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Poisson Structures on Projective Manifolds

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Introdução

Estamos interessados em estudar estruturas de Poisson em variedades Fano. Veremos que estudar tais estruturas implica em estudar estruturas de Poisson quadráticas em $\mathbb{C}^n$. Estas estruturas aparecem naturalmente em uma grande quantidade de exemplos, tais como as álgebras de Sklyanin e grupos de Lie-Poisson.

Um dos motivadores para estudarmos sobre este tema é o artigo de Polishchuk [**20**]. Neste artigo, foram definidas estruturas de Poisson em esquemas e boa parte do artigo foi uma tradução das propriedades básicas de estruturas de Poisson para a linguagem de geometria algébrica. A outra parte do artigo foi aplicações destas idéias para resolver problemas concretos. Dentre estes problemas, foi proposto estudar estruturas de Poisson não degeneradas em que o conjunto singular é um divisor de cruzamento normal e foi provado que a estrutura de Poisson define uma folheação de codimensão 1 em cada uma das componentes irredutíveis do conjunto singular e, se a variedade ambiente for $\mathbb{P}^{2n}$, $n \geq 2$, o conjunto singular possui pelo menos $2n-1$ componentes irredutíveis.

Nesta tese, generalizamos este resultado de Polishchuk e provamos.

TEOREMA A. *Seja X uma variedade Fano de dimensão $2n$, $n \geq 2$, com grupo de Picard cíclico. Suponha que Π é uma estrutura de Poisson não degenerada em X cujo conjunto singular de Π é reduzido e a cruzamentos normais. Então X é o espaço projetivo $\mathbb{P}^{2n}$ e o conjunto singular consiste de $2n+1$ hiperplanos em posição geral.*

Concentraremos nossos estudos para provar este teorema em variedades de dimensão 4. Ao estudarmos em dimensão superiores, nos restringiremos à interseção de duas hipersuperfícies que estão no conjunto singular da estrutura de Poisson. Veremos que se escolhermos bem as hipersuperfícies, teremos uma estrutura de Poisson não degenerada nesta interseção com singularidades tipo cruzamento normal e será possível aplicar indução.

Para provarmos o teorema em dimensão 4, utilizaremos o resultado de Kobayashi e Ochiai em [**15**], sobre caracterização do espaço projetivo. É sabido que para cada variedade Fano X de grupo de Picard cíclico, associa-se um índice $i(X)$, que é um número inteiro e positivo. Eles provaram que, se X é uma variedade de dimensão 4, $i(X) \leq 5$ e $i(X) = 5$ se e somente se X é $\mathbb{P}^4$. A prova do Teorema A será excluindo os índices anteriores.

É bem conhecido que a distribuição induzida pela estrutura de Poisson Π, cujo o conjunto singular $\operatorname{Sing}\Pi$ tem codimensão ≥ 2, é uma **distribuição involutiva** e possui fibrado canônico trivial. Em [**17**], Loray, Pereira e Touzet classificaram **folheações** com fibrado canônico trivial em variedades Fano de dimensão 3 com grupo de Picard cíclico. Juntando alguns resultados desta classificação e mais o fato que, sob nossas hipóteses, a folheação induzida no conjunto singular é sempre logarítmica, conseguiremos excluir cada um dos índices anteriores da variedades Fano.

Alertamos ao leitor que distinguiremos folheação de Poisson com distribuição de Poisson (na literatura, as duas nomenclaturas são tratadas como iguais). Ver obeservação 2.4 do capítulo 2 para maiores detalhes.

A família encontrada no teorema A é não vazia e cada elemento da família é conhecida, na literatura, como estrutura de Poisson diagonal.

O segundo teorema da tese é:

Teorema B. *Se tomarmos deformações suficientemente pequenas de uma estrutura de Poisson diagonal genérica em* $\mathbb{P}^{2n}$*, então as estruturas de Poisson resultantes ainda são estruturas de Poisson diagonais em* $\mathbb{P}^{2n}$*.*

A idéia da prova deste Teorema é usar o campo divergente da estrutura de Poisson e ver que, genericamente, este campo divergente tem parte linear "quase" não ressonante. Utilizamos resultados conhecidos de deformação formal de campos e, com algumas contas, veremos propriedades locais no conjunto singular da estrutura de Poisson deformada. Tais propriedades locais serão suficientes para provar que a estrutura de Poisson resultante é ainda diagonal.

Estudamos também um exemplo de estrutura de Poisson Π de posto 2 em $\mathbb{P}^{n+1}$ (ver seção 4 do capítulo 4 para o exemplo). O principal problema de estudar uma estrutura de Poisson de posto baixo é que, ao perturbar tal estrutura, a tendência é que o posto aumente. O exemplo que estudamos tem uma propriedade bem peculiar:

Teorema C. Π *é um ponto regular do espaço de estruturas de Poisson em* $\mathbb{P}^{n+1}$*. Mais ainda, se* Π_ϵ *é um perturbado de* Π*, então* Π_ϵ *possui as mesmas propriedades de* Π*. Ver teorema C da seção 4 do capítulo 4 para um enunciado mais preciso.*

A idéia da prova deste Teorema é descrever o espaço tangente da estrutura de Poisson de Π e, com a descrição obtida, verificar que a estrutura de Poisson resultante terá posto 2 e concluiremos que deformar tal estrutura de Poisson é o mesmo que deformar a folheação de Poisson. Com uma simples aplicação do resultado de Cukierman e Pereira em [**8**], obtem-se o Teorema.

A tese será dividida em quatro capítulos.

No capítulo **Teoria local**, daremos várias definições equivalentes de estruturas de Poisson e as propriedades básicas destas estruturas. A primeira definição dada é a mais intuitiva e que também nos permite fazer as contas e a última definição dada é ótima sob a perpectiva da geometria algébrica. Definiremos o campo divergente de uma estrutura de Poisson e algumas propriedades conhecidas sobre tal campo. Faremos também um breve resumo sobre o que já se sabe sobre estudo local de estruturas de Poisson.

No capítulo **Teoria global**, falaremos de aspectos globais das estruturas de Poisson. Definiremos distribuição de Poisson, folheação de Poisson, noção de subvariedades e subvariedades fortes de Poisson, módulos de Poisson, conexão de Poisson e o conjunto singular da estrutura de Poisson.

Falaremos também de estruturas de Poisson Π em espaços projetivos. Daremos uma representação para Π tanto em coordenadas homogêneas como em carta afim. E veremos que, em coordenadas homogêneas, Π é uma estrutura de Poisson quadrática.

Daremos alguns exemplos de folheações em $\mathbb{P}^n$ que são induzidas por estruturas de Poisson e exemplos de folheações em $\mathbb{P}^n$ que não são induzido por estruturas de Poisson. Veremos algumas restrições no feixe tangente de uma folheação de Poisson.

O capítulo **Estruturas de Poisson não-degenerada em variedade Fano** é o mais importante desta tese. O objetivo final é provar o Teorema A. Para isso, falaremos de variedades Fano com grupo de Picard cíclico; definiremos o índice de uma variedade Fano; enunciaremos os resultados que serão importantes para esta tese sobre tais variedades, dentre eles, o teorema de Kobayashi e Ochiai (ver [**15**]) e veremos algumas propriedades básicas de folheações em variedades Fano.

Enunciaremos o resultado de Polishchuk sobre estruturas de Poisson não degenerada com singularidade tipo cruzamento normal e falaremos de algumas propriedades de folheações induzidas pela estrutura de Poisson em variedades Fano. Veremos que, sob nossas hipóteses, a folheação de Poisson induzida no conjunto singular de um estrutura de Poisson é sempre logarítmica.

Juntando tudo o que foi feito, começamos de fato a provar o teorema A. Começamos excluindo os casos de índice ≤ 3 e veremos que o complicado será exclusão do índice 3. Para excluir este caso, provaremos que, sob nossas hipóteses, cada componente irredutível do conjunto singular possui um campo global tangente à folheação de Poisson induzida no conjunto singular. Cada componente irredutível do conjunto singular é uma variedade Fano de dimensão 3 e, utilizando a classificação de Loray, Pereira e Touzet em [**17**], verificamos que as folheações logarítmicas nestas variedades não possui subfolheação de dimensão 1. E portanto uma contradição, pois sabemos que a folheação de Poisson é, sob nossas hipóteses, logarítmica.

Para excluir o índice 4, utilizamos o teorema de Kobayashi e Ochiai, que nos garante que a variedade é uma hiperquádrica em $\mathbb{P}^5$ e estudamos singularidades isoladas de folheações logarítmicas. Provamos que nos casos que estamos interessados, a folheação sempre possuirá singularidades isoladas e que cada uma delas estará na interseção de três componentes do conjunto singular da estrutura de Poisson. Utilizando a geometria das folheações logarítmicas e as restrições obtidas pela existência da estrutura de Poisson, obtemos uma contradição. Basicamente, o motivo de tal contradição será que o conjunto singular possuirá no máximo 4 componentes irredutíveis. E com isto, concluímos que a variedade Fano é o espaço projetivo.

Para terminar a demonstração do teorema, utilizamos exatamente a mesma idéia no caso em que excluimos a quádrica para concluir que o conjunto singular são cinco hiperplanos em posição geral.

No capítulo **Deformações do colchete de Poisson em espaços projetivos**, dedicamos à prova dos Teoremas B e C. Começamos o capítulo definindo o que seria uma deformação de estrutura de Poisson em uma variedade projetiva. Falaremos também da relação entre a cohomologia de Poisson e deformação de estrutura de Poisson. Depois, começamos a provar o Teorema B.

Para provar o Teorema, estudaremos numa vizinhança de alguns pontos singulares "especiais" da estrutura de Poisson (onde a estrutura de Poisson tem posto 0). Utilizaremos o campo divergente e resultados já bem conhecidos de perturbação formal de campos. Com as equações obtidas do divergente do perturbado da estrutura de Poisson, teremos uma descrição completa do conjunto singular do perturbado da estrutura de Poisson na vizinhança dos pontos singulares "especiais". Pela descrição local, conseguimos provar que é a união de $2n+1$ hiperplanos.

Depois de provarmos o Teorema B, computaremos, com as descrições obtidas, o segundo grupo de cohomologia da estrutura de Poisson. Este grupo descreve o espaço tangente do espaço dos conjuntos da estrutura de Poisson.

Quanto ao Teorema C, começamos dando o exemplo de estrutura de Poisson Π no espaço projetivo que iremos estudar. Vemos que a folheação induzida tem feixe tangente totalmente decomponível. Para provar o Teorema, descreveremos o espaço tangente em Π do conjunto de estruturas de Poisson e, com esta descrição, veremos que a dimensão de folheação de Poisson de qualquer pequena perturbação de Π é 2. Concluiremos que deformar a estrutura de Poisson é o mesmo que deformar a folheação induzida e utilizaremos os resultados de estabilidade obtidas por Cukierman e Pereira em [**8**].

Introduction

We are interested in studying Poisson structures on Fano varieties. We will see that to study such structures is the same as to study quadratic Poisson structures in $\mathbb{C}^n$. Quadratic Poisson structures appear naturally in a great amount of important examples, such as Sklyanin algebra and Lie-Poisson group.

Polishchuk's paper [**20**] motivates us to study Poisson structures. In his article, it was defined Poisson structures in schemes and part of the article consists in a translation of the basic properties of Poisson brackets in the language of algebraic geometry. The other part of the article consists of applications of the ideas developed to solve concrete problems. Among them, it was studied nondegenerate Poisson structures such that the singular locus consists of a divisor with normal crossing singularity and it was proved that the Poisson structure induces a codimension 1 foliation in each of the irreducible component of the singular loci of the Poisson structure and, if the variety is the projective space $\mathbb{P}^{2n}$, the singular set has at least $2n-1$ irreducible components.

In this thesis, we generalized this result of Polishchuk, proving the following Theorem.

Theorem A. Let X be a even dimensional, $\dim X \geq 4$ Fano variety with cyclic Picard group. Suppose that Π is a nondegenerate Poisson structure on X such that the singular locus of Π is reduced smooth normal crossing. Then X is the projective space $\mathbb{P}^{2n}$ and the singular locus is $2n+1$ hyperplanes in general position.

The difficult part is to prove the Theorem in dimension 4. Because when we study in greater dimension, if we restrict ourselves in the intersection of two well chosen hypersurfaces which are in the singular set of Poisson structure, we have a nondegenerate Poisson structure there such that the singular loci are a divisor in normal crossing and we are able to use induction.

So as to prove the Theorem in dimension 4, we make use of a result of Kobayashi and Ochiai (see [**15**]), about a characterization of the projective space. For each Fano variety X with cyclic Picard group, we can associate an index $i(X)$, which is a positive integer, and they proved that, if the dimension of X is 4, then $i(X) \leq 5$ and $i(X) = 5$ if and only if X is $\mathbb{P}^4$. The proof of Theorem A consists in excluding the other index.

It is a well known fact that the **involutive distribution** induced by the Poisson structure Π has trivial canonical bundle whenever Sing Pi has codimension ≥ 2. In [**17**], Loray, Pereira and Touzet classified the **foliations** with trivial canonical bundle in Fano threefolds with cyclic Picard group. We will see that, under our hypothesis, the foliation induced in the singular set of the Poisson structure is always logarithmic.

We warn the reader that, in this thesis, we will distinguish Poisson foliations with Poisson distributions (in the literature, they are treated as the same). See remark 2.4 of chapter 2 for details.

The family we found in Theorem A is nonempty and each element of the family is known, in the literature, as diagonal Poisson structure.

The second Theorem of this thesis is:

Theorem B. If we take sufficiently small deformations of a generic diagonal Poisson structure in $\mathbb{P}^{2n}$ then the resulting Poisson structures are still diagonal Poisson structures in $\mathbb{P}^{2n}$.

The idea behind the proof of this Theorem is to use the curl vector field induced by Poisson structure and see that, generically, this vector field has linear part "almost" non-resonant. We make use of some well known results about formal deformation of vector fields and, with some computations, we will see some local properties in the singular set of the deformed Poisson structure. Such properties will be sufficient to prove that the resulting Poisson structure is still diagonal.

We also study an example of rank 2 Poisson structure Π in $\mathbb{P}^{n+1}$ (see chapter 4, section 4 for the example). The main problem of studying low rank Poisson structures is that, if we deform such structures, we expect that its rank grows up. Our example has a very nice property.

Theorem C. Π is a regular point of the space of Poisson structure in $\mathbb{P}^{n+1}$. Moreover, if Π_ϵ is a perturbation of Π, then Π_ϵ has the same properties as Π. See Theorem C of the chapter 4, section 4 for a more precise description of this Theorem.

The idea to prove this Theorem is to describe the tangent space in Π of the space of Poisson structures in $\mathbb{P}^{n+1}$ and, with the description, we verify that the deformed Poisson structure has rank 2 and we conclude that deforming Π is the same as deforming the Poisson foliation induced by Π. A simple application of the main result of Cukierman and Pereira in [**8**], we prove the Theorem.

We split this thesis in 4 chapters. We will give a brief description of each one of them.

In the chapter **Local theory**, we give many equivalent definitions of Poisson structures and we study the basic properties of these structures. The first definiton is more intuitive and it is very good to work in a local system of coordinates and the last definition is good under the perspective of algebraic geometry. We define the curl vector field and we give some known properties about this vector field. In this chapter, we also give a brief resume what is known about local study of Poisson structures.

In the chapter **Global theory**, we talk about the global properties of this theory. We define Poisson distribution, Poisson foliation, the notion of Poisson subvarieties and Strong Poisson subvarieties, Poisson modules, Poisson connection and the singular set of a Poisson structure.

Also, we talk about Poisson structures on projective spaces. We also do the calculations of Poisson structures in homogeneous coordinates and also in the chart of projective space. We will see that, in homogeneous coordinates, such Poisson structures is always quadratic.

We also give some examples of foliations in $\mathbb{P}^n$ which come from Poisson structures and we give some example of foliations in $\mathbb{P}^n$ which do not come from any Poisson structure. We also see some restrictions about the tangent sheaf of a Poisson foliation.

The chapter **Nondegenerate Poisson structures on Fano variety** is the most important of this thesis. The final objective is to prove Theorem A. To do so, we talk about Fano varieties with cyclic Picard group. We define the index of a Fano variety, we enunciate some important results for this thesis about these varieties, such as Kobayashi-Ochiai theorem (see [**15**]) and we see some basic properties of foliations in Fano varieties.

We enunciate the theorem of Polishchuk about nondegenerate Poisson structure such that the singular set is smooth normal crossing and we talk about some properties of the Poisson foliation in the singular set of the Poisson structures. We will see that, under our assumptions, in the singular set of the Poisson structure, the Poisson foliation is always logarithmic.

Then we join all the work done so far and we exclude some indexes of Fano varieties. We start excluding Fano varieties with indexes ≤ 3 and we see that the complicated case is the index 3. To exclude this case, we prove that, under our assumptions, each irreducible component of the singular loci of Poisson structure has a global vector field tangent to the Poisson foliation induced in the singular loci. Each irreducible component of the singular loci is a Fano variety of dimension 3 and, by Loray, Pereira and Touzet classification in [**17**], we verify that logarithmic foliations in such varieties do not have subfoliations by curves. This is a contradiction, because the Poisson foliation is, under our assumptions, logarithmic.

To exclude index 4, we use Kobayashi and Ochiai work, which says that the variety is exactly a quadric in $\mathbb{P}^5$, and we study isolated singularities of logarithmic foliations. We will see that, in the cases we are considering, the foliation has isolated singularities and we prove that each of the isolated singularities of the Poisson foliation is in the intersection of 3 components of the singular loci of the Poisson structure. We make use of the geometry of logarithmic foliations and the restrictions obtained by the existence of a Poisson structure so as to get a contradiction. Basically, the reason we get a contradiction is due to the fact that the singular set of the Poisson structure has at most 4 irreducible components. This proves that the Fano variety is the projective space.

To finish the proof of the Theorem, we use exactly the same idea when we excluded the quadric to conclude that the singular loci of a Poisson structure in $\mathbb{P}^4$ are five hiperplanes in general position.

In the chapter **Deformation of Poisson brackets in projective space**, we prove the Theorems B and C. We begin the chapter defining what a deformation of Poisson structure in the projective space is. We also talk about the relation between Poisson cohomology and deformation of Poisson structure. Then we start to prove the Theorem B.

In order to prove this Theorem, we study in a neighborhood of some special singular points of the Poisson structure Π (this points are the one where the rank of Π is 0). We use the curl vector field and some well known results about formal deformations of vector fields. With the equations obtained of the curl vector field of the deformed Poisson structure, we have a complete description of the singular set of the deformed Poisson structure in a neighborhood of each special singular point. With this local description, we are able to prove that the singular set of deformed Poisson structure is the union of $2n + 1$ hyperplanes.

After proving the Theorem B, we compute the second Poisson cohomology group of Π. This group describes the tangent space at Π of the set of Poisson structure.

About Theorem C, we start giving the example of the Poisson structure Π and we will see that the tangent sheaf of the Poisson foliation totally splits. So as to prove the Theorem, we describe the tangent space of Π of the set of Poisson structures and, with its description, we will see that the dimension of the Poisson foliation of any small deformation of

Π is 2. We conclude that deforming the Poisson structure is the same as deforming the foliation induced by the Poisson structure and we can apply the stability result of Cukierman and Pereira in [**8**].

CHAPTER 1

Local theory

1. Poisson structures

The goal of this section is to define and to review the basic properties of Poisson structures and to introduce some natural constructions that will be useful later on.

1.1. Definition and first examples. We will give the classical definition of Poisson structure. We are interested in the holomorphic category, but Poisson structures could also be defined in other categories.

Definition 1.1. A holomorphic Poisson structure on a holomorphic variety X is an $\mathbb{C}$-bilinear antisymmetric operation

$$\begin{aligned} \mathcal{O}_X \times \mathcal{O}_X &\to \mathcal{O}_X \\ (f,g) &\mapsto \{f,g\} \end{aligned}$$

which verifies the Jacobi identity

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$$

and the Leibniz identity

$$\{f,gh\} = \{f,g\}h + g\{f,h\}.$$

In other words, $\mathcal{O}_X$, equipped with $\{\cdot\,,\cdot\}$, is a Lie algebra whose Lie bracket satisfies the Leibniz identity. This bracket $\{\cdot\,,\cdot\}$ is called a **Poisson Bracket**. A variety equipped with such a bracket is called a **Poisson variety**.

Remark 1.2. It is important to have in mind that X could be a singular variety. In this chapter, every time that is said "let $(x_1,\ldots,x_n)$ be a local system of coordinates", it will be considered on a neighborhood of a regular point of X.

Example 1.3. One can define a trivial Poisson structure on any variety by putting $\{f,g\} = 0$ for all f, g local functions in X.

EXAMPLE 1.4. Take $X = \mathbb{C}^2$ with coordinates (x, y) and we consider $p : \mathbb{C}^2 \to \mathbb{C}$ an arbitrary holomorphic function. One can define a holomorphic Poisson structure on $\mathbb{C}^2$ by putting

$$\{f, g\} = \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right) p.$$

An important example is the Poisson structure attached to any holomorphic symplectic manifold.

DEFINITION 1.5. A symplectic manifold (M, ω) is a manifold M with a nondegenerate closed differential 2-form ω, which is called symplectic form.

The nondegeneracy of a holomorphic 2-form ω means that the corresponding homomorphism

$$\begin{aligned} \omega^\sharp :& TM \to T^*M \\ & X \mapsto i_X\omega \end{aligned}$$

is an isomorphism.

REMARK 1.6. By linear algebra, if the dimension of M is $2n$, it is possible to prove that ω in nondegenerate if and only if $(\omega^n)_x \neq 0$ for every $x \in M$. Here, ω^n means $\omega \wedge \ldots \wedge \omega$, n times.

If f is a local function on a symplectic manifold (M, ω), then we define its **Hamiltonian vector field**, denoted by X_f, as follows:

$$i_{X_f}\omega = -\mathrm{d}f.$$

We can also define on (M, ω) a natural bracket, called the Poisson bracket of ω, as follows:

$$\{f, g\} = \omega(X_f, X_g) = -X_g(f) = X_f(g).$$

PROPOSITION 1.7. *If (M, ω) is a holomorphic symplectic manifold, then the bracket $\{f, g\} = \omega(X_f, X_g)$ is a Poisson structure on M.*

PROOF. The Leibniz identity follows from $X_{fg} = fX_g + gX_f$. Let us show the Jacobi identity. Recall the following Cartan Formula for the differential k-form η

$$\begin{aligned} \mathrm{d}\eta\,(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i-1} X_i \left(\eta(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1})\right) \\ &+ \sum_{i \leq i < j \leq k+1} (-1)^{i+j} \eta\left([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}\right), \end{aligned}$$

where $X_1, \ldots, X_{k+1}$ are vector fields, and the hat means that the corresponding entry is omitted. Applying Cartan's formula to ω on X_f, X_g and X_h, we get:

$$\begin{aligned}
0 &= \mathrm{d}\omega(X_f, X_g, X_h) \\
&= X_f\left(\omega(X_g, X_h)\right) + X_g\left(\omega(X_h, X_f)\right) + X_h\left(\omega(X_f, X_g)\right) \\
&- \omega\left([X_f, X_g], X_h\right) - \omega\left([X_h, X_f], X_g\right) - \omega\left([X_g, X_h], X_f\right) \\
&= X_f\{g, h\} + X_g\{h, f\} + X_h\{f, g\} \\
&+ [X_f, X_g](h) + [X_h, X_f](g) + [X_g, X_h](f) \\
&= \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} + X_f(X_g(h)) - X_g(X_f(h)) \\
&+ X_h(X_f(g)) - X_f(X_h(g)) + X_g(X_h(f)) - X_h(X_g(f)) \\
&= 3(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}).
\end{aligned}$$

and this is sufficient to prove the proposition. □

Thus, any symplectic manifold is also a Poisson manifold, though the converse is not true.

We will give more examples later, after we express the Poisson bracket as 2-derivations.

1.2. Poisson tensor. Let X be a variety and q a positive integer. We denote by $\mathfrak{X}_X^q$ the sheaf of holomorphic q-derivations of X, i.e., the sheaf $\mathrm{Hom}(\Omega_X^q, \mathcal{O}_X)$, the dual of Ω_X^q. If $(x_1, \ldots, x_n)$ is a local system of coordinates at $x \in X$ then $\mathfrak{X}_X^q$ admits a linear basis consisting of the elements of the form

$$\frac{\partial}{\partial x_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_q}}(x) \text{ with } i_1 < \ldots < i_q.$$

In other words, if $\Pi \in H^0(X, \mathfrak{X}_X^q)$ is holomorphic q-derivation, then, in local coordinates, Π has the following expression

$$\Pi(x) = \sum_{i_1 < \ldots < i_q} \Pi_{i_1 \ldots i_q} \frac{\partial}{\partial x_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_q}}(x).$$

If Π is a q-derivation and α is a differential q-form, which in a system of coordinates are written as $\Pi(x) = \sum_{i_1 < \ldots < i_q} \Pi_{i_1 \ldots i_q} \frac{\partial}{\partial x_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_q}}(x)$ and $\alpha(x) = \sum_{i_1 < \ldots < i_q} \alpha_{i_1 \ldots i_q} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_q}(x)$, then the pairing $\langle \Pi, \alpha \rangle$ is defined by

$$\langle \Pi, \alpha \rangle = \sum_{i_1 < \ldots < i_q} \Pi_{i_1 \ldots i_q} \alpha_{i_1 \ldots i_q}.$$

A simple check shows that it is independent of the choice of local coordinates.

REMARK 1.8. One may think that $\mathfrak{X}_X^q = \bigwedge^q(TX)$, but this is not necessarily true. By definition, $(\Omega_X^q)^* = \mathfrak{X}_X^q$ and since X may be a singular variety, it may happen $\bigwedge^q TX \neq (\Omega_X^q)^*$. We always have $\mathfrak{X}_X^1 = TX$ and we have a natural morphism $\bigwedge^q T_X \to \mathfrak{X}_X^q$ for every q.

EXAMPLE 1.9. Consider $X = \mathbb{C}^2/\mathbb{Z}_2$ defined by the equivalence relation $(x,y) \sim (-x,-y)$. The map $\mathbb{C}(u,v,w) \to \mathbb{C}(x,y)$ sending u,v,w to x^2, xy, y^2 gives an isomorphism from X to the quadric $uw = v^2$ in $\mathbb{C}^3$, a surface which is singular at the origin. Consider the map $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$ sending (x,y) to $(-x,-y)$. Then the tangent sheaf TX at the origin can be identified with the set of germs of vector fields Z in $\mathbb{C}^2$ such that $\sigma_* Z = Z$. So, if $p = (0,0) \in X$, then $(TX)_p$ is generated by $\{x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}\}$ and the image of the morphism $(\bigwedge^2 TX)_p \to (\mathfrak{X}_X^2)_p$is generated by $\{x^2\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}, xy\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}, y^2\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\}$. But we note that the 2-derivation $Z = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ satisfies $\sigma_* Z = Z$ and it is not in the image of the morphism above.

A holomorphic q-derivation Π will define a $\mathbb{C}$-multilinear skewsymmetric map from $\mathcal{O}_X^q$ to $\mathcal{O}_X$ by the following formula:

$$\Pi(f_1, \ldots, f_q) := \langle \Pi, \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_q \rangle.$$

Conversely, we have:

LEMMA 1.10. *A $\mathbb{C}$-multilinear map $\Pi : \mathcal{O}_X^q \to \mathcal{O}_X$ arises from a holomorphic q-derivation if and only if Π is skewsymmetric and satisfies the Leibniz rule:*

$$\Pi(fg, f_2, \ldots, f_q) = f\Pi(g, f_2, \ldots, f_q) + g\Pi(f, f_2, \ldots, f_q).$$

PROOF. The "only if" part is straightforward. For the "if" part, we just have to check that the value of $\Pi(f_1, \ldots, f_q)(x)$ depends only on $\mathrm{d}f_1(x), \ldots, \mathrm{d}f_q(x)$. Equivalently, we have to check that if $\mathrm{d}f_1(x) = 0$ then $\Pi(f_1, \ldots, f_q)(x) = 0$. But this is a direct consequence of Leibniz rule. □

In particular, if $\{\cdot\,,\cdot\}$ is a Poisson structure on X, then there exists a unique 2-derivation Π, the **Poisson tensor**, such that

$$\{f,g\} = \Pi(f,g) = \langle \Pi, \mathrm{d}f \wedge \mathrm{d}g \rangle.$$

If Π is a 2-derivation on X and $(x_1, \ldots, x_n)$ is a local system of coordinates, we can write

$$\Pi = \sum_{i<j} \{x_i, x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

In order to study the Jacobi identity, we will use the following lemma.

LEMMA 1.11. *For any holomorphic* 2*-derivation* Π*, one can associate a* 3*-derivation* Λ *defined by*

$$\Lambda(f,g,h) = \{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\}$$

where $\{f,g\}$ *denotes* $\langle \Pi, \mathrm{d}f \wedge \mathrm{d}g \rangle$*.*

PROOF. It is clear that the right side of the formula above is $\mathbb{C}$-linear and antisymmetric. The Leibniz rule of the right side is a simple verification based on the Leibniz rule $\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\}f_2$ for the bracket of the 2-derivation Π. □

Direct calculations in local coordinates show that

$$\Lambda(f,g,h) = \sum_{i<j<k} \left(\oint_{ijk} \sum_s \frac{\partial \{x_i, x_j\}}{\partial x_s} \{x_s, x_k\} \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$$

where $\oint_{ijk} a_{ijk}$ means the cyclic sum $a_{ijk} + a_{jki} + a_{kij}$.

Clearly, the Jacobi identity for Π is equivalent to the condition $\Lambda = 0$. Thus we have:

PROPOSITION 1.12. *A* 2*-derivation* $\Pi = \sum_{i<j} \{x_i, x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ *expressed in terms of a given system of coordinates* $(x_1, \ldots, x_n)$ *is a Poisson tensor if and only if it satisfies the following system of equation:*

$$\oint_{ijk} \sum_s \frac{\partial \{x_i, x_j\}}{\partial x_s} \{x_s, x_k\} = 0 \ (\forall i, j, k)$$

In the next section, we give another interpretation of Jacobi identity using the Schouten bracket.

1.3. Lie and Schouten brackets. If we write $A = \sum_i a_i \frac{\partial}{\partial x_i}$ and $B = \sum_i b_i \frac{\partial}{\partial x_i}$, the Lie Bracket of A and B is

$$[A,B] = \sum_i \left(a_i \sum_j \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} - b_i \sum_j \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial x_j} \right).$$

We change the notation and rewrite $\frac{\partial}{\partial xi}$ as ζ_i and consider them as formal variables. We define the multiplication $\zeta_i \zeta_j := \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ and we have $\zeta_i \zeta_j = -\zeta_j \zeta_i$. We also ask that $x_i \zeta_j = \zeta_j x_i$ for arbitrary i and j.

So, we can write $X = \sum_i a_i \zeta_i$ and $Y = \sum_i b_i \zeta_i$ and consider them formally as functions of variables (x_i, ζ_i), which are linear in the formal variables (ζ_i). We can write $[X, Y]$ formally as

$$[A,B] = \sum_i \left(\frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} - \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i} \right)$$

If $V = \sum_{i_1<\ldots<i_p} V_{i_1\ldots i_p}\frac{\partial}{\partial x_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_p}}$ is a p-derivation, then we will consider it as a homogeneous polynomial of degree p in the formal variables (ζ_i):

$$V = \sum_{i_1<\ldots<i_p} V_{i_1\ldots i_p}\zeta_{i_1}\ldots\zeta_{i_p}$$

Due to the anti-commutativity of (ζ_i), one must be careful about the signs when dealing with multiplications and differentiations involving these formal variables. The differentiation rule we will adopt is as follows:

$$\frac{\partial(\zeta_{i_1}\ldots\zeta_{i_p})}{\partial\zeta_{i_p}} := \zeta_{i_1}\ldots\zeta_{i_{p-1}}.$$

Equivalently,

$$\frac{\partial(\zeta_{i_1}\ldots\zeta_{i_p})}{\partial\zeta_{i_k}} := (-1)^{p-k}\zeta_{i_1}\ldots\hat{\zeta}_{i_k}\ldots\zeta_{i_p}.$$

If

$$A = \sum_{i_1<\ldots<i_p} A_{i_1\ldots i_p}\frac{\partial}{\partial x_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_p}} = \sum_{i_1<\ldots<i_p} A_{i_1\ldots i_p}\zeta_{i_1}\ldots\zeta_{i_p}.$$

is a p-derivation, and

$$B = \sum_{i_1<\ldots<i_q} B_{i_1\ldots i_q}\frac{\partial}{\partial x_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_q}} = \sum_{i_1<\ldots<i_q} B_{i_1\ldots i_q}\zeta_{i_1}\ldots\zeta_{i_q}.$$

is a q-derivation, we can define the bracket of X and Y as follows:

$$(1) \qquad [A,B] = \sum_i \left(\frac{\partial A}{\partial\zeta_i}\frac{\partial B}{\partial x_i} - (-1)^{(p-1)(q-1)}\frac{\partial B}{\partial\zeta_i}\frac{\partial A}{\partial x_i}\right).$$

The bracket $[A,B]$ is a homogeneous polynomial of degree $p+q-1$ in the formal variables ζ_i, so it is a $(p+q-1)$-derivation.

Definition 1.13. If A is a p-derivation and B is a q-derivation, then a $(p+q-1)$-derivation defined by the bracket $[A,B]$ is called the **Schouten bracket** of A and B.

We will list, without any proof, the basic properties of the Schouten bracket. For the interested reader, we refer to [**11**], chapter 1.

Proposition 1.14. *If A is a p-derivation, B is a q-derivation and C is an r-derivation, then the Schouten bracket satisfies the following properties:*

(1) *Graded anti-commutative*

$$[A,B] = -(-1)^{(p-1)(q-1)}[B,A]$$

(2) *Graded Leibniz rule*

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(p-1)q} B \wedge [A, C]$$

$$[A \wedge B, C] = A \wedge [B, C] + (-1)^{(r-1)q} [A, C] \wedge B$$

(3) *Graded Jacobi identity*

$$(-1)^{(p-1)(r-1)}[A, [B, C]] + (-1)^{(q-1)(p-1)}[B, [A, C]] \\ +(-1)^{(r-1)(q-1)}[C, [A, B]] = 0$$

(4) *If X is a vector field and B is a q-derivation*

$$[X, B] = \mathcal{L}_X B$$

where $\mathcal{L}_X$ denotes the Lie derivative by X. In particular, if X and B are two vectors field then the Schouten Bracket coincides with their Lie bracket. If X is a vector field and $B = f$ is a function (i.e a 0-vector field) then we have

$$[X, f] = X(f).$$

(5) *If $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a biholomorphism, then*

$$\phi_*[A, B] = [\phi_* A, \phi_* B].$$

Note that item (5) of the last proposition shows that the computations of $[A, B]$ does not depend on the choice of local coordinates. It follows that if X is a variety, we have a well defined $\mathbb{C}$-bilinear map

$$\begin{array}{rll} [\cdot\,,\cdot\,]: & \mathfrak{X}^p_X \times \mathfrak{X}^q_X & \to \mathfrak{X}^{p+q-1}_X \\ & (A, B) & \mapsto [A, B]. \end{array}$$

The Schouten Bracket offers a very convenient way to characterize Poisson structures.

THEOREM 1.15. *A 2-derivation Π is a Poisson tensor if and only if the Schouten bracket of Π with itself vanishes:*

$$[\Pi, \Pi] = 0.$$

Proof. Write $\Pi = \sum_{i<j}\{x_i, x_j\}\zeta_i\zeta_j$ and use the formula (1) to get

$$\begin{aligned}
[\Pi, \Pi] &= \sum_s \left(\frac{\partial \Pi}{\partial \zeta_k}\frac{\partial \Pi}{\partial x_k} + \frac{\partial \Pi}{\partial \zeta_k}\frac{\partial \Pi}{\partial x_k}\right) \\
&= 2\sum_s \left(\frac{\partial \Pi}{\partial \zeta_k}\frac{\partial \Pi}{\partial x_k}\right) \\
&= 2\sum_{i<j} \left(\{x_i, x_j\}\zeta_i \frac{\partial \Pi}{\partial x_j} - \{x_i, x_j\}\zeta_j \frac{\partial \Pi}{\partial x_i}\right) \\
&= 2\sum_{\substack{i<j \\ k<l}} \left(\{x_i, x_j\}\zeta_i \frac{\partial \{x_k, x_l\}}{\partial x_j}\zeta_k\zeta_l - \{x_i, x_j\}\zeta_j \frac{\partial \{x_k, x_l\}}{\partial x_i}\zeta_k\zeta_l\right) \\
&= 2\sum_{\substack{i<s \\ k<l}} \{x_i, x_s\}\frac{\partial \{x_k, x_l\}}{\partial x_s}\zeta_i\zeta_k\zeta_l - 2\sum_{\substack{s<j \\ k<l}} \{x_s, x_j\}\frac{\partial \{x_k, x_l\}}{\partial x_s}\zeta_j\zeta_k\zeta_l \\
&= 2\sum_{\substack{i<s \\ k<l}} \{x_i, x_s\}\frac{\partial \{x_k, x_l\}}{\partial x_s}\zeta_i\zeta_k\zeta_l - 2\sum_{\substack{s<i \\ k<l}} \{x_s, x_i\}\frac{\partial \{x_k, x_l\}}{\partial x_s}\zeta_i\zeta_k\zeta_l \\
&= 2\sum_{\substack{i \\ k<l}} \left(\sum_s \left(\{x_i, x_s\}\frac{\partial \{x_k, x_l\}}{\partial x_s}\zeta_i\zeta_k\zeta_l\right)\right) \\
&= 2\sum_{i<k<l} \left(\oint_{ikl} \sum_s \{x_i, x_s\}\frac{\partial \{x_k, x_l\}}{\partial x_s}\right)\zeta_i\zeta_k\zeta_l,
\end{aligned}$$

then $[\Pi, \Pi] = 0$ if and only if $\oint_{ikl}\sum_s\{x_i, x_s\}\frac{\partial\{x_k, x_l\}}{\partial x_s}\zeta_i\zeta_k\zeta_l = 0\ (\forall i, k, l)$. We conclude the proof by Proposition 1.12. □

1.4. More examples. Expressing Poisson structures as 2-derivations has some advantages to give examples, as we shall see.

Example 1.16. Let (X, Π_X) and (Y, Π_Y) be Poisson varieties. Then their direct product $M = X \times Y$ can be equipped with the natural Poisson structure $\Pi_X + \Pi_Y$.

Example 1.17. Because there are no nontrivial 3-derivation on a two dimensional variety S, any 2-derivation on S is a Poisson tensor.

Example 1.18. Choose arbitrary constants Π_{ij}. By Proposition 1.12, we have that $\Pi = \sum \Pi_{ij}\zeta_i\zeta_j$ is a Poisson tensor on $\mathbb{C}^n$. This structure is called a constant Poisson structure.

EXAMPLE 1.19. Let V be a finite-dimensional vector space over $\mathbb{C}$. A **linear Poisson structure** on V is a Poisson structure on V for which the Poisson bracket of two linear functions is again a linear function. Since linear functions can be regarded as an element of V^*, then the operation $(f, g) \mapsto \{f, g\}$ restricted to the linear functions gives rise to an operation $[\cdot\,,\cdot\,] : V^* \times V^* \to V^*$, which is a Lie algebra structure on V^*.

Conversely, any Lie algebra structure on $(V^*, [\cdot\,,\cdot\,])$ determines a linear Poisson structure on V. Indeed, choose a linear basis $(e_1, \ldots, e_n)$ of V^*. We can view e_i as an linear function x_i on $V^{**} = V$. If $[e_i, e_j] = \sum c_{ij}^k e_k$, $c_{ij}^k \in \mathbb{C}$, then we write $\{x_i, x_j\} = \sum c_{ij}^k x_k$. Consider the 2-derivation $\Pi = \{x_i, x_j\}\zeta_i\zeta_j$, then the Jacobi identity for $[\cdot\,,\cdot\,]$ is equivalent to the Jacobi identity for Π.

Thus, there is a natural bijection between finite-dimensional linear Poisson structures and finite-dimensional Lie algebras.

Another natural example is the diagonal Poisson structure (see [**11**]).

EXAMPLE 1.20. The diagonal Poisson structure on $\mathbb{C}^n$ is defined by:

$$\Pi = \sum_{i<j} \lambda_{ij} x_i x_j \zeta_i \zeta_j,\ \lambda_{ij} \in \mathbb{C}.$$

We will study this example in detail later (see chapter 4, section 2).

1.5. The curl operator. Recall that, if A is a p-derivation and ω is a holomorphic q-form, with $q \geq p$, then the inner product of ω by A is a unique $(q-p)$-form, denoted by $i_A\omega$ such that

$$\langle i_A\omega, B\rangle = \langle \omega, A \wedge B\rangle$$

for any $(q-p)$-derivation B. If $q < p$, we put $i_A\omega = 0$ by convention. Similarly, when $p \geq q$, we can define the inner product of a p-derivation A by a q-form η to be a unique $(p-q)$-derivation, denoted by $i_\eta A$ such that

$$\langle \alpha, i_\eta A\rangle = \langle \alpha \wedge \eta, A\rangle$$

for any $(p-q)$-form α.

REMARK 1.21. We must be careful with the signs when dealing with inner products. Note that if Π is a 2-derivation, then $i_{\mathrm{d}f}\Pi(g) = \Pi(g, f)$.

LEMMA 1.22. *If f is a function and A a p-derivation then*

$$i_{\mathrm{d}f}A = [A, f].$$

PROOF. Just need to use induction on p and apply the graded Leibniz property for the Schouten Bracket (Proposition 1.14). ☐

Let Ω be a holomorphic volume form on a n-dimensional variety X. Then, for every $p = 0, 1, \ldots, n$, the map

$$\Omega : \mathfrak{X}^p_X \to \Omega^{n-p}_X$$

defined by $\Omega(A) = i_A(\Omega)$, is an $\mathcal{O}_X$-linear isomorphism from the space $\mathfrak{X}^p_X$ to Ω^{n-p}_X. The inverse map will be denoted as $\widehat{\Omega} : \Omega^{n-p}_X \to \mathfrak{X}^p_X$.

Denote by $D_\Omega : \mathfrak{X}^p_X \to \mathfrak{X}^{p-1}_X$ the linear operator defined by the composition $D_\Omega = \widehat{\Omega} \circ \mathrm{d} \circ \Omega$. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{X}^p_X & \xrightarrow{\ \Omega\ } & \Omega^{n-p}_X \\ {\scriptstyle D_\Omega}\downarrow & & \downarrow{\scriptstyle \mathrm{d}} \\ \mathfrak{X}^{p-1}_X & \xleftarrow{\ \widehat{\Omega}\ } & \Omega^{n-p+1}_X \end{array}$$

Definition 1.23. The operator D_Ω is called the curl operator with respect to Ω. If A is a p-derivation, then $D_\Omega A$ is called the curl of A.

In a local system of coordinates $(x_1, \ldots, x_n)$ with $\Omega = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$, we have the following convenient formula

$$D_\Omega A = \sum_i \frac{\partial^2 A}{\partial x_i \partial \zeta_i}.$$

We can recover the Schouten bracket from the curl operator via Koszul formula:

Theorem 1.24. *If A is a p-derivation, B is a q-derivation and Ω is a volume form then*

$$[A, B] = (-1)^q D_\Omega(A \wedge B) - (D_\Omega A) \wedge B - (-1)^q A \wedge (D_\Omega B)$$

Proof. The proof is obtained by expanding $(-1)^q D_\Omega(A \wedge B)$. □

Corollary 1.25. *We have the following formula:*

$$D_\Omega[A, B] = [A, D_\Omega B] + (-1)^{b-1}[D_\Omega A, B].$$

Proof. Just expand the right side by the formula of the last theorem, apply D_Ω and use $D_\Omega \circ D_\Omega = 0$. □

Corollary 1.26. *If Π is a Poisson tensor and Ω a volume form, then*

$$[D_\Omega \Pi, \Pi] = 0 \text{ and } \mathcal{L}_{(D_\Omega \Pi)} \Omega = 0.$$

The vector field $D_\Omega \Pi$ is called the curl vector field. Using the same notation of the subsection 1.3, Let

$$\Pi = \sum_{i,j,k,l} \Pi^{kl}_{ij} x_k x_l \zeta_i \zeta_j$$

be a quadratic Poisson structure ($\Pi_{ij}^{kl} \in \mathbb{C}$, $\Pi_{ij}^{kl} = \Pi_{ij}^{lk} = -\Pi_{ji}^{kl}$). Its curl vector field $X = D_\Omega \Pi$ with respect to the volume form $\Omega = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$ is a linear vector field which has the following expression:

$$X = \sum_{i,j,k} \Pi_{ij}^{kl} x_k \zeta_i$$

Recall that, if we change the coordinate system linearly, the corresponding volume form will be multiplied by a constant function and, therefore, the curl vector field, up to scalar, will not be changed. So we may write the curl vector field as $X = D\Pi$, without the reference to Ω.

If $c_1, \ldots, c_n$ are the eigenvalues of X, then Corollary 1.26 says that

$$\sum_{i=1}^{n} c_i = 0 \tag{2}$$

Definition 1.27. If Π is a quadratic Poisson structure, then the eigenvalues of its linear curl vector field $X = D\Pi$ are called the eigenvalues of Π. A quadratic Poisson structure is called non-resonant if its eigenvalues $c_1, \ldots, c_n$ do not satisfy any relation of resonance other than (2). In other words, if $\sum_{i=1}^{n} \alpha_i c_i = 0$ with $\alpha_i \in \mathbb{Z}$ then $\alpha_1 = \ldots = \alpha_n$.

One may argue that a "generic" (in a disputable sense) quadratic Poisson structure is nonresonant. The following proposition, due to Dufour and Haraki, says that "generic" quadratic Poisson structures are diagonalizable in the sense of example 1.20.

Proposition 1.28. *If the eigenvalues* $c_1, \ldots, c_n$ *of a quadratic Poisson structure* Π *do not verify any relation of the type*

$$c_i + c_j = c_r + c_s$$

with $i < j$ *and* $\{r, s\} \neq \{i, j\}$*, then* Π *is diagonalizable, i.e., there exists a linear coordinate system in which* Π *is diagonal. In particular, nonresonant quadratic Poisson structures are diagonalizable.*

Proof. Notice that the condition of the above theorem implies that the eigenvalues $c_1, \ldots, c_n$ are pairwise different (for example, $c_1 = c_2$ leads to $c_1 + c_2 = c_1 + c_1$). Thus the linear curl vector field $X = D\Pi$ is diagonalizable, i.e., there is a linear coordinate system in which X is diagonal:

$$X = \sum c_i x_i \frac{\partial}{\partial x_i}$$

Then the equation $[X, \Pi] = 0$ can be written as

$$\begin{aligned} 0 &= \left[\sum_i c_i x_i \frac{\partial}{\partial x_i}, \sum_{r,s,u,v} \Pi_{rs}^{uv} x_u x_v \frac{\partial}{\partial x_r} \frac{\partial}{\partial x_s} \right] \\ &= \sum_{r,s,u,v} \Pi_{rs}^{uv} (c_u + c_v - c_r - c_s) x_u x_v \frac{\partial}{\partial x_r} \wedge \frac{\partial}{\partial x_s}, \end{aligned}$$

whence $\Pi_{rs}^{uv} = 0$ for $\{r, s\} \neq \{u, v\}$, hence the result. □

2. Normal forms

Given a Poisson manifold Π on X, we can associate to it a natural homomorphism

$$\begin{aligned} \Pi^\sharp : \quad \Omega_X^1 &\to \mathfrak{X}_X^1 \\ \alpha &\mapsto i_\alpha \Pi \end{aligned}$$

the contraction of 2-derivation by 1-forms (see remark 1.21 to recall the convention adopted in this thesis).

For every $x \in X$, if $f \in \mathcal{O}_{X,x}$ is a function, then the **Hamiltonian vector field** of f is defined as $Z_f = \Pi^\sharp(\mathrm{d}f) = \{\cdot\, , f\}$. The $\mathbb{C}$-sheaf of all Hamiltonian vector fields will be denoted by $Ham(\Pi)$.

LEMMA 1.29. *The following identity holds:*

$$[Z_f, Z_g] = -Z_{\{f,g\}}$$

PROOF.

$$\begin{aligned} [Z_f, Z_g](h) &= Z_f(Z_g(h)) - Z_g(Z_f(h)) \\ &= Z_f(\{h, g\}) - Z_g(\{h, f\}) \\ &= \{\{h, g\}, f\} - \{\{h, f\}, g\} \\ &= \{\{f, g\}, h\} \quad \text{by Jacobi identity} \\ &= -Z_{\{f,g\}}(h) \end{aligned}$$

for every $h \in \mathcal{O}_{X,x}$. □

Since $\Pi^\sharp$ is antysimmetric, we see that the rank of Poisson structure is always even.

The map $\Pi^\sharp : \Omega_X^1 \to \mathfrak{X}_X^1$ defines naturally a holomorphic distribution $\mathcal{D}_\Pi$ generated by the Hamiltonian vector fields. By simple linear algebra, if $\mathrm{rk}(\Pi) = 2k$ then $\mathcal{D}_\Pi$ is a holomorphic distribution of dimension $2k$. By the lemma above, this distribution is involutive. We put this in a definition.

DEFINITION 1.30. Let (X, Π) be a Poisson variety and $x \in X$. Then the image $(\mathcal{D}_\Pi)_x = \mathcal{D}_x$ is called the characteristic space at x of the Poisson structure. The dimension $\dim \mathcal{D}_x$ is called the **rank of** Π **at** x, and $\max_{x \in X} \dim \mathcal{D}_x$ is called the **rank** of Π. When $\operatorname{rk} \Pi_x = \dim X$, we say that Π is **nondegenerate** at x. If $\operatorname{rk} \Pi_x$ does not depend on x, then Π is called **regular Poisson structure**.

EXAMPLE 1.31. In $\mathbb{C}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, consider the constant Poisson structure $\sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$, with $k \leq n$. It is a regular Poisson structure of rank $2k$. The case that $k = n$, this Poisson structure is symplectic with $\omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i$. This symplectic structure is the canonical one.

The characteristic space $\mathcal{D}_x$ admits a natural antisymmetric nondegenerate bilinear scalar product: if X and Y are two vectors of $\mathcal{D}_x$, then we put

$$(X, Y) := \langle \beta, X \rangle = \langle \Pi, \alpha \wedge \beta \rangle = -\langle \Pi, \beta \wedge \alpha \rangle = -\langle \alpha, Y \rangle = -(Y, X),$$

where $\alpha, \beta \in T_x^* X$ are two covectors such that $X = \Pi^\sharp(\alpha)$ and $Y = \Pi^\sharp(\beta)$.

About local forms of symplectic structure, we have the so called Darboux theorem.

THEOREM 1.32. *Let (M, ω) be a symplectic manifold and $x \in M$. Then there exists a neighborhood U of x and coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that $\omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i$ in U.*

2.1. Weinstein Theorem. In this subsection, we prove the splitting theorem of Weinstein [**24**], which says that locally a Poisson manifold is a direct product of a symplectic manifold with another Poisson manifold whose Poisson tensor vanishes at a point. This splitting theorem, together with Darboux theorem, which is proved at the same time, gives us local "good" coordinates for Poisson manifold. The proof we give of this theorem can be found in [**11**], Theorem 1.4.5.

THEOREM 1.33 (Splitting Theorem). *Centered at any point x in a Poisson manifold M, there are coordinates $(x_1, \ldots x_k, y_1, \ldots, y_k, z_1, \ldots, z_\ell)$ such that*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i,j=1}^{\ell} \varphi_{ij}(y) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \quad \textit{and} \quad \varphi_{ij}(0) = 0 \ .$$

PROOF. Let N be a submanifold of M transverse to the distribution in a neighborhood of x, i.e, $T_x N \oplus \mathcal{D}_x = T_x M$.

If $\Pi(x) = 0$ then $k = 0$ and there is nothing to prove. Suppose that $\Pi(x) \neq 0$. Let x_1 be a local function (defined in a small neighborhood of x in M) which vanishes on N and such that $\mathrm{d}x_1(x) \neq 0$. Since $\mathcal{D}_x$ is transversal to N, there is a Hamiltonian vector field $Z_g(x) \in \mathcal{D}_x$ such that $\langle Z(x), \mathrm{d}x_1(x)\rangle \neq 0$, or equivalently $Z_{x_1}(g)(x) \neq 0$. Then $Z_{x_1}(x) \neq 0$ and, since Z_{x_1} is not tangent to N, there is a local function y_1 such that $y_1(N) = 0$ and $Z_{x_1}(y_1) = 1$ in a neighborhood of x, in other words,

$$\{x_1, y_1\} = 1$$

We see that Z_{x_1} and Z_{y_1} are linearly independent, because, if $Z_{y_1} = \lambda Z_{x_1}$, λ a function, then $\{x_1, y_1\} = \lambda Z_{x_1}(x_1) = 0$. Moreover, we have

$$[Z_{x_1}, Z_{y_1}] = -Z_{\{x_1,y_1\}} = 0$$

Thus Z_{x_1} and Z_{y_1} are two linearly independent vector fields which commute. By Frobenius theorem, we can find a local system of coordinates $(z_1, \ldots, z_n)$ such that

$$Z_{x_1} = \frac{\partial}{\partial z_1}, \quad Z_{y_1} = \frac{\partial}{\partial z_2}.$$

In the coordinates $(z_1, \ldots, z_n)$, we have $\{y_1, z_i\} = Z_{y_1}(z_i) = 0$ and we have $\{x_1, z_i\} = Z_{x_1}(z_i) = 0$, for $i = 3, \ldots, n$. Jacobi identity implies that $\{x_1, \{z_i, z_j\}\} = \{y_1, \{z_i, z_j\}\} = 0$ for $i, j \geq 3$. Hence $\{z_i, z_j\}$ does not depend of (x_1, y_1) for $i, j \geq 3$.

Then, in the coordinates $(x_1, y_1, z_3, \ldots, z_n)$, Π has the form

$$\Pi = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i,j\geq 3} \varphi_{ij}(y) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

The above formula implies that our Poisson structure is locally the product of a canonical symplectic structure on the plane $\{(x_1, y_1)\}$ with a Poisson structure on a $(n-2)$-dimensional manifold $\{(z_3, \ldots, z_n)\}$. We prove the theorem by induction on rank of Π at x. □

2.2. Linearization of Poisson structures. The local classification question can be reduced, thanks to Weinstein splitting theorem, to the study of Poisson structures Π vanishing at x, i.e, $\Pi(x) = 0$.

Let Π be a Poisson structure which vanishes at a point x. Denote by

$$\Pi = \Pi^{(1)} + \Pi^{(2)} + \cdots$$

the Taylor expansion of Π in a local coordinate system centered at x, where $\Pi^{(k)}$ is a homogeneous 2-derivation of degree k. Recall that, the terms of degree k of the equation $[\Pi, \Pi]$ give

$$\sum_{i=1}^{k} \left[\Pi^{(i)}, \Pi^{(k+1-i)}\right] = 0$$

In particular $[\Pi^{(1)}, \Pi^{(1)}] = 0$, i.e., the linear part $\Pi^{(1)}$ of Π is a linear Poisson structure. One says that Π is locally analytically (resp. formally) *linearizable* if there exists a local analytic (resp. formal) diffeomorphism ϕ (a coordinate transformation) such that $\phi_* \Pi = \Pi^{(1)}$.

DEFINITION 1.34. A finite-dimensional Lie algebra $\mathfrak{g}$ is called analytically (resp. formally) nondegenerate if any analytic (resp. formal) Poisson structure Π which vanishes at a point and whose linear part at that point corresponds to $\mathfrak{g}$ is analytically (resp. formally) linearizable.

There are many interesting results about non-degeneracy of $\mathfrak{g}$ algebras. We will cite some of them. We refer to [**11**], chapter 4, for the proofs.

THEOREM 1.35. *Any semisimple Lie algebra is formally and analytically nondegenerate.*

The formal case was proved by Weinstein (1983) and the analytic case, by Conn (1984).

The next theorem is due to Dufour and Zung.

THEOREM 1.36. *For any natural number n, the Lie algebra $\mathfrak{aff}(n, \mathbb{C})$ of affine tranformations of $\mathbb{C}^n$ is formally and analytically nondegenerate.*

The next theorem, due to Dufour and Molinier, will be used in this thesis to prove a technical Lemma 3.21.

THEOREM 1.37. *The direct product*

$$\mathfrak{aff}(1, \mathbb{C}) \times \cdots \times \mathfrak{aff}(1, \mathbb{C})$$

of n copies of $\mathfrak{aff}(1, \mathbb{C})$ is formally and analytically nondegenerate for any natural number n.

CHAPTER 2

Global theory

1. Poisson structures on algebraic varieties

1.1. Poisson foliation. Let X be a variety. Denote by $\mathfrak{X}_X^q = (\Omega_X^q)^*$ the sheaf of q-derivations of X (see remark 1.8). A Poisson structure Π is a holomorphic section $\Pi \in H^0(X, \mathfrak{X}_X^2)$ such that $[\Pi, \Pi] = 0$. In this case, we say that (X, Π) is a Poisson variety. If $\Pi^k \neq 0$ and $\Pi^{k+1} = 0$, where Π^k is the k-th wedge power of Π, we say that the Poisson structure has **rank** $2k$. For each p in X, let m be the biggest integer such that $\Pi^m(p) \neq 0$. We say that Π has **rank** $2m$ at p.

In local coordinates $(x_1, \ldots, x_n)$ of a neighborhood of a regular point $p \in X$, we can write $\Pi = \sum_{ij}\{x_i, x_j\}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. We will use the following notation: $\Pi(f, g) = \Pi(\mathrm{d}f \wedge \mathrm{d}g) = \{f, g\}_\Pi = \{f, g\}$, if no confusion can arise.

We denote by $\Pi^\sharp : \Omega_X^1 \to \mathfrak{X}_X^1$ the $\mathcal{O}_X$-linear **anchor map** defined by

$$\Pi^\sharp(\alpha)(\beta) = \Pi(\beta \wedge \alpha)$$

A function $f \in \mathcal{O}_X(U)$, where U is an open subset of X, naturally generates a vector field $Z_f = \Pi^\sharp(\mathrm{d}f)$, the so called **Hamiltonian vector field**. The $\mathbb{C}$-sheaf of all Hamiltonian vector field is denoted by $Ham(\Pi)$.

Definition 2.1. Let X be a variety with $\dim X = n$. A **singular holomorphic distribution** $\mathcal{D}$ of codimension $n - 2k$ consists of a holomorphic global section of $\omega \in \mathbb{P}H^0(X, \mathcal{L}^* \otimes \Omega_X^{n-2k})$, where $\mathcal{L}$ is a line bundle. The singular set Sing$\mathcal{D}$ of the distribution $\mathcal{D}$ consists of points $x \in X$ such that $\omega(x) = 0$. A distribution $\mathcal{D}$ is involutive if $(ker\, \omega)_x := \{Z \in (TX)_x; i_Z\omega = 0\}$ is involutive, i.e., if $Z_1, Z_2 \in (ker \omega)_x$, then $[Z_1, Z_2] \in (ker\, \omega)_x$.

A foliation $\mathcal{F}$ is an involutive distribution such that the singular set Sing $\mathcal{F}$ has codimension ≥ 2. Clearly, every involutive distribution $\mathcal{D}$ induces a foliation, we call this process as **saturation of the distribution**.

We call $T\mathcal{F} := ker\, \omega$ the **tangent sheaf** of $\mathcal{F}$, its dual, denoted by $T^*\mathcal{F}$, is called the **cotangent sheaf** of $\mathcal{F}$. We call $N_\mathcal{F} := TX/T_\mathcal{F}$ the normal sheaf and its dual, denoted by $N^*\mathcal{F}$, is the conormal sheaf.

REMARK 2.2. The normal sheaf $N_{\mathcal{F}}$ is a line bundle if and only if the foliation $\mathcal{F}$ is regular, i.e., $\operatorname{Sing}\mathcal{F} = \emptyset$, but $N^*\mathcal{F}$ is always a line bundle (see [**3**] Proposition 1.33).

If $\omega \in \mathbb{P}H^0(X, \mathcal{L}^* \otimes \Omega^1_X)$ defines a codimension 1 foliation in X, the involutivity of $T\mathcal{F}$ is equivalent to $\omega \wedge \mathrm{d}\omega = 0$ and we have $\mathcal{L} = N^*\mathcal{F}$.

Since $H^0(X, \mathfrak{X}^{2k}_X) = H^0(X, K^*_X \otimes \Omega^{n-2k}_X)$, a rank $2k$ Poisson vector field Π induces a section $\omega \in H^0(X, K^*_X \otimes \Omega^{n-2k}_X)$. Let Z_f and Z_g be Hamiltonian vector fields in X, then, by Jacobi identity, $[Z_f, Z_g] = -Z_{\{f,g\}}$, and so $ker\,\omega$ is involutive.

DEFINITION 2.3. The $2n-2k$-form $\omega \in H^0(X, K^*_X \otimes \Omega^{n-2k}_X)$ induced by Π, where $\operatorname{rk}\Pi = 2k$, is the **Poisson distribution**. The saturation of the Poisson distribution is called **Poisson foliation**.

REMARK 2.4. Usually, in the literature, the image of the anchor map $\Pi^\sharp : \Omega^1_X \to TX$ is called the tangent sheaf of a **Foliation by Symplectic leaves**. In this thesis, we do not consider this sheaf. We call the $2n-2k$-form induced by Π the Poisson distribution and its saturation by **Poisson foliation**.

REMARK 2.5. If Π is a rank $2k$ Poisson structure in X, then $\Pi^k \wedge Z_f = 0$ for every Hamiltonian vector field Z_f. In particular, at the points where the Poisson distribution is regular, the tangent sheaf of the Poisson foliation is composed by Hamiltonian vector field.

1.2. Poisson structures on projective spaces. Let us consider the projective space $\mathbb{P}^n$ and denote $\phi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ the natural projection. The following is a description of the Poisson tensors of $\mathbb{P}^n$ in homogeneous coordinates.

Let $T\mathbb{P}^n$ be the tangent sheaf on $\mathbb{P}^n$ and let $R = \sum x_i \frac{\partial}{\partial x_i}$ be the radial vector field on $\mathbb{C}^{n+1}$, let $\mathfrak{X}_1$ be $\mathbb{C}$-vector space consisting of linear vector fields on $\mathbb{C}^{n+1}$. Recall also the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}^{n+1}_{\mathbb{P}^n}(1) \xrightarrow{P} T\mathbb{P}^n \to 0. \tag{3}$$

Remember that we can identify $H^0(\mathbb{P}^n, \mathcal{O}^{n+1}_{\mathbb{P}^n}(1))$ with $\mathfrak{X}_1$ and the first map of the exact sequence above is the wedge product by the radial field and the second map is the natural projection, which we denoted by P. Remark that $\mathfrak{X}_1$ consists of vector fields which are invariant under the action of $\mathbb{C}^{n+1}$ by homotheties.

Taking exterior products of (3), one obtains the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \wedge \mathcal{O}^{n+1}_{\mathbb{P}^n}(1) \to \bigwedge^2 \mathcal{O}^{n+1}_{\mathbb{P}^n}(1) \xrightarrow{P} \bigwedge^2 T\mathbb{P}^n \to 0.$$

Note that the first wedge product in the exact sequence above makes sense, since we can identify $\mathcal{O}_{\mathbb{P}^n}$ as a subsheaf of $\mathcal{O}_{\mathbb{P}^n}^{n+1}(1)$. With the natural identifications, $\mathcal{O}_{\mathbb{P}^n} \subseteq \mathcal{O}_{\mathbb{P}^n}^{n+1}(1)$ is a subsheaf generated by the radial vector field. Consider the following map

$$\begin{aligned} \xi : \mathcal{O}_{\mathbb{P}^n} \wedge \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) &\to T\mathbb{P}^n \\ R \wedge X &\mapsto P(X), \end{aligned}$$

where P is the morphism appearing in (3).

A simple check shows that this map is well defined and it is surjective. Comparing the rank of both sheaves, we conclude that the map ξ is an isomorphism (both have rank n).

So we have the following exact sequence

$$0 \to T\mathbb{P}^n \to \bigwedge^2 \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \to \bigwedge^2 T\mathbb{P}^n \to 0.$$

Let $V \in H^0(\mathbb{P}^n, \bigwedge^2 T\mathbb{P}^n)$ be a 2-derivation on $\mathbb{P}^n$. Since we have $H^1(\mathbb{P}^n, T\mathbb{P}^n) = 0$, then V can be lifted to a section of $\tilde{V} \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1))$. By the exact sequences above and with the natural identifications, if $V' \in H^0(\mathbb{P}^{n+1}, \bigwedge^2 \mathcal{O}_{\mathbb{P}^n}^{n+1}(1))$ is another lifting of V, then there exists a vector field $X \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1))$, such that $\tilde{V} = V' + X \wedge R$, where $\langle R \rangle = \mathcal{O}_{\mathbb{P}^n} \subseteq \mathcal{O}_{\mathbb{P}^n}^{n+1}(1)$ is the radial vector field.

By the same arguments, we have the following exact sequence

$$0 \to \bigwedge^2 T\mathbb{P}^n \to \bigwedge^3 \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \xrightarrow{P} \bigwedge^3 T\mathbb{P}^n \to 0,$$

where the first map is the wedge product with the radial vector field.

Let $\Pi \in H^0(\mathbb{P}^n, \bigwedge^2 T\mathbb{P}^n)$ be a Poisson bivector field on $\mathbb{P}^n$ and let $\widetilde{\Pi} \in \bigwedge^2 \mathfrak{X}_1$ be a lifting of Π. In homogeneous coordinates $\widetilde{\Pi}$ takes a form

$$\widetilde{\Pi} = \sum a_{ij}^{kl} x_k x_l \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

We conclude that Π is a Poisson structure if and only if $P([\widetilde{\Pi}, \widetilde{\Pi}]) = 0$, where $P : \bigwedge^3 \mathfrak{X}_1 \to \bigwedge^3 T\mathbb{P}^n$ is induced by the last map of (3).

The kernel of $P : \bigwedge^3 \mathfrak{X}_1 \to \bigwedge^3 T\mathbb{P}^n$ is generated by the vector fields of the form

$$X = Y \wedge R,$$

where R is the radial field and Y belongs to $\bigwedge^2 \mathfrak{X}_1$.

Consider the map $D_R : \bigwedge^\bullet \mathfrak{X}_1 \to \bigwedge^\bullet \mathfrak{X}_1$ defined by $D_R(Y) = Y \wedge R$. So $\widetilde{\Pi}$ defines a Poisson Bracket on $\mathbb{P}^n$ if and only if

$$D_R[\widetilde{\Pi}, \widetilde{\Pi}] = 0.$$

By Koszul formula (see Theorem 1.24 of chapter 1, we conclude that

$$D_R(D_\Omega(\widetilde{\Pi} \wedge \widetilde{\Pi}) - 2\widetilde{\Pi} \wedge D_\Omega(\widetilde{\Pi})) = 0,$$

where $\Omega = \mathrm{d}x_0 \wedge \ldots \wedge \mathrm{d}x_n$.

LEMMA 2.6. *We have the following formula*

$$D_R D_\Omega X + D_\Omega D_R X = (n+d)X$$

where X is a homogeneous multiderivation such that the degree of the coefficients is d.

PROOF. By Koszul Formula, we have

$$\begin{aligned} D_\Omega D_R X &= D_\Omega(X \wedge R) = \\ &= -[X, R] - (D_\Omega X) \wedge R + X \wedge D_\Omega R = \\ &= (d-1)X - D_R(D_\Omega X) + (n+1)X = \\ &= (n+d)X - D_R D_\Omega X. \end{aligned}$$

and this concludes the lemma. □

Now, we can prove the following theorem due to Bondal (see [**2**])

THEOREM 2.7. *Let Π be a Poisson bracket on $\mathbb{P}^n$. There exists a unique lifting $\widetilde{\Pi}$ of it on $\mathbb{C}^{n+1}$ with the following properties:*

(1) $\operatorname{rk} \Pi = \operatorname{rk} \widetilde{\Pi}$
(2) $D_\Omega \widetilde{\Pi} = 0$
(3) $[\widetilde{\Pi}, \widetilde{\Pi}] = 0$

PROOF. Let V be a lift of Π. By the last lemma, we can decompose $V = \widetilde{\Pi} + v'$ with $D_\Omega \widetilde{\Pi} = 0$, $D_R v' = 0$ and $\widetilde{\Pi}$ is a lifting of Π ($\widetilde{\Pi}$ is $\frac{1}{n+2} D_\Omega D_R V$ and v' is $\frac{1}{n+2} D_R D_\Omega V$). Since Π is a Poisson Bracket in $\mathbb{P}^n$, we have $D_R[\widetilde{\Pi}, \widetilde{\Pi}] = 0$ and, by Koszul formula, we have $D_\Omega[\widetilde{\Pi}, \widetilde{\Pi}] = 0$. Using lemma above, we conclude that $[\widetilde{\Pi}, \widetilde{\Pi}] = 0$.

If $\operatorname{rk} \Pi = 2k$, then $\Pi^k \neq 0$ and $\Pi^{k+1} = 0$. So $\widetilde{\Pi}^k \neq 0$ and $D_R \widetilde{\Pi}^{k+1} = 0$. By induction, we can prove that $D_\Omega \widetilde{\Pi}^{k+1} = 0$ and so, by lemma above, we conclude that $\operatorname{rk} \widetilde{\Pi} = 2k$. □

Now, we want to characterize a Poisson structure Π on the projective space $\mathbb{P}^n$ with homogeneous coordinates $(X_0, \ldots, X_n)$ by its restriction to the affine subset U_0 where $X_0 \neq 0$. Put

$$P_{ij} = \left\{ \frac{X_i}{X_0}, \frac{X_j}{X_0} \right\}$$

and write $\Pi = \sum_{i<j} \Pi_{ij} \frac{\partial}{\partial X_i} \wedge \frac{\partial}{\partial X_j}$. Since $\frac{\partial}{\partial X_0} = -\sum_i \frac{X_i}{X_0} \frac{\partial}{\partial X_i}$, then, by Leibniz rule, we have

$$P_{ij} = \frac{X_0}{X_0^3}\Pi_{ij} + \frac{X_j}{X_0^3}\Pi_{0i} - \frac{X_i}{X_0^3}\Pi_0 j.$$

Hence, we conclude that

(1) $X_0^3 P_{ij}$ is polynomial for every i, j;
(2) $X_0^2(X_i P_{jk} + X_j P_{ki} + X_k P_{ij})$ is polynomial for every i, j, k.

Considering P_{ij} as polynomial in the variables $x_i = \frac{X_i}{X_0}$, these conditions are equivalent to the following:

(1) $\deg P_{ij} \leq 3$;
(2) $\deg (x_i P_{jk} + x_j P_{ki} + x_k P_{ij}) \leq 3$.

If we write $P_{ij} = Q_{ij} + C_{ij}$ where $\deg Q_{ij} \leq 2$ and C_{ij} are homogeneous cubic polynomials, we have

$$x_i C_{jk} + x_j C_{ki} + x_k C_{ij} = 0$$

Note that from the formula of P_{ij} and the fact that Π_{ij} are homogeneous quadratic polynomials, we conclude that the hyperplane $x_0 = 0$ is invariant by the Poisson distribution defined by Π if and only if P_{ij} has zero cubic part. Thus, we can extend any (nonhomogeneous) quadratic Poisson structure in the affine space to the Poisson structure in the projective space of the same dimension such that the complementary hyperplane is invariant by the Poisson distribution.

Example 2.8. The diagonal quadratic Poisson structure in $\mathbb{C}^n$ defined in example 1.20 of chapter 1 extends to a Poisson structure in $\mathbb{P}^n$.

1.3. Examples of foliations in the projective spaces induced by Poisson structures. In this subsection, we give some examples of foliations in $\mathbb{P}^n$ induced by Poisson structures and some examples of foliations which are not induced by Poisson structures.

Example 2.9. There exists a natural bijection between Poisson structure in $\mathbb{P}^2$ and cubics in $\mathbb{P}^2$, because $\bigwedge^2 T\mathbb{P}^2 = \mathcal{O}_{\mathbb{P}^2}(3)$ and every bivector field in a surface is a Poisson bivector field.

The example above does not extend to $\mathbb{P}^{2n}$ for every $n \geq 2$. For example, if Π is a nondegenerate Poisson structure in $\mathbb{P}^4$, then we have $\Pi \wedge \Pi \in H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))$ and so the zero locus of $\Pi \wedge \Pi$ is a quintic. Is the converse true? More precisely, given a quintic Q in $\mathbb{P}^4$, does there exist a Poisson structure such that $\Pi \wedge \Pi$ vanishes along Q?

Theorem A says no. There exist quintics in $\mathbb{P}^4$ which are not the zero locus of $\Pi \wedge \Pi$ where Π is a Poisson structure: any normal crossing hypersurface of degree 5 different from five hyperplanes in general position cannot be the singular set of a nondegenerate Poisson structure in $\mathbb{P}^{2n}$.

Let $\mathcal{D}$ be a codimension q distribution in $\mathbb{P}^{2n+1}$ induced by the q-form $w \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n} \otimes \mathcal{L})$. We define the degree of $\mathcal{D}$, denote by deg $\mathcal{D}$, to be the degree of the zero locus of $i^*\omega$, where $i : \mathbb{P}^q \to \mathbb{P}^n$ is a linear embedding of a generic q-plane. Since $\Omega^q_{\mathbb{P}^q} = \mathcal{O}_{\mathbb{P}^q}(-q-1)$, it follows that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(\deg \mathcal{D} + q + 1)$. In particular, if $\mathcal{D}$ is a Poisson distribution, we have $\deg \mathcal{D} = n - q$. If $\mathcal{F}$ is a Poisson foliation, we conclude that $\deg \mathcal{F} \leq n - q$ and the equality holds if and only if the singular set of the Poisson distribution has codimension ≥ 2.

Definition 2.10. A holomorphic section $\omega \in H^0(X, \Omega^q_X \otimes \mathcal{L}^*)$ is integrable if and only if satisfies the Plucker equations:

$$i_v\omega \wedge \omega = 0, \text{ where } v \text{ is any } (q-1)\text{-derivation}$$

and the condition

$$i_v\omega \wedge \mathrm{d}\omega = 0, \text{ where } v \text{ is any } (q-1)\text{-derivation.}$$

The distribution $\mathcal{D}$ is involutive if and only if $\omega \in H^0(X, K^*_X \otimes \Omega^{n-2k}_X)$ is integrable.

A direct consequence of Koszul formula (see Theorem 1.24 of chapter 1) is the following:

Lemma 2.11. *There exists a natural bijection between nontrivial Poisson structures in $\mathbb{P}^3$ and the pair $(\mathcal{F}, D)$, where $\mathcal{F}$ is a codimension 1 foliation induced by an involutive distribution $\mathcal{D}$ with trivial canonical bundle and D is the effective divisor where the distribution $\mathcal{D}$ is singular.*

Proof. One side of the bijection is straitghtforward: for each Poisson structure Π, we asssociate the Poisson distribution $\mathcal{D}$. If $\operatorname{cod}\operatorname{Sing}\mathcal{D} = 1$, we associate the divisor $D = \operatorname{cod}\operatorname{Sing}\mathcal{D}$ and $\mathcal{F}$ is the Poisson foliation.

For each pair $(\mathcal{F}, D)$, we can associate $\omega \in H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(4))$ such that $\omega \wedge \mathrm{d}\omega = 0$ is a involutive distribution: if ω' is 1-form defining the foliation $\mathcal{F}$ and D is given by the homogeneous polynomial $f = 0$, then $\omega = f\omega'$. We see that D is the effective divisor where the distribution $\mathcal{D}$ induced by ω is singular. Since $\Omega^1_{\mathbb{P}^3}(4) = \Omega^1_{\mathbb{P}^3} \otimes K^*_{\mathbb{P}^3} = \bigwedge^2 T\mathbb{P}^3$, then, for each ω, we can associate to a bivector field Π. In local coordinates (x_1, x_2, x_3), we have $i_\Pi\Omega = \omega$, where $\Omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$. By the definition of curl operator, we have $i_{D_\Omega\Pi}\Omega = \mathrm{d}\omega$. Since $\omega \wedge \mathrm{d}\omega = 0$, we have $i_{D_\Omega\Pi}\Omega \wedge i_\Pi\Omega = 0$ and, so, $\Omega \wedge i_{(D_\Omega\Pi)\wedge\Pi}\Omega = 0$. So $(D_\Omega\Pi) \wedge \Pi = 0$ and, by Koszul formula (see Theorem 1.24 of chapter 1, we conclude that $[\Pi, \Pi] = 0$. □

Corollary 2.12. *Every foliation in $\mathbb{P}^3$ with degree ≤ 2 comes from a Poisson structure.*

The classification of foliation of degree ≤ 2 in $\mathbb{P}^3$ is well known (see [**7**] and [**17**]). In dimension 4, we have:

LEMMA 2.13. *Any integrable 2-form $\omega \in H^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4} \otimes \mathcal{O}_{\mathbb{P}^4}(5))$ comes from a Poisson structure.*

The proof is exactly the same of the last lemma, we just need to observe that, in $\mathbb{P}^4$, ω is integrable if and only if $\omega \wedge \omega = 0$ and $i_v\omega \wedge d\omega = 0$ for every v vector field in $\mathbb{P}^4$.

COROLLARY 2.14. *Every codimension 2 foliation in $\mathbb{P}^4$ with degree ≤ 2 comes from a Poisson structure.*

When the dimension of the variety gets bigger, it is harder to know if a foliation comes from a Poisson structure. We do not even know if any dimension 2 foliation with degree ≤ 2 comes from a Poisson structure.

Now, we want to determine a simple criteria if a codimension 1 foliation $\mathcal{F}$ in $\mathbb{P}^{2n+1}$ with trivial canonical sheaf is attached to a Poisson distribution.

Suppose that $\mathcal{F}$ admits an isolated singularity p. In local coordinates $(x_1, \ldots, x_{2n+1})$ of a neighborhood of p, $\mathcal{F}$ can be given by the 1-form $\omega = A_1 dx_1 + \ldots A_{2n+1} dx_{2n+1}$, where A_i are holomorphic function with $A_i(p) = 0$. We define the **Milnor number** of $\mathcal{F}$ in p by

$$n_p = \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{(A_1, \ldots, A_{2n+1})}.$$

It is a standard check that n_p independs of the choice of local coordinates and n_p is finite because p is an isolated singularity. We say that p is a **simple singularity** if $n_p = 1$.

We need the following interpretation of Corollary 7.7 of [**14**].

PROPOSITION 2.15. *Let Π be a Poisson structure in $\mathbb{P}^n$, $n \geq 3$, and suppose that the singular set of Poisson distribution has codimension ≥ 2. If the Poisson foliation admits a simple isolated singularity at $p \in X$, then $\Pi(p) = 0$, i.e., the rank of Π at p is 0.*

COROLLARY 2.16. *Let $\mathcal{F}$ be a codimension 1 foliation of degree $2n$ in $\mathbb{P}^{2n+1}$, $n \geq 2$ and suppose that $\mathcal{F}$ has a simple isolated singularity. Then $\mathcal{F}$ is not attached to any Poisson structure in $\mathbb{P}^{2n+1}$.*

PROOF. Suppose that $\mathcal{F}$ is attached to a Poisson structure Π. Since the degree of $\mathcal{F}$ is $2n$, the singular set of the Poisson distribution $\mathcal{D}$ has codimension ≥ 2. Let p be a simple isolated singularity of $\mathcal{F}$, then by proposition above, we have $\Pi(p) = 0$ and so in the local ring $\mathcal{O}_p$, we have $\Pi \in \mathfrak{m}_p \otimes (\bigwedge^{2n} T\mathbb{P}^{2n+1})_p$, where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_p$. We conclude that $\Pi^n \in \mathfrak{m}_p^n \otimes (\bigwedge^4 T\mathbb{P}^{2n+1})_p$. In a local system of coordinates $(x_1 \ldots, x_{2n+1})$ at a neighborhood of p, the Poisson foliation is given by $\omega = i_{\Pi^n}\Omega$, where $\Omega = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{2n+1}$. So, p is not a simple singularity of $\mathcal{F}$, a contradiction. □

When we study codimension 1 foliations in $\mathbb{P}^n$, $n \geq 3$, we find many works about irreducible components of the space of codimension 1 foliations in $\mathbb{P}^n$ (see [**5, 7, 8, 9**], for example). In these works, it appears naturally the so called logarithmic component. It is proved in [**5**] that if $\mathcal{F}$ is a degree d foliation in $\mathbb{P}^n$ and if H is a smooth normal crossing divisor of degree $d+2$ in $\mathbb{P}^n$ invariant by $\mathcal{F}$ (i.e., H is the Zariski closure of a finite number of leaves of $\mathcal{F}$), then H has at least two irreducible components and $\mathcal{F}$ is a logarithmic foliation (see Lemma 3.19 of chapter 3 for a partial proof). It is also proved (in the literature) that if $\mathcal{F}$ is a logarithmic foliation of degree d and H is a smooth normal crossing divisor invariant by $\mathcal{F}$ of degree $d+2$, then any small pertubation $\mathcal{F}_\epsilon$ admits a smooth normal crossing hypersurface H_ϵ of degree $d+2$ and each irreducible component of H and H_ϵ are linearly equivalent. For example, if H is composed by 2 quadrics and $d-2$ hyperplanes then H_ϵ is composed by 2 quadrics and $d-2$ hyperplanes. In particular, a generic element $\mathcal{F}$ of the space of logarithmic foliations of degree d has a smooth normal crossing divisor H of degree $d+2$ invariant by $\mathcal{F}$.

Putting everything together, a **logarithmic component** of the space of foliation of degree d is totally described by the smooth normal crossing divisor H of degree $d+2$. Usually, it is denoted the logarithmic component by *Log*, if H has at least 3 components, and *Rat*, if H has exactly two components. For example, a generic element $\mathcal{F} \in Log(2,1,1)$ is a foliation $\mathcal{F}$ which admits a smooth normal crossing divisor H of degree 4 composed by a quadric and 2 hyperplanes and $\mathcal{F}$ is a degree 2 foliation. A generic element $\mathcal{F} \in Rat(3,3)$ is a foliation $\mathcal{F}$ which admits a smooth normal crossing divisor of degree 6 composed by 2 cubics. In this case, $\mathcal{F}$ is a degree 4 foliation.

In [**9**], it is proved that if $\mathcal{F}$ is a generic logarithmic foliation of degree $\leq n$ in $\mathbb{P}^n$, then the singular set of $\mathcal{F}$ is composed by the intersection of 2 irreducible component of H and N isolated singularities counted with proper multiplicity. Moreover, $N = 0$ if and only if H is composed only by hyperplanes. So a direct consequence of Corollary 2.16 is

COROLLARY 2.17. *In $\mathbb{P}^5$, a generic element in any of the irreducible component $Rat(5,1)$, $Rat(4,2)$, $Rat(3,3)$, $Log(3,2,1)$ of the space of degree 4 foliation is not induced by a Poisson structure.*

EXAMPLE 2.18. If we consider the projection from the last coordinate $\phi : \mathbb{P}^{n+1} \to \mathbb{P}^n$ and if $\tilde{X} \in H^0(\mathbb{P}^n, T\mathbb{P}^n(1))$ is a quadratic global vector field in $\mathbb{P}^n$. Thinking of $\tilde{X}$ as a foliation of dimension 1, we may pull it back by ϕ, defining a foliation $\mathcal{F}$ of dimension 2 which is a Poisson structure. In the chart $x_0 = 1$, the vector field will be denoted by X and the Poisson structure will be $\Pi = \frac{\partial}{\partial x_{n+1}} \wedge X$. Note that $T\mathcal{F} = \mathcal{O}_{\mathbb{P}^{n+1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(1)$, i.e., the tangent sheaf of the Poisson distribution totally splits. This example will be studied in detail in the section 4 of chapter 4.

2. Basic concepts

2.1. Degeneracy locus. In this subsection, we start defining, perhaps, the most important subvariety of a Poisson variety, the so called degeneracy locus.

DEFINITION 2.19. Let (X, Π) be a Poisson variety. The $2k^{th}$ **degeneracy locus of** Π is the variety $D_{2k}(\Pi)$ where the morphism $\sigma^\sharp : \Omega^1_X \to TX$ has rank $\leq 2k$. If Π is a Poisson structure on X of rank $2k$, we call $D_{2k-2}(\Pi)$ the degeneracy locus. We will usually denote $D_{2k-2}(\Pi)$ by $\operatorname{Sing} \Pi$.

REMARK 2.20. Consider the ideal $\mathcal{I}$ defined by the image of the morphism

$$\Omega^{2k+2}_X \xrightarrow{\Pi^{k+1}} \mathcal{O}_X \ .$$

D_{2k} is the variety defined by $\sqrt{\mathcal{I}}$.

EXAMPLE 2.21. Consider the Poisson structure on $X = \mathbb{C}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, $n \geq 3$. And consider the Poisson structure

$$\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2} + x_3 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial y_3}.$$

Denoting $H_i = \{x_i = 0\}$, we see that it is a rank 6 Poisson structure with:

$$\begin{aligned} D_4(\Pi) &= H_1 + H_2 + H_3, \\ D_2(\Pi) &= H_1 \cap H_2 + H_2 \cap H_3 + H_3 \cap H_1, \\ D_0(\Pi) &= H_1 \cap H_2 \cap H_3. \end{aligned}$$

Note that the $D_4(\Pi)$ is invariant by Π and the Poisson distribution on $\{x_1 = 0\}$ has singular set of codimension 1 on $\{x_1 = 0\}$. Note that the Poisson distribution induced on $\{x_1 = 0\}$ is not a Poisson foliation.

In the next subsections, we will see some basic properties of the degeneracy locus. We will prove that $D_{2k}(\Pi)$ is a strong Poisson subvariety.

2.2. Poisson and strongly Poisson subvarieties. This section is devoted to give an important notion which is Poisson subvarieties.

DEFINITION 2.22. Let $(X, \{\cdot\,,\cdot\}_X)$ and $(Y, \{\cdot\,,\cdot\}_Y)$ be Poisson varieties. A morphism $\phi : Y \to X$ is a **Poisson morphism** if it preserves the Poisson brackets, i.e., the pull-back morphism $\phi^* : \mathcal{O}_X \to \phi_*\mathcal{O}_Y$ satisfies

$$\phi^*(\{f, g\}_X) = \{\phi^* f, \phi^* g\}_Y.$$

We say that $(Y, \{\cdot\,,\cdot\}_Y) \subseteq (X, \{\cdot\,,\cdot\}_X)$ is a **Poisson subvariety** if the inclusion $i : Y \to X$ is a Poisson morphism.

The following algebraic characterization of Poisson subvariety is due to Polishchuk (see [**20**]).

PROPOSITION 2.23. *Let (X, Π) a Poisson variety, $Y \subseteq X$ be a subvariety with ideal sheaf $\mathcal{I}_Y$. Then the following are equivalent*

(1) *Y admits the structure of a Poisson subvariety;*
(2) *$\mathcal{I}_Y$ is a sheaf of Poisson ideals, i.e., $\{\mathcal{I}_Y, \mathcal{O}_X\} \subseteq \mathcal{I}_Y$;*
(3) *$Z(\mathcal{I}_Y) \subseteq \mathcal{I}_Y$ for all Z Hamiltonian vector field.*

PROOF. (1) $\Leftrightarrow$ (2). Consider the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_{X_{|Y}} \to \mathcal{O}_Y \to 0.$$

We see that the bracket $\{\cdot\,,\cdot\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ defines naturally a bracket $\{\cdot\,,\cdot\} : \mathcal{O}_X \times \mathcal{O}_Y \to \mathcal{O}_Y$ if and only if $\{\mathcal{O}_X, \mathcal{I}_Y\} \subseteq \mathcal{I}_Y$.

(2) $\Leftrightarrow$ (3). Note that if Z is a Hamiltonian vector field, there exists a holomorphic function g such that $Z = \{\cdot\,, g\}$. The equivalence between (2) and (3) is immediate. □

In particular, we have:

COROLLARY 2.24. *If $Y \subseteq X$ is a Poisson subvariety, then every irreducible component of Y is a Poisson subvariety.*

We say that $\mathcal{I} \subseteq \mathcal{O}_X$ is a Poisson ideal if condition (2) of the item above is satisfied, i.e., $\{\mathcal{I}, \mathcal{O}_X\} \subseteq \mathcal{I}$. We have the following lemma:

LEMMA 2.25. *The radical of a Poisson ideal is a Poisson ideal.*

EXAMPLE 2.26. If $H = H_1 + \ldots + H_k$ is a smooth normal crossing hypersurface in $\mathbb{P}^4$, with degree 5 and $k \geq 3$, then there exists a rank 2 Poisson structure Π in $\mathbb{P}^4$ such that H is a Poisson subvariety. This happens because $\Omega^2_{\mathbb{P}^4}(\log H)$ has global sections and Deligne theorem says that every global logarithmic 2-form is closed and we can apply Lemma 2.13. In particular, the hypothesis of nondegeneracy of Theorem A is necessary.

DEFINITION 2.27. A vector field Z in X is **Poisson** with respect to Π if $\mathcal{L}_Z\Pi = 0$. We denote by $Pois(\Pi)$ the $\mathbb{C}$-sheaf consisting of Poisson vector fields.

LEMMA 2.28. *If* Π *is a Poisson structure then*

$$Ham(\Pi) \subseteq Pois(\Pi)$$

PROOF. Given $x \in X$, if $Z = \{\cdot\,, h\}$, $h \in \mathcal{O}_{X,x}$, then, by the identity

$$\mathcal{L}_Z(\Pi(\alpha)) = (\mathcal{L}_Z(\Pi))\alpha + \Pi(\mathcal{L}_Z\alpha),\ \alpha \in (\Omega^2_X)_x,$$

we have

$$\begin{aligned}(\mathcal{L}_Z\Pi)(f,g) &= \mathcal{L}_Z(\Pi(f,g)) - \Pi(\mathcal{L}_Z(\mathrm{d}f \wedge \mathrm{d}g)) \\ &= \mathcal{L}_Z\{f,g\} - \Pi(\mathrm{d}i_Z(\mathrm{d}f \wedge \mathrm{d}g)) \\ &= \{\{f,g\},h\} - \Pi(\mathrm{d}(Z(f)\mathrm{d}g)) + \Pi(\mathrm{d}(Z(g))\mathrm{d}f) \\ &= \{\{f,g\},h\} - \Pi(\{f,h\},g) + \Pi(\{g,h\},f) \\ &= \{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0\end{aligned}$$

for every $f, g \in \mathcal{O}_{X,x}$. □

Note that, with the same proof, we have:

LEMMA 2.29. *A vector field* Z *is Poisson if and only if the following identity holds:*

$$Z\{f,g\} = \{Z(f),g\} + \{f,Z(g)\},\ \ \forall f,g \in \mathcal{O}_{X,x}.$$

COROLLARY 2.30. *If* $X_f = \{\cdot\,, f\}$ *is a Hamiltonian vector field and* Z *is a Poisson vector field, then* $\mathcal{L}_Z X_f = X_{Z(f)} = \{\cdot\,, Z(f)\}$.

PROOF. We have

$$\begin{aligned}(\mathcal{L}_Z X_f)(g) &= \mathcal{L}_Z(X_f(g)) - X_f(\mathcal{L}_Z g) \\ &= \mathcal{L}_Z\{g,f\} - \{Z(g),f\} \\ &= \{g, Z(f)\} \text{ by the lemma above.}\end{aligned}$$

□

The corollary above says that the Poisson vector is not only an infinitesimal symmetry of the Poisson distribution, but it is also an infinitesimal symmetry of $Ham(\Pi)$.

We also observe that the inclusion $Ham(\Pi) \subseteq Pois(\Pi)$ is strict in general.

EXAMPLE 2.31. If $\Pi = 0$, then $Ham(\Pi) = 0$ and every vector field Z is a Poisson vector field.

We have the following definition due to Gualtieri and Pym [**14**].

DEFINITION 2.32. Let (X, Π) be a Poisson variety. A closed subvariety Y of X is a **strong Poisson subvariety** if its ideal sheaf $\mathcal{I}_Y$ is preserved by all (germs of) Poisson vector fields, i.e., if

$$Z(\mathcal{I}_Y) \subseteq \mathcal{I}_Y$$

for all Z Poisson vector field.

LEMMA 2.33. *Let (X, Π) be a Poisson variety. If Y_1 and Y_2 are strong Poisson subvarieties, then so are $Y_1 \cap Y_2$ and $Y_1 \cup Y_2$.*

PROOF. Simply notice that if Z is a vector field preserving the ideals $\mathcal{I}_{Y_1}$ and $\mathcal{I}_{Y_2}$ defining Y_1 and Y_2, it also preserves $\mathcal{I}_{Y_1} + \mathcal{I}_{Y_2}$ and $\mathcal{I}_{Y_1 \cap Y_2}$. Apply this observation to Poisson vector fields. □

A Poisson ideal $I \subseteq \mathcal{O}_X$ is said to be a strong Poisson ideal if $Z(\mathcal{I}) \subseteq \mathcal{I}$ for all Z Poisson vector field. We have the following simple lemma:

LEMMA 2.34. *The radical of a strong Poisson ideal is a strong Poisson ideal.*

PROPOSITION 2.35. *Let (X, Π) be a Poisson variety. Then for every k, with $0 \leq 2k \leq \dim X$, the degeneracy loci $D_{2k}(\Pi)$ is a strong Poisson subvariety of X.*

PROOF. Let $\mathcal{I}$ be the ideal sheaf of $D_{2k}(\Pi)$. We have the exact sequence

$$\Omega_X^{2k+2} \xrightarrow{\Pi^{k+1}} \mathcal{I} \to 0$$

Let $\alpha \in \Omega^{2k+2}$ and $Z \in Pois(\Pi)$, we have to prove that $Z(\Pi^{k+1}\alpha) \in \mathcal{I}$, i.e., $\mathcal{L}_Z(\Pi^{k+1}\alpha) \in \mathcal{I}$. We compute

$$\begin{aligned} \mathcal{L}_Z(\Pi^{k+1}\alpha) &= (\mathcal{L}_Z\Pi^{k+1})(\alpha) + \Pi^{k+1}(\mathcal{L}_Z\alpha) \\ &= \Pi^{k+1}(\mathcal{L}_Z\alpha) \end{aligned}$$

since $\mathcal{L}_Z\Pi = 0$. To finish the proof of the proposition, we just need to apply Lemma 2.34. □

The importance of the strong Poisson subvarieties is that they behave very well with respect to Poisson modules, which we will now define.

2.3. Poisson modules. Poisson modules naturally appear when we study the geometry of Poisson varieties. We start defining what a Poisson connection is.

DEFINITION 2.36. Let (X, Π) be a Poisson variety and let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules. A **Poisson connection** on $\mathcal{E}$ is a $\mathbb{C}$-linear morphism of sheaves $\nabla : \mathcal{E} \to TX \otimes \mathcal{E}$ satisfying the Leibniz rule

$$\nabla(fs) = \Pi^\sharp(\mathrm{d}f) \otimes s + f\nabla s$$

for all $f \in \mathcal{O}_{X,x}$, $s \in \mathcal{E}_x$ and $x \in X$.

Recall that an usual connection on a sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules is given by a $\mathbb{C}$-linear morphism $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ satisfying the Leibniz rule $\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$. In a Poisson connection the roles of TX and Ω^1_X are interchanged.

REMARK 2.37. If Π is a nonzero Poisson structure on X and let $(\mathcal{E}, \nabla)$ be a Poisson connection, then ∇ is nonzero. To see this, suppose that $\nabla = 0$ and let $f \in \mathcal{O}_{X,x}$ be a function. If s is any local section of $\mathcal{E}$, then

$$0 = \nabla(fs) = \Pi^\sharp(df) \otimes s + f\nabla s = \Pi^\sharp(df) \otimes s,$$

this proves that $\Pi = 0$. A contradiction.

Using a Poisson connection, we may differentiate a section of $\mathcal{E}$ along a 1-form: if $\alpha \in \Omega^1_X$, we set

$$\nabla_\alpha s = \nabla s(\alpha).$$

The Poisson connection ∇ is **flat** if

$$\nabla_{\mathrm{d}\{f,g\}} s = (\nabla_{\mathrm{d}f}\nabla_{\mathrm{d}g} - \nabla_{\mathrm{d}g}\nabla_{\mathrm{d}f})s$$

DEFINITION 2.38. A **Poisson module** $(\mathcal{E}, \nabla)$ is a sheaf of $\mathcal{O}_X$-modules equipped with a flat Poisson connection.

REMARK 2.39. If $\mathcal{L}$ is an invertible sheaf (i.e., a line bundle) equipped with a Poisson connection ∇, and for $x \in X$, if $s \in \mathcal{L}_x$, we obtain a unique vector field $v_s \in (T_X)_x$ (depending on s) such that

$$\nabla s = v_s \otimes s.$$

and ∇ is flat if and only if v_s is a Poisson vector field for every $s \in \mathcal{L}_x$ and for every $x \in X$. This is a direct consequence of Lemma 2.29.

We are ready to prove the following lemma (see [**14**]):

LEMMA 2.40. *Let (X, Π) be a Poisson variety and let $(\mathcal{L}, \bigtriangledown)$ be an invertible Poisson module. Then if $Y \subset X$ is any strong Poisson subvariety of X, the restriction $(\mathcal{L}_{|Y}, \bigtriangledown_{|Y})$ is a Poisson module on Y with respect to the induced Poisson structure, i.e., the map*

$$\bigtriangledown_{|Y} : \mathcal{L}_{|Y} \to \mathcal{L}_{|Y} \otimes T_{X_{|Y}}$$

lifts to

$$\bigtriangledown_{|Y} : \mathcal{L}_{|Y} \to \mathcal{L}_{|Y} \otimes TY$$

PROOF. Consider the exact sequence,

$$0 \to \mathcal{L}_{|Y} \otimes TY \to \mathcal{L}_{|Y} \otimes T_{X_{|Y}} \to (I_Y/I_Y^2)^* \otimes \mathcal{L}_{|Y}$$

Choosing a local trivialization s of $\mathcal{L}$, we have

$$\bigtriangledown s = v_s \otimes s$$

where v_s is a Poisson vector field. Since Y is a strong Poisson variety, we have $v_s(I_Y) \subset I_Y$, hence the image of $v_s \otimes s$ in $(I_Y/I_Y^2)^* \otimes \mathcal{L}_{|Y}$ is zero. □

Let $(\mathcal{L}, \bigtriangledown)$ be a Poisson module on X and let $E = E(\mathcal{L}^*)$ be the total space of $\mathcal{L}^*$. We have $\mathcal{O}_E = Sym(\mathcal{L}) = \bigoplus_{k=0}^{\infty} \mathcal{L}^k$. In [**20**], we can find:

PROPOSITION 2.41. *A flat Poisson connection $\bigtriangledown : \mathcal{L} \to TX \otimes \mathcal{L}$ induces a homogeneous bracket $\{\cdot, \cdot\}_{\bigtriangledown}$ on $Sym^k(\mathcal{L})$ by the following formula:*

$$\begin{aligned} \{f, g\}_{\bigtriangledown} &= \{f, g\}, \quad \forall f, g \text{ local sections of } \mathcal{O}_X \\ \{f, s\}_{\bigtriangledown} &= -\bigtriangledown_{\mathrm{d}f} s \\ \{fs^n, gs^m\}_{\bigtriangledown} &= (\{f, g\}s + mg\{f, s\}_{\bigtriangledown} - nf\{g, s\}_{\bigtriangledown})s^{n+m-1} \end{aligned}$$

In particular, the projection $E \to X$ is a Poisson morphism.

PROOF. The proof of Jacobi identity of $\{\cdot, \cdot\}_{\bigtriangledown}$ is just a tedious calculations based on Lemma 2.29. □

If $(\mathcal{E}, \bigtriangledown)$ is a Poisson module, then $\mathcal{E}^*$ inherits a natural Poisson connection $\bigtriangledown'$ by the formula:

$$\begin{aligned} \bigtriangledown'_{\mathrm{d}f}(\phi) : \mathcal{E} &\to \mathcal{O}_X \\ s &\mapsto \phi(\bigtriangledown_{\mathrm{d}f} s), \text{ where } \phi \text{ is a local section of } \mathcal{E}^*. \end{aligned}$$

We see that $\bigtriangledown'$ is a flat Poisson connection.

2.4. Modular connection. Whenever X is a smooth and irreducible Poisson variety, there exists a natural Poisson module structure on the canonical sheaf $\omega_X = \Omega^n_X$. For $\alpha \in \Omega^1_X$ and $\mu \in \omega_X$, the connection is defined by the formula

$$\nabla^{\mathrm{mod}}_{\mathrm{d}f}\mu = -\mathcal{L}_{\sigma^\sharp(\mathrm{d}f)}\mu.$$

This connection is variously referred as the modular representation, the Evens-Lu-Weinstein module or the canonical module.

LEMMA 2.42. *The modular connection ∇^{mod} is flat, i.e., $(\omega_X, \nabla^{\mathrm{mod}})$ is a Poisson module.*

PROOF. It is a simple computation

$$\begin{aligned}\nabla^{\mathrm{mod}}_{\mathrm{d}f}\nabla^{\mathrm{mod}}_{\mathrm{d}g}\mu &= \mathcal{L}_{\sigma^\sharp(\mathrm{d}f)}\mathcal{L}_{\sigma^\sharp(\mathrm{d}g)}\mu \\ &= \mathcal{L}_{\sigma^\sharp(\mathrm{d}f)}(\mathrm{d}(i_{\sigma^\sharp(\mathrm{d}g)}\mu)) \\ &= \mathrm{d}(i_{\sigma^\sharp(\mathrm{d}g)}\mathcal{L}_{\sigma^\sharp(\mathrm{d}f)}\mu + \mathrm{d}i_{[\sigma^\sharp(\mathrm{d}f),\sigma^\sharp(\mathrm{d}g)]}\mu \\ &= \mathcal{L}_{\sigma^\sharp(\mathrm{d}g)}\mathcal{L}_{\sigma^\sharp(\mathrm{d}f)}\mu - \mathrm{d}i_{\sigma^\sharp\{f,g\}}\mu \\ &= \nabla^{\mathrm{mod}}_{\mathrm{d}g}\nabla^{\mathrm{mod}}_{\mathrm{d}f}\mu + \nabla^{\mathrm{mod}}_{\mathrm{d}\{f,g\}}\mu\end{aligned}$$

where f, g are local sections of $\mathcal{O}_X$. □

2.5. Polishchuk connection. We will see that a Poisson divisor defines canonically a Poisson connection. This connection was defined by Polishchuk (see [**20**]).

DEFINITION 2.43. A divisor D in X is called Poisson divisor if each of the irreducible component is a Poisson subvariety. We denote the set of Poisson divisors by $PDiv$.

By Proposition 2.23, if $f = \{f_i\}$ is the section on $\mathcal{O}_X(D)$ defining the divisor D, then $\{\mathcal{O}_X, f_i\} \subset f_i\mathcal{O}_X$. In particular, we have the following vector field

$$\begin{aligned}X_{\log f_i} : \mathcal{O}_X \to\ & \mathcal{O}_X \\ g \mapsto\ & \frac{\{g, f_i\}}{f_i}.\end{aligned}$$

If D is a irreducible hypersurface, the vector field $X_{\log f_i}$ is a Hamiltonian vector field of Π away from D, but $X_{\log f_i}$ restricted to D is not necessarily tangent to the Poisson distribution. For example, if we consider the Poisson vector field in $\mathbb{C}^4$ defined by $\Pi = x_1\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}$. We see that $x_1 = 0$ is a strong Poisson subvariety and $X_{\log x_1} = \frac{\partial}{\partial x_2}$, but the Poisson structure induced in $x_1 = 0$ is given by $\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}$.

The proof we present of the proposition below can be found in [**20**].

PROPOSITION 2.44. *The group PDiv is isomorphic to the group of isomorphism class of triples* $(\mathcal{L}, \nabla, s)$*, where* $\mathcal{L}$ *is a line bundle,* ∇ *is a flat Poisson connection on* $\mathcal{L}$ *and* $s \neq 0$ *is a rational section of* $\mathcal{L}$ *which is horizontal with respect to* ∇*, i.e.,* $\nabla s = 0$*.*

PROOF. Let D be a Poisson divisor, $\mathcal{L} = \mathcal{O}_X(D)$ and consider a Čech representative for D, i.e., the collection of functions $f_i \in \mathcal{M}(U_i) \otimes \mathcal{L}(U_i)$ for some open covering U_i, such that $g_{ij} = \frac{f_i}{f_j} \in \mathcal{O}_X^*(U_i \cap U_j)$. Moreover, $D_{|U_i} = \{f_i = 0\}$. The corresponding line bundle is trivialized over U_i, that is, for each i there exists a nowhere vanishing section $s_i \in L(U_i)$ such that $s_i = g_{ij} s_j$ over the intersection. Now define the connection on $\mathcal{L} = \mathcal{O}_X(D)$ by the formula $\nabla(s_i) = -X_{\log f_i} \otimes s_i$ and extends via Leibniz rule. Then the formula $s_{|U_i} = \frac{s_i}{f_i}$ gives a well-defined rational horizontal section of $\mathcal{L}$.

Reciprocally, suppose we have a flat Poisson connection ∇ on $\mathcal{L}$ and a horizontal rational section s. Then trivializing $\mathcal{L}$ over an open covering as above, we can write $s = \frac{s_i}{f_i}$ for some rational functions f_i. Now, the condition $\nabla s = 0$ implies the equality $\nabla s_i = -X_{\log f_i} \otimes s_i$. It follows that the divisor defined by $\{f_i = 0\}$ is a Poisson divisor. □

Let X be a smooth variety of dimension $2n$ and let Π be a nondegenerate Poisson structure on X. Then $\Pi^n \in H^0(X, \bigwedge^{2n} TX) = H^0(X, \omega_X^*)$ is a nonzero section of ω_X^*. The zero locus of Π^n is a Poisson divisor and we have the Polishchuk connection and the modular connection on ω_X^*. We have the following proposition due to Polishchuk.

PROPOSITION 2.45. *The divisor of degeneration of a nondegenerate Poisson structure is a Poisson divisor. The connection on* ω_X *is the modular connection.*

PROOF. Denote by ∇^{mod} the modular connection ω_X. Let α be a rational n-form dual to Π^n, i.e, $\Pi^n(\alpha) = 1$. Clearly, α defines the divisor $-D_{2n-2}(\Pi)$. We just need to check that $\nabla^{\text{mod}}(\alpha) = 0$. Recall that $\mathcal{L}_{\Pi^\sharp(\mathrm{d}f)}\Pi^n = 0$, then, for any f local section of $\mathcal{O}_X$, we have

$$\begin{aligned}
\Pi^n(\nabla^{\text{mod}}_{\mathrm{d}f}\alpha) &= -\Pi^n(\mathcal{L}_{\Pi^\sharp(\mathrm{d}f)}\alpha) \\
&= -\mathcal{L}_{\Pi^\sharp(\mathrm{d}f)}(\Pi^n(\alpha)) + (\mathcal{L}_{\Pi^\sharp(\mathrm{d}f)}\Pi^n)\alpha \\
&= -\mathcal{L}_{\Pi^\sharp(\mathrm{d}f)}(1) = 0.
\end{aligned}$$

Since $\Pi^n(\nabla^{\text{mod}}_{\mathrm{d}f}\alpha) = 0$, we conclude that $\nabla^{\text{mod}}_{\mathrm{d}f}\alpha = 0$. This proves the proposition. □

CHAPTER 3

Nondegenerate Poisson structures on Fano varieties

1. Introduction

This is the most important chapter of this thesis, which is devoted to prove Theorem A. We restate it below for the convenience of the reader.

THEOREM A. Let X be an even dimensional, dim $X \geq 4$, Fano variety with cyclic Picard group. Suppose that Π is a nondegenerate Poisson structure on X such that the singular locus of Π is reduced smooth normal crossing. Then X is the projective space $\mathbb{P}^{2n}$ and the singular locus is $2n+1$ hyperplanes in general position.

The idea is to prove the Theorem in dimension 4 and observe that, in the case of dimension ≥ 4, we can choose two hypersurfaces in the singular locus such that the induced Poisson structure is nondegenerate. We prove that the intersection of these hypersurfaces is a Fano variety with cyclic Picard group and the singular locus of the nondegenerate Poisson structure induced in the intersection is smooth normal crossing. An induction argument on the dimension of the Fano variety, together with Kobayashi-Ochiai theorem (stated below), proves the Theorem A.

THEOREM 3.1. *Suppose that X is a Fano variety with $Pic\, X = \mathbb{Z}$ and* $\dim X = n$*. Let $i(X) = \deg K_X^*$ be the so called index of X. Then we have $i(X) \leq n+1$. Moreover, $i(X) = n+1$ if and only if $X = \mathbb{P}^n$ and $i(X) = n$ if and only if X is a hyperquadric on $\mathbb{P}^{n+1}$.*

In order to prove the Theorem in dimension 4, we use again Kobayashi-Ochiai theorem. The proof of our Theorem will be excluding, case by case, the index $1, 2, 3$ and 4. Excluding the cases $i(X) = 1$ and $i(X) = 2$ are quite simple. The complicated ones are the cases $i(X) = 3$ and $i(X) = 4$.

In order to exclude the case $i(X) = 3$, we study the Poisson foliation induced in H, where H is an irreducible component of the singular locus of the Poisson structure. We prove that there exist exactly 3 irreducible components and, under our assumptions, the Poisson foliation in H is logarithmic and has trivial canonical sheaf. However, Polishchuk connection induces a global vector field in H tangent to the Poisson foliation and the classification obtained by Loray, Pereira and Touzet in [**17**] says that logarithmic foliations with trivial canonical sheaf in Fano threefold of index 2 does not have any global vector field tangent to it. This contradiction excludes the case $i(X) = 3$.

To exclude the case $i(X) = 4$, we make use the fact that X is a hyperquadric in $\mathbb{P}^5$ and compute all the Chern classes. Since, under our assumptions, the Poisson foliation $\mathcal{F}$ is logarithmic and we prove that $\mathcal{F}$ has an isolated singularity. Using the classification of Lie algebra in dimension 4, this isolated singularity is always in 3 components of the degeneracy loci of the Poisson structure. Using the geometry of a logarithmic foliation and the fact that the singular loci of a nondegenerate Poisson structure in the hyperquadric have, at most, 4 components, we can get a contradiction. We conclude that the variety is $\mathbb{P}^4$.

The argument used to exclude the index 4 can be used, without any extra effort, to prove that the singular loci of the Poisson structure in $\mathbb{P}^4$ is composed by 5 hyperplanes in general position. An inductive argument proves that, under the assumptions of Theorem A, the singular loci of the Poisson structure in $\mathbb{P}^{2n}$ are $2n + 1$ hyperplanes in general position.

2. Fano varieties with cyclic Picard group

In this section, we gather some important results about nondegenerate Poisson structures in Fano manifolds. We recall some definitions.

A Poisson structure in Π in X is said to be regular Poisson structure if the rank of Π never drops, i.e., the Poisson distribution is a regular Poisson foliation. If $\mathcal{F}$ is a k-dimensional foliation in X, we define the canonical sheaf, $K_{\mathcal{F}} := (\bigwedge^k T^*\mathcal{F})^{**}$. The canonical bundle is a line bundle (see [**3**] Proposition 1.33). If the singular set of a Poisson distribution has codimension ≥ 2, we have that $K_{\mathcal{F}}$ is trivial by adjunction formula $K_X = K_{\mathcal{F}} \otimes (\det N^*\mathcal{F})^{**}$ and the fact that $(\det N^*\mathcal{F})^{**} = K_X$.

We say that Π is a nondegenerate (or generically symplectic) Poisson structure in X with $\dim X = 2n$ and $n \geq 2$ if $\Pi^n \neq 0$. The singular locus of Π, denoted by $\operatorname{Sing}\Pi$ is, by definition, the set of points $x \in X$ where $\Pi^n(x) = 0$.

A variety X is said to be a Fano variety if K_X is anti-ample. We are interested when X is a Fano variety with $Pic\, X = \mathbb{Z}$ and Π is a nondegenerate Poisson structure in X.

We start proving some general facts about foliations in Fano varieties with cyclic Picard group.

Definition 3.2. Let $\mathcal{L} \in Pic\, X$ be a line bundle on X. If $Pic\, X = \mathbb{Z}$, the integer number associated to $\mathcal{L}$ is called the degree of $\mathcal{L}$, denoted by deg $\mathcal{L}$.

Lemma 3.3. *There is no regular codimension* 1 *foliation* $\mathcal{F}$ *with trivial canonical bundle in variety* X *with* $Pic\, X = \mathbb{Z}$.

Proof. Suppose we have a regular foliation $\mathcal{F}$ in X. By Baum-Bott formula, we have that $c_1(N\mathcal{F})^n = 0$. Since $Pic\, X = \mathbb{Z}$, we conclude that $c_1(N\mathcal{F}) = 0$ and so $N\mathcal{F} = \mathcal{O}_X$. A regular foliation $\mathcal{F}$ can be given by the exact sequence

$$0 \to N^*\mathcal{F} \to \Omega^1_X \to T^*\mathcal{F} \to 0,$$

in particular, $H^1(X, \mathcal{O}_X) = H^0(X, \Omega^1_X) \neq 0$. Since $\frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})} \neq 0$, then, by the exponencial exact sequence

$$0 \to \frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})} \to Pic\, X \to H^2(X, \mathbb{Z}),$$

we have that $Pic\, X$ is not discrete. A contradiction. □

Proposition 3.4. *Let* $\mathcal{F}$ *be a codimension* 1 *foliation on the Fano variety* X *with* $Pic\, X = \mathbb{Z}$. *Then* $\deg (N_{\mathcal{F}})^{**} > 0$.

Proof. Suppose that deg $N^*_{\mathcal{F}} \geq 0$. The case deg $N^*_{\mathcal{F}} = 0$ was done in the last lemma. If deg $N^*_{\mathcal{F}} > 0$, then $N^*_{\mathcal{F}}$ is ample, so $(N^*_{\mathcal{F}})^m$ is very ample and, in particular, $h^0(X, (N^*_{\mathcal{F}})^m) > 0$. An $\mathcal{O}_X$-linear injection $N^*_{\mathcal{F}} \to \Omega^1_X$ defines, naturally, $0 \to (N^*_{\mathcal{F}})^m \to (\Omega^1_X)^{\otimes m}$. In particular, $h^0(X, (\Omega^1_X)^{\otimes m}) > 0$. Since X is Fano, according to [**16**], we have $h^0(X, (\Omega^1_X)^{\otimes m}) = 0$, a contradiction. This proves the proposition. □

About varieties with cyclic Picard group, recall the so called Lefschetz-Grothendieck (see [**13**]).

Theorem 3.5. *Let* X *be a nonsingular projective variety over* $\mathbb{C}$ *with* dim $X \geq 4$ *and let* $\mathcal{L}$ *be an ample line bundle on* X. *Let* $s \in H^0(X, \mathcal{L})$ *be a global section of* $\mathcal{L}$ *and let* $H = Z(s)$ *be the zero locus of* s. *If* H *is smooth, then the natural restriction map* $Pic\, X \to Pic\, H$ *is an isomorphism.*

In particular, we have:

Corollary 3.6. *Let* X *be a a smooth variety with* $Pic\, X = \mathbb{Z}$ *and* dim $X \geq 4$. *If* H *is a smooth hypersurface of* X, *then* $Pic\, H = \mathbb{Z}$.

Lemma 3.7. *Suppose* (X, Π) *is a generically symplectic Poisson Fano manifold with* $Pic\, X = \mathbb{Z}$, dim $X \geq 4$, *and suppose that the degeneration locus* $\mathrm{Sing}\, \Pi = H_1 \cup \ldots \cup H_k$ *is smooth normal crossing hypersurface, then* H_i *is Fano for every* i.

PROOF. Let H_1 be a smooth component of $\operatorname{Sing}\Pi$. By Lefschetz-Grothendieck's theorem, we have $Pic\, H_1 = \mathbb{Z}$. By Polishchuk's theorem, $\Pi_1 = \Pi_{|H_1}$ defines a codimension one foliation on H_1 and, so, we have the inclusion $\operatorname{Sing}\Pi_1 \subseteq (H_2 \cup \ldots \cup H_k) \cap H_1$.

By the exact sequence

$$0 \to N^*_{H_1} \to \Omega^1_{X_{|H_1}} \to \Omega^1_{H_1} \to 0;$$

the adjunction formula

$$N^*_{H_1} = \mathcal{O}_{H_1}(-H_1)$$

and the fact that

$$K_X = \mathcal{O}_X(-H_1 - \ldots - H_k),$$

we conclude

$$K_{H_1} = -\mathcal{O}_{H_1}(H_2 + \ldots + H_k)$$

To prove that H_1 is Fano, we just need to prove that $H_i \cap H_1$ is nonempty for some $i \neq 1$. But this is a direct consequence from the fact that $Pic\, X = \mathbb{Z}H$, where H is an ample divisor in X and, so $H^n \neq 0$. □

Polishchuk proved the following (see [**20**]):

THEOREM 3.8. *Suppose that the degeneration locus* $\operatorname{Sing}\Pi$ *of a non-degenerate Poisson structure* $\Pi \in H^0(X, \mathfrak{X}^2(X))$ *over a even dimensional projective variety* X *is smooth normal crossing. Let* $H^{(k)}$ *be the set consisting of points of* X *such that exactly* k *irreducible components of* $\operatorname{Sing}\Pi$ *meet. Then*

(1) *The induced Poisson structure on each connected component of* $H^{(k)}$ *is regular;*
(2) $2n - 2k \leq \operatorname{rk}\Pi_{|H^{(k)}} \leq 2n - k$, *where* $dim\, X = 2n$.

With these results in mind, we can prove the following proposition:

PROPOSITION 3.9. *Let* X *be a Fano* $2n$*-fold,* $n \geq 2$ *and cyclic Picard group. Let* $\Pi \in H^0(X, \mathfrak{X}^2_X)$ *be a nondegenerate Poisson structure with* $\operatorname{Sing}\Pi = H_1 \cup \ldots \cup H_k$ *smooth normal crossing. Let* $\Pi_1 = \Pi_{|H_1}$ *be the induced Poisson structure on* H_1*, then*

(1) $\operatorname{Sing}\Pi_1 \subsetneqq (H_2 \cup \ldots \cup H_k) \cap H_1$
(2) $\operatorname{Sing}\Pi_1$ *is non-empty.*

PROOF. We know that $\operatorname{Sing}\Pi_1 \subseteq (H_2 \cup \ldots \cup H_k) \cap H_1$ by Polishchuk's theorem 3.8. We have to prove that the inclusion is strict. Suppose that $\operatorname{Sing}\Pi_1 = (H_2 \cup \ldots \cup H_k)_{|H_1}$ and so $K^*_{H_1} = \mathcal{O}_{H_1}(\operatorname{Sing}\Pi_1)$, the Poisson distribution $\mathcal{D}_{\Pi_1}$ defines the injection

$$0 \to K_{H_1} \to \Omega^1_{H_1}.$$

So, the conormal bundle of the Poisson foliation $\mathcal{F}$ induced by Π_1 is given by

$$N^*\mathcal{F} = K_{H_1} \otimes K^*_{H_1} = \mathcal{O}_X.$$

This contradicts Proposition 3.4.

About item (2), as for $\operatorname{Sing}\Pi_1 = \emptyset$, Π_1 defines a regular codimension 1 foliation $\mathcal{F}$ on H_1 with $K_{\mathcal{F}}$ trivial. By Lemma 3.3, we get a contradiction. □

Corollary 3.10. *Under the assumptions of Proposition 3.9, if we have* $\operatorname{Sing}\Pi = H_1 \cup \ldots \cup H_k$*, then* $k \geq 3$*.*

Proof. If $k = 1$, then Π_1 is a regular Poisson structure by Polishchuk's theorem 3.8. So $\operatorname{Sing}\Pi_1 = \emptyset$. Contradicting Proposition 3.9 item (2). If $k = 2$, then Π_1 must vanish on $H_2 \cap H_1$. But this contradicts Proposition 3.9 item (1). □

We need the following lemma which is proved in section 4 (see Lemma 3.19).

Lemma 3.11. *Let $\mathcal{F}$ be a codimension 1 foliation in the projective variety X with $Pic\, X = \mathbb{Z}$. If H is a $\mathcal{F}$-invariant smooth normal crossing hypersurface, such that* $\deg \mathcal{O}_X(H) = \deg (N\mathcal{F})^{**}$*, then $\mathcal{F}$ is logarithmic and H can not be a smooth hypersurface.*

Corollary 3.12. *Under the assumptions of Proposition 3.9. If we write* $\operatorname{Sing}\Pi = H_1 \cup \ldots \cup H_k$*, then the foliation $\mathcal{F}_1$ on H_1 induced by the Poisson structure is logarithmic. In particular, if $k = 3$, then the singular set of a Poisson distribution has codimension ≥ 2.*

Proof. Let D be the codimension one set of $\operatorname{Sing}\Pi_1$. Suppose that $D = (H_2 \cup \ldots \cup H_m) \cap H_1$, with $m < k$. We have

$$N^*\mathcal{F}_1 = K_{H_1} \otimes \mathcal{O}_{H_1}(D) = -\mathcal{O}_{H_1}(H_{m+1} + \ldots + H_k).$$

Since $H = (H_{m+1} \cup \ldots \cup H_k) \cap H$ is $\mathcal{F}_1$-invariant smooth normal crossing hypersurface with deg $H = \deg (N_{\mathcal{F}_1})^{**}$, we have that $\mathcal{F}_1$ is logarithmic by lemma above. The last part of the corollary is a direct consequence of Proposition 3.9 item (1) and the lemma above. □

The next corollary is important for an inductive argument of Theorem A, it is just a restatement of Proposition 3.9 item (2):

Corollary 3.13. *Under the assumptions of Proposition 3.9, there exists two irreducible components H_1, H_2 of* $\operatorname{Sing}\Pi$*, such that the induced Poisson structure $\Pi_{|H_1 \cap H_2}$ is nondegenerate.*

3. Constraints on the index

We will give a very important step towards the proof of Theorem A.

PROPOSITION 3.14. *Let X be Fano four-fold with $Pic\, X = \mathbb{Z}$. Suppose that we have a nondegenerate Poisson structure Π on X with singular set* $\operatorname{Sing}\Pi = H_1 \cup \ldots \cup H_k$ *smooth normal crossing, then $i(X) \geq 4$.*

To prove the proposition, we construct a global vector field in H_1 tangent to the Poisson foliation in H_1. To do so, we need the following lemma.

LEMMA 3.15. *Let Π be a nondegenerate Poisson structure on X, with $dim\, X = 2n$. Let H_2 and H_3 be strong Poisson subvarieties defining the same line bundle, then H_2 and H_3 induces a vector field $Z \in H^0(X, TX)$ which is a Hamiltonian vector field in $X \setminus (H_2 \cup H_3)$.*

PROOF. Write $\mathcal{L} = \mathcal{O}_X(H_2) = \mathcal{O}_X(H_3)$ and consider the Polishchuk connection ∇ induced by H_2 on $\mathcal{L}$. If $f = \{f_i\}, g = \{g_i\} \in H^0(X, \mathcal{L})$ defines the hypersurfaces H_2 and H_3 respectively, then

$$\nabla f = 0, \text{ but } \nabla g \neq 0.$$

To see this, first note that $X_{\log f_i} - X_{\log g_i} \neq 0$ because Π is a nondegenerate Poisson structure and the hypersurfaces H_2 and H_3 are distincts, i.e, $\frac{f}{g}$ is not constant. If we restrict to a trivialization U_i, we write $f = f_i^{-1} s_i$ and $g = g_i^{-1} s_i$. If $\nabla f = \nabla g = 0$, we have $\nabla s_i = -X_{\log g_i} \otimes s_i = -X_{\log f_i} \otimes s_i$. So, $X_{\log f_i} - X_{\log g_i} = 0$. A contradiction.

Since H_3 is a strong Poisson subvariety, we have that ∇g restricted to H_3 is 0 (see lemma 2.40 of chapter 2), and, so, we can divide the global section $\nabla g \in H^0(X, TX \otimes \mathcal{L})$ by g, i.e.,

$$Z = \frac{\nabla g}{g} \in H^0(X, TX)$$

is a global vector field on X. Note that Z is, locally, the vector field $X_{log f_i} - X_{\log g_i} \neq 0$. □

Recall Wahl's theorem (see [**23**]):

THEOREM 3.16. *Let $\mathcal{L}$ be an ample line bundle on X. And suppose that $H^0(X, TX \otimes \mathcal{L}^*)$ is nonzero, then $X = \mathbb{P}^n$.*

PROOF OF THE PROPOSITION 3.14. Since $K_X^* = \mathcal{O}_X(H_1 + \ldots + H_k)$, then, by Corollary 3.12, $k \geq 3$. In particular, $i(X) \geq 3$ and we just need to exclude the case $i(X) = 3$ and $k = 3$.

Since X is Fano with $Pic\,X = \mathbb{Z}$, we have $H_1 \cap H_2 \cap H_3 \neq \emptyset$. Let $\Pi_1 = \Pi_{|H_1}$ be the induced Poisson structure on H_1. We have that H_1 is a Fano threefold with $Pic\,H_1 = \mathbb{Z}$. Since $K^*_{H_1} = \mathcal{O}_{H_1}(H_{2|H_1} + H_{3|H_1})$ and $H_1 \cap H_2 \cap H_3 \neq \emptyset$, we have, by Lefschetz-Grothendieck theorem (see Theorem 3.5), $i(H_1) = 2$, i.e., H_1 is a Fano threefold with index 2.

By Corollary 3.12, the Poisson distribution on H_1 has codimension 2 singular set.

Note that $\mathcal{O}_X(H_1) = \mathcal{O}_X(H_2) = \mathcal{O}_X(H_3)$, and $i(H_i) = 2$ for all i. By Lemma 3.15, we have a global vector field Z in the variety X induced by H_2 and H_3. Let $Y = Z_{|H_1}$ be a vector field in H_1. Y is not zero since, by Wahl's theorem, Z cannot have a divisor as singular set. We have that Y is Hamiltonian vector field in $H_1 \setminus (H_2 \cup H_3)$ and so Y is tangent to the Poisson foliation $\mathcal{F}_1$ in H_1.

Recall that the classification of Fano threefolds with index 2 is very precise.

Theorem 3.17. *Let X be a Fano 3-fold with $K^*_X = 2H$. Then, X fits in one of the following classes:*

1. $H^3 = 1$. *Hypersurface of degree* 6 *in* $\mathbb{P}(1, 1, 1, 2, 3)$*;*
2. $H^3 = 2$. *Hypersurface of degree* 4 *in* $\mathbb{P}(1, 1, 1, 1, 2)$*;*
3. $H^3 = 3$. *Cubic in* $\mathbb{P}^4$*;*
4. $H^3 = 4$. *Intersection of two quadrics in* $\mathbb{P}^5$*;*
5. $H^3 = 5$. *Intersection of Grassmannian* $Gr(2, 5) \subset \mathbb{P}^9$ *with a* $\mathbb{P}^6$.

Moreover, The 3*-folds falling in class* (5) *are all isomorphic. We denote by* X_5.

Loray, Pereira and Touzet in [**17**] proved the following:

Theorem 3.18. *Suppose that X is a Fano variety with index* 2*, then* $H^0(X, TX) \neq 0$ *if and only if* $X = X_5$. *If $\mathcal{F}$ is a logarithmic foliation in X_5, then $\mathcal{F}$ does not admit a vector field tangent to $\mathcal{F}$.*

Continuing the proof of the proposition, we have that H_1 is isomorphic to X_5, Y is tangent to the Poisson foliation $\mathcal{F}$ in H_1, which is logarithmic by Corollary 3.12. A contradiction.

So there is no nondegenerate Poisson structure on a Fano four-fold with $Pic = \mathbb{Z}$ and $i(X) = 3$. This proves the proposition. □

Now, to prove Theorem A, we will need to study logarithmic foliations on Q^3. The main idea is to prove that logarithmic foliation on Q^3 always have isolated singularity and apply Polishchuk's theorem.

4. Logarithmic foliations

In this section, we study logarithmic foliations in a quadric Q^3 and we show that it always has isolated singularities in the cases we consider.

Let $H = H_1 \cup \ldots \cup H_k$ be a smooth normal crossing hypersurface on a smooth variety X. We define the sheaf $\Omega^1_X(\log H)$ to be the sheaf of meromorphic 1-form ω such that ω and $\mathrm{d}\omega$ have pole of order 1 along H. If X is a projective variety, every element of $\Omega^1_X(\log H)$ is closed. If $p \in H_1 \cap \ldots \cap H_m$ but $p \notin H_{m+1} \cup \ldots \cup H_k$, and if $(x_1, \ldots, x_n)$ are local coordinates around p with $\{x_i = 0\} = H_i$ for $i = 1, \ldots, m$, then $\Omega^1_X(\log H)$ is generated by the following linearly independent elements:

$$\left\{\frac{\mathrm{d}x_1}{x_1}, \ldots, \frac{\mathrm{d}x_m}{x_m}, \mathrm{d}x_{m+1}, \ldots, \mathrm{d}x_n\right\}$$

i.e., we have the following exact sequence:

$$0 \to \Omega^1_X \to \Omega^1_X(\log H) \to \bigoplus_{i=1}^{m} \mathcal{O}_{H_i} \to 0,$$

where the last map is the residue map.

From this description, we see that $\Omega^1_X(\log H) \subset j_*\Omega^1_U$, $U = X \setminus H$, $j : U \to X$ the inclusion, is the subsheaf of 1-forms with logarithmic singularities along H.

We say that a codimension 1 foliation $\mathcal{F}$ on X is logarithmic with poles in $H = H_1 \cup \ldots \cup H_k$ if there exists ω meromorphic 1-form defining $\mathcal{F}$ such that ω has pole along each H_i, i.e., a global section $\omega \in H^0(X, \Omega^1_X(\log H))$ such that the image of ω in $\bigoplus_{i=1}^k \mathcal{O}_{H_i}$ is nonzero on each i. Note that each H_i is $\mathcal{F}$-invariant.

By the description of $\Omega^1_X(\log H)$, if p is an isolated singularity of ω, global section of $\Omega^1_X(\log H)$, then p is not in H_i for every i. In the section 2 of [**9**], they proved that if H_i is ample for each i, $H = H_1 \cup \ldots \cup H_k$ smooth normal crossing and if p is a nonisolated point of $\mathcal{F}$, then p lies in $H_i \cap H_j$ for some $i \neq j$. Putting everything together, we conclude that, if $\omega \in H^0(X, \Omega^1_X(\log H))$ and each of H_i is ample, then ω has only isolated singularities. In particular, if $c_n(\Omega^1_X(\log H)) \neq 0$, where $n = \dim X$ then ω has at least one singularity.

We have the following lemma which was used in the other sections.

LEMMA 3.19. *Let $\mathcal{F}$ be a codimension 1 foliation in a projective variety X with $Pic\, X = \mathbb{Z}$. If H is a $\mathcal{F}$-invariant smooth normal crossing hypersurface, such that $\deg \mathcal{O}_X(H) = \deg (N\mathcal{F})^{**}$, then $\mathcal{F}$ is logarithmic and H cannot be a smooth hypersurface.*

Proof. Let $\sigma \in H^0(X, \Omega^1_X \otimes N\mathcal{F}^{**})$ be the holomorphic section defining $\mathcal{F}$. We can look at σ as a global holomorphic section of $\Omega^1_X \otimes \mathcal{O}_X(-H) \otimes N\mathcal{F} \otimes \mathcal{O}_X(H)$, that is a meromorphic section ω of $\Omega^1 \otimes \mathcal{O}_X(-H) \otimes N\mathcal{F}$ with simple pole $(\omega)_\infty \subseteq H$. Since $\mathcal{O}_X(-H) \otimes N\mathcal{F} = \mathcal{O}_X$ by our hypothesis, we just need to prove that ω is logarithmic. Let $\{f = 0\}$ be a local reduced equation of H, then, since H is invariant by $\mathcal{F}$, we have that $\mathrm{d}f \wedge \sigma$ is identically 0 along H, i.e.,

$$\mathrm{d}f \wedge \sigma = f\theta,$$

where θ is a holomorphic section of $\Omega^2 \otimes N\mathcal{F}$. Hence $\mathrm{d}f \wedge \sigma_f = \mathrm{d}f \wedge \omega = \theta$ is a holomorphic section of Ω^2. Since $f\omega$ and $\mathrm{d}f \wedge \omega$ are holomorphic, we conclude that $f\mathrm{d}\omega$ is holomorphic. This proves that ω is logarithmic.

Lemma 2.2 of [**5**] proves the well known fact that H can not be smooth. We will not reproduce the prove here because it uses the theory of currents and it is beyond the scope of this thesis. □

Now, the variety we consider is the quadric Q^3 and $\mathcal{F}$ is a logarithmic foliation on Q^3. We only need to consider 3 possibilities for H.

(1) $H = H_1 + Z$, where deg $H_1 = 1$ and deg $Z = 2$;
(2) $H = H_1 + H_2$, where deg $H_1 =$ deg $H_2 = 1$;
(3) $H = H_1 + H_2 + H_3$, where deg $H_1 =$ deg $H_2 =$ deg $H_3 = 1$.

Proposition 3.20. *In notation above, in all of* 3 *cases,* $\mathcal{F}$ *has at least one isolated singularity.*

Proof. Let $i : Q^3 \to \mathbb{P}^4$ be the inclusion. We have the following exact sequence:

$$0 \to TQ^3 \to T\mathbb{P}^4_{|Q^3} \to \mathcal{O}_{Q^3}(2h_{|Q^3}) \to 0$$

since $N_{\mathbb{P}^4|Q^3}$ is $\mathcal{O}_{Q^3}(2h_{|Q^3})$ by adjunction formula, where h is the hyperplane section of $\mathbb{P}^4$. We will abuse the notation and write h in the place of $h_{|Q^3} = i^*h$.

Since $c(T\mathbb{P}^4_{|Q^3}) = 1 + 5h + 10h^2 + 10h^3$, $\mathcal{O}_{Q^3}(2h) = 1 + 2h$ and since $c(TQ^3) = c(T\mathbb{P}^4_{|Q^3})/\mathcal{O}_{Q^3}(2h)$, we have:

$$\begin{aligned} c(TQ^3) &= (1 + 5h + 10h^2 + 10h^3)(1 - 2h + 4h^2 - 8h^3) \\ &= 1 + 3h + 4h^2 + 2h^3 \end{aligned}$$

we conclude that

$$c(\Omega^1_{Q^3}) = 1 - 3h + 4h^2 - 2h^3.$$

If we are in the item (1) above, we have the following exact sequence:

$$0 \to \Omega^1_{Q^3} \to \Omega^1_{Q^3}(\log H) \to \mathcal{O}_{H_1} \oplus \mathcal{O}_Z \to 0$$

By the exact sequences

$$\begin{aligned} 0 \to &\mathcal{O}_{Q^3}(-h) \to \mathcal{O}_{Q^3} \to \mathcal{O}_{H_1} \to 0 \\ 0 \to &\mathcal{O}_{Q^3}(-2h) \to \mathcal{O}_{Q^3} \to \mathcal{O}_{Z} \to 0, \end{aligned}$$

we have that $c(\mathcal{O}_{H_1}) = 1 + h + h^2 + h^3$ and $c(\mathcal{O}_Z) = 1 + 2h + 4h^2 + 8h^3$. The product of these two expressions gives

$$c(\mathcal{O}_{H_1} \oplus \mathcal{O}_Z) = 1 + 3h + 7h^2 + 15h^3.$$

Then,

$$\begin{aligned} c_3(\Omega^1_{Q^3}(\log H)) &= (c(\Omega^1_{Q^3})c(\mathcal{O}_{H_1} \oplus \mathcal{O}_Z))_3 \\ &= ((1 - 3h + 4h^2 - 2h^3)(1 + 3h + 7h^2 + 15h^3))_3 \\ &= 4h^3 = 8 \end{aligned}$$

If we are in item (2), then

$$\begin{aligned} c(\mathcal{O}_{H_1} \oplus \mathcal{O}_{H_2}) &= (1 + h + h^2 + h^3)(1 + h + h^2 + h^3) \\ &= 1 + 2h + 3h^2 + 4h^3 \end{aligned}$$

and

$$\begin{aligned} c_3(\Omega^1_{Q^3}(\log H)) &= (c(\Omega^1_{Q^3})c(\mathcal{O}_{H_1} \oplus \mathcal{O}_{H_2}))_3 \\ &= ((1 - 3h + 4h^2 - 2h^3)(1 + 2h + 3h^2 + 4h^3))_3 \\ &= h^3 = 2\,. \end{aligned}$$

If we are in item (3), then

$$\begin{aligned} c(\mathcal{O}_{H_1} \oplus \mathcal{O}_{H_2} \oplus \mathcal{O}_{H_3}) &= (1 + h + h^2 + h^3)^2(1 + h + h^2 + h^3) \\ &= (1 + 2h + 3h^2 + 4h^3)(1 + h + h^2 + h^3) \\ &= 1 + 3h + 6h^2 + 10h^3 \end{aligned}$$

and

$$\begin{aligned} c_3(\Omega^1_{Q^3}(\log H)) &= (c(\Omega^1_{Q^3})c(\mathcal{O}_{H_1} \oplus \mathcal{O}_{H_2} \oplus \mathcal{O}_{H_3}))_3 \\ &= ((1 - 3h + 4h^2 - 2h^3)(1 + 3h + 6h^2 + 10h^3))_3 \\ &= 2h^3 = 4. \end{aligned}$$

This proves the proposition. □

To prove Theorem A, we will need a local study of nondegenerate Poisson structure which will occupy the next section.

5. Nondegenerate Lie algebra in dimension 4

We know that we have a natural bijection between linear Poisson structures on $\mathbb{C}^4$ and Lie algebras on $(\mathbb{C}^4)^*$ (see example 1.19 of chapter 1).

We have a complete classification of the Lie algebra structure in dimension 4 (see [**6**], Lemma 3) and, for our surprise, a simple check of the classification shows that we have few nondegenerate linear Poisson structures in $\mathbb{C}^4$.

We reproduce the table of linear Poisson structure in $\mathbb{C}^4$ for the convenience of the reader (ζ_i stands for $\frac{\partial}{\partial x_i}$, as in Chapter 1, section 1.3).

$\mathfrak{g}$	Poisson bracket Π
$\mathbb{C}^4$	0
$\mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$	$x_3\zeta_1\zeta_2$
$\mathfrak{aff}(\mathbb{C}) \oplus \mathbb{C}^2$	$x_1\zeta_1\zeta_2$
$\mathfrak{r}_3(\mathbb{C}) \oplus \mathbb{C}$	$x_2\zeta_1\zeta_2 + (x_2 + x_3)\zeta_1\zeta_3$
$\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathbb{C}$	$x_2\zeta_1\zeta_2 + (\lambda x_3)\zeta_1\zeta_3$, $\lambda \in \mathbb{C}$, $0 < \lvert\lambda\rvert \leq 1$
$\mathfrak{aff}(\mathbb{C}) \times \mathfrak{aff}(\mathbb{C})$	$x_1\zeta_1\zeta_2 + x_3\zeta_3\zeta_4$
$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$	$x_3\zeta_1\zeta_2 - 2x_1\zeta_1\zeta_3 + 2x_2\zeta_2\zeta_3$
$\mathfrak{n}_4(\mathbb{C})$	$x_3\zeta_1\zeta_2 + x_4\zeta_1\zeta_3$
$\mathfrak{g}_1(\alpha)$	$x_2\zeta_1\zeta_2 + x_3\zeta_1\zeta_3 + \alpha x_4\zeta_1\zeta_4$, $\alpha \in \mathbb{C}^*$
$\mathfrak{g}_2(\alpha, \beta)$	$x_3\zeta_1\zeta_2 + x_4\zeta_1\zeta_3 + (\alpha x_2 - \beta x_3 + x_4)\zeta_1\zeta_4$, $\alpha \in \mathbb{C}, \beta \in \mathbb{C}^*$ or $\alpha, \beta = 0$
$\mathfrak{g}_3(\alpha)$	$x_3\zeta_1\zeta_2 + x_4\zeta_1\zeta_3 + \alpha(x_2 + x_3)\zeta_1\zeta_4$, $\alpha \in \mathbb{C}$
$\mathfrak{g}_4$	$x_3\zeta_1\zeta_2 + x_4\zeta_1\zeta_3 + x_2\zeta_1\zeta_4$
$\mathfrak{g}_5$	$(\frac{1}{3}x_2 + x_3)\zeta_1\zeta_2 + \frac{1}{3}x_3\zeta_1\zeta_3 + \frac{1}{3}x_4\zeta_1\zeta_4$
$\mathfrak{g}_6$	$x_2\zeta_1\zeta_2 + x_3\zeta_1\zeta_3 + 2x_4\zeta_1\zeta_4 + x_4\zeta_2\zeta_3$
$\mathfrak{g}_7$	$x_3\zeta_1\zeta_2 + x_2\zeta_1\zeta_3 + x_4\zeta_2\zeta_3$
$\mathfrak{g}_8(\alpha)$	$x_3\zeta_1\zeta_2 - (\alpha x_2 - x_3)\zeta_1\zeta_3 + x_4\zeta_1\zeta_4 + x_4\zeta_2\zeta_3, \alpha \in \mathbb{C}^*$

Classification of 4-dimensional Lie algebras.

Now we can state the following lemma.

LEMMA 3.21. *Let* Π *be a nondegenerate Poisson structure on a Fano fourfold* X *with* $\operatorname{Sing}\Pi = H_1 \cup \ldots \cup H_k$ *smooth normal crossing hypersurface. Let* $\mathcal{F}_1$ *be the codimension* 1 *foliation on* H_1 *induced by Poisson* *structure* Π. *If* p *is an isolated singular point of* $\mathcal{F}_1$, *then* p *is in three components of* $\operatorname{Sing}\Pi$.

PROOF. If p is a singular point of $\mathcal{F}_1$, then $\Pi(p) = 0$. Locally, this means that $\Pi \in \mathfrak{m}_p \otimes \mathfrak{X}_X^2$. In particular, $\Pi \wedge \Pi \in \mathfrak{m}_p^2 \otimes \mathfrak{X}_X^4$. Since $\operatorname{Sing}\Pi$ is normal crossing, we can find local coordinates (x_1, x_2, x_3, x_4) in a neighborhood of $p = 0$, such that $\Pi \wedge \Pi = x_1 x_2 V$, where V is a 2-derivation. Write $\Pi = \Pi_1 + \Pi_2 + \dots$ the Taylor series of Π. To prove the lemma, we just need to check that $\Pi_1 \wedge \Pi_1 = 0$. A simple check of the table 5 shows that $\Pi_1 \wedge \Pi_1 \neq 0$ just in the cases $\mathfrak{aff}(\mathbb{C}) \times \mathfrak{aff}(\mathbb{C})$, $\mathfrak{g}_6$ and $\mathfrak{g}_8(\alpha)$. The last two cases are excluded because $\operatorname{Sing}\Pi$ is smooth normal crossing. To exclude the first case, we use the Theorem 1.37 of chapter 1, which states that we can find coordinates $(y_1, \dots, y_4)$, $H_1 = \{y_1 = 0\}$ such that

$$\Pi = y_1 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2} + y_3 \frac{\partial}{\partial y_3} \wedge \frac{\partial}{\partial y_4}.$$

In particular, the Poisson foliation induced on H_1 is regular at p. This proves the lemma. □

6. Excluding the quadric

The objective of this section is to prove the following:

PROPOSITION 3.22. *Let* Π *be a nondegenerate Poisson structure with normal crossing singularity in the Fano fourfold* X *with cyclic Picard group. Then* $i(X) \neq 4$.

PROOF. Let Π be a Poisson structure satisfying the hypothesis of the proposition. Since $i(X) = 4$, the, by Theorem 3.1, X is a quadric Q^4. Let H be the positive generator of X, then H is Q^3, since H is a hyperplane section of $\mathbb{P}^5$ intersected with X.

Suppose that X admits a nondegenerate Poisson structure Π with $\operatorname{Sing}\Pi = Y_1 + \dots + Y_k$ smooth normal crossing, then we have $k = 3$ or $k = 4$. Suppose that $k = 3$. Since $\deg Y_1 + \deg Y_2 + \deg Y_3 = 4$, we have that, after reordering the index, $\deg Y_1 = \deg Y_2 = 1$ and $\deg Y_3 = 2$. We will rewrite $Y_2 = Q^3$, $Y_{3|Q^3} = Z$ and $Y_{1|Q^3} = H_1$ so as to keep a notation of the section 4. If $\mathcal{F}$ is the logarithmic foliation on Q^3 associated to the Poisson distribution $\Pi_{|Q^3}$, then we are in the situation number (1) of last section and we have an isolated singularity p. By Lemma 3.21, we have that $p \in Y_1 \cap Y_2 \cap Y_3 = Z \cap H_1$. A contradiction, since the isolated singularities are never in the divisor of poles of a logarithmic foliation.

So $k = 4$ and we have that $\deg Y_1 = \deg Y_2 = \deg Y_3 = \deg Y_4 = 1$.

Rewrite $Y_4 = Q^3$, $Y_{i|Q^3} = H_i$ for $i = 1, 2, 3$ and let $\Pi_4 = \Pi_{|Q^3}$ be the Poisson structure on Q^3. By Proposition 3.20, we have only two possibilities:

(1) Π_4 has rank 0 along H_i for just one i, and we can suppose that $i = 3$, i.e., Π_4 defines a logarithmic foliation $\mathcal{F}$ along the divisor $H_1 + H_2$ or
(2) Π_4 defines a logarithmic foliation $\mathcal{F}$ along $H_1 + H_2 + H_3$.

In both case we have an isolated singularity p. By Lemma 3.21, we see that $p \in H_2$ or $p \in H_1$, a contradiction. □

We proved that a quadric does not admit a nondegenerate Poisson structure with normal crossing singularity. By Proposition 3.14 and by Kobayashi-Ochiai theorem 3.1, we proved.

Corollary 3.23. *If X is a Fano fourfold with $Pic\, X = \mathbb{Z}$ and Π is a nondegenerate Poisson structure with normal crossing singularity then $X = \mathbb{P}^4$.*

Corollary 3.24. *Suppose that X is a Fano variety with $\dim X = 2n$, $n \geq 2$, $Pic\, X = \mathbb{Z}$ and Π is a nondegenerate Poisson structure with normal crossing singularity, then $X = \mathbb{P}^{2n}$.*

Proof. The proof is by induction on n. For $n = 2$, it was proved in the last corollary. Suppose the result true for n. If X is Fano with cyclic Picard group, $\dim X = 2(n+1)$ and suppose that Π is a nondegenerate Poisson structure on X such that the singular locus $\operatorname{Sing} \Pi = H_1 \cup H_2 \cup \ldots \cup H_k$ is smooth normal crossing. By Corollary 3.13 and by Polishchuk's theorem, we have a nondegenerate Poisson structure on $Y = H_1 \cap H_2$ and the singular loci are smooth normal crossing contained in $(H_3 \cup \ldots \cup H_k) \cap Y$. Since $Pic\, Y = \mathbb{Z}$, we have, by the induction hypothesis, $Y = \mathbb{P}^{2n}$, i.e., $i(Y) = 2n + 1$. In particular, $\deg(H_3 + \ldots + H_k) = 2n + 1$. So $i(X) = \deg(H_1 + \ldots + H_k) \geq 2n + 3$. By Kobayashi-Ochiai theorem, we have $i(X) = 2n + 3$ and $X = \mathbb{P}^{2n+2}$. □

Note that, by the last line of the proof of the corollary, we have $\deg\, H_1 = \deg\, H_2 = 1$. So we proved:

Corollary 3.25. *Let Π be a nondegenerate Poisson structure in $\mathbb{P}^{2n}$ with smooth normal crossing singularity and $\operatorname{Sing} \Pi = H_1 + \ldots + H_k$. Let H_1 and H_2 be components of $\operatorname{Sing} \Pi$ such that Π induces a nondegenerate Poisson structure on $H_1 \cap H_2$. Then H_1 and H_2 are hyperplanes.*

7. Proof of Theorem A

So, in order to prove the Theorem A, we need to study the singular locus of a nondegenerate Poisson structure Π on $\mathbb{P}^4$. The idea is to prove that if $\operatorname{Sing}\Pi$ is normal crossing then $\operatorname{Sing}\Pi$ is the union of 5-hyperplanes. The result we discovered in this section was the motivation to conjecture Theorem A. We start giving an example.

EXAMPLE 3.26. Consider $\mathbb{C}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and let $\Pi = \sum_{i=1}^n x_i y_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$. Then Π is a nondegenerate Poisson structure on $\mathbb{C}^{2n}$ with singular locus $\operatorname{Sing}\Pi$ the hyperplanes $x_i = 0$ and $y_i = 0$. We saw in the subsection 1.2 of chapter 2 that this quadratic Poisson structure extends to $\mathbb{P}^{2n}$. Since $\deg\, K^*_{\mathbb{P}^{2n}} = 2n+1$, we conclude that the hyperplane at infinity is a component of the singular locus of Π.

Write $\operatorname{Sing}\Pi = H_1 \cup \ldots \cup H_k$, the singular locus of Π in $\mathbb{P}^4$. We know by Corollary 3.10 that $k \geq 3$. Since $\deg \operatorname{Sing}\Pi = 5$, we have the following possibilities:

(1) $k = 3$, H_1 is a hyperplane and H_2, H_3 are quadrics;
(2) $k = 3$, H_1, H_2 are hyperplanes and H_3 is a cubic;
(3) $k = 4$, H_1, H_2, H_3 are hyperplanes and H_4 is a quadric;
(4) $k = 5$ and all components are hyperplanes.

Excluding the cases (1) and (3) were done in the proof of the section *excluding the quadric*. We just need to consider the Poisson foliation on the quadric. The Poisson foliation is a generic element of $Log(1,1,1)$, $Rat(1,2)$ or $Rat(1,1)$. So there exists an isolated singularity and we can apply Proposition 3.9, Lemma 3.21 and the fact that isolated singularity cannot be at the invariant hypersurfaces of the logarithmic foliation and this is a contradiction.

We just need to exclude the second item. So, consider the Poisson distribution $\mathcal{D}_1$ on $H_1 = \mathbb{P}^3$. We have two possibilities: $\mathcal{D}_1$ has no divisorial component on the singular locus or $\mathcal{D}_1$ has a divisor component. In the second case, it cannot happen because we know that the associated foliation $\mathcal{F}_1$ is logarithmic. So $\mathcal{D}_1$ is a foliation with trivial canonical bundle which have a cubic and a hyperplane invariant, i.e., it is in the component $\operatorname{Rat}(1,3)$. A general element of this component has at least one isolated singularity p (see [**9**]). By Lemma 3.21, we have that $p \in H_1 \cap H_2 \cap H_3$. A contradiction, since $H_1 \cap H_2 \cap H_3$ is a curve contained in the singular locus of $\mathcal{D}_1$, i.e., p cannot be an isolated singularity.

To finish the proof of Theorem A, we just need to use induction on the dimension and apply the Corollary 3.25.

CHAPTER 4

Deformation of Poisson brackets on projective spaces

1. Deformation of Poisson structures and Poisson cohomology

Definition 4.1. We say that two Poisson structures Π and Π' are equivalent if there exists a $\lambda \in \mathbb{C}^*$ such that $\Pi = \lambda\Pi'$. By Theorem 2.7 of chapter 2, the set of equivalent Poisson structures in $\mathbb{P}^n$ is given by homogeneous quadratic Poisson structures Π in $\mathbb{C}^{n+1}$ such that $[\Pi, \Pi] = 0$ and $D_\Omega\Pi = 0$. So it is an algebraic subset of $\mathbb{P}(\bigwedge^2 T(\mathbb{C}^{n+1}))$ which we will denote by Λ.

Let S_k be the set of Poisson structure Π on $\mathbb{P}^n$ with $\operatorname{rk}\Pi = 2k$. It is an algebraic variety on the space $\mathbb{P}(\bigwedge^2(T\mathbb{C}^{n+1}))$.

In the chart $X_0 = 1$, we see that $T_\Pi\Lambda = \{\xi \in (\bigwedge^2 T\mathbb{C}^n); [\Pi, \xi] = 0\}$ and if $\operatorname{rk}\Pi = 2k$, then $T_\Pi S_{2k} = \{\xi \in T_\Pi\Lambda; \Pi^k \wedge \xi = 0\}$.

Let Π be a Poisson structure on a variety X. We say that Π_ϵ is a deformation of Π if we can write $\Pi_\epsilon = \Pi + \epsilon\Pi_1 + \epsilon^2\Pi_2 + \ldots$, where Π_i are biderivations (not necessarily Poisson) and we have $[\Pi_\epsilon, \Pi_\epsilon] = 0$, i.e., Π_ϵ is a Poisson biderivation. In other words, Π_ϵ is in Λ and "near" to Π and $\operatorname{rk}\Pi_\epsilon \geq \operatorname{rk}\Pi$.

There exists a cohomological interpretation of Poisson deformation, we will now describe, but first we need a simple lemma.

Lemma 4.2. *If Π is a Poisson tensor, then for any multi-vector field A, we have*

$$[\Pi, [\Pi, A]] = 0$$

Proof. Just use the graded Jacobi identity. □

Let (X, Π) be a Poisson variety. Consider $\delta : \mathfrak{X}^\bullet_X \to \mathfrak{X}^\bullet_X$ the $\mathbb{C}-$linear operator on the space of multi-vector fields on X, defined as follows

$$\delta(X) = [\Pi, X]$$

The lemma says that δ is a differential operator in the sense that $\delta \circ \delta = 0$. The corresponding differential complex $(\mathfrak{X}^\bullet_X, \delta)$

$$\cdots \longrightarrow \mathfrak{X}^{p-1}_X \xrightarrow{\delta} \mathfrak{X}^p_X \xrightarrow{\delta} \mathfrak{X}^{p+1}_X \longrightarrow \cdots$$

is called the Lichnerowics complex. The cohomology of this complex is called *Poisson Cohomology.*

By definition, the Poisson cohomology groups of (X, Π) are the quotient groups

$$H^p_\Pi(X) = \frac{ker(\delta : \mathfrak{X}^p_X \longrightarrow \mathfrak{X}^{p+1}_X)}{Im(\delta : \mathfrak{X}^{p-1}_X \longrightarrow \mathfrak{X}^p_X)}$$

REMARK 4.3. Poisson Cohomology can be very big, even infinite-dimensional.

The zeroth Poisson cohomology group $H^0_\Pi(X)$ is the group of functions f such that $Z_f = -[\Pi, f] = 0$. In the other words, $H^0_\Pi(X)$ is the space of Casimir functions of Π, i.e., the space of first integrals of the associated Poisson distribution.

The first cohomology group $H^1_\Pi(X)$ is the quotient of the space of Poisson vector fields by the space of Hamiltonian vector fields. Poisson vector fields are infinitesimal automorphisms of the Poisson structures, while Hamiltonian vector fields may be interpreted as *inner* infinitesimal automorphisms. Thus $H^1_\Pi(X)$ may be interpreted as the space of *outer inifinitesimal automorphism of* Π.

The second Poisson cohomology group $H^2_\Pi(X)$ is the quotient space of 2-vector fields ξ which satisfy the equation $[\Pi, \xi] = 0$ by the space of 2-vector fields of the type $\xi = [\Pi, X]$. If $[\Pi, \xi] = 0$ and ε is a formal parameter, then $v + \varepsilon\xi$ satisfies the Jacobi identity up to term of order ε^2

$$[\Pi + \varepsilon\xi, \Pi + \varepsilon\xi] = \varepsilon^2[\xi, \xi] = 0 \bmod \varepsilon^2$$

So one may view $\Pi + \varepsilon\xi$ as an infinitesimal deformation of v in the space of Poisson tensors. On the other hand, up to terms of order ε^2, $\Pi + \varepsilon[\Pi, X]$ is equal to $(\phi^\varepsilon_X)_*\Pi$, where ϕ^ε_X denotes the time-ε flow of X. Therefore, $\Pi + \varepsilon[\Pi, X]$ is a trivial deformation of Π up to an infinitesimal diffeomorphism. Thus $H^2_\Pi(X)$ is the quotient of the space of all possible infinitesimal deformations of Π by the space of trivial deformations. In other words, $H^2_\Pi(X)$ may be interpreted as the moduli space of formal infinitesimal deformations of Π.

2. Theorem B

If Π is a nondegenerate Poisson structure on $\mathbb{P}^{2n}$, with $2n + 1$ hyperplanes in general position as singular set (we will denote each hyperplane by H_i), then Π is a diagonal Poisson structure. To see this, consider the homogeneous coordinates $(X_0 : \ldots : X_{2n})$ in $\mathbb{P}^{2n}$ such that $H_i = \{X_i = 0\}$. In affine coordinates $(x_1, \ldots, x_{2n})$, we have that Π is a nonhomogeneous quadratic Poisson structure on $\mathbb{C}^{2n}$. If we write

$\Pi = \sum_{i=2}^{2n} \sum_{k,l} \left(\lambda_{kl} x_k x_l \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right) + \frac{\partial}{\partial x_1} \wedge X$ where X is a vector field, then since H_1 is a Poisson subvariety, i.e., $\Pi_{|H_1}$ is a bivector on H_1, we conclude that $\Pi_{|H_1}$ cannot have the component $\frac{\partial}{\partial x_1}$, i.e., $X_{|H_1} = 0$. This means that x_1 divides X. In other words, every time that occurs the component $\frac{\partial}{\partial x_1}$, we have that in fact occurs the component $x_1 \frac{\partial}{\partial x_1}$. This works for every coordinate x_i and since Π is quadratic, we conclude that $\Pi = \sum_{i,j} \lambda_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, i.e., Π is diagonal.

A more geometric description of the diagonal Poisson structure can be done in the following way: consider the universal cover $\phi : \mathbb{C}^n \to (\mathbb{C}^*)^n$ defined by $\phi(x_1, \ldots, x_n) = (e^{x_1}, \ldots, e^{x_n})$ and consider the constant Poisson structure $\Pi' = \sum_{i,j} \lambda_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. Then, with a simple check, we show that $\Pi = \phi_* \Pi' = \sum_{i,j} \lambda_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$.

Theorem B. If we take sufficiently small deformations of a generic diagonal Poisson structure in $\mathbb{P}^{2n}$ then the resulting Poisson structures are still diagonal Poisson structures in $\mathbb{P}^{2n}$.

Proof. Let $H_i = \{X_i = 0\}$ be the singular set of the diagonal Poisson structure Π. In affine coordinates $x_i = X_i/X_0$, we write $\Pi = \sum_{ij} \lambda_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. In $\mathbb{C}^{2n}$, we have the natural volume form $\Omega = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{2n}$ and the Poisson vector field $D_\Omega \Pi$. This vector field is diagonal with 0 trace. Let $c_1, \ldots, c_{2n}$ be the eigenvalues of $D_\Omega \Pi$, then $c_1 + \ldots + c_{2n} = 0$ and we say that Π is a nonresonant Poisson structure if there is no other resonance relation (see definition 1.27 of chapter 1).

The set of traceless nonresonant vector fields is open on the set of traceless vector fields, i.e., we have the following well known theorem

Theorem 4.4. *Let X be a linear vector field in $(\mathbb{C}^{2n}, 0)$ with isolated singularity at 0. Suppose that trace of X is 0 and the eigenvalues of X at 0 are nonresonant. If X_ϵ is a deformation of X, sufficiently near to X, then there exists $p_\epsilon \in \mathbb{C}^{2n}$ near to 0 such that p_ϵ is an isolated singularity of X_ϵ. If the trace of X_ϵ at p_ϵ is zero, then X_ϵ has semisimple linear part on a neighborhood of p_ϵ and the eigenvalues of X_ϵ at p_ϵ are nonresonant.*

We know that $D_\Omega \Pi = \sum_i c_i x_i \frac{\partial}{\partial x_i}$ and suppose that Π is a nonresonant diagonal Poisson structure. Let Π_ϵ be a deformation of Π. We can suppose that $D_\Omega \Pi_\epsilon$ is in a small neighborhood of $D_\Omega \Pi$. So, by theorem above, $D_\Omega \Pi_\epsilon$ has an isolated singularity at a point p_ϵ near to 0 and holomorphic coordinates $(y_1, \ldots, y_{2n})$ such that $X_\epsilon = D'_\Omega \Pi_\epsilon = \sum c'_i y'_i \frac{\partial}{\partial y_i} + \ldots$, with $\Omega' = \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_n$ and $\sum_i c'_i = 0$ is the only resonance of c'_i.

According to [**18**], we have a formal change of coordinates $(y_1,\dots,y_{2n})$, such that p_ϵ is 0,

$$X_\epsilon = \sum_i c'_i y_i \frac{\partial}{\partial y_i} + \sum_{i,j\geq 1} \lambda_{ij}(y_1\dots y_{2n})^j y_i \frac{\partial}{\partial y_i},\ \lambda_{ij}\in\mathbb{C}.$$

We observe that $\Pi_\epsilon(0)=0$. To see this, write $\Pi_\epsilon = \sum_{ij}\Pi_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}$ and, for example, expand the relation $0=[X_\epsilon,\Pi]$ and look at the term $\frac{\partial}{\partial y_1}\wedge\frac{\partial}{\partial y_2}$ at the point 0. We see that it is exactly $\Pi_{12}(0)(c'_1+c'_2)$. From the nonresonance of c'_i we deduce that $\Pi_{12}(0)=0$.

We claim that the hypersurfaces $\{y_i=0\}$ are in the singular locus of Π. To see this, write $\Pi^n = f(y_1,\dots,y_{2n})\frac{\partial}{\partial y_1}\wedge\dots\wedge\frac{\partial}{\partial y_{2n}}$ with $f(0)=0$. By Koszul formula (Corollary 1.25 of chapter 1), we have $[X_\epsilon,\Pi^n_\epsilon]=0$, then

$$\begin{aligned}
0 = [X_\epsilon,\Pi^n_\epsilon] &= [X_\epsilon, f]\frac{\partial}{\partial y_1}\wedge\dots\wedge\frac{\partial}{\partial y_{2n}} + f\left[X_\epsilon, \frac{\partial}{\partial y_1}\wedge\dots\wedge\frac{\partial}{\partial y_{2n}}\right]\\
&= \left(\sum_{i=1}^{2n} c'_i y_i\frac{\partial f}{\partial y_i} + \sum_{i,j\geq 1}\lambda_{ij}(y_1\dots y_{2n})^j y_i\frac{\partial f}{\partial y_i}\right)\frac{\partial}{\partial y_1}\wedge\dots\wedge\frac{\partial}{\partial y_{2n}}\\
&+ 2nf.\left(\sum_i c'_i + \sum_{ij} j\lambda_{ij}(y_1\dots y_{2n})^j\right)\frac{\partial}{\partial y_1}\wedge\dots\wedge\frac{\partial}{\partial y_{2n}}\\
&= \left(\sum_{i=1}^{2n} c'_i y_i\frac{\partial f}{\partial y_i} + \sum_{i,j\geq 1}\lambda_{ij}(y_1\dots y_{2n})^j y_i\frac{\partial f}{\partial y_i}\right)\frac{\partial}{\partial y_1}\wedge\dots\wedge\frac{\partial}{\partial y_{2n}}\\
&+ 2nf.\left(\sum_j j\lambda_{ij}(y_1\dots y_{2n})^j\right)\frac{\partial}{\partial y_1}\wedge\dots\wedge\frac{\partial}{\partial y_{2n}}
\end{aligned}$$

Restricting the equation above to the hypersurface $y_1=0$ and expanding $f(y_1,\dots y_{2n}) = a_0(y_2,\dots y_{2n}) + a_1(y_2,\dots y_{2n})y_1+\dots$, the Taylor series in y_1, we have

$$\sum_{i=2}^{2n} c'_i y_i\frac{\partial a_0}{\partial y_i} = 0$$

Since $c'_2,\dots,c'_n$ are nonresonant, expanding $a_0(y_2,\dots y_n)=0$, we conclude that $a_0(y_2,\dots,y_{2n})$ is constant. Since $\Pi(0)=0$, we have that $a_0(0)=0$ and y_1 divides f. Similarly, we conclude that y_i divides f for every i.

We note that we are in a neighborhood of the point p' which is a intersection of $2n$ components of $\operatorname{Sing}\Pi_\epsilon$, we remark that we had not proved that the singular set of Π_ϵ is smooth normal crossing. We proved that there exists $2n+1$ points, $p_1,\dots,p_{2n+1}$, in general position, such that, for each i, there exists a neighborhood U_i of p_i where $\operatorname{Sing}\Pi_\epsilon$, restricted to U_i, has at least $2n$ components intersecting in p_i. We conclude the proof of the Theorem B with the following lemma.

LEMMA 4.5. *Let D be an effective divisor in $\mathbb{P}^k$ of degree $k+1$. Suppose that D admits $k+1$ points $p_1, \ldots, p_{k+1}$ in general position, such that for each i, there exists a neighborhood U_i of p_i, where D restricted to U_i has at least k components intersecting only at p_i. Then D is the union of $k+1$ hyperplanes in general position.*

PROOF. First, we will do the case for $k = 2$, just to explain the idea. Consider the line l_{ij} which joins the points p_i, p_j, $1 \leq i < j \leq 3$. Since on each neighborhood of p_i, l_{ij} intersects D with multiplicity at least 2, we conclude that $(l_{ij}.D) \geq 4$ or l_{ij} is a component of D. Since deg $D = 3$, we have that l_{ij} is a component of D. Since we have 3 lines, we conclude the case $k = 2$.

For the general case, let $H \simeq \mathbb{P}^{k-1}$ be the hyperplane defined by the points $p_1, \ldots, p_{k-1}$. At the point p_1, let $Z_1, \ldots, Z_k$ be local components of D passing through p_1 and write $Y_i = Z_i \cap H$. We claim that $Y_i \neq Y_j$ for every $i \neq j$. To prove this, the interesting part is the case where Y_i and Y_j are not H.

Suppose $Y_i = Y_j$ for some $i \neq j$. For each $q \in Y_i$, let l_q be the line joining p_k and q. We have that $(l_q.D)_q \geq 2$ and $(l_q.D)_{p_k} \geq k$. Looking at the degree of Z_i, we conclude that l_q is invariant by Z_i, for each q. By the same argument, l_q is invariant by Z_j. The set V defined to be the union of the lines l_q intersected in a neighborhood U_1 of p_1 is a open subset of Z_i and Z_j (counting dimension). By the irreducibility of Z_i and Z_j, we conclude that $Z_i = Z_j$. This is a contradiction.

Note that it could happen that some Y_i is exactly H. So, if D is a divisor of degree $k + 1$ in $\mathbb{P}^k$, then D_{k-1} can be defined to be a divisor in H of degree l, with $l = k$ or $l = k + 1$, such that in each point p_i for $1 \leq i \leq k$, we have at least $l - 1$ local distincts irreducible components of D_{k-1} passing through p_i, for each i. Such local irreducible components are Y_i.

Iterating the process above, we reach to the plane $\mathbb{P}^2$ generated by p_1, p_2 and p_3 and a divisor D_2, deg $D_2 = l$ with $3 \leq l \leq k + 1$ such that in each point p_i, for $i = 1, 2, 3$, there exists at least $l - 1$ local distincts irreducible components of D_2 passing through p_i. Note that $l = 3$ if and only if one of Y_i is exactly H in the first process. We will prove that $l = 3$.

Let l_{ij} be the line joining the points p_i and p_j. Comparing the degrees, we have that l_{ij} is a component of D_2. Consider the effective divisor $D' = D_2 - (l_{12} + l_{13} + l_{23})$. We have that deg $D' = l - 3$ with $3 \leq l \leq k+1$

and on each point p_i, there exists at least $l-3$ local distincts irreducible components of D_2 passing through p_i. If $l-3>0$, consider, again, the line l_{ij} joining the points p_i and p_j. Comparing the degrees, we conclude that l_{ij} is a component of D' for every i,j. Iterating this process, we conclude that the lines joining p_i and p_j are the unique components of D_2. So, in each point p_i, there exists exactly 2 local distinct irreducible components of D_2 passing through p_i. This is sufficient to conclude that $l=3$.

So we that that, say, Y_k is exactly the hyperplane H. We can do the same argument for any of $k+1$ hyperplane H_i defined for k points p_j, $1 \le j \le k+1$ to conclude that H_i is an irreducible component of D.

Comparing degrees, we conclude that D is the union of $k+1$ hyperplanes. □

So Π_ϵ is a Poisson structure in $\mathbb{P}^{2n}$ with singular locus $\operatorname{Sing}\Pi_\epsilon$ exactly the union of $2n+1$ hyperplanes in general position. We conclude that Π_ϵ is a diagonal Poisson structure and we finish the proof of Theorem B. □

3. Computing the second Poisson cohomology

Let X be a variety and consider $G = Aut(X)$, the group of automorphism of X, with the Whitney topology. Then G is a Lie group.

Consider $\mathfrak{g} = T_e(G)$. We have a natural isomorphism between $\mathfrak{g}$ and $H^0(X,TX)$ which we will describe.

Since G is a Lie group, we have, for each $g \in G$, the left translation $L_g : G \to G$ which maps h to gh and it is well known that we can identify $\mathfrak{g}$ with the left invariant vector fields $X \in H^0(G,TG)$, i.e., $(L_g)_*X = X$, for every g. Consider the flow $\phi_t : G \to G$ of X. Since $\phi_{t+s} = \phi_t \circ \phi_s$, then it is a flow of a global vector field $Z \in H^0(X,TX)$. So for each element $\alpha \in \mathfrak{g}$, we associate a flow $\phi_t^\alpha : G \to G$ and, from this flow, we associate a vector field $Z_\alpha \in H^0(X,TX)$.

Conversely, for each vector field $Z \in H^0(X,TX)$, we associate a flow $\phi_t : G \to G$. We check that this flow is left invariant and, in particular, we can associate to an element $\alpha_Z \in \mathfrak{g}$. A simple check shows that one map is the inverse of the other.

The identification $\mathfrak{g} \simeq H^0(X,TX)$ is a Lie algebra morphism, with the natural Lie brackets.

Let Π be a nonresonant Poisson diagonal vector field in $\mathbb{P}^{2n}$.

If we write $\Pi = \sum_{ij} \lambda_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ and if Π_ϵ is a small deformation of Π, then Theorem B says that there exists an automorphism

$$\begin{aligned} \phi : \mathbb{P}^{2n} \quad &\longrightarrow \quad \mathbb{P}^{2n} \\ (x_0,\ldots,x_{2n}) &\mapsto (y_0,\ldots,y_{2n}) \end{aligned}$$

such that $\phi^*\Pi_\epsilon = \sum_{ij} \lambda'_{ij}(\epsilon) x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, for some $\lambda_{ij}(\epsilon) \in \mathbb{C}$, sufficiently near from λ_{ij}. We can see $\lambda_{ij}(\epsilon)$ as holomorphic functions $\lambda_{ij} : \mathbb{C} \to \mathbb{C}$ with $\lambda_{ij}(0) = \lambda_{ij}$.

Since $G = Aut(\mathbb{P}^{2n}) = \mathbb{P}SL(2n+1, \mathbb{C})$ is a connected Lie group and $H^0(\mathbb{P}^{2n}, T\mathbb{P}^{2n}) \simeq sl(2n+1, \mathbb{C})$, we can find, under natural identifications, a path $\alpha : [0,1] \to G$ which connects Id to ϕ and $\alpha(t+s) = \alpha(t) \circ \alpha(s)$. To see this, we use the well known fact that for every $A \in G$, there exists an element $B \in sl(2n+1, \mathbb{C})$ such that $A = \exp B$ and we just need to take $\alpha(t) = \exp tB$.

Consider $\Pi'_\epsilon = \sum_{ij} \lambda_{ij}(\epsilon) x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ and let $\alpha : [0,1] \to \bigwedge^2 T\mathbb{P}^{2n}$ be the path defined by $\alpha(t) = (1-t)\Pi + t\Pi'_\epsilon$ that connects Π to Π'_ϵ, i.e., we have that $\Pi_t = \Pi + t(\Pi'_\epsilon - \Pi)$. We see that Π_t is a Poisson structure for every t and $\Pi_1 = \Pi'_\epsilon$.

Consider the flow ϕ_s that connects Id to ϕ (note that $\phi_* \Pi'_\epsilon = \Pi_\epsilon$). So,

$$\lim_{s \to 0} \frac{\phi_{s*}\Pi'_\epsilon - \Pi'_\epsilon}{s} = \mathcal{L}_X \Pi' = [\Pi'_\epsilon, X]$$

where X is the vector field associated to the flow ϕ_s. Taking s and ϵ so small, we can suppose that $s = \epsilon + h.o.t$ and we can write, infinitesimaly,

$$\Pi_\epsilon - \Pi'_\epsilon = \epsilon[\Pi'_\epsilon, X] + h.o.t.$$

Expanding $\lambda_{ij}(\epsilon)$ in Taylor series, with $\lambda_{ij}(0) = \lambda_{ij}$, we have that $\Pi'_\epsilon = \Pi + \epsilon(\Pi'_\epsilon - \Pi) = \Pi + \epsilon \sum_{ij} c_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + h.o.t$, and so

$$\Pi_\epsilon = \Pi + \epsilon([\Pi, X] + \Pi'') + h.o.t.,$$

where $\Pi'' = \sum_{ij} c_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$.

A simple computation shows that Π'' is not 0 in $H^2_\Pi(\mathbb{P}^{2n})$ (we did this calculation in section 4).

So, by the interpretation of the cohomology, we conclude that we have a natural identification of $H^2_\Pi(\mathbb{P}^{2n})$ with the set of diagonal structures with the same singular locus as Π. This proves that $\dim H^2_\Pi(\mathbb{P}^{2n}) = n(2n-1)$.

We also note that if Λ_Π is the irreducible component of the space of Poisson structure containing Π, then

$$\begin{aligned} \dim \Lambda_\Pi &= \dim H^2_\Pi(\mathbb{P}^{2n}) + \dim sl(2n+1, \mathbb{C}) - 1 \\ &= n(2n-1) + (2n+1)^2 - 2 \\ &= 6n^2 + 3n - 1. \end{aligned}$$

4. Theorem C

In this section, we work on a specific example of rank 2 Poisson structure on $\mathbb{P}^n$ and we prove that it is stable under deformation.

EXAMPLE 4.6. Consider the projection from the last coordinate $\phi : \mathbb{P}^{n+1} \to \mathbb{P}^n$ and let $\tilde{X} \in H^0(\mathbb{P}^n, T\mathbb{P}^n(1))$ be a quadratic vector field in $\mathbb{P}^n$. Thinking of $\tilde{X}$ as a foliation of dimension 1, we may pull it back by ϕ, defining a foliation $\mathcal{F}$ of dimension 2 which is a Poisson structure. In the chart $x_0 = 1$, the vector field will be denoted by X and the Poisson structure will be $\Pi = \frac{\partial}{\partial x_{n+1}} \wedge X$. Note that $T\mathcal{F} = \mathcal{O}_{\mathbb{P}^{n+1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(1)$, i.e., the tangent sheaf of the Poisson foliation totally splits.

The Theorem we prove is:

THEOREM C. For a very generic quadratic vector field X in $\mathbb{P}^n$, consider the Poisson structure $\Pi = \frac{\partial}{\partial x_{n+1}} \wedge X$ on $\mathbb{P}^{n+1}$ and let Π_ϵ be a small deformation of Π. Then $\operatorname{rk} \Pi_\epsilon = 2$ and there exist coordinates $(x'_1, \ldots, x'_{n+1})$ and a quadratic vector field X_ϵ such that $\Pi_\epsilon = \frac{\partial}{\partial x'_{n+1}} \wedge X_\epsilon$, where X_ϵ does not depend on $\frac{\partial}{\partial x'_{n+1}}$ nor x'_{n+1}.

This Theorem is not trivial, because the rank of a small deformation of Poisson structure can grow up as the following example shows.

EXAMPLE 4.7. In $\mathbb{P}^4$ and in the coordinates $X_0 = 1$, consider the rank 2 diagonal Poisson structure $\Pi = x_1 x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$. We see that for any ϵ small enough, the deformation $\Pi_\epsilon = x_1 x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \epsilon x_3 x_4 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}$ is a small deformation of the Poisson structure Π and rank of Π is 4.

We will explain what properties of the quadratic vector field X we will ask.

DEFINITION 4.8. Let X be a holomorphic vector field in $(\mathbb{C}^n, 0)$ with isolated singularity at the origin. We say that X is in the Poincaré's domain if the eigenvalues λ_i of the linear part of X at 0 are non-resonant and 0 is not in the convex hull of $(\lambda_0, \ldots, \lambda_n)$.

The Theorem 3.5 of [**19**] implies the following:

THEOREM 4.9. *A very generic homogeneous quadratic vector field has only isolated singularities, one of the singularities satisfies the hypothesis of the Poincaré's Linearization theorem and the integral curves of the vector fields* $x_i \frac{\partial}{\partial x_i}$ *are Zariski dense in* $\mathbb{P}^n$.

The quadratic vector field of Theorem C has the properties of the Theorem 4.9. Before starting the proof the Theorem C, we state the well-known Poincarè's Linearization theorem.

THEOREM 4.10. *Let X be a holomorphic vector field in $(\mathbb{C}^n, 0)$ with isolated singularity at the origin. If X is in the Poincaré's domain, then there exists a holomorphic change of coordinates $(x_1, \dots, x_n)$ such that $X = \sum_i \lambda_i x_i \frac{\partial}{\partial x_i}$.*

PROOF OF THEOREM C. Let X be a quadratic vector field on $\mathbb{P}^n$ with the hypothesis of the Theorem 4.9. Let $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ be the projection of the last coordinate and Π the Poisson structure induced by X and the projection.

We will work in the chart $X_0 \neq 0$, $x_i = X_i/X_0$, where $(X_0 : \dots :, X_n)$ are the homogeneous coordinates of $\mathbb{P}^{n+1}$. Note that X is a cubic vector field in $\mathbb{C}^n$.

If $\xi \in T_\Pi \Lambda$ then $[\Pi, \xi] = 0$. We want to prove that $\Pi \wedge \xi = 0$, i.e., $T_\Pi \Lambda = T_\Pi S_2$ (see section 1 for the definition).

Writting $\xi = \alpha + \frac{\partial}{\partial x_{n+1}} \wedge \beta$ where α and β do not depend on $\frac{\partial}{\partial x_{n+1}}$, α is a polynomial bivector field with degree at most three, and β is a polynomial vector field with degree at most three. We compute $[\Pi, \xi] = 0$, using the graded Leibniz rule of the Schouten bracket:

$$\begin{aligned}
0 = [\Pi, \xi] &= \left[\frac{\partial}{\partial x_{n+1}} \wedge X, \xi\right] \\
&= \frac{\partial}{\partial x_{n+1}} \wedge [X, \xi] - \frac{\partial \xi}{\partial x_{n+1}} \wedge X \\
&= \frac{\partial}{\partial x_{n+1}} \wedge [X, \alpha] - \frac{\partial \xi}{\partial x_{n+1}} \\
&= \frac{\partial}{\partial x_{n+1}} \wedge [X, \alpha] - \frac{\partial \alpha}{\partial x_{n+1}} \wedge X - \frac{\partial}{\partial x_{n+1}} \wedge \frac{\partial \beta}{\partial x_{n+1}} \wedge X.
\end{aligned}$$

So, we have that $\frac{\partial \alpha}{\partial x_{n+1}} \wedge X = 0$. Recall the de Rham lemma

LEMMA 4.11. *Let X be a polynomial vector field in $\mathbb{C}^n$ and suppose that $\operatorname{cod} \operatorname{Sing} X \geq 3$, then for every polynomial k-vector field ξ such that $\xi \wedge X = 0$, there exists a $(k-1)$-vector field Y with $\xi = X \wedge Y$.*

Since $\frac{\partial \alpha}{\partial x_{n+1}} \wedge X = 0$, by the de Rham lemma 4.11, we conclude that $\frac{\partial \alpha}{\partial x_{n+1}} = X \wedge Y$, for some polynomial vector field Y. Comparing the degrees of the polynomials, we conclude that $\frac{\partial \alpha}{\partial x_{n+1}} = 0$, and α does not depend on x_{n+1}. Looking at the terms with $\frac{\partial}{\partial x_{n+1}}$ in the expression above, we have

$$[X, \alpha] = \frac{\partial \beta}{\partial x_{n+1}} \wedge X.$$

In particular, we have $[X, \alpha] \wedge X = 0$ and since α does not depend on $\frac{\partial}{\partial x_{n+1}}$ nor x_{n+1}, we can think of α as a bivector field in $\mathbb{C}^n$. The key fact is:

PROPOSITION 4.12. *If X satisfies the hypothesis of Theorem 4.9, then, in notation above, there exists a homogeneous vector field Y such that $\alpha = X \wedge Y$.*

PROOF. By Poincarè's Linearization theorem, write $X = \sum_i \lambda_i x_i \frac{\partial}{\partial x_i}$ in a neighborhood of the singularity. Consider the linear map $\Delta : \mathfrak{X}^2 \to \mathfrak{X}^2$ which sends α to $[X, \alpha]$. For every multi-index I, we have

$$\left[X, X^I \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\right] = (< \lambda, I > -\lambda_i - \lambda_j) X^I \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

Since λ_i are non-resonant, we conclude that $ker(\Delta)$ is generated by $x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, as a $\mathbb{C}$-vector space. Moreover, $ker(\Delta) \cap im(\Delta) = 0$ and $ker(\Delta) \oplus im(\Delta) = \mathfrak{X}^2$.

Since $[X, \alpha] \wedge X = 0$, by the de Rham Lemma 4.11, there exists a vector field Z such that $[X, \alpha] = Z \wedge X$. Since $\sum_{i,j} a_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ is not in the image of Δ, we can choose Z such that the diagonal of the linear part is identically zero, i.e., Z does not have the terms $x_i \frac{\partial}{\partial x_i}$, for all i.

Consider the application $\delta : \mathfrak{X}^1 \to \mathfrak{X}^1$ which sends Y to $[X, Y]$. In the same way, we can prove that $ker\, \delta$ is generated by the diagonal $x_i \frac{\partial}{\partial x_i}$, $im\, \delta \cap ker\, \delta = 0$ and $im\, \delta \oplus ker\, \delta = \mathfrak{X}^1$. So there exists $Y \in \mathfrak{X}^1$ such that $[X, Y] = Z$. Then,

$$[X, Y \wedge X] = [X, Y] \wedge X = Z \wedge X = [X, \alpha].$$

Since $ker(\Delta)$ is generated by $x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, we have

$$\alpha = Y \wedge X + \sum_{ij} a_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},\ a_{ij} \in \mathbb{C},$$

and

$$\alpha \wedge X = \sum_{ij} a_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge X.$$

Suppose for example $a_{12} \neq 0$, then the integral curve C defined by the vector field $x_1 \frac{\partial}{\partial x_1}$ is tangent to the foliation defined by the bivector field $\sum_{ij} a_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. In particular, for every point $p \in C$ we have that $\sum_{ij} a_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}(p) \wedge X(p) = 0$. Since the integral curve defined by the vector field $x_1 \frac{\partial}{\partial x_1}$ is Zariski dense in $\mathbb{P}^n$, we conclude that $\alpha \wedge X = 0$. We finish the proof of the proposition by applying de Rham lemma 4.11. □

So, if $\xi \in T_{\Pi}\Lambda$, then $\xi = X \wedge Y + \frac{\partial}{\partial x_{n+1}} \wedge \beta$ for some Y and β vector fields. In particular, $\xi \wedge \Pi = 0$, i.e., $\xi \in S_2$. So we proved that every Poisson deformation Π_ϵ and Π have the same rank. Lef $\mathcal{F}$ be the Poisson foliation induced by Π, then the Poisson foliation $\mathcal{F}_{\Pi_\epsilon}$ induced by Π_ϵ has dimension 2. In particular $\mathcal{F}_{\Pi_\epsilon}$ is a small deformation of the foliation $\mathcal{F}$. In [**8**], Cukierman and Pereira proved the following theorem.

THEOREM 4.13. *Let $\mathcal{F}$ be a codimension $q \geq 2$ foliation such that its tangent sheaf totally split, if $codSing(\mathcal{F}) \geq 3$, then there exists a Zariski open set U of the space of foliations, containing $\mathcal{F}$ such that for every $\mathcal{G} \in U$, $T\mathcal{G} \cong T\mathcal{F}$.*

Since the tangent sheaf $T\mathcal{F}_{\Pi} = \mathcal{O}_{\mathbb{P}^{n+1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(-1)$ splits, by Theorem 4.13, we have $T\mathcal{F}_{\Pi_\epsilon} = \mathcal{O}_{\mathbb{P}^{n+1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(-1)$. So the Poisson foliation is a direct sum of two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. The first one has degree 0 and the second one has degree 2.

We can choose homogeneous coordinates $(y_0 : \ldots : y_{n+1})$ of $\mathbb{P}^{n+1}$ such that $\mathcal{F}_1$ is given by $\frac{\partial}{\partial y_{n+1}}$. If $\mathcal{F}_2$ is given by quadratic vector field Y, we can choose Y in such way that Y does not depend on $\frac{\partial}{\partial y_{n+1}}$. The integrability condition shows that Y does not depend on y_{n+1} either.

So Y can be regarded as a quadratic vector field on $\mathbb{P}^n$, with coordinates $(y_0 : \ldots : y_n)$. This proves the Theorem C. □

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