ON THE AXIOM OF SEPARABILITY

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ABSTRACT. This work studies axiomatic systems for additive and separable preference structures on a product set, and explores the roles of different separability conditions in their corresponding utility representations. We show that the Thomsen condition is a necessary and sufficient condition for an independent preference structure is additive on a general two-dimensional domain. We develop a theorem to identify a large class of separable preference relations that can also admit additive utility representations, namely, a separable preference structure on a two-dimensional domain is additive if and only if its separability rule can represent an additive preference relation on the plane.

1. INTRODUCTION

Consider a general domain Σ, on which a rational preference relation can be defined. Suppose Σ can be formalized as a product set, namely, $\Sigma = \prod_{i=1}^{n} X_i$, where n is a positive integer. We plan to study rational preferences on such a product set, especially the property of separability in their utility measurement.

Examples. Let $\Sigma = X_1 \times X_2 \times \cdots \times X_n$, where $X_i \subseteq \mathbf{R}$ for all i, and $n \in \mathbf{Z}^+$. Σ can be a set of bundles of n different commodities, or a set of n independent stimulus variables affecting a certain attribute of an observed object.

Let $\Sigma = X \times X$, where X is a set of sure prospects, each $(x_1, x_2) \in \Sigma$ can represent an uncertain prospect of being x_1 and x_2 with equal probabilities.

Let $\Sigma = X \times T$, where X is a set of events, and T is a time domain. Any $(x, t) \in \Sigma$ is a realization x at the time $t \in T$.

Let $\Sigma = X \times Q$, where $X \subseteq \mathbf{R}^+$, and $Q \subseteq [0, 1]$. Each $(x, q) \in \Sigma$ can represent such a gambling, in which a gambler wins x and loses $\sqrt{x}$ with probability q and $1-q$, respectively, where $q \in Q$.

A utility function $u : \Sigma \to \mathbf{R}$ is called an *additive conjoint representation* of a preference relation on $\Sigma = \prod_{i=1}^{n} X_i$, if

$$u(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} u_i(x_i)$$

for all $(x_1, x_2, \ldots, x_n) \in \Sigma$, where $u_i : X_i \to \mathbf{R}$ for all i. A preference relation that admits an additive conjoint representation is called *additively separable* or *additive*.

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More generally, a utility function $u : \Sigma \to \mathbf{R}$ is called a *separable representation* of a preference relation on $\Sigma = \prod_{i=1}^{n} X_i$, if there exists a function $v : \mathbf{R}^n \to \mathbf{R}$, such that

$$u(x_1, x_2, \ldots, x_n) = v\big(u_1(x_1), u_2(x_2), \ldots, u_n(x_n)\big)$$

for all $(x_1, x_2, \ldots, x_n) \in \Sigma$, where $u_i : X_i \to \mathbf{R}$ for all i. The aggregation operator v in a separable utility function is called a *separability rule*. As a special case, the summation operator in an additive utility function is an additive separability rule. A preference relation that has a separable representation is called *generally separable* or *separable*.

The structure of our study is as follows. Section 2 connects the existing studies and our proposed study. We formally define the rational preference, the independent preference, the solvable preference, the Thomsen condition, the double cancellation condition, and the stationary (time) preference. We construct four axiomatic systems based on the solvability, the Thomsen condition, the double cancellation condition, and the stationarity, all of which can sufficiently generate additive preference structures. Among them, the system with the Thomsen condition is the weakest one, while the system with the stationary condition is the strongest, so it is limited to a special case on time preference. In Section 3, we study the Thomsen condition on the plane with the assistance of the web theory. We state a more intuitive definition of the Thomsen condition based on a transformation operator in a web-covered plane. We show a critical lemma to relate equivalent properties of two different mathematical objects, namely, a preference relation satisfies the Thomsen condition if and only if its related 3-web on the plane is hexagonal. With this lemma, we prove, in Section 4, that the Thomsen condition is also a necessary condition for the additivity of a preference structure on a general domain. Finally, we state a theorem that can generate an identification rule used to check whether a separable preference structure is actually additive.

2. Axiomatic Systems

In this section, we will summarize some developed axiomatic systems that can sufficiently ensure separable preference structures. Before we state these results, we have to make some formal definitions.

Let $\succsim$ be a binary relation on the product set $\Sigma = \prod_{i=1}^{n} X_i$. $\succsim$ is said to be

(i) *reflective*, if $\sigma \succsim \sigma$ for all $\sigma \in \Sigma$,
(ii) *transitive*, if $\sigma_1 \succsim \sigma_2$ and $\sigma_2 \succsim \sigma_3$ imply $\sigma_1 \succsim \sigma_3$ for all $\sigma_1, \sigma_2, \sigma_3 \in \Sigma$,
(iii) *antisymmetric*, if $\sigma_1 \succsim \sigma_2$ and $\sigma_2 \succsim \sigma_1$ imply $\sigma_1 = \sigma_2$ for all $\sigma_1, \sigma_2 \in \Sigma$,
(iv) *complete*, if either $\sigma_1 \succsim \sigma_2$ or $\sigma_2 \succsim \sigma_1$ for all $\sigma_1, \sigma_2 \in \Sigma$.

A binary relation is called a *partial order*, if it is reflective, transitive, and antisymmetric. If a partial order is also complete, it is said to be *simple*. A *preference relation* on a domain is essentially a binary relation $\succsim$ on that domain.

Definition. A preference relation $\succsim$ on a domain Σ is *rational*, if it is a simply partial order on Σ.

Recall that $\mathbf{R}$ is a poset embedded with the simply partial order $\geq$, and most of rational preference relations $\succsim$ on a domain Σ can be represented by $\geq$ on $\mathbf{R}$. Moreover, we have the following statement. If Σ is countable or Σ is uncountable but $\succsim$ is dense on it, there exists an isomorphism $u : \Sigma \to \mathbf{R}$ such that $\succsim$ on Σ can be preserved by $\geq$ on $\mathbf{R}$, or in terms of utility theory, there exists an equivalent utility function $u : \Sigma \to \mathbf{R}$, such that $\sigma_1 \succsim \sigma_2$ if and only if $u(\sigma_1) \geq u(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \Sigma$.

Suppose $\succsim$ is a rational preference on $\Sigma = \prod_{i=1}^n X_i$. If $\sigma_1 \succsim \sigma_2$ and $\sigma_2 \succsim \sigma_1$ at the same time, we write $\sigma_1 \sim \sigma_2$, where $\sim$ is called the indifference relation corresponding to $\succsim$. Let the rational preference on X_i induced from $\succsim$ on Σ be $\succsim_i$, and the rational preference on $\prod_{j \neq i} X_j$ induced from $\succsim$ on Σ be $\succsim_{-i}$ for all $i \in \{1, 2, \ldots, n\}$. Suppose $\sigma = (x_1, x_2, \ldots, x_n)$ for all $\sigma \in \Sigma$. We can also write $\sigma = (x_i, x_{-i})$ for all $i \in \{1, 2, \ldots, n\}$, where x_{-i} denotes the tuple of all the elements in σ except x_i.

Definition. A preference relation $\succsim$ on Σ is *independent*, if $(x_i, x_{-i}) \succsim (y_i, x_{-i})$ is equivalent to $x_i \succsim_i y_i$ for all $i \in \{1, 2, \ldots, n\}$.

Definition. A preference relation $\succsim$ on Σ is *solvable*, if $(x_i, x_{-i}) \succsim \sigma \succsim (y_i, x_{-i})$, then there exists a $z_i \in X_i$ such that $(z_i, x_{-i}) \sim \sigma$ for all $i \in \{1, 2, \ldots, n\}$.

Then we can establish a somehow restricted axiomatic system for an additive preference structure,

Theorem 1. If a preference relation $\succsim$ is rational, independent, and solvable on $\Sigma = \prod_{i=1}^n X_i$, then it is additive, and admits an additive conjoint representation.

Sketch of Proof. By the independence condition, $(x_i, x_{-i}) \succsim (y_i, x_{-i})$ if and only if $x_i \succsim_i y_i$ for all i. Since $\succsim$ is solvable on Σ, $\succsim_i$ is dense on X_i. Thus there exists a utility function $u_i : X_i \to \mathbf{R}$ representing $\succsim_i$. By the solvability condition, each x_i is essential, so for all $\sigma \in \Sigma$, we can have $(x_i, x_{-i}) \succsim \sigma \succsim (x_i, y_{-i})$. Then we can find a z_{-i} such that $\sigma \sim (x_i, z_{-i})$ by applying the solvability condition $(n-1)$ times. So there also exists a function $u_{-i} : \prod_{j \neq i} X_j \to \mathbf{R}$ representing $\succsim_{-i}$. We obtain $2n$ functions, say, $u_1, u_2, \ldots, u_n$ and $u_{-1}, u_{-2}, \ldots, u_{-n}$. Note that u_{-i} must be comprised of all the u_j's for $j \neq i$, otherwise, some x_j's would not be essential. The utility function $u : \Sigma \to \mathbf{R}$ representing $\succsim$ is thus determined by n functions, say, $u_1, u_2, \ldots, u_n$. Notice that the role of u_i in u_{-j} and that of u_j in u_{-i} is inverse to each other for all $i \neq j$, but the function form of u is constant, thus we must have $u_i + u_j$ in u. Therefore, we have $u = \sum_{i=1}^n u_i$, which completes the proof. $\square$

Note that the condition of solvability is a very strong axiom of separability, as any factor can be separated from the other $(n-1)$ factors. Debreu (1960) presented an axiomatic system with a weaker separability condition for preference relations on a two-dimensional domain. To follow this tradition and also to simplify our analysis, we will focus on two-dimensional domains in the remaining parts of this work.

Let $\Sigma = X_1 \times X_2$, where X_1 and X_2 are general topologically connected spaces, (as assumed by Debreu, 1960).

Definition. A preference relation $\succsim$ on $\Sigma = X_1 \times X_2$ satisfies the *Thomsen condition*, if $(x_1, y_2) \sim (y_1, x_2)$ and $(y_1, z_2) \sim (z_1, y_2)$ imply $(x_1, z_2) \sim (z_1, x_2)$ for all $x_1, y_1, z_1 \in X_1$ and $x_2, y_2, z_2 \in X_2$.

If a preference relation $\succsim$ on $X_1 \times X_2$ is rational, independent, and satisfies the Thomsen condition, then $\succsim$ on $X_1 \times X_2$ has an additive structure, and can be represented by an additive utility function. Fishburn (1970) generalized the two-dimensional domains in Debreu's axiomatic system to domains with finitely many factors, say, $\Sigma = \prod_{i=1}^n X_i$, in which the general Thomsen condition can be defined through the relation between any pair of factors, (see Theorem 5.5 in Fishburn, 1970, pp. 71–76). As a graphic illustration, Figure 1 shows a preference relation satisfying the Thomsen condition on $\mathbf{R}^2$.

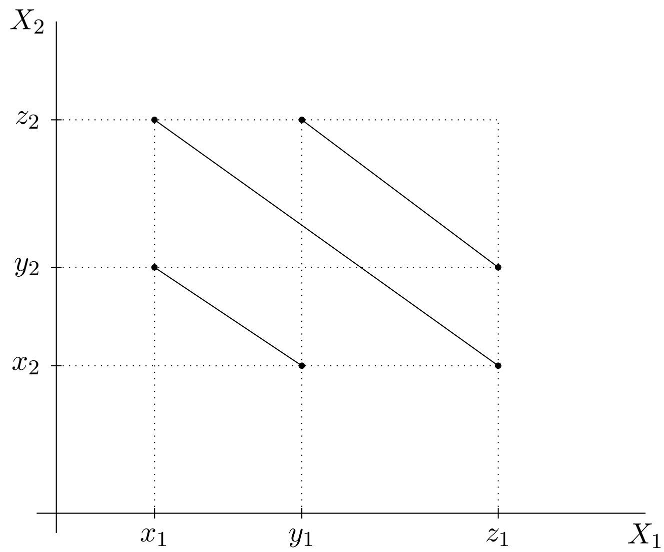

FIGURE 1. The Thomsen condition on the plane.

Definition. A preference relation $\succsim$ on $\Sigma = X_1 \times X_2$ satisfies the *double cancellation condition*, if $(x_1, y_2) \succsim (y_1, x_2)$ and $(y_1, z_2) \succsim (z_1, y_2)$ imply $(x_1, z_2) \succsim (z_1, x_2)$ for all $x_1, y_1, z_1 \in X_1$ and $x_2, y_2, z_2 \in X_2$.

Luce and Tukey (1964) proposed a similar axiomatic system to Debreu's by replacing the Thomsen condition with the double cancellation condition. The double cancellation condition is slightly stronger than the Thomsen condition, but again weaker than the solvability condition. So the range of preference relations that can be captured by the axiomatic system in Theorem 1 is smaller than the axiomatic systems proposed by Luce and Tukey, and even much smaller than the axiomatic system proposed by Debreu.

As a special case of Debreu's axiomatic system, Fishburn and Rubinstein (1982) stated an axiomatic system on the domain $X \times T$, where $X \subseteq \mathbf{R}$, and T is the time domain that is either discrete as $\mathbf{Z}$ or dense on $\mathbf{R}$. According to Debreu's theorem, if the time preference $\succsim$ on $X \times T$ is rational, independent, and satisfies the Thomsen condition, there should exist an additive conjoint representation $\mu : X \times T \to \mathbf{R}$,

$$\mu(x, t) = \nu(x) + \varrho(t)$$

for all $(x,t) \in X \times T$, where $\nu : X \to \mathbf{R}$, and $\varrho : T \to \mathbf{R}$.

Recall that an ordinal utility representation is invariant to any positive monotonic transformation, so we can have another utility representation $u = \exp(\mu)$, or equivalently,

$$u(x,t) = \rho(t)v(x)$$

for all $(x,t) \in X \times T$, where $\rho = \exp(\varrho)$ and $v = \exp(\nu)$.

Definition. A time preference $\succsim$ on $X \times T$ is *stationary*, if $(x,t) \sim (y,s)$ implies $(x, t+\tau) \sim (y, s+\tau)$ for all $x, y \in X$ and $t, s, t+\tau, s+\tau \in T$.

Notice that the stationarity condition means the induced preference $\succsim_t$ on the time dimension, (as a naive time preference), is linear and independent with the x dimension, so it is much stronger than the solvability condition.

If the time preference $\succsim$ on $X \times T$ is rational, independent, and stationary, the additive utility representation will be quasi-linear, namely, $\varrho(t) = \alpha t$, where $\alpha < 0$. Thus we have $\mu(x,t) = \nu(x) + \alpha t$. Let $\beta = e^{\alpha}$, so $\beta \in (0,1)$. Note that $\rho(t) = \exp(\alpha t) = \beta^t$, hence the exponentially scaled utility representation $u(x,t)$ becomes

$$u(x,t) = \beta^t v(x)$$

for all $(x,t) \in X \times T$, where β is the discounting rate to measure the impatience of the time preference.

To sum up, we state four axiomatic systems that are sufficient for the additivity of a preference relation on product sets, namely, the stationarity condition, the solvability condition, the double cancellation condition, and the Thomsen condition. Among them, the Thomsen condition is the weakest one, while the stationarity condition is the strongest one. Once we plan to study the necessary condition for the additivity and separability of preference relations on a product set, we should first study the Thomsen condition. If the Thomsen condition is not a necessary condition for an additive or separable preference structure, then all the remaining stronger conditions could not be, either.

3. Thomsen Condition

In this section, we want to reconstruct the Thomsen condition by the theory of web geometry. We will again consider a two-dimensional domain $\Sigma = X_1 \times X_2$, where X_1 and X_2 are general metric spaces.

Assume there always exists a diffeomorphism $f : \Sigma \to \mathbf{R}^2$ transforming Σ to an affine domain $\Sigma' = X \times Y$ of the plane $\mathbf{R}^2$. Under the diffeomorphism f, any point $(x_1, x_2) \in \Sigma$ is mapped uniquely to a corresponding point $(x,y) = f(x_1,x_2)$ in Σ', and also any smooth curve in Σ is transformed into a corresponding smooth curve in Σ'. A preference relation $\succsim$ on Σ is again a preference relation on Σ', as $\sigma_1 \succsim \sigma_2$ if and only if $f(\sigma_1) \succsim f(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \Sigma$, where $f(\sigma_1), f(\sigma_2) \in \Sigma'$.[1] Therefore, the Thomsen condition is satisfied by $\succsim$

[1]It means that an agent's preference relation as her subjective knowledge over Σ is shaped after her knowledge on the domain Σ and a generic transformation rule f on such a domain. So her preference relation can be kept over the domain $\Sigma' = f(\Sigma)$.

on the domain Σ, if and only if it can be satisfied by the same preference relation on the affine domain Σ'.

In $\Sigma' = X \times Y$, through a given point (x, y) there exist infinitely many smooth curves, each of which can be determined by a mapping $g : \Sigma' \to \mathbf{R}$, such that $g(x, y)$ is a constant for all (x, y) on the same curve. Thus any mapping $g : \Sigma' \to \mathbf{R}$ actually determines a family of smooth curves on Σ'. We denote a regular family of smooth curves on Σ' by γ. Let the collection of all the regular families of smooth curves on Σ' be

$$\Gamma(\Sigma') = \{\gamma_i : i \in I\},$$

where I is an index set. Denote the mapping that determines the family $\gamma_i \in \Gamma(\Sigma')$ by $g_i : \Sigma' \to \mathbf{R}$ for all $i \in I$, thus we have an equivalent collection of mappings $\{g_i : i \in I\}$, and hence $\Gamma(\Sigma') \simeq \{g_i : i \in I\}$.

$\{\gamma_1, \gamma_2, \gamma_3\}$ is called a 3-*web* on Σ', if $\gamma_1, \gamma_2, \gamma_3 \in \Gamma(\Sigma')$. Since γ_i is solely determined by the mapping $g_i(x, y)$ for all $i \in I$, we can denote a 3-web $\{\gamma_1, \gamma_2, \gamma_3\}$ by $\{g_1, g_2, g_3\}$ equivalently.

Examples. Suppose $g_1(x, y) = x + y$, $g_2(x, y) = x - y$, and $g_3(x, y) = xy$, we get a 3-web $\{x + y, x - y, xy\}$ on $X \times Y$.

Suppose $g_i(x, y) = x + a_i y$, for all $i = 1, 2, 3$, where $a_1 \neq a_2 \neq a_3 \neq a_1$, we obtain a linear 3-web $\{x + a_1 y, x + a_2 y, x + a_3 y\}$ on $X \times Y$, in which all the curves are lines.

A 3-web $\{\gamma_1, \gamma_2, \gamma_3\}$ is *complete*, if (i) there exists only one curve in the family γ_i passing any point $\sigma \in \Sigma'$, (ii) any two different curves in the same family γ_i are disjoint, and (iii) any curve in a family γ_i has only one intersection with any curve in a different family γ_j, for $i \neq j$ and $i, j \in \{1, 2, 3\}$. Note that in a complete 3-web $\{\gamma_1, \gamma_2, \gamma_3\}$, there exist exactly three different curves passing a given point in Σ', say, $L_i \in \gamma_i$ for $i = 1, 2, 3$.

Suppose we have three curves $L_i \in \gamma_i$, where $i = 1, 2, 3$, in a complete 3-web $\{\gamma_1, \gamma_2, \gamma_3\}$, such that $L_1 \cap L_2 \cap L_3 = \{\sigma\}$. For any point $\sigma' \neq \sigma$ on the curve L_i, we define a rule,

$$p_{ij} : L_i \to L_j,$$

such that a curve in γ_k passes both σ' and $p_{ij}(\sigma')$, where $i, j, k \in \{1, 2, 3\}$ are different. Notice that $p_{ij} = p_{ji}^{-1}$, so $p_{ij} \circ p_{ji}$ is an identity function on L_i for all $j \neq i$. Although both $p_{ij} \circ p_{jk} \circ p_{ki}$ and $p_{ik} \circ p_{kj} \circ p_{ji}$ will return back to a point in L_i, they are not necessarily the same.

For example, we start from a point $\sigma_1 \in L_1$, we have $p_{12}(\sigma_1) = \sigma_2 \in L_2$, $p_{23}(\sigma_2) = \sigma_3 \in L_3$, $p_{31}(\sigma_3) = \sigma_4 \in L_1$, $p_{12}(\sigma_4) = \sigma_5 \in L_2$, $p_{23}(\sigma_5) = \sigma_6 \in L_3$, and $p_{31}(\sigma_6) = \sigma_7 \in L_1$. We call the region shaped by the points $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7$ a *sequentially constructed hexagon*. If $\sigma_7 = \sigma_1$, such a hexagon is said to be *closed*. Actually, we can see if $p_{12} \circ p_{23} \circ p_{31} = p_{13} \circ p_{32} \circ p_{21}$, then any sequentially constructed hexagon is closed. In Figure 2, we show a closed sequentially constructed hexagon on the plane embedded with a linear 3-web.

Definition. A 3-web on Σ' is called *hexagonal*, if any sequentially constructed hexagon in Σ' is closed.

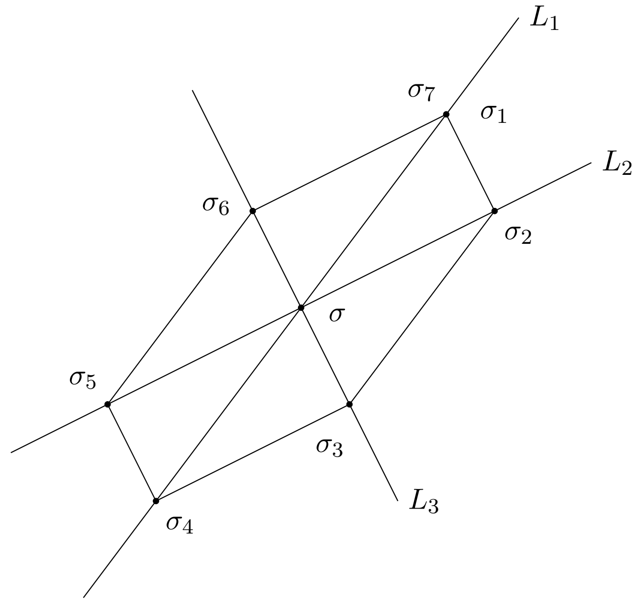

FIGURE 2. A sequentially constructed hexagon on the plane.

Thomsen (1927) proved that any planar 3-web is hexagonal if and only if it is equivalent to a linear 3-web on the plane. The Thomsen condition for a preference relation is proposed, because it is difficult to identify geometrically the equivalence condition between different 3-webs before we know exactly its utility representation.

Lemma 1. Assume the preference relation $\succsim$ on Σ' has a utility representation $u : \Sigma' \to \mathbf{R}$. $\succsim$ on Σ' satisfies the Thomsen condition, if and only if the 3-web $\{x, y, u(x, y)\}$ on Σ' is hexagonal.

Let $\{x, y, w(x, y)\}$ be a complete 3-web on $\Sigma' = X \times Y$, where $w : \Sigma' \to \mathbf{R}$. Define a transformation operator $\bowtie$, such that for all (x_1, y_1) and (x_2, y_2) in Σ', we have

$$(x_1, y_1) \bowtie (x_2, y_2) = (x_1, y_2),$$

where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Note that (x_1, y_2) and (x_1, y_1) are on a same curve $x = x_1$ in the family x, and (x_1, y_2) and (x_2, y_2) are on a same curve $y = y_2$ in the family y, so we have certain rules such that (x_1, y_1) on a curve in the family y moves to (x_1, y_2) on a curve in the family $w(x, y)$, and moves to (x_2, y_2) on a curve in the family x. Thus (x_1, y_1), (x_1, y_2), and (x_2, y_2) will be consecutive in a sequentially constructed hexagon on $\{x, y, w(x, y)\}$.

Proof of Lemma 1. We have three regular families of smooth curves $g_1(x, y) = x$, $g_2(x, y) = y$, and $g_3(x, y) = u(x, y)$, where x and y are coordinate bases of Σ', and $u(x, y)$ is determined by the preference relation $\succsim$ on Σ', thus $\{x, y, u(x, y)\}$ is a complete 3-web on Σ'.

Suppose the 3-web $\{x, y, u(x,y)\}$ on Σ' is hexagonal, and assume $(x_1, y_2) \sim (x_2, y_1)$ and $(x_2, y_3) \sim (x_3, y_2)$, so we need to prove $(x_1, y_3) \sim (x_3, y_1)$. Note that

$$(x_1, y_2) \bowtie (x_2, y_3) = (x_1, y_3) \text{ with } (x_2, y_3) \sim (x_3, y_2),$$

and also

$$(x_3, y_2) \bowtie (x_2, y_1) = (x_3, y_1) \text{ with } (x_2, y_1) \sim (x_1, y_2).$$

Thus the two sequences of consecutive points (x_1, y_2), (x_1, y_3), (x_2, y_3), and (x_3, y_2), as well as (x_3, y_2), (x_3, y_1), (x_2, y_1), and (x_1, y_2) are in certain sequentially constructed hexagons. Since $\{x, y, u(x,y)\}$ is hexagonal, those two sequences should form a closed hexagon in Σ'. Note that $(x_2, y_3) \sim (x_3, y_2)$ and $(x_2, y_1) \sim (x_1, y_2)$, so both pairs of points are on same curves in the family $u(x,y)$. Thus the remaining two points, say, (x_1, y_3) and (x_3, y_1), must be also on a same curve in $u(x,y)$, as any closed hexagon is shaped by 3 curves in x, 3 curves in y, and 3 curves in $u(x,y)$. Therefore, $(x_1, y_3) \sim (x_3, y_1)$, and the Thomsen condition is satisfied by $\gtrsim$ on Σ', which completes the proof of the sufficient part.

If $\gtrsim$ satisfies the Thomsen condition on Σ', we have

$$(x_1, y_2) \sim (x_2, y_1), \ (x_2, y_3) \sim (x_3, y_2), \text{ and } (x_1, y_3) \sim (x_3, y_1),$$

for all $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$. We again have

$$(x_1, y_2) \bowtie (x_2, y_3) = (x_1, y_3) \text{ and } (x_3, y_2) \bowtie (x_2, y_1) = (x_3, y_1).$$

So the sequence of (x_1, y_2), (x_1, y_3), and (x_2, y_3), and also the sequence of (x_3, y_2), (x_3, y_1), and (x_2, y_1) are on sequentially constructed hexagons.

Note that (x_2, y_1) and (x_3, y_1) are on a same curve in y, and (x_2, y_1) and (x_1, y_2) are on a same curve in $u(x,y)$, so (x_3, y_1), (x_2, y_1), and (x_1, y_2) are consecutive points in a sequentially constructed hexagon. Similarly, (x_2, y_3), (x_3, y_2), and (x_3, y_1) are also in a sequentially constructed hexagon. Thus the sequence of points

$$(x_1, y_2), \ (x_1, y_3), \ (x_2, y_3), \ (x_3, y_2), \ (x_3, y_1), \ (x_2, y_1), \text{ and } (x_1, y_2)$$

form a closed sequentially constructed hexagon. Since x_1, x_2, x_3 and y_1, y_2, y_3 are selected arbitrarily, the 3-web $\{x, y, u(x,y)\}$ must be hexagonal, which completes the proof of the necessary part. □

By using the transformation operator $\bowtie$ on Σ', we can have a more intuitive illustration of the Thomsen condition.

Let $\sigma_1 = (x_1, y_2)$, $\sigma_2 = (x_2, y_1)$, $\sigma_3 = (x_2, y_3)$, and $\sigma_4 = (x_3, y_2)$. So σ_1 and σ_4 are on the same curve $y = y_2$ in the family y, and σ_2 and σ_3 are on the same curve $x = x_2$ in the family x. A preference relation $\gtrsim$ on Σ' satisfies the Thomsen condition, if $\sigma_1 \sim \sigma_2$ and $\sigma_3 \sim \sigma_4$ imply

$$\sigma_1 \bowtie \sigma_3 \sim \sigma_4 \bowtie \sigma_2,$$

where $\sigma_1 \bowtie \sigma_3 = (x_1, y_3)$ and $\sigma_4 \bowtie \sigma_2 = (x_3, y_1)$. The graphic illustration is shown in Figure 3.

Note that

$$\sigma_3 \bowtie \sigma_1 = \sigma_2 \bowtie \sigma_4 = (x_2, y_2),$$

and also

$$\sigma_1 \bowtie \sigma_4 = \sigma_1,\ \sigma_3 \bowtie \sigma_2 = \sigma_2,\ \sigma_4 \bowtie \sigma_1 = \sigma_4,\ \text{and } \sigma_2 \bowtie \sigma_3 = \sigma_3.$$

Let $\delta \in \{\sigma_1, \sigma_2\}$, and $\delta' \in \{\sigma_1, \sigma_2\} \setminus \{\delta\}$. Let $\kappa \in \{\sigma_3, \sigma_4\}$, and $\kappa' \in \{\sigma_3, \sigma_4\} \setminus \{\kappa\}$. We have a more general invariant relation between those two indifference sets $\{\sigma_1, \sigma_2\}$ and $\{\sigma_3, \sigma_4\}$,

$$\delta \bowtie \kappa \sim \kappa' \bowtie \delta'$$

for all δ and κ, in which the specific implication of the Thomsen condition can be included.

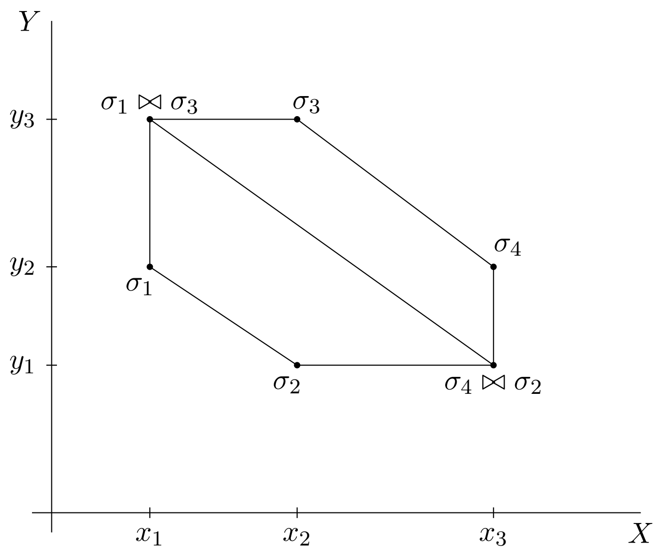

FIGURE 3. An alternative illustration of the Thomsen condition.

4. Necessary Condition

The main result of this section is the Thomsen condition on a general domain $\Sigma = X_1 \times X_2$, which is equivalent to $\Sigma' \subseteq \mathbf{R}^2$ through a diffeomorphism, is also a necessary condition for an additive preference structure. Thus a rational preference is additive on Σ can be fully captured by a pair of axioms, concretely, the independence condition, and the Thomsen condition.

Theorem 2. If a preference relation $\succsim$ on $\Sigma' = X \times Y$ is additive, then $\succsim$ must satisfy the Thomsen condition.

Note that any 3-web $\{\gamma_1, \gamma_2, \gamma_3\} \simeq \{x, y, u(x, y)\}$ on Σ' can represent a certain type of rational preferences on Σ', where $u : X \times Y \to \mathbf{R}$ can be seen as the utility function representing the family of indifference curves, say, γ_3. A 3-web $\{x, y, u(x, y)\}$ is hexagonal, if and only if its curvature is zero (see *e.g.*, Akivis and Goldberg, 2000, p. 207). Recall that the curvature of the 3-web $\{x, y, u(x, y)\}$ is

$$k(u) = -\frac{1}{u_x u_y} \frac{\partial^2}{\partial x \partial y} \log(u_x/u_y),$$

where $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$.

If $u(x, y)$ is additive, namely, $u(x, y) = u_1(x) + u_2(y)$, then

$$u_x/u_y = u_1'(x)/u_2'(y),$$

where $u_1' = du_1/dx$ and $u_2' = du_2/dy$, and thus its curvature $k(u) = 0$. However, if $u(x, y)$ is separable, namely, $u(x, y) = v(u_1(x), u_2(y))$, where $v : \mathbf{R}^2 \to \mathbf{R}$, $k(u)$ will not be necessarily zero. For instance,

$$u(x, y) = u_1(x) + \log\big(u_1(x) + u_2(y)\big),$$

in which $u_x/u_y = (u_1 + u_2 + 1)u_1'/u_2'$, and its curvature

$$k(u) = \frac{(u_1 + u_2)^2}{(u_1 + u_2 + 1)^3} \neq 0.$$

Proof of Theorem 2. By the definition of additive preference relation, $\succsim$ on $\Sigma' = X \times Y$ admits at least one additive conjoint representation. Let $u(x, y) = u_1(x) + u_2(y)$ be such a utility representation of $\succsim$, so $\{x, y, u_1(x) + u_2(y)\}$ is a 3-web on Σ'. Define a mapping g that maps $(x, y) \in \Sigma'$ to $(u_1(x), u_2(y))$. Note that g is bijective. Let the image of g be $\Sigma^* = Z_1 \times Z_2$. We have an equivalent 3-web $\{z_1, z_2, z_1 + z_2\}$ on $Z_1 \times Z_2$. The linear 3-web $\{z_1, z_2, z_1 + z_2\}$ is hexagonal, so $\{x, y, u_1(x) + u_2(y)\}$ is also hexagonal on Σ'. By Lemma 1, $\succsim$ satisfies the Thomsen condition on Σ', which completes our proof. □

Corollary 1. If a preference relation $\succsim$ is additive on Σ, and Σ can be transformed to an affine domain $\Sigma' \subseteq \mathbf{R}^2$ by a diffeomorphism, then $\succsim$ satisfies the Thomsen condition.

Proof. Suppose there exists a diffeomorphism $f : \Sigma \to \mathbf{R}^2$, which transforms Σ to $\Sigma' = X \times Y$. The preference relation $\succsim$ will be again additive on Σ'. By Theorem 2, $\succsim$ satisfies the Thomsen condition on Σ', thus $\succsim$ also satisfies the Thomsen condition on Σ, as f is bijective between Σ and Σ'. □

Therefore, given the condition that a rational preference relation $\succsim$ is independent on a general domain $\Sigma = X_1 \times X_2$, the Thomsen condition is a sufficient and also necessary condition for its additive structure. As we can see from the discussion above based on the curvature, the Thomsen condition on Σ' is too strong for its separability. However, there exists a relationship between the separability and the Thomsen condition. Concretely, we will show that a separable preference structure is additive if and only if its separability rule can represent an additive preference structure on a proper subset of $\mathbf{R}^2$.

Theorem 3. A separable preference relation $\succsim$ on $\Sigma = X_1 \times X_2$ with a separability rule $v : \mathbf{R}^2 \to \mathbf{R}$ is additive, if and only if v can represent a preference relation on $u_1(X_1) \times u_2(X_2)$ satisfying the Thomsen condition, where $u_i : X_i \to \mathbf{R}$ for $i = 1, 2$.

Proof. If $v : \mathbf{R}^2 \to \mathbf{R}$ can represent a preference relation $\succsim'$ on $\Sigma' = u_1(X_1) \times u_2(X_2)$ satisfying the Thomsen condition, $\succsim'$ must be additive on Σ'. So $\succsim'$ on Σ' admits an additive conjoint representation, say, $v_1(x)+v_2(y)$ for all $(x, y) \in \Sigma'$, where $v_i : u_i(X_i) \to \mathbf{R}$ for all $i = 1, 2$. There must exist a positively monotonic function $g : \mathbf{R} \to \mathbf{R}$, such that $v(x, y) = g(v_1(x) + v_2(y))$ can also represent $\succsim'$ on Σ' for all $(x, y) \in \Sigma'$.

Note that for all $x = u_1(x_1)$ and $y = u_2(x_2)$, we have the utility representation for $\succsim$,

$$g\big(v_1 \circ u_1(x_1) + v_2 \circ u_2(x_2)\big) = v(u_1(x_1), u_2(x_2))$$

for all $(x_1, x_2) \in \Sigma$. Scaling $v(u_1(x_1), u_2(x_2))$ by the positively monotonic function g^{-1}, we obtain another utility representation for $\succsim$,

$$u(x_1, x_2) = v_1 \circ u_1(x_1) + v_2 \circ u_2(x_2).$$

So $\succsim$ is actually additive on Σ.

On the other side, if $\succsim$ is separable on Σ with a separability rule v, then $\succsim$ should admit the specific utility representation $u(x_1, x_2) = v(u_1(x_1), u_2(x_2))$ for all $(x_1, x_2) \in X_1 \times X_2$. Suppose v is not additive on $\Sigma' = u_1(X_1) \times u_2(X_2)$, so it can not take any form $v_1(x) + v_2(y)$, up to any positively monotonic transformation, for all $(x, y) \in \Sigma'$. But then $\succsim$ on Σ can not admit any utility function $v_1 \circ u_1(x_1) + v_2 \circ u_2(x_2)$ up to any positively monotonic transformation, which directly means that $\succsim$ can not be additive, a contradiction. Therefore, v must be additive on $u_1(X_1) \times u_2(X_2)$. By Theorem 2, it should satisfy the Thomsen condition. □

Examples. Consider the Cobb-Douglas utility function $u(x_1, x_2) = x_1^\alpha x_2^\beta$ on the domain $\Sigma = \mathbf{R}^+ \times \mathbf{R}^+$, where $\alpha, \beta \in (0, 1)$. It can be expressed equivalently as

$$u(x_1, x_2) = \exp(\alpha \log x_1 + \beta \log x_2)$$

for all $(x_1, x_2) \in \Sigma$. Note that $\exp(x + y)$ is additive on $\Sigma' = \log(X_1) \times \log(X_2)$, as it is equivalent to $x + y$ on Σ' by the positively monotonic transformation log. Thus, the Cobb-Douglas utility function represents an additive preference relation on Σ. Evidently, we know it is equivalent to the utility function $\alpha \log x_1 + \beta \log x_2$.

Consider a utility function

$$u(x_1, x_2) = \min\{\alpha u_1(x_1), \beta u_2(x_2)\}$$

on a general domain $\Sigma = X_1 \times X_2$, where $\alpha, \beta \neq 0$, and $u_i : X_i \to \mathbf{R}$ for all $i = 1, 2$. It is a representation in the class of Leontief utility functions. We can observe that u is separable with a separability rule min. However, it is not additive, as $\min\{x, y\}$ can not be additive on $u_1(X_1) \times u_2(X_2)$.

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