Doctoral Thesis

Instituto de Matemática Pura e Aplicada
IMPA

Partial Sums of the Random Möbius Function

Marco Aymone

October 2013

Doctoral Thesis

Instituto de Matemática Pura e Aplicada
IMPA

Partial Sums of the Random Möbius Function

Marco Aymone

Thesis presented to the Post-graduate Program in Mathematics at Instituto de Matemática Pura e Aplicada as partial fulfillment of the requirements for the degree of Doctor in Mathematics.

Advisor: Vladas Sidoravicius.

The probabilistic way of thinking is a major contribution, which has reshaped the way of reasoning in many areas of mathematics and beyond.

Prof. Noga Alon
Erdős Centennial

Agradecimentos

Aos meus pais, irmãos e Flávia, por todo apoio e paciência mais do que suficientes.

Ao meu orientador e amigo Vladas Sidoravicius, pela amizade, pelos ensinamentos e por todo apoio nesses 6 anos.

Aos Professores e amigos Alexandre Baraviera, Carlos Gustavo (Gugu), Roberto Imbuzeiro e Sandra Prado.

Aos colegas e amigos, pela amizade e pelo apoio: Bruno, Eric, Guilherme, Lucas Ambrosio, Marcelo Hilario, Mauricio, Ricardo Misturini, Ricardo Turolla, Robertinho e Susana.

Abstract

To each prime number $p \in \mathcal{P}$ assign a random variable X_p taking values on the set $\{-1, 1\}$ and denote this sequence by $X = (X_p)_{p\in\mathcal{P}}$. We define the Random Möbius function μ_X to be the multiplicative function with support on the square free naturals such that at each prime p, $\mu_X(p) = X_p$. A Theorem due to A. Wintner [19] states that if the sequence $X = (X_p)_{p\in\mathcal{P}}$ is of independent and identically distributed random variables such that X_p has equal probability to be $+1$ and -1 then the partial sums of the Random Möbius function satisfies with probability one:

$$\mu_X(1) + ... + \mu_X(N) \ll N^{1/2+\varepsilon},\ \forall\varepsilon > 0,$$

and in [12] the term $\ll N^{1/2+\varepsilon}$ was improved to $\ll \sqrt{N}(\log\log N)^{3/2+\varepsilon}$ for all $\varepsilon > 0$. In this work we study the effect of the condition $\mathbb{P}(X_p = -1) > \mathbb{P}(X_p = +1)$ on the partial sums of the Random Möbius function.

Theorem. Assume that the random variables $X = (X_p)_{p\in\mathcal{P}}$ are independent and such that $\mathbb{P}(X_p = -1) = \frac{1}{2} + \frac{1}{2p^\alpha}$, for $\alpha > 0$. Denote by $S_\alpha(N) := \sum_{k=1}^{N} \mu_X(k)$. Then the Riemann Hypothesis is equivalent to

$$\lim_{\alpha\to 0} \mathbb{P}(S_\alpha(N) \ll N^{1/2+\varepsilon}, \forall\varepsilon > 0) = 1.$$

Moreover, we prove that the Random Möbius function as in this Theorem correlates with the one associated to the Wintner's model. Also we prove the following results:

Theorem ($\lim_{p\to\infty} \mathbb{P}(X_p = -1) = 1/2$)**.** Assume that the random variables $X = (X_p)_{p\in\mathcal{P}}$ are independent and such that $\mathbb{P}(X_p = -1) = \frac{1}{2} + \frac{\delta_p}{2}$. Let Δ be the event

$$\Delta = [\mu_X(1) + ... + \mu_X(N) \ll N^{1/2+\varepsilon},\ \forall\varepsilon > 0],$$

then:

i. $\sum_{p\in\mathcal{P}} \frac{\delta_p}{\sqrt{p}} < \infty$ implies that $\mathbb{P}(\Delta) = 1$;
ii. $\delta_p \to 0$ and $\frac{1}{p^\epsilon} \ll \delta_p$, for all $\epsilon > 0$ implies that $\mathbb{P}(\Delta) = 0$.

Theorem ($\lim_{p\to\infty} \mathbb{P}(X_p = -1) = 1$)**.** Assume that the random variables $X = (X_p)_{p\in\mathcal{P}}$ are independent and such that $\mathbb{P}(X_p = -1) = 1 - \delta_p$. Assume that for some $\alpha \in (0, 1/2)$, $\frac{1}{p^\alpha} \ll \delta_p$. Then

$$\mathbb{P}(\mu_X(1) + ... + \mu_X(N) \ll N^{1/2+\varepsilon}, \forall\varepsilon > 0) = 0.$$

Keywords: Random Multiplicative Functions, Probabilistic Number Theory.

Resumo

A cada número primo $p \in \mathcal{P}$ associe uma variável aleatória X_p tomando valores no conjunto $\{-1, 1\}$ e denote esta sequência por $X = (X_p)_{p \in \mathcal{P}}$. Definimos a Função de Möbius aleatória μ_X como sendo a função multiplicativa com suporte nos números livres de quadrados tal que a cada número primo p, $\mu_X(p) = X_p$. Um Teorema provado por A. Wintner [19] afirma que se a sequência $X = (X_p)_{p \in \mathcal{P}}$ é constituída por variáveis aleatórias independentes e identicamente distribuídas tais que cada X_p possui igual probabilidade de assumir $+1$ e -1 então as somas parciais da Função de Möbius aleatória satisfazem com probabilidade um:

$$\mu_X(1) + ... + \mu_X(N) \ll N^{1/2+\varepsilon},\ \forall \varepsilon > 0.$$

Recentemente, em [12] o termo $\ll N^{1/2+\varepsilon}$ no Teorema de Wintner foi melhorado para $\ll \sqrt{N}(\log \log N)^{3/2+\varepsilon}$ para todo $\varepsilon > 0$. Neste trabalho estudamos o efeito da condição $\mathbb{P}(X_p = -1) > \mathbb{P}(X_p = +1)$ nas somas parciais da Função de Möbius aleatória.

Teorema. Assuma que as variáveis aleatórias $X = (X_p)_{p \in \mathcal{P}}$ são independentes e tais que $\mathbb{P}(X_p = -1) = \frac{1}{2} + \frac{1}{2p^\alpha}$, para $\alpha > 0$. Denote $S_\alpha(N) := \sum_{k=1}^{N} \mu_X(k)$. Então a Hipótese de Riemann é equivalente a:

$$\lim_{\alpha \to 0} \mathbb{P}(S_\alpha(N) \ll N^{1/2+\varepsilon}, \forall \varepsilon > 0) = 1.$$

Além disso, provamos que a Função de Möbius aleatória como nesse Teorema correlaciona com a Função de Möbius aleatória associada ao modelo de Wintner. Também provamos os seguintes resultados:

Teorema ($\lim_{p \to \infty} \mathbb{P}(X_p = -1) = 1/2$)**.** Assuma que as variáveis aleatórias $X = (X_p)_{p \in \mathcal{P}}$ são independentes e tais que $\mathbb{P}(X_p = -1) = \frac{1}{2} + \frac{\delta_p}{2}$. Seja Δ o evento

$$\Delta = [\mu_X(1) + ... + \mu_X(N) \ll N^{1/2+\varepsilon},\ \forall \varepsilon > 0],$$

então:

i. $\sum_{p \in \mathcal{P}} \frac{\delta_p}{\sqrt{p}} < \infty$ implica que $\mathbb{P}(\Delta) = 1$;

ii. $\delta_p \to 0$ e $\frac{1}{p^\epsilon} \ll \delta_p$, para todo $\epsilon > 0$ implica que $\mathbb{P}(\Delta) = 0$.

Teorema ($\lim_{p \to \infty} \mathbb{P}(X_p = -1) = 1$)**.** Assuma que as variáveis aleatórias $X = (X_p)_{p \in \mathcal{P}}$ são independentes e tais que $\mathbb{P}(X_p = -1) = 1 - \delta_p$. Assume que para algum $\alpha \in (0, 1/2)$, $\frac{1}{p^\alpha} \ll \delta_p$. Então

$$\mathbb{P}(\mu_X(1) + ... + \mu_X(N) \ll N^{1/2+\varepsilon}, \forall \varepsilon > 0) = 0.$$

Keywords: Funções Multiplicativas Aleatórias, Teoria Probabilistica dos Números.

Contents

1 Introduction.

Let $\mu : \mathbb{N} \to \{-1, +1\}$ be the Möbius function, that is, $\mu(1) = 1$, $\mu(k) = (-1)^l$ if k is the product of l distinct primes and $\mu(k) = 0$ otherwise. The asymptotic behavior of the Mertens function $M : \mathbb{N} \to \mathbb{R}$ which is given by the sum $M(N) := \sum_{k=1}^{N} \mu(k)$ is of great importance in Analytic Number Theory. For example, the Prime Number Theorem is equivalent to $M(N) = o(N)$ and the Riemann Hypothesis is equivalent to $M(N) = o(N^{1/2+\varepsilon})$ for all $\varepsilon > 0$. Moreover, $M(N) = o(N^{1/2+\alpha})$ for some $\alpha > 0$ implies that the zeros of the Riemann zeta function have real part less or equal to $1/2 + \alpha$. In particular, the exponent $1/2$ inside this o-term is *optimal*, since the Riemann zeta function has many zeros with real part equals to $1/2$. The order $M(N) = o(N^{1/2+\alpha})$ for any $\alpha < 1/2$ still unknown up to date.

It is interesting to observe that the sum of independent random variables $(Y_k)_{k\in\mathbb{N}}$ assuming -1 or $+1$ with equal probability exhibits the behavior expected for $M(N)$ under the Riemann Hypothesis. Indeed, denoting $S_Y(N) = \sum_{k=1}^{N} Y_k$, with probability 1 one has $S_Y(N) = o(N^{1/2+\varepsilon})$, for all $\varepsilon > 0$ and also that the exponent $1/2$ is optimal.

This suggests a probabilistic approach to the Möbius function. It is necessary to observe that μ has multiplicative "dependencies", that is, for n and m naturals without primes in common, the knowledge of the values $\mu(n)$ and $\mu(m)$ implies the knowledge of $\mu(n \cdot m)$ which is equal to $\mu(n) \cdot \mu(m)$. A arithmetic function possessing this property is called multiplicative function. Thus, as observed by P. Lévy [13], independent random variables can not reproduce this effect. This motivates the following definitions:

Definition 1.1. Let $\mathcal{P}$ be the set of the prime numbers, μ the Möbius function and $X = (X_p)_{p\in\mathcal{P}}$ a sequence of independent random variables assuming values on $\{-1, 1\}$ defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The Random Möbius Function associated with X is the arithmetic random function given by

$$\mu_X(1) = 1,$$
$$\mu_X(n) = |\mu(n)| \prod_{p|n} X_p.$$

Definition 1.2. The random Mertens function is the arithmetic random function M_X which is given by $M_X(N) = \sum_{k=1}^{N} \mu_X(k)$.

1.1 Random Möbius Function: An Historical Overview.

The Wintner's model. The Definition 1.1 with a different terminology[1] was firstly introduced by A. Wintner [19]. Assuming that the random variables are distributed as $\mathbb{P}(X_p = -1) = \mathbb{P}(X_p = +1) = 1/2$, A. Wintner proved that the Random Mertens function satisfies

$$\mathbb{P}(M_X(N) = O(N^{1/2+\varepsilon}), \forall \varepsilon > 0) = 1$$

and also that the exponent 1/2 is optimal. This is a remarkable result since the random variables $(\mu_X(k))_{k\in\mathbb{N}}$ have multiplicative dependencies which turns on dependencies of infinite range between them.

Pairwise Independence. A sequence of random variables $Y = (Y_k)_{k\in\mathbb{N}}$ is said to be pairwise independent if for any $k \neq l$ the pair Y_k, Y_l is independent. Denoting by $\mathcal{S}$ the set of the square free naturals, the random variables $(\mu_X(k))_{k\in\mathcal{S}}$ are identically distributed and *pairwise independent*, that is, for any square free k, l with $k \neq l$, $\mu_X(k)$ is independent from $\mu_X(l)$. Thus, the Strong Law of Large Numbers for pairwise independent random variables (see Theorem A.3) gives another proof of Wintner's Theorem.

Below is a historical overview about improvements obtained in Wintner's model.

An Improvement by P. Erdős. P. Erdős [6] proved the existence of a constant c_1 such that

$$\mathbb{P}(M_X(N) = O(\sqrt{N}(\log N)^{c_1}) = 1.$$

An Improvement by G. Halász. G. Halász [9] substituted the Erdős term $(\log N)^{c_1}$ by $\exp(c_2\sqrt{\log\log N \log\log\log N})$ for some $c_2 > 0$.

Close to the Law of the Iterated Logarithm. The Law of the Iterated Logarithm states that for a sum $S(N) := \sum_{k=1}^{N} Z_k$ of independent random variables $(Z_k)_{k\in\mathbb{N}}$ uniformly distributed over $\{+1, -1\}$ with probability one:

$$\limsup_{N\to\infty} \frac{S_N}{\sqrt{N\log\log N}} = \frac{1}{\sqrt{2}}.$$

Recently, the Halász term was improved to $(\log\log N)^{c+\varepsilon}$ for all $\varepsilon > 0$. In [2] J. Basquin obtained $c = 2$ and in [12] Y. Lau, G. Tenenbaum and J. Wu obtained $c = 3/2$.

Central Limit Theorems for short Intervals. Denote $\mathcal{S}_{N,N+m}$ to be the quantity of square free naturals between N and $N + m$. Assuming that $N^{1/5}\log N \leq m = o(N/\log N)$, in

[1] In Wintner's paper [19] μ_X did not received the name Random Möbius Function. In the subsequent works [4, 10] μ_X was called Random Multiplicative function. In [11] Random Multiplicative function refer to a completely multiplicative function with random signs at the primes. We found conveniently to call μ_X by Random Möbius function to emphasize a Random Multiplicative function with support on the square free naturals.

[4] S. Chatterjee and K. Soundararajan proved that $\frac{1}{\sqrt{\mathcal{S}_{N,N+m}}}\sum_{k=N}^{N+m}\mu_X(k)$ converges in distribution to the gaussian distribution.

More Central Limit Theorems. Denote $d(k)$ the quantity of distinct primes that divide k,

$$M_X^L(N) := \sum_{\substack{k=1 \\ d(k)\leq L}}^{N} \mu_X(k),$$

and $\sigma_N := \mathbb{E}|M_X^L(N)|^2$. Assuming that $L = o(\log\log N)$, in [10] A. Harper proved gaussian approximation for $\frac{1}{\sqrt{\sigma_N}}M_X^L(N)$.

Consider $\overline{\mu}_X$ the completely multiplicative extension of μ_X, that is, for all k and l, $\overline{\mu}_X(k\cdot l) = \overline{\mu}_X(k)\cdot\overline{\mu}_X(l)$ and $\overline{\mu}_X(n) = \mu_X(n)$ for n square free. Denote $\overline{M}_X^L(N) := \sum_{\substack{k=1 \\ d(k)\leq L}}^{N}\overline{\mu}_X(k)$ and $\overline{\sigma}_N := \mathbb{E}|\overline{M}_X^L(N)|^2$. Assuming that $L = o(\log\log\log N)$, also in this case, in [11] B. Hough proved gaussian approximation for $\frac{1}{\sqrt{\overline{\sigma}_N}}\overline{M}_X^L(N)$.

1.2 Main Results.

The aim of this work is to study the effect of the condition $\mathbb{P}(X_p = +1) < \mathbb{P}(X_p = -1) < 1$ on the random Mertens function. It turns out that one loses the pairwise independence as in Wintner's model. Yet, this condition brings to the Random Möbius Function μ_X the parity effect of the quantity of distinct primes that divide a given number k, denoted by $d(k)$. This effect is also exhibited by the classical Möbius function. This because for a square free natural k, $\mu(k) = (-1)^{d(k)}$, and under the condition $\mathbb{P}(X_p = +1) < \mathbb{P}(X_p = -1) < 1$, $\mathbb{P}(\mu_X(k) = \mu(k)) > \mathbb{P}(\mu_X(k) \neq \mu(k))$.

β Random Möbius function. Let β be a parameter in the interval $[1/2, 1]$ and $X_\beta = (X_{p,\beta})_{p\in\mathcal{P}}$ be a sequence of independent identically distributed random variables with distribution $\mathbb{P}(X_{p,\beta} = -1) = \beta = 1 - \mathbb{P}(X_{p,\beta} = +1)$. Denote by μ_β and M_β the respective Random Möbius function and Random Mertens function associated with X_β. Observe that for $\beta = 1$ one recover the definition of the Classical Möbius and Mertens function.

Question 1. Is it true that for all $\beta \in (1/2, 1)$ we have that $\mathbb{P}(M_\beta(N) = O(N^{1/2+\varepsilon}), \forall\varepsilon > 0) = 1$?

Question 2. Assume that answer to the Question 1 is yes at least for all $\beta \in [\beta^*, 1)$ for some $\beta^* < 1$. There exists a continuity argument in which it implies the result in the case $\beta = 1$, that is, the Mertens Function $M(N) = O(N^{1/2+\varepsilon})$, $\forall\varepsilon > 0$ and hence that the Riemann Hypothesis is true?

Theorem 1.1. Answer to Question 1 is *no*. Precisely for $\beta \in (1/2, 1)$ we have that

$$\mathbb{P}(M_\beta(N) \ll N^{1-\varepsilon}, \text{ for some } \varepsilon > 0) = 0.$$

The proof of this result involve complex analytic and probabilistic results. In the special case where β is either of the form $1-\frac{1}{2^n}$ or $\frac{1}{2}+\frac{1}{2^n}$ the proof is done by combining coupling techniques such as uniform coupling with the concept of Interval Exchange Transformations. Moreover, a variant of the proof for this special case enables to construct the continuity argument inquired in Question 2.

Distance between multiplicative functions. One way to measure the distance between two multiplicative functions $f,g:\mathbb{N}\to\mathbb{C}$ with $|f|,|g|\leq 1$ was introduced in [8] by A. Granville and K. Soundararajan and is defined as follows: Denote $\mathcal{P}_N$ to be the set of the primes less than N and define the distance from f to g up to N by

$$\mathbb{D}(f,g,N)^2 := \sum_{p\in\mathcal{P}_N} \frac{1-Re(f(p)\overline{g}(p))}{p}.$$

Indeed, denoting by I the constant function equals to 1, one has $\mathbb{D}(f,g,N)=\mathbb{D}(I,fg,N)$; the triangle inequality

$$\mathbb{D}(I,fg,N)\leq \mathbb{D}(I,f,N)+\mathbb{D}(I,g,N)$$

and $\mathbb{D}(f,g,N)=0$ for all N if and only if $f(p)=g(p)$ and $|f(p)|=1$. Accordingly to a Theorem due to Wirsing (see Theorem A.14) for a multiplicative function $f:\mathbb{N}\to[-1,1]$ with support on the square free naturals such that $\mathbb{D}(I,f,N)^2$ is a convergent series then f possess a positive mean value, that is, $\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N f(k)>0$. Thus, the multiplicative function $f,g:\mathbb{N}\to[-1,1]$ with support on the square free naturals are said to be at a finite distance if $\mathbb{D}(f,g,N)^2=\mathbb{D}(I,fg,N)^2$ is a convergent series and we denote $\mathbb{D}(f,g)=\lim_{N\to\infty}\mathbb{D}(f,g,N)$. In this case, since the multiplicative function fg has a positive mean value,

$$\lim_{N\to\infty}\frac{\#\{k\leq N: f(k)=g(k)\}-\#\{k\leq N: f(k)\neq g(k)\}}{N}>0.$$

Now let $\mu_{1/2}=\mu_{X_{1/2}}$ be the β Random Möbius function with $\beta=1/2$ and $X_{1/2}=(X_{p,1/2})_{p\in\mathcal{P}}$ be the associated sequence of independent identically distributed random variables. Let $X=(X_p)_{p\in\mathcal{P}}$ be another sequence of independent random variables such that

$$\mathbb{P}(X_p=-1)=\frac{1}{2}+\frac{\delta_p}{2}=1-\mathbb{P}(X_p=+1),$$

with $\delta_p>0$ and $\lim_{p\to\infty}\delta_p=0$. Moreover, assume that the sequences $X_{1/2}$ and X are coupled in the following way: For each prime p, $\mathbb{P}(X_p\neq X_{p,1/2})=\frac{\delta_p}{2}$. Thus, as a consequence of Kolmogorov Two Series Theorem we can prove the following:

Proposition 1.1. If $\sum_{p\in\mathcal{P}}\frac{\delta_p}{p}<\infty$ then with probability one μ_X pretend to be $\mu_{1/2}$, that is, $\mathbb{P}(\mathbb{D}(\mu_X,\mu_{1/2})<\infty)=1$.

This motivates the following question.

Question 3. Suppose that $X = (X_p)_{p\in\mathcal{P}}$ are independent with distribution

$$\mathbb{P}(X_p = -1) = \frac{1}{2} + \frac{\delta_p}{2} = 1 - \mathbb{P}(X_p = +1),$$

where $\delta_p \in [0,1]$. Let Δ be the event $\Delta = [M_X(N) = O(N^{1/2+\varepsilon}), \forall\varepsilon > 0]$. Then we obtain that the probability $\mathbb{P}(\Delta)$ is function of the sequence $(\delta_p)_{p\in\mathcal{P}}$. When $\delta_p = 0$ for all prime p, by Wintner's result, $\mathbb{P}(\Delta) = 1$. When $\delta_p = \delta$ for all prime p for some small $\delta > 0$, by Theorem 1.1, $\mathbb{P}(\Delta) = 0$. Thus how exactly happens the transition $\mathbb{P}(\Delta) = 1 \leftrightarrow \mathbb{P}(\Delta) = 0$ in terms of the velocity in which $\delta_p \to 0$?

Theorem 1.2. Let Δ be as in Question 3 and $\delta_p \in [0,1]$ be such that either $\delta_p = \delta > 0$ for all prime p or $\delta_p \to 0$. Then

$$\mathbb{P}(\Delta) = \begin{cases} 1, & \text{if } \delta_p = O(p^{-1/2}); \\ ?, & \text{if } \delta_p = \frac{1}{p^\alpha}, \text{ for any } \alpha \in (0,1/2); \\ 0, & \text{if } \delta_p^{-1} = O(p^\epsilon) \text{ for all } \epsilon > 0 \text{ (ex. } \delta_p = (\log p)^{-A} \text{ for some } A > 0\text{).} \end{cases}$$

Theorem 1.3. Let $X^\alpha = (X_p^\alpha)_{p\in\mathcal{P}}$ be independent random variables with distribution

$$\mathbb{P}(X_p^\alpha = -1) = \frac{1}{2} + \frac{1}{2p^\alpha} = 1 - \mathbb{P}(X_p^\alpha = +1).$$

Then the Riemann Hypothesis is equivalent to

$$\lim_{\alpha\to 0} \mathbb{P}(M_{X^\alpha}(N) \ll N^{1/2+\varepsilon}, \forall\varepsilon > 0) = 1.$$

Theorems 1.2 and 1.3 gives, conditionally on the Riemann Hypothesis, a full answer to Question 3. In particular, the goal of Theorem 1.3 is to achieve a criteria for the Riemann Hypothesis based on a sum of random variables having the same multiplicative structure of the Möbius function and that pretend to be the Random Möbius function $\mu_{1/2}$ in the sense of Proposition 1.1.

On the other hand, in a similar random neighborhood of the Möbius function:

Theorem 1.4. Let $X = (X_p)_{p\in\mathcal{P}}$ be independent with distribution $\mathbb{P}(X_p = -1) = 1-\delta_p = \mathbb{P}(X_p = +1)$. Assume that $\delta_p \to 0$ and $\delta_p^{-1} \ll p^{-\alpha}$ for some $\alpha \in [0,1/2)$. Then

$$\mathbb{P}(M_X(N) = O(N^{1-\alpha-\varepsilon}), \text{ for some } \varepsilon > 0) = 0.$$

Is interesting to compare Theorems 1.2 and 1.4. In Theorem 1.3 μ_X is close to the Wintner's model while in Theorem 1.4 μ_X is close to the Möbius function. Also this both distributions share the same size of perturbation $\delta_p = p^{-\alpha}$. Theorem 1.4 tell us that $M_X(N)$ is not $O(N^{1/2+\varepsilon})$ without the knowledge of the order of the Mertens function itself. On the other hand, in Wintner's model, $M_X(N)$ exhibits a kind of Iterated Logarithm Law while Theorem 1.2 do not gives a completely description of the order of M_X in the neighborhood of this model. Then Theorem 1.3 tell us that in the gap of Theorem 1.2 one gets a criteria for the Riemann Hypothesis.

α Random Möbius Function. This terminology refers to the Random Möbius Function associated to the sequence X^α as in Theorem 1.3. We can prove that for any $\alpha \in (0, 1/2)$, with probability one $M_{X^\alpha}(N) = o(N^{1-\alpha+\varepsilon})$ for all $\varepsilon > 0$. On the other hand, at this moment we can only prove that, in the border case $\varepsilon = 0$, with probability one $M_{X^\alpha} = o(N^{1-\alpha})$ for $\alpha < 1/3$. Indeed we have

Proposition 1.2. If $\alpha < 1/3$, then

$$\mathbb{P}\left(\exists \lim_{N\to\infty} \sum_{k=1}^{N} \frac{\mu_{X^\alpha}(k)}{k^{1-\alpha+it}} \,\forall t \in \mathbb{R}\right) = 1.$$

If $\alpha \in (0, 1/2)$, then

$$\lim_{N\to\infty} \mathbb{E}\left|\sum_{k=1}^{N} \frac{\mu_{X^\alpha}(k)}{k^{1-\alpha}}\right|^2 = 0.$$

The reason $\alpha < 1/3$ is related to some probabilistic bounds and to the exponent $2/3$ appearing in the log term that describes the best zero free region known up to date for the Riemann zeta function, due to Vinogradov-Korobov: There exists a constant $A > 0$ such that for σ and t satisfying

$$1 - \sigma \leqslant \frac{A}{(\log t)^{2/3} (\log\log t)^{1/3}}$$

then $\zeta(\sigma + it) \neq 0$. The L^2 convergence is proved by using a recent Remark due to T. Tao [16]: For any $m \in \mathbb{N}$

$$\left| \sum_{\substack{k=1 \\ \gcd(k,m)=1}}^{N} \frac{\mu(k)}{k} \right| \leq 1.$$

As consequence obtained from Proposition 1.2 and it's proof we have

Proposition 1.3. There exists a real B such that for all $\alpha \in (0, 1/3)$, with probability one

$$M_{X^\alpha}(N) \ll N^{1-\alpha} \exp\left(- B\left(\frac{\log N}{\log\log N}\right)^{1/3}\right).$$

1.3 Organization.

I try to write this thesis in both Probability Theory and Analytic Number Theory languages. For this, in Section 2.1 I introduce some common notations and concepts developed here that will be used in the proofs. Also, intended to a fast reading, at the end I included an appendix containing some general results and some proofs that I found conveniently to left for the end. In Section 2.2 I prove Theorem 1.1 in the special case $\beta = 1 - \frac{1}{2^n}$ and $\beta = \frac{1}{2} + \frac{1}{2^n}$ for $n \geq 2$. The proof of Theorem 1.1 in it's generality is included in the proof of Theorem 1.2 in Section 2.3. The proof of Theorem 1.4 is found in Section 2.4. In Section 2.5 I prove a result in which Theorem 1.3 follows as a consequence in Section 2.6. In Section 3 I conclude with some interesting remarks and some open questions.

2 Main Results: The proofs.

2.1 The Probabilistic-Analytic setup.

Notations 2.1. $\mathcal{P}$ stands for the set of the prime numbers and p for a generic element of $\mathcal{P}$. $d|n$ and $d \nmid n$ means that d divides and that d do not divides n, respectively. $\mathcal{S} = \{k \in \mathbb{N} : p|k \Rightarrow p^2 \nmid k\}$ stands for the set of the square free naturals. μ is the canonical Möbius function:

$$\mu(k) = \begin{cases} (-1)^{|\{p\in\mathcal{P}:p|k\}|}, & \text{if } k \in \mathcal{S}, \\ 0, & \text{if } k \in \mathbb{N} \setminus \mathcal{S}. \end{cases}$$

$M : \mathbb{N} \to \mathbb{R}$ stands for the Mertens function: $M(N) = \sum_{k=1}^{N} \mu(k)$. Given a unbounded set $R \subset \mathbb{C}$ and functions $f, g : R \to \mathbb{C}$, the Vinogradov notation $f \ll g$ and big Oh notation $f = O(g)$ are used to mean that there exists a constant $C > 0$ such that $|f(z)| \leq C|g(z)|$.

Uniform Coupling. $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space: $\Omega = [0,1]^{\mathcal{P}}$, $\mathcal{F}$ is the Borel sigma algebra of Ω and $\mathbb{P}$ is the Lebesgue product measure. A generic element $\omega \in \Omega$ is denoted by $(\omega_p)_{p\in\mathcal{P}}$. A sequence $X = (X_p)_{p\in\mathcal{P}}$ of independent random variables such that $\mathbb{P}(X_p = -1) = a_p = 1 - \mathbb{P}(X_p = +1)$ is defined as follows: $X_p : \Omega \to \{-1, +1\}$ depends only in the coordinate ω_p for each prime p and

$$X_p(\omega_p) = -\mathbb{1}_{[0,a_p]}(\omega_p) + \mathbb{1}_{(a_p,1]}(\omega_p).$$

If $Y = (Y_p)_{p\in\mathcal{P}}$ is another sequence of independent random variables defined in the same way as X such that $\mathbb{P}(Y_p = -1) \leq \mathbb{P}(X_p = -1)$, then

$$X_p \leq Y_p, \ \forall p \in \mathcal{P}.$$

This property is called uniform coupling.

Notations 2.2. Given a *square integrable* random variable $X : \Omega \to \mathbb{R}$ denote:

$$\begin{aligned} \mathbb{E}X &:= \int_\Omega X(\omega)\mathbb{P}(d\omega), \\ \mathbb{V}X &:= \mathbb{E}X^2 - (\mathbb{E}X)^2. \end{aligned}$$

A measurable set is called *event*. Also, given a property Q such that the set $\{\omega \in \Omega : X(\omega) \text{ has property } Q\}$ is measurable, this set is abbreviated by $[X \text{ has property } Q]$.

Definition 2.1. Given a set S, a random function f is a map $f : S \times \Omega \to \mathbb{C}$ such that for each $s \in S$, $f(s)$ is a random variable.

Notations 2.3. If $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ is a random sequence and $f : S \times \mathbb{R}^{\mathbb{N}} \to \mathbb{C}$ is such that for each s, $\omega \mapsto f(s, X(\omega))$ is a random variable, $f_X : S \times \Omega \to \mathbb{C}$ stands for the random function $(s, \omega) \mapsto f(s, X(\omega))$ and $f_{X(\omega)}$ stands for the function $f(\cdot, X(\omega)) : S \to \mathbb{C}$.

Multiplicative functions. A function $f : \mathbb{N} \to \mathbb{C}$ such that $f(n \cdot m) = f(n)f(m)$ for all n and m coprime, that is, with greater common divisor $\gcd(n, m) = 1$, is called multiplicative function. Observe that for each $\omega \in \Omega$, $\mu_{X(\omega)}$ is a multiplicative function.

Notations 2.4. Given $a \in \mathbb{R}$, $\mathbb{H}_a$ stands for the half plane $\{s \in \mathbb{C} : Re(s) > Re(a)\}$.

Definition 2.2. Let X as in Definition 1.1. The random Riemann zeta function associated to X is a random function $\zeta_X : \mathbb{H}_1 \times \Omega \to \mathbb{C}$ given by:

$$\zeta_X(z) := \prod_{p \in \mathcal{P}} \frac{1}{1 + \frac{X_p}{p^z}}.$$

Proposition 2.1. *For all* $\omega \in \Omega$:

i. ζ_X and $1/\zeta_X$ are analytic functions in $\mathbb{H}_1$ and in particular these random functions never vanish on this half plane.

ii. For each $z \in \mathbb{H}_1$:

$$\frac{1}{\zeta_X(z)} = \sum_{k=1}^{\infty} \frac{\mu_X(k)}{k^z} = \prod_{p \in \mathcal{P}} \left(1 + \frac{X_p}{p^z}\right). \tag{1}$$

iii. For each $\omega \in \Omega$ there exists an analytic function $R_{X(\omega)} : \mathbb{H}_{1/2} \to \mathbb{C}$ and a *branch of the logarithm* $\log^* \zeta_{X(\omega)} : \mathbb{H}_1 \to \mathbb{C}$ such that:

$$\frac{1}{\zeta_{X(\omega)}(z)} = \exp(-\log^* \zeta_{X(\omega)}(z)) \tag{2}$$

$$-\log^* \zeta_{X(\omega)}(z) = \sum_{p \in \mathcal{P}} \frac{X_p(\omega)}{p^z} + R_{X(\omega)}(z). \tag{3}$$

Proof. See Proposition A.1 with parameters $C = 1$ and $\alpha = 0$. □

Definition 2.3. Let $R_1 \subset R_2$ be two connected regions of $\mathbb{C}$ and $h : R_1 \times \Omega \to \mathbb{C}$ be a random function such that for each $\omega \in \Omega$, $h(\cdot, \omega)$ is analytic. The function $h(\cdot, \omega)$ has analytic extension to the region R_2 if there exists an analytic function $\overline{h}_\omega : R_2 \to \mathbb{C}$ that coincides with $h(\cdot, \omega)$ in R_1.

We always keep the symbol h instead $\overline{h}$ to represent an analytic extension. This because when an analytic extension exists, it is unique.

Proposition 2.2. Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of independent random variables such that $\mathbb{V}X_k^2<\infty$ for all k. Let $z\in\mathbb{C}$ and $F_N(z):=\sum_{k=1}^N \frac{X_k}{k^z}$. Define

$$\begin{aligned}\sigma_1 &= \inf\big\{\sigma\in\mathbb{R}:\ \mathbb{E}F_N(\sigma)\text{ is a convergent sequence }\big\},\\ \sigma_2 &= \inf\big\{\sigma\in\mathbb{R}:\ \mathbb{V}F_N(\sigma)\text{ is a convergent sequence }\big\}.\end{aligned}$$

Let $\sigma_c=\max\{\sigma_1,\sigma_2\}$. Then with probability one there exists a random analytic function $F:\mathbb{H}_{\sigma_c}\times\Omega\to\mathbb{C}$ such that $F_N\to F$ in $\mathbb{H}_{\sigma_c}$.

Proof. The assumption that for all real $\sigma>\sigma_c$ both series $\mathbb{E}F_N(\sigma)$ and $\mathbb{V}F_N(\sigma)$ converge, by Kolmogorov Two-Series Theorem (see Theorem A.2), it implies that with probability one $F_N(\sigma)$ converges for all real $\sigma>\sigma_c$. By Theorem A.5, convergence of a series of the type $\sum_{k=1}^N \frac{c_k}{k^\sigma}$ implies that this series converges uniformly on compact subsets of $\mathbb{H}_\sigma$. Thus we conclude that with probability one, the pointwise limit $\lim_{N\to\infty}F_N(z)$ exists for all $z\in\mathbb{H}_{\sigma_c}$ and defines a random function that it is analytic in this region. □

Proposition 2.3. The random variable

$$\rho_X:=\inf\{1/2\leqslant c\leqslant 1: 1/\zeta_X \text{ has analytic extension to } \mathbb{H}_c\}$$

is measurable in the tail sigma algebra generated by the sequence X and hence, by Kolmogorov 0-1 Law, there exists a deterministic number υ such that $\mathbb{P}(\rho_X=\upsilon)=1$. Furthermore

$$\mathbb{P}(M_X(N)\ll N^c)\leq\mathbb{P}(\rho_X\leq c). \tag{4}$$

Proof. By Proposition 2.1, $1/\zeta_X$ is analytic in $\mathbb{H}_1$ for all $\omega\in\Omega$ and can be represented as a Dirichlet series on this half plane. For each $q=q_1+iq_2\in\mathbb{H}_1$ with $q_1,q_2\in\mathbb{Q}$ define:

$$R_q(\omega)=\limsup_{n\to\infty}\left|\frac{1}{n!}\frac{d^n}{dz^n}\frac{1}{\zeta_{X(\omega)}(q)}\right|^{\frac{1}{n}}.$$

Since for all $n\in\mathbb{N}$, $\frac{d^n}{dz^n}\frac{1}{\zeta_{X(\omega)}(q)}=(-1)^n\sum_{k=1}^\infty\frac{\mu_{X(\omega)}(k)}{k^q}(\log k)^n$ is an absolute convergent series for each $q\in\mathbb{H}_1$ it follows that R_q is measurable with respect to the sigma algebra generated by the random variables X_p. Since $1/R_q(\omega)$ is the radius of convergence of the Taylor series of the function $1/\zeta_{X(\omega)}$ at the point $q=q_1+iq_2$ (c.f. [5], chapter III), this function is analytic in the open ball with center q and radius $1/R_q(\omega)$. Therefore, if for all $q\in\mathbb{H}_1\cap(\mathbb{Q}+i\mathbb{Q})$ we assume that $1/R_q(\omega)\geqslant q_1-c$, then $1/\zeta_{X(\omega)}$ has analytic extension to the half plane $\mathbb{H}_c$. Reciprocally, if $1/\zeta_{X(\omega)}$ has analytic extension to $\mathbb{H}_c$, then for each $q=q_1+iq_2\in\mathbb{H}_1$ the radius of convergence $1/R_q(\omega)$ is greater or equal to the distance from the point q to the boundary of $\mathbb{H}_c$ (c.f [5], page 72, Theorem 2.8). This implies that $[\rho_X\leqslant c]=\bigcap_{q\in\mathbb{H}_1\cap(\mathbb{Q}+i\mathbb{Q})}[R_q\leqslant 1/(q_1-c)]$, and hence ρ_X is measurable in the sigma algebra generated by the independent random variables $(X_p)_{p\in\mathcal{P}}$. To show that ρ_X is a tail random variable, let D be a *finite* subset of primes and $\mathcal{F}_D$ be the sigma algebra generated by the random variables $(X_p)_{p\in D}$. Define $\theta_D : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C}$ by $\theta_D(z) := \prod_{p\in D}\left(1 + \frac{X_p}{p^z}\right)$. Clearly θ_D and $1/\theta_D$ are analytic for all $\omega \in \Omega$ since for each p, $1 + X_p p^{-z}$ is analytic and only vanish at the vertical strip $Re(z) = 0$. This implies that, since for each $z \in \mathbb{H}_1$

$$\frac{1}{\zeta_X(z)} = \theta_D(z) \prod_{p\in\mathcal{P}\setminus D}\left(1 + \frac{X_p}{p^z}\right),$$

$1/\zeta_X$ has analytic extension to $\mathbb{H}_c$ $(c < 1)$ if and only if $\frac{1}{\theta_D\cdot\zeta_X}$ has analytic extension to $\mathbb{H}_c$. Since $\frac{1}{\theta_D\cdot\zeta_X}$ depends only on the random variables $(X_p)_{p\in\mathcal{P}\setminus D}$ we conclude that the event $[\rho_X \le c]$ is measurable in the sigma algebra generated by this random variables and hence that this is indeed a tail event (see Definition A.1). This shows that ρ_X is a tail random variable. To prove Inequality (4), observe that by the summation by parts formula (see Lemma A.7) we have that $[M_X(N) \ll N^c] \subset [S_N(\sigma) = \sum_{k=1}^{N} \frac{\mu_X(k)}{k^\sigma}$ converges $\forall\, \sigma > c]$. By the analytic properties of Dirichlet Series (see the proof of Proposition 2.2) we obtain that $[M_X(N) \ll c] \subset [\rho_X \le c]$. □

2.2 β Random Möbius Function.

Let $\beta \in [1/2, 1]$. By our choose of the probability space (see uniform coupling in Section 2.1) the random variables $X_\beta = (X_{p,\beta})_{p\in\mathcal{P}}$ given by:

$$X_{p,\beta}(\omega) = -\mathbb{1}_{[0,\beta]}(\omega_p) + \mathbb{1}_{(\beta,1]}(\omega_p). \tag{5}$$

are independent, have common distribution $\mathbb{P}(X_{p,\beta} = -1) = \beta = 1 - \mathbb{P}(X_{p,\beta} = 1)$ and are *uniformly coupled.*

Notations 2.5. The random functions μ_β, M_β and ζ_β are the random Möbius function, Random Mertens Function and the random Riemann zeta function associated with the sequence $X_\beta = (X_{p,\beta})_{p\in\mathcal{P}}$ respectively. For $\gamma \in [0,1]$ denote $\theta_\gamma : \mathbb{H}_1 \times \Omega \to \mathbb{C}$ by $\theta_\gamma = \frac{\zeta}{\zeta_\gamma}$.

The aim of this section is to prove the following result:

Proposition 2.4. Assume that β is either of the form $1 - \frac{1}{2^n}$ or $\frac{1}{2} + \frac{1}{2^n}$ for $n \ge 2$. Then

$$\mathbb{P}(M_\beta(N) \ll N^{1-\varepsilon},\ \text{for some } \varepsilon > 0) = 0.$$

The following Lemma is a essential part of the proof Proposition 2.4 and it's construction will be used in the proof of Theorem 1.4 and also to construct the continuity argument inquired in Question 2 whose proof is left for the Remarks Section.

Lemma 2.1. Let $\beta \in [1/2, 1]$ and $\beta_n = 1 - \frac{1-\beta}{2^n}$ where $n \in \mathbb{N}$. Then there exists a measure preserve transformation $T_n : \Omega \to \Omega$ such that the following functional equations holds in $\mathbb{H}_1$ for all $\omega \in \Omega$:

$$\theta_\beta(z, \omega) = \prod_{k=1}^{2^n} \theta_{\beta_n}(z, T_n^k \omega). \tag{6}$$

Proof. Let E_n be a partition of the interval $[\beta, 1]$ into to 2^n subintervals $\{I_k : k = 1, ..., 2^n\}$ of length $\frac{1-\beta}{2^n}$ where $I_k = (a_{k-1}, a_k]$ with $a_k = \beta + \frac{k}{2^n}(1 - \beta)$. For $z \in \mathbb{H}_1$ and $\omega \in \Omega$ decompose:

$$\theta_\beta(z, \omega) = \prod_{k=1}^{2^n} \frac{\zeta_{a_k}(z, \omega)}{\zeta_{a_{k-1}}(z, \omega)}. \tag{7}$$

Proposition 2.1 implies that each random function $\zeta_{a_k}/\zeta_{a_{k-1}}$ is analytic in $\mathbb{H}_1$ for all $\omega \in \Omega$. Also these random functions have Euler product representation in $\mathbb{H}_1$. In fact, by using the uniform coupling we obtain:

$$\begin{aligned}\frac{\zeta_{a_k}(z, \omega)}{\zeta_{a_{k-1}}(z, \omega)} &= \lim_{N\to\infty} \prod_{p \in \mathcal{P}\cap[0,N]} \left(1 + \frac{X_{p,a_k}(\omega)}{p^z}\right)^{-1} \lim_{N\to\infty} \prod_{p \in \mathcal{P}\cap[0,N]} \left(1 + \frac{X_{p,a_{k-1}}(\omega)}{p^z}\right) \\ &= \prod_{p\in\mathcal{P}} \frac{p^z + \mathbb{1}_{I_k}(\omega_p)}{p^z - \mathbb{1}_{I_k}(\omega_p)}.\end{aligned}$$

This identity gives that the distribution of $\frac{\zeta_{a_k}(z)}{\zeta_{a_{k-1}}(z)}$ depends only on z and on the difference $a_k - a_{k-1}$ that do not depend on k. Hence for each $z \in \mathbb{H}_1$ and $k \in \{1, ..., 2^n\}$, $\zeta_{a_k}/\zeta_{a_{k-1}}$ equals in distribution to $\theta_{\beta_n}(z)$. In fact we can couple these random functions. Let $g : [0, 1] \to [0, 1]$ be a interval exchange transformation defined as follows: For each k, $g|_{I_k}$ is a translation; $g|_{[0,\beta]} = Id$; $g(I_1) = I_{2^n}$ and for $k \geq 2$, $g(I_k) = I_{k-1}$. It follows that g preserves Lebesgue measure and $g^k(I_k) = I_{2^n}$. Thus $\mathbb{1}_{I_k} = \mathbb{1}_{I_{2^n}} \circ g^k$. Define $T_n : \Omega \to \Omega$ to be the measure preserve transformation $(\omega_p) \mapsto (g(\omega_p))_{p\in\mathcal{P}}$. Then we can rewrite (7) as:

$$\theta_\beta(z, \omega) = \prod_{k=1}^{2^n} \theta_{\beta_n}(z, T_n^k \omega).$$

□

Proof. (Proposition 2.4)

Claim 2.1. For $n \in \mathbb{N}$ set $\beta_n = 1 - \frac{1}{2^n}$. Then for $n \geq 2$

$$\mathbb{P}(M_{\beta_n} \ll N^{1-\varepsilon} \text{ for some } \varepsilon > 0) = 0.$$

Proof of Claim 2.1. Lemma 2.1 applied to the parameter $\beta = 1/2$ gives a measure preserving transformation $T_n : \Omega \to \Omega$ such that

$$\theta_{1/2}(z,\omega) = \prod_{k=1}^{2^n} \theta_{\beta_n}(z, T_n^k \omega).$$

Recalling that $\theta_{\beta_n} = \frac{\zeta}{\zeta_{\beta_n}}$, simplifying equation above we obtain the following functional equation in $\mathbb{H}_1$ for all $\omega \in \Omega$:

$$\frac{1}{\zeta(z)^{2^n-1}} = \frac{\zeta_{1/2}(z,\omega)}{\prod_{k=1}^{2^n} \zeta_{\beta_n}(z, T_n^k\omega)}. \tag{8}$$

Claim 2.2. For $\beta > 1/2$, $\mathbb{P}\big(\lim_{k\to\infty} \frac{1}{\zeta_\beta(1+1/k)} = 0\big) = 1$.

Proof of the Claim 2.2. For each $k \in \mathbb{N}$ decompose:

$$\sum_{p\in\mathcal{P}} \frac{X_{p,\beta}}{p^{1+1/k}} = \sum_{p\in\mathcal{P}} \frac{X_{p,\beta} - \mathbb{E}X_{p,\beta}}{p^{1+1/k}} + \sum_{p\in\mathcal{P}} \frac{\mathbb{E}X_{p,\beta}}{p^{1+1/k}}. \tag{9}$$

Proposition 2.2 implies that with probability one the random function $z \mapsto \sum_{p\in\mathcal{P}} \frac{X_{p,\beta}-\mathbb{E}X_{p,\beta}}{p^z}$ is analytic in $\mathbb{H}_{1/2}$. In particular we obtain

$$\lim_{k\to\infty} \left| \sum_{p\in\mathcal{P}} \frac{X_{p,\beta} - \mathbb{E}X_{p,\beta}}{p^{1+1/k}} \right| = \left| \sum_{p\in\mathcal{P}} \frac{X_{p,\beta} - \mathbb{E}X_{p,\beta}}{p} \right| < \infty, \ \mathbb{P} - a.s. \tag{10}$$

On the other hand, since $-\mathbb{E}X_{p,\beta} = 2\beta - 1 > 0$ for each p and $\sum_p \frac{-\mathbb{E}X_{p,\beta}}{p} = \infty$, Fatou's Lemma implies that $\lim_{k\to\infty} \sum_p \frac{-\mathbb{E}X_{p,\beta}}{p^{1+1/k}} = \infty$. This divergence together with (10) and (9) gives that with probability one $\lim_{k\to\infty} \sum_{p\in\mathcal{P}} \frac{X_{p,\beta}}{p^{1+1/k}} = -\infty$. Since the term R_{X_β} in (3) is analytic in $\mathbb{H}_{1/2}$ for all $\omega \in \Omega$, this divergence combined with (3) finishes the proof of Claim 2.2.

Claim 2.3. For $\beta = 1/2$, $\mathbb{P}\big(\zeta_{1/2}$ and $\frac{1}{\zeta_{1/2}}$ have analytic extension to $\mathbb{H}_{1/2}\big) = 1$.

Proof of the Claim 2.3. Proposition 2.2 implies that, with probability one, the random function $z \mapsto \sum_{p\in\mathcal{P}} \frac{X_{p,1/2}}{p^z}$ is analytic in $\mathbb{H}_{1/2}$. This combined with (3) gives that with probability one $\log^* \zeta_{1/2}$ extends analytically to $\mathbb{H}_{1/2}$. Hence, the relation $\zeta_{1/2}^{\pm 1} = \exp(\pm \log^* \zeta_{1/2})$ also holds in $\mathbb{H}_{1/2}$, finishing the proof of Claim 2.3.

End of proof of the Claim 2.1. Let $\rho_\beta = \rho_{X_\beta}$ as in Proposition 2.3 and

$$E = [\rho_{\beta_n} < 1] \cap [1/\zeta_{\beta_n}(1) = 0] \cap [\rho_{1/2} = 1/2] \cap [\zeta_{1/2}(z) \neq 0, \forall z \in \mathbb{H}_{1/2}].$$

Claims 2.2 and 2.3 imply that $\mathbb{P}(E) = \mathbb{P}(\rho_{\beta_n} < 1)$. By contradiction assume that $\mathbb{P}(\rho_{\beta_n} < 1) = 1$. This implies that the event $E' := \bigcap_{k=1}^{2^n} T_n^{-k}(E)$ also has probability one, since T_n

preserves measure. In particular there exists $\omega' \in E'$. For this ω', the functions $\zeta_{1/2}(\cdot,\omega')$ and $1/\zeta_{\beta}(\cdot, T_n^k\omega')$ for $k=1,...,2^n$ are analytic in $\mathbb{H}_{1-\varepsilon}$ for some $\varepsilon=\varepsilon(\omega')>0$ and satisfy:

$$\zeta_{1/2}(1,\omega') \neq 0,$$
$$\frac{1}{\zeta_{\beta_n}(1,T_n^k\omega')} = 0, \ k=1,...,2^n.$$

Thus we obtain that the function on the right side of the (8) has the point $z=1$ as a zero of multiplicity at least 2^n. On the other hand the function on the left side of the same equation has the point $z=1$ as a zero of multiplicity 2^{n-1}, since the Riemann ζ function has a simple pole at the point $z=1$. This leads to a contradiction which implies that $E'=\varnothing$ and hence $\mathbb{P}(\rho_{\beta_n}<1)<1$. This together Proposition 2.3 imply that this probability actually is 0. Inequality (4) completes the proof of Claim 2.1.

Claim 2.4. For $\gamma_n=\frac{1}{2}+\frac{1}{2^{n+1}}$

$$\mathbb{P}(M_{\gamma_n} \ll N^{1-\varepsilon} \text{ for some } \varepsilon>0)=0.$$

Proof of Claim 2.4. The proof of this statement is done by induction on n. The case $n=1$ corresponds to the case $\gamma_1=3/4$ in which is included in the proof of Claim 2.1. Assume the induction hypothesis:
$\mathbb{P}(\rho_{\gamma_k}<1)=0$ for $1\leq k\leq n$. By adapting the proof of formula (8) there exists a transformation $T_n:\Omega\to\Omega$ that preserves measure and such that the following functional equation holds in $\mathbb{H}_1$ for all $\omega\in\Omega$:

$$\frac{1}{\zeta_{\gamma_n}(z,\omega)}=\frac{\zeta_{1/2}(z,T_n\omega)}{\zeta_{\gamma_{n+1}}(z,T_n\omega)\cdot\zeta_{\gamma_{n+1}}(z,\omega)}. \tag{11}$$

Indeed, let $h:[0,1]\to[0,1]$ be a interval exchange transformation such that $h((1/2,\gamma_{n+1}])=(\gamma_{n+1},\gamma_n]$ and $h((\gamma_{n+1},\gamma_n])=(1/2,\gamma_{n+1}]$ are translations and $h|_{[0,1]\setminus(1/2,\gamma_n]}=Id$. Let $T:\Omega\to\Omega$ be the measure preserve transformation given by $(\omega_p)_{p\in\mathcal{P}}\mapsto(h(\omega_p))_{p\in\mathcal{P}}$. Since

$$\frac{\zeta_{\gamma_{n+1}}(z,\omega)}{\zeta_{1/2}(z,\omega)}=\frac{\zeta_{\gamma_n}(z,T\omega)}{\zeta_{\gamma_{n+1}}(z,T\omega)},$$

we obtain

$$\frac{\zeta_{\gamma_n}(z,\omega)}{\zeta_{1/2}(z,\omega)}=\frac{\zeta_{\gamma_n}(z,T\omega)}{\zeta_{\gamma_{n+1}}(z,T\omega)}\cdot\frac{\zeta_{\gamma_n}(z,\omega)}{\zeta_{\gamma_{n+1}}(z,\omega)}$$

which gives (11) by denoting $T_n=T^{-1}$.

Define $E_{n+1}=[\rho_{\gamma_{n+1}}<1]\cap[\rho_{1/2}=1/2]$ and $E'_{n+1}=E_{n+1}\cap T_n^{-1}(E_{n+1})$. Then by (11) and the induction hypothesis

$$\mathbb{P}(E'_{n+1})\leq\mathbb{P}(\rho_{\gamma_n}<1)=0,$$

and hence $\mathbb{P}(E_{n+1})<1$. Thus, by Claim 2.3 we obtain $\mathbb{P}(\rho_{\gamma_{n+1}}<1)=\mathbb{P}(E_{n+1})<1$. Proposition 2.3 implies that this probability equals to 0, finishing the induction step. Inequality (4) completes the proof of Claim 2.4, finishing the proof of Proposition 2.4. □

2.3 Neighborhood of $\beta = 1/2$.

Let $\Delta = [M_X(N) \ll N^{1/2+\varepsilon}, \forall \varepsilon > 0]$. Then the probability $\mathbb{P}(\Delta)$ depends in the distribution of X. The aim of this section is to prove the following result:

Theorem 1.2. Let $X = (X_p)_{p\in\mathcal{P}}$ be independent with distribution

$$\mathbb{P}(X_p = -1) = \frac{1}{2} + \frac{\delta_p}{2} = 1 - \mathbb{P}(X_p = +1),$$

where $\delta_p \in [0, 1]$ and either $\delta_p = \delta > 0$ for all prime p or $\delta_p \to 0$. Then

$$\mathbb{P}(\Delta) = \begin{cases} 1, & \text{if } \delta_p = O(p^{-1/2}); \\ 0, & \text{if } \delta_p^{-1} = O(p^{\epsilon}) \text{ for all } \epsilon > 0 \text{ (ex. } \delta_p = (\log p)^{-A} \text{ for some } A > 0\text{).} \end{cases}$$

Proof.

Claim 2.5. Assume that $(\delta_p)_{p\in\mathcal{P}}$ are *non negative* and $\delta_p \ll p^{-\alpha}$ for some $\alpha > 0$. Then:

$$\mathbb{P}(M_X(N) \ll N^{\max\{1-\alpha,1/2\}+\varepsilon}, \ \forall \varepsilon > 0) = 1.$$

In particular, if $\alpha \geq 1/2$, then $\mathbb{P}(\Delta) = 1$.

Proof of the Claim 2.5 Define:

$$\begin{aligned} X_p(\omega) &:= -\mathbb{1}_{[0,\frac{1}{2}+\frac{\delta_p}{2}]}(\omega_p) + \mathbb{1}_{(\frac{1}{2}+\frac{\delta_p}{2},1]}(\omega_p), \\ Z_p(\omega) &:= -\mathbb{1}_{[0,\frac{1}{2}-\frac{\delta_p}{2})}(\omega_p) + \mathbb{1}_{(\frac{1}{2}+\frac{\delta_p}{2},1]}(\omega_p), \\ W_p(\omega) &:= -\mathbb{1}_{[\frac{1}{2}-\frac{\delta_p}{2},\frac{1}{2}+\frac{\delta_p}{2}]}(\omega_p). \end{aligned}$$

The identities $X_p = W_p + Z_p$ and $W_p \cdot Z_p = 0$ imply that $\mu_X = \mu_W * \mu_Z$, where $*$ stands for the Dirichlet convolution (see Definition A.3). Thus, to prove that with probability one both series $\sum_{k=1}^{\infty} \frac{|\mu_W(k)|}{k^x}$ and $\sum_{k=1}^{\infty} \frac{\mu_Z(k)}{k^x}$ converges for each for $x > \max\{1-\alpha, 1/2\}$ implies that, by Theorem A.8, the series $\sum_{k=1}^{\infty} \frac{\mu_X(k)}{k^x}$ converges for the same values of x which combined with Kronecker's Lemma (see Theorem A.6) gives that $\mathbb{P}(M_X(N) \ll N^{\max\{1-\alpha,1/2\}+\varepsilon}, \ \forall \varepsilon > 0) = 1$. Thus one needs only to show the convergence of the series $\sum_{k=1}^{\infty} \frac{|\mu_W(k)|}{k^x}$ and $\sum_{k=1}^{\infty} \frac{\mu_Z(k)}{k^x}$ for $x > \max\{1-\alpha, 1/2\}$.

For the convergence of $\sum_{k=1}^{\infty} \frac{|\mu_W(k)|}{k^x}$, the assumption $\delta_p \ll 1/p^{\alpha}$ implies that for each $x > \max\{1-\alpha, 1/2\}$:

$$\sum_{p\in\mathcal{P}} \mathbb{E}\frac{|W_p|}{p^x} \ll \sum_{p\in\mathcal{P}} \frac{1}{p^{x+\alpha}} < \infty,$$

$$\sum_{p\in\mathcal{P}} \mathbb{V}\frac{|W_p|}{p^x} \ll \sum_{p\in\mathcal{P}} \frac{1}{p^{2x}} < \infty.$$

Thus Kolmogorov 2-series Theorem imply that with probability one the random series $\sum_{p\in\mathcal{P}}^{\infty}\frac{W_p}{p^z}$ converges absolutely for these values of x. This combined with Theorem A.15 gives the desired absolute convergence of $\sum_{k=1}^{\infty}\frac{\mu_W(k)}{k^x}$. For the convergence of $\sum_{k=1}^{\infty}\frac{\mu_Z(k)}{k^x}$, observe that $Z_p = X_{p,1/2}\cdot(1+W_p)$ where

$$X_{p,1/2} = -\mathbb{1}_{[0,\frac{1}{2}]}(\omega_p) + \mathbb{1}_{(\frac{1}{2},1]}(\omega_p)$$

and that $X_{p,1/2}$ is independent from $1+W_p$. Let $\mathcal{S}$ be the set of the square free naturals and denote the random set $\mathcal{S}' = \{k\in\mathcal{S} : p|k \Rightarrow W_p = 0\}$. Being $\mu_{1/2}$ the random Möbius function as in Notations 2.5, we have

$$M_Z(N) = \sum_{k\in\mathcal{S}'\cap[0,N]}\mu_{1/2}(k).$$

Since $\mathcal{S}'\subset\mathcal{S}$, the random variables $(\mu_{1/2}(k))_{k\in\mathcal{S}'}$ are, conditioned on the sigma algebra $\sigma(W_p : p\in\mathcal{P})$, pairwise independent and identically distributed since this sigma algebra is independent from $\sigma(X_{p,1/2} : p\in\mathcal{P})$. Thus, by the Strong Law of Large numbers for pairwise independent random variables (Theorem A.3) we have that $\mathbb{P}(\Lambda_Z) = 1$ and hence, by Lemma A.7, with probability one the series $\sum_{k=1}^{\infty}\frac{\mu_Z(k)}{k^x}$ converges for all $x > 1/2$, finishing the proof of Claim 2.5.

Claim 2.6. Let the constants $\delta_p = -\mathbb{E}X_p$ be such that $\delta_p = \delta < 1$ for all prime p or $\delta_p\to 0$ and $\delta_p^{-1}\ll p^{-\epsilon}$ for all $\epsilon > 0$. Then either:

$$\text{a.} \sum_{p\in\mathcal{P}}\frac{\mathbb{E}X_p}{p} = -\infty \text{ and } \sum_{p\in\mathcal{P}}\frac{1+\mathbb{E}X_p}{p} = \infty;$$

$$\text{b.} -\sum_{p\in\mathcal{P}}\frac{\mathbb{E}X_p}{p^{1+\varepsilon}} < \infty \text{ iff } \varepsilon\geq 0.$$

Proof of Claim 2.6. For $0 < \delta_p = \delta < 1$ for all p or $\delta_p\to 0^+$ the series $\sum_{p\in\mathcal{P}}\frac{1-\delta_p}{p} = \infty$ and $\sum_{p\in\mathcal{P}}\frac{\delta_p}{p}\in\mathbb{R}\cup\{\infty\}$. If the the last sum above is ∞ then the constants $(\delta_p)_{p\in\mathcal{P}}$ satisfy a. If the last sum above is a real number, then condition $\delta_p^{-1}\ll p^{\epsilon}$, $\forall\epsilon > 0$ implies that $\sum_{p\in\mathcal{P}}\frac{\delta_p}{p^{1+\varepsilon}}$ converges if and only $\varepsilon\geq 0$ and hence the constants $(\delta_p)_{p\in\mathcal{P}}$ satisfy b, finishing the proof of the Claim 2.6.

The proof of the second part of Theorem 1.2 will be done accordingly the constants δ_p satisfy a. or b. of Claim 2.6.

Case a. By Proposition 2.1 there exists a random function R_X analytic in $\mathbb{H}_{1/2}$ for all $\omega\in\Omega$ such that

$$-\log^*\zeta_X(z) = \sum_{p\in\mathcal{P}}\frac{X_p}{p^z} + R_X(z), z\in\mathbb{H}_1, \tag{12}$$

Claim 2.7. $0 \le -\mathbb{E}X_p \le 1$ and $\sum_{p\in\mathcal{P}} \frac{\mathbb{E}X_p}{p} = -\infty$ imply that

$$\mathbb{P}(\lim_{k\to\infty} 1/\zeta_X(1+1/k) = 0) = 1.$$

Proof of Claim 2.7. The proof is done by an adaption of the proof of Claim 2.2.

Claim 2.8. $0 \le -\mathbb{E}X_p \le 1$ and $\sum_{p\in\mathcal{P}} \frac{1+\mathbb{E}X_p}{p} = \infty$ imply that

$$\mathbb{P}\Big(\lim_{k\to\infty} \frac{\zeta(1+1/k)}{\zeta_X(1+1/k)} = \infty\Big) = 1.$$

Proof of the Claim 2.8. (12) applied for the sequence $\xi = (-1,-1,...)$ gives for $z \in \mathbb{H}_1$

$$\log^* \zeta(z) = \sum_{p\in\mathcal{P}} \frac{1}{p^z} + R(z), \tag{13}$$

where $R(z) := \sum_{p\in\mathcal{P}} \sum_{m=2}^{\infty} \frac{1}{mp^{mz}}$ is analytic in $\mathbb{H}_{1/2}$. Proposition 2.2 implies that with probability one the random function $z \mapsto \sum_{p\in\mathcal{P}} \frac{X_p - \mathbb{E}X_p}{p^z}$ is analytic in $\mathbb{H}_{1/2}$. Thus the random function $H : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C}$ given by $H(z) = \sum_{p\in\mathcal{P}} \frac{X_p-\mathbb{E}X_p}{p^z} + R(z) + R_X(z)$ is, with probability one, analytic in $\mathbb{H}_{1/2}$. In particular

$$\mathbb{P}\big(\lim_{k\to\infty} H(1+1/k) \in \mathbb{R}\big) = 1. \tag{14}$$

On the other hand, Fatou's Lemma implies that

$$\lim_{k\to\infty} \sum_{p\in\mathcal{P}} \frac{1+\mathbb{E}X_p}{p^{1+1/k}} = \infty. \tag{15}$$

By adding (12) at the point $z = 1 + k^{-1}$ with (13) at the same point we obtain:

$$\log \frac{\zeta(1+1/k)}{\zeta_X(1+1/k)} = \sum_{p\in\mathcal{P}} \frac{1+\mathbb{E}X_p}{p^{1+1/k}} + H(1+1/k).$$

This combined with equations (14) and (15) complete the proof of Claim 2.8.

End of the proof of a. Let ρ_X as in Proposition 2.3 and by contradiction assume $\mathbb{P}(\rho_X < 1) = 1$. Then Claim 2.7 and Theorem A.16 imply that there exists a random variable $m \ge 1$ such that with probability one $\frac{1/\zeta_X(z)}{(z-1)^m}$ is analytic and does not vanish on a open (random) ball $B = B(\omega)$ centered at $z = 1$. This implies that, since the Riemann ζ function has a simple pole at $z = 1$, $\mathbb{P}\big(\lim_{k\to\infty} \frac{\zeta(1+1/k)}{\zeta_X(1+1/k)} = \infty\big) = 0$, which contradicts Claim 2.8. Thus $\mathbb{P}(\rho_X < 1) < 1$ and hence, by Proposition 2.3, this probability equals to 0. Inequality (4) completes the proof.

Case b. Since $\delta_p \in [0,1]$, the function $A : \mathbb{H}_1 \to \mathbb{C}$ defined by $A(z) := -\sum_{p\in\mathcal{P}} \frac{\delta_p}{p^z}$ is analytic, since this series converges absolutely in this half plane. The random function

$H_2(z) =: \sum_{p\in\mathcal{P}} \frac{X_p - \mathbb{E}X_p}{p^z} + R_X(z)$ has the same structure of the function $H(z) - R(z)$, where H and R are as in the proof of the Case a above and hence we obtain:

$$\mathbb{P}(H_2 \text{ is analytic in } \mathbb{H}_{1/2}) = 1. \tag{16}$$

In the half plane $\mathbb{H}_1$ decompose (12) in the form

$$-\log^* \zeta_X = A + H_2. \tag{17}$$

Assumption $\delta_p \geq 0$ for all p and $\sum_{p\in\mathcal{P}} \frac{\delta_p}{p} < \infty$ combined with the monotone convergence Theorem implies that $\lim_{k\to\infty} A(1+1/k) \in \mathbb{R}$ in which combined with (16) and (17) gives that $\mathbb{P}(\lim_{k\to\infty} 1/\zeta_X(1+k^{-1}) \neq 0) = 1$. This implies that

$$\mathbb{P}([\rho_X < 1] \cap [H_2 \text{ is analytic in } \mathbb{H}_{1/2}] \cap [1/\zeta_X(1) \neq 0]) = \mathbb{P}(\rho_X < 1).$$

By contradiction assume that $\mathbb{P}(\rho_X < 1) = 1$. In particular exists $\omega' \in \Omega$ and $\epsilon = \epsilon(\omega') > 0$ such that:

$$H_2(\cdot, \omega') : \mathbb{H}_{1/2} \to \mathbb{C} \text{ is analytic;} \tag{18}$$

$$1/\zeta_{X(\omega')} : \mathbb{H}_{1-\epsilon} \to \mathbb{C} \text{ is analytic;} \tag{19}$$

$$1/\zeta_{X(\omega')}(1) \neq 0.. \tag{20}$$

Thus, (19) and (20) imply the existence of an open ball $B_{\omega'} \subset \mathbb{H}_{1-\epsilon}$ centered in $z = 1$ such that $1/\zeta_{X(\omega')}(z) \neq 0$ for all $z \in B_{\omega'}$. Then, by (17), $1/\zeta_{X(\omega')}(z) \neq 0$ for every $z \in \mathbb{H}_1 \cup B_{\omega'}$. Since $\mathbb{H}_1 \cup B_{\omega'}$ is a simply connected region, by Theorem A.17, there exists a branch of the logarithm $\log^* \zeta_{X(\omega')} : \mathbb{H}_1 \cup B_{\omega'} \to \mathbb{C}$ which is given by (17) for $z \in \mathbb{H}_1$. Thus the function $-A$ extends analytically to the region $\mathbb{H}_1 \cup B_{\omega'}$ given by the formula

$$-A(z) = \log^* \zeta_{X(\omega')}(z) + H_2(z, \omega').$$

Since the function $-A$ is a Dirichlet series consisted of positive terms, by Landau's Theorem (see definition A.4 and Theorem A.10), assumption $\sum_{p\in\mathcal{P}} \frac{\delta_p}{p^{1+\varepsilon}} < \infty$, iff $\varepsilon \geq 0$ implies that the function A has singularity at $z = 1$ and hence it can not be analytic in $\mathbb{H}_1 \cup B_{\omega'}$. This leads to a contradiction which implies $\mathbb{P}(\rho_X < 1) < 1$ and hence by Proposition 2.3 this probability is zero. Inequality (4) completes the proof, finishing the proof of Theorem 1.2. □

2.4 Neighborhood of $\beta = 1$

Theorem 1.4. Let $X = (X_p)_{p\in\mathcal{P}}$ be independent with distribution $\mathbb{P}(X_p = -1) = 1 - \delta_p = \mathbb{P}(X_p = +1)$. Assume that $\delta_p \to 0$ and $\delta_p^{-1} \ll p^{-\alpha}$ for some $\alpha \in [0, 1/2)$. Then

$$\mathbb{P}(M_X(N) = O(N^{1-\alpha-\varepsilon}), \text{ for some } \varepsilon > 0) = 0.$$

Proof. Without loss of generality, assume that $\delta_p \to 0$ and $\delta_p = b_p p^{-\alpha}$ for some $\alpha \in [0, 1/2)$ with $b_p^{-1} \ll p^{\epsilon}$ for all $\epsilon > 0$. Let X, Y and Z be sequences of independent random variables uniformly coupled, that is,

$$\begin{aligned} X_p(\omega) &= -\mathbb{1}_{[0,1-\delta_p]}(\omega_p) + \mathbb{1}_{(1-\delta_p,1]}(\omega_p), \\ Y_p(\omega) &= -\mathbb{1}_{[0,\frac{1}{2}-\delta_p]}(\omega_p) + \mathbb{1}_{(\frac{1}{2}-\delta_p,1]}(\omega_p), \\ X_{p,\frac{1}{2}}(\omega) &= -\mathbb{1}_{[0,\frac{1}{2}]}(\omega_p) + \mathbb{1}_{(\frac{1}{2},1]}(\omega_p). \end{aligned}$$

Claim 2.9. Let Y as above and ρ_Y as in Proposition 2.3. The condition $\delta_p \to 0$, $\alpha \in [0, 1/2)$ and $\delta_p = b_p p^{-\alpha}$ with $b_p^{-1} \ll p^{\epsilon}$ for all $\epsilon > 0$ implies that $\mathbb{P}(\rho_Y < 1 - \alpha) = 0$.

Proof of Claim 2.9. Introduce new random variables for each prime p:

$$\begin{aligned} W_p(\omega_p) &= \mathbb{1}_{(1/2-\delta_p,1/2+\delta_p)}(\omega_p), \\ Z_p(\omega_p) &= \mathbb{1}_{[1/2+\delta_p,1]}(\omega_p) - \mathbb{1}_{[0,1/2-\delta_p]}(\omega_p). \end{aligned}$$

The identities $Y_p = W_p + Z_p$ and $W_p \cdot Z_p = 0$ imply that $\mu_Y = \mu_W * \mu_Z$. Thus, by Proposition 2.1 and Theorem A.8, for all $\omega \in \Omega$ and $z \in \mathbb{H}_1$:

$$\frac{1}{\zeta_{Y(\omega)}(z)} = \frac{1}{\zeta_{W(\omega)}(z)} \frac{1}{\zeta_{Z(\omega)}(z)}. \tag{21}$$

Observe that the random variables Z_p are independent, $Z_p \in \{-1, 0, 1\}$ and $\mathbb{E}Z_p = 0$. Hence, similarly to the proof of Claim 2.3 we obtain

$$\mathbb{P}(1/\zeta_Z \text{ and } \zeta_Z \text{ have analytic extension to } \mathbb{H}_{1/2}) = 1. \tag{22}$$

On the other hand, since each $W_p \geq 0$, Theorem A.15 assures that for each $\omega \in \Omega$, $1/\zeta_{W(\omega)}(z) = \sum_{k=1}^{\infty} \frac{\mu_{W(\omega)}(k)}{k^z}$ and $\sum_{p\in\mathcal{P}} \frac{W_p(\omega)}{p^z}$ share the same abscissa of convergence $\sigma_c(\omega)$. Theorem A.10 implies that the random function $1/\zeta_W$ has a singularity at the random variable σ_c. The independence of the random variables $\{W_p\}_{p\in\mathcal{P}}$ implies that σ_c is a tail random variable and hence is a constant with probability one. In fact, being σ_c' the abscissa of convergence of the Dirichlet series $\sum_{p\in\mathcal{P}} \mathbb{E}\frac{W_p}{p^x} = 2\sum_{p\in\mathcal{P}} \frac{\delta_p}{p^x}$, by Kolmogorov 2-Series Theorem, with probability one σ_c equals to $\max\{\sigma_c', \frac{1}{2}\}$. The condition $\alpha \in [0, 1/2)$, $\delta_p = b_p p^{-\alpha}$ with $b_p^{-1} \ll p^{\epsilon}$ for all $\epsilon > 0$ imply that $\sigma_c' = 1 - \alpha$ and hence, $\mathbb{P}(\sigma_c = 1 - \alpha) = 1$. This combined with (21) and (22) imply that:

$$\mathbb{P}(1/\zeta_Y \text{ has a singularity at } z = 1 - \alpha) = \mathbb{P}(1/\zeta_W \text{ has a singularity at } z = 1 - \alpha) = 1,$$

and hence $\mathbb{P}(\rho_Y < 1 - \alpha) = 0$, finishing the proof of the Claim 2.9.

End of the proof of Theorem 1.4. Let $I_p = (\frac{1}{2} - a_p, \frac{1}{2}]$ and $J_p = (1 - a_p, 1]$. By Proposition 2.1, the random functions $\theta_X := \frac{\zeta}{\zeta_X}$ and $\chi_Y := \frac{\zeta_{1/2}}{\zeta_Y}$ are analytic $\mathbb{H}_1$ and have Euler product representation in this half plane. Moreover, using that these sequences are uniformly coupled we obtain that

$$\theta_X(z,\omega)=\prod_{p\in\mathcal{P}}\frac{p^z+\mathbb{1}_{J_p}(\omega_p)}{p^z-\mathbb{1}_{J_p}(\omega_p)},$$
$$\chi_Y(z,\omega)=\prod_{p\in\mathcal{P}}\frac{p^z+\mathbb{1}_{I_p}(\omega_p)}{p^z-\mathbb{1}_{I_p}(\omega_p)}.$$

Let $g_p:[0,1]\to[0,1]$ be the interval exchange transformation such that $g|_{[0,1]\setminus I_p\cup J_p}=Id$ and $g_p(I_p)=J_p$ and $g_p(J_p)=I_p$ are translations. Thus g_p preserves Lebesgue measure for each prime p and $\mathbb{1}_{I_p}=\mathbb{1}_{J_p}\circ g_p$. Thus the map $T:\Omega\to\Omega$ such that $(\omega_p)_{p\in\mathcal{P}}\mapsto(g_p(\omega_p))_{p\in\mathcal{P}}$ preserves $\mathbb{P}$ and together with the product representation for θ_X and χ_Y yield the following functional equation in $\mathbb{H}_1$

$$\chi_Y(z,\omega)=\theta_X(z,T\omega). \tag{23}$$

Since $\delta_p\to 0$, $\sum_{p\in\mathcal{P}}\frac{\mathbb{E}X_p}{p}=-\sum_{p\in\mathcal{P}}\frac{1-2\delta_p}{p}=-\infty$. Hence, by adapting the proof of the Claim 2.2 we obtain $\mathbb{P}(\lim_{k\to\infty}1/\zeta_X(1+1/k)=0)=1$. Thus on the event $[\rho_X<1-\alpha]\cap[1/\zeta_X(1)=0]$, the random function θ_X has analytic extension to the same half plane where $1/\zeta_X$ has it, because by Theorem A.16, the zero of the function $1/\zeta_X$ at $z=1$ cancel with the simple pole at $z=1$ of the Riemann ζ function. This combined with the fact that T preserves measure, (23), Claim 2.3 and Claim 2.9 imply, respectively:

$$\begin{aligned}\mathbb{P}(\rho_X<1-\alpha)&=\mathbb{P}([\rho_X<1-\alpha]\cap[1/\zeta_X(1)=0])\\&\leq\mathbb{P}\big(\exists\,\varepsilon>0:\ \theta_X\text{ is analytic in }\mathbb{H}_{1-\alpha-\varepsilon}\big)\\&=\mathbb{P}\big(\exists\varepsilon>0:\ \chi_Y\text{ is analytic in }\mathbb{H}_{1-\alpha-\varepsilon}\big)\\&=\mathbb{P}\big(\exists\varepsilon>0:\ \chi_Y\cdot\zeta_{1/2}^{-1}\text{ is analytic in }\mathbb{H}_{1-\alpha-\varepsilon}\big)\\&=\mathbb{P}(\rho_Y<1-\alpha)=0.\end{aligned}$$

Inequality (4) completes the proof of Theorem 1.4. □

2.5 The main technical Theorem

Exceptionally in this section each random variable X_p on Definition 1.1 may assume arbitrary values on $\mathbb{R}$. The aim is to prove the following result:

Theorem 2.1. Let $(X_p)_{p\in\mathcal{P}}$ be independent random variables such that:

i. For some $C\geq 1$ and some $\alpha\in[0,\frac{1}{2})$:

$$|X_p|\leq\begin{cases}1 & ,\ \text{if } C>\sqrt{p};\\ Cp^\alpha & ,\ \text{if } C\leq\sqrt{p}.\end{cases}$$

ii. $\mathbb{E}X_p = -1$ except on a finite subset of primes and $\mathbb{E}|X_p| \leq 1$ for all p. Then

$$\mathbb{P}(M_X(N) \ll N^{1/2+\alpha+\varepsilon}, \forall \varepsilon > 0) = 1$$

if and only if the zeros of the Riemann Zeta function have real part less or equal than $1/2 + \alpha$.

In the next section the Theorem 1.3 will be obtained as a consequence from Theorem 2.1 and in Section 3 some interesting applications.

Being the random variables $(X_p)_{p\in\mathcal{P}}$ independent, the abscissa of absolute convergence (see definition A.4) of the random series $\sum_{p\in\mathcal{P}} \frac{X_p}{p^x}$ is a tail random variable and hence with probability one it is equal to a extended real number ρ. Denote

$$\Omega' = \left[\sum_{p\in\mathcal{P}} \frac{X_p}{p^x} \text{ converges absolutely in } \mathbb{H}_\rho \right]$$

and define the random Riemann zeta function $\zeta_X : \mathbb{H}_\rho \times \Omega \to \mathbb{C}$ as:

$$\zeta_{X(\omega)}(z) = \begin{cases} \prod_{p\in\mathcal{P}} \frac{1}{1+\frac{X_p(\omega)}{p^z}}, & \text{if } \omega \in \Omega' \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Lemma 2.2. Let $(X_p)_{p\in\mathcal{P}}$ be independent random variables satisfying conditions i. and ii. of Theorem 2.1. Then with probability one the random function $\theta_X := \frac{\zeta}{\zeta_X} : \mathbb{H}_{1/2+\alpha} \times \Omega \to \mathbb{C}$ is analytic and

$$\theta_X(z) = \exp\left(\sum_{p\in\mathcal{P}} \frac{X_p - \mathbb{E}X_p}{p^z} \right) \exp(A_X(z)), \tag{25}$$

where $A_X : \mathbb{H}_{1/2+\alpha} \times \Omega \to \mathbb{C}$ is analytic for all $\omega \in \Omega$ and satisfies:

$$A_X(z) \ll_\delta 1, \quad \text{in } \overline{\mathbb{H}}_{\frac{1}{2}+\alpha+\delta}.$$

Proof. Define $R : \mathbb{H}_{1/2} \to \mathbb{C}$ as $R(z) := \sum_{p\in\mathcal{P}} \sum_{m=2}^{\infty} \frac{1}{mp^{mz}}$ and $R_X : \mathbb{H}_{1/2+\alpha} \times \Omega \to \mathbb{C}$ as $R_X(z) := \sum_{p\in\mathcal{P}} \sum_{m=2}^{\infty} (-1)^{m+1} \frac{X_p^m}{mp^{mz}}$. By Proposition A.1, R and R_X are analytic for all $\omega \in \Omega$ and satisfy:

$$R + R_X \ll_\delta 1, \text{ in } \overline{\mathbb{H}}_{1/2+\alpha}. \tag{26}$$

Since the set $\mathcal{P}' = \{p : \mathbb{E}X_p \neq -1\}$ is finite and $\mathbb{E}|X_p| \leq 1$, the function $z \mapsto \sum_{p\in\mathcal{P}'} \frac{\mathbb{E}X_p+1}{p^z}$ is analytic in $\mathbb{H}_{1/2+\alpha}$ and satisfies in the closure $\overline{\mathbb{H}}_{1/2+\alpha}$:

$$\left| \sum_{p\in\mathcal{P}'} \frac{\mathbb{E}X_p + 1}{p^z} \right| \ll 1. \tag{27}$$

Defining $A_X : \mathbb{H}_{1/2+\alpha} \times \Omega \to \mathbb{C}$ by

$$A_X(z) = R(z) + R_X(z) + \sum_{p\in\mathcal{P}'} \frac{\mathbb{E}X_p + 1}{p^z},$$

(26) and (27) imply that $A_X \ll_\delta 1$, in $\overline{\mathbb{H}}_{1/2+\alpha+\delta}$. By Proposition A.1, for $z \in \mathbb{H}_{1+\alpha}$ $\log^* \zeta(z) = \sum_{p\in\mathcal{P}} \frac{1}{p^z} + R(z)$, $-\log^* \zeta_X(z) = \sum_{p\in\mathcal{P}} \frac{X_p}{p^z} + R_X(z)$. Thus

$$\log^* \frac{\zeta(z)}{\zeta_X(z)} = \sum_{p\in\mathcal{P}} \frac{X_p - \mathbb{E}X_p}{p^z} + A_X(z).$$

Since for each $x > 1/2 + \alpha$, $\sum_{p\in\mathcal{P}} \mathbb{V}\frac{X_p - \mathbb{E}X_p}{p^x} \ll \sum_{p\in\mathcal{P}} \frac{1}{p^{2(x-\alpha)}} < \infty$, Proposition 2.2 implies that with probability one the random function $z \mapsto \sum_{p\in\mathcal{P}} \frac{X_p - \mathbb{E}X_p}{p^z}$ is analytic in $\mathbb{H}_{1/2+\alpha}$. Thus, with probability one $\log^* \frac{\zeta(z)}{\zeta_X(z)}$ is analytic in $\mathbb{H}_{1/2+\alpha}$ and since $\theta_X = \exp\big(\log^* \frac{\zeta}{\zeta_X}\big)$ in $\mathbb{H}_{1+\alpha}$, this relation extends analytically to the half plane $\mathbb{H}_{1/2+\alpha}$. $\square$

Definition 2.4. Given $c \geq 0$ and a random function $f : \mathbb{H}_c \times \Omega \to \mathbb{C}$, $L_f : (c, \infty) \times \Omega \to \mathbb{R} \cup \{\infty\}$ stands for the random Lindellöff function, that is

$$L_f(\sigma, \omega) := \inf\{A \geq 0 : \ f(\sigma + it, \omega) \ll t^A\}.$$

Lemma 2.3. Let $X = (X_p)_{p\in\mathcal{P}}$ satisfy condition i of Lemma 2.2 and ρ_X be as in Proposition 2.3. Then ρ_X is a tail random variable. Moreover, for $c \in (1/2 + \alpha, 1 + \alpha)$,

$$U_X(c, A) := [\rho_X \leq c] \cap [L_{1/\zeta_X}(\sigma)\} \leq A, \ \forall \sigma > c]$$

also is a tail event and

$$\mathbb{P}(U_X(c, 0)) \leqslant \mathbb{P}(M_X(N) \ll N^{c+\delta}, \ \forall \delta > 0) \leq \mathbb{P}(\rho_X \leqslant c). \tag{28}$$

Proof. By Proposition A.1, for all $\omega \in \Omega$ the Dirichlet series $1/\zeta_{X(\omega)}$ is analytic in $\mathbb{H}_{1+\alpha}$. Thus, by the same arguments of the proof of Proposition 2.3, the set $[\rho_X \leqslant c]$ is measurable in the sigma algebra generated by the independent random variables $(X_p)_{p\in\mathcal{P}}$. Let D be a finite subset of prime numbers, $\mathcal{F}_D$ be the sigma algebra generated by the random variables $\{X_p : \ p \in D\}$ and $Y = (Y_p)_{p\in\mathcal{P}}$ be such that

$$Y_p = \begin{cases} X_p, & \text{if } p \in D; \\ 0, & \text{if } p \in \mathcal{P} \setminus D. \end{cases}$$

Observe that $\frac{1}{\zeta_X} = \frac{1}{\zeta_Y} \prod_{p\in\mathcal{P}\setminus D} \big(1 + \frac{X_p}{p^z}\big)$, and that the sequence Y satisfies the hypothesis of Proposition A.1. Hence, by statement iii. of this proposition:

$$-\log^* \zeta_{Y(\omega)}(z) = \sum_{p\in D} \frac{X_p(\omega)}{p^z} + R_{Y(\omega)}(z),$$

where R_Y is analytic in $\mathbb{H}_{1/2+\alpha}$ and satisfies $|R_Y| \ll_\delta 1$ in $\overline{\mathbb{H}}_{1/2+\alpha+\delta}$, for all $\omega \in \Omega$. Hence, since $|D| < \infty$, $-\log^* \zeta_Y$ also is analytic in this half plane for all $\omega \in \Omega$ and satisfies: $\log^* \zeta_Y \ll_\delta 1$, in $\overline{\mathbb{H}}_{1/2+\alpha+\delta}$. Subsequently:

$$\zeta_Y, \frac{1}{\zeta_Y} \ll_\delta 1, \text{ in } \overline{\mathbb{H}}_{1/2+\alpha+\delta}. \tag{29}$$

Thus $1/\zeta_X$ extends analytically to $\mathbb{H}_c$ if and only if ζ_Y/ζ_X extends analytically to the same half plane. Since ζ_Y/ζ_X depends only the random variables $(X_p)_{p\in\mathcal{P}\setminus D}$, we conclude that $[\rho_X \le c]$ is independent from $\mathcal{F}_D$ and hence ρ_X is a tail random variable.

By the continuity of the Lindelöff function (Theorem A.12):

$$U_X(c, A) = [\rho_X \le c] \bigcap_{\sigma\in\mathbb{Q}\cap(c,\infty)} [L_{1/\zeta_X}(\sigma) \le A].$$

Thus $U_X(c, A)$ is measurable in the sigma algebra generated by the independent random variables $(X_p)_{p\in\mathcal{P}}$. On the other hand, on the event $[\rho_X \le c]$, for $\sigma > c$ (29) implies: $L_{\frac{1}{\zeta_X}}(\sigma) = L_{\frac{\zeta_Y}{\zeta_X}}(\sigma)$. Since $\frac{\zeta_Y}{\zeta_X}$ is independent from $\mathcal{F}_D$, $L_{\frac{\zeta_Y}{\zeta_X}}$ also is independent from $\mathcal{F}_D$. Thus $U_X(c, A)$ is a tail event.

The second inequality in (28) follows from Theorem A.7. For the first inequality, if $\omega \in U_X(c, 0)$, by Theorem A.13 the series $\sum_{k=1}^\infty \frac{\mu_{X(\omega)}(k)}{k^z}$ converges in $\mathbb{H}_c$. In particular this series converges at every point $c + \delta$ for any $\delta > 0$. This combined with Kronecker's Lemma (Theorem A.6) gives that $\lim_{N\to\infty} \frac{M_{X(\omega)}(N)}{N^{c+\delta}} = 0$. This completes the proof. □

Lemma 2.4. Let $\{X_k\}_{k\in\mathbb{N}}$ be independent random variables with 0 expectation. Assume that for some $\alpha > 0$, $|X_n| \ll n^\alpha$. Let $\sigma > 1/2 + \alpha$. Then there exists a finite random variable C_σ such that for each $x = 1/2 + \alpha + \kappa \ge \sigma$ with $\kappa \le \frac{1}{2}$:

$$\mathbb{P}\left(\left| \sum_{k=1}^\infty \frac{X_k}{k^{x+it}} \right| \le C_\sigma(\log(|t| + e))^{\frac{1}{2}-\kappa}(\log\log(|t| + e^e))^{\frac{1}{2}+\kappa} \right) = 1.$$

In [3], F. Carlson proved that if $(X_k)_{k\in\mathbb{N}}$ are independent, uniformly bounded and *symmetric* then for all $\sigma > 1/2$

$$\mathbb{P}\left(\sum_{k=1}^\infty \frac{X_k}{k^{\sigma+it}} = o(\sqrt{\log t}\,) \right) = 1.$$

The Lemma 2.4 improve Carlson's result in two directions: Firstly the *symmetry* condition is weakened by $\mathbb{E}X_k = 0$ for all k and secondly the exponent $1/2$ is decreased. Indeed, this improvement is obtained by combining Carlson's ideas with the Central Limit Theorem encoded in the Marcinkiewicz-Zygmund inequality and the Hadamard Tree Circles Theorem (see Theorems A.4 and A.18, respectively).

Proof. Let $F(z) := \sum_{k=1}^{\infty} \frac{X_k}{k^z}$ and denote $F_\omega = F(\cdot,\omega)$. For each $x > \frac{1}{2}+\alpha$, $\sum_{k=1}^{\infty} \mathbb{V}\frac{X_k}{k^x} \ll \sum_{k=1}^{\infty} \frac{1}{k^{2(x-\alpha)}} < \infty$ and hence by Proposition 2.2 the event

$$\Omega^* := [F(z) \text{ converges and is analytic in } \mathbb{H}_{1/2+\alpha}]$$

has probability one. Let $\epsilon \in (0,1/3]$, $\sigma = 1/2+\alpha+\epsilon$ and $\sigma' = 1/2+\alpha+\epsilon/2$. Let R_1 and R_2 be the rectangles:

$$\begin{aligned} R_1 =& R_1(T,\epsilon) = [\sigma, 4/3+\alpha] \times [-e^{T-2}, e^{T-2}], \\ R_2 =& R_2(T,\epsilon,\omega) = [\sigma', \sigma'+1] \times [-\tau'(\omega), \tau(\omega)] \end{aligned}$$

whith $\tau,\tau' \in [e^{T-1}, e^T]$ to be chosen later. Observe that $R_1 \subset R_2$ and the distance from R_1 to R_2 equals to $\epsilon/2$. For $T \in \mathbb{N}$ and $\omega \in \Omega^*$, F_ω^T is analytic on $\mathbb{H}_{1/2+\alpha}$ and hence, by the Cauchy integral formula, for each $z \in R_1$:

$$|F_\omega(z)|^T = |F_\omega(z)^T| = \frac{1}{2\pi}\left| \int_{\partial R_2} \frac{F_\omega(s)^T}{s-z} ds \right| \leq \frac{1}{\pi\epsilon} \int_{\partial R_2} |F_\omega(s)|^T ds,$$

and hence:

$$\|F_\omega\|_{L^\infty(R_1)} \leq \frac{1}{\epsilon^{1/T}} \|F_\omega\|_{L^T(\partial R_2)}. \tag{30}$$

Claim 2.10. Let $\epsilon^{-1} = 3\log(T+e)$. There exists H_1 such that

$$\mathbb{P}(\Omega^* \cap [\ \|F\|_{L^\infty(R_1)} > H_1\sqrt{\epsilon^{-1}T}\ ]) \leq 2e^{-T}. \tag{31}$$

Proof of the Claim 2.10. The complex Marcinkiewicz-Zygmund inequality (Lemma A.1) combined with Fatou's Lemma gives for each $T \in \mathbb{N}$ and $y \in \mathbb{R}$:

$$\mathbb{E}|F(\sigma'+iy)|^T \leq (2B)^T T^{T/2} \mathbb{E}\left[\left(\sum_{k=1}^{\infty} \frac{|X_k|^2}{k^{2\sigma'}}\right)^{T/2}\right],$$

and since by hypothesis $X_n \ll n^\alpha$, there exists $\Lambda > 0$ such that:

$$\begin{aligned} \mathbb{E}|F(\sigma'+iy)|^T &\leq (\epsilon^{-1}\Lambda T)^{T/2}, \\ |F(\sigma'+1+iy)|^T &\leq (\Lambda)^{T/2}. \end{aligned} \tag{32}$$

Hence, by Fubini's Theorem:

$$\mathbb{E}\int_{-e^T}^{e^T} |F(\sigma'+iy)|^T dy = \int_{-e^T}^{e^T} \mathbb{E}|F(\sigma'+iy)|^T dy \leq (4e\epsilon^{-1}\Lambda T)^{T/2}$$

$$\mathbb{E}\int_{\sigma'}^{\sigma'+1}\int_{-e^T}^{e^T} |F(x+iy)|^T dxdy = \int_{\sigma'}^{\sigma'+1}\left(\mathbb{E}\int_{-e^T}^{e^T} |F(x+iy)|^T dy\right)dx \leq (4e\epsilon^{-1}\Lambda T)^{T/2}.$$

Thus, by Markov's inequality we obtain for $H = 4e^3\Lambda$:

$$\mathbb{P}(A_\epsilon^T) := \mathbb{P}\bigg(\Omega^* \cap \bigg[\int_{-e^T}^{e^T} |F(\sigma' + iy)|^T dy \geq (\epsilon^{-1}HT)^{T/2}\bigg]\bigg) \leq e^{-T},$$

$$\mathbb{P}(B_\epsilon^T) := \mathbb{P}\bigg(\Omega^* \cap \bigg[\int_{\sigma'}^{\sigma'+1}\int_{-e^T}^{e^T} |F(x + iy)|^T dxdy \geq (\epsilon^{-1}HT)^{T/2}\bigg]\bigg) \leq e^{-T},$$

Define $E_\epsilon^T = \Omega^* \cap (A_\epsilon^T \cup B_\epsilon^T)^c$. Then $\mathbb{P}(E_\epsilon^T) \geq 1 - 2e^{-T}$ and for each $\omega \in E_\epsilon^T$:

$$\int_{-e^T}^{e^T} |F_\omega(\sigma' + iy)|^T dy < (\epsilon^{-1}HT)^{T/2}, \tag{33}$$

$$\int_{\sigma'}^{\sigma'+1}\int_{-e^T}^{e^T} |F_\omega(x + iy)|^T dxdy < (\epsilon^{-1}HT)^{T/2}. \tag{34}$$

Decompose: $\partial R_2 = I_1 \cup I_2 \cup I_3 \cup I_4$, where I_1 and I_3 are the vertical lines at $Re(s) = \sigma' + 1$ and $Re(s) = \sigma'$ respectively and I_2 and I_4 are the horizontal lines at $Im(s) = -\tau'$ and $Im(s) = \tau$ respectively. Thus (32), (33) and (34) implies that for each $\omega \in E_\epsilon^T$ we can choose $\tau', \tau \in [e^{T-1}, e^T]$ such that:

$$\int_{I_j} |F_\omega(s)|^T ds \leq (\epsilon^{-1}HT)^{T/2},\ j = 1, 2, 3, 4. \tag{35}$$

In fact, for any choice of $\tau, \tau' \leq e^T$, (32) and (33) gives the bound for I_1 and I_3. To prove the bound for I_2 and I_4, assume by contradiction that do not exists τ and τ' as above. Then for all $y \in [e^{T-1}, e^T]$, $(\epsilon^{-1}HT)^{T/2} < \int_{\sigma'}^{\sigma'+1} |F_\omega(x + iy)|^T dx$, which combined with Fubini's Theorem and (34) implies that:

$$\begin{aligned} e^{T-1}(e-1)(\epsilon^{-1}HT)^{T/2} &\leq \int_{e^{T-1}}^{e^T}\int_{\sigma'}^{\sigma'+1} |F_\omega(x + iy)|^T dxdy \\ &\leq \int_{\sigma'}^{\sigma'+1}\int_{-e^T}^{e^T} |F_\omega(x + iy)|^T dxdy \leq (\epsilon^{-1}HT)^{T/2}, \end{aligned}$$

and hence $e^{T-1}(e-1) \leq 1$, which is an absurd for $T \geq 1$.

Let $H_1 = 4\theta\sqrt{H}$ where $\theta = \sup_{T\geq 1} \log(T + e)^{1/T}$. Thus (35) and (30) implies that for $\omega \in E_\epsilon^T$: $\|F_\omega\|_{L^\infty(R_1)} \leq \theta \|F_\omega\|_{L^T(\partial R_2)} \leq H_1\sqrt{\epsilon^{-1}T}$, finishing the proof of the Claim 2.10.

Claim 2.11. Let $t \in \mathbb{R}$, $\epsilon^{-1} = 3\log\log(|t| + e^e)$, $\sigma = 1/2 + \alpha + \epsilon$ and H_1 as in the Claim 2.10. Denote $R = R(t) := [\sigma, \frac{4}{3} + \alpha] \times [-t, t]$. Then there exists Ω' with $\mathbb{P}(\Omega') = 1$ such that for each $\omega \in \Omega'$ there exists a real number $t_0 = t_0(\omega)$ such that for $t \geq t_0$:

$$\|F_\omega\|_{L^\infty(R)} \leq H_1\sqrt{\epsilon^{-1}\log(|t| + e)}.$$

Proof of the Claim 2.11. Claim 2.10 implies that

$$\sum_{T=1}^{\infty}\mathbb{P}(\Omega^*\cap[\ \|F\|_{L^\infty(R_1)}>H_2\sqrt{T\log(T+e)}\ ])\leq\sum_{T=1}^{\infty}2e^{-T}\infty$$

and hence by the Borel-Cantelli Lemma:

$$\mathbb{P}(\Omega^*\cap[\ \|F\|_{L^\infty(R_1)}>H_1\sqrt{T\log(T+e)}\ ]\text{ for infinitely many }T's.\ )=0.$$

Thus there exists a event Ω' of probability one such that for each $\omega\in\Omega'$ exists $T_0=T_0(\omega)$ such that for all integer $T\geq T_0$

$$\|F_\omega\|_{L^\infty(R_1)}\leq H_1\sqrt{T\log(T+e)}. \tag{36}$$

Let $t_0=e^{T_0}$. For real t with $|t|\geq t_0$ let $T-5=\lfloor\log(|t|+e)+1\rfloor$ and $R=R(t)$ as in the statement of the Claim 2.11. Thus $T\ll\log(|t|+e)$, $\log(T+e^e)\ll\epsilon^{-1}$ and $R(t)\subset R_1(T,\epsilon)$. Hence by (36), $\|F\|_{L^\infty(R)}\leq\|F\|_{L^\infty(R_1)}\leq H_1\sqrt{\epsilon^{-1}\log(|t|+e)}$, finishing the proof of the Claim 2.11.

End of the Proof of Lemma 2.4. Let $\omega\in\Omega'$, $t_0(\omega)$ and $\epsilon=\epsilon(t)$ as in the Claim 2.11. Let $\sigma=\frac{1}{2}+\alpha+\epsilon$ and $x=\frac{1}{2}+\alpha+\kappa$ with $\kappa\in(0,1/2]$. Let t_1 be such that $\epsilon(t_1)<\kappa$ and hence $\sigma<x$ for $|t|\geq t_1$. Define $t_2=t_2(x,\omega)=\max\{t_0(\omega),t_1\}$. Since $F_\omega(\sigma-it)=F_\omega(\sigma+it)^*$, it is enough establish Lemma 2.4 for $t>t_2(\omega)$. Let $\lambda=o(t)$ be a large number to be chosen later and C_1,C_2,C_3 be concentric circles with center $\lambda+it$ and passing trough the points: $\sigma+\frac{1}{2}+it$, $x+it$ and $\sigma+it$ respectively (see the figure 1 below). Thus, the respective radius of C_1,C_2,C_3 are:

$$\begin{aligned}r_1=&\lambda-\sigma-\frac{1}{2},\\ r_2=&\lambda-x,\\ r_3=&\lambda-\sigma.\end{aligned}$$

Denote $M_j=M_j(\omega)=\max_{z\in C_j}|F_\omega(z)|$, $j=1,2,3$. Thus by the Hadamard Three-Circles Theorem (Theorem A.18):

$$M_2\leq M_1^{1-a}M_3^a, \tag{37}$$

where $a=\frac{\log(\frac{r_2}{r_1})}{\log(\frac{r_3}{r_1})}$. Assumption $X_n\ll n^\alpha$ implies that

$$M_1\leq\sum_{k=1}^{\infty}\frac{|X_k|}{k^{\sigma+\frac{1}{2}}}\ll\sum_{k=1}^{\infty}\frac{1}{k^{1+\epsilon}}\ll\frac{1}{\epsilon}. \tag{38}$$

Denote $\tau=\lambda^{-1}$. Hence

$$a=a(\tau)=\frac{\log\left(\frac{1-\tau x}{1-\tau(\sigma+\frac{1}{2})}\right)}{\log\left(\frac{1-\tau\sigma}{1-\tau(\sigma+\frac{1}{2})}\right)}=\frac{\log\left(1+\tau\frac{\frac{1}{2}-\kappa+\epsilon}{1-\tau(\sigma+\frac{1}{2})}\right)}{\log\left(1+\tau\frac{\frac{1}{2}}{1-\tau(\sigma+\frac{1}{2})}\right)}.$$

Using that for φ small, $\log(1+\varphi) = \varphi + O(\varphi^2)$, we obtain:

$$\begin{aligned} a(\tau) =& \left(\tau \frac{\frac{1}{2} - \kappa + \epsilon}{1 - \tau(\sigma + \frac{1}{2})} + O(\tau^2)\right) \cdot \left(\frac{2}{\tau} \cdot \frac{1 - \tau(\sigma + \frac{1}{2})}{1 + O(\tau^2)}\right) \\ =& 1 - 2\kappa + 2\epsilon + O(\lambda^{-1}). \end{aligned} \tag{39}$$

Let $R = R(t+\lambda)$ as in the Claim 2.11. Thus $C_3 \cap \mathbb{H}^c_{4/3+\alpha} \subset R$ (see the figure 1 below). Thus, since F converges absolutely in $\mathbb{H}_{1+\alpha}$:

$$M_3 \leq \max\{\|F\|_{L^\infty(R)}\,,\, \|F\|_{L^\infty(\mathbb{H}_{4/3+\alpha})}\,\} \ll \|F\|_{L^\infty(R)}.$$

Thus, by combining Claim 2.11 with (39) we obtain:

$$M_3^a \ll (\,\epsilon^{-1}\log(t+\lambda+e)\,)^{1/2-\kappa+\epsilon+O(\lambda^{-1})}.$$

Choose $\lambda = \epsilon^{-1}$. Since $\epsilon \to 0$, $\epsilon^{O(\epsilon)} = O(1)$ and since $\epsilon^{-1} = 3\log\log(t+e^e)$, also $(\log(|t+\lambda|+3))^{O(\epsilon)} = O(1)$. This and (38) gives respectively:

$$\begin{aligned} M_3^a \ll& \epsilon^{-(1/2-\kappa)}(\log(|t|+3))^{\frac{1}{2}-\kappa}, \\ M_1^{1-a} \ll& \epsilon^{-2\kappa}, \end{aligned}$$

and hence by (37):

$$|F_\omega(x+it)| \leq M_1^{1-a}M_3^a \ll (\log(|t|+3))^{\frac{1}{2}-\kappa}(\log\log(|t|+e^e))^{1/2+\kappa}.$$

Observe that the inequality above holds with the same constant for $F_\omega(y+it)$ with $y \in [x, 1+\alpha]$ and $t > t_2(\omega)$. Let $R'(\omega) = [x, 4/3+\alpha] \times [0, t_2(\omega)]$ (see the figure 1 below). Thus $R' \subset R$ and by our choice of $t_2 = t_2(x,\omega)$

$$\|F_\omega\|_{L^\infty(R')} \leq \sqrt{3}H_1\sqrt{\log(t_2+e)\log\log(t_2+e^e)},$$

where H_1 is the constant of Claim 2.11. Thus there exists a constant $C_x(\omega)$ such that for $y = \frac{1}{2} + \alpha + \kappa' \geq x$ with $\kappa' \leq 1/2$:

$$|F_\omega(y+it)| \leq C_x(\omega)(\log(|t|+3))^{\frac{1}{2}-\kappa'}(\log\log(|t|+e^e))^{1/2+\kappa'},$$

finishing the proof of Lemma 2.4. □

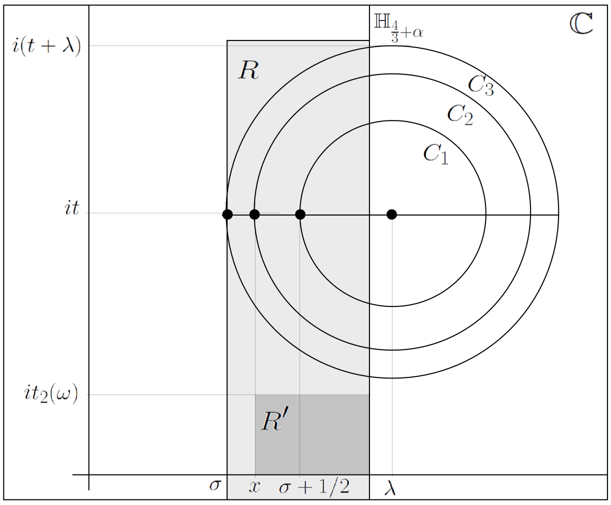

Figure 1: The concentric circles C_1, C_2, C_3 and the rectangles R and R'.

Proof of Theorem 2.1. The only if implication. Let $\theta_X = \zeta/\zeta_X$. By Lemma 2.2

$$\Omega' = [\,\theta_X \text{ and } 1/\theta_X \text{ have analytic extension to } \mathbb{H}_{1/2+\alpha}\,]$$

has probability one. Thus, hypothesis $\mathbb{P}(M_X(N) \ll N^{1/2+\alpha+\varepsilon}, \forall\varepsilon > 0) = 1$ combined with the second inequality in (28) imply that $\mathbb{P}(\rho_X = 1/2+\alpha) = 1$. Since for all $\omega \in \Omega$ we have in $\mathbb{H}_{1+\alpha}$

$$\frac{1}{\zeta(z)} = \frac{1}{\theta_X(z)} \cdot \frac{1}{\zeta_X(z)}, \tag{40}$$

there exists $\omega_0 \in \Omega' \cap [\rho_X = 1/2+\alpha]$ where the function in the right side of (40) is a product of two analytic functions in $\mathbb{H}_{1/2+\alpha}$ providing in this way one analytic extension of the function $1/\zeta$ to this half plane. Hence the zeros of the Riemann zeta function can only have real part less or equal than $1/2+\alpha$.

The if implication. Assuming that the Riemann zeta function does not have zeros in $\mathbb{H}_{1+\alpha}$, by (40) we obtain for each $\omega \in \Omega'$ that $1/\zeta_{X(\omega)}$ has analytic extension to the half plane $\mathbb{H}_{1/2+\alpha}$ given by the product:

$$\frac{1}{\zeta_{X(\omega)}} = \frac{1}{\zeta} \cdot \theta_{X(\omega)}, \tag{41}$$

and hence $\mathbb{P}(\rho_X = 1/2+\alpha) = 1$. Observe that the Lindelöff function (see Definition 2.4) L_{1/ζ_X} satisfies for each $\sigma > 1/2+\alpha$:

$$L_{1/\zeta_X}(\sigma,\omega) \leq L_{\theta_X(\sigma,\omega)} + L_{1/\zeta}(\sigma). \tag{42}$$

Littlewood's result ([14]) implies that the absence of zeros of ζ in $\mathbb{H}_{1/2+\alpha}$ gives that $L_{1/\zeta}(\sigma) = 0$ for $\sigma > 1/2+\alpha$. On the other hand, by Lemma 2.2, with probability one

$$\theta_X(z) \ll \exp\Big(\sum_{p\in\mathcal{P}} \frac{X_p - \mathbb{E}X_p}{p^z}\Big)$$

and by Lemma 2.4, also with probability one

$$\sum_{p\in\mathcal{P}} \frac{X_p - \mathbb{E}X_p}{p^{\sigma+it}} \ll (\,\log(t+e)\,)^{1-\sigma}(\,\log\log(t+e^e)\,)^{\sigma}$$

and hence $\mathbb{P}(L_{\theta_X}(\sigma) = 0, \forall\sigma > 1/2+\alpha) = 1$. By (42) also $\mathbb{P}(L_{1/\zeta_X}(\sigma) = 0, \forall\sigma > 1/2+\alpha) = 1$. Thus the event $U_X(1/2+\alpha, 0)$ of Lemma 2.3 has probability one and finally (28) completes the proof. □

2.6 α Random Möbius function

In this section we consider the sequence of independent random variables $X^\alpha = (X_p^\alpha)_{p\in\mathcal{P}}$ such that:

$$\mathbb{P}(X_p^\alpha = -1) = \frac{1}{2} + \frac{1}{2p^\alpha} = 1 - \mathbb{P}(X_p^\alpha = +1). \tag{43}$$

The aim is to prove the following result:

Theorem 1.3 The Riemann hypothesis is equivalent to:

$$\lim_{\alpha\to 0} \mathbb{P}(M_{X^\alpha}(N) \ll N^{1/2+\varepsilon}, \forall\varepsilon > 0) = 1. \tag{44}$$

Proof. Let $\alpha \in (0, 1/2)$. Consider $Y_p = p^\alpha X_p^\alpha$. Then $Y = (Y_p)_{p\in\mathcal{P}}$ satisfies the conditions i. and ii. of Theorem 2.1 with parameters $C = 1$ and same α. Thus the Riemann zeta function ζ is free of zeros on $\mathbb{H}_{1/2+\alpha}$ if and only if $\mathbb{P}(M_Y(N) \ll N^{1/2+\alpha+\varepsilon},\ \forall\varepsilon > 0) = 1$. Thus by Lemma A.7 and Kronecker's Lemma (Lemma A.6) this probability is one if and only if with probability one the series $\sum_{k=1}^{\infty} \frac{\mu_Y(k)}{k^{1/2+\alpha+\varepsilon}}$ converges for all $\varepsilon > 0$. Since $\mu_Y(k) = k^\alpha \mu_{X^\alpha}(k)$ for each k, this series converges with probability one for all $\varepsilon > 0$ if and only if with probability one the series $\sum_{k=1}^{\infty} \frac{\mu_{X^\alpha}(k)}{k^{1/2+\varepsilon}}$ converges for all $\varepsilon > 0$. Hence, by Lemma A.7 and Kronecker's Lemma this convergence is equivalent to

$$\mathbb{P}(M_{X^\alpha}(N) \ll N^{1/2+\varepsilon}, \forall\varepsilon > 0) = 1. \tag{45}$$

Thus (45) is equivalent to ζ be free of zeros on $\mathbb{H}_{1/2+\alpha}$. Hence

$$\lim_{\alpha\to 0} \mathbb{P}(M_{X^\alpha}(N) \ll N^{1/2+\varepsilon}, \forall\varepsilon > 0) = 1$$

is equivalent to ζ be free of zeros on $\mathbb{H}_{1/2}$ which is equivalent to the Riemann Hypothesis, since the zeros of ζ are symmetrically reflected over the critical line $Re(z) = 1/2$. □

By Claim 2.4 we obtain that

$$\mathbb{P}(M_{X^\alpha}(N) \ll N^{1/2+\alpha+\varepsilon},\ \forall\varepsilon > 0) = 1.$$

Thus Theorem 1.3 motivates, at least, to prove the bound $\mathbb{P}(M_{X^\alpha}(N) \ll N^{1-\alpha}) = 1$ for all $\alpha \in (0, 1/2)$. Below we prove this bound for $\alpha < 1/3$ and for $1/3 \leq \alpha < 1/2$ this bound still unknown. The reason $\alpha < 1/3$ is related to the bound obtained in Lemma 2.4 and the best zero free region for the Riemann Zeta function obtained by Vinogradov and Korobov ([18], Chapter VI). This zero free region R is constituted by the points $x + iy$ such that

$$1 - x \leq \frac{A}{(\log |y|)^{2/3}(\log \log |y|)^{1/3}}$$

for some constant $A > 0$. In this region R the Riemann zeta function satisfies uniformly for all $z = z + iy \in R$

$$\frac{1}{\zeta(x + iy)} \ll (\log |y|)^{2/3}(\log \log |y|)^{1/3}.$$

Proposition 1.2. If $\alpha < 1/3$, then

$$\mathbb{P}\left(\exists \lim_{N\to\infty} \sum_{k=1}^{N} \frac{\mu_{X^\alpha}(k)}{k^{1-\alpha+it}} \, \forall t \in \mathbb{R}\right) = 1.$$

If $\alpha \in (0, 1/2)$, then

$$\lim_{N\to\infty} \mathbb{E}\left|\sum_{k=1}^{N} \frac{\mu_{X^\alpha}(k)}{k^{1-\alpha}}\right|^2 = 0.$$

Proof. Proceeding equally as in the proof of Lemma 2.2 we conclude that with probability one the random Riemann zeta function $1/\zeta_{X^\alpha}$ satisfies in $\overline{\mathbb{H}}_{1-\alpha}$ the following functional equation:

$$\frac{1}{\zeta_{X^\alpha}(z)} = \frac{G_\alpha(z)}{\zeta(z + \alpha)},$$

where with probability one $G_\alpha : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C}$ is analytic and satisfies

$$G_\alpha(z) \ll_{\mathbb{H}_{\frac{1}{2}+\delta}} \exp\left(\sum_{p\in\mathcal{P}} \frac{X_p - \mathbb{E}X_p}{p^z}\right), \ \forall \delta > 0.$$

Thus, with probability one $1/\zeta_{X^\alpha}$ is analytic in the region $R_\alpha = \mathbb{H}_{1-\alpha} \cup (R - \alpha)$ where R is the best zero free region of the Riemann zeta function up to date. This together Lemma 2.4 and Vinogradov-Korobov bound for $1/\zeta$ imply that there exists a event Ω' of probability one such that for $\omega \in \Omega'$, for all $\sigma + it \in R_\alpha$:

$$\frac{1}{\zeta_{X^\alpha}(\sigma + it, \omega)} \ll_{\omega,\sigma} \exp\big(\,(\log(t + e)\,)^{1-\sigma}(\,\log\log(t + e^e)\,)^\sigma\big). \tag{46}$$

From now on $t \geq 0$ and $\alpha < 1/3$ will be fixed. The realization $\omega \in \Omega'$ will be implicit in the expressions. By the Perron's Formula (Theorem A.11) we obtain:

$$\sum_{k=1}^{N} \frac{\mu_{X^\alpha}(k)}{k^{1-\alpha+it}} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{1}{\zeta_{X^\alpha}(1 - \alpha + it + z)} \frac{N^z}{z} dz + O\left(\frac{N^{1/3}\log N}{T} + \frac{N^{1/3}\log T}{T}\right),$$

where $c = \alpha + 1/\log N$. Let $N = T \geq t + e^e$ and $\delta = A(\,(\log N)^{2/3}(\log\log N)^{1/3}\,)^{-1}$, where $A > 0$ is the Vinogradov-Korobov constant. Define the rectangle $S = [-\delta, c] \times [-N, N]$ and decompose $\partial S = I_1 \cup I_2 \cup I_3 \cup I_4$ where I_1 and I_3 are the vertical lines at the levels $\Re(z) = c$ and $\Re(z) = -\delta$ respectively and I_2 and I_4 are horizontal lines at the levels $Im(z) = -N$ and $Im(z) = +N$ respectively. On that way, the closed curve $\partial S + 1 - \alpha$ is contained in R_α which is the region where the bound (46) is applicable. By the cauchy integral formula: $\frac{1}{\zeta_{X^\alpha}(1-\alpha+it)} = \frac{1}{2\pi i}\int_{\partial S} \frac{1}{\zeta_{X^\alpha}(1-\alpha+it+z)}\frac{N^z}{z}dz$, and hence:

$$\left|\sum_{k=1}^{N} \frac{\mu_{X^\alpha}(k)}{k^{1-\alpha+it}} - \frac{1}{\zeta_{X^\alpha}(1-\alpha+it)}\right| \leq |J_2| + |J_3| + |J_4| + O\left(\frac{\log N}{N^{2/3}}\right), \tag{47}$$

where $J_k = \int_{I_k} \frac{1}{\zeta_{X^\alpha}(1-\alpha+it+z)}\frac{N^z}{z}dz$, $k = 1, 2, 3, 4$. Now observe that

$$\begin{aligned}
J_2, J_4 \ll & \|1/\zeta_{X^\alpha}(1-\alpha+it+\cdot)\|_{L^\infty(I_2)} \int_{-\delta}^{c} e^{x\log N}\frac{dx}{\sqrt{N^2+x^2}} \\
\ll & \frac{1}{N^{1-\alpha}} \exp\big((\,\log N\,)^{\alpha+\delta}(\,\log\log N)\,)^{1-\alpha}\big), \\
J_3 \ll & N^{-\delta}\|1/\zeta_{X^\alpha}(1-\alpha+it+\cdot)\|_{L^\infty(I_3)} \int_{-N}^{N} \frac{dy}{\sqrt{\delta^2+y^2}} \\
\ll & \frac{1}{N^{\delta}} \exp\big((\,\log N\,)^{\alpha+\delta}(\,\log\log N)\,)^{1-\alpha}\big) \\
\ll & \exp\left(\,(\,\log N\,)^{\alpha}(\,\log\log N)\,)^{1-\alpha} - A\left(\frac{\log N}{\log\log N}\right)^{\frac{1}{3}}\,\right) \\
= & \exp\left(\,-A\left(\frac{\log N}{\log\log N}\right)^{\frac{1}{3}}(1+o_{\omega,\alpha}(1))\,\right).
\end{aligned}$$

Inserting these upper bounds in (47), we conclude that:

$$\left|\sum_{k=1}^{N} \frac{\mu_{X^\alpha}(k,\omega)}{k^{1-\alpha+it}} - \frac{1}{\zeta_{X^\alpha}(1-\alpha+it,\omega)}\right| \ll_{\omega,\alpha} \exp\left(\,-A\left(\frac{\log N}{\log\log N}\right)^{\frac{1}{3}}(1+o_{\omega,\alpha}(1))\,\right). \tag{48}$$

Therefore:

$$\mathbb{P}\left(\lim_{N\to\infty}\sum_{k=1}^{N}\frac{\mu_{X^\alpha}(k,\omega)}{k^{1-\alpha+it}} = \frac{G_\alpha(1-\alpha+it)}{\zeta(1+it)}\right) = 1. \tag{49}$$

To prove the second statement, for each prime p define $Y_p = X_p^\alpha + p^{-\alpha}$ and consider μ_Y the respective Möbius function. Observe that the sequence $\{\mu_Y(k)\}_{k\geq 2}$ is orthogonal in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E}Y_p^2 = 1 - p^{-2\alpha} \leq 1$. Writing $X_p^\alpha = Y_p - p^{-\alpha}$ we have for each k square free:

$$\mu_{X^\alpha}(k) = \prod_{p|k}(Y_p - p^{-\alpha}) = \sum_{d|k}\mu_Y(d)\mu\left(\frac{k}{d}\right)\left(\frac{k}{d}\right)^{-\alpha}.$$

Therefore

$$\begin{aligned}\sum_{k=1}^{N}\frac{\mu_{X^\alpha}(k)}{k^{1-\alpha}} &= \sum_{k=1}^{N}\frac{\mathbb{1}_{\mathcal{S}}(k)}{k^{1-\alpha}}\sum_{d|k}\mu_Y(d)\mu\left(\frac{k}{d}\right)\left(\frac{k}{d}\right)^{-\alpha}\\ &= \sum_{k=1}^{N}\frac{\mathbb{1}_{\mathcal{S}}(k)}{k}\sum_{d=1}^{N}\mu_Y(d)d^{\alpha}\mu\left(\frac{k}{d}\right)\mathbb{1}_{d|k}\\ &= \sum_{d=1}^{N}\mu_Y(d)d^{\alpha}\sum_{k=1}^{N}\frac{1}{k}\mu\left(\frac{k}{d}\right)\mathbb{1}_{d|k}\mathbb{1}_{\mathcal{S}}(k).\end{aligned}$$

Since for k square free with $d|k$ we have $k = da$ with $\gcd(d,a) = 1$, we obtain:

$$\sum_{k=1}^{N}\frac{\mu_{X^\alpha}(k)}{k^{1-\alpha}} = \sum_{d=1}^{N}\frac{\mu_Y(d)}{d^{1-\alpha}}A_{d,N},$$

where

$$A_{d,N} = \sum_{\gcd(a,d)=1}^{[N/d]}\frac{\mu(a)}{a}.$$

One of the consequences of the Prime number Theorem is that $\lim_{N\to\infty} A_{d,N} = 0$, for all $d \in \mathbb{N}$. On the other hand, a result in [16] states that for all d and N, $|A_{d,N}| \leq 1$. Since

$$A_{1,N} = \mathbb{E}\sum_{k=1}^{N}\frac{\mu_{X^\alpha}(k)}{k^{1-\alpha}} = \sum_{k=1}^{N}\frac{\mu(k)}{k},$$

the orthogonality of $\{\mu_Y(k)\}_{k\geq 2}$ implies:

$$\begin{aligned}\mathbb{E}\left|\sum_{k=1}^{N}\frac{\mu_{X^\alpha}(k)-\mathbb{E}\mu_{X^\alpha}(k)}{k^{1-\alpha}}\right|^2 &\leq \sum_{d=2}^{N}\frac{A_{d,N}^2}{d^{2-2\alpha}} &(50)\\ &\leq \sum_{d=1}^{\infty}\frac{1}{d^{2-2\alpha}}. &(51)\end{aligned}$$

Defining $A_{d,N} = 0$ for $d > N$, by the dominated convergence Theorem for series, (51) implies that

$$\lim_{N\to\infty}\sum_{d=2}^{\infty}\frac{A_{d,N}^2}{k^{2-2\alpha}} = \sum_{d=2}^{\infty}\lim_{N\to\infty}\frac{A_{d,N}^2}{k^{2-2\alpha}} = 0$$

which combined with (50) and $\lim_{N\to\infty} A_{1,N} = 0$ gives that

$$\lim_{N\to\infty}\mathbb{E}\left|\sum_{k=1}^{N}\frac{\mu_{X^\alpha}(k)}{k^{1-\alpha}}\right|^2 = 0.$$

□

Proposition 1.3. There exists a real B such that for all $\alpha \in (0, 1/3)$, with probability one

$$M_{X^\alpha}(N) \ll N^{1-\alpha} \exp\left(- B\left(\frac{\log N}{\log\log N}\right)^{1/3}\right).$$

Proof. For $\alpha < 1/3$, by (49), with probability one $\lim_{N\to\infty} \sum_{k=1}^{N} \frac{\mu_{X^\alpha}(k)}{k^{1-\alpha}} = 0$, since also with probability one $|G_\alpha(1-\alpha)| < \infty$ and $\frac{1}{\zeta(1)} = 0$. Thus, by (48) we obtain that with probability one

$$T_N := \sum_{k=1}^{N} \frac{\mu_{X^\alpha}(k)}{k^{1-\alpha}} \ll \exp\left(- A\left(\frac{\log N}{\log\log N}\right)^{1/3}\right) := K_N. \tag{52}$$

Let T_N and K_N as in (52). Hence we obtain for $M = N^{1/3}$

$$\begin{aligned} M_{X^\alpha}(N) &= M_{X^\alpha}(M) + \sum_{k=M+1}^{N} k^{1-\alpha}(T_k - T_{k-1}) \\ &= M_{X^\alpha}(M) - (M+1)^{1-\alpha} T_M + N^{1-\alpha} T_N + \sum_{k=M+1}^{N-1} T_k (k^{1-\alpha} - (k+1)^{1-\alpha}) \\ &\ll |M_{X^\alpha}(M)| + 2 \max_{M \le k \le N} |T_k| (M^{1-\alpha} + N^{1-\alpha}) \\ &\ll N^{1/3} + N^{1-\alpha} K_{N^{1/3}} \\ &\ll N^{1-\alpha}\left(N^{-1/3} + \exp\left(- A\left(\frac{\log N^{1/3}}{\log\log N^{1/3}}\right)^{1/3}\right)\right) \\ &\ll N^{1-\alpha} \exp\left(- \frac{A}{4}\left(\frac{\log N}{\log\log N}\right)^{1/3}\right). \end{aligned}$$

Thus by taking $B = A/4$ we conclude the proof of Proposition 1.3. □

3 Concluding Remarks

3.1 Back to the β Random Möbius function.

Let μ_β, M_β and ζ_β as in Notations 2.5.

Claim 3.1 (Answer to Question 2)**.** The assumption that

$$\mathbb{P}(M_\beta(N) \ll N^{1/2+\varepsilon}, \forall\varepsilon >)) = 1$$

for all $\beta \in [\beta^*, 1)$ for some $\beta^* < 1$ implies the Riemann Hypothesis.

Claim 3.1 is a conditional result and it's proof utilizes the construction of Lemma 2.1. Even that the basic assumption of this result is false by Theorem 1.1, the intuition that motivated it is of theoretical interest: Denoting $d(k)$ the number of *distinct* primes that divide k and ν_β an approximation of μ_β defined as $\nu_\beta(k) = (2\beta - 1)^{-\omega(k)}\mu_\beta(k)$, we have

Proposition 3.1. The Riemann hypothesis is equivalent to

$$\mathbb{P}\bigg(\sum_{k=1}^{N}\nu_\beta(k) \ll N^{1/2+\varepsilon} = 0, \forall\varepsilon > 0\bigg) = 1,$$

for all $\beta \in [\frac{1}{2} + \frac{1}{2\sqrt{2}}, 1)$.

Observe that for β close to 1, $(2\beta-1)^{-d(k)}$ is *uniformly closed* to 1 for all k with $\omega(k) \leq \lambda$ for fixed $\lambda \in \mathbb{N}$. Thus ν_β and μ_β are uniformly close on the set $\{k \in \mathbb{N} : \omega(k) \leq \lambda\}$.

Proof of Claim 3.1. By Lemma 2.1, for each n there exists a measure preserving transformation T_n such that

$$\theta_{\beta^*}(z,\omega) = \prod_{k=1}^{2^n}\theta_{\beta_n^*}(z, T_n^k\omega).$$

Let $\Lambda_0 = [1/\zeta_{\beta^*}$ extends analytically to $\mathbb{H}_{1/2} \setminus \{1\}]$ and for $n \geq 1$

$$\Lambda_n = \bigcap_{k=1}^{2^n} T_n^{-k}\big(\,[1/\zeta_{\beta_n^*} \text{ extends analytically to } \mathbb{H}_{1/2} \setminus \{1\}]\,\big).$$

Since $\theta_{\beta_n^*} = \zeta(z)\frac{1}{\zeta_{\beta_n^*}}$ and that ζ is analytic in $\mathbb{C} \setminus \{1\}$, our assumption combined with inequality 4 gives for each $n \geq 0$, with probability one, $\theta_{\beta_n^*}$ and $\theta_{\beta_n^*}$ extends analytically to $\mathbb{H}_{1/2} \setminus \{1\}$. Since T_n preserves measure, $\mathbb{P}(\Lambda_n) = 1$ for all $n \geq 0$ and hence $\mathbb{P}(\cap_{n=0}^{\infty}\Lambda_n) = 1$. Now let $\omega^* \in \cap_{n=0}^{\infty}\Lambda_n$, $K \subset \mathbb{H}_{1/2} \setminus \{1\}$ a compact set and $m_K(\omega^*)$ the quantity of zeros counted with multiplicities of $\theta_{\beta^*}(\cdot,\omega^*)$ in K. Since $\theta_{\beta^*}(\cdot,\omega^*)$ is given by the Euler product representation in $\mathbb{H}_1$, this function does not vanishes in this half plane. Thus, by analyticity, Theorem A.16 implies that the set of points in the complex plane for which $\theta_{\beta^*}(\cdot, \omega^*)$ equals to zero is discrete an hence $m_K(\omega^*) < \infty$. Now choose $t = t(\omega^*)$ such that $2^t > m_K(\omega^*)$. Then the functional relation

$$\theta_{\beta^*}(z, \omega^*) = \prod_{k=1}^{2^t} \theta_{\beta_t^*}(z, T^k \omega^*)$$

implies that there exists $\tau \in \{1, ..., 2^t\}$ such that $\theta_{\beta_t^*}(z, T_t^\tau \omega^*) \neq 0$ for all $z \in K$. Since $\theta_{\beta_t^*}(z, T^\tau \omega^*) = \zeta(z) \frac{1}{\zeta_{\beta_t}(z, T_t^\tau \omega^*)}$ and these functions are analytic in K, it follows that $\zeta(z) \neq 0$ for all $z \in K$. Repeating the procedure for a sequence of compacts $K_N \subset \mathbb{H}_{1/2} \setminus \{1\}$ such that $\bigcup_{N=1}^\infty K_N = \{z \in \mathbb{C} : \frac{1}{2} < Re(z) < 1\}$, we conclude that the zeros of ζ all have real part less or equal to $1/2$ in which together Riemann's functional equation $\zeta(s) = \zeta(1-s) \cdot A(s)$, where $A(s)$ is analytic in $\{z \in \mathbb{N} : Re(z) \in (0,1)\}$, implies that the zeros of ζ can only have real part equals to $1/2$, finishing the proof of Claim 3.1. □

Proof of Proposition 3.1. For $\beta \in [\frac{1}{2} + \frac{1}{2\sqrt{2}}, 1)$, at each prime p

$$|\nu_\beta(p)| = \frac{1}{2\beta - 1} |\mu_\beta(p)| \leq \sqrt{2}.$$

Thus the sequence $((2\beta - 1)^{-1} \mu_\beta(p))_{p \in \mathcal{P}}$ satisfies conditions i. and ii. of Theorem 2.1 with parameters $C = \sqrt{2}$ and $\alpha = 0$. □

3.2 Random Dirichlet Series and the Lindelöf Hypothesis.

Let $\eta : \mathbb{H}_0 \to \mathbb{C}$ be the alternating series:

$$\eta(z) = \sum_{k=1}^\infty \frac{(-1)^k}{k^z}.$$

This function is analytic and satisfies on the half plane $\mathbb{H}_0$ the following relation

$$\zeta(z) = \frac{1}{1 - 2^{1-z}} \eta(z). \tag{53}$$

The Lindelöf Hypothesis due to E. Lindelöf states that the Riemann zeta function ζ satisfies on the critical line $\zeta(1/2 + it) = o((t+1)^\epsilon)$ for all $\epsilon > 0$ which is equivalent to $\zeta(\sigma + it) = o((t+1)^\epsilon)$ for all $\epsilon > 0$ and $\sigma > 1/2$. This Hypothesis is known to be a consequence of the Riemann Hypothesis yet is not equivalent. Indeed it has many consequences related to the zeros of ζ in the critical line (see [18], Chapter XIII). By (53) this is equivalent to the alternating series $\eta(\sigma + it) = o((t+1)^\epsilon)$ for all $\epsilon > 0$ and $\sigma > 1/2$.

Changing the sequence $((-1)^k)_{k \in \mathbb{N}}$ by independent random variables, we have as a direct consequence from Lemma 2.4 and Proposition 2.2:

Theorem 3.1. Assume that $(X_k)_{k\in\mathbb{N}}$ are independent, uniformly bounded and that the series $\sum_{k=1}^{\infty}\frac{\mathbb{E}X_k}{k^x}$ converges for all $x>1/2$. Then with probability one, for $x=1/2+\kappa$ with $\kappa>0$

$$\sum_{k=1}^{\infty}\frac{X_k}{k^{\sigma+it}}=\sum_{k=1}^{\infty}\frac{\mathbb{E}X_k}{k^{\sigma+it}}+O((\log t)^{\frac{1}{2}-\kappa}(\log\log t)^{\frac{1}{2}+\kappa}).$$

I would like to call this result by Complex Strong Law of Large Numbers and abbreviate it by CSLLN.

Now let $(X_k)_{k\in\mathbb{N}}$ be independent, uniformly distributed over $\{-1,+1\}$ and $(Y_k)_{k\in\mathbb{N}}$ be independent with distribution $\mathbb{P}(Y_k=+1)=\frac{1}{2}+(-1)^k\delta=1-\mathbb{P}(Y_k=-1)$. Thus $\mathbb{E}Y_k=2\delta(-1)^k$ and hence $|\mathbb{E}(Y_1+...+Y_N)|\leq 1$. Denote $S_X(N):=X_1+...+X_N$ and similarly $S_Y(N)$. By the Central Limit Theorem both $N^{-1/2}S_X(N)$ and $N^{-1/2}S_Y(N)$ converge in distribution to the gaussian distribution. Moreover, define a continuous random function $B_X^N:[0,1]\to\mathbb{R}$ as follows: For $k=1,...,N$ and $t=k/N$, $B_X^N(t)=N^{-1/2}S_X(k)$ and for other values of t, $B_X^N(t)$ is given by the linear interpolation of these points. Similarly define B_Y^N. The Donsker Invariance Principle states that both B_X^N and B_Y^N converge in distribution to a continuous random function $B:[0,1]\to[0,1]$ called Brownian Motion. Thus at least in this sense, the sequences X and Y are indistinguishable. This motivates the following concept: Say that sequences of independent random variables X and Y are indistinguishable in the sense of CSLLN if, with probability one $\sum_{k=1}^{\infty}\frac{Y_k}{k^{\sigma+it}}=o(\tau^{\epsilon})$ for all $\epsilon>0$. As a consequence of Theorem 3.1:

The Lindelöf hypothesis is equivalent to the sequences X and Y being indistinguishable in the sense of CSLLN.

A Appendix

A.1 Probability Theory

Definition A.1. Let $X = (X_k)_{k\in\mathbb{N}}$ be a sequence of independent $\mathbb{R}$-valued random variables and $\mathcal{F}_n^\infty$ be the sigma algebra $\sigma(X_k : k \geq n)$. The sigma algebra $\mathcal{F} = \bigcap_{n=1}^\infty \mathcal{F}_n^\infty$ is called *tail* sigma algebra and an element $A \in \mathcal{F}$ is called tail event.

Theorem A.1 (Kolmogorv 0-1 Law)**.** Every tail event has either probability 0 or probability 1.

Theorem A.2 (Kolmogorov Two-Series Theorem)**.** Let $X = (X_k)_{k\in\mathbb{N}}$ be a sequence of independent $\mathbb{R}$-valued random variables. Assume that $|X_k| \leq C$ for all k. Then the random series $\sum_{k=1}^\infty X_k$ converges if and only if $\sum_{k=1}^\infty \mathbb{E}X_k$ and $\sum_{k=1}^\infty \mathbb{V}X_k$ are convergent series.

Theorem A.3 (SLLN for pairwise independence, [15])**.** Let $(V_k)_{k\in\mathbb{N}}$ be identically distributed and pairwise independent. For $\gamma \in (1,2)$ and $\tau > 0$ and $\tau > 4\gamma - 6$ assume

$$\mathbb{E}|V_1|^\gamma (\log(|X|+2))^\tau < \infty.$$

Then $S_N := V_1 + ... + V_N$ satisfies

$$\mathbb{P}(S_N = o(N^{1/\gamma})) = 1.$$

Theorem A.4 (The Marcinkiewicz-Zygmund Inequality)**.** If $(X_k)_{k\in\mathbb{N}}$ are independent, $\mathbb{E}|X_k|^T < \infty$ for some $T > 1$ and $\mathbb{E}X_k = 0$, then there exists $A > 0$ and $B > 0$ such that

$$A^T T^{T/2}\mathbb{E}\Bigg[\bigg(\sum_{k=1}^N |X_k|^2\bigg)^{T/2}\Bigg] \leq \mathbb{E}\Bigg[\bigg(\sum_{k=1}^N X_k\bigg)^T\Bigg] \leq B^T T^{T/2}\mathbb{E}\Bigg[\bigg(\sum_{k=1}^N |X_k|^2\bigg)^{T/2}\Bigg].$$

Lemma A.1. Let $T \geq 1$ and $\{X_k\}_{k\in\mathbb{N}}$ be a family of independent real random variables such that each $X_k \in L^T(\Omega, \mathcal{F}, \mathbb{P})$ has $\mathbb{E}X_k = 0$. For $z = x + iy$:

$$\mathbb{E}\bigg|\sum_{k=1}^N \frac{X_k}{k^z}\bigg|^T \leq (2B)^T T^{T/2}\mathbb{E}\bigg(\sum_{k=1}^N \frac{|X_k|^2}{k^{2x}}\bigg)^{T/2}$$

Proof. Let $T \geq 1$ and for $z = x + iy$ define:

$$a = \sum_{k=1}^N \frac{X_k}{k^x}\cos(y\log k), \quad b = -\sum_{k=1}^N \frac{X_k}{k^x}\sin(y\log k).$$

Thus, by the Marcinkiewicz-Zygmund inequality we obtain

$$\mathbb{E}|a|^T \leq B^T T^{T/2} \mathbb{E}\left(\sum_{k=1}^{N} \frac{|X_k|^2}{k^{2x}} \cos^2(y \log k) \right)^{T/2} \leq B^T T^{T/2} \mathbb{E}\left(\sum_{k=1}^{N} \frac{|X_k|^2}{k^{2x}} \right)^{T/2},$$

and similarly we obtain the same inequality for $\mathbb{E}|b|^T$. Since $\sum_{k=1}^{N} \frac{X_k}{k^z} = a + ib$, by the inequality $|a+ib|^T \leq 2^T(|a|^T + |b|^T)$, we obtain:

$$\mathbb{E}\left| \sum_{k=1}^{N} \frac{X_k}{k^z} \right|^T \leq 2^T(\mathbb{E}|a|^T + \mathbb{E}|b|^T) \leq (2B)^T \mathbb{E}\left(\sum_{k=1}^{N} \frac{|X_k|^2}{k^{2x}} \right)^{T/2}.$$

□

A.2 Dirichlet Series.

Definition A.2. Given a function $f : \mathbb{N} \to \mathbb{C}$ a Dirichlet series associated to f at the point $z \in \mathbb{C}$ is denoted by $F(f,z)$ and is given by $F(f,z) = \sum_{k=1}^{\infty} \frac{f(k)}{k^z}$.

$\mathbb{H}_a$ is the half plane $\{z \in \mathbb{C} : Re(z) > Re(a)\}$.

Theorem A.5 ([1], Theorems 11.8 and 11.11)**.** If a Dirichlet series converges at $z_0 \in \mathbb{C}$ then converges at every point of the half plane $\mathbb{H}_{z_0}$ and $F(f,\cdot) : \mathbb{H}_{z_0} \to \mathbb{C}$ is an analytic function. The complex derivatives of all orders of this analytic function are also convergent Dirichlet series on the same half plane and are given by:

$$\frac{d^n}{dz^n} \sum_{k=1}^{\infty} \frac{f(k)}{k^z} = (-1)^n \sum_{k=1}^{\infty} \frac{f(k)}{k^z} (\log k)^n.$$

Theorem A.6 (Kronecker's Lemma)**.** Suppose that $x > 0$ and the Series $\sum_{k=1}^{\infty} \frac{f(k)}{k^x}$ converges. Then

$$\lim_{N \to \infty} \frac{f(1) + ... + f(N)}{N^x} = 0.$$

Theorem A.7. Suppose that the Dirichlet series $F(z) := \sum_{k=1}^{\infty} \frac{f(k)}{k^z}$ converges absolutely for $z \in \mathbb{H}_a$ and that for some $c < a$

$$\sup_N \frac{|f(1) + ... + f(N)|}{N^c} = C < \infty.$$

Then the function $F : \mathbb{H}_a \to \mathbb{C}$ has analytic extension to $\mathbb{H}_c$.

Proof. Let $\delta > 0$ and denote $S_N := f(1) + ... + f(N)$. By summation by parts we have that

$$\Big|\sum_{k=M}^{N} \frac{f(k)}{k^{c+\delta}}\Big| = \Big|\frac{f(M)}{M^{c+\delta}} + \frac{S_N}{N^{c+\delta}} - \frac{S_M}{(M+1)^{c+\delta}} - \sum_{k=M+1}^{N-1} S_k((k+1)^{-c-\delta} - k^{-c-\delta})\Big|.$$

Observe that by hypothesis the first, second and third term on the right side of the equation above goes to 0 when N, M goes to infinity. For the fourth term, the mean value inequality implies that for some $C' > 0$, $|(k+1)^{-c-\delta} - k^{-c-\delta}| \leqslant C' k^{-c-\delta-1}$ and this allow us to bound:

$$\Big|\sum_{k=M+1}^{N-1} S_k((k+1)^{-c-\delta} - k^{-c-\delta})\Big| \leqslant CC' \sum_{k=M}^{N} \frac{1}{k^{1+\delta}}.$$

Hence the referred fourth term also goes to 0 when N, M goes to infinity. Hence the Dirichlet series $F(c+\delta)$ is convergent for every $\delta > 0$ and by Theorem A.5 this series gives the desired analytic extension. □

Definition A.3. Given functions $f, g : \mathbb{N} \to \mathbb{C}$, the Dirichlet Convolution is given by

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

Theorem A.8 ([1], Theorem 11.5)**.** Let $f, g : \mathbb{N} \to \mathbb{C}$ and $\mathbb{H}$ be a half plane where the Dirichlet Series associated to f converges and the Dirichlet series associated to g converges absolutely. Then the Dirichlet series associated to $f * g$ converges at each $z \in \mathbb{H}$ and:

$$\left(\sum_{k=1}^{\infty} \frac{f(k)}{k^z}\right)\left(\sum_{k=1}^{\infty} \frac{g(k)}{k^z}\right) = \sum_{k=1}^{\infty} \frac{(f*g)(k)}{k^z}.$$

Definition A.4. Given a Dirchlet series $\sum_{k=1}^{\infty} \frac{f(k)}{k^x}$, the numbers

$$\sigma_a = \inf\{x \in \mathbb{R} : \sum_{k=1}^{\infty} \frac{f(k)}{k^x} \text{ converges absolutely}\}$$

and

$$\sigma_c = \inf\{x \in \mathbb{R} : \sum_{k=1}^{\infty} \frac{f(k)}{k^x} \text{ converges }\}$$

are called abscissa of absolute convergence and convergence respectively. $\mathbb{H}_{\sigma_a}$ and $\mathbb{H}_{\sigma_c}$ are called half plane of absolute convergence and half plane of convergence respectively.

Theorem A.9 ([1], Theorem 11.10)**.** For every Dirichlet series such that σ_c is finite,

$$\sigma_a - 1 \leqslant \sigma_c \leqslant \sigma_a.$$

Theorem A.10 (Landau, [17], page 111, Theorem 6)**.** A Dirichlet series of positive terms always has the point σ_a as a singularity.

Theorem A.11 (Perron's Formula, [17] Corollary 2.1, pg 133)**.** Let $F(f,s)$ be a Dirichlet series with $\sigma_a<\infty$. Suppose that there exists a real number $\eta\geq 0$ and a non-decreasing function $B:\mathbb{N}\to\mathbb{R}$ such that

$$i)\ \sum_{k=1}^{\infty}\frac{|f(k)|}{k^\sigma}=O\bigg(\frac{1}{(\sigma-\sigma_a)^\eta}\bigg),\sigma>\sigma_a$$
$$ii)\ |f(n)|\leq B(n)$$

Then for $N\geq 2$, $T\geq 2$, $\sigma=\Re(z)\leq\sigma_a$, $\delta=\sigma_a-\sigma+1/\log N$, we have

$$\sum_{k=1}^{N}\frac{f(k)}{k^z}=\frac{1}{2\pi i}\int_{\delta-iT}^{\delta+iT}F(f,z+s)N^s\frac{ds}{s}$$
$$+O\bigg(N^{\sigma_a-\sigma}\frac{(\log N)^\eta}{T}+\frac{B(2N)}{N^\sigma}\bigg(1+N\frac{\log T}{T}\bigg)\bigg)$$

Definition A.5. Let $c\in\mathbb{R}$ and $f:\mathbb{H}_c\times\Omega\to\mathbb{C}$. The Lindelöf function associated to f, $L_f:(c,\infty)\to\mathbb{R}\cup\{\infty\}$ is the function

$$L_f(\sigma):=\inf\bigg\{A>0:\lim_{\tau\to\infty}\frac{f(\sigma+i\tau)}{\tau^A}=0\bigg\}$$

where the infimum over the set above is taken to be ∞ if this set is empty.

Theorem A.12 ([17], pg.120 Theorem 16)**.** The Lindelöff function L_f is convex and continuous in the interval (σ_1,σ_2) if is finite on the closure of this interval

Theorem A.13 ([17], (Schnee-Landau), pg. 133, Theorem 4)**.** Suppose that the Dirichlet series $F(z):=\sum_{k=1}^\infty\frac{f(k)}{k^z}$ has finite abscissa of absolute convergence σ_a. Suppose that the function F has analytic extension to $\mathbb{H}_c$ with $c<\sigma_a$ and for all $\sigma>c$ has Lindelöf function $L_F(\sigma)=0$. Then this Dirichlet series converges at every point of $\mathbb{H}_c$.

A.3 Multiplicative Functions.

A arithmetic function $f:\mathbb{N}\to\mathbb{C}$ is called multiplicative if for all n and m coprime, that is, with $\gcd(n,m)=1$, $f(n\cdot m)=f(n)\cdot f(m)$. When this relation holds for all pairs n and m, is called completely multiplicative.

Theorem A.14 (Wirsing,[17], Theorem 5 pg. 336)**.** Let $f:\mathbb{N}\to[-1,+1]$ be multiplicative. Then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}f(k)=\prod_{p\in\mathcal{P}}\bigg(\bigg(1+\frac{1}{p}\bigg)\cdot\sum_{m=0}^{\infty}\frac{f(p^m)}{p^m}\bigg).$$

Theorem A.15 ([1], Theorem 11.6)**.** A Dirichlet series associated to a multiplicative function $f : \mathbb{N} \to \mathbb{C}$ has the same half plane of absolute convergence of the double series $\sum_{p\in\mathcal{P}} \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{mz}}$. For every z in this half plane this Dirichlet series admits representation in Eulerian product:

$$\sum_{k=1}^{\infty} \frac{f(k)}{k^z} = \prod_{p\in\mathcal{P}} \left(1 + \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{mz}}\right).$$

A.4 Complex analysis

Theorem A.16 ([5], Corollary 3.8 and 3.9)**.** Let $G \subset \mathbb{C}$ be an connected region. Then the zeros of a non constant analytic function $f : G \to \mathbb{C}$ are isolated. Furthermore for each z_0 such that $f(z_0) = 0$, there exists an integer m, called multiplicity of the zero z_0, such that $\frac{f(z)}{(z-z_0)^m}$ is analytic in G and is not zero at z_0.

Theorem A.17 ([5], Corollary 6.16)**.** If G is simply connected and $f : G \to \mathbb{C}$ is analytic and does not vanish at every point in G, then there exists an analytic function $g : G \to \mathbb{C}$ such that

$$f(z) = \exp(g(z)).$$

A function g satisfying the equation above is unique up to a constant and is called branch of the logarithm.

Theorem A.18 (Hadamard Three-Circles Theorem)**.** Let $f : \{R_1 \le |z| \le R_2\} \to \mathbb{C}$ be continuous and analytic in $\{R_1 < |z| < R_2\}$. For $R_1 < r < R_2$, denote

$$M(r) := \max_{|z|=r} |f(z)|,$$
$$a := \frac{\log(\frac{r}{R_1})}{\log(\frac{R_2}{R_1})}.$$

Then:

$$M(r) \le M(R_1)^{1-a} M(R_2)^a.$$

A.5 Decomposition of the Random Riemann zeta function.

Proposition A.1. Let $(X_p)_{p\in\mathcal{P}}$ be independent random variables such that:

i. For some $C \ge 1$ and some $\alpha \ge 0$:

$$|X_p| \le \begin{cases} 1 & , \text{ if } C > \sqrt{p}; \\ Cp^{\alpha} & , \text{ if } C \le \sqrt{p}. \end{cases}$$

Then for all $\omega \in \Omega$:

i. $\zeta_X(z) := \prod_{p\in\mathcal{P}} \frac{1}{1+\frac{X_p}{p^z}}$ and $1/\zeta_X$ are analytic in $\mathbb{H}_{1+\alpha}$ and in particular these random functions never vanish on this half plane.

ii. For each $z \in \mathbb{H}_{1+\alpha}$:

$$\frac{1}{\zeta_X(z)} = \sum_{k=1}^{\infty} \frac{\mu_X(k)}{k^z} = \prod_{p\in\mathcal{P}} \left(1 + \frac{X_p}{p^z}\right). \tag{54}$$

iii. For each $\omega \in \Omega$ there exists an analytic function $R_{X(\omega)} : \mathbb{H}_{1/2+\alpha} \to \mathbb{C}$ such that $R_X \ll_\delta 1$ in $\overline{\mathbb{H}}_{\frac{1}{2}+\alpha+\delta}$ and a *branch of the logarithm* $\log^* \zeta_{X(\omega)} : \mathbb{H}_{1+\alpha} \to \mathbb{C}$ such that:

$$-\log^* \zeta_{X(\omega)}(z) = \sum_{p\in\mathcal{P}} \frac{X_p(\omega)}{p^z} + R_{X(\omega)}(z). \tag{55}$$

Proof. The condition $|X_p| \leq Cp^\alpha$ implies that $\sum_{p\in\mathcal{P}} \frac{X_p(\omega)}{p^x}$ is absolutely convergent on the half plane $\mathbb{H}_{1+\alpha}$ for all $\omega \in \Omega$. Therefore the product $\prod_{p\in\mathcal{P}} \left(1 + \frac{X_p}{p^z}\right)$ is absolutely convergent in $\mathbb{H}_{1+\alpha}$ and hence never vanishes in this half plane for all $\omega \in \Omega$ ([5], chapter VII, sub. 5). This implies that ζ_X is analytic in $\mathbb{H}_{1+\alpha}$ for all $\omega \in \Omega$, since is defined to be the inverse of this product. Also the absolute convergence of this product together Theorem A.15 implies that for all $\omega \in \Omega$ and $z \in \mathbb{H}_{1+\alpha}$

$$\frac{1}{\zeta_X(z)} = \sum_{k=1}^{\infty} \frac{\mu_X(k)}{k^z} = \prod_{p\in\mathcal{P}} \left(1 + \frac{X_p}{p^z}\right).$$

Thus both $1/\zeta_X$ and ζ_X are analytic functions in $\mathbb{H}_{1+\alpha}$ for all $\omega \in \Omega$ and this proves i. and ii.

iii. Since $\zeta_{X(\omega)}(z) \neq 0$ for all $z \in \mathbb{H}_{1+\alpha}$ and all $\omega \in \Omega$ and that a half plane is a simply connected region, Theorem A.17 implies that for each $\omega \in \Omega$ exists a branch of the logarithm $\log^* \zeta_{X(\omega)} : \mathbb{H}_{1+\alpha} \to \mathbb{C}$. For each realization we can construct a branch coinciding with the canonical logarithm of the real function $1/\zeta_{X(\omega)}\big|_{\mathbb{R}}$. In fact let $x = 1 + \alpha + \varepsilon$ for some $\varepsilon > 0$. Observe that

$$\frac{|X_p|}{p^x} \leq \frac{Cp^\alpha}{p^x} \leq \frac{1}{p^{x-\alpha-\frac{1}{2}}} = \frac{1}{p^{\frac{1}{2}+\varepsilon}} < 1.$$

Thus the Taylor expansion $\log(1+y) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{y^m}{m}$ for $|y| < 1$, gives for each prime p:

$$\log\left(1 + \frac{X_p}{p^x}\right) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{X_p^m(\omega)}{mp^{mx}}. \tag{56}$$

Claim A.1. The double series $R_{X(\omega)}(z) = \sum_{p\in\mathcal{P}}\sum_{m=2}^{\infty}(-1)^{m+1}\frac{X_p^m(\omega)}{mp^{mz}}$ converges absolutely in $\mathbb{H}_{\frac{1}{2}+\alpha}$ and $\sum_{p\in\mathcal{P}}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{X_p^m(\omega)}{mp^{mz}}$ converges absolutely on $\mathbb{H}_{1+\alpha}$. Moreover

$$R_X \ll_\delta 1 \text{ in } \overline{\mathbb{H}}_{\frac{1}{2}+\alpha+\delta}.$$

Proof of the Claim: Let $z = x+iy$ and decompose:

$$R_X(z) = \sum_{\sqrt{p}<C}\sum_{m=2}^{\infty}(-1)^{m+1}\frac{X_p^m(\omega)}{mp^{mz}} + \sum_{\sqrt{p}\geq C}\sum_{m=2}^{\infty}(-1)^{m+1}\frac{X_p^m(\omega)}{mp^{mz}}.$$

For $\sqrt{p}<C$ we use that $|X_p|\leq 1$ and hence for each $z\in\mathbb{H}_0$:

$$\Big|\sum_{\sqrt{p}<C}\sum_{m=2}^{\infty}(-1)^{m+1}\frac{X_p^m(\omega)}{mp^{mz}}\Big| \leq C^2\sum_{m=2}^{\infty}\frac{1}{2^{mx}} = \frac{C^2}{2^x(2^x-1)}. \tag{57}$$

For $\sqrt{p}\geq C$ we use that $|X_p|\leq Cp^{\alpha}$ and hence for each $z\in\mathbb{H}_{\frac{1}{2}+\alpha}$:

$$\begin{aligned}\Big|\sum_{\sqrt{p}\geq C}\sum_{m=2}^{\infty}(-1)^{m+1}\frac{X_p^m(\omega)}{mp^{mz}}\Big| &\leq \sum_{\sqrt{p}\geq C}\sum_{m=2}^{\infty}\left(\frac{C}{p^{x-\alpha}}\right)^m\\ &= C^2\sum_{\sqrt{p}\geq C}\frac{1}{p^{x-\alpha}(p^{x-\alpha}-C)}\\ &\ll_{x,C}\sum_{\sqrt{p}\geq C}\frac{1}{p^{2(x-\alpha)}}<\infty. \end{aligned}\tag{58}$$

Thus $R_{X(\omega)}$ converges absolutely in $\mathbb{H}_{\frac{1}{2}+\alpha}$ and since $\sum_{p\in\mathcal{P}}\frac{X_p}{p^z}$ converges absolutely in $\mathbb{H}_{1+\alpha}$ we conclude that $\sum_{p\in\mathcal{P}}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{X_p^m(\omega)}{mp^{mz}}$ also converges absolutely on $\mathbb{H}_{1+\alpha}$. To conclude, (57) and (58) implies:

$$R_X \ll_\delta 1 \text{ in } \overline{\mathbb{H}}_{\frac{1}{2}+\alpha+\delta}$$

finishing the proof of the Claim A.1.

End of the proof of iii. By (54), (56) and Claim A.1 we have for each $x>1+\alpha$ respectively:

$$\begin{aligned}-\log\zeta_{X(\omega)}(x) &= \log\prod_{p\in\mathcal{P}}\left(1+\frac{X_p}{p^z}\right)\\ &= \sum_{p\in\mathcal{P}}\log\left(1+\frac{X_p}{p^x}\right)\\ &= \sum_{p\in\mathcal{P}}\frac{X_p}{p^x}+R_X(x).\end{aligned}$$

Also, Claim A.1 implies that R_X is analytic in $\mathbb{H}_{\frac{1}{2}+\alpha}$ and $z \mapsto \sum_{p\in\mathcal{P}} \frac{X_p}{p^z}$ is analytic in $\mathbb{H}_{1+\alpha}$. Hence the random function $-\log^* \zeta_X : \mathbb{H}_{1+\alpha} \times \Omega \to \mathbb{C}$ given by

$$-\log^* \zeta_{X(\omega)}(z) = \sum_{p\in\mathcal{P}} \frac{X_p}{p^z} + R_X(z)$$

is analytic for all ω and satisfies for each real $x > 1 + \alpha$:

$$\frac{1}{\zeta_{X(\omega)}(x)} = \exp(-\log^* \zeta_{X(\omega)}(x)).$$

Assuming by contradiction that for some $z_0 \in \mathbb{H}_{1+\alpha}$

$$\frac{1}{\zeta_{X(\omega)}(z_0)} \neq \exp(-\log^* \zeta_{X(\omega)}(z_0)),$$

we have that $g_\omega := \frac{1}{\zeta_{X(\omega)}} - \exp(-\log^* \zeta_{X(\omega)})$ is a non constant analytic function in $\mathbb{H}_{1+\alpha}$ and hence by Theorem A.16 the zeros of g_ω are isolated. Since the function $\frac{1}{\zeta_{X(\omega)}}$ and $\exp(-\log^* \zeta_{X(\omega)})$ are analytic and equal in the line segment $(1 + \alpha, \infty)$ we have that $g_\omega(x) = 0$ for all $x > 1 + \alpha$ which contradicts the fact that the zeros of g_ω are isolated. Thus we conclude that $g_\omega = 0$ for all $\omega \in \Omega$, finishing the proof of Proposition 54. $\square$

References

[1] T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York, 1976. Undergraduate Texts in Mathematics.

[2] J. Basquin, *Sommes friables de fonctions multiplicatives aléatoires*, Acta Arith., 152 (2012), pp. 243–266.

[3] F. Carlson, *Contributions à la théorie des séries de Dirichlet. III*, Ark. Mat. Astr. Fys., 23A, 19 (1933).

[4] S. Chatterjee and K. Soundararajan, *Random multiplicative functions in short intervals*, Int. Math. Res. Not. IMRN, (2012), pp. 479–492.

[5] J. B. Conway, *Functions of one complex variable*, vol. 11 of Graduate Texts in Mathematics, Springer-Verlag, New York, second ed., 1978.

[6] P. Erdős, *Some unsolved problems*, Magyar Tud. Akad. Mat. Kutató Int. Közl., 6 (1961), pp. 221–254.

[7] P. Erdös and M. Kac, *The Gaussian law of errors in the theory of additive number theoretic functions*, Amer. J. Math., 62 (1940), pp. 738–742.

[8] A. Granville and K. Soundararajan, *Pretentious multiplicative functions and an inequality for the zeta-function*, Anatomy of integers, volume 46 of *CRM Proc. Lecture Notes*, pages 191–197. Amer. Math. Soc., Providence, RI, 2008.

[9] G. Halász, *On random multiplicative functions*, in Hubert Delange colloquium (Orsay, 1982), vol. 83 of Publ. Math. Orsay, Univ. Paris XI, Orsay, 1983, pp. 74–96.

[10] A. Harper, *On the limit distributions of some sums of a random multiplicative function*, arXiv:1012.0207, (2010).

[11] B. Hough, *Summation of a random multiplicative function on numbers having few prime factors*, Math. Proc. Cambridge Philos. Soc., 150 (2011), pp. 193–214.

[12] Y.-K. Lau, G. Tenenbaum, and J. Wu, *On mean values of random multiplicative functions*, Proc. Amer. Math. Soc., 141 (2013), pp. 409–420.

[13] P. Lévy, *Sur les séries dont les termes sont des variables éventuelles indépendents*, Studia Mathematica, vol. 3 (1931), pp. 119–155.

[14] J. E. Littlewood, *Quelques conséquences de l'hypothèse que la fonction* $\zeta(s)$ *de riemann n'a pas de zéros dans le demi-plan* $\Re(s) > \frac{1}{2}$, Comptes rendus de l'Acadmie des Sciences, 154 (1912), pp. 263–266.

[15] A. Martíkainen, *On the strong law of large numbers for sums of pairwise independent random variables*, Statist. Probab. Lett., 25 (1995), pp. 21–26.

[16] T. Tao, *A remark on partial sums involving the Möbius function*, Bull. Aust. Math. Soc., 81 (2010), pp. 343–349.

[17] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, vol. 46 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1995. Translated from the second French edition (1995) by C. B. Thomas.

[18] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, The Clarendon Press Oxford University Press, New York, second ed., 1986. Edited and with a preface by D. R. Heath-Brown.

[19] A. Wintner, *Random factorizations and Riemann's hypothesis*, Duke Math. J., 11 (1944), pp. 267–275.