Interior Hybrid Proximal Extragradient Methods for the Linear Monotone Complementarity Problem.

Abstract

We present new infeasible path-following methods for linear monotone complementarity problems based on Auslander, Tebboulle, and Ben-Tiba's log-quadratic barrier functions. The central paths associated with these barriers are always well defined and, for those problems which have a solution, convergent to a pair of complementary solutions. Starting points in these paths are easy to compute. The theoretical iteration-complexity of these new path-following methods is derived and improved by a strategy which uses relaxed hybrid proximal-extragradient steps to control the quadratic term. Encouraging preliminary numerical experiments are presented.

1 Introduction.

In this work we are concerned with the Linear Monotone Complementarity Problem (LMCP). Interior point methods are efficient tools for solving this class of problems (see, for example, [13, 19, 16, 14, 20, 29, 27] and the references therein). Among these methods, feasible path-following interior point methods are based on the *central path*. The primal (logarithmic) central path for the LMCP assigns to each $\lambda > 0$ the solution of the original problem perturbed by λ^{-1} times the gradient of the logarithmic barrier, while the primal-dual central path is obtained by a reformulation of the primal central path.

Feasible primal and primal-dual path-following interior point methods for the LMCP generate sequences of points in suitable neighborhoods of the primal and primal-dual central paths, respectively. These iterates shall be feasible and positive. However, when the solution set of the LMCP is unbounded, neither these feasible and positive points nor the central path do exist [15, 11]. Even if the solution set is bounded and such points do exist, sometimes the computation of the first one, to be used as initial iterate, is expensive. Several methods were proposed for dealing with these two issues, such as *infeasible interior point methods* [17, 30, 22] and *non interior point methods* [9, 10].

We address these two issues considering infeasible path-following methods based on log-quadratic barrier functions. These barrier functions were proposed by Auslander, Teboulle, and Ben-Tiba [1, 2], in the context of generalized proximal point methods for monotone complementarity problems. In the context of path-following methods, one of the advantages of using these barrier functions is that the associated central paths are always well defined. Another practical advantage is that starting points in suitable neighborhoods of these central paths are easy to construct.

In the complexity analysis of "pure" path following methods for log-quadratic barriers, the distance to the solution set appears multiplicatively in the complexity estimation, and in our preliminary numerical experiments on such methods the quadratic term seems to slow down the algorithms. These problems are circumvented by combining classical path-following steps with (relaxed) hybrid proximal extragradient steps [24] that push the base point of the quadratic term closer to the solution set. In our preliminary numerical tests, this combination improves the performance of infeasible path-following methods based on log-quadratic barriers as compared with those that do not updated the base point of the quadratic term.

This paper is organized as follows. In Section 2 we introduce some notations, define the log-quadratic central path, define an error measure, and analyze centering and predictor Newton steps. In Section 3 we discuss the advantages and disadvantages of "pure" path-following schemes based on log-quadratic barriers. In Section 4 we discuss the application of the hybrid proximal extragradient method to our specific problem. In Sections 5 and 6 we present and study two interior hybrid proximal extragradient algorithms, and in Section 7 we present some numerical results. In the Appendix we present some properties of log-quadratic central paths.

The notation used in this work is standard. In particular, $\|\cdot\|$ denotes the quadratic norm in $\mathbb{R}^n$; given a non-empty, closed, and convex set $C \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $d(x, C) = \min_{y \in C} \|x - y\|$ denotes the distance of x to C, $P_C(x)$ denotes the Euclidean projection of x on C, and N_C denotes the normal cone operator of C; $\lfloor t \rfloor$ stands for the greatest integer smaller or equal than t; $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes a point-to-set operator T; given a vector, say for example, u, U denotes a diagonal matrix with components of vector u in the main diagonal, e indicates vector $(1, 1, \ldots, 1)$; $r_1 = O(r_2)$ indicates that $\dfrac{|r_1|}{|r_2|}$ is bounded above for some constant.

2 The Log-Quadratic Central Path.

In this section we review the definition of the Linear Monotone Complementarity Problem, introduce a family of log-quadratic central paths for this problem, and adapt the classical tools of path-following methods for these central paths.

The Linear Monotone Complementarity Problem (LMCP) is that of finding $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Mx + b \geq 0, \quad \langle x, Mx + b \rangle = 0, \tag{1}$$

given $M \in \mathbb{R}^{n \times n}$ a positive semi-definite matrix, $b \in \mathbb{R}^n$, and the notation $x \geq 0$ meaning that all components of x are non-negative. From now on S^* will denote the solution set of the LMCP (1) and S^*_{CP} the set of pairs of complementary solutions, that is, all pairs (x, y) with $x \in S^*$ and $y = Mx + b$.

The (logarithmic) central path assigns to each $\lambda > 0$ the point $x(\lambda)$ solution of the problem

$$x > 0, \quad \lambda(Mx + b) - \nabla \left[\sum_{i=1}^{n} \log x_i \right] = 0,$$

where $x > 0$ means that $x \in \mathbb{R}^n_{++}$. The addition of a quadratic-regularization term to the logarithmic barrier function in the above equation yields the new problem: Find x such that

$$x > 0, \quad \lambda(Mx + b) + \nabla \left[\frac{\nu}{2} \|x - \bar{x}\|^2 - \sum_{i=1}^{n} \log x_i \right] = 0, \tag{2}$$

where $\bar{x}$ is an arbitrary point and $\lambda > 0$, $\nu > 0$. Introducing variable $y = \lambda^{-1} x^{-1}$ we can reformulate (2) as:

$$\begin{aligned} x > 0, \quad y > 0, \quad \lambda(Mx + b - y) + \nu(x - \bar{x}) &= 0, \\ \lambda X y - e &= 0. \end{aligned} \tag{3}$$

The *log-quadratic central path* for problem (1) assigns to each $\lambda > 0$ the point $\big(x(\lambda;\nu,\bar{x}), y(\lambda;\nu,\bar{x})\big)$ solution of (3). This curve depends also on ν and $\bar{x}$; therefore, strictly speaking, we have a *family* of log-quadratic central paths. Alternatively, one could consider the log-quadratic central path as a function of λ, ν, and $\bar{x}$. Since the left hand-side of the equality in (2) is, as a function of $x > 0$, maximal monotone and strongly monotone, this equation has a unique solution, which proves well-definiteness of the log-quadratic central path. Additional properties of log-quadratic central paths are discussed in the Appendix.

Now we extend the path-following methodology to the log-quadratic central path. Let $R_1(x, y; \lambda, \nu)$ and $R_2(x, y; \lambda)$ denote the residuals of system (3) at $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, that is,

$$R_1(x, y; \lambda, \nu, \bar{x}) = \lambda(Mx + b - y) + \nu(x - \bar{x}), \tag{4}$$

$$R_2(x, y; \lambda) = \lambda X y - e. \tag{5}$$

We define the error measure

$$\delta_{\nu,\bar{x}}(x, y; \lambda) = \frac{\|R_1(x, y; \lambda, \nu)\|}{\nu^{1/2}} + \|R_2(x, y; \lambda)\|, \tag{6}$$

and, for $\beta \in [0, 1]$, the associated *β-neighborhood* of the log-quadratic central path,

$$\mathcal{N}_{\beta,\nu,\bar{x}} = \left\{(x, y) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \mid \ \delta_{\nu,\bar{x}}(x, y; \lambda) \le \beta, \ \text{for some } \lambda > 0\right\}.$$

The next lemma analyzes how an increment in λ affects the residuals and the error measure above defined.

Lemma 2.1. *Consider $\lambda > 0$, $\nu > 0$, $\bar{x} \in \mathbb{R}^n$, $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. If $\rho \ge 0$ and $\lambda' = \lambda(1 + \rho)$, then*

$$R_1(x, y; \lambda', \nu, \bar{x}) = (1 + \rho)R_1(x, y; \lambda, \nu, \bar{x}) - \rho\nu(x - \bar{x}), \tag{7}$$

$$R_2(x, y; \lambda') = (1 + \rho)R_2(x, y; \lambda) + \rho e, \tag{8}$$

$$\delta_{\nu,\bar{x}}(x, y; \lambda') \le \delta_{\nu,\bar{x}}(x, y; \lambda) + \rho\bigg(\delta_{\nu,\bar{x}}(x, y; \lambda) + \nu^{1/2}\|x - \bar{x}\| + \sqrt{n}\bigg). \tag{9}$$

Proof 2.2. *The definition of λ' and the definitions* (4) *and* (5) *yield, after algebraic manipulations, relations* (7) *and* (8)*. Inequality* (9) *follows from* (7)*,* (8)*, definition* (6)*, and triangle inequality.*

The next technical lemma will be used in the analysis of Newton steps in β-neighborhoods of the log-quadratic central path.

Lemma 2.3. *Take $\lambda > 0$, $\nu' > 0$, $\bar{x} \in \mathbb{R}^n$, and $(x, y) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$. The linear system*

$$\begin{bmatrix} \lambda M + \nu' I & -\lambda I \\ \lambda Y & \lambda X \end{bmatrix} \begin{bmatrix} d^x \\ d^y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

has a unique solution for any $b_1,\ b_2 \in \mathbb{R}^n$. If, additionally,

$$\delta_{\nu',\bar{x}}(x, y; \lambda) \le \beta < 1,$$

then

$$0 < 1 - \beta \le \lambda x_i y_i \le 1 + \beta, \quad i = 1, \ldots, n \tag{10}$$

and

$$\nu'\|d^x\|^2 + \lambda\|D^x d^y\| \le \langle d^x, b_1\rangle + \frac{\|b_2\|^2}{2(1 - \beta)}. \tag{11}$$

Proof 2.4. *Since M is positive semi-definite, $\lambda > 0$, $\nu' > 0$, $x > 0$ and $y > 0$, the matrix of coefficients of the linear system is non-singular. Hence, d^x and d^y are well defined.*

Suppose that $\delta_{\nu,\bar{x}}(x, y; \lambda) \le \beta < 1$. The bounds for $\lambda x_i y_i$ follow immediately from relations $|\lambda x_i y_i - 1| \le \|\lambda X y - e\| \le \beta$, $i = 1, 2, \ldots, n$, and triangle inequality.

Multiplying the first block of equations on the linear system by vector d^x and using the positive semi-definiteness of M we conclude that

$$\nu'\|d^x\|^2 - \lambda\langle d^x, d^y\rangle \le \langle d^x, b_1\rangle.$$

Observe that

$$\begin{aligned}\lambda\|D^x d^y\| \le \lambda\sum_{i=1}^n |d_i^x d_i^y| &\le \frac{\lambda}{2}\left(\sum_{i=1}^n \frac{y_i}{x_i}(d_i^x)^2 + \frac{x_i}{y_i}(d_i^y)^2\right) \\ &= \frac{\lambda}{2}\|(X^{-1}Y)^{1/2}d^x + (X^{-1}Y)^{-1/2}d^y\|^2 - \lambda\langle d^x, d^y\rangle \\ &= \frac{1}{2}\|(\lambda XY)^{-1/2}b_2\|^2 - \lambda\langle d^x, d^y\rangle \le \frac{1}{2(1-\beta)}\|b_2\|^2 - \lambda\langle d^x, d^y\rangle,\end{aligned}$$

using in the second equality the definition of (d^x, d^y) and in the last inequality the bounds on $\lambda x_i y_i$, $i = 1\ldots, n$. To end the proof, add the above inequalities.

From now on in this section we will consider

$$\lambda > 0, \quad \nu > 0, \quad \bar{x} \in \mathbb{R}^n, \quad (x, y) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}, \tag{12}$$

and (d^x, d^y) will denote the solution of the linear system

$$\begin{bmatrix}\lambda M + \sigma_1\nu I & -\lambda I \\ \lambda Y & \lambda X\end{bmatrix}\begin{bmatrix}d^x \\ d^y\end{bmatrix} = -\begin{bmatrix}R_1(x, y; \lambda, \nu, \bar{x}) + (\sigma_1 - 1)\nu(x - \bar{x}) \\ R_2(x, y; \lambda) + (1 - \sigma_2)e\end{bmatrix}, \tag{13}$$

where $\sigma_1 \in (0, 1]$ and $\sigma_2 \in [0, 1]$ are given parameters. Observe that (13) defines a generic (perturbed) Newton iteration for system (3). Define also

$$(x_\alpha, y_\alpha) = (x + \alpha d^x, y + \alpha d^y), \ \alpha \in [0, 1]. \tag{14}$$

In the rest of this section we will consider

$$\beta_1 \in (0, 2/3), \quad \beta_2 = \frac{\beta_1^2}{2(1 - \beta_1)}. \tag{15}$$

Due to the above assumptions $\beta_1 > \beta_2$. These parameters will be used to define outer and inner neighborhoods of the log-quadratic central path. In the next proposition we prove that a centering step at a point $(x, y) \in \mathcal{N}_{\beta_1,\nu,\bar{x}}$ generates a new point in $\mathcal{N}_{\beta_2,\nu,\bar{x}}$.

Proposition 2.5 (Centering step). *Take $\sigma_1 = \sigma_2 = 1$. Consider λ, ν, $\bar{x}$, and (x, y) as in* (12)*, β_1 and β_2 as in* (15)*, and (d^x, d^y) as in* (13)*. If*

$$\delta_{\nu,\bar{x}}(x, y; \lambda) \le \beta_1,$$

then

$$x + d^x > 0, \quad y + d^y > 0, \quad \delta_{\nu,\bar{x}}(x + d^x, y + d^y; \lambda) \le \beta_2. \tag{16}$$

Proof 2.6. *From* (4), (5), (13), *and* (14) *it follows*

$$R_1(x_\alpha, y_\alpha; \lambda, \nu, \bar{x}) = (1-\alpha) R_1(x, y; \lambda, \nu, \bar{x}),$$
$$R_2(x_\alpha, y_\alpha; \lambda) = (1-\alpha) R_2(x, y; \lambda) + \alpha^2 \lambda D^x d^y, \ \forall\, \alpha \in [0,1],$$

which, combined with definition (6) *and triangle inequality, yield*

$$\delta_{\nu,\bar{x}}(x_\alpha, y_\alpha; \lambda) \leq (1-\alpha)\delta_{\nu,\bar{x}}(x, y; \lambda) + \alpha^2 \|\lambda D^x d^y\|.$$

From the definition of (d^x, d^y) *and Lemma 2.3, with* $\nu' = \nu$, $\beta = \beta_1$, $b_1 = R_1(x, y; \lambda, \nu, \bar{x})$, *and* $b_2 = R_2(x, y; \lambda)$, *it follows*

$$\begin{aligned}\lambda \|D^x d^y\| &\leq \langle d^x, b_1 \rangle - \frac{\nu \|d^x\|^2}{2} + \frac{\|b_2\|^2}{2(1-\beta)} \\ &\leq \frac{\|b_1\|^2}{2\nu} + \frac{\|b_2\|^2}{2(1-\beta)} \leq \frac{1}{2(1-\beta)} \left(\frac{\|b_1\|}{\nu^{1/2}} + \|b_2\| \right)^2 \leq \frac{\beta_1^2}{2(1-\beta_1)},\end{aligned}$$

using in the second inequality Cauchy-Schwartz inequality, in the third inequality relation $0 < \beta < 1$ *and simple algebraic manipulations, and in the last inequality definition* (6) *and the first assumption. Thus, it holds*

$$\delta_{\nu,\bar{x}}(x_\alpha, y_\alpha; \lambda) \leq (1-\alpha)\beta_1 + \alpha^2 \frac{\beta_1^2}{2(1-\beta_1)}, \ \forall\, \alpha \in [0,1]. \tag{17}$$

Taking $\alpha = 1$ *in* (17) *and using the definition of* β_2 *we obtain the third inequality in* (16). *The right hand-side of* (17) *is a convex function in variable* α, *which takes values smaller than* 1 *for* $\alpha = 0$ *and* $\alpha = 1$. *Thus,*

$$\delta_{\nu,\bar{x}}(x_\alpha, y_\alpha; \lambda) < 1, \ \forall \alpha \in [0,1],$$

and, from Lemma 2.3, it follows that each component of vector $X_\alpha y_\alpha$ *remains positive for* $\alpha \in [0,1]$. *Since* $(x_0, y_0) = (x, y) > 0$ *and the components of vector* (x_α, y_α) *are continuous in* α, *we conclude that* $(x + d^x, y + d^y) = (x_1, y_1) > 0$, *which ends the proof of* (16).

The next technical lemma is an auxiliary result to be used in the analysis of the predictor/affine scaling step.

Lemma 2.7. *Take* $\sigma_2 = 0$ *and* $\sigma_1 \in (0,1]$. *Consider* λ, ν, $\bar{x}$, *and* (x, y) *as in* (12), (d^x, d^y) *as in* (13), *and* (x_α, y_α) *as in* (14). *If* $\lambda' = \lambda(1+\rho)$, *with* $\rho \geq 0$, *then*

$$\begin{aligned}\delta_{\nu,\bar{x}}(x_\alpha, y_\alpha; \lambda') \leq &(1+\rho)(1-\alpha)\delta_{\nu,\bar{x}}(x, y; \lambda) + \big|\alpha(1+\rho)(1-\sigma_1) - \rho\big|\, \nu^{1/2}\|x - \bar{x}\| \\ &+ \alpha\big|(1+\rho)(1-\sigma_1) - \rho\big|\, \nu^{1/2}\|d^x\| + (1+\rho)\alpha^2 \|\lambda D^x d^y\| \\ &+ |\rho - (1+\rho)\alpha|\sqrt{n}.\end{aligned}$$

Proof 2.8. *Direct combination of* (4), (5), (13), *and* (14) *yields*

$$R_1(x_\alpha, y_\alpha; \lambda, \nu, \bar{x}) = (1-\alpha)R_1 + \alpha(1-\sigma_1)\nu(x - \bar{x}) + \alpha(1-\sigma_1)\nu d^x,$$
$$R_2(x_\alpha, y_\alpha; \lambda) = (1-\alpha)R_2 - \alpha e + \alpha^2 \lambda D^x d^y.$$

Now, using (7) and (8), with $x = x_\alpha$ and $y = y_\alpha$, it follows

$$\begin{aligned} R_1(x_\alpha, y_\alpha; \lambda', \nu, \bar{x}) &= (1+\rho) R_1(x_\alpha, y_\alpha; \lambda, \nu, \bar{x}) - \rho\nu(x_\alpha - \bar{x}) \\ &= (1+\rho)(1-\alpha) R_1 + (\alpha(1+\rho)(1-\sigma_1) - \rho)\, \nu(x - \bar{x}) \\ &\quad + \alpha\left((1+\rho)(1-\sigma_1) - \rho\right) \nu d^x, \\ R_2(x_\alpha, y_\alpha; \lambda') &= (1+\rho) R_2(x_\alpha, y_\alpha; \lambda) + \rho e \\ &= (1+\rho)(1-\alpha) R_2 + (1+\rho)\alpha^2 \lambda D^x d^y + (\rho - (1+\rho)\alpha)\, e. \end{aligned}$$

To end the proof use definition (6), the above expressions for the residuals, and triangle inequality.

In the next proposition we analyze the predictor/affine scaling step, which is a Newton step aimed at getting $Xy = 0$. This step is used to accelerate the reduction of the duality gap; however, in most cases, this Newton step do not preserve positivity, and a relaxed step shall be used instead. Since our penalization/regularization parameter is λ, instead of the duality gap, we include in the analysis a particular scheme for updating this parameter.

Proposition 2.9 (Predictor/Affine scaling step)**.** *Take $\sigma_2 = 0$ and $\sigma_1 \in (0, 1]$. Consider λ, ν, $\bar{x}$, and (x, y) as in (12), (d^x, d^y) as in (13), (x_α, y_α) as in (14), and β_1 and β_2 as in (15). If*

$$\delta_{\nu,\bar{x}}(x, y; \lambda) \le \beta_2$$

and

$$\rho_\alpha = \frac{\alpha}{1-\alpha}, \quad \lambda_\alpha = (1+\rho_\alpha)\lambda, \quad \alpha \in [0, \alpha^*], \text{ where } \alpha^* = \frac{\sigma_1(\beta_1 - \beta_2)}{5(\beta_2 + \nu^{1/2}\|x - \bar{x}\| + \sqrt{n})^2},$$

then

$$x_\alpha > 0, \quad y_\alpha > 0, \quad \delta_{\nu,\bar{x}}(x_\alpha, y_\alpha; \lambda_\alpha) \le \beta_1. \tag{18}$$

Proof 2.10. *Take any $\alpha \in [0, \alpha^*] \subset [0, 1/2]$. Note from the definitions that it hold*

$$(1+\rho_\alpha)(1-\alpha) = 1, \quad \rho_\alpha = (1+\rho_\alpha)\alpha \le 2\alpha, \quad |1 - \sigma_1 - \alpha| \le 1, \ \forall \alpha \in [0, \alpha^*].$$

Combining the above relations with Lemma 2.7 it follows

$$\begin{aligned} \delta_{\nu,\bar{x}}(x_\alpha, y_\alpha; \lambda_\alpha) &\le \beta_2 + \rho_\alpha \sigma_1 \nu^{1/2}\|x - \bar{x}\| + \rho_\alpha \nu^{1/2}\|d^x\| + \rho_\alpha \alpha \|\lambda D^x D^y e\| \\ &\le \beta_2 + 2\alpha\left(\nu^{1/2}\|x - \bar{x}\| + \nu^{1/2}\|d^x\| + \|\lambda D^x D^y e\|\right). \end{aligned} \tag{19}$$

Let $b_1 = \lambda(Mx + b - y) + \sigma_1\nu(x - \bar{x})$, $b_2 = \lambda X y$, $R_1 = R_1(x, y; \lambda, \nu, \bar{x})$, and $R_2 = R_2(x, y; \lambda)$. From the definition of (d^x, d^y), Lemma 2.3, and Cauchy-Schwartz inequality it follows

$$\sigma_1\nu\|d^x\|^2 + \lambda\|D^x d^y\| \le \langle d^x, b_1\rangle + \frac{\|b_2\|^2}{2(1-\beta)} \le \frac{1}{2}\left(\sigma_1\nu\|d^x\|^2 + \frac{\|b_1\|^2}{\sigma_1\nu}\right) + \frac{\|b_2\|^2}{2(1-\beta_2)},$$

which, combined with relations $\beta_2 \in [0, 1)$ and $\sigma_1 \in (0, 1]$, yields

$$\frac{\sigma_1\nu\|d^x\|^2}{2} + \lambda\|D^x d^y\| \le \frac{\|b_1\|^2}{2\sigma_1\nu} + \frac{\|b_2\|^2}{2(1-\beta_2)} \le \frac{1}{2\sigma_1(1-\beta_2)}\left(\frac{\|b_1\|^2}{\nu} + \|b_2\|^2\right). \tag{20}$$

Hence,

$$\begin{aligned}
\nu^{1/2}\|d^x\| + \lambda\|D^x d^y\| &= \left(\nu^{1/2}\|d^x\| - \frac{\sigma_1\nu\|d^x\|^2}{2}\right) + \left(\frac{\sigma_1\nu\|d^x\|^2}{2} + \lambda\|D^x d^y\|\right) \\
&\le \frac{1}{2\sigma_1} + \frac{1}{2\sigma_1(1-\beta_2)}\left(\frac{\|b_1\|^2}{\nu} + \|b_2\|^2\right) \\
&= \frac{1}{2\sigma_1} + \frac{1}{2\sigma_1(1-\beta_2)}\left(\frac{\|R_1 + (1-\sigma_1)\nu(x-\bar{x})\|^2}{\nu} + \|R_2 + e\|^2\right) \\
&\le \frac{1}{2\sigma_1} + \frac{1}{2\sigma_1(1-\beta_2)}\left(\frac{\|R_1|}{\nu^{1/2}} + \|R_2\| + (1-\sigma_1)\nu^{1/2}\|x-\bar{x}\| + \sqrt{n}\right)^2 \\
&\le \frac{1}{2\sigma_1} + \frac{1}{2\sigma_1(1-\beta_2)}\left(\beta_2 + \nu^{1/2}\|x-\bar{x}\| + \sqrt{n}\right)^2, \qquad (21)
\end{aligned}$$

using in the first inequality relation $t - \sigma_1 t^2/2 \le 1/(2\sigma_1)$, $\forall\, t \in \mathbb{R}$, *and* (20), *in the second inequality triangle inequality and simple algebraic manipulations, and in the last inequality* (6), *the first assumption, and relation* $\sigma_1 \in (0,1]$. *Combining* (19) *and* (21) *we obtain*

$$\begin{aligned}
\delta_{\nu,\bar{x}}(x_\alpha, y_\alpha; \lambda_\alpha) &\le \beta_2 + 2\alpha\left(\nu^{1/2}|x-\bar{x}\| + \frac{1}{2\sigma_1} + \frac{\left(\beta_1 + \nu^{1/2}\|x-\bar{x}\| + \sqrt{n}\right)^2}{2\sigma_1(1-\beta)}\right) \\
&\le \beta_2 + \frac{5\alpha}{\sigma_1}\left(\beta_2 + \nu^{1/2}\|x-\bar{x}\| + \sqrt{n}\right)^2,
\end{aligned}$$

where in the last inequality we used relations $\sigma_1 \in (0,1]$, $\beta_2 \le 2/3$, *and* $t + 3t^2/2 \le 5t^2/2$, $\forall t \ge 1$, *together with simple algebraic manipulations. The third relation in* (18) *follows immediately from the above inequality and the definition of* α^*. *Since* $\delta_{\nu,\bar{x}}(x_\alpha, y_\alpha; \lambda_\alpha) < 1$, $\forall\, \alpha \in [0, \alpha^*]$, *the first two relations in* (18) *can be proved using similar arguments as those used in Proposition 2.5 to prove the first two relations in* (16).

3 Advantages and Disadvantages of Pure Path-Following Methods Based on the Log-Quadratic Barrier.

In this section we show that it is easy to obtain initial iterates in β_2-neighborhoods of log-quadratic central paths. However, as we also show, a basic, pure path-following method for the log-quadratic barrier has a serious drawback: the distance of $\bar{x}$ to the solution set appears multiplicatively in the complexity estimation. Practical implications of this theoretical result are discussed in the end of this section.

As previously discussed, the log-quadratic central path for problem (1) is always well defined. A practical advantage of the log-quadratic barrier is that starting points in β-neighborhoods are easy to compute. Next we present two procedures for computing starting points: Choose $\nu > 0$ and $\rho \in \mathbb{R}$ and

procedure 1: define

$$\begin{aligned}
&\bar{x} = \rho e, \quad x^0 \text{ as the positive solution of } -x^{-1} + \nu(x-\bar{x}) = 0, \\
&\text{if } \|Mx^0 + b\| \ne 0,\ \lambda_0 = \frac{\beta_2\nu^{1/2}}{\|Mx^0 + b\|}, \quad y^0 = \frac{1}{\lambda_0}(x^0)^{-1}; \qquad (22)
\end{aligned}$$

procedure 2: define

$$x^0 = |\rho| e, \quad \lambda_0 = 1, \quad y^0 = (x^0)^{-1}, \quad \bar{x} = x^0 + \frac{1}{\nu}\left(Mx^0 + b - (x^0)^{-1}\right). \tag{23}$$

It is easy to verify that $\delta_{\nu,\bar{x}}(x^0, y^0; \lambda_0) \le \beta_2$ in both procedures. Observe that these initializations are *generic* and do not take in to account particularities of the problem at hand. So, in a specific setting, better choices may be available.

Now we discuss a basic short-step path-following algorithm for the log-quadratic barrier, LQ1.

LQ1 algorithm (a basic short-step method for the log-quadratic barrier).

Initialization: Take $\beta_1 \in (0, 2/3)$, $\beta_2 = \beta_1^2/2(1-\beta_1)$, and $\nu > 0$. Consider $\lambda_0 > 0$, $\bar{x} \in \mathbb{R}^n$ and $(x^0, y^0) > 0$, such that $\delta_{\nu,\bar{x}}(x^0, y^0; \lambda_0) \le \beta_2$.

Iterations: For $k = 0, 1, \ldots$

(step k)

1. Set $\lambda_{k+1} = \max\left\{\lambda > 0 \mid \delta_{\nu,\bar{x}}(x^k, y^k, \lambda) \le \beta_1\right\}$, (update λ)
2. Compute (d^x, d^y), the solution of the linear system (centering step)

$$\begin{bmatrix} \lambda_{k+1} M + \nu I & -\lambda_{k+1} I \\ \lambda_{k+1} Y^k & \lambda_{k+1} X^k \end{bmatrix} \begin{bmatrix} d^x \\ d^y \end{bmatrix} = -\begin{bmatrix} R_1(x^k, y^k; \lambda_{k+1}, \nu, \bar{x}) \\ R_2(x^k, y^k; \lambda_{k+1}) \end{bmatrix},$$

and set

$$(x^{k+1}, y^{k+1}) = (x^k + d^x, y^k + d^y).$$

The above algorithm is being discussed to show that a "pure" path-following method for the log-quadratic central path is not efficient. Therefore, we will present a summary analysis of its complexity. Using induction in k, Proposition 2.5, Lemma 2.3, and Lemma 9.1 it is easy to prove that

1. the above algorithm is well defined,
2. $x^k > 0$, $y^k > 0$, $\delta_{\nu,\bar{x}}(x^k, y^k; \lambda_k) \le \beta_2$, $\forall\, k \in \mathbb{N}$,
3. $\{\lambda_k\}$ is an increasing sequence,
4. setting $H_k = \max\left\{\nu^{1/2}\beta_2 + \nu\|x^k - \bar{x}\|, (1+\beta_2)n\right\}$ it holds

$$\|Mx^k + b - y^k\| \le \frac{H_k}{\lambda_k}, \quad \langle x^k, y^k\rangle \le \frac{H_k}{\lambda_k},$$

5. if the LMCP (1) has solutions, then the sequence $\{(x^k, y^k)\}$ is bounded.

Thus, the complexity of the LQ1 algorithm for attaining ϵ-feasibility and ϵ-complementarity depends on how fast (λ_k) grows. From Lemma 2.1 (see (9)) it follows

$$\frac{\lambda_{k+1}}{\lambda_k} \ge 1 + \frac{\beta_1 - \beta_2}{\beta_2 + \nu^{1/2}\|x^k - \bar{x}\| + \sqrt{n}}. \tag{24}$$

Hence, we need to bound $\nu^{1/2}\|x^k - \bar{x}\|$.

In the rest of the section we will assume that the LMCP (1) has solutions. From item 5 above and (24) it follows that

$$\lambda_k \to \infty, \text{ as } k \to \infty.$$

From this relation, item 2 above, and Lemma 9.3 it follows that all the accumulation points of $\{(x^k, y^k)\}$ are pairs of complementary solutions. Hence,

$$\liminf_{k\to\infty} \|x^k - \bar{x}\| \geq \tau = d(\bar{x}, S^*),$$

which, combined with (24), item 4 above, and a standard complexity analysis (see, for example, [28, Theorem 3.2]), yields a complexity estimation *not better* than

$$O\left(\nu^{1/2}\tau + \sqrt{n}\right) \log_2 \left(\frac{\nu^{1/2}\tau + n}{\lambda_0 \epsilon}\right), \tag{25}$$

to reach ϵ-feasibility and ϵ-complementarity. Thus, if $\|x^k - \bar{x}\|$ (which lim inf is bounded by $d(\bar{x}, S^*)$) is too large, it may slows down the growth of λ_k, deteriorating the performance of the algorithm.

In accordance with these theoretical reasonings, in our preliminary numerical tests the LQ1 algorithm had a poor performance as compared to: a combination of a similar method with an extragradient scheme (see Section 5) and the infeasible method of Roos et.al. [22]. These numerical tests are are summarized in Table 1, in Section 7. Similar considerations and results hold for LQ2, a pure short-step path-following predictor-corrector algorithm for the log-quadratic barrier.

4 The Hybrid Proximal Extragradient Step.

Our aim in this section is to bound $\nu^{1/2}\|x^k - \bar{x}\|$ by applying (in the variable $\bar{x}$) relaxed steps of the hybrid proximal-extragradient method proposed in [25, 24, 26].

Consider the maximal monotone point-to-set operator

$$\widehat{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, \quad \widehat{F}(x) = Mx + b + N_{\mathbb{R}^n_+}(x).$$

The LMCP (1) is equivalent to the monotone inclusion problem: find $x^* \in \mathbb{R}^n$ such that

$$0 \in \widehat{F}(x^*). \tag{26}$$

The proximal point method [21] is a classical algorithm for finding zeros of maximal monotone operators. Applied to problem (26), in its exact formulation, it generates iteratively a sequence $\{x^k\}$ by solving, in each iteration, the problem: given $c_k > 0$ and $x^k \in \mathbb{R}^n$, find $x^{k+1} \in \mathbb{R}^n$ such that

$$0 \in c_k \widehat{F}(x^{k+1}) + x^{k+1} - x^k.$$

The classical inexact version of the proximal point method requires summable errors.

The hybrid proximal extragradient method (HPE) [24] is a modification of the classical proximal point method that uses, instead, a relative error tolerance. The constructive nature of the error tolerance of the HPE method will allow us to use (relaxed) steps of this methods in our scheme. First we discuss the error tolerance of the HPE method. A generic iteration of the proximal point method is: find x such that

$$0 \in c\widehat{F}(x) + x - \bar{x},$$

which can be reformulated as the proximal system: find $(x, w) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w \in \widehat{F}(x), \quad cw + x - \bar{x} = 0. \tag{27}$$

The inclusion in (27) will be relaxed using the concept of ϵ-enlargement of a maximal monotone operator [5]. Given $\epsilon \geq 0$, the ϵ-enlargement of $\widehat{F}$ is the operator $\widehat{F}^{[\epsilon]} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$

$$\widehat{F}^{[\epsilon]}(x) = \left\{ w \mid \langle w - w', x - x' \rangle \geq -\epsilon, \ \forall\, x' \in D(\widehat{F}), \ w' \in \widehat{F}(x') \right\}.$$

As $\widehat{F}$ is maximal monotone, it holds $\widehat{F}(x) \subset \widehat{F}^{[\epsilon]}(x), \ \forall\, \epsilon \geq 0, \ x \in \mathbb{R}^n$, with equality for $\epsilon = 0$ (for other properties and applications see [6, 7, 4]). Following [24], we say that a pair $(x, w) \in \mathbb{R}^n \times \mathbb{R}^n$ is an approximate solution of (27), with error tolerance $\eta \in [0, 1)$, if for some $\epsilon \geq 0$ it hold

$$w \in \widehat{F}^{[\epsilon]}(x), \quad \|cw + x - \bar{x}\|^2 + 2c\epsilon \leq \eta \|x - \bar{x}\|^2. \tag{28}$$

Next we recall a key property of approximate solutions of (27), which is the basis of the proximal extragradient algorithm presented in [24]. Since we will consider algorithms in which the iterates remain in β-neighborhoods of the central path, we will analyze under-relaxed hybrid proximal-extragradient steps.

Lemma 4.1. *Consider $c > 0$, $\eta \in [0, 1)$, and $\bar{x} \in \mathbb{R}^n$. If x, w, and ϵ satisfy* (28) *and*

$$\bar{x}(\theta) = \bar{x} - \theta c w, \ \theta \in [0, 1],$$

then, for any x^* solution of (26),

$$\|x^* - \bar{x}(\theta)\|^2 \leq \|x^* - \bar{x}\|^2 - \theta(1 - \eta^2)\|x - \bar{x}\|^2. \tag{29}$$

Proof 4.2. *In [24, Lemma 2.3] it was proved that, under the above assumptions,* (29) *holds for $\theta = 1$. Since this inequality holds trivially for $\theta = 0$ and the norm square is convex,* (29) *holds for any $\theta \in [0, 1]$.*

Lemma 4.1 shows how to use approximate solutions of (27) in the sense of (28) for displacing $\bar{x}$ along direction $-w$ and towards the solution set. This lemma also estimates how much smaller is the square of the distance to the solution set of the "displaced" $\bar{x}_\theta$ as compared to $\bar{x}$. This quantitative estimation will be used in our complexity analysis to bound the number of relaxed hybrid extragradient steps in our scheme.

Now we recall how the duality gap of a pair $(x, y) \geq 0$ is related with $\widehat{F}^{[\epsilon]}$.

Lemma 4.3. *If $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ and $\varepsilon \geq \langle x, y \rangle$, then*

$$Mx + b - y \ \in \widehat{F}^{[\varepsilon]}(x).$$

Proof 4.4. *See [8, Lemma 2.2], using that $D(\widehat{F}) = \mathbb{R}^n_+$.*

In the next proposition we prove that a pair (x, y) in the inner neighborhood of a log-quadratic central path, with a "large" value of $\nu^{1/2}\|x - \bar{x}\|$, provides an inexact solution of (27) in the sense of (28). We also present a relaxation parameter $\hat{\theta}$ for performing an under-relaxed hybrid extragradient step, in variable $\bar{x}$, as described in Lemma 4.1, which keeps (x, y) in the outer neighborhood of the (updated) log-quadratic central path. In the last part of the proposition we evaluate the rightmost term of inequality (29) for $\theta = \hat{\theta}$.

Proposition 4.5. *Take $\beta_2,\ \beta_1 \in [0,1]$, with $\beta_2 < \beta_1$. Consider λ, ν, $\bar{x}$, and (x,y) as in* (12)*. Suppose that*

$$\delta_{\nu,\bar{x}}(x,y;\lambda) \le \beta_2, \quad (4n+1) \le \nu\|x-\bar{x}\|^2, \tag{30}$$

and define

$$w = Mx + b - y, \quad \epsilon = \frac{(1+\beta_2)n}{\lambda}, \quad c = \frac{\lambda}{\nu}, \quad \eta = \sqrt{\frac{1+\beta_2}{2}}.$$

Then

$$w \in \widehat{F}^{[\epsilon]}(x), \quad \|cw + x - \bar{x}\|^2 + 2c\epsilon \le \eta^2\|x-\bar{x}\|^2. \tag{31}$$

Moreover defining

$$\bar{x}(\hat{\theta}) = \bar{x} - \hat{\theta}\, cw, \quad \hat{\theta} = \frac{\beta_1 - \beta_2}{2\nu^{1/2}\|x-\bar{x}\|}$$

it holds

$$0 \le \hat{\theta} \le 1, \quad \delta_{\nu,\bar{x}(\hat{\theta})}(x,y;\lambda) \le \beta_1, \tag{32}$$

and

$$\hat{\theta}(1-\eta^2)\|x-\bar{x}\|^2 \ge \frac{(1-\beta_2)(\beta_1-\beta_2)\sqrt{4n+1}}{4\nu}. \tag{33}$$

Proof 4.6. *From the first inequality in* (30)*, definition* (6)*, and Lemma 2.3 it follows*

$$\frac{\|\lambda w + \nu(x-\bar{x})\|}{\nu^{1/2}} \le \beta_2, \quad \langle x, y\rangle \le \frac{(1+\beta_2)n}{\lambda}.$$

The first relation in (31) *follows from the second above inequality and Lemma 4.3. From the first above inequality, the definitions of c and ϵ, and the second inequality in* (30) *it follows*

$$\|cw + x - \bar{x}\|^2 + 2c\epsilon \le \frac{\beta_2^2}{\nu} + \frac{2(1+\beta_2)n}{\nu} \le \frac{(1+\beta_2)(4n+1)}{2\nu} \le \frac{(1+\beta_2)}{2}\|x-\bar{x}\|^2,$$

where in the second inequality we used the assumption $\beta_2 \le 1$ and simple algebraic manipulations. The second relation in (31) *follows from the above inequalities and the definition of η.*

The first inequality in (32) *follows from the assumptions on β_1 and β_2 and the second inequality in* (30)*. Note that*

$$\begin{aligned} R_1(x,y;\lambda,\nu,\bar{x}(\hat{\theta})) &= \lambda(Mx+b-y) + \nu(x - \bar{x}(\hat{\theta})) \\ &= \lambda(Mx+b-y) + \nu(x-\bar{x}) + \hat{\theta}\lambda(Mx+b-y) \\ &= (1+\hat{\theta})\big(\lambda(Mx+b-y) + \nu(x-\bar{x})\big) - \nu\hat{\theta}(x-\bar{x}). \end{aligned}$$

The second residual $R_2(x,y;\lambda)$ does not depend on $\bar{x}$; therefore, using (6) *it follows*

$$\begin{aligned} \delta_{\nu,\bar{x}(\hat{\theta})}(x,y;\lambda) &= \frac{\|R_1(x,y;\lambda,\nu,\bar{x}(\hat{\theta}))\|}{\nu^{1/2}} + \|R_2(x,y;\lambda)\| \\ &\le \frac{(1+\hat{\theta})\|R_1(x,y;\lambda,\nu,\bar{x})\|}{\nu^{1/2}} + \|R_2(x,y;\lambda)\| + \hat{\theta}\nu^{1/2}\|x-\bar{x}\| \\ &\le (1+\hat{\theta})\delta_{\nu,\bar{x}}(x,y;\lambda) + \hat{\theta}\nu^{1/2}\|x-\bar{x}\| \le \beta_2 + \hat{\theta}\left(\beta_2 + \nu^{1/2}\|x-\bar{x}\|\right), \end{aligned}$$

which, combined with the definition of $\hat{\theta}$, proves the second inequality in (32)*.*

Inequality (33) *follows from the definitions of $\hat{\theta}$ and η and the second inequality in* (30)*.*

Observe in the above proposition that, after the relaxed hybrid extragradient step, $(x, y) \in \mathcal{N}_{\beta_1,\nu,\bar{x}_{\hat{\theta}}}$ Thus, for β_1 and β_2 as in (15), one centralizing full Newton step is enough to return to $\mathcal{N}_{\beta_2,\nu,\bar{x}_{\hat{\theta}}}$, the inner neighborhood of the updated log-quadratic central path.

5 An Interior Hybrid Proximal Extragradient Algorithm.

In this section we present an algorithm for the LMCP (1), which is based on the combination of a short-step path-following scheme for the log-quadratic barrier with an hybrid proximal extragradient scheme. If the solution set of the LMCP (1) is non-empty, the algorithm performs a finite number of proximal extragradient steps and generates bounded sequences, with all their accumulation points in the set of pairs of complementary solutions. The extragradient steps control the quadratic regularization terms, and prevent the distance of point $\bar{x}$ to the solution set from appearing multiplicatively in the complexity estimation of the algorithm. For the initialization of the algorithm we could use any of the procedures described in (22) and (23). The algorithm will be denoted as the short-step interior-hybrid proximal extragradient (IHPE1) algorithm.

IHPE1 algorithm

Initialization: Take $\beta_1 \in (0, 2/3)$, $\beta_2 = \beta_1^2/2(1-\beta_1)$, and $\nu > 0$. Consider $\lambda_0 > 0$, $\bar{x}^0 \in \mathbb{R}^n$, and $(x^0, y^0) > 0$, such that $\delta_{\nu,\bar{x}^0}(x^0, y^0; \lambda_0) \leq \beta_2$.

Iterations: For $k = 0, 1, \ldots$

(step k)

1. **(a)** If $\nu\|x^k - \bar{x}^k\|^2 < 4n + 1$, then set (interior point: update λ)

$$\lambda_{k+1} = \max\left\{\lambda > 0 \mid \delta_{\nu,\bar{x}^k}(x^k, y^k, \lambda) \leq \beta_1\right\},$$
$$\bar{x}^{k+1} = \bar{x}^k.$$

(b) Else, set (relaxed hybrid proximal extragradient step)

$$\bar{x}^k(\theta) = \bar{x}^k - \theta\frac{\lambda_k}{\nu}w^k, \quad \text{where} \quad w^k = Mx^k + b - y^k,$$
$$\theta_k = \max\left\{\theta \in [0, 1] \mid \delta_{\nu,\bar{x}^k(\theta)}(x^k, y^k, \lambda_k) \leq \beta_1\right\},$$
$$\bar{x}^{k+1} = \bar{x}^k(\theta_k),$$
$$\lambda_{k+1} = \lambda_k.$$

2. Compute (d^x, d^y), the solution of the linear system (centering step)

$$\begin{bmatrix} \lambda_{k+1}M + \nu I & -\lambda_{k+1}I \\ \lambda_{k+1}Y^k & \lambda_{k+1}X^k \end{bmatrix} \begin{bmatrix} d^x \\ d^y \end{bmatrix} = -\begin{bmatrix} R_1(x^k, y^k; \lambda_{k+1}, \nu, \bar{x}^{k+1}) \\ R_2(x^k, y^k; \lambda_{k+1}) \end{bmatrix},$$

and set

$$(x^{k+1}, y^{k+1}) = (x^k + d^x, y^k + d^y).$$

To simplify the analysis of the algorithm we define

$$\hat{h} = \frac{(1-\beta_2)(\beta_1-\beta_2)\sqrt{4n+1}}{2\nu}, \quad \hat{\rho} = \frac{\beta_1-\beta_2}{\beta_2+\sqrt{4n+1}+\sqrt{n}}. \tag{34}$$

In the next result we prove that the algorithm IHPE1 is well defined and generates sequences in β_2-neighborhoods of certain log-quadratic central paths; also, we analyze the outcomes of steps 1.(a) and 1.(b).

Theorem 5.1. *The algorithm IHPE1 is well defined;*

$$x^k > 0, \quad y^k > 0, \quad \lambda_k > 0, \quad \delta_{\nu,\bar{x}^k}(x^k, y^k; \lambda_k) \le \beta_2, \ \forall\, k \in \mathbb{N}; \tag{35}$$

and

i) *if at iteration k an interior point step 1.(a) is performed, then*

$$\lambda_{k+1} \ge \lambda_k(1+\hat{\rho}),$$

ii) *if at iteration k a relaxed hybrid proximal extragradient step 1.(b) is performed, then for any x^* solution of the LMCP (1)*

$$\|\bar{x}^{k+1} - x^*\|^2 \le \|\bar{x}^k - x^*\|^2 - \hat{h}.$$

Proof 5.2. *First observe that the initialization condition of β_1 and β_2 is the same condition considered in (15) and*

$$0 < \beta_2 < \beta_1 < 1.$$

We prove the first part of the theorem by induction. Assume that the algorithm is well defined up to any index $i < k$ and that (35) holds for index k. In view of (35) and the above inequalities, step 1 (which is either 1.(a) or 1.(b)) is well defined and

$$x^k > 0, \quad y^k > 0, \quad \lambda_{k+1} > 0, \quad \delta_{\nu,\bar{x}^{k+1}}(x^k, y^k; \lambda_{k+1}) \le \beta_1.$$

Hence, from Lemma 2.3 it follows that step 2 is well defined. From the above inequalities, the definition of step 2, and Proposition 2.5 we conclude that

$$x^{k+1} > 0, \quad y^{k+1} > 0, \quad \lambda_{k+1} > 0, \quad \delta_{\nu,\bar{x}^{k+1}}(x^{k+1}, y^{k+1}; \lambda_{k+1}) \le \beta_2,$$

which means that (35) holds for $k+1$. To conclude the induction proof, observe that (35) holds for $k = 0$.

To prove i), assume that step 1.(a) is performed at iteration k, which means that

$$\nu^{1/2}\|x^k - \bar{x}^k\| < \sqrt{4n+1},$$

and the algorithm increases parameter λ_k. Using (35), Lemma 2.1 (relation (9)), with $x = x^k$, $y = y^k$, $\bar{x} = \bar{x}^k$, and $\lambda = \lambda_k$, and the above inequality it follows that

$$\lambda_{k+1} = (1+\rho_k)\lambda_k, \quad \textit{with} \ \ \rho_k \ge \frac{\beta_1-\beta_2}{\beta_2+\nu^{1/2}\|x^k-\bar{x}^k\|+\sqrt{n}} \ge \hat{\rho} > 0,$$

which ends the proof.

To prove ii), let x^* be a solution of (1) *and suppose that step* 1.*(b) is performed at iteration k, which means that*

$$\sqrt{4n+1} \leq \nu^{1/2}\|x^k - \bar{x}^k\|.$$

Define

$$\epsilon_k = \frac{(1+\beta_2)n}{\lambda_k}, \quad c_k = \frac{\lambda_k}{\nu}, \quad \eta = \sqrt{\frac{1+\beta_2}{2}}, \quad \hat{\theta}_k = \frac{\beta_1 - \beta_2}{2\nu^{1/2}\|x^k - \bar{x}^k\|}.$$

Using (35)*, the above inequality, and Proposition 4.5, with $x = x^k$, $y = y^k$, $\lambda = \lambda_k$, and $w = w^k$, it follows that*

$$w_k \in \hat{F}^{[\epsilon_k]}(x^k), \quad \|c_k w^k + x^k - \bar{x}^k\|^2 + 2c_k\epsilon_k \leq \eta^2\|x^k - \bar{x}^k\|^2, \quad \hat{\theta}_k \leq \theta_k \leq 1.$$

Therefore, using the above relations and Lemma 4.1 we conclude that

$$\begin{aligned}\|x^* - \bar{x}^{k+1}\|^2 = \|x^* - \bar{x}^k(\theta_k)\|^2 &\leq \|x^* - \bar{x}^k\|^2 - \theta_k(1-\eta^2)\|x^k - \bar{x}^k\|^2 \\ &\leq \|x^* - \bar{x}^k\|^2 - \hat{\theta}_k(1-\eta^2)\|x^k - \bar{x}^k\|^2 \\ &\leq \|x^* - \bar{x}^k\|^2 - \hat{h},\end{aligned}$$

where the last inequality follows from (33)*.*

In the next result we bound the number of hybrid extragradient steps of algorithm IHPE1 and study its global convergence.

Theorem 5.3. *If S^*, the solution set of the LMCP* (1)*, is non empty, then*

i) *at most K relaxed hybrid proximal extragradient steps* 1.(b) *are performed, where*

$$K = \left\lfloor \frac{d(\bar{x}^0, S^*)^2}{\hat{h}} \right\rfloor,$$

ii) *the sequence $\{1/\lambda_k\}$ converges Q-linearly to zero as $k \to +\infty$,*

iii) *the sequence $\{(x^k, y^k)\}$ is bounded and its accumulation points are pairs of complementary solutions of the LMCP* (1).

Proof 5.4. *To prove i), let be $x^* \in S^*$ and let $k_j, j = 1, \ldots, m$, denote the indexes corresponding to the first m hybrid steps. From ii) of Theorem 5.1 it follows*

$$\|\bar{x}^{k_j+1} - x^*\|^2 \leq \|\bar{x}^{k_j} - x^*\|^2 - \hat{h}, \; j = 1, \ldots, m.$$

Since $\bar{x}^k$ does not change at the interior-point iterations, $\bar{x}^{k_{j+1}} = \bar{x}^{k_j+1}$ for $j = 1, \ldots, m-1$. Thus, adding the above inequalities and taking x^* as the projection of $\bar{x}^0$ in S^* (which is a non-empty, closed, and convex set) we obtain

$$0 \leq \|\bar{x}^{k_m+1} - x^*\|^2 \leq \|\bar{x}^0 - x^*\|^2 - m\hat{h} = d(\bar{x}^0, S^*)^2 - m\hat{h},$$

which proves that $m \leq K$, ending the proof of i).

From i), the definition of step 1.*(a), and i) of Theorem 5.1 it follows that for k large enough step* 1.*(a) is performed and*

$$\nu^{1/2}\|x^k - \bar{x}^k\| < \sqrt{4n+1}, \quad \bar{x}^{k+1} = \bar{x}^k, \quad \lambda_{k+1} \geq \lambda_k(1+\hat{\rho}).$$

Therefore, ii) holds, the sequence $\{\bar{x}^k\}$ assumes a finite number of distinct values, and the sequence $\{x^k\}$ is bounded. Note that for any k,

$$Mx^k + b - y^k + \frac{\nu}{\lambda_k}(x^k - \bar{x}^k) = 0, \quad \|\lambda_k X^k y^k - e\| \leq \beta_2. \tag{36}$$

From the first above relation, the boundedness of the sequences $\{\bar{x}^k\}$ and $\{x^k\}$, and ii) it follows that the sequence $\{y^k\}$ is bounded. We just proved that the sequences $\{\bar{x}^k\}$ and $\{(x^k, y^k)\}$ are bounded. Combining this result with the above equation, ii), and the first two inequalities in (35) *we conclude that all the accumulation points of the sequence $\{(x^k, y^k)\}$ are pairs of complementary solutions.*

In the next result we present a complexity analysis of the algorithm IHPE1. To this purpose, given $\epsilon > 0$, we define an *ϵ-solution* of the LMPC as a pair $(x, y) \geq 0$ satisfying

$$\|Mx + b - y\| \leq \epsilon, \quad \langle x, y\rangle \leq \epsilon. \tag{37}$$

Theorem 5.5. *Take $\epsilon > 0$. Define $H = \max\left\{(1+\beta_2)n, \nu^{1/2}\sqrt{4n+1}\right\}$ and let $\hat{h}$ and $\hat{\rho}$ be defined as in* (34)*. Set*

$$K(\epsilon) = 1 + \left\lfloor \left(1 + \frac{1}{\hat{\rho}}\right)\log_2\left(\frac{H}{\lambda_0 \epsilon}\right)\right\rfloor + \left\lfloor \frac{d\left(\bar{x}^0, S^*\right)^2}{\hat{h}}\right\rfloor. \tag{38}$$

If the LMCP (1) *has solutions, then algorithm IHPE*1 *generates an ϵ-solution after at most $K(\epsilon)$ iterations.*

Proof 5.6. *Define k as the iteration in which the algorithm performs its $\tilde{k}$-th interior point step* 1.*(a), where*

$$\tilde{k} = 2 + \left\lfloor \left(1 + \frac{1}{\hat{\rho}}\right)\log_2\left(\frac{H}{\lambda_0 \epsilon}\right)\right\rfloor.$$

Since in each iteration either one interior step or one hybrid step is performed, from i) of Theorem 5.3 and (38) *it follows that*

$$k \leq \tilde{k} + \left\lfloor \frac{d\left(\bar{x}^0, S^*\right)^2}{\hat{h}}\right\rfloor = K(\epsilon) + 1.$$

To end the proof it suffices to prove that (x^k, y^k) satisfy (37)*.*

Since parameter $\lambda_{(\cdot)}$ does not change in the hybrid extragradient steps, using i) of Theorem 5.1, the above definition of k, and the definition of the interior point step 1.*(a) we have*

$$\lambda_k \geq \lambda_0(1+\hat{\rho})^{\tilde{k}-1}, \quad \nu^{1/2}\|x^k - \bar{x}^k\| < \sqrt{4n+1}. \tag{39}$$

Relations (36) *and* (39) *yield*

$$\|Mx^k + b - y^k\| = \frac{\nu}{\lambda_k}\|x^k - \bar{x}^k\| < \frac{\nu^{1/2}}{\lambda_k}\sqrt{4n+1} \leq \frac{H}{(1+\hat{\rho})^{\tilde{k}-1}\lambda_0}.$$

The last relation in (35)*, Lemma 2.3, and* (39) *yield*

$$\langle x^k, y^k \rangle \leq \frac{(1+\beta_2)n}{\lambda_k} \leq \frac{H}{(1+\hat{\rho})^{\tilde{k}-1}\lambda_0}.$$

Using the definition of $\tilde{k}$ *and inequality* $\ln(1+h) \geq h/(1+h)$, $\forall h \geq 0$, *it follows that*

$$\frac{H}{(1+\hat{\rho})^{\tilde{k}-1}\lambda_0} \leq \epsilon,$$

which, combined with the above inequalities, implies that (x^k, y^k) *is an* ϵ*-solution of the LMCP* (1)*. Therefore, the IHPE*1 *algorithm finds an* ϵ*-solution after at most* $K(\epsilon)$ *iterations.*

To compare the IHPE1 algorithm with the LQ1 algorithm (presented in Section 3), consider, for example, $\bar{x}^0 = 0$ and the initialization procedure described in (22). With these initializations, the complexity estimate (25) for LQ1 yields the bound

$$O\left(\left(\nu^{1/2}d(0,S^*) + \sqrt{n}\right)\log_2\left(\frac{(\nu^{1/2}d\,(0,S^*) + n)\|Mx^0 - b\|}{\nu^{1/2}\epsilon}\right)\right), \tag{40}$$

while the complexity estimate for IHP1, presented in Theorem 5.5, yields the bound

$$O\left(\sqrt{n}\log_2\left(\frac{(\nu^{1/2}\sqrt{n} + n)\|Mx^0 - b\|}{\nu^{1/2}\epsilon}\right) + \frac{\nu d\,(0,S^*)^2}{\sqrt{n}}\right),$$

which improves the main term of (40).

We end this section showing that, as a direct consequence of Theorem 5.5, we can verify the existence of a solution in some regions.

Proposition 5.7. *Take* $\epsilon > 0$*. Let* H *be defined as in Theorem 5.5 and let* $\hat{\rho}$ *and* $\hat{h}$ *be as define in* (34)*. Set*

$$K_\epsilon = 1 + \left\lfloor \left(1 + \frac{1}{\hat{\rho}}\right)\log_2\left(\frac{H}{\lambda_0\epsilon}\right)\right\rfloor.$$

If no ϵ*-solution is found by the algorithm* $IHPE1$ *after* $\hat{K} > K_\epsilon$ *iterations, then the LMCP* (1) *has no solution on the ball* $B(\bar{x}_0, r)$*, with*

$$r = \sqrt{(\hat{K} - K_\epsilon)\hat{h}}.$$

Proof 5.8. *Assume that there exists a solution of the LMCP* (1) *on the ball* $B(\bar{x}_0, r)$*. This is equivalent to*

$$d(\bar{x}_0, S^*)^2 \leq (\hat{K} - K_\epsilon)\hat{h},$$

and also to

$$K_\epsilon + \left\lfloor \frac{d(\bar{x}^0, S^*)^2}{\hat{h}} \right\rfloor < \hat{K}.$$

From the inequality above, applying i) of Theorem 5.5, we conclude that before $\hat{K}$ *iterations an* ϵ*-solution is found.*

6 A Predictor-Corrector Interior-Hybrid Proximal Extragradient Algorithm.

In this section we present an algorithm that combines an interior, short-step, predictor-corrector path-following scheme with an hybrid proximal extragradient scheme. This algorithm will be denoted as the predictor-corrector interior hybrid proximal extragradient (IHPE2) algorithm. For the initialization of the algorithm we can use any of the procedures described in (22) and (23).

IHPE2 algorithm

Initialization: Take $\beta_1 \in (0, 2/3)$, $\beta_2 = \beta_1^2/2(1-\beta_1)$, $\nu > 0$, and $\sigma_1 \in (0,1]$. Consider $\lambda_0 > 0$, $\bar{x}_0 \in \mathbb{R}^n$, and $(x_0, y_0) > 0$, such that $\delta_{\nu,\bar{x}_0}(x_0, y_0; \lambda_0) \le \beta_2$.

Iteration: For $k = 0, 1, \ldots$

(step k)

1. (a) If $\nu\|x^k - \bar{x}^k\|^2 < 4n+1$, then (interior point: predictor/affine-scaling step)

compute (d^x, d^y), the solution of the linear system

$$\begin{bmatrix} \lambda_k M + \sigma_1 \nu I & -\lambda_k I \\ \lambda_k Y^k & \lambda_k X^k \end{bmatrix} \begin{bmatrix} d^x \\ d^y \end{bmatrix} = - \begin{bmatrix} \lambda_k(Mx^k + b - y^k) + \sigma_1\nu(x^k - \bar{x}^k) \\ \lambda_k X^k y^k \end{bmatrix}.$$

Set

$$\begin{aligned} &(x_\alpha, y_\alpha) = (x, y) + \alpha(d^x, d^y), \quad \rho_\alpha = \frac{\alpha}{1-\alpha}, \quad \lambda_\alpha = \lambda(1+\rho_\alpha), \\ &\alpha_k = \max\left\{\alpha \in [0,1] \mid (x_\alpha, y_\alpha) > 0,\ \delta_{\nu,\bar{x}^k}\big(x_\alpha, y_\alpha; \lambda(1+\rho_\alpha)\big) \le \beta_1\right\}, \\ &(x^k, y^k) = (x(\alpha_k), y(\alpha_k)), \\ &\lambda_{k+1} = \lambda(\alpha_k), \\ &\bar{x}^{k+1} = \bar{x}^k. \end{aligned}$$

(b) Else, set (relaxed hybrid proximal extragradient step)

$$\begin{aligned} &\bar{x}^k(\theta) = \bar{x}^k - \theta\frac{\lambda_k}{\nu} w^k, \quad \text{where} \quad w^k = Mx^k + b - y^k, \\ &\theta_k = \max\left\{\theta \in [0,1] \mid \delta_{\nu,\bar{x}^k_\theta}(x^k, y^k, \lambda_k) \le \beta_1\right\}, \\ &\bar{x}^{k+1} = \bar{x}^k(\theta_k), \\ &\lambda_{k+1} = \lambda_k. \end{aligned}$$

2. Compute (d^x, d^y), the solution of the system (interior point: corrector/centering step)

$$\begin{bmatrix} \lambda_{k+1} M + \nu I & -\lambda_{k+1} I \\ \lambda_{k+1} Y^k & \lambda_{k+1} X^k \end{bmatrix} \begin{bmatrix} d^x \\ d^y \end{bmatrix} = - \begin{bmatrix} R_1(x^k, y^k; \lambda_{k+1}, \nu, \bar{x}^{k+1}) \\ R_2(x^k, y^k; \lambda_{k+1}) \end{bmatrix},$$

and set

$$(x^{k+1}, y^{k+1}) = (x^k + d^x, y^k + d^y).$$

Analogous results to Theorems 5.1, 5.3, and 5.5 and Proposition 5.7 can be proved for the IHPE2 algorithm using Proposition 2.9 and similar reasonings as those used in the analysis of the IHPE1. For the sake of brevity we skip these results and just present its complexity estimation.

Theorem 6.1. *Take* $\epsilon > 0$. *Define* $H = \max\left\{(1+\beta_2)n, \nu^{1/2}\sqrt{4n+1}\right\}$, $\bar{\rho} = \dfrac{\sigma_1(\beta_1-\beta_2)}{5(\beta_2+\sqrt{4n+1}+\sqrt{n})^2}$, *and let* $\hat{h}$ *be defined as in* (34). *Set*

$$\widehat{K}(\epsilon) = 1 + \left\lfloor \left(1+\frac{1}{\bar{\rho}}\right)\log_2\left(\frac{H}{\lambda_0\epsilon}\right)\right\rfloor + \left\lfloor \frac{d\left(\bar{x}^0, S^*\right)^2}{\hat{h}} \right\rfloor. \tag{41}$$

If the LMCP (1) *has solutions, then algorithm* $IHPE2$ *reaches an* ϵ*-solution after at most* $\widehat{K}(\epsilon)$ *iterations.*

7 Numerical Tests.

We test the algorithms considering LMCPs as in (1) defined by operators of the form

$$F: \mathbb{R}^{n+m} \to R^{n+m}, \quad F(x,s) = \begin{bmatrix} 0 & -A^t \\ A & 0 \end{bmatrix}\begin{bmatrix} x \\ s \end{bmatrix} + \begin{bmatrix} c \\ -b \end{bmatrix}.$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. These LMCPs arise from the KKT optimality conditions for linear programming problems in the standard form:

$$\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax \geq b, \\ & x \geq 0. \end{array} \tag{42}$$

As testing problems we consider the well known example of Beale [12] and some problems of small size of the Netlib library. We reduced all the problems to the standard form (42) using the procedure described in Appendix D of [23]. We got the best results using parameters $\beta_1 = .64$ and β_2 defined as in (15) and, for algorithm IHPE2, $\sigma_1 = 0.4$, and the initialization procedure (22), with $\bar{x}_0 = -n^{.25}e$ and $\nu = \frac{1}{\|\bar{x}_0\|}$. As stopping rule we used

$$\frac{|c^t x^k - b^t y^k|}{1+|c^t x^k|} + \frac{\|Ax^k - b - v^k\|_1}{1+\|b\|_1} + \frac{\|A^t s^k - c - u^k\|_1}{1+\|c\|_1} \leq 10^{-8},$$

where $y = (u, v)$ denotes dual variable in the primal-dual formulation of the LMCP.

We implemented in MATLAB the two hybrid interior-extragradient algorithms IHPE1 and IHPE2. Also we implemented their corresponding "pure interior" versions, that do not use the proximal extragradient step, denoted by LQ1 and LQ2. We implemented a modified version of the algorithm IHPE2, indicated by IHPE2+, that allows over-relaxed centering steps, and proved to be more efficient. The convergence and complexity analysis of this last algorithm can be carried out in a similar way as for the algorithm IHPE2. Also, we implemented the infeasible interior-point algorithm of Roos et.al described in [22], with parameters (in the notation of the authors) $\theta = 1/n$ and $\tau = 1/8$.

In Table 1 we present the dimensions of each solved problem, denoted by n, and the number of iterations required by each of the algorithms to satisfy the stopping rule. Between parenthesis we

Table 1:

Name	n	Roos	LQ1	IHPE1	LQ2	IHPE2	IHPE2 +
Beale	7	186	300	300 (0)	101	101 (0)	70 (0)
Afiro	51	1449	2116	1008 (71)	785	370 (71)	246 (51)
Sc50b	78	2397	2215	1260 (51)	781	428 (52)	290 (42)
Sc50a	78	2364	2289	1253 (58)	794	431 (58)	281 (42)
Adlittle	138	4437	> 5000	4004 (2263)	> 5000	2826 (2257)	1793 1363
Share2b	162	> 5000	1306	1306 (0)	329	329 (0)	213 (0)
Sc105	163	> 5000	4130	1933 (122)	1537	710 (122)	426 (80)
Stocfor1	165	> 5000	> 5000	3202 (1346)	> 5000	1952 (1347)	1553 (793)
Share1b	253	> 5000	> 5000	4534 (1738)	> 5000	2663 (1680)	1916 (1112)
Sc205	317	> 5000	> 5000	3170 (414)	> 5000	1316 (42)	957 (253)

present the number of extragradient steps performed by algorithms IHPE1, IHPE2, and IHPE2+. The notation > 5000 indicates that the algorithm was stopped after reaching 5000 iterations. Note that at each iteration algorithms LQ1 and IHPE1 solve one Newton system, while algorithms LQ2, IHPE2, and IHPE2+ solve two Newton systems.

As expected, these short-steps algorithms, confined to tight neighborhoods of the log-quadratic central path, are not efficient. However, we observed that using the proximal-extragradient improved the performance of the pure interior point algorithms. The hybrid algorithms, in particular the algorithm IHPE2+, showed the best behavior. We must point out that in our numerical tests, using other starting procedures, the algorithm of Roos matched the best performance of the hybrid algorithms, showing a more robust behavior in relation to the selection of the initial iterate.

8 Conclusions.

In this work we developed infeasible short-steps path-following algorithms for solving the LMCP (1). For this purpose, we define and study central paths associated to log-quadratic barriers. These paths are well defined and, whenever the solution set is not empty, convergent to a pair of complementary solutions. To improve the theoretical and practical performance of "pure" path-following algorithms based on log-quadratic barriers, we devised hybrid algorithms using the machinery of the HPE method for finding zeros of maximal monotone operators. Encouraging preliminary numerical experiments were presented. Extension of these ideas to devise similar algorithms for solving Nonlinear Monotone Complementarity Problems and more efficient long-step path-following HPE algorithms for solving the LMCP is object of further research.

9 Appendix

In this section we present the main features of the log-quadratic central path. For this purpose, we need some previous definitions and technical results.

Define the sets

$$I = \{i \,|\, \hat{x}_i > 0 \text{ for some } (\hat{x}, \hat{y}) \in S^*_{CP}\}\,, \quad J = \{j \,|\, \hat{y}_j > 0 \text{ for some } (\hat{x}, \hat{y}) \in S^*_{CP}\}\,. \tag{43}$$

and the affine subspace

$$\widehat{S}_{CP} = \{(x, y) \mid Mx + b - y = 0,\ x_j = 0,\ j \in \{1, \ldots, n\} \backslash I,\ y_i = 0,\ i \in \{1, \ldots, n\} \backslash J\}. \tag{44}$$

The following relations are well known [18]:

$$S^*_{CP} \subset \widehat{S}_{CP}, \quad \langle \hat{x}, \hat{y} \rangle = 0\ \forall\, (\hat{x}, \hat{y}) \in \widehat{S}_{CP}.$$

Lemma 9.1. *Take $\beta \geq 0$, $\bar{\nu} > 0$, $\tau > 0$, and $\bar{x} \in \mathbb{R}^n$. If the LMCP (1) has solutions, then the set*

$$\mathcal{L}(\beta, \bar{\nu}, \tau, \bar{x}) = \{(x, y) > 0 \,|\, \delta_{\nu,\bar{x}}(x, y; \lambda) \leq \beta,\ \textit{for some } \lambda,\ \nu \textit{ such that } \nu \geq \bar{\nu},\ \lambda \geq \tau\nu\}$$

is bounded.

Proof 9.2. *Take $\beta \geq 0$, $\bar{\nu} > 0$, $\tau > 0$ and $\bar{x} \in \mathbb{R}^n$. Let $\lambda \in \mathbb{R}$, $\nu \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ be such that $\nu \geq \bar{\nu}$, $\lambda \geq \tau\nu$ and $\delta_{\nu,\bar{x}}(x, y; \lambda) \leq \beta$. Define*

$$r = \lambda(Mx + b - y) + \nu(x - \bar{x}), \quad a = \lambda X y. \tag{45}$$

In view of relation $\delta_{\nu,\bar{x}}(x, y; \lambda) = \|r\|/\nu^{1/2} + \|a - e\| \leq \beta$, it follow

$$\frac{\|r\|}{\nu} \leq \frac{\beta}{\nu^{1/2}} \leq \frac{\beta}{\bar{\nu}^{1/2}}, \quad \|a\| \leq \beta + \sqrt{n}. \tag{46}$$

Take $(\hat{x}, \hat{y}) \in \widehat{S}_{CP}$. Since M is positive semi-definite, we have

$$\langle y + \frac{1}{\lambda}(r - \nu(x - \bar{x})) - \hat{y}, x - \hat{x} \rangle = \langle Mx - M\hat{x}, x - \hat{x} \rangle \geq 0.$$

Multiplying by λ and rearranging terms we obtain

$$\langle a, e \rangle \geq \langle \nu(x - \bar{x}) - r, x - \hat{x} \rangle + \langle \hat{x}, \lambda y \rangle + \langle \hat{y}, \lambda x \rangle. \tag{47}$$

Assuming additionally that $(\hat{x}, \hat{y}) \geq 0$, that is $(\hat{x}, \hat{y}) \in S^*_{CP}$, it follows

$$\langle a, e \rangle \geq \langle \nu(x - \bar{x}) - r, x - \hat{x} \rangle = \frac{\nu}{2} \left(\|\bar{x} - x + \frac{r}{\nu}\|^2 + \|x - \hat{x}\|^2 - \|\bar{x} - \hat{x} + \frac{r}{\nu}\|^2 \right), \tag{48}$$

which, combined with the assumption on ν, implies that

$$\|x - \hat{x}\|^2 \leq \|\bar{x} - \hat{x} + \frac{r}{\nu}\|^2 + \frac{2}{\bar{\nu}} \langle a, e \rangle.$$

From (45), the assumptions on λ and ν, and triangle inequality it follows

$$\|y\| = \|Mx + b + \frac{\nu}{\lambda} \left(x - \bar{x} - \frac{r}{\nu} \right) \| \leq \|Mx + b\| + \frac{1}{\tau} \|x - \bar{x} - \frac{r}{\nu}\|.$$

Since the above inequalities hold for any $(x, y) \in \mathcal{L}(\beta, \bar{\nu}, \tau)$, it follows from these relations and (46) that there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\|x\| \leq C_1, \quad \|y\| \leq C_2, \quad \forall\, (x, y) \in \mathcal{L}(\beta, \bar{\nu}, \tau),$$

which ends the proof.

In the next result, in the spirit of Lemma 4.2 of [3], we characterize the limit points of some sequences of points of a β-neighborhood of a log-quadratic central path. Its proof is based on the techniques presented in [18].

Lemma 9.3. *Take $\beta \in [0,1)$. Let $\{(x^k, y^k, \lambda_k, \nu_k)\} \subset \mathbb{R}^{2n}_{++} \times \mathbb{R}^2_{++}$ be such that*

$$\delta_{\nu_k,\bar{x}}(x^k, y^k; \lambda_k) \leq \beta, \ \forall k \in \mathbb{N}. \tag{49}$$

Define

$$r^k = \lambda_k(Mx^k + b - y^k) + \nu_k(x^k - \bar{x}), \quad a^k = \lambda_k X^k y^k, \quad k \in \mathbb{N}. \tag{50}$$

If the LMCP (1) has solutions and

$$\lambda_k \longrightarrow +\infty, \quad \nu_k \longrightarrow \bar{\nu} > 0, \quad r_k \longrightarrow r, \quad a_k \longrightarrow a, \tag{51}$$

then the sequence $\{(x^k, y^k)\}$ converges to a pair of complementary solutions.

Proof 9.4. *From Lemma 9.1 it follows that $\{(x^k, y^k)\}$ is a bounded sequence. Let (x^*, y^*) be an accumulation point of $\{(x^k, y^k)\}$. Trivially $(x^*, y^*) \geq 0$. Dividing (50) by λ_k and taking limits (along suitable subsequences), when $k \to +\infty$, it follows*

$$Mx^* + b - y^* = 0, \quad \langle x^*, y^* \rangle = 0.$$

Thus, $(x^*, y^*) \in S^*_{CP}$. To end the proof it suffices to prove that this accumulation point is unique.

Note that (47) holds, with $a = a^k$, $x = x^k$, $y = y^k$, $\lambda = \lambda_k$, $r = r^k$, and $\nu = \nu_k$, for any $k \in \mathbb{N}$. Hence, from this relation and the definition of a^k it follows

$$\langle a^k, e \rangle \geq \langle \nu_k(x^k - \bar{x}) - r^k, x^k - \hat{x} \rangle + \sum_{i \in I} (a_k)_i \frac{\hat{x}_i}{(x^k)_i} + \sum_{j \in J} (a_k)_j \frac{\hat{y}_j}{(y^k)_j}, \quad \forall\, (\hat{x}, \hat{y}) \in \widehat{S}_{CP}, \ \forall\, k \in \mathbb{N}. \tag{52}$$

From (49) and Lemma 2.3 it follows that $(1-\beta)e \leq a_k \leq (1+\beta)e$, $\forall\, k \in \mathbb{N}$. Thus, making $k \to +\infty$, we obtain

$$0 < (1-\beta)e \leq a.$$

Taking limits in (52), using the above inequality and the definitions of I and J we conclude that

$$x^*_i > 0, \ \forall\, i \in I, \quad y^*_j > 0, \ \forall\, j \in J.$$

and

$$\langle a, e \rangle \geq \langle \bar{\nu}(x^* - \bar{x}) - r, x^* - \hat{x} \rangle + \sum_{i \in I} a_i \frac{\hat{x}_i}{x^*_i} + \sum_{j \in J} a_j \frac{\hat{y}_j}{y^*_j}, \ \forall\, (\hat{x}, \hat{y}) \in \widehat{S}_{CP}.$$

The right-hand side of the above relation is a linear function in variables $\hat{x}$ and $\hat{y}$, bounded above on the affine subspace $\widehat{S}_{CP}$. Therefore, it must be constant on $\widehat{S}_{CP}$. Since $(x^*, y^*) \in \widehat{S}_{CP}$, it holds

$$0 = \langle \bar{\nu}(x^* - \bar{x}) - r, x^* - \hat{x} \rangle + \sum_{i \in I} a_i \frac{\hat{x}_i - x^*_i}{x^*_i} + \sum_{j \in J} a_j \frac{\hat{y}_j - y^*_j}{y^*_j}, \ \forall\, (\hat{x}, \hat{y}) \in \widehat{S}_{CP}.$$

Convexity of function $\phi(t) = -\ln t$ and the above relation yield

$$\begin{aligned}\sum_{i\in I} -a_i \ln \hat{x}_i + \sum_{j\in J} -a_j \ln \hat{y}_j &\geq \sum_{i\in I} -a_i \ln x_i^* + \sum_{j\in J} -a_j \ln y_j^* + \langle \bar{\nu}(x^* - \bar{x}) - r, x^* - \hat{x}\rangle \\ &= \sum_{i\in I} -a_i \ln x_i^* + \sum_{j\in J} -a_j \ln y_j^* \\ &\quad + \frac{\bar{\nu}}{2}\Big(\|\bar{x} - x^*\|^2 + \|\hat{x} - x^*\|^2 - \|\bar{x} - \hat{x}\|^2\Big) + \langle r, \hat{x}\rangle - \langle r, x^*\rangle.\end{aligned}$$

Thus, (x^*, y^*) is a solution of the convex programming problem

$$\begin{aligned}&\min h(x,y) = \sum_{i\in I} -a_i \ln x_i + \sum_{j\in J} -a_j \ln y_j - \langle r, x\rangle + \frac{\bar{\nu}}{2}\|x - \bar{x}\|^2 \\ &s.t.\ (x,y) \in S^*_{CP}.\end{aligned} \tag{53}$$

Since h is a strictly convex function, the above convex programming problem has a unique solution, which implies the convergence of the sequence.

We end this section presenting the main features of the log-quadratic central path.

Proposition 9.5. *For any $\nu > 0$ and $\bar{x} \in \mathbb{R}^n$, the log-quadratic central path*

$$\lambda \mapsto \big(x(\lambda;\nu,\bar{x}), y(\lambda;\nu,\bar{x})\big)$$

is well defined and differentiable in parameter λ. Moreover, if the LMCP (1) has solutions, then

i) *for any $\bar{\nu} > 0$ and $\tau > 0$, the set*

$$\Big\{\big(x(\lambda;\nu,\bar{x}), y(\lambda;\nu,\bar{x})\big) \mid \nu \geq \bar{\nu},\ \frac{\lambda}{\nu} \geq \tau\Big\}$$

is bounded,

ii) *if $\lambda \to +\infty$ and $\nu \to \bar{\nu} > 0$, then $(x(\lambda;\nu,\bar{x}), y(\lambda;\nu,\bar{x}))$ converges to the unique solution of the convex programming problem*

$$\begin{aligned}&\min\ h(x,y) = \sum_{i\in I} -\ln x_i + \sum_{j\in J} -\ln y_j + \frac{\bar{\nu}}{2}\|x - \bar{x}\|^2 \\ &s.t.\ (x,y) \in S^*_{CP},\end{aligned}$$

iii) *if $\lambda \to +\infty$, $\nu \to +\infty$ and $\frac{\nu}{\lambda} \to 0$, then $(x(\lambda;\nu,\bar{x}), y(\lambda;\nu,\bar{x}))$ converges to the pair of complementary solutions $(x^*, y^*) = (P_{S^*}(\bar{x}), Mx^* + b)$.*

Proof 9.6. *For any $\lambda > 0$ and $\nu > 0$, the operator*

$$T_{(\lambda,\nu)} : \mathbb{R}^n_{++} \to \mathbb{R}^n, \quad T_{(\lambda,\nu)}(x) = \lambda(Mx+b) + \nabla\Big[\frac{\nu}{2}\|x-\bar{x}\|^2 - \sum_{i=1}^n \log x_i\Big] = \lambda(Mx+b) + \nu(x-\bar{x}) - x^{-1}$$

is a strongly monotone, maximal monotone operator. Hence, there exist a unique $x(\lambda, \nu; \bar{x}) > 0$, *solution of the equation* $T_{(\lambda,\nu)}(x) = 0$. *Differentiability of* $x(\lambda, \nu; \bar{x})$ *on parameter* $\lambda > 0$ *follows from the fact that* $\partial_x T_{(\lambda,\nu)}(x) = \lambda M + \nu I + X^{-2}$ *is non-singular, for all* $x > 0$, *and the implicit function theorem. From relation*

$$y(\lambda, \nu; \bar{x}) = Mx(\lambda, \nu; \bar{x}) + b + \left(\frac{\nu}{\lambda}\right)\left(x(\lambda, \nu; \bar{x}) - \bar{x}\right), \ \lambda > 0,$$

it follows the well definiteness, uniqueness, and differentiability in parameter λ *of the dual log-quadratic central path.*

Since $\delta_{\nu,\bar{x}}\left((x(\lambda; \nu, \bar{x}), y(\lambda; \nu, \bar{x}); \lambda\right) = 0$, *i) follows directly from Lemma 9.1.*

Consider arbitrary sequences of positive parameters $\{\lambda_k\}$ *and* $\{\nu_k\}$ *and let* $x^k = x(\lambda_k, \nu_k; \bar{x})$, $y^k = y(\lambda_k, \nu_k; \bar{x})$, $\forall\, k \in \mathbb{N}$. *The following hold*

$$r^k = \lambda_k(Mx^k + b - y^k) + \nu_k(x^k - \bar{x}) = 0, \quad a^k = \lambda_k X^k y^k = e, \ \forall\, k \in \mathbb{N}. \tag{54}$$

To prove ii), assume that

$$\lambda_k \to +\infty, \quad \nu_k \to \bar{\nu} > 0.$$

From (54) *and Lemma 9.3 it follows the convergence of* $\{(x^k, y^k)\}$ *to the (unique) solution of the convex programming problem* (53) *(with* $r = 0$ *and* $a = e$*), which ends the proof.*

To prove iii), assume that

$$\lambda_k \to +\infty, \quad \nu_k \to +\infty, \quad \frac{\nu_k}{\lambda_k} \to 0,$$

From i) it follows that $\{(x^k, y^k)\}$ *is a bounded sequence. Let* (x^*, y^*) *be an accumulation point of* $\{(x^k, y^k)\}$. *Trivially* $(x^*, y^*) \geq 0$. *Dividing* (54) *by* λ_k *and taking limits, along suitable subsequences, we obtain*

$$Mx^* + b - y^* = 0, \quad \langle x^*, y^* \rangle = 0. \tag{55}$$

Thus, $(x^*, y^*) \in S^*_{CP}$. *To end the proof it suffices to prove that this accumulation point is unique. Using* (48), *with* $a = e$, $r = 0$, $x = x^k$, $y = y^k$, *and* $\nu = \nu_k$, *it follows*

$$n \geq \frac{\nu_k}{2}\left(\|\bar{x} - x^k\|^2 + \|x^k - \hat{x}\|^2 - \|\bar{x} - \hat{x}\|^2\right), \ \forall\, \hat{x} \in S^*.$$

Dividing the above relation by ν_k *and taking limits, along suitable subsequences, we obtain*

$$\|\bar{x} - \hat{x}\|^2 \geq \|\bar{x} - x^*\|^2 + \|\hat{x} - x^*\|^2, \ \forall\, \hat{x} \in S^*.$$

Hence, x^* *is the (unique) Euclidean projection of* $\bar{x}$ *on the (closed and convex) solution set of the LMCP* (1), *which implies the convergence of* $x(\lambda, \nu; \bar{x})$ *to* $P_{S^*}(\bar{x})$. *The convergence of* $y(\lambda, \nu; \bar{x})$ *follows immediately from the first relation in* (55).

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