Instituto Nacional de Matemática Pura e Aplicada

Thesis

Markov and Lagrange Dynamical Spectra for Geodesic Flows in Surfaces with Negative Curvature.

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Abstract

The current work consists in two parts, both of them related to the study of the fractal geometry.

The first part focuses on showing that the Lagrange and the Markov dynamical spectrum has a non-empty stable interior. First, we study the Lagrange and the Markov dynamical spectrum for diffeomorphisms in surface that have a horseshoe with Hausdorff dimension greater than 1 and the property V. We show that for a "large" set of real functions on the surface and for a diffeomorphism with a horseshoe associated and Hausdorff dimension greater than 1, with the property V, both, the Lagrange and the Markov dynamical spectrum have persistently non-empty interior. Then, we find hyperbolic sets for the geodesic flow of surfaces of pinched negative curvature and finite volume, with Hausdorff dimension close to 3. Associated with this hyperbolic set, we find a horseshoe of Hausdorff dimension close to 2 for Poincaré map. Finally, we prove that the Lagrange and the Markov dynamical spectrum (associated to geodesic flow) have persistently non-empty interior.

The second part focuses on showing that the Marstrand's thoerem is true in surfaces simply connected and non-positive curvature.

Keywords: Lagrange and Markov dynamical spectrum, Horseshoe, Hausdorff dimension, pinched curvature negative, Poincaré map, persistently non-empty interior, the Marstrand's theorem.

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Chapter 1

Introduction

Regular Cantor set on the line play a fundamental role in dynamical systems and notably also in some problems in number theory. They are defined by expansive maps and have some kind of self similarity property: Small parts of them are diffeomorphic to big parts with uniformly bounded distortion (see precise definition in section 2.2). Some background on the regular Cantor sets with are relevant to our work can be found in [CF89], [PT93], [MY01] and [MY10].

An example intimately related to our work (cf. [CF89]), is the following: Given an irrational number α, according to Dirichlet's theorem the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$ has infinite rational solutions $\frac{p}{q}$. Markov and Hurwitz improved this result, proving that, for all irrational α, the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$ has infinitely many rational solutions $\frac{p}{q}$.

Meanwhile, fixed α irrational can expect better results, which leads us to associate with each α, its best constant of approximation (Lagrange value of α) is given by

$$\begin{aligned} k(\alpha) &= \sup\left\{k > 0 : \left|\alpha - \frac{p}{q}\right| < \frac{1}{kq^2} \text{ has infinite rational solutions } \frac{p}{q}\right\} \\ &= \limsup_{\substack{p,q\to\infty \\ p,q\in\mathbb{N}}} |q(q\alpha - p)|^{-1} \in \mathbb{R}\cup\{+\infty\}. \end{aligned}$$

Then, always holds that $k(\alpha) \geq \sqrt{5}$. Consider the set

$$L = \{k(\alpha) : \alpha \in \mathbb{R}\setminus\mathbb{Q} \text{ and } k(\alpha) < \infty\}$$

known as *Lagrange Spectrum* (for properties of L cf. [CF89]).

In 1947 Marshall Hall (cf.[Hal47]) proved that the regular Cantor set $C(4)$ the real number of $[0,1]$ whose continued fraction only appears coefficients $1, 2, 3, 4$, then

$$C(4) + C(4) = [\sqrt{2} - 1, 4(\sqrt{2} - 1)].$$

Put α irrational in continued fractions by $\alpha = [a_0, a_1, \dots]$, for $n \in \mathbb{N}$, defined $\alpha_n = [a_n, a_{n+1}, \dots]$ and $\beta_n = [0, a_{n-1}, a_{n-2}, \dots]$, then using continued fractions techniques is proved that

$$k(\alpha) = \limsup_{n\to\infty}(\alpha_n + \beta_n).$$

With this latter characterization Lagrange spectrum and from of Hall's result it follows that $L \supset [6, +\infty)$ of *Hall's ray* of the Lagrange spectrum.

In 1975, G. Freiman (cf. [Fre75] and [CF89]) proved some difficult results showing that the arithmetic sum of certain (regular) Cantor sets, related to continued fractions contain intervals, and used them to determine the precise beginning of Hall's ray (The biggest half-line contained in L) which is

$$\frac{2221564096 + 283748\sqrt{462}}{491993569} \cong 4,52782956616\ldots.$$

There are several characterizations of the Lagrange spectrum. We will give some of them that are of interest to our work.

Let's begin with the following dynamic interpretation:

Let $\Sigma = (\mathbb{N}^*)^{\mathbb{Z}}$ and $\sigma\colon \Sigma \to \Sigma$ the shift defined by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$. If $f\colon \Sigma \to \mathbb{R}$ is defined by $f((a_n)_{n\in\mathbb{Z}}) = \alpha_0 + \beta_0 = [a_0, a_1, \ldots] + [0, a_{-1}, a_{-2}, \ldots]$, then

$$L = \left\{ \limsup_{n\to\infty} f(\sigma^n(\underline{\theta})) : \underline{\theta} \in \Sigma \right\}.$$

Another interesting set is the *Markov Spectrum* defined by

$$M = \left\{ \sup_{n\in\mathbb{Z}} f(\sigma^n(\underline{\theta})) : \underline{\theta} \in \Sigma \right\}.$$

This last interpretation in terms of shift admits a natural generalization of Lagrange and Markov spectrum in the context of hyperbolic dynamics (at least in dimension 2, which is our interest).

We will define the Lagrange and Markov dynamical spectrum as follows. Let $\varphi\colon M^2 \to M^2$ be a diffeomorphism with $\Lambda \subset M^2$ a hyperbolic set for φ. Let $f\colon M^2 \to \mathbb{R}$ be a continuous real function, then the *Lagrange Dynamical Spectrum* is defined by

$$L(f, \Lambda) = \left\{ \limsup_{n\to\infty} f(\varphi^n(x)) : x \in \Lambda \right\},$$

and the *Markov Dynamical Spectrum* is defined by

$$M(f, \Lambda) = \left\{ \sup_{n\in\mathbb{Z}} f(\varphi^n(x)) : x \in \Lambda \right\}.$$

Using techniques of stable intersection of two regular Cantor sets whose sum of Hausdorff dimension is greater than 1, found in [MY01] and [MY10], we obtain the first result of this work (cf. Chapter 2; more precisely Theorem 3).

Theorem: *Let Λ be a horseshoe associated to a C^2-diffeomorphism φ such that $HD(\Lambda) > 1$. Then there is, arbitrarily close to φ a diffeomorphism φ_0 and a C^2-neighborhood W of φ_0 such that, if Λ_ψ denotes the continuation of Λ associated to $\psi \in W$, there is an open and dense set $H_\psi \subset C^1(M, \mathbb{R})$ such that for all $f \in H_\psi$, we have*

$$int\ L(f, \Lambda_\psi) \neq \emptyset \text{ and } int\ M(f, \Lambda_\psi) \neq \emptyset,$$

where $intA$ denotes the interior of A.

There is also a geometric interpretation of the Lagrange spectrum which is the main focus of our work (cf. [CF89]). Consider the modular group, $SL(2, \mathbb{Z})$; that is, the set of all 2×2 integer matrices with determinant equal one, and $PSL(2, \mathbb{Z})$ the projectivization of $SL(2, \mathbb{Z})$. Given any $V \in SL(2, \mathbb{Z})$, $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we define the associated transformation by $V(z) = \frac{az+b}{cz+d}$. Note that if $W = \lambda V$ with $\lambda \in \mathbb{Z}^*$, then $V(z) = W(z)$.

Remember that for an irrational number α the Lagrange value of α is

$$k(\alpha) = \sup\{k : |q(q\alpha - p)| \leq k^{-1} \text{ for infinitely many } (p, q) = 1\}.$$

We note that if $(p, q) = 1$, there exist integers p', q' such that $q'p - p'q = 1$, so for $V = \left(\begin{smallmatrix} q' & -p' \\ -q & p \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ and $V(z) = \frac{q'z-p'}{-qz+p}$, we have

$$k(\alpha) = \sup\{k : |V(\infty) - V(\alpha)|^{-1} = |q(q\alpha - p)| \leq k^{-1} \text{ for infinitely many } V \in SL(2, \mathbb{Z})\}.$$

Let $\mathbb{H}^2$ be of upper half-plane model of the real hyperbolic plane, with the Poincaré metric, and let $N := \mathbb{H}^2/PSL(2, \mathbb{Z})$ the modular orbifold. Let e be an end of $\mathbb{H}^2/PSL(2, \mathbb{Z})$ (cf. [HP02] and [PP10]), define *the asymptotic height spectrum of the pair* (N, e) by

$$LimsupSp(N, e) = \left\{ \limsup_{t \to \infty} ht_e(\gamma(t)) : \gamma \in SM \right\}$$

where ht_e is the height associated to the end e of N, defined by

$$ht_e(x) = \lim_{t \to +\infty} d(x, \Gamma(t)) - t,$$

being Γ a ray that defines the end e.

Using the latter interpretation of the Lagrange spectrum, the asymptotic height spectrum $LimsupSp(N, e)$ of the modular orbifold N is the image of the Lagrange spectrum by the map $t \to \log \frac{t}{2}$ (see for instance [[HP02] theorem 3.4]). Marshall Hall (cf. [Hal47]) showed that the Lagrange spectrum contains the interval $[c, +\infty)$ for some $c > 0$. The maximal such interval $[\mu, +\infty)$ (which is closed as the Lagrange is closed (cf. [CF89])). The Hall's ray was determined by Freiman [Fre75] (cf. also Sloane [Slo]). The geometric interpretation of Freiman's result in our context is that $LimsupSp(N, e)$ contains the maximal interval $[\mu, +\infty)$ with

$$\mu = \log\left(\frac{2221564096 + 283748\sqrt{462}}{2 \cdot 491993569}\right) \approxeq 0.817\ .$$

In 1986, similar results were obtained by A. Haas and C. Series (cf. [HS86]) to the quotient of $\mathbb{H}^2$ by a fuchsian group of $SL(2,\mathbb{R})$. In particular Hecke group G_q defined by

$$G_q = \left\langle \begin{pmatrix} 1 & 2\cos\pi/q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \quad \text{for } q \geq 3.$$

In the same year, Andrew Haas [Haa86] obtained results in this direction for hyperbolic Riemann surfaces. Then 11 years later, in 1997, Thomas A. Schmidt and Mark Sheigorn (cf. [SS97]) proved that Riemann surfaces have Hall's ray in every cusp. Recently, (in 2012) P. Hubert, L. Marchese and C. Ulcigrai (cf.[HMU12]) showed that there exist Hall's rays in Teichmüller dynamics, more precisely in moduli surfaces, using renormalization.

All results mentioned above are on surfaces; let us see a bit of what is know in dimension greater than or equal to 3, for generalizations of both the Lagrange spectrum and Markov spectrum.

In fact: The classical Markov spectrum can also be expressed as (cf. [CF89])

$$M = \left\{ \inf_{(x,y)\in\mathbb{Z}^2\setminus(0,0)} |f(x,y)|^{-1} : f(x,y) = ax^2 + bxy + cy^2 \text{ with } b^2 - 4ac = 1 \right\}. \qquad (*)$$

We could think of the following natural generalization of Markov spectrum:
Let $B(x) = \sum_{1\leq i,j\leq n} b_{ij}x_ix_j$, $b_{ij} = b_{ji}$ be a real non-degenerate indefinite quadratic form in n variables and let us denote by Φ_n the set of all such forms. Let $d(B)$ denote the determinant of the matrix (b_{ij}). Let us set

$$m(B) = \inf_{(x,y)\in\mathbb{Z}^2\setminus(0,0)} |B(x)| \quad \text{and} \quad \mu(B) = \frac{m(B)^n}{|d(B)|}.$$

Let M_n denote $\mu(\Phi_n)$, Grigorii A. Margulis in [Mar] showed that for $n \geq 3$ and $\epsilon > 0$, then the set $M_n \cap (\epsilon, +\infty)$ is a finite set. Since the Lagrange spectrum L, satisfies that $L \subset M$ (cf. [CF89]), then M contains the Hall's ray, but by the foregoing and $(*)$ implies this phenomenon only happens in $n = 2$.

Returning to the geometry, let M be a complete connected Riemannian manifold with sectional curvature at most -1 and let e be an end; we defined *the Lagrange and Markov Spectra* respectively by

$$LimsupSp(M,e) = \left\{ \limsup_{t\to\infty} ht_e(\gamma(t)) : \gamma \in SM \right\}$$

and

$$MaxSp(M,e) = \left\{ \sup_{t\in\mathbb{R}} ht_e(\gamma(t)) : \gamma \in SM \right\}.$$

In this case, J. Parkkonen and F. Paulin [PP10], using purely geometric arguments showed the following theorems:

Theorem *If M has finite volume, dimension $n \geq 3$ and e is an end of M, then $MaxSp(M, e)$ contains the interval* $[4.2, +\infty]$.

Theorem(The Ubiquity of Hall's rays) *If M has finite volume, dimension $n \geq 3$ and e is and end of M, then $LimsupSp(M, e)$ contains the interval* $[6.8, +\infty]$.

These last two theorems can be true in the constant negative curvature 2-dimensional case, in [[PP10] page 278] J. Parkkonen and F. Paulin expected to be false in variable curvature and dimension 2.

Inspired in this last expectation is based our second part of work: There is hope that the two previous theorems be true for variable negative curvature in 2-dimensional case. In contrast with the work of J. Parkkonen and F. Paulin [PP10], we use purely dynamical arguments, Theorem 3, the techniques of [MY01] and [MY10] for regular Cantor set and combinatorial techniques similar to [MPV01]. More precisely we prove the following theorems (cf. Chapter 4):

Let M be a complete noncompact surface M with metric $\langle \, , \rangle$ and such that the Gaussian curvature is bounded between two negative constants and the Gaussian volume is finite. Denote by K_M the Gaussian curvature, thus there are constants $a, b > 0$ such that

$$-a^2 \leq K_M \leq -b^2 < 0.$$

Let ϕ be the vector field in SM defining the geodesic flow of the metric $\langle \, , \rangle$, (here SM denotes the unitary tangent bundle of M).

Theorem: *Let M be as above, let ϕ be the geodesic flow, then there is X a vector field sufficiently close to ϕ such that*

$$intM(f, X) \neq \emptyset \text{ and } intL(f, X) \neq \emptyset$$

for a dense and C^2-open subset $\mathcal{U}$ of $C^2(SM, \mathbb{R})$. Moreover, the above holds for a neighborhood of $\{X\} \times \mathcal{U}$ in $\mathfrak{X}^1(SM) \times C^2(SM, \mathbb{R})$, where $\mathfrak{X}^1(SM)$ is the space of C^1 vector field on SM.

The previous result can be extended by the following Theorem, which requires more sophisticated techniques:

Theorem: *Let M be as above, then there is a metric g close to $\langle \, , \, \rangle$ and a dense and C^2-open subset $\mathcal{H} \subset C^2(SM, \mathbb{R})$ such that*

$$intM(f, \phi_g) \neq \emptyset \text{ and } intL(f, \phi_g) \neq \emptyset$$

where ϕ_g is the vector field defining the geodesic flow of the metric g.

(see Definition 8 for $M(f, X)$ and $L(f, X)$).

Working the problem of two-dimensional spectra, is also motivated by the reason that none of the above references (related to the geometric spectra), has mentioned the expression “Cantor set”. But the problem of finding intervals in the classical Lagrange and Markov spectra, is closely related to the study of the fractal geometry of Cantor sets, for this reason we believe that fractal geometry of Cantor set could be the key to solve the problems about dynamical Lagrange and Markov spectra associated to geodesic flows in negative curvature. In fact, the techniques used in [MY01], [MY10] and the theorems stated above show that this is possibly.

It is worth noting that other students of IMPA, as R. Mañe [Mn97] and G. Contreras [Con10], also obtained interesting results related to the geodesic flow, they developed techniques of dynamical systems and differential geometry.

Another important related theorem of fractal geometry was proved by Marstrand in 1954 (cf. [Mar54], also cf. [PT93] page 64); it is as follows:

Theorem[Marstrand] *Let $K \subset \mathbb{R}^2$ be such that $HD(K) > 1$ and $\pi_\theta \colon \mathbb{R}^2 \to \mathbb{R}$ be the orthogonal projection on the line of direction θ. Then $\pi_\theta(K)$ has positive Lebesgue measure for almost every $\theta \in (-\pi/2, \pi/2)$ (in the Lebesgue measure sense).*

Making a simple observation on the geometry of the previous theorem, we get (cf. Chapther 5) a geometric Marstrand's theorem, using techniques of potential theory, that is: changing the canonical metric of $\mathbb{R}^2$, for a metric of nonpositive curvature.

Theorem[Geometric Marstrand] *Let M be a Hadamard surface, let $K \subset M$ and $p \in M$, such that $HD(K) > 1$, then for almost every line l coming from p, $\pi_l(K)$ has positive Lebesgue measure, where π_l is the orthogonal projection on l.*

Then using Hadamard's theorem (cf. [PadC08]), the theorem can be stated as follows:

Theorem *Let $\mathbb{R}^2$ be with a metric g of non-positive curvature, let $K \subset \mathbb{R}^2$ with $HD(K) > 1$, then for almost every $\theta \in (-\pi/2, \pi/2)$, we have that $m(\pi_\theta(K)) > 0$, where π_θ is the projection with the metric g on the line l_θ, of initial velocity $v_\theta = (\cos\theta, \sin\theta) \in T_p\mathbb{R}^2$.*

Part I

Lagrange and Markov Dynamical Spectra: For Geodesic Flows in Surfaces with Negative Curvature.

Chapter 2

The Lagrange and Markov Dynamical Spectrum

Contents

Let $\varphi : M \to M$ be a diffeomorphism on a compact 2-manifold and let Λ be a basic set for φ. That is Λ a compact, invariant, hyperbolic set which is transitive and contains a dense subset of periodic orbits. Moreover, Λ has local product structure or, equivalently, Λ is the maximal invariant set in some neighbourhood of it. We suppose that Λ is not just a periodic orbit, in which case we say that Λ is not *trivial*.

Definition 1. *Let $f : M \to \mathbb{R}$ be any continuous function, then the dynamical Markov spectrum associated to (f,Λ) is defined by*

$$M(f,\Lambda) = \left\{ \sup_{n\in\mathbb{Z}} f(\varphi^n(x)) : x \in \Lambda \right\}$$

and the dynamical Lagrange spectrum associated to (f,Λ) by

$$L(f,\Lambda) = \left\{ \limsup_{n\to\infty} f(\varphi^n(x)) : x \in \Lambda \right\}.$$

Observation: Of the definition 1 we have that $L(f,\Lambda) \subset M(f,\Lambda)$ for any $f \in C^0(M,\mathbb{R})$. In fact:
Let $a \in L(f,\Lambda)$, then there is $x_0 \in \Lambda$ such that $a = \limsup_{n\to+\infty} f(\varphi^n(x_0))$, since Λ is a compact set, then there is a subsequence $(\varphi^{n_k}(x_0))$ of $(\varphi^n(x_0))$ such that $\lim_{k\to+\infty} \varphi^{n_k}(x_0) = y_0$ and

$$a = \limsup_{n\to+\infty} f(\varphi^n(x_0)) = \lim_{k\to+\infty} f(\varphi^{n_k}(x_0)) = f(y_0).$$

Affirmation: $f(y_0) \geq f(\varphi^n(y_0))$ for all $n \in \mathbb{Z}$, otherwise, suppose there is $n_0 \in \mathbb{Z}$ such that $f(y_0) < f(\varphi^{n_0}(y_0))$, put $\epsilon = f(\varphi^{n_0}(y_0)) - f(y_0)$, then, as f is a continuous function, then there is a neighborhood U of y_0 such that

$$f(y_0) + \frac{\epsilon}{2} < f(\varphi^{n_0}(z)) \text{ for all } z \in U.$$

Thus, since $\varphi^{n_k}(x_0) \to y_0$, then there is $k_0 \in \mathbb{N}$ such that $\varphi^{n_k}(x_0) \in U$ for $k \geq k_0$, therefore,

$$f(y_0) + \frac{\epsilon}{2} < f(\varphi^{n_0+n_k}(x_0)) \text{ for all } k \geq k_0.$$

This contradicts the definition of $a = f(y_0)$.

In this section we show that for a "large" C^1-functions, the sets $M(f,\Lambda)$ and $L(f,\Lambda)$ have non empty interior.

2.1 The "Large" Subset of $C^1(M,\mathbb{R})$

A "Large" set in the following sense.

Theorem 1. *The set*

$$H_\varphi = \left\{ f \in C^1(M,\mathbb{R}) : \#M_f(\Lambda) = 1 \ \ \text{and for} \ \ z \in M_f(\Lambda),\ Df_z(E_z^{s,u}) \neq 0 \right\}$$

is open and dense, where $M_f(\Lambda) = \{z \in \Lambda : f(z) \geq f(y) \ \ \forall\, y \in \Lambda\}$.

The purpose of this section is to prove the previous theorem.

From the local product structure, we know that for $x, x' \in \Lambda$ sufficiently close, $W^u_\epsilon(x)$ and $W^s_\epsilon(x')$ have a unique point of intersection and that this point also belongs to Λ.

We say that x is a boundary point of Λ in the unstable direction, if x is a boundary point of $W^u_\epsilon(x) \cap \Lambda$, *i.e.* if x is an accumulation point only from one side by points in $W^u_\epsilon(x) \cap \Lambda$. If x is a boundary point of Λ in the unstable direction, then, due to the local product structure, the same holds for all points in $W^s(x) \cap \Lambda$. So the boundary points in the *unstable* direction are local intersections of local *stable* manifolds with Λ. For this reason we denote the set of boundary points in the unstable direction by $\partial_s\Lambda$. The boundary points in the stable direction are defined similarly. The set of these boundary points is denoted by $\partial_u\Lambda$.

The following theorem is due to S. Newhouse an J. Palis (cf. [PT93, pp170]).

Theorem [PN] *For a basic set* Λ *as above there is a finite number of (periodic) saddle points* $p_1^s, ..., p_{n_s}^s$ *such that*

$$\Lambda \cap \left(\bigcup_i W^s(p_i^s) \right) = \partial_s \Lambda.$$

Similarly, there is a finite number of (periodic) saddle points $p_1^u, ..., p_{n_u}^u$ *such that*

$$\Lambda \cap \left(\bigcup_i W^u(p_i^u) \right) = \partial_u \Lambda.$$

Moreover, both $\partial_s \Lambda$ *and* $\partial_u \Lambda$ *are dense in* Λ*.*

A consequence of this theorem and local product structure is:

Corollary 1. *The set* $\partial_s \Lambda \cap \partial_u \Lambda$ *is dense in* Λ*.*

Proof. Remember that $\partial_s \Lambda$ and $\partial_u \Lambda$ are dense in Λ, so it suffices to prove that $\partial_s \Lambda \cap \partial_u \Lambda$ is dense in $\partial_s \Lambda$, in fact:

Let $\delta > 0$, of the definition of local product structure. Let $z \in \partial_s \Lambda$, then for all $0 < r < \delta$, there is $w_r \in \partial_u \Lambda$ with $d(z, w_r) < r$, is easy to see the local product structure that the point $W_\epsilon^s(z) \cap W_\epsilon^u(w_r) \in \partial_s \Lambda \cap \partial_u \Lambda$, and is close to z if r is small. □

Given $f \in C^0(M, \mathbb{R})$, denote $M_f(\Lambda) = \{z \in \Lambda : f(z) \geq f(y) \ \ \forall\ y \in \Lambda\}$.

Remember that $T_\Lambda M = E^s \oplus E^u$ is the splitting in the definition of hyperbolicity.

Lemma 1. *The set*

$$\mathscr{A}' = \left\{ f \in C^2(M, \mathbb{R}) : \text{there is } \ z \in M_f(\Lambda) \text{ and } Df_z(E_z^{s,u}) \neq 0 \right\}$$

is dense in $C^2(M, \mathbb{R})$*.*

Before doing proof we introduce the concept of Morse functions (cf. [Hir76]).

Definition 2. *Let* $f \colon M \to \mathbb{R}$*,* C^r*,* $r \geq 2$ *is a Morse function, if for all* $x \in M$ *such that* $Df_x = 0$ *we have that*

$$D^2 f(0) \colon T_x M \times T_x M \to \mathbb{R}$$

is nondegenerate, i.e. *if* $D^2 f(0)(v, w) = 0$ *for all* $w \in T_x M$ *implies* $v = 0$*. Denote this set by* $\mathscr{M}$*.*

Theorem: *The set of Morse functions is open and dense in* $C^2(M, \mathbb{R})$*,* $r \geq 2$*.*

If f is a Morse function, then $Crit(f) = \{x \in M : Df_x = 0\}$ is a discrete set. In particular, since Λ is a compact set, we have that $\#\,(Crit(f) \cap \Lambda) < \infty$.

Proof of Lemma 1. By theorem above let's show simply that $\mathscr{A}'$ is dense in $\mathscr{M}$. Let $f_1 \in \mathscr{M}$, then $\#\,(M_{f_1}(\Lambda) \cap Crit(f)) < \infty$, so we can find $f \in \mathscr{M}$, C^2-close to f_1 such that $M_f(\Lambda) \cap Crit(f) = \emptyset$. Therefore, if $z \in M_f(\Lambda)$, we have $Df_z(E^s_z) \neq 0$ or $Df_z(E^u_z) \neq 0$. If both $Df_z(E^s_z)$ and $Df_z(E^u_z)$ are nonzero, then $f \in \mathscr{A}'$.

In other case, suppose that $Df_z(E^s_z) = 0$ and $Df_z(E^u_z) \neq 0$, then there are C^2-neighborhood $\mathcal{V}$ of f and neighborhood U of z, such that if $x \in U \cap \Lambda$ and $g \in \mathcal{V}$, then $Dg_x(e^u_x) \neq 0$.

Let $\epsilon > 0$ be small, such that the fundamental neighborhood of f,

$$V_\epsilon(f) := \{g : d_{C^2}(f, g) < \epsilon\} \subset \mathcal{V}.$$

Let $\widetilde{U}_z \subset U$ neighborhood of z such that $d(f(x), f(z)) < \epsilon/2$, and by Corollary 1 there is $z' \in \widetilde{U}_z \cap (\partial_s\Lambda \cap \partial_u\Lambda)$. Let $\varphi_{z'} \in C^2(M, \mathbb{R})$ a C^2-function such that $\varphi_{z'}$ is C^2-close to the constant function 1, in fact, $d_{C^2}(\varphi_{z'}, 1) < \epsilon$ and $\varphi_{z'} = 1$ in $M \setminus \widetilde{U}_z$ and satisfies the following properties:

1. $\varphi_{z'}(z) \geq 1$;
2. $f(x) - f(z') + \epsilon > \varphi_{z'}(z') - \varphi_{z'}(x) > f(x) - f(z')$ for all $x \in U \cap (\Lambda \setminus \{z'\})$;
3. $D\varphi_{z'}(e^s_{z'}) \neq -Df_{z'}(e^s_{z'})$;

where $e^s_{z'} \in E^s_{z'}$ is unit vector.
Thus, consider the function $g_{z'} = f + \varphi_{z'} - 1$ is C^2-close to f and by the property 2 we have $g_{z'}(z') > g_{z'}(x)$ for all $x \in \widetilde{U}_z \cap (\Lambda \setminus \{z'\})$.
If $x \in \Lambda \setminus U$, then by the property 1 and 2, follows that

$$g_{z'}(z') = f(z') + \varphi_{z'}(z') - 1 \geq f(z) + \varphi_{z'}(z) - 1 \geq f(x) = f(x) + \varphi_{z'}(x) - 1 = g_{z'}(x).$$

Moreover, since $d_{C^2}(g_{z'}, f) < \epsilon$, then $g_{z'} \in V_\epsilon(f)$, thus $Dg_{z'}(e^u_{z'}) \neq 0$ and the property 3 we have that $g_{z'} \in \mathscr{A}'$.

Case $Df_z(e^u_z) = 0$ and $Df_z(E^s_z) \neq 0$ is obtained analogously a function C^2-close to f and in $\mathscr{A}'$.
This concludes the proof of Lemma.

□

Lemma 2. *Let $f \in C^1(M, \mathbb{R})$ and $z \in M_f(\Lambda)$ such that $Df_z(E^{s,u}_z) \neq 0$, then $z \in \partial_s\Lambda \cap \partial_u\Lambda$.*

Proof. Using local coordinates in z, we can assume that we are in $U \subset \mathbb{R}^2$ containing 0. The hypothesis of the lemma implies that $Df_z \neq 0$, *i.e.* $f(z)$ is a regular value of f, then $\alpha := f^{-1}(f(z))$ is a C^1-curve transverse to $W^s_\epsilon(z)$ and $W^u_\epsilon(z)$ in z, also, the gradient vector $\nabla f(z)$ is orthogonal a α in z.

Let U be a small neighborhood z, then α subdivided into two regions U, say U_1, U_2 (see Figure 2.1). Now suppose that $\nabla f(z)$ is pointing in the direction of U_1, then in the region I, II, III, IV and V, (see Figure 2.1), there are no points of Λ, in fact:

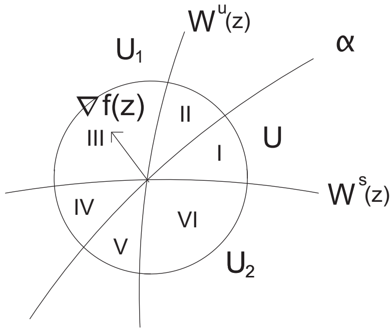


Figure 2.1: Localization of $z \in M_f(\Lambda)$

As the function increases in the direction of the gradient, then in the regions II, III and IV, there are no points of Λ, because $z \in M_f(\Lambda)$.
If there are points in I of Λ, then by the local product structure, there are points in II of Λ, which we know can not happen.
Analogously, if there are points in V of Λ, then there are points in IV of Λ, which we know can not happen.
In conclusion, the only region where there are points of Λ is VI, so $z \in \partial_s\Lambda \cap \partial_u\Lambda$.

□

Remark 1. *Since, $C^s(M,\mathbb{R})$, $1 \leq s \leq \infty$ is dense in $C^r(M,\mathbb{R})$, $0 \leq r < s$, then the Lemma 1 implies that $\mathscr{A}'$ is dense in $C^1(M,\mathbb{R})$.*

Lemma 3. *The set*

$$H_1 = \left\{f \in C^2(M,\mathbb{R}) : \#M_f(\Lambda) = 1 \ \ and\ for \ \ z \in M_f(\Lambda),\ Df_z(E_z^{s,u}) \neq 0\right\}$$

is dense in $C^2(M,\mathbb{R})$, therefore dense in $C^1(M,\mathbb{R})$.

Proof. By Lemma 1, it is sufficient to show that H_1 is dense in $\mathcal{A}'$.
Let $f \in \mathcal{A}'$, then there is $z \in M_f(\Lambda)$ such that $Df_z(e_z^{s,u}) \neq 0$. Take U a small neighborhood of z. Thus, given $\epsilon > 0$ small, consider the function $\varphi_\epsilon \in C^2(M,\mathbb{R})$ such that φ_ϵ is C^2-close to constant function 1, also $\varphi_\epsilon = 1$ in $M \setminus U$, $\varphi_\epsilon(z) = 1+\epsilon$ and z is a single maximum of φ_ϵ. Also, $\varphi_\epsilon \overset{C^2}{\to} 1$ as $\epsilon \to 0$.
Define $g_\epsilon = f + \varphi_\epsilon - 1$, clearly $g_\epsilon \overset{C^2}{\to} f$ as $\epsilon \to 0$, since $z \in M_f(\Lambda)$ we have $g_\epsilon(z) = f(z) + \varphi_\epsilon(z) - 1 > f(x) + \varphi_\epsilon(x) - 1 = g_\epsilon(x)$ for all $x \in \Lambda$, this is $z \in M_{g_\epsilon}(\Lambda)$ and $\#M_{g_\epsilon}(\Lambda) = 1$.
Also, $D(g_\epsilon)_z(e_z^{s,u}) = Df_z(e_z^{s,u}) \neq 0$, that is, $g_\epsilon \in H_1$. □

Lemma 4. *The set*

$$H_\varphi = \left\{f \in C^1(M,\mathbb{R}) : \#M_f(\Lambda) = 1 \ \ and\ for \ \ z \in M_f(\Lambda),\ Df_z(E_z^{s,u}) \neq 0\right\}$$

is open.

Proof. Let $f \in H_\varphi$ and $z \in M_f(\Lambda)$ with $Df_z(e_z^{s,u}) \neq 0$, where $e_z^{s,u} \in E_z^{s,u}$ is a unit vector, respectively. Suppose that $\dfrac{\partial f}{\partial e_z^{s,u}} = \langle \nabla f(z), e_z^{s,u} \rangle = Df_z(e_z^{s,u}) > 0$ and $\nabla f(z)$ is the gradient vector of f at z.
(If we have to, $Df_z(e_z^s) > 0$ and $Df_z(e_z^u) < 0$, we consider the basis $\{e_z^s, -e_z^u\}$ of T_zM or vice versa).

Let $\mathcal{U} \subset C^1(M,\mathbb{R})$ an open neighborhood of f such that, for all $g \in \mathcal{U}$ we have $\dfrac{\partial g}{\partial e_z^{s,u}} > 0$. The set $\{e_z^s, e_z^u\}$ is basis of T_zM. Let

$$V = \{v \in T_zM : v = a_v e_z^s + b_v e_z^u, \quad a_v, b_v \geq 0\}.$$

Let $v \in V \setminus \{0\}$, then $\dfrac{\partial g}{\partial v}(z) = Dg_z(v) > 0$, for any $g \in \mathcal{U}$. Since by Lemma 2 we have that $z \in \partial_s\Lambda \cap \partial_u\Lambda$, this implies that, there is an open set U of z such that $g(z) > g(x)$, for all $g \in \mathcal{U}$ and all $x \in U \cap \Lambda \setminus \{z\}$.

Let $\epsilon > 0$ such that $|f(z) - f(x)| > \frac{\epsilon}{2}$ for $x \in \Lambda \setminus U$. Let

$$V_{\frac{\epsilon}{8}}(f) = \left\{ g \in C^1(M,\mathbb{R}) : \|f - g\|_\infty < \frac{\epsilon}{8} \text{ and } \|Df - Dg\|_\infty < \frac{\epsilon}{8} \right\}$$

a fundamental neighborhood of f, then we claim that, for all $g \in V_{\frac{\epsilon}{8}}(f)$, the set $M_g(\Lambda) \subset U$. In fact: Let $x \in \Lambda \setminus U$, then

$$\begin{aligned} |g(z) - g(x)| &= |g(z) - f(z) + f(z) - g(x) - f(x) + f(x)| \\ &\geq f(z) - f(x) - |g(z) - f(z)| - |g(x) - f(x)| \\ &\geq \frac{\epsilon}{2} - 2\frac{\epsilon}{8} = \frac{\epsilon}{4}. \end{aligned}$$

Suppose that, there is $z_g \in M_g(\Lambda) \cap (\Lambda \setminus U)$, then

$$\begin{aligned} \frac{\epsilon}{8} \geq |g(z) - f(z)| &\geq |g(z) - g(z_g) + f(z_g) - f(z)| - |g(z_g) - f(z_g)| \\ &= g(z_g) - g(z) + f(z) - f(z_g) - |g(z_g) - f(z_g)| \\ &\geq \frac{\epsilon}{4} + \frac{\epsilon}{2} - \frac{\epsilon}{8} = \frac{5\epsilon}{8}. \end{aligned}$$

This is a contradiction. Therefore, we have the assertion.
Consider the open set $\mathcal{U}_1 = \mathcal{U} \cap V_{\frac{\epsilon}{8}}(f)$, then clearly, $\mathcal{U}_1 \subset H_\varphi$. □

Now we are in condition to prove Theorem 1

Proof of Theorem 1. Since, $H_1 \subset H_\varphi$ and by Lemma 3 the set H_1 is dense in $C^1(M,\mathbb{R})$, then $H_\varphi \subset C^1(M,\mathbb{R})$ is dense and open in $C^1(M,\mathbb{R})$. □

2.2 Regular Cantor Set and Expanding Maps Associated to a Horseshoe

2.2.1 Regular Cantor Set

Let $\mathbb{A}$ be a finite alphabet, $\mathbb{B}$ a subset of $\mathbb{A}^2$, and $\Sigma_{\mathbb{B}}$ the subshift of finite type of $\mathbb{A}^{\mathbb{Z}}$ with allowed transitions $\mathbb{B}$. We will always assume that $\Sigma_{\mathbb{B}}$ is topologically mixing, and that every letter in $\mathbb{A}$ occurs in $\Sigma_{\mathbb{B}}$.
An *expansive map of type* $\Sigma_{\mathbb{B}}$ is a map g with the following properties:

(i) the domain of g is a disjoint union $\bigcup_{\mathbb{B}} I(a,b)$. Where for each (a,b), $I(a,b)$ is a compact subinterval of $I(a) := [0,1] \times \{a\}$;

(ii) for each $(a,b) \in \mathbb{B}$, the restriction of g to $I(a,b)$ is a smooth diffeomorphism onto $I(b)$ satisfying $|Dg(t)| > 1$ for all t.

The *regular Cantor set* associated to g is the maximal invariant set

$$K = \bigcap_{n \geq 0} g^{-n}\left(\bigcup_{\mathbb{B}} I(a,b)\right).$$

Let $\Sigma_{\mathbb{B}}^+$ be the unilateral subshift associated to $\Sigma_{\mathbb{B}}$. There exists a unique homeomorphism $h\colon \Sigma_{\mathbb{B}}^+ \to K$ such that

$$h(\underline{a}) \in I(a_0), \text{ for } \underline{a} = (a_0, a_1, \dots) \in \Sigma_{\mathbb{B}}^+ \ \textit{and} \ \ h \circ \sigma = g \circ h,$$

where, $\sigma^+\colon \Sigma_{\mathbb{B}}^+ \to \Sigma_{\mathbb{B}}^+$, is defined as follows $\sigma^+((a_n)_{n\geq 0}) = (a_{n+1})_{n \geq 0}$. For $(a,b) \in \mathbb{B}$, let

$$f_{a,b} = \left[g|_{I(a,b)}\right]^{-1}.$$

This is a contracting diffeomorphism from $I(b)$ onto $I(a,b)$. If $\underline{a} = (a_0, \cdots, a_n)$ is a word of $\Sigma_{\mathbb{B}}$, we put

$$f_{\underline{a}} = f_{a_0,a_1} \circ \cdots \circ f_{a_{n-1},a_n}$$

this is a diffeomorphism from $I(a_n)$ onto a subinterval of $I(a_0)$ that we denote by $I(\underline{a})$, with the property that if z in the domain of $f_{\underline{a}}$ we have that

$$f_{\underline{a}}(z) = h(\underline{a}h^{-1}(z)).$$

2.2.2 Expanding Maps Associated to a Horseshoe

Let φ be a diffeomorphism of class C^2 on a surface M, and Λ is a horseshoe of φ. Consider a finite collection $(R_a)_{a\in\mathbb{A}}$ of disjoint rectangles of M, which are a Markov partition of Λ (cf. [Shu86]). The set $\mathbb{B} \subset \mathbb{A}^2$ of admissible transitions consist of pairs (a_0, a_1) such that $\varphi(R_{a_0}) \cap R_{a_1} \neq \emptyset$.
So we can define the following transition matrix B which induces the same transitions that $\mathbb{B} \subset \mathbb{A}^2$

$$b_{ij} = 1 \ \text{ if } \ \varphi(R_i) \cap R_j \neq \emptyset, \quad b_{ij} = 0 \quad \text{otherwise, for } (i,j) \in \mathbb{A}^2.$$

There is a homeomorphism $\Pi\colon \Sigma_B \to \Lambda$ such that, the following diagram commutes

$$\begin{array}{ccc} \Sigma_B & \xrightarrow{\sigma} & \Sigma_B \\ \Pi\downarrow & & \downarrow\Pi \\ \Lambda & \xrightarrow{\varphi} & \Lambda \end{array}$$

$\varphi \circ \Pi = \Pi \circ \sigma$.

Given $x, y \in \mathbb{A}$, we denote by $N_n(x, y, B)$ the number of admissible strings for B of length $n+1$, beginning at x and ending with y. Then the following holds

$$N_n(x, y, B) = b^n_{xy}.$$

In particular, since $\varphi|_\Lambda$ is transitive, then given $x, y \in \mathbb{A}$ always exists a increasing sequence containing arithmetic progression $n_l(x, y) > 0$ such that

$$N_{n_l(x,y)}(x, y, B) = b^{n_l(x,y)}_{xy} > 0.$$

The dynamics of φ on Λ is topologically conjugate to sub-shifts $\Sigma_{\mathbb{B}}$ defined by $\mathbb{B}$.
Put

$$W^s(\Lambda, R) = \bigcap_{n\geq 0} \varphi^{-n}(\bigcup_{a\in\mathbb{A}} R_a)$$

$$W^u(\Lambda, R) = \bigcap_{n\leq 0} \varphi^{-n}(\bigcup_{a\in\mathbb{A}} R_a).$$

There is a $r > 1$ and a collection of C^r-submersions $(\pi_a : R_a \to I(a))_{a\in\mathbb{A}}$, satisfying the following properties:

If $z, z' \in R_{a_0} \cap \varphi^{-1}(R_{a_1})$ and $\pi_{a_0}(z) = \pi_{a_0}(z')$, then we have

$$\pi_{a_1}(\varphi(z)) = \pi_{a_1}(\varphi(z')).$$

In particular, the connected components of $W^s(\Lambda, R) \cap R_a$ are the level lines of π_a. Then we define a mapping g^u of class C^r (expansive of type $\Sigma_{\mathbb{B}}$) by the formula

$$g^u(\pi_{a_0}(z)) = \pi_{a_1}(\varphi(z))$$

for $(a_0, a_1) \in \mathbb{B}$, $z \in R_{a_0} \cap \varphi^{-1}(R_{a_1})$. The regular Cantor set K^u defined by g^u, describes the geometry transverse of the stable foliation $W^s(\Lambda, R)$. Moreover, there exists a unique homeomorphism $h^u\colon \Sigma^+_{\mathbb{B}} \to K^u$ such that

$$h^u(\underline{a}) \in I(a_0),\ \text{for } \underline{a} = (a_0, a_1, \dots) \in \Sigma^+_{\mathbb{B}} \ \ \textit{and}\ \ h^u \circ \sigma^+ = g^u \circ h^u,$$

where $\sigma^+\colon \Sigma^+_{\mathbb{B}} \to \Sigma^+_{\mathbb{B}}$, is defined as follows $\sigma^+((a_n)_{n\geq 0}) = (a_{n+1})_{n\geq 0}$.

Given a finite word $\underline{a} = (a_0, \cdots, a_n)$, denote $f^u_{\underline{a}}$ as in previous section, such that

$$f^u_{\underline{a}}(z) = h^u(\underline{a}(h^u)^{-1}(z)).$$

Analogously, we can describe the geometry transverse of the unstable foliation $W^u(\Lambda, R)$, using a regular Cantor set K^s define by a mapping g^s of class C^r (expansive of type $\Sigma_{\mathbb{B}}$). Moreover, there exists a unique homeomorphism $h^s\colon \Sigma_{\mathbb{B}}^- \to K^s$ such that

$$h^s(\underline{a}) \in I(a_0), \text{ for } \underline{a} = (\ldots, a_1, a_0) \in \Sigma_{\mathbb{B}}^- \quad and \quad h^s \circ \sigma^- = g^s \circ h^s,$$

where $\sigma^-\colon \Sigma_{\mathbb{B}}^- \to \Sigma_{\mathbb{B}}^-$, is defined as follows $\sigma^-((a_n)_{n\leq 0}) = (a_{n-1})_{n\leq 0}$.
Given a finite word $\underline{a} = (a_{-n}, \cdots, a_0)$, denote $f^s_{\underline{a}}$ as in previous section, such that

$$f^s_{\underline{a}}(z) = h^s((h^s)^{-1}(z)\underline{a}).$$

2.3 The Interior of the Spectrum

Recall that the set

$$H_\varphi = \left\{f \in C^1(M, \mathbb{R}) : \#M_f(\Lambda) = 1 \text{ for } z \in M_f(\Lambda),\ Df_z({E_z}^{s,u}) \neq 0\right\}$$

is open and dense.
Let $f \in H_\varphi$ and $x_M \in M_f(\Lambda)$, then by Lemma 2 we have that $x_M \in \partial_s\Lambda \cap \partial_u\Lambda$, by Theorem [PN], we have that there are $p, q \in \Lambda$ periodic points such that

$$x_M \in W^s(p) \cap W^u(q).$$

Suppose that $x_M \notin per(\varphi)$, that is, $p \neq q$. Assume that p and q have the symbolic representation

$$(\cdots, a_1, \cdots, a_r, a_1, \cdots, a_r, \cdots) \text{ and } (\cdots, b_1, \cdots, b_s, b_1, \cdots, b_s, \cdots)$$

respectively.
So, there are l symbols $c_1, \cdots, c_l$ such that x_M is symbolically of the form

$$\Pi^{-1}(x_M) = (\cdots, b_1, \cdots, b_s, b_1, \cdots, b_s, c_1, \cdots, c_t, \cdots, c_l, a_1, \cdots, a_r, a_1, \cdots, a_r, \cdots)$$

where c_t is the zero position of $\Pi^{-1}(x_M)$.

Let $s \in \mathbb{N}$ sufficiently large and $\underline{q} = (-q_s, \cdots, q_0, \cdots, q_s)$ a admissible word such that $x_M \in R_{\underline{q}} = \bigcap_{i=-s}^{s} \varphi^{-i}(R_{q_i})$, as in the Figure 2.2, and let U be an open set such that $U \cap \Lambda = \Lambda \setminus R_{\underline{q}}$, then we define

$$\tilde{\Lambda} := \bigcap_{n\in\mathbb{Z}} \varphi^n(U).$$

Take $d \in \tilde{\Lambda}$, call $\underline{d} = (\cdots, d_{-n}, \cdots, d_0, \cdots, d_n, \cdots)$ the symbolic representation. For $n_0 \in \mathbb{N}$, let $\underline{d}_{n_0} = (d_{-n_0}, \cdots, d_{n_0})$ an admissible finite word. Denote the cylinder
$C^{n_0}_{d_{n_0}} = \left\{\underline{w} \in \mathbb{A}^{\mathbb{Z}} : w_i = d_i \text{ for } i = -n_0, \cdots, n_0\right\}$. Then, the set

$$C^{n_0}_{d_{n_0},B} := \Sigma_B \cap C^{n_0}_{d_{n_0}} = \{\underline{w} \in \Sigma_B : w_i = d_i \text{ for } i = -n_0, \cdots, n_0\}$$

is not empty and contains a periodic point.

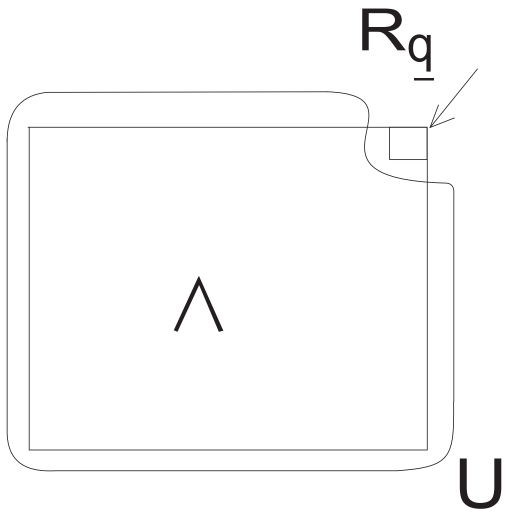


Figure 2.2: Removing the maximum point

Let $n_0(d_0, b_1) := k_0$ and $n_0(a_r, d_1) := j_0$ the minimum of $n_l(d_0, b_1)$ and $n_l(a_r, d_1)$, respectively. So, $b^{k_0}_{d_0 b_1} > 0$ and $b^{j_0}_{a_r d_1} > 0$, therefore there are admissible strings $\underline{e} = (e_1, \cdots, e_{k_0-1})$ and $\underline{f} = (f_1, \cdots, f_{j_0-1})$ joined d_0 with b_1 and a_r with d_1, respectively.

Let $k \in \mathbb{N}$, $k > \max\{n_0(x, y) : (x, y) \in \mathbb{A}^2\}$, then given the word $(a_1, \cdots, a_r)$ and $(b_1, \cdots, b_s)$, we defined the word

$$(a_1, \cdots, a_r)^k = \underbrace{(a_1, \cdots, a_r, \cdots\cdots, a_1, \cdots, a_r)}_{k-times}$$

and

$$(b_1, \cdots, b_s)^k = \underbrace{(b_1, \cdots, b_s, \cdots\cdots, b_1, \cdots, b_s)}_{k-times}.$$

Put the word

$$\alpha = ((b_1, \cdots, b_s)^k, c_1, \cdots, c_t, \cdots, c_l, (a_1, \cdots, a_r)^k),$$

where c_t is the zero position of the word α.
So, fixed the words $\underline{e}$ and $\underline{f}$, we can define the following application, defined for all $\underline{x} \in C^{n_0}_{d_{n_0}, B}$ by

$$A(\underline{x}) = (\cdots, x_{-1}, x_0, e_1, \cdots, e_{k_0-1}, (b_1, \cdots, b_s)^k, c_1, \cdots, \\ , c_t, \cdots, c_l, (a_1, \cdots, a_r)^k, f_1, \cdots, f_{j_0-1}, x_1, x_2, \cdots).$$

Given a finite word $\underline{a} = (a_1, \cdots, a_n)$, denote by $|\underline{a}| = n$, the length of the word $\underline{a}$. Then, since $k > \max\{n_0, j_0\}$, we have

$$|\underline{e}|, |\underline{f}| < |\alpha| = k(s + r) + l.$$

where $\underline{e} = (e_1, \cdots, e_{k_0-1})$ and $\underline{f} = (f_1, \cdots, f_{j_0-1})$. It is easy to see that for all $\underline{x} \in C^{n_0}_{d_{n_0}, B} \cap \Pi^{-1}(\tilde{\Lambda})$, we have

$$\sup_{n \in \mathbb{Z}} \tilde{f}(\sigma^n(A(\underline{x}))) = \tilde{f}(A(\underline{x})) \tag{2.1}$$

where $\tilde{f} = f \circ \Pi$. In fact:
Observe that $\sigma^{l-t+kr+j_0-1}(A(\underline{x})) \in W^s_{\frac{1}{3}}(\sigma(\underline{x}))$ and $\sigma^{-(t+sk+k_0-1)}(A(\underline{x})) \in W^u_{\frac{1}{3}}(\underline{x})$ and $\underline{x} \in \Pi^{-1}(\tilde{\Lambda})$. Since $M_f(\Lambda) = \{x_M\}$ and $|e|, |f| < |\alpha|$, we have the desired (2.1).

Suppose now that $x_M \in Per(\varphi)$, then there is a finite word $\underline{a} = (a_1, \cdots, a_m)$ such that $\Pi^{-1}(x_M) = (\cdots, \underline{a}, \underline{a}, \cdots, \underline{a}, \cdots)$. Let k, d and $C^{n_0}_{d_{n_0,B}}$ be as above. Also, there are admissible strings $\underline{e}' = (e'_1, \cdots, e'_{m_0})$ and $\underline{f}' = (f'_1, \cdots, f'_{r_0})$ joined d_0 with a_1 and a_m with d_1, respectively.
So, we can define the following application, defined for all $\underline{x} \in C^{n_0}_{d_{n_0,B}}$ by

$$A(\underline{x}) = (\cdots, x_{-1}, x_0, e'_1, \cdots, e'_{m_0}, (a_1, \cdots, a_m)^k, f'_1, \cdots, f'_{r_0}, x_1, x_2, \cdots).$$

Note that $|\underline{e}'|, |\underline{f}'| < km = |\underline{a}^k|$, so there is a finite number of positions $p_1, \ldots, p_l$

$$\underline{e}'\,\overbrace{\underline{a}, \cdots, \underline{a}, \underbrace{\underline{a}}_{p_1}, \cdots, \underbrace{\underline{a}}_{p_l}, \underline{a}, \cdots, \underline{a}}^{k-times}\,\underline{f}'$$

such that for each $\underline{x} \in C^{n_0}_{d_{n_0,B}} \cap \Pi^{-1}(\tilde{\Lambda})$ there is p_i such that

$$\sup_{n\in\mathbb{Z}} \tilde{f}(\sigma^n(A(\underline{x}))) = \tilde{f}(\sigma^{p_i}(A(\underline{x}))). \tag{2.2}$$

Put, $\Pi^{-1}(x) = \underline{x}$, define the set

$$\tilde{\Lambda}_i := \{x \in (\tilde{\Lambda} \cap \Pi(C^{n_0}_{d_{n_0,B}})) : \sup_{n\in\mathbb{Z}} \tilde{f}(\sigma^n(A(\underline{x}))) = f(\sigma^{p_i}(A(\underline{x})))\}.$$

Thus,

$$\tilde{\Lambda} \cap \Pi(C^{n_0}_{d_{n_0,B}}) = \bigcup_{i}^{l} \tilde{\Lambda}_i. \tag{2.3}$$

The equation (2.3) implies that there is $i_0 \in \{1, \ldots, l\}$ such that $\tilde{\Lambda}_{i_0}$ is open in $\tilde{\Lambda} \cap \Pi(C^{n_0}_{d_{n_0,B}})$, so

$$HD(\tilde{\Lambda} \cap \Pi(C^{n_0}_{d_{n_0,B}})) = HD(\tilde{\Lambda}_{i_0}). \tag{2.4}$$

On $\tilde{\Lambda}_{i_0}$ we have (2.2) for $i = i_0$.

The next objective is to show that $\tilde{A} = \Pi \circ A \circ \Pi^{-1}$ extends to a local diffeomorphism.

First we show that $\tilde{A}$ extends to a local diffeomorphism in stable and unstable manifolds of d, $W^s_{loc}(d)$ and $W^u_{loc}(d)$.

As Λ is symbolically the product $\Sigma^-_{\mathbb{B}} \times \Sigma^+_{\mathbb{B}}$, put β the finite word ($\beta = \underline{e}\alpha\underline{f}$). By the previous section, we have that for

$$x^u \in W^u_{loc}(d) \cap \Lambda, \quad \text{then} \quad f^u_\beta(x^u) \in W^u(d) \cap \Lambda \quad \text{and} \quad (\Pi^{-1}(f^u_\beta(x^u)))^+ = \beta(\Pi^{-1}(x^u))^+,$$

also

$$x^s \in W^s_{loc}(d) \cap \Lambda, \text{ then } f^s_\beta(x^s) \in W^s(d) \cap \Lambda \text{ and } (\Pi^{-1}(f^s_\beta(x^s)))^- = (\Pi^{-1}(x^s))^-\beta.$$

The position zero of $\Pi^{-1}\left(\varphi^{-|\beta|+1}\left(f^s_\beta(x^s)\right)\right)$, is equal to $(\beta)_0 = e_1$, this is

$$\left(\Pi^{-1}\left(\varphi^{-|\beta|+1}\left(f^s_\beta(x^s)\right)\right)\right)_0 = (\beta)_0 = \left(\Pi^{-1}\left(f^u_\beta(x^u)\right)\right)_0.$$

So, we can define the bracket

$$\left[\Pi^{-1}\left(f^u_\beta(x^u)\right), \Pi^{-1}\left(\varphi^{-|\beta|+1}\left(f^s_\beta(x^s)\right)\right)\right] = \left(\Pi^{-1}(x^s)\right)^- \beta \left(\Pi^{-1}(x^u)\right)^+ = A\left[\Pi^{-1}(x^u), \Pi^{-1}(x^s)\right].$$

Note that for x^u, x^s sufficiently close to d the bracket $[\Pi^{-1}(x^u), \Pi^{-1}(x^s)]$ is well defined. As Π is a morphism of the local product structure, then

$$\begin{aligned} \left[f^u_\beta(x^u), \varphi^{-|\beta|+1}\left(f^s_\beta(x^s)\right)\right] &= \Pi\left(\left[\Pi^{-1}\left(f^u_\beta(x^u)\right), \Pi^{-1}\left(\varphi^{-|\beta|+1}\left(f^s_\beta(x^s)\right)\right)\right]\right) \\ &= \Pi(A\left[\Pi^{-1}(x^u), \Pi^{-1}(x^s)\right]) = \tilde{A}\left[x^u, x^s\right]. \end{aligned} \tag{2.5}$$

Put $\tilde{A}_1(x^u) = f^u_\beta(x^u)$ and $\tilde{A}_1(x^s) = \varphi^{-|\beta|+1}(f^s_\beta(x^s))$, therefore, $\tilde{A}\left[x^u, x^s\right] = [\tilde{A}_1(x^u), \tilde{A}_2(x^s)]$. Thus, we have the following lemma.

Lemma 5. *If φ is a C^2-diffeomorphism, then $\tilde{A}$ extends to a local C^1-diffeomorphism defined in neighborhood U_d of d.*

Proof. As φ is a C^2-diffeomorphism of a closed surface, then the stable and unstable foliations of the horseshoe Λ, $\mathscr{F}^s(\Lambda)$ and $\mathscr{F}^u(\Lambda)$ can be extended to C^1 invariant foliations defined on a full neighborhood of Λ. Also, if φ is a C^2-diffeomorphism, then f^s_β and f^u_β are at least C^1, then by (2.5) we have the result. $\square$

An immediate consequence of this Lemma **??** and the (2.1) (in the case that $x_M \in (\partial_s\Lambda \cap \partial_u\Lambda) \setminus Per(\varphi)$) is:

Corollary 2. *If $x \in \tilde{\Lambda} \cap U_d$, then $\sup_{n\in\mathbb{Z}} f(\varphi^n(\tilde{A}(x))) = f(\tilde{A}(x))$.*

This Corollary implies that $f(\tilde{A}(\tilde{\Lambda} \cap U_d)) \subset M(f, \Lambda)$.

Another consequence of this Lemma 5 and the (2.2) (in the case that $x_M \in Per(\varphi)$) is:

Corollary 3. *If $x \in \tilde{\Lambda}_{i_0}$, then $\sup_{n\in\mathbb{Z}} f(\varphi^n(\tilde{A}(x))) = f(\varphi^{p_{i_0}}(\tilde{A}(x)))$.*

This Corollary implies that $f(\varphi^{p_{i_0}}(\tilde{A}(x))) \subset M(f, \Lambda)$.

Remark 2. *The local diffeomorphism $\tilde{A}$ depend of f, moreover for $f \in H_\varphi$, we have $Df_{x_M}(e^{s,u}_{x_M}) \neq 0$, so this property is true in neighborhood of x_M, so there is $x \in U_d$ such that $Df_{\tilde{A}(x)}(e^{s,u}_{\tilde{A}(x)}) \neq 0$ and since by construction of $\tilde{A}$, we have that $\dfrac{\partial \tilde{A}}{\partial e^{s,u}_x} \parallel e^{s,u}_{\tilde{A}(x)}$, then there is $x \in U_d$ such that $\nabla(f \circ \tilde{A})(x) \not\parallel e^{s,u}_x$.*

Now we will prove the same for the Lagrange spectrum.

First suppose that $x_M \notin Per\varphi$. Then, using the above notation, let $x \in \tilde{\Lambda} \cap \Pi(C^{n_0}_{d_{n_0},B})$ and $\Pi^{-1}(x) = (\cdots, x_{-n}, \cdots, x_0, \cdots, x_n, \cdots)$, let $n_0(x_i, x_{-i}) := m_i$ be such that $b^{m_i}_{x_i x_{-i}} > 0$, then there is admissible string $E_i = (e^i_1, \cdots, e^i_{s_i})$ joined x_i with x_{-i} the length $|E_i| = m_i - 1$.
So, we can define the following application, defined for all $\underline{x} \in C^{n_0}_{d_{n_0},B}$ by

$$\begin{aligned} A_1(\underline{x}) = (\cdots, x_3, E_3, x_{-3}, x_{-2}, x_{-1}, x_0, \beta, x_1, x_2, E_2, x_{-2}, x_{-1}, x_0, \beta, x_1, E_1, x_{-1}, x_0, \\ , \beta, x_1, E_1, x_{-1}, x_0, \beta, x_1, x_2, E_2, x_{-2}, x_{-1}, x_0, \beta, x_1, x_2, x_3, E_3, x_{-3}, \cdots) \end{aligned}$$

where $\beta = \underline{e}\alpha\underline{f}$.

Therefore, it is easy to see that for all $\underline{x} \in C^{n_0}_{d_{n_0},B} \cap \Pi^{-1}(\tilde{\Lambda})$, we have

$$\limsup_{n\to\infty} \tilde{f}(\sigma^n(A_1(\underline{x}))) = \tilde{f}(A(\underline{x})) \tag{2.6}$$

where $\tilde{f} = \Pi^{-1} \circ f \circ \Pi$. In fact:
Remember that $|\underline{e}|, |\underline{f}|, |E_i| < |\alpha|$ for all $i \geq 1$ and $M_f(\Lambda) = \{x_M\}$, then

$$\limsup_{n\to\infty} \tilde{f}(\sigma^n(A_1(\underline{x}))) = \sup_k \tilde{f}(\sigma^{n_k}(A_1(\underline{x})))$$

where n_k is such that $(\sigma^{n_k}(A_1(\underline{x})))^+ = (c_t, \cdots c_l, (a_1, \cdots, a_r)^k, f_1 \cdots, f_{j_0-1}, \cdots)$, that is, the positive part of $\sigma^{n_k}(A_1(\underline{x}))$, begins with $(c_t, \cdots, c_l, (a_1, \cdots, a_r)^k)$, where $c_t = \alpha_0$ and $\alpha = ((b_1, \cdots, b_s)^k, c_1, \cdots, c_t, \cdots, c_l, (a_1, \cdots, a_r)^k)$. Therefore, there is a subsequence n_{k_j} with $n_{k_j} \to \infty$, as $j \to \infty$ such that

$$\sup_k \tilde{f}(\sigma^{n_k}(A_1(\underline{x}))) = \lim_{j\to\infty} \tilde{f}(\sigma^{n_{k_j}}(A_1(\underline{x}))).$$

By construction of A_1, it is true that

$$\lim_{j\to\infty} \sigma^{n_{k_j}}(A_1(\underline{x})) = A(\underline{x}),$$

where $A(\underline{x})$ is defined as before.

Therefore,

$$\limsup_{n\to\infty} \tilde{f}(\sigma^n(A_1(\underline{x}))) = \tilde{f}(A(\underline{x})).$$

As an immediate consequence (in the case that $x_M \in \partial_s\Lambda \cap \partial_u\Lambda \setminus Per(\varphi)$) we have.

Corollary 4. *If $x \in \tilde{\Lambda} \cap U_d$, then*

$$\limsup_{n\to\infty} f(\varphi^n(\tilde{A}_1(x))) = f(\tilde{A}(x)), \quad \textit{where} \quad \tilde{A}_1 = \Pi \circ A_1 \circ \Pi^{-1}.$$

This Corollary implies that $f(\tilde{A}(\tilde{\Lambda} \cap U_d)) \subset L(f, \Lambda)$.
Analogously, as in (2.2) in the case $x_M \in Per(\varphi)$ we have

Corollary 5. *If* $x \in \tilde{\Lambda}_{i_0}$*, then* $\limsup_{n\to\infty} f(\varphi^n(\tilde{A}_1(x))) = f(\varphi^{p_{i_0}}(\tilde{A}(x)))$.

This Corollary implies that $f(\varphi^{p_{i_0}}(\tilde{A}(\tilde{\Lambda}_{i_0}))) \subset L(f,\Lambda)$.

Remark 3. *Let* φ *be a* C^2*-diffeomorphism, and* Λ *a horseshoe associate to* φ*, denoted* $T_\Lambda M = E^s \oplus E^u$ *the splitting of the definition of hyperbolicity of* Λ*, then* Λ *is locally the product of two regular Cantor sets* K^s*,* K^u*.*

Lemma 6. *If* Λ *is a horseshoe associated to a* C^2*-diffeomorphism* φ *and* $HD(\Lambda) > 1$*, then* $HD(\tilde{\Lambda} := \bigcap_{n\in\mathbb{Z}} \varphi^n(U)) > 1$*.*

The equation (2.4) implies that

Corollary 6. *Assuming the Lemma 6, we have* $HD(\tilde{\Lambda}_{i_0}) > 1$*.*

Remark 4. *By Corollary 6, 5 and 3 we can assume that if* $f \in H_\varphi$*, then* $x_M \in (\partial_s\Lambda \cap \partial_u\Lambda) \setminus Per(\varphi)$*. Therefore, henceforth we assume that* $x_M \in \partial_s\Lambda \cap \partial_u\Lambda \setminus Per(\varphi)$*.*

Recalling that, as φ is a C^2-diffeomorphism, Λ is locally the product of stable and unstable regular Cantor set, $K^s \times K^u$. Then the previous lemma will be a consequence of the following lemma.

Let K be a regular Cantor set, with expanding map ψ and Markov partition $\mathcal{R} = \{K_1, \cdots, K_k\}$ and $K = \bigcap_{n\geq 0} \psi^{-n}(\bigcup_{i=1}^{k} K_i)$, consider the transition matrix $A = (a_{ij})_{k\times k}$ associated to the partition $\mathcal{R}$, define by

$$a_{ij} = \begin{cases} 1 & \text{if } \psi(K_i) \supset K_j; \\ 0 & \text{if } \psi(K_i) \cap K_j = \emptyset \end{cases} .$$

Given a finite word admissible of length m, $\underline{b} = (b_1, \cdots, b_m)$, such that $a_{b_i b_{i+1}} = 1$, we associate the interval $I_{\underline{b}} = I_{b_1} \cap \phi^{-1}(I_{b_2}) \cap \phi^{-2}(I_{b_3}) \cdots \cap \phi^{-(n-1)}(I_{b_m})$.

Lemma 7. *Let* K *be a regular Cantor set, with expanding map* ψ *and Markov partition* $\mathcal{R} = \{K_1, \cdots, K_k\}$ *and* $K = \bigcap_{n\geq 0} \psi^{-n}(\bigcup_{i=1}^{k} K_i)$*. Given a finite word admissible of arbitrarily large length* m*,* $\underline{b} = (b_1, \cdots, b_m)$*, then there is* $\epsilon > 0$ *small, such that*

$$HD(K_{\underline{b}}) \geq HD(K) - \epsilon, \quad \textit{where} \quad K_{\underline{b}} = \bigcap_{n\geq 0} \psi^{-n}(\bigcup_{i=1}^{k} Ki \setminus I_{\underline{b}}).$$

Proof. Let $\mathcal{R} = \{K_1, \cdots, K_k\}$ be a Markov partition for K and, for $n \geq 2$. Let $\mathcal{R}^n$ denote the set of connected components of $\psi^{-(n-1)}(K_i)$, $K_i \in \mathcal{R}$. For $R \in \mathcal{R}^n$ take $\lambda_{n,R} = \inf |(\psi^n)'|_R|$ and $\Lambda_{n,R} = \sup |(\psi^n)'|_R|$. Define $\alpha_n, \beta_n > 0$ by

$$\sum_{R\in\mathcal{R}^n} (\Lambda_{n,R})^{-\alpha_n} = C \quad \text{and} \quad \sum_{R\in\mathcal{R}^n} (\lambda_{n,R})^{-\beta_n} = 1.$$

In [PT93, pg. 69-70], was shown that, $HD(K) \geq \alpha_n$ and $d(K) \leq \beta_n$, where $d(K)$ is a box dimension. Let a be such that $\Lambda_{n,R} \leq a\lambda_{n,R}$, for all $n \geq 1$ and $R \in \mathcal{R}^n$ and $\lambda = \inf|\psi'| > 1$, then

$$\beta_n - \alpha_n \leq \frac{HD(K)\log a + \log C}{n\log\lambda - \log a}.$$

Put $A_n = \sup_{R\in\mathcal{R}^n} \lambda_{n,R}$ and $B_n = \inf_{R\in\mathcal{R}^n} \lambda_{n,R}$, then

$$(\#\mathcal{R}^n)B_n^{-\beta_n} \geq \sum_{R\in\mathcal{R}^n} (\lambda_{n,R})^{-\beta_n} = 1 \geq (\#\mathcal{R}^n)A_n^{-\beta_n}$$

which implies that

$$\frac{\log\#\mathcal{R}^n}{\log B_n} \geq \beta_n \geq \frac{\log\#\mathcal{R}^n}{\log A_n}.$$

We can do the above for the regular Cantor set $K_{\underline{b}}$. Note that $I_{\underline{b}}$ is a element of Markov partition $\mathcal{R}^m = \bigvee_{r=0}^{m-1}\psi^{-r}\mathcal{R}$. Let $\mathcal{R}_{\underline{b}} = (\bigvee_{r=0}^{n-1}\psi^{-r}\mathcal{R}) \setminus I_{\underline{b}}$ a Markov partition for $K_{\underline{b}}$. Without loss of generality, the numbers $\lambda_{n,R}$, $\Lambda_{n,R}$, α_n, β_n, A_n, B_n will be called $\tilde{\lambda}_{n,R}$, $\tilde{\Lambda}_{n,R}$, $\tilde{\alpha}_n$, $\tilde{\beta}_n$, $\tilde{A}_n$, $\tilde{B}_n$ for $K_{\underline{b}}$. Clearly, $\beta_n \geq \tilde{\beta}_n$.

Analogously, for $\mathcal{R}_{\underline{b}}^n$ denote the set of connected components of $\psi^{-(n-1)}(J)$, $J \in \mathcal{R}_{\underline{b}}$ we have

$$\frac{\log\#\mathcal{R}_{\underline{b}}^n}{\log\tilde{B}_n} \geq \tilde{\beta}_n \geq \frac{\log\#\mathcal{R}_{\underline{b}}^n}{\log\tilde{A}_n}.$$

Therefore,

$$\beta_n - \tilde{\beta}_n \geq \frac{\log\#\mathcal{R}^n}{\log A_n} - \frac{\log\#\mathcal{R}_{\underline{b}}^n}{\log\tilde{B}_n}$$

As, $\#\mathcal{R}^n \approx \#(\mathcal{R})^n = k^n$, $\#\mathcal{R}_{\underline{b}}^n \approx \#(\mathcal{R}_{\underline{b}})^n = (k^m - 1)^n$ and $A_n \approx \lambda^n$, $\tilde{B}_n \approx \lambda^{nm}$, this implies that

$$\beta_n - \tilde{\beta}_n \approx \frac{n\log k}{n\log\lambda} - \frac{n\log(k^m-1)}{nm\log\lambda} = \left(\frac{\log\frac{k^m}{k^m-1}}{m\log\lambda}\right).$$

Since $\beta_n \to HD(K)$ and $\tilde{\beta}_n \to HD(K_{\underline{b}})$.
This completes the proof of Lemma. □

Proof of Lemma 6. Apply the previous Lemma to K^s and K^u and then use the fact that for regular Cantor sets, it is true that the Hausdorff dimension of the product is the sum of the Hausdorff dimensions (cf. [PT93]). □

Note that by Lemma 6 and the local structure of $\tilde{\Lambda}$, we have $HD(\tilde{\Lambda} \cap U_d) > 1$.

2.3.1 Intersections of Regular Cantor Sets

Let r be a real number > 1, or $r = +\infty$. The space of C^r expansive maps of type Σ (cf. Section 2.2.1), endowed with the C^r topology, will be denoted by Ω^r_Σ. The union $\Omega_\Sigma = \bigcup_{r>1} \Omega^r_\Sigma$ is endowed with the inductive limit topology.

Let $\Sigma^- = \{(\theta_n)_{n\leq 0}, (\theta_i, \theta_{i+1}) \in \mathbb{B} \text{ for } i < 0\}$. We equip Σ^- with the following ultrametric distance: for $\underline{\theta} \neq \underline{\widetilde{\theta}} \in \Sigma^-$, set

$$d(\underline{\theta}, \underline{\widetilde{\theta}}) = \begin{cases} 1 & \text{if } \theta_0 \neq \widetilde{\theta}_0; \\ & \\ |I(\underline{\theta} \wedge \underline{\widetilde{\theta}})| & \text{otherwise} \end{cases}.$$

where $\underline{\theta} \wedge \underline{\widetilde{\theta}} = (\theta_{-n}, \ldots, \theta_0)$ if $\widetilde{\theta}_{-j} = \theta_{-j}$ for $0 \leq j \leq n$ and $\widetilde{\theta}_{-n-1} \neq \theta_{-n-1}$.
Now, let $\underline{\theta} \in \Sigma^-$; for $n > 0$, let $\underline{\theta}^n = (\theta_{-n}, \ldots, \theta_0)$, and let $B(\underline{\theta}^n)$ be the affine map from $I(\underline{\theta}^n)$ onto $I(\theta_0)$ such that the diffeomorphism $k^{\underline{\theta}}_n = B(\underline{\theta}^n) \circ f_{\underline{\theta}^n}$ is orientation preserving.

We have the following well-known result (cf. [Sul87]):

Proposition. *Let $r \in (1, +\infty)$, $g \in \Omega^r_\Sigma$.*

1. *For any $\underline{\theta} \in \Sigma^-$, there is a diffeomorphism $k^{\underline{\theta}} \in Diff^r_+(I(\theta_0))$ such that $k^{\underline{\theta}}_n$ converge to $k^{\underline{\theta}}$ in $Diff^{r'}_+(I(\theta_0))$, for any $r' < r$, uniformly in $\underline{\theta}$. The convergence is also uniform in a neighborhood of g in Ω^r_Σ.*

2. *If r is an integer, or $r = +\infty$, $k^{\underline{\theta}}_n$ converge to $k^{\underline{\theta}}$ in $Diff^r_+(I(\theta_0))$. More precisely, for every $0 \leq j \leq r-1$, there is a constant C_j (independent on $\underline{\theta}$) such that*

$$\left| D^j \log D \left[k^{\underline{\theta}}_n \circ (k^{\underline{\theta}})^{-1}\right](x)\right| \leq C_j |I(\underline{\theta}^n)|.$$

   It follows that $\underline{\theta} \to k^{\underline{\theta}}$ is Lipschitz in the following sense: for $\theta_0 = \widetilde{\theta}_0$, we have

$$\left| D^j \log D\left[k^{\underline{\widetilde{\theta}}} \circ (k^{\underline{\theta}})^{-1}\right](x)\right| \leq C_j \, d(\underline{\theta}, \underline{\widetilde{\theta}}).$$

Let $r \in (1, +\infty]$. For $a \in \mathbb{A}$, denote by $\mathcal{P}^r(a)$ the space of C^r-embeddings of $I(a)$ into $\mathbb{R}$, endowed with the C^r topology. The affine group $Aff(\mathbb{R})$ acts by composition on the left on $\mathcal{P}^r(a)$, the quotient space being denoted by $\overline{\mathcal{P}}^r(a)$. We also consider $\mathcal{P}(a) = \bigcup_{r>1} \mathcal{P}^r(a)$ and $\overline{\mathcal{P}}(a) = \bigcup_{r>1} \overline{\mathcal{P}}^r(a)$, endowed with the inductive limit topologies.

Remark 5. *In [MY01] is considered $\mathcal{P}^r(a)$ for $r \in (1, +\infty]$, but all the definitions and results involving $\mathcal{P}^r(a)$ can be obtained considering $r \in [1, +\infty]$.*

Let $\mathcal{A} = (\underline{\theta}, A)$, where $\underline{\theta} \in \Sigma^-$ and A is now an *affine* embedding of $I(\theta_0)$ into $\mathbb{R}$. We have a canonical map

$$\begin{array}{rcl} \mathcal{A} & \to & \mathcal{P}^r = \bigcup_{\mathbb{A}} \mathcal{P}^r(a) \\ (\underline{\theta}, A) & \mapsto & A \circ k^{\underline{\theta}} \ \ (\in \mathcal{P}^r(\theta_0)). \end{array}$$

Now we assume we are given two sets of data $(\mathbb{A}, \mathbb{B}, \Sigma, g)$, $(\mathbb{A}', \mathbb{B}', \Sigma', g')$ defining regular Cantor sets K, K'.
We define as in the previous the spaces $\mathcal{P} = \bigcup_{\mathbb{A}} \mathcal{P}(a)$ and $\mathcal{P}' = \bigcup_{\mathbb{A}'} \mathcal{P}(a')$.
A pair (h, h'), $(h \in \mathcal{P}(a), h' \in \mathcal{P}'(a'))$ is called a *smooth configuration* for $K(a) = K \cap I(a)$, $K'(a') = K' \cap I(a')$. Actually, rather than working in the product $\mathcal{P} \times \mathcal{P}'$, it is better to go to the quotient Q by the diagonal action of the affine group $Aff(\mathbb{R})$. Elements of Q are called *smooth relative configurations* for $K(a)$, $K'(a')$.
We say that a smooth configuration $(h, h') \in \mathcal{P}(a) \times \mathcal{P}(a')$ is

- *linked* if $h(I(a)) \cap h'(I(a')) \neq \emptyset$;
- *intersecting* if $h(K(\underline{a})) \cap h'(K(\underline{a}')) \neq \emptyset$, where $K(\underline{a}) = K \cap I(\underline{a})$ and $K(\underline{a}') = K \cap I(\underline{a}')$;
- *stably intersecting* if it is still intersecting when we perturb it in $\mathcal{P} \times \mathcal{P}'$, and we perturb (g, g') in $\Omega_\Sigma \times \Omega_{\Sigma'}$.

All these definitions are invariant under the action of the affine group, and therefore make sense for smooth relative configurations.

As in previous, we can introduce the spaces $\mathcal{A}$, $\mathcal{A}'$ associated to the limit geometries of g, g' respectively. We denote by $\mathcal{C}$ the quotient of $\mathcal{A} \times \mathcal{A}'$ by the diagonal action on the left of the affine group. An element of $\mathcal{C}$, represented by $(\underline{\theta}, A) \in \mathcal{A}$, $(\underline{\theta}', A') \in \mathcal{A}'$, is called a relative configuration of the limit geometries determined by $\underline{\theta}$, $\underline{\theta}'$. We have canonical maps

$$\begin{array}{rcl} \mathcal{A} \times \mathcal{A}' & \to & \mathcal{P} \times \mathcal{P}' \\ \mathcal{C} & \to & Q \end{array}$$

which allow to define linked, intersecting, and stably intersecting configurations at the level of $\mathcal{A} \times \mathcal{A}'$ or $\mathcal{C}$.

Remark: For a configuration $((\underline{\theta}, A), (\underline{\theta}', A'))$ of limit geometries, one could also consider the *weaker* notion of stable intersection, obtained by considering perturbations of g, g' in $\Omega_\Sigma \times \Omega_{\Sigma'}$ and perturbations of $(\underline{\theta}, A)$, $(\underline{\theta}', A')$ in $\mathcal{A} \times \mathcal{A}'$. We do not know of any example of expansive maps g, g', and configurations $(\underline{\theta}, A)$, $(\underline{\theta}', A')$ which are stably intersecting in the weaker sense but not in the stronger sense.

We consider the following subset V of $\Omega_\Sigma \times \Omega_{\Sigma'}$. A pair (g, g') belongs to V if for any $[(\underline{\theta}, A), (\underline{\theta}', A')] \in \mathcal{A} \times \mathcal{A}'$ there is a translation R_t (in $\mathbb{R}$) such that $(R_t \circ A \circ k^{\underline{\theta}}, A' \circ k'^{\underline{\theta}'})$ is a stably intersecting configuration.

Theorem [cf. [MY01]]:

1. *V is open in $\Omega_\Sigma \times \Omega_{\Sigma'}$, and $V \cap (\Omega^\infty_\Sigma \times \Omega^\infty_{\Sigma'})$ is dense (for the C^∞-topology) in the set $\{(g, g'), HD(K) + HD(K') > 1\}$.*

2. *Let $(g,g')\in V$. There exists $d^*<1$ such that for any $(h,h')\in\mathcal{P}\times\mathcal{P}'$, the set*

$$\mathcal{I}_s=\{t\in\mathbb{R},(R_t\circ h,h') \text{ is a stably intersecting smooth configuration for } (g,g')\}$$

is (open and) dense in

$$\mathcal{I}=\{t\in\mathbb{R},(R_t\circ h,h') \text{ is an intersecting smooth configuration for } (g,g')\}$$

and moreover $HD(\mathcal{I}-\mathcal{I}_s)\le d^*$. The same d^* is also valid for $(\tilde g,\tilde g')$ in a neighborhood of (g,g') in $\Omega_\Sigma\times\Omega_{\Sigma'}$.

2.4 The Image of the Product of Two Regular Cantor Sets by a Real Function and Behavior of the Spectrum

Theorem 2. *Let K, K' are two regular Cantor sets with expanding map g, g'. Suppose that $HD(K)+HD(K')>1$ and $(g,g')\in V$. Let f be a C^1-function $f\colon U\to\mathbb{R}$ with $K\times K'\subset U\subset\mathbb{R}^2$ such that, in some point of $K\times K'$ its gradient is not parallel to any of the two coordinate axis, then*

$$int f(K\times K')\neq\emptyset.$$

Proof. By hypothesis, and by continuity of df, we find a pair of periodic points p_1, p_2 of K and K', respectively, with addresses $\overline{a}_1=\underline{a_1a_1a_1}...$ and $\overline{a}_2=\underline{a_2a_2a_2}...$, where $\underline{a}_1$ and $\underline{a}_2$ are finite sequences, such that $df(p_1,p_2)$ is not a real multiple of dx nor of dy. There are increasing sequences of natural number (m_k), (n_k) such that the intervals $I_{\underline{a}_1^{m_k}}$ and $I_{\underline{a}_2^{n_k}}$ defined by the finite words $\underline{a}_1^{m_k}$ and $\underline{a}_2^{n_k}$, satisfy

$$\frac{|I_{\underline{a}_1^{m_k}}|}{|I_{\underline{a}_2^{n_k}}|}\in(C^{-1},C) \text{ for some } C>1.$$

Thus, we can assume that $\frac{|I_{\underline{a}_1^{m_k}}|}{|I_{\underline{a}_2^{n_k}}|}\to\lambda\in[C^{-1},C]$ as $k\to\infty$, define $\tilde\lambda:=-\dfrac{\frac{\partial f}{\partial y}(p_1,p_2)}{\frac{\partial f}{\partial x}(p_1,p_2)}\lambda$.

Put $k^{\overline{a}_1}(K\cap I_{\underline{a}_1^{m_k}}):=k_{\overline{a}_1}$ and $k^{\overline{a}_2}(K'\cap I_{\underline{a}_2^{n_k}}):=k'_{\overline{a}_2}$ for k large. As $(K,K')\in V$, then there is $t\in\mathbb{R}$ such that $(k_{\overline{a}_1},\tilde\lambda k'_{\overline{a}_2}+t)$ have stable intersection, therefore there is neighborhood $U(k_{\overline{a}_1},k'_{\overline{a}_2})$ of $(k_{\overline{a}_1},k'_{\overline{a}_2})$ and a C^1-neighborhood $W(\tilde\lambda x+t)$ of $\tilde\lambda x+t$ (by Remark 5) such that

$$\tilde K\cap h(\tilde K')\neq\emptyset \text{ for any } h\in W \text{ and } (\tilde K,\tilde K')\in U.$$

Observe that the map $k^{\overline{a}_1}\circ g^{m_k}\colon K\cap I_{\underline{a}_1^{m_k}}\to k_{\overline{a}_1}$ is almost homothety, *i.e.*
if $B_k\colon[0,1]\to I_{\underline{a}_1^{m_k}}$ affine map orientation-preserving, then $k^{\overline{a}_1}\circ g^{m_k}\circ B_k$ is C^2-close to the identity $I\colon[0,1]\to[0,1]$.
Analogously, the map $k^{\overline{a}_2}\circ(g')^{n_k}\colon K'\cap I_{\underline{a}_2^{n_k}}\to k'_{\overline{a}_2}$ is almost homothety (in the same sense).

Let $z \in k_{\underline{a}_1} \cap (\tilde{\lambda} k'_{\underline{a}_2} + t)$, then

$$w = (w_1, w_2) = \left((k^{\overline{a}_1} \circ g^{m_k})^{-1}(z), (k^{\overline{a}_2} \circ (g')^{n_k})^{-1} \left(\frac{z-t}{\tilde{\lambda}} \right) \right) \in (K \cap I_{\underline{a}_1^{m_k}}) \times (K' \cap I_{\underline{a}_2^{n_k}}).$$

So, we can suppose that, $df(w_1, w_2)$ is not a real multiple of dx nor of dy, in particular $df(w_1, w_2) \neq 0$, then there is α the level line of $f(w_1, w_2)$, this is, $\alpha = f^{-1}(f(w_1, w_2))$, is a submanifold of U.
Therefore, by the implicit function theorem, there is a neighborhood U_{w_2} of w_2 and C^1-function ξ defined in U_{w_2}, such that

$$f(\xi(y), y) = f(w_1, w_2).$$

Note that $\xi'(w_2) = -\dfrac{\frac{\partial f}{\partial y}(w_1, w_2)}{\frac{\partial f}{\partial x}(w_1, w_2)}$, then $\xi|_{I_{\underline{a}_2^{n_k}}}$ is C^1-close, up to a composition on the left by an affine map, to the map $\tilde{\lambda} x + t$.

Now consider a vertical segment $l_{w_1} = \{(w_1, w_2 + s) : |s| < \epsilon\}$, for $\epsilon > 0$ small, such that if $p \in B_\epsilon(w_1, w_2) := \{x : \|x - p\| < \epsilon\}$, then $\nabla f(p) \nparallel e_i$ for $i = 1, 2$.
Let ξ_s be a C^1-diffeomorphisms defined in a neighborhood of w_2, such that $f(\xi_s(x), x) = f(w_1, w_2 + s)$, we can assume that the domain of ξ_s is U_{w_2}.

Call $\xi_0 = \xi$, clearly, we have that ξ_s is C^1-close of ξ_0 is a continuous family. Suppose that for s small, ξ_s is C^1-close to ξ, therefore C^1-close up to a composition on the left by an affine map, to the map $\tilde{\lambda} x + t$.

Thus, $k_{\underline{a}_1} \cap \xi_s(k'_{\underline{a}_2}) \neq \emptyset$, let $z_s \in k_{\underline{a}_1} \cap \xi_s(k'_{\underline{a}_2})$, then

$$w_s = (w_1^s, w_2^s) = \left((k^{\overline{a}_1} \circ g^{m_k})^{-1}(z_s), (k^{\overline{a}_2} \circ (g')^{n_k})^{-1} \left(\xi_s^{-1}(z_s) \right) \right) \in (K \cap I_{\underline{a}_1^{m_k}}) \times (K' \cap I_{\underline{a}_2^{n_k}}).$$

We are interested in the image of w_s by f, in fact

$$f(w_s) = \frac{\partial f}{\partial x}(p_1, p_2) \left(w_1^s - \tilde{\lambda} w_2^s \right) + \widetilde{C}.$$

As z_s varies continuously with s, then have the results. □

The following example show that the property V in the Theorem 2 is fundamental.

Example: Consider the regular Cantor set $K_\alpha := \bigcap_{n \geq 0} \psi^{-n}(K_1 \cup K_2)$, where

$$\psi(x) = \begin{cases} \frac{2}{1-\alpha} x & \text{if } \ x \in K_1 := [0, \frac{1-\alpha}{2}]; \\ \\ -\frac{2}{1-\alpha} x + \frac{2}{1-\alpha} & \text{if } \ x \in K_2 := [\frac{1+\alpha}{2}, 1]. \end{cases}$$

Since, $HD(K_\alpha) = -\frac{\log 2}{\log(\frac{1-\alpha}{2})}$ (cf. [PT93]). Then, if $\alpha < 1/2$, then $HD(K_\alpha) > 1/2$ and for $1/3 < \alpha < 1/2$ hold that $K_\alpha - K_\alpha$ has measure zero (cf. [Mor99]).

Moreover, $HD(K_\alpha \times K_\alpha) > 1$ and $f(x,y) = x - y$, satisfies the hypothesis of previous Theorem, but for $1/3 < \alpha < 1/2$ we have that

$$int\ f(K_\alpha \times K_\alpha) = \emptyset.$$

Corollary 7. *Let φ be a C^2-diffeomorphism, and Λ a horseshoe associate to φ, suppose that K^s, K^u satisfy the hypotheses of theorem above, put*

$$\mathcal{A}_\Lambda = \{f \in C^1(M,\mathbb{R}) : \exists z = (z^s, z^u) \in \Lambda \ \ such\ that\ \ \nabla f(z) \nparallel e_z^{s,u}\ \}.$$

Then, for all $f \in \mathcal{A}_\Lambda$, we have $int f(\Lambda) \neq \emptyset$.

It is easy to prove that $\mathcal{A}_\Lambda$ given the above Corollary is an open and dense set in $C^1(M,\mathbb{R})$.

A fundamental result due to Moreira-Yoccoz in [MY10] on the existence of elements in V associated to the pairs regular Cantor sets (K^s, K^u) defined by g^s, g^u where g^s describes the geometry transverse of the unstable foliation $W^u(\Lambda, R)$ and g^u describes the geometry transverse of the stable foliation $W^s(\Lambda, R)$ given in the section 2.2.2 is the following:

Theorem[cf. [MY10]] *Suppose the sum of the Hausdorff dimensions of the Cantor regular set K^s, K^u defined by g^s, g^u are > 1. So, if the neighborhood $\mathcal{U}$ of φ_0 in $Diff^\infty(M)$ is sufficiently small, there is an open and dense $\mathcal{U}^* \subset \mathcal{U}$ such that for $\varphi \in \mathcal{U}^*$ the corresponding pair of expanding applications (g, g') belongs to V.*

Definition 3. *Let φ be a C^2-diffeomorphism φ with a horseshoe Λ of $HD(\Lambda) > 1$, the pair (φ, Λ) is said to have the property V, if the pair of expanding map (g^s, g^u) associated to pair of regular Cantor sets (K^s, K^u) associate to (φ, Λ) is in V.*

The previous theorem implies that given a horseshoe Λ_0 associate to diffeomorphism φ_0 with $HD(\Lambda_0) > 1$, there is a diffeomorphism φ close to φ_0 with a horseshoe Λ (Hyperbolic Continuation of Λ_0), such that (φ, Λ) satisfies the hypotheses of Corollary 7 for all $f \in \mathcal{A}_\Lambda$.

We use the above results to show that the Markov and Lagrange spectra has non-empty interior.

Given a pair (φ, Λ) of a diffeomorphism φ and a horseshoe associate to φ with $HD(\Lambda) > 1$, In the section 2.1 (cf. Lemma 4 and Theorem 1) was defined the open dense set H_φ in $C^1(M,\mathbb{R})$ by

$$H_\varphi = \left\{f \in C^1(M,\mathbb{R}) : \#M_f(\Lambda) = 1 \ \ for \ \ z \in M_f(\Lambda),\ Df_z(e_z^{s,u}) \neq 0\right\}.$$

Remember that $\tilde{\Lambda}$ is a sub-horseshoe of Λ as in Lemma 6 with $HD(\tilde{\Lambda}) > 1$ and $HD(\tilde{\Lambda} \cap U_d) > 1$, then by theorem [MY10], there is a diffeomorphism φ_0 close to φ, and horseshoe Λ_0 associate to φ_0 and sub-horseshoe $\tilde{\Lambda}_0 \subset \Lambda_0$ with $HD(\tilde{\Lambda}_0) > 1$ and such that $\tilde{\Lambda}_0$ (we are perturbing the sub-horseshoe) satisfies the hypotheses of Theorem 2.

Given $f \in H_{\varphi_0}$, we can define $\tilde{A}_{\varphi_0}(f)$ a local diffeomorphism, in coordinates given by the stable and unstable foliation, we can write, $\tilde{A}_{\varphi_0}(f)(x,y) = (\tilde{A}^1_{\varphi_0}(f)(x), \tilde{A}^2_{\varphi_0}(f)(y))$ (see

(2.5)) as in the section 2.3.

Note that, if $f \in H_{\varphi_0}$, Let $\tilde{d} \in \tilde{\Lambda}_0$ close to d and $U_{\tilde{d}}$ is a small neighborhood of $\tilde{d}$. Then, by Remark 2 there is $x \in U_{\tilde{d}}$ such that $\nabla(f \circ \tilde{A}_{\varphi_0}(f))(x) \nparallel \tilde{e}_x^{s,u}$, where $\tilde{e}_x^{s,u}$ are the unit vectors in stable and unstable bundle of hyperbolic set $\tilde{\Lambda}_0$, respectively. So, the function $f \circ \tilde{A}_{\varphi_0}(f) \in \mathcal{A}_{\Lambda_0}$. Therefore, by Corollary 7

$$int(f \circ \tilde{A}_{\varphi_0}(f))(\tilde{\Lambda}_0 \cap U_{\tilde{d}}) \neq \emptyset. \tag{2.7}$$

Then, as in Corollary 2, we have that

$$\sup_{n \in \mathbb{Z}} f(\varphi_0^n(\tilde{A}_{\varphi_0}(f)(x)) = (f \circ \tilde{A}_{\varphi_0}(f))(x)$$

for all $x \in \tilde{\Lambda}_0 \cap U_{\tilde{d}}$. This implies that $(f \circ \tilde{A}_{\varphi_0}(f))(\tilde{\Lambda}_0 \cap U_{\tilde{d}}) \subset M(f, \Lambda_0)$. Thus, by (2.7) we have that $intM(f, \Lambda_0) \neq \emptyset$.

By Corollary 4 and 5 we have the analogue to the Lagrange spectrum. Thus, we have the following theorem.

Then, as in Corollary 2, we have that

$$\sup_{n \in \mathbb{Z}} f(\varphi_0^n(\tilde{A}_{\varphi_0}(f)(x)) = (f \circ \tilde{A}_{\varphi_0}(f))(x)$$

for all $x \in \tilde{\Lambda}_0 \cap U_{\tilde{d}}$. This implies that $(f \circ \tilde{A}_{\varphi_0}(f))(\tilde{\Lambda}_0 \cap U_{\tilde{d}}) \subset M(f, \Lambda_0)$. Thus, by (2.7) we have that $intM(f, \Lambda_0) \neq \emptyset$.

Theorem 3. *Let Λ be a horseshoe associated to a C^2-diffeomorphism φ such that $HD(\Lambda) > 1$. Then there is, arbitrarily close to φ a diffeomorphism φ_0 and a C^2-neighborhood W of φ_0 such that, if Λ_ψ denotes the continuation of Λ associated to $\psi \in W$, there is an open and dense set $H_\psi \subset C^1(M, \mathbb{R})$ such that for all $f \in H_\psi$, we have*

$$intL(f, \Lambda_\psi) \neq \emptyset \text{ and } intM(f, \Lambda_\psi) \neq \emptyset.$$

where $intA$ denotes the interior of A.

Chapter 3

Hausdorff Dimension Greater than 1 for Hyperbolic Set in Cross-section for Geodesic Flow

Contents

We will work with a complete noncompact surface M such that the Gaussian curvature is bounded between two negative constants and the Gaussian volume is finite. Denote by K_M the Gaussian curvature, thus there are constants $a, b > 0$ such that

$$-a^2 \leq K_M \leq -b^2 < 0$$

Next we will be considering some theorems that will be used in our arguments.

Definition 4. *Let X_1, X_2 be metric spaces with metrics d_1 and d_2 respectively, a sequence of maps $f_i : X_1 \longrightarrow X_2$, $i = 1, 2, ...$ is said to be uniformly bi-Lipschitz if there exists a $C > 1$ such that*

$$C^{-1}d_1(x, y) \leq d_2(f_i(x), f_i(y)) \leq Cd_1(x, y)$$

for all $x, y \in X_1$ and $i = 1, 2, ...$; a $C > 1$ for which the relation holds is called a uniform bi-Lipschitz constant for the sequence.

Definition 5. *A subset S of a metric space X is said to be incompressible if for any nonempty open subset Ω of X and any sequence f_i of uniformly bi-Lipschitz maps from Ω onto (possibly different) open subsets of X, the subset*

$$\bigcap_{i=1}^{\infty} f_i^{-1}(S)$$

has the same Hausdorff dimension as X.

The following theorem was proved by S.G. Dani (cf. [Dan86] and [DV89])

Theorem: *Let M be a complete noncompact Riemannian manifold such that all the sectional curvatures are bounded between two negative constants and the Riemannian volume is finite. Let $p \in M$ and S_p the space of unit tangent vectors at p, let C be the subset of S_p consisting of all elements u such that the geodesic rays starting at p in the direction u is a bounded subset of M. Then C is an incompressible subset of S_p.*

In other words, let M be a manifold as in the theorem and consider the geodesic flow corresponding to M, defined on the unit tangent bundle, *i.e.*,

$$SM = \{(p, u) : p \in M, u \in S_p\}$$

equipped with the usual Riemannian metric. Then, the above theorem implies the following result on the dynamics of the flow.

Corollary: *Let the notation be as above and let C be the subset of SM consisting of all elements (p, u) whose orbit under the geodesic flow is a bounded subset of SM, then C is a subset of SM, which has Hausdorff dimension equal to the dimension of SM, with respect to the distance induced by the Riemannian metric.*

In particular, if M is a surface, then $HD(C) = dim(SM) = 2(2) - 1 = 3$.

Consider now a family of bounded open subsets indexed by $\mathbb{R}$, with the following properties:

1. If $\alpha < \beta$, then $\Omega_\alpha \subset \Omega_\beta$.
2. $\Omega_\alpha \nearrow SM$, this is, $\bigcup_{\alpha \in \mathbb{R}} \Omega_\alpha = SM$.

For example, $\Omega_\alpha = B_\alpha(p)$, the ball of radius α and center p.
Let

$$\begin{aligned} \phi^t : \ \mathbb{R} \times SM &\longrightarrow SM \\ (t, x) &\longmapsto \phi^t(x) \end{aligned}$$

be the geodesic flow.
Put $\widetilde{\Omega}_\alpha = \bigcap_{t \in \mathbb{R}} \phi^t(\Omega_\alpha)$, then we have the following statement:

$$C \subset \bigcup_{\alpha \in \mathbb{R}} \widetilde{\Omega}_\alpha,$$

where C is given in the previous Corollary.

In fact, let $x \in C$, then there exists a compact set K_x such that the orbit of x, $O(x) \subset K_x \subset \Omega_{\alpha_x}$ for some $\alpha_x \in \mathbb{R}$, this implies that $\phi^t(x) \in \Omega_{\alpha_x}$ for all $t \in \mathbb{R}$, therefore $x \in \widetilde{\Omega}_{\alpha_x}$

and the statement is proved.

Let α_n be a sequence in $\mathbb{R}$ such that $\alpha_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and $\alpha_n < \alpha_{n+1}$, then $\widetilde{\Omega}_{\alpha_n} \subset \widetilde{\Omega}_{\alpha_{n+1}}$, since $\Omega_{\alpha_n} \subset \Omega_{\alpha_{n+1}}$. Hence

$$C \subset \bigcup_{n=1}^{\infty} \widetilde{\overline{\Omega}}_{\alpha_n},$$

where $\overline{\Omega}$ is the closure of Ω. Since $HD(C) = 3$, then $\sup_n HD(\widetilde{\overline{\Omega}}_{\alpha_n}) = 3$, therefore there exists n such that $HD(\widetilde{\overline{\Omega}}_{\alpha_n})$ is very close to 3.

Definition 6. *[cf. [CL77]]*
Let M be a C^∞ manifold of dimension m. A C^r foliation of dimension n of M is a C^r atlas $\mathcal{F}$ on M which is maximal with the following properties:

- *If $(U, \varphi) \in \mathcal{F}$ then $\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^{n-m}$ where U_1 and U_2 are open disks in $\mathbb{R}^n$ and $\mathbb{R}^{n-m}$ respectively.*
- *If (U, φ) and $(V, \psi) \in \mathcal{F}$ are such that $U \cap V \neq \emptyset$, then the change of coordinates map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is the form, $\psi \circ \varphi^{-1}(x, y) = (h_1(x, y), h_2(y))$. We say that M is foliated by $\mathcal{F}$, or that $\mathcal{F}$ is a foliated structure of dimension n and class C^r on M.*

The charts $(U, \varphi) \in \mathcal{F}$ will be called foliation charts. Let $\mathcal{F}$ be a C^r foliation of dimension n, $0 < n < m$, of a manifold M^m. Consider a local chart (U, φ) of $\mathcal{F}$ such that $\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^{n-m}$. The set $\varphi^{-1}(U_1 \times \{c\})$, $c \in U_2$ are called *plaques* of U, or else *plaques* of $\mathcal{F}$. Fixing $c \in U_2$, the map $f = \varphi^{-1}/U_1 \times \{c\} : U_1 \times \{c\} \to U$ is a C^r embedding, so the plaques are connected n-dimensional C^r submanifolds of M. Further if α and β are plaques of U then $\alpha \cap \beta = \emptyset$ or $\alpha = \beta$.

A path of plaques of $\mathcal{F}$ is a sequence $\alpha_1, ..., \alpha_k$ of plaques of $\mathcal{F}$ such that $\alpha_j \cap \alpha_{j+1} \neq \emptyset$ for $\{1, ..., k\}$. Since M is covered by plaques of $\mathcal{F}$, we can define on M the following equivalence relation: "pRq if there exists a path of plaques $\alpha_1, ..., \alpha_k$ with $p \in \alpha_1$, $q \in \alpha_k$". The equivalence class of the relation R are called *leaves* of $\mathcal{F}$.

A classic result due to D. Anosov (cf. [Ano69], [Kli82] and [KH95]) states that for complete manifolds of curvature bounded between two negative constants, the geodesic flow ϕ on SM is Anosov, that is, there exists a continuous splitting $T(SM) = E^{ss} \oplus \phi \oplus E^{uu}$, invariant under the derivative of the flow $D\phi$ on $T(SM)$, such that ϕ is the subbundle spanned by the direction of geodesic flow, $D\phi$ exponentially expands E^{uu}, and $D\phi$ exponentially contracts E^{ss}, that is, there are constants $C, c > 0$, $\lambda > 1$ such that

$$\begin{aligned} \left|D\phi^t(x)\right| &\geq c\lambda^t |x| \quad \text{if} \quad x \in E^{uu} \quad \text{and} \quad t \geq 0, \\ \left|D\phi^t(x)\right| &\leq C\lambda^{-t} |x| \quad \text{if} \quad x \in E^{ss} \quad \text{and} \quad t \geq 0. \end{aligned}$$

The subbundles E^{ss} and E^{uu} are known to be uniquely integrable. They are tangent to the strong stable foliation (W^{ss}) and strong unstable foliation (W^{uu}).

Now notice that $\Lambda := \overline{\widetilde{\Omega}}_{\alpha_n}$ is compact and ϕ^t-invariant, and since ϕ^t is an Anosov flow on SM, Λ is hyperbolic set for geodesic flow ϕ^t.

3.1 Cross-sections and Poincaré Maps

This section is adapted from [AP10] Chapter 6.

Now let Σ be a cross-section to the flow, that is a C^1-embedded compact disk transverse a ϕ^t at every point $z \in \Sigma$: We have $T_z\Sigma \oplus \langle\phi(z)\rangle = T_zSM$ (recall that $\langle\phi(z)\rangle$ is the 1-dimensional subspace $\{s\phi(z) : s \in \mathbb{R}\}$). For every $x \in \Sigma$ we define $W^s(x, \Sigma)$ to be the connected component of $W^{cs}(x) \cap \Sigma$ that contains x. This defines a foliation $\mathcal{F}^s_\Sigma$ of Σ into codimension 1 submanifolds of class C^1 (cf. [AP10]).

Remark 6. *Given any cross-section Σ and a point x in its interior, we may always find a smaller cross-section also with x in its interior and which is the image of the square* $[0,1] \times [0,1]$, *by a C^2 diffeomorphism h that sends horizontal lines inside leaves of $\mathcal{F}^s_\Sigma$. Thus, the cross section that we consider are those that are image of the square* $[0,1] \times [0,1]$ *by a C^2 diffeomorphism h that sends horizontal lines inside leaves of $\mathcal{F}^s_\Sigma$. In this case, we denote by $int(\Sigma)$ the image of* $(0,1) \times (0,1)$ *under the above-mentioned diffeomorphism, which we call the interior of Σ*

3.1.1 Hyperbolicity of Poincaré Maps

Let $\Xi = \bigcup \Sigma_i$ be finite union of cross-sections to the flow ϕ^t and let $\mathcal{R}\colon \Xi \to \Xi$ be a Poincaré map or the map of first return to Ξ, $\mathcal{R}(y) = \phi^{t_1(y)}(y)$, where $t_1(y)$ correspond to the first time that the orbits of $y \in \Xi$ encounter Ξ.

The splitting $E^{ss} \oplus \phi \oplus E^{uu}$ over U_0 neighborhood of Λ defines a continuous splitting $E^s_\Sigma \oplus E^u_\Sigma$ of the tangent bundle $T\Sigma$ with $\Sigma \in \{\Sigma_i\}_i$, defined by

$$E^s_\Sigma(y) = E^{cs}_y \cap T_y\Sigma \text{ and } E^u_\Sigma(y) = E^{cu}_y \cap T_y\Sigma \tag{3.1}$$

where $E^{cs}_y = E^{ss}_y \oplus \langle\phi(y)\rangle$ and $E^{cu}_y = E^{uu}_y \oplus \langle\phi(y)\rangle$.

We now show that for a sufficiently large iterated of $\mathcal{R}$, $\mathcal{R}^n$, then (3.1) define a hyperbolic splitting for transformation $\mathcal{R}^n$ on the cross-sections, at last restricted to Λ.

Remark 7.

1. *In what follows we use $K \geq 1$ as a generic notation for large constants depending only on a lower bound for the angles between the cross-sections and the flow direction, and on upper and lower bounds for the norm of the vector field on the cross-sections.*

2. *Let us consider unit vectors, $e_x^{ss} \in E_x^{ss}$ and $\hat{e}_x^s \in E_\Sigma^s(x)$, and write*

$$e_x^{ss} = a_x \hat{e}_x^s + b_x \frac{\phi(x)}{\|\phi(x)\|}. \tag{3.2}$$

Since the angle between E_x^{ss} and $\phi(x)$, $\angle(E_x^{ss}, \phi(x))$ is greater than or equal to the angle between E_x^{ss} and E_x^{cu}, $\angle(E_x^{ss}, E_x^{cu})$, because $\phi(x) \in E_x^{cu}$ and the latter is uniformly bounded from zero, we have $|a_x| \geq \kappa$ for some $\kappa > 0$ which depends only on the flow. It is clear from (3.2) and the fact that the above angle is uniformly bounded from zero.

Let $0 < \lambda < 1$ be, then there is $t_1 > 0$ such that $\lambda^{t_1} < \dfrac{\kappa}{K}\lambda$ and $\lambda^{t_1} < \dfrac{\lambda}{K^3}$, take n, such that $t_n(x) := \sum_{i=1}^n t_i(x) > t_1$ for all $x \in \Xi$, where $t_i(x)$ is such that $\mathcal{R}^i(x) = \phi^{t_i(x)}(\mathcal{R}^{i-1}(x))$.

So, we have the following proposition:

Proposition 1. *Let $\mathcal{R}\colon \Xi \to \Xi$ be a Poincaré map and n as before. Then $D\mathcal{R}_x^n(E_\Sigma^s(x)) = E_{\Sigma'}^s(\mathcal{R}^n(x))$ at every $x \in \Sigma \in \{\Sigma_i\}_i$ and $D\mathcal{R}_x^n(E_\Sigma^u(x)) = E_{\Sigma'}^u(\mathcal{R}^n(x))$ at every $x \in \Lambda \cap \Sigma$ where $\mathcal{R}^n(x) \in \Sigma' \in \{\Sigma_i\}_i$.*
Moreover, we have that

$$\left\|D\mathcal{R}^n|_{E_\Sigma^s(x)}\right\| < \lambda \text{ and } \left\|D\mathcal{R}^n|_{E_\Sigma^u(x)}\right\| > \tfrac{1}{\lambda}$$

at every $x \in \Sigma \in \{\Sigma_i\}_i$.

Proof. The differential of the map $\mathcal{R}^n$ at any point $x \in \Sigma$ is given by

$$D\mathcal{R}^n(x) = P_{\mathcal{R}^n(x)} \circ D\phi^{t_n(x)}|_{T_x\Sigma},$$

where $P_{\mathcal{R}^n(x)}$ is the projection onto $T_{\mathcal{R}^n(x)}\Sigma'$ along the direction of $\phi(\mathcal{R}^n(x))$.

Note that E_Σ^s is tangent to $\Sigma \cap W^{cs} \supset W^s(x, \Sigma)$. Since the center stable manifold $W^{cs}(x)$ is invariant, we have invariance of the stable bundle:

$$D\mathcal{R}^n(x)(E_\Sigma^s(x)) = E_{\Sigma'}^s(\mathcal{R}^n(x)).$$

Moreover, for all $x \in \Sigma$ we have

$$D\phi^{t_n(x)}(E_\Sigma^u(x)) \subset D\phi^{t_n(x)}(E_x^{cu}) = E_{\mathcal{R}^n(x)}^{cu},$$

since $P_{\mathcal{R}^n(x)}$ is the projection along the vector field, it sends $E_{\mathcal{R}^n(x)}^{cu}$ to $E_{\Sigma'}^u(\mathcal{R}^n(x))$.
This proves that the unstable bundle is invariant restricted to Λ, that is, $D\mathcal{R}^n(x)(E_\Sigma^u(x)) = E_{\Sigma'}^u(\mathcal{R}^n(x))$, because has the same dimension 1.

Next we prove the expansion and contraction statements. We start by noting that $\left\|P_{\mathcal{R}^n(x)}\right\| \leq K$, with $K \geq 1$, then we consider the basis $\left\{\frac{\phi(x)}{\|\phi(x)\|}, e_x^u\right\}$ of E_x^{cu}, where e_x^u is a unit vector in the direction of $E_\Sigma^u(x)$ and $\phi(x)$ is the direction of flow. Since the flow direction is invariant, the matrix of $D\phi^t|E_x^u$ relative to this basis is upper triangular:

$$D\phi^{t_n(x)}|_{E^{cu}_x} = \begin{bmatrix} \frac{\|\phi(\mathcal{R}^n(x))\|}{\|\phi(x)\|} & * \\ & \\ 0 & a \end{bmatrix}$$

this is due to fact that $D\phi^{t_n(x)}(\phi(x)) = \phi(\phi^{t_n(x)}(x)) = \phi(\mathcal{R}^n(x))$.
Then,

$$\begin{aligned} \|D\mathcal{R}^n(x)e^u_x\| &= \left\|P_{\mathcal{R}^n(x)}(D\phi^{t_n(x)}(x))e^u_x\right\| = \left\|ae^u_{\mathcal{R}^n(x)}\right\| = |a| \\ &\geq \frac{1}{K}\frac{\|\phi(x)\|}{\|\phi(\mathcal{R}^n(x))\|}\left|det(D\phi^{t_n(x)}|_{E^{cu}_x}\right| \geq \frac{1}{K^3}\lambda^{-t_n(x)} \geq K^{-3}\lambda^{-t_1} > \frac{1}{\lambda}. \end{aligned}$$

To prove that $\left\|D\mathcal{R}^n|_{E^s_\Sigma(x)}\right\| < \lambda$, let us consider unit vectors, $e^{ss}_x \in E^{ss}_x$ and $\hat{e}^s_x \in E^s_\Sigma(x)$, and write as in (3.2)

$$e^{ss}_x = a_x\hat{e}^s_x + b_x\frac{\phi(x)}{\|\phi(x)\|}.$$

We have $|a_x| \geq \kappa$ for some $\kappa > 0$ which depends only on the flow.

Then, since $P_{\mathcal{R}^n(x)}\left(\frac{\phi(\mathcal{R}^n(x))}{\|\phi(x)\|}\right) = 0$ we have that

$$\begin{aligned} \|D\mathcal{R}^n(x)\hat{e}^s_x\| &= \left\|P_{\mathcal{R}^n(x)}(D\phi^{t_n(x)}(x))\hat{e}^s_x\right\| \\ &= \left\|P_{\mathcal{R}^n(x)}(D\phi^{t_n(x)}(x))\left(\frac{1}{a_x}\left(e^{ss}_x - b_x\frac{\phi(x)}{\|\phi(x)\|}\right)\right)\right\| \\ &= \frac{1}{|a_x|}\left\|P_{\mathcal{R}^n(x)}(D\phi^{t_n(x)}(x))\left(e^{ss}_x - b_x\frac{\phi(x)}{\|\phi(x)\|}\right)\right\| \\ &= \frac{1}{|a_x|}\left\|P_{\mathcal{R}^n(x)}(D\phi^{t_n(x)}(x))(e^{ss}_x) - b_xP_{\mathcal{R}^n(x)}\left(\frac{\phi(R^n(x))}{\|\phi(x)\|}\right)\right\| \\ &\leq \frac{K}{\kappa}\left\|D\phi^{t_n(x)}(x)(e^{ss}_x)\right\| \leq \frac{K}{\kappa}\lambda^{t_n(x)} \leq \frac{K}{\kappa}\lambda^{t_1} < \lambda\ . \end{aligned} \tag{3.3}$$

□

3.2 Good Cross-Sections

For each $x \in \Lambda = \widetilde{\widetilde{\Omega}}_{\alpha_n}$, we can take cross-section in x, and using a tubular neighborhood construction in each cross-section Σ, we linearize the flow in an open set $U_\Sigma = \phi^{(-\epsilon,\epsilon)}(int\Sigma)$ for a small $\epsilon > 0$, containing x the interior of the cross section.
This provides an open covering of the compact set Λ by tubular neighborhoods.
We let $\{U_{\Sigma_i} : i = 1, 2, ...l\}$ be a finite covering of Λ, this is

$$\Lambda \subset \bigcup_{i=1}^{l} U_{\Sigma_i} = \bigcup_{i=1}^{l}\phi^{(-\epsilon,\epsilon)}(int\Sigma_i). \tag{3.4}$$

Using a result on the differentiability of the strong stable foliation, we can choose these cross-sections Σ_i in such a way that they do not intersect.

Now introduce the tools to prove the above claims.

The following result is due to Morris W. Hirsch & Charles C. Pugh (cf. [HP75]).

Theorem(**Smoothness Theorem**)
Let M be a complete surface with Gaussian curvature bounded between two negative constants, then the Anosov splitting $T(SM) = E^{ss} \oplus \phi \oplus E^{uu}$ for the geodesic flow is of class C^1. In particular, the strong stable foliations and strong unstable foliations are of class C^1.

Let $\mathcal{F}^{ss}$ the strong stable foliations and $\mathcal{F}^{uu}$ the strong unstable foliations, this is $\mathcal{F}^i(x) = W^i(x)$ for $i = ss, uu$, are foliations of dimension one. Then we have the following Lemma.

Lemma 8. *Let $x \in SM$ and L be a C^1-embedded curve of dimension one, containing x and transverse to the foliation $\mathcal{F}^{ss}$, then the set*

$$S_L := \bigcup_{z \in L} \mathcal{F}^{ss}(z)$$

contains a surface S_x that is C^1-embedded, which contains x in the interior and if L is transverse to the foliation W^{cs} then, S_x is transverse to the geodesic flow.

Proof. Let (U, φ) be a chart of the foliation $\mathcal{F}^{ss}$, with $x \in U$, since the dimension of foliation $\mathcal{F}^{ss}$ is equal to 1 and $dim(SM) = 3$, there are disks $U_1 \subset \mathbb{R}$ and $U_2 \subset \mathbb{R}^2$ such that $\varphi : U \to U_1 \times U_2$, put $\Pi_2 : U_1 \times U_2 \to U_2$ the projection on the second coordinate. Let $f = \Pi_2 \circ \varphi$ function of class C^1, clearly f is a submersion.

Now $f(L) \subset U_2$, is a C^1-submanifold of dimension one, in fact, suppose that $D\varphi_x(v) = (w, (0,0)) \in \mathbb{R} \times \mathbb{R}^2$, then $v \in T_x\mathcal{F}^{ss}(x)$, indeed, let $\alpha(t)$ be any curve in U, such that $\alpha(0) = x$ and $\alpha'(0) = v$, put $\varphi(x) = (x_1, x_2)$ and $\varphi(\alpha(t)) = (\alpha_1(t), \alpha_2(t))$, therefore $\alpha_i(0) = x_i$ for $i = 1, 2$ and $\alpha_1'(0) = w$, $\alpha_2'(0) = (0,0)$, then $D\varphi^{-1}_{\varphi(x)}(w, (0,0)) = \dfrac{d}{dt}\varphi^{-1}(\tilde{\alpha}_1(t), \tilde{\alpha}_2(t)) \mid_{t=0}$, where $\tilde{\alpha}_1(t) = x_1 + tw$ and $\tilde{\alpha}_2(t) = x_2$, by the properties of the chart (U, φ) $\varphi^{-1}(\tilde{\alpha}_1(t), \tilde{\alpha}_2(t)) = \varphi^{-1}(\tilde{\alpha}_1(t), x_2) \subset \mathcal{F}^{ss}(x)$ and since $\varphi^{-1}(\tilde{\alpha}_1(0), \tilde{\alpha}_2(0)) = \varphi^{-1}(x_1, x_2) = x$, then

$$v = D\varphi^{-1}_{\varphi(x)}(w, (0,0)) = \frac{d}{dt}\varphi^{-1}(\tilde{\alpha}_1(t), \tilde{\alpha}_2(t))|_{t=0} \in T_x\mathcal{F}^{ss}(x),$$

as wanted.

Let $\beta : (-\epsilon, \epsilon) \to SM$ a C^1-embedding on L in some $y \in L \cap U$, with $\beta(0) = y$, then as L is transverse to the foliation $\mathcal{F}^{ss}$ and demonstrated above, we have

$$(f \circ \beta)'(t) = D(\Pi_2)_{\varphi(\beta(t))}(D\varphi_{\beta(t)})(\beta'(t)) \neq 0$$

for all t, as L is a C^1-embedded, then the above implies that $f(L)$ is a C^1-submanifold of U_2.

Now since f is a submersion and $f(L)$ is a submanifold, then $f^{-1}(f(L))$ is a C^1-submanifold of SM, with the following property, if $z \in f(L)$, then $f^{-1}(z) = \varphi^{-1}(\Pi_2^{-1}(z)) = \varphi^{-1}(U_1 \times \{z\}) = \mathcal{F}^{ss}(y) \cap U$ where $z = f(y)$ and $y \in L$, and follows the Lemma.

□

In particular, taking $L = W_\epsilon^{uu}(x)$ with ϵ given by the stable and unstable manifolds theorem, we call $S_x := \Sigma_x$.
Note that an analogous Lemma holds for the foliation $\mathcal{F}^{uu}$.
Without loss of generality we can assume that Σ_x is diffeomorphic to the square $[0,1] \times [0,1]$, put $\Sigma_x = \Sigma$, with the horizontal lines $[0,1] \times \eta$ being mapped to stable sets $W^s(y, \Sigma_x) = W^{ss}(y) \cap \Sigma_x$. The stable-boundary $\partial^s\Sigma$ is the image of $[0,1] \times \{0,1\}$, the unstable-boundary $\partial^u\Sigma$ is the image of $\{0,1\} \times [0,1]$.

Therefore we have the following definition.

Definition 7. *A cross sections is said* ***δ-Good Cross-Section*** *for some $\delta > 0$, if satisfies the following:*

$$d(\Lambda \cap \Sigma, \partial^u\Sigma) > \delta \text{ and } d(\Lambda \cap \Sigma, \partial^s\Sigma) > \delta$$

where d is the intrinsic distance in Σ, (see Figure 3.1).
A cross-section which is δ-Good Cross-Section for some $\delta > 0$ is said a ***Good Cross-Section-GCS****.*

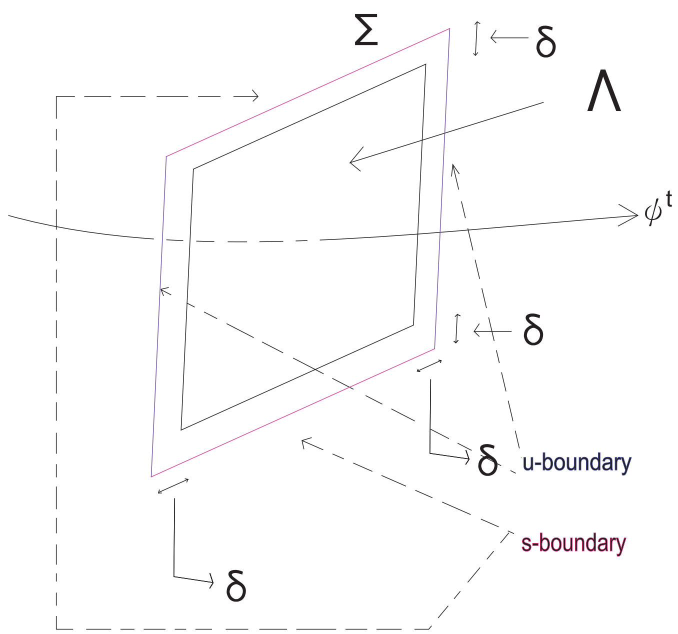


Figure 3.1: Good Cross-section

Lemma 9. *Let Σ be a δ-Good Cross-Section, then given $0 < \delta' < \delta$ there is a δ'-Good Cross-Section $\Sigma' \subset int(\Sigma)$ and such that $\partial\Sigma' \cap \partial\Sigma = \emptyset$.*

Proof. Call γ_i, $i = 1, 2, 3, 4$ the C^1-curves which form the boundary of Σ. Let γ_i'' be a C^1-curve contained in Σ and satisfies $d(\gamma_i, x) = \delta$ for any $x \in \gamma_i''$ (see Figure 3.2).
Therefore $\Lambda \cap \Sigma$ is contained in the region bounded by the curves γ_i'' in Σ. Now consider the C^1-curves $\gamma_i' \subset \Sigma$ with the property $d(\gamma_i, x) = \delta'$ for any $x \in \gamma_i'$, then the region bounded by the curves γ_i' is a δ'-Good Cross-Section, $\Sigma' \subset \Sigma$ (see Figure 3.2).

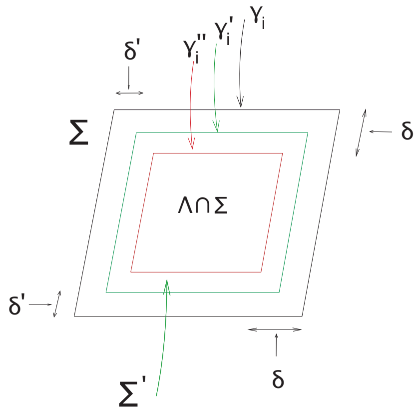


Figure 3.2: Reduction of GCS

□

Now we prove that for any $x \in \Lambda$ there exists a Good Cross-Section which contains x for some $\delta > 0$.

We use the following result (cf. [Ano69], [Pat99] and [[Kli82] chapter 3]).

We know that, since there are $a, b > 0$ such that $-a^2 \leq K_M \leq -b^2 < 0$ where K_M os the sectional curvature of M and volume of M is finite, then the non wandering set $\Omega(\phi^t) = SM$. Since the geodesic flow is Anosov, the spectral decomposition theorem implies the geodesic flow is transitive, then for any $(x, v) \in SM$, $\overline{W^{cs}(x, v)} = SM$ and $\overline{W^{cu}(x, v)} = SM$. Thus, we have the following Lemma.

Lemma 10. *For any $x \in \Lambda$ there exist points $x^+ \notin \Lambda$ and $x^- \notin \Lambda$ in distinct connected components of $W^{ss}(x) - \{x\}$.*

Proof. Let $x \in \Lambda$, suppose otherwise there would exists a whole segment of the strong stable manifold entirely contained in Λ and containing x in the interior, called the segment of γ, without loss of generality, we can assume that $W^{ss}_{loc}(x) \subset \gamma$. Now take t_k a sequence such that $t_k \to \infty$ as $k \to \infty$, then as Λ is a compact set, we can assume that $\phi^{-t_k}(x) \to y \in \Lambda$ as $k \to \infty$.
Let us prove now that $W^{ss}(y) \subset \Lambda$, in fact:
Let $z \in W^{ss}(y)$, as $W^{ss}(y) = \bigcup_{t \geq 0} \phi^{-t}(W^{ss}_{loc}(\phi^t(y))$, then there is $T \geq 0$, such that $\phi^T(z) \in W^{ss}_{loc}(\phi^T(y))$, by Stable Manifold Theorem $W^{ss}_{loc}(\phi^T(y))$ is accumulated by points $W^{ss}_{loc}(\phi^{(-t_k+T)}(x))$ for large k. Let k be sufficiently large such that $(-t_k + T) < 0$ and $W^{ss}_{loc}(\phi^{(-t_k+T)}(x)) \subset \phi^{(-t_k+T)}(\gamma) \subset \Lambda$, as well Λ is a invariant set and $\gamma \subset \Lambda$, hence as Λ is closed, we have that $W^{ss}_{loc}(\phi^T(y)) \subset \Lambda$, this implies $z \in \Lambda$, this proves the assertion.

The above statement implies that $\Lambda \supset W^{cs}(y) = \bigcup_{t\in\mathbb{R}} W^{ss}(\phi^t(y))$, in fact:
Let $w \in W^{cs}(y)$, then there is $t_0 \in \mathbb{R}$ such that, $w \in W^{ss}(\phi^{t_0}(y))$, hence there is $T \geq 0$ such that $\phi^T(w) \in W^{ss}_\epsilon(\phi^{T+t_0}(y))$, then $\phi^{T+r}(w) \in W^{ss}_{K\epsilon e^{-\lambda r}}(\phi^{T+r+t_0}(y))$ for $r > 0$, so we can assume that $T + t_0 > 0$, therefore

$$\phi^{-t_0}(w) = \phi^{-(T+t_0)}(\phi^T(w)) \in \phi^{-(T+t_0)}\left(W^{ss}_\epsilon(\phi^{T+t_0}(y))\right) \subset W^{ss}(y) \subset \Lambda,$$

since Λ is invariant, then $w \in \Lambda$, this implies that $W^{cs}(y) \subset \Lambda$ and by the previous observations $SM = \overline{W^{cs}(y)} \subset \Lambda$ and this is a contradiction.
This conclude the proof of Lemma. □

Similarly, we have,

Lemma 11. *For any $y \in \Lambda$ there exist points $y^+ \notin \Lambda$ and $y^- \notin \Lambda$ in distinct connected components of $W^{uu}(x) - \{x\}$.*

Proof. Similarly to Lemma 10. □

Lemma 12. *Let $x \in \Lambda$, then there is $\delta > 0$ and a δ-Good Cross-Section Σ at x.*

Proof. Fix $\epsilon > 0$ as in the Stable Manifold Theorem, and consider the cross section Σ_x given by the Lemma 8 containing a segment of $W^{ss}_\epsilon(x)$ and $W^{uu}_\epsilon(x)$ with x in the interior, by the Lemma 10 and Lemma 11, we may find points $x^\pm \notin \Lambda$ in each of the connected components of $W^{ss}_\epsilon(x) \cap \Sigma_x$ and points $z^\pm \notin \Lambda$ in each of the connected components of $W^{uu}_\epsilon(x) \cap \Sigma_x$. Since Λ is closed, there are neighborhoods $V^\pm$ of $x^\pm$ and $V_1^\pm$ of $z^\pm$ respectively disjoint from Λ (see Figure 3.3).

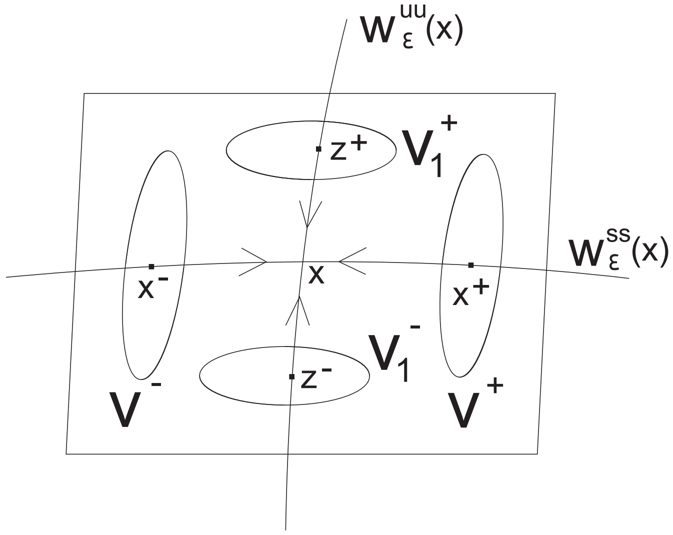


Figure 3.3: First step to construct GCS for $x \in \Lambda$

In Figure 3.3, it can happen that $V^\pm$, $V_1^\pm$ enclose a region homeomorphic to a square, in this case there is nothing to be done.

If this is not the case in the first instance, we prove that the above can be obtained. Let t_k be a sequence, such that $t_k \to +\infty$ as $k \to +\infty$ and $\phi^{-t_k}(x) \to y \in \Lambda$ as $k \to +\infty$, then by Lemma 10, there are $y^{\pm}$ in each of the connected components of $W^{ss}_{\epsilon}(y)$ such that $y^{\pm} \notin \Lambda$ and there are neighborhoods $J^{\pm}$ of $y^{\pm}$ respectively with $J^{\pm} \cap \Lambda = \emptyset$.
Now for $z \in W^{uu}_{\epsilon}(x)$, we have

$$d(\phi^{-t_k}(z), y) \leq d(\phi^{-t_k}(z), \phi^{-t_k}(x)) + d(\phi^{-t_k}(x), y)$$

converges to zero as $k \to \infty$, using the continuity of $W^{ss}_{\epsilon}(x)$ with $x \in SM$, given by the Stable Manifold Theorem, we have for sufficiently large k, say $k \geq k_0$, $W^{ss}_{\epsilon}(\phi^{-t_k}(z))$ is close to $W^{ss}_{\epsilon}(y)$, for all $z \in W^{uu}_{\epsilon}(x)$, this implies that $J^{\pm} \cap W^{ss}_{\epsilon}(\phi^{-t_k}(z)) \neq \emptyset$. Hence, there are $z^{\pm}_k \in (J^{\pm} \cap W^{ss}_{\epsilon}(\phi^{-t_k}(z)))$, for all $z \in W^{uu}_{\epsilon}(x)$, (see Figure 3.4).

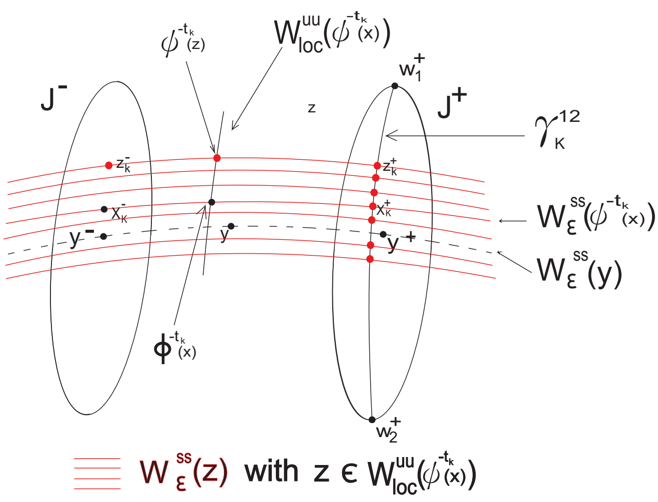


Figure 3.4: Second step to construct GCS for $x \in \Lambda$

We want to see now that for sufficiently large k, $\phi^{t_k}(J^{\pm})$ and $V^{\pm}_1$ has the property of enclosing a region homeomorphic to a square. In fact: Consider the points w^+_i in J^+ $i = 1, 2$ as in Figure 3.4, with $d(w^+_1, w^+_2) > 0$, let $\gamma^{12}_k \subset J^+$ a segment joining w^+_1 with w^+_2 that contains x^+_k and transverse to $W^{ss}_{\epsilon}(\phi^{-t_k}(x))$, it suffices to prove that $\phi^{t_k}(\gamma^{12}_k)$ has length greater than or equal to ϵ for sufficiently large k, now we can assume that $w^+_i \in W^{uu}_{\epsilon}(\phi^{-t_k}(x))$ for $i = 1, 2$, then

$$d(\phi^{t_k}(z^+_1), \phi^{t_k}(z^+_2)) \geq K e^{\frac{1}{\lambda} t_k} d(w^+_1, w^+_2)$$

and for $k \geq k_0$ the expression on the right in the above inequality is greater than equal to ϵ as desired.

Note also that as $z^+_k \in W^{ss}_{\epsilon}(\phi^{-t_k}(z))$, for all $z \in W^{ss}_{\epsilon}(x)$ then

$$\begin{aligned} d(\phi^{t_k}(z^+_k), z) = d(\phi^{t_k}(z^+_k), \phi^{t_k}(\phi^{-t_k})(z)) &\leq K e^{-\lambda t_k} d(z^+_k, \phi^{-t_k}(z)) \\ &\leq K e^{-\lambda t_k} \epsilon. \end{aligned}$$

for all $z \in W^{ss}_\epsilon(x)$.

So for sufficiently large k, say $k \geq k_0$ the expression on the right in the above inequality is very small, so that $\phi^{t_k}(J^+)$ cross $V_1^\pm$ and is close to $W^{uu}_\epsilon(x)$. Analogously, one can obtain k_0 such that $\phi^{t_k}(J^-)$ cross $V_1^\pm$ and is close to $W^{uu}_\epsilon(x)$ for $k \geq k_0$.

On the other hand, we know that there is $z_k^\pm \in J^\pm \cap W^{ss}_\epsilon((\phi^{-t_k}(z)) \neq \emptyset$ for any $z \in W^{uu}_\epsilon(x)$ respectively. Hence, for sufficiently large k_0, $\phi^{t_{k_0}}(J^+)$ and $\phi^{t_{k_0}}(J^-)$ crossing $V^\pm$. Moreover, $\phi^{t_{k_0}}(z_{k_0}^\pm) \in W^{ss}_{Ke^{-\lambda t_{k_0}}}(z) \cap \phi^{t_{k_0}}(J^\pm) \subset W^{ss}_\epsilon(z) \cap \phi^{t_{k_0}}(J^\pm) \subset \Sigma_x \cap \phi^{t_{k_0}}(J^\pm)$ for any $z \in W^{uu}_\epsilon(x)$ with $\phi^{t_{k_0}}(J^\pm) \cap \Lambda = \emptyset$, then the open sets $V_1^\pm$ and $\phi^{t_{k_0}}(J^\pm)$ have the desired property.

Let $\beta^\pm$ be a segment of $W^{ss}(z^\pm)$ contained in $V_1^\pm$ respectively, take k_0 large enough such that the endpoints of $\beta^\pm$, $\beta_i^\pm$ for $i = 1, 2$ is contained in $\phi^{t_{k_0}}(J^\pm)$, (see Figure 3.4). Let $\eta^\pm$ be a C^1-curve transverse to the foliation $\mathcal{F}^{ss}$ contained in $\phi^{t_{k_0}}(J^\pm) \cap \Sigma_x$ and joining $\beta_1^\pm$ with $\beta_2^\pm$, respectevely. Finally, good cross-section it is the section determined by the curves $\beta^\pm$ and $\eta^\pm$, (see Figure 3.5).

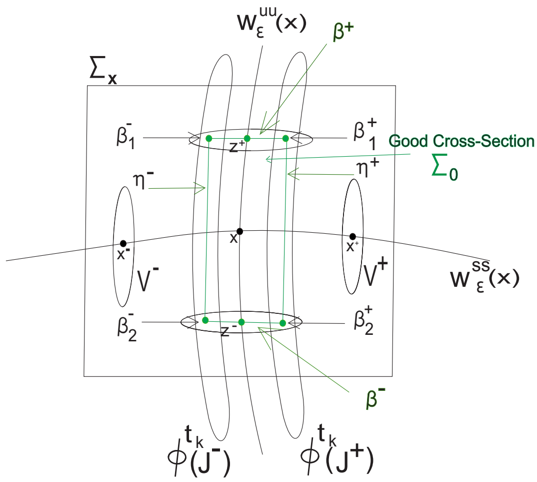


Figure 3.5: The construction of GCS for $x \in \Lambda$ using positive iterated

And this concludes the proof of the Lemma. □

Remark 8. *Note that if $k \geq k_0$, is as in the proof of the Lemma 12, this is, we have the Figure 3.5, then for $k' \geq k \geq k_0$, we have the same Figure 3.5, but the open $\phi^{t_{k'}}(J^{\pm})$, has diameter much greater than ϵ, (see Figure 3.6).*

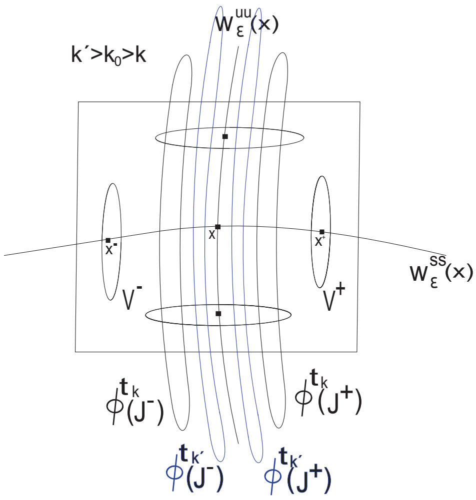


Figure 3.6: Small GCS

Remark 9. *In the proof of Lemma 12, we could consider an accumulation point $\phi^t(x)$ for $t > 0$, and get the same result, but in this case crossed $V^{\pm}$, consequently satisfies Remark 8 in this case, (see Figure 3.7).*

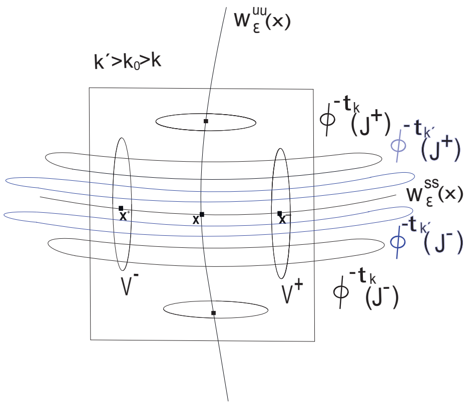


Figure 3.7: The construction of GCS for $x \in \Lambda$ using negative iterated

Given $x \in \Lambda = \tilde{\tilde{\Omega}}_{\alpha_n}$, from now on, we call Σ_x the Good Cross-Section given by the previous Lemma associated to x.

Corollary 8. *Given $x, y \in \Lambda$, if the interior of the Good Cross-Sections Σ_x and Σ_y given in the Lemma 12 intersect transversal to foliation $\mathcal{F}^{ss}$, that is, $(int(\Sigma_x) \cap int(\Sigma_y)) \pitchfork \mathcal{F}^{ss}$, then $int(\Sigma_x) \cap int(\Sigma_y)$ is an open set of Σ_x and Σ_y.*

Proof. Let $\gamma \subset int(\Sigma_x) \cap int(\Sigma_y)$ a C^1-curve transverse to $\mathcal{F}^{ss}$, then for all $z \in \gamma$, there are $x' \in W^{uu}_\epsilon(x)$ and $y' \in W^{uu}_\epsilon(y)$ such that $z \in W^{ss}(x') \cap \Sigma_x$ and $z \in W^{ss}(y') \cap \Sigma_y$, then there is $\delta > 0$ such that the set

$$B = \bigcup_{z\in\gamma} W^{ss}_\delta(z) \subset int(\Sigma_x) \cap int(\Sigma_y)$$

Thus we have the Corollary. □

Remark 10. *Suppose that Σ_1, Σ_2 are GCS and $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, but $int\Sigma_1 \cap int\Sigma_2 = \emptyset$, then as both are GCS, there are two GCS $\widetilde{\Sigma}_i \subset \Sigma_i$ for $i = 1, 2$ such that $\widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2 = \emptyset$ with*

$$\Lambda \cap \bigcup_{i=1}^{2} \phi^{(-\epsilon,\epsilon)}(int\Sigma_i) = \Lambda \cap \bigcup_{i=1}^{2} \phi^{(-\epsilon,\epsilon)}(int\widetilde{\Sigma}_i). \tag{3.5}$$

In fact:
By Lemma 9, there are two GCS, $\tilde{\Sigma}_i \subset int(\Sigma_i)$ such that $\partial\tilde{\Sigma}_i \cap \partial\Sigma_i = \emptyset$ for $i = 1, 2$, as $int\Sigma_1 \cap int\Sigma_2 = \emptyset$, then $\tilde{\Sigma}_1 \cap \tilde{\Sigma}_2 = \emptyset$, also the Lemma 9 implies that $\Lambda \cap int(\Sigma_i) \subset int(\tilde{\Sigma}_i)$, thus we have $\Lambda \cap int(\Sigma_i) = \Lambda \cap int(\tilde{\Sigma}_i)$ and as Λ is ϕ^t invariant, then

$$\Lambda \cap \phi^{(-\epsilon,\epsilon)}(int(\Sigma_i)) = \phi^{(-\epsilon,\epsilon)}(\Lambda \cap int(\Sigma_i)) = \phi^{(-\epsilon,\epsilon)}(\Lambda \cap int(\tilde{\Sigma}_i)) = \Lambda \cap \phi^{(-\epsilon,\epsilon)}(int(\tilde{\Sigma}_i)).$$

Therefore we have (3.5).

So, we can assume that if two GCS has nonempty intersections, then their interiors have nonempty intersection.

3.3 Separation of GCS

At each point of $x \in \Lambda$, we can find a Good Cross-Section Σ_x as in Lemma 12. Since Λ is a compact set then as in (3.4), there are a finite number of points $x_i \in \Lambda$, putting $\Sigma_{x_i} = \Sigma_i$ for $i = 1, ..., l$, we have

$$\Lambda \subset \bigcup_{i=1}^{l} \phi^{(-\frac{\epsilon}{2},\frac{\epsilon}{2})}(int\Sigma_i) \subset \bigcup_{i=1}^{l} \phi^{(-\epsilon,\epsilon)}(int\Sigma_i) = \bigcup_{i=1}^{l} U_{\Sigma_i}. \tag{3.6}$$

Lemma 13. *If $\Sigma_i \cap \Sigma_j \neq \emptyset$ for some $i, j \in \{1, ...l\}$, Σ_i and Σ_j as in the Corollary 8, then there is $\delta' > 0$ such that $\phi^\delta(\Sigma_i) \cap \Sigma_j = \emptyset$ for all $0 < \delta \leq \delta'$.*

Proof. Suppose otherwise, then for all n sufficiently large, there is $z^n_i \in \Sigma_i$ such that $\phi^{\frac{1}{n}}(z^n_i) \in \Sigma_j$. Since Σ_i is a compact set, we can assume that z^n_i converge to z_i as n tends to infinity, then $\phi^{\frac{1}{n}}(z^n_i)$ converge to z_i as n tends to infinity, this implies that $z_i \in \Sigma_i \cap \Sigma_j$. Suppose that $z_i \in int\Sigma_j$, as the vector field which generates the geodesic flow has no singularities, then by the Tubular Flow Theorem, there are $r > 0$ and $\eta > 0$ such that $B_r(z_i)$ the open ball of radius r and center z_i satisfies

$$\phi^t(B_r(z_i) \cap \Sigma_j) \cap \Sigma_j = \emptyset$$

for all $0 < t \leq \eta$.
Then by the Corollary 8, we have $(B_r(z_i) \cap \Sigma_i) \setminus \{z_i\} \subset \Sigma_j$, take n large enough such that $z_i^n \in B_r(z_i) \cap \Sigma_i$ and $\frac{1}{n} < \eta$, so $\phi^{\frac{1}{n}}(z_i^n) \notin \Sigma_j$ which is a contradiction.

Suppose now that $z_i \in \partial\Sigma_j$, then we can find a new GCS $\Sigma_j' \supset \Sigma_j$ as in the Lemma 12 and such that $z_i \in int\Sigma_j'$, so Σ_i and Σ_j' behave as in the previous case and again to obtain a contradiction. Thus we conclude the Lemma. □

The following Lemma proves that the GCS in (3.6) can be taken disjoint if all possible intersections of Σ_i with Σ_j are in the hypothesis of the Corollary 8.

Lemma 14. *Assuming (3.6) there are GCS $\widetilde{\Sigma}_i$ such that $\Lambda \subset \bigcup_{i=1}^{l} \phi^{(-\epsilon,\epsilon)}(\widetilde{\Sigma}_i)$ with the property $\widetilde{\Sigma}_i \cap \widetilde{\Sigma}_j = \emptyset$ for all $i, j \in \{1, ..., l\}$.*

Proof. We will do the proof by induction on l. If $l = 1$, is clearly true. For $l = 2$, can happen two cases:

1. $\Sigma_1 \cap \Sigma_2 = \emptyset$, in this case take $\widetilde{\Sigma}_k = \Sigma_k$ for $k = 1, 2$.
2. $\Sigma_1 \cap \Sigma_2 \neq \emptyset$ and $int\Sigma_1 \cap int\Sigma_2 = \emptyset$, then by Remark 10 and (3.5) have the desired.
3. $int(\Sigma_1) \cap int(\Sigma_2) \neq \emptyset$, then by the Lemma 13, for $0 < \delta < \delta'$ and $\delta < \frac{\epsilon}{2}$, by (3.6) putting $\widetilde{\Sigma}_1 = \phi^\delta(\Sigma_1)$, clearly $\phi^\delta(\Sigma_1)$ is a GCS and as $\delta < \frac{\epsilon}{2}$, then $\phi^{(-\frac{\epsilon}{2},\frac{\epsilon}{2})}(\Sigma_1) \subset \phi^{(-\epsilon,\epsilon)}(\widetilde{\Sigma}_1)$ and satisfies the Lemma.

Suppose the Lemma is true for all $k < l$ and we show to hold for $k = l$. In fact: Suppose that given any number $k < l$ of GCS as in (3.6) there are a number $k < l$ of new GCS such that

$$\bigcup_{s=1}^{k} \phi^{(-\frac{\epsilon}{2},\frac{\epsilon}{2})}(\Sigma_{i_s}) \subset \bigcup_{s=1}^{k} \phi^{(-\epsilon,\epsilon)}(\widetilde{\Sigma}_{i_s}) \tag{3.7}$$

and $\widetilde{\Sigma}_{i_s} \cap \widetilde{\Sigma}_{i_r} = \emptyset$ for $s, r \in \{1, ..., k\}$.
Note also by Remark 10, we can suppose that, $\Sigma_i \cap \Sigma_j = \emptyset \Leftrightarrow d(\Sigma_i, \Sigma_j) = \delta_{ij} > 0$, where d is the distance between the two cross-section.

Statements:

1. If $\Sigma_i \cap \Sigma_j = \emptyset$ and $\Sigma_k \cap \Sigma_i \neq \emptyset$, then there is $\delta > 0$ such that $\phi^\delta(\Sigma_i) \cap \Sigma_k = \emptyset$ and $\phi^\delta(\Sigma_i) \cap \Sigma_j = \emptyset$. In fact:

   Let Σ_k be such that $\Sigma_i \cap \Sigma_k \neq \emptyset$, then take $\delta < \min\{\delta_{ij}, \frac{\epsilon}{2}\}$ in Lemma 13 such that $\phi^\delta(\Sigma_i) \cap \Sigma_k = \emptyset$. Moreover, if $z \in \phi^\delta(\Sigma_i) \cap \Sigma_j$, then $\phi^{-\delta}(z) \in \Sigma_i$ and $d(\phi^{-\delta}(z), z) = |\delta| < \delta_{ij} = d(\Sigma_i, \Sigma_j)$, which is absurd, therefore $\phi^\delta(\Sigma_i) \cap \Sigma_j = \emptyset$.

2. Given $i \in \{1, ..., l\}$, call $B_i = \{j : \Sigma_i \cap \Sigma_j \neq \emptyset\}$, then there is $\delta > 0$ such that $\phi^{\delta}(\Sigma_i) \cap \Sigma_j = \emptyset$ for all j. In fact:

   If $r \notin B_i$, then $d(\Sigma_i, \Sigma_r) = \delta_{ir} > 0$, then by Lemma 13 for each $s \in B_i$, there is $\delta_s < \min\{\min_{r \notin B_i} \delta_{ir}, \frac{\epsilon}{2}\}$ such that $\phi^{\delta_s}(\Sigma_i) \cap \Sigma_s = \emptyset$ and by the choice of δ_s, we also have to $\phi^{\delta_s}(\Sigma_i) \cap \Sigma_r = \emptyset$ for any $r \notin B_i$.
   So for $\delta = \min_{s \in B_j} \delta_s$ we have the statements.

Fix the GCS Σ_1, suppose that $\#(\{j : \Sigma_j \cap \Sigma_1 \neq \emptyset\} := C_1) \leq l - 1$, then from statements 2 above, there is δ such that $\phi^{\delta}(\Sigma_1) \cap \Sigma_j = \emptyset$ for all $j \neq 1$, then by induction hypothesis applied to $\{\Sigma_j : j \neq 1\}$, we obtain new GCS $\widetilde{\Sigma}_j$ and satisfy (3.7) and calling $\phi^{\delta}(\Sigma_1) = \widetilde{\Sigma}_1$, the set $\widetilde{\Sigma}_j : j = 1, ..., l$ satisfies the Lemma.
Note that since $d(\widetilde{\Sigma}_1, \Sigma_j) = \delta_{1j} > 0$ for all $j \neq 1$, then $\widetilde{\Sigma}_j$ may be obtained such that $d(\widetilde{\Sigma}_1, \widetilde{\Sigma}_j) > 0$.

Suppose now that $\#C_1 = l$, then for all $j \neq 1$ there is $\delta_j > 0$ given by the Lemma 13, such that $\phi^t(\Sigma_1) \cap \Sigma_j = \emptyset$ for all $0 < t \leq \delta_j$, put $0 < \delta < \min\{\min_{j \neq 1} \delta_j, \frac{\epsilon}{2}\}$, therefore $\widetilde{\Sigma}_1 := \phi^{\delta}(\Sigma_1)$, satisfies $\widetilde{\Sigma}_1 \cap \Sigma_j = \emptyset$ for all $j \neq 1$. Considering $\{\widetilde{\Sigma}_1, \Sigma_2, ..., \Sigma_l\}$, we have $\#\{j : \Sigma_j \cap \Sigma_2 \neq \emptyset\} \leq l - 1$, as done previously, we have the result of Lemma. $\square$

Let Σ, Σ' are GCS as in the Lemma 12 with $\Sigma \cap \Sigma' \neq \emptyset$, suppose that $\Sigma \cap \Sigma'$ is non-transverse to $\mathcal{F}^{ss}$, then since Σ, Σ' are transverse to flow, then we can assume that Σ and Σ' intersect transversely.

Suppose now that two GCS Σ_x, Σ' as in the Lemma 12 intersect transversely, then $\Sigma_x \pitchfork \Sigma'$ is a finite number of C^1-curve γ_i for $i = 1, ..., k$ and by Corollary 8 these curves is contained in a finite number of leaves of $\mathcal{F}^{ss} \cap \Sigma_x$, say $\mathcal{F}^{ss}(z_i) \cap \Sigma_x$ with $z_i \in \mathcal{F}^{uu}(x) \cap \Sigma_x$ for $i = 1, ..., k$.

Let $\overline{\Sigma}_i$ be surface contained in Σ_x, containing $\mathcal{F}^{ss}(z_i) \cap \Sigma_x$ and saturate by $\mathcal{F}^{ss}$, *i.e.* there is a interval I_i contained in $\mathcal{F}^{uu}(x)$ and centered in z_i such that

$$\overline{\Sigma}_i = \bigcup_{z \in I_i} \mathcal{F}^{ss}(z) \cap \Sigma_x \text{ for } i = 1, ..., k$$

with $\overline{\Sigma}_i \cap \overline{\Sigma}_j = \emptyset$ for $i \neq j$.
Note that if $\mathcal{F}^{ss}(z_i) \cap \Sigma_x \cap \Lambda = \emptyset$ for some i, then since Λ is a compact set there is an open set U_i containing $\mathcal{F}^{ss}(z_i) \cap \Sigma_x$ and $U_i \cap \Lambda = \emptyset$, therefore Σ_x is subdivided into two GCS Σ_x^1 and Σ_x^2, such that Σ_x^r and Σ' satisfies the above for $r = 1, 2$.
This implies that without loss of generality we can assume that for any $i \in \{1, ..., k\}$ there is $p_i \in \mathcal{F}^{ss}(z_i) \cap \Sigma_x \cap \Lambda$ (see Figure 3.8).

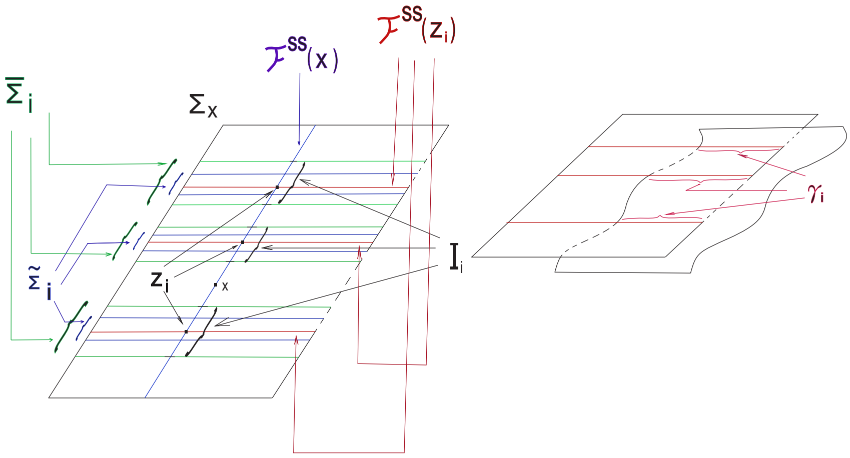


Figure 3.8: Separation of GCS

So, we have the following Lemma.

Lemma 15. *If Σ_x and Σ' are two GCS as in the Lemma 12 which intersect transversely, let $\gamma_i \subset \Sigma_x \cap \Sigma'$, $\mathcal{F}^{ss}(z_i) \cap \Sigma_x$ and $\overline{\Sigma}_i$ for $i = 1, ..., k$ as above. Given $\delta > 0$, $0 < \delta < \frac{\epsilon}{2}$ with ϵ as in (3.6). Then there are GCS $\widetilde{\Sigma}_i \subset \overline{\Sigma}_i$ containing $\mathcal{F}^{ss}(z_i) \cap \Sigma_x$, such that Σ_x is subdivided into $2k+1$ GCS disjoint, including $\widetilde{\Sigma}_i$ for $i \in \{1, ..., k\}$, denoted by $\Sigma_x^{\#}$ the complement of the set $\bigcup_{i=1}^{k} \widetilde{\Sigma}_i$ in the subdivision above of Σ_x and such that*

1. *$\phi^{\delta}(\widetilde{\Sigma}_i) \cap \Sigma' = \emptyset$ for all $i \in \{1, ..., k\}$ and $\Sigma' \cap \Sigma = \emptyset$ for any $\Sigma \in \Sigma_x^{\#}$.*
2. *$\phi^{\delta}(\widetilde{\Sigma}_i) \cap \phi^{\delta}(\widetilde{\Sigma}_j) = \emptyset$ for $i \neq j$ and $\phi^{\delta}(\widetilde{\Sigma}_i) \cap \Sigma_x = \emptyset$ for all $i \in \{1, ..., k\}$.*
3. $$\Lambda \cap \phi^{(-\frac{\epsilon}{2}, \frac{\epsilon}{2})}(int(\Sigma_x)) \subset \Lambda \cap \left(\bigcup_{i=1}^{k} \phi^{(-\epsilon,\epsilon)} \left(\phi^{\delta}(int(\widetilde{\Sigma}_i)) \right) \cup \bigcup_{\Sigma \in \Sigma_x^{\#}} \phi^{(-\frac{\epsilon}{2}, \frac{\epsilon}{2})}(int(\Sigma)) \right).$$

Proof.
Given $\delta < \frac{\epsilon}{2}$ small, by transversality we have $\phi^{\delta}(\mathcal{F}^{ss}_{loc}(z_i) \cap \Sigma_x) \cap \Sigma' = \emptyset$, also $\phi^{\delta}(\mathcal{F}^{ss}_{loc}(z_i) \cap \Sigma_x) \cap \phi^{\delta}(\mathcal{F}^{ss}_{loc}(z_j) \cap \Sigma_x) = \emptyset$ for $i \neq j$. So, by continuity of ϕ^{δ}, for each i there is an interval $\overline{I}_i \subset I_i \subset \mathcal{F}^{uu}(x)$ centered in z_i such that the surface

$$\overline{\overline{\Sigma}}_i = \bigcup_{z \in \overline{I}_i} \mathcal{F}^{ss}(z) \cap \Sigma_x$$

satisfies $\phi^{\delta}(\overline{\overline{\Sigma}}_i) \cap \Sigma' = \emptyset$ for any i and $\phi^{\delta}(\overline{\overline{\Sigma}}_i) \cap \phi^{\delta}(\overline{\overline{\Sigma}}_j) = \emptyset$ for $i \neq j$.

We can assume that for any $i \in \{1, ..., k\}$ there is $p_i \in \mathcal{F}^{ss}(z_i) \cap \Sigma_x \cap \Lambda$. Consider $\mathcal{F}^{uu}_{loc}(p_i)$, then by Remark 9 we can find open sets V_i^+ and V_i^- in each side of $\mathcal{F}^{uu}_{loc}(p_i)$ sufficiently close to $\mathcal{F}^{ss}_{loc}(p_i)$ and diameter sufficiently large and $V_i^{\pm} \cap \Lambda = \emptyset$. Denoted by $\widetilde{V}_i^{\pm}$ the projection by the flow of $V_i^{\pm}$ over Σ_x, respectively. Therefore, by Remark 9 we can take $\widetilde{V}_i^{\pm}$ such that $\widetilde{V}_i^{\pm} \cap \Sigma_x \subset \overline{\overline{\Sigma}}_i$ and $\widetilde{V}_i^{\pm}$, $\overline{\overline{\Sigma}}_i$ crosses Σ_x (see Figure 3.8). Using $\widetilde{V}_i^{\pm}$ we can construct the GCS $\widetilde{\Sigma}_i$ such that $\widetilde{\Sigma}_i \subset \overline{\Sigma}_i$ and satisfies 1 and 2 of Lemma.

To prove 3 note simply that $\delta < \frac{\epsilon}{2}$ and $\Lambda \cap \phi^{(-\frac{\epsilon}{2}, \frac{\epsilon}{2})}(int(\Sigma_x)) = \phi^{(-\frac{\epsilon}{2}, \frac{\epsilon}{2})}(\Lambda \cap int(\Sigma_x))$, which is a consequence of Λ be invariant by the flow (see Remark 10). □

Remark 11. *Let Σ'' be a GCS as in the Lemma 12 such that $\Sigma_x \cap \Sigma'' = \emptyset$. Taking $\delta < d(\Sigma_x, \Sigma'')$, we have $\phi^{\delta}(\widetilde{\Sigma}_i) \cap \Sigma'' = \emptyset$ for any $i \in \{1, ..., k\}$, $\widetilde{\Sigma}_i$ as in Lemma 15.*

Now remember (3.6)

$$\Lambda \subset \bigcup_{i=1}^{l} \phi^{(-\frac{\epsilon}{2}, \frac{\epsilon}{2})}(int\Sigma_i) \subset \bigcup_{i=1}^{l} \phi^{(-\epsilon, \epsilon)}(int\Sigma_i) = \bigcup_{i=1}^{l} U_{\Sigma_i}.$$

In the Lemma 14 was proved that the GCS in (3.6) can be taken disjoint if all possible intersections of Σ_i with Σ_j are in the hypothesis of the Corollary 8.

Now we will prove that the GCS in (3.6) can be taken disjoint, even if some of them intersect transversely.

Lemma 16. *Let Σ_i be a GCS as in 3.6, let $B_i = \{j : \Sigma_i \pitchfork \Sigma_j\}$. Then, Σ_i can be subdivided in a finite number of GCS $\{\Sigma_i^s : s = 1, ..., m\}$ and for each s there is $0 < \delta_s < \frac{\epsilon}{2}$ such that*

1. *$\phi^{\delta_s}(\Sigma_i^s) \cap \Sigma_j = \emptyset$ for any $j \in B_i$ and $\phi^{\delta_s}(\Sigma_i^s) \cap \phi^{\delta_{s'}}(\Sigma_i^{s'}) = \emptyset$ for $s \neq s'$.*

2. $\Lambda \cap \bigcup_{j \in B_i \cup \{i\}} \phi^{(-\frac{\epsilon}{2}, \frac{\epsilon}{2})}(int\Sigma_j) \subset \Lambda \cap \left(\bigcup_{j \in B_i} \phi^{(-\frac{\epsilon}{2}, \frac{\epsilon}{2})}(int(\Sigma_j)) \cup \bigcup_{s=1}^{k} \phi^{(-\epsilon, \epsilon)} \left(int(\phi^{\delta_s}(\Sigma_i^s)) \right) \right).$

Proof. The proof is by induction on $\#B_i$.
The case $\#B_i = 1$ is true by the Lemma 15. Suppose true for $\#B_i < q$ and we prove for $\#B_i = q$, in fact:
Let $k \in B_i$, then by Lemma 15, given $0 < \delta < \frac{\epsilon}{2}$, there are a finite number of GCS $\left\{ \widetilde{\Sigma}_k^r \subset \Sigma_k : r \in \{1, ...r_k\} \right\}$ such that

$$\phi^{\delta}(\widetilde{\Sigma}_k^r) \cap \Sigma_k = \emptyset, \text{ also } \phi^{\delta}(\widetilde{\Sigma}_k^r) \cap \Sigma_i = \emptyset \text{ for any } r, \; \Sigma_i \cap \Sigma = \emptyset \text{ for any } \Sigma \in \Sigma_k^{\#}. \quad (3.8)$$

$$\Lambda \cap \phi^{(-\frac{\epsilon}{2}, \frac{\epsilon}{2})}(int(\Sigma_k)) \subset \Lambda \cap \left(\bigcup_{r=1}^{r_k} \phi^{(-\epsilon, \epsilon)} \left(\phi^{\delta}(int(\widetilde{\Sigma}_k^r)) \right) \cup \bigcup_{\Sigma \in \Sigma_k^{\#}} \phi^{(-\epsilon, \epsilon)}(int(\Sigma)) \right) \quad (3.9)$$

$\Sigma_k^{\#}$ as in Lemma 15.

Consider now the set of GCS $\Sigma_i \cup \{\Sigma_j : j \in B_i \setminus \{k\}\} \cup \Big\{\phi^{\delta}(\widetilde{\Sigma}_k^r) : r \in \{1, ..., r_k\}\Big\} \cup \Sigma_k^{\#}$, for this new set of GCS, we have $\# B_i < q$, therefore by the induction hypothesis, the Lemma is true for $\{\Sigma_j : j \in B_i \cup \{i\} \setminus \{k\}\}$ and by (3.8) and (3.9) we have the Lemma. □

Remark 12. *Let Σ_p be a GCS as in (3.6) such that $\Sigma_p \cap \Sigma_i = \emptyset$, then by Remark 11, δ_s can be taken less than $d(\Sigma_i, \Sigma_p)$, so $\phi^{\delta_s}(\Sigma_i^s) \cap \Sigma_p = \emptyset$ for any $s \in \{1, ..., m\}$.*

Let Σ_i be GCS as in (3.6) where $\Lambda \subset \bigcup_{i=1}^{l} \phi^{-(\frac{\epsilon}{2},\frac{\epsilon}{2})}(int(\Sigma_i))$. We can assume that the possible intersections of $int(\Sigma_i)$ with $int(\Sigma_i)$ are as in Corollary 8 or transverse. Then

Lemma 17. *There are GCS $\widetilde{\Sigma}_i$ such that*

$$\Lambda \subset \bigcup_{i=1}^{m(l)} \phi^{(-2\epsilon,2\epsilon)}(int(\widetilde{\Sigma}_i))$$

with $\widetilde{\Sigma}_i \cap \widetilde{\Sigma}_j = \emptyset$.

Proof. If all the possible intersections are as in Corollary 8 and the result follows from Lemma 14. Then, we can suppose that there is i, such that $B_i = \{j : \Sigma_i \pitchfork \Sigma_j\} \neq \emptyset$. Without loss of generality, assume that $B_1 \neq \emptyset$. The proof is by induction on l in (3.6).
For $l = 1$ ok.
The Lemma 15 implies the case $l = 2$.
Suppose true for $k < l$ and we prove for $k = l$, in fact:
Fix Σ_1, call $T_1 = \{j : \Sigma_j$ intersect Σ_1 as in Corollary 8 $\}$, then by statement 2 in the proof of Lemma 14, there is $0 < \delta < \frac{\epsilon}{4}$ small, such that $\phi^{\delta}(\Sigma_1) \cap \Sigma_j = \emptyset$ for any $j \in T_1$.

Consider now the GCS $\phi^{\delta}(\Sigma_1)$ as in Lemma 12, call still $B_1 = \{j : \phi^{\delta}(\Sigma_1) \pitchfork \Sigma_j\}$ then by Lemma 16, $\phi^{\delta}(\Sigma_1)$ can be subdivided in a finite number of GCS $\{\Sigma_1^s : s = 1, ..., m\}$ and for each s there is $0 < \delta_s < \frac{\epsilon}{2}$ such that holds 1 and 2 of Lemma 16. Also by Remark 12 we can assume that $\phi^{\delta_s}(\Sigma_1^s) \cap \Sigma_j = \emptyset$ for any $s \in \{1, ..., m\}$ and any $j \in T_1 \setminus \{1\}$.

Now take the set $\{\Sigma_j : j \in T_1 \setminus \{1\}\} \cup \{\Sigma_k : k \in B_1\}$, as $\# (T_1 \setminus \{1\} \cup B_1) < l$, then by the induction hypothesis there are GCS $\widetilde{\Sigma}_i$ such that

$$\Lambda \cap \bigcup_{i \in T_1 \cup B_1 \setminus \{1\}} \left(\phi^{-(\frac{\epsilon}{2},\frac{\epsilon}{2})}(int(\Sigma_i))\right) \subset \Lambda \cap \bigcup_{i=2}^{n(l)} \phi^{-(\epsilon,\epsilon)}(int(\widetilde{\Sigma}_i)) \tag{3.10}$$

Since $\phi^{\delta_s}(\Sigma_1^s) \cap \Sigma_j = \emptyset$ for any $j \in T_1 \cup B_1 \setminus \{1\}$ and any $s \in \{1, ..., m\}$, then the $\widetilde{\Sigma}_j$ may be taken such that $\phi^{\delta_s}(\Sigma_1^s) \cap \widetilde{\Sigma}_i = \emptyset$ for any $s \in \{1, ..., m\}$ and any $i \in \{2, ..., n(l)\}$.
So, by 2 of Lemma 16 and (3.10) we have that

$$\Lambda = \Lambda \cap \bigcup_{j=1}^{l} \phi^{-(\frac{\epsilon}{2},\frac{\epsilon}{2})}(int(\Sigma_j)) \subset \Lambda \cap \left(\bigcup_{j=2}^{l} \phi^{(-\frac{\epsilon}{2},\frac{\epsilon}{2})}(int(\Sigma_j)) \cup \phi^{(-\epsilon,\epsilon)}\left(int(\phi^{\delta}(\Sigma_1))\right)\right) =$$

$$= \Lambda \cap \left(\bigcup_{j \in B_1} \phi^{(-\frac{\epsilon}{2},\frac{\epsilon}{2})}(int(\Sigma_j)) \cup \bigcup_{j \in T_1 \setminus \{1\}} \phi^{(-\frac{\epsilon}{2},\frac{\epsilon}{2})}(int(\Sigma_j)) \cup \phi^{(-\epsilon,\epsilon)}\left(int(\phi^{\delta}(\Sigma_1))\right)\right)$$

$$\subset \Lambda \cap \left(\bigcup_{i=2}^{n(l)} \phi^{-(\epsilon,\epsilon)}(int(\widetilde{\Sigma}_i)) \cup \bigcup_{s=1}^{k} \phi^{(-2\epsilon,2\epsilon)}\left(int(\phi^{\delta_s}(\Sigma_i^s))\right) \right).$$

This concludes the proof of Lemma.

□

3.4 Global Poincaré Map

Let $\mathcal{R}\colon \Xi \to \Xi$ be a Poincaré map as in the section 3.1.1 with $\Xi = \bigcup_{i=1}^{l} \Sigma_i$, where Σ_i are GCS and $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$. We are going to show that, if the cross-section of Ξ are δ-Good Cross-Section (GCS), we have the invariance property

$$\mathcal{R}^n(W^s(x,\Sigma)) \subset W^s(\mathcal{R}^n(x),\Sigma'),$$

for some n sufficiently large.
Given Σ , $\Sigma' \in \Xi$ we set $\Sigma(\Sigma')_n = \{x \in \Sigma : \mathcal{R}^n(x) \in \Sigma'\}$ the domain of the map $\mathcal{R}^n$from Σ to Σ'.

Remember relation (3.3) from the proof of proposition 1, the tangent direction to each $W^s(x,\Sigma)$ is contracted at an exponential rate $\|D\mathcal{R}^n(x)\hat{e}^s_x\| \leq Ce^{-\beta t_n(x)}$, with $C = \frac{K}{\kappa}$ and $\beta = -\log\lambda > 0$.

Suppose that the cross-sections in Ξ are δ-GCS. Take n such that $t_n(x) > t_1$ as in proposition 1 with t_1 satisfying

$$Ce^{-\beta t_1}\sup\{l(W^s(x,\Sigma)) : x \in \Sigma\} < \delta \text{ and } \ Ce^{-\beta t_1} < \tfrac{1}{2}.$$

Lemma 18. *Let n be satisfying the above. Given δ-Good Cross-Sections, $\Sigma, \Sigma' \in \{\Sigma_i\}_i$ if $\mathcal{R}^n\colon \Sigma(\Sigma')_n \to \Sigma'$ defined by $\mathcal{R}^n(z) = \phi^{t_n(z)}(z)$. Then,*

1. *$\mathcal{R}^n(W^s(x,\Sigma)) \subset W^s(\mathcal{R}^n(x),\Sigma')$ for every $x \in \Sigma(\Sigma')_n$, and also*
2. *$d(\mathcal{R}^n(y),\mathcal{R}^n(z)) \leq \frac{1}{2}d(y,z)$ for every $y,z \in W^s(x,\Sigma)$ and $x \in \Sigma(\Sigma')_n$.*

We let $\{U_{\Sigma_i} : i = 1,...,l\}$ be a finite cover of Λ, as in the Lemma 17 where Σ_i are GCS, and we set T_3 to be an upper bound for the time it takes any point $z \in U_{\Sigma_i}$ to leave this tubular neighborhood under the flow, for any $i = 1,...,l$. We assume without loss of generality that $t_1 > T_3$.

To define the Poincaré map $\mathcal{R}$, for any point z in one of the cross-sections in

$$\Xi = \bigcup_{i=1}^{l} \Sigma_i.$$

Let t_1 be as in the Lemma 18 and consider $\mathcal{R}^n$. If the point z never returns to one of the cross-sections, then the map $\mathcal{R}$ is not defined at z. Moreover, by the Lemma 18, if $\mathcal{R}^n$ is defined for $x \in \Sigma$ on some $\Sigma \in \Xi$, then $\mathcal{R}$ is defined for every point in $W^s(x, \Sigma)$. Hence, the domain of $\mathcal{R}^n|\Sigma$ consists of strips of Σ. The smoothness of $(t, x) \longrightarrow \phi^t(x)$ ensure that the strips

$$\Sigma(\Sigma')_n = \{x \in \Sigma : \mathcal{R}^n(x) \in \Sigma'\}$$

have non-empty interior in Σ for every $\Sigma, \Sigma' \in \Xi$.

Note that $\mathcal{R}$ is locally smooth for all points $x \in int\Sigma$ such that $\mathcal{R}(x) \in int(\Xi)$, by the Tubular Flow Theorem and the smoothness of the flow, where $int(\Xi) = \bigcup_{i=1}^{l} int\Sigma_i$. Denote $\partial^j \Xi = \bigcup_{i=1}^{l} \partial^j \Sigma_i$ for $j = s, u$.

Lemma 19. *The set of discontinuities of $\mathcal{R}$ in $\Xi \setminus (\partial^s \Xi \cup \partial^u \Xi)$ is contained in the set of point $x \in \Xi \setminus (\partial^s \Xi \cup \partial^u \Xi)$ such that, $\mathcal{R}(x)$ is defined and belongs to $(\partial^s \Xi \cup \partial^u \Xi)$.*

Proof. Let x be a point in $\Sigma \setminus (\partial^s \Sigma \cup \partial^u \Sigma$ for some $\Sigma \in \Xi$, not satisfying the condition, then $\mathcal{R}(x)$ is defined and $\mathcal{R}(x)$ belongs to the interior of some cross-section Σ'. By the smoothness of the flow we have that $\mathcal{R}$ is smooth in a neighborhood of x in Σ. Hence, any discontinuity point for $\mathcal{R}$ must be in the condition of the Lemma. □

Let $D_j \subset \Sigma_j$ be the set of points sent by $\mathcal{R}^n$ into stable boundary points of some Good Cross-Section of Ξ is such that the set

$$L_j = \{W^s(x, \Sigma_j) : x \in D_j\},$$

by Lemma 18, we have $L_j = D_j$. Let $B_j \subset \Sigma_j$ be the set of points sent by $\mathcal{R}^n$ into unstable boundary points of some Good Cross-Section of Ξ.
Denote

$$\Gamma_j = \bigcup_{x \in D_j} W^s(x, \Sigma_j) \cup B_j \quad and \quad \Gamma = \bigcup \Gamma_j \cup (\partial^s \Xi \cup \partial^u \Xi).$$

Then, in the complement $\Xi \setminus \Gamma$ of Γ, $\mathcal{R}^n$ is smooth. Observe that if $x \in D_j$ for some $j \in \{1, ..., l\}$, then

$$\mathcal{R}^n(W^s(x, \Sigma_j)) \subset \partial^s \Sigma' \quad for\ some \quad \Sigma' \in \Xi.$$

We know that $\partial^s \Xi \cap \Lambda = \emptyset$, then $\mathcal{R}^n(W^s(x, \Sigma_j)) \cap \Lambda = \emptyset$. This implies that $W^s(x, \Sigma_j) \cap \Lambda = \emptyset$ for all $x \in D_j$. Moreover, if $x \in B_j$, then $\mathcal{R}^n(x) \in \partial^u \Sigma'$ for some $\Sigma' \in \Xi$, we know that $\partial^u \Xi \cap \Lambda = \emptyset$, this implies that $B_j \cap \Lambda = \emptyset$. Therefore, $\Gamma_j \cap \Lambda = \emptyset$ for all $j \in \{1, ..., l\}$, so $\Gamma \cap \Lambda = \emptyset$.

Clearly, if $x \in \Lambda \cap \Xi$, then $\mathcal{R}(x)$ is defined and $\mathcal{R}(x) \in int(\Xi)$.

Let $x \in \Lambda \cap \Sigma_j$, then $x \in \Sigma_j \setminus (\Gamma_j \cup \partial^s \Sigma_j \cup \partial^u \Sigma_j)$ and $\mathcal{R}^n(x)$ is defined and $\mathcal{R}^n(x) \in \Sigma_i \cap \Lambda$ for some $i \in \{1, ..., l\}$. The above implies that $\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i$ is an invariant set for $\mathcal{R}^n$ and by Proposition 1, $\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i$ is hyperbolic set for $\mathcal{R}^n$ and since $\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i$ is invariant for $\mathcal{R}$, then $\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i$ is hyperbolic for $\mathcal{R}$, and

$$\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i \subset \bigcap_{n\in\mathbb{Z}} \mathcal{R}^{-n}(\bigcup_{i=1}^{l} \Sigma_i).$$

3.5 Hausdorff Dimension of $\bigcap_{n\in\mathbb{Z}} \mathcal{R}^{-n}(\bigcup_{i=1}^{l} \Sigma_i)$

Now we are going to estimate of $HD(\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i)$.

Lemma 20. *The set Λ satisfies*

$$\Lambda \subset \bigcup_{t\in\mathbb{R}} \phi^t(\bigcap_{n\in\mathbb{Z}} \mathcal{R}^{-n}(\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i)) = \bigcup_{t\in\mathbb{R}} \phi^t(\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i) \subset \bigcup_{t\in\mathbb{R}} \phi^t(\bigcap_{n\in\mathbb{Z}} \mathcal{R}^{-n}(\bigcup_{i=1}^{l} \Sigma_i)).$$

Proof. Remember that $\Lambda \subset \bigcup_{i=1}^{l} U_{\Sigma_i}$ where $U_i = \phi^{(-\epsilon,\epsilon)}(int\Sigma_i)$. Let $z \in \Lambda$, then there is i_z such that $z = \phi^{i_z}(x)$ with $x \in int(\Sigma_{i_z})$. This implies $x \in \Lambda \cap \bigcup_{i=1}^{l} \Sigma_i$ and $\mathcal{R}(x) \in int(\Sigma_j)$ for some j and $\mathcal{R}(x) \in int(\Xi)$. Analogously, $\mathcal{R}^n(x) \in int(\Xi)$, this is $R^n(x) \in \Lambda \cap \bigcup_{i=1}^{l} \Sigma_i$ for all $n \in \mathbb{Z}$. Hence, $x \in \bigcap_{n\in\mathbb{Z}} \mathcal{R}^{-n}(\Lambda \cap (\bigcup_{i=1}^{l} \Sigma_i))$, therefore $z \in \phi^{t_z}\left(\bigcap_{n\in\mathbb{Z}} \mathcal{R}^{-n}(\Lambda \cap (\bigcup_{i=1}^{l} \Sigma_i))\right)$. $\square$

Lemma 21. *The Hausdorff Dimension of $\Lambda \cap (\bigcup_{i=1}^{l} \Sigma_i)$ and $\bigcap_{n\in\mathbb{Z}} \mathcal{R}^{-n}(\bigcup_{i=1}^{l} \Sigma_i)$ satisfies,*

$$HD(\bigcap_{n\in\mathbb{Z}} \mathcal{R}^{-n}(\bigcup_{i=1}^{l} \Sigma_i)) \geq HD(\Lambda \cap (\bigcup_{i=1}^{l} \Sigma_i)) \geq HD(\Lambda) - 1$$

and thus $HD(\Lambda \cap (\bigcup_{i=1}^{l} \Sigma_i)) \sim 2$.

Proof. Take a *bi*-infinite sequence

$$... < t_{-k} < t_{-k+1} < ... < t_0 < t_1 < ...t_k < ...$$

such that $|t_k - t_{k+1}| < \alpha$ with α is very small, then

$$\Lambda \subset \bigcup_{k=-\infty}^{+\infty} \phi^{[t_k,t_{k+1}]}(\Lambda \cap (\bigcup_{i=1}^{l} \Sigma_i)) := \bigcup_{k=-\infty}^{+\infty} A_k$$

since $HD(\Lambda) \sim 3$ and $HD(\Lambda) \leq \sup_k HD(A_k)$, then there exists k_0 such that

$$HD(A_{k_0}) \sim 3.$$

For α very small, the map

$$\begin{aligned} \psi :\ \left(\Lambda \cap (\textstyle\bigcup_{i=1}^{l} \Sigma_i)\right) \times [t_k, t_{k+1}] &\longrightarrow A_k \ \text{ defined by} \\ (x,t) &\longmapsto \phi^t(x) \end{aligned}$$

is Lipschitz, we see this since $\psi = \phi/_{(\Lambda\cap(\bigcup_{i=1}^{l} \Sigma_i))\times[t_k,t_{k+1}]}$ where $\phi/_{(\bigcup_{i=1}^{l} \Sigma_i)\times[t_k,t_{k+1}]}$ is a diffeomorphism, for $|t_{k+1} - t_k| < \alpha$ and α very small.

Therefore, $HD(\Lambda) \sim HD(A_{k_0}) \leq HD(\left(\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i\right) \times [t_{k_0}, t_{k_0+1}])$, call $I_k = [t_k, t_{k+1}]$, we have the following affirmation:

Affirmation: The following inequality is true,

$$HD((\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i) \times I_k) \leq HD(\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i) + D(I_k)$$

where D is a upper box counting dimension of I_k defined by

$$D(I_k) = \limsup_{\delta \to 0} \frac{logN_\delta(I_k)}{-log\delta}$$

and $N_\delta(I_k)$ is the smallest number of sets of diameter at most δ which can cover I_k (cf. [Fal85]), is easy to see that $D(I_k) = 1$ for all k, then

$$HD(\Lambda) \sim HD((\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i) \times [t_{k_0}, t_{k_0+1}]) \leq HD(\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i) + D(I_{k_0}) = HD(\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i) + 1.$$

Hence, $HD(\Lambda \cap \bigcup_{i=1}^{l} \Sigma_i) \sim 2$. □

Chapter 4

Markov and Lagrange Spectrum For Geodesic Flow

Contents

Let M be a complete noncompact surface M such that the Gaussian curvature is bounded between two negative constants and the Gaussian volume is finite. Denote by K_M the Gaussian curvature, thus there are constants $a, b > 0$ such that

$$-a^2 \leq K_M \leq -b^2 < 0.$$

Definition 8. *Let X be a complete vector field on SM and $f \in C^0(SM, \mathbb{R})$, then the dynamical Markov spectrum associated to (f, X) is defined by*

$$M(f, X) = \left\{ \sup_{t \in \mathbb{R}} f(X^t(x)) : x \in SM \right\}$$

and the dynamical Lagrange spectrum associate to (f, X) by

$$L(f, X) = \left\{ \limsup_{t \to \infty} f(X^t(x)) : x \in SM \right\}$$

where $X^t(x)$ is the integral curve of the field X in x.

4.1 The Interior of Spectrum for Perturbations of ϕ

Let ϕ be the vector field defining the geodesic flow.

The objective of this section is to prove the following theorem.

Theorem 4. *Let M be as above, let ϕ be the geodesic flow, then there is X a vector field sufficiently close to ϕ such that*

$$intM(f,X) \neq \emptyset \text{ and } intL(f,X) \neq \emptyset$$

for a dense and C^2-open subset $\mathcal{U}$ of $C^2(SM,\mathbb{R})$. Moreover, the above holds for a neighborhood of $\{X\} \times \mathcal{U}$ in $\mathfrak{X}^1(SM) \times C^2(SM,\mathbb{R})$, where $\mathfrak{X}^1(SM)$ is the space of C^1 vector field on SM.

To prove this theorem we use the results of Chapter 2, 3 and a construction of obtaining property V of Section 2.3.1, found in [MY10].

In chapter 3 it was proven that there are a finite number of C^1-GCS, Σ_i pairwise disjoint and such that the Poincaré map $\mathcal{R}$ (map of first return) of $\Xi := \bigcup_{i=1}^{l} \Sigma_i$

$$\mathcal{R} \colon \Xi \to \Xi$$

satisfies:

- $\bigcap_{n\in\mathbb{Z}} \mathcal{R}^{-n}(\Xi) := \Delta$ is hyperbolic set for $\mathcal{R}$.
- $HD\left(\bigcap_{n\in\mathbb{Z}} \mathcal{R}^{-n}(\Xi)\right) \sim 2.$

We can assume without loss of generality that the GCS Σ_i are C^∞-GCS.

4.1.1 The Family of Perturbation

Now we describe the family of perturbations of $\mathcal{R}$ for which we can find the property V of Section 2.3.1 (cf. [MY10] page 19).

Let R be a Markov partition of $\Delta = \bigcap_{n\in\mathbb{Z}} \mathcal{R}^{-n}(\Xi)$.
Is selected once and for all a constant $c_0 > 1$. For all $0 < \rho < 1$, then we denote $R(\rho)$ the set of words $\underline{a}$ of R such that ${c_0}^{-1}\rho \leq |I(\underline{a})| \leq c_0\rho$ (cf. 2.2.2).

We consider a partition $\widetilde{R}_1$ of Δ in rectangle whose two sides are approximately sized $\rho^{2/m}$. A rectangle denotes here a part of Δ consisting of points with itinerary prescribed for a certain time interval; a word of R, which prescribes the route is associated with said rectangle. Among the rectangles on $\widetilde{R}_1$ preserves only those for which no word in $R(\rho^{\frac{1}{2m}})$ appears no more than once in the associated word. Called R_1 the set of associated words.

For each $\underline{a} \in R_1$ denotes $R(\underline{a})$ the associated rectangle; construct a vector field $X_{\underline{a}}$ having the following properties:

- On $R(\underline{a})$, $X_{\underline{a}}$ is constant, the size of the order of approximately $\rho^{1+1/m}$ directed the unstable direction;

- $X_{\underline{a}}$ is the size $\rho^{1/m}$ in the $C^{m/2}$-topology.

Clearly, the condition latter ensures that time one of the flow of X is, if m is large and ρ small, in a neighborhood of id_{Ξ} beforehand prescribed in C^{∞}-topology.

We equip $\Omega = [-1,+1]^{R_1}$ the normalized Lebesgue measure; for $\underline{w} \in \Omega$, let

$$X_{\underline{w}} = -c_X \sum_{\underline{a}} w(\underline{a}) X_{\underline{a}},$$

$$\mathcal{R}^{\underline{w}} = \mathcal{R} \circ \Phi^{\underline{w}},$$

where $\Phi^{\underline{w}}$ denotes the time one of the flow $X_{\underline{w}}$. Note that the "sum" in the definition of $X_{\underline{w}}$ has at each point of $\bigcup R(\underline{a})$ at most one nonzero term.

4.1.2 Realization of the Perturbation

In [MY10] was proved that for c_X large enough, appropriately chosen, there are many parameters $\underline{w}$ such that $(\mathcal{R}^{\underline{w}}, \Delta_{\underline{w}})$ has the property V (cf. Definition 3), where $\Delta_{\underline{w}}$ is the continuation of the hyperbolic set Δ for $\mathcal{R}^{\underline{w}}$.

Lemma 22. *Let $\mathcal{R}$ be as above and $\underline{w} \in \Omega$ and the vector field $X_{\underline{w}}$, then there is a vector field $G_{\underline{w}}$ sufficiently close to ϕ such that $\mathcal{R}^{\underline{w}}$ is the Poincaré map, (map of first return) to Ξ by the flow of $G_{\underline{w}}$.*

Proof. The argument is made on the $R(\underline{a})$ with $R(\underline{a}) \in R_1$. Fix $\underline{w} \in \Omega = [-1,+1]^{R_1}$, $\underline{w} = (w(\underline{a}))_{\underline{a} \in R_1}$, then on sufficiently small neighborhood of $R(\underline{a})$ in Ξ is defined the field vector $-C_X w_{\underline{a}} X_{\underline{a}} := Y_{\underline{a}}$.
Now we can extend this vector field in a neighborhood of $R(\underline{a})$ in SM as follows:
Suppose that $R(\underline{a}) \subset \Sigma \in \Xi$, let $\beta_{\underline{a}} > 0$ such that $\phi^t(V_{\underline{a}}) \cap (\Xi \setminus \Sigma_i) = \emptyset$ for all $t \in [0, \beta_{\underline{a}})$, where $V_{\underline{a}} \supset R(\underline{a})$ neighborhood of $R(\underline{a})$ in Σ and such that $Y_{\underline{a}}$ is defined and $Y_{\underline{a}} = 0$ in $\Sigma \setminus V_{\underline{a}}$, put $\widetilde{V}_{\underline{a}} := \phi^{[0,\beta_{\underline{a}})}(V_{\underline{a}})$ is a neighborhood of $R(\underline{a})$ in SM, this neighborhood can be seen as $V_{\underline{a}} \times [0, \beta_{\underline{a}})$.
Define the vector field $\widetilde{Y}_{\underline{a}}$ on $\widetilde{V}_{\underline{a}}$ by

$$\widetilde{Y}_{\underline{a}}(\phi^t(z)) = D\phi^t_z(Y_{\underline{a}}(z)).$$

Let $\varphi_{\underline{a}}$ be a smooth real function defined in $V_{\underline{a}} \times [0, \beta_{\underline{a}})$ such that

$$\varphi_{\underline{a}} = \begin{cases} 1 & \text{in } V_{\underline{a}} \times [0, \frac{\beta_{\underline{a}}}{4}); \\ 0 & \text{in } V_{\underline{a}} \times [\frac{\beta_{\underline{a}}}{2}, \beta_{\underline{a}}) \end{cases}.$$

Put the vector field $Z_{\underline{a}} = \varphi_{\underline{a}} \widetilde{Y}_{\underline{a}}$ defined in $V_{\underline{a}} \times [0, \beta_{\underline{a}})$. Note that by definition

$$Z_{\underline{a}} = 0 \text{ in } \Sigma \times [0, \beta_{\underline{a}}) \setminus V_{\underline{a}} \times [0, \frac{\beta_{\underline{a}}}{2}). \tag{4.1}$$

We will describe the relation between the time one diffeomorphism of $Z_{\underline{a}}$ say $\Phi_{Z_{\underline{a}}}$ and the time one diffeomorphism of $Y_{\underline{a}}$, say $\Phi_{Y_{\underline{a}}}$ in $V_{\underline{a}}$. In fact:
Let $0 \leq t_0 < \frac{\beta_{\underline{a}}}{4}$, $z \in V_{\underline{a}}$, take $\alpha(t)$ an integral curve of the vector field $Y_{\underline{a}}$ with $\alpha(0) = z$ contained in $V_{\underline{a}}$, then we consider the curve $\eta(t) = \phi^{t_0}(\alpha(t))$, therefore, as $\varphi_{\underline{a}}(\eta(t)) = 1$, we have

$$\begin{aligned} \eta'(t) &= D\phi^{t_0}_{\alpha(t)}(\alpha'(t)) = D\phi^{t_0}_{\alpha(t)}\,(Y_{\underline{a}}(\alpha(t))) \\ &= \widetilde{Y}_{\underline{a}}\left(\phi^{t_0}(\alpha(t))\right) = \widetilde{Y}_{\underline{a}}\,(\eta(t)) = \varphi_{\underline{a}} \widetilde{Y}_{\underline{a}}(\eta(t)) = Z_{\underline{a}}(\eta(t)). \end{aligned}$$

So,

$$\Phi_{Z_{\underline{a}}}(\phi^{t_0}(z)) = \phi^{t_0}\left(\Phi_{Y_{\underline{a}}}(z)\right) \tag{4.2}$$

Varying $t_0 \in [0, \frac{\beta_{\underline{a}}}{4})$ and differentiating the equation (4.2) with respect to t_0, implies

$$D(\Phi_{Z_{\underline{a}}})_{\phi^t(z)}\left(\phi(\phi^t(z))\right) = \phi\left(\phi^t\left(\Phi_{Y_{\underline{a}}}(z)\right)\right) \quad \text{for} \quad z \in V_{\underline{a}},$$

where ϕ is the vector field defining the geodesic flow.
Note that $\Phi_{Z_{\underline{a}}} = \Phi_{Y_{\underline{a}}}$ in $V_{\underline{a}}$, call $h_{\underline{a}} := \Phi_{Z_{\underline{a}}}$, then

$$D(h_{\underline{a}})_{\phi^t(z)}(\phi(\phi^t(z))) = \phi\left(\phi^t(h_{\underline{a}}(z))\right) \quad \text{for} \quad (z,t) \in V_{\underline{a}} \times [0, \frac{\beta_{\underline{a}}}{4}). \tag{4.3}$$

Put the vector field $G_{\underline{a}}(x) = (Dh_{\underline{a}})^{-1}_{h_{\underline{a}}(x)}\,(\phi(h_{\underline{a}}(x)))$ for $x \in \widetilde{V}_{\underline{a}}$, then by (4.2) and (4.3) we have

$$G_{\underline{a}}(x) = \phi(x) \quad \text{for any} \quad x \in V_{\underline{a}} \times [0, \frac{\beta_{\underline{a}}}{4}).$$

And by (4.1)

$$G_{\underline{a}}(x) = \phi(x) \quad \text{for any} \quad x \in \Sigma \times [0, \beta_{\underline{a}}) \setminus V_{\underline{a}} \times [0, \frac{\beta_{\underline{a}}}{2}).$$

These last two relations implies that $G_{\underline{a}}$ is a smooth field that coincides with ϕ outside of $V_{\underline{a}} \times [\frac{\beta_{\underline{a}}}{4}, \frac{\beta_{\underline{a}}}{2})$.
Let $\beta(t)$ be the geodesic $\phi^t\,(h_{\underline{a}}(z))$ with $z \in V_{\underline{a}}$, define $\alpha(t) = h_{\underline{a}}^{-1}(\beta(t))$, then

$$\begin{aligned} \alpha'(t) &= (Dh_{\underline{a}})^{-1}_{\beta(t)}\,(\beta'(t)) = (Dh_{\underline{a}})^{-1}_{\beta(t)}\,(\phi(\beta(t))) \\ &= (Dh_{\underline{a}})^{-1}_{h_{\underline{a}}(\alpha(t))}\,(\phi\,(h_{\underline{a}}(\alpha(t)))) = G_{\underline{a}}(\alpha(t)) \end{aligned}$$

this $\alpha(t)$ is an integral curve of $G_{\underline{a}}$ in z.

Since $G_{\underline{a}} = \phi$ outside of neighborhood of $V_{\underline{a}} \times [\frac{\beta_{\underline{a}}}{4}, \frac{\beta_{\underline{a}}}{2})$ in SM, then the integral curve of vector field $G_{\underline{a}}$ passing by z coincides with the orbit of geodesic flow of $h_{\underline{a}}(z)$ outside of neighbourhood $V_{\underline{a}} \times [\frac{\beta_a}{4}, \frac{\beta_a}{2})$ of z in SM.

In particular, denoting $\mathcal{R}_{G_{\underline{a}}} : \Xi \to \Xi$ the Poincaré map of vector field $G_{\underline{a}}$, we have that if $z \in R(\underline{a})$ and $h_{\underline{a}}(z) \in (R(\underline{a}))$ then

$$(\mathcal{R} \circ h_{\underline{a}})(z) = \mathcal{R}_{G_{\underline{a}}}(z).$$

Now for each $R(\underline{a}) \in R_1$ we can assume that $suppY_{\underline{a}} = \overline{\{x : Y_{\underline{a}}(x) = 0\}}$ the support vector field $Y_{\underline{a}}$, are disjoint, so for each $R(\underline{a}) \in R_1$ the vector field $G_{\underline{a}}$ can be constructed such that the sets $supp_{\phi} G_{\underline{a}} = \overline{\{z \in SM : G_{\underline{a}}(z) \neq \phi(z)\}}$ are disjoint.
Define the smooth vector field

$$G_{\underline{w}}(z) = \begin{cases} G_{\underline{a}}(z) & \text{if } z \in supp_{\phi} G_{\underline{a}}; \\ \\ \phi(z) & \text{otherwise} \end{cases}.$$

and satisfies

$$\mathcal{R}^{\underline{w}} = \mathcal{R} \circ \Phi^{\underline{w}} = \mathcal{R}_{G_{\underline{w}}},$$

where $\mathcal{R}_{G_{\underline{w}}}$ is the Poincaré map of vector field $G_{\underline{w}}$. □

Remark 13. *Note that since $X_{\underline{a}}$ is small size, then taking $\beta_{\underline{a}}$ sufficiently small, then $G_{\underline{a}}$ can be constructed can be constructed close to ϕ, therefore $G_{\underline{w}}$ is close to ϕ.*

The following Lemma is combinatorial and will be used to show the Lemma 24.

Lemma 23. *Let $A = (a_{ij})_{1 \le i,j \le n}$ a matrix such that $a_{ij} \in \{0, 1\}$ for any i, j and $|\{(i, j) : a_{ij} = 1\}| \ge \frac{99}{100} n^2$, then $tr(A^k) \ge \left(\frac{n}{2}\right)^k$ for all $k \ge 2$.*

Remember that if $B = (b_{ij})_{1 \le i,j \le n}$ is a square matrix, then $tr(B) = \sum_{i=1}^{n} b_{ii}$ denotes the trace of B.

Proof.
There is $X \subset \{1, 2, ..., n\}$ with $|X| \ge \frac{9n}{10}$ such that, for any $i \in X$, $|\{j \le n : a_{ij} = 1\}| \ge \frac{9n}{10}$.
In fact: If there are more of $\frac{n}{10}$ lines in the matrix, each with at least $\frac{n}{10}$ null entries, the number of null entries of the matrix is greater that $\frac{n^2}{100}$, therefore $|\{(i, j) : a_{ij} = 1\}| < n^2 - \frac{n^2}{100} = \frac{99n}{100}$ which is a contradiction.
Analogously, there is $Y \subset \{1, 2, ..., n\}$ with $|Y| \ge \frac{9n}{10}$ such that, for any $j \in Y$, $|\{i \le n : a_{ij} = 1\}| \ge \frac{9n}{10}$.
Let $Z = X \cap Y$, we have $|Z| \ge \frac{9n}{10} + \frac{9n}{10} - n = \frac{4n}{5}$. If $i, j \in Z$, then

$$(A^2)_{ij} = \sum_{r=1}^{n} a_{ir} a_{rj} \ge \sum_{r \in A_i \cap B_j} a_{ir} a_{rj} = |A_i \cap B_j| \ge \frac{9n}{10} + \frac{9n}{10} - n = \frac{4n}{5},$$

where $A_i = \{j \le n : a_{ij} = 1\}$ and $B_j = \{i \le n : a_{ij} = 1\}$.
we will show by induction that if $i, j \in Z$, then

$$A_{ij}^k \ge \frac{4}{5} \left(\frac{3}{5}\right)^{k-2} .n^{k-1} \text{ for all } k \ge 2.$$

In fact:

$$\begin{aligned}(A^{k+1})_{ij} &= \sum_{r=1}^{n}(A^k)_{ir}.a_{rj} \geq \sum_{r\in Z}(A^k)_{ir}.a_{rj} \geq |Z\setminus\{r\in Z: a_{rj}=0\}|.\frac{4}{5}\left(\frac{3}{5}\right)^{k-2}.n^{k-1}\\ &\geq \left(\frac{4n}{5}-\frac{n}{10}\right)\frac{4}{5}\left(\frac{3}{5}\right)^{k-2}.n^{k-1} > \frac{4}{5}\left(\frac{3}{5}\right)^{k-1}.n^k.\end{aligned}$$

since $|\{r\in Z: a_{rj}=0\}|\leq \frac{n}{10}$. Thus, for all $k\geq 2$

$$tr(A^k)\geq \sum_{i\in Z}(A^k_{ii})\geq \frac{4n}{5}.\frac{4}{5}\left(\frac{3}{5}\right)^{k-2}.n^{k-1} > \left(\frac{3}{5}\right)^k.n^k > \left(\frac{n}{2}\right)^k.$$

□

Remark 14. *Suppose that the matrix A as in Lemma 23, is the matrix of transitions for a regular Cantor set K with Markov partition $R=\{R_1,R_2,\cdots R_N\}$ by intervals of size approximately ϵ. Since for each $k\geq 2$, denote by a^k_{ij} the element ij of the matrix A^k, which represents the numbers of transitions of size $k+1$ of i for j, moreover these transitions represent intervals of size ϵ^{k+1}.*
As the matrix A satisfies the Lemma 23, then there exists a subset B of $\{1,2,\cdots N\}$ such that that submatrix $\tilde{A}=(a_{ij})$ for $i,j\in B$ has all nonzero entries, so each transition R_iR_j with $i,j\in B$ is admissible.
Put the regular Cantor set

$$\tilde{K} := \{\cdots R_{j_1}R_{j_2}\cdots R_{j_k}\cdots / R_{j_i}\in B\}\subset K.$$

Thus,

$$HD(\tilde{K})\sim \frac{\sum_{i,j\in B}a^k_{ij}}{-\log\epsilon^{k+1}}\geq \frac{\log\left(\frac{N}{2}\right)^k}{-\log\epsilon^{k+1}}\geq \frac{k}{k+1}\frac{(\log N-\log 2)}{-\log\epsilon}\sim HD(K).$$

Therefore, $HD(\tilde{K})\sim HD(K)$.

The following Lemma says as is the behavior of the horseshoe Δ when it is intersected by a finite number of C^1-curves.

Lemma 24. ***Intersection of curves with Δ***
Let α_i be a finite numbers of C^1-curves, $i\in\{1,\ldots,m\}$, then for all $\epsilon>0$ there are sub-horseshoe Δ^s_α, Δ^u_α of Δ such that $\Delta^{s,u}_\alpha\cap\alpha_i=\emptyset$ for any $i\in\{1,\ldots,m\}$ and

$$HD(K^s_\alpha)\geq HD(K^s)-\epsilon \text{ and } HD(K^u_\alpha)\geq HD(K^u)-\epsilon,$$

where K^s_α, K^s are of regular Cantor set that describe the geometry transverse of the unstable foliation $W^u(\Delta^s_\alpha)$, $W^u(\Delta)$ respectively, and K^u_α, K^u are of regular Cantor set that describe the geometry transverse of the stable foliation $W^s(\Delta^u_\alpha)$, $W^s(\Delta)$ respectively.

We will prove for the Cantor set stable, for the unstable Cantor set is analogous.

Before starting with the proof of Lemma we introduce some definitions and observations.

Let us fix Markov partition R of Δ as above, call K^s the regular Cantor set associate to Δ as in the section 2.2 and sub-section 2.2.2.
Given $R(\underline{a}) \in R$ for $\underline{a} = (a_{i_1}, \cdots, a_{i_r})$ denote $|a_{i_1}, \cdots, a_{i_r}|$ the diameter of the projection in W^s_{loc} of $R(\underline{a})$ by $\mathcal{F}^u$ (see sub-section 2.2.2 in the construction of K^s). Fix $a_r,\ a_s$ such that the pair (a_r, a_s) is admissible.

Let $\epsilon > 0$, we have the following definition.

Definition 9. *A piece $(a_{i_1}, \cdots, a_{i_k})$ (in the construction of K^s) is called ϵ-piece if*

$$|a_{i_1}, \cdots, a_{i_k}| < \epsilon \text{ and } |a_{i_1}, \cdots, a_{i_{k-1}}| \geq \epsilon.$$

Put

$$X_\epsilon = \{\epsilon\text{-piece } (a_{i_1}, \cdots, a_{i_k}) : i_1 = s \text{ and } i_k = r\} = \{\theta_1, ..., \theta_N\}.$$

with $N \sim \epsilon^{-d_s}$ where $d_s = HD(K^s)$.

Note now that $\theta_{j_1}\theta_{j_2}$ is a admissible word, thus, we define the set

$$K(X_\epsilon) := \{\cdots \theta_{j_1}\theta_{j_2} \cdots \theta_{j_k} \cdots \ /\ \theta_{j_i} \in X_\epsilon\} \subset K^s.$$

Since for regular Cantor set, the Hausdorff dimension ($HD(.)$) is equal to box dimension ($d(.)$), then $d_s = HD(K^s) \geq d(K(X_\epsilon))$ and $d(K(X_\epsilon)) \leq \frac{\log N}{-\log \epsilon}$ since θ_i is a covering of diameter less that ϵ, therefore $HD(K(X_\epsilon))$ is close to $HD(K^s) = d_s$.

We can assume that the finite family α only has a unique curve, we still call α. Divide the family as curves α, in curves that are graphs of C^1-functions of $W^s(\Delta)$ on $W^u(\Delta)$ or from $W^u(\Delta)$ on $W^s(\Delta)$, (cf. sub-section 2.2.2 for definition of $W^{s,u}(\Delta)$).

Denote by I_{θ_i} the interval associated with θ_i in the construction of K^s, let $C > 1$ such that

$$C^{-1}\epsilon < |I_{\theta_i}| < C\epsilon.$$

For each I_{θ_i}, with $\theta_i = (a_{i_1}, \cdots, a_{i_k})$, we associate the interval transposed $I'_{\theta_i^t}$ (associated to the word $(a_{i_k}, \cdots, a_{i_1})$) in the construction of K^u (unstable Cantor set). Then, since Δ is horseshoe there exists $\eta > 1$ such that

$$|I_{\theta_i}|^\eta < |I'_{\theta_i^t}| < |I_{\theta_i}|^{1/\eta}.$$

Proof of Lemma 24.

- <u>In the first case</u> (graph of a C^1-function from $W^s(\Delta)$ on $W^u(\Delta)$), in this case, consider the image P of I_{θ_i} by this function, then C and ϵ can be take such that $|P| \leq C^2\epsilon$. Let

$P' \subset P$, the largest interval of the construction of K^u contained in P, then iterating forward the interval P', we obtain an interval J in $W^s(\Delta)$ with $|J| \leq (C^2\epsilon)^{1/\eta}$, then

$$\#\{I_{\theta_i} : I_{\theta_i} \subset J\} \leq C\left(\frac{(C^2\epsilon)^{1/\eta}}{\epsilon}\right)^{d_s} = \tilde{C}\epsilon^{d_s(1/\eta-1)}.$$

Thus,

$$\#\{(I_{\theta_i}, I'_{\theta_j^t}) : I_{\theta_i} \times I'_{\theta_j^t} \text{ is cut by the curve } \alpha\} \leq \epsilon^{-d_s}\tilde{C}\epsilon^{d_s(1/\eta-1)} = \tilde{C}\epsilon^{d_s(1/\eta-2)} \ll \epsilon^{-2d_s}.$$

- <u>In the second case</u> (graph of a C^1-function from $W^u(\Delta)$ on $W^s(\Delta)$), in this case, consider the image K of $I'_{\theta_i^t}$, then, $|K| \leq c|I'_{\theta_i^t}| \leq c(C\epsilon)^{1/\eta}$, ($K$ is the image by a C^1-function), so we have analogously

$$\#\{I_{\theta_i} : I_{\theta_i} \subset K\} \leq \hat{C}\epsilon^{d_s(1/\eta-1)}.$$

And

$$\#\{(I_{\theta_i}, I'_{\theta_j^t}) : I_{\theta_i} \times I'_{\theta_j^t} \text{ is cut by the curve } \alpha\} \leq \epsilon^{-d_s}\tilde{C}\epsilon^{d_s(1/\eta-1)} = \hat{C}\epsilon^{d_s(1/\eta-2)} \ll \epsilon^{-2d_s}.$$

Note that $\epsilon^{-2d_s} \sim N^2 =$ total number of transitions $\theta_i\theta_j$.

We say that $\theta_U\theta_V$ is a prohibited transition iff the curve α intersects the rectangle $I_{\theta_U} \times I'_{\theta_V^t}$.

Consider the admissible word $\theta_i\theta_j\theta_k\theta_s$ with $\theta_i, \theta_j, \theta_k, \theta_s \in X_\epsilon$, this word generates an interval of size ϵ^4 in the construction of K^s.

We say that $\theta_i\theta_j\theta_k\theta_s$ is a prohibited word, if within there is a prohibited transition $\theta_U\theta_V$

$$\overbrace{-\,-\,-}^{\theta_i}\overbrace{-\,-\,-}^{\theta_j}\overbrace{-\,-\,-}^{\theta_k}\overbrace{-\,-\,-}^{\theta_s}$$

$$\underbrace{-\,-}_{\rho}\underbrace{-\,-\,-\,-\,-\,-}_{\theta_V\theta_V}\underbrace{-\,-\,-\,-}_{\beta}.$$

Denote by PW the set of the prohibited words $\theta_i\theta_j\theta_k\theta_s$. We want to now estimate $|PW|$. In fact: $|I_\rho||I_\beta| \sim \epsilon^2 \sim 2^{-2n}$, then there is $t \leq 2n$ such that $|I_\rho| \sim 2^{-t}$ and $|I_\beta| \sim 2^{t-2n}$. Thus, $\#\{I_\rho\} \sim (2^{-t})^{-d_s} = 2^{td_s}$ and $\#\{I_\beta\} \sim (2^{-(2n-t)})^{-d_s} = 2^{(2n-t)d_s}$, therefore for some constant $\chi > 1$ (as in the first part of the proof), we have that

$$|PW| \leq \chi(2n)2^{td_s}2^{(2n-t)d_s}\epsilon^{d_s(1/\eta-2)} \leq 2\chi \log\epsilon^{-1}\epsilon^{d_s(1/\eta-4)} \ll \epsilon^{-4d_s}$$

the last inequality is by $2\chi\log\epsilon^{-1}\epsilon^{d_s/\eta} \ll 1$.

Then, the total of prohibited words $\theta_i\theta_j\theta_k\theta_s$ is much less than $\epsilon^{-4d_s} \sim N^4$ the total of the words $\theta_i\theta_j\theta_k\theta_s$.
Consider $A = (a_{(i,j)(k,s)})$ for $(i,j), (k,s) \in \{1, ..., N\}^2$ the matrix defined by

$$a_{(i,j),(s,k)} = \begin{cases} 1 & \text{if } \theta_i\theta_j\theta_k\theta_s \text{ is not prohibited;} \\ \\ 0 & \text{if } \theta_i\theta_j\theta_k\theta_s \text{ is prohibited for some } \theta_U\theta_V. \end{cases}.$$

by the previous we have $\#\{a_{(i,j)(s,k)} : a_{(i,j)(s,k)} = 1\} \geq \frac{99}{100} N^{2^2}$, so by Lemma 23 holds that the trace of matrix A^m, $tr(A^m)$, satisfies

$$tr(A^m) \geq \left(\frac{N^2}{2}\right)^m \text{ for any } m \geq 2.$$

This implies that there are many transitions not prohibited.

Thus, there exists a subset B of N^2 such that, the submatrix $\widetilde{A} = (a_{(i,j)(k,s)})$ for $(i,j),(k,s) \in B$ has all nonzero entries, so each transition $\theta_i\theta_j\theta_k\theta_s$ with $(i,j),(k,s) \in B$ is not prohibited.

Put $\tilde{\theta}_{ij} = \theta_i\theta_j$ for $(i,j) \in B$.
Define $\widetilde{K}$ the regular Cantor set

$$\widetilde{K} := \{\cdots \tilde{\theta}_{i_1j_1}\tilde{\theta}_{i_2j_2}\cdots\tilde{\theta}_{i_nj_n}\cdots \ / \ (i_k, j_k) \in B\} \subset K^s.$$

Moreover, by Remark 14 we have $HD(\widetilde{K}) \sim K(X_\epsilon) \sim HD(K^s)$.

Consider the sub-horseshoe of Δ defined by $\Delta_\alpha^s := \bigcap_{n\in\mathbb{Z}} \mathcal{R}^n \left(\bigcup_{(i,j)\in B} \tilde{\theta}_{ij}\right)$. Since, $K_\alpha^s = \widetilde{K}$ the stable regular Cantor set, described by geometry transverse of the unstable foliation $W^u(\Delta_\alpha^s)$, then by the above we have that

$$HD(K_\alpha^s) \sim HD(K^s).$$

And by definition of Δ_α^s we have that $\Delta_\alpha^s \cap \alpha = \emptyset$.

□

4.1.3 Regaining the Spectrum

Given $F \in C^0(SM, \mathbb{R})$, we can define the function $maxF_\phi \colon \Xi \to \mathbb{R}$ by

$$maxF_\phi(x) := \max_{t_-(x)\leq t\leq t_+(x)} F(\phi^t(x))$$

where $t_-(x), t_+(x)$ are such that $\mathcal{R}^{-1}(x) = \phi^{t_-(x)}(x)$ and $\mathcal{R}(x) = \phi^{t_+(x)}(x)$.

Note that this definition depends on the geodesic flow ϕ^t, or equivalently the vector field ϕ. Note also that $maxF_\phi$ is always a continuous function, even if F is C^∞, $maxF_\phi$ can be only a continuous function. In what follows we try to give some "differentiability" to $maxF_\phi$ at least for $F \in C^2(SM, \mathbb{R})$ (see Lemma 25).

Consider the set

$$\mathcal{O} = \left\{F \in C^\infty(SM, \mathbb{R}) : maxF(x) = F(\phi^{t(x)}(x)) \text{ and } t_-(x) < t(x) < t_+(x) \text{ for all } x \in \Xi\right\}.$$

The set $\mathcal{O}$ is open and dense subset of $C^\infty(SM, \mathbb{R})$.

Let $x \in int(\Sigma)$ with $\Sigma \in \Xi$ such that $\mathcal{R}(x) = \phi^{t_x}(x) \in int(\Xi)$, by The Large Tubular Flow Theorem, there exists a neihgborhood $U_x \subset \Sigma$ of x a diffeomorphism $\varphi\colon U_x \times (-\epsilon, t_x+\epsilon) \to \varphi(U_x \times (-\epsilon, t_x+\epsilon)) \subset SM$ such that $D\varphi_{(z,t)}(0,0,1) = \phi(\varphi(z,t))$. Moreover, as the elements of the Markov partition are disjoint, has small diameter and Δ is compact, then, we can suppose that there is a finite number an open set U_{x_i} such that $U_{x_i} \cap U_{x_j} = \emptyset$ and $\Delta \subset \bigcup U_{x_i}$ for some $x_i \in \Delta$. Denote $\varphi_i\colon U_{x_i} \times (-\epsilon, t_{x_i}+\epsilon) \to \varphi_i\left(U_{x_i} \times (-\epsilon, t_{x_i}+\epsilon)\right) \subset SM$ such that $(D\varphi_i)_{(z,t)}(0,0,1) = \phi(\varphi_i(z,t))$.

Remark 15. *Let $F \in \mathcal{O}$, consider the function $f(x_1,x_2,x_3) = F \circ \varphi_i(x_1,x_2,x_3)$, we want to see the behavior of the critical points of $F \circ \varphi_i|_{\{z\}\times(-\epsilon,t_z+\epsilon)}$. Let δ be small regular value of $\frac{\partial f}{\partial x_3}(z_1,z_2,z_3)$, then $f_\delta(x_1,x_2,x_3) = f(x_1,x_2,x_3) - \delta x_3$ has 0 as regular value, so $\left(\frac{\partial f_\delta}{\partial x_3}\right)^{-1}(0) := S_\delta$ is a surface. Later we want this surface does not contain an open consisting of orbits of the flow.*
Also, observe that if $(0,0,1) \in T_zS_\delta$ for $z = (z_1,z_2,z_3)$, then $D\left(\frac{\partial f_\delta}{\partial x_3}\right)_z(0,0,1) = 0$, this implies that $\dfrac{\partial^2 f_\delta}{\partial x_3^2}(z) = 0$ so, $z \in \left\{x : \frac{\partial^2 f_\delta}{\partial x_3^2}(x) = 0\right\}$, so the critical points of $f_\delta|_{\{z\}\times(-\epsilon,t_z+\epsilon)}$ can not be non-degenerate.

Lemma 25. *There exists a dense $\mathcal{B}_\phi \subset C^\infty(SM,\mathbb{R})$ and C^2-open such that given $\beta > 0$, then for any $F \in \mathcal{B}_\phi$ there are sub-horseshoe $\Delta_F^{s,u}$ of Δ with $HD(K_F^s) \geq HD(K^s) - \beta$, $HD(K_F^u) \geq HD(K^u) - \beta$ (as in Lemma 24) and a Markov partition $R_F^{s,u}$ of $\Delta_F^{s,u}$, respectively, such that the function $maxF|_{\Xi\cap R_F^{s,u}} \in C^1(\Xi \cap R_F^{s,u},\mathbb{R})$, where $K_F^{s,u}$, $K^{s,u}$ as in Lemma 24.*

Proof. We prove the Lemma for Δ_F^s, for Δ_F^u is analogue.
Let $F \in C^\infty(SM,\mathbb{R})$ and $f = F \circ \varphi_i$ as above, by Remark 15, we want to perturb f by a f_δ such that $\left\{z : \frac{\partial f_\delta}{\partial x_3}(z) = 0\right\}$ and $\left\{z : \frac{\partial^2 f_\delta}{\partial x_3^2}(z) = 0\right\}$ are surface and

$$J_\delta(x_i) := \left\{z : \frac{\partial f_\delta}{\partial x_3}(z) = 0\right\} \pitchfork \left\{z : \frac{\partial^2 f_\delta}{\partial x_3^2}(z) = 0\right\}. \tag{4.4}$$

In fact:
Let δ be small regular value of $\frac{\partial^2 f}{\partial x_3^2}$, put $f_\delta(x_1,x_2,x_3) = f(x_1,x_2,x_3) - \dfrac{\delta x_3^2}{2} - cx_3$, so $\left\{z : \frac{\partial^2 f_\delta}{\partial x_3^2}(z) = 0\right\} := \widetilde{S}_\delta$ is surface for all $c \in \mathbb{R}$. Therefore, consider the function $\left(\frac{\partial f}{\partial x_3} - \delta x_3\right)|_{\widetilde{S}_\delta}$, $\frac{\partial f}{\partial x_3} - \delta x_3$ restrict to $\widetilde{S}_\delta$, thus taking c small a regular value of $\left(\frac{\partial f}{\partial x_3} - \delta x_3\right)|_{\widetilde{S}_\delta}$, so, we have that f_δ satisfies (4.4). Thus, by Remark 15 the surface $\left\{z : \frac{\partial f_\delta}{\partial x_3}(z) = 0\right\}$ does not contain an open consisting of orbits of the flow. Call α_{x_i} the projection of curve $J_\delta(x_i)$ along to the flow on Σ. Thus, considering the finite family of curve $\alpha := \{\alpha_{x_i}\}$, then by Lemma 24, given $\beta > 0$ small there is a sub-horseshoe Δ_α such that

$$HD(K_\alpha^s) \geq HD(K^s) - \beta$$

and $\Delta_\alpha \cap \alpha_i = \emptyset$.
For $x \in \Delta_\alpha$ holds that the critical points of $f_\delta|_{\{x\}\times(-\epsilon,t_x+\epsilon)}$ are non-degenerates and therefore finite. Thus, the critical points are locally graphs of a finite number of functions ψ_j,

that is, locally $\{(x_1, x_2, \psi_j(x_1, x_2)); 1 \leq j \leq k\}$.
Since we want the function $maxf_\delta$ be C^1, we have to rid of the points (x_1, x_2) such that for $k \neq j$

$$f_\delta(x_1, x_2, \psi_j(x_1, x_2)) = f_\delta(x_1, x_2, \psi_k(x_1, x_2)).$$

In fact:
Let $g_{kj}(x_1, x_2) = f_\delta(x_1, x_2, \psi_j(x_1, x_2)) - f_\delta(x_1, x_2, \psi_k(x_1, x_2))$ and let $\gamma > 0$ small regular value of g_{kj} for all $j \neq k$. Take ξ a C^∞-function close to the constant function 0 and equal to γ in neighborhood of $\{z = \psi_i(x_1, x_2)\}$ and 0 outside. So, the function $f_\delta + \xi$ is close to f_δ. Define the function

$$g_{kj}^\gamma(x_1, x_2) = (f_\delta + \xi)(x_1, x_2, \psi_j(x_1, x_2)) - (f_\delta + \xi)(x_1, x_2, \psi_k(x_1, x_2)) = g_{kj}(x_1, x_2) - \gamma.$$

Then, we have that $\gamma_{kj}^i := (g_{kj}^\gamma)^{-1}(0)$ is a curve in $U_i \subset \Sigma$. So, consider of finite family of curves $\Gamma = \{\gamma_{kj}^i\}$, then by Lemma 24 there is a sub-horseshoe Δ_Γ of Δ_α such that

$$HD(K_\Gamma^s) \geq HD(K_\delta^s) - \beta \geq HD(K^s) - 2\beta. \tag{4.5}$$

As the tube $U_i \times (-\epsilon, t_{x_i} + \epsilon)$ are disjoint and each is defined a function $f_\delta^i + \xi^i$ close to $f = F \circ \varphi_i$, then we have define a function $G \in \mathcal{O} \subset C^\infty(SM, \mathbb{R})$ close to F, with the following properties:
-$G = f_\delta + \xi^i$ on $U_i \times (-\epsilon, t_{x_i} + \epsilon)$ and $G = F$ outside of neighborhood of $\bigcup_i (U_i \times (-\epsilon, t_{x_i} + \epsilon))$.
-Take a Markov partition R_Γ of Δ_Γ with diameter small, then $maxG_\phi|_{\Xi \cap R_\Gamma}$ is a C^1-function.
The above holds by construction of G, which satisfies that the critical point of $G|_{\{x\} \times (-\epsilon, t_x + \epsilon)}$ is a unique point for $x \in R_\Gamma$, since $G \in \mathcal{O}$, we have the second item.
Note that by construction of G, we have that

$$\frac{\partial G}{\partial x_3}(x_1, x_2, \psi_k(x_1, x_2)) = 0 \text{ and } \frac{\partial^2 G}{\partial x_3^2}(x_1, x_2, \psi_k(x_1, x_2)) \neq 0 \text{ in } U_i \cap R_\Gamma.$$

And this condition implies that, if H is C^2 close to G, then there exists $\widetilde{\psi}_k$ C^1-close to ψ_k and holds

$$\frac{\partial H}{\partial x_3}(x_1, x_2, \widetilde{\psi}_k(x_1, x_2)) = 0 \text{ and } \frac{\partial^2 H}{\partial x_3^2}(x_1, x_2, \widetilde{\psi}_k(x_1, x_2)) \neq 0 \text{ in } U_i \cap R_\Gamma.$$

This last condition implies that there is a single maximum of $H|_{\{(x_1,x_2)\} \times (-\epsilon, t_{(x_1,x_2)} + \epsilon)}$, thus $max_\phi H|_{\Xi \cap R_\Gamma}$ is a C^1-function. □

Keeping the notation of the previous Lemma we have:

Corollary 9. *The above property is robust in the following sense: If X is a vector field C^1-close to ϕ, then $\mathcal{B}_\phi = \mathcal{B}_X$ and for any $F \in \mathcal{B}_X$, holds that $maxF_X \in C^1(\Xi \cap R_F^{s,u}, \mathbb{R})$.*

Proof. As time t_x of return of $x \in \Xi$ to Ξ by the flow ϕ is bounded, so for any vector field X sufficiently C^1-close to ϕ, then for $x \in \Xi$ the orbits $\phi^t(x)$ and $X^t(x)$ of the vector field ϕ and X respectively, are close.
Let φ, $\tilde{\varphi}$ be diffeomorphism given by The Large Tubular Flow Theorem (for ϕ and X,

respectively, as above), let $F \in C^{\infty}(SM, \mathbb{R})$, then $F \circ \varphi$, $F \circ \tilde{\varphi}$ are C^0-close, moreover $\frac{\partial}{\partial x_3} F \circ \varphi(x) = DF_{\varphi(x)} D\varphi_x(0,0,1) = DF_{\varphi(x)}(\phi(\varphi(x)))$ and $\frac{\partial}{\partial x_3} F \circ \tilde{\varphi}(x) = DF_{\tilde{\varphi}(x)} D\tilde{\varphi}_x(0,0,1) = DF_{\tilde{\varphi}(x)}(X(\tilde{\varphi}(x)))$, since ϕ is C^1-close of X, then $F \circ \varphi|_{\{(x_1,x_2)\}\times(a,b)}$ is C^2-close of $F \circ \tilde{\varphi}|_{\{(x_1,x_2)\}\times(a,b)}$.
Suppose that $F \in B_{\phi}$, then by Lemma 25 there is a horseshoe Δ_F of Δ and Markov partition R_F of Δ_F such that $maxF_{\phi}|_{\Xi\cap R_F}$ is C^1, thus by construction in proof of Lemma 25 we have that $maxF_X|_{\Xi\cap R_F}$ is C^1. □

The following proposition is found in ([MY10, pg. 21]).

Proposition 2. *Let Λ be a horseshoe and let $L \subset \Lambda$ an invariant proper subset of Λ. Then, for all $\epsilon > 0$, there is a sub-horseshoe $\widetilde{\Lambda} \subset \Lambda$ such that $\widetilde{\Lambda} \cap L = \emptyset$ and*

$$HD(\widetilde{K}) \geq HD(K) - \epsilon,$$

where, K, $\widetilde{K}$ are of regular cantor set that describe the geometry transverse of the stable foliation $W^s(\Lambda)$, $W^s(\widetilde{\Lambda})$, respectively.

Proof of theorem 4. Let $F \in B_{\phi}$, $\Delta_F^{s,u}$ and $R_F^{s,u}$ as in Lemma 25 with

$$HD(K_F^{s,u}) \geq HD(K^{s,u}) - \beta.$$

Put $L = \Delta_F^s \cap \Delta_F^u \subset \Delta_F^s$ a $\mathcal{R}$-invariant set, then by Proposition 2 (applied to $\mathcal{R}^{-1}$), there is a sub-horseshoe ${}_1\Delta_F^s$ of Δ_F^s such that ${}_1\Delta_F^s \cap L = \emptyset$, that implies ${}_1\Delta_F^s \cap \Delta_F^u = \emptyset$. Moreover,

$$HD({}_1K_F^s) \geq HD(K_F^s) - \beta \geq HD(K^s) - 2\beta.$$

Define the sub-horseshoe Δ_F of Δ by $\Delta_F := {}_1\Delta_F^s \cup \Delta_F^u$, denote this by $\Delta_F := ({}_1\Delta_F^s, \Delta_F^u)$, put $R_F := R_F^s \cup R_F^u$ and consider the open and dense set

$$H_1(\mathcal{R}, \Delta_F) = \left\{ f \in C^1(\Xi \cap R_F, \mathbb{R}) : \#M_f(\Delta_F) = 1 \ \ for \ \ z \in M_f(\Delta_F), \ D\mathcal{R}_z(e_z^{s,u}) \neq 0 \right\}.$$

as in the Section 2.3.
Let $f \in H_1(\mathcal{R}, \Delta_F)$, then there is a unique $z_f \in M_f(\Delta_F)$. Let $\widetilde{\Delta}_F$ be a sub-horseshoe of Δ_F as in Section 2.3, such that $HD(\widetilde{\Delta}_F) \sim HD(\Delta_F)$ and $z_f \notin \widetilde{\Delta}_F$.

Since ${}_1\Delta_F^s \cap \Delta_F^u = \emptyset$, we can suppose that $z_f \in {}_1\Delta_F^s$, thus as in Section 2.3, let ${}_1\widetilde{\Delta}_F^s$ sub-horseshoe of ${}_1\Delta_F^s$ and $z_f \notin {}_1\widetilde{\Delta}_F^s$, then

$$\widetilde{\Delta}_F = ({}_1\widetilde{\Delta}_F^s, \Delta_F^u).$$

Moreover, since $HD(K^s) + HD(K^u) = HD(\Delta) \sim 2$ and β is small, then

$$HD({}_1\widetilde{K}_F^s) + HD(K_F^u) > 1,$$

Where ${}_1\widetilde{K}_F^s$ is of regular cantor set that describe the geometry transverse of the unstable foliation $W^u({}_1\widetilde{\Delta}_F^s)$.

Hence, by [MY1] it is sufficient perturb ${}_1\Delta_F^s$ as in sub-section 4.1.1, to obtain property V (see Definition 3). Let $\underline{w} \in \Omega$ such that $(\mathcal{R}^{\underline{w}}, \widetilde{\Delta}_F^{\underline{w}})$ has the property V, where $\widetilde{\Delta}_F^{\underline{w}} = ({}_1\widetilde{\Delta}_{\underline{w}}, \Delta_F^u)$ and ${}_1\widetilde{\Delta}_{\underline{w}}$ is the continuation of the hyperbolic set ${}_1\widetilde{\Delta}_F^s$ for $\mathcal{R}^{\underline{w}}$, thus by Lemma 22, $\mathcal{R}^{\underline{w}} = \mathcal{R}_{G_{\underline{w}}}$, then $(\mathcal{R}_{G_{\underline{w}}}, \widetilde{\Delta}_F^{\underline{w}})$ has the property V so by Theorem 3 we have that

$$intM(f, \widetilde{\Delta}_F^{\underline{w}}) \neq \emptyset \ \text{ and } \ intL(f, \widetilde{\Delta}_F^{\underline{w}}) \neq \emptyset,$$

for any $f \in H_1(\mathcal{R}, \Delta_F)$.

Now by Corollary 9 the function $maxF_{G_{\underline{w}}}|_{\Xi \cap (R_F^s \cup R_F^u)}$ is C^1, using local coordinates as in Remark 15 respect to the field $G_{\underline{w}}$, we can find $g \in C^1(\Xi, \mathbb{R})$ such that

$$maxF_{G_{\underline{w}}}|_{\Xi \cap (R_F^s \cup R_F^u)}(x_1, x_2, x_3) + g(x_1, x_2) \in H_1(\mathcal{R}, \Delta_F).$$

Put $h(x_1, x_2, x_3) = F(x_1, x_2, x_3) + g(x_1, x_2)$, then $maxh_{G_{\underline{w}}} = maxF_{G_{\underline{w}}} + g \in H_1(\mathcal{R}, \Delta_F)$. Therefore, since $M(h, \widetilde{\Delta}_F^{\underline{w}}) = \left\{\sup_{n\in\mathbb{Z}} h(\mathcal{R}_{G_{\underline{w}}}^n(x)) : x \in \widetilde{\Delta}_F^{\underline{w}}\right\} \subset M(h, G_{\underline{w}})$. So,

$$intM(h, G_{\underline{w}}) \neq \emptyset.$$

Analogously, $L(h, \widetilde{\Delta}_F^{\underline{w}}) \subset L(h, G_{\underline{w}})$, therefore $intL(h, G_{\underline{w}}) \neq \emptyset$. □

An important observation is that the vector field $G_{\underline{w}}$ is not necessarily a geodesic field for some Riemannian metric near the initial Riemannian metric.

In the next section will prove a version of Theorem 4, but where the vector field X is the geodesic field to some Riemannian metric near the initial Riemannian metric.

4.2 The Interior of Spectrum for Geodesic Flow

The main problem to obtain X in Theorem 4 as being a geodesic field is independence in the perturbation of the diffeomorphism $\mathcal{R}$, to obtain property V (see Definition 3), *i.e.*, in the proof of Theorem 4 we could perturb $\mathcal{R}$ in each $R(\underline{a}) \in R_1$ without affecting the dynamics out.

If we want perturb $\mathcal{R}$ to obtain property V and still be an application of first return of the geodesic flow for Riemannian metric near the initial Riemannian metric. We must keep in mind that upsetting a metric in a neighborhood of a point in the manifold M, then we affect the metric (of Sasaki in SM) in the points of the fiber of the neighborhood perturbed, that is, if the metric is perturbed in U, then the Sasaki metric is perturbed in $\pi^{-1}(U)$, where $\pi\colon SM \to M$ is the canonical projection $\pi(x, v) = x$. Therefore, if we want to perturb $\mathcal{R}$ in $R(\underline{a}) \subset R_1$ as an application of first return of a geodesic flow, then such perturbation is not necessarily independent of $R(\underline{a})$.

So what we do is obtain a sub-horseshoe $\bar{\Delta}$ of Δ with $HD(\bar{\Delta}) > 1$ and such that the perturbation in the metric, induces a perturbation in $\mathcal{R}$ as an application of a first return of the geodesic flow for a metric near and that the perturbation be independent.

4.2.1 The Set of Geodesics With Transversal Self-Intersection

Let $(x_0, v_0) \in SM$ such that the geodesic $\pi(\phi^t(x_0, v_0)) = \gamma_{v_0}(t)$ (with $\gamma_{v_0}(0) = x_0$ and $\gamma'_{v_0}(0) = v_0$) has a point of transversal self-intersection, that is, there is $t_0 \in \mathbb{R}$ such that $\phi^{t_0}(x_0, v_0) \in \pi^{-1}(x_0)$ and $\{v_0, \gamma'_{v_0}(t_0)\}$ is basis of T_xM.

Remark: Since the Liouville measure is invariant by the geodesic flow and we are assuming that M have finite volume, then the set of geodesics with transverse self-intersection is not empty.

Let $\mathcal{L}$ be a section transverse to flow and to the fiber $\pi^{-1}(x_0)$, define the following function

$$\begin{array}{rccl} f: & \mathcal{L} \times \mathbb{R} & \longrightarrow & M \\ & ((x,v),t) & \longmapsto & \pi(\phi^t(x,v)) \end{array}$$

with $f((x_0, v_0), 0) = x_0$ and $f((x_0, v_0), t_0) = x_0$, let I_0, I_{t_0} are small intervals containing 0 and t_0 respectively. Denoted $f_0 = f|_{\mathcal{L}\times I_0}$ and $f_{t_0} = f|_{\mathcal{L}\times I_{t_0}}$.
Let $\varphi\colon U_0 \subset T_{x_0}M \to U_{x_0}$ normal coordinates in x_0, where U_{x_0} is neighborhood of x_0, that is, let $\{e_1, e_2\}$ be orthonormal basis of $T_{x_0}M$ and $\varphi(x_1, x_2) = \varphi(x_1e_1 + x_2e_2) = exp_{x_0}(x_1e_1 + x_2e_2)$.
We define

$$H\colon \mathcal{L} \times I_{t_0} \times I_0 \longrightarrow V_0 \subset T_{x_0}M$$

by $H((x,v),t,s) = \left(\varphi^{-1} \circ f_{t_0}\right)((x,v),t) - \left(\varphi^{-1} \circ f_0\right)((x,v),s)$ satisfies $H((x_0, v_0), t_0, 0) = 0$.
Then,

$$\begin{aligned} \frac{\partial H}{\partial t}((x_0,v_0),t_0,0) &= (D\varphi^{-1})_{f_{t_0}((x_0,v_0),t_0)}\left(\frac{\partial f_{t_0}}{\partial t}((x_0,v_0),t_0)\right) \\ &= (Dexp_{x_0}^{-1})_{x_0}(\gamma'_{v_0}(t_0)) = \gamma'_{v_0}(t_0) \end{aligned}$$

the last equality is due to the fact that $(Dexp_{x_0}^{-1})_{x_0} = Id\colon T_{x_0}M \to T_{x_0}M$ identity of $T_{x_0}M$. Also,

$$\begin{aligned} \frac{\partial H}{\partial s}((x_0,v_0),t_0,0) &= -(D\varphi^{-1})_{f_0((x_0,v_0),0)}\left(\frac{\partial f_0}{\partial s}((x_0,v_0),0)\right) \\ &= -(Dexp_{x_0}^{-1})_{x_0}(\gamma'_{v_0}(0)) = -(Dexp_{x_0}^{-1})_{x_0}(v_0) \\ &= -v_0. \end{aligned}$$

Since $\{-v_0, \gamma'_{v_0}(t_0)\}$ are linearly independent, then, $\frac{\partial H}{\partial(t,s)}$ is an isomorphism, therefore by the Implicit Function Theorem, there is and open $U_{\mathcal{L}}$ of (x_0, v_0) in $\mathcal{L}$ and a difeomorphism $\xi\colon U_{\mathcal{L}} \to V_{(t_0,0)}$ with $V_{(t_0,0)}$ open set containing $(t_0, 0)$ in $\mathbb{R}\times\mathbb{R}$ and $H((y,w), \xi(y,w)) = 0$. Without loss of generality we can assume that $V_{(t_0,0)} = \widetilde{I}_{t_0} \times \widetilde{I}_0$ and

$$\xi(y,w) = (\xi_1(y,w), \xi_2(y,w)),$$

with ξ_1 close to t_0 and ξ_2 close to 0, this implies

$$exp_{x_0}^{-1}(\pi(\phi^{\xi_1(y,w)}(y,w))) = exp_{x_0}^{-1}(\pi(\phi^{\xi_2(y,w)}(y,w))),$$

so $\pi(\phi^{\xi_1(y,w)}(y,w)) = \pi(\phi^{\xi_2(y,w)}(y,w))$ equivalently $\gamma_w(\xi_1(y,w)) = \gamma_w(\xi_2(y,w))$, where $\pi(\phi^t(y,w)) = \gamma_w(t)$ for any $(y,w) \in U_{\mathcal{L}}$.
Consider the new section transverse to flow

$$\widetilde{U}_{\mathcal{L}} = \left\{\phi^{\xi_2(y,w)}(y,w) : (y,w) \in U_{\mathcal{L}}\right\}.$$

Note that $\xi_1(x_0,v_0) = t_0$ and $\xi_2(x_0,v_0) = 0$, so $(x_0,v_0) \in \widetilde{U}_{\mathcal{L}}$.

Let $(x,v) \in \widetilde{U}_{\mathcal{L}}$, then there exists a unique $(y,w) \in U_{\mathcal{L}}$ such that $(x,v) = \phi^{\xi_2(y,w)}(y,w)$, so there exist a unique $\xi_1(y,w)$ such that

$$\begin{aligned} x = \pi(x,v) &= \pi(\phi^{\xi_2(y,w)}(y,w)) = \pi(\phi^{\xi_1(y,w)}(y,w)) \\ &= \pi\left(\phi^{\xi_1(y,w)-\xi_2(y,w)}\left(\phi^{\xi_2(y,w)}(y,w)\right)\right) \\ &= \pi\left(\phi^{\eta(y,w)}(x,v)\right), \end{aligned}$$

where $\eta(y,w) = \xi_1(y,w) - \xi_2(y,w)$ is close to t_0.
This implies that for any $(x,v) \in \widetilde{U}_{\mathcal{L}}$ there is $\eta(y,w)$ such that $\phi^{\eta(y,w)}(x,v) \in \pi^{-1}(x)$ and $\{v, \gamma'_v(\eta(y,w))\}$ are linearly independent.

Denote (*HPTSI* := Has a Point of Transverse Self-Intersection) and (*LI* := Linearly Independent), from the above we have that the set,

$$\begin{aligned} \mathcal{S} &= \{(x,v) \in SM : \exists \ \ t(x,v) \ \text{ such that } \ \gamma_v(t) \ \text{ HPTSI in } \ \gamma_v(t(x,v))\} \\ &= \{(x,v) \in SM : \exists \ \ t(x,v) \ \text{ such that } \gamma_v(t(x,v)) = x \ \text{ and } \{v, \gamma'_v(t(x,v))\} \text{ are } LI\ \} \end{aligned}$$

is a submanifold of SM of dimension 2.
Put $\mathcal{S}_n = \{(x,v) : \exists \ |t(x,v)| < n \ \text{ and } \ \gamma_v(t) \ \text{ HPTSI in } \ \gamma_v(t(x,v))\ \}$, $\mathcal{S}_n \subset \mathcal{S}_{n+1}$. Given $(x,v) \in \mathcal{S}_n$, there is a neighborhood U of (x,v) in $\mathcal{S}$ such that $U \subset \mathcal{S}_{n+1}$, therefore we can consider that $\mathcal{S}_n$ is a surface, submanifold of SM.

4.2.2 Perturbation of the Metric

In Lemma 17 was proven that there are GCS Σ_i such that

$$\Lambda \subset \bigcup_{i=1}^{m(l)} \phi^{(-2\epsilon,2\epsilon)}(int(\Sigma_i)),$$

with $\Sigma_i \cap \Sigma_j = \emptyset$.

Then, the hyperbolic set $\Delta = \bigcap_{n\in\mathbb{Z}} \mathcal{R}^n(\bigcup_i \Sigma_i)$ where $\mathcal{R}$ is the Poincaré map (map of first return) of $\bigcup_i \Sigma_i := \Xi$, satisfies by Lemma 21 that $d := HD(\Delta) \sim 2$.

The next purpose is to prove the following Lemma:

Lemma 26. *Let $\Delta_1 \subset \Delta$ a sub-horseshoe with $0 < HD(\Delta_1) := \lambda < \frac{1}{2}$, then there exists another sub-horseshoe Δ_2 of Δ with the following properties:*

1. *$\Delta_1 \cap \Delta_2 = \emptyset$ and $HD(K_2^u) \sim HD(K^u)$, where K_2^u, K^u are the regular Cantor sets that describe the geometry transverse of the unstable foliation $W^s(\Delta_2)$, $W^s(\Delta)$, respectively.*

2. *Δ_1 and Δ_2 are independent,* i.e., *there are Markov partitions $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of Δ_1 and Δ_2, respectively, such that*

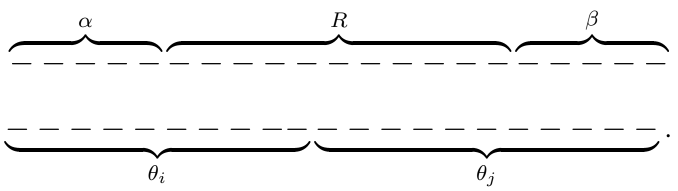
$$T_{R_2} \cap \tau_{R_1} = \emptyset \quad \textit{for any } R_1 \in \mathfrak{R}_1 \textit{ and } R_2 \in \mathfrak{R}_2,$$

where $T_{R_2} = \{\phi^t(x) : x \in R_2$ and $0 \le t \le t(x)\}$ and $\tau_{R_1} = \pi^{-1}(\pi(R_1))$.

Proof. Consider a Markov partition $\mathfrak{R}$ of Δ for squares of side ϵ (remember that $\mathcal{R}$ is conservative). Then, we have approximately ϵ^{-d} squares, where $d = HD(\Delta)$. Consider now the set $\widetilde{\mathfrak{R}}_{\Delta_1}$ of $\epsilon^{-\lambda}$ squares determined by Δ_1.
We say that a square $R_j \in \mathfrak{R}$ is prohibited for $R_i \in \mathfrak{R}$, if $T_{R_i} \cap \tau_{R_j} \neq \emptyset$. Observe also that, each square prohibits at most ϵ^{-1} other squares. Thus, each square of $\widetilde{\mathfrak{R}}_{\Delta_1}$, prohibit at most $\epsilon^{-\lambda} \cdot \epsilon^{-1} \le \epsilon^{-3/2}$, call these prohibited square and denote by $\mathfrak{R}_{\Delta_1}$. Therefore, we have $\ge [\epsilon^{-d} - \epsilon^{-3/2}] := N$ squares non prohibited.

The idea now is to understand what happens in the maximal invariant of these remaining squares.

Let $\{\theta_1, \ldots, \theta_N\}$ be the words associate with the remaining squares, which generate intervals of length ϵ^2 in $W^u(\Delta)$, in the construction of the unstable regular Cantor set. Without loss of generality we can assume that the transitions $\theta_i\theta_j$ is admissible for all $i, j \in \{1, \ldots, N\}$. A transition $\theta_i\theta_j$ is said prohibited, if there exists $R \in \mathfrak{R}_{\Delta_1}$ inside $\theta_i\theta_j$



Since the interval in $W^u(\Delta)$ generated by the word $\alpha\beta$ has length $\sim \epsilon^2$, then $\#\{\alpha\beta\} \sim N$. Also as the size of each word θ_i is $\sim \log \epsilon^{-d} \sim \log N$ (because $d > 3/2$), then each $R \in \mathfrak{R}_{\Delta_1}$ (prohibited square) prohibits $\le kN \log N$ transitions. So we have in total $\le k\epsilon^{-3/2} N \log N$ prohibited transition, where $k > 1$ is constant.

Affirmation : $\epsilon^{-3/2} N \log N \le C\epsilon^{-(d+3/2)} \log \epsilon^{-1} \ll N^2 = O(\epsilon^{-2d})$, for some constant $C > 0$.

In fact: Since $d > 3/2$, then

- $\lim_{\epsilon \to 0} \dfrac{\epsilon^{-3/2}(\epsilon^{-d} - \epsilon^{-3/2}) \log(\epsilon^{-d} - \epsilon^{-3/2})}{\epsilon^{-(d+3/2)} \log \epsilon^{-1}} = d,$

- $\lim_{\epsilon\to 0} \dfrac{\epsilon^{-(d+3/2)}\log \epsilon^{-1}}{(\epsilon^{-d}-\epsilon^{-3/2})^2} = 0$ and $\lim_{\epsilon\to 0}\dfrac{\epsilon^{-d}-\epsilon^{-3/2}}{\epsilon^{-d}} = 1.$

The previous affirmation says that the number of prohibited transitions is less than the total number of transitions. So, we consider the following matrix A for $i, j \in \{1, \dots, N\}$

$$a_{ij} = \begin{cases} 1 & \text{if } \theta_i\theta_j \text{ is not prohibited;} \\ \\ 0 & \text{if } \theta_i\theta_j \text{ is prohibited for some } R \in \mathfrak{R}_{\Delta_1} \end{cases}.$$

By the previous we have $\#\{a_{ij} : a_{ij} = 1\} \geq \frac{99}{100}N^2$, so by Lemma 23 holds that the trace of matrix A^m, $tr(A^m)$ satisfies

$$tr(A^m) \geq \left(\frac{N}{2}\right)^m \text{ for any } m \geq 2.$$

This implies that there are many transitions not prohibited.

Thus, there exists a subset $B \subset \{1, \dots, N\}$ such that, the submatrix $\widetilde{A} = (a_{ij})$ for $i, j \in B$ has all nonzero entries, so each transition $\theta_i\theta_j$ with $i, j \in B$ is not prohibited. Therefore, the sub-horseshoe $\Delta_2 = \bigcap_{n\in\mathbb{Z}} \mathcal{R}^n(\bigcup_{i\in B}\theta_i)$ satisfies the conditions 1 and 2 of lemma.

□

Now we create independence in the perturbation of $\mathcal{R}$ using the sub-horseshoe Δ_1.

Fix $n \in \mathbb{N}$ large and let $\mathcal{S}_n$ be as in the subsection 4.2.2. Since the transversality condition is open and dense, then we can suppose that the GCS Σ_i are transverse to surface $\mathcal{S}_n$. This last implies that $\alpha_n := \bigcup_i \Sigma_i \pitchfork \mathcal{S}_n$ is a finite family of smooth curves. Now by Lemma 24 applied to the family of curves α_n and the sub-horseshoe Δ_1, we have that given $\epsilon > 0$ there are sub-horseshoe Δ_0^s of Δ_1 such that $\Delta_0^s \cap \alpha = \emptyset$ for any $\alpha \in \alpha_n$ and

$$HD(K_0^s) \geq HD(K_1^s) - \epsilon, \tag{4.6}$$

where K_0^s, K_1^s are of regular Cantor sets that describe the geometry transverse of unstable foliation $W^u(\Delta_0^s)$, $W^u(\Delta_1)$, respectively.

Observe also that Δ_1 in Lemma 26 can be take such that

$$HD(K_1^s) \sim 1/4. \tag{4.7}$$

Remark 16. *By definition of Δ_0^s, we can take a Markov partition $\mathfrak{R}_0$ such that for each $R_a \in \mathfrak{R}_0$ there is a neihgborhood U_a of R_a with the property $\mathcal{R}^r(U_a) \cap \tau_{U_a} = \emptyset$ for $0 < r \leq \inf_x \sum_i t_i(x) < n$, $x \in \overline{U}_a$ and $\mathcal{R}^j(x) = \phi^{\sum_{i=0}^{j} t_i(x)}(x)$, where $\overline{U}_a$ is the closure of U_a and $\tau_{U_a} = \pi^{-1}(\pi(U_a))$.*

Let $\mathfrak{R}_0 = \{R_1, \dots, R_N\}$ a Markov partition by squares of side $\epsilon^{1/2}$ of Δ_0^s as in Remark 16. Note that since $\mathcal{R}$ is conservative, then the Markov partition $\mathfrak{R}_0$, generates N interval of size ϵ in $W^s(\Delta_0^s)$ and $W^u(\Delta_0^s)$ (in the construction of stable and unstable regular Cantor set). We call $\{\theta_1, \ldots, \theta_N\}$ the words associate to the intervals generated by $\mathfrak{R}_0$ in $W^s(\Delta_0^s)$ (in the construction of stable regular Cantor set). We said that the word θ_i prohibits the word θ_j, if $T_{\theta_i} \cap \tau_{\theta_j} \neq \emptyset$. Define $P_{\theta_i} = \{\theta_j : \theta_j$ is prohibited by $\theta_i\}$, then there is a constant $C > 0$ such that $\#P_{\theta_i} \leq CN^{1/2}$. Let $\alpha < 1/2$ and $0 < \delta \leq 1/10$, without loss of generality we can assume that the transitions $\theta_i\theta_j$ is admissible. Given θ_i, denote $P_{\theta_i} = \{\theta_{r_1(i)}, \cdots, \theta_{r_{p_i}(i)}\}$ the set of prohibited words by θ_i.

Definition 10. *We say that θ_i prohibits the transition $\theta_j\theta_k$, if within $\theta_j\theta_k$, appears some word $\theta_{r_l(i)} \in P_{\theta_i}$.*

Lemma 27. *Let $L = \{i : \theta_i$ prohibited more than δN transitions of type $\theta_i\theta_j$ or $\theta_j\theta_i\}$, then $\#L < \delta N$.*

Proof. From the definition of Δ_0^s and Remark 16, we can assume that if θ_i prohibited the transition $\theta_i\theta_j$, then there is a unique $\theta_{r_l(i)} \in P_{\theta_i}$ within $\theta_i\theta_j$. Each word θ_i prohibited no more than $\tilde{C}$ word, this is $|P_{\theta_i}| \leq \tilde{C}$ for any θ_i. Also, there is m' such that the number of possible beginnings with m letters of a word θ_j, is greater than or equal to λ^m, where $3/2 > \lambda > 1$ is uniform in m and $\lambda^{m'} \geq 100\tilde{C}$.
Given a word θ_i, then the proportion of θ_j such that the transition $\theta_i\theta_j$ is prohibited and such that the m first letters of θ_j with $m \geq m'$ is

$$\leq \sum_{m \geq m'} \frac{\tilde{C}}{\lambda^m} = \frac{\tilde{C}}{\lambda^{m'}} \sum_{j=0} \frac{1}{\lambda^j} \leq \frac{1}{100}\left(\frac{1}{1-\lambda^{-1}}\right) < \frac{1}{10}.$$

□

Lemma 28. *Let $f\colon \{1, ..., [N^\alpha]\} \to X_N := \{\theta_1, \ldots, \theta_N\} \setminus \{\theta_i : i \in L\}$ a random function, then the cardinal of the image f, $|Imf| > \frac{1}{2}N^\alpha$ with probability $1 - O_N(1)$.*

Proof. Given $P \subset X_N$ with cardinality $|P| = [\frac{1}{2}N^\alpha]$, then

$$\#\{g\colon \{1, ..., [N^\alpha]\} \to P\} \leq |P|^{N^\alpha} = \left[\frac{1}{2}N^\alpha\right]^{N^\alpha}.$$

Thus, using the fact $\binom{N}{k} \leq \frac{N^k}{k!} \leq (\frac{eN}{k})^k$ and $[\frac{x}{2}] \geq \frac{[x]}{4}$ for $x \geq 2$, we have

$$\begin{aligned}
\#\left\{f : |Imf| \leq \left[\frac{1}{2}N^\alpha\right]\right\} &\leq \binom{|X_N|}{[\frac{1}{2}N^\alpha]}\left[\frac{1}{2}N^\alpha\right]^{N^\alpha} \\
&\leq \left(\frac{e|X_N|}{[\frac{1}{2}N^\alpha]}\right)^{[\frac{1}{2}N^\alpha]}\left[\frac{1}{2}N^\alpha\right]^{N^\alpha} \\
&\leq \left(\frac{e|X_N|}{\frac{1}{4}|X_N|^\alpha}\right)^{\frac{1}{2}[N^\alpha]}\left(\frac{1}{2}N^\alpha\right)^{[N^\alpha]} \\
&= \left((4e)^{1/2}|X_N|^{\frac{1-\alpha}{2}}\frac{1}{2}N^\alpha\right)^{[N^\alpha]} \\
&< \left(2|X_N|^{\frac{1-\alpha}{2}}N^\alpha\right)^{[N^\alpha]} \lll |X_N|^{[N^\alpha]}
\end{aligned}$$

and $|X_N|^{[N^\alpha]}$ is the total number of functions f. □

The following question arises: Given $f\colon \{1, ..., [N^\alpha]\} \to X_N$ a random function, we want to know the probability of the transition $f(j)f(r)$ be prohibited by $f(i)$?

In fact: Given $i \in \{1, ..., [N^\alpha]\}$, such that $f(i)$ prohibited $\theta_j\theta_r$

$$\overbrace{-\,-\,-\,-\,-}^{\alpha}\overbrace{-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-}^{f(i)}\overbrace{-\,-\,-\,-\,-}^{\beta}$$

$$\underbrace{-\,-\,-\,-\,-\,-\,-\,-\,-\,-}_{\theta_j}\underbrace{-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-}_{\theta_r}.$$

Then, $\#\{\alpha\beta\} \sim N$ and since $\#\{P_{f(i)}\} \leq CN^{1/2}$, so

$$\#\{\theta_j\theta_r : \theta_j\theta_r \text{ is prohibited by } f(i)\} \leq CN^{1/2}N.$$

Therefore, given $f\colon \{1, ..., [N^\alpha]\} \to X_N$ a random function, then the probability of the transition $f(j)f(r)$ be prohibited by $f(i)$ is

$$P\,(\ f(j)f(r) \text{ be prohibited the transition } f(i)) \leq \frac{CN^{1/2}N}{|X_N|^2} < \frac{C}{(1-\delta)^2 N^{1/2}}. \tag{4.8}$$

So, since $\alpha < 1/2$, then the expected number of prohibited transitions is

$$< (N^\alpha)^3 . \frac{C}{(1-\delta)^2 N^{1/2}} = \frac{C}{(1-\delta)^2 N^{\frac{1}{2}-\alpha}} (N^\alpha)^2 \ll \left(\frac{N^\alpha}{2}\right)^2 \leq |Imf|^2.$$

That is, the expected number of prohibited transitions is $\ll |Imf|^2$.

We consider the complete regular Cantor set $K(Imf)$ and consider $A = (a_{ij})$ for $i, j \in \{1, ..., |Imf|\}$ the matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \theta_i\theta_j \text{ is not prohibited;} \\ \\ 0 & \text{if } \theta_i\theta_j \text{ is prohibited for some } \theta_k \in \ Imf \end{cases}.$$

By the previous we have $\#\{a_{ij} : a_{ij} = 1\} \geq \frac{99}{100}|Imf|^2$, so by Lemma 23 holds that the trace of matrix A^m, $tr(A^m)$, satisfies

$$tr(A^m) \geq \left(\frac{|Imf|}{2}\right)^m > \left(\frac{[N^\alpha]}{4}\right)^m \text{ for any } m \geq 2.$$

This implies that there are many transitions not prohibited.

Thus, there exists a subset B of Imf such that, the submatrix $\widetilde{A} = (a_{ij})$ for $i, j \in B$ has all nonzero entries, so each transition $\theta_i\theta_j$ with $\theta_i, \theta_j \in B$ is not prohibited.

Put $\widetilde{K}$ the regular Cantor set

$$\widetilde{K} := \{\cdots, \theta_{j_1}, \theta_{j_2}, \cdots, \theta_{j_k}, \cdots \ / \ \theta_{j_i} \in B\} \subset K^s.$$

Moreover, by Remark 14

$$HD(\widetilde{K}) \sim \frac{\log|Imf|}{-\log\epsilon} > \frac{\log(N)^{\alpha} - \log 2}{-\log\epsilon} \sim \alpha HD(K_0^s).$$

Thus,

$$HD(\widetilde{K}) \sim \alpha HD(K_0^s).$$

Consider the sub-horseshoe of Δ_0^s defined by $\Delta_3 := \bigcap_{n\in\mathbb{Z}} \mathcal{R}^n(\cup_{i\in B}\theta_i)$. Since, $K_3^s = \widetilde{K}$ the stable regular Cantor set, described by THE transverse geometry of unstable foliation $W^u(\Delta_3)$, then by the above we have that

$$HD(K_3^s) \sim \alpha HD(K_0^s). \tag{4.9}$$

As $HD(\Delta) \sim 2$, then $HD(K^u) \sim 1$, then by Lemma 26 the sub-horseshoe Δ_2 satisfies that $\Delta_2 \cap \Delta_3 = \emptyset$ and $HD(K_2^u) \sim 1$. Also, combining the equations (4.6), (4.7) and (4.9) we have that $HD(K_3^s) \sim \alpha\frac{1}{4}$. Therefore, since α can be take equal to $\frac{1}{2} - 4\epsilon$, with $\epsilon > 0$ small, then $HD(K_3^s) \sim \frac{1}{8} - \epsilon$, thus

$$HD(K_2^u) + HD(K_3^s) > 1.$$

From section 4.1, we described the family of perturbations given in [MY1 page 19-20], in which it is possible to obtain the property V (see Definition 3).

In [MY10] is proved that if $\mathcal{R}$ is a diffeomorphism with two horseshoe Δ_2, Δ_3 disjoint, we can perturb $\mathcal{R}$ in a Markov partition of Δ_3 without altering the dynamics in Δ_2 as in the section 4.1 and such that the new dynamics has a horseshoe with the property V (see Definition 3). Let us now make the perturbation $\mathcal{R}$ in the Markov partition $\{\theta_i : i \in B\}$ of Δ_3. Call this partition of $\mathfrak{R}_3$.

Let $i, j \in B$, since $\theta_i\theta_j$ is not prohibited, then $\pi(T_{\theta_i}) \cap \pi(\theta_j) = \emptyset$. This implies that, if we perturb of metric g in $\pi(\theta_j)$, then this perturbation is independent, *i.e*, the dynamics of $\mathcal{R}$ in θ_i for $j \neq i$ not changed. Also, the dynamic of $\mathcal{R}$ in Markov partition of Δ_2 given Lemma 26 also does not change. We want to perturb of metric g in a neighborhood of $\pi(\theta_i)$ for $i \in B$. Since the diameter of θ_i is sufficiently small, we can assume that $\pi(\theta_i)$ is contained in a normal coordinate system, *i.e.*, there is a point $p \in \pi(\theta_i)$, an orthonormal basis $\{e_1, e_2\}$ of T_pM and open set $\widetilde{U} \subset T_pM$ such that the function

$$\varphi \colon \widetilde{U} \to U_i$$

defined by $\varphi(x, y) = exp_p(xe_1 + ye_2)$ is a diffeomorphism and $\pi(\theta_i) \subset U_i$.

Let $g_{ij}(x, y)$ denote the components of the metric g in the chart $(\varphi, \widetilde{U})$. Let $\alpha^w(x, y)$ be a continuous family of C^∞ real function with support contained $\widetilde{U}$, C^∞-close to constant function 0 and $\alpha^0(x, y) \equiv 0$.
We can define a new Riemannian metric g_i^w by setting

$$\begin{aligned}(g_i^w)_{00}(x, y) &= g_{00}(x, y) + \alpha^w(x, y)\\ (g_i^w)_{ij}(x, y) &= g_{ij}(x, y) \quad (i, j) \neq (0, 0).\end{aligned}$$

For w small, the rectangles θ_i are transverse to geodesic flow ${}_i\phi^t_w$ of g^w_i, so denote $\mathcal{R}^w_i$ the Poincaré map of the geodesic flow ${}_i\phi^t_w$. Define the following application Φ^w_i on θ_i by

$$\Phi^w_i(x,v) := \mathcal{R}^{-1} \circ \mathcal{R}^w_i(x,v) \ \text{ for } \ (x,v) \in \theta_i.$$

Lemma 29. *If $(x,v) \in W^s_{\mathcal{R}}(z)$ with $z \in \Delta_3$, then $\Phi^w_i(x,v) \notin W^s_{\mathcal{R}}(z)$ for $w \neq 0$ small.*

Proof. Since $\mathcal{R}(W^s_{\mathcal{R}}(z)) \subset W^s_{\mathcal{R}}(\mathcal{R}(z))$, it suffices see that $\mathcal{R}^w_i(x,v) \notin W^s_{\mathcal{R}}(\mathcal{R}(z))$. □

We define the new metric g_w on M close to the metric g by

$$g_w(x) = \begin{cases} g^w_i(x) & \text{if } x \in U_i; \\ \\ g & \text{otherwise} \end{cases}.$$

Put $\Phi^w(x,v) := \Phi^w_i(x,v)$ if $(x,v) \in \theta_i$. This Lemma implies that the perturbation of $\mathcal{R}$, given by $\mathcal{R} \circ \Phi^w$ satisfies the condition on the family of perturbation to get the property V (cf. subsection 4.1.1).

Let us apply Lemma 25 to Δ_2 and Δ_3, we find a set dense $\mathcal{B}_\phi \in C^\infty(SM,\mathbb{R})$, C^2-open, such that given $\epsilon > 0$, then for any $F \in \mathcal{B}_\phi$, there are sub-horseshoes Δ^s_F of Δ_3 and Δ^u_F of Δ_2 with $HD(K^s_F) \geq HD(K^s_3) - \epsilon$ and $HD(K^u_F) \geq HD(K^u_2) - \epsilon$ (as in Lemma 24), also there are Markov partitions $R^{s,u}_F$ of $\Delta^{s,u}_F$, respectively, such that the function $maxF|_{\Xi \cap R^{s,u}_F} \in C^1(\Xi \cap R^{s,u}_F, \mathbb{R})$. Since $\Delta_2 \cap \Delta_3 = \emptyset$, the above implies that $\Delta^s_F \cap \Delta^u_F = \emptyset$ and

$$HD(\Delta^s_F) + HD(\Delta^u_F) > 1.$$

Hence, by [MY1] it is sufficient perturb Δ^s_F as in subsection 4.1.1, to obtain property V (see Definition 3). By Lemma 29 there is w small such that $(\mathcal{R}^w, \Delta^w_F)$ has the property V, where $\Delta^w_F = ((\Delta^s_F)^w, \Delta^u_F)$ and $(\Delta^s_F)^w$ is the continuation of the hyperbolic set Δ^s_F for $\mathcal{R}^w$.

Continuing analogously as in the proof of Theorem 4, we have the following theorem:

Theorem 5. *Let M be as above, then there is a metric g_0 close to g and a dense and C^2-open subset $\mathcal{H} \subset C^2(SM,\mathbb{R})$ such that for any $f \in \mathcal{H}$,*

$$intM(f,\phi_g) \neq \emptyset \text{ and } intL(f,\phi_g) \neq \emptyset,$$

where ϕ_{g_0} is the vector field defining the geodesic flow of the metric g_0.

Part II

Geometric Marstrand

Chapter 5

The Marstrand Theorem in Nonposisive Curvature

Contents

5.1 Introduction

Consider $\mathbb{R}^2$ as a metric space with a metric d. If U is a subset of $\mathbb{R}^2$, the diameter of U is $|U| = \sup\{d(x,y) : x, y \in U\}$ and, if $\mathcal{U}$ is a family of subsets of $\mathbb{R}^2$, the diameter of $\mathcal{U}$ is defined by

$$\|U\| = \sup_{U \in \mathcal{U}} |U|.$$

Given $s > 0$, the Hausdorff d-measure of a subset K of $\mathbb{R}^2$ is

$$m_s(K) = \lim_{\epsilon \to 0} \left(\inf_{\substack{\mathcal{U} \text{ covers } K \\ \|U\| < \epsilon}} \sum_{U \in \mathcal{U}} |U|^s \right).$$

In particular, when d is the Euclidean metric and $s = 1$, then $m = m_1$ is the Lebesgue measure. It is not difficult to show that there exists a unique $d_0 \geq 0$ for which $m_d(K) = +\infty$ if $d < d_0$ and $m_d(K) = 0$ if $d > d_0$. We define the Hausdorff dimension of K as $HD(K) = d_0$. Also, for each $\theta \in \mathbb{R}$, let $v_\theta = (\cos\theta, \sin\theta)$, L_θ the line in $\mathbb{R}^2$ through of the origin containing v_θ and $\pi_\theta : \mathbb{R}^2 \to L_\theta$ the orthogonal projection.

In 1954, J. M. Marstrand [Mar54] proved the following result on the fractal dimension of plane sets.

Theorem[Marstrand]: *If $K \subset \mathbb{R}^2$ such that $HD(K) > 1$, then $m(\pi_\theta(K)) > 0$ for m-almost every $\theta \in \mathbb{R}$.*

The proof is based on a qualitative characterization of the "bad" angles θ for which the result is not true.

Many generalizations and simpler proofs have appeared since. One of them came in 1968 by R. Kaufman, who gave a very short proof of Marstrand's Theorem using methods of potential theory. See [Kau68] for his original proof and [PT93], [Fal85] for further discussion. Another recent proof of the theorem (2011), which uses combinatorial techniques is found in [LM11].

In this article, we consider M a simply connected surface with a Riemannian metric of non-positive curvature, and using the potential theory techniques of Kaufman [Kau68], we show the following more general version of the Martrand's Theorem.

The Geometric Marstrand Theorem: *Let M be a Hadamard surface, let $K \subset M$ and $p \in M$, such that $HD(K) > 1$, then for almost every line l coming from p, we have $\pi_l(K)$ has positive Lebesgue measure, where π_l is the orthogonal projection on l.*

Then using the Hadamard's theorem (cf. [PadC08]), the theorem above can be stated as follows:

Main Theorem: *Let $\mathbb{R}^2$ be with a metric g of non-positive curvature, let $K \subset \mathbb{R}^2$ be with $HD(K) > 1$, then for almost every $\theta \in (-\pi/2, \pi/2)$, we have that $m(\pi_\theta(K)) > 0$, where π_θ is the orthogonal projection with the metric g on the line l_θ, of initial velocity $v_\theta = (\cos\theta, \sin\theta) \in T_p\mathbb{R}^2$.*

5.2 Preliminaries

Let M be a Riemannian manifold with metric $\langle \ , \ \rangle$, a line in M is a geodesic defined for all parameter values and minimizing distance between any of its points, that is, $\gamma : \mathbb{R} \to M$ is a isometry. If M is a manifold of dimension n, simply connected and non-positive curvature, then the space of lines leaving of a point p can be seen as a sphere of dimension $n-1$. So, in the case of surfaces the set of lines agrees with S^1 in the space tangent T_pM of the point p. Therefore, in each point on the surface the set of lines can be oriented and parametrized by $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, we can talk about almost every line in the point with the Lebesgue measure using the above identification (cf. [BH99]). In the conditions above, Hadamard's theorem states that M is diffeomorphic to $\mathbb{R}^n$, (cf. [PadC08]).

Moreover, given a geodesic triangle ΔABC with sides , $\vec{BC}$ and $\vec{AC}$ denote by $\angle A$ the angle between geodesic segments $\vec{AB}$ and $\vec{AC}$, then *the law of cosines* says

$$|\vec{BC}|^2 \geq |\vec{AB}|^2 + |\vec{AC}|^2 - 2|\vec{AB}||\vec{AC}|\cos\angle A,$$

where $|\vec{ij}|$ is the distance between the points i, j for $i, j \in \{A, B, C\}$.

Gauss's Lemma: Let $p \in M$ and let $v, w \in B_\epsilon(0) \in T_vT_pM \approx T_pM$ and $M \ni q = exp_pv$. Then,

$$\langle d(exp_p)_v v, d(exp_p)_v w\rangle_q = \langle v, w\rangle_p .$$

5.2.1 Projections

Let M be a manifold simply connected and of non-positive curvature. Let C be a complete convex set in M. *The orthogonal projection* (or simply '*projection*') is the name given to the map $\pi\colon M \to C$ constructed in the follows, (cf. [BH99, pp 176]).

Proposition 3. *The projection π satisfies the following properties:*

1. *For any $x \in M$ there is a unique point $\pi(x) \in C$ such that $d(x, \pi(x)) = d(x, C) = \inf_{y\in C} d(x, y)$.*
2. *If x_0 is in the geodesic segment $[x, \pi(x)]$, then $\pi(x_0) = \pi(x)$.*
3. *Given $x \notin C$, $y \in C$ and $y \neq \pi(x)$, then $\angle_{\pi(x)}(x, y) \geq \frac{\pi}{2}$.*
4. *$x \longmapsto \pi(x)$ is a retraction on C.*

Corollary 10. *Let M, C be as above and define $d_C(x) := d(x, C)$, then*

1. *d_C is a convex function, this is, if $\alpha(t)$ is a geodesic parametrized proportional to arc length, then*

$$d_C(\alpha(t)) \leq (1-t)d_C(\alpha(0)) + td_C\alpha(1) \quad para \quad t \in [0, 1].$$

2. *For all $x, y \in M$, has $|d_C(x) - d_C(y)| \leq d(x, y)$.*
3. *The restriction of d_C the sphere of center x and radius $r \leq d_C(x)$ reaches the* inf *in a unique point y with*

$$d_C(x) = d_C(y) + r.$$

Here we consider $\mathbb{R}^2$ with a Riemannian metric g, such that the curvature $K_{\mathbb{R}^2}$ is non-positive, *i.e.*, $K_{\mathbb{R}^2} \leq 0$. Recall that a line γ in $\mathbb{R}^2$ is a geodesic defined for all parameter values and minimizing distance between any of its points, that is, $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ and $d(\gamma(t), \gamma(s)) = |t - s|$, where d is the distance induced by the Riemannian metric g, in other words, a parametrization of γ is a isometry. Then, given $x \in \mathbb{R}^2$ $\exists!$ $\gamma(t_x)$ such that $\pi_\gamma(x) = \gamma(t_x)$, then without loss of generality we may call $\pi_\gamma(x) = t_x$.

Fix $p \in \mathbb{R}^2$ and put $\{e_1, e_2\}$ a positive orthogonal basis of $T_p\mathbb{R}^2$, *i.e.*, the basis $\{e_1, e_2\}$ has the induced orientation of $\mathbb{R}^2$. Then, call $v_t = (\cos t, \sin t)$ in coordinates the unit vector $(\cos t)e_1 + (\sin t)e_2 \in T_p\mathbb{R}^2$. Denote by l_t the line through p with velocity v_t, given by $l_t(s) = exp_p s v_t$ and by π_t the projection on l_t. Then, given $\theta \in [0, 2\pi)$, we can define $\pi\colon [0, 2\pi) \times T_p\mathbb{R}^2 \to \mathbb{R}$ by the unique parameter s such that $\pi_\theta(exp_p w) = \exp_p s v_\theta$ *i.e.*, $\pi(\theta, w) := \pi_\theta(w)$ and

$$\pi_\theta(exp_p w) = exp_p \pi(\theta, w) v_\theta.$$

5.3 Behavior of the Projection π

In this section we will prove some lemmas that will help to understand the projection π.

5.3.1 Differentiability of π in θ and w.

Lemma 30. *The projection π is differentiable in θ and w.*

Proof. Fix w and call $q = exp_{\gamma(0)} w = exp_p w$. Let $\alpha_v(t) \subset T_q\mathbb{R}^2$ such that $exp_q \alpha_v(t) = \gamma_v(t)$, where γ_v is the line such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$, then, for all $v \in S^1$, $\exists!\ t_v$ such that $d(q, \gamma_v(\mathbb{R})) = d(q, \gamma_v(t_v))$ and satisfies

$$\left\langle d(exp_q)_{\alpha_v(t_v)}(\alpha_v'(t_v)), d(exp_q)_{\alpha_v(t_v)}(\alpha_v(t_v)) \right\rangle = \left\langle \gamma_v'(t_v), d(exp_q)_{\alpha_v(t_v)}(\alpha_v(t_v)) \right\rangle = 0.$$

By Gauss Lemma, we have

$$\frac{1}{2}\frac{\partial}{\partial t} \|\alpha_v(t)\|^2 (t_v) = \langle \alpha_v'(t_v), \alpha_v(t_v) \rangle = 0.$$

We define the real function

$$\begin{aligned} \eta : S^1 &\times \mathbb{R} \longrightarrow \mathbb{R} \\ \eta(v, t) &= \frac{1}{2}\frac{\partial}{\partial t} \|\alpha_v(t)\|^2, \end{aligned}$$

we have $\eta(v, t) \in C^\infty$ and satisfies $\eta(v_0, t_{v_0}) = 0$, also $\frac{\partial}{\partial t}\eta(v, t) = \frac{1}{2}\frac{\partial^2}{\partial t^2} \|\alpha_v(t)\|^2$.

Put $g(t) = \|\alpha_{v_0}(t)\|^2$, then $\frac{\partial}{\partial t}\eta(v_0, t_0) = \frac{1}{2}g''(t_0)$. Also, $g(t) = d(q, \gamma_{v_0}(t))^2$ is differentiable and has a global minimum at t_{v_0}, as $K_{\mathbb{R}^2} \leq 0$, g is convex. In fact, for $s \in [0, 1]$

$$\begin{aligned} g(sx + (1-s)y) &= d(q, \gamma_{v_0}(sx + (1-s)y))^2 \leq (sd(q, \gamma_{v_0}(x)) + (1-s)d(q, \gamma_{v_0}(y)))^2 \\ &\leq sd(q, \gamma_{v_0}(x))^2 + (1-s)d(q, \gamma_v(y))^2 = sg(x) + (1-s)g(y) \end{aligned}$$

by the law of cosines and using the fact $\angle_{\pi_{\gamma_{v_0}(t_0)}}(q, \gamma_{v_0}(\mathrm{t})) = \frac{\pi}{2}$ at the point of projection

$$d(q, \gamma_{v_0}(t_{v_0}))^2 + d(\gamma_{v_0}(t_{v_0}), \gamma_{v_0}(t))^2 \leq d(q, \gamma_{v_0}(t))^2$$

equivalently

$$g(t_{v_0}) + (t - t_{v_0})^2 \leq g(t).$$

Therefore, as $g'(t_{v_0}) = 0$, then $g''(t_{v_0}) > 0$. This implies $\frac{\partial \eta}{\partial t}(v_0, t_0) \neq 0$ and by Theorem of Implicit Functions, there is an open U containing (v_0, t_{v_0}), a open $V \subset S^1$ containing v_0 and $\xi : V \longrightarrow \mathbb{R}$, a class function C^∞ with $\xi(v_0) = t_{v_0}$ such that

$$\{(v, t) \in U : \eta(v, t) = 0\} \Longleftrightarrow \{v \in V : t = \xi(v)\}.$$

If $\eta(v,\xi(v)) = 0$ implies $\pi(v,q) = \xi(v)$ is differentiable, in fact it is C^∞. The above shows that π is differentiable in θ.

Analogously, is proven that π is differentiable in w.

□

Let $w \in T_p\mathbb{R}^2 \setminus \{0\}$ and put $\theta_w^\perp \in [0, 2\pi)$ such that w and $v_{\theta_w^\perp}$ are orthogonal, that is $\langle w, v_{\theta_w^\perp} \rangle = 0$, where the $\langle\ ,\ \rangle$ is the inner product in $T_p\mathbb{R}^2$ and the set $\{w, v_{\theta_w^\perp}\}$ is a positive basis of $T_p\mathbb{R}^2$.

Lemma 31. *The projection π satisfies,*

$$\frac{\partial \pi}{\partial \theta}\left(\theta_w^\perp, w\right) = -\left\|w\right\|.$$

Moreover, there exists $\epsilon > 0$ such that, for all w

$$-\left\|w\right\| \leq \frac{\partial \pi}{\partial \theta}\left(\theta, w\right) \leq -\frac{1}{2}\left\|w\right\| \ \ \textit{and} \ \ \left|\frac{\partial^2 \pi}{\partial^2 \theta}\left(\theta, w\right)\right| \leq \left\|w\right\|,$$

whenever $\left|\theta - \theta_w^\perp\right| < \epsilon$.

Before proving Lemma 31 we will seek to understand the function $\pi(\theta, w)$.

Let π_{l_θ} be the projection on the line l_θ generated by the vector v_θ in $T_p\mathbb{R}^2$, in this case, $\pi_{l_\theta}(w) = \|w\| \cos(arg(w) - \theta)$, where $arg(w)$ is the argument of w with relation to e_1 and the positivity of basis $\{e_1, e_2\}$.
Now using the law of cosines

$$d(p, \pi_\theta(exp_p w))^2 + d(exp_p w, \pi_\theta(exp_p w))^2 \leq \|w\|^2 = \pi_{l_\theta}(w)^2 + d(\pi_{l_\theta}(w)v_\theta, w)^2,$$

Since, $K \leq 0$, then $d(exp_p w, \pi_\theta(exp_p w)) \geq d(w, \pi_\theta(w)v_\theta) \geq d(w, \pi_{l_\theta}(w)v_\theta)$.
Joining the previous expressions we obtain

$$d(p, \pi_\theta(exp_p w))^2 \leq \pi_{l_\theta}(w)^2 \iff \pi_\theta(w)^2 \leq \pi_{l_\theta}(w)^2.$$

Thus, since $\pi_\theta(w)$ has the same sign as $\pi_{l_\theta}(w)$, then

$$\pi_\theta(w) \geq 0 \implies \pi_\theta(w) \leq \pi_{l_\theta}(w) = \|w\| \cos(arg(w) - \theta); \tag{5.1}$$
$$\pi_\theta(w) \leq 0 \implies \pi_\theta(w) \geq \pi_{l_\theta}(w) = \|w\| \cos(arg(w) - \theta). \tag{5.2}$$

Proof of Lemma 31.
As $\langle w, \theta_w^\perp \rangle = 0$, then $arg(w) - \theta_w^\perp = \pi/2$, thus $\pi(\theta_w^\perp, w) = 0 = \|w\| \cos(\pi/2)$. Moreover, as $\pi(\theta_w^\perp - h, w) \geq 0$ and $\pi(\theta_w^\perp + h, w) \leq 0$ for $h > 0$ small, then

$$\frac{\pi(\theta_w^\perp - h, w)}{h} \leq \frac{\|w\| \cos\left(arg(w) - (\theta_w^\perp - h)\right)}{h}$$

and

$$\frac{\pi(\theta_w^\perp + h, w)}{h} \geq \frac{\|w\| \cos\left(arg(w) - (\theta_w^\perp + h)\right)}{h}.$$

If $h \to 0$ in the two previous inequalities we have

$$\frac{\partial \pi}{\partial \theta}(\theta_w^{\perp}, w) \leq -\|w\| \sin(arg(w) - \theta_w^{\perp}) = -\|w\|$$

and

$$\frac{\partial \pi}{\partial \theta}(\theta_w^{\perp}, w) \geq -\|w\| \sin(arg(w) - \theta_w^{\perp}) = -\|w\|.$$

Therefore,

$$\frac{\partial \pi}{\partial \theta}(\theta_w^{\perp}, w) = -\|w\|. \tag{5.3}$$

Moreover, for $h > 0$ small and by the equation (5.2), we have

$$\begin{aligned}
\pi(\theta_w^{\perp} + h, w) &= \frac{\partial \pi}{\partial \theta}(\theta_w^{\perp}, w)h + \frac{1}{2}\frac{\partial^2 \pi}{\partial^2 \theta}(\theta_w^{\perp}, w)h^2 + r(h) \\
&\geq \|w\| \cos\big(arg(w) - (\theta_w^{\perp} + h)\big) \\
&= \|w\| \left(\frac{\partial}{\partial \theta} \cos(\theta - arg(w))|_{\theta_w^{\perp}} h + \frac{1}{2}\frac{\partial^2}{\partial^2 \theta} \cos(\theta - arg(w))|_{\theta_w^{\perp}} h^2 + R(h) \right).
\end{aligned}$$

The above inequality implies that

$$\frac{\partial^2 \pi}{\partial^2 \theta}(\theta_w^{\perp}, w)h^2 + r(h) \geq \|w\| \left(\frac{\partial^2}{\partial^2 \theta} \cos(\theta - arg(w))|_{\theta_w^{\perp}} h^2 + R(h) \right).$$

Since $\frac{\partial^2}{\partial^2 \theta} \cos(\theta - arg(w))|_{\theta_w^{\perp}} = 0$, then $\frac{\partial^2 \pi}{\partial^2 \theta}(\theta_w^{\perp}, w) \geq 0$. Analogously, using $\pi(\theta_w^{\perp} - h, w)$ and equation (5.1) we have that $\frac{\partial^2 \pi}{\partial^2 \theta}(\theta_w^{\perp}, w) \leq 0$. So,

$$\frac{\partial^2 \pi}{\partial^2 \theta}(\theta_w^{\perp}, w) = 0. \tag{5.4}$$

Using Taylor's expansion of third order for $\pi(\theta_w^{\perp} + h, w)$ and $h > 0$, the equations (5.2) and (5.4) and the fact that $\frac{\partial^3}{\partial^3 \theta} \cos(\theta - arg(w))|_{\theta_w^{\perp}} = 1$, implies that

$$\frac{\partial^3 \pi}{\partial^3 \theta}(\theta_w^{\perp}, w)\frac{h^3}{6} + r_3(h) \geq \frac{h^3}{6} + R_3(h).$$

Thus,

$$\frac{\partial^3 \pi}{\partial^3 \theta}(\theta_w^{\perp}, w) \geq 1. \tag{5.5}$$

EquationS (5.4) and (5.5) implies that, for any $w \in T_p\mathbb{R}^2$, the function $\frac{\partial \pi}{\partial \theta}(\cdot, w)$ has a minimum in $\theta = \theta_w^{\perp}$, therefore there is $\epsilon_1 > 0$ such that

$$-\|w\| \leq \frac{\partial \pi}{\partial \theta}(\theta, w) \quad \text{for all} \quad \left|\theta - \theta_w^{\perp}\right| < \epsilon_1. \tag{5.6}$$

The lemma will be proved if we show the following statements:

1. There is $\delta_1 > 0$, such that for all $\|w\| \geq 1$,
$$\frac{\partial \pi}{\partial \theta}(\theta, w) \leq -\frac{1}{2}\|w\|, \quad \text{whenever } \left|\theta - \theta_w^{\perp}\right| < \delta_1.$$
In fact: Let $1/2 > \beta > 0$, then by continuity of $\frac{\partial \pi}{\partial \theta}$, there is δ_1 such that
$$\text{if } \left|\theta - \theta_w^{\perp}\right| < \delta_1, \quad \text{then } \frac{\partial \pi}{\partial \theta}(\theta, w) - \frac{\partial \pi}{\partial \theta}(\theta_w^{\perp}, w) < \beta.$$
Thus, $\frac{\partial \pi}{\partial \theta}(\theta, w) < \beta - \|w\| < -\frac{1}{2}\|w\|$ for any $\|w\| \geq 1$.

2. There is $\epsilon_2 > 0$, such that for all $\|w\| = 1$ and $t \in [0, 1]$
$$\frac{\partial \pi}{\partial \theta}(\theta, tw) \leq -\frac{1}{2}t, \quad \text{whenever } \left|\theta - \theta_w^{\perp}\right| < \epsilon_2.$$
In fact: Suppose by contradiction that for all $n \in \mathbb{N}$, there are w_n, t_n, θ_n, $\|w_n\| = 1$ such that $\left|\theta_{w_n}^{\perp} - \theta_n\right| < \frac{1}{n}$ and $\frac{\partial \pi}{\partial \theta}(\theta_n, t_n w_n) > -\frac{1}{2}t_n$. Without loss of generality, we can assume that $w_n \to w$, $\theta_n \to \theta_w^{\perp}$ and $t_n \to t$. If $t \neq 0$, the above implies a contradiction. Thus, suppose that $t = 0$, then consider the C^1-function $H(\theta, t, w) = \frac{\partial \pi}{\partial \theta}(\theta, tw)$, then $\frac{\partial H}{\partial t}(\theta_w^{\perp}, 0, w) = -\|w\| = -1$. Since H is C^1, then
$$\lim_{n\to\infty} \frac{H(\theta_n, t_n, w_n)}{t_n} = \lim_{t \to 0} \frac{H(\theta, t, w)}{t} = -1 < -1/2 \leq \lim_{n\to\infty} \frac{H(\theta_n, t_n, w_n)}{t_n}.$$
Which is absurd, so the assertion 2 is proved.

Take $\epsilon = \min\{\epsilon_1, \epsilon_2, \delta_1\}$, then by the equation (5.6) and the statements 1 and 2 we have the second part of Lemma 31. The third part is analogous, just consider that $\frac{\partial^2 \pi}{\partial^2 \theta}(\theta_w^{\perp}, w) = 0$. So we conclude the proof of Lemma. □

Lemma 32. *Let $w \neq 0$ and $\theta \neq \theta_w^{\perp}$, then* $\lim_{t\to 0^+} \frac{\pi_\theta(tw)}{t} \neq 0$.

proof. Suppose that $\lim_{t\to 0^+} \frac{\pi_\theta(tw)}{t} = 0$, put $w(t) = exp_p tw$, let $v(t) \in T_{w(t)}\mathbb{R}^2$ the unit vector such that $exp_{w(t)} s(t)v(t) = \pi_\theta(exp_p tw)$ for some $s(t) \geq 0$. Let $J(t) \in T_{w(t)}\mathbb{R}^2$ such that $exp_{w(t)} J(t) = p$, that is $J(t) = -d(exp_p)_{tw} w$. Then, putting $\alpha(t)$ the oriented angle between $v(t)$ and $J(t)$ (cf. Figure 5.1).

By the law of cosines and using that $d(p, w(t)) = \|J(t)\| = t\|w\|$ for $t > 0$, and $\pi_\theta(tw) = d(p, \pi_\theta(w(t)))$, we obtain

$$\pi_\theta(tw)^2 \geq \|J(t)\|^2 + d(w(t), \pi_\theta(w(t)))^2 - 2\|J(t)\| d(w(t), \pi_\theta(w(t))) \cos\alpha(t).$$

Put $\lim_{t\to 0^+} \frac{d(w(t), \pi_\theta(w(t)))}{t} = B$, then dividing by t^2 and when $t \to 0$ we have

$$\begin{aligned} 0 = \left(\lim_{t\to 0^+} \frac{\pi_\theta(tw)}{t}\right)^2 &\geq \|w\|^2 + B^2 - 2\|w\| B \lim_{t\to 0^+} \cos\alpha(t) \\ &\geq \|w\|^2 + B^2 - 2\|w\| B = (\|w\| - B)^2 \geq 0. \end{aligned}$$

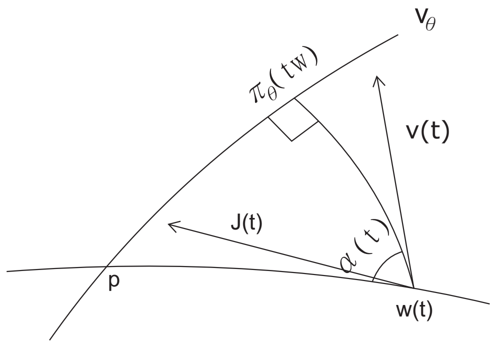


Figure 5.1: Convergence of geodesics

Thus, we conclude that $B = \|w\|$ and $\lim_{t\to 0^+} \cos\alpha(t) = 1$. Therefore, $\lim_{t\to 0^+} \alpha(t) = 0$, this implies the following geodesic convergence

$$exp_{w(t)} sv(t) \overset{t\to 0^+}{\longrightarrow} exp_p s\frac{-w}{\|w\|},$$

given that $w(t) \to 0$ and $v(t) \to -\frac{w}{\|w\|}$ when $t \to 0^+$.
Moreover, by definition of $s(t)$, we have that

$$\left\langle d\left(exp_{w(t)}\right)_{s(t)v(t)} v(t), d\left(exp_p\right)_{\pi_\theta(tw)v_\theta} v_\theta)\right\rangle_{\pi_\theta(tw)v_\theta} = 0,$$

when $t \to 0^+$ and using the fact that $d(exp_p)_0 = I$, where I is the identity of $T_p\mathbb{R}^2$, we conclude that $\left\langle -\frac{w}{\|w\|}, v_\theta\right\rangle = 0$ and this is a contradiction as $\theta \neq \theta_w^\perp$. □

Now we subdivide $T_p\mathbb{R}^2$ in three regions: Consider ϵ given by the Lemma 31, then

$$\begin{aligned}
R_1 &= \left\{ w \in T_p\mathbb{R}^2 \colon \text{the angle } \angle(w, e_1) \leq \frac{\pi}{2} - \frac{3}{2}\epsilon \text{ and } \angle(w, e_1) \geq \frac{3\pi}{2} + \frac{3}{2}\epsilon \right\};\\
R_2 &= \left\{ w \in T_p\mathbb{R}^2 \colon \text{the angle } \frac{\pi}{4} + \frac{3}{2}\epsilon \leq \angle(w, e_1) \leq \frac{5\pi}{4} - \frac{3}{2}\epsilon \right\};\\
R_3 &= \left\{ w \in T_p\mathbb{R}^2 \colon \text{the angle } \frac{3\pi}{4} + \frac{3}{2}\epsilon \leq \angle(w, e_1) \leq \frac{7\pi}{4} - \frac{3}{2}\epsilon \right\}.
\end{aligned}$$

For $w \in T_p\mathbb{R}^2$, putting $a_w^\perp = \theta_w^\perp - \epsilon$ and $\tilde{a}_w^\perp = \theta_w^\perp + \epsilon$, where ϵ is given in Lemma 31.

Lemma 33. *For the function $\pi_\theta(w)$ we have that*

1. (a) *There is $C_1 > 0$ such that for all $w \in R_1$ with $\|w\| = 1$ and all $t \in [0, 1]$ we have*

$$\pi_\theta(tw) \geq C_1 t \quad \textit{for} \quad \theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi].$$

(b) *There is* $C_1' > 0$ *such that for all* $w \in R_1$ *with* $\|w\| \geq 1$

$$\pi_\theta(w) \geq C_1' \quad \textit{for} \quad \theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi].$$

2. (a) *There is* $C_2 > 0$ *such that for all* $w \in R_2$ *with* $\|w\| = 1$ *and all* $t \in [0,1]$ *we have*

$$\pi_\theta(tw) \geq C_2 t \quad \textit{for} \quad \theta \in \left[\frac{3}{4}\pi, a_w^\perp\right] \cup \left[\tilde{a}_w^\perp, \frac{7}{4}\pi\right].$$

(b) *There is* $C_2' > 0$ *such that for all* $w \in R_1$ *with* $\|w\| \geq 1$

$$\pi_\theta(w) \geq C_2' \quad \textit{for} \quad \theta \in \left[\frac{3}{4}\pi, a_w^\perp\right] \cup \left[\tilde{a}_w^\perp, \frac{7}{4}\pi\right].$$

3. (a) *There is* $C_3 > 0$ *such that for all* $w \in R_2$ *with* $\|w\| = 1$ *and all* $t \in [0,1]$ *we have*

$$\pi_\theta(tw) \geq C_3 t \quad \textit{for} \quad \theta \in \left[\frac{5}{4}\pi, a_w^\perp\right] \cup \left[\tilde{a}_w^\perp, \frac{9}{4}\pi\right].$$

(b) *There is* $C_3' > 0$ *such that for all* $w \in R_1$ *with* $\|w\| \geq 1$

$$\pi_\theta(w) \geq C_3' \quad \textit{for} \quad \theta \in \left[\frac{5}{4}\pi, a_w^\perp\right] \cup \left[\tilde{a}_w^\perp, \frac{9}{4}\pi\right].$$

We prove the part 1, the parts 2 and 3 are analogous.

proof.
(a) By contradiction, suppose that for all $n \in \mathbb{N}$ the existe w_n with $\|w_n\| = 1$, $t_n \in [0,1]$ and $\theta_n \in [0, a_{w_n}^\perp] \cup [\tilde{a}_{w_n}^\perp, \pi]$ such that $\pi_{\theta_n}(t_n w_n) < \frac{1}{n} t_n \|w_n\|$. We can assume that $w_n \to w$, $\theta_n \to \theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi]$ and $t_n \to t$ as $n \to \infty$. If $t \neq 0$, then since for $w \in R_1$ and $\theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi]$, $\pi_\theta(tw) \geq 0$, we have $0 \leq \pi_\theta(tw) \leq 0$, so $\theta = \theta_w^\perp$ and this is a contradiction, because $\theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi]$ and ϵ is fixed.
If $t = 0$, consider the C^1-function $F(\theta, t, w) = \pi_\theta(tw)$, then

$$0 = \lim_{n \to \infty} \frac{F(\theta_n, t_n, w_n)}{t_n} = \lim_{t \to 0} \frac{F(\theta, t, w)}{t},$$

by Lemma 32 we know that $\lim_{t \to 0} \dfrac{F(\theta, t, w)}{t} \neq 0$, and this is a contradiction with the above, so (a) is proven.

(b) Since $\theta_w^\perp = \theta_{tw}^\perp$ for $t > 0$ and $\pi_\theta(w) \geq \pi_\theta(\frac{w}{\|w\|})$ for $\|w\| \geq 1$, then is sufficient to prove that there is $C_1' > 0$ such that for all $w \in R_1$ with $\|w\| = 1$

$$\pi_\theta(w) \geq C_1' \quad \text{for} \quad \theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi].$$

Assume tha contrary, that is for any $n \in \mathbb{N}$ there are $w_n \in R_1$, $\theta_n \in [0, a_{w_n}^\perp] \cup [\tilde{a}_{w_n}^\perp, \pi]$ with $\|w_n\| = 1$, such that $0 \leq \pi_{\theta_n}(w_n) \leq \frac{1}{n}$. We can assume that $w_n \to w$ and $\theta_n \to \theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi]$ as $n \to \infty$, then the above implies that $\pi_\theta(w) = 0$, that is, $\theta = \theta_w^\perp$, this is absurd with $\theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi]$. □

5.3.2 The Bessel Function Associated to $\pi_\theta(w)$

For $w \in T_p\mathbb{R}^2$ consider the Bessel function

$$\tilde{J}_w(z) = \int_0^{2\pi} \cos(z\pi_\theta(w))d\theta.$$

Observe that we can consider $\pi_\theta(w)$ as a periodic function in θ of period 2π. Moreover, $\tilde{J}_w(z)$ has the following properties:

1. $\tilde{J}_w(z) = \tilde{J}_w(-z)$;
2. $\tilde{J}_w(z) = \int_0^{2\pi} \cos(z\pi_\theta(w))d\theta = \int_t^{2\pi+t} \cos(z\pi_\theta(w))d\theta$ for any $t \in \mathbb{R}$.
3. As $\pi_{\theta+\pi}(\exp_p(w)) = -\pi_\theta(\exp_p(w))$, then

$$\begin{aligned}\int_t^{\pi+t} \cos(z\pi_\theta w)d\theta &= \int_{\pi+t}^{2\pi+t} \cos(z\pi_{\theta-\pi} w)d\theta = \int_{\pi+t}^{2\pi+t} \cos(-z\pi_\theta w)d\theta \\ &= \int_{\pi+t}^{2\pi+t} \cos(z\pi_\theta(w)d\theta,\end{aligned}$$

Thus,

$$\tilde{J}_w(z) = 2\int_t^{\pi+t} \cos(z\pi_\theta(w))d\theta := 2J_w^t(z). \tag{5.7}$$

Remark 17. *To fix ideas we consider*

$$\begin{aligned} t &= 0 \quad \textit{for} \quad w \in R_1; \\ t &= \frac{3}{4}\pi \quad \textit{for} \quad w \in R_2; \\ t &= \frac{5}{4}\pi \quad \textit{for} \quad w \in R_3. \end{aligned}$$

Proposition 4. *For any $w \in T_p\mathbb{R}^2$ we have that $\int_{-\infty}^{\infty} \tilde{J}_w(z)dz < \infty$.*

proof. We divide the proof in three parts.

1. If $w \in R_1$, in this case, by Remark 17 and equation (5.7) is it suffices to prove the Lemma for $J_w^0(z) := J_w(z)$.
2. If $w \in R_2$, in this case, by Remark 17 and equation (5.7) is it suffices to prove the Lemma for $J_w^{3\pi/4}(z)$.
3. If $w \in R_3$, in this case, by Remark 17 and equation (5.7) is it suffices to prove the Lemma for $J_w^{5\pi/4}(z)$.

We will prove 1, the proof of 2 and 3 are analogous. In fact: Since $J_w(z) = J_w(-z)$, then

$$\int_{-\infty}^{\infty} J_w(z)dz = 2\int_0^{\infty} J_w(z)dz,$$

so, the proof is reduced to prove that $\displaystyle\int_0^{\infty} J_w(z)dz < \infty$.
Let $w \in R_1$ and $x > 0$, then

$$\begin{aligned}\int_0^x J_w(z)dz &= \int_0^{\pi}\int_0^x \cos(z\pi_\theta(w))dzd\theta = \int_0^{\pi} \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)}d\theta \\ &= \int_0^{\theta_w^\perp} \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)}d\theta + \int_{\theta_w^\perp}^{\pi} \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)}d\theta := I_w^1(x) + I_w^2(x).\end{aligned}$$

The next step is to estimate $I_w^1(x)$ and $I_w^2(x)$.

$$I_w^1(x) = \int_0^{a_w^\perp} \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)}d\theta + \int_{a_w^\perp}^{\theta_w^\perp} \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)}d\theta, \tag{5.8}$$

where $a_w^\perp = \theta_w^\perp - \epsilon$.
Now, by Lemma 33.1 we have that if $\theta \in [0, a_w^\perp]$, then for $\|w\| \leq 1$, $\pi_\theta(w) \geq C_1 \|w\|$ and for $\|w\| \geq 1$, $\pi_\theta(w) \geq C_1'$.

Since, $\sin(x\pi_\theta(w)) \leq 1$, then the first integral on the right side (5.8) is bounded in x. In fact:

$$\int_0^{a_w^\perp} \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)}d\theta \leq \begin{cases} \frac{\pi}{C_1\|w\|} & \text{if } \ 0 < \|w\| \leq 1; \\ \frac{\pi}{C_1'} & \text{if } \ \ \|w\| > 1. \end{cases} \tag{5.9}$$

Now we estimate the second integral on the right side of (5.8).

Put $f_w(\theta) = \pi_\theta(w)$, then $f_w(\theta_w^\perp) = 0$ and $f_w(\theta) > 0$ for $\theta < \theta_w^\perp$. Moreover, recall that by Lemma 31, $\frac{\partial \pi}{\partial \theta}(\theta_w^\perp, w) = -\|w\| \neq 0$, then $f_w'(\theta_w^\perp) \neq 0$, and

$$\int_{a_w^\perp}^{\theta_w^\perp} \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)}d\theta = -\int_0^{f_w(a_w^\perp)} \frac{\sin(xs)}{sf_w'(f_w^{-1}(s))}ds = -\int_0^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)}ds, \tag{5.10}$$

where $g_w(s) = f_w'(f_w^{-1}(s))$ is C^∞.

Now by definition of s, if $s \in [0, f_w(a_w^\perp)]$, then $f_w^{-1}(s) \in [0, a_w^\perp]$. Thus by Lemma 31 we have

$$-\|w\| \leq g_w(s) \leq -\frac{1}{2}\|w\| \quad \text{for all} \ \ s \in [0, f_w(a_w^\perp)]. \tag{5.11}$$

For large x

$$-\int_0^{\frac{2\pi}{x}} \frac{\sin(xs)}{sg_\alpha(s)}ds = -\int_0^{\frac{\pi}{x}} \frac{\sin(xs)}{sg_\alpha(s)}ds - \int_{\pi/x}^{2\pi/x} \frac{\sin(xs)}{sg_\alpha(s)}ds$$

Since $\sin(xs) \geq 0$ in $\left[0, \frac{\pi}{x}\right]$ then

$$-\int_0^{\frac{\pi}{x}} \frac{\sin(xs)}{sg_\alpha(s)}ds \leq \frac{2}{\|w\|}\int_0^{\frac{\pi}{x}} \frac{\sin(xs)}{s}ds = \frac{2}{\|w\|}\int_0^{\pi} \frac{\sin y}{y}dy. \tag{5.12}$$

As well $-sin(xs) \geq 0$ for $s \in \left[\frac{\pi}{x}, \frac{2\pi}{x}\right]$, then $-\displaystyle\int_{\pi/x}^{2\pi/x} \frac{\sin(xs)}{sg_w(s)}ds \leq 0$. So, by (5.12)

$$-\int_0^{\frac{2\pi}{x}} \frac{\sin(xs)}{sg_w(s)}ds \leq \frac{2}{\|w\|}\int_0^{\pi} \frac{\sin y}{y}dy. \tag{5.13}$$

Let $n \in \mathbb{N}$ such that $n \leq \dfrac{xf_w(a_w^\perp)}{2\pi} \leq n+1$, then

$$\int_0^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)}ds = \int_0^{\frac{2\pi}{x}} \frac{\sin(xs)}{sg_w(s)}ds + \sum_{k=1}^{k=n-1}\int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{sg_w(s)}ds + \int_{\frac{2\pi n}{x}}^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)}ds.$$

If $\frac{2\pi n}{x} \leq f_w(a_w^\perp) \leq \frac{\pi(2n+1)}{x}$, then $\sin(xs) \geq 0$ and by Lemma 31, we have

$$\frac{\sin(xs)}{s\|w\|} \leq -\frac{\sin(xs)}{sg_w(s)} \leq \frac{2\sin(xs)}{s\|w\|} \quad \text{and} \quad \frac{2\sin(xs)}{s\|w\|} \leq \frac{2x\sin(xs)}{\|w\|\,2\pi n}.$$

This implies

$$\begin{aligned}
-\int_{\frac{2\pi n}{x}}^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)}ds &\leq \int_{\frac{2\pi n}{x}}^{f_w(a_w^\perp)} \frac{x\sin(xs)}{\|w\|\,\pi n}ds \leq \frac{x}{\|w\|\,\pi n}\int_{\frac{2\pi n}{x}}^{f_w(a_w^\perp)} \sin(xs)ds \\
&\leq \frac{x}{\|w\|\,\pi n}\left(f_w\left(a_w^\perp\right) - \frac{2\pi n}{x}\right) = \frac{2}{\|w\|}\left(\frac{xf_w(a_w^\perp)}{2\pi n} - 1\right) \\
&\leq \frac{2}{\|w\|}\left(\frac{2\pi(n+1)}{2\pi n} - 1\right) = \frac{2}{\|w\|}\frac{1}{n}.
\end{aligned}$$

In the case that $f_w(a_w^\perp) \geq \frac{\pi(2n+1)}{x}$, then $-\displaystyle\int_{\frac{\pi(2n+1)}{x}}^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)}ds \leq 0$, so

$$\begin{aligned}
-\int_{\frac{2\pi n}{x}}^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)}ds &\leq -\int_{\frac{\pi(2n+1)}{x}}^{\frac{\pi(2n+1)}{x}} \frac{\sin(xs)}{sg_w(s)}ds \leq \frac{x}{\|w\|\,2\pi n}\left(\frac{\pi(2n+1)}{x} - \frac{2\pi n}{x}\right) \\
&= \frac{1}{\|w\|}\frac{1}{n}.
\end{aligned}$$

In any case, we have

$$\int_{\frac{2\pi n}{x}}^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)}ds \leq \frac{2}{\|w\|}\frac{1}{n}. \tag{5.14}$$

Now we only need to estimate $\displaystyle\sum_{k=1}^{k=n-1}\int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{sg_w(s)}ds.$

Put $s_0 = \frac{2\pi k}{x}$, then

$$\int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{s g_w(s)} ds = \int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{s_0 g_w(s_0)} ds + \int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \sin(xs) \left(\frac{1}{s g_w(s)} - \frac{1}{s_0 g_w(s_0)} \right) ds.$$

The first integral on the right term of the above equality is zero.

Now we estimate the second integral on the right side of the above equation.

By Lemma 31 we have $g_w(s) g_w(s_0) > \dfrac{\|w\|^2}{4}$, also $s s_0 \geq \left(\frac{2\pi k}{x}\right)^2$.
Thus, $\dfrac{1}{s s_0 g_w(s) g_w(s_0)} < \dfrac{1}{\|w\|^2} \dfrac{x^2}{\pi^2 k^2}$. Moreover,

$$\begin{aligned}
|s_0 g_w(s_0) - s g_w(s)| &= |(s_0 - s) g_w(s_0) + s(g_w(s_0) - g_w(s))| \\
&\leq |s_0 - s| \, |g_w(s_0)| + s \, |g_w(s) - g_w(s_0)| \\
&\leq |s - s_0| \left(|g_w(s_0)| + s \sup_{s \in [0, f_w(a_w^\perp)]} |g_w'(s)| \right) \quad \downarrow \text{ by Lemma 31} \\
&\leq \frac{2\pi}{x} \left(\|w\| + \frac{2\pi(k+1)}{x} \|w\| \right) \\
&\leq \frac{2\pi}{x} \|w\| \left(1 + \frac{2\pi n}{x} \right) \\
&\leq \frac{2\pi}{x} \|w\| \left(1 + f_w(a_w^\perp) \right) \\
&\leq \frac{2\pi}{x} \|w\| \left(1 + \|w\| \right)
\end{aligned}$$

as, $f_w(a_w^\perp) \leq \|w\|$. Therefore,

$$\left| \frac{1}{s g_w(s)} - \frac{1}{s_0 g_w(s_0)} \right| = \left| \frac{s g_w(s) - s_0 g_w(s_0)}{s s_0 g_w(s) g_w(s_0)} \right| \leq \frac{2(1 + \|w\|)}{\pi \|w\|} \left(\frac{x}{k^2} \right).$$

Then,

$$\left| \int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{s g_w(s)} ds \right| \leq \frac{2(1 + \|w\|)}{\pi \|w\|} \left(\frac{x}{k^2} \right) \left(\frac{2\pi(k+1)}{x} - \frac{2\pi k}{x} \right) = \frac{4(1 + \|w\|)}{\|w\|} \left(\frac{1}{k^2} \right).$$

Therefore,

$$\left| \sum_{k=1}^{k=n-1} \int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{s g_w(s)} ds \right| \leq \sum_{k=1}^{k=n-1} \left| \int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{s g_w(s)} ds \right| \leq A(\|w\|) \sum_{k=1}^{k=n-1} \frac{1}{k^2}, \tag{5.15}$$

where $A(\|w\|) = \dfrac{4(1 + \|w\|)}{\|w\|}$.

Since $\sum_{k=1}^{\infty} \frac{1}{k^2} := a < \infty$ and put $b = \int_0^\pi \frac{\sin y}{y}$, then the equations (5.13), (5.14), and (5.15) imply

$$- \int_0^{f_w(a_w^\perp)} \frac{\sin(xs)}{s g_w(s)} ds \leq \frac{2}{\|w\|} b + \frac{2}{\|w\|} \frac{1}{n} + A(\|w\|) a. \tag{5.16}$$

Thus, by the equation (5.8), (5.9), (5.10), and (5.16) we have

$$I_w^1(x) \leq \begin{cases} \dfrac{\pi}{C_1\|w\|} + \dfrac{2}{\|w\|}b + \dfrac{2}{\|w\|}\dfrac{1}{n} + A(\|w\|)a & \text{if } 0 < \|w\| \leq 1; \\ \\ \dfrac{\pi}{C_1'} + \dfrac{2}{\|w\|}b + \dfrac{2}{\|w\|}\dfrac{1}{n} + A(\|w\|)a & \text{if } \|w\| > 1. \end{cases} \tag{5.17}$$

Completely analogous using $\tilde{a}_w^\perp$ instead of $a_w^\perp$ and taking n' such that $-\frac{\pi(2n'+1)}{x} \leq f_w(\tilde{a}_w^\perp) \leq -\frac{2\pi n'}{x}$, we also obtain

$$I_w^2(x) \leq \begin{cases} \dfrac{\pi}{C_1\|w\|} + \dfrac{2}{\|w\|}b + \dfrac{2}{\|w\|}\dfrac{1}{n'} + A(\|w\|)a & \text{if } 0 < \|w\| \leq 1; \\ \\ \dfrac{\pi}{C_1'} + \dfrac{2}{\|w\|}b + \dfrac{2}{\|w\|}\dfrac{1}{n'} + A(\|w\|)a & \text{if } \|w\| > 1. \end{cases} \tag{5.18}$$

Since $n, n' \to \infty$ as $x \to \infty$, then (5.17) and (5.18) implies

$$\int_0^\infty J_w(z)dz \leq \begin{cases} \dfrac{2\pi}{C_1\|w\|} + \dfrac{4}{\|w\|}b + 2A(\|w\|)a & \text{if } 0 < \|w\| \leq 1; \\ \\ \dfrac{2\pi}{C_1'} + \dfrac{2}{\|w\|}b + 2A(\|w\|)a & \text{if } \|w\| > 1. \end{cases} \tag{5.19}$$

Thus, we conclude the proof of Proposition 4. □

Put $j_1 = 0$, $j_2 = \dfrac{3\pi}{4}$ and $j_3 = \dfrac{5\pi}{4}$, then it is also easy to see that for $w \in R_i$,

$$\int_0^\infty J_w^{j_i}(z)dz \leq \begin{cases} \dfrac{2\pi}{C_i\|w\|} + \dfrac{4}{\|w\|}b + 2A(\|w\|)a & \text{if } 0 < \|w\| \leq 1; \\ \\ \dfrac{2\pi}{C_i'} + \dfrac{2}{\|w\|}b + 2A(\|w\|)a & \text{if } \|w\| > 1, \end{cases} \tag{5.20}$$

$i = 2, 3$, where C_i, C_i' are given in Lemma 33.

5.4 Proof of the Main Theorem

As in the Kaufman's proof of Marstrand's theorem (cf. [Kau68]), we use the potential theory.
Put $d = HD(K) > 1$, assume that $0 < M_d(K) < \infty$ and for some $C > 0$, we have

$$m_d(K \cap B_r(x)) \leq Cr^d$$

for $x \in \mathbb{R}^2$ and $0 < r \leq 1$ (cf. [Fal85]). Let μ be the finite measure on $\mathbb{R}^2$ defined by $\mu(A) = m_d(K \cap A)$, A a measurable subset of $\mathbb{R}^2$. For $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, let us denote by μ_θ the (unique) measure on $\mathbb{R}$ such that $\int f d\mu_\theta = \int (f \circ \pi_\theta) d\mu$ for every continuous function f.

The theorem will follow, if we show that the support of μ_θ has positive Lebesgue measure for almost all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, since this support is clearly contained in $\pi_\theta(K)$. To do this we use the following fact.

Lemma 34. (cf. [PT93, pg. 65]) *Let η be a finite measure with compact support on $\mathbb{R}$ and*

$$\hat{\eta}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixp} d\eta(x),$$

for $p \in \mathbb{R}$ ($\hat{\eta}$ is the fourier transform of η). If $0 < \int_{-\infty}^{\infty} |\hat{\eta}(p)|^2 dp < \infty$ then the support of η has positive Lebesgue measure.

Proof of the Main Theorem.
We now show that, for almost any $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\hat{\mu}_\theta$ is square-integrable. From the definitions we have

$$|\hat{\mu}_\theta(p)|^2 = \frac{1}{2\pi} \int\int e^{i(y-x)p} d\mu_\theta(x) d\mu_\theta(y) = \frac{1}{2\pi} \int\int e^{ip(\pi_\theta(v)-\pi_\theta(u))} d\mu(u) d\mu(v)$$

as $\pi_{\theta+\pi}(u) = -\pi_\theta(u)$, then

$$|\hat{\mu}_\theta(p)|^2 + |\hat{\mu}_{\theta+\pi}(p)|^2 = \frac{1}{\pi} \int\int cos(p(\pi_\theta(v) - \pi_\theta(u))) d\mu(u) d\mu(v).$$

And so

$$\begin{aligned} \int_0^{2\pi} |\hat{\mu}_\theta(p)|^2 d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \int\int cos(p(\pi_\theta(v) - \pi_\theta(u))) d\mu(u) d\mu(v) d\theta \\ &= \frac{1}{2\pi} \int\int \left(\int_0^{2\pi} cos(p(\pi_\theta(v) - \pi_\theta(u))) d\theta \right) d\mu(u) d\mu(v). \end{aligned}$$

Observe now that for all $x > 0$ and for all u, v there are $L \in \mathbb{N}$ and $w(u, v)$ such that

$$\int_0^x \int_0^{2\pi} cos(p(\pi_\theta(u) - \pi_\theta(v))) d\theta dp \leq L \left| \int_0^x \int_0^{2\pi} cos(x\pi_\theta(w(u, v)) d\theta dp \right|$$

$w(u, v)$ can be taken such that $d(p, w) = d(u, v)$. So, we have for $x > 0$

$$\int_{-x}^x \int_0^{2\pi} |\hat{\mu}_\theta(p)|^2 d\theta dp \leq \frac{2L}{2\pi} \int\int \left| \int_0^x \tilde{J}_{w(u,v)}(p) dp \right| d\mu(u) d\mu(v).$$

Follows

$$\frac{\pi}{L} \int_{-\infty}^{\infty} \int_0^{2\pi} |\hat{\mu}_\theta(p)|^2 d\theta dp \leq \int\int \left| \int_0^\infty \tilde{J}_{w(u,v)}(p) dp \right| d\mu(u) d\mu(v) =$$

$$= \int\int_{\{\|w\|>1\}} \left| \int_0^\infty \tilde{J}_{w(u,v)}(p) dp \right| d\mu(u) d\mu(v) + \int\int_{\{\|w\|\leq 1\}} \left| \int_0^\infty \tilde{J}_{w(u,v)}(p) dp \right| d\mu(u) d\mu(v)$$

$$=: I + II. \tag{5.21}$$

By (5.7) and Remark 17

$$\begin{aligned} I &= \int\int_{\{\|w\|>1\}} \left|\int_0^\infty \tilde{J}_w(p)dp\right| d\mu(u)d\mu(v) = \sum_{i=1}^{3}\int\int_{\{\|w\|>1\}\cap R_i} \left|\int_0^\infty \tilde{J}_w(p)dp\right| d\mu(u)d\mu(v) \\ &= 2\sum_{i=1}^{3}\int\int_{\{\|w\|>1\}\cap R_i} \left|\int_0^\infty J_w^{j_i}(p)dp\right| d\mu(u)d\mu(v). \end{aligned}$$

Now by (5.19) and (5.20), we have

$$I \leq 2\sum_{i=1}^{3}\int\int_{\{\|w\|>1\}\cap R_i} \left(\frac{2\pi}{C_i'} + \frac{2}{\|w\|}b + 2A(\|w\|)a\right) d\mu(u)d\mu(v).$$

If $\|w\| > 1$, then $\frac{1}{\|w\|} < 1$ and $A(\|w\|) = \frac{4(1+\|w\|)}{\|w\|} < 8$, moreover, as the support of the measure $\mu \times \mu$ is contained in $K \times K$ which is compact, then

$$I \leq 6\left(2\pi \max\left\{\frac{1}{C_i'}\right\} + 2b + 16a\right)\mu(K)^2. \tag{5.22}$$

We now estimate II, in fact: By (5.7) and Remark 17,

$$\begin{aligned} II &= \int\int_{\{\|w\|\leq 1\}} \left|\int_0^\infty \tilde{J}_w(p)dp\right| d\mu(u)d\mu(v) = \sum_{i=1}^{3}\int\int_{\{\|w\|\leq 1\}\cap R_i} \left|\int_0^\infty \tilde{J}_w(p)dp\right| d\mu(u)d\mu(v) \\ &= 2\sum_{i=1}^{3}\int\int_{\{\|w\|\leq 1\}\cap R_i} \left|\int_0^\infty J_w^{j_i}(p)dp\right| d\mu(u)d\mu(v). \end{aligned}$$

Now by (5.19) and (5.20), we have

$$\begin{aligned} II &\leq 2\sum_{i=1}^{3}\int\int_{\{\|w\|\leq 1\}\cap R_i} \left(\frac{2\pi}{C_i\,\|w\|} + \frac{4}{\|w\|}b + 2A(\|w\|)a\right) d\mu(u)d\mu(v) \\ &\leq 6\int\int_{\{\|w\|\leq 1\}} \left(\left(\max\left\{\frac{2\pi}{C_i}\right\} + 4b + 8a\right)\frac{1}{\|w\|} + 8a\right) d\mu(u)d\mu(v). \end{aligned} \tag{5.23}$$

Remember that $\|w(u,v)\| = d(u,v)$, then

$$\int\int_{\{\|w\|\leq 1\}} \frac{1}{\|w\|} d\mu(u)d\mu(v) = \int\int_{\{d(u,v)\leq 1\}} \frac{1}{d(u,v)} d\mu(u)d\mu(v).$$

Now, for some $0 < \beta < 1$

$$\begin{aligned} \int_{\{\|w\|\leq 1\}} \frac{1}{d(u,v)} d\mu(v) &= \sum_{n=1}^{\infty}\int_{\beta^n\leq d(u,v)\leq \beta^{n-1}} \frac{d\mu(v)}{d(u,v)} \leq \sum_{n=1}^{\infty}\beta^{-n}\mu(B_{\beta^{n-1}}(u)) \\ &\leq C\sum_{n=1}^{\infty}\beta^{-n}(\beta^{n-1})^d \\ &\leq C\sum_{n=1}^{\infty}\beta^{-d}(\beta^{d-1})^n \text{ with } d>1 \\ &= C\beta^{-d}\left(\frac{1}{1-\beta^{d-1}} - 1\right) = \frac{C}{\beta-\beta^d}. \end{aligned}$$

Therefore,

$$\int\int_{\{\|w\|\leq 1\}} \frac{1}{\|w\|} d\mu(u)d\mu(v) \leq \mu(\mathbb{R}^2)\frac{C}{\beta - \beta^d}.$$

Also, $\displaystyle\int\int_{\{\|w\|\leq 1\}} 48 d\mu(u)d\mu(v) \leq 8a\mu(K)^2 < \infty.$

Using these last two inequalities and the equation (5.23) we have that

$$II \leq 6\left(\left(\max\left\{\frac{2\pi}{C_i}\right\} + 4b + 8a\right)\frac{C}{\beta - \beta^d} + 8a\mu(K)^2\right). \tag{5.24}$$

Using Fubini, the by equations (5.21), (5.22) and (5.24) we have

$$\frac{\pi}{L}\int_0^{2\pi}\int_{-\infty}^{\infty} |\hat{\mu}_\theta(p)|^2 dp d\theta \leq I + II \leq 6\left(2\pi \max\left\{\frac{1}{C_i'}\right\} + 2b + 16a\right)\mu(K)^2 + 6\left(\left(\max\left\{\frac{2\pi}{C_i}\right\} + 4b + 8a\right)\frac{C}{\beta - \beta^d} + 8a\mu(K)^2\right) < \infty.$$

Therefore, $\displaystyle\int_{-\infty}^{\infty} |\hat{\mu}_\theta(p)|^2 dp < \infty$ for almost all $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

If exists $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\displaystyle\int_{-\infty}^{\infty} |\hat{\mu}_\theta(p)|^2 dp = 0$, then $\displaystyle\int_{-\infty}^{\infty} |\varphi(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{\mu}_\theta(p)|^2 dp = 0$ where $\displaystyle\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixp}\hat{\mu}_\theta(p)dp$. This implies that $\varphi \equiv 0$ almost every where, but $d\mu_\theta = \varphi dx$. This is $\mu_\theta(\mathbb{R}) = \int_{-\infty}^{\infty}\varphi(x)dx = 0$ and this implies that $\mu(\mathbb{R}^2) = 0$, this contradicts the fact that d-measure of Haussdorff of K is positive.

The result follows of Lemma 34, in the case $0 < m_d(K) < \infty$.

In the general case, we take $0 < m_{d'}(K') < \infty$ with $1 < d' < d$ and $K' \subset K$ (cf. [Fal85]). Then, by the same argument $\pi_\theta(K')$ has positive measure for almost all θ, and since $\pi_\theta(K') \subset \pi_\theta(K)$, then the same is true for $\pi_\theta(K)$. □

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