Anti-Self-Dual Geometry, Twistor Spaces and Painlevé VI Equation

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Introduction

It is classically known in the 2-dimensional setting the correspondence that exists between conformal geometry and complex analysis on a surface, i.e. there is a well established 'dictionary' between both theories. In attempting a higher dimensional analogue, we consider over an oriented Riemannian 4-manifold M the bundle of compatible complex structures, the so called *Twistor Space* of M denoted by $\mathcal{Z}$. It is a 6-dimensional manifold which comes endowed with a canonical almost-complex structure. *Twistor Theory* concerns with transforming questions about the differential geometry of M into questions about the complex geometry of $\mathcal{Z}$. The fundamental example of this theory is given by the following theorem, due to Penrose:

Theorem 1 ([Pen], [AHS]) *Let M be an oriented Riemannian 4-manifold and $\mathcal{Z}$ its twistor space. Then the almost-complex structure on $\mathcal{Z}$ is integrable (i.e. $\mathcal{Z}$ becomes a complex manifold) if and only if the Weyl curvature tensor W of the metric on M is anti-self-dual with respect to the Hodge-star operator, i.e. $*W = -W$.*

Metrics satisfying the property above stated are called *Anti-Self-Dual metrics.* Extra information about the geometry of M provides us extra information about the holomorphic structure of $\mathcal{Z}$ and viceversa. The most important feature about the twistor space $\mathcal{Z}$ is that it supports a real analytic family of projective lines (the fibres of the projection $\mathcal{Z} \to M$), known as the *twistor lines.*

The link between 4-dimensional Riemannian geometry and 3-dimensional complex geometry given by the theorem above is called the *Penrose Transform.* It has several and remarkable applications. Recall that every finitely presentable group is the fundamental group of a compact smooth 4-manifold. Theorem 1, combined with a result of Taubes [Tau] about the existence of anti-self-dual metrics, has as consequence that such a group is also the fundamental group of a compact complex 3-manifold (a twistor space). There is no known proof of this fact using complex geometry techniques. Since the fundamental groups of complex algebraic manifolds satisfy highly non-trivial constraints, twistor spaces open up a new world of complex *transcendental* manifolds. In fact, a theorem of Hitchin [Hit2] asserts that the only compact anti-self-dual manifolds with Kählerian twistor spaces are the round sphere S^4 (in which $\mathcal{Z} = \mathbb{C}P^3$) and the reverse-oriented projective plane $\overline{\mathbb{C}P^2}$ with the Fubini-Study metric (where $\mathcal{Z}$ is given by the flag manifold F_3). On the other hand, in the context of Mathematical Physics, the solutions of many field equations on an anti-self-dual manifold, notably the *Yang-Mills equations*, can be translated into holomorphic data on the twistor space. Also, in Four-manifold Topology, the *Donaldson's polynomial invariants* of a 4-manifold M, in the anti-self-dual case, give no more information than is already contained in the holomorphic structure of $\mathcal{Z}$. Motivated by these variety of results involving anti-self-dual manifolds, we are interested in the following problem:

Problem 1 *To find explicit anti-self-dual metrics on 4-manifolds, i.e. to solve the anti-self-dual equation* $*W = -W$.

The results of Taubes and others about the existence of anti-self-dual metrics are based on deep results of the qualitative theory of non-linear elliptic PDEs, as the outcome of a large and very difficult theory. Thus, it will be of great interest finding anti-self-dual metrics by using alternative methods. When the geometric structure of M is invariant by a free action of the group SU_2, the lifted action of SU_2 on $\mathcal{Z}$ induces a family of meromorphic connections on the twistor lines [Hit1]. The connection varies as the lines vary, but the monodromy remains the same. A family of connections with constant monodromy is called an *isomonodromic deformation*. The differential equation that determines the behaviour of an isomonodromic deformation is the *Painlevé VI equation*:

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)\frac{dy}{dx} + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right),$$

where $\alpha, \beta, \gamma, \delta$ are parameters. It turns out that, by using twistor methods, we can go from complex analysis back to differential geometry and obtain explicit anti-self-dual metrics from explicit solutions of Painlevé VI equation. In analogy with the work of Hitchin [Hit1], in this work we obtain the following result:

Theorem 2 *For $\lambda \in \mathbb{R}\backslash\{0\}$ given, let y be a real solution of the Painlevé VI equation with parameters $(\alpha,\, \beta,\, \gamma,\, \delta) = \left(\frac{1}{2}(1-2\lambda)^2, -2\lambda^2\,, \, 2\lambda^2\,, \, \frac{1}{2}(1-4\lambda^2)\right)$, defined on the interval (a,b). Then, for $\{\mu_1, \mu_2, \mu_3\}$ an orthonormal basis of $\mathcal{SU}_2$, the metric*

$$g = \frac{(x-1)\, d\mu_1^2}{\lambda^2 + u_2 u_3 - \frac{y-x}{2x(y-1)}(\lambda^2 - u_2^2) - \frac{x(y-1)}{2(y-x)}(\lambda^2 - u_3^2)} - \frac{x\, d\mu_2^2}{\lambda^2 + u_1 u_3 + \frac{y-x}{2(x-1)y}(\lambda^2 - u_1^2) + \frac{(x-1)y}{2(y-x)}(\lambda^2 - u_3^2)} + \frac{d\mu_3^2}{\lambda^2 + u_1 u_2 - \frac{x(y-1)}{2(x-1)y}(\lambda^2 - u_1^2) - \frac{(x-1)y}{2x(y-1)}(\lambda^2 - u_2^2)} + \frac{dx^2}{2x(x-1)}\,,$$

is an ASD metric on $\mathrm{SU}_2 \times (a,b)$, where

$$u_1 = \frac{1}{4\lambda}\frac{x(x-1)^2}{(y-1)(y-x)}\left(\frac{dy}{dx}\right)^2 - \frac{x-1}{y-x}\left(\frac{1-2\lambda}{2\lambda}\, y - \frac{1}{2}x\right)\frac{dy}{dx} - \frac{1-2\lambda}{2}\,\frac{y(y-1)}{y-x} + \frac{(1-2\lambda)^2}{4\lambda}\,\frac{y^2(y-1)}{x(y-x)} - \lambda\,\frac{y-x}{x(y-1)}\,,$$

$$u_2 = -\frac{1}{4\lambda}\frac{x^2(x-1)}{y(y-x)}\left(\frac{dy}{dx}\right)^2 + \frac{x}{y-x}\left(\frac{1-2\lambda}{2\lambda}(y-1) - \frac{1}{2}(x-1)\right)\frac{dy}{dx} + \frac{1-2\lambda}{2}\,\frac{y(y-1)}{y-x} - \frac{(1-2\lambda)^2}{4\lambda}\,\frac{y(y-1)^2}{(x-1)(y-x)} - \lambda\,\frac{y-x}{(x-1)y}\,,$$

$$u_3 = \frac{1}{4\lambda}\frac{x(x-1)}{y(y-1)}\left(\frac{dy}{dx}\right)^2 - \frac{1-2\lambda}{2\lambda}\,\frac{dy}{dx} + \frac{(1-2\lambda)^2}{4\lambda}\frac{y(y-1)}{x(x-1)} - \lambda\frac{(y-x)^2}{x(x-1)y(y-1)} - \lambda\,.$$

Hitchin's list of Einstein anti-self-dual metrics founded in [Hit1] corresponds to solutions of the Painlevé VI equation above with $\lambda = \frac{1}{4}$. Thus, in the search of new explicit examples of anti-self dual metrics we must consider other values for λ. As part of this work we obtain a couple of new explicit examples:

Theorem 3 *The solutions of*

$$-y^2(2y-3) \pm 2y(y-1)\sqrt{y(y-1)} = x\,,$$

satisfies the Painlevé VI equation with parameters $(\alpha,\,\beta,\,\gamma,\,\delta) = \left(\frac{9}{32}, -\frac{1}{32}, \frac{1}{32}, \frac{15}{32}\right)$, *i.e.* $\lambda = \frac{1}{8}$. *They generate anti-self-dual metrics which are not Einstein.*

The first part of this work is devoted to the theory of connections over fiber bundles (the so called *Gauge Theory*) and to the self-duality phenomena that arises in this context when the base space is a Riemannian 4-manifold. We develop here all the rudiments necessaries for the study of twistor spaces to be realized in the subsequent chapters. Standard references for Gauge Theory are [KoNo] and [DFN]. Self-duality theory can be found in [DoKr] and [FrMo]. However, there is some full-detailed accounts here (which are useful in other parts of this work) that are not present in these references.

The second part deals with the theory of twistor spaces and their properties. Classical proofs of Theorem 1 involves rather sophisticated concepts and details, like projective spinors and group representations in tensorial algebras, to give an example. Thus, we consider that it is worth to provide a more elementary (though more extensive) proof. This is done here. This part concludes by describing a procedure for retrieve the anti-self-dual structure on a 4-manifold in terms of its twistor space. This procedure is known as the *Reverse Penrose Transform.* It is of fundamental importance for finding anti-self-dual metrics in the last part of this work.

The last part of this work establishes the link before mentioned between anti-self-dual metrics and solutions of Painlevé VI equation. The study of SU_2-invariant anti-self-dual metrics is developed here (based in the reference [Hit1]) and full details are given. This part finishes with the proof of Theorem 2 and Theorem 3.

Chapter 1

Gauge Theory and Self-Duality

1.1 Connections

1.1.1 Definitions and Fundamental Facts

Let G be a Lie group with identity element e and Lie algebra $\mathcal{G}$. Denote by Ad the *adjoint* representation of G in $\mathrm{Aut}\,\mathcal{G}$: $\mathrm{Ad}(g).X = g.X.g^{-1}$. Denote by ω_{MC} the *Maurer-Cartan form* of G, i.e. the only left-invariant 1-form on G with values in $\mathcal{G}$ such that $(\omega_{MC})_e$ is the identity linear map $\mathrm{Id} : T_eG \to \mathcal{G}$ (in other words, $(\omega_{MC})_g X = g^{-1}.X$). Let M be a smooth manifold and let $\pi : P \to M$ (or simply P) be a smooth principal G-bundle over M. As usual, T_pP denotes the tangent space of P at p, T^v_pP denotes the *vertical* tangent space of P at p (i.e. the tangent space of the fibre $\pi^{-1}(\pi(p))$ at the point p) and $\Omega^k_M(E)$ denotes the space of the k-forms on M with values in a vector bundle E (the case $k = 0$ is just the space of sections of E).

A *connection* A on P can be defined in three equivalent ways:

(1) As a distribution of 'horizontal' subspaces $\mathcal{H}_A \subset TP$ transverse to the fibres of π (i.e. for each $p \in P$ we have a decomposition $T_pP = T^v_pP \oplus (\mathcal{H}_A)_p$) which is preserved by the right-action of G on P.

(2) As a 'connection form' on P, i.e. a 1-form ω_A on P with values in $\mathcal{G}$ satisfying the following properties:

(a) For any $p \in P$ the embedding $L_p : G \to P$ given by $L_p(g) = p.g$ is such that $L_p^* \omega_A = \omega_{MC}$.

(b) The right-multiplication by G transforms the 1-form ω_A in the same way that it transforms the Maurer-Cartan form ω_{MC}, i.e. via the adjoint representation: $(R_g^* \omega_A)_p X = \mathrm{Ad}(g^{-1}).((\omega_A)_p X) = g^{-1}.(\omega_A)_p X.g$.

(3) As a covariant derivative ∇_A on a vector bundle $\pi_E : E \to M$ associated to P and to a linear (locally faithful) representation $G \xrightarrow{\rho} \mathrm{Aut}\, V$ for some vector space V. In other words, E is the vector bundle with fibre V whose transition functions are the same as those of P after composing with ρ and ∇_A is a linear map $\nabla_A : \Omega^0_M(E) \to \Omega^1_M(E)$ satisfying the Leibniz rule $\nabla_A(f.s) = f.\nabla_A s + df.s,\ \forall f \in C^\infty(M),\ s \in \Omega^0_M(E)$. This covariant derivative must be compatible with the induced 'G-structure' on E in the following sense: the action $G \times V \to V$ induces in a natural way an application $P \times V \to E$. By considering this application $P \times V \to E$ as an 'action' itself, the G-compatibility condition of ∇_A means that the induced 'orbits' $\mathcal{O}_v := \{p.v,\ p \in P\}$ must be preserved by the parallel transport of ∇_A.

We see the equivalence of these three viewpoints as follows:

To go from (1) to (2) we define $(\omega_A)_p : T_pP \to \mathcal{G}$ as the composition

$$T_pP \xrightarrow{\pi_v} T^v_pP \xrightarrow{(d(L_p)_e)^{-1}} \mathcal{G}\ ,$$

where the application π_v is the *vertical* projection corresponding to the decomposition $T_pP = T^v_pP \oplus (\mathcal{H}_A)_p$ of (1). The verification for ω_A of properties (a) and (b) of (2) is a straightforward computation.

To go from (2) to (1) we consider $\omega_A \in \Omega^1_P(\mathcal{G})$ as in (2) and for any $p \in P$ we define $(\mathcal{H}_A)_p = Ker(\omega_A)_p$. Since $L_p^*\omega_A = \omega_{MC}$, the restriction of $(\omega_A)_p$ to T^v_pP is an isomorphism on $\mathcal{G}$, therefore $T_pP = T^v_pP \oplus (\mathcal{H}_A)_p$. From the naturality of the pull-back it follows that R_g takes $Ker(\omega_A)_p$ on $Ker(R_g^*\omega_A)_{pg}$. On the other hand, property (b) gives us $Ker(R_g^*\omega_A)_{pg} = Ker(\mathrm{Ad}(g).(\omega_A)_{pg}) = Ker(\omega_A)_{pg}$. So, R_g takes $(\mathcal{H}_A)_p$ on $(\mathcal{H}_A)_{pg}$ and therefore $\mathcal{H}_A$ is a connection in the sense of (1).

It is not hard to see that the process (2)$\to$(1) is the inverse process of (1)$\to$(2) and vice versa.

In order to establish the identification (2)$\leftrightarrow$(3), we shall see what happens at the particular case of the trivial bundle $P = M \times G$:

Let $\omega_A \in \Omega^1_{M\times G}(\mathcal{G})$ be a connection on $M \times G$ in the sense of (2) and let $p = (x, h)$ be any point of $M \times G$.

Applying the property (a) in p we have for any $g \in G$ and $\tau \in T_gG$,

$$(\omega_{MC})_g(\tau) = (L_p^*\omega_A)_g(\tau) = (\omega_A)_{(x,hg)}(0, dL_h(\tau)) \ .$$

In particular, taking $p = (x, e)$, we obtain

$$(\omega_A)_{(x,g)}(0, \tau) = (\omega_{MC})_g(\tau) \ . \tag{1.1}$$

Applying the property (b) in p we have for any $g \in G$ and $X \in T_p(M \times G)$,

$$(R_g^*\omega_A)_pX = g^{-1}.(\omega_A)_pX.g \ .$$

In particular, taking $p = (x, e)$, $X = (v, 0)$, $v \in T_xM$ and observing that $(R_g^*\omega_A)_{(x,e)}(v, 0) = (\omega_A)_{(x,g)}(v, 0)$, we obtain $(\omega_A)_{(x,g)}(v, 0) = g^{-1}.(\omega_A)_{(x,e)}(v, 0).g$. Consider the inclusion $i : M \to M \times G$, $i(x) = (x, e)$ and define $A \in \Omega^1_M(\mathcal{G})$ as $A := i^*\omega_A$. The last expression can thus be written as

$$(\omega_A)_{(x,g)}(v, 0) = g^{-1}.A_x(v).g \ . \tag{1.2}$$

Putting (1.1) and (1.2) together, we obtain for any $p = (x, g) \in M \times G$ and $(v, \tau) \in T_{(x,g)}(M \times G)$ the formula

$$(\omega_A)_{(x,g)}(v, \tau) = (\omega_{MC})_g(\tau) + g^{-1}.A_x(v).g \ . \tag{1.3}$$

Thus, all the data of the connection ω_A on $M \times G$ is determined by the 1-form A on M. So we get an identification

$$\{Connection\ forms\ on\ M \times G\} \longleftrightarrow \Omega^1_M(\mathcal{G}) \tag{1.4}$$

by taking ω_A on $i^*\omega_A = A$ in one direction and by noticing, in the other direction, that for any $A \in \Omega^1_M(\mathcal{G})$ the 1-form $(\omega_A)_{(x,g)}(v,\tau) := (\omega_{MC})_g(\tau) + g^{-1}.A_x(v).g$ on $M \times G$ is actually a connection form.

On the other hand, for a given linear locally faithful representation $G \xrightarrow{\rho} \operatorname{Aut} V$, the vector bundle E associated to $M \times G$ and to this representation is the trivial bundle $M \times V$. This representation also induces the Lie algebra inclusion $\mathcal{G} \xhookrightarrow{d\rho_e} \operatorname{End} V$, and from now on we shall identify $\mathcal{G}$ with its image in $\operatorname{End} V$ via $d\rho_e$. From this inclusion, we can construct for each $A \in \Omega^1_M(\mathcal{G})$ a covariant derivative ∇_A on $M \times V$ as $\nabla_A := d + A$, where A is considered as an element of $\Omega^1_M(\operatorname{End} V)$, d is the usual differential operator and a section $s \in \Omega^0_M(M \times V)$ is considered as a vector-valued function $s : M \to V$, so $\nabla_A s = ds + A.s$. It is straightforward to verify that ∇_A is linear and satisfies the Leibniz rule, so ∇_A is in fact a covariant derivative. It can be proved by means of parallel transport arguments that for $A \in \Omega^1_M(\operatorname{End} V)$, the more restrictive condition $A \in \Omega^1_M(\mathcal{G})$ is equivalent to $\nabla_A = d + A$ be compatible with the G-structure on $M \times V$. Also, it is not hard to see by using the Leibniz rule that for a covariant derivative ∇ on $M \times V$ and for $f \in C^\infty(M)$ we have $(\nabla - d)(f.s) = f.(\nabla - d)(s)$, so $(\nabla - d) : \Omega^0_M(V) \to \Omega^1_M(V)$ actually defines a 1-form $A \in \Omega^1_M(\operatorname{End} V)$ by setting for any $(x_0, v_0) \in TM$ and $s_0 \in V$, $A_{x_0}(v_0).s_0 := (\nabla s)_{x_0}(v_0) - ds_{x_0}(v_0)$, where s is any section such that $s(x_0) = s_0$. If the G-compatibility condition is imposed on ∇, we have already noticed that this implies $A \in \Omega^1_M(\mathcal{G})$.

We have therefore obtained an identification

$$\Omega^1_M(\mathcal{G}) \longleftrightarrow \{G\text{-}compatible\ covariant\ derivatives\ on\ M \times V\} \tag{1.5}$$

taking A on $\nabla_A = d + A$ in one direction and ∇ on $A = \nabla - d$ in the other direction.

Everything we have done above allows us to establish the correspondence $(2) \leftrightarrow (3)$ for the case of the trivial bundle $P = M \times G$ and will enable us to establish the correspondence $(2) \leftrightarrow (3)$ for the general case:

To go from definition (2) to (3), we take $\omega_A \in \Omega^1_P(\mathcal{G})$ as in (2), a given linear locally faithful representation $G \xrightarrow{\rho} \operatorname{Aut} V$ and the respective associated vector bundle E. Define the covariant derivative ∇_A on E locally: for any local trivialization $\varphi_\alpha : U_\alpha \times G \to P$ defined on an open set $U_\alpha \subset M$, the 1-form $\varphi_\alpha^* \omega_A \in \Omega^1_{U_\alpha \times G}(\mathcal{G})$ is a connection form on the trivial bundle $U_\alpha \times G$. On behalf of the identification (1.4) above we can consider the 1-form $A^\alpha \in \Omega^1_{U_\alpha}(\mathcal{G})$ associated to $\varphi_\alpha^* \omega_A$ and define the covariant derivative ∇_A on E in the associated trivialization $\overline{\varphi}_\alpha : U_\alpha \times V \to E$ as

$$\nabla_{A,\alpha} := d + A^\alpha \ . \tag{1.6}$$

In order to prove that the covariant derivative $\nabla_A : \Omega^0_M(E) \to \Omega^1_M(E)$ is a globally well-defined operator, we shall consider the two local trivializations $\varphi_\alpha : U_\alpha \times G \to P$ and $\varphi_\beta : U_\beta \times G \to P$ and prove that $\nabla_{A,\alpha}$ and $\nabla_{A,\beta}$ actually define the same operator ∇_A by finding the relationship between A^α and A^β in $U_\alpha \cap U_\beta$:

According to formula (1.3), for the connection forms $\varphi_\alpha^* \omega_A$ and $\varphi_\beta^* \omega_A$ we have the expressions

$$(\varphi_\alpha^* \omega_A)_{(x,g)}(v, \tau) = (\omega_{MC})_g(\tau) + g^{-1}.A^\alpha_x(v).g \ ,$$

for any $p = (x, g) \in U_\alpha \times G$, $(v, \tau) \in T_{(x,g)}(U_\alpha \times G)$, and

$$(\varphi_\beta^* \omega_A)_{(y,h)}(u, \sigma) = (\omega_{MC})_h(\sigma) + h^{-1}.A^\beta_y(u).h \ ,$$

for any $q = (y, h) \in U_\beta \times G$, $(u, \sigma) \in T_{(y,h)}(U_\beta \times G)$.

Thus, in $U_\alpha \cap U_\beta$ we have

$$A^\beta_y(u) = (\varphi_\beta^* \omega_A)_{(y,e)}(u, 0) = ((\varphi_\alpha^{-1} \varphi_\beta)^* (\varphi_\alpha^* \omega_A))_{(y,e)}(u, 0) \ ,$$

where the application $\varphi_\alpha^{-1}\varphi_\beta : (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G$ is of the form $(\varphi_\alpha^{-1}\varphi_\beta)(y, h) = (y, \varphi_{\beta\alpha}(y).h)$ for some $\varphi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$. So,

$$\begin{aligned} A^\beta_y(u) &= ((\varphi_\alpha^{-1} \varphi_\beta)^* (\varphi_\alpha^* \omega_A))_{(y,e)}(u, 0) = (\varphi_\alpha^* \omega_A)_{(y,\varphi_{\beta\alpha}(y))}(u, d(\varphi_{\beta\alpha})_y(u)) \\ &= (\omega_{MC})_{\varphi_{\beta\alpha}(y)}(d(\varphi_{\beta\alpha})_y(u)) + \varphi_{\beta\alpha}(y)^{-1}.A^\alpha_y(u).\varphi_{\beta\alpha}(y) \\ &= \varphi_{\beta\alpha}(y)^{-1}.d(\varphi_{\beta\alpha})_y(u) + \varphi_{\beta\alpha}(y)^{-1}.A^\alpha_y(u).\varphi_{\beta\alpha}(y) \ . \end{aligned}$$

Hence

$$A^\beta = \varphi_{\beta\alpha}^{-1}.d\varphi_{\beta\alpha} + \varphi_{\beta\alpha}^{-1}.A^\alpha.\varphi_{\beta\alpha} \ . \tag{1.7}$$

Formula (1.7) is known as *change of gauge* formula. We say that A^β was obtained from A^α after performing a 'change of gauge'.

Now it only remains to prove that for a given covariant derivative $\nabla : \Omega^0_M(E) \to \Omega^1_M(E)$ and for the associated trivializations $\overline{\varphi}_\alpha : U_\alpha \times V \to E$ and $\overline{\varphi}_\beta : U_\beta \times V \to E$, the relationship in $U_\alpha \cap U_\beta$ between the 1-forms $A^\alpha \in \Omega^1_{U_\alpha}(\mathcal{G})$ and $A^\beta \in \Omega^1_{U_\beta}(\mathcal{G})$, the 1-forms associated by the identification (1.5) to the local representations ∇_α and ∇_β of ∇ in $U_\alpha \times V$ and $U_\beta \times V$, is the same as in formula (1.7):

In $U_\alpha \times V$ we have for $s \in \Omega^0_{U_\alpha}(U_\alpha \times V)$,

$$(\nabla_\alpha s)_x(v) = \overline{\varphi}_\alpha^{-1}(\nabla(\overline{\varphi}_\alpha s)_x(v)) \ .$$

In $U_\beta \times V$ we have for $r \in \Omega^0_{U_\beta}(U_\beta \times V)$,

$$(\nabla_\beta r)_y(u) = \overline{\varphi}_\beta^{-1}(\nabla(\overline{\varphi}_\beta r)_y(u)) \ .$$

It follows that in $(U_\alpha \cap U_\beta) \times V$,

$$\nabla_\beta r = (\overline{\varphi}_\beta^{-1}\overline{\varphi}_\alpha).(\nabla_\alpha(\overline{\varphi}_\alpha^{-1}\overline{\varphi}_\beta)r) \ .$$

Therefore

$$\nabla_\beta r = \varphi_{\beta\alpha}^{-1}.\nabla_\alpha(\varphi_{\beta\alpha}.r) \ ,$$

with $\varphi_{\beta\alpha} : U_\alpha \cap U_\beta \to G \xrightarrow{\rho} \operatorname{Aut} V$. From $\nabla_\alpha = d + A^\alpha$ with $A^\alpha \in \Omega^1_{U_\alpha}(\operatorname{End} V)$ and $\nabla_\beta = d + A^\beta$ with $A^\beta \in \Omega^1_{U_\beta}(\operatorname{End} V)$, the last equation becomes

$$\begin{aligned} dr + A^\beta.r &= \nabla_\beta r = \varphi_{\beta\alpha}^{-1}.\nabla_\alpha(\varphi_{\beta\alpha}.r) = \varphi_{\beta\alpha}^{-1}.(d.(\varphi_{\beta\alpha}.r) + A^\alpha.(\varphi_{\beta\alpha}.r)) \\ &= \varphi_{\beta\alpha}^{-1}.(d\varphi_{\beta\alpha}.r + \varphi_{\beta\alpha}.dr + A^\alpha.\varphi_{\beta\alpha}.r) = \varphi_{\beta\alpha}^{-1}.d\varphi_{\beta\alpha}.r + dr + \varphi_{\beta\alpha}^{-1}.A^\alpha.\varphi_{\beta\alpha}.r \ . \end{aligned}$$

So

$$A^\beta = \varphi_{\beta\alpha}^{-1}.d\varphi_{\beta\alpha} + \varphi_{\beta\alpha}^{-1}.A^\alpha.\varphi_{\beta\alpha} \ ,$$

as in (1.7).

Thus, in the process (2)$\to$(3) the local covariant derivatives $\nabla_{A,\alpha}$ match to form the global covariant derivative ∇_A. This covariant derivative ∇_A is in fact a G-compatible one because the property of G-compatibility is a local property and we have already noticed in the local case that this property is equivalent to the condition $A^\alpha \in \Omega^1_{U_\alpha}(\mathcal{G})$, which is the case.

Conversely, to go from (3) to (2), for a covariant derivative ∇_A on E we can define a 1-form ω_A on P with values in $\operatorname{End} V$ locally (according to (1.3)) as

$$(\omega_{A,\alpha})_{(x,g)}(v,\tau) := (\omega_{MC})_g(\tau) + g^{-1}.A^{\alpha}_x(v).g$$

for $p = (x,g) \in U_\alpha \times G$ and $(v,\tau) \in T_{(x,g)}(U_\alpha \times G)$, where $A^{\alpha} := \nabla_{A,\alpha} - d$.

On behalf of we have done above, the local connection forms $\omega_{A,\alpha}$ actually define a global 1-form ω_A on P with values in $\operatorname{End} V$ which satisfies conditions (a) and (b) of (2). Moreover, from the G-compatibility condition for ∇_A we have already observed that the local forms A^{α} actually have values in $\mathcal{G}$, so ω_A also has values in $\mathcal{G}$ and therefore ω_A is in fact a connection form on P.

There is a fourth approach for a connection, which is an extension of our first approach:

(4) Given a vector bundle $\pi_E : E \to M$ associated to P and to a linear representation $G \xrightarrow{\rho} \operatorname{Aut} V$, a connection A can be thought as a distribution of horizontal subspaces $\mathcal{H}^E_A \subset TE$ transverse to the fibres of π_E. This distribution $\mathcal{H}^E_A$ must be compatible with the G-structure on E in such a way integral curves of $\mathcal{H}^E_A$ are always contained in orbits of $P \times V \to E$.

It will be important later to establish how we arrive to this definition of a connection from definition (1) and definition (3):

In order to obtain this definition from definition (1), let $s_0 \in E$ be fixed and let $p_0 \in P$ and $v_0 \in V$ be such that $P \times V \to E$ maps (p_0, v_0) to s_0. Consider the application

$$\begin{aligned} R_{v_0} : P &\longrightarrow E \\ p &\longmapsto p\,.\,v_0\ , \end{aligned}$$

and define $(\mathcal{H}^E_A)_{s_0} := d(R_{v_0})_{p_0}(\mathcal{H}_A)$. It is readily seen that the definition of $(\mathcal{H}^E_A)_{s_0}$ is independent of the choice of p_0 and v_0. Also, since the image of integral curves of $\mathcal{H}_A$ are integral curves of $\mathcal{H}^E_A$, clearly $\mathcal{H}^E_A$ satisfies the required conditions.

Now, to obtain such a connection from definition (3), consider ∇_A a G-covariant derivative on E, $s_0 \in E$ and $x_0 := \pi_E(s_0)$. Define $(\mathcal{H}^E_A)_{s_0}$ as the set of tangent vectors at s_0 of curves in E obtained by parallel transport of s_0 along curves in M passing through x_0. In a local trivialization $\overline{\varphi}_\alpha : U_\alpha \times V \to E$, the equation of the curve $\overline{s} : (-\varepsilon, \varepsilon) \to V$ with $\overline{s}(0) = \overline{s}_0$ ($\overline{s}_0$ such that $\overline{\varphi}_\alpha^{-1}(s_0) = (x_0, \overline{s}_0)$) obtained by parallel transport along $\gamma : (-\varepsilon, \varepsilon) \to U_\alpha$, $(\gamma(0), \gamma'(0)) = (x_0, v_0)$, is

$$(\nabla_{A,\alpha}\overline{s})_{\gamma(t)}(\gamma'(t)) = \overline{s}'(t) + A^\alpha_{\gamma(t)}(\gamma'(t)).\overline{s}(t) = 0 \ .$$

Therefore, the tangent vector $(\gamma'(0), \overline{s}'(0)) \in T_{(x_0,\overline{s}_0)}(U_\alpha \times V) = T_{x_0}U_\alpha \times V$ is of the form $(v_0, -A^\alpha_{x_0}(v_0).\overline{s}_0)$. Thus, in this local trivialization, $(\mathcal{H}^E_A)_{s_0}$ can be expressed as

$$(\mathcal{H}^E_{A,\alpha})_{(x_0,\overline{s}_0)} = \{(v, -A^\alpha_{x_0}(v).\overline{s}_0) \, ; \, v \in T_{x_0}U_\alpha\} \ , \tag{1.8}$$

i.e. as the graphic of the linear application

$$\begin{aligned} T_{x_0}U_\alpha &\longrightarrow \ V \\ \cdot \ &\mapsto \ -A^\alpha_{x_0}(\cdot).\overline{s}_0 \ . \end{aligned}$$

It is easy to see from this expression that the distribution $(\mathcal{H}^E_A)$ satisfies the required conditions.

1.1.2 The Space of Connections and the Action of the Gauge Group

Let us begin by modelling the space of connections of a bundle on a manifold. Once the equivalence between the given definitions of a connection was already established, we shall use the most convenient one depending on the situation.

Notice that so far we have not yet shown that connections exist at all, however it can be easily proved by considering a connection according to definition (2): the identification (1.4) gives us a lot of connection forms on the trivial bundle $M \times G$, as many as the space $\Omega^1_M(\mathcal{G})$. A connection form ω_A on an arbitrary principal bundle P can be obtained by choosing arbitrarily local connection forms $\omega_{A,\alpha}$ on each of the local trivializations $\varphi_\alpha : U_\alpha \times G \to P$ and by gluing all of them together via a partition of unity

$\{\lambda_\alpha\}$ subordinate to the covering $\{U_\alpha\}$ of M: $\omega_A = \sum_\alpha \lambda_\alpha \omega_{A,\alpha}$. It is not hard to see that the 1-form ω_A satisfies properties (a) and (b) of (2), so ω_A is in fact a connection form on P.

In order to model the space of connections, we shall consider a connection according to definition (3), i.e. we suppose given a vector bundle E associated to a principal bundle P on M and to a linear (locally faithful) representation $G \xrightarrow{\rho} \operatorname{Aut} V$, and a connection will be a G-covariant derivative ∇_A on E. Denote by $\operatorname{Ad} P$ (or by $\operatorname{Ad} E$) the *adjoint bundle* of P, i.e. the Lie algebra bundle associated to P and the adjoint representation Ad of G in $\operatorname{Aut} \mathcal{G}$. Observe that this bundle can be seen as a subbundle of $\operatorname{End} E$ because the Lie algebra inclusion $\mathcal{G} \xhookrightarrow{d\rho_e} \operatorname{End} V$ induces in a natural way a Lie algebra bundle inclusion $\operatorname{Ad} E \hookrightarrow \operatorname{End} E$.

Once connections do in fact exist, let ∇_0 be a fixed G-covariant derivative on E. Considering another G-covariant derivative ∇_A on E, we observe by using the Leibniz rule that the difference $(\nabla_A - \nabla_0) : \Omega^0_M(E) \to \Omega^1_M(E)$ satisfies

$$(\nabla_A - \nabla_0)(f.s) = f.(\nabla_A - \nabla_0)(s) \quad \text{for any } f \in C^\infty(M) \ .$$

Hence $(\nabla_A - \nabla_0)$ actually defines a 1-form $a \in \Omega^1_M(\operatorname{End} E)$ by setting for any $(x_0, v_0) \in TM$ and $s_0 \in E_{x_0}$ (the fibre of E on x_0), $a_{x_0}(v_0).s_0 := (\nabla_A s)_{x_0}(v_0) - (\nabla_0 s)_{x_0}(v_0)$, where s is any section of E such that $s(x_0) = s_0$.

In local coordinates we have

$$\nabla_{A,\alpha} - \nabla_{0,\alpha} = (d + A^\alpha) - (d + A_0^\alpha) = A^\alpha - A_0^\alpha = a^\alpha \ ,$$

so since $A^\alpha, A_0^\alpha \in \Omega^1_{U_\alpha}(\mathcal{G})$ we have $a^\alpha \in \Omega^1_{U_\alpha}(\mathcal{G})$, therefore a actually lies in $\Omega^1_M(\operatorname{Ad} E)$ and $\nabla_A = \nabla_0 + a$.

Conversely, for each $a \in \Omega^1_M(\operatorname{Ad} E)$ we can define a linear operator $\nabla_A : \Omega^0_M(E) \to \Omega^1_M(E)$ on E by setting $\nabla_A := \nabla_0 + a$. It is straightforward to verify that ∇_A is a covariant derivative on E and since in local coordinates we have $A^\alpha = A_0^\alpha + a^\alpha$ then $A^\alpha \in \Omega^1_{U_\alpha}(\mathcal{G})$, therefore ∇_A is compatible with the G-structure on E.

Thus, for a G-covariant derivative ∇_0 on E fixed we have obtained an identification

$$\{G\text{-}covariant\ derivatives\ on\ E\} \longleftrightarrow \Omega^1_M(\operatorname{Ad} E) \tag{1.9}$$

taking ∇_A on $a = \nabla_A - \nabla_0$ in one direction and a on $\nabla_A = \nabla_0 + a$ in the other direction. In particular, the space of G-connections on E is an affine space whose underlying vector space is the infinite-dimensional space $\Omega^1_M(\mathrm{Ad}\, E)$.

Next we shall describe the action of the group of bundle automorphisms on the space of connections. Observe that $\Omega^0_M(\mathrm{Aut}\, E)$, the set of vector bundle automorphisms $u : E \to E$, is a group and acts on the space of covariant derivatives on E (which will be denoted by $\mathcal{C}(E)$) via the pull-back transformation:

$$\begin{array}{rcl} \Omega^0_M(\mathrm{Aut}\, E) \times \mathcal{C}(E) & \longrightarrow & \mathcal{C}(E) \\ (u\,, \nabla_A) & \longmapsto & \nabla_{u(A)} \end{array}$$

where $\nabla_{u(A)} s := u.\nabla_A(u^{-1}.s),\ \forall s \in \Omega^0_M(E)$.

It is readily seen that $\nabla_{u(A)}$ is in fact a covariant derivative. However, if ∇_A is a G-covariant derivative we cannot say the same about $\nabla_{u(A)}$ for any u in general. In order to preserve the space of G-covariant derivatives, u must preserve the G-structure on E first, i.e. it must leave fixed the 'orbits' $\mathcal{O}_v = \{p.v,\ p \in P\}$ of the 'action' $P \times V \to E$. From now on we shall restrict our attention to this type of automorphisms on E. To accomplish this, consider the *adjoint* representation ad of G in $\mathrm{Aut}\, G$: $\mathrm{ad}(g).h = g.h.g^{-1}$. Denote by $\mathrm{ad}\, P$ (or by $\mathrm{ad}\, E$) the group bundle with fibre G associated to P and to this representation, its space of sections $\Omega^0_M(\mathrm{ad}\, E)$ is called the *gauge group* of E. The homomorphism $G \xrightarrow{\rho} \mathrm{Aut}\, V$ induces in a natural way a group bundle map $\mathrm{ad}\, E \to \mathrm{Aut}\, E$ (which is also denoted by ρ). By considering the homomorphism $\Omega^0_M(\mathrm{ad}\, E) \to \Omega^0_M(\mathrm{Aut}\, E)$ (still denoted by ρ) we can see the elements of the gauge group $\Omega^0_M(\mathrm{ad}\, E)$ as elements of $\Omega^0_M(\mathrm{Aut}\, E)$. According to this point of view, it can be proved passing to local coordinates that the elements of $\Omega^0_M(\mathrm{ad}\, E)$ preserve the G-structure on E. In particular, noting that the action of $\Omega^0_M(\mathrm{Aut}\, E)$ on $\mathcal{C}(E)$ preserves the parallel transport operation (i.e. denoting by $T^\gamma_A : E_{\gamma(0)} \to E_{\gamma(1)}$ the parallel transport of ∇_A along $\gamma : [0,1] \to M$, for any $u \in \Omega^0_M(\mathrm{Aut}\, E)$ we have $T^\gamma_{u(A)}.u = u.T^\gamma_A$), it follows easily according to definition (3) that the elements of the gauge group $\Omega^0_M(\mathrm{ad}\, E)$ preserve the space of G-covariant derivatives on E (which will be denoted by $\mathcal{C}_G(E)$). So we have an action

$$\begin{array}{rcl} \Omega^0_M(\mathrm{ad}\,E)\times\mathcal{C}_G(E) & \longrightarrow & \mathcal{C}_G(E) \\ (u\,,\nabla_A) & \longmapsto & \nabla_{u(A)} \end{array}$$

where the elements of $\Omega^0_M(\mathrm{ad}\,E)$ are seen as elements of $\Omega^0_M(\mathrm{Aut}\,E)$.

The expression $\nabla_{u(A)}s = u.\nabla_A(u^{-1}.s)$ is not an explicit expression for $\nabla_{u(A)}$ in terms of ∇_A and u. In order to obtain such an expression, passing to local coordinates we have

$$\begin{aligned} \nabla_{u(A),\alpha}s &= u.\nabla_{A,\alpha}(u^{-1}.s) = u.(d(u^{-1}.s) + A^\alpha.u^{-1}.s) \\ &= u.(u^{-1}.ds + d(u^{-1}).s + A^\alpha.u^{-1}.s) = ds + u.d(u^{-1}).s + u.A^\alpha.u^{-1}.s \\ &= ds - du.u^{-1}.s + u.A^\alpha.u^{-1}.s = ds - (du - u.A^\alpha).u^{-1}.s \\ &= \nabla_{A,\alpha}s - A^\alpha.s - (du - u.A^\alpha).u^{-1}.s \\ &= \nabla_{A,\alpha}s - (A^\alpha.u + du - u.A^\alpha).u^{-1}.s = \nabla_{A,\alpha}s - (du + [A^\alpha, u]).u^{-1}.s\ , \end{aligned}$$

so

$$\nabla_{u(A),\alpha} = \nabla_{A,\alpha} - (du + [A^\alpha, u]).u^{-1}\ . \tag{1.10}$$

On the other hand, since E is the vector bundle associated to P and to the representation $G \xrightarrow{\rho} \mathrm{Aut}\,V$, it follows easily that $\mathrm{End}\,E$ is the vector bundle associated to P and to the representation of G in $\mathrm{Aut}(\mathrm{End}\,V)$ given by the composition

$$\begin{array}{rcl} G \xrightarrow{\rho} \mathrm{Aut}\,V & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathrm{End}\,V) \\ T & \longmapsto & (S \mapsto T.S.T^{-1})\ . \end{array}$$

It also follows easily that the Lie algebra homomorphism induced by this representation is the composition

$$\begin{array}{rcl} \mathcal{G} \xhookrightarrow{d\rho_e} \mathrm{End}\,V & \xrightarrow{d(\mathrm{Ad})_{\mathrm{Id}}} & \mathrm{End}(\mathrm{End}\,V) \\ T & \longmapsto & (S \mapsto [T, S])\ . \end{array}$$

Thus, according the equivalence between the given definitions of a connection, we obtain a G-covariant derivative on $\mathrm{End}\,E$ (still denoted by ∇_A) which is locally expressed in function of the local forms $A^\alpha \in \Omega^1_{U_\alpha}(\mathcal{G})$ of ∇_A on E as

$$\nabla_{A,\alpha} := d + [A^\alpha, \,.\,]\ . \tag{1.11}$$

In particular, since $\Omega^0_M(\mathrm{Aut}\, E) \subset \Omega^0_M(\mathrm{End}\, E)$, for sections in E we obtain from equation (1.10) that

$$\nabla_{u(A),\alpha} = \nabla_{A,\alpha} - (du + [A^\alpha, u]).u^{-1} = \nabla_{A,\alpha} - \nabla_{A,\alpha} u\, .u^{-1}\ .$$

Therefore, we have globally the formula

$$\nabla_{u(A)} = \nabla_A - \nabla_A u\, .u^{-1}\ . \tag{1.12}$$

1.2 Curvature

In this section we shall define another fundamental concept in gauge theory: the curvature of a connection.

For a covariant derivative $\nabla_A : \Omega^0_M(E) \to \Omega^1_M(E)$, we shall extend the ordinary de Rham complex

$$\Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \dots \xrightarrow{d} \Omega^k_M \xrightarrow{d} \Omega^{k+1}_M \dots\ ,$$

to a complex of the form

$$\Omega^0_M(E) \xrightarrow{d_A} \Omega^1_M(E) \xrightarrow{d_A} \dots \xrightarrow{d_A} \Omega^k_M(E) \xrightarrow{d_A} \Omega^{k+1}_M(E) \dots\ ,$$

where the operators d_A are uniquely determined by the properties:

(1) $d_A = \nabla_A$ on $\Omega^0_M(E)$,

(2) $d_A(\omega \wedge \theta) = (d_A\omega) \wedge \theta + (-1)^k \omega \wedge d\theta$, $\forall \omega \in \Omega^k_M(E)$, $\theta \in \Omega^l_M$.
(The wedge product $\wedge : \Omega^k_M \times \Omega^l_M \to \Omega^{k+l}_M$ extends in a natural way to $\wedge : \Omega^k_M(E) \times \Omega^l_M \to \Omega^{k+l}_M(E)$).

We proceed to define the operators d_A as follows: consider the trivialization $\overline{\varphi}_\alpha : U_\alpha \times V \to E$, a chart $\psi_\alpha : U_\alpha \to \mathbb{R}^n$, the local basis $\{e_1, e_2, \dots, e_n\}$ of tangent vectors for U_α which comes from the canonical basis of $\mathbb{R}^n$ via the chart ψ_α and $\{dx_1, dx_2, \dots, dx_n\}$ the dual basis of $\{e_1, e_2, \dots, e_n\}$. Define $d_A : \Omega^k_M(E) \to \Omega^{k+1}_M(E)$ in the trivialization $\overline{\varphi}_\alpha : U_\alpha \times V \to E$ as $d_{A,\alpha}(s.dx_{i_1} \wedge \dots \wedge dx_{i_k}) := \nabla_{A,\alpha} s \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$ and extend linearly. It is readily seen that the local operators $d_{A,\alpha} : \Omega^k_{U_\alpha}(V) \to \Omega^{k+1}_{U_\alpha}(V)$ satisfy property (2) and (using this fact) that these operators commute with the pull-back operation, so the definition of $d_{A,\alpha}$ does not depend on the chart $\psi_\alpha : U_\alpha \to \mathbb{R}^n$ chosen and the local operators $d_{A,\alpha}$ match to form the global operator d_A which obviously satisfy properties (1) and (2). The unicity of the family of operators $d_A : \Omega^k_M(E) \to \Omega^{k+1}_M(E)$ satisfying (1) and (2) can be proved by induction on k.

Unlike the case of the de Rham complex, it is not necessarily true that the operators d_A satisfy $d_A d_A = 0$. Instead the Leibniz rule for d_A tell us that the operator $d_A d_A : \Omega^0_M(E) \to \Omega^2_M(E)$ in fact can be seen as a tensor: for $f \in C^\infty(M)$ and $s \in \Omega^0_M(E)$ we have

$$\begin{aligned} d_A d_A(f.s) &= d_A(df.s + f.\nabla_A s) = d_A(df \wedge s + f \wedge \nabla_A s) \\ &= ddf \wedge s + (-1) df \wedge \nabla_A s + df \wedge \nabla_A s + f \wedge d_A \nabla_A s \\ &= f.d_A d_A s \ . \end{aligned}$$

Thus, $d_A d_A : \Omega^0_M(E) \to \Omega^2_M(E)$ actually defines a 2-form $F_A \in \Omega^2_M(\operatorname{End} E)$ by setting for any $x_0 \in M$, $v_0, w_0 \in T_{x_0}M$ and $s_0 \in E_{x_0}$, $(F_A)_{x_0}(v_0, w_0).s_0 := (d_A d_A s)_{x_0}(v_0, w_0)$, where s is any section of E such that $s(x_0) = s_0$ (more briefly, $d_A d_A s = F_A.s$). This 2-form F_A is called the *curvature* of the connection ∇_A. Recalling that the inclusion $\mathcal{G} \xhookrightarrow{d\rho_e} \operatorname{End} V$ induces the vector bundle inclusion $\operatorname{Ad} E \hookrightarrow \operatorname{End} E$, we shall see later that in fact $F_A \in \Omega^2_M(\operatorname{Ad} E)$.

Now we shall obtain the expression of the curvature with respect to local coordinates for E and for M: let $\overline{\varphi}_\alpha : U_\alpha \times V \to E$, $\psi_\alpha : U_\alpha \to \mathbb{R}^n$, $\{e_1, e_2, \ldots, e_n\}$ and $\{dx_1, dx_2, \ldots, dx_n\}$ be as before. For a local section $s \in \Omega^0_{U_\alpha}(V)$ denote by $\nabla_i s$ the covariant derivative of s in the direction e_i, $\nabla_i s := (\nabla_{A,\alpha} s)(e_i)$, thus $\nabla_{A,\alpha} s = \sum_i \nabla_i s \, dx_i$.

In these local coordinates we have

$$\begin{aligned} d_{A,\alpha} d_{A,\alpha} s &= d_{A,\alpha} \nabla_{A,\alpha} s = d_{A,\alpha}(\sum_i \nabla_i s \, dx_i) = \sum_i \nabla_{A,\alpha}(\nabla_i s) \wedge dx_i \\ &= \sum_i (\sum_j \nabla_j(\nabla_i s) \, dx_j) \wedge dx_i = \sum_{i,j} \nabla_j \nabla_i s \, dx_j \wedge dx_i \\ &= \sum_{i<j} (\nabla_i \nabla_j s - \nabla_j \nabla_i s) \, dx_i \wedge dx_j = \sum_{i<j} [\nabla_i \, , \nabla_j] s \, dx_i \wedge dx_j \ . \end{aligned} \tag{1.13}$$

On the other hand, from $\nabla_{A,\alpha} = d + A^\alpha$ we have $\nabla_i = \frac{\partial}{\partial x_i} + A^\alpha(e_i)$, and denoting $A^\alpha(e_i)$ by A_i^α we can obtain an expression for $[\nabla_i, \nabla_j]$ in terms of the 'connection matrices' A_i^α:

$$\begin{aligned}
[\nabla_i, \nabla_j]s &= (\nabla_i\nabla_j s - \nabla_j\nabla_i s) = (\frac{\partial}{\partial x_i} + A_i^\alpha)(\frac{\partial s}{\partial x_j} + A_j^\alpha s) - (\frac{\partial}{\partial x_j} + A_j^\alpha)(\frac{\partial s}{\partial x_i} + A_i^\alpha s) \\
&= \frac{\partial^2 s}{\partial x_i \partial x_j} + A_i^\alpha . \frac{\partial s}{\partial x_j} + \frac{\partial(A_j^\alpha s)}{\partial x_i} + A_i^\alpha A_j^\alpha s \\
&\quad - \frac{\partial^2 s}{\partial x_j \partial x_i} - A_j^\alpha . \frac{\partial s}{\partial x_i} - \frac{\partial(A_i^\alpha s)}{\partial x_j} - A_j^\alpha A_i^\alpha s \\
&= A_i^\alpha . \frac{\partial s}{\partial x_j} + \frac{\partial A_j^\alpha}{\partial x_i}.s + A_j^\alpha . \frac{\partial s}{\partial x_i} - A_j^\alpha . \frac{\partial s}{\partial x_i} - \frac{\partial A_i^\alpha}{\partial x_j}.s - A_i^\alpha . \frac{\partial s}{\partial x_j} + [A_i^\alpha, A_j^\alpha].s \\
&= \frac{\partial A_j^\alpha}{\partial x_i}.s - \frac{\partial A_i^\alpha}{\partial x_j}.s + [A_i^\alpha, A_j^\alpha].s \\
&= \left(\frac{\partial A_j^\alpha}{\partial x_i} - \frac{\partial A_i^\alpha}{\partial x_j} + [A_i^\alpha, A_j^\alpha] \right).s \ ,
\end{aligned}$$

so

$$[\nabla_i, \nabla_j] = \frac{\partial A_j^\alpha}{\partial x_i} - \frac{\partial A_i^\alpha}{\partial x_j} + [A_i^\alpha, A_j^\alpha] \ . \tag{1.14}$$

Combining everything we have done above with the equality $d_A d_A s = F_A.s$, we obtain the local expression of the curvature F_A:

$$F_{A,\alpha} = \sum_{i<j} [\nabla_i, \nabla_j]\, dx_i \wedge dx_j = \sum_{i<j} (\frac{\partial A_j^\alpha}{\partial x_i} - \frac{\partial A_i^\alpha}{\partial x_j} + [A_i^\alpha, A_j^\alpha])\, dx_i \wedge dx_j \ .$$

In particular, since the connection matrices $A_i^\alpha \in \mathcal{G}$, we have $F_{A,\alpha} \in \Omega^2_{U_\alpha}(\mathcal{G})$, so in fact $F_A \in \Omega^2_M(\mathrm{Ad}\, E)$.

Next, we shall see how the curvature varies with the connection. Let ∇_A be a G-covariant derivative on E. According to (1.9), any other G-covariant derivative can be written in the form $\nabla_{A+a} := \nabla_A + a$ for some $a \in \Omega^1_M(\mathrm{Ad}\, E)$. In order to obtain F_{A+a} in function of F_A and a, passing to local coordinates we have

$$F_{A+a,\alpha} = \sum_{i<j} [(\nabla_{A+a})_i, (\nabla_{A+a})_j]\, dx_i \wedge dx_j \ ,$$

and in this setting we write $a^\alpha = \sum_i a_i^\alpha dx_i$. Then, according to (1.14),

$$
\begin{aligned}
[(\nabla_{A+a})_i\,,(\nabla_{A+a})_j] &= \frac{\partial}{\partial x_i}(A_j^\alpha + a_j^\alpha) - \frac{\partial}{\partial x_j}(A_i^\alpha + a_i^\alpha) + [A_i^\alpha + a_i^\alpha\,, A_j^\alpha + a_j^\alpha] \\
&= (\frac{\partial A_j^\alpha}{\partial x_i} - \frac{\partial A_i^\alpha}{\partial x_j} + [A_i^\alpha\,, A_j^\alpha]) \\
&\quad + (\frac{\partial a_j^\alpha}{\partial x_i} - \frac{\partial a_i^\alpha}{\partial x_j}) + [A_i^\alpha\,, a_j^\alpha] - [A_j^\alpha\,, a_i^\alpha] + [a_i^\alpha, a_j^\alpha] \\
&= [(\nabla_A)_i\,,(\nabla_A)_j] + (\frac{\partial a_j^\alpha}{\partial x_i} - \frac{\partial a_i^\alpha}{\partial x_j}) + [A_i^\alpha\,, a_j^\alpha] - [A_j^\alpha\,, a_i^\alpha] + [a_i^\alpha, a_j^\alpha]\ .
\end{aligned}
$$

So,

$$
\begin{aligned}
F_{A+a,\alpha} &= F_{A,\alpha} + \sum_{i<j}(\frac{\partial a_j^\alpha}{\partial x_i} - \frac{\partial a_i^\alpha}{\partial x_j})\, dx_i \wedge dx_j \\
&\quad + \sum_{i<j}[A_i^\alpha\,, a_j^\alpha]\, dx_i \wedge dx_j - \sum_{i<j}[A_j^\alpha\,, a_i^\alpha]\, dx_i \wedge dx_j + \sum_{i<j}[a_i^\alpha, a_j^\alpha]\, dx_i \wedge dx_j \\
&= F_{A,\alpha} + da^\alpha + \sum_{i<j}[A_i^\alpha\,, a_j^\alpha]\, dx_i \wedge dx_j + \sum_{i>j}[A_i^\alpha\,, a_j^\alpha]\, dx_i \wedge dx_j \\
&\quad + \frac{1}{2}\sum_{i<j}[a_i^\alpha, a_j^\alpha]\, dx_i \wedge dx_j + \frac{1}{2}\sum_{i>j}[a_i^\alpha, a_j^\alpha]\, dx_i \wedge dx_j \\
&= F_{A,\alpha} + da^\alpha + \sum_{i,j}[A_i^\alpha\,, a_j^\alpha]\, dx_i \wedge dx_j + \frac{1}{2}\sum_{i,j}[a_i^\alpha, a_j^\alpha]\, dx_i \wedge dx_j\ .
\end{aligned}
$$

Combining the Lie algebra multiplication $[\,,] : \operatorname{Ad} E \times \operatorname{Ad} E \to \operatorname{Ad} E$ with the wedge product $\wedge : \Omega_M^k \times \Omega_M^l \to \Omega_M^{k+l}$ we obtain the product $[\,,] : \Omega_M^k(\operatorname{Ad} E) \times \Omega_M^l(\operatorname{Ad} E) \to \Omega_M^{k+l}(\operatorname{Ad} E)$. So, considering this product we have

$$
\begin{aligned}
F_{A+a,\alpha} &= F_{A,\alpha} + da^\alpha + \sum_j [\sum_i A_i^\alpha dx_i\,, a_j^\alpha] \wedge dx_j + \frac{1}{2}[\sum_i a_i^\alpha dx_i\,, \sum_j a_j^\alpha dx_j] \\
&= F_{A,\alpha} + \sum_j da_j^\alpha \wedge dx_j + \sum_j [A^\alpha, a_j^\alpha] \wedge dx_j + \tfrac{1}{2}[a^\alpha, a^\alpha] \\
&= F_{A,\alpha} + \sum_j (da_j^\alpha + [A^\alpha, a_j^\alpha]) \wedge dx_j + \tfrac{1}{2}[a^\alpha, a^\alpha] \\
&= F_{A,\alpha} + \sum_j \nabla_{A,\alpha} a_j^\alpha \wedge dx_j + \tfrac{1}{2}[a^\alpha, a^\alpha] = F_{A,\alpha} + d_{A,\alpha} a^\alpha + \tfrac{1}{2}[a^\alpha, a^\alpha]\ .
\end{aligned}
$$

Therefore globally we obtain the formula

$$F_{A+a} = F_A + d_A a + \tfrac{1}{2}[a, a] \ . \tag{1.15}$$

As an application of this formula, we can derive the *Second Bianchi Identity* for the curvature: $d_A F_A = 0$. We proceed as follows: consider the local covariant derivatives $\nabla_{A,\alpha}$ and d, use (1.15) to obtain

$$\begin{aligned} F_{A,\alpha} &= F_{d+A^\alpha} = F_d + dA^\alpha + \tfrac{1}{2}[A^\alpha, A^\alpha] \\ &= dd + dA^\alpha + \tfrac{1}{2}[A^\alpha, A^\alpha] \\ &= dA^\alpha + \tfrac{1}{2}[A^\alpha, A^\alpha] \ . \end{aligned} \tag{1.16}$$

Thus,

$$\begin{aligned} d_{A,\alpha} F_{A,\alpha} &= dF_{A,\alpha} + [A^\alpha, F_{A,\alpha}] = d(dA^\alpha + \tfrac{1}{2}[A^\alpha, A^\alpha]) + [A^\alpha, dA^\alpha + \tfrac{1}{2}[A^\alpha, A^\alpha]] \\ &= \tfrac{1}{2} d([A^\alpha, A^\alpha]) + [A^\alpha, dA^\alpha] + \tfrac{1}{2}[A^\alpha, [A^\alpha, A^\alpha]] \\ &= \tfrac{1}{2}([dA^\alpha, A^\alpha] + (-1)[A^\alpha, dA^\alpha]) + [A^\alpha, dA^\alpha] + \tfrac{1}{2}[A^\alpha, [A^\alpha, A^\alpha]] \\ &= \tfrac{1}{2}([dA^\alpha, A^\alpha] + [dA^\alpha, A^\alpha]) - [dA^\alpha, A^\alpha] + \tfrac{1}{2}[A^\alpha, [A^\alpha, A^\alpha]] \\ &= \tfrac{1}{2}[A^\alpha, [A^\alpha, A^\alpha]] \ . \end{aligned}$$

Finally,

$$\begin{aligned} [A^\alpha, [A^\alpha, A^\alpha]] &= [\sum_i A_i^\alpha dx_i , [\sum_j A_j^\alpha dx_j , \sum_k A_k^\alpha dx_k]] \\ &= \sum_{i,j,k} [A_i^\alpha, [A_j^\alpha, A_k^\alpha]] \, dx_i \wedge dx_j \wedge dx_k \\ &= \sum_{i<j<k} ([A_i^\alpha, [A_j^\alpha, A_k^\alpha]] + [A_j^\alpha, [A_k^\alpha, A_i^\alpha]] + [A_k^\alpha, [A_i^\alpha, A_j^\alpha]] \\ &\qquad -[A_i^\alpha, [A_k^\alpha, A_j^\alpha]] - [A_j^\alpha, [A_i^\alpha, A_k^\alpha]] - [A_k^\alpha, [A_j^\alpha, A_i^\alpha]]) \, dx_i \wedge dx_j \wedge dx_k \ . \end{aligned}$$

Then, from the Jacobi identity for the Lie multiplication, $[A^\alpha, [A^\alpha, A^\alpha]] = 0$.

Thus, combining everything we have

$$d_A F_A = 0 \ , \tag{1.17}$$

as asserted. In local coordinates, this identity becomes $dF_{A,\alpha} = [F_{A,\alpha}\,, A^{\alpha}]$.

We finish this section by describing how the curvature transforms under bundle automorphisms. Given $u \in \Omega^0_M(\mathrm{ad}\, E)$ we have

$$\begin{aligned} F_{u(A)}.s &= d_{u(A)}(d_{u(A)}s) = u.d_A(u^{-1}.d_{u(A)}s) \\ &= u.d_A(u^{-1}.u.d_A(u^{-1}s)) = u.d_A(d_A(u^{-1}s)) \\ &= u.F_A.u^{-1}.s\ . \end{aligned}$$

Therefore

$$F_{u(A)} = u.F_A.u^{-1}\,, \tag{1.18}$$

i.e. the curvature transforms as a tensor under bundle automorphisms. In particular the space of connections with zero curvature, called $flat$ connections, is preserved by the gauge group. There is another class of connections, satisfying a weaker condition than vanishing of the curvature, that are still preserved by the action of the gauge group. Such connections are known as *self-dual* connections and they will be studied in the next section.

1.3 Self-Duality and Hodge Theory

We need some preliminaries of linear algebra. Let V be an *oriented* real vector space of dimension n and $\langle\,,\rangle$ an inner product on V. Denote by $\Lambda^k V$ the space of k-forms on V. It is already known that $\langle\,,\rangle$ induces an isomorphism between V and its dual space V^*, and the inner product on V induces via this isomorphism an inner product on V^*, which is also denoted by $\langle\,,\rangle$.

We can then obtain an inner product on all the spaces $\Lambda^k V$ (which is still denoted by $\langle\,,\rangle$) that satisfies and is uniquely determined by the property

$$\langle\, \alpha_1 \wedge \ldots \wedge \alpha_k\,, \beta_1 \wedge \ldots \wedge \beta_k\,\rangle = \det(\langle\, \alpha_i\,, \beta_j\,\rangle)_{i,j}\ ,\ \ \forall \alpha_i\,, \beta_j \in \Lambda^1 V\ . \tag{1.19}$$

We proceed to define this inner product as follows: considering $\{\beta_1, \ldots, \beta_n\}$ a basis for V^*, the k-forms

$$\beta_{i_1} \wedge \ldots \wedge \beta_{i_k} \quad \text{with } 1 \le i_1 < i_2 < \ldots < i_k \le n$$

constitute a basis for $\Lambda^k V$. Define then

$$\langle \beta_{i_1} \wedge \ldots \wedge \beta_{i_k} , \beta_{j_1} \wedge \ldots \wedge \beta_{j_k} \rangle := \det(\langle \beta_{i_l} , \beta_{j_m} \rangle)_{l,m}$$

and extend bilinearly on $\Lambda^k V$.

In order to prove that this definition does not depend on the basis $\{\beta_1, \ldots, \beta_n\}$ chosen, it suffices to prove that our definition of $\langle \, , \rangle$ satisfies (1.19). This last statement is proved by observing that (1.19) holds when all the entries of this equality are constituted by elements of the basis $\{\beta_1, \ldots, \beta_n\}$ and that equality (1.19) is preserved after performing linear operations in each of its entries. The same kind of arguments allows us to conclude that our definition of $\langle \, , \rangle$ is also symmetric. Finally, for the positivity condition of $\langle \, , \rangle$, we observe that for an orthonormal basis $\{b_1, \ldots, b_n\}$ of V, its dual basis $\{db_1, \ldots, db_n\}$ is an orthonormal basis of V^* and more generally (using (1.19)) the basis

$$\{db_I := db_{i_1} \wedge \ldots \wedge db_{i_k};\ I = (i_1, \ldots, i_k) \text{ with } 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$$

satisfies $\langle db_I , db_J \rangle = \delta_{IJ}$, so $\langle \, , \rangle$ is actually positive definite on $\Lambda^k V$.

Next, for $0 \leq k \leq n$ we shall define the linear *Hodge-star* operator

$$* : \Lambda^k V \to \Lambda^{n-k} V$$

which can be obtained by comparing the inner product on the forms with the wedge product, i.e. (as before) the $*$-operator can be defined and is uniquely determined by satisfying the property

$$\alpha \wedge *\beta = \langle \alpha , \beta \rangle \, d\mu, \quad \forall \alpha, \beta \in \Lambda^k V \ , \tag{1.20}$$

where $d\mu$ is the (oriented) volume element of V.

We define then the $*$-operator in the following way: considering the orthonormal basis $\{b_1, \ldots, b_n\}$ of V, define for each element of the orthonormal basis $\{db_{i_1} \wedge \ldots \wedge db_{i_k}$, $1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$ of $\Lambda^k V$ the $*$-operator as

$$*(db_{i_1} \wedge \ldots \wedge db_{i_k}) := db_{j_1} \wedge \ldots \wedge db_{j_{n-k}}$$

(where the indices $j_1, \ldots, j_{n-k}$ are selected such that $b_{i_1}, \ldots, b_{i_k}, b_{j_1}, \ldots, b_{j_{n-k}}$ is a positive basis of V), and extend linearly on $\Lambda^k V$.

Again, in order to prove that this definition does not depend on the orthonormal basis $\{b_1, \ldots, b_n\}$ chosen, it suffices to prove that our definition of the $*$-operator satisfies (1.20), because once it happens we have immediately that

$$*(d\bar{b}_{i_1} \wedge \ldots \wedge d\bar{b}_{i_k}) = d\bar{b}_{j_1} \wedge \ldots \wedge d\bar{b}_{j_{n-k}}$$

for any orthonormal basis $\{\bar{b}_1, \ldots, \bar{b}_n\}$ of V. Since (1.20) holds for α, β elements of the form $db_{i_1} \wedge \ldots \wedge db_{i_k}$ and equality (1.20) is preserved after performing linear operations in each of its entries, then equality (1.20) holds for any $\alpha, \beta \in \Lambda^k V$.

We remark that the composition

$$** : \Lambda^k V \to \Lambda^k V$$

consists simply of the multiplication by the factor $(-1)^{k(n-k)}$, because they are needed $k(n-k)$ transpositions to pass from the ordering $b_{i_1}, \ldots, b_{i_k}, b_{j_1}, \ldots, b_{j_{n-k}}$ to the ordering $b_{j_1}, \ldots, b_{j_{n-k}}, b_{i_1}, \ldots, b_{i_k}$.

Also observe that $*$ is actually a linear orthogonal transformation, because by the property (1.20) for $\alpha, \beta \in \Lambda^k V$,

$$\langle *\alpha \, , *\beta \rangle \, d\mu = *\alpha \wedge **\beta = (-1)^{k(n-k)} *\alpha \wedge \beta = \beta \wedge *\alpha = \langle \beta \, , \alpha \rangle \, d\mu = \langle \alpha \, , \beta \rangle \, d\mu \ ,$$

so $\langle *\alpha \, , *\beta \rangle = \langle \alpha \, , \beta \rangle$.

We shall need later to know how the $*$-operator is affected by a conformal change of the metric $\langle \, , \rangle$ on V: let $\langle \, , \rangle_1$ and $\langle \, , \rangle_2 = \lambda^2 \langle \, , \rangle_1$ be two conformal metrics on V, and let $*_1$ and $*_2$ be their respective Hodge-star operators. Observe that if $\{b_1, \ldots, b_n\}$ is a positive $\langle \, , \rangle_1$-orthonormal basis of V then $\{\frac{1}{\lambda} b_1, \ldots, \frac{1}{\lambda} b_n\}$ is a positive $\langle \, , \rangle_2$-orthonormal basis of V and $\{\lambda db_1, \ldots, \lambda db_n\}$ is a $\langle \, , \rangle_2$-orthonormal basis of $\Lambda^1 V$. Therefore, $\{\lambda^k db_{i_1} \wedge \ldots \wedge db_{i_k}, 1 \le i_1 < i_2 < \ldots < i_k \le n\}$ is a $\langle \, , \rangle_2$-orthonormal basis for $\Lambda^k V$ and

$$*_2(\lambda^k db_{i_1} \wedge \ldots \wedge db_{i_k}) = \lambda^{n-k} db_{j_1} \wedge \ldots \wedge db_{j_{n-k}} = \lambda^{n-k} . *_1 (db_{i_1} \wedge \ldots \wedge db_{i_k}) \ .$$

Hence in $\Lambda^k V$,

$$*_2 = \lambda^{n-2k} . *_1 \ . \tag{1.21}$$

Now we shall consider the particular case when V is a four-dimensional vector space. In this case the $*$-operator on the space of the 2-forms,

$$* : \Lambda^2 V \to \Lambda^2 V$$

satisfies $*^2 = \mathrm{Id}$, so we can consider the subspaces of the *self-dual* and *anti-self-dual* forms in $\Lambda^2 V$,

$$\Lambda^+ V := \{\omega \in \Lambda^2 V ; *\omega = \omega\} \quad \text{and} \quad \Lambda^- V := \{\omega \in \Lambda^2 V ; *\omega = -\omega\},$$

to be the ± 1 eigenspaces of $*$. Clearly any $\omega \in \Lambda^2 V$ can be expressed as

$$\omega = \tfrac{1}{2}(\omega + *\omega) + \tfrac{1}{2}(\omega - *\omega)$$

with $\frac{1}{2}(\omega + *\omega) \in \Lambda^+ V$ and $\frac{1}{2}(\omega - *\omega) \in \Lambda^- V$, so we have the decomposition

$$\Lambda^2 V = \Lambda^+ V \oplus \Lambda^- V$$

which is invariant by $*$. This decomposition is indeed orthogonal because for $\alpha \in \Lambda^+ V$ and $\beta \in \Lambda^- V$ we have

$$\langle\, \alpha \,,\beta \,\rangle = \langle\, *\alpha \,, *\beta \,\rangle = \langle\, \alpha \,, -\beta \,\rangle = -\langle\, \alpha \,,\beta \,\rangle \ ,$$

so $\langle\, \alpha \,,\beta \,\rangle = 0$.

Selecting a positive orthonormal basis $\{b_1, b_2, b_3, b_4\}$ for V, it is straightforward to verify that

$$\left\{ \tfrac{1}{\sqrt{2}}(db_1 \wedge db_2 + db_3 \wedge db_4),\ \tfrac{1}{\sqrt{2}}(db_1 \wedge db_3 + db_4 \wedge db_2),\ \tfrac{1}{\sqrt{2}}(db_1 \wedge db_4 + db_2 \wedge db_3) \right\} \qquad (1.22)$$

constitutes an orthonormal basis for $\Lambda^+ V$, and

$$\left\{ \tfrac{1}{\sqrt{2}}(db_1 \wedge db_2 - db_3 \wedge db_4),\ \tfrac{1}{\sqrt{2}}(db_1 \wedge db_3 - db_4 \wedge db_2),\ \tfrac{1}{\sqrt{2}}(db_1 \wedge db_4 - db_2 \wedge db_3) \right\} \qquad (1.23)$$

constitutes an orthonormal basis for $\Lambda^- V$.

We conclude by remarking that if $\langle\, ,\rangle_1$ and $\langle\, ,\rangle_2 = \lambda^2 \langle\, ,\rangle_1$ are two conformal metrics on V and $*_1 : \Lambda^2 V \to \Lambda^2 V$, $*_2 : \Lambda^2 V \to \Lambda^2 V$ are their respective Hodge-star operators then according to (1.21) we have $*_1 = *_2$, i.e. in the four-dimensional case the Hodge-star operator on the 2-forms and therefore the self-dual and anti-self-dual spaces are conformally invariant objects.

Finally we can pass to the manifold situation. Here M^n is an oriented smooth manifold with a Riemannian metric $\langle\,,\rangle$. In this context, for any $0 \leq k \leq n$, all we have done above can be carried to each of the fibres of the vector bundle Λ^k_M of k-forms on M, so $\langle\,,\rangle$ induces a metric on the vector bundles Λ^k_M and we can consider the Hodge-star operator as a vector bundle map

$$* : \Lambda^k_M \to \Lambda^{n-k}_M \ ,$$

or as a linear operator defined on Ω^k_M (the space of sections of Λ^k_M),

$$* : \Omega^k_M \to \Omega^{n-k}_M \ .$$

More generally, for a given vector bundle E on M we can extend the definition of the Hodge-star operator to the bundle of k-forms on M with values in E to obtain a vector bundle map

$$* : \Lambda^k_M(E) \to \Lambda^{n-k}_M(E)$$

which can be defined and is uniquely determined by satisfying the property

$$*(s.\omega) = s.{*\omega} \ , \text{ for all } s \in E \text{ and } \omega \in \Lambda^k_M$$

(from a more technical viewpoint, since $\Lambda^k_M(E) = \Lambda^k_M \otimes E$, the Hodge-star operator in this context is defined as $* \otimes \mathrm{id}$). We also have the linear operator induced on $\Omega^k_M(E)$ (the space of sections of $\Lambda^k_M(E)$),

$$* : \Omega^k_M(E) \to \Omega^{n-k}_M(E) \ .$$

Now, supposing that M is 4-dimensional, the decomposition of the 2-forms on M into self-dual and anti-self-dual parts gives us the vector bundle decomposition

$$\Lambda^2_M = \Lambda^+_M \oplus \Lambda^-_M \ , \tag{1.24}$$

and more generally, given E a vector bundle on M, the vector bundle decomposition

$$\Lambda^2_M(E) = \Lambda^+_M(E) \oplus \Lambda^-_M(E) \ . \tag{1.25}$$

In particular, we have also the vector space decompositions $\Omega^2_M = \Omega^+_M \oplus \Omega^-_M$ and $\Omega^2_M(E) = \Omega^+_M(E) \oplus \Omega^-_M(E)$.

A connection A on E is said to be *self-dual* (resp. *anti-self-dual*) and called a *SD connection* (resp. an *ASD connection*) if for the decomposition $F_A = F^+_A + F^-_A$ of the curvature of A obtained from the splitting $\Omega^2_M(\mathrm{Ad}\, E) = \Omega^+_M(\mathrm{Ad}\, E) \oplus \Omega^-_M(\mathrm{Ad}\, E)$ we have $F^-_A = 0$ (resp. $F^+_A = 0$).

This notion of self-duality, which a priori depends on the Riemannian metric $\langle\,,\rangle$ on M, is in fact a conformally invariant notion (i.e. it only depends on the conformal class of $\langle\,,\rangle$), as we have already noticed. Also, for $u \in \Omega^0_M(\mathrm{ad}\, E)$ it is readily seen that $*F_{u(A)} = u.(*F_A).u^{-1}$. In particular $F^+_{u(A)} = u.F^+_A.u^{-1}$ and $F^-_{u(A)} = u.F^-_A.u^{-1}$, so the spaces of SD and ASD connections are preserved by the action of the gauge group.

Chapter 2

Twistor Spaces and the Penrose Transform

It is classically known in the 2-dimensional setting the correspondence that exists between conformal geometry and complex analysis on a surface, i.e. there is a well established 'dictionary' between both theories. The Penrose Transform is the 4-dimensional analogous of this correspondence, where the conformal structure of a 4-manifold is encoded in the holomorphic structure of its 'Twistor space'. In this chapter we will develop the rudiments of this theory.

2.1 A Glimpse of Riemannian Geometry. The Special Four-Dimensional Case

Let us begin with some general considerations in the n-dimensional case, so for the time being M is an oriented smooth n-manifold with a Riemannian metric $\langle\, ,\rangle$. In this setting, according to chapter 1 we can consider the tangent bundle TM as a SO_n - vector bundle on M, the *Levi-Civita* connection of M (which will be denoted simply by ∇) as a SO_n - covariant derivative on TM and the curvature tensor (which in this context is denoted by R) as an element of $\Omega^2_M(\mathrm{Ad}\, TM)$.

Recall that the bundle $\mathrm{Ad}\, TM$ is the Lie algebra bundle (with fibre the Lie algebra $\mathcal{SO}_n$) on M whose transition functions are the same as those of TM (seen as a SO_n - vector bundle) after composing with the adjoint representation $\mathrm{Ad} : \mathrm{SO}_n \to \mathrm{Aut}\,\mathcal{SO}_n$, and that there is a natural Lie algebra bundle inclusion $\mathrm{Ad}\,TM \hookrightarrow \mathrm{End}\,TM$. Since $\mathcal{SO}_n$ is just the Lie algebra of skew-symmetric $n \times n$ matrices, it is readily seen that the bundle $\mathrm{Ad}\,TM$, considered as a sub-bundle of $\mathrm{End}\,TM$, is simply the bundle of skew-symmetric endomorphisms of TM.

A local trivialization $\varphi_\alpha : U_\alpha \times \mathbb{R}^n \to TM$ of the tangent bundle of M as a SO_n - vector bundle consists of the choice of a local positive orthonormal frame field $\{b_1, \dots, b_n\}$ of tangent vectors on U_α. Still denoting by ∇ the local representation of the Levi-Civita connection in φ_α and by A the associated local matrix connection, for a local vector field s we have, according (1.6),

$$\nabla_{b_i} s := (\nabla s)(b_i) = \frac{\partial s}{\partial b_i} + A(b_i).s \ .$$

In particular

$$\nabla_{b_i} b_j = A(b_i).b_j \ ,$$

and defining

$$\Gamma^i_{jk} := \langle \nabla_{b_i} b_j, b_k \rangle = \langle A(b_i).b_j, b_k \rangle \tag{2.1}$$

we have

$$\nabla_{b_i} b_j = \sum_k \Gamma^i_{jk}\, b_k \ ,$$

where $A(b_i) = \left(\Gamma^i_{jk}\right)_{j,k} \in \mathcal{SO}_n$, so $\Gamma^i_{jk} = -\Gamma^i_{kj}$. Notice that our 'symbols' Γ^i_{jk} *do not agree* with the usual Christoffel symbols of Riemannian Geometry. This justifies the classical calculations presented below (of course adapted to our context).

Since we are dealing with the Levi-Civita connection, we have a local expression for the Lie bracket of vector fields in M as a function of the coefficients Γ^i_{jk}:

$$[b_i, b_j] = \nabla_{b_i} b_j - \nabla_{b_j} b_i = \sum_k (\Gamma^i_{jk} - \Gamma^j_{ik})\, b_k \ . \tag{2.2}$$

Now, about the curvature tensor R, in a similar way to the computations carried out in (1.13) we have

$$R_{ij} := R(b_i, b_j) = [\nabla_{b_i}, \nabla_{b_j}] - \nabla_{[b_i, b_j]} \ .$$

Thus, denoting $\frac{\partial}{\partial b_i}$ simply by ∂_i , we get

$$\begin{aligned}
R_{ij}(b_k) &= \nabla_{b_i}(\nabla_{b_j} b_k) - \nabla_{b_j}(\nabla_{b_i} b_k) - \nabla_{[b_i,b_j]} b_k \\
&= \nabla_{b_i}(\sum_s \Gamma^j_{ks} b_s) - \nabla_{b_j}(\sum_s \Gamma^i_{ks} b_s) - \sum_s (\Gamma^i_{js} - \Gamma^j_{is}) \nabla_{b_s} b_k \\
&= \sum_s \Gamma^j_{ks} \nabla_{b_i} b_s + \sum_s \partial_i \Gamma^j_{ks}\, b_s - \sum_s \Gamma^i_{ks} \nabla_{b_j} b_s - \sum_s \partial_j \Gamma^i_{ks}\, b_s - \sum_s (\Gamma^i_{js} - \Gamma^j_{is}) \nabla_{b_s} b_k \\
&= \sum_s \Gamma^j_{ks} \nabla_{b_i} b_s - \sum_s \Gamma^i_{ks} \nabla_{b_j} b_s + \sum_s (\partial_i \Gamma^j_{ks} - \partial_j \Gamma^i_{ks})\, b_s - \sum_s (\Gamma^i_{js} - \Gamma^j_{is}) \nabla_{b_s} b_k \ .
\end{aligned}$$

Therefore

$$R_{ijkl} := \langle R_{ij}(b_k), b_l \rangle = \partial_i \Gamma^j_{kl} - \partial_j \Gamma^i_{kl} + \sum_s (\Gamma^j_{ks} \Gamma^i_{sl} - \Gamma^i_{ks} \Gamma^j_{sl}) - \sum_s (\Gamma^i_{js} - \Gamma^j_{is}) \Gamma^s_{kl} \ . \tag{2.3}$$

On the other hand, from a more global point of view, for an inner product vector space $(V, \langle\, ,\rangle)$ there is a natural isomorphism

$$\{\, \textit{Skew-symmetric endomorphisms of } (V, \langle\, ,\rangle) \,\} \longleftrightarrow \Lambda^2 V \tag{2.4}$$

taking S on $\omega_S := \langle S\, .\, , . \rangle$. This isomorphism extends in a natural way to a vector bundle isomorphism

$$\mathrm{Ad}\; TM \xleftrightarrow{S \mapsto \langle S\, .\, , . \rangle} \Lambda^2_M \ . \tag{2.5}$$

These considerations enables us to see the curvature tensor R (or any other similar tensor, as the Weyl tensor W) as a 2-form on M with values in the vector bundle of 2-forms on M or, more conveniently, as a vector bundle endomorphism

$$R : \Lambda^2_M \to \Lambda^2_M \ ,$$

which is completely determined by the relations

$$\langle R(db_i \wedge db_j), db_k \wedge db_l \rangle = R_{ijkl} \ .$$

We conclude from the *Bianchi relations* $R_{ijkl} = R_{klij}$ (see [Be] § 1.85e) that R is actually a symmetric endomorphism.

Now we can move to the four-dimensional situation. The special feature in dimension four is the orthogonal decomposition (1.24) of the 2-forms on M in their self-dual and anti-self-dual parts,

$$\Lambda^2_M = \Lambda^+_M \oplus \Lambda^-_M \ .$$

This decomposition yields the decomposition

$$\mathrm{End}\,\Lambda^2_M = \mathrm{End}\,\Lambda^+_M \oplus \mathrm{Hom}(\Lambda^+_M, \Lambda^-_M) \oplus \mathrm{Hom}(\Lambda^-_M, \Lambda^+_M) \oplus \mathrm{End}\,\Lambda^-_M \ ,$$

such that (with respect to the subspaces Λ^+_M , Λ^-_M) a tensor $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \in \mathrm{End}\,\Lambda^2_M$ has the block matrix form

$$T = \left(\begin{array}{c|c} T_1 & T_3 \\ \hline T_2 & T_4 \end{array}\right) \ .$$

The good news in four dimensions arise by noticing that in this setting the Weyl curvature tensor W turns out to have a diagonal block matrix form,

$$W = \left(\begin{array}{c|c} W^+ & 0 \\ \hline 0 & W^- \end{array}\right) \ ,$$

where W^+ and W^- correspond to the splitting $\Omega^2_M(\mathrm{Ad}\ TM) = \Omega^+_M(\mathrm{Ad}\ TM) \oplus \Omega^-_M(\mathrm{Ad}\ TM)$ as in (1.25). The full curvature tensor R also has a simple block matrix form:

$$R = \left(\begin{array}{c|c} W^+ + \frac{s}{12}\,\mathrm{Id} & B \\ \hline B^{\intercal} & W^- + \frac{s}{12}\,\mathrm{Id} \end{array}\right) \ , \tag{2.6}$$

where s is the usual scalar curvature and B is the traceless Ricci tensor (in an unusual guise). See ([Be] § 1.128) for further reference and details.

2.2 The Bundle of Orthogonal Complex Structures

Let $(V, \langle\,,\rangle)$ be an inner product vector space of dimension $n = 2m$. An *orthogonal complex structure* on $(V, \langle\,,\rangle)$ is an endomorphism $J \in \mathrm{End}\,V$ that preserves the inner product $\langle\,,\rangle$ and satisfies $J^2 = -\mathrm{Id}$. It is clear that J automatically induces a structure of complex vector space on V, where J acts as the multiplication by the complex scalar i. The tensor J also induces an orientation on V by choosing any complex basis $\{\mu_1, \mu_2, \ldots, \mu_m\}$ of V and by declaring as a positive basis the real basis $\{\mu_1, J\mu_1, \mu_2, J\mu_2, \ldots, \mu_m, J\mu_m\}$. Since $J^2 = -\mathrm{Id}$ we have $J^{-1} = -J$ and since J is orthogonal we have $J^{-1} = J^{\intercal}$. In particular $J^{\intercal} = -J$, so an orthogonal complex structure is always a skew-symmetric endomorphism.

Again, in the four-dimensional case, the self-duality phenomena gives us some special features: it turns out that if $(V, \langle\,,\rangle)$ is an oriented inner product 4-dimensional vector space, the correspondence (2.4) identifies in particular the set of orthogonal complex structures on V inducing the given orientation with the 2-sphere $S_{\sqrt{2}}(\Lambda^+V)$, the set of the self-dual forms with $\sqrt{2}$ norm. This fact is resumed in the following diagram:

$$\begin{tikzcd}
\{\, \textit{Skew-symmetric endomorphisms of } (V,\langle\,,\rangle)\,\} \arrow[r, leftrightarrow, "S\mapsto\langle S\,.\,,\,.\,\rangle"] & \Lambda^2V \\
\left\{\begin{matrix}\textit{Orthogonal complex structures on } (V,\langle\,,\rangle) \\ \textit{inducing the given orientation}\end{matrix}\right\} \arrow[u, hook] \arrow[r, leftrightarrow, dashed] & S_{\sqrt{2}}(\Lambda^+V) \arrow[u, hook]
\end{tikzcd} \tag{2.7}$$

We shall remain in the four-dimensional setting. Consider a positive orthonormal basis $\{b_1, b_2, b_3, b_4\}$ for V. Skew-symmetric endomorphisms of V will be considered as skew-symmetric 4×4 matrices with respect to this basis. Define

$$\begin{aligned}
\mathbf{w_1} &:= db_1 \wedge db_2 + db_3 \wedge db_4\ , \\
\mathbf{w_2} &:= db_1 \wedge db_3 + db_4 \wedge db_2\ , \\
\mathbf{w_3} &:= db_1 \wedge db_4 + db_2 \wedge db_3\ .
\end{aligned} \tag{2.8}$$

We have from (1.22) that $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ constitutes an orthogonal vector basis for Λ^+V with $\|\mathbf{w_1}\| = \|\mathbf{w_2}\| = \|\mathbf{w_3}\| = \sqrt{2}$. In this setting the identification (2.7) between orthogonal complex structures and self-dual 2-forms becomes

$$\begin{pmatrix} 0 & -y_1 & -y_2 & -y_3 \\ y_1 & 0 & -y_3 & y_2 \\ y_2 & y_3 & 0 & -y_1 \\ y_3 & -y_2 & y_1 & 0 \end{pmatrix} \longleftrightarrow (y_1\mathbf{w_1} + y_2\mathbf{w_2} + y_3\mathbf{w_3})\ , \tag{2.9}$$

with $y_1^2 + y_2^2 + y_3^2 = 1$.

From now on, in virtue of (2.4), we shall consider skew-symmetric endomorphisms and 2-forms as being the same. Observe that for an orthogonal complex structure J and for a skew-symmetric endomorphism A we have

$$[A, J\,] = (d(\mathrm{Ad})_{\mathrm{Id}}.A)(J) \in T_J(S_{\sqrt{2}}(\Lambda^+V)) \subset \Lambda^+V\ , \tag{2.10}$$

because for any orthogonal automorphism T of V the automorphism $\mathrm{Ad}\,T.J = TJT^{-1}$ is an orthogonal complex structure, i.e. $TJT^{-1} \in S_{\sqrt{2}}(\Lambda^+V)$.

Consider the matrix form of the skew-symmetric endomorphism A,

$$A = \begin{pmatrix} 0 & -a_{12} & -a_{13} & -a_{14} \\ a_{12} & 0 & -a_{23} & a_{42} \\ a_{13} & a_{23} & 0 & -a_{34} \\ a_{14} & -a_{42} & a_{34} & 0 \end{pmatrix}\ ,$$

and $J = y_1\mathbf{w_1}+y_2\mathbf{w_2}+y_3\mathbf{w_3}$ as in (2.9). After some calculation, we obtain the expression of $[A, J\,]$ as a 2-form:

$$\begin{aligned}[A, J\,] = {} & ((a_{13} + a_{42})y_3 - (a_{14} + a_{23})y_2)\mathbf{w_1} \\ & +((a_{14} + a_{23})y_1 - (a_{12} + a_{34})y_3)\mathbf{w_2} + ((a_{12} + a_{34})y_2 - (a_{13} + a_{42})y_1)\mathbf{w_3}\ .\end{aligned} \tag{2.11}$$

Since $T_J(S_{\sqrt{2}}(\Lambda^+V)) = \{\tau \in \Lambda^+V; \langle\, \tau, J\,\rangle = 0\}$, we have $\langle\, [A, J\,]\,, J\,\rangle = 0\,$.

As a final observation, note that the spaces of skew-symmetric endomorphisms of V and orthogonal complex structures on V are *conformally invariant* objects, i.e. they do not change when considering the metrics $\langle\,,\rangle_1$ and $\langle\,,\rangle_2 = \lambda^2\langle\,,\rangle_1$ on V. However, the isomorphism (2.4) is not conformally invariant.

All we have done above can be carried to the manifold setting, working simultaneously on each tangent space. So let M be an oriented smooth Riemannian 4-manifold. The bundle of orthogonal complex structures on the tangent spaces of M inducing the given orientation is called the *Twistor Space* of M and is denoted by $\mathcal{Z}_M$ or simply $\mathcal{Z}$. We have already remarked that the bundle $\mathcal{Z}$ (as well as $\mathrm{Ad}\,TM$, Λ^+_M and Λ^-_M) only depends on the conformal structure of M and that for each prefered choice of a metric in the conformal class we have, extending (2.7), a bundle isomorphism between $\mathcal{Z}$ and the 2-sphere bundle $S_{\sqrt{2}}(\Lambda^+_M)$:

$$\begin{tikzcd} \mathrm{Ad}\, TM \arrow[r, leftrightarrow, "S \mapsto \langle S\,.\,,\,.\,\rangle"] & \Lambda^2_M \\ \mathcal{Z} \arrow[u, hook] \arrow[r, leftrightarrow, "J \mapsto \langle J.\,,\,.\,\rangle"] & S_{\sqrt{2}}(\Lambda^+_M) \arrow[u, hook] \end{tikzcd} \tag{2.12}$$

In particular, $\mathcal{Z}$ is a 6-dimensional manifold.

We know that, unlike the 2-dimensional case, an oriented 4-manifold M do not need to have any globally defined almost-complex structures at all. However, its twistor space $\mathcal{Z}$ carries a natural almost-complex structure which only depends on the conformal structure of M, and the integrability condition for this almost-complex structure is encoded in the conformal part of the curvature of M. An explanation of all of this will be done in the next section.

2.3 The Penrose Construction

In this section we shall proceed with the construction, due to R. Penrose (see [Pen]), of a natural almost-complex structure on the twistor space $\mathcal{Z}_M$ of an oriented Riemannian 4-manifold M. It turns out that the obstruction to the integrability of this complex structure lies in W^+, the self-dual part of the Weyl curvature tensor of M.

The starting point for this construction is to consider the horizontal distribution $\mathcal{H} \subset T(\mathrm{Ad}\, TM)$ induced by the Levi-Civita connection ∇ of M on the vector bundle $\mathrm{Ad}\, TM$, according (4) in pg. 13 . Recall that there is a natural inclusion $\mathcal{Z} \hookrightarrow \mathrm{Ad}\, TM$ and notice that for $J \in \mathcal{Z}$ we have $\mathcal{H}_J \subset T_J\mathcal{Z}$ (in other words, $\mathcal{H}$ preserves $\mathcal{Z}$), because the parallel transport in End TM preserves both orthogonal linear transformations and skew-symmetric endomorphisms. Thus, $\mathcal{H}$ can be also seen as the horizontal distribution induced by ∇ on the bundle $\mathcal{Z}$.

The next step consists in to identify $\mathcal{Z}$ with the sphere bundle $S_{\sqrt{2}}(\Lambda^+_M)$ according (2.12). It is not hard to see that the diagram (2.12) preserves the parallel transport on each bundle involved. Therefore the horizontal distribution on the bundle Λ^2_M induced by ∇ can be identified with $\mathcal{H}$ and, as before, it preserves Λ^+_M , $S_{\sqrt{2}}(\Lambda^+_M)$ and can be also seen as the horizontal distribution induced on these bundles. In this way, given J_x a point in $\mathcal{Z}$ (i.e. J_x is an orthogonal complex structure on T_xM), we have the decomposition

$$T_{J_x}\mathcal{Z} = T_{J_x}(S_{\sqrt{2}}(\Lambda^+_M)) = T_{J_x}(S_{\sqrt{2}}\big(\Lambda^+_M\big)_x) \oplus \mathcal{H}_{J_x}\ , \tag{2.13}$$

where $S_{\sqrt{2}}(\Lambda^+_M)_x$ is the fibre of $S_{\sqrt{2}}(\Lambda^+_M)$ on x.

The final step of the construction consists in to observe that the subspaces of $T_{J_x}\mathcal{Z}$ arising in the decomposition (2.13) admit natural complex structures:

The projection $\pi : \mathcal{Z} \to M$ induces an isomorphism $\pi_* : \mathcal{H}_{J_x} \to T_xM$. Therefore, we can carry the complex structure J_x on T_xM through π_* to obtain a complex structure on $\mathcal{H}_{J_x}$. On the other hand, choosing $\{b_1, b_2, b_3, b_4\}$ a positive orthonormal basis of T_xM and declaring the basis $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ (defined in (2.8)) a positive basis for $(\Lambda^+_M)_x$ makes Λ^+_M an oriented vector bundle. Therefore the sphere $S_{\sqrt{2}}(\Lambda^+_M)_x$, seen as an orientable surface in the 3-dimensional euclidean vector space $(\Lambda^+_M)_x$, carries an almost-complex structure which is given in each tangent space by the vector cross product in $(\Lambda^+_M)_x$ with the outward unit normal vector to the sphere. In other words, for $J_x \in S_{\sqrt{2}}(\Lambda^+_M)_x$ and $\tau \in T_{J_x}(S_{\sqrt{2}}(\Lambda^+_M)_x)$, the almost-complex structure is given by $\frac{1}{\sqrt{2}} J_x \times \tau$.

Thus, the almost-complex structure on $\mathcal{Z}$ is defined as an endomorphism $\mathbf{J}$ of $T\mathcal{Z}$ whose restriction to the subspaces $T_{J_x}(S_{\sqrt{2}}(\Lambda^+_M)_x)$ and $\mathcal{H}_{J_x}$ is given as above.

At this point is good to remark that the almost-complex structure $\mathbf{J}$ defined on $\mathcal{Z}$ is actually *conformally invariant*, though the distribution $\mathcal{H}$ is not. This fact will be proved in the next section. In particular, the obstruction to the integrability of $\mathbf{J}$ can be encoded in the conformal structure of M, as is described in the following theorem:

Theorem 2.3.1 (Penrose [Pen], [AHS]) *Let M be an oriented Riemannian 4-manifold and $\mathcal{Z}$ its twistor space. Then the almost-complex structure $\mathbf{J}$ on $\mathcal{Z}$ is integrable (i.e. $\mathcal{Z}$ becomes a complex manifold) if and only if the Weyl curvature tensor W of M satisfies $W^+ = 0$.*

Classical proofs of Theorem (2.3.1) (like [AHS] or [Be] § 13.46) involves rather sophisticated concepts and details. Thus, we consider that it is worth to provide a more elementary (though more extensive) proof. This will be done in the next section.

An oriented Riemannian 4-manifold $(M, \langle\, , \rangle)$ whose Weyl tensor W satisfies $W^+ = 0$ is called an *anti-self-dual manifold*. The metric $\langle\, , \rangle$ is called an *ASD metric*, and its conformal equivalence class is called an *ASD conformal structure*. Since the Weyl tensor $W \in \Lambda^2_M(\mathrm{Ad}\, TM)$ and the decomposition $\Lambda^2_M(\mathrm{Ad}\, TM) = \Lambda^+_M(\mathrm{Ad}\, TM) \oplus \Lambda^-_M(\mathrm{Ad}\, TM)$ are conformally invariants, any metric within an ASD conformal structure is clearly an ASD metric. From now on, we shall consider the twistor space $\mathcal{Z}$ of an anti-self-dual 4-manifold M as a complex 3-manifold. The twistor space, as a complex manifold, carries an extra structure which is settled in the next theorem:

Theorem 2.3.2 ([Be] § 13.63) *The twistor space $\mathcal{Z}$ of an anti-self-dual 4-manifold M satisfies the following properties:*

(1) *The fibres of $\pi : \mathcal{Z} \to M$ are holomorphic curves in $\mathcal{Z}$. Each is a rational curve $\mathbb{C}P^1$ whose normal bundle in $\mathcal{Z}$ is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$.*

(2) *$\mathcal{Z}$ possesses a free antiholomorphic involution (real structure) $i : \mathcal{Z} \to \mathcal{Z}$ which in each fibre corresponds to the classical antiholomorphic involution $z \mapsto -1/\overline{z}$ on the Riemann sphere $\mathbb{C}P^1 \simeq \overline{\mathbb{C}}$.*

The fibres of $\pi : \mathcal{Z} \to M$ are called the *twistor lines* of $\mathcal{Z}$. The antiholomorphic involution $i : \mathcal{Z} \to \mathcal{Z}$ in the preceding theorem restricted to the fibre corresponding to $x \in M$ is nothing but the antipodal map on the sphere $S_{\sqrt{2}}(\Lambda^+_M)_x$. The proof that this involution is in fact antiholomorphic readily follows from a direct computation in local coordinates using formula (2.14) of section (2.4) below. A proof of property (1) will be given in section (2.5).

It turns out that the properties of Theorem (2.3.2) completely characterize $\mathcal{Z}$ as a twistor space. This is provided by the following converse to Theorem (2.3.1):

Theorem 2.3.3 (Penrose [Pen]) *Let $\mathcal{Z}$ be a complex 3-manifold such that*

(1) *$\mathcal{Z}$ is fibred by projective lines whose normal bundles are isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$.*

(2) *$\mathcal{Z}$ possesses a free antiholomorphic involution i which transforms each fibre to itself.*

Then $\mathcal{Z}$ is the twistor space of some anti-self-dual 4-manifold M.

The manifold M in the preceding theorem is nothing but the space of fibres of $\mathcal{Z}$. We shall describe in section (2.5) how to obtain the ASD conformal structure on M which inverts the Penrose construction above.

Putting Theorems (2.3.1) and (2.3.3) together, we see that there is a well-defined correspondence between ASD 4-manifolds M up to conformal diffeomorphisms and complex 3-manifolds $\mathcal{Z}$ as in Theorem (2.3.3) up to structure-preserving biholomorphisms. This correspondence is known as the *Penrose Transform*.

Finally, we have already commented that natural geometrical properties of the 4-manifold M are often reflected in equally natural holomorphic properties of the twistor space $\mathcal{Z}$. Some relevant cases of this correspondence are illustrated in the theorem below:

Theorem 2.3.4 ([Hit1]) *Let $\mathcal{Z}$ be the twistor space of an anti-self-dual 4-manifold M. Then:*

(1) *An Einstein metric in the conformal class of M is defined by a holomorphic section θ of $\Omega^{1,0}_{\mathcal{Z}}(K^{-1/2})$ which is compatible with i and restricts to a non-zero form on each fibre of $\mathcal{Z}$.*

(2) *An scalar-flat Kähler metric in the conformal class of M is defined by a holomorphic section s of $K^{-1/2}$ compatible with i and non-zero on each fibre of $\mathcal{Z}$.*

(3) *A hypercomplex structure in the conformal class of M is defined by a holomorphic projection $p : \mathcal{Z} \to \mathbb{C}P^1$ compatible with i and for which p is an isomorphism on each fibre of $\mathcal{Z}$.*

Here $K^{-1/2}$ is a distinguished square root of the anticanonical bundle K^{-1} of $\mathcal{Z}$.

2.4 Proof of Penrose Theorem

2.4.1 Computation of the Nijenhuis Tensor

Our proof of Theorem (2.3.1) relies on the computation of the *Nijenhuis tensor* $\mathbf{N}$ of the almost-complex manifold $(\mathcal{Z}, \mathbf{J})$:

$$\mathbf{N}(X,Y) = [\mathbf{J}X, \mathbf{J}Y] - \mathbf{J}[\mathbf{J}X, Y] - \mathbf{J}[X, \mathbf{J}Y] - [X,Y] \ ,$$

where $[\![\,,\,]\!]$ denotes the Lie bracket for vector fields (the notation is slightly different of the matrix Lie bracket $[\,,]$). By the Newlander-Niremberg Theorem [NeNi], the vanishing of the tensor $\mathbf{N}$ implies that the almost-complex structure $\mathbf{J}$ on $\mathcal{Z}$ is induced by a (unique) honest complex structure.

In accomplishing this, consider a local trivialization $\varphi_\alpha : U_\alpha \times \mathbb{R}^4 \to TM$ of the tangent bundle of M as a SO_4 - vector bundle, i.e. choose a local positive orthonormal frame field $\{b_1, b_2, b_3, b_4\}$ of tangent vectors on U_α. This trivialization induces the local trivialization $\overline{\varphi}_\alpha : U_\alpha \times \Lambda^+\mathbb{R}^4 \to \Lambda^+_M$ of the bundle of self-dual forms, which takes the local positive orthonormal frame $\{\frac{1}{\sqrt{2}}\mathbf{w_1}, \frac{1}{\sqrt{2}}\mathbf{w_2}, \frac{1}{\sqrt{2}}\mathbf{w_3}\}$ of Λ^+_M (defined in (2.8)) on the standard orthonormal basis of self-dual forms of $\mathbb{R}^4$. Since $\mathcal{Z} \subset \Lambda^+_M$ and $\overline{\varphi}_\alpha$ is a diffeomorphism of $U_\alpha \times \Lambda^+\mathbb{R}^4$ on an open set in Λ^+_M, we can carry the calculus of the Nijenhuis tensor to the manifold $U_\alpha \times \Lambda^+\mathbb{R}^4$:
For $x \in U_\alpha$ and $J = y_1\mathbf{w_1} + y_2\mathbf{w_2} + y_3\mathbf{w_3} \in S_{\sqrt{2}}(\Lambda^+_M)_x$ a complex structure in T_xU_α, the point $p = (x, J) \in S_{\sqrt{2}}(\Lambda^+_M)$ has coordinates $(x_1, x_2, x_3, x_4, y_1, y_2, y_3)$ in $U_\alpha \times \Lambda^+\mathbb{R}^4$. Observe that $\{b_1, b_2, b_3, b_4, \mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$, via the identification $\overline{\varphi}_\alpha$, constitutes an orthogonal vector basis of $T_xU_\alpha \oplus \Lambda^+\mathbb{R}^4 = T_{(x,J)}(U_\alpha \times \Lambda^+\mathbb{R}^4)$. Notice also that $[\![b_i, \mathbf{w_j}]\!] = 0$ and $[\![\mathbf{w_i}, \mathbf{w_j}]\!] = 0$.

Adopting the notation of section (2.1) and considering skew-symmetric endomorphisms and 2-forms as being the same, the horizontal distribution $\mathcal{H}$ on $\mathrm{Ad}\, TM = \Lambda^2_M$ can be expressed in this local trivialization, according (1.8) and (1.11), as

$$\mathcal{H}_{(x,J)} = \{(v, -[A(v), J])\,;\, v \in T_xU_\alpha\}\ .$$

In this way, a vector $v + \tau \in T_xU_\alpha \oplus \Lambda^+\mathbb{R}^4 = T_{(x,J)}(U_\alpha \times \Lambda^+\mathbb{R}^4)$ decomposes as

$$(v - [A(v), J\,]) + ([A(v), J\,] + \tau) \in \mathcal{H}_{(x,J)} \oplus \Lambda^+\mathbb{R}^4 = T_{(x,J)}(U_\alpha \times \Lambda^+\mathbb{R}^4)\,,$$

because according (2.10) we have $[A(v), J\,] \in \Lambda^+\mathbb{R}^4$. We also have from (2.10) that $\langle\, [A(v), J\,]\,, J\,\rangle = 0$, so $\langle\, \tau, J\,\rangle = 0 \Leftrightarrow \langle\, [A(v), J\,] + \tau, J\,\rangle = 0$. Therefore

$$T_{(x,J)}(U_\alpha \times S_{\sqrt{2}}(\Lambda^+\mathbb{R}^4)) = T_xU_\alpha \oplus T_J(S_{\sqrt{2}}(\Lambda^+\mathbb{R}^4)) = \mathcal{H}_{(x,J)} \oplus T_J(S_{\sqrt{2}}(\Lambda^+\mathbb{R}^4))\ .$$

Thus, we finally arrive to a local formula for the complex structure $\mathbf{J}$:

For $v+\tau \in T_xU_\alpha \oplus T_J(S_{\sqrt{2}}(\Lambda^+\mathbb{R}^4))$ we have

$$\begin{aligned}\mathbf{J}(v+\tau) &= \mathbf{J}(v-[A(v),J\,]) + \mathbf{J}([A(v),J\,]+\tau)\\ &= (Jv-[A(Jv),J\,]) + \tfrac{1}{\sqrt{2}}J \times ([A(v),J\,]+\tau)\\ &= Jv + \Big(-[A(Jv),J\,] + \tfrac{1}{\sqrt{2}}J\times[A(v),J\,] + \tfrac{1}{\sqrt{2}}J\times\tau\Big)\end{aligned} \tag{2.14}$$

in $T_xU_\alpha \oplus T_J(S_{\sqrt{2}}(\Lambda^+\mathbb{R}^4))$.

In order to determine the tensor $\mathbf{N}$, we only need to compute the values of $\mathbf{N}(b_i,b_j)$ and $\mathbf{N}(b_i,\tau)$, where τ is a tangent vector field in $S_{\sqrt{2}}(\Lambda^+\mathbb{R}^4)$. In fact, by re-ordering the terms of the basis $\{b_1,b_2,b_3,b_4\}$, we only need to compute the values of $\mathbf{N}(b_1,b_2)$ and $\mathbf{N}(b_1,\tau)$.

So consider the matrix form of $A(b_i)$ and J in the basis $\{b_1,b_2,b_3,b_4\}$ (see (2.9)),

$$A(b_i) = \begin{pmatrix} 0 & -\Gamma^i_{12} & -\Gamma^i_{13} & -\Gamma^i_{14}\\ \Gamma^i_{12} & 0 & -\Gamma^i_{23} & \Gamma^i_{42}\\ \Gamma^i_{13} & \Gamma^i_{23} & 0 & -\Gamma^i_{34}\\ \Gamma^i_{14} & -\Gamma^i_{42} & \Gamma^i_{34} & 0\end{pmatrix}\ ,\quad J = \begin{pmatrix} 0 & -y_1 & -y_2 & -y_3\\ y_1 & 0 & -y_3 & y_2\\ y_2 & y_3 & 0 & -y_1\\ y_3 & -y_2 & y_1 & 0\end{pmatrix}\ .$$

We have already seen in (2.11) that the expression of $[A(b_i),J\,]$ as a 2-form is

$$\begin{aligned}[A(b_i),J\,] = {}&((\Gamma^i_{13}+\Gamma^i_{42})y_3 - (\Gamma^i_{14}+\Gamma^i_{23})y_2)\mathbf{w_1}\\ &+((\Gamma^i_{14}+\Gamma^i_{23})y_1 - (\Gamma^i_{12}+\Gamma^i_{34})y_3)\mathbf{w_2} + ((\Gamma^i_{12}+\Gamma^i_{34})y_2 - (\Gamma^i_{13}+\Gamma^i_{42})y_1)\mathbf{w_3}\ .\end{aligned}$$

Thus, after some calculation, the vector cross product $\frac{1}{\sqrt{2}}J\times[A(b_i),J\,]$ has the expression

$$\begin{aligned}\tfrac{1}{\sqrt{2}}J\times[A(b_i),J\,] = {}&((\Gamma^i_{12}+\Gamma^i_{34})(y_2^2+y_3^2) - (\Gamma^i_{13}+\Gamma^i_{42})y_1y_2 - (\Gamma^i_{14}+\Gamma^i_{23})y_1y_3)\mathbf{w_1}\\ &+((\Gamma^i_{13}+\Gamma^i_{42})(y_1^2+y_3^2) - (\Gamma^i_{12}+\Gamma^i_{34})y_1y_2 - (\Gamma^i_{14}+\Gamma^i_{23})y_2y_3)\mathbf{w_2}\\ &+((\Gamma^i_{14}+\Gamma^i_{23})(y_1^2+y_2^2) - (\Gamma^i_{12}+\Gamma^i_{34})y_1y_3 - (\Gamma^i_{13}+\Gamma^i_{42})y_2y_3)\mathbf{w_3}\ .\end{aligned}$$

Therefore, according formula (2.14),

$$\begin{aligned}\mathbf{J}b_1 &= Jb_1 + (-[A(Jb_1),J\,] + \tfrac{1}{\sqrt{2}}J\times[A(b_1),J\,])\\ &= y_1b_2+y_2b_3+y_3b_4 + (-y_1[A(b_2),J\,] - y_2[A(b_3),J\,] - y_3[A(b_4),J\,] + \tfrac{1}{\sqrt{2}}J\times[A(b_1),J\,])\\ &= y_1b_2+y_2b_3+y_3b_4 + C_{11}\mathbf{w_1} + C_{12}\mathbf{w_2} + C_{13}\mathbf{w_3}\ .\end{aligned} \tag{2.15}$$

Where the explicit expressions for C_{11}, C_{12}, C_{13} are

$$C_{11}=(\Gamma_{12}^1+\Gamma_{34}^1+\Gamma_{14}^3+\Gamma_{23}^3)y_2^2+(\Gamma_{12}^1+\Gamma_{34}^1-\Gamma_{13}^4-\Gamma_{42}^4)y_3^2+(\Gamma_{14}^2+\Gamma_{23}^2-\Gamma_{13}^1-\Gamma_{42}^1)y_1y_2-(\Gamma_{14}^1+\Gamma_{23}^1+\Gamma_{13}^2+\Gamma_{42}^2)y_1y_3+(\Gamma_{14}^4+\Gamma_{23}^4-\Gamma_{13}^3-\Gamma_{42}^3)y_2y_3$$

$$C_{12}=(\Gamma_{13}^1+\Gamma_{42}^1-\Gamma_{14}^2-\Gamma_{23}^2)y_1^2+(\Gamma_{13}^1+\Gamma_{42}^1+\Gamma_{12}^4+\Gamma_{34}^4)y_3^2-(\Gamma_{12}^1+\Gamma_{34}^1+\Gamma_{14}^3+\Gamma_{23}^3)y_1y_2+(\Gamma_{12}^2+\Gamma_{34}^2-\Gamma_{14}^4-\Gamma_{23}^4)y_1y_3+(\Gamma_{12}^3+\Gamma_{34}^3-\Gamma_{14}^1-\Gamma_{23}^1)y_2y_3$$

$$C_{13}=(\Gamma_{14}^1+\Gamma_{23}^1+\Gamma_{13}^2+\Gamma_{42}^2)y_1^2+(\Gamma_{14}^1+\Gamma_{23}^1-\Gamma_{12}^3-\Gamma_{34}^3)y_2^2+(\Gamma_{13}^3+\Gamma_{42}^3-\Gamma_{12}^2-\Gamma_{34}^2)y_1y_2+(\Gamma_{13}^4+\Gamma_{42}^4-\Gamma_{12}^1-\Gamma_{34}^1)y_1y_3-(\Gamma_{13}^1+\Gamma_{42}^1+\Gamma_{12}^4+\Gamma_{34}^4)y_2y_3$$

In the same way,

$$\begin{aligned}\mathbf{J}b_2 &= Jb_2+\left(-[A(Jb_2),J]+\tfrac{1}{\sqrt{2}}J\times[A(b_2),J]\right)\\ &= -y_1b_1+y_3b_3-y_2b_4+\left(y_1[A(b_1),J]-y_3[A(b_3),J]+y_2[A(b_4),J]+\tfrac{1}{\sqrt{2}}J\times[A(b_2),J]\right)\\ &= -y_1b_1+y_3b_3-y_2b_4+C_{21}\mathbf{w_1}+C_{22}\mathbf{w_2}+C_{23}\mathbf{w_3}\ .\end{aligned}$$

Where the explicit expressions for C_{21}, C_{22}, C_{23} are

$$C_{21}=(\Gamma_{12}^2+\Gamma_{34}^2-\Gamma_{14}^4-\Gamma_{23}^4)y_2^2+(\Gamma_{12}^2+\Gamma_{34}^2-\Gamma_{13}^3-\Gamma_{42}^3)y_3^2-(\Gamma_{14}^1+\Gamma_{23}^1+\Gamma_{13}^2+\Gamma_{42}^2)y_1y_2+(\Gamma_{13}^1+\Gamma_{42}^1-\Gamma_{14}^2-\Gamma_{23}^2)y_1y_3+(\Gamma_{14}^3+\Gamma_{23}^3+\Gamma_{13}^4+\Gamma_{42}^4)y_2y_3$$

$$C_{22}=(\Gamma_{14}^1+\Gamma_{23}^1+\Gamma_{13}^2+\Gamma_{42}^2)y_1^2+(\Gamma_{13}^2+\Gamma_{42}^2+\Gamma_{12}^3+\Gamma_{34}^3)y_3^2+(\Gamma_{14}^4+\Gamma_{23}^4-\Gamma_{12}^2-\Gamma_{34}^2)y_1y_2-(\Gamma_{12}^1+\Gamma_{34}^1+\Gamma_{14}^3+\Gamma_{23}^3)y_1y_3-(\Gamma_{14}^2+\Gamma_{23}^2+\Gamma_{12}^4+\Gamma_{34}^4)y_2y_3$$

$$C_{23}=(\Gamma_{14}^2+\Gamma_{23}^2-\Gamma_{13}^1-\Gamma_{42}^1)y_1^2+(\Gamma_{14}^2+\Gamma_{23}^2+\Gamma_{12}^4+\Gamma_{34}^4)y_2^2+(\Gamma_{12}^1+\Gamma_{34}^1-\Gamma_{13}^4-\Gamma_{42}^4)y_1y_2+(\Gamma_{13}^3+\Gamma_{42}^3-\Gamma_{12}^2-\Gamma_{34}^2)y_1y_3-(\Gamma_{13}^2+\Gamma_{42}^2+\Gamma_{12}^3+\Gamma_{34}^3)y_2y_3$$

For a tangent vector field $\tau(y)=\tau_1(y)\mathbf{w_1}+\tau_2(y)\mathbf{w_2}+\tau_3(y)\mathbf{w_3}$ in $S_{\sqrt{2}}(\Lambda^+\mathbb{R}^4)$, i.e. τ satisfies $\tau_1y_1+\tau_2y_2+\tau_3y_3=0$, we have according (2.14),

$$\begin{aligned}\mathbf{J}\tau &= \tfrac{1}{\sqrt{2}}J\times\tau\\ &= \tfrac{1}{\sqrt{2}}(y_1\mathbf{w_1}+y_2\mathbf{w_2}+y_3\mathbf{w_3})\times(\tau_1\mathbf{w_1}+\tau_2\mathbf{w_2}+\tau_3\mathbf{w_3})\\ &= (\tau_3y_2-\tau_2y_3)\mathbf{w_1}+(\tau_1y_3-\tau_3y_1)\mathbf{w_2}+(\tau_2y_1-\tau_1y_2)\mathbf{w_3}\ .\end{aligned}\tag{2.16}$$

Now, in order to compute the Lie bracket $[\mathbf{J}b_1,b_2]$, recall that we have $[b_i,\mathbf{w_j}]=0$, $[\mathbf{w_i},\mathbf{w_j}]=0$ and there exists the following formula (cf. [dCa] pg. 27):

$$[fX,gY]=fg\,[X,Y]+f\,\frac{\partial g}{\partial X}\,Y-g\,\frac{\partial f}{\partial Y}\,X\ .$$

Thus, denoting $\partial_i:=\frac{\partial}{\partial b_i}$, we have

$$\begin{aligned}[\mathbf{J}b_1,b_2] &= [y_1b_2,b_2]+[y_2b_3,b_2]+[y_3b_4,b_2]+[C_{11}\mathbf{w_1},b_2]+[C_{12}\mathbf{w_2},b_2]+[C_{13}\mathbf{w_3},b_2]\\ &= y_2\,[b_3,b_2]+y_3\,[b_4,b_2]-\partial_2C_{11}\mathbf{w_1}-\partial_2C_{12}\mathbf{w_2}-\partial_2C_{13}\mathbf{w_3}\ .\end{aligned}$$

Since $\mathbf{N}$ is a tensor, we only need to know his value at the point $p=(x,J)$. Thus, to simplify calculations, let us suppose that the frame $\{b_1,b_2,b_3,b_4\}$ is chosen to be geodesic at x, i.e. $\forall\, i, j$ we have $\nabla_{b_i} b_j\,(x) = 0$. So, at the point x, formulas (2.1), (2.2) and (2.3) reduce to

$$\begin{aligned} \Gamma^i_{jk}(x) &= 0 \ . \\ [b_i, b_j]\,(x) &= 0 \ . \\ R_{ijkl}(x) &= \partial_i \Gamma^j_{kl}(x) - \partial_j \Gamma^i_{kl}(x) \ . \end{aligned} \tag{2.17}$$

Therefore, at the point p we have

$$[\mathbf{J}b_1, b_2]\,(p) = -\partial_2 C_{11}(p)\mathbf{w_1} - \partial_2 C_{12}(p)\mathbf{w_2} - \partial_2 C_{13}(p)\mathbf{w_3} \ .$$

It follows that

$$\begin{aligned} \mathbf{J}\,[\mathbf{J}b_1, b_2]\,(p) &= \tfrac{1}{\sqrt{2}} J \times (-\partial_2 C_{11}(p)\mathbf{w_1} - \partial_2 C_{12}(p)\mathbf{w_2} - \partial_2 C_{13}(p)\mathbf{w_3}) \\ &= (\partial_2 C_{12}\, y_3 - \partial_2 C_{13}\, y_2)(p)\mathbf{w_1} \\ &\quad + (\partial_2 C_{13}\, y_1 - \partial_2 C_{11}\, y_3)(p)\mathbf{w_2} + (\partial_2 C_{11}\, y_2 - \partial_2 C_{12}\, y_1)(p)\mathbf{w_3} \ . \end{aligned}$$

Analogously, we obtain

$$[\mathbf{J}b_2, b_1]\,(p) = -\partial_1 C_{21}(p)\mathbf{w_1} - \partial_1 C_{22}(p)\mathbf{w_2} - \partial_1 C_{23}(p)\mathbf{w_3} \ .$$

So,

$$\begin{aligned} \mathbf{J}\,[\mathbf{J}b_2, b_1]\,(p) &= \tfrac{1}{\sqrt{2}} J \times (-\partial_1 C_{21}(p)\mathbf{w_1} - \partial_1 C_{22}(p)\mathbf{w_2} - \partial_1 C_{23}(p)\mathbf{w_3}) \\ &= (\partial_1 C_{22}\, y_3 - \partial_1 C_{23}\, y_2)(p)\mathbf{w_1} \\ &\quad + (\partial_1 C_{23}\, y_1 - \partial_1 C_{21}\, y_3)(p)\mathbf{w_2} + (\partial_1 C_{21}\, y_2 - \partial_1 C_{22}\, y_1)(p)\mathbf{w_3} \ . \end{aligned}$$

On the other hand,

$$\begin{aligned} [\mathbf{J}b_1, \mathbf{J}b_2]\,(p) &= [\, C_{11}\mathbf{w_1} + C_{12}\mathbf{w_2} + C_{13}\mathbf{w_3}\,, -y_1 b_1 + y_3 b_3 - y_2 b_4\,]\,(p) \\ &\quad - [\, C_{21}\mathbf{w_1} + C_{22}\mathbf{w_2} + C_{23}\mathbf{w_3}\,, y_1 b_2 + y_2 b_3 + y_3 b_4\,]\,(p) \\ &= (\partial_1 C_{11}\, y_1 - \partial_3 C_{11}\, y_3 + \partial_4 C_{11}\, y_2 + \partial_2 C_{21}\, y_1 + \partial_3 C_{21}\, y_2 + \partial_4 C_{21}\, y_3)(p)\mathbf{w_1} \\ &\quad + (\partial_1 C_{12}\, y_1 - \partial_3 C_{12}\, y_3 + \partial_4 C_{12}\, y_2 + \partial_2 C_{22}\, y_1 + \partial_3 C_{22}\, y_2 + \partial_4 C_{22}\, y_3)(p)\mathbf{w_2} \\ &\quad + (\partial_1 C_{13}\, y_1 - \partial_3 C_{13}\, y_3 + \partial_4 C_{13}\, y_2 + \partial_2 C_{23}\, y_1 + \partial_3 C_{23}\, y_2 + \partial_4 C_{23}\, y_3)(p)\mathbf{w_3} \ . \end{aligned}$$

Finally, we have already done all the necessary computation for determine the Nijenhuis tensor at p:

$$\begin{aligned}
\mathbf{N}(b_1,\tau)(p) &= [\mathbf{J}b_1,\mathbf{J}\tau](p) - \mathbf{J}[\mathbf{J}b_1,\tau](p) - \mathbf{J}[b_1,\mathbf{J}\tau](p) - [b_1,\tau](p) \\
&= [\mathbf{J}b_1,\mathbf{J}\tau](p) - \mathbf{J}[\mathbf{J}b_1,\tau](p) \\
&= [\,y_1b_2+y_2b_3+y_3b_4\,,(\tau_3y_2-\tau_2y_3)\mathbf{w_1}+(\tau_1y_3-\tau_3y_1)\mathbf{w_2}+(\tau_2y_1-\tau_1y_2)\mathbf{w_3}\,](p) \\
&\quad -\mathbf{J}[\,y_1b_2+y_2b_3+y_3b_4\,,\tau_1\mathbf{w_1}+\tau_2\mathbf{w_2}+\tau_3\mathbf{w_3}\,](p) \\
&= (-(\tau_3y_2-\tau_2y_3)b_2-(\tau_1y_3-\tau_3y_1)b_3-(\tau_2y_1-\tau_1y_2)b_4)\,(p) \\
&\quad -\mathbf{J}(-\tau_1b_2-\tau_2b_3-\tau_3b_4)(p) \\
&= -\,((\tau_3y_2-\tau_2y_3)b_2+(\tau_1y_3-\tau_3y_1)b_3+(\tau_2y_1-\tau_1y_2)b_4)\,(p) \\
&\quad +(\tau_1J(b_2)+\tau_2J(b_3)+\tau_3J(b_4))(p) \\
&= -\,((\tau_3y_2-\tau_2y_3)b_2+(\tau_1y_3-\tau_3y_1)b_3+(\tau_2y_1-\tau_1y_2)b_4)\,(p) \\
&\quad +(\tau_1(-y_1b_1+y_3b_3-y_2b_4)+\tau_2(-y_2b_1-y_3b_2+y_1b_4)+\tau_3(-y_3b_1+y_2b_2-y_1b_3))(p) \\
&= 0\ .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbf{N}(b_1,b_2)(p) &= [\mathbf{J}b_1,\mathbf{J}b_2](p) - \mathbf{J}[\mathbf{J}b_1,b_2](p) - \mathbf{J}[b_1,\mathbf{J}b_2](p) - [b_1,b_2](p) \\
&= [\mathbf{J}b_1,\mathbf{J}b_2](p) - \mathbf{J}[\mathbf{J}b_1,b_2](p) + \mathbf{J}[\mathbf{J}b_2,b_1](p) \\
&= ((\partial_1C_{11}+\partial_2C_{21})y_1+(\partial_4C_{11}+\partial_3C_{21}+\partial_2C_{13}-\partial_1C_{23})y_2+(-\partial_3C_{11}+\partial_4C_{21}-\partial_2C_{12}+\partial_1C_{22})y_3)(p)\mathbf{w_1} \\
&\quad +((\partial_1C_{12}+\partial_2C_{22}-\partial_2C_{13}+\partial_1C_{23})y_1+(\partial_4C_{12}+\partial_3C_{22})y_2+(-\partial_3C_{12}+\partial_4C_{22}+\partial_2C_{11}-\partial_1C_{21})y_3)(p)\mathbf{w_2} \\
&\quad +((\partial_1C_{13}+\partial_2C_{23}+\partial_2C_{12}-\partial_1C_{22})y_1+(\partial_4C_{13}+\partial_3C_{23}-\partial_2C_{11}+\partial_1C_{21})y_2+(-\partial_3C_{13}+\partial_4C_{23})y_3)(p)\mathbf{w_3} \\
&= N_1(b_1,b_2)(p)\mathbf{w_1}+N_2(b_1,b_2)(p)\mathbf{w_2}+N_3(b_1,b_2)(p)\mathbf{w_3}\ .
\end{aligned}$$

Since the quantities $C_{\mu\nu}$ are expressed as a function of the symbols Γ^i_{jk} and the coordinates (y_1,y_2,y_3), the calculation of the coefficients $N_i(b_1,b_2)(p)$ is straightforward (but extensive): each of the coefficients $N_i(b_1,b_2)(p)$ can be expressed in the form

$$N_i(b_1,b_2)(p)=\sum_{j+k+l=3} N_{ijkl}(x)\,y_1^jy_2^ky_3^l\ .$$

At this point the special matrix form (2.6) of the curvature tensor R arises: a direct computation (re-ordering the terms after simplifying) and formula (2.17) gives:

$$N_{1300} = 0\,.$$
$$N_{1210} = 0\,.$$
$$N_{1201} = 0\,.$$

$$\begin{aligned}
N_{1120} &= (\partial_1\Gamma_{14}^3 - \partial_3\Gamma_{14}^1) + (\partial_1\Gamma_{13}^4 - \partial_4\Gamma_{13}^1) + (\partial_1\Gamma_{23}^3 - \partial_3\Gamma_{23}^1) + (\partial_2\Gamma_{13}^3 - \partial_3\Gamma_{13}^2) \\
&\qquad +(\partial_4\Gamma_{14}^2 - \partial_2\Gamma_{14}^4) + (\partial_1\Gamma_{42}^4 - \partial_4\Gamma_{42}^1) + (\partial_4\Gamma_{23}^2 - \partial_2\Gamma_{23}^4) + (\partial_2\Gamma_{42}^3 - \partial_3\Gamma_{42}^2) \\
&= R_{1314} + R_{1413} + R_{1323} + R_{2313} + R_{4214} + R_{1442} + R_{4223} + R_{2342} \\
&= 2(R_{1314} + R_{1323} + R_{4214} + R_{4223}) \\
&= 2R(db_1 \wedge db_3 + db_4 \wedge db_2\,,\ db_1 \wedge db_4 + db_2 \wedge db_3) \\
&= 2R(\mathbf{w_2}, \mathbf{w_3}) = 2W(\mathbf{w_2}, \mathbf{w_3}) \\
&= 2W^+(\mathbf{w_2}, \mathbf{w_3})\,.
\end{aligned}$$

$$\begin{aligned}
N_{1102} &= (\partial_3\Gamma_{14}^1 - \partial_1\Gamma_{14}^3) + (\partial_4\Gamma_{13}^1 - \partial_1\Gamma_{13}^4) + (\partial_3\Gamma_{23}^1 - \partial_1\Gamma_{23}^3) + (\partial_3\Gamma_{13}^2 - \partial_2\Gamma_{13}^3) \\
&\qquad +(\partial_2\Gamma_{14}^4 - \partial_4\Gamma_{14}^2) + (\partial_4\Gamma_{42}^1 - \partial_1\Gamma_{42}^4) + (\partial_2\Gamma_{23}^4 - \partial_4\Gamma_{23}^2) + (\partial_3\Gamma_{42}^2 - \partial_2\Gamma_{42}^3) \\
&= -R_{1314} - R_{1413} - R_{1323} - R_{2313} - R_{4214} - R_{1442} - R_{4223} - R_{2342} \\
&= -N_{1120} = -2W^+(\mathbf{w_2}, \mathbf{w_3})\,.
\end{aligned}$$

$$\begin{aligned}
N_{1021} &= (\partial_1\Gamma_{13}^2 - \partial_2\Gamma_{13}^1) + (\partial_1\Gamma_{12}^3 - \partial_3\Gamma_{12}^1) + (\partial_1\Gamma_{42}^2 - \partial_2\Gamma_{42}^1) + (\partial_4\Gamma_{12}^2 - \partial_2\Gamma_{12}^4) \\
&\qquad +(\partial_3\Gamma_{13}^4 - \partial_4\Gamma_{13}^3) + (\partial_1\Gamma_{34}^3 - \partial_3\Gamma_{34}^1) + (\partial_3\Gamma_{42}^4 - \partial_4\Gamma_{42}^3) + (\partial_4\Gamma_{34}^2 - \partial_2\Gamma_{34}^4) \\
&= R_{1213} + R_{1312} + R_{1242} + R_{4212} + R_{3413} + R_{1334} + R_{3442} + R_{4234} \\
&= 2(R_{1213} + R_{1242} + R_{3413} + R_{3442}) \\
&= 2R(db_1 \wedge db_2 + db_3 \wedge db_4\,,\ db_1 \wedge db_3 + db_4 \wedge db_2) \\
&= 2R(\mathbf{w_1}, \mathbf{w_2}) = 2W(\mathbf{w_1}, \mathbf{w_2}) \\
&= 2W^+(\mathbf{w_1}, \mathbf{w_2})\,.
\end{aligned}$$

$$
\begin{aligned}
N_{1003} &= (\partial_1\Gamma^2_{13} - \partial_2\Gamma^1_{13}) + (\partial_1\Gamma^3_{12} - \partial_3\Gamma^1_{12}) + (\partial_1\Gamma^2_{42} - \partial_2\Gamma^1_{42}) + (\partial_4\Gamma^2_{12} - \partial_2\Gamma^4_{12}) \\
&\qquad +(\partial_3\Gamma^4_{13} - \partial_4\Gamma^3_{13}) + (\partial_1\Gamma^3_{34} - \partial_3\Gamma^1_{34}) + (\partial_3\Gamma^4_{42} - \partial_4\Gamma^3_{42}) + (\partial_4\Gamma^2_{34} - \partial_2\Gamma^4_{34}) \\
&= R_{1213} + R_{1312} + R_{1242} + R_{4212} + R_{3413} + R_{1334} + R_{3442} + R_{4234} \\
&= N_{1021} = 2W^+(\mathbf{w_1}, \mathbf{w_2})\,.
\end{aligned}
$$

$$
\begin{aligned}
N_{1012} &= (\partial_2\Gamma^1_{14} - \partial_1\Gamma^2_{14}) + (\partial_4\Gamma^1_{12} - \partial_1\Gamma^4_{12}) + (\partial_2\Gamma^1_{23} - \partial_1\Gamma^2_{23}) + (\partial_3\Gamma^2_{12} - \partial_2\Gamma^3_{12}) \\
&\qquad +(\partial_4\Gamma^3_{14} - \partial_3\Gamma^4_{14}) + (\partial_4\Gamma^1_{34} - \partial_1\Gamma^4_{34}) + (\partial_4\Gamma^3_{23} - \partial_3\Gamma^4_{23}) + (\partial_3\Gamma^2_{34} - \partial_2\Gamma^3_{34}) \\
&= -R_{1214} - R_{1412} - R_{1223} - R_{2312} - R_{3414} - R_{1434} - R_{3423} - R_{2334} \\
&= -2(R_{1214} + R_{1223} + R_{3414} + R_{3423}) \\
&= -2R(db_1 \wedge db_2 + db_3 \wedge db_4\,,\ db_1 \wedge db_4 + db_2 \wedge db_3) \\
&= -2R(\mathbf{w_1}, \mathbf{w_3}) = -2W^+(\mathbf{w_1}, \mathbf{w_3})\,.
\end{aligned}
$$

$$
\begin{aligned}
N_{1030} &= (\partial_2\Gamma^1_{14} - \partial_1\Gamma^2_{14}) + (\partial_4\Gamma^1_{12} - \partial_1\Gamma^4_{12}) + (\partial_2\Gamma^1_{23} - \partial_1\Gamma^2_{23}) + (\partial_3\Gamma^2_{12} - \partial_2\Gamma^3_{12}) \\
&\qquad +(\partial_4\Gamma^3_{14} - \partial_3\Gamma^4_{14}) + (\partial_4\Gamma^1_{34} - \partial_1\Gamma^4_{34}) + (\partial_4\Gamma^3_{23} - \partial_3\Gamma^4_{23}) + (\partial_3\Gamma^2_{34} - \partial_2\Gamma^3_{34}) \\
&= -R_{1214} - R_{1412} - R_{1223} - R_{2312} - R_{3414} - R_{1434} - R_{3423} - R_{2334} \\
&= N_{1012} = -2W^+(\mathbf{w_1}, \mathbf{w_3})\,.
\end{aligned}
$$

$$
\begin{aligned}
N_{1111} &= 2((\partial_1\Gamma^4_{14} - \partial_4\Gamma^1_{14}) + (\partial_1\Gamma^4_{23} - \partial_4\Gamma^1_{23}) + (\partial_2\Gamma^3_{14} - \partial_3\Gamma^2_{14}) + (\partial_2\Gamma^3_{23} - \partial_3\Gamma^2_{23}) \\
&\qquad +(\partial_3\Gamma^1_{13} - \partial_1\Gamma^3_{13}) + (\partial_3\Gamma^1_{42} - \partial_1\Gamma^3_{42}) + (\partial_2\Gamma^4_{13} - \partial_4\Gamma^2_{13}) + (\partial_2\Gamma^4_{42} - \partial_4\Gamma^2_{42})) \\
&= 2(R_{1414} + R_{1423} + R_{2314} + R_{2323} - R_{1313} - R_{1342} - R_{4213} - R_{4242}) \\
&= 2(R(db_1 \wedge db_4 + db_2 \wedge db_3\,,\ db_1 \wedge db_4 + db_2 \wedge db_3) \\
&\qquad -R(db_1 \wedge db_3 + db_4 \wedge db_2\,,\ db_1 \wedge db_3 + db_4 \wedge db_2)) \\
&= 2(R(\mathbf{w_3}, \mathbf{w_3}) - R(\mathbf{w_2}, \mathbf{w_2})) = 2(W(\mathbf{w_3}, \mathbf{w_3}) - W(\mathbf{w_2}, \mathbf{w_2})) \\
&= 2(W^+(\mathbf{w_3}, \mathbf{w_3}) - W^+(\mathbf{w_2}, \mathbf{w_2}))\,.
\end{aligned}
$$

In order to compute more easily the values of N_{2jkl}, observe that $\mathbf{N}(b_1, b_2)(p) \in T_p(U_\alpha \times S_{\sqrt{2}}(\Lambda^+\mathbb{R}^4)) = T_x U_\alpha \oplus T_J(S_{\sqrt{2}}(\Lambda^+\mathbb{R}^4))$, so $\langle\, \mathbf{N}(b_1, b_2)(p)\,, J\,\rangle = 0$, i.e.

$$
N_1(b_1, b_2)(p)\, y_1 + N_2(b_1, b_2)(p)\, y_2 + N_3(b_1, b_2)(p)\, y_3 = 0\,. \tag{2.18}
$$

The left side of (2.18) is a polynomial in the variables (y_1, y_2, y_3), so

$N_{2030} = 0$, because the coefficient of y_2^4 is N_{2030}.
$N_{2300} = 0$, because the coefficient of $y_1^3 y_2$ is $N_{1210} + N_{2300}$.
$N_{2120} = 2W^+(\mathbf{w_1}, \mathbf{w_3})$, because the coefficient of $y_1 y_2^3$ is $N_{1030} + N_{2120}$.
$N_{2210} = -2W^+(\mathbf{w_2}, \mathbf{w_3})$, because the coefficient of $y_1^2 y_2^2$ is $N_{1120} + N_{2210}$.

Again, direct calculation gives:

$$N_{2111} = 0\,.$$

$$\begin{aligned}
N_{2102} &= (\partial_1\Gamma^2_{14} - \partial_2\Gamma^1_{14}) + (\partial_1\Gamma^4_{12} - \partial_4\Gamma^1_{12}) + (\partial_1\Gamma^2_{23} - \partial_2\Gamma^1_{23}) + (\partial_2\Gamma^3_{12} - \partial_3\Gamma^2_{12}) \\
&\qquad + (\partial_3\Gamma^4_{14} - \partial_4\Gamma^3_{14}) + (\partial_1\Gamma^4_{34} - \partial_4\Gamma^1_{34}) + (\partial_3\Gamma^4_{23} - \partial_4\Gamma^3_{23}) + (\partial_2\Gamma^3_{34} - \partial_3\Gamma^2_{34}) \\
&= R_{1214} + R_{1412} + R_{1223} + R_{2312} + R_{3414} + R_{1434} + R_{3423} + R_{2334} \\
&= 2(R_{1214} + R_{1223} + R_{3414} + R_{3423}) \\
&= 2R(db_1 \wedge db_2 + db_3 \wedge db_4\,,\ db_1 \wedge db_4 + db_2 \wedge db_3) \\
&= 2R(\mathbf{w_1}, \mathbf{w_3}) = 2W(\mathbf{w_1}, \mathbf{w_3}) \\
&= 2W^+(\mathbf{w_1}, \mathbf{w_3})\,.
\end{aligned}$$

$$\begin{aligned}
N_{2012} &= (\partial_3\Gamma^1_{14} - \partial_1\Gamma^3_{14}) + (\partial_4\Gamma^1_{13} - \partial_1\Gamma^4_{13}) + (\partial_3\Gamma^1_{23} - \partial_1\Gamma^3_{23}) + (\partial_3\Gamma^2_{13} - \partial_2\Gamma^3_{13}) \\
&\qquad + (\partial_2\Gamma^4_{14} - \partial_4\Gamma^2_{14}) + (\partial_4\Gamma^1_{42} - \partial_1\Gamma^4_{42}) + (\partial_2\Gamma^4_{23} - \partial_4\Gamma^2_{23}) + (\partial_3\Gamma^2_{42} - \partial_2\Gamma^3_{42}) \\
&= -R_{1314} - R_{1413} - R_{1323} - R_{2313} - R_{4214} - R_{1442} - R_{4223} - R_{2342} \\
&= -2(R_{1314} + R_{1323} + R_{4214} + R_{4223}) \\
&= -2R(db_1 \wedge db_3 + db_4 \wedge db_2\,,\ db_1 \wedge db_4 + db_2 \wedge db_3) \\
&= -2R(\mathbf{w_2}, \mathbf{w_3}) = -2W(\mathbf{w_2}, \mathbf{w_3}) \\
&= -2W^+(\mathbf{w_2}, \mathbf{w_3})\,.
\end{aligned}$$

$$\begin{aligned}
N_{2201} &= (\partial_1\Gamma^3_{13} - \partial_3\Gamma^1_{13}) + (\partial_1\Gamma^3_{42} - \partial_3\Gamma^1_{42}) + (\partial_4\Gamma^2_{13} - \partial_2\Gamma^4_{13}) + (\partial_4\Gamma^2_{42} - \partial_2\Gamma^4_{42}) \\
&\quad +(\partial_4\Gamma^1_{14} - \partial_1\Gamma^4_{14}) + (\partial_4\Gamma^1_{23} - \partial_1\Gamma^4_{23}) + (\partial_3\Gamma^2_{14} - \partial_2\Gamma^3_{14}) + (\partial_3\Gamma^2_{23} - \partial_2\Gamma^3_{23}) \\
&= R_{1313} + R_{1342} + R_{4213} + R_{4242} - R_{1414} - R_{1423} - R_{2314} - R_{2323} \\
&= R(db_1 \wedge db_3 + db_4 \wedge db_2\,,\ db_1 \wedge db_3 + db_4 \wedge db_2) \\
&\qquad -R(db_1 \wedge db_4 + db_2 \wedge db_3\,,\ db_1 \wedge db_4 + db_2 \wedge db_3) \\
&= R(\mathbf{w_2}, \mathbf{w_2}) - R(\mathbf{w_3}, \mathbf{w_3}) = W(\mathbf{w_2}, \mathbf{w_2}) - W(\mathbf{w_3}, \mathbf{w_3}) \\
&= W^+(\mathbf{w_2}, \mathbf{w_2}) - W^+(\mathbf{w_3}, \mathbf{w_3})\,.
\end{aligned}$$

$$\begin{aligned}
N_{2021} &= (\partial_1\Gamma^4_{14} - \partial_4\Gamma^1_{14}) + (\partial_1\Gamma^4_{23} - \partial_4\Gamma^1_{23}) + (\partial_2\Gamma^3_{14} - \partial_3\Gamma^2_{14}) + (\partial_2\Gamma^3_{23} - \partial_3\Gamma^2_{23}) \\
&\quad +(\partial_2\Gamma^1_{12} - \partial_1\Gamma^2_{12}) + (\partial_2\Gamma^1_{34} - \partial_1\Gamma^2_{34}) + (\partial_4\Gamma^3_{12} - \partial_3\Gamma^4_{12}) + (\partial_4\Gamma^3_{34} - \partial_3\Gamma^4_{34}) \\
&= R_{1414} + R_{1423} + R_{2314} + R_{2323} - R_{1212} - R_{1234} - R_{3412} - R_{3434} \\
&= R(db_1 \wedge db_4 + db_2 \wedge db_3\,,\ db_1 \wedge db_4 + db_2 \wedge db_3) \\
&\qquad -R(db_1 \wedge db_2 + db_3 \wedge db_4\,,\ db_1 \wedge db_2 + db_3 \wedge db_4) \\
&= R(\mathbf{w_3}, \mathbf{w_3}) - R(\mathbf{w_1}, \mathbf{w_1}) = W(\mathbf{w_3}, \mathbf{w_3}) - W(\mathbf{w_1}, \mathbf{w_1}) \\
&= W^+(\mathbf{w_3}, \mathbf{w_3}) - W^+(\mathbf{w_1}, \mathbf{w_1})\,.
\end{aligned}$$

$$\begin{aligned}
N_{2003} &= (\partial_1\Gamma^3_{13} - \partial_3\Gamma^1_{13}) + (\partial_1\Gamma^3_{42} - \partial_3\Gamma^1_{42}) + (\partial_4\Gamma^2_{13} - \partial_2\Gamma^4_{13}) + (\partial_4\Gamma^2_{42} - \partial_2\Gamma^4_{42}) \\
&\quad +(\partial_2\Gamma^1_{12} - \partial_1\Gamma^2_{12}) + (\partial_2\Gamma^1_{34} - \partial_1\Gamma^2_{34}) + (\partial_4\Gamma^3_{12} - \partial_3\Gamma^4_{12}) + (\partial_4\Gamma^3_{34} - \partial_3\Gamma^4_{34}) \\
&= R_{1313} + R_{1342} + R_{4213} + R_{4242} - R_{1212} - R_{1234} - R_{3412} - R_{3434} \\
&= R(db_1 \wedge db_3 + db_4 \wedge db_2\,,\ db_1 \wedge db_3 + db_4 \wedge db_2) \\
&\qquad -R(db_1 \wedge db_2 + db_3 \wedge db_4\,,\ db_1 \wedge db_2 + db_3 \wedge db_4) \\
&= R(\mathbf{w_2}, \mathbf{w_2}) - R(\mathbf{w_1}, \mathbf{w_1}) = W(\mathbf{w_2}, \mathbf{w_2}) - W(\mathbf{w_1}, \mathbf{w_1}) \\
&= W^+(\mathbf{w_2}, \mathbf{w_2}) - W^+(\mathbf{w_1}, \mathbf{w_1})\,.
\end{aligned}$$

Finally, equation (2.18) gives us:

$N_{3003} = 0$, because the coefficient of y_3^4 is N_{3003}.
$N_{3300} = 0$, because the coefficient of $y_1^3 y_3$ is $N_{1201} + N_{3300}$.
$N_{3111} = 0$, because the coefficient of $y_1 y_2 y_3^2$ is $N_{1012} + N_{2102} + N_{3111}$.

$N_{3102} = -2W^{+}(\mathbf{w_1}, \mathbf{w_2})$, because the coefficient of $y_1 y_3^3$ is $N_{1003} + N_{3102}$.

$N_{3120} = -2W^{+}(\mathbf{w_1}, \mathbf{w_2})$, because the coefficient of $y_1 y_2^2 y_3$ is $N_{1021} + N_{2111} + N_{3120}$.

$N_{3021} = 2W^{+}(\mathbf{w_2}, \mathbf{w_3})$, because the coefficient of $y_2^2 y_3^2$ is $N_{2012} + N_{3021}$.

$N_{3201} = 2W^{+}(\mathbf{w_2}, \mathbf{w_3})$, because the coefficient of $y_1^2 y_3^2$ is $N_{1102} + N_{3201}$.

$N_{3030} = W^{+}(\mathbf{w_1}, \mathbf{w_1}) - W^{+}(\mathbf{w_3}, \mathbf{w_3})$, because the coefficient of $y_2^3 y_3$ is $N_{2021} + N_{3030}$.

$N_{3210} = W^{+}(\mathbf{w_2}, \mathbf{w_2}) - W^{+}(\mathbf{w_3}, \mathbf{w_3})$, because the coefficient of $y_1^2 y_2 y_3$ is $N_{1111} + N_{2201} + N_{3120}$.

$N_{3012} = W^{+}(\mathbf{w_1}, \mathbf{w_1}) - W^{+}(\mathbf{w_2}, \mathbf{w_2})$, because the coefficient of $y_2 y_3^3$ is $N_{2003} + N_{3012}$.

This ends the proof of Theorem (2.3.1).

2.4.2 The Conformal Invariance of J

The computations realized in section (2.4.1) enables us to prove the conformal invariance of $\mathbf{J}$: let $(M, \langle\, ,\rangle)$ be an oriented Riemannian 4-manifold, $\langle\, ,\rangle_{\lambda} = \lambda^2 \langle\, ,\rangle$ a metric conformal to $\langle\, ,\rangle$ and $\mathbf{J}^{\lambda}$ the almost-complex structure on $\mathcal{Z}$ associated to $\langle\, ,\rangle_{\lambda}$. Fixing a point $(x, J) \in \mathcal{Z}$, observe that if $\{b_1, b_2, b_3, b_4\}$ is a local positive $\langle\, ,\rangle$-orthonormal basis of tangent vectors, then $\{b_1^{\lambda}, b_2^{\lambda}, b_3^{\lambda}, b_4^{\lambda}\}$ is a local positive $\langle\, ,\rangle_{\lambda}$-orthonormal basis of tangent vectors and $\{\mathbf{w_1^{\lambda}}, \mathbf{w_2^{\lambda}}, \mathbf{w_3^{\lambda}}\}$ is a local positive $\langle\, ,\rangle_{\lambda}$-orthonormal basis of self-dual forms, where $b_i^{\lambda} := \frac{1}{\lambda} b_i$ and $\mathbf{w_i^{\lambda}} := \lambda^2 \mathbf{w_i}$.

It is readily seen that if a self-dual form $\tau \in (\Lambda_M^+)_x$ has coordinates (τ_1, τ_2, τ_3) with respect to $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ and coordinates $(\tau_1^{\lambda}, \tau_2^{\lambda}, \tau_3^{\lambda})$ with respect to $\{\mathbf{w_1^{\lambda}}, \mathbf{w_2^{\lambda}}, \mathbf{w_3^{\lambda}}\}$, then $\tau_i^{\lambda} = \frac{1}{\lambda^2} \tau_i$. However, the complex structure J on $T_x U_{\alpha}$, seen as a self-dual form, has the same coordinates (y_1, y_2, y_3) with respect to the basis $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ and $\{\mathbf{w_1^{\lambda}}, \mathbf{w_2^{\lambda}}, \mathbf{w_3^{\lambda}}\}$. It happens because the isomorphism between skew-symmetric endomorphism and 2-forms changes with the metric according to λ^2. Thus, from formula (2.16) we obtain $\mathbf{J}^{\lambda}(\tau) = \mathbf{J}(\tau)$. It only rests to prove the invariance of $\mathbf{J} b_1$, by re-ordering the basis $\{b_1, b_2, b_3, b_4\}$ the result follows for the other indices. Observe that formula (2.15) expresses $\mathbf{J} b_1$ as a function of the coordinates (y_1, y_2, y_3) of J and the symbols Γ_{jk}^i. These symbols depend intimately on the orthonormal frame $\{b_1, b_2, b_3, b_4\}$ and the Levi-Civita connection of $\langle\, ,\rangle$ according (2.1). Notice that in this formula is not assumed that the frame $\{b_1, b_2, b_3, b_4\}$ is geodesic at x. In particular, if $\{b_1, b_2, b_3, b_4\}$ is geodesic at x we have $\mathbf{J}b_1 = Jb_1$. Denote by $\overline{\Gamma}^i_{jk}$ the symbols of the Levi-Civita connection of $\langle\, ,\rangle_\lambda$ associated to the $\langle\, ,\rangle_\lambda$-orthonormal frame $\{b_1^\lambda, b_2^\lambda, b_3^\lambda, b_4^\lambda\}$. Now, if we assume that the frame $\{b_1, b_2, b_3, b_4\}$ is geodesic at x with respect to $\langle\, ,\rangle$, then we have according [Hit3] that the values of $\overline{\Gamma}^i_{jk}$ at x are given by the formula

$$\overline{\Gamma}^i_{jk}(x) = \frac{1}{\lambda^2(x)}(\delta_{ik}\,\partial_j\lambda(x) - \delta_{ij}\,\partial_k\lambda(x)) \ , \tag{2.19}$$

where δ_{ij} denotes the *Kronecker symbol* ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise).

Thus, a direct calculation of formula (2.15) applied to the metric $\langle\, ,\rangle_\lambda$ (with the symbols $\overline{\Gamma}^i_{jk}$ as in (2.19)) and to the almost-complex structure $\mathbf{J}^\lambda$ gives us

$$\mathbf{J}^\lambda(b_1^\lambda) = Jb_1^\lambda \ .$$

In particular $\mathbf{J}^\lambda(b_1) = Jb_1 = \mathbf{J}b_1$, so $\mathbf{J}$ is conformally invariant.

Let us conclude this section with a final remark which will be useful in the next chapter. Let $(M, \langle\, ,\rangle)$ be an oriented Riemannian 4-manifold and $f : M \to M$ a conformal orientation-preserving diffeomorphism. Observe that for $x \in M$ and J_x an orthogonal complex structure on T_xM, $\tilde{f}(J_x) := df_x\, J_x\, (df_x)^{-1}$ is an orthogonal complex structure on $T_{f(x)}M$. Thus, a conformal diffeomorphism f on M has a natural lifting to a diffeomorphism $\tilde{f} : \mathcal{Z} \to \mathcal{Z}$ on its twistor space. This diffeomorphism in fact preserves the almost-complex structure $\mathbf{J}$ on $\mathcal{Z}$. In order to prove this, observe that since f is a conformal diffeomorphism we have already proved that $\mathbf{J}^f = \mathbf{J}$, where $\mathbf{J}^f$ is the almost-complex structure on $\mathcal{Z}$ associated to the push-forward metric $f_*\langle\, ,\rangle$. On the other hand, since $f : (M, \langle\, ,\rangle) \to (M, f_*\langle\, ,\rangle)$ is clearly an isometry and all of the objects in the Penrose construction (section (2.3)) are preserved by isometries, we conclude that $\tilde{f}$ takes $\mathbf{J}$ on $\mathbf{J}^f$: $\tilde{f}_*\,\mathbf{J} = \mathbf{J}^f$. Therefore $\tilde{f}_*\,\mathbf{J} = \mathbf{J}$, i.e. $\tilde{f}$ preserves $\mathbf{J}$.

In particular, if M is anti-self-dual then $\tilde{f} : \mathcal{Z} \to \mathcal{Z}$ is a biholomorphism that preserves the twistor lines and the real structure i on $\mathcal{Z}$.

2.5 Reversing the Penrose Construction

In this section we shall describe a procedure for retrieve the anti-self-dual conformal structure on a 4-manifold in terms of its twistor space, inverting thus the Penrose construction of section (2.3).

Let us begin by considering M an oriented smooth 4-manifold with a fixed conformal structure and $\pi : \mathcal{Z} \to M$ its twistor space. For each point $x \in M$, denote by ℓ_x the twistor line on x. Observe that from a vector $v \in T_xM$ we can obtain in a natural way a smooth section $\tilde{v}$ of $N = T\mathcal{Z}|_{\ell_x}/T\ell_x$ (the normal bundle of ℓ_x in $\mathcal{Z}$), whose value at each point $p \in \ell_x$ is given by the 'inverse' of the mapping $\pi_{*p} : T_p\mathcal{Z} \to T_xM$ evaluated at v. In fact, since a metric g in the conformal class of M induces a distribution $\mathcal{H} \subset T\mathcal{Z}$ transverse to the twistor lines, this section $\tilde{v}$ can be realized as a section $\tilde{v}^g$ of $T\mathcal{Z}|_{\ell_x}$, where $\tilde{v}^g(p) = (\pi_{*p}|_{\mathcal{H}_p})^{-1}(v)$.

When the conformal structure on M is anti-self-dual, it turns out that all of the objects presented above have very special properties. It is described in the following theorem:

Theorem 2.5.1 *If the conformal structure on M is anti-self-dual (i.e $\mathcal{Z}$ is a complex manifold), then we have*

(1) *The sections $\tilde{v}$ are in fact holomorphic, i.e. $\tilde{v} \in H^0(\ell_x, N)$. Furthermore, given a metric g in the conformal class of M we have $\tilde{v}^g \in H^0(\ell_x, T\mathcal{Z}|_{\ell_x})$. Thus, $\mathcal{H}|_{\ell_x}$ is a holomorphic subbundle of $T\mathcal{Z}|_{\ell_x}$ isomorphic to N.*

(2) *The real structure i on $\mathcal{Z}$ induces a real structure ($\mathbb{C}$-antilinear involution) i_* on $H^0(\ell_x, N)$, which leaves the sections $\tilde{v}$ invariant. Conversely, if a section $\sigma \in H^0(\ell_x, N)$ is real invariant then $\sigma = \tilde{v}$ for some $v \in T_xM$.*

(3) *The complexification of the linear correspondence $v \mapsto \tilde{v}$ induces an isomorphism $T_x^{\mathbb{C}}M \xrightarrow{\sim} H^0(\ell_x, N)$ that preserves the respective real structures. Thus, T_xM is a real form of $H^0(\ell_x, N)$.*

(4) *Every non-zero section $\sigma \in H^0(\ell_x, N)$ vanishes in at most one point of ℓ_x. Sections coming from T_xM are nowhere vanishing.*

(5) *The metric g on T_xM induces a quadratic form $\tilde{g}$ on $H^0(\ell_x, N)$ whose null cone $C = \{\sigma \in H^0(\ell_x, N), \tilde{g}(\sigma, \sigma) = 0\}$ is constituted by the sections $\sigma \in H^0(\ell_x, N)$ that vanishes at some point of ℓ_x.*

The proof of item (1) of Theorem (2.5.1) is much at the same spirit of Theorem (2.3.1), working in local coordinates, using formulas (2.14) and (2.15), and considering a geodesic frame at x to simplify the calculations. Each of items (2)-(5) can be proved using the previous one.

As a corollary, let us prove item (1) of Theorem (2.3.2), i.e. $N \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$: by a classical result of Grothendieck [Gro], every holomorphic bundle over $\mathbb{C}P^1$ is isomorphic to a direct sum of powers of the tautological line bundle. Thus, $N \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$ for some $a, b \in \mathbb{Z}$. From item (4) above we conclude that in fact $a, b \leq 1$ (otherwise, if $a \geq 2$ say, the bundle $\mathcal{O}(a)$ admits non-zero holomorphic sections vanishing in at least two points). On the other hand, from item (3) above we have $\dim_{\mathbb{C}} H^0(\ell_x, N) = 4$, so the only possibility left is $a = b = 1$, proving the result.

Now, with Theorem (2.5.1) in hand, we proceed to reverse the Penrose construction. Let $\mathcal{Z}$ be a complex 3-manifold as in Theorem (2.3.3), i.e. it satisfies the following properties:

(1) $\mathcal{Z}$ is fibred by projective lines whose normal bundles are isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

(2) $\mathcal{Z}$ possesses a free antiholomorphic involution $\imath$ which transforms each fibre to itself.

Consider M the space of fibres of $\mathcal{Z}$, so M is a smooth 4-manifold. For each $x \in M$ denote by ℓ_x the fibre on x and for each $v \in T_x M$ denote by $\tilde{v}$ the section of $N = T\mathcal{Z}|_{\ell_x}/T\ell_x$ associated to v (it is only smooth a priori). Observe from (1) that the normal bundle N of ℓ_x has the property $H^1(\ell_x, N) = 0$. Thus, by a theorem of Kodaira [Kod], the space $\mathbf{M}$ of projective lines in $\mathcal{Z}$ having the above normal bundle is a non-singular complex manifold whose tangent space at a point $x \simeq \ell_x$ is canonically isomorphic to the space of holomorphic sections of N. As consequence of this fact we have

(1) $\mathbf{M}$ has complex dimension 4.

(2) The sections $\tilde{v}$ are in fact holomorphic, i.e. $\tilde{v} \in H^0(\ell_x, N)$.

(3) We have an inclusion $M \subset \mathbf{M}$ whose differential map at $x \in M$ is given precisely by the correspondence $v \mapsto \tilde{v}$.

(4) The real structure on $\mathcal{Z}$ induces a real structure on $\mathbf{M}$ of which M is the fixed point set.

(5) For a point $x \in M$, the complexification of the correspondence $v \mapsto \tilde{v}$ induces an isomorphism $T_x^{\mathbb{C}} M \xrightarrow{\sim} H^0(\ell_x, N)$ that preserves the real structures. Thus, $T_x M$ is a real form of $H^0(\ell_x, N)$.

(6) The 'null cone' $C = \{\sigma \in H^0(\ell_x, N),\ \sigma(p) = 0 \text{ for some } p \in \ell_x\}$ is real (i.e. $C = \overline{C}$). It also has no real points (i.e. elements of $T_x M$) other than 0.

Concluding our procedure, since by hypothesis $N \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$, a section $\sigma \in H^0(\ell_x, N)$ consists of a pair of linear functions $(az + b, cz + d)$ in an affine parameter z on $\mathbb{C}P^1$. The vanishing of σ at some point is therefore given by the quadratic condition $ad - bc = 0$. Thus, C defines a quadratic form on $H^0(\ell_x, N)$ up to a scalar multiple, which restricts to a real quadratic form on $T_x M$ because C is real according (6). Again, according (6), C has no real points, so this quadratic form on $T_x M$ defined up to a scalar multiple is in fact positive definite. This is the conformal structure on M that we are looking for.

Proving that this conformal structure is anti-self-dual and inverts the Penrose construction is by no means an easy task and it will not be done here (for a complete proof see [Be] § 13.69). Everything we have done above will suffice for the applications in the next chapter.

Chapter 3

ASD Metrics and the Painlevé VI Equation

In this chapter we shall use the Penrose Transform to establish a link between ASD conformal structures on 4-manifolds and solutions of the Painlevé VI equation in the complex plane,

$$\begin{aligned}\frac{d^2y}{dx^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)\frac{dy}{dx} \\ + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right) ,\end{aligned} \tag{3.1}$$

where $\alpha, \beta, \gamma, \delta$ are parameters. This is done with the purpose of obtaining explicit ASD metrics on a 4-manifold from explicit solutions of the Painlevé VI equation. The conformal structures on a 4-manifold which are amenable to this approach are those which admit SU_2 as a symmetry group, with certain generic properties.

3.1 The Twistor Approach to $\mathbf{SU_2}$-invariant ASD Conformal Structures

The basic geometrical object we shall focus on is an oriented 4-manifold M with an anti-self-dual conformal structure preserved by a (left) action of the Lie group SU_2,

$$\mathrm{SU}_2 \times M \longrightarrow M ,$$

all of the orbits being 3-dimensional. Since this group is compact, we can, if required, take it to preserve a metric in the conformal equivalence class (which is of course an ASD metric). Since SU_2 preserves the conformal structure on M, we have already seen in the final remark of section (2.4.2) that the natural lift of this action to an action on the twistor space $\mathcal{Z}$,

$$\begin{tikzcd} \mathrm{SU}_2 \times \mathcal{Z} \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] \\ \mathrm{SU}_2 \times M \arrow[r] & M \end{tikzcd}$$

preserves the complex structure, the real structure i and the twistor lines. In particular, each element of the Lie algebra $\mathcal{SU}_2$ defines a holomorphic vector field on $\mathcal{Z}$. Thus, once the complexification of the Lie algebra $\mathcal{SU}_2$ is identified with $\mathcal{SL}_2(\mathbb{C})$ (the Lie algebra of the complex Lie group $\mathrm{SL}_2(\mathbb{C})$), we have a homomorphism of holomorphic vector bundles:

$$\alpha : \mathcal{SL}_2(\mathbb{C}) \times \mathcal{Z} \longrightarrow T\mathcal{Z}\,.$$

This homomorphism α is compatible with the real structures on $\mathcal{Z}$ and $\mathcal{SL}_2(\mathbb{C})$. If $\alpha(a)$ denotes the vector field on $\mathcal{Z}$ defined by $a \in \mathcal{SL}_2(\mathbb{C})$ then

$$i_*\alpha(a) = \alpha(\overline{a})\,, \tag{3.2}$$

where the complex conjugation on $\mathcal{SL}_2(\mathbb{C})$ is induced from the identification $\mathcal{SU}_2^{\mathbb{C}} = \mathcal{SL}_2(\mathbb{C})$, i.e. $\overline{a} := -a^*$. The homomorphism α is also compatible with the SU_2-action on $\mathcal{Z}$ in the following way:

$$g_*\alpha(a) = \alpha(gag^{-1})\,, \tag{3.3}$$

for all $a \in \mathcal{SL}_2(\mathbb{C})$ and $g \in \mathrm{SU}_2$. The trivial vector bundle $\mathcal{SL}_2(\mathbb{C}) \times \mathcal{Z}$ and the tangent bundle $T\mathcal{Z}$ are both 3-dimensional. If we fix $\{\mu_1, \mu_2, \mu_3\}$ a basis of $\mathcal{SU}_2$, the determinant of the homomorphism α with respect to this basis and local holomorphic basis of $T\mathcal{Z}$ defines a holomorphic section $\det\alpha$ of the line bundle $\det T\mathcal{Z} = K^{-1}$, the anticanonical bundle of $\mathcal{Z}$. The homomorphism α fails to be an isomorphism in the set Y where $\det\alpha = 0$, which is a SU_2-invariant set by (3.3). Again, $\det\alpha$ is compatible with the real structure on $\mathcal{Z}$ in the following sense: if $p \in \mathcal{Z}$ and U is a neighborhood of p then the value of $\det\alpha$ at p with respect to the local basis induced by a holomorphic chart $\varphi : U \hookrightarrow \mathbb{C}^3$ is the complex conjugate value of $\det \alpha$ at $i(p)$ with respect to the local basis induced by the holomorphic chart $\overline{\varphi \circ i} : i(U) \hookrightarrow \mathbb{C}^3$. Therefore the zero set Y is real invariant. Since the tangent bundle of a twistor line ℓ is isomorphic to $\mathcal{O}(2)$ and its normal bundle N is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (cf. Theorem (2.3.2)), we have

$$K^{-1}|_\ell = \det T\mathcal{Z}|_\ell = \det(T\ell \oplus N) \simeq \mathcal{O}(4).$$

In particular, Y is always non-empty and has degree 4 on each twistor line. Thus from the real invariance we have that $\det \alpha$ is either identically zero on a twistor line, vanishes with multiplicity 2 at a pair of antipodal points, or vanishes non-degenerately at four points, forming antipodal pairs.

The (non-generic) case $\det \alpha \equiv 0$ on $\mathcal{Z}$ has already been developed in [Hit1], so from now on we shall restrict ourselves to the case where α is generically an isomorphism. In particular, for a generic twistor line we have a non-trivial intersection with the divisor Y. The (non-generic) case where Y meets a generic twistor line in two double points has also been treated in [Hit1], so here we shall focus on the (generic) case where Y meets a generic twistor line in four simple points. On the complement of Y, α is an isomorphism. We set

$$A := -\alpha^{-1} : T(\mathcal{Z}\backslash Y) \longrightarrow \mathcal{SL}_2(\mathbb{C}) \times \mathcal{Z}\backslash Y\,.$$

Thus A is a holomorphic 1-form on $\mathcal{Z}\backslash Y$ with values in $\mathcal{SL}_2(\mathbb{C})$ and can therefore be considered as a $\mathrm{SL}_2(\mathbb{C})$-connection on the trivial vector bundle $\mathbb{C}^2 \times \mathcal{Z}\backslash Y$. It is just this connection which establishes the link we have mentioned above between ASD metrics and the Painlevé VI equation. We shall see all of this with more detail in the next sections.

Let us conclude this section with a final remark which will be useful later. Recall that for a point $p \in \mathcal{Z}\backslash Y$, the image of α_p is the whole tangent space $T_p\mathcal{Z}$. It is not true anymore for a point $p \in Y$. Since we have an action $\mathrm{SU}_2 \times \mathcal{Z} \to \mathcal{Z}$ with 3-dimensional real orbits, for every point of $\mathcal{Z}$ the image of α contains at least three real independent directions, so the rank of α is at least 2 in each point. In fact, since we are supposing that Y is an analytic hypersurface of $\mathcal{Z}$, we soon realize that for a point p in the smooth part of Y, the image of α_p is the tangent space T_pY. It is because the orbit of a point $p \in Y$ is all contained in Y, so α_p takes $\mathcal{SU}_2$ (and hence $\mathcal{SL}_2(\mathbb{C})$) inside T_pY. On the other hand, $\dim_{\mathbb{C}} \alpha_p(\mathcal{SL}_2(\mathbb{C})) = \dim_{\mathbb{C}} T_pY = 2$. Therefore we conclude $\alpha_p(\mathcal{SL}_2(\mathbb{C})) = T_pY$, as asserted.

3.2 Isomonodromic Deformations along the Twistor Lines

We shall proceed to describe the connection A defined in the last section more carefully. The striking point about this connection is that it is in fact a *flat connection*. There is a natural reason to arrive at this conclusion. If the SU_2-action extends to a holomorphic $\mathrm{SL}_2(\mathbb{C})$-action on $\mathcal{Z}$, then Y is simply the union of lower-dimensional orbits and $\mathcal{Z}\backslash Y$ is an open orbit $\mathrm{SL}_2(\mathbb{C})/\Gamma$ for some discrete closed subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$. So there is an action-preserving isomorphism

$$\begin{array}{ccc} \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})/\Gamma & \longrightarrow & \mathrm{SL}_2(\mathbb{C})/\Gamma \\ \downarrow & & \downarrow \\ \mathrm{SL}_2(\mathbb{C}) \times \mathcal{Z}\backslash Y & \longrightarrow & \mathcal{Z}\backslash Y \end{array}$$

from which we can obtain A looking at the action $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})/\Gamma \to \mathrm{SL}_2(\mathbb{C})/\Gamma$. Now, for a Lie group G and a discrete closed subgroup Γ the left action $G \times G/\Gamma \to G/\Gamma$ is such that the associated homomorphism $\alpha : \mathcal{G} \times G/\Gamma \to T(G/\Gamma)$ is actually an isomorphism and $A = -\alpha^{-1} : T(G/\Gamma) \to \mathcal{G} \times G/\Gamma$ is of the form $A(X) = -X.g^{-1}$, which is simply the usual Maurer-Cartan form $\omega_{MC}(X) = g^{-1}.X$ after a change of gauge. Since the Maurer-Cartan form is classically known to be flat (the zero-curvature equation $dA + \frac{1}{2}[A\,, A] = 0$ for ω_{MC} becomes the Maurer-Cartan equation, see [KoNo]), we are done. In the general setting, though we have no more a holomorphic action on $\mathcal{Z}$, we soon realize that the calculations for the vanishing of the curvature tensor $dA + \frac{1}{2}[A\,, A]$ work the same as in the holomorphic case.

The connection A becomes singular on Y. In fact, since we are supposing that Y meets generically the twistor lines in four simple points, A has a simple pole along (the smooth part of) Y. In order to proof this, observe that if α is represented in local coordinates by the holomorphic 3×3 matrix B then locally A has the form

$$A = -\alpha^{-1} = -\frac{B^\vee}{\det B}\,, \tag{3.4}$$

where $B^\vee$ denotes the transpose of the matrix of cofactors of B. Thus, since in a generic twistor line $\det B$ has non-degenerate zeros only, A has a simple pole along Y.

We have therefore on the twistor space $\mathcal{Z}$ a meromorphic $\mathrm{SL}_2(\mathbb{C})$-connection with a simple pole along Y. We can restrict it to a generic twistor line to obtain a flat meromorphic connection on $\mathbb{C}P^1$ with simple poles at $\{x_1, x_2, x_3, x_4\}$, the points x_i being the intersection with Y. In particular we have a holonomy (monodromy) representation

$$\pi_1(\mathbb{C}P^1\backslash\{x_1, x_2, x_3, x_4\}) \to \mathrm{SL}_2(\mathbb{C})$$

of A restricted to each generic twistor line. In principle, *any* meromorphic connection on $\mathcal{Z}$ with a pole along Y induces a *flat* meromorphic connection when restricted to a generic twistor line, but the essential fact about A concerns its (global) flatness feature: it turns out that for a connected family of twistor lines in $\mathcal{Z}$, each of which meets Y transversally, the monodromy representation of A restricted to each line of the family is the same. This assertion is clearly true because since A is flat on $\mathcal{Z}\backslash Y$ we have the holonomy representation of A,

$$\pi_1(\mathcal{Z}\backslash Y) \to \mathrm{SL}_2(\mathbb{C})\,.$$

Thus, restricted to a generic twistor line the monodromy representation factors as

$$\pi_1(\mathbb{C}P^1\backslash\{x_1, x_2, x_3, x_4\}) \hookrightarrow \pi_1(\mathcal{Z}\backslash Y) \to \mathrm{SL}_2(\mathbb{C})\,.$$

In a connected family of generic twistor lines, the homotopy class of the inclusion of the punctured projective line in $\mathcal{Z}\backslash Y$ is unchanged, so the first homomorphism is independent of the twistor line in the family. Since the second homomorphism is fixed, the monodromy representation in the family is therefore unchanged.

Such a family of connections with fixed monodromy is called an *isomonodromic deformation*. In a more general setting, we consider a meromorphic $\mathrm{GL}_m(\mathbb{C})$-connection on the vector bundle $\mathbb{C}^m \times \mathbb{C}P^1$ with simple poles at $\{x_1, x_2, \ldots, x_n\}$:

$$A = \sum_{i=1}^{n} \frac{A_i}{z - x_i}\, dz\,,$$

the A_i's being $m \times m$ matrices. An isomonodromic deformation

$$A(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{A_i(x_1, \ldots, x_n)}{z - x_i}\, dz \tag{3.5}$$

for $(x_1, x_2, \ldots, x_n) \in U \subset \mathbb{C}^n$, is a family with constant monodromy. According to [Mal], such a family necessarily satisfies the following set of equations known as the *Schlesinger equations*:

$$dA_i + \sum_{j \neq i} [A_j\,, A_i]\, \frac{dx_j - dx_i}{x_j - x_i} = 0\,, \quad 1 \leq i \leq n\,. \tag{3.6}$$

We shall see in the next section that the Schlesinger equations (3.6) arise in our context essentially as a restatement of the flatness property of A. We shall also see how the Painlevé VI equation (3.1) can be readily derived from them.

3.3 Relation with Painlevé VI

3.3.1 Deformations of Connections with Logarithmic Poles

There is another special feature of the connection A that will be important for us: the nature of its singularity along Y. In fact, A has a *logarithmic singularity* along Y. This means that the connection A, seen as a meromorphic 1-form on $\mathcal{Z}$, has a simple pole along Y whose residue vanishes as a 1-form restricted to it. This is equivalent to say that in a local coordinate system (z, w_1, w_2) of $\mathcal{Z}$ where Y is defined by $\{z = 0\}$, A has the form

$$A = M\, \frac{dz}{z} + N_1\, dw_1 + N_2\, dw_2\,,$$

with the matrices M, N_1, N_2 being holomorphic through Y. We proceed to prove our assertion. We consider then a local coordinate system (z, w_1, w_2) of $\mathcal{Z}$ where Y is defined by $\{z=0\}$ and let $\{\mu_1,\mu_2,\mu_3\}$ be a basis of $\mathcal{SU}_2$. The matrix

$$B=\begin{pmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix},$$

is the local matrix representation of α. We observe from the last remark of section (3.1) that b_{11}, b_{12}, b_{13} vanish non-degenerately along Y. According to equation (3.4), the matrix

$$-\frac{B^\vee}{\det B}=-\frac{1}{\det B}\begin{pmatrix} b^\vee_{11} & b^\vee_{21} & b^\vee_{31}\\ b^\vee_{12} & b^\vee_{22} & b^\vee_{32}\\ b^\vee_{13} & b^\vee_{23} & b^\vee_{33}\end{pmatrix},$$

is the local matrix representation of A. Seeing A as a meromorphic 1-form on $\mathcal{Z}$, it has locally the expression

$$\begin{aligned}A=-\frac{1}{\det B}\big(&(b^\vee_{11}\mu_1+b^\vee_{12}\mu_2+b^\vee_{13}\mu_3)dz\\ &+(b^\vee_{21}\mu_1+b^\vee_{22}\mu_2+b^\vee_{23}\mu_3)dw_1+(b^\vee_{31}\mu_1+b^\vee_{32}\mu_2+b^\vee_{33}\mu_3)dw_2\big)\,.\end{aligned}$$

Its residue along Y is, up to a non-vanishing function,

$$\mathrm{Res}_Y A=(b^\vee_{11}\mu_1+b^\vee_{12}\mu_2+b^\vee_{13}\mu_3)dz+(b^\vee_{21}\mu_1+b^\vee_{22}\mu_2+b^\vee_{23}\mu_3)dw_1+(b^\vee_{31}\mu_1+b^\vee_{32}\mu_2+b^\vee_{33}\mu_3)dw_2\,.$$

Since the quantities $b^\vee_{21}$, $b^\vee_{22}$, $b^\vee_{23}$, $b^\vee_{31}$, $b^\vee_{32}$, $b^\vee_{33}$ vanish along Y (e.g. $b^\vee_{21}=b_{13}b_{32}-b_{12}b_{33}$), the restriction $\mathrm{Res}_Y A\,|_Y$ clearly vanishes, as asserted.

Next, we shall consider holomorphic deformations of a twistor line ℓ in $\mathcal{Z}$, i.e. parametrized families of lines

$$f:\mathbb{C}P^1\times U\to\mathcal{Z}$$

that contain ℓ and with the property that all of the lines of the deformation meet Y transversally. Since Y meets each line of such a deformation in four points, there exists a cross-ratio x defined on each line. We choose now a deformation f in such a way this cross-ratio x works as the deformation parameter, which varies in $U\subset\mathbb{C}$. Performing a Möbius transformation on the lines if necessary, we can take f such that $f(0)$, $f(1)$, $f(x)$, $f(\infty)$ are the points of intersection with Y. Observe also that the cross-ratio of four points forming antipodal pairs in $S^2 \simeq \mathbb{C}P^1$ is always real, so the twistor lines of $\mathcal{Z}$ correspond to real points of U.

Pulling back the connection A on $\mathcal{Z}$, we obtain a $\mathrm{SL}_2(\mathbb{C})$-connection f^*A on $\mathbb{C}P^1 \times U$ with a pole along the divisors $z = 0$, $z = 1$, $z = x$, $z = \infty$. Define

$$\begin{aligned} \mathit{\Lambda}_1(x) &:= \mathrm{Res}_0(f^*A\,|_{\mathbb{C}P^1\times\{x\}})\,,\\ \mathit{\Lambda}_2(x) &:= \mathrm{Res}_1(f^*A\,|_{\mathbb{C}P^1\times\{x\}})\,,\\ \mathit{\Lambda}_3(x) &:= \mathrm{Res}_x(f^*A\,|_{\mathbb{C}P^1\times\{x\}})\,,\\ \mathit{\Lambda}_4(x) &:= \mathrm{Res}_\infty(f^*A\,|_{\mathbb{C}P^1\times\{x\}}) = -\mathit{\Lambda}_1(x) - \mathit{\Lambda}_2(x) - \mathit{\Lambda}_3(x)\,. \end{aligned} \tag{3.7}$$

Because the singularity is logarithmic, it is readily seen that the meromorphic rest

$$R = f^*A - \Big(\mathit{\Lambda}_1(x)\,\frac{dz}{z} + \mathit{\Lambda}_2(x)\,\frac{dz}{z-1} + \mathit{\Lambda}_3(x)\,\frac{dz-dx}{z-x}\Big)$$

is in fact holomorphic through $z = 0$, $z = 1$, $z = x$, $z = \infty$. Since every holomorphic 1-form on $\mathbb{C}P^1 \times U$ necessarily has the form $R(x)\,dx$ with $R(x)$ holomorphic, then

$$f^*A = \mathit{\Lambda}_1(x)\,\frac{dz}{z} + \mathit{\Lambda}_2(x)\,\frac{dz}{z-1} + \mathit{\Lambda}_3(x)\,\frac{dz-dx}{z-x} + R(x)\,dx\,, \tag{3.8}$$

the matrices $\mathit{\Lambda}_1$, $\mathit{\Lambda}_2$, $\mathit{\Lambda}_3$ and R being in $\mathcal{SL}_2(\mathbb{C})$.

Finally, we observe that a change of gauge

$$\Omega \mapsto S.\Omega.S^{-1} - dS.S^{-1}\,,$$

with $S = S(x) \in \mathrm{SL}_2(\mathbb{C})$, transforms f^*A into

$$\widetilde{f^*A} = \tilde{\mathit{\Lambda}}_1(x)\,\frac{dz}{z} + \tilde{\mathit{\Lambda}}_2(x)\,\frac{dz}{z-1} + \tilde{\mathit{\Lambda}}_3(x)\,\frac{dz-dx}{z-x} + \tilde{R}(x)\,dx\,,$$

where $\tilde{\mathit{\Lambda}}_i(x) = S(x).\mathit{\Lambda}_i(x).S^{-1}(x)$ and $\tilde{R}(x) = S(x).R(x).S^{-1}(x) - S'(x).S^{-1}(x)$. Thus, once the differential equation $S'(x) = S(x).R(x)$ on U is solved, there exists a $\mathrm{SL}_2(\mathbb{C})$-automorphism u on $\mathbb{C}^2 \times (\mathbb{C}P^1 \times U)$ such that f^*A assumes the canonical form

$$uf^*A = A_1(x)\,\frac{dz}{z} + A_2(x)\,\frac{dz}{z-1} + A_3(x)\,\frac{dz-dx}{z-x}\,. \tag{3.9}$$

It is from this canonical expression for f^*A that we obtain the Schlesinger equations (3.6): the zero-curvature condition $d(uf^*A) + \frac{1}{2}[uf^*A\,, uf^*A] = 0$ is readily seen to be equivalent to the following system of equations

$$\begin{aligned} \frac{dA_1}{dx} &= \frac{1}{x}\,[A_1\,, A_3]\,, \\ \frac{dA_2}{dx} &= \frac{1}{x-1}\,[A_2\,, A_3]\,, \\ \frac{dA_3}{dx} &= -\frac{1}{x}\,[A_1\,, A_3] - \frac{1}{x-1}\,[A_2\,, A_3]\,. \end{aligned} \tag{3.10}$$

This system, according (3.5), is nothing but the Schlesinger system associated to the isomonodromic deformation property of the family of connections

$$A_x(z)\,dz = \Big(\frac{A_1(x)}{z} + \frac{A_2(x)}{z-1} + \frac{A_3(x)}{z-x}\Big)\,dz\,. \tag{3.11}$$

The Schlesinger differential equations (3.10) are the analytical key to finding SU_2-invariant ASD conformal structures. We shall be able to write down ASD metrics in terms of the matrices A_1, A_2, A_3, as we shall see in section (3.4).

3.3.2 The Painlevé VI Equation as an Isomonodromic Deformation Equation

Let us proceed to derive the Painlevé VI equation (3.1) from system (3.10). First, in search of conserved quantities, we observe that the last equation of (3.10) is equivalent to

$$A_1 + A_2 + A_3 = -A_4 = const. \tag{3.12}$$

We take advantage of this for remove A_3 from system (3.10), obtaining thus the reduced system

$$\begin{aligned} \frac{dA_1}{dx} &= -\frac{1}{x}\,[A_1\,, A_2 + A_4]\,, \\ \frac{dA_2}{dx} &= -\frac{1}{x-1}\,[A_2\,, A_1 + A_4]\,. \end{aligned} \tag{3.13}$$

From (3.10) we also obtain

$$\begin{aligned} \frac{d}{dx}(\det A_1) &= \operatorname{tr}\Big(\frac{dA_1}{dx}.A_1^{-1}\Big)\det A_1 \\ &= \frac{1}{x}\operatorname{tr}\,([A_1\,, A_3].A_1^{-1})\det A_1 = \frac{1}{x}\operatorname{tr}\,(A_1.A_3.A_1^{-1} - A_3)\det A_1 = 0\,. \end{aligned}$$

Thus $\det A_1$, and similarly $\det A_2$ and $\det A_3$, are conserved quantities. Let $\pm\lambda_1$, $\pm\lambda_2$, $\pm\lambda_3$ and $\pm\lambda_4$ be the eigenvalues of the matrices A_1, A_2, A_3 and A_4 respectively. Since $\det A_i = -\lambda_i^2$, we have then that λ_1, λ_2, λ_3 and λ_4 are constant parameters. So the conserved quantities for the reduced system (3.13) are

$$\begin{aligned} \det A_1 &= -\lambda_1^2\,, \\ \det A_2 &= -\lambda_2^2\,, \\ \det(A_1 + A_2 + A_4) = \det(-A_3) &= -\lambda_3^2\,. \end{aligned} \tag{3.14}$$

Since the identity $\det(A+B) = \det A + \det B - \operatorname{tr}(AB)$ holds in $\mathcal{SL}_2(\mathbb{C})$, we can replace the last equality of (3.14) by

$$\operatorname{tr} A_1A_2 + \operatorname{tr} A_1A_4 + \operatorname{tr} A_2A_4 = \lambda_3^2 - \lambda_1^2 - \lambda_2^2 - \lambda_4^2\,, \tag{3.15}$$

which will be more useful for our purposes.

From now on, let us suppose $\lambda_4 \neq 0$ and let us work in a basis of $\mathbb{C}^2$ where the constant matrix A_4 takes the diagonal form $\begin{pmatrix} -\lambda_4 & 0 \\ 0 & \lambda_4 \end{pmatrix}$. In this basis, the component $(1,2)$ of the matrix $A_x(z) = \dfrac{A_1(x)}{z} + \dfrac{A_2(x)}{z-1} + \dfrac{A_3(x)}{z-x}$ in (3.11) writes down as

$$A_x(z)_{12} = \frac{w(x)(z-y(x))}{z(z-1)(z-x)}\,,$$

where

$$\begin{aligned} w(x) &= -xA_1(x)_{12} - (x-1)A_2(x)_{12}\,, \\ y(x) &= \frac{xA_1(x)_{12}}{xA_1(x)_{12} + (x-1)A_2(x)_{12}}\,. \end{aligned}$$

The function $y(x)$ is indeed the function on which we will focus our attention. The reason for this is resumed in the theorem below:

Theorem 3.3.1 :

(1) $A_1,\ A_2,\ A_3 \in \mathcal{SL}_2(\mathbb{C})$ *satisfy Schlesinger equations* (3.10)

$$\Updownarrow$$

$y(x)$ satisfies Painlevé VI equation (3.1) *with parameters*
$(\alpha,\, \beta,\, \gamma,\, \delta) = \left(\tfrac{1}{2}(1-2\lambda_4)^2, -2\lambda_1^2\,, 2\lambda_2^2\,, \tfrac{1}{2}(1-4\lambda_3^2)\right)$.

(2) *The matrices* A_1, A_2, A_3 *can be retrieved as functions of* y *and the parameters* $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$.

The proof of Theorem (3.3.1) is not difficult, but requires several computations. Some account can be founded in [JiMi]. In order to be able to perform our original task of finding ASD metrics from solutions of Painlevé VI, we shall need explicit formulas for the matrices A_1, A_2, A_3 in terms of y. Thus, we consider that it is worth to provide a proof of Theorem (3.3.1) and to do some calculations.

First, let us parametrize the matrices $A_1(x)$ and $A_2(x)$ in the following way:

$$A_i(x) = \begin{pmatrix} u_i(x) & v_i(x) \\ \dfrac{\lambda_i^2 - u_i^2(x)}{v_i(x)} & -u_i(x) \end{pmatrix} .$$

Therefore we have

$$\begin{aligned} w &= -xv_1 - (x-1)v_2 \,, \\ y &= \frac{xv_1}{xv_1 + (x-1)v_2} \,. \end{aligned} \tag{3.16}$$

Another quantities that will appear naturally in the calculations are

$$\begin{aligned} \mu &= u_1 + u_2 \,, \\ \nu &= -x(y-1)u_1 - (x-1)yu_2 \,. \end{aligned} \tag{3.17}$$

In these new variables, after some calculation the first integral (3.15) can be written as

$$\lambda_4\,\mu = -\frac{\nu^2}{x(x-1)y(y-1)} + (y-x)\left(\frac{\lambda_1^2}{(x-1)y} - \frac{\lambda_2^2}{x(y-1)}\right) + \lambda_3^2 - \lambda_4^2 \,. \tag{3.18}$$

On the other hand, each matrix equation of the reduced system (3.13) provides us with two scalar equations. Choosing the components $(1,1)$ and $(1,2)$ of both equations, after some calculation we get the following system

$$\frac{du_1}{dx} = \frac{Z}{x} \,, \tag{3.19}$$

$$\frac{du_2}{dx} = -\frac{Z}{x-1} \,, \tag{3.20}$$

$$\frac{dy}{dx} = \frac{(1-2\lambda_4)y(y-1) - 2\nu}{x(x-1)} \,, \tag{3.21}$$

$$w^{-1}\frac{dw}{dx} = -(1-2\lambda_4)\frac{y-x}{x(x-1)} \,, \tag{3.22}$$

where

$$Z=\left(\frac{1}{x(y-1)}+\frac{1}{(x-1)y}\right)\frac{\nu^2}{y-x}+\frac{2\nu\mu}{y-x}-\lambda_1^2\,\frac{x(y-1)}{(x-1)y}+\lambda_2^2\,\frac{(x-1)y}{x(y-1)}\,. \tag{3.23}$$

Now we differentiate both sides of (3.21) with respect to x, apply (3.17), (3.19) and (3.20) to obtain the intermediate result

$$\begin{aligned}\frac{d^2y}{dx^2}=-\frac{2}{x(x-1)(y-x)}\Bigg(&(y-x)Z-2\nu\mu-2\lambda_4\,y(y-1)\mu+\left(\frac{dy}{dx}-1\right)\nu \\ &+\frac{1}{2}(y-x)\Big((2\lambda_4-1)(2y-1)+(2x-1)\Big)\frac{dy}{dx}\Bigg)\,.\end{aligned} \tag{3.24}$$

Finally, by using (3.23) for eliminate Z, (3.18) for eliminate μ and (3.21) for eliminate ν from (3.24), we arrive to the differential equation satisfied by $y(x)$:

$$\begin{aligned}\frac{d^2y}{dx^2}=&\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx} \\ &+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\frac{(1-2\lambda_4)^2}{2}-2\lambda_1^2\,\frac{x}{y^2}+2\lambda_2^2\,\frac{x-1}{(y-1)^2}+\frac{1-4\lambda_3^2}{2}\,\frac{x(x-1)}{(y-x)^2}\right)\,,\end{aligned} \tag{3.25}$$

it is the Painlevé VI equation (3.1).

Let us complete the proof of Theorem (3.3.1). Once y is known by integration of (3.25), w is calculated by integrating (3.22) and ν by using (3.21). Therefore we can obtain v_1 and v_2 from (3.16). On the other hand, μ is given by the first integral (3.18) and once μ and ν are known, u_1 and u_2 are easily calculated from (3.17). Since the matrices A_1, A_2 and A_3 are directly obtained from $\{u_1, v_1\}$ and $\{u_2, v_2\}$, we are done.

Explicit formulas for A_1, A_2 and A_3 in terms of y are given below:

$$A_i(x)=\begin{pmatrix} u_i(x) & v_i(x) \\ \dfrac{\lambda_i^2-u_i^2(x)}{v_i(x)} & -u_i(x)\end{pmatrix}\,,$$

where

$$v_1=-\frac{y}{x}\,e^{-(1-2\lambda_4)\int\frac{y-x}{x(x-1)}dx}\,,$$

$$v_2=\frac{y-1}{x-1}\,e^{-(1-2\lambda_4)\int\frac{y-x}{x(x-1)}dx}\,,$$

$$v_3=-\frac{y-x}{x(x-1)}\,e^{-(1-2\lambda_4)\int\frac{y-x}{x(x-1)}dx}\,,$$

$$u_1 = \frac{1}{4\lambda_4}\frac{x(x-1)^2}{(y-1)(y-x)}\left(\frac{dy}{dx}\right)^2 - \frac{x-1}{y-x}\left(\frac{1-2\lambda_4}{2\lambda_4}y - \frac{1}{2}x\right)\frac{dy}{dx} - \frac{1-2\lambda_4}{2}\frac{y(y-1)}{y-x}$$
$$+ \frac{1}{\lambda_4}\frac{(x-1)y}{y-x}\left[\frac{(1-2\lambda_4)^2}{4}\frac{y(y-1)}{x(x-1)} - (y-x)\left(\frac{\lambda_1^2}{(x-1)y} - \frac{\lambda_2^2}{x(y-1)}\right) - (\lambda_3^2 - \lambda_4^2)\right],$$

$$u_2 = -\frac{1}{4\lambda_4}\frac{x^2(x-1)}{y(y-x)}\left(\frac{dy}{dx}\right)^2 + \frac{x}{y-x}\left(\frac{1-2\lambda_4}{2\lambda_4}(y-1) - \frac{1}{2}(x-1)\right)\frac{dy}{dx} + \frac{1-2\lambda_4}{2}\frac{y(y-1)}{y-x}$$
$$- \frac{1}{\lambda_4}\frac{x(y-1)}{y-x}\left[\frac{(1-2\lambda_4)^2}{4}\frac{y(y-1)}{x(x-1)} - (y-x)\left(\frac{\lambda_1^2}{(x-1)y} - \frac{\lambda_2^2}{x(y-1)}\right) - (\lambda_3^2 - \lambda_4^2)\right],$$

$$u_3 = \frac{1}{4\lambda_4}\frac{x(x-1)}{y(y-1)}\left(\frac{dy}{dx}\right)^2 - \frac{1-2\lambda_4}{2\lambda_4}\frac{dy}{dx}$$
$$+ \frac{1}{\lambda_4}\left[\frac{(1-2\lambda_4)^2}{4}\frac{y(y-1)}{x(x-1)} - (y-x)\left(\frac{\lambda_1^2}{(x-1)y} - \frac{\lambda_2^2}{x(y-1)}\right) - \lambda_3^2\right].$$

This completes the proof of Theorem (3.3.1).

3.4 Retrieving the ASD Conformal Structure

3.4.1 SU_2-invariant Metrics in Diagonal Form

Finally, we are ready to obtain explicit ASD metrics from explicit solutions of Painlevé VI equation. Coming back to our original setting, we have an oriented 4-manifold M with an ASD conformal structure preserved by an action of SU_2 with 3-dimensional orbits. The manifold M is therefore topologically locally a product

$$M \simeq \mathrm{SU}_2/\Gamma \times (a, b),$$

for some finite subgroup $\Gamma \subset \mathrm{SU}_2$. We may take the conformal structure to be defined by an invariant metric g and thus on each orbit it is a left invariant metric. Hence g is given by an inner product B_t on the Lie algebra $\mathcal{SU}_2$ for each $t \in (a, b)$ parametrizing the set of orbits. Furthermore, if the parametrization of the set of orbits is done via a unit vector field normal to the orbits, then the metric on $\mathrm{SU}_2/\Gamma \times (a, b)$ has the form

$$g = B_t + dt^2.$$

Using a standard orthonormal basis $\{\mu_1, \mu_2, \mu_3\}$ of $\mathcal{SU}_2$, we have

$$B_t = \sum_{i,j=1}^{3} b_{ij}(t)\, d\mu_i \otimes d\mu_j \,.$$

If the basis $\{\mu_1, \mu_2, \mu_3\}$ can be chosen such that the matrix $(b_{ij}(t))$ is diagonal for all $t \in (a, b)$, then we say that g can be put in *diagonal form*:

$$g = b_1^2(t)\, d\mu_1^2 + b_2^2(t)\, d\mu_2^2 + b_3^2(t)\, d\mu_3^2 + dt^2. \tag{3.26}$$

This is precisely the class of SU_2-invariant ASD metrics that we shall be able to explicit in this work. We can obtain extra information about the twistor space when the metric can be put in diagonal form. However some preliminaries are needed. In the twistor setting, we have the homomorphism $\alpha : \mathcal{SL}_2(\mathbb{C}) \times \mathcal{Z} \to T\mathcal{Z}$ and the connection $A = -\alpha^{-1} : T(\mathcal{Z}\backslash Y) \to \mathcal{SL}_2(\mathbb{C}) \times \mathcal{Z}\backslash Y$ introduced in section (3.1). Let ℓ be a twistor line in $\mathcal{Z}$. The choice of an identification $\ell \simeq \mathbb{C}P^1$ allows us to write the restriction to ℓ of the connection A under the form

$$A = \left(\frac{\Lambda_1}{z - x_1} + \frac{\Lambda_2}{z - x_2} + \frac{\Lambda_3}{z - x_3} + \frac{\Lambda_4}{z - x_4} \right) dz \,, \tag{3.27}$$

where $\ell \cap Y = \{x_1, x_2, x_3, x_4\}$. Notice that the matrices Λ_1, Λ_2, Λ_3, Λ_4 do not depend on the identification $\ell \simeq \mathbb{C}P^1$ chosen since the residues are naturally invariant. So, each $x \in M$ defines a 4-tuple $\Lambda_1(x)$, $\Lambda_2(x)$, $\Lambda_3(x)$, $\Lambda_4(x)$ in accordance with (3.27) (the same 4-tuple already found in (3.7)). As seen in chapter 2, for $x \in M$ the corresponding twistor line ℓ_x is given by the sphere $S_{\sqrt{2}}(\Lambda^+_M)_x$. The real structure i on $\mathcal{Z}$ arises from the antipodal map on ℓ_x . Recall that $S_{\sqrt{2}}(\Lambda^+_M)_x$ can be thought of as the 2-dimensional sphere embedded in the 3-dimensional euclidean space $(\Lambda^+_M)_x$. If we now choose the identification of ℓ_x with $\mathbb{C}P^1 \simeq \overline{\mathbb{C}}$ given by the standard stereographic projection, the antipodal map becomes $z \mapsto -1/\overline{z}$. Thus, when ℓ_x intersects Y transversally, the real invariance of Y yields

$$\ell_x \cap Y = \{x_1, -1/\overline{x}_1, x_2, -1/\overline{x}_2\} \,.$$

Hence the connection A restricted to ℓ_x can be written as

$$A = \left(\frac{\Lambda_1(x)}{z - x_1} + \frac{\Lambda_2(x)}{z + 1/\overline{x}_1} + \frac{\Lambda_3(x)}{z - x_2} + \frac{\Lambda_4(x)}{z + 1/\overline{x}_2} \right) dz \,.$$

Therefore, at a point $z \in \mathbb{C}P^1 \backslash \{x_1, -1/\overline{x}_1, x_2, -1/\overline{x}_2\}$,

$$\alpha\left(\frac{\Lambda_1(x)}{z-x_1} + \frac{\Lambda_2(x)}{z+1/\overline{x}_1} + \frac{\Lambda_3(x)}{z-x_2} + \frac{\Lambda_4(x)}{z+1/\overline{x}_2}\right) = -d/dz\,.$$

The compatibility of α with respect to i and to the conjugation $\overline{a} = -a^*$ on $\mathcal{SL}_2(\mathbb{C})$ imposes nontrivial relations on the matrices $\Lambda_i(x)$. It follows from identity (3.2) and the equality above that

$$\Lambda_2(x) = -\Lambda_1^*(x)\,, \qquad \Lambda_4(x) = -\Lambda_3^*(x)\,.$$

Because the matrices $\Lambda_i(x)$ are conjugate to the Schlesinger matrices $A_i(x)$ considered in (3.9)-(3.10), it follows that they have the same eigenvalues. Thus,

$$\lambda_2^2 = \overline{\lambda}_1^2\,, \qquad \lambda_4^2 = \overline{\lambda}_3^2\,. \tag{3.28}$$

Also, A is a meromorphic differential on $\mathbb{C}P^1$ so that the sum of the residues must be zero. Therefore we obtain

$$\Lambda_1(x) - \Lambda_1^*(x) + \Lambda_2(x) - \Lambda_2^*(x) = 0\,.$$

Finally, considering the SU_2-action on the twistor space $\mathcal{Z}$, it follows from equality (3.3) that

$$\Lambda_i(gx) = g\Lambda_i(x)g^{-1}\,, \tag{3.29}$$

for all $x \in M$ and $g \in \mathrm{SU}_2$. In particular the eigenvalues λ_i are constant along the orbits in M and therefore constant in M by the results of section (3.3.2).

The following result gives us an extra relation on the matrices Λ_i when the conformal structure can be put in diagonal form.

Theorem 3.4.1 ([Hit1]) *Let M be a SU_2-invariant anti-self-dual 4-manifold with 3-dimensional orbits such that $\det\alpha$ is non-degenerate. If the conformal structure can be put in diagonal form, then the matrices Λ_1, Λ_2, Λ_3, Λ_4 are conjugate, i.e.*

$$\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda_4^2 = \lambda^2 \in \mathbb{R}\,.$$

(Note that once the square of the eigenvalues are assumed to be equal, it follows from (3.28) that they are real).

Observe that the statement above is of local nature so that, to prove it, we can suppose that $M = \mathrm{SU}_2/\Gamma \times (a,b)$. Actually we can indeed suppose that $M = \mathrm{SU}_2 \times (a,b)$. Identifyng SU_2 with the group of unit quaternions, and thanks to (3.29), it is enough to prove the theorem at a point $x \in M$ of the form $x = (1,t)$. For these points we proceed as follows. Suppose that the conformal structure has diagonal form for some basis $\{\mu_1, \mu_2, \mu_3\}$ of $\mathcal{SU}_2$ (orthonormal with respect to the standard inner product). Performing a conjugation on SU_2 if necessary, we can suppose $\{\mu_1, \mu_2, \mu_3\} = \{i, j, k\}$. Now consider the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $M = \mathrm{SU}_2 \times (a,b)$ induced by the conjugation action of the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$. It fixes x and, for example, i acts on the Lie algebra by conjugation sending (i,j,k) to $(i,-j,-k)$. This action clearly preserves the metric (3.26), as do the corresponding actions of j and k. Taking the orthonormal basis $\{b_1(t)i\,, b_2(t)j\,, b_3(t)k\,, d/dt\}$, the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on the tangent space T_xM via the diagonal matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, as seen in section (1.3) of chapter 1, we conclude that $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on the 3-dimensional space $(\Lambda^+_M)_x$ via the diagonal matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with respect to the orthogonal basis

$$\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\} = \left\{ \frac{di \wedge dj}{b_1(t)b_2(t)} + \frac{dk \wedge dt}{b_3(t)}, \frac{di \wedge dk}{b_1(t)b_3(t)} - \frac{dj \wedge dt}{b_2(t)}, \frac{dj \wedge dk}{b_2(t)b_3(t)} + \frac{di \wedge dt}{b_1(t)} \right\}.$$

So, apart from the vectors $\pm\mathbf{w_i}$, each orbit in $S_{\sqrt{2}}(\Lambda^+_M)_x = \ell_x$ has lenght 4. The right action of $\{\pm 1, \pm i, \pm j, \pm k\}$ on M is isometric and commutes with the left multiplication, therefore preserving the twistor space and the SU_2-action on it. Thus, this right action preserves α as well as the divisor Y and the connection A. Since the divisor Y is SU_2-invariant, it is also preserved by $\mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, the intersection $\ell_x \cap Y$ is preserved by this group, although the set $\ell_x \cap Y$ is not pointwisely fixed by it. For a generic $x \in M$, the twistor line ℓ_x intrersects Y in four distinct points. The group $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts transitively on these points unless some of the vectors $\pm\mathbf{w_i}$ belongs to $\ell_x \cap Y$. However, in this case, the only possibility is to have $\ell_x \cap Y = \{\mathbf{w_i}, \mathbf{w_j}, -\mathbf{w_i}, -\mathbf{w_j}\}$. In particular, since these points form the vertices of a square, their cross-ratio is -1. We shall be able to prove in the next section that the cross-ratio of the four points $\ell_x \cap Y$ is non-constant in M. So for $x \in M$ generic, the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts transitively on the four points $\ell_x \cap Y$. The left multiplication acts on the residues Λ_i by conjugation and the right action of $\{\pm 1, \pm i, \pm j, \pm k\}$ preserves the residues, thus $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on the residues by conjugation. Since the singularities of A in ℓ_x are fully permuted by this action, we obtain the desired result.

3.4.2 The Cross-Ratio as Parameter for the Orbits

Consider the SU_2-action on the twistor space $\mathcal{Z}$. This action maps one twistor line ℓ biholomorphically to another ℓ' and, since the divisor Y is invariant, makes the intersections $\ell \cap Y$ and $\ell' \cap Y$ correspond. If we take a point $\overline{x} \in M$ such that the twistor line $\ell_{\overline{x}}$ meets Y transversally, the same holds in a neighboorhood of $\overline{x}$. Moreover, because the SU_2-invariance of Y, this property remains true in a neighborhood $SU_2/\Gamma \times (a, b)$ of the orbit of $\overline{x}$. Thus, for each $\overline{x} \in SU_2/\Gamma \times (a, b)$, the cross-ratio x of the four points $\ell_{\overline{x}} \cap Y$ defines a function on $SU_2/\Gamma \times (a, b)$ which, by projective invariance, is constant along the orbits. Recall that for a twistor line ℓ the points of $\ell \cap Y$ occurs in antipodal pairs, therefore the function x is in fact a real function. So the idea is to use this cross-ratio x to parametrize the set of orbits in M (and hence the manifold M itself). Thus, considering M as a real submanifold of $\mathbf{M}$ (where $\mathbf{M}$ is the deformation space of lines introduced in section (2.5)), this would enable us to immediately relate the parametrization in question with the deformation of lines discussed in section (3.3.1). Since the cross-ratio x has a holomorphic extension to a neighborhood of $SU_2/\Gamma \times (a, b)$ in $\mathbf{M}$, it suffices that x is a non-constant function to allow us to obtain the desired parametrization.

Let us prove then that x is, in fact, a non-constant function. Let ℓ be a twistor line on $\mathcal{Z}$ that meets Y transversally. According to section (2.5), the tangent space of $\mathbf{M}$ at ℓ is given by the space $H^0(\ell, N)$, where N is the normal bundle $T\mathcal{Z}|_\ell/T\ell$. Observe that if a section $\sigma \in H^0(\ell, N)$ is represented by a deformation $f : \mathbb{C}P^1 \times U \to \mathcal{Z}$ of ℓ that preserves the cross-ratio x, then we have (after reparametrizing the lines by a Möbius transformation) that the intersection with Y is given by the horizontals $f(0)$, $f(1)$, $f(x)$, $f(\infty)$. Thus, the derivative $f' \in H^0(\ell, T\mathcal{Z}|_\ell)$ is tangent to Y at $\ell \cap Y$ and σ is therefore realized by such a section. However, not every section $\sigma \in H^0(\ell, N)$ possesses this property. Indeed, if a section $\sigma \in H^0(\ell, N)$ can be realized as a section of $H^0(\ell, T\mathcal{Z}|_\ell)$ that is tangent to Y at $\ell \cap Y$, it can be proved that $\sigma = \alpha(a)$ for some $a \in \mathcal{SL}_2(\mathbb{C})$. Thus, since $\dim_{\mathbb{C}} \mathcal{SL}_2(\mathbb{C}) = 3$ and $\dim_{\mathbb{C}} H^0(\ell, N) = 4$, the cross-ratio x is non-constant along directions not coming from $\mathcal{SL}_2(\mathbb{C})$. Proving the characterization above demands some effort. Begin by recalling that the homomorphism $\alpha : \mathcal{SL}_2(\mathbb{C}) \times \mathcal{Z} \to T\mathcal{Z}$ is an isomorphism on the complement of Y. At a smooth point $p \in Y$, the image of α_p is T_pY, as it was observed in the final remark of section (3.1). Since ℓ meets Y transversally, this means that the composition $\beta = \pi \circ \alpha$,

$$\mathcal{SL}_2(\mathbb{C}) \xrightarrow{\alpha} T\mathcal{Z}|_\ell \xrightarrow{\pi} N\,,$$

is surjective. This surjection β allows us to define a natural cohomological obstruction to the problem of finding a representant of $\sigma \in H^0(\ell, N)$ as a section of $H^0(\ell, T\mathcal{Z}|_\ell)$ that is tangent to Y at $\ell \cap Y$. This obstruction lies in $H^1(\ell, T\ell \otimes \mathcal{I}_{\ell\cap Y})$, where $T\ell \otimes \mathcal{I}_{\ell\cap Y}$ is the sheaf of sections of the tangent bundle $T\ell$ that vanish at $\ell \cap Y$. Observe also that the kernel of β is a line bundle of degree $-\deg N = -2$, so we have a short exact sequence of sheaves

$$0 \to \mathcal{O}(-2) \to \mathcal{SL}_2(\mathbb{C}) \xrightarrow{\beta} N \to 0.$$

Moreover, α maps the kernel $\mathcal{O}(-2)$ isomorphically to the sheaf $T\ell \otimes \mathcal{I}_{\ell\cap Y}$. From the long exact cohomology sequence we have

$$0 \to \mathcal{SL}_2(\mathbb{C}) \xrightarrow{\beta} H^0(\ell, N) \xrightarrow{\delta} H^1(\ell, \mathcal{O}(-2)) \to 0.$$

But, under the isomorphism $\mathcal{O}(-2) \simeq T\ell \otimes \mathcal{I}_{\ell \cap Y}$, it is readily seen that the boundary map δ provides precisely the obstruction described above. It finishes the proof.

Let us close this section by deriving the mentioned parametrization. Consider a point $\overline{x} \in M$ and $\mathrm{SU}_2/\Gamma \times (a,b)$ a neighborhood of the orbit of $\overline{x}$. Recall from Theorem (2.5.1) that the space $H^0(\ell_{\overline{x}}, N)$ is spanned by the normal sections associated to the elements of $T^{\mathbb{C}}_{\overline{x}}M$, i.e. to the elements a of the Lie algebra $\mathcal{SU}_2^{\mathbb{C}} = \mathcal{SL}_2(\mathbb{C})$ and to the vector field d/dt on (a,b). Moreover, the normal section associated to an element $a \in \mathcal{SL}_2(\mathbb{C})$ is realized by the vector field $\alpha(a)$. Therefore, the cross-ratio is non-constant along d/dt. So let $\Sigma \subset \mathbf{M}$ be a local holomorphic disc tangent to d/dt at $\overline{x}$. Consider the inverse of the cross-ratio $x : \Sigma \to \mathbb{C}$,

$$f = x^{-1} : U = x(\Sigma) \to \mathbf{M}\,. \tag{3.30}$$

It corresponds to the deformation of lines discussed in section (3.3.1). Observe that the real points of U correspond to the intersection $\Sigma \cap M$, which is transverse to the orbits in M. Thus, we can use this transverse section $\Sigma \cap M$ to obtain a local parametrization $M \simeq \mathrm{SU}_2/\Gamma \times (a,b)$, where the parameter $x \in (a,b)$ represents the cross-ratio of $\ell_{\overline{x}} \cap Y$ and possesses f as holomorphic extension.

3.4.3 The Explicit Expression of a SU_2-invariant ASD Metric

The expression for the metric g on M with respect to the local parametrization $M \simeq \mathrm{SU}_2/\Gamma \times (a,b)$ constructed above is

$$g = B_x + T_x \otimes dx + c_x\, dx^2\,,$$

where for each $x \in (a,b)$, B_x is an inner product and T_x is a 1-form on the Lie algebra $\mathcal{SU}_2$. In the twistor setting, recall that $A = -\alpha^{-1} : T(\mathcal{Z}\backslash Y) \to \mathcal{SL}_2(\mathbb{C}) \times \mathcal{Z}\backslash Y$ identifies the tangent bundle $T\mathcal{Z}$ with the trivial bundle $\mathcal{SL}_2(\mathbb{C}) \times \mathcal{Z}$ on $\mathcal{Z}\backslash Y$. So we shall use this identification to describe tangent vectors and, according section (2.5), determine the conformal class of g by only considering the generic sections of $H^0(\ell, N)$ which does not vanish at Y. Take $f : \mathbb{C}P^1 \times U \to \mathcal{Z}$ as in (3.30) and normalize the lines such that

$f(0)$, $f(1)$, $f(x)$, $f(\infty)$ are the points of intersection with Y. Thus, according formula (3.8),

$$f^*A = \varLambda_1(x)\,\frac{dz}{z} + \varLambda_2(x)\,\frac{dz}{z-1} + \varLambda_3(x)\,\frac{dz-dx}{z-x} + R(x)\,dx\,,$$

the tangent vector to a twistor line is identified with

$$f_*(d/dz) = \frac{\varLambda_1(x)}{z} + \frac{\varLambda_2(x)}{z-1} + \frac{\varLambda_3(x)}{z-x} = \varLambda_x(z)\,.$$

The normal section associated to d/dx acquires the form

$$f_*(d/dx) = -\frac{\varLambda_3(x)}{z-x} + R(x)\,,$$

and, of course, for $a \in \mathcal{SL}_2(\mathbb{C})$ the normal section $\alpha(a)$ is identified with $-a$. Thus, a normal section $\sigma \in H^0(\ell, N)$ has the form

$$\sigma(z) = -a - r\Big(\frac{\varLambda_3(x)}{z-x} - R(x)\Big), \quad a \in \mathcal{SL}_2(\mathbb{C})\,,\; r \in \mathbb{C}\,.$$

The null cone $g(\sigma,\sigma) = 0$ is given by

$$B_x(a,a) + T_x(a)r + c_x r^2 = 0\,. \tag{3.31}$$

Also, according Theorem (2.5.1), a section $\sigma \in H^0(\ell, N)$ is a null vector if exists $z \in \mathbb{C}P^1$ such that

$$-a - r\Big(\frac{\varLambda_3(x)}{z-x} - R(x)\Big) = s\varLambda_x(z)\,, \tag{3.32}$$

for some $s \in \mathbb{C}$.

Recall that the matrices $\varLambda_i(x)$ and the Schlesinger matrices $A_i(x)$ are conjugate, for such a reason they will make their appearance later. For the time being, let us proceed to evaluate B_x and T_x. Note that, for $z \in \mathbb{C}P^1$ fixed, the section σ given by $a = \varLambda_x(z)$ and $r = 0$ satisfies condition (3.32), so

$$B_x(\varLambda_x(z), \varLambda_x(z)) = 0$$

for all $z \in \mathbb{C}P^1$. But,

$$B_x(\varLambda_x(z), \varLambda_x(z)) = \frac{B_{11}(x)}{z^2} + \frac{B_{22}(x)}{(z-1)^2} + \frac{B_{33}(x)}{(z-x)^2} + 2\frac{B_{12}(x)}{z(z-1)} + 2\frac{B_{13}(x)}{z(z-x)} + 2\frac{B_{23}(x)}{(z-1)(z-x)}\,,$$

where $B_{ij}(x) = B_x(\Lambda_i(x), \Lambda_j(x))$. Thus, we obtain

$$B_x(\Lambda_1(x), \Lambda_1(x)) = B_x(\Lambda_2(x), \Lambda_2(x)) = B_x(\Lambda_3(x), \Lambda_3(x)) = 0\,, \tag{3.33}$$

$$B_x(\Lambda_1(x), \Lambda_2(x)) = -\frac{1}{x} B_x(\Lambda_1(x), \Lambda_3(x)) = \frac{1}{x-1} B_x(\Lambda_2(x), \Lambda_3(x))\,.$$

If $B_x(\Lambda_i(x), \Lambda_j(x)) = 0$ for all i, j, since B is non-degenerate, we must have $\Lambda_1(x)$, $\Lambda_2(x)$, $\Lambda_3(x)$ null and proportional. But then the complex dimension of the null cone is 2, which is a contradiction. So, since we are interested in the conformal class, we can normalize g in order to have

$$\begin{aligned} B_x(\Lambda_1(x), \Lambda_2(x)) &= 1\,, \\ B_x(\Lambda_1(x), \Lambda_3(x)) &= -x\,, \\ B_x(\Lambda_2(x), \Lambda_3(x)) &= x-1\,. \end{aligned} \tag{3.34}$$

In particular, $\{\Lambda_1(x), \Lambda_2(x), \Lambda_3(x)\}$ constitutes a basis of $\mathcal{SL}_2(\mathbb{C})$. Now, for r, s and z fixed, take a defined by the equation (3.32) and substitute in the equation of the null cone (3.31) to obtain

$$\begin{aligned} B_x\Big(r\Big(\frac{\Lambda_3(x)}{z-x} - R(x)\Big) + s\Lambda_x(z), r\Big(\frac{\Lambda_3(x)}{z-x} - R(x)\Big) + s\Lambda_x(z)\Big) \\ - T_x\Big(r\Big(\frac{\Lambda_3(x)}{z-x} - R(x)\Big) + s\Lambda_x(z)\Big)r + c_x r^2 = 0\,, \end{aligned}$$

for all r, s and z. Since $B_x(\Lambda_x(z), \Lambda_x(z)) = 0$ and $B_x(\Lambda_3(x), \Lambda_3(x)) = 0$, this equation reduces to

$$\begin{aligned} &\Big(2\frac{B_x(\Lambda_3(x), \Lambda_x(z))}{z-x} - 2B_x(R(x), \Lambda_x(z)) - T_x(\Lambda_x(z))\Big)rs \\ &\quad + \Big(B_x(R(x), R(x)) + T_x(R(x)) + c_x - \frac{2B_x(R(x), \Lambda_3(x)) + T_x(\Lambda_3(x))}{z-x}\Big)r^2 = 0\,. \end{aligned}$$

Therefore,

$$B_x(R(x), R(x)) = -T_x(R(x)) - c_x\,, \tag{3.35}$$

$$B_x(R(x), \Lambda_3(x)) = -\frac{T_x(\Lambda_3(x))}{2}\,, \tag{3.36}$$

and using (3.34), we obtain

$$-2\frac{x}{z(z-x)}+2\frac{(x-1)}{(z-1)(z-x)} \\ -\frac{2B_x(R(x),\varLambda_1(x))+T_x(\varLambda_1(x))}{z}-\frac{2B_x(R(x),\varLambda_2(x))+T_x(\varLambda_2(x))}{z-1}=0\,,$$

so

$$B_x(R(x),\varLambda_1(x))=1-\frac{T_x(\varLambda_1(x))}{2}\,, \tag{3.37}$$

$$B_x(R(x),\varLambda_2(x))=-1-\frac{T_x(\varLambda_2(x))}{2}\,. \tag{3.38}$$

With these informations in hand, we can eliminate $R(x)$ from the discussion by proceeding as follows. Observe that equations (3.35)-(3.38) are equivalent to

$$g(d/dx+R(x),d/dx+R(x))=0\,, \tag{3.39}$$

$$g(d/dx+R(x),\varLambda_1(x))=1\,,\quad g(d/dx+R(x),\varLambda_2(x))=-1\,,\quad g(d/dx+R(x),\varLambda_3(x))=0\,.$$

Thus,

$$\varGamma(x)=R(x)+\frac{1}{2x}\varLambda_1(x)+\frac{1}{2(x-1)}\varLambda_2(x)+\frac{2x-1}{2x(x-1)}\varLambda_3(x)\,,$$

satisfies $g(d/dx+\varGamma(x),\varLambda_i(x))=0$, i.e.

$$g(d/dx+\varGamma(x),a)=0\,,\ \text{for all } a\in\mathcal{SL}_2(\mathbb{C})\,.$$

In particular, $\varGamma(x)$ is real (i.e. $\varGamma(x)\in\mathcal{SU}_2$) and the vector field $d/dx+\varGamma(x)$ is normal to the orbits in M. On the other hand, from (3.39) we obtain

$$g(d/dx+\varGamma(x),d/dx+\varGamma(x))=-\frac{1}{2x(x-1)}\,.$$

Thus, parametrizing the space of orbits in M via an integral curve of $d/dx+\varGamma(x)$, the metric acquires the form

$$g=\tilde{B}_x-\frac{dx^2}{2x(x-1)}\,,$$

where $\tilde{B}_x$ is conjugate to B_x.

Summarizing, the information so far collected about g consists of the above expression together with the fact that in a suitable basis $\{\tilde{A}_1(x), \tilde{A}_2(x), \tilde{A}_3(x)\}$ of $\mathcal{SL}_2(\mathbb{C})$ we have

$$\tilde{B}_x = \begin{pmatrix} 0 & 1 & -x \\ 1 & 0 & x-1 \\ -x & x-1 & 0 \end{pmatrix} .$$

Also we have the property that the matrices $\tilde{A}_1(x), \tilde{A}_2(x), \tilde{A}_3(x)$ are conjugate to the Schlesinger matrices $A_1(x), A_2(x), A_3(x)$ and, furthermore, the diagonalization of $\tilde{B}_x$ depends only on the conjugacy class of $\tilde{A}_1(x), \tilde{A}_2(x), \tilde{A}_3(x)$. Then for the case where the metric can be put in diagonal form, an explicit expression for g can be obtained. This can be carried out as follows: the diagonalization of $\tilde{B}_x$ is given by the roots of

$$\det(\tilde{B}_x - tI) = 0 \,,$$

where $I(a,b) = -\frac{1}{2}\mathrm{tr}\, ab$ is the inner product on $\mathcal{SU}_2$. Since we are supposing that g can be put in diagonal form, Theorem (3.4.1) provides us

$$\mathrm{tr}\, A_1^2 = \mathrm{tr}\, A_2^2 = \mathrm{tr}\, A_3^2 = 2\lambda^2 \,.$$

The expression of I in the basis $\{\tilde{A}_1(x), \tilde{A}_2(x), \tilde{A}_3(x)\}$ is therefore

$$I = -\frac{1}{2}\begin{pmatrix} 2\lambda^2 & \mathrm{tr}\, \tilde{A}_1\tilde{A}_2 & \mathrm{tr}\, \tilde{A}_1\tilde{A}_3 \\ \mathrm{tr}\, \tilde{A}_1\tilde{A}_2 & 2\lambda^2 & \mathrm{tr}\, \tilde{A}_2\tilde{A}_3 \\ \mathrm{tr}\, \tilde{A}_1\tilde{A}_3 & \mathrm{tr}\, \tilde{A}_2\tilde{A}_3 & 2\lambda^2 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 2\lambda^2 & \mathrm{tr}\, A_1A_2 & \mathrm{tr}\, A_1A_3 \\ \mathrm{tr}\, A_1A_2 & 2\lambda^2 & \mathrm{tr}\, A_2A_3 \\ \mathrm{tr}\, A_1A_3 & \mathrm{tr}\, A_2A_3 & 2\lambda^2 \end{pmatrix} .$$

So a direct calculation of $\det(\tilde{B}_x - tI)$, using identity (3.15)

$$\mathrm{tr}\, A_1A_2 + \mathrm{tr}\, A_1A_3 + \mathrm{tr}\, A_2A_3 = -2\lambda^2 \,,$$

leads us to

$$\det(\tilde{B}_x - tI) = 2\big(1 + t(\lambda^2 + \tfrac{1}{2}\mathrm{tr}\, A_1A_2)\big)\big(-x + t(\lambda^2 + \tfrac{1}{2}\mathrm{tr}\, A_1A_3)\big)\big(x - 1 + t(\lambda^2 + \tfrac{1}{2}\mathrm{tr}\, A_2A_3)\big) \,.$$

Therefore $\tilde{B}_x$ is given by the formula

$$\tilde{B}_x = \frac{(x-1)\, d\mu_1^2}{\lambda^2 + \frac{1}{2}\mathrm{tr}\, A_2A_3} - \frac{x\, d\mu_2^2}{\lambda^2 + \frac{1}{2}\mathrm{tr}\, A_1A_3} + \frac{d\mu_3^2}{\lambda^2 + \frac{1}{2}\mathrm{tr}\, A_1A_2} \,,$$

for some orthonormal basis $\{\mu_1, \mu_2, \mu_3\}$ of $\mathcal{SU}_2$.

Thus, by using the explicit expressions for A_1, A_2, A_3 obtained at the end of section (3.3.2), we finally arrive to

Theorem 3.4.2 *For $\lambda \in \mathbb{R}\backslash\{0\}$ given, let y be a real solution of the Painlevé VI equation* (3.1) *with parameters* $(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}(1-2\lambda)^2, -2\lambda^2, 2\lambda^2, \frac{1}{2}(1-4\lambda^2)\right)$, *defined on the interval (a,b). Then, for $\{\mu_1, \mu_2, \mu_3\}$ an orthonormal basis of $\mathcal{SU}_2$, the metric*

$$g = \frac{(x-1)\, d\mu_1^2}{\lambda^2 + u_2 u_3 - \frac{y-x}{2x(y-1)}(\lambda^2 - u_2^2) - \frac{x(y-1)}{2(y-x)}(\lambda^2 - u_3^2)} - \frac{x\, d\mu_2^2}{\lambda^2 + u_1 u_3 + \frac{y-x}{2(x-1)y}(\lambda^2 - u_1^2) + \frac{(x-1)y}{2(y-x)}(\lambda^2 - u_3^2)} + \frac{d\mu_3^2}{\lambda^2 + u_1 u_2 - \frac{x(y-1)}{2(x-1)y}(\lambda^2 - u_1^2) - \frac{(x-1)y}{2x(y-1)}(\lambda^2 - u_2^2)} + \frac{dx^2}{2x(x-1)},$$

is an ASD metric on $\mathrm{SU}_2 \times (a,b)$, *where*

$$u_1 = \frac{1}{4\lambda}\frac{x(x-1)^2}{(y-1)(y-x)}\left(\frac{dy}{dx}\right)^2 - \frac{x-1}{y-x}\left(\frac{1-2\lambda}{2\lambda}y - \frac{1}{2}x\right)\frac{dy}{dx} - \frac{1-2\lambda}{2}\frac{y(y-1)}{y-x} + \frac{(1-2\lambda)^2}{4\lambda}\frac{y^2(y-1)}{x(y-x)} - \lambda\frac{y-x}{x(y-1)},$$

$$u_2 = -\frac{1}{4\lambda}\frac{x^2(x-1)}{y(y-x)}\left(\frac{dy}{dx}\right)^2 + \frac{x}{y-x}\left(\frac{1-2\lambda}{2\lambda}(y-1) - \frac{1}{2}(x-1)\right)\frac{dy}{dx} + \frac{1-2\lambda}{2}\frac{y(y-1)}{y-x} - \frac{(1-2\lambda)^2}{4\lambda}\frac{y(y-1)^2}{(x-1)(y-x)} - \lambda\frac{y-x}{(x-1)y},$$

$$u_3 = \frac{1}{4\lambda}\frac{x(x-1)}{y(y-1)}\left(\frac{dy}{dx}\right)^2 - \frac{1-2\lambda}{2\lambda}\frac{dy}{dx} + \frac{(1-2\lambda)^2}{4\lambda}\frac{y(y-1)}{x(x-1)} - \lambda\frac{(y-x)^2}{x(x-1)y(y-1)} - \lambda.$$

Let us point out that focusing attention on real solutions of the Painlevé VI equation does not confine us to any specific 'smaller' group of them. In fact, when the paremeters of the Painlevé VI equation are all real, the condition for a solution to be real depends solely on the initial values (they all have to be real). Hence we are actually considering solutions of the Painlevé VI equation in full generality.

3.5 Some New Examples

Set

$$\mathrm{PVI}\,(\alpha,\beta,\gamma,\delta)\,[y] = \frac{d^2y}{dx^2} - \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 + \left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx}$$
$$- \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right).$$

Thus, the Painlevé VI equation can be denoted as $\mathrm{PVI}\,(\alpha,\beta,\gamma,\delta)\,[y]=0$. Let us consider the special case of PVI given by the parameters $(\alpha,\,\beta,\,\gamma,\,\delta) = \left(\frac{r^2\theta^2}{2}, -\frac{\theta^2}{2}, \frac{\theta^2}{2}, \frac{1-\theta^2}{2}\right)$. The interest in this case arises from the fact that, when $\theta = \frac{1}{1+r}$, we recover the same equation obtained in our previous discussion of ASD metrics with $\lambda = \frac{1}{2(1+r)}$. Also, for these values of the parameters, we have the following useful decomposition

$$\mathrm{PVI}\left(\frac{r^2\theta^2}{2}, -\frac{\theta^2}{2}, \frac{\theta^2}{2}, \frac{1-\theta^2}{2}\right)[y] = \mathrm{PVI}\,(0,0,0,\tfrac{1}{2})\,[y] - \frac{\theta^2}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(r^2-\frac{x}{y^2}+\frac{x-1}{(y-1)^2}-\frac{x(x-1)}{(y-x)^2}\right). \tag{3.40}$$

In particular, we may search special explicit solutions of $\mathrm{PVI}\left(\frac{r^2\theta^2}{2}, -\frac{\theta^2}{2}, \frac{\theta^2}{2}, \frac{1-\theta^2}{2}\right)$ that happen to annihilate both summands on the right-hand side of the equation above. In this direction we have the following result:

Theorem 3.5.1 *Let $\mathcal{C}_r$ be the curve given by $r^2 - \frac{x}{y^2} + \frac{x-1}{(y-1)^2} - \frac{x(x-1)}{(y-x)^2} = 0$.*

(1) *For $r=1$, the curve $\mathcal{C}_1$ has three irreducible components: $y^2-x=0$, $y^2-2y+x=0$ and $y^2-2yx+x=0$. The respective solutions $y=\pm\sqrt{x}$, $y=1\pm\sqrt{1-x}$ and $y=x\pm\sqrt{x(x-1)}$ are also solutions of* $\mathrm{PVI}\,(0,0,0,\frac{1}{2})$. *Hence they are solutions of* $\mathrm{PVI}\left(\frac{\theta^2}{2}, -\frac{\theta^2}{2}, \frac{\theta^2}{2}, \frac{1-\theta^2}{2}\right)$ *for every value of θ.*

(2) *For $r=3$, the curve $\mathcal{C}_3$ has two irreducible components: $3y^2-2y-2yx+x=0$ and $3y^4-4y^3-4y^3x+6y^2x-x^2=0$. The first one leads us to solutions which do not satisfy* $\mathrm{PVI}\,(0,0,0,\frac{1}{2})$. *The second one leads us to the solutions*

$$-y^2(2y-3)\pm 2y(y-1)\sqrt{y(y-1)} = x\,,$$

which do satisfy $\mathrm{PVI}\,(0,0,0,\frac{1}{2})$. *Hence they are solutions of* $\mathrm{PVI}\left(\frac{9\theta^2}{2}, -\frac{\theta^2}{2}, \frac{\theta^2}{2}, \frac{1-\theta^2}{2}\right)$ *for every value of θ.*

Again, the proof of Theorem (3.5.1) amounts to straightforward computation. The calculus for determine the irreducible components of $\mathcal{C}_3$ and verify if they are solutions of PVI$\,(0,0,0,\frac{1}{2})$ is huge. In the ASD case, the solutions of (1) for $\theta = \frac{1}{2}$ (i.e. $\lambda = \frac{1}{4}$) already appears in Hitchin's work [Hit1] about ASD Einstein metrics. Nevertheless, the solutions of (2) for $\theta = \frac{1}{4}$ (i.e. $\lambda = \frac{1}{8}$) yields ASD metrics that are not Einstein and therefore are not captured by Hitchin's list.

In order to obtain new explicit examples of ASD metrics, it would be useful to know if Theorem (3.5.1) holds for other values of r. So we would like to close this work by posing the following question:

For what values of r, there is an irreducible component of the curve $r^2 - \dfrac{x}{y^2} + \dfrac{x-1}{(y-1)^2} - \dfrac{x(x-1)}{(y-x)^2} = 0$ *that satisfies* PVI$\,(0,0,0,\frac{1}{2})$*?*

The above question may sound slightly naïve in the sense that the study of the left-hand side of decomposition (3.40) seems more involved than the individual analysis of each summand in the right-hand side. Yet, Theorem (3.5.1) provides a partial affirmative answer to our question indicating that the mentioned decomposition might have more subtle implications.

Bibliography

[AHS] M.F. Atiyah, N.J. Hitchin and I.M. Singer: *Self-Duality in Four-Dimensional Riemannian Geometry*. Proc. Roy. Soc. London A 362 (1978), 425-461.

[Be] A.L. Besse: *Einstein Manifolds*. Springer, Berlin - Heildelberg - New York, 1987.

[dCa] M. do Carmo: *Riemannian Geometry*. Birkhäuser, Boston, 1992.

[DoKr] S. Donaldson and P. Kronheimer: *The Geometry of Four-Manifolds*. Oxford University Press, USA, 1990.

[DFN] B. Dubrovin, A. Fomenko and S. Novikov: *Modern Geometry - Methods and Applications, Volumes I, II*. Springer, Berlin - Heildelberg - New York, 1985.

[FrMo] R. Friedman and J. Morgan: *Gauge Theory and the Topology of Four-Manifolds*. IAS/Park City Mathematics Series 4. American Mathematical Society, 1997.

[Fuc] R. Fuchs: *Sur Quelques Équations Differéntielles Linéaires du Second Ordre*. C. R. Acad. Sci. Paris 141 (1905), 555-558.

[Gro] A. Grothendieck: *Sur la Classification des Fibrés Holomorphes sur la Sphère de Riemann*. Amer. J. of Math. 79 (1957), 121-138.

[Hit1] N. Hitchin: *Twistor Spaces, Einstein Metrics and Isomonodromic Deformations*. J. Diff. Geometry 42 (1995), 30-112.

[Hit2] N. Nitchin: *Kählerian Twistor Spaces*. Proc. London Math. Soc. 43 (1981), 133-150.

[Hit3] N. Hitchin: *Polygons and Gravitons.* Math. Proc. Camb. Phil. Soc. 85 (1979), 465-476.

[JiMi] M. Jimbo and T. Miwa: *Monodromy Preserving Deformations of Linear Differential Equations with Rational Coeficients II.* Physica 2D (1981), 407-448.

[KoNo] S. Kobayashi and K. Nomizu: *Foundations of Differential Geometry, Volume I.* Wiley-Interscience, New York, 1963.

[Kod] K. Kodaira: *A Theorem of Completeness of Characteristic Systems for Analytic Families of Compact Submanifolds of Complex Manifolds.* Ann. of Math. 75 (1962), 146-162.

[Mal] B. Malgrange: *Sur les Déformations Isomonodromiques I: Singularités Régulières.* Mathématique et Physique, Séminaire ENS, Progr. in Math. 37, 401-426. Birkhäuser, Boston, 1983.

[NeNi] A. Newlander and L. Nirenberg: *Complex Analytic Coordinates in Almost Complex Manifolds.* Ann. of Math. 65 (1957), 391-404.

[Pen] R. Penrose: *Techniques of Differential Topology in Relativity.* CBMS-NSF Regional Conference Series in Applied Mathematics 7. Society for Industrial and Applied Mathematics, USA, 1972.

[Tau] C. Taubes: *The Existence of Anti-Self-Dual Conformal Structures.* J. Diff. Geometry 36 (1992), 163-253.