



Instituto Nacional de Matemática Pura e Aplicada

**Decaimento exponencial do Fluxo de
Teichmüller no espaço de módulos de
diferenciais quadráticas**

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Abstract

We study the Veech flow with involution in the moduli space of zippered rectangles and prove that it is exponentially mixing for a natural class of observables. Using this fact and the canonical $SL(2, \mathbb{R})$ -action we conclude that the Teichmüller flow is exponentially mixing for the Ratner class of observables, restricted to an arbitrary connected component of each stratum of the moduli space. This generalizes a result of Avila, Gouëzel, and Yoccoz, who considered the case of strata of squares.

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Contents

Introduction	4
1 Excellent hyperbolic semi-flows	12
2 The Veech flow with involution	14
2.1 Rauzy classes and interval exchange transformations with involution	14
2.1.1 Interval exchange transformations with involution	14
2.1.2 Rauzy classes with involution	15
2.1.3 Linear action	16
2.2 Rauzy algorithm with involution	17
2.3 Zippered rectangles	19
2.3.1 Extension of induction to the space of zippered rectangles	25
2.3.2 The Veech flow with involution	26
3 The distortion estimate	26
4 Recurrence estimates	31
5 Construction of an excellent hyperbolic semi-flow	32
5.1 The Veech flow with involution as a suspension over the Rauzy renormalization	32
5.2 Precompact section	33
5.3 Hyperbolic properties	34
5.4 Hilbert metric	34
5.5 Uniform expansion and contraction	35
5.6 Properties of the roof function	36
6 The Teichmüller flow	40
6.1 Half-translation surfaces	40
6.2 Translation surfaces	40
6.3 Translation surfaces with involution	41
6.4 Moduli spaces	41
6.4.1 Moduli space of marked translation surfaces	41
6.4.2 Moduli space of marked translation surfaces with involution	42
6.4.3 Moduli space of marked half-translation surfaces	43
6.5 Teichmüller Flow	44
7 From the model to the Teichmüller flow	44
7.1 Zippered rectangles construction	44
7.2 Coordinates	48
7.2.1 Homology and cohomology	50
7.3 Teichmüller flow is exponential mixing	50
References	51

Introduction

In this work we analyze the dynamics of the Teichmüller flow on the moduli space of quadratic differentials. This flow preserves the canonical volume on each stratum of the moduli space. Moreover, after Masur and Veech, it is ergodic on every connected component of strata, if one normalizes the norm of the quadratic differentials. Our goal is to prove that this dynamical system also exhibits fast loss of memory:

Theorem A. *The Teichmüller flow is exponentially mixing, for observables in the Ratner class, with respect to every ergodic component of the canonical volume.*

In the sequel we introduce the main notions involved in this statement and review a number of important results on the dynamics of the Teichmüller flow. Near the end of this introduction we discuss some of the ingredients in the proof.

Quadratic differentials

This notion is central to Teichmüller theory, where quadratic differentials are seen to describe the ‘directions’ along which the conformal structure of a Riemann surface can be deformed. Let us briefly explain this, referring the reader to Ahlfors [Ah], Gardiner [G], Lehto [Le] for details and proofs.

Quadratic differentials: A *quadratic differential* q on a Riemann surface S assigns to each point of the surface a complex quadratic form on the corresponding tangent space, depending meromorphically on the point. Given any local coordinate z on S , the quadratic differential may be written as $q_z = \phi(z)dz^2$ where the coefficient $\phi(z)$ is a meromorphic function; then the expression $q_w = \phi'(w)dw^2$ with respect to any other local coordinate w is determined by

$$\phi'(w) = \phi(z) \left(\frac{dz}{dw} \right)^2$$

on the intersection of the domains. The *norm* of a quadratic differential is defined by $\|q\| = \int |\phi| dz d\bar{z}$ (the integral does not depend on the choice of the local coordinates). Quadratic differentials with finite norm are called *integrable*.

Each non-zero quadratic differential comes with a pair of transverse foliations, \mathcal{F}^h and \mathcal{F}^v , defined as follows. The horizontal direction and the vertical direction at a point z are defined by

$$q_z(u) > 0 \quad \text{and} \quad q_z(u) < 0$$

respectively. Integrating these directions, one obtains the *horizontal foliation* \mathcal{F}^h and the *vertical foliation* \mathcal{F}^v , respectively. Zeros and poles of the differential, where this definition breaks down, correspond to singularities of the two foliations. They are isolated points, since quadratic differentials are meromorphic.

Half-translation surfaces: Each quadratic differential q induces a special geometric structure on S , as follows. Near any non-singular point one can choose *adapted coordinates* ζ for which the local expression of q reduces to $q_\zeta = d\zeta^2$. Given any pair ζ_1 and ζ_2 of such adapted coordinates,

$$(d\zeta_1)^2 = (d\zeta_2)^2 \quad \text{or, equivalently,} \quad \zeta_1 = \pm\zeta_2 + \text{const.} \quad (1)$$

Thus, we say that the set of adapted coordinates is a *half-translation atlas* on the complement of the singularities and S is a *half-translation surface*. In particular, $S \setminus \{\text{singularities}\}$ is endowed with a flat Riemannian metric imported from the plane via the half-translation atlas. The total area of this metric coincides with the norm $\|q\|$. Adapted coordinates ζ may also be constructed in the neighborhood of each singularity z_i such that

$$q_\zeta = \zeta^{l_i} d\zeta^2$$

with $|l_i| = \text{order of the singularity}$ ($l_i > 0$ at zeros and $l_i < 0$ at poles). Through them, the flat metric can be completed with a conical singularity of angle equal to $\pi(l_i + 2)$ at z_i .

A quadratic differential q is *orientable* if it is the square of some *Abelian differential*, that is, some holomorphic complex 1-form ω . Notice that square roots can always be chosen locally, at least far from the singularities, so that orientability has mostly to do with having a globally consistent choice. In the orientable case adapted coordinates may be chosen so that $\omega_\zeta = d\zeta$. Changes between such coordinates are given by

$$d\zeta_1 = d\zeta_2 \quad \text{or, equivalently,} \quad \zeta_1 = \zeta_2 + \text{const} \quad (2)$$

instead of (1). One speaks of *translation atlas* and *translation surface* in this case. A point z_i is a singularity of q , of order l_i , if and only if z_i is a zero of ω , of multiplicity m_i , with $l_i = 2m_i$. We shall be particularly interested in the case when q is *not* orientable.

Teichmüller space: For simplicity, the Riemann surface S will be taken to be compact. Let g be its genus. The Teichmüller space $\mathcal{T}(S)$ is the space of all quasi-conformal homeomorphisms from S to any other Riemann surface, modulo the equivalence relation that identifies $f_1 : S \rightarrow R_1$ and $f_2 : S \rightarrow R_2$ if and only if $f_2 \circ f_1^{-1}$ is homotopic to some conformal homeomorphism $h : R_1 \rightarrow R_2$. Recall that an orientation preserving homeomorphism $f : S \rightarrow R$ is *quasi-conformal* if it is differentiable at almost every point and the *dilatation*

$$\kappa_f(z) = \frac{|\partial_z f| + |\partial_{\bar{z}} f|}{|\partial_z f| - |\partial_{\bar{z}} f|}(z).$$

is bounded. The supremum $\kappa(f) = \sup_z \kappa_f(z)$ is the *global dilatation* of f .

There is a natural distance in $\mathcal{T}(S)$, the *Teichmüller metric*, defined by

$$d_{\mathcal{T}}([f_1], [f_2]) = \frac{1}{2} \inf \{ \log \kappa(h) : h \in [f_2 \circ f_1^{-1}] \}.$$

The metric space $(\mathcal{T}(S), d_{\mathcal{T}})$ is complete. Furthermore, $\mathcal{T}(S)$ has the structure of a complex manifold of dimension $3g - 3$ (dimension 1 if the genus $g = 1$) and $d_{\mathcal{T}}$ is a Finsler metric on this manifold.

Teichmüller theorem: The Teichmüller theorem states that every equivalence class includes some map, unique up to post-composition with a conformal mapping, that minimizes the global dilatation in the class. Moreover, this extremal homeomorphism f admits a very precise description in terms of certain quadratic differentials:

- there exist integrable quadratic differentials ϕ on S and ψ on R such that f sends the horizontal foliation of ϕ to the horizontal foliation of ψ and the vertical foliation of ϕ to the vertical foliation of ψ ;
- moreover, in (horizontal, vertical) coordinates the map f has the form

$$f(x, y) = (e^t x, e^{-t} y) \quad \text{for some constant } t \in \mathbb{R}.$$

In particular, zeros and poles of ϕ are mapped to zeros and poles of ψ preserving the orders, the dilatation $\kappa_f(z) \equiv e^{2|t|}$, and f is real analytic outside the zeros and poles. One calls ϕ and ψ , respectively, the *initial* and *final differentials* of f .

Teichmüller flow: Given any quasi-conformal map $f : S \rightarrow R$ and any integrable quadratic differential ψ on the Riemann surface R , let ϕ be the pull-back of ψ under f . By definition, ϕ is a quadratic differential for the conformal structure pulled-back to S by the map f . For each number $t \in \mathbb{R}$, define

$$\mu_t = k_t \frac{|\psi|}{\psi} \quad \text{with} \quad k_t = \frac{e^t - e^{-t}}{e^t + e^{-t}} \in (-1, 1).$$

Let $g_t : R \rightarrow R_t$ be a quasi-conformal homeomorphism such that

$$\partial_z g_t(\xi) = \mu_t(\xi) \partial_z g_t(\xi).$$

In other words, μ_t is the *Beltrami differential* of g_t . Then g_t is an extremal map in the sense of the Teichmüller theorem, with ψ as the initial differential and some ψ_t as the final differential on R_t . Then $[g_t \circ f]$ is a curve in the Teichmüller space $\mathcal{T}(S)$, and the pull-back ϕ_t of each ψ_t is a quadratic differential for the conformal structure pulled-back to S by the map $g_t \circ f$. Then

$$\mathcal{T}^t : ([f], \phi) \mapsto ([g_t \circ f], \phi_t), \quad t \in \mathbb{R},$$

defines a flow in the space $\mathcal{Q}(S)$ of pairs $([f], \phi)$ such that $[f] \in \mathcal{T}(S)$ and ϕ is an integrable quadratic differential for the conformal structure associated to $[f]$.

From the definition we have that the Teichmüller flow preserves the norm of quadratic differentials. The projection of each flow trajectory down to $\mathcal{T}(R)$ is a geodesic relative to the Teichmüller metric: since g_t is an extremal map,

$$d_{\mathcal{T}}([f], [g_t \circ f]) = \frac{1}{2} \log \kappa(g_t) = \frac{1}{2} \log \frac{1 + k_t}{1 - k_t} = t$$

for all $t \in \mathbb{R}$. Thus, one often speaks of the *Teichmüller geodesic flow*. Observe also that $\mathcal{Q}(S)$ is a fiber bundle over the Teichmüller space; in fact, it may be interpreted as the cotangent bundle of $\mathcal{T}(S)$.

Moduli spaces

We are particularly interested in the moduli space \mathcal{Q}_g of quadratic differentials on Riemann surfaces of genus $g \geq 0$. A crucial feature of \mathcal{Q}_g is that it is naturally stratified according to the number and orders of the singularities, and the strata have a rich geometric structure.

Mapping class group: The *mapping class group* (or *modular group*) of an orientable manifold is the group of homotopy classes of orientation preserving homeomorphisms of the manifold. In the case of compact Riemann surfaces every homotopy class contains some quasi-conformal homeomorphism. The mapping class group $\text{Mod}(S)$ acts on the Teichmüller space $\mathcal{T}(S)$ through

$$\langle g \rangle [f] = [f \circ g^{-1}],$$

where $\langle g \rangle$ is the homotopy class of g , and $[f]$ and $[f \circ g]$ are the Teichmüller equivalence classes of $f : S \rightarrow R$ and $f \circ g : S \rightarrow R$, respectively. This action is by isometries of the Teichmüller metric:

$$d_{\mathcal{T}}([f_1 \circ g], [f_2 \circ g]) = \frac{1}{2} \inf \{ \log \kappa(h) : h \in [f_2 \circ f_1^{-1}] \} = d_{\mathcal{T}}([f_1], [f_2]).$$

The quotient space $\mathcal{T}(S)/\text{Mod}(S)$ coincides with the moduli space \mathcal{M}_g of complex structures on a surface of genus g . Indeed, let $f_1 : S \rightarrow R_1$ and $f_2 : S \rightarrow R_2$ represent the same point on \mathcal{M}_g , that is, there exists a conformal homeomorphism $h : R_1 \rightarrow R_2$. Then $g = f_2^{-1} \circ h \circ f_1$ is a quasi-conformal homeomorphism of S and $[f_2] = \langle g \rangle [f_1] = \langle g \rangle [f_1]$. Conversely, if $[f_2] = \langle g \rangle [f_1] = [f_1 \circ g^{-1}]$ for some quasi-conformal homeomorphism g , then there exists some conformal homeomorphism $h : R_1 \rightarrow R_2$. In particular, f_1 and f_2 represent the same point in the moduli space \mathcal{M}_g .

Similarly, the quotient of $\mathcal{Q}(S)$ by the mapping class group is identified to the moduli space \mathcal{Q}_g of integrable quadratic differentials on a surface of genus g , that is, the space of pairs (S, q) modulo the equivalence relation that identifies (S_1, q_1) to (S_2, q_2) whenever there exists a conformal homeomorphism $f : S_1 \rightarrow S_2$ such that $f_* q_1 = q_2$. Both \mathcal{M}_g and \mathcal{Q}_g are complex *varieties* with the same dimensions as the complex manifolds $\mathcal{T}(S)$ and $\mathcal{Q}(S)$, respectively. The quotient maps $\mathcal{T}(S) \rightarrow \mathcal{M}_g$ and $\mathcal{Q}(S) \rightarrow \mathcal{Q}_g$ have branching points, where the target spaces fail to be manifolds.

Strata: Each moduli space \mathcal{Q}_g is naturally stratified according to the number σ and orders l_1, \dots, l_σ of the zeros and poles. We abide to the usual convention that each $l_i \geq 1$ corresponds to a zero of multiplicity l_i whereas each $l_j \leq -1$ corresponds to a pole of order $|l_j|$. A quadratic differential on a compact manifold is integrable if and only if all poles are simple. So, in what follows we always consider $l_j \in \{-1\} \cup \mathbb{N}$. The orders of the singularities are related to the genus of the surface through the Gauss-Bonnet formula

$$\sum_{i=1}^{\sigma} l_i = 4g - 4. \quad (3)$$

We denote by $\mathcal{Q}_g(l_1, \dots, l_\sigma, \varepsilon)$ the stratum of quadratic differentials having exactly σ singularities, of orders $l_1 \leq \dots \leq l_\sigma$, where the additional variable ε

distinguishes between orientable ($\varepsilon = 1$) and non-orientable ($\varepsilon = -1$) quadratic differentials. As observed before, in the orientable case $q = \omega^2$ every l_i is even. Then (3) becomes

$$\sum_{i=1}^{\sigma} m_i = 2g - 2, \quad (4)$$

where the m_i are the multiplicities of the zeros of the Abelian differential ω .

Affine structure: Each stratum $\mathcal{Q}_g(l_1, \dots, l_\sigma, \varepsilon)$ comes with a canonical structure of complex affine variety of dimension $d = 2g + \sigma + (\varepsilon - 3)/2$. Let us explain this, focussing on the orientable case $\varepsilon = 1$. Let ω_0 be any Abelian differential and Σ be the set of its singularities. By definition, the number of singularities and their multiplicities are constant on the whole stratum. Thus, for Abelian differentials in a neighborhood of ω_0 we can try and view their singularities as ‘analytic’ continuations of the singularities of ω_0 , so that the singular sets are all identified to Σ ; if ω_0 admits non-trivial symmetries (e.g. making some of its singularities indistinguishable) these must first be ‘unraveled’ by going to some finite covering of a neighborhood of ω_0 . Then we can consider the map

$$U \rightarrow H^1(S, \Sigma, \mathbb{C}), \quad \omega \mapsto [\omega] \quad (5)$$

assigning to each Abelian differential ω in a neighborhood U of ω_0 (possibly up to finite covering) the element it represents in the relative complex cohomology of (S, Σ) . The relative cohomology $H^1(S, \Sigma, \mathbb{C})$ is a complex vector space of dimension $d = 2g + \sigma - 1$: for any choice $\{\gamma_j : j = 1, \dots, d\}$ of a basis for the homology $H_1(S, \Sigma, \mathbb{C})$, the corresponding *period map*

$$H^1(S, \Sigma, \mathbb{C}) \rightarrow \mathbb{C}^d, \quad [\omega] \mapsto \left(\int_{\gamma_j} \omega \right)_{j=1, \dots, d} \quad (6)$$

is an isomorphism. The map (5) is a homeomorphism (at the finite covering level), as long as U is small enough. Through these homeomorphisms we can import the standard affine structure of $H^1(S, \Sigma, \mathbb{C})$ to the stratum, thus defining the affine variety structure we announced.

Canonical volume: Let us consider the standard volume measure on the space $H^1(S, \Sigma, \mathbb{C})$, normalized so that the integer lattice $H^1(S, \Sigma, \mathbb{Z})$ has covolume 1. Pulling this volume measure back through the charts (5) one obtains a canonical volume measure ν on each stratum $\mathcal{Q}_g(l_1, \dots, l_\sigma, \varepsilon)$. This measure is invariant under the Teichmüller flow and homogeneous, in the sense that the multiplication by any constant $c > 0$ on the stratum has Jacobian $\equiv c^d$. Thus, ν induces a measure $\nu^{(1)}$ on the subset $\mathcal{Q}_g^{(1)}(l_1, \dots, l_\sigma, \varepsilon)$ of quadratic differentials with norm 1 and this measure $\nu^{(1)}$ is invariant under the restricted Teichmüller flow.

Connected components: Somewhat surprisingly, strata are not all connected. Kontsevich, Zorich [KZ] and Lanneau [L1, L3] have given a complete catalogue of the connected components of, respectively, orientable and non-orientable quadratic differentials, together with complete sets of invariants to distinguish between all components. It turns out that there can be up to 3

connected components in the orientable case and up to 2 connected components in the non-orientable case.

Dynamics of the Teichmüller flow

The first main results on the dynamics of this flow were due to Masur and Veech, motivated by its connections to the theory of interval exchange transformations. Their approaches highlighted the principle that the Teichmüller flow orbit of a quadratic differential encodes important information about the differential itself, and its horizontal and vertical foliations. This fueled further interest on the dynamical behavior of this flow and a lot of progress has indeed been obtained in recent years, especially in the orientable setting.

Ergodicity theorem: Keane [Ke1, Ke2] conjectured that almost every interval exchange transformation admits a unique invariant probability, namely, normalized Lebesgue measure. Masur and Veech, independently, realized this would follow if one could prove that the Teichmüller flow orbit of almost every (orientable) quadratic differential returns infinitely many times to some compact part of phase space. It is clear from the definition that the Teichmüller flow leaves every stratum $\mathcal{Q}_g(l_1, \dots, l_\sigma, \varepsilon)$ invariant. Then the subset $\mathcal{Q}_g^{(1)}(l_1, \dots, l_\sigma, \varepsilon)$ of quadratic differentials with unit norm is also invariant. Moreover, as mentioned before, the canonical volume measure $\nu^{(1)}$ on every normalized stratum $\mathcal{Q}_g^{(1)}(l_1, \dots, l_\sigma, \varepsilon)$ is also preserved by the Teichmüller flow. Masur [M] and Veech [Ve2] proved that $\nu^{(1)}$ is always finite and, using this fact, went on to prove that the Teichmüller flow is ergodic on every connected component of every $\mathcal{Q}_g^{(1)}(l_1, \dots, l_\sigma, \varepsilon)$. This implies recurrence to compact parts of phase space which, as mentioned before, leads to a proof of the Keane conjecture.

An important refinement of the Masur-Veech theorem was obtained a few years later by Kerckhoff, Masur, Smillie [KMS]. In particular, they showed that the set of orbits with good recurrence properties contains a full (arc-length) measure subset of every circle $\{e^{i\theta}q : \theta \in \mathbb{R}\}$ in the moduli space of quadratic differentials. This had a number of applications including, for instance, the first examples of ergodic polygonal billiards. In fact, one gets that a residual subset of polygonal billiards are ergodic.

Lyapunov exponents: More recently, Avila, Viana [AV1, AV2, AV3] proved that the Lyapunov spectrum of the Teichmüller flow in any stratum of Abelian differentials is simple. It is well known that the Lyapunov exponents of the Teichmüller flow have the form

$$\{\pm 2\} \cup \{\pm 1 \pm \nu_j : j = 2, \dots, g\} \cup \{\pm 1\}, \quad (7)$$

for numbers $1 \geq \nu_2 \geq \dots \geq \nu_g \geq \nu_{g+1} = 0$ that depend only on the connected component of the stratum (by ergodicity). Zorich and Kontsevich conjectured that all the inequalities in this last relation are strict, so that all the Lyapunov exponents in (7) have multiplicity 1 (except for the trivial exponents ± 1 whose multiplicity is always one less than the number of singularities). Veech [Ve3] noted that $\nu_2 < 1$ and so the Teichmüller geodesic flow \mathcal{T}^t is a *non-uniformly hyperbolic flow*: all Lyapunov exponents are different from zero. More recently,

Forni [F] proved that $\nu_g > 0$. This settled case $g = 2$ of the conjecture and also provided enough information on the dynamics of the Teichmüller flow that Avila, Forni [AF] could use to settle another old problem: almost all interval exchange transformations are weak mixing (or else irrational circle rotations). Then Avila, Viana [AV1, AV2, AV3] used a general simplicity criterium for linear cocycles to obtain the full statement of the Zorich-Kontsevich conjecture. The corresponding statement for non-orientable strata has yet to be proven.

Asymptotic flag: The conjecture itself was motivated by Zorich's discovery of a surprising link between the Lyapunov spectra of the Teichmüller flow on strata of Abelian differentials and the asymptotic distribution of geodesics on typical translation surfaces. For each long geodesic γ on a translation surface S , consider the element $[\gamma]$ of the first homology $H_1(S, \mathbb{R})$ represented by the closed curve one obtains when the endpoints of γ are joined by some segment with uniformly bounded length. Zorich observed that the asymptotic behavior of $[\gamma]$ as the length of γ goes to infinity seems to be governed by a certain flag in homology: computer evidence suggested that for almost every translation surface and initial direction there are vector subspaces $0 < L_1 < L_2 < \dots < L_g < H_1(S, \mathbb{R})$ with $\dim L_j = j$ and numbers $1 > \eta_2 > \dots > \eta_g > \eta_{g+1} = 0$ such that the deviation of $[\gamma]$ from each L_j is $\lesssim |\gamma|^{\eta_{j+1}}$. The case $j = g$ just means that $[\gamma]$ remains within bounded distance from the subspace L_g as the length goes to infinity. Moreover, Zorich [Z1, Z3] was able to reduce the proof of this fact to the aforementioned conjecture that the Lyapunov spectrum of the Teichmüller flow is simple (one gets $\eta_j = \nu_j$ for all j).

Exponential mixing

It is now well known that the Teichmüller flow is mixing on each ergodic component:

$$\int \phi(\psi \circ \mathcal{T}^t) d\nu^{(1)} - \int \phi d\nu^{(1)} \int \psi d\nu^{(1)} \rightarrow 0 \quad (8)$$

for any L^2 observables ϕ and ψ . This can be proven from the Masur-Veech ergodicity theorem together with the fact that \mathcal{T}^t embeds into a certain volume preserving $\mathrm{SL}(2, \mathbb{R})$ action on each stratum. A quick way to define this action is by post-composition in the charts (5):

$$A \cdot [\omega] = [A \circ \omega]$$

for each $A \in \mathrm{SL}(2, \mathbb{R})$. The Teichmüller flow is just the restriction to the one-parameter subgroup

$$\left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}$$

Our main object of interest is the speed of mixing. As usual, estimates on the speed of mixing are possible only if one consider fairly regular observable functions. In our case this will be the so-called *Ratner class*.

Ratner class: This issue was first addressed by Avila, Gouëzel, Yoccoz [AGY], who proved that the Teichmüller flow in the space of Abelian differentials is exponentially mixing: the convergence in (8) is exponentially fast for all observable functions in the Ratner class. An L^2 function $\xi : \mathcal{Q}_g^{(1)}(l_1, \dots, l_\sigma, \varepsilon) \rightarrow \mathbb{C}$ is

in the *Ratner class* if the map

$$\begin{aligned} \mathbb{S}^1 &\mapsto L^2(\nu^{(1)}) \\ \theta &\mapsto \xi \circ R_\theta \end{aligned}$$

is Hölder continuous, where R_θ is the rotation of angle $\theta \in \mathbb{S}^1$. This is the case, in particular, for all functions which are Hölder continuous along the fibers of \mathcal{Q}_g . The reason to consider this particular class of observables is that exponential mixing in the Ratner class is equivalent to the ‘spectral gap’ property for the $\mathrm{SL}(2, \mathbb{R})$ action. The hard part of this equivalence is due to Ratner; see [AGY, Appendix B] for a discussion and references.

Outline of the proof: Essentially all of the results mentioned previously make important use of a combinatorial formalism that was initiated by Rauzy [R] and Veech [Ve1] and perfected over the years. For instance, to each Abelian differential one can associate a combinatorial object called extended Rauzy class. Veech [Ve1, Ve3] observed that there is a one-to-one correspondence between connected components of strata of Abelian differentials and extended Rauzy classes, which is how he was led to discover the existence of non-connected strata. Due, perhaps to the applications in the theory of interval exchanges, this combinatorial formalism was developed quite more rapidly in the orientable case (e.g. in [AGY], [AV3], [KZ]).

Our strategy for proving Theorem A starts from the observation that the space of (non-orientable) quadratic differentials can be viewed as a subset of the space of Abelian differentials with involution. Indeed, it is well known that given any quadratic differential q on a Riemann surface S of genus g there exists a double covering $\pi : \tilde{S} \rightarrow S$, branched over the singularities of odd order, and there is an Abelian differential ω on the surface \tilde{S} such that $\pi_*(\omega^2) = q$. In other words, q lifts to an orientable quadratic differential on \tilde{S} . In this construction,

- to each zero of q with even multiplicity $l_i \geq 1$ corresponds a pair of zeros of ω with multiplicity $m_j = l_i/2$;
- to each zero of q with odd multiplicity $l_i \geq 1$ corresponds a zero of ω with multiplicity $m_j = l_i + 1$;
- to each pole of q with $l_i = -1$ corresponds a removable (that is, order 0) singularity of ω .

The surface \tilde{S} is connected if and only if q is non-orientable. Notice $i_*(\omega) = \pm\omega$, where $i : \tilde{S} \rightarrow \tilde{S}$ is the involution permuting the points in each fiber of the double cover π .

Thus, we consider moduli spaces of Abelian differentials with involution and a certain combinatorial marking. The combinatorial marking includes the order of the zeros at the singularities, but also a distinguished singularity with a fixed eastbound separatrix. This moduli space \mathcal{M} is a finite cover of \mathcal{Q}_g where $\mathrm{SL}(2, \mathbb{R})$ is still acting, and thus it is enough to prove the result on this space.

We parametrize the moduli space as a moduli space of zippered rectangles with involution as follows. Choosing a convenient segment I inside the separatrix, we look at the first return map under the northbound flow to the union of the I and its image under the involution. This map is an *interval exchange*

transformation with involution, and the original northbound flow becomes a suspension flow, living in the union of some rectangles. The original surface can be obtained from the rectangles by gluing appropriately. This construction can be carried out in a large open set of \mathcal{M} (with complement of codimension 2, see [Ve3]).

Once this combinatorial model is derived, one can view the Teichmüller flow on \mathcal{M} as a suspension flow over a (weakly) hyperbolic transformation, which is itself a skew-product over a (weakly) expanding transformation, the Rauzy algorithm with involution (RAI). We then consider some appropriate compact subset of the domain of the RAI: the induced transformation is automatically expanding and the Teichmüller flow is thus modelled on an “excellent hyperbolic flow” in the language of [AGY]. Two properties need to be verified to deduce exponential mixing: the return time should not be co-homologous to locally constant, and it should have exponential tails. The first property is transparent from the zippered rectangle construction. The second depends essentially on proving some distortion estimate. Both proofs of the distortion estimate in [AGY] depend heavily on certain properties of the usual Rauzy induction (simple description of transition probabilities for a random walk) which seem difficult to generalize to our setting. We provide here an alternative proof which is less dependent on precise estimates for the random walk.

1 Excellent hyperbolic semi-flows

In [AGY], an abstract result for exponential mixing was proved for the class of so-called *excellent hyperbolic semi-flows*, following the work of Baladi-Vallée [BV] based on the foundational work of Dolgopyat [D]. This result can be directly used in our work. In this section we state precisely this result, which will need several definitions.

By definition, a Finsler manifold is a smooth manifold endowed with a norm on each tangent space, which varies continuously with the base point.

Definition 1.1. A *John domain* Δ is a finite dimensional connected Finsler manifold, together with a measure Leb on Δ , with the following properties:

1. For $x, x' \in \Delta$, let $d(x, x')$ be the infimum of the length of a C^1 path contained in Δ and joining x and x' . For this distance, Δ is bounded and there exist constants C_0 and ϵ_0 such that, for all $\epsilon < \epsilon_0$, for all $x \in \Delta$, there exists $x' \in \Delta$ such that $d(x, x') \leq C_0\epsilon$ and such that the ball $B(x', \epsilon)$ is compactly contained in Δ .
2. The measure Leb is a fully supported finite measure on Δ , satisfying the following inequality: for all $C > 0$, there exists $A > 0$ such that, whenever a ball $B(x, r)$ is compactly contained in Δ , $\text{Leb}(B(x, Cr)) \leq A\text{Leb}(B(x, r))$.

Definition 1.2. Let L be a finite or countable set, let Δ be a John domain, and let $\{\Delta^{(l)}\}_{l \in L}$ be a partition into open sets of a full measure subset of Δ . A map $T : \bigcup_l \Delta^{(l)} \rightarrow \Delta$ is a *uniformly expanding Markov map* if

1. For each l , T is a C^1 diffeomorphism between $\Delta^{(l)}$ and Δ , and there exist constants $\kappa > 1$ (independent of l) and $C_{(l)}$ such that, for all $x \in \Delta^{(l)}$ and all $v \in T_x\Delta$, $\kappa \|v\| \leq \|DT(x) \cdot v\| \leq C_{(l)} \|v\|$.

2. Let $J(x)$ be the inverse of the Jacobian of T with respect to Leb. Denote by \mathcal{H} the set of inverse branches of T . The function $\log J$ is C^1 on each set $\Delta^{(l)}$ and there exists $C > 0$ such that, for all $h \in \mathcal{H}$, $\|D((\log J) \circ h)\|_{C^0(\Delta)} \leq C$.

Such a map T preserves a unique absolutely continuous measure μ . Its density is bounded from above and from below and is C^1 .

Definition 1.3. Let $T : \bigcup_l \Delta^{(l)} \rightarrow \Delta$ be a uniformly expanding Markov map on a John domain. A function $r : \bigcup_l \Delta^{(l)} \rightarrow \mathbb{R}_+$ is a *good roof function* if

1. There exists $\epsilon_1 > 0$ such that $r \geq \epsilon_1$.
2. There exists $C > 0$ such that, for all $h \in \mathcal{H}$, $\|D(r \circ h)\|_{C^0} \leq C$.
3. It is not possible to write $r = \psi + \phi \circ T - \phi$ on $\bigcup_l \Delta^{(l)}$, where $\psi : \Delta \rightarrow \mathbb{R}$ is constant on each set $\Delta^{(l)}$ and $\phi : \Delta \rightarrow \mathbb{R}$ is C^1 .

If r is a good roof function for T , we will write $r^{(n)}(x) = \sum_{k=0}^{n-1} r(T^k x)$.

Definition 1.4. A good roof function r as above has *exponential tails* if there exists $\sigma_0 > 0$ such that $\int_{\Delta} e^{\sigma_0 r} d\text{Leb} < \infty$.

If $\widehat{\Delta}$ is a Finsler manifold, we will denote by $C^1(\widehat{\Delta})$ the set of functions $u : \widehat{\Delta} \rightarrow \mathbb{R}$ which are bounded, continuously differentiable, and such that $\sup_{x \in \widehat{\Delta}} \|Du(x)\| < \infty$. Let

$$\|u\|_{C^1(\widehat{\Delta})} = \sup_{x \in \widehat{\Delta}} |u(x)| + \sup_{x \in \widehat{\Delta}} \|Du(x)\| \quad (9)$$

be the corresponding norm.

Definition 1.5. Let $T : \bigcup_l \Delta^{(l)} \rightarrow \Delta$ be a uniformly expanding Markov map, preserving an absolutely continuous measure μ . A *hyperbolic skew-product* over T is a map \widehat{T} from a dense open subset of a bounded connected Finsler manifold $\widehat{\Delta}$, to $\widehat{\Delta}$, satisfying the following properties:

1. There exists a continuous map $\pi : \widehat{\Delta} \rightarrow \Delta$ such that $T \circ \pi = \pi \circ \widehat{T}$ whenever both members of this equality are defined.
2. There exists a probability measure ν on $\widehat{\Delta}$, giving full mass to the domain of definition of \widehat{T} , which is invariant under \widehat{T} .
3. There exists a family of probability measures $\{\nu_x\}_{x \in \Delta}$ on $\widehat{\Delta}$ which is a disintegration of ν over μ in the following sense: $x \mapsto \nu_x$ is measurable, ν_x is supported on $\pi^{-1}(x)$ and, for every measurable set $A \subset \widehat{\Delta}$, $\nu(A) = \int \nu_x(A) d\mu(x)$.

Moreover, this disintegration satisfies the following property: there exists a constant $C > 0$ such that, for any open subset $O \subset \bigcup \Delta^{(l)}$, for any $u \in C^1(\pi^{-1}(O))$, the function $\bar{u} : O \rightarrow \mathbb{R}$ given by $\bar{u}(x) = \int u(y) d\nu_x(y)$ belongs to $C^1(O)$ and satisfies the inequality

$$\sup_{x \in O} \|D\bar{u}(x)\| \leq C \sup_{y \in \pi^{-1}(O)} \|Du(y)\|.$$

4. There exists $\kappa > 1$ such that, for all $y_1, y_2 \in \widehat{\Delta}$ with $\pi(y_1) = \pi(y_2)$, holds

$$d(\widehat{T}y_1, \widehat{T}y_2) \leq \kappa^{-1}d(y_1, y_2).$$

Let \widehat{T} be an hyperbolic skew-product over a uniformly expanding Markov map T . Let r be a good roof function for T , with exponential tails. It is then possible to define a space $\widehat{\Delta}_r$ and a semi-flow \widehat{T}_t over \widehat{T} on $\widehat{\Delta}$, using the roof function $r \circ \pi$, in the following way. Let $\widehat{\Delta}_r = \{(y, s) : y \in \bigcup_l \widehat{\Delta}_l, 0 \leq s < r(\pi y)\}$. For almost all $y \in \widehat{\Delta}$, all $0 \leq s < r(\pi y)$ and all $t \geq 0$, there exists a unique $n \in \mathbb{N}$ such that $r^{(n)}(\pi y) \leq t + s < r^{(n+1)}(\pi y)$. Set $\widehat{T}_t(y, s) = (\widehat{T}^n y, s + t - r^{(n)}(\pi y))$. This is a semi-flow defined almost everywhere on $\widehat{\Delta}_r$. It preserves the probability measure $\nu_r = \nu \otimes \text{Leb} / (\nu \otimes \text{Leb})(\widehat{\Delta}_r)$. Using the canonical Finsler metric on $\widehat{\Delta}_r$, namely the product metric given by $\|(u, v)\| := \|u\| + \|v\|$, we define the space $C^1(\widehat{\Delta}_r)$ as in (9). Notice that $\widehat{\Delta}_r$ is not connected, and the distance between points in different connected components is infinite.

Definition 1.6. A semi-flow \widehat{T}_t as above is called an *excellent hyperbolic semi-flow*.

Theorem 1.7 ([AGY]). *Let \widehat{T}_t be an excellent hyperbolic semi-flow on a space $\widehat{\Delta}_r$, preserving the probability measure ν_r . There exist constants $C > 0$ and $\delta > 0$ such that, for all functions $U, V \in C^1(\widehat{\Delta}_r)$, for all $t \geq 0$,*

$$\left| \int U \cdot V \circ \widehat{T}_t d\nu_r - \left(\int U d\nu_r \right) \left(\int V d\nu_r \right) \right| \leq C \|U\|_{C^1} \|V\|_{C^1} e^{-\delta t}.$$

2 The Veech flow with involution

2.1 Rauzy classes and interval exchange transformations with involution

2.1.1 Interval exchange transformations with involution

Let \mathcal{A} be an alphabet on $2d \geq 4$ letters with an involution $i : \mathcal{A} \rightarrow \mathcal{A}$ and let $*$ $\notin \mathcal{A}$. When considering objects modulo involution, we will use underline: for instance the involution class of an element $\alpha \in \mathcal{A}$ will be denoted by $\underline{\alpha} \in \underline{\mathcal{A}} = \mathcal{A}/i$. An *interval exchange transformation with involution* of type $(\mathcal{A}, i, *)$ depends on the specification of the following data:

Combinatorial data: Let $\pi : \mathcal{A} \cup \{*\} \rightarrow \{1, \dots, 2d + 1\}$ be a bijection such that neither $i(\mathcal{A}_l) \subset \mathcal{A}_r$ nor $i(\mathcal{A}_r) \subset \mathcal{A}_l$, where $\mathcal{A}_l = \{\alpha \in \mathcal{A}, \pi(\alpha) < \pi(*)\}$ and $\mathcal{A}_r = \{\alpha \in \mathcal{A}, \pi(\alpha) > \pi(*)\}$. The combinatorial data can be viewed as a row where the elements of $\mathcal{A} \cup \{*\}$ are displayed in the order $(\pi^{-1}(1), \dots, \pi^{-1}(2d + 1))$.

Length data: Let $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ be a vector satisfying

$$\sum_{\pi(\alpha) < \pi(*)} \lambda_{\underline{\alpha}} = \sum_{\pi(\alpha) > \pi(*)} \lambda_{\underline{\alpha}} \quad (10)$$

(it is easy to find such a vector λ).

Let $\mathfrak{S} = \mathfrak{S}(\mathcal{A}, i, *)$ be the set of all bijections π as above.

The transformation is then defined as follows:

1. Let $I \subset \mathbb{R}$ be the interval (all intervals will be assumed to be closed at the left and open at the right) centered on 0 and of length $|I| \equiv \sum_{\alpha \in \mathcal{A}} \lambda_{\alpha}$ (notice that $|I| = 2 \sum_{\alpha \in \mathcal{A}} \lambda_{\alpha}$).
2. Let $\bar{\pi} : \mathcal{A} \cup \{*\} \rightarrow \{1, \dots, 2d + 1\}$ be defined by $\bar{\pi}(*) = 2d + 2 - \pi(*)$ and $\bar{\pi}(\alpha) = 2d + 2 - \pi(i(\alpha))$.
3. Break I into $2d$ subintervals I_{α} of length λ_{α} , ordered according to π .
4. Rearrange the subintervals inside I in the order given by $\bar{\pi}$.

Example 2.1. To simplify the notation, given a letter we will represent its image by involution with an inverted letter (i.e. symmetric relatively to the horizontal or the vertical direction).

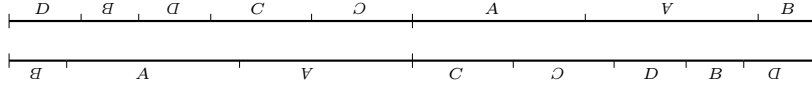


Figure 1: Interval exchange transformation with involution.

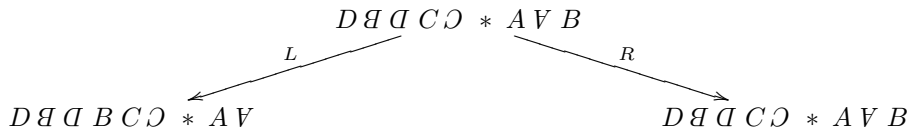
As we can see in this example, the bottom row of letters is equal to the top one rotated by 180 degrees around the central point of the interval. In fact, this always occurs by definition of $\bar{\pi}$.

2.1.2 Rauzy classes with involution

We define two operations, the *left* and the *right* on \mathfrak{S} as follows. Let α and β be the leftmost and the rightmost letters of the row representing π , respectively. If $\beta \neq i(\alpha)$ and taking β and putting it into the position immediately after $i(\alpha)$ results in a row representing an element π' of \mathfrak{S} , we say that the left operation is defined at π , and it takes π to π' . In this case, we say that α wins and β loses. Similarly, if $\alpha \neq i(\beta)$ and taking α and putting it into the position immediately before $i(\beta)$ results in a row representing an element π' of \mathfrak{S} , we say that the right operation is defined at π , and it takes π to π' . In this case, we say that β wins and α loses.

Remark 2.2. Notice that $\beta \neq i(\alpha)$ and $\alpha \neq i(\beta)$ are equivalent conditions since the involution i is a bijection. But to define left (respectively right) operation we also ask that the row obtained after moving β (respectively α) represents an element of \mathfrak{S} . So we can have either just one of the operations defined at some permutation or both, as we can see in the next example.

Example 2.3. Given the permutation π defined in the example 2.1, both operations (left denoted by L and right denoted by R) are defined at π .



But the left operation is not defined at $\pi' = D \mathcal{E} \mathcal{C} B C \mathcal{D} * A V$. In fact, if it was defined, we would have:

$$\begin{array}{c} \pi' = D \mathcal{E} \mathcal{C} B C \mathcal{D} * A V \\ \quad \quad \quad \downarrow L \\ \pi'' = D \mathcal{E} \mathcal{C} V B C \mathcal{D} * A \end{array}$$

and the associated length vector $\lambda'' \in \mathbb{R}_+^{\mathcal{A}}$, by condition (10), would satisfy $\lambda''_A = \lambda''_A + \lambda''_B + \lambda''_C + \lambda''_D$, which is impossible, so $\pi'' \notin \mathfrak{S}$.

Consider an oriented diagram with vertices which are the elements of \mathfrak{S} and oriented arrows representing the operations left and right starting and ending at two vertices of \mathfrak{S} . We will say that such an arrow has type left or right, respectively. We will call this diagram by *Rauzy diagram with involution*. A path γ of length $m \geq 0$ is a sequence of m arrows, a_1, \dots, a_m , joining $m+1$ vertices, v_0, \dots, v_m , respectively. In this case we say that γ starts at v_0 , it ends at v_m and pass through v_1, \dots, v_{m-1} . Let γ_1 and γ_2 be two paths such that the end of γ_1 is the start of γ_2 . We define their concatenation denoted by $\gamma_1 \gamma_2$, which also is a path. A path of length zero is identified with a vertex and if it has length one we identify it with an arrow.

A *Rauzy class with involution* \mathfrak{R} is a minimal non-empty subset of \mathfrak{S} , which is invariant under the left and the right operations, and such that any involution class admits a representative which is the winner of some arrow starting (and ending) in \mathfrak{R} . Elements of Rauzy classes with involution are said to be *irreducible*. We denote by $\mathfrak{S}^0 = \mathfrak{S}^0(\mathcal{A}) \subset \mathfrak{S}$ the set of irreducible permutations and let $\Pi(\mathfrak{R})$ be the set of all paths.

Lemma 2.4. *If π is irreducible then the left operation (respectively the right operation) is defined at π if and only if there exists $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ satisfying (10) such that $\lambda_{\underline{\alpha}} > \lambda_{\underline{\beta}}$ (respectively $\lambda_{\underline{\beta}} > \lambda_{\underline{\alpha}}$) where α and β are the leftmost and the rightmost elements of π .*

Proof. Assume that the left operation is defined at π and let π' be the image of π . Let $\lambda' \in \mathbb{R}_+^{\mathcal{A}}$ be a vector satisfying $\sum_{\pi'(\xi) < \pi'(*)} \lambda'_\xi = \sum_{\pi'(\xi) > \pi'(*)} \lambda'_\xi$. Let $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ be given by $\lambda_{\underline{\alpha}} = \lambda'_{\underline{\alpha}} + \lambda'_{\underline{\beta}}$ and $\lambda_{\underline{\xi}} = \lambda'_{\underline{\xi}}$, $\underline{\xi} \neq \underline{\alpha}$. Then λ satisfies (10) and we have $\lambda_{\underline{\alpha}} > \lambda_{\underline{\beta}}$.

Assume that $\lambda_{\underline{\alpha}} > \lambda_{\underline{\beta}}$. Let $\lambda'_{\underline{\alpha}} = \lambda_{\underline{\alpha}} - \lambda_{\underline{\beta}}$. Let $\pi'(x) = \pi(x)$ for $\pi(x) \leq \pi(i(\alpha))$, $\pi'(\beta) = \pi(i(\alpha)) + 1$ and $\pi'(x) = \pi(x) + 1$ for $\pi(i(\alpha)) < \pi(x) < 2d + 1$. We need to show that $\pi' \in \mathfrak{S}$.

Let $\mathcal{A}_l = \{\pi(\xi) < \pi(*)\}$, $\mathcal{A}_r = \{\pi(\xi) > \pi(*)\}$, $\mathcal{A}'_l = \{\pi'(\xi) < \pi'(*)\}$, $\mathcal{A}'_r = \{\pi'(\xi) > \pi'(*)\}$. Notice that $\sum_{\pi'(\xi) < \pi'(*)} \lambda'_\xi = \sum_{\pi'(\xi) > \pi'(*)} \lambda'_\xi$, so $i(\mathcal{A}'_l)$ can not be properly contained or properly contain \mathcal{A}'_r . If $i(\mathcal{A}'_l) = \mathcal{A}'_r$, then $\pi'(i(\alpha)) > \pi'(*)$, so $\pi(i(\alpha)) > \pi(*)$ as well. This implies that $\mathcal{A}'_l = \mathcal{A}_l$ and $\mathcal{A}'_r = \mathcal{A}_r$, and since $\pi \in \mathfrak{S}$ we have $i(\mathcal{A}_l) \neq \mathcal{A}_r$. \square

2.1.3 Linear action

Given a Rauzy class \mathfrak{R} , we associate to each path $\gamma \in \Pi(\mathfrak{R})$ a linear map $B_\gamma \in \mathrm{SL}(\mathcal{A}, \mathbb{Z})$. If γ is a vertex we take $B_\gamma = \mathrm{id}$. If γ is an arrow with winner α

and loser β then we define $B_\gamma \cdot e_\xi = e_\xi$ for $\xi \in \mathcal{A} \setminus \{\alpha\}$, $B_\gamma \cdot e_\alpha = e_\alpha + e_\beta$, where $\{e_\xi\}_{\xi \in \mathcal{A}}$ is the canonical basis of $\mathbb{R}^{\mathcal{A}}$. If γ is a path, of the form $\gamma = \gamma_1 \dots \gamma_n$, where γ_i are arrows for all $i = 1, \dots, n$, we take $B_\gamma = B_{\gamma_1 \dots \gamma_n} = B_{\gamma_n} \dots B_{\gamma_1}$.

2.2 Rauzy algorithm with involution

Given a Rauzy class $\mathfrak{R} \subset \mathfrak{S}$, consider the set

$$\mathcal{S}_\pi = \left\{ \lambda \in \mathbb{R}^{\mathcal{A}} : \sum_{\pi(\alpha) < \pi(*)} \lambda_\alpha = \sum_{\pi(\alpha) > \pi(*)} \lambda_\alpha \right\}$$

We define

$$\mathcal{S}_\pi^+ = \mathcal{S}_\pi \cap \mathbb{R}_+^{\mathcal{A}}, \quad \Delta_\pi = \mathcal{S}_\pi^+ \times \{\pi\}, \quad \Delta_{\mathfrak{R}}^0 = \bigcup_{\pi \in \mathfrak{R}} \Delta_\pi.$$

Let (λ, π) be an element of $\Delta_{\mathfrak{R}}^0$. We say that we can apply Rauzy algorithm with involution (RAI) to (λ, π) if $\lambda_\alpha \neq \lambda_\beta$, where $\alpha, \beta \in \mathcal{A}$ are the leftmost and the rightmost elements of π , respectively. Then we define (λ', π') as follows:

1. Let $\gamma = \gamma(\lambda, \pi)$ be an arrow representing the left or the right operation at π , according to whether $\lambda_\alpha > \lambda_\beta$ or $\lambda_\beta > \lambda_\alpha$.
2. Let $\lambda'_\xi = \lambda_\xi$ if ξ is not the class of the winner of γ , and $\lambda'_\xi = |\lambda_\alpha - \lambda_\beta|$ if ξ is the class of the winner of γ , i.e., $\lambda = B_\gamma^* \cdot \lambda'$ (here and in the following we will use the notation A^* to the transpose of a matrix A).
3. Let π' be the end of γ .

We say that (λ', π') is obtained from (λ, π) by applying RAI, of type left or right depending on whether the operation is left or right. We have $(\lambda', \pi') \in \Delta_{\mathfrak{R}}^0$. In this way we define a map $Q : (\lambda, \pi) \mapsto (\lambda', \pi')$ which is called *Rauzy induction map with involution*. Its domain of definition is the set of all $(\lambda, \pi) \in \Delta_{\mathfrak{R}}^0$ such that $\lambda_\alpha \neq \lambda_\beta$ (where α and β are the leftmost and the rightmost letters of π) and we denote it by $\Delta_{\mathfrak{R}}^1$. The connected components $\Delta_\pi \subset \Delta_{\mathfrak{R}}^0$ are naturally labeled by elements of \mathfrak{R} and the connected components Δ_γ of $\Delta_{\mathfrak{R}}^1$ are naturally labeled by arrows, i.e., paths in $\Pi(\mathfrak{R})$ of length 1.

We associate to (λ, π) and to (λ', π') two interval exchange transformations with involution $f : I \rightarrow I$ and $f' : I' \rightarrow I'$, respectively. The relation between (λ, π) and (λ', π') implies a relation between the interval exchange transformations with involution, namely, the map f' is the first return map of f to a subinterval of I , obtained by cutting two subintervals from the beginning and from the end of I with the same length λ_ξ , where ξ is the loser of γ .

Let $\Delta_{\mathfrak{R}}^n$ be the domain of Q^n , $n \geq 2$. The connected components of $\Delta_{\mathfrak{R}}^n$ are naturally labeled by paths in $\Pi(\mathfrak{R})$ of length n : if γ is obtained by following a sequence of arrows $\gamma_1, \dots, \gamma_n$, then $\Delta_\gamma = \{x \in \Delta_{\mathfrak{R}}^0 : Q^{k-1}(x) \in \Delta_{\gamma_k}, 1 \leq k \leq n\}$. Notice that if γ starts at π and ends at π' then $\Delta_\gamma = (B_\gamma^* \cdot \mathcal{S}_{\pi'}^+) \times \{\pi\}$.

If γ is a path in $\Pi(\mathfrak{R})$ of length n ending at $\pi \in \mathfrak{R}$, let

$$Q^\gamma = Q^n : \Delta_\gamma \rightarrow \Delta_\pi.$$

Let $\Delta_{\mathfrak{R}}^\infty = \bigcap_{n \geq 0} \Delta_{\mathfrak{R}}^n$.

Definition 2.5. A path γ is said to be *complete* if all involution classes $\underline{\alpha} \in \underline{\mathcal{A}}$ are winners of some arrow composing γ .

The concatenation of k complete paths is said to be *k-complete*.

A path $\gamma \in \Pi(\mathfrak{R})$ is *positive* if B_γ is given, in the canonical basis of $\mathbb{R}_+^{\underline{\mathcal{A}}}$, by a matrix with all entries positive.

Lemma 2.6. A $(2d - 3)$ -complete path $\gamma \in \Pi(\mathfrak{R})$ is positive.

Proof. Let $\gamma = \gamma^1 \gamma^2 \dots \gamma^N$ where γ^i is an arrow starting at π^{i-1} and ending at π^i . Since γ is l -complete we also can represent it as $\gamma = \gamma_{(1)} \gamma_{(2)} \dots \gamma_{(l)}$ where each $\gamma_{(i)}$ is a complete path passing through $\pi_1^{(i-1)}, \dots, \pi_{n(i)}^{(i-1)}, \pi_1^{(i)}$.

Let $B_{\gamma_{(i)}}^*$ be the matrix such that $\lambda_1^{(i)} = B_{\gamma_{(i)}}^* \cdot \lambda_1^{(i+1)}$. And let $B^*(\underline{\alpha}, \underline{\beta}, i)$ be the coefficient on row $\underline{\alpha}$ and on column $\underline{\beta}$ of the matrix $B_{\gamma_{(i)}}^*$. Fix $k < l$. We denote $C^*(k) = B_{\gamma_{(1)}}^* \dots B_{\gamma_{(k)}}^*$. Let $C^*(\underline{\alpha}, \underline{\beta}, k)$ be the coefficient on row $\underline{\alpha}$ and on column $\underline{\beta}$ of the matrix $C^*(k)$. We want to prove that for all $\underline{\alpha}, \underline{\beta} \in \underline{\mathcal{A}}$ we have $C^*(\underline{\alpha}, \underline{\beta}, l) > 0$. For $r \geq 0$ denote $\widehat{C}(r) = B_{\gamma^1}^* \dots B_{\gamma^r}^*$.

Since the diagonal elements of the matrices $B_{\tilde{\gamma}}^*$, where $\tilde{\gamma}$ is an arrow, are one and all other terms are non-negative integers, we obtain that the sequence $C^*(\underline{\alpha}, \underline{\beta}, k)$ is non-decreasing in k , thus:

$$C^*(\underline{\alpha}, \underline{\beta}, k) > 0 \Rightarrow C^*(\underline{\alpha}, \underline{\beta}, k+1) > 0 \quad (11)$$

Fix any $\underline{\alpha}, \underline{\beta} \in \underline{\mathcal{A}}$. We will reorder the involution classes of $\underline{\mathcal{A}}$, as $\underline{\alpha} = \underline{\alpha}_1, \underline{\alpha}_2, \dots, \underline{\alpha}_d = \underline{\beta}$ with associate numbers $0 = r_1 < r_2 < \dots < r_d$ such that

$$C^*(\underline{\alpha}_1, \underline{\alpha}_j, r) > 0 \quad \forall r \geq r_j \quad (12)$$

If $\underline{\alpha} = \underline{\beta}$ we take $s = 1$ and $r_1 = 0$ and therefore we have (12). Otherwise we choose the smallest positive integer r_2 such that the winner of γ^{r_2} is $\underline{\alpha}_1 = \underline{\alpha}$ and let $\underline{\alpha}_2$ be the loser of the same arrow. Notice that $\underline{\alpha}_1 \neq \underline{\alpha}_2$ by irreducibility, and $B^*(\underline{\alpha}_1, \underline{\alpha}_2, r_2) = 1$, hence $C^*(\underline{\alpha}_1, \underline{\alpha}_2, r) > 0$ for every $r \geq r_2$. This gives the result for $d = 2$.

Now we will see the general case. Assume that $\underline{\alpha}_1, \dots, \underline{\alpha}_j$ and r_1, \dots, r_j have been constructed with $\underline{\beta} \neq \underline{\alpha}_m$ for $1 \leq m \leq j$. Let r'_j be the smallest integer greater than r_j such that the winner of $\gamma^{r'_j}$ does not belong to $\{\underline{\alpha}_1, \dots, \underline{\alpha}_j\}$ and let r_{j+1} be the smallest integer greater than r'_j such that the winner of $\gamma^{r_{j+1}}$ is in $\{\underline{\alpha}_1, \dots, \underline{\alpha}_j\}$. Let $\underline{\alpha}_{j+1}$ be the loser of $\gamma^{r_{j+1}}$. Then $\underline{\alpha}_{j+1}$ is the winner of $\gamma^{r_{j+1}-1}$ and therefore $\underline{\alpha}_{j+1} \notin \{\underline{\alpha}_1, \dots, \underline{\alpha}_j\}$. Thus, for some $1 \leq m \leq j$ we have $B^*(\underline{\alpha}_m, \underline{\alpha}_{j+1}, r_{j+1}) = 1$ and $C^*(\underline{\alpha}_1, \underline{\alpha}_m, r_{j+1} - 1) > 0$, since $r_{j+1} > r_m$. Thus

$$C^*(\underline{\alpha}_1, \underline{\alpha}_{j+1}, r) > 0 \quad \forall r \geq r_{j+1}.$$

Following this process, we will obtain $\underline{\alpha}_s = \underline{\beta}$. Now, we will see how many complete paths we need until define r_d .

We need a complete path to define each r_j and another one to define each r'_j , for $3 \leq j \leq d$. And we need another complete path to define r_2 . Thus we need at most $2(d-2) + 1 = 2d - 3$ complete paths composing γ to conclude it is positive. \square

2.3 Zippered rectangles

Let π be a permutation in a Rauzy class $\mathfrak{R} \subset \mathfrak{S}^0$. Let $\Theta_\pi \subset \mathcal{S}_\pi$ be the set of all τ such that

$$\begin{aligned} \sum_{\pi(*) < \pi(\xi) \leq k_r} \tau_{\underline{\xi}} &> 0 \quad \text{for all } \pi(*) < k_r < 2d+1 \\ \sum_{k_l \leq \pi(\xi) < \pi(*)} \tau_{\underline{\xi}} &< 0 \quad \text{for all } 1 < k_l < \pi(*) \end{aligned} \quad (13)$$

Observe that Θ_π is an open convex polyhedral cone and we will see later it is non-empty.

Given a letter $\alpha \in \mathcal{A}$, we define $M_\pi(\alpha) = \max\{\pi(\alpha), \pi(i(\alpha))\}$ and $m_\pi(\alpha) = \min\{\pi(\alpha), \pi(i(\alpha))\}$.

Define the linear operator $\Omega(\pi)$ on $\mathbb{R}^{\mathcal{A}}$ as follows:

$$\Omega(\pi)_{\underline{x}, \underline{y}} = \begin{cases} 2 & \text{if } M_\pi(x) < m_\pi(y), \\ -2 & \text{if } M_\pi(y) < m_\pi(x), \\ 1 & \text{if } m_\pi(x) < m_\pi(y) < M_\pi(x) < M_\pi(y), \\ -1 & \text{if } m_\pi(y) < m_\pi(x) < M_\pi(y) < M_\pi(x), \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

If $\lambda \in \mathbb{R}^{\mathcal{A}}$, then

$$(\Omega(\pi) \cdot \lambda)_\alpha = \sum_{\pi(\xi) > \pi(i(\alpha))} \lambda_\xi - \sum_{\pi(\xi) < \pi(\alpha)} \lambda_\xi. \quad (15)$$

Suppose that $\pi(\alpha) < \pi(i(\alpha))$. Thus,

$$\begin{aligned} (\Omega(\pi) \cdot \lambda)_{i(\alpha)} &= \sum_{\pi(\xi) > \pi(\alpha)} \lambda_\xi - \sum_{\pi(\xi) < \pi(i(\alpha))} \lambda_\xi = \\ &= \sum_{\pi(\alpha) < \pi(\xi) \leq \pi(i(\alpha))} \lambda_\xi + \sum_{\pi(\xi) > \pi(i(\alpha))} \lambda_\xi - \sum_{\pi(\alpha) \leq \pi(\xi) < \pi(i(\alpha))} \lambda_\xi - \sum_{\pi(\xi) < \pi(\alpha)} \lambda_\xi = \\ &= (\Omega(\pi) \cdot \lambda)_\alpha. \end{aligned} \quad (16)$$

The case $\pi(i(\alpha)) < \pi(\alpha)$ is analogous. So, we conclude that $(\Omega(\pi) \cdot \lambda)_\alpha = (\Omega(\pi) \cdot \lambda)_{i(\alpha)}$ for all $\alpha \in \mathcal{A}$. Therefore $\Omega(\pi)$ is well-defined.

We define the vector $w \in \mathbb{R}^{\mathcal{A}}$ by $w = \Omega(\pi) \cdot \lambda$ and the vector $h \in \mathbb{R}^{\mathcal{A}}$ by $h = -\Omega(\pi) \cdot \tau$. For each $\underline{\alpha} \in \underline{\mathcal{A}}$ define $\zeta_{\underline{\alpha}} = \lambda_{\underline{\alpha}} + i\tau_{\underline{\alpha}}$.

Lemma 2.7. *If γ is an arrow between (λ, π) and (λ', π') , then $w' = B_\gamma \cdot w$.*

Proof. We will consider the case when γ is a left arrow. The other case is entirely analogous. Let $\underline{\alpha}(l)$ and $\underline{\alpha}(r)$ be the leftmost and the rightmost letters in π , respectively. Thus $\underline{\alpha}(l)$ is the winner and $\underline{\alpha}(r)$ is the loser.

By definition,

$$w'_{\underline{\alpha}} = \sum_{\pi'(\xi) > \pi'(i(\alpha))} \lambda'_\xi - \sum_{\pi'(\xi) < \pi'(\alpha)} \lambda'_\xi$$

Since $\lambda'_{\underline{\alpha}} = \lambda_{\underline{\alpha}}$ for all $\underline{\alpha} \neq \underline{\alpha}(l)$ and $\lambda'_{\underline{\alpha}(l)} = \lambda_{\underline{\alpha}(l)} - \lambda_{\underline{\alpha}(r)}$, it is easy to verify that:

$$w'_{\underline{\alpha}} = \sum_{\pi(\xi) > \pi(i(\alpha))} \lambda_{\xi} - \sum_{\pi(\xi) < \pi(\alpha)} \lambda_{\xi} = w_{\underline{\alpha}} \quad \text{if } \underline{\alpha} \neq \underline{\alpha}(r).$$

And if $\underline{\alpha} = \underline{\alpha}(r)$, by (16), we have:

$$\begin{aligned} w'_{\underline{\alpha}(r)} &= \sum_{\pi'(\xi) > \pi'(i(\alpha(r)))} \lambda'_{\xi} - \sum_{\pi'(\xi) < \pi'(\alpha(r))} \lambda'_{\xi} \\ &= \sum_{\pi'(\xi) > \pi'(\alpha(r))} \lambda'_{\xi} - \sum_{\pi'(\xi) < \pi'(i(\alpha(r)))} \lambda'_{\xi} = \sum_{\pi(\xi) > \pi(i(\alpha(l)))} \lambda_{\xi} - \sum_{\pi(\xi) < \pi(i(\alpha(r)))} \lambda_{\xi} \\ &= \sum_{\pi(\xi) > \pi(i(\alpha(l)))} \lambda_{\xi} - \sum_{\pi(\xi) < \pi(\alpha(l))} \lambda_{\xi} + \sum_{\pi(\xi) > \pi(\alpha(r))} \lambda_{\xi} - \sum_{\pi(\xi) < \pi(i(\alpha(r)))} \lambda_{\xi} \\ &= w_{\underline{\alpha}(l)} + w_{\underline{\alpha}(r)} \end{aligned}$$

Therefore, $w' = B_{\gamma} \cdot w$. \square

Let $H(\pi) = \Omega(\pi) \cdot \mathcal{S}_{\pi}$. According to the previous lemma, given a path $\gamma \in \Pi(\mathfrak{R})$ starting at π and ending at π' , we have $B_{\gamma} \cdot H(\pi) = H(\pi')$.

Lemma 2.8. *If $\pi \in \mathfrak{R}$ and $\tau \in \Theta_{\pi}$ then $h \in \mathbb{R}_{\mp}^A$.*

Proof. Let $\alpha \in \mathcal{A}$. We have

$$h_{\underline{\alpha}} = \sum_{\pi(\xi) < \pi(\alpha)} \tau_{\xi} - \sum_{\pi(\xi) > \pi(i(\alpha))} \tau_{\xi}.$$

Suppose $\pi(\alpha), \pi(i(\alpha)) < \pi(*)$:

$$\begin{aligned} h_{\underline{\alpha}} &= \sum_{\pi(\xi) < \pi(\alpha)} \tau_{\xi} - \sum_{\pi(\xi) > \pi(i(\alpha))} \tau_{\xi} = \\ &= \sum_{\pi(\xi) < \pi(\alpha)} \tau_{\xi} - \sum_{\pi(i(\alpha)) < \pi(\xi) < \pi(*)} \tau_{\xi} - \sum_{\pi(*) < \pi(\xi) \leq 2d+1} \tau_{\xi} = \\ &= - \sum_{\pi(\alpha) \leq \pi(\xi) < \pi(*)} \tau_{\xi} - \sum_{\pi(i(\alpha)) < \pi(\xi) < \pi(*)} \tau_{\xi} > 0. \end{aligned}$$

Analogously, if $\pi(\alpha), \pi(i(\alpha)) > \pi(*)$ we have $h_{\underline{\alpha}} > 0$.

Now we will suppose that $\pi(\alpha) < \pi(*) < \pi(i(\alpha))$. In this case, we have:

$$\begin{aligned} h_{\underline{\alpha}} &= \sum_{\pi(\xi) < \pi(\alpha)} \tau_{\xi} - \sum_{\pi(\xi) > \pi(i(\alpha))} \tau_{\xi} = \\ &= \sum_{1 \leq \pi(\xi) < \pi(*)} \tau_{\xi} - \sum_{\pi(\alpha) \leq \pi(\xi) < \pi(*)} \tau_{\xi} - \sum_{\pi(*) < \pi(\xi) \leq 2d+1} \tau_{\xi} + \sum_{\pi(*) < \pi(\xi) \leq \pi(i(\alpha))} \tau_{\xi} = \\ &= - \sum_{\pi(\alpha) < \pi(\xi) < \pi(*)} \tau_{\xi} + \sum_{\pi(*) < \pi(\xi) < \pi(i(\alpha))} \tau_{\xi} > 0 \end{aligned}$$

So, $h \in \mathbb{R}_{\mp}^A$. \square

Lemma 2.9. *If $\gamma \in \Pi(\mathfrak{R})$ is an arrow starting at π and ending at π' then $(B_\gamma^*)^{-1} \cdot \Theta_\pi \subset \Theta_{\pi'}$.*

Proof. We will suppose that γ is a right arrow and the other case is entirely analogous. Let $\tau \in \Theta_\pi$ and let $\underline{\alpha} \in \underline{\mathcal{A}}$ be the winner and $\underline{\beta} \in \underline{\mathcal{A}}$ be the loser of γ .

Notice we have $\tau'_\xi = \tau_\xi$ for all $\xi \in \underline{\mathcal{A}} \setminus \{\underline{\alpha}\}$. Let $m = \pi(i(\alpha))$. Notice that

$$h_{\underline{\alpha}} = h_{i(\alpha)} = \sum_{\pi(\xi) < \pi(i(\alpha))} \tau_\xi - \sum_{\pi(\xi) > \pi(\alpha)} \tau_\xi = \sum_{\pi(\xi) < m} \tau_\xi \quad (17)$$

Suppose $m < \pi(*)$. Since $\pi'(\xi) = \pi(\xi)$ for all $\xi \in \mathcal{A}$ such that $\pi(\xi) \geq m$ we have that the first inequalities of (13) are satisfied and

$$\sum_{k_l \leq \pi'(\xi) < \pi'(*)} \tau'_\xi = \sum_{k_l \leq \pi(\xi) < \pi(*)} \tau_\xi < 0 \quad \text{for all } m < k_l < \pi(*)$$

Thus, it remains to prove the last inequalities to $1 < k_l \leq m$. Let $1 < k_l < m$. Since $\tau'_\alpha = \tau_\alpha - \tau_\beta$,

$$\sum_{k_l \leq \pi'(\xi) < \pi'(*)} \tau'_\xi = \sum_{k_l \leq \pi(\xi) < \pi(*)} \tau_\xi < 0 \quad \text{for all } 1 < k_l < m.$$

If $k_l = m$, by (17)

$$\sum_{m \leq \pi'(\xi) < \pi'(*)} \tau'_\xi = \sum_{m \leq \pi(\xi) < \pi(*)} \tau_\xi - \tau_\beta = \sum_{2 \leq \pi(\xi) < \pi(*)} \tau_\xi - h_{\underline{\alpha}} < 0 \quad (18)$$

Now suppose $m > \pi(*)$. This case is analogous to the first one. We will just do the part corresponding to (18).

$$\sum_{\pi'(*) < \pi'(\xi) \leq m-1} \tau'_\xi = \sum_{\pi(*) < \pi(\xi) \leq m-1} \tau_\xi - \tau_\beta = h_{\underline{\alpha}} - \sum_{2 \leq \pi(\xi) < \pi(*)} \tau_\xi > 0$$

Thus $\tau' \in \Theta_{\pi'}$, as we wanted to prove. \square

Definition 2.10. Let $\Theta'_\pi \subset \mathcal{S}_\pi$ be the set of all $\tau \neq 0$ such that

$$\begin{aligned} \sum_{\pi(*) < \pi(\xi) \leq k_r} \tau_\xi &\geq 0 \quad \text{for all } \pi(*) < k_r < 2d+1 \\ \sum_{k_l \leq \pi(\xi) < \pi(*)} \tau_\xi &\leq 0 \quad \text{for all } 1 < k_l < \pi(*) \end{aligned} \quad (19)$$

Let $\gamma \in \Pi(\mathfrak{R})$ be a path starting at π_s and ending at π_e . In the same way we showed that $(B_\gamma^*)^{-1} \cdot \Theta_{\pi_s} \subset \Theta_{\pi_e}$ in the previous lemma, one sees that $(B_\gamma^*)^{-1} \cdot \Theta'_{\pi_s} \subset \Theta'_{\pi_e}$.

Definition 2.11. Let $\pi \in \Pi(\mathfrak{R})$ and $\alpha \in \mathcal{A}$. We say that α is a *simple letter* if $\pi(\alpha) < \pi(*) < \pi(i(\alpha))$ or if $\pi(i(\alpha)) < \pi(*) < \pi(\alpha)$. We say that α is a *double letter* if $\pi(\alpha)$ and $\pi(i(\alpha))$ are either both smaller or either both greater than $\pi(*)$. If $\pi(\alpha), \pi(i(\alpha)) < \pi(*)$ we say that α is a *left double letter* or has *left type*, otherwise we say that α is a *right double letter* or has *right type*.

Lemma 2.12. *If π is irreducible then Θ'_π is non-empty.*

Proof. By invariance and irreducibility, it is enough to find some $\pi \in \mathfrak{R}$ such that Θ'_π is non-empty.

Given $\alpha, \beta \in \mathcal{A}$ suppose we have $\pi \in \mathfrak{R}$ with one of the two following forms:

$$\cdot \cdot \alpha \cdot \cdot i(\alpha) \cdot \beta \cdot i(\beta) \cdot * \cdot \cdot \quad (20)$$

or

$$\cdot \cdot \alpha \cdot \cdot \beta \cdot i(\alpha) \cdot i(\beta) \cdot * \cdot \cdot \quad (21)$$

We can define $\tau \in \Theta'_\pi$ by choosing $\tau_\alpha = -\tau_\beta = 1$ and $\tau_\xi = 0$ for all $\xi \in \mathcal{A} \setminus \{\alpha, \beta\}$.

Let us show that there exists some $\pi \in \mathfrak{R}$ satisfying this property.

By definition of permutation in \mathfrak{S} , there exist at least one double letter of each one of the types, i.e., there exist $\alpha, \beta \in \mathcal{A}$ such that α is left double letter and β is right double letter.

If there exists more than one double letter of both types, we can obtain another irreducible permutation π' which has at most one double letter of each one of the types, as follows. First we apply left or right operations until we obtain one double letter in the leftmost or the rightmost position, which is possible by irreducibility. We will assume, without loss of generality, that such a letter is at rightmost position. If there is at most one left double letter, we take the permutation obtained to be π' . But, if there are more than one left double letter, we apply right operations, until we find a permutation with just one left double letter. Those right operations are well-defined since we have more than one double letter of both types.

Suppose that $\alpha \in \mathcal{A}$ is the unique left double letter. Then, if it is necessary, we apply right operations until obtain $\pi(\alpha) = 1$.

Let $\beta \in \mathcal{A}$ such that $\pi(\beta) = 2d + 1$, i.e.,

$$\alpha \cdot \cdot \cdot i(\alpha) \cdot * \cdot \cdot \beta$$

If β is simple applying the left operation we obtain a permutation of type (20) or (21) depending on $\pi(i(\beta)) > \pi(i(\alpha))$ or $\pi(i(\beta)) < \pi(i(\alpha))$, respectively. If β is double we apply the left operation until we obtain a simple letter in the rightmost position of π and we are in the same conditions as in the previous case. \square

Definition 2.13. Let us say that a path $\gamma \in \Pi(\mathfrak{R})$, starting at π_s and ending at π_e , is *strongly positive* if it is positive and $(B_\gamma^*)^{-1} \cdot \Theta'_{\pi_s} \subset \Theta_{\pi_e}$.

Lemma 2.14. *Let γ be a $(4d - 6)$ -complete path. Then γ is strongly positive.*

Proof. Let $d = \#\mathcal{A}$. Fix $\tau \in \Theta'_{\pi_s} \setminus \{0\}$. Write γ as a concatenation of arrows $\gamma = \gamma_1 \dots \gamma_n$, and let π^{i-1} and π^i denote the start and the end of γ_i . Let $\tau^0 = \tau$, $\tau^i = (B_{\gamma_i}^*)^{-1} \cdot \tau^{i-1}$. We must show that $\tau^n \in \Theta_{\pi^n}$.

Let $h^i = -\Omega(\pi^i) \cdot \tau^i$. Notice that $\tau \in \Theta'_{\pi_0} \setminus \{0\}$ implies that $h^0 \in \overline{\mathbb{R}}_+^{\mathcal{A}} \setminus \{0\}$. Indeed, since $\tau \in \Theta'_{\pi_0}$, for every $\xi \in \mathcal{A}$, we have $\sum_{\pi^0(*) < \pi^0(\alpha) < \pi^0(\xi)} \tau_\alpha \geq 0$, $\sum_{\pi^0(\xi) < \pi^0(\alpha) < \pi^0(*)} \tau_\alpha \leq 0$. Moreover, since $\tau \neq 0$, there exist $1 \leq k^l \leq \pi^0(*)$ maximal and $\pi^0(*) \leq k^r \leq 2d + 1$ minimal such that $\tau_{(\pi^0)^{-1}(k^l)} \neq 0$ and

$\tau_{(\pi^0)^{-1}(k^r)} \neq 0$. Since π^0 is irreducible, $k^r - k^l < 2d - 1$. Remember that $h_{\xi}^0 \geq 0$ for all ξ and the inequality is strict if $\pi^0(\xi) = k^r + 1$ and $k^r < 2d + 1$ or if $\pi^0(\xi) = k^l - 1$ and $1 < k^l < \pi^0(*)$. Since $h^i = B_{\gamma_i} \cdot h^{i-1}$ we can consider a positive path $\gamma_1 \dots \gamma_i$ and then $h^i \in \mathbb{R}_+^A$.

Let $\pi^i(*) \leq k_i^r \leq 2d$ be maximal and $2 \leq k_i^l \leq \pi^i(*)$ be minimal such that

$$\sum_{\pi^i(*) < \pi^i(\xi) \leq k} \tau_{\xi}^i > 0 \quad \text{for all } \pi^i(*) < k \leq k_i^r,$$

$$\sum_{k \leq \pi^i(\xi) < \pi^i(*)} \tau_{\xi}^i < 0 \quad \text{for all } k_i^l \leq k < \pi^i(*) .$$

We claim that

1. If $h^{i-1} \in \mathbb{R}_+^A$ then $k_i^r - \pi^i(*) \geq k_{i-1}^r - \pi^{i-1}(*)$ and $\pi^i(*) - k_i^l \geq \pi^{i-1}(*) - k_{i-1}^l$, in particular $k_i^r - k_i^l \geq k_{i-1}^r - k_{i-1}^l$;
2. If $h^{i-1} \in \mathbb{R}_+^A$ and the winner of γ_i is one of the first $k_{i-1}^r + 1$ letters after $*$ in π^{i-1} then $k_i^r - k_i^l \geq \min\{k_{i-1}^r - k_{i-1}^l + 1, 2d - k_{i-1}^l\}$;
3. If $h^{i-1} \in \mathbb{R}_+^A$ and the winner of γ_i is one of the last $k_{i-1}^l - 1$ letters before $*$ in π^{i-1} then $k_i^r - k_i^l \geq \min\{k_{i-1}^r - k_{i-1}^l + 1, k_{i-1}^r - 2\}$.

Notice that $2 < \pi^i(*) < 2d$ for all i . Let us see that (1), (2) and (3) imply the result, which is equivalent to the statement that $k_n^r - k_n^l \geq 2d - 2$. Let us write $\gamma = \gamma_{(1)} \dots \gamma_{(4d-7)}$ where $\gamma_{(j)}$ is complete and each $\gamma_{(j)} = \gamma_{s_j} \dots \gamma_{e_j}$. By Lemma 2.6, $h^k \in \mathbb{R}_+^A$ for $k \geq e_{2d-3}$. From the definition of a complete path, for each $j > 2d - 3$, there exists $e_j < i_1 \leq e_{j+1}$ such that the winner of γ_{i_1} is one of the letters in position m_1 at π^{i_1-1} such that $\pi^{i_1-1}(*) < m_1 < k_{e_j}^r + 1$. It follows that $k_{e_{j+1}}^r - k_{e_{j+1}}^l \geq \min\{k_{i_1-1}^r - k_{i_1-1}^l + 1, 2d - k_{i_1-1}^l\}$, so

$$k_{e_{j+1}}^r - k_{e_{j+1}}^l \geq \min\{k_{e_j}^r - k_{e_j}^l + 1, 2d - k_{e_j}^l\}. \quad (22)$$

In the same way there exists $e_{j-1} < i_2 \leq e_j$ such that the winner of γ_{i_2} is one of the letters in position m_2 at π^{i_2-1} such that $k_{e_j}^r + 1 < m_2 < \pi^{i_2-1}(*)$. It follows that $k_{e_j}^r - k_{e_j}^l \geq \min\{k_{i_2-1}^r - k_{i_2-1}^l + 1, k_{i_2-1}^r - 2\}$, thus

$$k_{e_j}^r - k_{e_j}^l \geq \min\{k_{e_{j-1}}^r - k_{e_{j-1}}^l + 1, k_{e_{j-1}}^r - 2\}. \quad (23)$$

By (22) and (23), we see that:

$$k_{e_{j+1}}^r - k_{e_{j+1}}^l \geq \min\{k_{e_{j-1}}^r - k_{e_{j-1}}^l + 2, 2d - (k_{e_{j-1}}^l - 1), (k_{e_{j-1}}^r + 1) - 2, 2d - 2\}.$$

Therefore, we obtain $k_n^r - k_n^l = k_{e_{2d-3+2d-4}}^r - k_{e_{2d-3+2d-4}}^l \geq \min\{k_{e_{2d-3}}^r - k_{e_{2d-3}}^l + 2d - 2, 2d - (k_{e_{2d-3}}^l - 2d - 4), (k_{e_{j-1}}^r + 2d - 4) - 2, 2d - 2\} = 2d - 2$.

We now check (1), (2) and (3). Assume that $h^{i-1} \in \mathbb{R}_+^A$, and that γ_i is a right arrow, the other case being analogous. Let α be the rightmost letter of π^{i-1} which is the winner of γ_i , and let β be the leftmost letter of π^{i-1} which is the loser of γ_i .

Case 1: Suppose $\pi^{i-1}(i(\alpha)) < \pi^{i-1}(*)$.

If the winner of γ_i is not one of the $k_{i-1}^l - 1$ last letters in the left side of $*$ in π^{i-1} , then for every $\xi \in \mathcal{A}$ such that $k_{i-1}^l \leq \pi^{i-1}(\xi) \leq 2d + 1$, we have $\pi^{i-1}(\xi) = \pi^i(\xi)$ and $\tau_{\xi}^{i-1} = \tau_{\xi}^i$ for all $k_{i-1}^l \leq \pi^{i-1}(\xi) \leq 2d$. Hence $k_i^r - \pi^i(*) \geq k_{i-1}^r - \pi^{i-1}(*)$ and $\pi^i(*) - k_i^l \geq \pi^{i-1}(*) - k_{i-1}^l$.

If the winner $\underline{\alpha}$ of γ_i appears in the k -th position counting from $*$ to the left in π^{i-1} with $k_{i-1}^l - 1 \leq k < \pi^i(*)$, then

$$\begin{aligned} \sum_{j \leq \pi^i(\xi) < \pi^i(*)} \tau_{\xi}^i &= \sum_{j \leq \pi^{i-1}(\xi) < \pi^{i-1}(*)} \tau_{\xi}^{i-1} < 0 \quad \text{for all } k+1 \leq j < \pi^i(*), \\ \sum_{j \leq \pi^i(\xi) < \pi^i(*)} \tau_{\xi}^i &= \sum_{j+1 \leq \pi^{i-1}(\xi) < \pi^{i-1}(*)} \tau_{\xi}^{i-1} < 0 \quad \text{for all } k_{i-1}^l - 1 \leq j \leq k-1, \\ \sum_{k \leq \pi^i(\xi) < \pi^i(*)} \tau_{\xi}^i &= \sum_{2 \leq \pi^{i-1}(\xi) < \pi^{i-1}(*)} \tau_{\xi}^{i-1} - h_{\underline{\alpha}}^{i-1} \leq -h_{\underline{\alpha}}^{i-1} < 0, \end{aligned}$$

which implies that $\pi^i(*) - k_i^l \geq \min\{\pi^{i-1}(*) - 2, \pi^{i-1}(*) + 1 - k_{i-1}^l\}$, hence $k_i^r - k_i^l \geq \min\{k_{i-1}^r - k_{i-1}^l + 1, k_{i-1}^r - 2\}$.

This shows that (1) holds and (3) holds. Moreover, (2) also holds since its hypothesis can only be satisfied if $k_{i-1}^r = 2d$.

Case 2: Suppose $\pi^{i-1}(i(\alpha)) > \pi^{i-1}(*)$.

If the winner of γ_i is not one of the $k_{i-1}^r + 1$ first letters in the right side of $*$ in π^{i-1} , then for every $\xi \in \mathcal{A}$ such that $1 < \pi^{i-1}(\xi) \leq k_{i-1}^r$, we have $\pi^i(\xi) = \pi^{i-1}(\xi) - 1$ and $\tau_{\xi}^{i-1} = \tau_{\xi}^i$, so $k_i^r - \pi^i(*) \geq k_{i-1}^r - \pi^{i-1}(*)$ and $\pi^i(*) - k_i^l \geq \pi^{i-1}(*) - k_{i-1}^l$.

If the winner $\underline{\alpha}$ of γ_i appears in the k -th in π^{i-1} with $\pi^{i-1}(*) < k \leq k_{i-1}^r + 1$, then

$$\begin{aligned} \sum_{\pi^i(*) < \pi^i(\xi) \leq j} \tau_{\xi}^i &= \sum_{\pi^{i-1}(*) < \pi^{i-1}(\xi) \leq j-1} \tau_{\xi}^{i-1} > 0 \quad \text{for all } \pi^i(*) \leq j < k-1, \\ \sum_{\pi^i(*) < \pi^i(\xi) \leq j} \tau_{\xi}^i &= \sum_{\pi^{i-1}(*) < \pi^{i-1}(\xi) \leq j} \tau_{\xi}^{i-1} > 0 \quad \text{for all } k \leq j \leq k_{i-1}^r + 1, \\ \sum_{\pi^i(*) < \pi^i(\xi) \leq k-1} \tau_{\xi}^i &= - \sum_{2 \leq \pi^{i-1}(\xi) < \pi^{i-1}(*)} \tau_{\xi}^{i-1} + h_{\underline{\alpha}}^{i-1} \geq h_{\underline{\alpha}}^{i-1} > 0, \end{aligned}$$

which implies that $k_i^r - \pi^i(*) \geq \min\{k_{i-1}^r + 1 - \pi^{i-1}(*), 2d - \pi^{i-1}(*)\}$, hence $k_i^r - k_i^l \geq \min\{k_{i-1}^r - k_{i-1}^l + 1, 2d - k_{i-1}^l\}$.

This shows that both (1) and (2) holds. Moreover, (3) also holds since its hypothesis can only be satisfied if $k_{i-1}^l = 2$. \square

Corollary 2.15. *If π is irreducible then Θ_{π} is non-empty.*

Proof. Let $\gamma \in \Pi(\mathfrak{A})$ be a strongly positive path starting and ending at π , which exists by Lemma 2.14. Then $(B_{\gamma}^*)^{-1} \cdot \Theta'_{\pi} \subset \Theta_{\pi}$ and by Lemma 2.12 the set Θ'_{π} is non-empty. Therefore Θ_{π} is non-empty. \square

Given that Θ_{π} is non-empty, it is easy to see that $\Theta'_{\pi} \cup \{0\}$ is in fact just the closure of Θ_{π} .

2.3.1 Extension of induction to the space of zippered rectangles

Let $\gamma \in \Pi(\mathfrak{R})$ be a path starting at π and define Θ_γ satisfying:

$$B_\gamma^* \cdot \Theta_\gamma = \Theta_\pi.$$

If γ is a right arrow ending at π' , then $\Theta_\gamma = \{\tau \in \Theta_{\pi'} \mid \sum_{x \in \mathcal{A}} \tau_x < 0\}$, and if γ is a left arrow ending at π' , then $\Theta_\gamma = \{\tau \in \Theta_{\pi'} \mid \sum_{x \in \mathcal{A}} \tau_x > 0\}$.

Thus, the map

$$\widehat{Q}^\gamma : \Delta_\gamma \times \Theta_\pi \rightarrow \Delta_{\pi'} \times \Theta_{\pi'}, \quad \widehat{Q}^\gamma(\lambda, \pi, \tau) = (Q(\lambda, \pi), (B_\gamma^*)^{-1} \cdot \tau)$$

is invertible. With this we define an invertible skew-product \widehat{Q} over Q considering all \widehat{Q}^γ for every arrow γ . So, we obtain a map from $\bigcup(\Delta_\gamma \times \Theta_\pi)$ (where the union is taken over all $\pi \in \mathfrak{R}$ and all arrows γ starting at π) to $\bigcup(\Delta_{\pi'} \times \Theta_{\pi'})$ (where the union is taken over all $\pi' \in \mathfrak{R}$ and all arrows ending at π'). Denote $\widehat{\Delta}_\mathfrak{R} = \bigcup_{\pi \in \mathfrak{R}} (\Delta_\pi \times \Theta_\pi)$.

Let $(e_\alpha)_{\alpha \in \mathcal{A}}$ be the canonical basis of $\mathbb{R}^{\mathcal{A}}$. We will consider a measure in $\widehat{\Delta}_\mathfrak{R}$ defined as follows.. Let $\{v_1, \dots, v_d\}$ be a basis of \mathbb{R}^d . We have a volume form given by $\omega(v_1, \dots, v_d) = \det(v_1, \dots, v_d)$. We want to define a volume form in $\widehat{\Delta}_\mathfrak{R}$ coherent with ω . Given the subspace \mathcal{S}_π define the orthogonal vector $v_\pi = \sum_{\pi(x) < \pi(*)} e_x - \sum_{\pi(x) > \pi(*)} e_x$. We can view this vector like a linear functional $\psi_\pi : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by $\psi_\pi(x) = \langle v_\pi, x \rangle$. Now we define a form ω^{v_π} on \mathcal{S}_π such that $\omega = \omega^{v_\pi} \wedge \psi_\pi$ (where \wedge denotes the exterior product). We have that ω^{v_π} is a $(d-1)$ -form and it is well defined in the orthogonal complement of ψ_π (i.e., $\psi_\pi(\mathbb{R}^d)^\perp = \ker(\psi_\pi) = \mathcal{S}_\pi$). Notice that, given $\pi, \pi' \in \mathfrak{R}$ and a path $\gamma \in \Pi(\mathfrak{R})$ joining π to π' , $B_\gamma \cdot v_\pi = v_{\pi'}$ and

$$(B_\gamma^*)^{-1} \cdot \left(\frac{v_\pi}{\langle v_\pi, v_\pi \rangle} \right) - \frac{v_{\pi'}}{\langle v_{\pi'}, v_{\pi'} \rangle} \in \mathcal{S}_{\pi'}.$$

So we have:

$$\begin{aligned} & [((B_\gamma^*)^{-1})^* \omega^{v_{\pi'}}](u_1, \dots, u_{d-1}) = \omega^{v_{\pi'}}((B_\gamma^*)^{-1} \cdot u_1, \dots, (B_\gamma^*)^{-1} \cdot u_{d-1}) \\ &= \frac{\omega(v_{\pi'}, (B_\gamma^*)^{-1} \cdot u_1, \dots, (B_\gamma^*)^{-1} \cdot u_{d-1})}{\psi_{\pi'}(v_{\pi'})} \\ &= \omega \left(\frac{v_{\pi'}}{\langle v_{\pi'}, v_{\pi'} \rangle}, (B_\gamma^*)^{-1} \cdot u_1, \dots, (B_\gamma^*)^{-1} \cdot u_{d-1} \right) \\ &= \omega \left(\frac{v_\pi}{\langle v_\pi, v_\pi \rangle}, u_1, \dots, u_{d-1} \right) = \frac{\omega(v_\pi, u_1, \dots, u_{d-1})}{\psi_\pi(v_\pi)} \\ &= \omega^{v_\pi}(u_1, \dots, u_{d-1}). \end{aligned}$$

So, the pull-back of $\omega^{v_{\pi'}}$ is equal to ω^{v_π} , i.e.,

$$[(B_\gamma^*)^{-1}]^* \omega^{v_{\pi'}} = \omega^{v_\pi}.$$

Consider the volume form ω^{v_π} and the corresponding Lebesgue measure Leb_π on \mathcal{S}_π . So we have a natural volume measure $\widehat{m}_\mathfrak{R}$ on $\widehat{\Delta}_\mathfrak{R}$ which is a product of a counting measure on \mathfrak{R} and the restrictions of Leb_π on \mathcal{S}_π^+ and Θ_π .

Let $\phi(\lambda, \pi, \tau) = \|\lambda\| = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha$. The subset $\mathcal{U}_\mathfrak{R} \subset \widehat{\Delta}_\mathfrak{R}$ of all x such that either

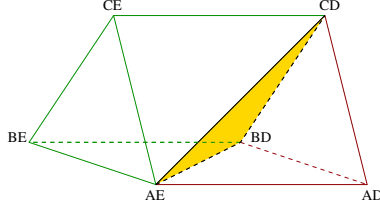


Figure 3: Polyhedrons with different number of vertices.

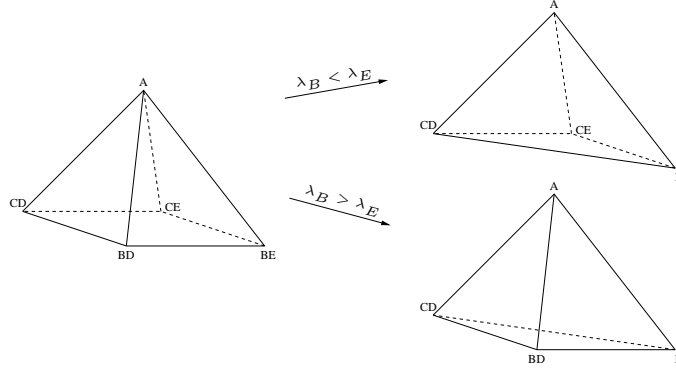


Figure 4: Simplex and subsimplices

A set of non-trivial vertices of $\Lambda_{\pi,q}$ is contained in the set of vertices of some elementary subsimplex if and only if the vertices are linearly independent. If α is simple then any elementary subsimplex must have a vertex of type $\underline{\alpha}$ and if α is double then any elementary subsimplex must have a vertex of type $\{\underline{\alpha}, \underline{x}\}$ for some $x \neq \alpha$ with x double. If Λ is an elementary subsimplex with simple vertices of type $\underline{\alpha}_i$ and double vertices of type $\{\underline{\beta}_{j_1}, \underline{\xi}_{j_2}\}$ then $\nu_{\pi,q}(\Lambda) = k(\pi, \Lambda) \prod q_{\alpha_i}^{-1} \prod (q_{\beta_{j_1}} + q_{\xi_{j_2}})^{-1}$ where $k(\pi, \Lambda)$ is a positive integer only depending on v_π and on the types of the double vertices of Λ . In particular there is an integer $C_2(d)$ such that $k(\pi, \Lambda) \leq C_2(d)$.

Let $\gamma \in \Pi(\mathfrak{R})$ be an arrow starting at π and ending at π' . If $\Gamma \subset \Pi(\mathfrak{R})$ then we define

$$P_q(\Gamma | \gamma) = \frac{\nu_{\pi', B_{\gamma \cdot q}}(\Lambda_{\Gamma, q})}{\nu_{\pi, q}(\Lambda_{\gamma, q})}$$

and

$$P_q(\gamma | \pi) = \frac{\nu_{\pi', B_{\gamma \cdot q}}(\Lambda_{\gamma, q})}{\nu_{\pi, q}(\Lambda_{\pi, q})}.$$

We have that $P_q(\Gamma | \gamma) = P_{B_{\gamma \cdot q}}(\Gamma_\gamma | \pi')$.

We define a partial order in the set of paths as follows. Let $\gamma, \gamma_s \in \Pi(\mathfrak{R})$ be two paths. We say that $\gamma_s \leq \gamma$ if and only if γ_s is the start of γ . A family $\Gamma_s \subset \Pi(\mathfrak{R})$ is called *disjoint* if no two elements are comparable by the partial order defined before. If Γ_s is disjoint and $\Gamma \subset \Pi(\mathfrak{R})$ is a family such that any $\gamma \in \Gamma$ starts by some element $\gamma_s \in \Gamma_s$, then for every $\pi \in \mathfrak{R}$

$$P_q(\Gamma | \pi) = \sum_{\gamma_s \in \Gamma_s} P_q(\Gamma | \gamma_s) P_q(\gamma_s | \pi) \leq P_q(\Gamma_s | \pi) \sup_{\gamma_s \in \Gamma_s} P_q(\Gamma | \gamma_s).$$

Lemma 3.4. *There exists $C_3(d) < 1$ with the following property. Let $q \in \mathbb{R}_+^A$, $\gamma \in \Pi(\mathfrak{R})$ be an arrow starting at π with loser β . If $C \geq 1$ is such that $q_{\underline{\beta}} > C^{-1}M(q)$ then*

$$P_q(\gamma | \pi) > C_3(d)C^{-(d-1)}.$$

Proof. Let α be the winner of γ and let π' be the end of γ . Let Λ be an elementary subsimplex of $\Lambda_{\pi, q}$. We are going to show that there exists an elementary subsimplex $\Lambda' \subset B_\gamma(\Lambda)$ of $\Lambda_{\pi', B_\gamma \cdot q}$ such that $\text{Leb}_{\pi'}(\Lambda') \geq C'^{-1} \text{Leb}_\pi(\Lambda)$, which implies the result by decomposition.

We will separate the proof in four cases depending on whether the winner and the loser are simple or double.

Suppose that α and β are simple. Let Λ be an elementary subsimplex of $\Lambda_{\pi, q}$. Then $\Lambda' = B_\gamma \cdot (\Lambda_{\pi, q} \cap \Lambda)$ is an elementary subsimplex of $\Lambda_{\pi', B_\gamma \cdot q}$. The set of vertices of Λ' differs from the set of vertices of Λ just by replacing the vertex $q_{\underline{\beta}}^{-1} e_{\underline{\beta}}$ by $(q_{\underline{\alpha}} + q_{\underline{\beta}})^{-1} e_{\underline{\beta}}$. It follows that $\text{Leb}_{\pi'}(\Lambda') / \text{Leb}_\pi(\Lambda) = q_{\underline{\beta}} / (q_{\underline{\alpha}} + q_{\underline{\beta}}) > 1/(C+1)$. By considering a decomposition into elementary subsimplices we conclude that $P_q(\gamma | \pi) = q_{\underline{\beta}} / (q_{\underline{\alpha}} + q_{\underline{\beta}}) > 1/(C+1)$.

Suppose that the winner is simple and the loser is double. Let Λ be an elementary subsimplex of $\Lambda_{\pi, q}$. Then $\Lambda' = B_\gamma \cdot (\Lambda_{\pi, q} \cap \Lambda)$ is an elementary subsimplex of $\Lambda_{B_\gamma \cdot q}$. The set of vertices of Λ' differs from the set of vertices of Λ just by replacing the vertices $(q_{\underline{x}} + q_{\underline{\beta}})^{-1} (e_{\underline{x}} + e_{\underline{\beta}})$ by $(q_{\underline{\alpha}} + q_{\underline{x}} + q_{\underline{\beta}})^{-1} (e_{\underline{x}} + e_{\underline{\beta}})$. It follows that $\text{Leb}_{\pi'}(\Lambda') / \text{Leb}_\pi(\Lambda) = \prod (q_{\underline{x}} + q_{\underline{\beta}}) / (q_{\underline{\alpha}} + q_{\underline{x}} + q_{\underline{\beta}}) > 1/(1+C)$, where the product is over all \underline{x} such that Λ has a vertex of type $\{\underline{x}, \underline{\beta}\}$. Thus $P_q(\gamma | \pi) > 1/(C+1)$.

If the winner is double and the loser is simple, let γ' be the other arrow starting at π . Analogous to the previous case, $P_q(\gamma' | \pi) = \prod (q_{\underline{x}} + q_{\underline{\alpha}}) / (q_{\underline{\alpha}} + q_{\underline{x}} + q_{\underline{\beta}}) < \prod (1 - q_{\underline{\beta}} / (q_{\underline{\alpha}} + q_{\underline{x}} + q_{\underline{\beta}})) < 2C / (1 + 2C)$, so $P_q(\gamma | \pi) > 1/(1 + 2C)$.

Finally, suppose that the winner and the loser are both double. Let Λ be an elementary subsimplex with $\text{Leb}_\pi(\Lambda) \geq \text{Leb}_\pi(\Lambda_{\pi, q}) / C_1(d)$. Let Z be the set of vertices of Λ and let $\tilde{Z} \subset Z$ be the set of double vertices of type $\{q_{\underline{x}}, q_{\underline{\beta}}\}$ with $\underline{x} \neq \underline{\alpha}$. Notice that $B_\gamma \cdot (Z \setminus \tilde{Z})$ is a subset of the set of vertices of $\Lambda_{\pi', B_\gamma \cdot q}$. Since $Z \setminus \tilde{Z}$ is linearly independent, $B_\gamma \cdot (Z \setminus \tilde{Z})$ is also. Thus there exists an elementary subsimplex Λ' of $\Lambda_{\pi', B_\gamma \cdot q}$ whose set Z' of vertices contains $B_\gamma \cdot (Z \setminus \tilde{Z})$. Let $\tilde{Z}' = Z' \setminus B_\gamma \cdot (Z \setminus \tilde{Z})$. The weight of a vertex $v \in Z \setminus \tilde{Z}$ is the same weight as the weight of $B_\gamma \cdot v$. Notice that each vertex of \tilde{Z} has weight at least $C^{-1}M(q)$ and each vertex of \tilde{Z}' has weight at most $2M(B_\gamma \cdot q) \leq 4M(q)$. Thus

$$\frac{\text{Leb}_{\pi'}(\Lambda')}{\text{Leb}_\pi(\Lambda)} = \frac{k(\pi', \Lambda') \prod_{v \in \tilde{Z}} w(v)}{k(\pi, \Lambda) \prod_{v \in \tilde{Z}'} w(v)} > C_2(d)^{-1} (4C)^{1-d}.$$

Thus $P_q(\gamma | \pi) > C_1(d)^{-1} C_2(d)^{-1} (4C)^{1-d}$. \square

The proof of the recurrence estimates is based on the analysis of the Rauzy renormalization map. The key step involves a control on the measure of sets which present big distortion after some long (Teichmüller) time.

Theorem 3.5. *There exists $C_4(d) > 1$ with the following property. Let $q \in \mathbb{R}_+^A$. Then for every $\pi \in \mathfrak{A}$,*

$$P_q(\gamma \in \Pi(\mathfrak{A}), M(B_\gamma \cdot q) > C_4(d)M(q) \text{ and } m(B_\gamma \cdot q) < M(q) | \pi) < 1 - C_4(d)^{-(d-1)}.$$

Proof. For $1 \leq k \leq d$, let $m_k(q) = \max m_{\mathcal{A}'}(q)$ where the maximum is taken over all involution invariant sets $\mathcal{A}' \subset \mathcal{A}$ such that $\#\mathcal{A}' = 2k$. In particular $m = m_d$. We will show that for $1 \leq k \leq d$ there exists $D > 1$ such that

$$P_q(\gamma \in \Pi(\mathfrak{A}), M(B_\gamma \cdot q) > DM(q) \text{ and } m_k(B_\gamma \cdot q) < M(q) | \pi) < 1 - D^{-(d-1)}. \quad (24)$$

(the case $k = d$ implies the desired statement). The proof is by induction on k . For $k = 1$ it is obvious, by lemma 3.4. Assume that it is proved for some $1 \leq k < d$ with $D = D_0$. Let Γ be the set of minimal paths γ starting at π with $M(B_\gamma \cdot q) > D_0M(q)$. Then there exists $\Gamma_1 \subset \Gamma$ with $P_q(\Gamma_1 | \pi) > D_1^{-(d-1)}$ and an involution invariant set $\mathcal{A}' \subset \mathcal{A}$ with $\#\mathcal{A}' = 2k$ such that if $\gamma \in \Gamma_1$ then $m_{\mathcal{A}'}(B_\gamma \cdot q) \geq M(q)$.

For $\gamma_s \in \Gamma_1$, choose a path $\gamma = \gamma_s \gamma_e$ with minimal length such that γ ends at a permutation π_e such that either the first or the last element of π_e is an element of $\mathcal{A} \setminus \mathcal{A}'$. Let Γ_2 be the collection of the $\gamma = \gamma_s \gamma_e$ thus obtained. Then $P_q(\Gamma_2 | \pi) > D_2^{-(d-1)}$ and $M(B_\gamma \cdot q) < D_2M(q)$ for $\gamma \in \Gamma_2$.

Let Γ_3 be the space of minimal paths $\gamma = \gamma_s \gamma_e$ with $\gamma_s \in \Gamma_2$ and $M(B_\gamma \cdot q) > 2dD_2M(q)$. Let $\Gamma_4 \subset \Gamma_3$ be the set of all $\gamma = \gamma_s \gamma_e$ where all the arrows of γ_e have as loser an element of \mathcal{A}' . For each $\gamma_s \in \Gamma_2$, there exists at most one $\gamma = \gamma_s \gamma_e \in \Gamma_4$, and if $P_q(\Gamma_4 | \gamma_s) < \frac{1}{2d}$, it follows that $P_q(\Gamma_3 \setminus \Gamma_4 | \pi) > (1 - \frac{1}{2d}) D_2^{-(d-1)}$. It remains to prove that $P_q(\Gamma_4 | \gamma_s) < \frac{1}{2d}$.

Let $\gamma = \gamma_s \gamma_e \in \Gamma_4$ such that $\gamma_s \in \Gamma_2$. Let $\pi_e \in \Pi(\mathfrak{A})$ be the end of γ_s and let α and β be the winner and the loser of π_e , respectively. By definition, we have that $\alpha \in \mathcal{A} \setminus \mathcal{A}'$ and $\beta \in \mathcal{A}'$. Besides, all losers of γ_e are in \mathcal{A}' .

We claim that α is simple. Suppose this is not the case. Assume, without lost of generality, $\pi_e(\alpha) = 1$ and $\pi_e(\beta) = 2d + 1$. Applying RAI one time we would obtain $\pi_e'(\beta) < \pi_e'(*)$ and to keep the same winner α we just can apply RAI at most $2d - 4$ times. But even if we could apply RAI those number of times, we will have

$$M(B_\gamma \cdot q) < (2d - 3)D_2M(q) < 2dD_2M(q)$$

what contradicts that $\gamma \in \Gamma_3$. Then α is simple as we claim.

Suppose $\gamma = \gamma_1 \dots \gamma_n$ where $\gamma_i \in \Pi(\mathfrak{A})$ are arrows joining $\pi_e^{(i-1)}$ and $\pi_e^{(i)}$. We have $\gamma_s \in \Gamma_2$, so, to obtain $M(B_\gamma \cdot q) > 2dD_2M(q)$ we need $n \geq 2d + 1$. We also have $\mathcal{S}_{\pi_e} = \mathcal{S}_{\pi_e^{(i)}}$ for all $i \in \{1, \dots, n\}$.

Let $\Lambda = \Lambda_{\pi_e, B_{\gamma_s} \cdot q}$ which is a finite union of elementary simplices Λ_j . Let $\beta_0 = \beta$ and β_i be the loser of $\pi_e^{(i)}$ for $i = 0, \dots, n$. For each $i \in \{0, \dots, n\}$ if β_i is a simple vertex then all Λ_j has a vertex of type $\underline{\beta}_i$ and if β_i is double then all Λ_j has a vertex of type $\{\underline{\beta}_i, \underline{x}\}$ for some $\underline{x} \notin \{\underline{\alpha}, \underline{\beta}_i\}$. Let $\Lambda_j^{(n)} = B_{\gamma_e}(\Lambda_j)$. Notice that the type of vertices of $\Lambda_j^{(n)}$ coincides with type of vertices of Λ_j . Let Z be the set of vertices of Λ_j and $Z^{(n)}$ be the set of vertices of $\Lambda_j^{(n)}$. Then

$$\frac{\text{Leb}_{\pi_e^{(n)}}(\Lambda_j^{(n)})}{\text{Leb}_{\pi_e}(\Lambda_j)} = \frac{\prod_{v \in Z} w(v)}{\prod_{v \in Z^{(n)}} w(v)} \quad (25)$$

But $M(B_{\gamma_s} \cdot q) < D_2M(q)$ and $M(B_\gamma \cdot q) > 2dD_2M(q)$, so there is one term in (25) which is less than $\frac{1}{2d}$. Thus

$$P_q(\Gamma_4 | \gamma_s) < \frac{1}{2d}$$

Let $\gamma = \gamma_s \gamma_e \in \Gamma_3 \setminus \Gamma_4$. Let us show that $m_{k+1}(B_\gamma \cdot q) > M(q)$, which implies (24) with $k+1$ in place of k and $D = 2dD_2$. Assume that this is not the case. In this case, the last arrow composing γ_e must have as loser an element of \mathcal{A}' . Moreover, no arrow composing γ_e has as winner an element of \mathcal{A}' (otherwise, the loser β of the first such arrow does not belong to \mathcal{A}' and is such that $m_{\mathcal{A}' \cup \{\beta\}}(B_\gamma \cdot q) > M(q)$). Let $\gamma_e = \gamma_{e,s} \gamma_{e,e}$ where $\gamma_{e,s}$ is maximal such that all losers of $\gamma_{e,s}$ are in \mathcal{A}' . Then all losers in $\gamma_{e,s}$ are distinct and $M(B_{\gamma_s \gamma_{e,s}} \cdot q) < 2D_2M(q)$. Let $\gamma_{e,e} = \gamma_{e,1} \dots \gamma_{e,l}$ where $\gamma_{e,j} = \gamma_{e,j,s} \gamma_{e,j,e}$ with $\gamma_{e,j,s}$ and $\gamma_{e,j,e}$ non-trivial such that all the losers of $\gamma_{e,j,s}$ are in $\mathcal{A} \setminus \mathcal{A}'$ and all the losers of $\gamma_{e,j,e}$ are in \mathcal{A}' . Let $\gamma_j = \gamma_s \gamma_{e,s} \gamma_{e,1} \dots \gamma_{e,j}$, $0 \leq j \leq l$. Notice that for each j , $\gamma_{e,j,e}$ has distinct losers, and the same winner $\alpha \in \mathcal{A} \setminus \mathcal{A}'$ which is also the last winner of $\gamma_{e,j,s}$. Let $\beta \in \mathcal{A} \setminus \mathcal{A}'$ be the last loser of $\gamma_{e,j,s}$. Then

$$M(B_{\gamma_{j+1}} \cdot q) - M(B_{\gamma_j} \cdot q) \leq M_{\underline{\beta}}(B_{\gamma_{j+1}} \cdot q) - M_{\underline{\beta}}(B_{\gamma_j} \cdot q)$$

which implies that

$$\begin{aligned} (2d-1)D_2M(q) &\leq M(B_\gamma \cdot q) - M(B_{\gamma_s} \cdot q) \\ &\leq \sum_{\beta \in \mathcal{A} \setminus \mathcal{A}'} M_\beta(B_\gamma \cdot q) - M_\beta(B_{\gamma_s} \cdot q) \leq dD_2M(q) \end{aligned}$$

which is a contradiction. \square

4 Recurrence estimates

Lemma 4.1. *For every $\hat{\gamma} \in \Pi(\mathfrak{R})$, there exist $M \geq 0$, $\rho < 1$ such that for every $\pi \in \mathfrak{R}$, $q \in \mathbb{R}_+^A$,*

$$P_q(\gamma \text{ can not be written as } \gamma_s \hat{\gamma} \gamma_e \text{ and } M(B_\gamma \cdot q) > 2^M M(q) | \pi) \leq \rho.$$

Proof. Fix $M_0 \geq 0$ large and let $M = 2M_0$. Let Γ be the set of all minimal paths γ starting at π which can not be written as $\gamma_s \hat{\gamma} \gamma_e$ and such that $M(B_\gamma \cdot q) > 2^M M(q)$. Any path $\gamma \in \Gamma$ can be written as $\gamma = \gamma_1 \gamma_2$ where γ_1 is minimal with $M(B_{\gamma_1} \cdot q) > 2^{M_0} M(q)$. Let Γ_1 collect the possible γ_1 . Then Γ_1 is disjoint, by minimality. Let $\tilde{\Gamma}_1 \subset \Gamma_1$ be the set of all γ_1 such that $M_{\mathcal{A}'}(B_{\gamma_1} \cdot q) \geq M(q)$ for all invariant involution set $\mathcal{A}' \subset \mathcal{A}$ non-empty. By Theorem 3.5, if M_0 is sufficiently large we have

$$P_q(\Gamma_1 \setminus \tilde{\Gamma}_1 | \pi) < \frac{1}{2}.$$

For $\pi_e \in \mathfrak{R}$, let γ_{π_e} be a shortest possible path starting at π_e with $\gamma_{\pi_e} = \gamma_s \hat{\gamma}$. Let π_f be the end of γ_{π_e} . If M_0 is sufficiently large then $\|B_{\gamma_{\pi_e}}\| < \frac{1}{d-1} 2^{M_0-1}$. It follows that if $\gamma_1 \in \Gamma_1$ ends at π_e then

$$P_q(\Gamma | \gamma_1) \leq 1 - P_{B_{\gamma_1} \cdot q}(\gamma_{\pi_e} | \pi_e).$$

Let $\Lambda \subset \Lambda_{\pi_e, B_{\gamma_1} \cdot q}$ be an elementary subsimplex with

$$\text{Leb}_{\pi_e}(\Lambda) \geq \frac{\text{Leb}_{\pi_e}(\Lambda_{\pi_e, B_{\gamma_1} \cdot q})}{C_1(d)}.$$

Choose an elementary subsimplex Λ' of $\Lambda_{\pi_f, B_{\gamma_1 \gamma_{\pi_e}} \cdot q}$, such that for all $\underline{\alpha} \in \underline{\mathcal{A}}$ there exists a vertex v' of Λ' of type $\underline{\alpha}$ or of type $\{\underline{\alpha}, \underline{\xi}\}$ for some $\underline{\xi} \in \underline{\mathcal{A}}$ and $\text{Leb}_{\pi_f}(\Lambda') \geq \text{Leb}_{\pi_f}(B_{\gamma_{\pi_e}} \cdot \Lambda_{\pi_e, B_{\gamma_1} \cdot q})/C_1(d)$. Let Z and Z' be the set of vertices of Λ and Λ' , respectively. If furthermore $\gamma_1 \in \tilde{\Gamma}_1$ then

$$\frac{\text{Leb}_{\pi_f}(\Lambda')}{\text{Leb}_{\pi_e}(\Lambda)} = \frac{k(\pi_f, \Lambda') \prod_{v \in Z} w(v)}{k(\pi_e, \Lambda) \prod_{v \in Z'} w(v)} \geq \frac{M(q)^{d-1}}{(2^{2M_0} M(q))^{d-1}} = 2^{-2(d-1)M_0}.$$

So, $P_{B_{\gamma_1} \cdot q}(\gamma_{\pi_e} | \pi_e) \geq 2^{-2(d-1)M_0}$ and $P_q(\Gamma | \pi) \leq 1 - 2^{-2(d-1)M_0-1}$. \square

Proposition 4.2. *For every $\hat{\gamma} \in \Pi(\mathfrak{R})$, there exist $\delta > 0$, $C > 0$ such that for every $\pi \in \mathfrak{R}$, $q \in \mathbb{R}_+^{\mathcal{A}}$ and for every $T > 1$*

$$P_q(\gamma \text{ can not be written as } \gamma_s \hat{\gamma} \gamma_e \text{ and } M(B_\gamma \cdot q) > TM(q) | \pi) \leq CT^{-\delta}.$$

Proof. Let M and ρ be as in the previous lemma. Let k be maximal with $T \geq 2^{k(M+1)}$. Let Γ be the set of minimal paths γ such that γ is not of the form $\gamma_s \hat{\gamma} \gamma_e$ and $M(B_\gamma \cdot q) > 2^{k(M+1)}M(q)$. Any path $\gamma \in \Gamma$ can be written as $\gamma_1 \dots \gamma_k$ where $\gamma_{(i)} = \gamma_1 \dots \gamma_i$ is minimal with $M(B_{\gamma_{(i)}} \cdot q) > 2^{i(M+1)}M(q)$. Let $\Gamma_{(i)}$ collect the $\gamma_{(i)}$. Then the $\Gamma_{(i)}$ are disjoint. Moreover, by Lemma 4.1, for all $\gamma_{(i)} \in \Gamma_{(i)}$,

$$P_q(\Gamma_{(i+1)} | \gamma_{(i)}) \leq \rho.$$

This implies that $P_q(\Gamma | \pi) \leq \rho^k$. The result follows. \square

Remark 4.3. Notice that in the case of [AGY], they obtain a better recurrence estimate. In fact, they obtain $T^{-(\delta-1)}$ instead $T^{-\delta}$. But our estimate will be enough.

5 Construction of an excellent hyperbolic semi-flow

5.1 The Veech flow with involution as a suspension over the Rauzy renormalization

Let $\hat{\Upsilon}_{\mathfrak{R}}$ be the subset of $\mathcal{U}_{\mathfrak{R}}$ of all (λ, π, τ) with $\phi(\lambda, \pi, \tau) = \|\lambda\| = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha = 1$. We denote by $\hat{\Upsilon}_\pi$ the connected components of $\hat{\Upsilon}_{\mathfrak{R}}$. Consider $\hat{\Upsilon}_{\mathfrak{R}}^{(1)} = \mathcal{U}_{\mathfrak{R}}^{(1)} \cap \hat{\Upsilon}_{\mathfrak{R}}$ and $\hat{\Upsilon}_\pi^{(1)} = \mathcal{U}_\pi^{(1)} \cap \hat{\Upsilon}_\pi$. Let $\hat{m}_{\mathfrak{R}}^{(1)}$ be the induced Lebesgue measure to $\hat{\Upsilon}_{\mathfrak{R}}^{(1)}$.

We have that $\hat{\Upsilon}_{\mathfrak{R}}^{(1)}$ is transverse to the Veech flow with involution on $\mathcal{U}_{\mathfrak{R}}^{(1)}$ which is given by a certain iterate of the Rauzy induction with involution, after applying the flow \mathcal{TV}_t . We are interested in the first return map \hat{R} to this section. The domain of \hat{R} is the intersection of $\hat{\Upsilon}_{\mathfrak{R}}^{(1)}$ with the domain of definition of \hat{Q} , and we have

$$\hat{R}(\lambda, \pi, \tau) = (e^r \lambda', \pi', e^{-r} \tau'),$$

where $(\lambda', \pi', \tau') = \widehat{Q}(\lambda, \pi, \tau)$ and $r = r(\lambda, \pi) = -\log \|\lambda'\| = -\log \phi(\widehat{Q}(\lambda, \pi, \tau))$ is the first return time. Notice that the map \widehat{R} is a skew-product $\widehat{R}(\lambda, \pi, \tau) = (R(\lambda, \pi), e^{-r}\tau')$ over the non-invertible map R defined by $R(\lambda, \pi) = (e^r\lambda', \pi)$. The map \widehat{R} is called the *Rauzy renormalization map with involution* and it preserves the measure $\widehat{m}_{\mathfrak{R}}^{(1)}$. The renormalization map \widehat{R} is an “invertible extension” of the map R .

We can see the Veech flow with involution as a suspension over the renormalization map \widehat{R} . In this suspension model, we lose the control of the orbits which do not return to $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)}$. However, this do not cause any problem to our considerations because the set of such orbits has zero Lebesgue measure.

5.2 Precompact section

The suspension model for the Veech flow with involution presented above is obtained over a discrete transformation \widehat{R} which is not sufficiently hyperbolic. In general, \widehat{R} can not be expected to be uniformly hyperbolic, in fact, it does not even have appropriate distortion properties. This is related to the fact that the domain is not compact. The approach taken in [AGY] and other recent works as [AF] and [AV3] is to introduce a class of suitably small (precompact in $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)}$) sections, and to prove that the corresponding return maps have good distortion properties.

So, we will choose a specific precompact section which is the intersection of $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)}$ with (finite unions of) sets of the form $\Delta_\gamma \times \Theta_{\gamma'}$. Let γ be a path starting at π_s and ending at π_e . Precompactness in the λ direction is equivalent to having $B_\gamma^* \cdot (\overline{\mathcal{S}_{\pi_e}^+} \setminus \{0\}) \subset \mathcal{S}_{\pi_s}^+$, which is a necessary condition if γ is a positive path. To take care of both the λ and the τ direction, we have already introduced the notion of strongly positive.

Let $\Pi(\pi) \subset \Pi(\mathfrak{R})$ be the set of paths starting and ending at the same $\pi \in \mathfrak{R}$. Let $\pi \in \mathfrak{R}$ and let $\gamma_* \in \Pi(\pi)$ be a strongly positive path. Assume further that if $\gamma_* = \gamma_s \gamma = \gamma \gamma_e$ then either $\gamma = \gamma_*$ or γ is trivial. We will say that γ_* is *neat*. If for example γ_* ends by a left arrow and starts by a sufficiently long (at least half the length of γ_*) sequence of right arrows then the last condition of being neat is automatically satisfied.

Let $\widehat{\Xi} = \widehat{\Upsilon}_{\mathfrak{R}}^{(1)} \cap (\Delta_{\gamma_*} \times \Theta_{\gamma_*})$ and let $\Xi = \Upsilon_{\mathfrak{R}}^0 \cap \Delta_{\gamma_*}$. We will study the first return map $T_{\widehat{\Xi}}$ to the section $\widehat{\Xi}$ under the Veech flow with involution. Notice that the connected components of its domain are given by $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)} \cap (\Delta_{\gamma\gamma_*} \times \Theta_{\gamma_*})$, where γ is either γ_* , or a minimal path of the form $\gamma_*\gamma_0\gamma_*$ not beginning by $\gamma_*\gamma_*$. The restriction of $T_{\widehat{\Xi}}$ to each connected component of its domain has the expression

$$T_{\widehat{\Xi}}(\lambda, \pi, \tau) = \left(\frac{(B_\gamma^*)^{-1} \cdot \lambda}{\|(B_\gamma^*)^{-1} \cdot \lambda\|}, \pi, \|(B_\gamma^*)^{-1} \cdot \lambda\| (B_\gamma^*)^{-1} \cdot \tau \right).$$

and the return time function is given by

$$r_{\widehat{\Xi}}(\lambda, \pi, \tau) = r_{\Xi}(\lambda, \pi) = -\log \|(B_\gamma^*)^{-1} \cdot \lambda\|.$$

The map $T_{\widehat{\Xi}}(\lambda, \pi, \tau) = (\lambda', \pi, \tau')$ is a skew-product over a non-invertible transformation $T_{\Xi}(\lambda, \pi) = (\lambda', \pi)$.

Analogously to the case of the renormalization map, the Veech flow with involution can be seen as a suspension over $T_{\widehat{\Xi}}$, with roof function $r_{\widehat{\Xi}}$. But considering this suspension model, we lose the control of many more orbits which do not come back to $\widehat{\Xi}$. Still, due to ergodicity of the Veech flow with involution, almost every orbit is captured by the suspension model.

5.3 Hyperbolic properties

The reason to choose the section $\widehat{\Xi}$ is because the transformation $T_{\widehat{\Xi}}$ has better hyperbolic properties than transformations considering larger sections and we can also describe easily the connected components of its domain.

Lemma 5.1. *$T_{\widehat{\Xi}}$ is a hyperbolic skew-product over T_{Ξ} .*

Recalling the definition 1.5, we observe that associated to a hyperbolic skew-product we have: a probability measure $\widehat{\nu}$ (which we chose as the normalized restriction of $\widehat{m}_{\mathfrak{R}}^{(1)}$ to $\widehat{\Xi}$) and a Finsler metric $\|\cdot\|_{\widehat{\Xi}}$ (which we will choose in order to obtain the hyperbolic properties we want from $T_{\widehat{\Xi}}$). At first we will introduce a complete Finsler metric on $\widehat{\Upsilon}_{\pi}^{(1)}$, and then we will consider its restriction denoting it by $\|\cdot\|_{\widehat{\Xi}}$. Since γ_* is strongly positive, the section $\widehat{\Xi}$ is a precompact open subset of $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)}$, therefore $\widehat{\Xi}$ will have bounded diameter with respect to such metric.

5.4 Hilbert metric

Now, we will introduce the Hilbert projective metric and state some of its properties which we will use after. This notion can be defined for a general convex cone C in any vector space, but in our case we only need $C = \mathbb{R}_+^A$.

We call the Hilbert pseudo-metric on \mathbb{R}_+^2 the function $\text{dist}_{\mathbb{R}_+^2}$ defined by $\text{dist}_{\mathbb{R}_+^2}(x, y) = \log \max_{1 \leq i, j \leq 2} \frac{x_i y_j}{x_j y_i}$, for each $x, y \in \mathbb{R}_+^2$. Given a linear operator $B \in \text{GL}(2, \mathbb{R})$ such that $B \cdot \mathbb{R}_+^2 \subset \mathbb{R}_+^2$, we have $\text{dist}_{\mathbb{R}_+^2}(B \cdot x, B \cdot y) \leq \text{dist}_{\mathbb{R}_+^2}(x, y)$ for all $x, y \in \mathbb{R}_+^2$, or equivalently, such that all coefficients of the matrix B are non-negative, which means that B contracts weakly the Hilbert pseudo-metric. In particular, the Hilbert pseudo-metric is invariant under linear isomorphisms of \mathbb{R}_+^2 .

In general, if we consider an open convex cone $C \subset \mathbb{R}^A \setminus \{0\}$ whose closure does not contain any one-dimensional subspace of \mathbb{R}^A , we define the Hilbert pseudo-metric on C by $\text{dist}_C(x, y) = 0$ if x and y are collinear and $\text{dist}_C(x, y) = \text{dist}_{\mathbb{R}_+^2}(\psi(x), \psi(y))$ where ψ is any isomorphism between the intersection of \mathbb{R}_+^2 and the subspace generated by x and y (this isomorphism exist since x and y are not collinear). If $C = \mathbb{R}_+^A$ then we define $\text{dist}_C(x, y) = \max_{\alpha, \beta \in A} \log \frac{x_{\alpha} y_{\beta}}{x_{\beta} y_{\alpha}}$.

Given two convex cones C and C' such that $C' \subset C$ then $\text{dist}_C(x, y) \leq \text{dist}_{C'}(x, y)$, i.e., the inclusion map $C' \rightarrow C$ is a weak contraction of the respective Hilbert pseudo-metrics. But if the diameter of C' with respect to dist_C is bounded by some M then we have an uniform contraction by some constant $\delta = \delta(M) < 1$, i.e., $\text{dist}_C(x, y) \leq \delta \text{dist}_{C'}(x, y)$.

It is clear that $\text{dist}_C(x, y) = 0$ if and only if there exists $t > 0$ such that $y = tx$. If we restrict the Hilbert pseudo-metric on a convex cone C to the

space of rays $\{t\alpha : t \in \mathbb{R}_+\} \subset C$ we have the Hilbert metric, which is a complete Finsler metric.

5.5 Uniform expansion and contraction

Recall that $\widehat{\Upsilon}_\pi^{(1)}$ is contained in $\Delta_\pi \times \Theta_\pi$, which is a product of two convex cones. In $\Delta_\pi \times \Theta_\pi$, we have the product Hilbert pseudo-metric $\text{dist}((\lambda, \pi, \tau), (\lambda', \pi, \tau')) = \text{dist}_{\Delta_\pi}((\lambda, \pi), (\lambda', \pi)) + \text{dist}_{\Theta_\pi}(\tau, \tau')$. Each product of rays $\{(a\lambda, \pi, b\tau) : a, b \in \mathbb{R}_+\} \subset \Delta_\pi \times \Theta_\pi$ intersects transversely $\widehat{\Upsilon}_\pi^{(1)}$ in a unique point. It follows that the product Hilbert pseudo-metric induces a metric dist on $\widehat{\Upsilon}_\pi^{(1)}$. It is a complete Finsler metric.

Lemma 5.2. *Given $\pi \in \mathfrak{A}$, define $\Delta_\pi^1 = \{(\lambda, \pi) \in \Delta_\pi : \|\lambda\| = 1\}$. Let $g^\pi : \Delta_\pi^1 \rightarrow \Delta_\pi^1$ be a functional defined by $g^\pi(\lambda, \pi) = \left(\sum_{\underline{\beta}} g_{\underline{\beta}}^\pi \lambda_{\underline{\beta}}, \pi\right)$, where $g_{\underline{\beta}}^\pi \geq 0$ for all $\underline{\beta} \in \mathcal{A}$. Then $\log g^\pi(\lambda, \pi)$ is 1-Lipschitz relative to the Hilbert metric.*

Proof. Given $(\lambda, \pi), (\lambda', \pi) \in \Delta_\pi^1$, we have

$$\frac{g^\pi(\lambda, \pi)}{g^\pi(\lambda', \pi)} = \frac{\left(\sum_{\underline{\beta}} g_{\underline{\beta}}^\pi \lambda_{\underline{\beta}}, \pi\right)}{\left(\sum_{\underline{\beta}} g_{\underline{\beta}}^\pi \lambda'_{\underline{\beta}}, \pi\right)} \leq \sup_{\underline{\beta}} \frac{\left(\lambda_{\underline{\beta}}, \pi\right)}{\left(\lambda'_{\underline{\beta}}, \pi\right)} \leq e^{\text{dist}_{\Delta_\pi}((\lambda, \pi), (\lambda', \pi))}.$$

Thus $\log g^\pi(\lambda, \pi)$ is 1-Lipschitz with respect to dist_{Δ_π} . \square

Proof of Lemma 5.1. Let us first show that T_Ξ is a uniformly expanding Markov map (the underlying Finsler metric being the restriction of dist_{Δ_π} , and the underlying measure Leb being the induced Lebesgue measure). It is clear that Ξ is a John domain.

Condition (1) of Definition 1.2 is easily verified, except for the definite contraction of inverse branches. To check this property, we notice that an inverse branch can be written as $h(\lambda, \pi) = \left(\frac{B_\gamma^* \cdot \lambda}{\|B_\gamma^* \cdot \lambda\|}, \pi\right)$. Since γ_* is neat, we can write $B_\gamma^* = B_{\gamma_*}^* B_{\gamma_0}^*$ for some γ_0 . Thus h can be written as (the restriction of) the composition of two maps $\Delta_\pi^1 \rightarrow \Delta_\pi^1$, $h = h_* \circ h_0$, where h_0 is weakly contracting and h_* is uniformly contracting by precompactness of Ξ in Δ_π (which is a consequence of strong positivity of γ_*).

To check condition (2) of Definition 1.2, let $h(\lambda, \pi)$ be an inverse branch of T_Ξ .

Let $V = \{v \in \mathcal{S}_\pi : \sum v_{\underline{\alpha}} = 0\}$ be the hyperplane tangent to Δ_π^1 at a point $(\lambda, \pi) \in \Delta_\pi^1$. Since the coordinate π is fixed by h we can dismiss it. Thus the simplified expression of h is $h(\lambda) = \frac{B_\gamma^* \cdot \lambda}{\|B_\gamma^* \cdot \lambda\|}$. We will denote $\phi(B_\gamma^* \cdot \lambda) = \|B_\gamma^* \cdot \lambda\| = \sum_{\underline{\alpha}, \underline{\beta}} (B_\gamma^*)_{\underline{\alpha}, \underline{\beta}} \lambda_{\underline{\beta}}$, where $(B_\gamma^*)_{\underline{\alpha}, \underline{\beta}}$ is the coefficient of B_γ^* in the line $\underline{\alpha}$ and the column $\underline{\beta}$. So,

$$Dh(\lambda) \cdot v = \frac{B_\gamma^* \cdot v}{\phi(B_\gamma^* \cdot \lambda)} - \frac{B_\gamma^* \cdot \lambda}{\phi(B_\gamma^* \cdot \lambda)} \frac{\sum_{\underline{\alpha}} (B_\gamma^* \cdot v)_{\underline{\alpha}}}{\phi(B_\gamma^* \cdot \lambda)}.$$

So $Dh(\lambda) = P_\lambda \circ \phi(B_\gamma^* \cdot \lambda)^{-1} \circ B_\gamma^*$, where $B_\gamma^* : V \rightarrow B_\gamma^* \cdot V$, $\phi(B_\gamma^* \cdot \lambda)^{-1}$ is the division by the scalar $\phi(B_\gamma^* \cdot \lambda)$ on $B_\gamma^* \cdot V$ and $P_\lambda : B_\gamma^* \cdot V \rightarrow V$ is the projection on

V along the direction $B_\gamma^* \cdot \lambda$. The Jacobian of h at (λ, π) is $J \circ h(\lambda) = \det Dh(\lambda)$, so,

$$\log J \circ h = \log \det P_\lambda - (d-2) \log \det \phi(B_\gamma^* \cdot \lambda) + \log \det B_\gamma^*$$

We want to prove that $\log J \circ h$ is Lipschitz relative to the Hilbert metric. We have that $\log \det B_\gamma^*$ is constant and, by lemma 5.2, $\log \det \phi(B_\gamma^* \cdot \lambda)$ is 1-Lipschitz. Now we have to verify what happens with $\log \det P_\lambda$. We have that

$$\det P_\lambda = \frac{\langle n_1, B_\gamma^* \cdot \lambda \rangle}{\langle n_0, B_\gamma^* \cdot \lambda \rangle}$$

where n_0 and n_1 are unit vectors in \mathcal{S}_π orthogonal to the hyperplanes V and $B_\gamma^* \cdot V$. Indeed, the vector n_0 and the vector $B_\gamma \cdot n_1$ are collinear with the orthogonal projection of $(1, \dots, 1)$ on \mathcal{S}_π . Note that n_0 has non-negative coefficients, so, neither $B_\gamma \cdot n_0$ and $B_\gamma \cdot n_1$ have. Once again by lemma 5.2, we have that each $\log(\langle n_i, B_\gamma^* \cdot \lambda \rangle)$ is 1-Lipschitz. Therefore, $\log J \circ h$ is d -Lipschitz with respect to dist_{Δ_π} .

To see that $T_{\widehat{\Xi}}$ is a hyperbolic skew-product over T_{Ξ} , one checks the conditions (1-4) of Definition 1.5. Condition (1) is obvious, and condition (4) follows from precompactness of $\widehat{\Xi}$ in $\Delta_\pi \times \Theta_\pi$ as before. Since $T_{\widehat{\Xi}}$ is a first return map, the restriction of $\widehat{m}_{\mathfrak{R}}^{(1)}$ to $\widehat{\Xi}$ is $T_{\widehat{\Xi}}$ -invariant. Its normalization is the probability measure $\widehat{\nu}$ of condition (2). In order to check condition (3), it is convenient to trivialize $\widehat{\Xi}$ to a product (via the natural diffeomorphism $\widehat{\Xi} \rightarrow \Xi \times \mathbb{P}\Theta_{\gamma_*}$, where $\mathbb{P}V$ denotes the projective space of V). Since $\widehat{\nu}$ has a smooth density with respect to the product of the Lebesgue measure on the factors, condition (3) follows by the Leibniz rule. \square

Our results give the finiteness of the measure and the integrability of the cocycle.

Proposition 5.3. *The space $\mathcal{U}_{\mathfrak{R}}^{(1)}$ has finite volume.*

Proof. Consider the section $\widehat{\Xi}$. Notice that this section has positive measure and almost every orbit return to $\widehat{\Xi}$. We have already mention the probability measure $\widehat{\nu}$ on $\widehat{\Xi}$, which is the normalized restriction of $\widehat{m}_{\mathfrak{R}}^{(1)}$ to $\widehat{\Xi}$.

We want to compute $\int_{\widehat{\Xi}} \log \|(B_\gamma^*)^{-1} \cdot x\| d\widehat{\nu}$.

A connected component of the domain of $T_{\widehat{\Xi}}$ which intersects the set $\{x \in \widehat{\Xi} : r_{\widehat{\Xi}}(x) > T\}$ is of the form $(\Delta_{\gamma_1} \times \Theta_{\gamma_2}) \cap \widehat{\Upsilon}_{\mathfrak{R}}^{(1)}$ where γ_1 can not be written as $\gamma_s \widehat{\gamma} \gamma_e$ with $\widehat{\gamma} = \gamma_* \gamma_* \gamma_* \gamma_*$, $M(B_\gamma \cdot q_0) \geq D^{-1}T$, where $q_0 = (1, \dots, 1)$ and $D = D(\gamma_*)$ is some constant.

The projection of $\widehat{\nu}|_{(\Delta_{\gamma_1} \times \Theta_{\gamma_2}) \cap \widehat{\Upsilon}_{\mathfrak{R}}^{(1)}}$ on $\Upsilon_{\mathfrak{R}}^{(1)}$ is absolutely continuous with a bounded density, so

$$\begin{aligned} \widehat{\nu}\{x \in (\Delta_{\gamma_1} \times \Theta_{\gamma_2}) \cap \widehat{\Upsilon}_{\mathfrak{R}}^{(1)} : r_{\widehat{\Xi}}(x) > T\} \leq \\ CP_{q_0}(\gamma \text{ can not be written as } \gamma_s \widehat{\gamma} \gamma_e \text{ and } M(B_\gamma \cdot q_0) \geq D^{-1}T | \pi) \end{aligned}$$

and the result follows by Proposition 4.2. \square

5.6 Properties of the roof function

Recall $H(\pi) = \Omega(\pi) \cdot \mathcal{S}_\pi$. As we have observed, given a path $\gamma \in \Pi(\pi)$, $H(\pi)$ is invariant under the map B_γ . By Lemma 2.8, if $\tau \in \Theta_\pi$ then $-\Omega(\pi) \cdot \tau \in \mathbb{R}_+^A$ and by Corollary 2.15, Θ_π is a non-empty set, so $H(\pi) \cap \mathbb{R}_+^A \neq \emptyset$.

Lemma 5.4. *Let π be an irreducible permutation and $\gamma \in \Pi(\mathfrak{A})$. The subspace $H(\pi)$ has dimension greater than one.*

Proof. Let A be a minimal double letter in the sense that A is a left double letter and there is no double letter Z such that $\pi(Z) < \pi(A)$ or $\pi(i(Z)) < \pi(A)$. Let B be a maximal double letter in the sense that B is a right double letter and there is no double letter Z such that $\pi(B) < \pi(Z)$ or $\pi(B) < \pi(i(Z))$.

We have that:

$$\mathcal{S}_\pi = \left\{ \lambda \in \mathbb{R}^{\mathfrak{A}} : \lambda_{\underline{A}} = \lambda_{\underline{B}} + \sum \epsilon_\alpha \lambda_\alpha \right\}$$

where $\epsilon_\alpha = 0$ if α is simple, $\epsilon_\alpha = -1$ if α is left double letter and $\epsilon_\alpha = 1$ if α is right double letter.

Case I: Let C be a simple letter. Denote $x_C = \Omega(\pi)_{AC} \in \{0, 1, 2\}$ and $y_C = \Omega(\pi)_{CB} \in \{0, 1, 2\}$. Thus the matrix $\Omega(\pi)$ has a submatrix of the form:

$$\begin{pmatrix} 0 & 2 & x_C \\ -2 & 0 & -y_C \\ -x_C & y_C & 0 \end{pmatrix}.$$

So, the matrix $\Omega(\pi) \cdot \mathcal{S}_\pi$ has a submatrix of the form:

$$\begin{pmatrix} 2 & x_C \\ -2 & -y_C \\ -x_C + y_C & 0 \end{pmatrix}.$$

In this case the matrix $\Omega(\pi) \cdot \mathcal{S}_\pi$ has rank 2, except if $x_C = y_C$.

Case II: Let D be a left double letter (the other case is analogous). Denote $z_D = \Omega(\pi)_{AD} \in \{-2, -1, 0, 1, 2\}$. Notice that $\Omega(\pi)_{DB} = 2$. In this case, the matrix $\Omega(\pi)$ has a submatrix of the form:

$$\begin{pmatrix} 0 & 2 & z_D \\ -2 & 0 & -2 \\ -z_D & 2 & 0 \end{pmatrix},$$

therefore, the matrix $\Omega(\pi) \cdot \mathcal{S}_\pi$ has a submatrix of the form:

$$\begin{pmatrix} 2 & 2 + z_D \\ -2 & -2 \\ 2 - z_D & 2 \end{pmatrix}.$$

In this case the matrix $\Omega(\pi) \cdot \mathcal{S}_\pi$ has rank 2, except if $z_D = 0$.

Suppose that if C is a simple letter of the permutation π then $x_C = y_C$ and if D is a double letter of the permutation π then $z_D = 0$. Since π is irreducible, $x_C = y_C = 1$. Thus π has the form

$$A \cdot \cdot \cdot \cdot i(A) * i(B) \cdot \cdot \cdot B$$

If we apply the right or the left operation we obtain a reducible permutation.

So, there exists a simple letter C such that $x_C \neq y_C$ or there exists a double letter D such that $z_D \neq 0$. \square

Recall that $v_\pi = \sum_{\pi(x) < \pi(*)} e_x - \sum_{\pi(x) > \pi(*)} e_x$ is the orthogonal vector to \mathcal{S}_π .

Lemma 5.5. *Let π be an irreducible permutation and $\gamma \in \Pi(\mathfrak{R})$. If $v_\pi \in H(\pi)$, then the subspace $H(\pi)$ has dimension greater than two.*

Proof. Let A and B be the leftmost and the rightmost letters of π .

Since $H(\pi)$ and v_π are invariant under B_γ , we can suppose that A is simple.

If B is simple, then π has the form:

$$A \cdot \cdot \cdot i(B) \cdot \cdot \cdot * \cdot \cdot i(A) \cdot \cdot \cdot B$$

We have that $\Omega(\pi) \cdot e_A$, $\Omega(\pi) \cdot e_B$ and v_π are linearly independent, since $(v_\pi)_A = (v_\pi)_B = 0$.

If $\pi(i(B)) < \pi(i(A))$ and B is double, we take a left double letter C , i.e., π has the form:

$$A \cdot C \cdot \cdot \cdot i(C) \cdot \cdot \cdot * \cdot \cdot i(B) \cdot i(A) \cdot \cdot \cdot B$$

And in the case that $\pi(i(B)) > \pi(i(A))$, π has the form:

$$A \cdot C \cdot \cdot \cdot i(C) \cdot \cdot \cdot * \cdot \cdot \cdot i(A) \cdot i(B) \cdot \cdot \cdot B$$

In these last two cases, we have $\Omega(\pi) \cdot e_A$, $\Omega(\pi) \cdot (e_B + e_C)$ and v_π are linearly independent, since $(v_\pi)_A = 0$ and $(v_\pi)_B = -(v_\pi)_C = 1$. \square

Lemma 5.6. *The roof function r_Ξ is good (in the sense of Definition 1.3).*

Proof. Let $\Gamma \subset \Pi(\mathfrak{R})$ be the set of all γ such that γ is either γ_* , or a minimal path of the form $\gamma_*\gamma_0\gamma_*$ not beginning by $\gamma_*\gamma_*$. Notice that Γ consists of positive paths.

The set \mathcal{H} of inverse branches h of T_Ξ is in bijection with Γ , since each inverse branch is of the form $h(\lambda, \pi) = \left(\frac{B_{\gamma_h}^* \cdot \lambda}{\|B_{\gamma_h}^* \cdot \lambda\|}, \pi \right)$ for some $\gamma_h \in \Gamma$.

Let $h \in \mathcal{H}$. Then $r_\Xi(h(\lambda, \pi)) = \log \|B_{\gamma_h}^* \cdot \lambda\|$. Since γ_h is positive, $r_\Xi \geq \log 2$, which implies condition (1). By lemma 5.2, $r_\Xi(h(\lambda, \pi))$ is 1-Lipschitz with respect to dist_{Δ_π} , so (2) follows.

Let us check condition (3). We identify the tangent space to Ξ at a point $(\lambda, \pi) \in \Xi$ with $V = \{\lambda \in \mathcal{S}_\pi : \sum \lambda_\alpha = 0\}$. Assume that we can write $r_\Xi = \psi + \phi \circ T_\Xi - \phi$ with ϕ C^1 and ψ locally constant. Write $r^{(n)}(\lambda, \pi) = \sum_{j=0}^{n-1} r_\Xi(T_\Xi^j(\lambda, \pi))$. Then $D(r^{(n)} \circ h^n) = D\phi - D(\phi \circ h^n)$, which can be rewritten as

$$\frac{\|(B_{\gamma_h}^*)^n \cdot v\|}{\|(B_{\gamma_h}^*)^n \cdot \lambda\|} = D\phi(\lambda, \pi) \cdot v - D(\phi \circ h^n)(\lambda, \pi) \cdot v, \quad (\lambda, \pi) \in \Xi, v \in V. \quad (26)$$

If we define

$$w_{n,h} = \frac{B_{\gamma_h}^n \cdot (1, \dots, 1)}{\langle \lambda, B_{\gamma_h}^n \cdot (1, \dots, 1) \rangle},$$

we replace (26) by

$$\langle v, w_{n,h} \rangle = D\phi(\lambda, \pi) \cdot v - D(\phi \circ h^n)(\lambda, \pi) \cdot v, \quad (\lambda, \pi) \in \Xi, v \in V.$$

We have that $\langle \lambda, w_{n,h} \rangle = 1$ for all $\lambda \in \mathcal{S}_\pi$ and $w_{n,h}$ are vectors with all coordinates positive. By the Perron-Frobenius Theorem $w_{n,h}$ converges to some w_h collinear with the unique positive eigenvector of B_{γ_h} (which also corresponds

to the largest eigenvalue). And $w_h = w_{0,h} + t_h v_\pi$, where $w_{0,h} \in \mathcal{S}_\pi \setminus \{0\}$ is the orthogonal projection of w_h in \mathcal{S}_π and v_π is the orthogonal vector of \mathcal{S}_π , which is invariant under B_{γ_h} for all γ_h .

Since $Dh^n \rightarrow 0$, we conclude that

$$\langle v, w_h \rangle = D\phi(\lambda, \pi) \cdot v, \quad (\lambda, \pi) \in \Xi, v \in V.$$

Since $\langle \lambda, w_{0,h} \rangle = 1$, we have $w_{0,h} = w_0$. Thus $w_h = w_0 + t_h v_\pi$ where $w_0 \in \mathcal{S}_\pi \setminus \{0\}$.

Recalling that $H(\pi)$ is invariant under B_{γ_h} , and intersects \mathbb{R}_+^A , it follows that $w_h \in H(\pi)$ and $\mathbb{R}_+ w_{n,h}$ is converging to $\mathbb{R}_- (\Omega(\pi) \cdot \tau)$.

Let $W = \mathbb{R}w_0 \oplus \mathbb{R}v_\pi$. We have that $W \cap H(\pi) \neq \emptyset$ is closed and invariant by B_{γ_h} . By the previous two lemmas, there exists $\tau \in \Theta_\pi$ such that $\Omega(\pi) \cdot \tau \notin W$. But, given such τ , we can construct paths γ_n such that $B_{\gamma_n} \cdot \Omega(\pi) \cdot \Theta_\pi$ converges to $\mathbb{R}_- (\Omega(\pi) \cdot \tau)$ as follows. We have already observed that \widehat{Q}^{-1} is recurrent, so given $(\lambda, \pi, \tau) \in \widehat{\Xi}$, we apply \widehat{Q}^{-1} until obtain $(\lambda^{(-1)}, \pi, \tau^{(-1)}) \in \widehat{\Xi}$. We denote by γ_1 the path obtained previously, starting at $(\lambda^{(-1)}, \pi, \tau^{(-1)})$ and ending at (λ, π, τ) . We follow the same procedure to obtain γ_n starting at $(\lambda^{(-n)}, \pi, \tau^{(-n)})$ and ending at (λ, π, τ) .

By definition of $\widehat{\Xi}$, we have that such paths γ_n are strongly positive, so the image of $B_{\gamma_n}^* \cdot \Theta_{\gamma_n}$ is contracted, relatively to the Hilbert metric. Thus we have that $B_{\gamma_n} \cdot \Omega(\pi) \cdot \Theta_\pi$ is converging to $\mathbb{R}_- (\Omega(\pi) \cdot \tau)$. Thus we have a contradiction. \square

Theorem 5.7. *The roof function r_Ξ has exponential tails.*

Proof. Let π be the start of γ_* . The push-forward under radial projection of the Lebesgue measure on Λ_{π, q_0} onto $\Delta_\pi \cap \Upsilon_{\mathfrak{R}}^{(1)}$ yields a smooth measure $\tilde{\nu}$. It is enough to show that $\tilde{\nu}\{x \in \Xi : r_\Xi(x) \geq T\} \leq CT^{-\delta}$, for some $C > 0$, $\delta > 0$. A connected component of the domain of T_Ξ that intersects the set $\{x \in \Xi : r_\Xi(x) \geq T\}$ is of the form $\Delta_\gamma \cap \Upsilon_{\mathfrak{R}}^{(1)}$ where γ can not be written as $\gamma_s \hat{\gamma} \gamma_e$ with $\hat{\gamma} = \gamma_* \gamma_* \gamma_* \gamma_*$ and $M(B_\gamma \cdot q_0) \geq C^{-1}T$, where $q_0 = (1, \dots, 1)$ and C is a constant depending on γ_* . Thus

$$\begin{aligned} \tilde{\nu}\{x \in \Xi : r_\Xi(x) \geq T\} \\ \leq P_{q_0}(\gamma \text{ can not be written as } \gamma_s \hat{\gamma} \gamma_e \text{ and } M(B_\gamma \cdot q_0) \geq C^{-1}T | \pi). \end{aligned}$$

The result follows from Proposition 4.2. \square

Using both the map $T_{\widehat{\Xi}}$ and the roof function r_Ξ we will define a flow \widehat{T}_t on the space $\widehat{\Delta}_r = \{(x, y, s) : (x, y) \in \widehat{\Xi}, T_{\widehat{\Xi}}(x, y) \text{ is defined and } 0 \leq s < r_\Xi(x)\}$. Since $T_{\widehat{\Xi}}$ is a hyperbolic skew-product (Lemma 5.1), and r_Ξ is a good roof function (Lemma 5.6) with exponential tails (Theorem 5.7), \widehat{T}_t is an excellent hyperbolic semi-flow. By Theorem 1.7, we get exponential decay of correlations

$$C_t(\tilde{f}, \tilde{g}) = \int \tilde{f} \cdot \tilde{g} \circ \widehat{T}_t \, d\nu - \int \tilde{f} \, d\nu \int \tilde{g} \, d\nu,$$

for C^1 functions \tilde{f}, \tilde{g} , that is

$$|C_t(\tilde{f}, \tilde{g})| \leq Ce^{-3\delta t} \|\tilde{f}\|_{C^1} \|\tilde{g}\|_{C^1}, \quad (27)$$

for some $C > 0$, $\delta > 0$.

6 The Teichmüller flow

6.1 Half-translation surfaces

Let S be a compact oriented surface of genus $g \geq 0$, let Σ be a finite non-empty subset of S , which we call the *singular set*. Let $l = \{l_x\}_{x \in \Sigma}$ (the *multiplicity vector*) be such that $l_x \in \{-1\} \cup \mathbb{N}$ and $\sum l_x = 4g - 4$. We say that l_x is the multiplicity of the singular point x . Consider a maximal atlas $\mathfrak{A} = \{(\mathcal{U}_\lambda, \phi_\lambda : \mathcal{U}_\lambda \rightarrow \mathcal{V}_\lambda \subset \mathbb{C})\}$ of orientation preserving charts on $S \setminus \Sigma$ such that for all λ_1, λ_2 with $\mathcal{U}_{\lambda_1} \cap \mathcal{U}_{\lambda_2} \neq \emptyset$ we have $\phi_{\lambda_1} \phi_{\lambda_2}^{-1}(z) = \pm z + \text{constant}$, i.e., coordinate changes are compositions of rotations by 180° and translations. We call these coordinates the *regular charts*. We also assume that each singular point x has an open neighborhood \mathcal{U} which is isomorphic to the $\frac{l_x+2}{2}$ -folded cover of an open neighborhood $\mathcal{V} \subset \mathbb{C}$ of 0, that is, there exists a homeomorphism, called a *singular chart*, $\phi : \mathcal{U} \rightarrow \mathcal{V}$ such that any branch of $z \mapsto \phi(z)^{(l_x+2)/2}$ is a regular chart. Under these conditions, we say that the atlas \mathfrak{A} defines a *half-translation structure* on (S, Σ) with multiplicity vector l , and we call S a *half-translation surface*.

Since the change of coordinates preserves families of parallel lines in the plane, we have a well-defined singular foliation \mathcal{F}_θ of S , for each direction $\theta \in \mathbb{P}\mathbb{R}^2$ (the projective space of \mathbb{R}^2). In particular, we have well-defined vertical and horizontal directions. Notice that we can pullback the Euclidean metric in \mathbb{R}^2 by the regular charts to define a flat metric on $S \setminus \Sigma$. This flat metric does not extend smoothly to Σ except at points with $l_x = 0$. The other points of Σ are genuine *conical singularities* with total angle $\pi(l_x + 2)$, and are thus responsible for any curvature. The corresponding volume form on $S \setminus \Sigma$ has finite total mass.

Notice that from each $x \in \Sigma$, there are $l_x + 2$ *horizontal separatrices* alternating with $l_x + 2$ *vertical separatrices* emanating from x . A half-translation surface together with the choice of some $x_0 \in \Sigma$ and of one of the horizontal separatrices X emanating from x_0 is called a *marked half-translation surface*.

6.2 Translation surfaces

If there exists a compatible atlas such that the coordinate changes are just translations, then any maximal such atlas is said to define a *translation structure* on (S, Σ) compatible with the half-translation structure, and we call S a *translation surface*. A half-translation surface has thus either 0 or 2 compatible translation structures. Locally, each half-translation structure is compatible with a translation structure, but in general it is not true globally. Given a half-translation surface S , we can associate a number ε where $\varepsilon = 1$ or $\varepsilon = -1$ according to whether the half-translation structure is, or is not, compatible with a translation structure (on the other hand, obviously each translation structure is compatible with a unique half-translation structure). Notice that if $\varepsilon = 1$ then $l_x \in 2\mathbb{N}$ for every $x \in \Sigma$ (and thus necessarily $g \geq 1$), but the converse is not generally true.

Given a translation surface S , each oriented direction $\theta \in \mathbb{S}^1$ determines a singular oriented foliation \mathcal{F}_θ on S . From every singularity thus emanate $(l_x + 2)/2$ eastbound (respectively, northbound, westbound, southbound) oriented separatrices. A translation surface together with the choice of some $x_0 \in \Sigma$ and of an eastbound separatrix X emanating from x_0 is called a *marked translation surface*.

6.3 Translation surfaces with involution

Let \tilde{S} be a compact oriented surface of genus $g \geq 1$, let $\tilde{\Sigma}$ be a finite non-empty subset of \tilde{S} , and let $I : \tilde{S} \rightarrow \tilde{S}$ be an involution preserving $\tilde{\Sigma}$ and whose fixed points are contained in $\tilde{\Sigma}$. A *translation structure with involution* on $(\tilde{S}, \tilde{\Sigma}, I)$ is a translation structure such that for every regular chart ϕ of the translation structure, $-\phi \circ I$ is also a regular chart.

Notice that given $(\tilde{S}, \tilde{\Sigma}, I)$ we can consider the canonical projection $p : \tilde{S} \rightarrow S = \tilde{S}/I$. Denote $\Sigma = \tilde{\Sigma}/I$. We see that any translation structure with involution on $(\tilde{S}, \tilde{\Sigma}, I)$ induces by p a half-translation structure on (S, Σ) , with $\varepsilon = -1$ if \tilde{S} is connected.

Conversely, given (S, Σ) and a multiplicity vector l such that there exists a half-translation structure on (S, Σ) with such multiplicity vector and $\varepsilon = -1$, there exists a ramified double covering $p : (\tilde{S}, \tilde{\Sigma}) \rightarrow (S, \Sigma)$ which is unramified in $\tilde{S} \setminus \tilde{\Sigma}$. Indeed, given such a half-translation structure, we can define $\tilde{S} \setminus \tilde{\Sigma}$ to be the set of pairs (z, α) where $z \in S \setminus \Sigma$ and α is an orientation of the horizontal direction through z (the assumption that $\varepsilon = -1$ guarantees that \tilde{S} is connected). It is then easy to define the missing set $\tilde{\Sigma}$ necessary to compactify: each $x \in \Sigma$ with odd l_x giving rise to a single point of $\tilde{\Sigma}$ with multiplicity $2l_x + 2$ and each $x \in \Sigma$ with even l_x giving rise to a pair points of $\tilde{\Sigma}$ with multiplicity l_x each one. To each half-translation surface we can associate a combinatorial data \tilde{l} , which is the multiplicity vector considered up to labelling. The construction above gives rise to a translation surface \tilde{S} with singularity set $\tilde{\Sigma}$ and there is a natural involution defined, interchanging points (z, α) with fixed $z \in \tilde{S}$.

A translation surface with involution together with the choice of some $\tilde{x}_0 \in \tilde{\Sigma}$ and of one of the horizontal separatrices \tilde{X} emanating from \tilde{x}_0 to east is called a *marked translation surface with involution*. We say that \tilde{x}_0 is the start point of \tilde{X} . It is obvious that fixing \tilde{x}_0 and \tilde{X} we also fix $I(\tilde{x}_0)$ and $I(\tilde{X})$.

Notice that when we do the double covering construction above we can do it in such a way that $p(\tilde{X}) = X$ and $p(\tilde{x}_0) = x_0$, i.e., the marked separatrix and its start point are preserved.

As we will see in section 7.2, we can obtain combinatorial and length data (λ, π, τ) (as in the section 2.1) associated to a marked translation surface with involution $(\tilde{S}, \tilde{\Sigma}, I)$.

6.4 Moduli spaces

Let S be a surface with singular set Σ and genus g . To consider the space of surfaces with fixed genus, singularity set, multiplicity vector and the marked separatrix, we can define equivalence relations on those surfaces, obtaining moduli spaces. Although moduli spaces are not manifolds, we can see them as a quotient of a less restricted space, which has a complex affine manifold structure, and the modular group of (S, Σ) , i.e., the group of orientation preserving diffeomorphisms of S fixing Σ modulo those isotopic to the identity. Thus, moduli spaces are complex affine orbifolds.

6.4.1 Moduli space of marked translation surfaces

Given $g \geq 1$, a function $\kappa : \mathbb{N} \rightarrow 2\mathbb{N}$ with finite support and $\sum_{i \geq 0} i\kappa(i) = 4g - 4$, and an integer $j \geq 0$ with $\kappa(j) \geq 0$, we let $\mathcal{MH}(g, \kappa, j)$ to be the *moduli space of*

marked translation surfaces (S, Σ, x_0, X) with genus g , $\#\{x \in \Sigma : l_x = i\} = \kappa(i)$ and $l_{x_0} = j$. Thus two surfaces (S, Σ, x_0, X) and (S', Σ', x'_0, X') are equivalent if there exists a homeomorphism $\phi : (S, \Sigma, x_0, X) \rightarrow (S', \Sigma', x'_0, X')$ preserving the translation structure, the marked point and the given preferred separatrix.

An alternative way to view $\mathcal{MH}(g, \kappa, j)$ is as follows. Given a fixed surface S , with finite singular set Σ , a multiplicity vector l satisfying $\sum l_i = 2g - 2$, a fixed point $x_0 \in \Sigma$ and some horizontal separatrix X starting from x_0 going to east, consider the space $\mathcal{TH}(S, \Sigma, x_0, X)$ of all marked translation surfaces modulo the following equivalence relation: two surfaces (S, Σ, x_0, X) and (S', Σ', x'_0, X') are equivalent if there exists a homeomorphism $\phi : (S, \Sigma, x_0, X) \rightarrow (S', \Sigma', x'_0, X')$ isotopic to the identity relatively to Σ , which preserves the translation structure. The space $\mathcal{MH}(g, \kappa, j)$ is recovered in this way by taking the quotient by an appropriate modular group, i.e., the group of orientation preserving diffeomorphisms of S , fixing Σ modulo those isotopic to the identity. The advantage of seeing the moduli space as a quotient like this, is that it inherits a structure of complex affine orbifold, since charts in $\mathcal{TH}(S, \Sigma, x_0, X)$ are complex affine. Indeed, given a path $\gamma \in C^0([0, T], S)$, we can lift it in \mathbb{C} . Since we have the translation structure, we can do this lifting everywhere. Thus, we can obtain a linear map $H_1(S, \Sigma; \mathbb{Z}) \rightarrow \mathbb{C}$, which we can see as an element of the relative cohomology group $H^1(S, \Sigma; \mathbb{C})$. This map is a local homeomorphism, thus it is a local coordinate chart. So $\mathcal{TH}(S, \Sigma, x_0, X)$ has a complex affine manifold structure.

The Lebesgue measure on space $H^1(S, \Sigma; \mathbb{C})$ (normalized so that the integer lattice $H^1(S, \Sigma; \mathbb{Z}) \oplus iH^1(S, \Sigma; \mathbb{Z})$ has covolume one) can be pulled back via these local coordinates, and we obtain a smooth measure on the space $\mathcal{TH}(S, \Sigma, x_0, X)$. In charts, the modular group acts (discretely and properly discontinuously) by complex affine maps preserving the integer lattice (and hence the Lebesgue measure). This exhibits $\mathcal{MH}(g, \kappa, j)$ as a complex affine orbifold, with a canonical absolutely continuous measure $\nu_{\mathcal{MH}}$. Denoting $\mathcal{MH}^{(1)}$ the moduli space of marked translation surfaces with area one, we obtain the respective induced measure $\nu_{\mathcal{MH}}^{(1)}$.

The moduli spaces \mathcal{MH} are also called strata and they can be disconnected. Kontsevich and Zorich ([KZ]) classified these connected components and they proved that they are at most three, for each strata.

6.4.2 Moduli space of marked translation surfaces with involution

Given $g \geq 1$, functions $\tilde{\kappa}, \eta : \mathbb{N} \rightarrow \mathbb{N}$ with finite support and $\sum_{i \geq 0} i\tilde{\kappa}(i) = 4\tilde{g} - 4$, and an integer $\tilde{j} \geq 0$ with $\tilde{\kappa}(\tilde{j}) \geq 0$, we let $\mathcal{MH}\mathcal{I}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j})$ to be the *moduli space of marked translation surfaces with involution* $(\tilde{S}, \tilde{\Sigma}, I, \tilde{x}_0, \tilde{X})$ with genus \tilde{g} , an involution $I : \tilde{S} \rightarrow \tilde{S}$ preserving $\tilde{\Sigma}$ and whose fixed points are contained in $\tilde{\Sigma}$, $\#\{x \in \tilde{\Sigma} : l_x = i\} = \tilde{\kappa}(i)$, $l_{x_0} = \tilde{j}$, and $\#\{x \in \tilde{\Sigma} : l_x = 2i \text{ and } I(x) = x\} = \eta(2i)$. Thus two surfaces $(\tilde{S}, \tilde{\Sigma}, I, \tilde{x}_0, \tilde{X})$ and $(\tilde{S}', \tilde{\Sigma}', I', \tilde{x}'_0, \tilde{X}')$ are equivalent if there exists a homeomorphism $\phi : (\tilde{S}, \tilde{\Sigma}, I, \tilde{x}_0, \tilde{X}) \rightarrow (\tilde{S}', \tilde{\Sigma}', I', \tilde{x}'_0, \tilde{X}')$ preserving the translation structure and preserving the involution, in the sense that $\phi \circ I = I' \circ \phi$. The marked point and the chosen separatrix are also preserved.

Analogous to the previous case, we will consider the moduli space of marked translation surfaces with involution $\mathcal{MH}\mathcal{I}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j})$ as a larger space, which has an affine complex manifold structure, quotiented by a modular group. Consider a fixed translation surface \tilde{S} , an associated involution $I : \tilde{S} \rightarrow \tilde{S}$, a finite sin-

gular set $\tilde{\Sigma}$ invariant by I , with a multiplicity vector \tilde{l} satisfying $\sum l_i = 4\tilde{g} - 4$, together with some fixed $\tilde{x}_0 \in \tilde{\Sigma}$ and one fixed horizontal separatrix \tilde{X} emanating from \tilde{x}_0 . Let $\mathcal{THI}(\tilde{S}, \tilde{\Sigma}, I, \tilde{x}_0, \tilde{X})$ be the set of $(\tilde{S}, \tilde{\Sigma}, I, \tilde{x}_0, \tilde{X})$ modulo homeomorphism ϕ isotopic to the identity relatively to Σ , which preserves the translation structure with involution, in particular $\phi \circ I = I \circ \phi$.

Let $I : \tilde{S} \rightarrow \tilde{S}$ be the involution as defined before. Consider the induced involution $I^* : H^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C}) \rightarrow H^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$ on the relative cohomology group. We can decompose the cohomology group into a direct sum $H^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C}) = H_+^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C}) \oplus H_-^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$, where $H_+^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$ and $H_-^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$ are, respectively, the invariant and the anti-invariant subspaces of I^* . Observe that, since the involution changes the orientation of regular charts, the element of $H^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$ which represents \tilde{S} is in $H_-^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$ and a small neighborhood of it gives a local coordinate chart of a neighborhood of \tilde{S} in $\mathcal{THI}(\tilde{S}, \tilde{\Sigma}, I, \tilde{x}_0, \tilde{X})$. Notice that we are considering that the translation surface with involution \tilde{S} can have some regular points in $\tilde{\Sigma}$. But if we consider the set $\hat{\Sigma} \subset \tilde{\Sigma}$ such that $\hat{\Sigma}$ has no regular points, we have that the canonical homomorphism $H_-^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C}) \rightarrow H_-^1(\tilde{S}, \hat{\Sigma}; \mathbb{C})$ induced by the inclusion $\hat{\Sigma} \hookrightarrow \tilde{\Sigma}$ is an isomorphism. So we can choose $\hat{\Sigma}$ or $\tilde{\Sigma}$ to define the coordinate charts (see [MZ]). Since the modular group acts discretely and properly discontinuously, we obtain a complex affine structure of orbifold to $\mathcal{MHI}(\tilde{g}, \tilde{\kappa}, \tilde{\eta}, \tilde{j})$. The space $H^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$ has a smooth standard measure which we can transport to $\mathcal{THI}(\tilde{S}, \tilde{\Sigma}, I, \tilde{x}_0, \tilde{X})$ obtaining a smooth measure in this space. Hence, the space \mathcal{MHI} inherits a smooth measure $\mu_{\mathcal{MHI}}$ and the moduli space of surfaces with area one $\mathcal{MHI}^{(1)}$ inherits the induced measure $\mu_{\mathcal{MHI}}^{(1)}$.

6.4.3 Moduli space of marked half-translation surfaces

Given $g \geq 0$, a function $\kappa : \mathbb{N} \cup \{-1\} \rightarrow \mathbb{N}$ with finite support and $\sum_{i \geq -1} i\kappa(i) = 4g - 4$, $\varepsilon \in \{-1, 1\}$, and an integer $j \geq -1$ with $\kappa(j) > 0$, we let $\mathcal{MHQ}(g, \kappa, \varepsilon, j)$ to be the *moduli space of marked half-translation surfaces* (S, Σ, x_0, X) with genus g , $\#\{x \in \Sigma : l_x = i\} = \kappa(i)$ and $l_{x_0} = j$. Two surfaces (S, Σ, x_0, X) and (S', Σ', x'_0, X') are equivalent if there exists a homeomorphism $\phi : (S, \Sigma, x_0, X) \rightarrow (S', \Sigma', x'_0, X')$ preserving the half-translation structure, the marked point and the fixed separatrix.

If $\varepsilon = 1$, the half-translation structure is compatible with two translation structures (corresponding to both possible orientations) and there is a natural map $\mathcal{MH}(g, \kappa, j) \rightarrow \mathcal{MHQ}(g, \kappa, 1, j)$ which forgets the polarization. This map is a ramified double cover of orbifolds.

Now given a half-translation structure which is not compatible with a translation structure, we will associate a translation structure using the (ramified) double covering construction. Define $\tilde{\kappa} : \mathbb{N} \cup \{-1\} \rightarrow \mathbb{N}$ by $\tilde{\kappa}(2i - 1) = 0$, $\tilde{\kappa}(4i) = 2\kappa(4i) + \kappa(2i - 1)$, $\tilde{\kappa}(4i + 2) = 2\kappa(4i + 2)$. Let $\tilde{g} = 4 + \sum_{i \geq -1} i\tilde{\kappa}(i) = 2g - 1 + \frac{1}{2} \sum_{i \geq 0} \kappa(2i - 1)$, $\tilde{j} = j$ if j is even or $\tilde{j} = 2j + 2$ if j is odd. Thus, we obtain a canonical injective map $\mathcal{MHQ}(g, \kappa, -1, j) \rightarrow \mathcal{MH}(\tilde{g}, \tilde{\kappa}, \tilde{\eta}, \tilde{j})$. In fact, by construction, the image of this map is $\mathcal{MHI}(\tilde{g}, \tilde{\kappa}, \tilde{\eta}, \tilde{j})$, where the map $\tilde{\eta} : \mathbb{N} \cup \{-1\} \rightarrow \mathbb{N}$ is such that $\tilde{\eta}(2i) = \#\{x \in \tilde{\Sigma} : l_x = 2i \text{ and } I(x) = x\}$ and $\tilde{\eta}(2i + 1) = 0$.

We also can define the quotient map $\mathcal{MHI}(\tilde{g}, \tilde{\kappa}, \tilde{\eta}, \tilde{j}) \rightarrow \mathcal{MHQ}(g, \kappa, -1, j)$, such that, to each structure $(\tilde{S}, \tilde{\Sigma}, I, \tilde{x}_0, \tilde{X})$ associates the quotient structure

$(S, \Sigma, x_0, X) = (\tilde{S}/I, \tilde{\Sigma}/I, \tilde{x}_0/I, \tilde{X}/I)$. Notice that this map is well-defined and it is injective, since \tilde{S} is connected. Thus, we have a bijection between marked half-translation surfaces which are not translation surfaces and connected marked translation surfaces with involution.

As in the case of translation surfaces, the moduli space of marked half-translation surfaces is called strata. Laneeau classified the connected components of each strata, which are at most two ([L1], [L3]).

6.5 Teichmüller Flow

The group $\mathrm{SL}(2, \mathbb{R})$ acts on \mathcal{MHI} (or more generally, on the space of marked translation surfaces with involution) by postcomposition in the charts. This action preserves the hypersurface $\mathcal{MHI}^{(1)}$ and measures $\mu_{\mathcal{MHI}}$ on \mathcal{MHI} and $\mu_{\mathcal{MHI}}^{(1)}$ on $\mathcal{MHI}^{(1)}$.

The *Teichmüller Flow* is the particular action of the diagonal subgroup $\mathcal{TF}_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ and it is measure-preserving.

Theorem 6.1 (Masur, Veech). *The Teichmüller flow is mixing on each connected component of each stratum of the moduli space $\mathcal{MHI}^{(1)}$, with respect to the finite equivalent Lebesgue measure, $\mu^{(1)}$.*

The Theorem A, in the setting of translation surfaces was proved by Avila, Gouëzel and Yoccoz [AGY]. So, we will restrict the proof just to the case of half-translation surfaces which are not translation surfaces. Thus, we can prove it, just considering marked translation surfaces with involution. In this setting, the Theorem A is equivalent to:

Theorem 6.2. *The Teichmüller flow is exponential mixing on each connected component of the moduli space $\mathcal{MHI}^{(1)}$ with respect to the measure $\mu_{\mathcal{MHI}}^{(1)}$ for observables in the Ratner class.*

7 From the model to the Teichmüller flow

7.1 Zippered rectangles construction

Consider an irreducible permutation $\pi \in \mathfrak{A}$ and length data $\lambda \in \mathcal{S}_\pi$, $\tau \in \Theta_\pi$ and $h \in \mathbb{R}_+^{\mathcal{A}}$ defined by $h = -\Omega(\pi) \cdot \tau$. Notice that $h = (h_\alpha)_{\alpha \in \mathcal{A}}$ is such that $h_\alpha > 0$ for all $\alpha \in \mathcal{A}$. Let $\alpha(l)$ and $\alpha(r)$ be the leftmost and the rightmost letters of π , i.e., $\pi(\alpha(l)) = 1 = \bar{\pi}(i(\alpha(r)))$ and $\pi(\alpha(r)) = 2d + 1 = \bar{\pi}(i(\alpha(l)))$.

Define the sets:

$$\mathcal{B}_\pi(\alpha) = \begin{cases} \{\beta \in \mathcal{A} : \pi(\alpha) < \pi(\beta) < \pi(*)\} & \text{if } 1 \leq \pi(\alpha) < \pi(*), \\ \{\beta \in \mathcal{A} : \pi(*) < \pi(\beta) < \pi(\alpha)\} & \text{if } \pi(*) < \pi(\alpha) \leq 2d + 1 \end{cases}$$

$$\mathcal{B}'_\pi(\alpha) = \begin{cases} \{\beta \in \mathcal{A} : \pi(\alpha) \leq \pi(\beta) < \pi(*)\} & \text{if } 1 \leq \pi(\alpha) < \pi(*), \\ \{\beta \in \mathcal{A} : \pi(*) < \pi(\beta) \leq \pi(\alpha)\} & \text{if } \pi(*) < \pi(\alpha) \leq 2d + 1 \end{cases}$$

$$\mathcal{B}_{\bar{\pi}}(\alpha) = \begin{cases} \{\beta \in \mathcal{A} : \bar{\pi}(\alpha) < \bar{\pi}(\beta) < \bar{\pi}(*)\} & \text{if } 1 \leq \bar{\pi}(\alpha) < \bar{\pi}(*), \\ \{\beta \in \mathcal{A} : \bar{\pi}(*) < \bar{\pi}(\beta) < \bar{\pi}(\alpha)\} & \text{if } \bar{\pi}(*) < \bar{\pi}(\alpha) \leq 2d + 1 \end{cases}$$

$$\mathcal{B}'_{\bar{\pi}}(\alpha) = \begin{cases} \{\beta \in \mathcal{A} : \bar{\pi}(\alpha) \leq \bar{\pi}(\beta) < \bar{\pi}(\ast)\} & \text{if } 1 \leq \bar{\pi}(\alpha) < \bar{\pi}(\ast), \\ \{\beta \in \mathcal{A} : \bar{\pi}(\ast) < \bar{\pi}(\beta) \leq \bar{\pi}(\alpha)\} & \text{if } \bar{\pi}(\ast) < \bar{\pi}(\alpha) \leq 2d+1 \end{cases}$$

For each $\alpha \in \mathcal{A}$ consider the rectangles with horizontal sides λ_α and vertical sides h_α defined by:

$$\begin{aligned} R_\alpha^{t,r} &= \left(\sum_{\beta \in \mathcal{B}_\pi(\alpha)} \lambda_{\underline{\beta}}, \sum_{\beta \in \mathcal{B}'_\pi(\alpha)} \lambda_{\underline{\beta}} \right) \times [0, h_\alpha], \\ R_\alpha^{t,l} &= \left(- \sum_{\beta \in \mathcal{B}'_\pi(\alpha)} \lambda_{\underline{\beta}}, - \sum_{\beta \in \mathcal{B}_\pi(\alpha)} \lambda_{\underline{\beta}} \right) \times [0, h_\alpha], \\ R_\alpha^{b,r} &= \left(\sum_{\beta \in \mathcal{B}_{\bar{\pi}}(\alpha)} \lambda_{\underline{\beta}}, \sum_{\beta \in \mathcal{B}'_{\bar{\pi}}(\alpha)} \lambda_{\underline{\beta}} \right) \times [-h_\alpha, 0], \\ R_\alpha^{b,l} &= \left(- \sum_{\beta \in \mathcal{B}'_{\bar{\pi}}(\alpha)} \lambda_{\underline{\beta}}, - \sum_{\beta \in \mathcal{B}_{\bar{\pi}}(\alpha)} \lambda_{\underline{\beta}} \right) \times [-h_\alpha, 0]. \end{aligned}$$

If $\alpha \notin \{\alpha(l), \alpha(r)\}$, also consider the vertical segments:

$$\begin{aligned} S_\alpha^{t,r} &= \left\{ \sum_{\beta \in \mathcal{B}'_\pi(\alpha)} \lambda_{\underline{\beta}} \right\} \times \left[0, \sum_{\beta \in \mathcal{B}'_\pi(\alpha)} \tau_{\underline{\beta}} \right] \\ S_\alpha^{t,l} &= \left\{ - \sum_{\beta \in \mathcal{B}'_\pi(\alpha)} \lambda_{\underline{\beta}} \right\} \times \left[0, - \sum_{\beta \in \mathcal{B}'_\pi(\alpha)} \tau_{\underline{\beta}} \right] \\ S_\alpha^{b,r} &= \left\{ \sum_{\beta \in \mathcal{B}'_{\bar{\pi}}(\alpha)} \lambda_{\underline{\beta}} \right\} \times \left[\sum_{\beta \in \mathcal{B}'_{\bar{\pi}}(\alpha)} \tau_{\underline{\beta}}, 0 \right] \\ S_\alpha^{b,l} &= \left\{ - \sum_{\beta \in \mathcal{B}'_{\bar{\pi}}(\alpha)} \lambda_{\underline{\beta}} \right\} \times \left[- \sum_{\beta \in \mathcal{B}'_{\bar{\pi}}(\alpha)} \tau_{\underline{\beta}}, 0 \right] \end{aligned}$$

If $L_\pi = \sum_{\pi(\ast) < \pi(\beta) \leq \pi(\alpha(r))} \tau_{\underline{\beta}} > 0$ we define

$$\begin{aligned} S_{\alpha(r)}^{t,r} &= -S_{i(\alpha(r))}^{b,l} = \left\{ \sum_{\pi(\ast) < \pi(\beta) \leq \pi(\alpha(r))} \lambda_{\underline{\beta}} \right\} \times [0, L_\pi] \\ S_{\alpha(l)}^{t,l} &= S_{i(\alpha(l))}^{b,r} = \emptyset \end{aligned}$$

If $L_\pi < 0$ we define

$$\begin{aligned} S_{\alpha(l)}^{t,l} &= -S_{i(\alpha(l))}^{b,r} = \left\{ \sum_{\pi(\alpha(l)) \leq \pi(\beta) < \pi(\ast)} \lambda_{\underline{\beta}} \right\} \times [0, L_\pi] \\ S_{\alpha(r)}^{b,r} &= S_{i(\alpha(r))}^{t,l} = \emptyset \end{aligned}$$

Otherwise, if $L_\pi = 0$ we take

$$S_{\alpha(l)}^{t,l} = S_{i(\alpha(r))}^{b,l} = \left\{ - \sum_{\pi(\alpha(l)) \leq \pi(\beta) < \pi(*)} \lambda_{\underline{\beta}} \right\} \times \{0\}$$

$$S_{\alpha(r)}^{t,r} = S_{i(\alpha(l))}^{b,r} = \left\{ \sum_{\pi(*) < \pi(\beta) \leq \pi(\alpha(r))} \lambda_{\underline{\beta}} \right\} \times \{0\}$$

Notice that, for each $\alpha \in \mathcal{A}$, the labels l and r in $X_\alpha^{\epsilon,l}$ and $X_\alpha^{\epsilon,r}$, where $\epsilon \in \{t, b\}$ and $X \in \{R, S\}$, are just to make clear when $\pi(\alpha) < \pi(*)$ or $\pi(*) < \pi(\alpha)$. When it does not lead to confusion, we will omit l and r .

Example 7.1. The next figure represents a zippered rectangle associated to $\pi = (D \mathcal{A} C C * A V B)$.

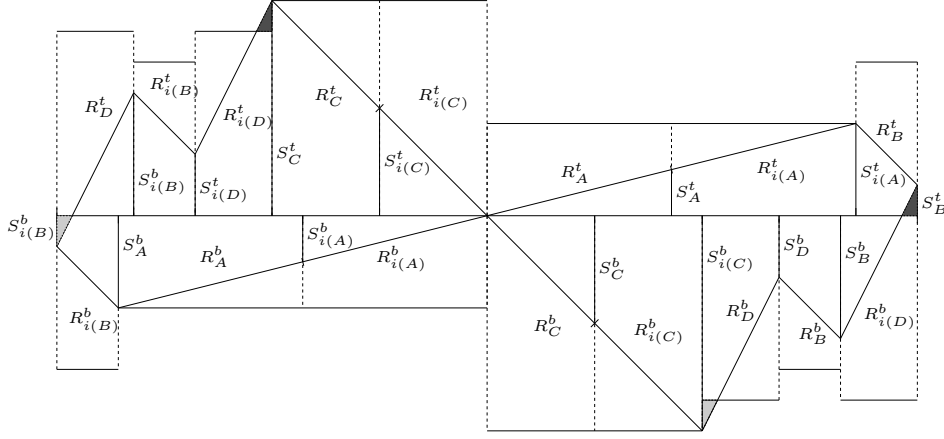


Figure 5: Zippered rectangle

Define the set

$$R_{(\lambda, \pi, \tau)} = \bigcup_{\alpha \in \mathcal{A}} \bigcup_{\epsilon \in \{l, r\}} (R_\alpha^\epsilon \cup S_\alpha^\epsilon)$$

We will identify, by translation, the rectangle R_α^t with R_α^b for all $\alpha \in \mathcal{A}$.

If $L_\pi > 0$ we identify $S_{\alpha(r)}^t$ with the vertical segment S_1 of length L_π at the bottom of the right side of the rectangle $R_{\alpha(r)}^b$ if $\alpha(r)$ is the winner of π or at the top of the right side of the rectangle $R_{i(\alpha(l))}^t$ if $\alpha(r)$ is the loser of π . Symmetrically, we identify $S_{i(\alpha(r))}^b$ with $-S_1$.

If $L_\pi < 0$ we identify $S_{i(\alpha(l))}^b$ with the vertical segment S_2 of length $-L_\pi$ in the bottom of the right side of the rectangle $R_{\alpha(r)}^b$ if $\alpha(r)$ is the winner of π or in the top of the right side of the rectangle $R_{i(\alpha(l))}^t$ if $\alpha(r)$ is the loser of π . Symmetrically, we identify $S_{\alpha(l)}^t$ with $-S_2$.

Let $\tilde{S}^*(\lambda, \pi, \tau)$ be the topological space obtained from $R_{(\lambda, \pi, \tau)}$ by these identifications. Thus, $\tilde{S}^*(\lambda, \pi, \tau)$ inherits from $\mathbb{R}^2 = \mathbb{C}$ the structure of a Riemann surface and also a holomorphic 1-form ω (given by dz).

For each $\alpha \in \mathcal{A}$ recall $\zeta_\alpha = \lambda_\alpha + i\tau_\alpha$. We call *vertices* the extreme points in the top of segments S_α^t and the extremes in the bottom of segments S_α^b , for all $\alpha \in \mathcal{A}$. So, the vertices are points with following coordinates:

$$\xi_\alpha^t = \begin{cases} \sum_{\pi(\alpha) \leq \pi(\beta) < \pi(*)} -\zeta_\beta & \text{if } \pi(\alpha) < \pi(*) \\ \sum_{\pi(*) < \pi(\beta) \leq \pi(\alpha)} \zeta_\beta & \text{if } \pi(*) < \pi(\alpha) \end{cases}$$

$$\xi_\alpha^b = \begin{cases} \sum_{\bar{\pi}(\alpha) \leq \bar{\pi}(\beta) < \bar{\pi}(*)} -\zeta_\beta & \text{if } \bar{\pi}(\alpha) < \bar{\pi}(*) \\ \sum_{\bar{\pi}(*) < \bar{\pi}(\beta) \leq \bar{\pi}(\alpha)} \zeta_\beta & \text{if } \bar{\pi}(*) < \bar{\pi}(\alpha) \end{cases}$$

Now we will define a relation to identify vertices between them. Define the set of all pairs (α, Y) with $\alpha \in \mathcal{A} \cup \{*\}$ and $Y \in \{L, R\}$. Consider the following identification:

$$\begin{aligned} (\pi(\pi^{-1}(*)) + 1, L) &\sim (*, R) \sim (\bar{\pi}(\bar{\pi}^{-1}(*)) + 1, L) \\ (\pi(\pi^{-1}(*)) - 1, R) &\sim (*, L) \sim (\bar{\pi}(\bar{\pi}^{-1}(*)) - 1, R) \\ (\alpha(r), R) &\sim (i(\alpha(l)), R) \\ (\alpha(l), L) &\sim (i(\alpha(r)), L) \end{aligned}$$

We say that these pairs are *irregular* and all other pairs we call *regular*. We also identify:

$$\begin{aligned} (\alpha(r), R) &\sim (\beta, L) \quad \text{if } \pi(\alpha) + 1 = \pi(\beta) \\ (\alpha(r), R) &\sim (\beta, L) \quad \text{if } \bar{\pi}(\alpha) + 1 = \bar{\pi}(\beta) \end{aligned}$$

We can extend \sim to an equivalence relation in the set of pairs (α, Y) . This equivalence relation describes how half-planes are identified when one winds around an end of $\tilde{S}^*(\lambda, \pi, \tau)$. Let $\tilde{\Sigma}$ be the set of equivalence classes relative to the relation \sim . Thus to each $c \in \tilde{\Sigma}$ we have one, and only one, end v_c of $\tilde{S} = \tilde{S}^*(\lambda, \pi, \tau)$. When it does not lead to confusion we will use \tilde{S} to mean $\tilde{S}(\lambda, \pi, \tau)$. From the local structure around v_c , the compactification

$$\tilde{S}(\lambda, \pi, \tau) = \tilde{S}^*(\lambda, \pi, \tau) \cup (\cup_{\tilde{\Sigma}} \{v_c\})$$

is a compact Riemann surface with marked points $\{v_c\}$. The 1-form ω extends to a holomorphic 1-form on $\tilde{S}(\lambda, \pi, \tau)$ such that at the points v_c we have marked zeroes of angle $2k_c\pi$ where $2k_c$ is the cardinality of the equivalence class of c .

Given $(x, 0)$ on the bottom side of the rectangle R_α^t , we can transport this point vertically and when we reach the top side, which is the point (x, h_α) we identify it with the point $(x + \omega_\alpha, 0)$, where $\omega = \Omega(\pi) \cdot \lambda$, in the top side of R_α^b . So, we have the vertical flow well-defined almost everywhere (except in the points which reach singularities in finite time). It is clear that the return time of points in the rectangle R_α^t is equal to $h_{\underline{\alpha}}$ and the area of the surface $\tilde{S}(\lambda, \pi, \tau)$ is $A(\lambda, \pi, \tau) = -2\langle \lambda, \Omega(\pi) \cdot \tau \rangle$.

When we constructed the surface \tilde{S} , we have an implicit relation between the horizontal coordinates λ_α and $\lambda_{i(\alpha)}$ and the vertical coordinates τ_α and $\tau_{i(\alpha)}$. Indeed, we have an involution $I : \tilde{S} \rightarrow \tilde{S}$, with a fixed point at the origin,

defined as follows. Given any point $x \in \tilde{S}$ there exists $\alpha \in \mathcal{A}$ such that $x \in R_\alpha^t$ or $x \in S_\alpha^t \cup S_\alpha^b$. Thus $-x \in R_{i(\alpha)}^b$ or $-x \in S_{i(\alpha)}^b \cup S_{i(\alpha)}^t$, respectively. So $I(x)$ is identified with $-x$.

Let $S(\lambda, \pi, \tau)$ be the surface $\tilde{S}(\lambda, \pi, \tau)$ quotiented by the involution I and let Σ to be the set $\tilde{\Sigma}$ quotiented by the involution I . We can see that this identification by involution implies that, for each $\alpha \in \mathcal{A}$, the rectangle R_α^t is identified with the rectangle $R_{i(\alpha)}^t$ by a translation composed with a rotation of 180 degrees. So, the top side (resp. the bottom side) of the rectangle R_α^t is identified with the bottom side (resp. the top side) of the rectangle $R_{i(\alpha)}^t$.

7.2 Coordinates

Let $(\tilde{S}, \tilde{\Sigma}, I, \tilde{x}_0, \tilde{X})$ be a marked translation surface with an involution I . The marked separatrix \tilde{X} starts at \tilde{x}_0 and it goes to east. A segment σ adjacent to \tilde{x}_0 contained in \tilde{X} is called *admissible* if the vertical geodesic Y passing through the right endpoint \tilde{z} of σ meets a singularity in the positive or in the negative direction before returning to $\sigma \cup I(\sigma)$. Symmetrically, if σ is an admissible segment then $I(\sigma)$ starting at $I(\tilde{x}_0)$ going to west and ending at $I(\tilde{z})$ (which has a vertical geodesic meeting a singularity in the negative or in the positive direction before return to $\sigma \cup I(\sigma)$), also is an admissible segment if we consider the marked separatrix $I(\tilde{X})$ instead consider X .

We call a separatrix *incoming* if its natural orientation points towards the associated singularity and we call it *outgoing* otherwise. Let σ^+ be the set of points of the first intersection of incoming vertical separatrices with $\sigma \cup I(\sigma)$ and σ^- be the set of points of the first intersection of outgoing vertical separatrices with $\sigma \cup I(\sigma)$. Notice that \tilde{x}_0 and $I(\tilde{x}_0)$ are in both sets σ^+ and σ^- . If Y is incoming (resp. outgoing) we extend it to the past (resp. future) until intersect $\sigma \cup I(\sigma)$ again and the second intersection point is an element of σ^+ (resp. σ^-). Thus \tilde{z} is an element of both σ^+ and σ^- . By the same argument applied to $I(Y)$ we conclude that $I(\tilde{z})$ also is an element of both σ^+ and σ^- .

Notice that $p \in \sigma^-$ if and only if $I(p) \in \sigma^+$, for all $p \in \sigma^-$. Thus we will consider just the set σ^+ which determines the set σ^- by involution.

Let $|\lambda|$ be the length of σ and of $I(\sigma)$. Let $\phi_r : [0, |\lambda|] \rightarrow \tilde{S}$ and $\phi_l : [-|\lambda|, 0] \rightarrow \tilde{S}$ be the parametrizations of σ and $I(\sigma)$, respectively, by arc-length with $\phi_r(0) = \tilde{x}_0$ and $\phi_l(0) = I(\tilde{x}_0)$.

We can write:

$$\sigma^+ = \{I(\tilde{z}) = p_{-l}^+ > \dots > p_{-1}^+ > p_{0-}^+ = I(\tilde{x}_0)\} \cup \{\tilde{x}_0 = p_{0+}^+ < p_1^+ < \dots < p_r^+ = \tilde{z}\}.$$

where $<$ and $>$ refer to the natural orientation on σ and $I(\sigma)$, respectively.

Therefore, we have numbers

$$-|\lambda| = a_{-l}^+ < \dots < a_{-1}^+ < a_{0-}^+ = 0 = a_{0+}^+ < a_1^+ < \dots < a_r^+ = |\lambda|$$

such that $\phi_r(p_j^+) = a_j^+$, for all $j \in \{-l, \dots, -1, 1, \dots, r\}$, $\phi_r(p_{0-}^+) = a_{0-}^+ = 0$ and $\phi_r(p_{0+}^+) = a_{0+}^+ = 0$.

Let $a_0^+ = a_{0+}^+ = a_{0-}^+$. Define $\lambda_j = |I_j| = a_{j+1}^+ - a_j^+$ for $-l \leq j \leq r-1$. Let τ_j be the length of the vertical segment from the horizontal section to the singularity corresponding to the point p_j^+ . Notice that $\tau_0 = 0$.

It is possible to verify that the first return map to the cross-section $\sigma \cup I(\sigma)$ is well-defined except at the points p_j^+ . Moreover the first return time is constant

on each open interval (a_j^+, a_{j+1}^+) . So, we can consider the interval exchange transformation with involution π associated to the cross-section $\sigma \cup I(\sigma)$ where the points defined before are the points of discontinuity.

Let h_j be the first return time of the points in the interval (a_{j-1}, a_j) . We can define the zippered rectangle which represents $(\tilde{S}, \tilde{\Sigma}, I, \tilde{x}_0, \tilde{X})$ by:

$$ZR(\lambda, \pi, \tau, h) = \cup_j (a_{j-1}, a_j) \times [0, h_j].$$

Lemma 7.2. *If two admissible segments σ and $\tilde{\sigma}$ with the same left extreme point \tilde{x}_0 are such that $\tilde{\sigma} \subset \sigma$, then the corresponding zippered rectangles (λ, π, τ, h) and $(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}, \tilde{h})$ satisfy:*

$$\exists n \in \mathbb{N} \text{ such that } (\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}) = \widehat{Q}^n(\lambda, \pi, \tau)$$

Proof. Let σ and $\tilde{\sigma}$ be admissible segments and let the respective zippered rectangles representations $ZR(\lambda, \pi, \tau, h)$ and $ZR(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}, \tilde{h})$.

Consider a sequence of maximal admissible segments σ^{i+1} strictly contained in σ^i such that $\sigma^1 = \sigma$. Let z_i be the right endpoint of σ^i . The right endpoint z_2 of σ^2 corresponds to a discontinuity point of the first return map of the vertical flow to the section σ^1 and there is no other discontinuity point between z_2 and z_1 . By maximality, we conclude that, up to relabeling, $\widehat{Q}(\lambda, \pi, \tau)$ is the representation of such first return map. We follow this process until obtain $\sigma^n = \tilde{\sigma}$ for some $n \in \mathbb{N}$. For such n we have $\widehat{Q}^n(\lambda, \pi, \tau) = (\tilde{\lambda}, \tilde{\pi}, \tilde{\tau})$. \square

Corollary 7.3. *Let $(\tilde{S}, \tilde{\Sigma}, I, \tilde{x}_0, \tilde{X})$ be a marked translation surface with involution and let $ZR(\lambda, \pi, \tau, h)$ and $ZR(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}, \tilde{h})$ be two zippered rectangle representations of the surface. Then there exists $n \in \mathbb{Z}$ such that $(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}) = \widehat{Q}^n(\lambda, \pi, \tau)$.*

Proof. Let σ and $\tilde{\sigma}$ be admissible segments of $ZR(\lambda, \pi, \tau, h)$ and $ZR(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}, \tilde{h})$, respectively. By definition, the initial points of σ and $\tilde{\sigma}$ are the same \tilde{x}_0 . Suppose, without loss of generality, that $\tilde{\sigma} \subset \sigma$.

By the previous lemma there exists $n \in \mathbb{N}$ such that $(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}) = \widehat{Q}^n(\lambda, \pi, \tau)$. Thus, the result follows. \square

Given a marked translation surface with involution $(\tilde{S}, \tilde{\Sigma}, I, \tilde{x}, \tilde{X})$ with zippered rectangle representation $ZR(\lambda, \pi, \tau, h)$, we can cut and paste it appropriately until we obtain a surface $(\tilde{S}', \tilde{\Sigma}', I', \tilde{x}', \tilde{X}')$ which representation in zippered rectangles is an iterated by Rauzy induction with involution of the first marked translation surface with involution. Since these operations preserve the relation between parallel sides, then \tilde{S} and \tilde{S}' are isomorphic and the marked separatrix is mapped to one another. Moreover if we have a marked translation surface with involution $(\tilde{S}_1, \tilde{\Sigma}_1, I_1, \tilde{x}_1, \tilde{X}_1)$ which is near from $(\tilde{S}, \tilde{\Sigma}, I, \tilde{x}, \tilde{X})$, by the continuity of the marked separatrix and of the singularities, we will obtain a zippered rectangle construction $ZR(\lambda_1, \pi_1, \tau_1, h_1)$ near, up to relabel, from $ZR(\lambda, \pi, \tau, h)$. So, the zippered rectangle construction, gives a system of local coordinates in each stratum of the moduli space.

Using the zippered rectangles construction, we obtain a finite covering ZR , of a stratum $\mathcal{MH}\mathcal{I}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j})$ of the moduli space of marked translation surfaces with involution. Under the condition $\langle \lambda, h \rangle = 1$, we get the space of zippered rectangles of area one covering the space $\mathcal{MH}\mathcal{I}^{(1)}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j})$. We have a bijection between Rauzy classes with involution and a connected component of a stratum

of the moduli space of translation surfaces with involution (see [BL]). Thus we have a well-defined map $\text{proj} : \mathcal{U}_{\mathfrak{R}} \rightarrow \mathcal{C}$, where $\mathcal{C} = \mathcal{C}(\mathfrak{R})$ is a connected component of $\mathcal{MHI}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j})$ and $\text{proj} \circ \widehat{Q} = \text{proj}$. The fibers of this map are almost everywhere finite (with constant cardinality). The projection of the standard Lebesgue measure on $\mathcal{U}_{\mathfrak{R}}$ is (up to scaling) the standard volume form on \mathcal{C} .

The subset $\mathcal{U}_{\mathfrak{R}}^{(1)} = \text{proj}^{-1}(\mathcal{C}^{(1)})$ of surfaces with area one is invariant by the Veech flow. So, the restriction $\mathcal{TV}_t(x) : \mathcal{U}_{\mathfrak{R}}^{(1)} \rightarrow \mathcal{U}_{\mathfrak{R}}^{(1)}$ leaves invariant the volume form that projects, up to scaling, to the invariant volume form on $\mathcal{C}^{(1)}$. It was proved by Veech that this volume form is finite using the lift measure on $\mathcal{U}_{\mathfrak{R}}^{(1)}$.

7.2.1 Homology and cohomology

For each $\alpha \in \mathcal{A}$ consider the curve c_α which is a path in $R_{(\lambda, \pi, \tau)}$ joining $\xi_\alpha^t - \zeta_\alpha$ to ξ_α^t if $\pi(\alpha) > \pi(*)$ or joining ξ_α^t to $\xi_\alpha^t + \zeta_\alpha$ if $\pi(\alpha) < \pi(*)$. Note that $I(c_\alpha) = -c_{i(\alpha)}$.

Consider the relative homology group $H_1(\tilde{S}, \tilde{\Sigma}; \mathbb{Z})$ of the surface \tilde{S} relative to the finite set of singularities $\tilde{\Sigma}$. We have a decomposition of the relative homology group into an invariant subgroup $H_1^+(\tilde{S}, \tilde{\Sigma}; \mathbb{Z})$ and an anti-invariant subgroup $H_1^-(\tilde{S}, \tilde{\Sigma}; \mathbb{Z})$, with respect to the involution I . Following Masur and Zorich ([MZ]), we can choose a basis in $H_1^-(\tilde{S}, \tilde{\Sigma}; \mathbb{Z})$ which has dimension $d - 1$, where d is the number of classes of $\underline{\mathcal{A}}^*$. The elements of the basis will be lifts of a collection of saddle connections on S , where S is the surface \tilde{S} quotiented by involution.

Analogously, the first (de Rham) cohomology group $H^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$, is decomposed into an invariant subspace $H_+^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$ and an anti-invariant subspace $H_-^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$, under the induced involution $I^* : H^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C}) \rightarrow H^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$. Notice that $[\omega]$ is anti-invariant under the induced involution, so $[\omega] \in H_-^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$ and we also have:

$$\int_{c_\alpha} \omega = \zeta_\alpha$$

In section 6.4.2 we have observed that $H_-^1(\tilde{S}, \tilde{\Sigma}; \mathbb{C})$ yields local coordinates of an element of a stratum of the moduli space of translation surfaces with involution. So if we consider the set of $\zeta_\alpha = \lambda_\alpha + i\tau_\alpha$ such that $\lambda, \tau \in \mathcal{S}_\pi$ we obtain coordinates which describe $\tilde{S}(\lambda, \pi, \tau)$. And as we have seen in section 7.2, for any other pair (\tilde{S}', ω') in a neighborhood of (\tilde{S}, ω) we can find coordinates (λ', π', τ') of (\tilde{S}', ω') , and so we can define the vectors ζ'_α as in section 7.1. For more details in the construction of coordinates see [Ve3].

7.3 Teichmüller flow is exponential mixing

In section 2.3.2 we have seen the relation between the Teichmüller flow and the Veech flow, which is naturally identified with the first return map of the renormalization operator to the section $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)}$.

*The complex dimension of the moduli space of half-translation surfaces of genus g with σ singularities is $2g + \sigma - 2$ and $d = 2g + \sigma - 1$.

We will identify $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)} \times \mathbb{R}$ with a connected component $\mathcal{C}^{(1)}$ by the map $P : \widehat{\Upsilon}_{\mathfrak{R}}^{(1)} \times \mathbb{R} \rightarrow \mathcal{C}^{(1)}$ defined by $P(z, s) = \mathcal{TF}_s(\text{proj}(z))$, where $z = (\lambda, \pi, \tau)$ and $\text{proj} : \widehat{\Delta}_{\mathfrak{R}} \rightarrow \mathcal{C}$ is the natural projection.

Lemma 7.4. *Let $f : \mathcal{C}^{(1)} \rightarrow \mathbb{R}$ be a C^1 compactly supported function and let $\delta > 0$ be as in (27). There exists $\epsilon_0 > 0$ and $C > 0$ such that for every $t > 0$, there exists a C^1 function $f^{(t)} : \widehat{\Delta}_r \rightarrow \mathbb{R}$, such that $\|f \circ P - f^{(t)}\|_{L^2(\nu)} \leq Ce^{-\epsilon_0 t}$ and $\|f^{(t)}\|_{C^1(\widehat{\Delta}_r)} \leq Ce^{\delta t}$.*

Proof. Let $\delta_0 > 0$ be small and let $Y_t \subset \widehat{\Delta}_r$ be the union of connected components of $\widehat{\Delta}_r$ which contain points (λ, π, τ, s) with $s > \delta_0 t$. Let $f^{(t)} = 0$ in Y_t and $f^{(t)} = f \circ P$ in the complement. The estimate $\|f \circ P - f^{(t)}\|_{L^2(\nu)} \leq Ce^{-\epsilon_0 t}$ is then clear since $\|f \circ P - f^{(t)}\|_{C^0} \leq \|f\|_{C^0}$, while the support of $f \circ P - f^{(t)}$ has exponentially small ν measure (since the roof function has exponential tails).

For the other estimate, it is enough to show that if $(z, s) \in \widehat{\Delta}_r$ and $P(z, s)$ belongs to any fixed compact set $K \subset \mathcal{C}^{(1)}$ then P is locally Lipschitz near (z, s) , with constant bounded by $C(K)e^{C(K)s}$. Here we fix some arbitrary Finsler metric in $\mathcal{C}^{(1)}$ (the precise choice is irrelevant since K is compact). This result is obvious if we impose some bound on s , say $0 \leq s \leq 1$, since P is smooth. If $s_0 > 0$ is such that $s_0 < s < s_0 + 1$, notice that for (z', s') in a neighborhood of (z, s) , $P(z', s')$ is obtained from $P(z, s' - s_0)$ by applying the Teichmüller flow for time s_0 . Thus, it is enough to show that if x and $\mathcal{TF}_{s_0}(x)$ belong to some fixed compact set of $\mathcal{C}^{(1)}$ then \mathcal{TF}_{s_0} is locally Ce^{Cs_0} Lipschitz in a neighborhood of x . This is a well known estimate, for instance, we can define a Finsler metric on $\mathcal{C}^{(1)}$ such that \mathcal{TF}_{s_0} is globally Lipschitz with Lipschitz constant e^{2s_0} (see [AGY] §2.2.2 for the construction of a metric in the whole strata of squares, the Finsler metric we need here being just the restriction to the substrata). \square

Lemma 7.5. *If $f : \mathcal{C}^{(1)} \rightarrow \mathbb{R}$ is C^1 and compactly supported with $\int f d\nu_{\mathcal{C}^{(1)}} = 0$ then there exists $C > 0$, $\epsilon > 0$ such that for $t > 0$,*

$$\int f \cdot (f \circ \mathcal{TF}_t) d\nu_{\mathcal{C}^{(1)}} \leq Ce^{-\epsilon t}. \quad (28)$$

Proof. We can estimate (28) with exponentially small error by comparison with the correlations $\int f^{(t)} \cdot f^{(t)} \circ \widehat{T}_t d\nu - (\int f^{(t)} d\nu)^2$, where $f^{(t)}$ is provided by the previous lemma. Those decays exponentially by (27). \square

Finally, we are in a position to prove the main theorem:

Proof of Theorem A. Let H be the Hilbert space of $\text{SO}(2, \mathbb{R})$ invariant $L^2(\nu_{\mathcal{C}^{(1)}})$ functions with zero mean. As shown in Appendix B of [AGY], exponential decay of correlations for the Ratner class follows from the existence of a dense set of f in H such that (28) decays exponentially fast. Since compactly supported smooth functions are dense in H , the result follows. \square

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