structure with poles, namely a translation structure [Sc]. On the other hand, a foliation of the form $\mathcal{F} = f^*(\mathcal{G})$ admits such a structure, if and only if, $\mathcal{G}$ admits [C.Ln.L.P.T], which is not always the case: a foliation of $\mathbb{P}^2$ which admits a projective or affine transverse structure has always algebraic leaves, whereas for any $k \geq 2$, the generic foliations on $\mathbb{P}^2$ of degree $k \geq 2$ has no algebraic invariant curves.

On the other hand there are known irreducible components in which the typical element is a pull-back of a foliation on $\mathbb{P}^2$ by a rational map, like the following result associated to "Non Linear (generic) Pull-Backs" [C.Ln.E],[Ln].

Let us introduce the objects needed to give the precise statement: Given a generic rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ of degree $\nu \geq 1$, it can be written in homogeneous coordinates as $f = (F_0, F_1, F_2)$ where F_0, F_1 and F_2 are homogeneous polynomials of degree ν. Also consider foliation $\mathcal{G}$ on $\mathbb{P}^2$ of degree $d \geq 2$. We can associate to the pair $(f, \mathcal{G})$ the pull-back foliation $\mathcal{F} = f^*\mathcal{G}$. The degree of the foliation $\mathcal{F}$ is $\nu(d+2) - 2$ as proved in [C.Ln.E]. Denote by $PB(d, \nu; n)$ the closure in $\mathbb{F}\mathrm{ol}\,(\nu(d+2)-2, n)$, $n \geq 3$ of the set of foliations $\mathcal{F}$ of the form $f^*\mathcal{G}$, where f has degree ν and $\mathcal{G} \in \mathbb{F}\mathrm{ol}\,(d, 2)$. Since the map $(f, \mathcal{G}) \to f^*\mathcal{G}$ is an algebraic parametrization of $PB(d, \nu; n)$, it follows that $PB(d, \nu; n)$ is an irreducible algebraic subset of $\mathbb{F}\mathrm{ol}\,(\nu(d+2)-2, n)$, $n \geq 3$.

The main result contained in [C.Ln.E] is as follows:

Theorem *$PB(d, \nu; n)$ is a unirational irreducible component of* $\mathbb{F}\mathrm{ol}\,(\nu(d+2)-2, n)$; $n \geq 3$, $\nu \geq 1$ and $d \geq 2$.

We remind that the previous result contains the result of [Ca.Ln].

This motivates the following conjecture which is attributed to different authors (Brunella, Lins-Neto among others.)

Main Conjecture: ***Any codimension one holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$ is either a pull-back of a foliation $\mathcal{G}$ on $\mathbb{P}^2$ by a Rational Map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ or admits a transverse projective structure with poles, in a codimension 1 algebraic submanifold.***

A particular case of this conjecture was proved in [C.Ln.L.P.T]. Recently, in a paper to appear, Cerveau and Lins-Neto [C.Ln2] proved that this conjecture is also true for foliations of degree 3.

3. The Present Work

Two distinct families of irreducible components of "Pull-Back type" are known:

(1) "Linear Pull-Backs" [Ca.Ln],[C.Ln],[Ln].

(2) "Non Linear (Generic) Pull-Backs" [C.Ln.E],[Ln].

In the present work, we will see a generalization of case (2).

What we will do is described as follows:

(1) Describe some irreducible subvariety $\mathcal{A}(k; n) \subset \mathbb{F}\mathrm{ol}\,(k; n)$.
(2) Study a neighborhood $\mathcal{U} \subset \mathbb{F}\mathrm{ol}\,(k; n)$ of a generic member of the family $\mathcal{A}(k; n)$.

In some cases, one can show that all such foliations on the neighborhood $\mathcal{U} \subset \mathbb{F}\mathrm{ol}\,(k;n)$, also belong to the family $\mathcal{A}(k;n)$. In this case, the closure $\mathcal{A}(k;n)$ will be an irreducible component of the space $\mathbb{F}\mathrm{ol}\,(k;n)$.

Without details, let us state some results that we will prove in this thesis:

Denote by $\mathcal{H}(d,2)$ the subset of foliations on $\mathbb{P}^2$ of degree d with all singularities of Hyperbolic-Type and $Il_3(d,2)$ the set of foliations on $\mathbb{P}^2$ of degree d having 3 invariant lines in general position (later we will discuss in detail this set).

Set also $A\,(d) = Il_3\,(d,2) \cap \mathcal{H}\,(d,2)$. Our first result is the following:

Theorem A. *Let $d \geq 2$. There exists an open and dense subset $M_1\,(d) \subset A\,(d)$, such that if $\mathcal{G} \in M_1\,(d)$ then the only algebraic invariant curves of $\mathcal{G}$ are the three lines.*

Let us describe the type of pull-back foliation that we will consider.
Let $\mathcal{G}$ be a foliation on $\mathbb{P}^2$ with three invariant straight lines in general position, say ℓ_0, ℓ_1 and ℓ_2.
Consider coordinates $(X,Y,Z) \in \mathbb{C}^3$ such that

$$\ell_0 = \Pi_2(X=0), \ell_1 = \Pi_2(Y=0) \text{ and } \ell_2 = \Pi_2(Z=0).$$

The foliation $\mathcal{G}$ can be represented in these coordinates by a polynomial 1-form of the type

$$\Omega = YZA\,(X,Y,Z)\,dX + XZB\,(X,Y,Z)\,dY + XYC\,(X,Y,Z)\,dZ$$

where $A+B+C=0$ by (1).
Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ be a rational map represented in homogeneous coordinates $W \in \mathbb{C}^{n+1}$ and $(X,Y,Z) \in \mathbb{C}^3$ by $\tilde{f} = (F_0^\alpha, F_1^\beta, F_2^\gamma)$, where F_0, F_1 and $F_2 \in \mathbb{C}[W]$ are homogeneous polynomials without common factors satisfying

$$\alpha.dg(F_0) = \beta.dg(F_1) = \gamma.dg(F_2) = \nu.$$

The pull back foliation $f^*(\mathcal{G})$ is then defined in homogeneous coordinates W by $\tilde{\eta}_{[f,\mathcal{G}]}\,(W)$, where

$$\begin{aligned}
\tilde{f}^*\Omega &= d\left(F_0^\alpha\right).F_1^\beta.F_2^\gamma.\left(A \circ F\right) + F_0^\alpha.d\left(F_1^\beta\right).F_2^\gamma.\left(B \circ F\right) + F_0^\alpha.F_1^\beta.d\left(F_2^\gamma\right).\left(C \circ F\right) \\
&= \left[F_0^{\alpha-1}.F_1^{\beta-1}.F_2^{\gamma-1}\right]\tilde{\eta}_{[f,\mathcal{G}]}\,(W).
\end{aligned}$$

Note that,

$$\tilde{\eta}_{[f,\mathcal{G}]}\,(W) = \left[\alpha.F_1.F_2.\left(A \circ F\right)dF_0 + \beta.F_0.F_2.\left(B \circ F\right)dF_1 + \gamma.F_0.F_1.\left(C \circ F\right)dF_2\right],$$

and if there are no more common terms to simplify we have that the degree of the 1− form $\tilde{\eta}_{[f,\mathcal{G}]}\,(W)$ is equal to the degree of any term. And the value of the degree is $\Gamma(d,\nu,\alpha,\beta,\gamma)$ is the degree's sum of one of the terms of the above expression, since $\tilde{F}^*\Omega$ is a homogeneous polynomial 1-form, that is, where the coefficients of $\tilde{\eta}_{[f,\mathcal{G}]}\,(W)$ are homogeneous of degree

$$\Gamma = \nu\left[(d-1) + \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right] - 1.$$

Remark 1. The crucial point here is that the mapping f sends the three hypersurfaces $(F_i = 0)$ contained in its critical set over the three $\mathcal{G}$ invariant lines.

Let $PB\,(\Gamma - 1, \nu, \alpha, \beta, \gamma)$ be the closure in $\mathbb{F}\mathrm{ol}\,(\Gamma - 1, n)$ of the following set:

$$\left\{ \left[\tilde{\eta}_{[f,\mathcal{G}]}\right],\ \text{where } \tilde{\eta}_{[f,\mathcal{G}]} \text{ is as before} \right\}.$$

This set is an irreducible algebraic subset of $\mathbb{F}\mathrm{ol}\,(\Gamma - 1, n)$. We will return to this point in section 6.

Let us state our main theorem.

Theorem B. $PB(\Gamma - 1, \nu, \alpha, \beta, \gamma)$ is a unirational irreducible component of $\mathbb{F}\mathrm{ol}\,(\Gamma - 1, n)$ for all $n \geq 3$, $deg(F_0).\alpha = deg(F_1).\beta = deg(F_2).\gamma = \nu \geq 2$, $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ such that $1 \leq \alpha < \beta < \gamma$, g.c.d $(\beta.\gamma, \alpha.\gamma, \alpha.\beta) = 1$ and $d \geq 2$.

In order to state and prove precisely our results we have to go through several concepts.

4. **Rational Maps**

As we know from projective geometry the concept of regular maps is very restrictive, since points of indeterminacy are quite natural. This leads to the concept of rational maps, which are regular on a dense Zariski-open set and need not be defined outside.

More precisely, let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be irreducible algebraic varieties. Then a *rational map* $f : X \dashrightarrow Y$ is an equivalence class of regular maps $f_U : U \to Y$ defined on some dense Zariski-open $U \subset X$, where f_U and f_V are equivalent if $f_U = f_V$ on $U \cap V$. In particular, if $X = \mathbb{P}^n$, there are homogeneous polynomials $F_0, ..., F_m \in \mathbb{C}\,[T_0, ..., T_m]$ of the same degree such that $U := \mathbb{P}^n \backslash V\,(F_0, ..., F_m) \neq \emptyset$ and $f\,(x) = (F_0\,(x)\,, ..., F_m\,(x)) \in Y$ for every $x \in U$. Obviously, there is a maximal open set $Def\,(f) \subset \mathbb{P}^n$, where f is regular, which is called the domain of definition. The closed set $I\,(f) := \mathbb{P}^n \backslash\ Def\,(f)$ is called the indeterminacy set. A rational map f is regular iff $Def\,(f) = \mathbb{P}^n$, that is $I\,(f) = \emptyset$.
It is important to note that when we are working in algebraic geometry we have two natural topologies: the **Zariski topology** and the **Strong topology**.

When we are working with the Zariski topology we use the term Regular and when we are using the Strong topology we use the term Holomorphic. In many situations, the strong topology will be more useful.
We will use the following interpretation of a rational map. Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ be a rational map and $\tilde{f} : \mathbb{C}^{n+1} \to \mathbb{C}^3$ a lifting of f, that is a homogeneous polynomial map such that the diagram below commutes:

$$\begin{array}{ccc}
\mathbb{C}^{n+1} \backslash \tilde{f}^{-1}\,(0) & \xrightarrow{\tilde{f}} & \mathbb{C}^3 \backslash \{0\} \\
{\scriptstyle \Pi_n} \big\downarrow & & {\scriptstyle \Pi_2} \big\downarrow \\
\mathbb{P}^n \backslash \Pi_n \left(\tilde{f}^{-1}\,(0)\right) & \xrightarrow{f} & \mathbb{P}^2
\end{array}$$

The *indeterminacy locus* of f is by definition the set $I(f) = \Pi_n\left(\tilde{f}^{-1}(0)\right)$. Observe that the restriction $f|_{\mathbb{P}^n \setminus I(f)}$ is holomorphic.

4.1. **Branched Rational Maps**

Definition 4.1. *Denote by* $BRM(n,\nu,\alpha,\beta,\gamma)$ *the set of maps* $\{f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2\}$ *of degree* ν *given by* $f = \left(F_0^\alpha : F_1^\beta : F_2^\gamma\right)$ *where* F_0, F_1 *and* F_2 *are homogeneous polynomials without common factors, with* $deg(F_0).\alpha = deg(F_1).\beta = deg(F_2).\gamma = \nu$, $\nu \geq 2$, $(\alpha,\beta,\gamma) \in \mathbb{N}^3$ *such that* $1 \leq \alpha < \beta < \gamma$ *and* $g.c.d(\beta.\gamma, \alpha.\gamma, \alpha.\beta) = 1$.

Let us fix some coordinates $(z_0, ..., z_n)$ on $\mathbb{C}^{n+1}$ and (X, Y, Z) on $\mathbb{C}^3$ and denote by $\left(F_0^\alpha, F_1^\beta, F_2^\gamma\right)$ the components of f relative to these coordinates.
We recall that the indeterminacy locus $I(f)$ is the intersection of the 3 hypersurfaces $(F_0 = 0)$, $(F_1 = 0)$ and $(F_2 = 0)$.

Definition 4.2. *We say that* $f \in BRM(n,\nu,\alpha,\beta,\gamma)$ *is generic if for all* $p \in \tilde{f}^{-1}(0) \setminus \{0\}$ *we have* $dF_0(p) \wedge dF_1(p) \wedge dF_2(p) \neq 0$.

This is equivalent to say that $f \in BRM(n,\nu,\alpha,\beta,\gamma)$ is *generic* if $I(f)$ is the transverse intersection of the 3 hypersurfaces $(F_0 = 0)$, $(F_1 = 0)$ and $(F_2 = 0)$ i.e. the three hypersurfaces intersect transversely at each point of $I(f)$, or equivalently, $dF_0(p) \wedge dF_1(p) \wedge dF_2(p) \neq 0$ for each $p \in I(f)$.

This implies that the set $I(f)$ is smooth.
For instance, if $n = 3$, f is generic and $deg(f) = \nu$, then by *Bezout's* theorem $I(f)$ consists of $\frac{\nu^3}{\alpha\beta\gamma}$ distinct points with multiplicity $\alpha\beta\gamma$. If $n = 4$, then $I(f)$ is a smooth connected algebraic curve in $\mathbb{P}^4$ of degree $\frac{\nu^3}{\alpha\beta\gamma}$. In general, for $n \geq 4$, $I(f)$ is a smooth connected algebraic submanifold of $\mathbb{P}^n$ of degree $\frac{\nu^3}{\alpha\beta\gamma}$ and codimension 3.
The critical set of $\tilde{f}$ is given by the points of $\mathbb{C}^{n+1}\setminus\{0\}$ where the rank of the following derivative matrix is smaller than 3

$$\begin{bmatrix} \alpha\left(F_0^{\alpha-1}\right)\nabla F_0 \\ \beta\left(F_1^{\beta-1}\right)\nabla F_1 \\ \gamma\left(F_2^{\gamma-1}\right)\nabla F_2 \end{bmatrix}$$

where $\nabla F_k = (\frac{\partial F_k}{\partial z_0}, ..., \frac{\partial F_k}{\partial z_n})$.

Note that the critical set is the union of two sets. The first is given by the set of $\{Z \in \mathbb{C}^{n+1} \setminus 0\} = X_1$ such that the rank of the following matrix

$$\begin{bmatrix} \nabla F_0 \\ \nabla F_1 \\ \nabla F_2 \end{bmatrix}$$

is smaller than 3. And the second is the subset

$$X_2 = \left\{Z \in \mathbb{C}^{n+1} \setminus \{0\} \mid \left(F_0^{\alpha-1}.F_1^{\beta-1}.F_2^{\gamma-1}\right)(Z) = 0\right\}.$$

Denote $P(f) = \Pi_n(X_1 \cup X_2)$ where X_1 and X_2 are the first and the second sets described previously.

The set of generic maps will be denoted by $Gen\,(n,\nu,\alpha,\beta,\gamma)$.
Let us prove the following proposition,

Proposition 4.3. *$Gen\,(n,\nu,\alpha,\beta,\gamma)$ is a Zariski dense subset of $BRM\,(n,\nu,\alpha,\beta,\gamma)$.*

Proof. Consider the following set

$$X = \{(f,p)\,;p \in I\,(f) \text{ and } dF_0\,(p)\wedge dF_1\,(p)\wedge dF_2\,(p) = 0\}\,,$$

Note that X is an algebraic subset of $BRM\,(n,\nu,\alpha,\beta,\gamma)\times\mathbb{P}^n$. If we consider the first projection

$$\pi_1 : BRM(n,\nu,\alpha,\beta,\gamma)\times\mathbb{P}^n \to BRM(n,\nu,\alpha,\beta,\gamma),$$

since $\mathbb{P}^n$ is a projective variety, $\pi_1\,(X)$ is closed in $BRM\,(n,\nu,\alpha,\beta,\gamma)$. Then if its complement is not empty it is an open and dense Zariski's subset of $BRM\,(n,\nu,\alpha,\beta,\gamma)$.

To conclude that the set $Gen(n,\nu,\alpha,\beta,\gamma)$ is non-empty we take three smooth projective hypersurfaces of degree $\frac{\nu}{\alpha}$, $\frac{\nu}{\beta}$ and $\frac{\nu}{\gamma}$ that intersect transverselly two by two. The existence of hypersufaces that satisfy these properties is a well known fact from differential topology [Hi].

The corresponding homogeneous polynomials that define each one of them we call F_0, F_1 and F_2. Hence we define lots of branched generic rational mappings. □

Remark 2. Let us observe that inside the set of all rational maps $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ with degree ν, $BRM\,(n,\nu,\alpha,\beta,\gamma)$ is a small subset. This is because irreducible polynomials are generic.

5. Foliations with 3 invariant lines

Denote by $I(d,2)$ the set of the holomorphic foliations on $\mathbb{P}^2$ of degree $d \geq 2$ that leaves the lines $X = 0$, $Y = 0$ and $Z = 0$ invariant. An element of this set can be represented by the following homogeneous polynomial 1−form

$$\Omega = YZA(X,Y,Z)dX + XZB(X,Y,Z)dY + XYC(X,Y,Z)dZ,$$

where the polynomials A, B and C are homogeneous of degree $d-1$. The condition $i_R\Omega = 0$, where R is the radial vector field,

$$R(X,Y,Z) = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z},$$

implies that $A+B+C = 0$. Moreover, any foliation which has 3 invariant straight lines in general position can be carried to one of these by a linear automorphism of $\mathbb{P}^2$. And what we do now is to let the group of linear automorphisms of $\mathbb{P}^2$ act on $I(d,2)$. After this procedure we obtain a set of foliations of degree d that we denote by $Il_3(d,2)$.
The relation

$$A+B+C = 0$$

enables to parametrize $I(d,2)$ as follows

$$\begin{aligned} \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1))^{\times 2} &\to \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1))^{\times 3} \\ (A,B) &\mapsto (A,B,-A-B) \end{aligned}$$

We are interested in making deformations of foliations and for our purposes we need a subset of $Il_3(d,2)$ with good properties (foliations having few algebraic invariant curves and only hyperbolic singularities). These properties will be explained in the next subsection.

5.1. A Jouanoulou's type theorem for foliations inside $Il_3(d,2)$

Let $q \in U$ be an isolated singularity of a foliation $\mathcal{G}$ defined on an open subset of $U \subset \mathbb{C}^2$. We say that q is *nondegenerate* if there exists a holomorphic vector field X tangent to $\mathcal{G}$ in a neighborhood of q, such that $DX(q)$ is nonsingular. In particular q is an isolated singularity of X.

Let q be a nondegenerate singularity of $\mathcal{G}$. The *characteristic numbers* of q are the quotients λ and λ^{-1} of the eingenvalues of $DX(q)$, which do not depend on the vector field X chosen as above. If $\lambda \notin \mathbb{Q}_+$ then $\mathcal{G}$ exhibits exactly two (smooth and transverse) *local separatrices* at q, say, S_q^+ and S_q^-, which are tangent to the characteristic directions of a vector field X as above, and with eigenvalues λ_q^+ and λ_q^-, respectively.
The Camacho-Sad index, for short, *characteristic numbers*, of these local separatrices are given by

$$I(\mathcal{G}, S_q^+) = \frac{\lambda_q^-}{\lambda_q^+} \text{ and } I(\mathcal{G}, S_q^-) = \frac{\lambda_q^+}{\lambda_q^-},$$

respectively.

The singularity is *hyperbolic* if the characteristic numbers are nonreal. We introduce the following spaces of foliations:

(1) $ND(d,2) = \{\mathcal{G} \in \mathbb{F}\mathrm{ol}(d,2)$ such that the singularities of $\mathcal{G}$ are nondegenerate$\}$,

(2) $\mathcal{H}(d,2) = \{\mathcal{G} \in ND(d,2)$ such that any characteristic number λ of $\mathcal{G}$ satisfies $\lambda \in \mathbb{C}\backslash\mathbb{R}\}$.

It is a well-known fact that $\mathcal{H}(d,2)$ contains an open and dense subset of $\mathbb{F}\mathrm{ol}(d,2)$, see [Ln1]. Denote by $A(d) = Il_3(d,2) \cap \mathcal{H}(d,2)$. Observe that $A(d)$ is a Zariski dense subset of $Il_3(d,2)$.

For the reader's convenience let us remember **Theorem A.**

Theorem A. *Let $d \geq 2$. There exists an open and dense subset $M_1(d) \subset A(d)$, such that if $\mathcal{G} \in M_1(d)$ then the only algebraic invariant curves of $\mathcal{G}$ are the three lines.*

We begin with a preliminary result:

Proposition 5.1. *Let $\mathcal{G}_0 \in ND(d,2)$. Then $\#sing(\mathcal{G}_0) = d^2+d+1 = N(d)$. Moreover if $sing(\mathcal{G}_0) = \{p_1^0, ..., p_N^0\}$ where $p_i^0 \neq p_j^0$ if $i \neq j$, then there are connected neighborhoods $U_j \ni p_j$, pairwise disjoint, and holomorphic maps $\phi_j : \mathcal{U} \subset ND(d,2) \to U_j$, where $\mathcal{U} \ni \mathcal{G}_0$ is an open neighborhood, such that for $\mathcal{G} \in \mathcal{U}$, $(sing(\mathcal{G}) \cap U_j) = \phi_j(\mathcal{G})$ is a nondegenerate singularity. In particular, $ND(d,2)$ is open in $\mathbb{F}\mathrm{ol}(2,d)$.*

Moreover, if $\mathcal{G}_0 \in \mathcal{H}(d,2)$ then the two local separatrices as well as their associated eigenvalues depend analytically on $\mathcal{G}$.

This result was proved in [Ln1] as a consequence of the Implicit Function Theorem for holomorphic mappings.

Regarding the position of the singularities of an element $\mathcal{G} \in A(d)$ we have the next proposition.

Proposition 5.2. *Let $\mathcal{G}$ be in $A(d)$, the localization of its singular set is:*

(1) *On the intersection of two of these lines we have one singularity.*

(2) *Over each line excluding the singularities which are in the intersection of two of them we have $(d-1)$ singularities.*

(3) *We have $(d-1)^2$ singularities outside of these 3 lines.*

We will use the intersection formulas of line bundles with curves. Let $\mathcal{G}$ be as above and consider its normal line bundle. When C is an invariant compact curve under $\mathcal{G}$ we have that:

$$N_{\mathcal{G}}.C = C.C + GSV(\mathcal{G}, C)$$

where

$$GSV(\mathcal{G}, C) = \sum_{p \in C} GSV(\mathcal{G}, C, p)$$

and $GSV(\mathcal{G}, C, p)$ is the Gomez-Mont-Seade-Verjovsky Index.

Let us prove the proposition 5.2:

Proof. Let C be an invariant straight line. It is known that index $GSV(\mathcal{G}, C, p) = 1$. This is due to the fact that our singularities are of the Hyperbolic-type [Bru] and the straight line has only one branch through p. As we know our foliation $\mathcal{G}$ has a normal bundle given by $N_{\mathcal{G}} = \mathcal{O}_{\mathbb{P}^2}(d+2)$ where d is the foliation's degree. Applying the intersection formula to the straight line we have that

$$(d+2) \cdot 1 = 1 + (\#sing(\mathcal{G})_{|C})$$

so

$$\#sing(\mathcal{G})_{|C} = d+1.$$

Excluding the singular points that are at the corners we have $(d-1)$ singularities on each straight line. This implies that the three lines contain $3(d-1)+3 = 3d$ singularities. Therefore, by Darboux's Theorem, outside the three lines we have

$$(d^2+d+1)-3-3(d-1) = (d-1)^2$$

singularities. $\square$

Let us fix a coordinate system on $\mathbb{P}^2$ and denote by ℓ_0, ℓ_1, and ℓ_2 the straight lines that corresponds to the planes $X=0$, $Y=0$ and $Z=0$ in $\mathbb{C}^3$, respectively.

Taking out the separatrices of the singularities of a foliation $\mathcal{G} \in A(d)$ that are at the corners, we will divide the subset of the remaining singularities and separatrices in 4 new subsets as follows,

(1) $\mathcal{S}_W(\mathcal{G}) = \{p_i; 1 \leq i \leq (d-1)^2\}$, that correspond to the singularities of $\mathcal{G}$ outside of the three invariant straight lines,

(2) $\mathcal{S}_{\ell_r}(\mathcal{G}) = \{p_{r,j}; 0 \leq r \leq 2, 1 \leq j \leq (d-1)\}$ that correspond to the singularities of $\mathcal{G}$ over the straight line ℓ_r different from the vertices,

(3) We also have $[0:0:1], [0:1:0]$ and $[1:0:0]$ that correspond to the intersection two by two of those invariant lines.

We enumerate the separatrices of the singularity p_i by S_i^+ and S_i^- for $\{1 \leq i \leq (d-1)^2\}$. The separatrix of the singularity $p_{r,j}$, $0 \leq r \leq 2$, $1 \leq j \leq (d-1)$ that is transverse to ℓ_r we will denote by S_j^r. We denote by $I(\mathcal{G}, S_i^-)$ and $I(\mathcal{G}, S_i^+)$ the characteristic numbers associated to the local separatrices S_i^+ and S_i^-, and by $I(\mathcal{G}, S_j^0)$, $I(\mathcal{G}, S_j^1)$ and $I(\mathcal{G}, S_j^2)$ the characteristic numbers associated to the local separatrices S_j^0, S_j^1 and S_j^2 respectively.

Let us choose a neighborhood $\mathcal{U}$ of $\mathcal{G}$ inside $A(d)$ as described previously, in such a way that

(1) $\mathcal{U} \ni \mathcal{G} \to I(\mathcal{G}, S_i^-)$,

(2) $\mathcal{U} \ni \mathcal{G} \to I(\mathcal{G}, S_i^+)$,

(3) $\mathcal{U} \ni \mathcal{G} \to I(\mathcal{G}, S_j^k)$,

are holomorphic maps. We denote by $S(\mathcal{G}) = \{S_i^-, S_i^+, S_j^k\}$ where i, j, k are as before.

Definition 5.3. *A configuration is a subset $C \subset S(\mathcal{G})$. Given a configuration C we define*

$$I(\mathcal{G}, C) = \sum_{S_i^- \in C} I(\mathcal{G}, S_i^-) + \sum_{S_i^+ \in C} I(\mathcal{G}, S_i^+) + \sum_{S_j^r \in C} I(\mathcal{G}, S_j^r).$$

Let C be a configuration. Then we can split C in three parts

$$C = L \cup M \cup N,$$

where

(1) $L = \{S_j^0, S_j^1, S_j^2 \in C\}$,

(2) $M = \{S_i^+ \in C | S_i^- \notin C\} \cup \{S_i^- \in C | S_i^+ \notin C\}$,

(3) $N = \{S_i^+ \in C | S_i^- \in C\} \cup \{S_i^- \in C | S_i^+ \in C\}$.

We define $l = \#L$, $m = \#M$ and $n = \#N$.

If $C = \emptyset$ then $I(\mathcal{G}, C) = 0$. If $S \subset \mathbb{P}^2$ is an invariant irreducible algebraic curve then we define the configuration of S as the configuration $C(S)$ defined by the local separatrices of $\mathcal{G}$. Denote by $\tilde{k}$ the degree of S.

Proposition 5.4. *Let $\mathcal{G}$ be as above, and let S be an irreducible algebraic curve which is different from the 3 invariant straight lines. Write $C(S) = L \cup M \cup N$ as above. Then $C(S)$ satisfies the following properties:*

(1) $l = \#L = 3\tilde{k}$, *where* $1 \leq \tilde{k} \leq (d-1)$.

(2) $I(\mathcal{G}, C(S)) = \frac{2l^2+9n}{18}$
where $l = 3deg(S)$ *and* $n = 2(\#nodes)$.

Proof. Part (1) follows from Bezout's Theorem. Note that the number of intersection points between S and anyone of the invariant straight lines is the same (it coincides with the degree of the curve). In order to prove (2) we recall [Ln1] where it is shown that

$$0 < I(\mathcal{G}, S) = l - \chi(\tilde{S})$$

where $\chi(\tilde{S})$ is the Euler characteristic of the normalization $\tilde{S}$ of the curve S. Since S has only nodal singularities, which correspond to local separatrices in N which meet transversely, it follows from Hurwitz formula that

$$\chi(\tilde{S}) = 2 - 2\frac{(\frac{l}{3}-1)(\frac{l}{3}-2)}{2} - \frac{1}{2}n = \frac{-2l^2 + 18l - 9n}{18}$$

so that $I(\mathcal{G}, S) = \frac{2l^2+9n}{18}$.

□

Definition 5.5. *Let $d \geq 2 \in \mathbb{N}$, we define the subset $M_1(d)$ consisting of those foliations $\mathcal{G} \in A(d)$ such that for all configurations $C \subseteq S(\mathcal{G})$, with $l(C) = 3\tilde{k}$, where $1 \leq \tilde{k} \leq (d-1)$ we have $I(\mathcal{G}, C(S)) \neq \frac{2l(C)^2+9n(C)}{18}$.*

REMARKS.

(1) If $\mathcal{G} \in M_1(d)$ then $\mathcal{G}$ admits no irreducible algebraic invariant curve except for the 3 invariant straight lines.

(2) $A(d) \backslash M_1(d)$ is an analytic subset of $A(d)$, because it is defined (locally) by a finite number of equations of the form $I(\mathcal{G}, C(S)) = \frac{2l(C)^2+9n(C)}{18}$.

(3) $M_1(d)$ is open in $A(d)$.

Now we will prove **Theorem A**:

Proof. Since $A(d)\backslash M_1(d)$ is an analytic subset of $A(d)$, it suffices to prove that $M_1(d) \neq \emptyset$ (see also [Ln1]).

Let $U_0 = \{\mathbb{C}^2, (x, y)\}$, $U_1 = \{\mathbb{C}^2, (u, v)\}$ and $U_2 = \{\mathbb{C}^2, (\varsigma, s)\}$ be a covering of $\mathbb{P}^2$ by affine coordinate systems such that:

$$\begin{cases} x = \dfrac{1}{u} & y = \dfrac{v}{u}, \\ x = \dfrac{s}{\varsigma} & y = \dfrac{1}{\varsigma}, \\ \varsigma = \dfrac{u}{v} & s = \dfrac{1}{v} \end{cases}$$

Consider the polynomial vector field $X_0(x, y)$ on $U_0 = \{\mathbb{C}^2, (x, y)\}$ given by:

$$\begin{cases} \dot{x} = x(cx^{d-1} + ay^{d-1} + \lambda), \\ \dot{y} = y(bx^{d-1} + ey^{d-1} + \mu). \end{cases}$$

We demand that the constants a, b, c, e, λ and μ are all nonzero and $a \neq e$, $b \neq c$.
In the coordinates $U_1 = \{\mathbb{C}^2, (u, v)\}$ this differential equation turns into $X_1(u, v)$ which is given by

$$\begin{cases} \dot{u} = u(\lambda u^{d-1} + av^{d-1} + c), \\ \dot{v} = v[(b - c) + (\mu - \lambda)u^{d-1} + (e - a)v^{d-1}] \end{cases}$$

In the coordinates $U_2 = \{\mathbb{C}^2, (\varsigma, s)\}$ this differential equation turns into $X_2(\varsigma, s)$ which is given by

$$\begin{cases} \dot{\varsigma} = \varsigma[(e - a) + (b - c)\varsigma^{d-1} + (\mu - \lambda)s^{d-1}], \\ \dot{s} = s(e + \mu s^{d-1} + b\varsigma^{d-1}) \end{cases}$$

Observe that this foliation can be seen on homogeneous coordinates of $\mathbb{C}^3$ by the following homogeneous polynomial $1-$form:

$$\Omega = YZA\,(X, Y, Z)\,dX + XZB\,(X, Y, Z)\,dY + XYC\,(X, Y, Z)\,dZ,$$

where:

(1) $A(X, Y, Z) = bX^{d-1} + eY^{d-1} + \mu Z^{d-1}$,

(2) $B(X, Y, Z) = -(cX^{d-1} + aY^{d-1} + \lambda Z^{d-1})$,

(3) $C(X, Y, Z) = (c - b)\,X^{d-1} + (a - e)\,Y^{d-1} + (\lambda - \mu)\,Z^{d-1}$.

The singular set of the previous foliation is decomposed in 7 orbits by the action of the following finite automorphism group of $\mathbb{P}^2_{\mathbb{C}}$,

$$\begin{bmatrix} \delta^K & 0 & 0 \\ 0 & \delta^l & 0 \\ 0 & 0 & \delta^m \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \delta^k X \\ \delta^l y \\ \delta^m Z \end{bmatrix}$$

where δ is a $(d-1)-^{th}$ root of 1 and $1 \leq k, j, l \leq d-1$.

In local coordinates, for example in chart $U_0 = \{\mathbb{C}^2, (x,y)\}$ the foliation is invariant by the finite group generated by the two linear automorphisms of $\mathbb{C}^2$, $T(x,y) = (\delta x, y)$ and $H(x,y) = (x, \delta y)$ where δ a $(d-1)^{th}$ root of unity. For the other local coordinate charts is analogous.

Let $\mathcal{G}$ be the foliation which is defined by the $1-$form Ω of the previous example. The action under this group enables to divide the singular set of $\mathcal{G}$ into 7 distinct orbits:

(1) Each corner $\ell_i \cap \ell_j, i \neq j$ is a fixed point of H and T.

(2) The set of $(d-1)$ singularities over the straight line $\ell_r, 0 \leq r \leq 2$, (which are not corners) is an orbit of G.

(3) The set of $(d-1)^2$ singularities outside the 3 invariant straight lines constitute another orbit of G.

In our example we fix $a = (1-i)$, $b = 1$, $c = i$, $e = 1$, $\mu = i$ and $\lambda = 1$.

Let us analyze the singular set of $\mathcal{G}$.

The singularities of $\mathcal{G}$ in U_0 are given by:

(1) (0,0) which has characteristic numbers $\frac{\lambda}{\mu} = \frac{1}{i}$ and $\frac{\mu}{\lambda} = i$ which implies that this singularity is hyperbolic.

(2) the characteristic numbers of the singularities outside the 2 axes are obtained from the matrix

$$DX_0(x_k, y_j) = \begin{pmatrix} c(d-1)x_k^{d-1} & a(d-1)x_k y_j^{d-2} \\ b(d-1)y_j x_k^{d-2} & e(d-1)y_j^{d-1} \end{pmatrix}$$

where $1 \leq j \leq (d-1)$ and $1 \leq k \leq (d-1)$, that is, these characteristic numbers are roots of

$$\sigma + \sigma^{-1} + 2 = \frac{(Tr)^2}{D} = \frac{-i}{2}$$

where (Tr) is the trace and D is the determinant of the matrix $DX_0(x_k, y_j)$. Therefore solving this equation we obtain

$$\sigma^- = \frac{1}{4}\left[(-4-i) - \sqrt{(-1+8i)}\right] = \frac{1}{4}\left[-(4+\zeta) - i(\theta+1)\right]$$

and
$$\sigma^+ = \frac{1}{4}\left[(-4-i)+\sqrt{(-1+8i)}\right] = \frac{1}{4}\left[(\zeta-4)+i(\theta-1)\right]$$

where

$$\zeta = \sqrt{\frac{-1+\sqrt{65}}{2}} \text{ and } \theta = \frac{4}{\zeta}.$$

Again we get hyperbolic singularities.

(3) The singularities over the line $x=0$, excluding the origin are obtained solving the following equations

$$\begin{cases} x = 0 \\ 0 = (ey^{d-1}+\mu). \end{cases}$$

The linear part on the singularities is given by

$$DX_0(0,y_j) = \begin{pmatrix} ay_j^{d-1}+\lambda & 0 \\ 0 & e(d-1)y_j^{d-1} \end{pmatrix}$$

where $1 \leq j \leq (d-1)$, and the characteristic value is

$$\frac{e(d-1)y_j^{d-1}}{ay_j^{d-1}+\lambda} = \frac{(d-1)}{5} + \frac{2i(d-1)}{5}.$$

Because $d \geq 2$ these singularities are hyperbolic.

(4) By an analogous argument the characteristic value of the singularities over the line $y=0$ is

$$\frac{c(d-1)x_k^{d-1}}{bx_k^{d-1}+\lambda} = \frac{-2(d-1)}{5} + \frac{i(d-1)}{5}.$$

Again we get hyperbolic singularities.

The singularities of $\mathcal{G}$ in U_1 are given by:

(1) (0,0) which has characteristic numbers $\frac{b-c}{c} = -1-i$ and $\frac{c}{b-c} = \frac{1}{-1-i}$ which implies that this singularity is hyperbolic.

(2) the singularities over the line $u=0$, excluding the origin are obtained solving the following equations

$$\begin{cases} u = 0 \\ 0 = [(b-c)+(e-a)v^{d-1}]. \end{cases}$$

The linear part on the singularities is given by

$$DX_1(0,v_j) = \begin{pmatrix} av_j^{d-1}+c & 0 \\ 0 & (b-c)+d(e-a)v_j^{d-1} \end{pmatrix}$$

as before we calculate the (characteristic number) corresponding to the transversal separatrix passing through the singularity. And we have

$$\frac{(b-c)+d(e-a)v_j^{d-1}}{av_j^{d-1}+c} = \frac{(d-1)}{5} + \frac{3i(d-1)}{5}.$$

The singularity of $\mathcal{G}$ in U_2 is given by:

(1) (0,0) which has characteristic numbers $\frac{e-a}{e} = i$ and $\frac{e}{e-a} = \frac{1}{i}$ which implies that this singularity is hyperbolic.

Since $d \geq 2$ this already shows that all the singularities of $\mathcal{G}$ are hyperbolic.

To finish the proof, it remains to prove that $\mathcal{G}$ has no other invariant algebraic curve. This will be done using the Lins Neto's version of Camacho-Sad Index Theorem.

In fact we will sum (to combine) all the possible configurations.

The possible values for the Camacho-Sad Index are the following :

$$\begin{cases} \mathtt{r}(\frac{1}{4}[(-4-\theta)+i(\zeta+1)])+ \\ \mathtt{s}(\frac{1}{4}[(-4+\theta)+i(-\zeta+1)])+ \\ \frac{6\tilde{k}}{5}(d-1)i \end{cases}$$

where $\tilde{k} = deg(S) \leq (d-1)$. We also have that $0 \leq \mathtt{r} \leq (d-1)^2$ and $0 \leq \mathtt{s} \leq (d-1)^2$. Now we divide the previous sum in its real and imaginary parts.

(1) Imaginary Part:

$$\mathcal{I} = \frac{1}{4}[(\zeta+1)\mathtt{r} + \mathtt{s}(1-\zeta)] + \frac{\tilde{k}}{5}(6d-6)$$

(2) Real Part

$$\mathcal{R} = \frac{1}{4}[(-\theta-4)\mathtt{r} + \mathtt{s}(\theta-4)]$$

Since $\mathcal{R} \in \mathbb{Z}$ and $\theta \notin \mathbb{Q}$ we have $\mathtt{r}=\mathtt{s}$ and $\mathcal{R} = -2\mathtt{r} \leq 0$. This is a contradiction because $I(\mathcal{G}, C(S)) = \frac{2l(C)^2+9n(C)}{18} > 0$.

Hence, for this foliation, there is no possibility to have an algebraic invariant curve different from those three invariant straight lines. This finishes the proof of the theorem. $\square$

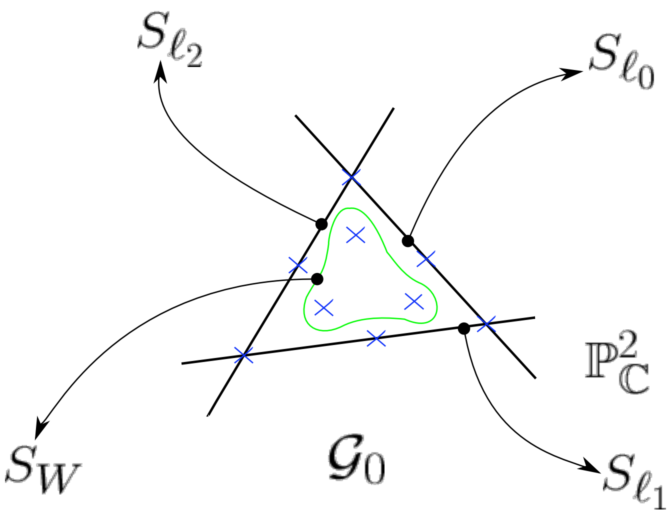

FIGURE 1. A singular hyperbolic holomorphic foliation on $\mathbb{P}^2_{\mathbb{C}}$ leaving 3 invariant lines.

6. Ramified Pull-Back Components - Generic Conditions -

Once we have a good set of foliations, that is, $M_1(d)$ and a good set of rational maps, $Gen(n, \nu, \alpha, \beta, \gamma)$, we will join these objects in the next definition.

Let us fix a coordinate system on $\mathbb{P}^2$ and denote by ℓ_0, ℓ_1 and ℓ_2 the straight lines that correspond to the planes $X = 0$, $Y = 0$ and $Z = 0$ in $\mathbb{C}^3$, respectively. Let us denote by $\tilde{M}_1(d)$ the subset $M_1(d) \cap I(d, 2)$.

Definition 6.1. *Let $f \in Gen(n, \nu, \alpha, \beta, \gamma)$. We say that $\mathcal{G} \in M_1(d)$ is in generic position with respect to f if $[Sing(\mathcal{G}) \cap Y_2] = \emptyset$, where:*

$$Y_2(f) = Y_2 := \Pi_2\left[\tilde{f}\left\{w \in \mathbb{C}^{n+1} | dF_0(w) \wedge dF_1(w) \wedge dF_2(w) = 0\right\}\right]$$

and ℓ_0, ℓ_1 and ℓ_2 are $\mathcal{G}$-invariant.

In this case we say that $(f, \mathcal{G})$ is a generic pair.

In particular when we fix a map $f \in Gen(n, \nu, \alpha, \beta, \gamma)$ the set

$$\mathcal{A} = \{\mathcal{G} \in M_1(d) \,|Sing(\mathcal{G}) \cap Y_2(f) = \emptyset\}$$

is an open and dense subset in $M_1(d)$ see[Ln.Sc], because $VC(f)$ is an algebraic curve in $\mathbb{P}^2$.

The set

$$U_1 := \{(f, \mathcal{G}) \in Gen(n, \nu, \alpha, \beta, \gamma) \times \tilde{M}_1(d) \,|Sing(\mathcal{G}) \cap Y_2(f) = \emptyset\}$$

is an open and dense subset in $Gen(n, \nu, \alpha, \beta, \gamma) \times \tilde{M}_1(d)$.

Hence the set

$$\mathcal{W} := \left\{\tilde{\eta}_{[f,\mathcal{G}]} | \, (f, \mathcal{G}) \in U_1\right\}$$

is an open and dense subset of $PB\left(\Gamma - 1, \nu, \alpha, \beta, \gamma\right)$.

As we will see later, if $(f, \mathcal{G})$ is a generic pair then the foliation $f^{*}\left(\mathcal{G}\right)$ has degree $(\Gamma - 1)$. (Section 7.)

Denote by $BRM\left(n, \nu, \alpha, \beta, \gamma\right)$ the set of maps $\{f : \mathbb{P}^{n} \dashrightarrow \mathbb{P}^{2}\}$ of degree ν given by $f = \left(F_0^{\alpha} : F_1^{\beta} : F_2^{\gamma}\right)$ where F_0, F_1 and F_2 are homogeneous polynomials without common factors, with $deg\left(F_0\right).\alpha = deg\left(F_1\right).\beta = deg\left(F_2\right).\gamma = \nu$, $\nu \geq 2$, $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ such that $1 \leq \alpha < \beta < \gamma$ and g.c.d $(\beta.\gamma, \alpha.\gamma, \alpha.\beta) = 1$.

Consider the set of foliations $Il_3\left(d, 2\right)$, $d \geq 2$, and the following map:

$$\begin{aligned} \Phi : BRM\left(n, \nu, \alpha, \beta, \gamma\right) \times Il_3\left(d, 2\right) &\rightarrow \mathbb{F}\mathrm{ol}\left(\Gamma - 1, n\right) \\ (f, \mathcal{G}) &\rightarrow f^{*}\left(\mathcal{G}\right) = \Phi\left(f, \mathcal{G}\right). \end{aligned}$$

The image of the mapping Φ can be written as:

$$\Phi\left(f, \mathcal{G}\right) = \left[\alpha.F_1.F_2.\left(A \circ F\right) dF_0 + \beta.F_0.F_2.\left(B \circ F\right) dF_1 + \gamma.F_0.F_1.\left(C \circ F\right) dF_2\right].$$

Remember that $\Phi\left(f, \mathcal{G}\right) = \tilde{\eta}_{[f,\mathcal{G}]}$.
More precisely, let $PB(\Gamma - 1, \nu, \alpha, \beta, \gamma)$ be the closure in $\mathbb{F}\mathrm{ol}\left(\Gamma - 1, n\right)$ of the set of foliations $\mathcal{F}$ of the form $f^{*}\left(\mathcal{G}\right)$, where $f \in BRM\left(n, \nu, \alpha, \beta, \gamma\right)$ and $\mathcal{G} \in Il_3(2, d)$.

Since $BRM\left(n, \nu, \alpha, \beta, \gamma\right)$ and $Il_3(2, d)$ are irreducible algebraic sets and the map $(f, \mathcal{G}) \rightarrow f^{*}\left(\mathcal{G}\right) \in \mathbb{F}\mathrm{ol}\left(\Gamma - 1, n\right)$ is an algebraic parametrization of $PB(\Gamma - 1, \nu, \alpha, \beta, \gamma)$. It follows that $PB(\Gamma - 1, \nu, \alpha, \beta, \gamma)$ is an irreducible algebraic subset of $\mathbb{F}\mathrm{ol}\left(\Gamma - 1, n\right)$, which contains the set of generic pull-backs foliations
$\{\mathcal{F}; \mathcal{F} = f^{*}(\mathcal{G}),$ where $(f, \mathcal{G})$ is a generic pair$\} \subset PB(\Gamma - 1, \nu, \alpha, \beta, \gamma)$ as an open(not Zariski) and dense subset of $PB(\Gamma - 1, \nu, \alpha, \beta, \gamma)$ for $\nu \geq 2$, $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ such that $1 \leq \alpha < \beta < \gamma$, g.c.d $(\beta.\gamma, \alpha.\gamma, \alpha.\beta) = 1$ and $d \geq 2$.

Once we have described all the ingredients let us recall **Theorem B**.

Theorem B. *$PB(\Gamma - 1, \nu, \alpha, \beta, \gamma)$ is a unirational irreducible component of $\mathbb{F}\mathrm{ol}\left(\Gamma - 1, n\right)$ for all $n \geq 3$, $deg(F_0).\alpha = deg(F_1).\beta = deg(F_2).\gamma = \nu \geq 2$, $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ such that $1 \leq \alpha < \beta < \gamma$ and g.c.d $(\beta.\gamma, \alpha.\gamma, \alpha.\beta) = 1$ and $d \geq 2$.*

7. Description of generic pull-back foliations on $\mathbb{P}^n$

7.1. The Kupka-Reeb Phenomenon

In this section we will consider an important class of singularities, which have stability properties under deformations. Moreover, these singularities appear in a Zariski's open subset of $\mathbb{F}\mathrm{ol}(k; n)$.

Definition 7.1. *Let $\Omega \in \mathbb{F}\mathrm{ol}(k;n)$. The Kupka singular set of the foliation Ω consists of the points*

$$K_{\mathcal{F}} = \{p \in \mathbb{P}^n | \Omega(p) = 0; d\Omega(p) \neq 0\}.$$

Remark 3. This condition does not depends of the holomorphic 1-form that expresses the foliation $\mathcal{F}$.

The main properties of the Kupka set, are summarized in the following result.

Theorem 7.2. *Let Ω and $K_{\mathcal{F}}$ be as above, then:*

(1) *$\mathcal{F}$ has a local product structure along $K_{\mathcal{F}}$: For every connected component $K \subset K_{\mathcal{F}}$ there exist a holomorphic 1-form, $\tilde{\omega} = A(x,y)dx + B(x,y)dy$ called the transversal type at K, defined on a neighborhood V of $0 \in \mathbb{C}^2$ and vanishing only at 0, an open cover U_α of a neighborhood of K in $\mathbb{P}^n$ and a family of submersions $\phi_\alpha : U_\alpha \to \mathbb{C}^2$ such that*

  [a] $\phi_\alpha^{-1}(0) = K \cap U_\alpha$ and
  [b] $\Omega_\alpha = \phi_\alpha^* \tilde{\omega}$ defines $\mathcal{F}$ in U_α

(2) *$K_{\mathcal{F}}$ is persistent under perturbations of $\mathcal{F}$. More precisely, let $K \subset K_{\mathcal{F}}$ be a connected component and $\tilde{K} \subset K$ be a compact subset. Then $\tilde{K}$ is stable under small deformations of $\mathcal{F}$. Let $(\mathcal{F}_t)_{t\in(\mathbb{C},0)}$ be an analytic deformation of $\mathcal{F}$ in $\mathbb{F}\mathrm{ol}(k;n)$. Then $(K_{\mathcal{F}_t})_{t\in(\mathbb{C},0)}$ contains a compact subset $(\tilde{K}_t)_{t\in(\mathbb{C},0)}$ such that:*

  (a) *$(\tilde{K}_t)_{t\in(\mathbb{C},0)}$ is a holomorphic deformation of $\tilde{K}$ in $\mathbb{P}^n$.*

  (b) *There exists a unique 1-form $(\tilde{\omega}_t)_{t\in(\mathbb{C},0)}$ in a neighborhood of $\mathbb{C}^2$ such that for all $p \in (\tilde{K}_t)_{t\in(\mathbb{C},0)}$ there exists a holomorphic submersion $\phi_p : V_p \to \mathbb{C}^2$, V_p a neighborhood of p, such that $(\mathcal{F}_t)_{t\in(\mathbb{C},0)}|_{V_p}$ is represented by $(\phi_p^* \tilde{\omega}_t)_{t\in(\mathbb{C},0)}$.*

Let X be the dual vector field of $\tilde{\omega}$, since $d\Omega \neq 0$, we have that $tr(DX(0)) \neq 0$, thus the linear part $D = DX(0)$ which is well defined up to linear conjugation and multiplication by scalars, has at least one non-zero eigenvalue. We will say that D is the linear type of K. Normalizing, we may assume that the eigenvalues are 1 and μ. We will distinguish four possible types of Kupka type singularities:

(1) Sadle-node: If $\mu = 0$, in this case, the transversal type has the normal form
$$\tilde{\omega}(x,y) = (x(1+\lambda y^p) + yR(x,y))dy - y^{p+1}dx,$$
where $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$.
(2) Semisimple: If $\mu \neq 0$ and D is semisimple.
(3) Non-semisimple: This is the case when $\mu = 1$ and D is not semisimple.
(3) Hyperbolic: If $\mu \notin \mathbb{R}$.

A consequence of the previous theorem is the following:

Proposition 7.3. *Let $\mathcal{F}$ be a holomorphic foliation on a complex manifold M of dimension $n \geq 3$ such that $cod(sing(\mathcal{F})) \geq 2$. Let $K_{\mathcal{F}} = \cup_{j\in J} K_j$ be the decomposition of $K_{\mathcal{F}}$ in its connected components. If $p, q \in K_j$ then the transversal type of $\mathcal{F}$ at p and q coincide.*

We also note that $K(\mathcal{F}) = Sing(\mathcal{F}) - W(\mathcal{F})$ where $W(\mathcal{F}) = \cup_{\alpha\in A} W_\alpha$, where
$W_\alpha = \{p \in U_\alpha | \omega_\alpha(p) = 0, d\omega_\alpha(p) = 0\}$. Observe that $W(\mathcal{F})$ is an analytic subset of M.

7.2. Codimension 2 part and the Kupka set of $\mathcal{F} = f^*(\mathcal{G})$

Let τ be a singularity of $\mathcal{G}$ and $V_\tau = \overline{f^{-1}(\tau)}$. We will show that if $(f, \mathcal{G})$ is a generic pair then $V_\tau \backslash I(f)$ is contained in the Kupka set of $\mathcal{F}$.

Consider coordinates (X, Y, Z) such that $\ell_0 = (X = 0)$, $\ell_1 = (Y = 0)$ and $\ell_2 = (Z = 0)$. The singular set of $\mathcal{G}$ consists of the points: $a = [0:0:1]$, $b = [0:1:0]$, $c = [1:0:0]$, $\mathcal{S}_W(\mathcal{G})$, $\mathcal{S}_{\ell_r}(\mathcal{G})$, $0 \le r \le 2$. We know that $\#\mathcal{S}_W(\mathcal{G}) = (d-1)^2$, $\#\mathcal{S}_{\ell_r}(\mathcal{G}) = (d-1)$, $0 \le r \le 2$. (See 5.1)

Fix $p \in V_\tau \backslash I(f)$. We have three possibilities:

(1) The case where τ is a corner, for instance $a = [0:0:1]$.

In this case f is not a submersion at p, but there exist analytic coordinate systems $(U, (x, y, z)), (x, y) : U \to \mathbb{C}^2, z : U \to \mathbb{C}^{n-2}$, and $(V, (u, v))$, at p and $a = f(p)$, respectively, such that $u(a) = v(a) = 0$, $f(x, y, z) = (x^\alpha, y^\beta) = (u, v)$. Suppose that $\mathcal{G}$ is represented by the 1-form

$$\omega = P(u, v)dv - Q(u, v)du$$

in a neighborhood of a. Then $\mathcal{F}$ is represented by

$$\tilde{\omega} = f^*(\omega) = P(x^\alpha, y^\beta)d(y^\beta) - Q(x^\alpha, y^\beta)d(x^\alpha).$$

Moreover, the hypothesis of $\mathcal{G}$ be of Hyperbolic-type implies that we can suppose

$$\omega(u, v) = \lambda_1 u(1 + R(u, v))dv - \lambda_2 v du,$$

where $\frac{\lambda_2}{\lambda_1} \in \mathbb{C} \backslash \mathbb{R}$.

We obtain $\tilde{\omega}(x, y) = (x^{\alpha-1}.y^{\beta-1})\hat{\omega}(x, y)$ where

$$\hat{\omega}(x, y) = \lambda_1 \beta x(1 + R(x^\alpha, y^\beta)dy - \alpha\lambda_2 y dx,$$

and so $d\hat{\omega}(p) \neq 0$.

Therefore p is in the Kupka-set of $\mathcal{F}$.

For the other corners the argumentation is analogous.

(2) At the points $\tau \in \mathcal{S}_W(\mathcal{G})$:

In this case f is a submersion at p and there exist analytic coordinate systems $(U, (x, y, z)), (x, y) : U \to \mathbb{C}^2, z : U \to \mathbb{C}^{n-2}$, and $(V, (u, v))$, at p and $\tau = f(p)$ respectively, such that, $u(\tau) = v(\tau) = 0$ and $f(x, y, z) = (x, y)$. In this case $\mathcal{G}$ can be represented in a neighborhood of 0 by a 1-form ω such that $d\omega \neq 0$. Since f is a submersion in a neighborhood of p, $\mathcal{F}$ is represented by $f^*\omega$ on U and so $d(f^*(\omega))(p) \neq 0$. Therefore p is in the Kupka-set of $\mathcal{F}$.

Remark 4. In this case the transversal type of $\mathcal{F}$ at p is the same of the germ of $\mathcal{G}$ at τ.

(3) At the points $\tau \in \mathcal{S}_{\ell_r}, 0 \le r \le 2$. For instance when $\tau \in \mathcal{S}_{\ell_0}$.

In this case f is not a submersion at p, but there exist analytic coordinate systems $(U, (x, y, z)), (x, y) : U \to \mathbb{C}^2, z : U \to \mathbb{C}^{n-2}$, and $(V, (u, v))$, at p and $\tau = f(p)$ respectively, such that, $u(\tau) = v(\tau) = 0$ and $f(x, y, z) = (x^\alpha, y)$. Suppose that $\mathcal{G}$ is represented by the 1-form

$$\omega = P(u,v)dv - Q(u,v)du$$

in a neighborhood of τ.

Then $\mathcal{F}$ is represented by

$$\tilde{\omega} = f^*(\omega) = P(x^\alpha, y)dy - Q(x^\alpha, y)d(x^\alpha).$$

Moreover, the hypothesis of $\mathcal{G}$ be of Hyperbolic-type implies that we can suppose

$$\omega(u,v) = \lambda_1 u(1 + R(u,v))dv - \lambda_2 v du,$$

where $\frac{\lambda_2}{\lambda_1} \in \mathbb{C}\backslash\mathbb{R}$.

We obtain $\tilde{\omega}(x,y) = (x^{\alpha-1})\hat{\omega}(x,y)$ where

$$\hat{\omega}(x,y) = \lambda_1 x(1 + R(x^\alpha, y)dy - \alpha\lambda_2 y dx,$$

and so $d\hat{\omega}(p) \neq 0$.

Therefore p is in the Kupka-set of $\mathcal{F}$.

The other cases are similar.

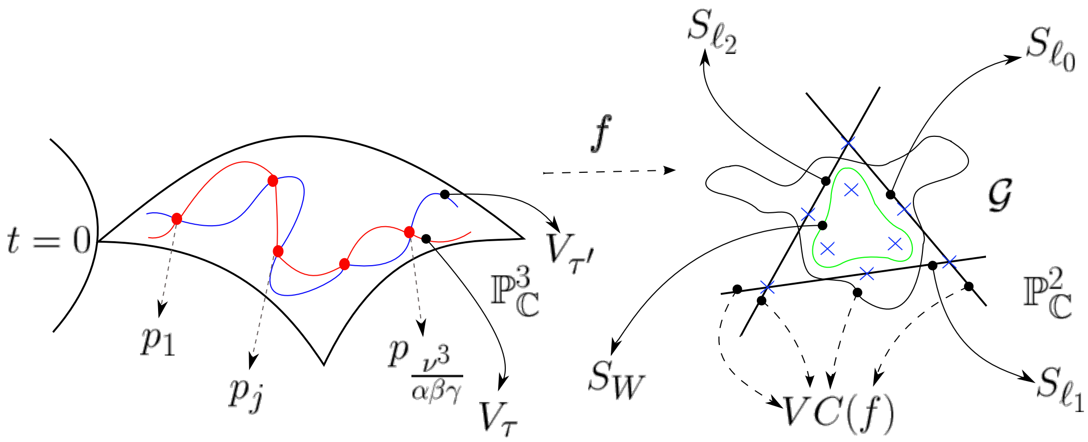

FIGURE 2. A typical singular set of a generic pull-back foliation in $\mathbb{P}^3$.

7.3. **Quasi-homogeneous singularities**

In this part we will study some special type of singularities: quasi-homogeneous singularities. They appear in the indeterminacy set of f. Remember that the indeterminacy set of f is contained in the singular set of $\mathcal{F}$. And they will play a central role in great part of the proof of Theorem B.

7.3.1. *Generalized Kupka and quasi-homogeneous singularities.* The next lines have been extracted from [C.CA.G.LN]. We refer the reader also to [Ln0].

Definition 7.4. *The affine Lie Algebra, $\mathfrak{aff}(\mathbb{C})$, is the Lie Algebra of dimension 2 generated by the vectors $\{e_1, e_2, [e_2, e_1] = e_2\}$.*

A representation

$$\rho : \mathfrak{aff}(\mathbb{C}) \to \chi(\mathbb{C}^3)$$

can be defined by a pair of two holomorphic vector fields in $\mathbb{C}^3$, namely S and $\mathcal{X}$ that satisfies an equality of the following type $[S, \mathcal{X}] = \ell\mathcal{X}$ for some $\ell \in \mathbb{C}^*$.

Definition 7.5. *Let ω be an holomorphic integrable 1-form defined in a neighborhood of $p \in \mathbb{C}^3$. We say that p is a Generalized Kupka(GK) singularity of ω if $\omega(p) = 0$ and either $d\omega(p) \neq 0$ or p is an isolated zero of $d\omega$.*

When p is an isolated singularity of $d\omega$, the singularity is logarithmic, degenerate or quasi-homogeneous. These singularities will be explained in the next lines.

Let $p_0 \geq p_1 \geq p_2 > 0$ be relatively prime integers and S be the semi-simple vector-field on $\mathbb{C}^3$ given by

$$S = p_0 x_0 \frac{\partial}{\partial x_0} + p_1 x_1 \frac{\partial}{\partial x_1} + p_2 x_2 \frac{\partial}{\partial x_2}.$$

We say that a vector-field $\mathcal{X}$ holomorphic in a neighborhood of $0 \in \mathbb{C}^3$, is S-quasi-homogeneous of weight ℓ, if we have the following Lie-Bracket identity $[S, \mathcal{X}] = \ell\mathcal{X}$. Notice that necessarily $\ell + p_2$ is a non-negative integer and $\mathcal{X}$ is a polynomial vector field. In fact, if

$$\mathcal{X} = P_0 \frac{\partial}{\partial x_0} + P_1 \frac{\partial}{\partial x_1} + P_2 \frac{\partial}{\partial x_2},$$

the condition that $\mathcal{X}$ is S quasi-homogeneous of weight ℓ is equivalent to the fact that, after giving weights p_0, p_1 and p_2 to the variables x_0, x_1 and x_2, the polynomials P_0, P_1 and P_2 are weighted homogeneous of degrees $\ell + p_0$, $\ell + p_1$ and $\ell + p_2$ respectively.

Moreover, S and $\mathcal{X}$ give a representation of the affine Lie algebra on the algebra of polynomial vector-fields. If we suppose that S and $\mathcal{X}$ are linearly independent at generic points, then these vector fields generate an algebraic foliation on $\mathbb{C}^3$, which is given by the integrable 1-form

$$\eta = i_S i_{\mathcal{X}}(dx_0 \wedge dx_1 \wedge dx_2).$$

The geometrical description of the foliation induced by η is as follows: the singular set of $\mathcal{F}(\eta)$, denoted by $sing(\mathcal{F}(\eta))$, is invariant under the flow of S, $exp(tS) := S_t$. This follows from the relation

$$L_S(\eta) = m\eta, \text{ where } m = \ell + tr(S) = \ell + p_0 + p_1 + p_2,$$

as the reader can check. The relation before also implies that if $q \in sing(\mathcal{F}(\eta)) \backslash \{0\}$, then $\mathcal{F}(\eta)$ is in a neighborhood of q, equivalent to the product of a foliation in dimension two by a one-dimensional disk.

In the affine chart $\mathbb{C}^3$, where S is as before, the leaves of $\mathcal{F}(\eta)$ are "S-cones" with vertex at $0 \in \mathbb{C}^3$, that is, immersed surfaces invariant by the flow of S. If $sing(\mathcal{F}(\eta))$

has codimension two, then each of its components is the closure of an orbit of S. Now we impose a condition which implies the local stability of this kind of singularity by small perturbations of the form defining the foliation.

Let ω be an integrable 1-form in a neighborhood of $p \in \mathbb{C}^3$ and (μ=volume form) be a holomorphic 3-form such that $\mu(p) \neq 0$. Then $d\omega = i_{\mathcal{Z}}(\mu)$ where $\mathcal{Z}$ is a holomorphic vector field. The integrability of ω is equivalent to $i_{\mathcal{Z}}(\omega) = 0$. It is not difficult to see that if p is a GK singularity of ω, then we have two possibilities as follows.

(1) $\mathcal{Z}(p) \neq 0$. In this case we have a singularity of Kupka type.

(2) $\mathcal{Z}(p) = 0$ and p is an isolated singularity of $\mathcal{Z}$. In this case, there exists a neighborhood U of p such that the singularities of ω in $U \backslash \{p\}$ are of the Kupka type. Let $L := D\mathcal{Z}(p)$ be the linear part of $\mathcal{Z}$ at p and $\lambda_0, \lambda_1, \lambda_2$ be the eigenvalues of L. Note that $\lambda_0 + \lambda_1 + \lambda_2 = 0$. This implies that we have three sub-cases.

(2.a) $\lambda_0, \lambda_1, \lambda_2 \neq 0$. In this case, if we take $p = 0$, the second jet of ω at p is of the form

$$j^2(\omega)_0 = ax_1x_2dx_0 + bx_0x_2dx_1 + cx_0x_1dx_2 = x_0x_1x_2(a\frac{dx_0}{x_0} + b\frac{dx_1}{x_1} + c\frac{dx_2}{x_2}),$$

where $\lambda_0 = c - b, \lambda_1 = a - c, \lambda_2 = b - a$. When $a, b, c \neq 0$ it is proven in [C.Ln3], that there exists a germ of vector field $\mathcal{X}$ at p such that $[\mathcal{X}, \mathcal{Z}] = 0$ and

$$i_{\mathcal{X}} i_{\mathcal{Z}}(dx_0 \wedge dx_1 \wedge dx_2) = f\omega$$

where $f(p) \neq 0$, so that the foliation is locally generated by an action of $\mathbb{C}^2$. It is also proven in [C.Ln3] that if the triple (a, b, c) satisfies some conditions of non-resonance, then there exists a local coordinate system (x_0, x_1, x_2) such that

$$\omega = x_0x_1x_2(a\frac{dx_0}{x_0} + b\frac{dx_1}{x_1} + c\frac{dx_2}{x_2}),$$

for this reason we say that the singularity is of the *logarithmic type* (even if ω is not equivalent to its 2-jet).

(2.b) One of the eigenvalues, say λ_2 is zero and the other two satisfy $\lambda_0 = -\lambda_1 \neq 0$. We call this type of singularity *degenerate*. An example of this situation is

$$\omega = x_0x_1dx_2 + x_2^n(ax_0dx_1 + bx_1dx_0),$$

where $a \cdot b \cdot (a - b) \neq 0$ and $n \geq 2$. In this case, if we take $\mu = dx_0 \wedge dx_1 \wedge dx_2$ then we get $d\omega = i_{\mathcal{Z}}\mu$ where

$$\mathcal{Z} = x_0(1 - bnx_2^{n-1})\frac{\partial}{\partial x_0} - x_1(1 - anx_2^{n-1})\frac{\partial}{\partial x_1} + (b - a)x_2^n\frac{\partial}{\partial x_2}.$$

Note that $0 \in \mathbb{C}^3$ is an isolated singularity of $\mathcal{Z}$ with multiplicity $mult(\mathcal{Z}, 0) = n$ and the eigenvalues of $D\mathcal{Z}(0)$ are $1, -1, 0$.

(2.c) $\lambda_0 = \lambda_1 = \lambda_2 = 0$. In this case, the germ of $\mathcal{X}$ at p is nilpotent (as a derivation in the local ring of formal power series at p.)

Definition 7.6. *We say that p is a quasi-homogeneous singularity of ω if p is an isolated singularity of $\mathcal{Z}$ and the germ of $\mathcal{Z}$ at p is nilpotent.*

This definition is justified by the following result that can be found in [Ln0] or [C.CA.G.LN]:

Theorem 7.7. *Let p be a quasi-homogeneous singularity of an holomorphic integrable 1-form ω. Then there exists two holomorphic vector fields S and $\mathcal{Z}$ and a local chart*

$U := (x_0, x_1, x_2)$ around p such that $x_0(p) = x_1(p) = x_2(p) = 0 \in (\mathbb{C}^3, 0)$; and germs of holomorphic vector fields S, $\mathcal{Z} \in (\mathbb{C}^3, 0)$ such that:

(a) *$\omega = \lambda i_S i_{\mathcal{Z}}(dx_0 \wedge dx_1 \wedge dx_2)$, $\lambda \in \mathbb{Q}_+$ $d\omega = i_{\mathcal{Z}}(dx_0 \wedge dx_1 \wedge dx_2)$ and $\mathcal{Z} = (rot(\omega))$,*
(b) *$S = p_0 x_0 \frac{\partial}{\partial x_0} + p_1 x_1 \frac{\partial}{\partial x_1} + p_2 x_2 \frac{\partial}{\partial x_2}$, where, p_0, p_1, p_2 are positive integers with $g.c.d(p_0, p_1, p_2) = 1$,*
(c) *p is an isolated singularity for $\mathcal{Z}$, $\mathcal{Z}$ is polynomial in the chart $U := (x_0, x_1, x_2)$ and $[S, \mathcal{Z}] = \ell\mathcal{Z}$, where $\ell \geq 1$*

Definition 7.8. *Let p be a quasi-homogeneous singularity of ω. We say that it is of the type $(p_0 : p_1 : p_2; \ell)$, if for some local chart and vector fields S and $\mathcal{Z}$ the properties (a), (b) and (c) of the previous theorem are satisfied.*

Remark 5. Let $p \in \mathbb{C}^3$ be a quasi-homogeneous singularity of ω. If S and $\mathcal{Z}$ are as in the previous theorem, then the multiplicity of $\mathcal{Z}$ at the singularity p, $mult(\mathcal{Z}, p)$, (also called Milnor's number), is given by

$$mult(\mathcal{Z}, p) = \frac{(\ell + p_0)(\ell + p_1)(\ell + p_2)}{p_0 p_1 p_2}.$$

In particular, $p_0 p_1 p_2$ must divide $(\ell + p_0)(\ell + p_1)(\ell + p_2)$.

The proof of this fact can be found in [Ln0]. We can now state the stability result:

Proposition 7.9. *Let $(\omega_s)_{s \in \Sigma}$ be a holomorphic family of integrable $1 - forms$ defined in a neighborhood of a compact ball $B = \{z \in \mathbb{C}^3; |z| \leq \rho\}$, where Σ is a neighborhood of $0 \in \mathbb{C}^k$. Suppose that all singularities of ω_0 in B are GK and that $sing(d\omega_0) \subset int(B)$. Then there exists $\epsilon > 0$ such that if $s \in B(0, \epsilon) \subset \Sigma$, then all singularities of ω_s in B are GK. Moreover, if $0 \in B$ is a logarithmic or quasi-homogeneous singularity of type $(p_0 : p_1 : p_2; \ell)$ then there exists a holomorphic map $B(0, \epsilon) \ni s \mapsto z(s)$, such that $z(0) = 0$ and $z(s)$ is a GK singularity of ω_s of the same type (logarithmic or quasi-homogeneous of the type $(p_0 : p_1 : p_2; \ell)$, according to the case).*

The proof can be found in [C.CA.G.LN].

Remark 6. Let p be a quasi-homogeneous singularity of ω. Since the singular set of ω is formed by a finite number of solutions of the vector field S, which is of the Poincaré-type, there exists an $\epsilon_0 > 0$ such that every component of $Sing(\omega)$ is transversal to the 5-sphere of radius $\epsilon < \epsilon_0$, $\mathbb{S}^5_\epsilon$.

7.4. Quasi-homogeneous singularities of $\mathcal{F} = f^*(\mathcal{G})$

Let $(f, \mathcal{G})$ be a generic pair. As before, let us fix a homogeneous coordinate system in $\mathbb{P}^2$ and denote by ℓ_0, ℓ_1, and ℓ_2 the $\mathcal{G}$-invariant straight lines that correspond to the planes $X = 0$, $Y = 0$ and $Z = 0$ in $\mathbb{C}^3$, respectively.
In this coordinate system, $\mathcal{G}$ is written as

$$\Omega = YZA(X, Y, Z)dX + XZB(X, Y, Z)dY + XYC(X, Y, Z)dZ.$$

Let us describe $\mathcal{F} = f^*(\mathcal{G})$ in a neighborhood of a point $p \in I(f)$.

Lemma 7.10. *There exists a local chart $(U, (x_0, x_1, x_2, y) \in \mathbb{C}^3 \times \mathbb{C}^{n-2})$ around p such that the lifting $\tilde{f}$ of f is of the form $\tilde{f}|_U = (x_0^\alpha, x_1^\beta, x_2^\gamma) : U \to \mathbb{C}^3$. In particular $\mathcal{F}|_{U(p)}$ is* *represented by the* 1*-form*

$$\begin{aligned}\eta(x_0,x_1,x_2,y) &= \alpha.x_1.x_2.A(x_0^\alpha,x_1^\beta,x_2^\gamma)dx_0\\ &+ \beta.x_0.x_2.B(x_0^\alpha,x_1^\beta,x_2^\gamma)dx_1\\ &+ \gamma.x_0.x_1.C(x_0^\alpha,x_1^\beta,x_2^\gamma)dx_2.\end{aligned}$$

Proof. We will consider only the case $n=3$. We can suppose $p=[0:0:0:1]\in I(f)$, so that $p=(0,0,0)$ in the affine plane $\Sigma=\{z_3=1\}$. Note that we can write the lifting of f, $\tilde{f}:\mathbb{C}^4\to\mathbb{C}^3$ as the composition of two mappings: $\tilde{f}=g\circ\overline{f}$ where $\overline{f}:\mathbb{C}^4\to\mathbb{C}^3$ is given by $\overline{f}(Z)=(F_0(Z),F_1(Z),F_2(Z))$ and $g:\mathbb{C}^3\to\mathbb{C}^3$ is given by $g(x_0,x_1,x_2)=(x_0^\alpha,x_1^\beta,x_2^\gamma)$. Since F_0, F_1, F_2 are homogeneous, it follows that $\tilde{p}=(0,0,0,1)\in(\overline{f})^{-1}(0)$ and $dF_0(\tilde{p})\wedge dF_1(\tilde{p})\wedge dF_2(\tilde{p})\neq 0$, and so $dF_0|_\Sigma(\tilde{p})\wedge dF_1|_\Sigma(\tilde{p})\wedge dF_2|_\Sigma(\tilde{p})\neq 0$. In particular, $\overline{f}|_\Sigma$ is a biholomorphism in a neighborhood of $\tilde{p}$ in Σ. This implies that there exists a coordinate system $\psi=(x_0,x_1,x_2):U(\tilde{p})\to\mathbb{C}^3$ such that $\overline{f}|_\Sigma\circ\psi=(x_0,x_1,x_2)$. After composing with the mapping g we obtain the local expression of $\tilde{f}$:

$$\tilde{f}(x_0,x_1,x_2)=g\circ\overline{f}|_\Sigma\circ\psi=(x_0,x_1,x_2)=(x_0^\alpha,x_1^\beta,x_2^\gamma).$$

Remark 7. For the case $n\geq 4$ the argument is similar.

Now, $\mathcal{F}=f^*(\mathcal{G})$ is represented in U by the 1-form:

$$\begin{aligned}f^*(\Omega)(x_0,x_1,x_2) &= x_1^\beta.x_2^\gamma.A(x_0^\alpha,x_1^\beta,x_2^\gamma)d(x_0^\alpha)\\ &+ x_0^\alpha.x_2^\gamma.B(x_0^\alpha,x_1^\beta,x_2^\gamma)d(x_1^\beta)\\ &+ x_0^\alpha.x_1^\beta.C(x_0^\alpha,x_1^\beta,x_2^\gamma)d(x_2^\gamma)\\ f^*(\Omega)(x_0,x_1,x_2) &= (x_0^{\alpha-1}.x_1^{\beta-1}.x_2^{\gamma-1})[\eta(x_0,x_1,x_2)]\end{aligned}$$

Extracting the codimension one term $(x_0^{\alpha-1}.x_1^{\beta-1}.x_2^{\gamma-1})$ we obtain

$$\begin{aligned}\eta(x_0,x_1,x_2) &= \alpha.x_1.x_2.A(x_0^\alpha,x_1^\beta,x_2^\gamma)dx_0\\ &+ \beta.x_0.x_2.B(x_0^\alpha,x_1^\beta,x_2^\gamma)dx_1\\ &+ \gamma.x_0.x_1.C(x_0^\alpha,x_1^\beta,x_2^\gamma)dx_2.\end{aligned}$$

□

And there are no other common factors to extract. Let us give a proof of this fact. We will divide the proof in two cases.

(1) First: We can suppose without loss of generality that we can write $\eta(x_0,x_1,x_2)=x_0^{m.\alpha}\tilde{\eta}(x_0,x_1,x_2)$ where $1\leq m\leq(d-1)$. This would imply that $A(X,Y,Z)=X^m\tilde{A}(X,Y,Z)$. Which would imply that $\operatorname{codim}(sing(\Omega))=1$, contradiction.
(2) Second: In fact, if for instance we could extract from η a polynomial $h(x_0^\alpha,x_1^\beta,x_2^\gamma)$, that we can suppose without loss of generality that h is irreducible, this would imply that the polynomial h(X,Y,Z) would have to divide A, B and C but this also would imply that $\operatorname{codim}(sing(\Omega))=1$, contradiction. In both cases as we know this is impossible because we are working with $\mathcal{G}$ satisfying $\operatorname{codim}(sing(\mathcal{G}))\geq 2$.

The previous discussion give us the following:

Corollary 7.11. *If $\mathcal{F}$ comes from a generic pair, then the degree of $\mathcal{F}$ is*

$$\nu\left[(d-1)+\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right]-2.$$

Remark 8. This corollary can be obtained also using the proposition 2.1 contained in [Fa.Pe].

Let us now obtain the vector field S as in Theorem 7.7.

Consider the radial vector field $R = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z}$ and observe that

$$(g \circ \overline{f}|_{\Sigma} \circ \psi)^* R = \frac{1}{\alpha}x_0\frac{\partial}{\partial x_0} + \frac{1}{\beta}x_1\frac{\partial}{\partial x_1} + \frac{1}{\gamma}x_2\frac{\partial}{\partial x_2}.$$

Since the eigenvalues of S have to be integers, multiplying $(g \circ f|_{\Sigma} \circ \psi) * R$ by $\alpha\beta\gamma$ we obtain

$$S = (\beta\gamma)x_0\frac{\partial}{\partial x_0} + (\alpha\gamma)x_1\frac{\partial}{\partial x_1} + (\alpha\beta)x_2\frac{\partial}{\partial x_2}.$$

Let us describe the orbits of the vector field S.
In fact the closure of the orbits of the vector field S can be parametrized as follows:

$$\mathbb{C} \ni s \to (js^{\beta\gamma}, ns^{\alpha\gamma}, ks^{\alpha\beta})$$

where $(j,n,k) \in \mathbb{C}^3$.

Let us concentrate in the case $n = 3$.

Lemma 7.12. *If $p \in I(f)$ then p is a quasi-homogeneous singularity of η.*

Proof. First of all note that $i_S\eta = 0$. Let us calculate $L_S\eta$. By standard computations with the Lie derivative we have

$L_S\eta = m\eta$, where $m = [(\beta\gamma+\alpha\beta+\alpha\beta)+(\alpha\beta\gamma)(d-1)]$. This implies that the singular set of η is invariant under the flow of S.

The vector field $\mathcal{Z}$ such that $\eta = i_S i_{\mathcal{Z}}(dx_0 \wedge dx_1 \wedge dx_2)$ is given by

$$\mathcal{Z} = \mathcal{Z}_0(x_0,x_1,x_2)\frac{\partial}{\partial x_0} + \mathcal{Z}_1(x_0,x_1,x_2)\frac{\partial}{\partial x_1} + \mathcal{Z}_2(x_0,x_1,x_2)\frac{\partial}{\partial x_2}$$

where:

(a) $\mathcal{Z}_0(x_0,x_1,x_2) = x_0.\tilde{A}(x_0^{\alpha}, x_1^{\beta}, x_2^{\gamma})$
(b) $\mathcal{Z}_1(x_0,x_1,x_2) = x_1.\tilde{B}(x_0^{\alpha}, x_1^{\beta}, x_2^{\gamma})$
(c) $\mathcal{Z}_2(x_0,x_1,x_2) = x_2.\tilde{C}(x_0^{\alpha}, x_1^{\beta}, x_2^{\gamma})$

Where the polynomials $\tilde{A}(X,Y,Z), \tilde{B}(X,Y,Z)$ and $\tilde{C}(X,Y,Z)$ are homogeneous of degree $(d-1)$ and they are not unique!

We have to show that 0 is an isolated singularity of $\mathcal{Z}$ and all eigenvalues of $D\mathcal{Z}(0)$ are 0. It is not difficult to show that the Jacobian matrix $D\mathcal{Z}(0)$ is the null matrix(by standard computations) which implies that it has null eigenvalues. To conclude that the 0 is an isolated singularity of $\mathcal{Z}$ if follows from the fact that all singular curves of $\mathcal{F}$ in a neighborhood $(U,(x_0,x_1,x_2))$ of 0 are of Kupka type, as we have proved in section 7.2. Note that the unique singularities of η in the neighborhood $(U,(x_0,x_1,x_2))$ of 0 come from $\tilde{f}^*Sing(\mathcal{G})$, this follows from the fact that $\{Sing\,(\mathcal{G}) \cap VC(f) \backslash \ell_0 \cup \ell_1 \cup \ell_2 = \emptyset\}$. On the other hand we have seen that $(g \circ f|_{\Sigma} \circ \psi)^{-1}(sing(\mathcal{G})) \backslash I(f)$ is contained in the Kupka set of $\mathcal{F}$. Hence the point p is an isolated singularity of $d\eta$ and hence it is an isolated singularity of $\mathcal{Z}$. □

Therefore, in the case $n = 3$ any $p \in I(f)$ is a quasi-homogeneous singularity of type

$$[\beta\gamma : \alpha\gamma : \alpha\beta] .$$

From any of components of the vector $\mathcal{Z}$ we can find the value of ℓ. For instance, it follows from the expression in the item (a) that

$$\ell + p_0 = \beta\gamma + \alpha\beta\gamma(d-1),$$

since $p_0 = \beta\gamma$ the value of ℓ is

$$\alpha\beta\gamma(d-1).$$

We can use Remark 5. to calculate the Milnor's number of the quasi-homogeneous singularity. This is the content of the next corollary.

Corollary 7.13. *Let $0 \in \mathbb{C}^3$ be a quasi-homogeneous singularity of η. If S and $\mathcal{Z}$ are as in the previous discussion, then the multiplicity of $\mathcal{Z}$ at 0 is:*

$$(\alpha(d-1)+1)(\beta(d-1)+1)(\gamma(d-1)+1).$$

Proof. The multiplicity of $\mathcal{Z}$ at 0 is given by

$$mult(\mathcal{Z}, 0) = \frac{(\ell + p_0)(\ell + p_1)(\ell + p_2)}{p_0 p_1 p_2}$$

where

(a) $p_0 = \beta\gamma$,
(b) $p_1 = \alpha\gamma$,
(c) $p_2 = \alpha\beta$.

Since

$$\ell = (d-1)(\alpha\beta\gamma)$$

it follows that

$$mult(\mathcal{Z}, q = 0) = (\alpha(d-1)+1)(\beta(d-1)+1)(\gamma(d-1)+1).$$

□

In the case $n \geq 4$ the explanation is analogous. In fact, in this case we will have a local structure product near any point $p \in I(f)$.

The local product structure follows from the next theorem:

Theorem 7.14. *If η has a simple singularity at $0 \in \mathbb{C}^n, n \geq 4$ then $rk(\eta, 0) \leq 3$. In particular, $\mathcal{F}(\eta)$ is equivalent to the product of a foliation of codimension one in $(\mathbb{C}^3, 0)$ by a regular foliation of codimension 3.*

Remark 9. By $rk(\eta, 0)$ we mean the minimum number of variables that we can write η in a neighborhood of $0 \in \mathbb{C}^n$.

The proof of this theorem and the definition of simple singularity can be found in [Ln] pp 43-44.

In fact in the case $n \geq 4$ we have:

Corollary 7.15. *Let $(f, \mathcal{G})$ be a generic pair. Let $p \in I(f)$ and η an 1-form defining $\mathcal{F}$ in a neighborhood of p. Then there exists a 3-plane $\Pi \subset \mathbb{C}^n$ such that $d(\eta)|_{\Pi}$ has an isolated singularity at $0 \in \Pi$.*

Proof. Immediate from the local product structure. □

7.5. Quasi-Homogeneous Foliations and Weighted Projective Spaces

If we observe the quasi-homogeneous 1-form η in detail, we see that each one of its components has a different degree. Let us write the quasi-homogeneous 1-form η in its simplified form:

$$\eta(x_0, x_1, x_2) = \omega_0(x_0, x_1, x_2)dx_0 + \omega_1(x_0, x_1, x_2)dx_1 + \omega_2(x_0, x_1, x_2)dx_2.$$

Let us calculate the degree of each one of its terms with respect to the weights $(\beta\gamma, \alpha\gamma, \alpha\beta)$. By standard computations we have:

(1) $\omega_0(x_0, x_1, x_2)$ is a quasi-homogeneous polynomial of degree

$$\alpha\beta\gamma(d-1) + \alpha\gamma + \alpha\beta,$$

(2) $\omega_1(x_0, x_1, x_2)$ is a quasi-homogeneous polynomial of degree

$$\alpha\beta\gamma(d-1) + \beta\gamma + \alpha\beta,$$

(3) $\omega_2(x_0, x_1, x_2)$ is a quasi-homogeneous polynomial of degree

$$\alpha\beta\gamma(d-1) + \beta\gamma + \alpha\gamma,$$

and following the discussion of [So.Cor]. Since

$$i_S\eta = 0$$

we have that the quasi-homogeneous 1-form η defines naturally a holomorphic foliation on the weighted projective plane $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$. Here the vector field S is what the authors in [So.Cor] call a Radial adapted vector field.

In the case $n = 3$, the previous discussions means that the germ $(f^*\mathcal{G}, p)$ of $f^*\mathcal{G}$ at $p \in I(f)$ is almost equivalent to the germ $\Pi_2^*(\mathcal{G}), 0$. More precisely, if we make a weighted blow-up with weights $(\beta\gamma, \alpha\gamma, \alpha\beta)$ at the point $p \in \mathbb{P}^3$, $p \in I(f)$, say $\pi_w : \tilde{\mathbb{P}^3} \to \mathbb{P}^3$, then the strict transform of $\pi_w^*(\mathcal{F})$ of $f^*\mathcal{G}$ is transversal to the exceptional divisor $E \cong \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$. Moreover, using that $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ is isomorphic to the standard projective plane $\mathbb{P}^2$ (see for instance [B.R]), we can use the isomorphism between them to obtain up to automorphism, the foliation $\mathcal{G}$. We will come back to this topic later.

7.6. Deformations of the singular set of $\mathcal{F}_0 = f_0^*(\mathcal{G}_0)$ - Auxiliary Lemmas

As done in [C.Ln.E], to prove the main theorem we have to use the following result from Complex Analytic Geometry.

Lemma 7.16. *Let $X_1 \subset X_2$ be irreducible analytic subsets of a complex manifold $\mathcal{M}$. Suppose that there exists an open subset $U \neq \emptyset$ of X_1 such that U is also an open subset of X_2. Then $X_1 = X_2$.*

Remark 10. For more details we refer the reader to [Seb], or [Ln] where it appears as Lemma 2.2.1 on page 75.

Here we are thinking in X_2 as being the irreducible component of $\mathbb{F}\mathrm{ol}\,(\Gamma-1;n)$ and $X_1 = PB(\Gamma-1, \nu, \alpha, \beta, \gamma)$. We have constructed an open and dense subset $\mathcal{W}$ inside $PB(\Gamma-1, \nu, \alpha, \beta, \gamma)$ containing the generic pull-back foliations. We will show that for any rational foliation $\mathcal{F}_0 \in \mathcal{W}$ and any germ of a holomorphic family of foliations $(\mathcal{F}_t)_{t\in(\mathbb{C},0)}$ such that $\mathcal{F}_0 = \mathcal{F}_o$, we have $\mathcal{F}_t \in PB(\Gamma-1, \nu, \alpha, \beta, \gamma)$ for all $t \in (\mathbb{C}, 0)$.

As before, let us fix a coordinate system on $\mathbb{P}^2$ and denote by ℓ_0, ℓ_1, and ℓ_2 the $\mathcal{G}$-invariant straight lines that correspond to the planes $X = 0$, $Y = 0$ and $Z = 0$ in $\mathbb{C}^3$, respectively.

Let $\mathcal{G} \in \mathcal{A}$ and suppose that $\mathcal{G}$ is given in homogeneous coordinates by a $1-$form

$$\Omega = YZA(X,Y,Z)dX + XZB(X,Y,Z)dY + XYC(X,Y,Z)dZ.$$

Let us describe the singular set of the foliation $\mathcal{F} = f^*\mathcal{G}$ in a neighborhood of a point $p \in I(f)$. First, let us consider the case $n = 3$. Let $\tilde{f}$ be a lift of f. Let $\mathbb{C}^3 \simeq E \subset \mathbb{C}^4$ be an affine plane ($0 \notin E$) such that E cuts transversely the line $\Pi_3^{-1}(p)$ at $q \in E$. Since $\tilde{f}(q) = 0$ and $dF_0(q) \wedge dF_1(q) \wedge dF_2(q) \neq 0$ by hypothesis, we have seen before that there exists a local coordinate system around q, $(W, x \in \mathbb{C}^3)$ such that $x(q) = 0$ and $\tilde{f}|_{W(x)} = (x_0^\alpha, x_1^\beta, x_2^\gamma)$. In particular, in this coordinate system $\mathcal{F}$ is described by the 1-form η and it has a quasi-homogeneous singularity of the type $[\beta\gamma : \alpha\gamma : \alpha\beta]$. Assuming $n = 3$, we also have $(d^2 + d + 1)$ special analytic curves passing through the point p in this neighborhood. These curves are common orbits of the two vector fields S and $\mathcal{Z}$. Moreover, we know that in $\mathbb{C}^3 \backslash \{0\}$ they are singularities of Kupka-type having local product structure with a constant transversal type. In an analogous way, for the case $n = r + 3 > 3$, the foliation $\mathcal{F}$ is locally the product of a regular foliation and the foliation defined by the $1-$form η (see Thm 7.14).

Now let us fix $\mathcal{F}_0 = f_0^*(\mathcal{G}_0)$ in $\mathcal{W}$ and a germ of a holomorphic family $(\mathcal{F}_t)_{t \in (\mathbb{C},0)}$ of foliations such that $\mathcal{F}_o = \mathcal{F}_0$.

Lemma 7.17. *There exists a germ of isotopy of class C^∞, $(I(t))_{t \in (\mathbb{C},0)}$ having the following properties:*

(I) *$I(0) = I(f_0)$ and $I(t)$ is algebraic and smooth of codimension* 3 *for all $t \in (\mathbb{C}, 0)$.*

(II) *For all $p \in I(t)$, there exists a neighborhood $U(p,t) = U$ of p such that $\mathcal{F}_t$ is equivalent to the product of a regular foliation of codimension* 3 *and a singular foliation $\mathcal{F}_{p,t}$ of codimension one given by the 1-form $\eta_{p,t}$. The family of* 1*-forms $\eta_{p,t}$, represents the quasi-homogeneous foliation given by the Lins Neto's stability theorem of quasi-homogeneous singularities* [Ln0] *and* [C.CA.G.LN].

Remark 11. For the definition and properties of isotopies we refer the reader to [Hi].

Let us denote $Z = (x, y)$ where $x = (z_1, ..., z_{n-3}) \in \mathbb{C}^{n-3}$ and $y = (z_{n-2}, z_{n-1}, z_n) \in \mathbb{C}^3$.

Proof. In the case $n = 3$ it follows from Proposition 7.9 since $I(f_0)$ is finite. Let us concentrate on the case $n \geq 4$. We can consider a smooth tubular neighborhood $\pi : U \to I(f_0)$, of $I(f_0)$ with fibers $B_Z := \pi^{-1}(Z) \simeq B^3$, where B^3 is a complex ball of dimension 3. This is possible because the variety $I(f_0)$ is smooth of codimension three [Hi]. The main idea is to construct a smooth germ of the mapping, $\psi : (\mathbb{C}, 0) \times I(f_0) \to U$, such that for all $t \in (\mathbb{C}, 0)$ we have $\psi(t, Z) \in B_Z$. We will work with a representative of the germ $(\mathcal{F}_t)_t$, defined on a disk $D_\epsilon = (|t| < \epsilon) \subset \mathbb{C}$.

Fix $p \in I(f_0)$ and a holomorphic local chart at p, $\Phi = (x, y) : V \to \mathbb{C}^{n-3} \times \mathbb{C}^3$, such that $V_1 := V \cap I(f_0) = (y = 0), \Phi(p) = (0, 0)$ and $V \subset U$. Take $Z_0 = (x_0, 0) \in V \cap I(f_0)$, and consider $F_{Z_0} = (Z = x_0) \subset \{x_0\} \times \mathbb{C}^3$. For $Z = (x, 0) \in V$ fixed, we know that $(\mathcal{F}_0)|B_Z$ has a quasi-homogeneous singularity of type $[\beta\gamma : \alpha\gamma : \alpha\beta]$. By the Proposition 7.9, there exists $0 < \epsilon_1 \leq \epsilon$ and a holomorphic mapping $\psi_Z : (|t| < \epsilon_1) \to F_Z$ such that $(\mathcal{F}_t)|_{F_Z}$ has a quasi-homogeneous singularity of the type $[\beta\gamma : \alpha\gamma : \alpha\beta]$ in $\psi_Z(t)$.

We can write $\phi_Z(t) = (x, Y(t,x))$, where $t \to Y(t,x) \in \mathbb{C}^3$ is holomorphic. The holomorphic local product structure for $\mathcal{F}_t$ at the point $(x, Y(t,x))$ implies that the germ $(t,x) \to Y(t,x)$ is holomorphic. We can define Y as a holomorphic function in a neighborhood C of $\{0\} \times V$. For fixed t, the graph gr_{Yt} of the mapping $x \in C \cap (\{t\} \times \mathbb{C}^{n-3}) \to Y(t,x)$ is a holomorphic submanifold of U.

Consider now the fibration given by the tubular neighborhood $\pi : U \to I(f_0)$. Since the fibers are transverse to $I(f_0)$, for small $|t|$ the fiber B_W intersects gr_{Yt} exactly in one point(by proposition 7.9), which we will denote by $\psi(t,W)$. For $t = 0$, we have $Y(0,x) = 0$, so that $\psi(0,w) = w$, which implies that ψ is defined on a neighborhood of $\{0\} \times V_1$. Note that the map ψ is smooth on its domain of definition.

To finish the proof, let us observe that the map ψ does not depend on the choice of local coordinates (Φ, V) about the point p, since the point $\psi(t,W)$ can be considered as the unique quasi-homogeneous singularity of type $[\beta\gamma : \alpha\gamma : \alpha\beta]$ on B_W of $\mathcal{F}_t$. If we take a covering of $I(f_0)$ by local charts as above, we can extend ψ to a germ of the mapping $\psi : (\mathbb{C}, 0) \times I(f_0) \to U$ such that $\psi(t,W)$ on B_W is the unique quasi-homogeneous singularity of $\mathcal{F}_t$ in B_W for all $t \in (\mathbb{C}, 0)$. Since $I(f_0)$ is compact, this germ has a representative, which we denote again by $\psi : (|t| < \delta) \times I(f_0) \to U, \delta > 0$. Putting $I(t) = \psi((\{t\} \times I(f_0))$, we have the desired isotopy.

The assertion (II) is a consequence of Thm.(7.14), since the foliation is regular of rank 3. This theorem also implies that $I(t)$ is a holomorphic submanifold of codimension 3. □

Remark 12. In the case $n > 3$, the variety $I(t)$ is connected since $I(f_0)$ is connected (Lefschetz's Theorem). The local product structure in $I(t)$ implies that the transversal type of $\mathcal{F}_t$ is constant. In particular, $\mathcal{F}_{p,t}$, does not depends on $p \in I(t)$. In the case $n = 3$, $I(t) = p_1(t), ..., p_j(t), ..., p_{\frac{\nu^3}{\alpha\beta\gamma}}(t)$ and we can not guarantee a priori that $\mathcal{F}_{p_i,t} = \mathcal{F}_{p_j,t}$, if $i \neq j$.

Consider coordinate system on $\mathbb{P}^2$ and denote by ℓ_0, ℓ_1, and ℓ_2 the $\mathcal{G}$-invariant straight lines that correspond to the planes $X = 0$, $Y = 0$ and $Z = 0$ in $\mathbb{C}^3$, respectively.

The singular set of $\mathcal{G}_0$ consists of the points: $a = [0 : 0 : 1]$, $b = [0 : 1 : 0]$, $c = [1 : 0 : 0]$, $\mathcal{S}_W(\mathcal{G}_0)$, $\mathcal{S}_{\ell_r}(\mathcal{G}_0)$, $0 \le r \le 2$. We know that $\#\mathcal{S}_W(\mathcal{G}_0) = (d-1)^2$, $\#\mathcal{S}_{\ell_r}(\mathcal{G}_0) = (d-1)$, $0 \le r \le 2$. Let $\tau \in sing(\mathcal{G}_0)$ and $K(\mathcal{F}_0) = \cup_{\tau \in sing(\mathcal{G}_0)} V_\tau \backslash I(f_0)$ where $V_\tau = \overline{f_0^{-1}(\tau)}$. As in the previous lemma, let us consider a representative of the germ $(\mathcal{F}_t)_t$, defined in a disc $D_\delta := (|t| < \delta)$.

Lemma 7.18. *There exists $\epsilon > 0$ and smooth isotopies $\phi_\tau : D_\epsilon \times V_\tau \to \mathbb{P}^n, \tau \in Sing(\mathcal{G}_0)$ such that $V_\tau(t) = \phi_\tau(\{t\} \times V_\tau)$ satisfies:*

(a) *$V_\tau(t)$ is an algebraic subvariety of codimension two of $\mathbb{P}^n$ and $V_\tau(0) = V_\tau$ for all $\tau \in Sing(\mathcal{G}_0)$ and for all $t \in D_\epsilon$.*

(b) *$I(t) \subset V_\tau(t)$ for all $\tau \in Sing(\mathcal{G}_0)$ and for all $t \in D_\epsilon$. Moreover, if $\tau \neq \tau'$, and $\tau, \tau' \in Sing(\mathcal{G}_0)$, we have $V_\tau(t) \cap V_{\tau'}(t) = I(t)$ for all $t \in D_\epsilon$ and the intersection* *is transversal.*

(c) $V_\tau(t)\backslash I(t)$ *is contained in the Kupka-set of* $\mathcal{F}_t$ *for all* $\tau \in Sing(\mathcal{G}_0)$ *and for all* $t \in D_\epsilon$*. In particular, the transversal type of* $\mathcal{F}_t$ *is constant along* $V_\tau(t)\backslash I(t)$*.*

Proof. We consider only the case $n \geq 4$, the proof in the case $n = 3$ is analogous (the only difference being that $I(t)$ consists of discrete points and so is not connected). Here we can consider the same smooth tubular neighborhood $\pi : U \to I(f_0)$ of $I(f_0)$ as before. This tubular neighborhood has the following properties:

(1) The boundary ∂U is a smooth submanifold of real dimension $2n - 1$.

(2) For each $\tau \in Sing(\mathcal{G}_0)$, V_τ intersects ∂U transversely in a real codimension 4 submanifold $S_\tau = \partial U \cap V_\tau$.

Fix $\tau \in \mathcal{S}_W(\mathcal{G}_0)$. Since $V_\tau\backslash U$ is compact and smooth of codimension 2, there exists a smooth tubular neighborhood $\pi_\tau : A_\tau \to V_\tau\backslash U$ such that the fiber $F_Z := \pi_1^{-1}(Z)$ is diffeomorphic to the complex ball B^2 of complex dimension 2. The argument now proceeds as in the proof of the previous lemma. We will construct a germ of the smooth mapping,

$$\phi : ((\mathbb{C}, 0) \times V_\tau) \to A_\tau$$

such that $\phi(t, Z) \in F_Z$ for all t. Since V_τ is compact, ϕ has a representative

$$\phi : D_\epsilon \times V_\tau \to A_\tau$$

and we define $V_\tau(t) = \phi(\{t\} \times V_\tau)$.

Let us fix $Z_0 \in V_\tau\backslash U$. We claim that there exists a germ of the mapping $\phi_{Z_0} : (\mathbb{C}, 0) \to F_{Z_0}$ such that $\phi_{Z_0}(t)$ is a Kupka-singularity of $\mathcal{F}_t$ for all t (where it is defined). Indeed, let $(\omega_t)_t$ be a holomorphic family of integrable 1-forms such that (ω_t) represents $\mathcal{F}_t$ on a fixed neighborhood U_{Z_0} of Z_0 in $\mathbb{P}^n$.

Since $d\omega_0(Z_0) \neq 0$, we can take U_{Z_0} in such a way that $d\omega_0(p) \neq 0$ for all $p \in U_{Z_0}$. Take $\epsilon(Z_0)$ such that if $|t| < \epsilon(Z_0)$, then $d\omega_t(p) \neq 0$ for all $p \in U_{Z_0}$. Hence, if $p \in sing(\mathcal{F}_t) \cap U_{Z_0}$, p is a Kupka-singularity for $\mathcal{F}_t$ when $|t| < \epsilon(Z_0)$. Let us now use the local transversal product structure for $\mathcal{F}_0$. Since F_{Z_0} is transversal to $V_\tau\backslash U$ and $\mathcal{F}_0$ has a Kupka-singularity in Z_0, $\omega_0|_{F_{Z_0}}$ has a Kupka-singularity of multiplicity one in $Z_0 \in F_{Z_0} \simeq B^2$.

Since these singularities are stable under small perturbations, there exists a smooth mapping $\phi_{Z_0} : (\mathbb{C}, 0) \to F_{Z_0}$, such that $\phi_{Z_0}(t)$ is a singularity of multiplicity one of $\omega_t|_{F_{Z_0}}$ for all t. Since the point $\phi_{Z_0}(t)$ is a singularity of $\mathcal{F}_t$ for all t such that $\phi_{Z_0}(t) \in U_{Z_0}$, it is Kupka. Then we have constructed a germ of mapping

$$\phi : ((\mathbb{C}, 0) \times (V_\tau\backslash U)) \to A_\tau$$

such that $\phi(t, Z) \in F_Z$ and $\phi(t, Z)$ is a Kupka singularity of $\mathcal{F}_t$ for all t. Let us now extend the germ ϕ to the points $((\mathbb{C}, 0) \times (V_\tau \cap U))$.

Fix $Z_0 \in I(f)$. As we have seen before, there exists a neighborhood U_{Z_0} of Z_0 and a change of coordinates $g : U_{Z_0} \to \mathbb{C}^3$ such that $\mathcal{F}_0|U_{Z_0}$ is represented by η_0. Moreover, the Lins Neto's Stability Theorem [Ln0] guarantees that there exists a holomorphic family of integrable holomorphic 1-forms η_t such that $\eta_0 = \eta$ and η_t represents $\mathcal{F}_t|U_{Z_0}$ for all $|t| < \delta$. By the construction in the previous lemma, $\psi(t, Z_0)$ is a quasi-homogeneous singularity of type $[\beta\gamma : \alpha\gamma : \alpha\beta]$ of $\eta_t|B_{Z_0}$ for all $t \in D_\epsilon$. In some holomorphic coordinate system in $\psi(t, Z_0)$, the 1-form $\eta_t|B_{Z_0}$ represents a foliation

$$\mathcal{F}_t|U_{Z_0} = \eta_t$$

where η_t defines a holomorphic family of foliations on $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$.

This implies that $Sing(\eta_t)|B_{z_0}$ has a smooth irreducible component $S_\tau(t)$ of codimension two that corresponds to the singularity $\kappa_\tau(t) \in \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$, where $S_\tau(0) = V_\tau \cap B_{Z_0}$. Since for each Kupka component the transversal type is constant, it follows that the $d^2 + d + 1$ algebraic subvarieties extend to the interior of the tubular neighborhood and, by continuity, connect to $I(t)$. The family $(S_\tau(t))_{t\in D_\epsilon}$ is an analytic deformation of the germ $V_\tau \cap B_{Z_0}$ in Z_0.

Let us now use the compatibility condition. Since $S_\tau(0) = V_\tau \cap B_{Z_0}$ for all $Z \in V_\tau \cap B_{Z_0}$, $S_\tau(0)$ is transversal to $F_Z \subset B_{Z_0}$ in B_{Z_0}. Hence there exists a neighborhood C of Z_0 in $S_\tau(0)$ and $\epsilon_1 > 0$ such that $S_\tau(t)$ is transversal to F_Z for all $Z \in C$ and $t \in D_{\epsilon_1}$. If we fix $Z \in C$ and let $|t|$ be sufficiently small, $S_\tau(t) \cap F_Z$ has a unique point $\zeta(t)$ and the germ of function $\zeta : (\mathbb{C} \times C) \to B_{z_0}$ is smooth. Note that the germs ϕ and ζ coincide along $\{0\} \times (C\backslash\{Z_0\})$, since if $Z \in (C\backslash\{Z_0\})$ and $|t|$ is sufficiently small, then $\phi(t, Z) \in F_Z$ and $\zeta(t, Z) \in F_Z$ are Kupka singularities of $\mathcal{F}_t$ and there exists only one singularity in F_Z. Since $Z_0 \in I(F)$ is arbitrary, we get a smooth extension of the germ ϕ along $\{0\} \times V_\tau$. $\square$

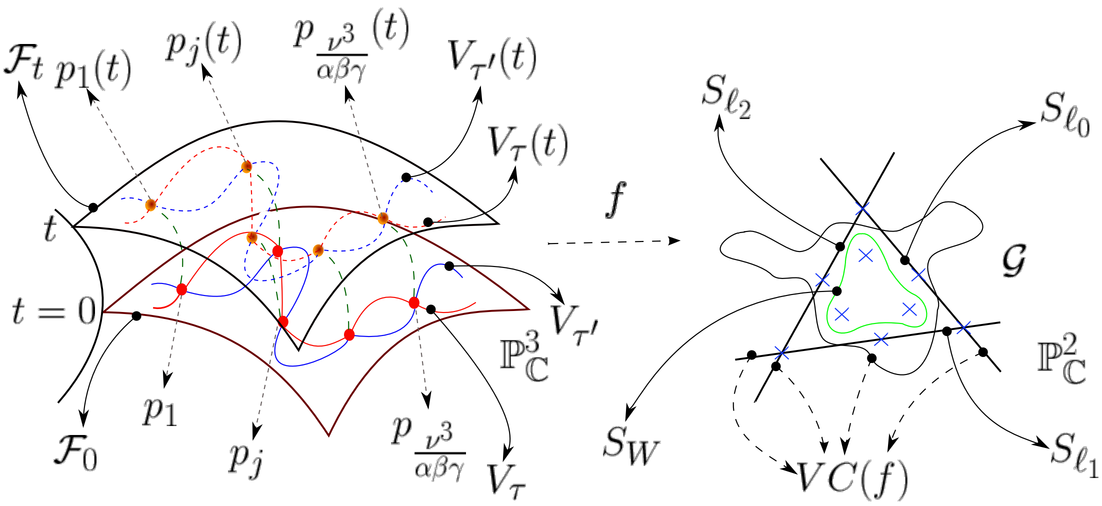

FIGURE 3. Global deformation of the singular set of a generic pull-back foliation, case $n = 3$.

7.7. **End of the Proof of Theorem B**

We now want to prove that the subvarieties $V_a(t), V_b(t)$ and $V_c(t)$ are fibers of $(f_t)_{t\in D_{\epsilon'}}$ are fibers of a rational map $f_t : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$, $f_t \in Gen(n,\nu,\alpha,\beta,\gamma)$, in such a way that $(f_t)_{t\in D_\epsilon}$ is a deformation of f_0. We also want to prove that there exists a family of foliations $(\mathcal{G}_t)_{t\in D_\epsilon}, \mathcal{G}_t \in \mathcal{A}$ **(see section 6 pp.25)** such that $\mathcal{F}_t = f_t^*(\mathcal{G}_t)$ for all $t \in D_\epsilon$. In fact, we will prove that every fiber $V_\tau(t)$ is a fiber of $(f_t)_{t\in D_\epsilon}$ at Appendix **section** 8.7. We consider the case $n = 3$, then show how the general case $n \geq 4$ reduces to this case in the end.

For convenience we remind the reader the description of the quasi-homogenous singular set of $\mathcal{F}_t$:

(1) In the case $n = 3$, $I(t)$ consists of $\frac{\nu^3}{\alpha\beta\gamma}$ distinct points $p_j(t)$, and for each $j \in \{1,...,\frac{\nu^3}{\alpha\beta\gamma}\}$ there is an $\varepsilon > 0$ and an analytic coordinate system around $p_j(t)$, say (U_t^j, Z_t^j), such that $Z_T^j(p_j(t)) = 0 \in (\mathbb{C}^3, 0)$ and $\mathcal{F}_t|(U_t^j, Z_t^j)$ can be represented by $\eta_{p_j(t)}$, a holomorphic family of integrable 1-forms such that:

   (a) $Sing(d\eta_{p_j(t)}) = p_j(t)$ for all $|t| < \varepsilon$

   (b) $p_j(t)$ is quasi-homogeneous singularity of type
$$[\beta.\gamma : \alpha.\gamma : \alpha.\beta]$$
   of $\eta_{p_j}(t)$ for all $|t| < \varepsilon$.

(2) If $n \geq 4$, $I(t)$ is a codimension-three smooth and connected submanifold of $\mathbb{P}^n$, and $\mathcal{F}_t$ has a local product structure near all points of $I(t)$. Namely, it is given by the previous homogeneous foliation times a regular foliation of codimension 3. The family of integrable 1-forms $\eta_{p_j(t)}$ naturally defines a holomorphic family of foliations on the weighted projective plane $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$.

Let us define the family of candidates that will be a deformation of the mapping f_0. Set $V_a = \overline{f_0^{-1}(a)}$, $V_b = \overline{f_0^{-1}(b)}$, $V_c = \overline{f_0^{-1}(c)}$, where $a = [0 : 0 : 1]$, $b = [0 : 1 : 0]$ and $c = [1 : 0 : 0]$ and denote by $V_{\tau^*} = \overline{f_0^{-1}(\tau^*)}$, where $\tau^* \in Sing(\mathcal{G}_0)\backslash\{a, b, c\}$.

Proposition 7.19. *Let $(\mathcal{F}_t)_{t\in D_\epsilon}$ be a deformation of $\mathcal{F}_0 = f_0^*(\mathcal{G}_0)$, where $(f_0, \mathcal{G}_0)$ is a generic pair, with $\mathcal{G}_0 \in \mathcal{A}$, $f_0 \in Gen\,(n,\nu,\alpha,\beta,\gamma)$ and $deg(f_0) = \nu \geq 2$. Then there exists a deformation $(f_t)_{t\in D_\epsilon}$ of f_0 in the set $Gen\,(n,\nu,\alpha,\beta,\gamma)$ such that:*

(i) *$V_a(t), V_b(t)$ and $V_c(t)$ are fibers of $(f_t)_{t\in D_{\epsilon'}}$.*
(ii) *$I(t) = I(f_t), \forall t \in D_{\epsilon'}$.*

In the Appendix **section** 8.7 we will prove that the others curves $V_\tau(t)$ where τ is different of a, b and c are also fibers of the mapping f_t for fixed t **(See Lemma** 8.4 **and Lemma** 8.5**)**.

Proof. Let $\tilde{f}_0 = (F_0^\alpha, F_1^\beta, F_2^\gamma) : \mathbb{C}^{n+1} \to \mathbb{C}^3$ be the homogeneous expression of f_0. Then V_c, V_b, and V_a appear as the complete intersections $(F_1 = F_2 = 0)$, $(F_0 = F_2 = 0)$, and $(F_0 = F_1 = 0)$ respectively, so $I(f_0) = V_a \cap V_b = V_a \cap V_c = V_b \cap V_c$. With this in

mind, it follows from [Ser1] (see section 4.6 pp 235-236) that $V_a(t)$ is a complete intersection, say $V_a(t) = (F_0(t) = F_1(t) = 0)$ in homogeneous coordinates, where $(F_0(t))_{t\in D_{\epsilon'}}$ and $(F_1(t))_{t\in D_{\epsilon'}}$ are deformations of F_0 and F_1 respectively, and $D_{\epsilon'}$ is a possibly smaller neighborhood of 0. Moreover, $F_0(t) = 0$ and $F_1(t) = 0$ meet transversely along $V_a(t)$. In the same way, it is possible to define $V_c(t)$ and $V_b(t)$ as the complete intersections, say $(\hat{F}_1(t) = F_2(t) = 0)$ and $(\hat{F}_0(t) = \hat{F}_2(t) = 0)$, where $(F_j(t))_{t\in D_{\epsilon'}}$ and $(\hat{F}_j(t))_{t\in D_{\epsilon'}}$ are deformations of F_j, $0 \leq j \leq 2$. We will use these families of polynomials to define a family of rational maps that satisfy the required properties.
We are going to prove that we can find polynomials $P_0(t)$, $P_1(t)$ and $P_2(t)$ in such a way that $V_c(t) = (P_1(t) = P_2(t) = 0)$, $V_b(t) = (P_0(t) = P_2(t) = 0)$ and $V_a(t) = (P_0(t) = P_1(t) = 0)$. Observe first that since $F_0(t), F_1(t)$ and $F_2(t)$ are near F_0, F_1 and F_2 respectively, they meet as a regular complete intersection at:

$$\begin{aligned} J(t) &:= (F_0(t) = F_1(t) = F_2(t) = 0) \\ &= (F_0(t) = F_1(t) = 0) \cap (F_2(t) = 0) \\ &= V_a(t) \cap (F_2(t) = 0), \end{aligned}$$

so that $J(t) \cap (\hat{F}_1(t) = 0) = V_c(t) \cap V_a(t) = I(t)$, which implies that $I(t) \subset J(t)$. Since $I(t)$ and $J(t)$ have $\frac{\nu^3}{\alpha\beta\gamma}$ points, we have that $I(t) = J(t)$ for all $t \in D_{\epsilon'}$.

Remark 13. In the case $n \geq 4$, both sets are codimension-three smooth and connected submanifolds of $\mathbb{P}^n$, implying again that $I(t) = J(t)$. In particular, we obtain that

$$I(t) = (F_0(t) = F_1(t) = F_2(t) = 0) \subset (\hat{F}_j(t) = 0), 0 \leq j \leq 2.$$

Let us now use Noether's Theorem, which can be stated as follows:

Lemma 7.20. *(Noether's Theorem) Let $G_0, ..., G_k \in \mathbb{C}[z_1, ..., z_m]$ be homogeneous polynomials where $0 \leq k \leq m$ and $m \geq 2$, and $X = (G_0 = ... = G_k = 0)$. Suppose that the set $Y := \{p \in X | dG_0(p) \wedge ... \wedge dG_k(p) = 0\} = 0$ or $\emptyset$. If $G \in \mathbb{C}[z_1, ..., z_m]$ satisfies $G|_X \equiv 0$, then $G \in < G_0, ..., G_k >$, the ideal generated by $G_0, ..., G_k$.*

Remark 14. We have taken the above lemma in this form from [Ln] (pp. 86).

Back to the proof of Proposition 7.19.
Take $k = 2$, $G_0 = F_0(t)$, $G_1 = F_1(t)$ and $G_2 = F_2(t)$. In this case, we have $Y = \{0\}$, and we can use the Noether's Theorem. Using the fact that all polynomials involved are homogeneous, we have $\hat{F}_1(t) \in < F_0(t), F_1(t), F_2(t) >$, and since $deg(F_0(t)) > deg(F_1(t)) > deg(F_2(t))$, we conclude that $\hat{F}_1(t) = F_1(t) + g(t)F_2(t)$, where $g(t)$ is a homogeneous polynomial of degree $deg(F_1(t)) - deg(F_2(t))$. Observe also that $V_c(t) = V(\hat{F}_1(t), F_2(t)) = V(F_1(t), F_2(t))$, where $V(H_1, H_2)$ denotes the projective algebraic variety defined by $(H_1 = H_2 = 0)$! Similarly for $V_b(t)$ we have that $\hat{F}_2(t) \in < F_0(t), F_1(t), F_2(t) >$. On the other hand, since $\hat{F}_2(t)$ has the lowest degree, we can assume that $\hat{F}_2(t) = F_2(t)$. In an analogous way we have that $\hat{F}_0(t) = F_0(t) + m(t)F_1(t) + n(t)F_2(t)$ for the polynomial $\hat{F}_0(t)$. Now observe that $V(\hat{F}_0(t), \hat{F}_2(t)) = V(F_0(t) + m(t)F_1(t), F_2(t))$. Hence we can define $f_t = (P_0^\alpha(t), P_1^\beta(t), P_2^\gamma(t))$ where $P_0(t) = F_0(t) + m(t)F_1(t)$, $P_1(t) = F_1(t)$ and $P_2(t) = F_2(t)$. This defines a family of mappings $(f_t)_{t\in D_{\epsilon'}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$, and $V_a(t)$, $V_b(t)$ and $V_c(t)$ are fibers of f_t for fixed t. Observe that, for ϵ' sufficiently small, $(f_t)_{t\in D_{\epsilon'}}$ is generic in the sense of definition 4.2, and its indeterminacy locus $I(f_t)$ is precisely $I(t)$.

Moreover, since $Gen(3,\nu,\alpha,\beta,\gamma)$ is open, we can suppose that this family $(f_t)_{t\in D_{\epsilon'}}$ is in $Gen\,(3,\nu,\alpha,\beta,\gamma)$. This concludes the proof of proposition 7.19. $\square$

Let us now define a family of foliations $(\mathcal{G}_t)_{t\in D_\epsilon}, \mathcal{G}_t \in \mathcal{A}$ **(see section 6 pp.25)** such that $\mathcal{F}_t = f_t^*(\mathcal{G}_t)$ for all $t \in D_\epsilon$. We consider first the case $n = 3$.

Let $M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t)$ be the family of "complex algebraic threefolds" obtained from $\mathbb{P}^3$ by weighted blowing-up with weights $(\beta\gamma,\alpha\gamma,\alpha\beta)$ (see Appendix section 8.4) at the $\frac{\nu^3}{\alpha\beta\gamma}$ points $p_1(t),...,p_j(t),...,p_{\frac{\nu^3}{\alpha\beta\gamma}}(t)$ corresponding to $I(t)$ of $\mathcal{F}_t$; and denote by

$$\pi_w(t) : M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t) \to \mathbb{P}^3$$

the blowing-up map. The exceptional divisor of $\pi_w(t)$ consists of $\frac{\nu^3}{\alpha\beta\gamma}$ orbifolds $E_j(t) = \pi_w(t)^{-1}(p_j(t))$, $1 \le j \le \frac{\nu^3}{\alpha\beta\gamma}$, that are weighted projective planes of the type $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$. Each of them has three lines of singular points of $M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t)$, all isomorphic to weighted projective lines, but these singularities will not disturb our arguments. (See [Ma-Mor] ex. 3.6 pp 21.)

More precisely, if we blow-up $\mathcal{F}_t$ at the point $p_j(t)$, then the restriction of the strict transform of $\pi_w^*\mathcal{F}_t$ to the exceptional divisor $E_j(t) = \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ is the same quasi-homogeneous 1-form that defines $\mathcal{F}_t$ at the point $p_j(t)$ (see Appendix sections 8.4, 8.5 and 8.6 for computations for case $t = 0$). Using the isomorphism between $E_j(t)$ and $\mathbb{P}^2$ we can push-forward the foliation to $\mathbb{P}^2$. With this process we produce a family of holomorphic foliations in $\mathcal{A}$. This family is the "holomorphic path" of candidates to be a deformation of $\mathcal{G}_0$. In fact, since $\mathcal{A}$ is an open set we can suppose that this family is inside $\mathcal{A}$ (see Appendix section 8.6 for computations for case $t = 0$). Let us denote this family of candidates by $(\mathcal{G}_t)_{t\in D_\epsilon}$, and choose the exceptional divisor $E_1(t)$ to work with. Observe that with this process we are producing foliations in $\mathcal{A}$ up to a linear automorphism of $\mathbb{P}^2$.

Consider the family of mappings $f_t : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$, $t \in D_{\epsilon'}$ defined in the proposition 7.19.

We wish to consider the family $(f_t)_{t\in D_\epsilon}$ as a family of rational maps $f_t : \mathbb{P}^3 \dashrightarrow E_1(t)$. We can decrease ϵ if necessary. Note that the map

$$f_t \circ \pi_w(t) : M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t) \backslash \cup_j E_j(t) \to E_1(t) \simeq \mathbb{P}^2$$

extends holomorphically, that is, as an orbifold mapping, to

$$\hat{f}_t : M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t) \to \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]} \simeq E_1(t) \simeq \mathbb{P}^2$$

This is due to the fact that each orbit of the vector field S_t in the coordinate system where it is linear, determines an equivalence class in $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ and the orbits are fibers of the map

$$(x_0(t), x_1(t), x_2(t)) \to (x_0^\alpha(t), x_1^\beta(t), x_2^\gamma(t)).$$

The mapping f_t can be interpreted as follows. Each fiber of f_t meets $p_j(t)$ once, which implies that each fiber of $\hat{f}_t$ cuts $E_1(t)$ once outside of the three singular curves in $[M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t) \cap E_1(t)]$. Since $M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t) \backslash \cup_j E_j(t)$ is biholomorphic to $\mathbb{P}^3 \backslash I(t)$, then after identifying $E_1(t)$ with $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$, we can imagine that if $q \in M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t) \backslash \cup_j E_j(t)$, then $\hat{f}_t(q)$ is the intersection point of the fiber $\hat{f}_t^{-1}(\hat{f}_t(q))$ with $E_1(t)$. Using the isomorphism between $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ and $\mathbb{P}^2$, we obtain a mapping

$$\hat{f}_t : M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t) \to \mathbb{P}^2.$$

It can be extended over the singular set of $M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t)$ using Riemann's Extension Theorem because the orbifold $M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t)$ has singular set of codimension 2 and theses singularities are of the quotient type and therefore it is a normal complex space. We shall denote this extension also by $\hat{f}_t$ to simplify the exposition. In the appendix section 8.8 we show that the weighted blowing-up with weights $(\beta\gamma, \alpha\gamma, \alpha\beta)$ can solve completely the indeterminacy set of f_t for each t. With all these ingredients we can define the foliation: $\tilde{\mathcal{F}}_t = f_t^*(\mathcal{G}_t) \in PB(\Gamma - 1, \nu, \alpha, \beta, \gamma)$. This foliation is a deformation of $\mathcal{F}_0$. Based on the previous discussion let us denote $\mathcal{F}_1(t) = \pi_w(t)^*(\mathcal{F}_t)$ and $\hat{\mathcal{F}}_1(t) = \pi_w(t)^*(\tilde{\mathcal{F}}_t)$.

We will prove the following

Lemma 7.21. *If $\mathcal{F}_1(t)$ and $\hat{\mathcal{F}}_1(t)$ are the foliations defined previously, we have that*

$$\mathcal{F}_1(t)|_{E_1(t)\simeq\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}} = \hat{\mathcal{G}}_t = \hat{\mathcal{F}}_1(t)|_{E_1(t)\simeq\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}}$$

where $\hat{\mathcal{G}}_t$ is the foliation induced on $E_1(t) \simeq \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ by the quasi-homogeneous 1-form $\eta_{p_1(t)}$.

Proof. The first equality, that is, $\mathcal{F}_1(t)|_{E_1(t)\simeq\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}} = \hat{\mathcal{G}}_t$, follows from the fact that $\mathcal{F}_t$ is represented in a neighborhood of $p_1(t) \in I(t)$ by the quasi-homogeneous 1-form $\eta_{p_1(t)}$, that satisfies $i_{S_t}\eta_{p_1(t)} = 0$ and therefore it defines naturally a foliation on the weighted projective space $E_1(t) \simeq \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$. In the appendix, section 8.4 and 8.5 we show that $\mathcal{F}_1(t)|_{E_1(t)\simeq\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}} = \hat{\mathcal{G}}_t$ is up to a linear automorphism of $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ the 1-form $\eta_{p_1(t)}$. The second equality $\hat{\mathcal{F}}_1(t)|_{E_1(t)\simeq\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}} = \hat{\mathcal{G}}_t$, follows from the geometrical interpretation of the mapping $\hat{f}_t : M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t) \to \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]} \simeq \mathbb{P}^2$. □

So now we use the fact that $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]} \simeq \mathbb{P}^2$ to obtain the equality

$$\mathcal{G}_t = \mathcal{F}_1(t)|_{E_1(t)\simeq\mathbb{P}^2} = \hat{\mathcal{F}}_1(t)|_{E_1(t)\simeq\mathbb{P}^2},$$

where $\mathcal{G}_t$ is obtained from $\hat{\mathcal{G}}_t$ by push-forward by the mapping $f_w : \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]} \to \mathbb{P}^2$. (See Appendix section 8.6). Observe that if after this procedure we do not obtain yet the equality we just have to compose with a linear automorphism of $\mathbb{P}^2$ and in the we will obtain the required equality. We remind that this equality we be fundamental to the next argument.

Now let $\tau_1(t)$ be a singularity of $\mathcal{G}_t$ outside the three invariant straight lines. Since the map $t \to \tau_1(t) \in \mathbb{P}^2$ is holomorphic, there exists a holomorphic family of automorphisms of $\mathbb{P}^2$, $t \to H(t)$ such that $\tau_1(t) = [a : b : c] \in E_1(t) \simeq \mathbb{P}^2$ is kept fixed. Observe that such a singularity has non algebraic separatrices at this point. Fix a local analytic coordinate system (x_t, y_t) at $\tau_1(t)$ such that the local separatrices are $(x_t = 0)$ and $(y_t = 0)$, respectively. Observe that the local smooth hypersurfaces along $\hat{V}_{\tau_1(t)} = \hat{f}_t^{-1}(\tau_1(t))$ defined by $\hat{X}_t := (x_t \circ \hat{f}_t = 0)$ and $\hat{Y}_t := (y_t \circ \hat{f}_t = 0)$ are invariant for $\hat{\mathcal{F}}_1(t)$. Furthermore, they meet transversely along $\hat{V}_{\tau_1(t)}$. On the other hand, $\hat{V}_{\tau_1(t)}$ is also contained in the Kupka set of $\mathcal{F}_1(t)$. Therefore there are two local smooth hypersurfaces say $X_t := (x_t \circ \hat{f}_t = 0)$ and $Y_t := (y_t \circ \hat{f}_t = 0)$ are invariant for $\mathcal{F}_1(t)$, such that:

(1) X_t and Y_t meet transversely along $\hat{V}_{\tau_1(t)}$.
(2) $X_t \cap \pi_w(t)^{-1}(p_1(t)) = (x_t = 0) = \hat{X}_t \cap \pi_w(t)^{-1}(p_1(t))$ and $Y_t \cap \pi_w(t)^{-1}(p_1(t)) = (y_t = 0) = \hat{Y}_t \cap \pi_w(t)^{-1}(p_1(t))$ (because $\mathcal{F}_1(t)$ and $\hat{\mathcal{F}}_1(t)$) coincide on $E_1(t) \simeq \mathbb{P}^2$).
(3) X_t and Y_t are deformations of $X_0 = \hat{X}_0$ and $Y_0 = \hat{Y}_0$, respectively.

Lemma 7.22. *$X_t = \hat{X}_t$ for small t.*

Proof. Let us consider the projection $\hat{f}_t : M_{[\beta\gamma,\alpha\gamma,\alpha\beta]}(t) \to \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]} \simeq \mathbb{P}^2$ on a neighborhood of the regular fibre $\hat{V}_{\tau_1(t)}$, and fix local coordinates x_t, y_t on $\mathbb{P}^2$ such that $X_t := (x_t \circ \hat{f}_t = 0)$. Let $H_\epsilon = (y_t \circ \hat{f}_t = \epsilon)$, ϵ small, so that $\hat{\Sigma}_\epsilon = \hat{X}_t \cap H_\epsilon$ are (vertical) compact curves, deformations of $\hat{\Sigma}_0 = \hat{V}_{\tau_1(t)}$. Set $\Sigma_\epsilon = X_t \cap \hat{H}_\epsilon$. Like the $\hat{\Sigma}'_\epsilon s$ also $\Sigma'_\epsilon s$ are compact curves (for t and ϵ small), because X_t and $\hat{X}_t$ are both deformations of the same X_0 and so X_t is close to $\hat{X}_t$, for t small. It follows that $\hat{f}_t(\Sigma_\epsilon)$ is an analytic curve contained in a small neighborhood of $\tau_1(t)$, for small ϵ. By the maximum principle, we must have that $\hat{f}_t(\Sigma_\epsilon)$ is a point, so that $\hat{f}_t(X_t) = \hat{f}_t(\cup_\epsilon \Sigma_\epsilon)$ is a curve C, i.e., $X_t = \hat{f}_t^{-1}(C)$. But X_t and $\hat{X}_t$ intersects along the exceptional divisor $E_1(t) \simeq \mathbb{P}^2$ along the separatrix $(x_t = 0)$ of $\mathcal{G}_t$ through $\tau_1(t)$. This implies that $X_t = \hat{f}_t^{-1}(C) = \hat{f}_t^{-1}(x_t = 0) = \hat{X}_t$. $\square$

We have prooved that the foliations $\mathcal{F}_t$ and $\tilde{\mathcal{F}}_t$ have a common local leaf: the leaf that contains $\pi_w(t)\left(X_t \backslash \hat{V}_{\tau_1(t)}\right)$ which is not algebraic. Let $D(t) := Tang(\mathcal{F}(t), \hat{\mathcal{F}}(t))$ be the set of tangencies between $\mathcal{F}(t)$ and $\hat{\mathcal{F}}(t)$. This set can be defined by $D(t) = \{Z \in \mathbb{C}^4; \Omega(t) \wedge \hat{\Omega}(t) = 0\}$, where $\Omega(t)$ and $\hat{\Omega}(t)$ define $\mathcal{F}(t)$ and $\hat{\mathcal{F}}(t)$, respectively. Hence it is an algebraic set. Since this set contains a immersed non-algebraic surface X_t, we have necessarily that $D(t) = \mathbb{P}^3$. This proves the Theorem in the case $n = 3$.

Suppose now that $n \geq 4$. The previous argument implies that if Υ is a generic $3-$plane in $\mathbb{P}^n$, we have $\mathcal{F}(t)_{|\Upsilon} = \hat{\mathcal{F}}(t)_{|\Upsilon}$. In fact, such planes cut transversely every strata of the singular set, and $I(t)$ consists of $\frac{\nu^3}{\alpha\beta\gamma}$ points. This implies that f_t is generic for $|t|$ sufficiently small. We can then apply the previous argument again, finishing the proof of the lemma and also that of Theorem B.

Using the same techniques used in the proof of the previous theorem and lemma, we can prove the following result:

Theorem C. *Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ be a generic rational map of degree ν given by $f = (F_0^\alpha : F_1^\beta : F_2^\gamma)$, where $deg(F_0).\alpha = deg(F_1).\beta = deg(F_2).\gamma = \nu$, $\nu \geq 2$, $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ such that $1 \leq \alpha < \beta < \gamma$ and g.c.d $(\beta.\gamma, \alpha.\gamma, \alpha.\beta) = 1$, $I(f)$ its indeterminacy locus and $\mathcal{F}$ a foliation on $\mathbb{P}^n, n \geq 3$.*
Suppose that the following conditions hold:

(1) *At any point $p_j \in I(f)$, $\mathcal{F}$ has the following local structure: In the case $n = 3$, there exists an analytic coordinate system around p_j, say (U^{p_j}, Z^{p_j}), such that $Z^{p_j}(p_j) = 0 \in (\mathbb{C}^3, 0)$ and $\mathcal{F}|_{(U^{p_j}, Z^{p_j})}$ can be represented by η_{p_j} a quasihomogeneous 1-form, as described in the lemma 7.10 p.32 and 33, such that:*

(a) *$sing(d\eta_{p_j}) = 0$,*

(b) *0 is a quasi-homogeneous singularity of the type*

$$[\beta.\gamma : \alpha.\gamma : \alpha.\beta]$$

of η_{p_j}.

In the case, $n \geq 4$, $\mathcal{F}$ has a local structure product, given as before, times a regular foliation in $\mathbb{C}^{n-3}$.

(2) *There exists a fibre $f^{-1}(q) = V(q)$ such that $V(q) = f^{-1}(q)\backslash I(f)$ is contained in the Kupka-Set of $\mathcal{F}$ and $V(q)$ is not contained inside of the 3 hypersurfaces, $\bigcup_{i=0}^{i=2}(F_i = 0)$.*

(3) *$V(q)$ has a transversal type X, where X is a germ of vector field on $(\mathbb{C}^2, 0)$, with a non-degenerated singularity at $0 \in \mathbb{C}^2$, having eigenvalues λ_1 and λ_2, where $\frac{\lambda_2}{\lambda_1} \notin \mathbb{R}$ and with a non-algebraic separatrix.*

Then $\mathcal{F}$ is a pull back foliation, $\mathcal{F} = f^*(\mathcal{G})$, where $\mathcal{G}$ is of degree $d \geq 2$ on $\mathbb{P}^2$ with three invariant lines in general position.

Proof. The proof of this theorem is essentially the previous lemma. □

Remark 15. Note that when $\mathcal{G}$ does not have invariant algebraic curves and we make the pull-back by a map as above, the indeterminacy set of f does not satisfy the hypothesis of the theorem. Take $\mathcal{G}$ for example as being the Jouanoulou's foliation and make the pull-back by f as above. In the case $n = 3$ the indeterminancy set of f is not a quasi-homogenious singularity of $\mathcal{F} = f^*(\mathcal{G})$.

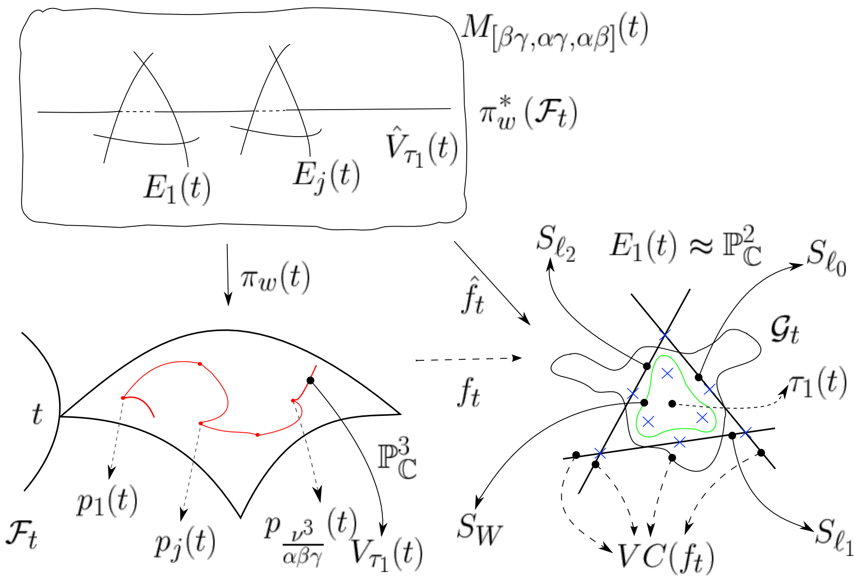

FIGURE 4. Weighted Blow-up and Tang Argument related to lemmas 7.21 and 7.22.

8. APPENDIX

8.1. Complete Intersections

In 1958 Kodaira and Spencer showed, that any smooth projective hypersurface which is not a K_3 surface remains projective algebraic under small deformations. This same type result has been done in the case of complete intersections of dimension ≥ 2 (see [Ser] and [Weh]), and later extended to the case when the complete intersection is an algebraic curve (see [Ser1]). We say that a k-dimensonal algebraic variety $V \subset \mathbb{P}^n$ is a global complete intersection if its homogeneous ideal $I(V) \subset \mathbb{C}[Z_0, ..., Z_n]$ is generated by $n-k$ elements, say, $f_1, ..., f_{n-k}$ of degrees $d_1, ...d_{n-k}$ respectively. We will always order the $d_{j,s}$ so that $d_j \leq d_{j+1}$, $j = 1, ..., n-k$, and call $d = (d_1, ..., d_{n-k})$ the multidegree of V.

We now state the stability theorem for complete intersections:

Theorem 8.1. *(Sernesi-Wehler) Let V be a global complete intersection in $\mathbb{P}^n$ of multidegree d, possibly with singularities, and assume $dim_{\mathbb{C}}V \geq 1$. If V is not a K_3 surface, then all sufficiently small deformations of V are again global complete intersections of multidegree d.*

A complete proof of this theorem can be found in [Ser1] *Deformation of Algebraic Schemes, Section 4.6 - Examples and Applications pages 235 and 236.*

Another important result is

Proposition 8.2. *Let V be a complete intersection on $\mathbb{P}^n$, we have the following:*

(1) *If $dim_{\mathbb{C}}V \geq 2$, then V is simply-connected,*

(2) *If $dim_{\mathbb{C}}V = 1$, then V is connected.*

8.2. Orbifolds and Foliations - A glimpse into the theory

This section is motivated by some of the techniques used to justify arguments in the proof of the main theorem (Theorem B). As we have seen, we needed to blow-up the points of the indeterminacy set of f. During this process there appears a new category of algebraic variety, that we call an **orbifold**, and the exceptional divisors obtained are **Weighted Projective Spaces**, which are singular spaces in general.

Primary examples of orbifolds are quotient spaces of smooth manifolds by a smooth finite group action. Here we consider that the quotient space is uniformized (or modeled) by a manifold with the finite group action. Hence a notion of smoothness for the quotient space is inherited from the manifold through those objects which are invariant under the group action. We require that any element of the group either acts trivially or has fixed-point set of codimension at least two.This requirement has the consequence that the non-fixed-point set is locally connected. Indeed, the first known examples of these objects where obtained as quotients of manifolds under the action of finite groups of automorphisms. Analogous to the definition of manifold, a complex orbifold atlas is locally modeled on open subsets of $\mathbb{C}^n$ modulo finite groups of biholomorphisms acting on it, such that a suitable condition of compatibility is satisfied in the intersection of any pair of "charts." There is a well defined notion of "map" between orbifolds, fiber bundle theory (Orbibundles) and Chern-Weil theory, and much more.

In this thesis, we use the following fundamental concepts:

- Foliations on Weighted Projective Spaces
- Weighted Blow-ups
- Mappings between Orbifolds
- Weighted Projective Spaces
- Chern-Weill Theory

8.3. **Foliations on weighted projective spaces**

Foliations on weighted projective spaces often appears in higher dimensional questions related to the resolution of singularities, for example [Pan]. They are also similar to foliations on Hopf surfaces. A polynomial P in n variables $(x_1, ..., x_n)$ is said to be quasi-homogeneous with weights $(k_1, ..., k_n)$ and degree d if for every $\lambda \in \mathbb{C}^*$ one has

$$P(\lambda^{k_1}x_1, ..., \lambda^{k_n}x_n) = \lambda^d P(x_1, ..., x_n).$$

For example, in $\mathbb{C}[x, y, z]$ the polynomial $P(x, y, z) = xz + y^2$ is not only quadratic (i.e. homogeneous of degree 2) but also quasi-homogeneous of degree 4 relative to the weights $(1, 2, 3)$.

Given a set of weights $(k_1, ..., k_n)$, we have a natural action of $\mathbb{C}^*$ on $\mathbb{C}^n\backslash\{0\}$ given by

$$\lambda.(x_1, ..., x_n) = (\lambda^{k_1}x_1, ..., \lambda^{k_n}x_n).$$

Consider the quotient space $\mathbb{C}^n\backslash\{0\}$, where two points are identified if and only if they belong to the same orbit of $\mathbb{C}^*$. The resulting space is a compact manifold with singularities called a weighted projective space, whose dimension is obviously equal to $(n-1)$.

Whether it has singularities or not, this type of manifold can be given an algebraic structure since it can be realized as a Zariski-closed set of a complex projective space with sufficiently high dimension. The existence of this embedding can be shown by means of Plücker coordinates. Alternatively, the quotient of this $\mathbb{C}^*$-action can also be realized as the quotient of the projective space of dimension $(n-1)$ by some finite group of automorphisms.

A polynomial vector field in $\mathbb{C}^n$

$$P_1\frac{\partial}{\partial x_1} + ... + P_n\frac{\partial}{\partial x_n}$$

is said to be quasi- homogeneous with weights $(k_1, ..., k_n)$ and degree d if for every $\lambda \in \mathbb{C}^*$, one has $\Lambda^* X = \lambda^{(d-1)} X$, where Λ stands for the map

$$(x_1, ..., x_n) \to (\lambda^{k_1} x_1, ..., \lambda^{k_n} x_n).$$

An example of a quasi-homogeneous vector field with weights $(1, 2, 3)$ and degree 4 in $\mathbb{C}[x, y, z]$ is

$$(xz + y^2)\frac{\partial}{\partial x} + (2zy + 3x^5)\frac{\partial}{\partial y} + (xy^3 z - y^3 + 2z^2)\frac{\partial}{\partial z}.$$

If X is quasi-homogeneous with weights $(k_1, ..., k_n)$, then the definition above implies that X defines a complex direction at each point of the projective space with the same weights $(k_1, ..., k_n)$. Being of complex dimension 1, these directions can naturally be integrated to form a singular holomorphic foliation. Therefore, quasi-homogeneous vector fields give rise to singular holomorphic foliations on weighted projective spaces.
Henceforth, we will concentrate on the case of weighted projective planes. Let $w := (w_0, w_1, w_2) \in \mathbb{N}^3$. Consider the action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^3 \backslash \{0\}$ given by

$$(x_0, x_1, x_2) \mapsto (t^{w_0} x_0, t^{w_1} x_1, t^{w_2} x_2).$$

The set of orbits $\mathbb{C}^3 \backslash \{0\}$ under this action is the weighted projective plane of type w,

$$\mathbb{P}^2_{[w_0:w_1:w_2]} := \mathbb{P}_w.$$

The class of a non-zero element $(x_0, x_1, x_2) \in \mathbb{C}^3$ is denoted by $[x_0 : x_1 : x_2]_w$ and the weight vector is omitted if no ambiguity seems likely to arise. When $w := (w_0, w_1, w_2) = (1, 1, 1)$ one obtains the usual projective plane and the weight vector is always omitted. For $Z \in \mathbb{C}^3 \backslash \{0\}$, the closure of $[Z]_w \in \mathbb{C}^3$ is obtained by adding the origin and it is an algebraic curve.

Definition 8.3. *We say that $\mathbb{P}_w$ is well-formed if g.c.d$(w_i, w_j) = 1$ for $i \neq j$.*

In the paper [So.Cor], the authors include hypotheses regarding the weights. In a private communication, Maurício Corrêa Júnior noted that the hypothesis g.c.d$(w_i, w_j) = 1$ is not necessary for the proof of the following statements:

According to [Mann E.], we have a natural orbifold map

$$\begin{array}{rcl} f_w : \mathbb{P}^2 & \to & \mathbb{P}_w \\ {[x_0 : x_1 : x_2]} & \to & [x_0^{w_0} : x_1^{w_1} : x_2^{w_2}]_w \end{array}$$

which allows us to show that there exists a unique (up to isomorphism) rank 1 complex $\mathbb{Q}$-bundle $\mathcal{O}_{\mathbb{P}_w}(1)$ over $\mathbb{P}_w$, such that

$$f_w^* \mathcal{O}_{\mathbb{P}_w}(1) = \mathcal{O}_{\mathbb{P}^2}(1).$$

The Euler sequence carries over to the weighted case and we have the exact sequence of $\mathbb{Q}$-bundles over $\mathbb{P}_w$:

$$0 \to \underline{\mathbb{C}} \to \mathcal{O}_{\mathbb{P}_w}(w_0) \oplus \mathcal{O}_{\mathbb{P}_w}(w_1) \oplus \mathcal{O}_{\mathbb{P}_w}(w_2) \to T\mathbb{P}_w \to 0$$

where $\underline{\mathbb{C}}$ is the trivial orbifold bundle and $T\mathbb{P}_w$ is the orbifold tangent bundle of $\mathbb{P}_w$. Additionally, from [Mann E.], the Chern-Weil theory of Chern classes holds in $\mathbb{P}_w$ as it does in projective spaces, and denoting $\zeta = c_1(\mathcal{O}_{\mathbb{P}_w}(1))$ from the previous exact sequence, we have:

$$c(T\mathbb{P}_w) = (1+w_0\zeta)(1+w_1\zeta)(1+w_2\zeta).$$

Hence

$$c_i(T\mathbb{P}_w) = \sigma_i(w_0, w_1, w_2)\zeta^i$$

where σ_i is the *i-th* elementary symmetric function. Now let X be a quasi-homogeneous vector field of type (w_0, w_1, w_2) and degree d in $\mathbb{C}^3$. Writing $X = \sum_{i=0}^{2} P_i(x_0, x_1, x_2)\frac{\partial}{\partial z_i}$, we have that

$$P_i(\lambda^{w_0}x_0, \lambda^{w_1}x_1, \lambda^{w_2}x_2) = \lambda^{d+w_i-1}P_i(x_0, x_1, x_2),$$

which descend to $\mathbb{P}_w$. In fact, tensorizing the first exact sequence by $\mathcal{O}_{\mathbb{P}_w}(d-1)$ we get

$$0 \to \mathcal{O}_{\mathbb{P}_w}(d-1) \to \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}_w}(d+w_i-1) \to T\mathbb{P}_w \otimes \mathcal{O}_{\mathbb{P}_w}(d-1) \to 0.$$

It follows that a quasi-homogeneous vector field X induces a foliation $\mathcal{F}$ of $\mathbb{P}_w$ and that $gR_w + X$ define the same foliation as X, where R_w is the adapted radial vector field

$$R_w = w_0x_0\frac{\partial}{\partial x_0} + w_1x_1\frac{\partial}{\partial x_1} + w_2x_2\frac{\partial}{\partial x_2},$$

with g a quasi-homogeneous polynomial of type (w_0, w_1, w_2) and degree $d-1$. Dually, noting $|w| = w_0 + w_1 + w_2$, we have the exact sequence

$$0 \to \Omega^1_{\mathbb{P}_w} \otimes \mathcal{O}_{\mathbb{P}_w}(d+|w|-1) \to \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}_w}(d+|w|-w_i-1) \to \mathcal{O}_{\mathbb{P}_w}(d+|w|-w_i-1) \to 0.$$

Hence, a foliation $\mathcal{F}$ of $\mathbb{P}_w$ is also induced by a 1-form

$$\eta = A_0dx_0 + A_1dx_1 + A_2dx_2,$$

with A_i a quasi-homogeneous polynomial of type (w_0, w_1, w_2), degree $d+|w|-w_i-1$ and $i_{R_w}\eta = w_0x_0A_0 + w_1x_1A_1 + w_2x_2A_2 \equiv 0$. In the situation that are interested in, we take the weight vector

$$w := (w_0, w_1, w_2) = (\beta\gamma, \alpha\gamma, \alpha\beta)$$

where $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ such that $1 \leq \alpha < \beta < \gamma$ and g.c.d $(\beta.\gamma, \alpha.\gamma, \alpha.\beta) = 1$.

Following [A.M.O-G] and [B.R], our wighted projective plane is not well-formed, thus we have another natural orbifold map between $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ and $\mathbb{P}^2$ as follows:

$$\begin{array}{rcl} f_w : \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]} & \to & \mathbb{P}^2 \\ [x_0 : x_1 : x_2]_w & \to & \left[x_0^\alpha : x_1^\beta : x_2^\gamma\right]. \end{array}$$

In fact, this map defines an isomorphism between them, as explained in [B.R] Proposition 3.C.5 page 128.

8.4. **Weighted Blowing-up**

In Singularity Theory, resolution of a singularity is one of the most important tools. In the embedded case (standard procedure), the starting point is a singular hypersurface. After a sequence of suitable blow-ups this hypersurface is replaced by a long list of smooth hypersurfaces (the strict transform and the exceptional divisors) which intersect in the simplest way (coordinate hyperplanes in general position for suitable local coordinates). This process can be very expensive from the computational point of view and, moreover, little of the data obtained is used for understanding the singularity.

Experimental work shows that most of these data can be recovered if one allows some mild singularities to survive in the process (the quotient singularities). These partial resolutions, called embedded $\mathbb{Q}$-resolutions, can be obtained as a sequence of weighted blow-ups and their computational complexity is much lower compared with standard resolutions. Moreover, the process is optimal in the sense that only useful data is obtained. For more details about weighted blow-up we refer the reader to [Ma-Mor] [A.M.O-G] and [A.M.O-G1] and for the application to the study of foliations we refer [Pan].

Let us fix a type of quasi-homogeneity, *i.e*, a set of exponents k_i. We are interested in blow-up points on $\mathbb{P}^3$. In this way we can proceed locally, that is, on $\mathbb{C}^3$. As before we are assuming that $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ with the conditions $1 \leq \alpha < \beta < \gamma$ and g.c.d $(\beta.\gamma, \alpha.\gamma, \alpha.\beta) = 1$. In blowing-up $\mathbb{C}^3$ at p with weights $(\beta.\gamma, \alpha.\gamma, \alpha.\beta)$, the idea is to leave $\mathbb{C}^3$ unaltered except at the point p, which is replaced by the set of "algebraic curves" through p, a copy of $\mathbb{P}^2_{[\beta\gamma, \alpha\gamma, \alpha\beta]}$. To make this precise, let us choose a suitable coordinate system for $\mathbb{C}^3$ so that the point p may be assumed to be the "origin". Consider $(\mathbb{C}^3, 0)$ with coordinates (x_0, x_1, x_2) and denote by $\widetilde{(\mathbb{C}^3, 0)}$ the closure of the graph of

$$(\mathbb{C}^3, 0) \to \mathbb{P}^2_{[\beta\gamma, \alpha\gamma, \alpha\beta]} \subset (\mathbb{C}^3, 0) \times \mathbb{P}^2_{[\beta\gamma, \alpha\gamma, \alpha\beta]}.$$

We have that the exceptional divisor is $\mathbb{P}^2_{[\beta\gamma, \alpha\gamma, \alpha\beta]}$. This $\widetilde{(\mathbb{C}^3, 0)}$ is a singular variety, that is, an orbifold. In general, $\widetilde{(\mathbb{C}^3, 0)}$ has three lines (cyclic quotient) of singular points located at the 3 axes of the exceptional divisor (the fixed points of the action).
As in the standard blow-up procedure, we also have a birational map

$$\pi_w : \widetilde{(\mathbb{C}^3, 0)} \to (\mathbb{C}^3, 0),$$

and orbifold charts covering $\widetilde{(\mathbb{C}^3, 0)}$ as follows:

(1) $U_0 = \pi_w^{-1}(\{x_0 \neq 0\})$;

Orbifold chart: $(\tilde{U_0}, \mathbb{Z}_{\beta\gamma}, \pi_{\beta\gamma})$ where:

$\mathbb{Z}_{\beta\gamma}$ is the finite group of order $\beta\gamma$, $\tilde{U_0} = \mathbb{C}^3$. The action is given by

$$(y_0, y_1, y_2) \to (\xi_{\beta\gamma} y_0, \xi_{\beta\gamma}^{-\alpha\gamma} y_1, \xi_{\beta\gamma}^{-\alpha\beta} y_2)$$

where $\xi_{\beta\gamma}$ is a $\beta\gamma$-th root of 1.

The exceptional divisor is given by $y_0 = 0$. The expression of $\pi_w \circ \pi_{\beta\gamma}$ is:

(a) $x_0 = y_0^{\beta\gamma}$

(b) $x_1 = y_0^{\alpha\gamma} y_1$

(c) $x_2 = y_0^{\alpha\beta} y_2$

(2) $U_1 = \pi_w^{-1}(\{x_1 \neq 0\})$;

Orbifold chart: $(\tilde{U_1}, \mathbb{Z}_{\alpha\gamma}, \pi_{\alpha\gamma})$ where:

$\mathbb{Z}_{\alpha\gamma}$ is the finite group of order $\alpha\gamma$, $\tilde{U_0} = \mathbb{C}^3$. The action is given by

$$(y_0, y_1, y_2) \to (\xi_{\alpha\gamma}^{-\beta\gamma} y_0, \xi_{\alpha\gamma} y_1, \xi_{\alpha\gamma}^{-\alpha\beta} y_2)$$

where $\xi_{\alpha\gamma}$ is a $\alpha\gamma$-th root of 1.

The exceptional divisor is given by $y_1 = 0$. The expression of $\pi_w \circ \pi_{\alpha\gamma}$ is:

(a) $x_0 = y_1^{\beta\gamma} y_0$

(b) $x_1 = y_1^{\alpha\gamma}$

(c) $x_2 = y_1^{\alpha\beta} y_2$

(3) $U_2 = \pi_w^{-1}(\{x_2 \neq 0\})$;

Orbifold chart: $(\tilde{U_2}, \mathbb{Z}_{\alpha\beta}, \pi_{\alpha\beta})$ where:

$\mathbb{Z}_{\alpha\beta}$ is the finite group of order $\alpha\beta$, $\tilde{U_0} = \mathbb{C}^3$. The action is given by

$$(y_0, y_1, y_2) \to (\xi_{\alpha\beta}^{-\beta\gamma} y_0, \xi_{\alpha\beta}^{-\alpha\gamma} y_1, \xi_{\alpha\beta} y_2)$$

where $\xi_{\alpha\beta}$ is a $\alpha\beta$-th root of 1.

The exceptional divisor is given by $y_2 = 0$. The expression of $\pi_w \circ \pi_{\alpha\gamma}$ is:

(a) $x_0 = y_2^{\beta\gamma} y_0$

(b) $x_1 = y_2^{\alpha\gamma} y_1$

(c) $x_2 = y_2^{\alpha\beta}$

In this part we will make the weighted blow-up with weights $(\beta\gamma, \alpha\gamma, \alpha\beta)$ at $0 \in \mathbb{C}^3$ of the quasi-homogeneous 1-form η and we will restrict $\pi_w^*\eta$ to the exceptional divisor $\pi_w^{-1}(0) \equiv \mathbb{P}_w^2$.

We know that

$$\begin{aligned} \eta(x_0, x_1, x_2) &= \alpha.x_1.x_2.A(x_0^\alpha, x_1^\beta, x_2^\gamma)dx_0 \\ &+ \beta.x_0.x_2.B(x_0^\alpha, x_1^\beta, x_2^\gamma)dx_1 \\ &+ \gamma.x_0.x_1.C(x_0^\alpha, x_1^\beta, x_2^\gamma)dx_2. \end{aligned}$$

Let us make the weighted blow-up of the 1-form η at the point $0 \in \mathbb{C}^3$ with weights

$$[\beta\gamma, \alpha\gamma, \alpha\beta].$$

On the first chart:

(1)
$$d(x_0) = d(y_0^{\beta\gamma}) = (\beta\gamma)y_0^{\beta\gamma-1}dy_0$$
(2)
$$d(x_1) = d(y_0^{\alpha\gamma}y_1) = [y_0^{\alpha\gamma}dy_1 + (\alpha\gamma)y_1y_0^{\alpha\gamma-1}dy_0]$$
(3)
$$d(x_2) = d(y_0^{\alpha\beta}y_2) = [y_0^{\alpha\beta}dy_2 + (\alpha\beta)y_2y_0^{\alpha\beta-1}dy_0]$$

(1)
$$\alpha y_0^{\alpha\gamma}y_1\tilde{A}(y_0^{\alpha\beta\gamma}, y_0^{\alpha\beta\gamma}y_1^{\beta}, y_0^{\alpha\beta\gamma}y_2^{\gamma}).y_0^{\alpha\beta}y_2 d(y_0^{\beta\gamma}) = (\beta\gamma)\alpha y_1y_2\tilde{A}(1, y_1^{\beta}, y_2^{\gamma})[y_0^{\alpha\beta+\beta\gamma+\alpha\gamma-1+(d-1)\alpha\beta\gamma}]dy_0$$

(2)
$$\begin{aligned}
&\beta y_0^{\alpha\gamma}y_0^{\alpha\beta}y_2\tilde{B}(y_0^{\alpha\beta\gamma}, y_0^{\alpha\beta\gamma}y_1^{\beta}, y_0^{\alpha\beta\gamma}y_2^{\gamma}).y_0^{\alpha\beta}y_2(y_0^{\alpha\gamma}dy_1 + (\alpha\gamma)y_1y_0^{\alpha\gamma-1}dy_0)\\
&= [y_0^{\alpha\beta+\beta\gamma+\alpha\gamma-1+(d-1)\alpha\beta\gamma}](\beta y_2\tilde{B}(1, y_1^{\beta}, y_2^{\gamma})y_0dy_1 + \alpha\gamma y_1dy_0)\\
&= y_0\beta y_2\tilde{B}(1, y_1^{\beta}, y_2^{\gamma})d_1 + \alpha\beta\gamma y_1y_2B(1, y_1^{\beta}, y_2^{\gamma})dy_0
\end{aligned}$$

(3)
$$\begin{aligned}
&\gamma(y_0^{\alpha\gamma})y_1y_0^{\beta\gamma}\tilde{B}(y_0^{\alpha\beta\gamma}, y_0^{\alpha\beta\gamma}y_1^{\beta}, y_0^{\alpha\beta\gamma}(y_0^{\alpha\beta}dy_2 + (\alpha\beta)y_0^{\alpha\beta-1}y_2y_0)\\
&= [y_0^{\alpha\beta+\beta\gamma+\alpha\gamma-1+(d-1)\alpha\beta\gamma}][\gamma y_1\tilde{C}(1, y_1^{\beta}, y_2^{\gamma})(y_0dy_2 + \alpha\beta y_2dy_0)]\\
&= [\gamma y_1\tilde{C}(1, y_1^{\beta}, y_2^{\gamma})(y_0dy_2 + \alpha\beta y_2dy_0)].
\end{aligned}$$

Extracting $[y_0^{\alpha\beta+\beta\gamma+\alpha\gamma-1+(d-1)\alpha\beta\gamma}]$ from the prior expressions and then adding them up we obtain:
$$\begin{aligned}
&\alpha\beta\gamma y_1y_2\tilde{A}(1, y_1^{\beta}, y_2^{\gamma})dy_0 + y_0\beta y_2\tilde{B}(1, y_1^{\beta}, y_2^{\gamma})dy_1 + \alpha\beta\gamma y_1y_2\tilde{B}(1, y_1^{\beta}, y_2^{\gamma})dy_0\\
&+y_0\gamma y_1\tilde{C}(1, y_1^{\beta}, y_2^{\gamma})dy_2 + \alpha\beta\gamma y_1y_2\tilde{C}(1, y_1^{\beta}, y_2^{\gamma})dy_0
\end{aligned}$$
Using the fact that
$$\alpha\beta\gamma y_1y_2\tilde{A}(1, y_1^{\beta}, y_2^{\gamma})dy_0 + \alpha\beta\gamma y_1y_2\tilde{B}(1, y_1^{\beta}, y_2^{\gamma})dy_0 + \alpha\beta\gamma y_1y_2\tilde{C}(1, y_1^{\beta}, y_2^{\gamma})dy_0 \equiv 0,$$

we can simplify (extract the y_0 from the two remaining expressions) to obtain the expression of $\pi_w^*\eta$ restricit to the exceptional divisor $\pi_w^{-1}(0) \equiv \mathbb{P}_w^2$.
$$\pi_w^*\eta|_{y_0=0} = \beta y_2\tilde{B}(1, y_1^{\beta}, y_2^{\gamma})dy_1 + \gamma y_1\tilde{C}(1, y_1^{\beta}, y_2^{\gamma})dy_2.$$

This is the local expression of the foliation in the first coordinate chart of the weighted projective plane $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$. In the second and third coordinate systems the computations are similar. Summarized as follows:

(1) In the second chart:
Using the previous coordinates we have:

$$d(x_0) = d(y_0 y_1^{\beta\gamma}) = (\beta\gamma) y_0 y_1^{\beta\gamma-1} dy_1 + y_1^{\beta\gamma} dy_0$$

$$d(x_1) = d(y_1^{\alpha\gamma}) = (\alpha\gamma) y_1^{\alpha\gamma-1} dy_1$$

$$d(x_2) = d(y_1^{\alpha\beta} y_2) = [y_1^{\alpha\beta} dy_2 + (\alpha\beta) y_2 y_1^{\alpha\beta-1} dy_1].$$

Making the pull-back we obtain:

$$\pi_w^\star \eta|_{[y_1=0]} = \alpha y_2 A(y_0^\alpha, 1, y_2^\gamma) dy_2 + \gamma y_0 B(y_0^\alpha, 1, y_2^\gamma) dy_2$$

and the exceptional divisor corresponds to $y_1 = 0$.

(2) In the third chart:
Using the previous coordinates we have:

$$d(x_0) = d(y_0 y_2^{\beta\gamma}) = (\beta\gamma) y_0 y_2^{\beta\gamma-1} dy_2 + y_2^{\beta\gamma} dy_0$$

$$d(x_1) = d(y_2^{\alpha\gamma} y_1) = (\alpha\gamma) y_1 y_2^{\alpha\gamma-1} dy_2 + y_2^{\alpha\gamma} dy_1$$

$$d(x_2) = d(y_2^{\alpha\beta}) = (\alpha\beta) y_2^{\alpha\beta-1} dy_2$$

Making the pull-back we obtain:

$$\pi_w^* \eta|_{[y_2=0]} = \alpha y_0 A(y_0^\alpha, y_1^\beta, 1) dy_1 + \beta y_1 B(y_0^\alpha, y_1^\beta, 1) dy_0$$

and the exceptional divisor is given by $y_2 = 0$.

Remark 16. Observe that this blow-up is being done at time $t = 0$.

After blowing-up $\mathbb{P}^3$ in $\frac{v^3}{\alpha\beta\gamma}$ points, the new space that appears has $\frac{v^3}{\alpha\beta\gamma}$ exceptional divisors, each of which is isomorphic to $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$. Each divisor contains have three lines (each of which is isomorphic to $\mathbb{P}^1$), which are in fact weighted projective lines of singular points. Note that although the quotient spaces are written in their normalized form, the exceptional divisors can be simplified. This is due to the fact that each weighted projective plane that appears is not well-formed. In the next section we will use the orbifold charts of $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ to conclude that this procedure is the correct way to recover the quasi-homogeneous 1-form η up to automorphisms.

8.5. **Recovering the original foliation from the blowing-up process**

Let us give an orbifold structure of the weighted projective plane $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ (for a detailed description see [Mann E.1] pp 51). We can consider the sets

$$V_i = \{[x_0 : x_1 : x_2]_w \in \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]} : x_i \neq 0\} \subset \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$$

and the bijective maps ϕ_i from V_i to $\mathbb{C}^2/G_i$, where G_i is a finite group of biholomorphisms of $\mathbb{C}^2$. For the orbifold chart V_0, we have coordinates

(1) $(1, y_1, y_2)$,

where $y_1 = \dfrac{x_1}{x_0^{\left(\frac{\alpha}{\beta}\right)}}$ and $y_2 = \dfrac{x_2}{x_0^{\left(\frac{\alpha}{\gamma}\right)}}$. For the chart V_1 we have coordinates

(1) $(y_0, 1, y_2)$,

where $y_0 = \dfrac{x_0}{x_1^{\left(\frac{\beta}{\alpha}\right)}}$ and $y_2 = \dfrac{x_2}{x_1^{\left(\frac{\beta}{\gamma}\right)}}$. For V_2 we have coordinates

(1) $(y_0, y_1, 1)$

where $y_0 = \dfrac{x_0}{x_2^{\left(\frac{\gamma}{\alpha}\right)}}$ and $y_1 = \dfrac{x_1}{x_2^{\left(\frac{\gamma}{\beta}\right)}}$. Let us examine the restriction to $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ of the foliation obtained via the weighted blow-up in coordinates (x_0, x_1, x_2) in $\mathbb{C}^3$. For this purpose we only need to analyze one case:
Let us take the 1-form

$$\Sigma_0 = \pi_w^*\eta|_{y_0=0} = \beta y_2 \tilde{B}(1, y_1^\beta, y_2^\gamma)dy_1 + \gamma y_1 \tilde{C}(1, y_1^\beta, y_2^\gamma)dy_2$$

that defines the foliation in V_0. Consider the mapping $\pi_c : \mathbb{C}^3 \to V_0$ and the 1-form $\pi_c^*\Sigma_0$. In the coordinates on V_0 we have

$$d(y_1) = d\left(\frac{x_1}{x_0^{\left(\frac{\alpha}{\beta}\right)}}\right) = \frac{x_0^{\frac{\alpha}{\beta}+1}dx_1 - \frac{\alpha}{\beta}x_1 x_0^{\frac{\alpha}{\beta}}dx_0}{x_0^{2\frac{\alpha}{\beta}+1}}$$

and

$$d(y_2) = d\left(\frac{x_2}{x_0^{\left(\frac{\alpha}{\gamma}\right)}}\right) = \frac{x_0^{\frac{\alpha}{\gamma}+1}dx_2 - \frac{\alpha}{\gamma}x_2 x_0^{\frac{\alpha}{\gamma}}dx_0}{x_0^{2\frac{\alpha}{\gamma}+1}}.$$

By standard computations and using the relation $i_S\eta = 0$, we obtain that

$$\left(\pi_c^*\Sigma_0\right)(x_0, x_1, x_2) = \frac{1}{p(x_0, x_1, x_2)}\eta(x_0, x_1, x_2),$$

where $p(x_0, x_1, x_2) = y_0^{\alpha\beta\gamma(d-1)+\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}+1}$. Now we extract the factor $\frac{1}{p(x_0,x_1,x_2)}$ and recover the 1-form η as we wanted.

Remark 17. If we make the same procedure using another coordinate chart we obtain the same thing.

8.6. **Push-Forward**

Following [A.M.O-G], since our weighted projective plane is not well-formed we have an another natural orbifold map between $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ and $\mathbb{P}^2$ as follows:

$$\begin{aligned} f_w : \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]} &\to \mathbb{P}^2 \\ (x_0 : x_1 : x_2)_w &\to (x_0^\alpha : x_1^\beta : x_2^\gamma) = (X, Y, Z). \end{aligned}$$

The lifting of the map f_w can be seen as follows:

$$\begin{array}{ccc} \mathbb{C}^3\backslash\{0\} & \xrightarrow{\tilde{f_w}} & \mathbb{C}^3\backslash\{0\} \\ {\scriptstyle \Pi_{2_w}}\big\downarrow & & {\scriptstyle \Pi_2}\big\downarrow \\ \mathbb{P}_{[\beta\gamma,\alpha\gamma,\alpha\beta]} & \xrightarrow{f_w} & \mathbb{P}^2. \end{array}$$

We are denoting by Π_{2_w} the quotient mapping.

$$\begin{aligned} \tilde{f_w} : \mathbb{C}^3 &\rightarrow \mathbb{C}^3 \\ (x_0, x_1, x_2) &\rightarrow (x_0^\alpha, x_1^\beta, x_2^\gamma) = (X, Y, Z). \end{aligned}$$

The pushforward of the foliation in $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ to $\mathbb{P}^2$ is given as follows. We have that the foliation on $\mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$ is given by

$$\eta = i_S i_{\mathcal{Z}}(dx_0 \wedge dx_1 \wedge dx_2)$$

where

$$S = (\beta.\gamma)x_0\frac{\partial}{\partial x_0} + (\alpha.\gamma)x_1\frac{\partial}{\partial x_1} + (\alpha.\beta)x_2\frac{\partial}{\partial x_2}$$

and $\mathcal{Z}$ is given by

$$\mathcal{Z} = \mathcal{Z}_0(x_0, x_1, x_2)\frac{\partial}{\partial x_0} + \mathcal{Z}_1(x_0, x_1, x_2)\frac{\partial}{\partial x_1} + \mathcal{Z}_2(x_0, x_1, x_2)\frac{\partial}{\partial x_2}$$

where:

(a) $\mathcal{Z}_0(x_0, x_1, x_2) = x_0.\tilde{A}(x_0^\alpha, x_1^\beta, x_2^\gamma)$
(b) $\mathcal{Z}_1(x_0, x_1, x_2) = x_1.\tilde{B}(x_0^\alpha, x_1^\beta, x_2^\gamma)$
(c) $\mathcal{Z}_2(x_0, x_1, x_2) = x_2.\tilde{C}(x_0^\alpha, x_1^\beta, x_2^\gamma)$.

Remark 18. The polynomials $\tilde{A}(X, Y, Z)$, $\tilde{B}(X, Y, Z)$ and $\tilde{C}(X, Y, Z)$ are homogeneous of degree $(d - 1)$ and they are not unique!

Let us take the pushforward of the two vector fields S and $\mathcal{Z}$ under the mapping $\tilde{f_w}$:

$$(\tilde{f_w})_*(S)(p) = (D\tilde{f_w})(\tilde{f_w}^{-1}(p))S(\tilde{f_w}^{-1}(p))$$

where

$$(D\tilde{f_w})(\tilde{f_w}^{-1}(x_0^\alpha, x_1^\beta, x_2^\gamma)) = \begin{bmatrix} \alpha x_0^{\alpha-1} & 0 & 0 \\ 0 & \beta x_1^{\beta-1} & 0 \\ 0 & 0 & \gamma x_2^{\gamma-1} \end{bmatrix}$$

Hence

$$(\tilde{f_w})_*(S)(p) = (D\tilde{f_w})(\tilde{f_w}^{-1}(p))S(\tilde{f_w}^{-1}(p))$$

is the vector

$$\begin{bmatrix} \alpha\beta\gamma x_0^\alpha \\ \alpha\beta\gamma x_1^\beta \\ \alpha\beta\gamma x_2^\gamma \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} \alpha\beta\gamma X \\ \alpha\beta\gamma Y \\ \alpha\beta\gamma Z \end{bmatrix}.$$

Hence, it is a multiple of the radial vector field.

Applying the same process to the vector field $\mathcal{Z}$ we obtain

$$\begin{bmatrix} x_0^\alpha \tilde{A}(x_0^\alpha, x_1^\beta, x_2^\gamma) \\ x_1^\beta \tilde{B}(x_0^\alpha, x_1^\beta, x_2^\gamma) \\ x_2^\gamma \tilde{C}(x_0^\alpha, x_1^\beta, x_2^\gamma) \end{bmatrix} \tag{3}$$

which is equivalent to

$$\begin{bmatrix} X\tilde{A}(X,Y,Z) \\ Y\tilde{B}(X,Y,Z) \\ Z\tilde{C}(X,Y,Z) \end{bmatrix} \tag{4}$$

hence it is a homogeneous vector field. Now we can define a foliation on $\mathbb{P}^2$ using these two vectors obtained in the pushforward process.
In fact

$$i_{(\tilde{f_w})_*(S)} i_{(\tilde{f_w})_*(\mathcal{Z})} dX \wedge dY \wedge dZ$$

is equivalent to the determinant of the following matrix

$$\begin{bmatrix} dX & dY & dZ \\ X\tilde{A}(X,Y,Z) & Y\tilde{B}(X,Y,Z) & Z\tilde{C}(X,Y,Z) \\ \alpha\beta\gamma X & \alpha\beta\gamma Y & \alpha\beta\gamma Z \end{bmatrix}.$$

Hence we have the 1−form

$$YZ[\tilde{B}-\tilde{C}](X,Y,Z)dX + XZ[\tilde{C}-\tilde{A}](X,Y,Z)dY + XY[\tilde{A}-\tilde{B}](X,Y,Z)dZ$$

of degree $(d+1)$. Also note that

$$\{[\tilde{B}-\tilde{C}] + [\tilde{C}-\tilde{A}] + [\tilde{A}-\tilde{B}]\}(X,Y,Z) \equiv 0,$$

and therefore it is in $Il_3(d,2)$, moreover if we begin with $\mathcal{G} \in M_1(d,2)$ we re-obtain up to a linear automorphism of $\mathbb{P}^2$ a foliation in $M_1(d,2)$. This also holds for a foliation $\mathcal{G} \in \mathcal{A}$.

It follows from the previous discussions, that if we make a weighted punctual blow-up at a indeterminacy point with weights $(\beta\gamma, \alpha\gamma, \alpha\beta)$ on the quasi-homogeneous 1-form defining the holomorphic foliation on a neighborhood of a indeterminacy point of f. Then the strict transform of the foliation restricted to the exceptional divisor is (up to a linear automorphism of $\mathbb{P}^2 \simeq \mathbb{P}^2_{[\beta\gamma,\alpha\gamma,\alpha\beta]}$) the same foliation. Hence, if we begin with a foliation $\mathcal{G} \in M_1(d,2)$ we re-obtain (again, up to a linear automorphism of $\mathbb{P}^2$) a foliation in $M_1(d,2)$.

8.7. **Proof that the curves $V_\tau(t)$ are fibers of f_t.**

We have seen in proposition 7.18 that we can define a family of rational mappings $(f_t)_{t\in D_{\epsilon'}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$, in such a way that the singular curves $V_a(t)$, $V_b(t)$ and $V_c(t)$ of the foliation $\mathcal{F}_t$ are fibers of f_t for fixed t. Observe that, for ϵ' sufficiently small, $(f_t)_{t\in D_{\epsilon'}}$ is generic in the sense of definition 4.2, and its indeterminacy locus $I(f_t)$ is precisely $I(t)$. Moreover, since $Gen(3,\nu,\alpha,\beta,\gamma)$ is open, we can suppose that this family $(f_t)_{t\in D_{\epsilon'}}$ is in $Gen\,(3,\nu,\alpha,\beta,\gamma)$. In this section we will show that the remaining singular curves, that we have denoted by $V_\tau(t)$ also are fibers of f_t for fixed t. In fact, in the local coordinates $X(t)=(x_0(t),x_1(t),x_2(t))$ near some point of $I(t)$, where the vector field S is diagonal we have that the components of the map f_t are written as follows:

(1) $P_0(t)=u_{0t}x_0(t)+x_1(t)x_2(t)h_{0t}$

(2) $P_1(t)=u_{1t}x_1(t)+x_0(t)x_2(t)h_{1t}$

(3) $P_2(t)=u_{2t}x_2(t)+x_0(t)x_1(t)h_{2t}$

where the functions $u_{it}\in\mathcal{O}^*(\mathbb{C}^3,0)$ and $h_{it}\in\mathcal{O}(\mathbb{C}^3,0), 0\leq i\leq 2$.

Observe that when the parameter t goes to 0 the functions $h_i(t), 0\leq i\leq 2$ also goes to 0. We want to show that an orbit of the vector field S in the coordinate system $X(t)$ that extends globally like a singular curve of the foliation $\mathcal{F}_t$ is a fiber of f_t.

Observe that the conditon $\alpha<\beta<\gamma$ implies that $\alpha\gamma<\beta(\alpha+\gamma)$ and also $\alpha\beta<\gamma(\alpha+\beta)$.

We will prove first that if $\beta\gamma\leq\alpha(\beta+\gamma)$ then any generic orbit of the vector field S that extends globally as singular curve of the foliations $\mathcal{F}_t$ is also a fiber of f_t for fixed t. On the other hand, if we have the situation $\beta\gamma>\alpha(\beta+\gamma)$ then we will prove that any orbit of the vector field S that is contained in the coordinate planes that extends globally as singular curve of the foliations $\mathcal{F}_t$ are fibers of the mapping f_t. Using this fact, we can prove that any generic orbit of the vector field S that extends globally as singular curve of the foliations $\mathcal{F}_t$ is also a fiber of f_t in this case.

Lemma 8.4. *If $\beta\gamma\leq\alpha(\beta+\gamma)$ then any generic orbit of the vector field S that extends globally as singular curve of the foliations $\mathcal{F}_t$ is also a fiber of f_t for fixed t.*

To simplify, in the notation we will omit the index t.

Proof. Let us consider a generic orbit of the vector field S. We will denote it by $\delta(s)$ (here by a generic orbit we mean an orbit that is not contained in any coordinate plane). We can parametrize the orbit as

$$s\to(as^{\beta\gamma},bs^{\alpha\gamma},cs^{\alpha\beta}),a\neq 0,b\neq 0,c\neq 0.$$

Without loss of generality we can suppose that $a=b=c=1$. We have

$$f_t(\delta(s))=[(s^{\beta\gamma}u_0+s^{\alpha(\beta+\gamma)}h_0)^\alpha:(s^{\alpha\gamma}u_1+s^{\beta(\alpha+\gamma)}h_1)^\beta:(s^{\alpha\beta}u_2+s^{\gamma(\alpha+\beta)}h_2)^\gamma]$$

If we have the condition $\beta\gamma\leq\alpha(\beta+\gamma)$ this implies that we can extract the factor $s^{\alpha\beta\gamma}$ from $f_t(\delta(s))$ since we are considering projective coordinates.

Hence we obtain

$$f_t(\delta(s))=[(u_0+s^kh_0)^\alpha:(u_1+s^lh_1)^\beta:(u_2+s^mh_2)^\gamma](**)$$

where $k=\alpha(\beta+\gamma)-\beta\gamma$, $l=\beta(\gamma+\alpha)-\alpha\gamma$ and $m=\gamma(\alpha+\beta)-\alpha\beta$.

Observe that V_τ is a fiber and so $f_0(V_\tau)=[d:e:f]\in\mathbb{P}^2$ where $d\neq 0,e\neq 0,f\neq 0$. If we take a covering of $I(f)=\{p_1,...,p_{\frac{\nu^3}{\alpha\beta\gamma}}\}$ by small open balls $B_j(p_j)$, $1\leq j\leq\frac{\nu^3}{\alpha\beta\gamma}$ the set $V_\tau\backslash\cup_j B_j(p_j)$ is compact. For a small deformation f_t of f_0 then $f_t[V_\tau(t)\backslash\cup_j B_j(p_j)(t)]$

stays near to $f[V_\tau \setminus \cup_j B_j(p_j)]$, and hence for t sufficiently small the components of the previous expression (**) does not vanish and when we are outside of the neighborhood $\cup_j B_j(p_j)(t)$ the components of f_t also does not vanish.

This implies that the components of f_t do not vanish along each generic fiber that extends locally as a singular curve of the foliation $\mathcal{F}_t$. This is possible only if f_t is constant along this curves. In fact, $f_t(V_\tau(t))$ is either a curve or a point. If it is a curve then it cuts all lines of $\mathbb{P}^2$ and therefore the components should be zero somewhere.

Hence $f_t(V_\tau(t))$ is constant and we conclude that $V_\tau(t)$ is a fiber.

Observe also that when we make a blow-up with weights $(\beta\gamma, \alpha\gamma, \alpha\beta)$ at the points of $I(f_t)$ we solve completely the indeterminacy points of the mappings f_t in the case $\beta\gamma \leq \alpha(\beta+\gamma)$ for each t. □

When $\beta\gamma > \alpha(\beta+\gamma)$ the situation is more difficult. Let us suppose that the orbits that are contained in the coordinate planes that extends globally as singular curves of the foliation $\mathcal{F}_t$ are fibers of f_t. This fact will be proved at the **Lemma** 8.5.
To simplify the argumentation let us suppose also that the numbers g.c.d$(\alpha,\beta) = 1$, g.c.d$(\alpha,\gamma) = 1$ and g.c.d$(\gamma,\beta) = 1$. The general case is similar.

We can assume without loss of generality that this orbit is contained in the coordinate plane $x_0(t) = 0$ and we will use the hypothesis g.c.d$(\gamma,\beta) = 1$.

In this case the orbit is of the form $(x_0 = x_1^\beta - cx_2^\gamma = 0)$. Since we are assuming by hypothesis that the previous curve is a fiber of the mapping f_t we have that the germ of $f_{0,t}$ at the point belongs to the ideal generated by $x_0(t)$ and $(x_1^\beta - cx_2^\gamma)(t)$ hence we can write the function h_{0t} as follows:

$$h_{0t} = x_0(t)h_{01t} + (x_1^\beta(t) - cx_2^\gamma(t))h_{02t},$$

where $h_{01t}, h_{02t} \in \mathcal{O}_2$. Hence we can repeat the argument at the first situation, and then we can extract the factor $s^{\alpha\beta\gamma}$. In fact, making the computations we have:

$$f_t(\delta(s)) = [(s^{\beta\gamma}[u_0 + s^{\alpha(\beta+\gamma)}(h_{01} + s^{\alpha\beta\gamma})(1-c)h_{02}])^\alpha : s^{\alpha\beta\gamma}(u_1 + s^l h_1)^\beta : s^{\alpha\beta\gamma}(u_2 + s^m h_2)^\gamma]$$

where $l = \beta(\gamma+\alpha) - \alpha\gamma$ and $m = \gamma(\alpha+\beta) - \alpha\beta$.
Hence we can extract from the previous expression the term $s^{\alpha\beta\gamma}$ obtaining the term

$$f_t(\delta(s)) = ([u_0 + s^{\alpha(\beta+\gamma)}(h_{01} + s^{\alpha\beta\gamma})(1-c)h_{02}])^\alpha : (u_1 + s^l h_1)^\beta : (u_2 + s^m h_2)^\gamma]$$

In this way we can proceed the argumentation as at the end of the first situation. We conclude that $V_\tau(t)$ is also a fiber when we have $\beta\gamma > \alpha(\beta+\gamma)$.

□

Lemma 8.5. *If $\beta\gamma > \alpha(\beta+\gamma)$ then any orbit of the vector field S that is contained in some coordinate plane at $p_j(t)$ where the vector field S is linear, that extends globally as a singular curve of the foliations $\mathcal{F}_t$ is a fiber of the mapping f_t for fixed t.*

Denote $(f_t)_{t\in D_{\epsilon'}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ by $f_t = [P_0^\alpha(t) : P_1^\beta(t) : P_2^\gamma(t)]$. As previously, let us consider an orbit of the vector field S on a small neighborhood of an indeterminacy point of f_t, $B_j(p_j(t))$, $1 \leq j \leq \frac{\nu^3}{\alpha\beta\gamma}$ and denote by $V_\tau(t)$ the global extension of this orbit to $\mathbb{P}^3$. Without loss of generality we can assume that the orbit is contained in the plane $(x_0(t) = 0)$ and we can suppose that it can be parametrized as

$$s \to (0, s^\gamma, s^\beta).$$

To simplify in the notation we will omit the index t in some expressions. After evaluating the mapping f_t on this orbit, on a neighborhood of $p_j(t)$ we obtain:

$$f_t(\delta(s)) = [s^{\alpha(\beta+\gamma)}h_0^\alpha : s^{\beta\gamma}u_1^\beta : s^{\beta\gamma}u_2^\gamma].$$

This can be written as

$$[s^{\alpha(\beta+\gamma)}\tilde{h_0} : s^{\beta\gamma}u_1^\beta : s^{\beta\gamma}u_2^\gamma] = [X(s) : Y(s) : Z(s)] \tag{5}$$

and we will initially prove that $f_t(V_\tau(t))$ is contained in a line of the form $(Y - \lambda Z = 0)$ of $\mathbb{P}^2$. Let us consider the meromorphic function in $B_j(p_j(t))$ given by $g_t(s) = \frac{Z(s)}{Y(s)} = \frac{u_2^\gamma}{u_1^\beta}$. When $s \to 0$ this function goes to a constant $\lambda \neq 0, \lambda \neq \infty$. Observe that for small t the function $\frac{P_1^\beta}{P_2^\gamma}(t) : V_\tau(t)\setminus \cup_j B_j(p_j(t))$ stays near $\frac{P_1^\beta}{P_2^\gamma}(0) : V_\tau(0)\setminus \cup_j B_j(p_j(0))$ which also does not vanish because $V_\tau(0)$ is a fiber. We conclude that $f_t(V_\tau(t)) \subset (Y - \lambda Z = 0) \simeq \mathbb{P}^1$.

If $\beta\gamma > \alpha(\beta+\gamma)$ we can write eq.(5) as

$$[\tilde{h}_0(s) : s^m u_1^\beta : s^m u_2^\gamma]$$

where $m = \beta\gamma - \alpha(\beta+\gamma)$.
Observe when $s = 0$ that the function $\tilde{h}_0(s)$ could vanish and in this case such a point corresponds to a indeterminacy point $p_j(t)$ of f_t for some j. At $p_j(t)$ we can write the first component of eq.(5) as $\tilde{h}_0(s) = s^{\rho_j}\tilde{h}_j(s)$ where $\tilde{h}_j(s) \in \mathcal{O}^*(\mathbb{C}, 0)$ or $\tilde{h}_0 \equiv 0$ but in this case we are done, that is $V_\tau(t)$ is a fiber of f_t.

At a point $p_j(t)$ we have two possibilities:

First case: if $\rho_j < m$.
In this case we can write the expression in eq.(5) as:

$$[\tilde{h}_j(s) : s^{m-\rho_j}u_1^\beta : s^{m-\rho_j}u_2^\gamma] \tag{6}$$

and if $s \to 0$ the image goes to $[1 : 0 : 0]$ which implies that $f_t|_{V_\tau(t)}(p_j(t)) = [1 : 0 : 0]$.

Second case, if $\rho_j \geq m$ we can write the expression in eq.(6) as:

$$[s^{\rho_j - m}\tilde{h}_j(s) : u_1^\beta : u_2^\gamma] \tag{7}$$

and if $s \to 0$ the image goes to $[a : \lambda : 1]$ where $a \in \mathbb{C}$. This is because the image of such a point belongs to the curve $(Y - \lambda Z = 0) \simeq \mathbb{P}^1$ and we can write it as $[a : \lambda : 1]$.

Indeed, let us suppose that $f_t|_{V_\tau(t)}$ is not constant and consider the mapping $f_t|_{V_\tau(t)} : V_\tau(t) \to f_t(V_\tau(t)) \subset (Y - \lambda Z = 0)$ for fixed t.

Denote $A = \{j|\rho_j < m\}$ and observe that $p \in V_\tau(t)$ and $f_t|_{V_\tau(t)}(p) = [1 : 0 : 0]$ implies that $p = p_j(t)$ for some $j \in A$; that is $(f_t|_{V_\tau(t)})^{-1}[1 : 0 : 0] = \{p_j(t), j \in A\}$. Moreover, by eq.(6) we have $mult(f_t|_{V_\tau(t)}, p_j(t)) = m - \rho_j$. In particular, the degree of $f_t|_{V_\tau(t)}$ is

$$degree(f_t|_{V_\tau(t)}) = \sum_j (m - \rho_j)$$

On the other hand, if $p \in (f_t|_{V_\tau(t)})^{-1}[0 : \lambda : 1]$ then $(P_0^\alpha(p) = 0)$ and so $mult(f_t|_{V_\tau(t)}, p) =$ the intersection number of $(P_0^\alpha(t) = 0)$ with $V_\tau(t)$ at p. Hence

$$degree(f_t|_{V_\tau(t)}) = V_\tau(t).P_0^\alpha(t) = deg(V_\tau(t)) \times deg(P_0^\alpha(t)) = \frac{\nu^3}{\alpha} = \sum_j (m - \rho_j)$$

But, $(m - \rho_j) \leq m = \beta\gamma - \alpha(\beta+\gamma)$ and so

$$\sum_{j\in A}(m-\rho_j) \leq \#A \times m \leq \frac{\nu^3}{\alpha\beta\gamma} \times (\beta\gamma - \alpha(\beta+\gamma)) = \nu^3(\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma})$$

which implies that

$$\frac{1}{\alpha} \leq \frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma},$$

a contradiction. Therefore, $A = \emptyset$ and $f_t|_{V_\tau(t)}$ is a constant and $V_\tau(t)$ is a fiber of f_t. □

8.8. **Solving the indeterminacy of f_t**

We saw that in a neighborhood of a point of indeterminacy of the map $f_t : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ most of its fibers are locally the orbits of the vector field S. Moreover, many of these orbits are singular with the exception of three. Due to this fact a standard punctual blow-up does not solve the indeterminacy point of the map f_t. Since the orbits are generally cusp an usual blow-up would produce tangencies between the strict transform of this curves with exceptional divisor. We will proceed locally, that is at each point $p_j(t)$. In fact, in the local coordinates $X(t) = (x_0(t), x_1(t), x_2(t))$ near some point of $I(t)$, where the vector field S is diagonal we have that the components of the map f_t are written as follows:

(1) $P_0(t) = u_{0t}x_0(t) + x_1(t)x_2(t)h_{0t}$

(2) $P_1(t) = u_{1t}x_1(t) + x_0(t)x_2(t)h_{1t}$

(3) $P_2(t) = u_{2t}x_2(t) + x_0(t)x_1(t)h_{2t}$

where the functions $u_{it} \in \mathcal{O}^*(\mathbb{C}^3, 0)$ and $h_{it} \in \mathcal{O}(\mathbb{C}^3, 0), 0 \leq i \leq 2$.

For each fixed t we can write the mapping f_t as

$$f_t = [u_{0t}x_0(t) + x_1(t)x_2(t)h_{0t})^\alpha : (u_{1t}x_1(t) + x_0(t)x_2(t)h_{1t})^\beta : (u_{2t}x_2(t) + x_0(t)x_1(t)h_{2t})^\gamma]$$

As we will see it is sufficient to do the blowing-up only in one chart. Let us take the first one. In local coordinates as described in section 8.4 the weighted blow-up with weights $(\beta\gamma, \alpha\gamma, \alpha\beta)$ can be written as:

(1) $x_0(t) = y_0(t)^{\beta\gamma}$

(2) $x_1(t) = y_0(t)^{\alpha\gamma}y_1(t)$

(3) $x_2(t) = y_0(t)^{\alpha\beta}y_2(t)$

To simplify the notation we will omit the parameter t from the next expressions. Observe that the condition $\alpha < \beta < \gamma$ implies that $\alpha\gamma < \beta(\alpha+\gamma)$ and also $\alpha\beta < \gamma(\alpha+\beta)$.

We will prove first that if $\beta\gamma \leq \alpha(\beta+\gamma)$ we can solve completely the indeterminacy set of f_t.

In fact, after standart computations we obtain

$$\pi_w(t)^* f_t = y_0^{\alpha\beta\gamma}[(u_0 + y_1y_2y_0^{2\alpha}h_0)^\alpha : (u_1y_1 + y_0^{\beta\gamma}y_2h_1)^\beta : (u_2y_2 + y_0^{2\gamma}y_1h_2)^\gamma](**).$$

If we have the condition $\beta\gamma \leq \alpha(\beta+\gamma)$ this implies that we can extract the factor $y_0^{\alpha\beta\gamma}$ from $\pi_w(t)^* f_t$ since we are considering projective coordinates. If we make this process at the $\frac{\nu^3}{\alpha\beta\gamma}$ indeterminacy points of f_t for each t we can solve completely the indeterminacy set of f_t. And we are done.

When $\beta\gamma > \alpha(\beta+\gamma)$ the situation is more difficult. In the previous section we have seen that the orbits that are contained in the coordinate planes that extends globally as singular curves of the foliation $\mathcal{F}_t$ are fibers of f_t. This fact was proved at the **Lemma** 8.5.

To simplify the argumentation let us suppose also that the numbers g.c.d$(\alpha,\beta) = 1$, g.c.d$(\alpha,\gamma) = 1$ and g.c.d$(\gamma,\beta) = 1$. The general case is similar. We can assume without loss of generality that this orbit is contained in the coordinate plane $x_0(t) = 0$ and we will use the hypothesis g.c.d$(\gamma,\beta) = 1$. In this case the orbit is of the form $(x_0 = x_1^\beta - cx_2^\gamma = 0)$.

Since we know that the previous curve is a fiber of the mapping f_t we have that the germ of $f_{0,t}$ at the point belongs to the ideal generated by $x_0(t)$ and $(x_1^\beta - cx_2^\gamma)(t)$ hence we can write the function h_{0t} as follows:

$$h_{0t} = x_0(t)h_{01t} + (x_1^\beta(t) - cx_2^\gamma(t))h_{02t},$$

where $h_{01t}, h_{02t} \in \mathcal{O}_2$. Hence we can repeat the argument at the first situation, and then we can extract the factor $y_0^{\alpha\beta\gamma}$. In fact, making the computations we have:

$$\pi_w(t)^* f_t = y_0^{\alpha\beta\gamma}[(u_0 + y_1y_2y_0^{2\alpha}[y_0h_{01} + y_0^{\alpha\beta\gamma}(y_1^\beta - cy_2\gamma))^\alpha : (u_1y_1 + y_0^{\beta\gamma}y_2h_1)^\beta : (u_2y_2 + y_0^{2\gamma}y_1h_2)^\gamma](**).$$

This implies that we can extract the factor $y_0^{\alpha\beta\gamma}$ from $\pi_w(t)^* f_t$ since we are considering projective coordinates. If we make this process at the $\frac{\nu^3}{\alpha\beta\gamma}$ indeterminacy points of f_t for each t we can solve completely the indeterminacy set of f_t. And we are done.

8.9. **Extension Theorem**

For complex spaces and orbifolds, holomorphic functions can often be extended to larger open sets. This is the content of the Riemann Removable Singularity Theorem for orbifolds and complex spaces.

Theorem 8.6. *Let V be a normal complex space and $A \subset V$ an analytic subset of codimension at least 2 in every point. Then every holomorphic function in $V \backslash A$ has a unique holomorphic extension to V.*

This result can be found in [G.M1] page 126.

References

[A.M.O-G] E. Artal, J. Martín-Morales, and J. Ortigas-Galindo, Intersection theory on abelian-quotient V -surfaces and Q-resolutions Preprint, 2011.

[A.M.O-G1] E. Artal, J. Martín-Morales, and J. Ortigas-Galindo, Cartier and Weil divisors on varieties with quotient singularities Preprint available at arXiv:1104.5628 [math.AG], 2011.

[Bea] A.Beauville. Complex Algebraic Surfaces. London Mathematical Society - Lecture Note Series 68 - Cambridge University Press, 1983.

[B.R] M. Beltrametti, L. Rabbiano Introduction to the theory of weighted projective spaces. *Expo. Math.*, n^{o}4, 111-162, (1986).

[Bru] M.Brunella. Birational geometry of foliations. Monografias de Matemática. [Mathematical Monographs]. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, (2000).

[C.A] O.Calvo-Andrade. Irreducible components of the space of holomorphic foliations. *Math. Ann.,* 299, n^{o}4, 751-767, (1994).

[C.A1] O.Calvo-Andrade. Positivity, vanishing theorems and rigidity of codimension one holomorphic foliations, *Annalles de la Facult é des Sciences de Tolouse.* 299, n^{o}Vol XVIII 4, 811-854, (2009).

[C.CA.G.LN] O.Calvo-Andrade, D.Cerveau, L.Giraldo, A.Lins Neto. Irreducible components of the space of foliations associated to the affine Lie algebra. *Ergodic Theory Dynam. Systems* 24, n^{o}4, 987-1014, (2004).

[Ca.Ln] C.Camacho, A.Lins Neto. The topology of integrable differential forms near a singularity. *Inst. Hautes Études Sci. Publ. Math.* n^{o}55, 5-35, (1982).

[Cartan.H] H. Cartan Quotient d'une espace analytique par un group d'automorphismes. *in Algebraic Geometry and Topology, Princeton Mathematical Series 12, Princeton 1957* 90-102.

[C.Ln0] D.Cerveau, A.Lins-Neto. Holomorphic foliations in $\mathbb{P}^2$ having an invariant algebraic curve. *Ann. Inst. Fourrier* 41, 883-903, (1991).

[C.Ln] D.Cerveau, A.Lins-Neto. Irreducible components of the space of holomorphic foliations of degree two in $\mathbb{P}^n$. *Annals of Mathematics* 143, 577-612, (1996).

[C.Ln1] D.Cerveau, A. Lins-Neto. Codimension one Foliations in $\mathbb{P}^n$, $n \geq 3$, with Kupka Components. *Astérisque* 224, (1994).

[C.Ln2] D.Cerveau, A. Lins-Neto. A structural theorem for codimension one foliations on $\mathbb{P}^n$, $n \geq 3$. with an application to degree three foliations, *To appear in Ann. Scuola Norm. Sup. Pisa Cl. Sci*

[C.Ln3] D. Cerveau and A. Lins Neto. Formes tangentes à des actions commutatives. *Ann. Fac. Sci. Toulouse 6 (1984), 51-85.*

[C.Ln.E] D.Cerveau, A.Lins-Neto, S.J. Edixhoven. Pull-back components of the space of holomorphic foliations on $\mathbb{CP}^n$, $n \geq 3$ *J. Algebraic Geom.,* 10, n^{o}4, 695-711, (2001).

[C.Ln.L.P.T] D.Cerveau, A.Lins-Neto, F.Loray, J.V.Pereira, F.Touzet. Algebraic reduction theorem for complex codimension one singular foliations, *Comment. Math. Helv.* 81, n^{o}1, 157-169, (2006).

[C.M] D.Cerveau, J.F.Mattei. Formes intègrables holomorphes singulières, *Astérisque* 97, (1982).

[Dim] Dimca, Alexandru. Singularities and Topology of Hypersurfaces. *Springer-Verlag*, New York, (1992) Univerexts in Mathematics.

[Fa.Pe] C.Favre, J.V.Pereira. Foliations invariant by rational maps. To appear in $Mathematische$ $Zeitschrift.$

[G.M] X.Gomez-Mont. Foliations by curves of complex analytic spaces.; Contemporary Mathematics vol 58, Part III 1987, 123-141.

[G.M1] X.Gomez-Mont. Integrals for holomorphic foliations with singularities having all leaves compact. ; Lect.Notes in Math. No. 1345, 129-162. *Ann. Inst. Fourier (Grenoble)* Vol. 39 (1989), no. 2, 451?458.

[G.Ln] X.Gomez-Mont, A.Lins Neto. Structural stability of singular holomorphic foliations having a meromorphic first integral. *Topology,* 30, n^{o}3, 315-334, (1991).

[Gri.Har] P.Griffiths, J.Harris. Principles of algebraic geometry. *Wiley Classics Library.* John Wiley & Sons Inc., New York, (1994). Reprint of the 1978 original.

[Har] R.Hartshorne. Algebraic geometry. *Springer-Verlag*, New York, (1977). Graduate Texts in Mathematics, No. 52.

[Hi] Morris W. Hirsch. Differential Topology. *Springer-Verlag*, New York, (1976). Graduate Texts in Mathematics, No. 33.

[Jou] J.P. Jouanolou. quations de Pffaf algèbriques,. *Lect. Notes in Math.*, 708, (1979).

[K] I.Kupka. The singularities of integrable structurally stable Pfaffian forms, Proc. Nat. Acad. Sci. U.S.A, 52, 1431-1432, (1964).

[Ln] A.Lins Neto; Componentes Irredutíveis dos Espaços de Folheações 26 Colóquio Brasileiro de Matemática. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, (2007).

[Ln0] A.Lins Neto, Finite determinacy of germs of integrable 1-forms in dimension 3 (a special case) *Geometric Dynamics, Springer Lect. Notes in Math.* Vol. 1007, No. 6, pp. 480-497, (1981).

[Ln1] A.Lins Neto. Algebraic solutions of polynomial differential equations and foliations in dimension two; Lect.Notes im Math. No. 1345, 192-231. *Holomorphic Dynamics in Springer Lect. Notes in Math.* Vol. 1345, 480-497, (1988).

[Ln.Sc] A.Lins Neto, B.A.Scárdua. Folheações Algébricas Complexas. *21º Colóquio Brasileiro de Matemática*, IMPA (1997).

[Ln.S.Sc] A.Lins Neto, P.Sad, B.A.Scárdua. On Topological Rigidity of Projective Foliations. *Bulletin de la Société Mathématique de France.* vol. 126, p. 381-406, (1998).

[Ma-Mor] J. Martín-Morales. Monodromy Zeta Function Formula for Embedded Q-Resolutions Preprint, 2011.

[Mann E.] Mann E. Orbifold quantum cohomology of weighted projective spaces. J. Algebraic Geom. 17 (2008), 137-166.

[Mann E.1] Mann E. Cohomologie quantique orbifold des espaces projectifs á poids. arXiv:math.AG/0510331 v1 16 Oct 2005

[Pan] Panazzolo,D. Resolution of Singularities of Vector Fields in Dimension Three. *Acta Math. 197 (2006), no. 2, 167289.*

[Prill] Prill, D. Local Classification of quotients of complex manifolds by discontinuous Groups Duke Math. Jr vol. 34 pp 375-386, (1967).

[Rei] Reid, M. Graded rings and varieties in weighted projective space 1-14, (Jan 2002).

[Sat] Satake, I. On a generalization of the notion of manifold Proc. Nat. Acad. Sci. U.S.A. 42, 359-363, (1956)

[Sc] Scárdua, B. A. Transversely affine and transversely projective holomorphic foliations Ann. Sci. École Norm. Sup. (4) 30 (1997), no. 2. pp. 169-204.

[Seb] M.Sebastiani. Introdução à Geometria Analítica Complexa Projeto Euclides. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, (2004).

[Ser] E.Sernesi. Small Deformations of Global Complete Intersections Bolletino U.M.I., 4, 12 138-146, (1975).

[Ser1] E.Sernesi. Deformations of Algebraic Schemes Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol 334. Springer-Verlag, Berlin, 2006. xii+339 pp

[So.Cor] M. G. Soares, M. Corrêa J.R.. A note on Poincaré Problem for quasi-homogeneous Foliations. *To appear in Proceedings of The American Mathematical Society.*

[So.Mo] M. G. Soares, R. S. Mol. Índices de Campos Holomorfos e Aplicações . *23 º Colóquio Brasileiro de Matemática*, IMPA (2001).

[Weh] Wehler, Joachim. Deformation of complete intersections with singularities. *Mathematische Zeitschrift,* 4, 473-491, (1982).