# Multiplicative Dirac structures

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In this thesis we introduce multiplicative Dirac structures on Lie groupoids, generalizing both multiplicative Poisson bivectors (i.e., Poisson group(oid)s) and closed 2-forms (e.g., symplectic groupoids). We prove that for every source simply connected Lie groupoid $G$ with Lie algebroid $AG$, there exists a one-to-one correspondence between multiplicative Dirac structures on $G$ and Dirac structures on $AG$, which are compatible with both the linear and algebroid structures of $AG$. This extends the integration of Lie bialgebroids to Poisson groupoids carried out in [48]. In the case of multiplicative 2-forms, our approach gives a new, simpler proof of the integration of Dirac manifolds of [10].

In the special case of multiplicative Dirac structures on Lie groups, we prove that the characteristic foliation of a multiplicative Dirac structure is given by the cosets of a normal Lie subgroup, and whenever this subgroup is closed, the space of characteristic leaves inherits the structure of a Poisson-Lie group. We use Drinfeld's correspondence between Poisson-Lie groups and Lie bialgebras to describe multiplicative Dirac structures on Lie groups infinitesimally.

We also explain the connection between multiplicative Dirac structures and Mackenzie theory of double geometric structures.

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# Contents

# Chapter 1

# Introduction

The natural geometric object describing phase spaces of mechanical systems is a symplectic manifold. More precisely, a symplectic manifold is a pair $(M, \omega)$ where $M$ is a smooth manifold and $\omega \in \Omega^2(M)$ is a nondegenerate 2-form satisfying the integrability condition

$$d\omega = 0,$$

where $d : \Omega^\bullet(M) \longrightarrow \Omega^{\bullet+1}(M)$ is the de Rham differential. Due to the physical interpretation of a symplectic manifold, there are two operations of special interest, namely restriction to submanifolds and quotients by a Lie group of symmetries. On one hand, given a submanifold $i_Q : Q \longrightarrow M$ one can consider the restriction of the symplectic form $\omega$ to the submanifold $Q$, that is we consider the pull back form $\omega_Q = i_Q^*\omega$. It is obvious that $\omega_Q$ is a closed 2-form, but it may have a non trivial kernel. On the other hand, given a Lie group $G$ acting on $(M, \omega)$ by diffeomorphism that preserve the symplectic structure, we can look at the orbit space $M/G$. Assuming that the $G$-action on $M$ is free and proper, the orbit space $M/G$ is a smooth manifold and one observes that it is generally not symplectic, but it inherits a Poisson structure. A **Poisson manifold** is a pair $(M, \pi)$ where $M$ is a smooth manifold and $\pi \in \Gamma(\bigwedge^2(TM))$ is a smooth bivector satisfying the integrability condition

$$[\pi, \pi] = 0,$$

where $[\cdot, \cdot] : \Gamma(\bigwedge^p(TM)) \times \Gamma(\bigwedge^q(TM)) \longrightarrow \Gamma(\bigwedge^{p+q-1}(TM))$ denotes the Schouten bracket of multivector fields on $M$. In summary, the property of a 2-form being symplectic may be lost under the operations of restricting to submanifolds and taking quotients by symplectic actions. Indeed, we are led to two different geometries: the geometry of closed 2-forms and the geometry of Poisson bivectors. This suggests that we need to go further and define a more general geometric structure which includes closed 2-forms and Poisson bivectors. This was exactly what T. Courant did in his thesis, defining what nowadays is called a Dirac manifold [17]. One observes that a closed 2-form $\omega$ on $M$ induces a bundle map $\omega^\sharp : TM \longrightarrow T^*M$ via $\omega^\sharp(X)(Y) = \omega(X,Y)$, and similarly a Poisson bivetor $\pi$ on $M$ defines a bundle map $\pi^\sharp : T^*M \longrightarrow TM$ by $\pi^\sharp(\alpha)(\beta) = \pi(\alpha,\beta)$. It follows that the graphs of the bundle maps $\omega^\sharp$ and $\pi^\sharp$ define natural subbundles $L \subseteq \mathbb{T}M := TM \oplus T^*M$, which are maximal isotropic with respect to the nondegenerate symmetric pairing on $\mathbb{T}M$,

$$\langle (X,\alpha),(Y,\beta)\rangle = \alpha(Y) + \beta(X),$$

and that satisfy the integrability condition

$$[\![\Gamma(L),\Gamma(L)]\!] \subseteq \Gamma(L),$$

with respect to the Courant bracket $[\![\cdot,\cdot]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \longrightarrow \Gamma(\mathbb{T}M)$,

$$[\![(X,\alpha),(Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - i_Y d\alpha).$$

The integrability in the sense of Courant interpolates the integrability conditions defining closed 2-forms and Poisson bivectors.

The main objective of this thesis is to study Dirac structures defined on Lie groupoids, satisfying a suitable compatibility condition with the groupoid multiplication. Recall that a **groupoid** is a small category in which every morphism is invertible. More specifically, a groupoid consists of a set $G$ of arrows, a set $M$ of objects, and structure mappings $s,t : G \longrightarrow M$ called **source** and **target** maps, a partially defined **multiplication** map $m : G_{(2)} \longrightarrow G$, where $G_{(2)} = \{(g,h) \in G \times G \mid s(g) = t(h)\}$ is the set of **composable** groupoid pairs, a **unit** section $\epsilon : M \longrightarrow G$ and an **inversion** map $i : G \longrightarrow G$, satisfying the axioms of a category (see e.g. [13, 41]). A **Lie groupoid** is a groupoid where $G$ and $M$ are smooth manifolds, all the structure mappings are smooth maps and $s$ and $t$ are surjective submersions.

Our study is motivated by a variety of geometrical structures compatible with group or groupoid structures, including:

i) *Poisson-Lie groups*: these structures consist of a Lie group $G$ with a Poisson structure $\pi$, which are compatible in the sense that the multiplication map $m : G \times G \longrightarrow G$ is a Poisson map. Equivalently, the Poisson bivector $\pi$ is **multiplicative**, that is

$$\pi_{gh} = (l_g)_* \pi_h + (r_h)_* \pi_g,$$

for every $g, h \in G$. Here $l_g$ and $r_h$ denote the left and right multiplication by $g$ and $h$, respectively. Poisson-Lie groups arise as semiclassical limit of quantum groups, and they are infinitesimally described by *Lie bialgebras*. See e.g. [23].

ii) *Symplectic groupoids*: a symplectic groupoid is a Lie groupoid $G$ with a symplectic structure $\omega$, which is compatible with the groupoid multiplication in the sense that the graph

$$\mathrm{Graph}(m) \subseteq G \times G \times \overline{G}$$

is a Lagrangian submanifold with respect to the symplectic structure $\omega \oplus \omega \ominus \omega$. This compatibility condition is equivalent to saying that $\omega$ is **multiplicative**, that is

$$m^* \omega = pr_1^* \omega + pr_2^* \omega,$$

where $pr_1, pr_2 : G_{(2)} \longrightarrow G$ are the canonical projections. Symplectic groupoids arise in the context of quantization of Poisson manifolds [63, 65], connecting Poisson geometry to noncommutative geometry. In [14], symplectic groupoids appeared as phase spaces of certain sigma models. The infinitesimal description of symplectic groupoids is given by Poisson structures, see e.g. [63, 16].

iii) *Poisson groupoids*: these objects were introduced by A. Weinstein [64] as a common generalization of Poisson-Lie groups and symplectic groupoids. A Poisson groupoid is a Lie groupoid $G$ equipped with a Poisson structure $\pi$, which is compatible with the groupoid multiplication in the sense that

$$\mathrm{Graph}(m) \subseteq G \times G \times \overline{G}$$

is a coisotropic submanifold. These structures are related to the geometry of the classical dynamic Yang-Baxter equation, see for instance [24]. At the infinitesimal level, Poisson groupoids are described by *Lie bialgebroids* [46].

iv) *Presymplectic groupoids*: Lie groupoids equipped with a multiplicative closed 2-form were studied in [10]. A presymplectic groupoid [10] is a Lie groupoid $G$ with a multiplicative closed 2-form $\omega$ satisfying suitable nondegeneracy conditions. These objects arise in connection with equivariant cohomology and generalized moment maps [9]. The infinitesimal description of presymplectic groupoids is given by Dirac structures, extending the infinitesimal description of symplectic groupoids. More generally, Lie groupoids endowed with arbitrary multiplicative closed 2-forms are infinitesimally described by bundle maps $\sigma : AG \longrightarrow T^*M$ called IM-2-forms. Here $AG$ denotes the Lie algebroid of $G$ and $T^*M$ is the cotangent bundle of the base of $G$.

The first goal of this work is to find a suitable definition of multiplicative Dirac structure that include both multiplicative Poisson bivectors and multiplicative closed 2-forms, and hence encompasses all examples above. This is obtained by observing that given a Lie groupoid $G$ over $M$ with Lie algebroid $AG$, the tangent bundle $TG$ and the cotangent bundle $T^*G$ inherit natural Lie groupoid structures over $TM$ and $A^*G$, respectively. One observes that a bivector $\pi$ is multiplicative if and only if the bundle map $\pi^\sharp : T^*G \longrightarrow TG$ is a groupoid morphism [46]. Similarly, a 2-form $\omega$ is multiplicative if and only if the bundle map $\omega^\sharp : TG \longrightarrow T^*G$ is a morphism of Lie groupoids. It turns out that the direct sum vector bundle $TG \oplus T^*G$ is a Lie groupoid over $TM \oplus A^*G$, and graphs of both multiplicative Poisson bivectors and closed 2-forms define Lie subgroupoids of $TG \oplus T^*G$. We say that a Dirac structure $L_G$ on a Lie groupoid $G$ is **multiplicative** if $L_G \subseteq TG \oplus T^*G$ is a Lie subgroupoid. A Lie groupoid $G$ equipped with a multiplicative Dirac structure is referred to as a **Dirac groupoid**.

Our main purpose is to describe multiplicative Dirac structures infinitesimally, that is, in terms of Lie algebroid data. We prove that, for every Lie groupoid $G$ with Lie algebroid $AG$, multiplicative Dirac structures correspond to Dirac structures on $AG$ suitably compatible with both the linear and Lie algebroid structures on $AG$. In the particular case of multiplicative Poisson bivectors and multiplicative 2-forms, we explain how this is equivalent to the known infinitesimal descriptions. Along the way, we develop techniques that can treat all multiplicative structures above in a unified manner, often simplifying existing results and proofs. The organization of this thesis and results are as follows.

**Lie groupoids and Dirac structures**

Here we review the basics of Lie groupoids and Lie algebroids. We also recall the definition and main properties of Dirac structures on smooth manifolds, as well as the notion of morphism of Dirac manifolds. We also review the main properties of Poisson groupoids and Lie bialgebroids, as well as multiplicative forms and IM-2-forms. In the last section of chapter 2 we define our main object of study, multiplicative Dirac structures and we discuss basic examples of these objects.

**Multiplicative 2-forms and their infinitesimal counterparts**

This chapter presents the detailed study of multiplicative Dirac structures in the case of multiplicative 2-forms, giving new, simpler proofs of the results in [10]. We use tangent lifts of differential forms [28] to understand the effect of the Lie functor on multiplicative forms. We show that every multiplicative 2-form $\omega_G$ on a Lie groupoid $G$ is infinitesimally described by a 2-form $\omega_{AG}$ on the Lie algebroid $AG$ of $G$, which is morphic in the sense that the natural map $\omega^{\sharp}_{AG} : T(AG) \longrightarrow T^*(AG)$ is a morphism of Lie algebroids. We show that when $\omega_G$ is closed relative to a 3-form $\phi \in \Omega^3(M)$, that is

$$d\omega_G = s^*\phi - t^*\phi,$$

then the induced morphic 2-form on $AG$ is given by

$$\omega_{AG} = -(\sigma^*\omega_{can} + \rho^*(\tau(\phi))),$$

where $\sigma : AG \longrightarrow T^*M$ is defined by $\sigma(u) = (i_u\omega_G)|_{TM}$, $\omega_{can}$ denotes the canonical symplectic form on $T^*M$, $\rho : AG \longrightarrow TM$ is the anchor map of $AG$, and $\tau(\phi) \in \Omega^2(TM)$ is defined at every $X \in TM$ by $\tau(\phi)_X = p_M^*(i_X\phi)$. The main result of this chapter establishes that, on an abstract Lie algebroid $A$ with anchor map $\rho : A \longrightarrow TM$, the 2-form $\Lambda := -(\sigma^*\omega_{can} + \rho^*(\tau(\phi)))$ is morphic if and only if $\sigma : A \longrightarrow T^*M$ defines an IM-2-form with respect to $\phi$. This characterization of IM-2-forms together with Lie's second theorem provide a new proof of the main result of [10], avoiding the path space construction of Lie groupoids.

### The case of Lie groups

This chapter is concerned with the study of multiplicative Dirac structures on Lie groups. We observe that a Dirac structure $L_G$ on a Lie group $G$ is multiplicative if and only if the multiplication map $m : G \times G \longrightarrow G$ is a forward Dirac map. In particular, Dirac-Lie groups provide a natural extension of Poisson-Lie groups. We show that the characteristic foliation of a Dirac-Lie group is given by cosets of a normal Lie subgroup, and whenever this subgroup is closed the space of characteristic leaves inherits the structure of a Poisson-Lie group. In particular, using Drinfeld's correspondence we find the infinitesimal picture of Dirac-Lie groups.

### Natural functors on Dirac groupoids

In this chapter we study the effect of two natural functors on Dirac groupoids, namely the tangent functor and the Lie functor. First, for an arbitrary Dirac manifold $(M, L_M)$ we construct a tangent Dirac structure $L_{TM}$ on the tangent bundle $TM$ via Mackenzie and Xu's method for prolongating Lie algebroid structures to tangent bundles [46]. Our procedure gives an alternative description of tangent Dirac structures studied before by T. Courant [18] and I. Vaisman [61]. In [28] it was proved that for every Poisson Lie group $(G, \pi_G)$ the tangent group $TG$ equipped with the tangent Poisson structure $\pi_{TG}$ is a Poisson Lie group as well. We extend this result to the Dirac groupoids setting. We prove that given a Dirac groupoid $(G, L_G)$ the tangent groupoid $TG \rightrightarrows TM$ endowed with the tangent Dirac structure $L_{TG}$ is also a Dirac groupoid.

The second functor acting on a Dirac groupoid $(G, L_G)$ is the Lie functor. We answer the main question of this thesis:

*What are the infinitesimal counterparts of multiplicative Dirac structures?*

We show that the multiplicativity of $L_G \subseteq TG \oplus T^*G$ translates into the linearity of a Dirac structure $L_{AG}$ on $AG$ which also defines a Lie subalgebroid $L_{AG} \subseteq T(AG) \oplus T^*(AG)$.

Moreover, we show that the Dirac structure $L_{AG}$ coincides with the Lie algebroid $A(L_G)$ of $L_G$, up to natural identifications. Conversely, on an integrable Lie algebroid $A$, every linear Dirac structure $L_A$ on $A$ which is also a subalgebroid of $TA \oplus T^*A$ can be integrated to a multiplicative Dirac structure $L_G \subseteq TG \oplus T^*G$ on the source simply connected Lie groupoid $G$ integrating the Lie algebroid $A$. This result is a natural extension of the integration of Lie bialgebroids [48], where the linear Dirac structures involved there are just graphs of Lie algebroid morphisms. We finish chapter 5 by studying multiplicative Dirac structures defined by $B$-fields transformations of Poisson groupoids. We also describe these structures infinitesimally.

**Dirac groupoids and $\mathcal{LA}$-groupoids**

This chapter is concerned with an alternative construction of the linear Dirac structure $L_{AG}$ on $AG$ determined in chapter 5. We use the second order geometry introduced by K. Mackenzie [42] to show that every Dirac groupoid $(G, L_G)$ may be thought of as a Lie groupoid object in the category of Lie algebroids. In the terminology of K. Mackenzie this is an $\mathcal{LA}$-groupoid [42]. The Lie functor applied to an arbitrary $\mathcal{LA}$-groupoid yields a double Lie algebroid [43]. In particular the induced Dirac structure $L_{AG}$ associated to a Dirac groupoid $(G, L_G)$ arises as the double Lie algebroid of the $\mathcal{LA}$-groupoid representing $(G, L_G)$.

**New research directions**

Chapter 7 describes natural new research directions. First, we briefly discuss the connection between the results shown in chapter 3 and the Van Est isomorphism between the Bott-Shulman complex of a Lie groupoid and the Weil algebra of its Lie algebroid, constructed recently by C. Arias Abad and M. Crainic in [2, 3]. We also explain how the theory of graded supermanifolds could give a different perspective on the infinitesimal invariant of a Dirac groupoid. This approach is based on Roytenberg's correspondence between Courant algebroids and certain degree 2 symplectic supermanifolds. We also explain how the underlying Courant algebroid where multiplicative Dirac structure lie would provide the prototype of new interesting structure, which might be called a Courant groupoid.

In addition, we have included an appendix with some double structures which are used throughout this work. Along this thesis we use Einstein's summation convention consistently.

# Chapter 2

# Lie groupoids and Dirac structures

## 2.1 Basic Lie theory of Lie algebroids and groupoids

A **groupoid** over a set $M$ is a set $G$ together with structure mappings

$$s, t : G \longrightarrow M,$$

called **source** and **target** maps, a partially defined **multiplication** map

$$\begin{aligned} m : G_{(2)} &\longrightarrow G \\ (g,h) &\mapsto gh \end{aligned}$$

where $G_{(2)} = \{(g,h) \in G \times G \mid s(g) = t(h)\}$ is the set of **composable** groupoid pairs, a **unit** section $\epsilon : M \longrightarrow G$ and an **inversion** map $i : G \longrightarrow G$, satisfying the following compatibility conditions:

1. $s(gh) = s(h), \quad t(gh) = t(g).$
2. $(gh)k = g(hk)$, whenever $s(g) = t(h)$ and $s(h) = t(k)$.
3. $\epsilon(t(g))g = g = g\epsilon(s(g))$
4. $gi(g) = \epsilon(t(g)), \quad i(g)g = \epsilon(s(g)).$

Equivalently, a groupoid is a small category in which every morphism is invertible. See for instance [13, 41]. We use the notation $G \rightrightarrows M$ to indicate that $G$ is a groupoid over $M$.

A **Lie groupoid** is a groupoid $G$ over $M$, where $G$ and $M$ are smooth manifolds, all the structure mappings are smooth maps and $s$ and $t$ are surjective submersions.

**Example 2.1.1.** Every Lie group $G$ can be viewed as a Lie groupoid over a point.

**Example 2.1.2.** Let $M$ be a smooth manifold. Consider the space

$$\Pi(M) = \{[\gamma] \mid \gamma \text{ is a curve in } M\},$$

here $[\gamma]$ denotes the homotopy class of $\gamma$ with fixed end-points. There is a natural groupoid structure on $\Pi(M)$ with source and target maps defined by

$$s([\gamma]) = \gamma(0); \quad t([\gamma]) = \gamma(1),$$

the multiplication is given by $[\gamma_1][\gamma_2] = [\gamma_1\gamma_2]$, where $\gamma_1\gamma_2$ is the path obtained via the concatenation of $\gamma_1$ and $\gamma_2$. The smooth structure on $\Pi(M)$ is the unique smooth structure making the map $(s,t) : \Pi(M) \longrightarrow M \times M$ into a surjective submersion. The groupoid $\Pi(M)$ is called the **fundamental groupoid** of $M$.

**Example 2.1.3.** Let $\mathcal{F}$ be a regular foliation on a smooth manifold $M$. We define a Lie groupoid $M(\mathcal{F})$ over $M$ as follows. If $x, y \in M$ are on different leaves, then there are no arrows from $x$ to $y$. If $x$ and $y$ are on the same leaf $\mathcal{L}$, then the arrows from $x$ to $y$ in $M(\mathcal{F})$ are homotopy classes of paths from $x$ to $y$ inside the leaf $\mathcal{L}$. The source and target maps are the obvious ones and the multiplication is given by the homotopy class of the concatenation of paths. We refer to $M(\mathcal{F}) \rightrightarrows M$ as the **monodromy groupoid** of the foliated manifold $(M, \mathcal{F})$. For details see [50].

**Example 2.1.4.** Given a foliated manifold $(M, \mathcal{F})$ we define a Lie groupoid $H(\mathcal{F}) \rightrightarrows M$ in a similar way to the definition of $M(\mathcal{F})$, except that we replace *homotopy* classes of paths by *holonomy* classes of paths. This Lie groupoid is referred to as the **holonomy** groupoid associated to the foliated manifold $(M, \mathcal{F})$. For a detailed explanation see [50].

**Example 2.1.5.** Let $H$ be a Lie group acting on a smooth manifold $M$. We endow $H \times M$ with a Lie groupoid structure over $M$ as follows. The source and target maps are defined by

$$s(h, x) = x, \quad t(h, x) = hx.$$

The multiplication is defined by $(h, h'x)(h', x) = (hh', x)$. The unit section is $\epsilon(x) = (e, x)$ where $e \in H$ is the identity element. Finally the inversion map is defined by $i(h, x) = (h^{-1}, hx)$. These maps define a Lie groupoid structure on $H \times M$, called the **transformation groupoid**. We usually denote the transformation groupoid by $H \ltimes M$. See [50] for more details.

**Definition 2.1.1.** Let $G_1$ and $G_2$ be Lie groupoids over $M_1$ and $M_2$, respectively. A **morphism** of Lie groupoids is a pair $(\Phi, \varphi)$ of smooth maps $\Phi : G_1 \longrightarrow G_2$, $\varphi : M_1 \longrightarrow M_2$, commuting with all structure maps (in the sense that they define a functor between the categories $G_1$ and $G_2$).

As in the case of Lie groups, every Lie groupoid has a natural infinitesimal invariant. In order to find this invariant we recall the definition of an abstract Lie algebroid.

**Definition 2.1.2.** A **Lie algebroid** over a smooth manifold $M$ is a vector bundle $A \xrightarrow{q_A} M$ with a Lie bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ and a bundle map, called the **anchor map**, $\rho_A : A \longrightarrow TM$ satisfying the Leibniz rule

$$[u, fv]_A = f[u, v]_A + (\mathcal{L}_{\rho_A(u)} f)v$$

where $u, v \in \Gamma(A)$ and $f \in C^\infty(M)$.

Given a Lie algebroid $A \xrightarrow{q_A} M$, the **Lie algebroid differential** is the operator $d_A : \Gamma(\bigwedge^k A^*) \longrightarrow \Gamma(\bigwedge^{k+1} A^*)$ defined by

$$d_A\xi(u_1, ..., u_{k+1}) = \sum_{i=1}^{k} (-1)^i \rho_A(u_i)\xi(u_1, ..., \hat{u}_i, ..., u_{k+1}) + \tag{2.1}$$

$$+ \sum_{i<j} (-1)^{i+j} \xi([u_i, u_j]_A, u_1, ..., \hat{u}_i, ..., \hat{u}_j, ..., u_{k+1}), \tag{2.2}$$

where $\xi \in \Gamma(\bigwedge^k A^*)$ and $u_i \in \Gamma(A)$ with $i = 1, ..., k+1$. The operator $d_A$ satisfies $d_A^2 = 0$, so we can talk about the Lie algebroid *cohomology*. One easily checks that the anchor map $\rho_A$ and the Lie bracket $[\cdot,\cdot]_A$ are completely determined by $d_A$ and the property $d_A^2 = 0$. See [13] for more details. Another characterization of Lie algebroid structures is via linear Poisson bivectors. More specifically, every Lie algebroid $A$ induces a Poisson structure on its dual bundle $A^*$ which is *linear* in the sense that the space of fiberwise linear functions $C^\infty_{lin}(A^*) \cong \Gamma(A) \subseteq C^\infty(A^*)$ is a Poisson subalgebra. More explicitly, if $(x^1, ..., x^m)$ is a system of local coordinates on $M$ and $\{e_1, ..., e_r\}$ is a basis of local sections of $A$, we induce coordinates $(x^i, u^a)$ on $A$. There are structure functions $\rho_a^j$, $C_{ab}^c$ for the Lie algebroid $A$, determined by

i) $\rho_A(e_a) = \rho_a^j \frac{\partial}{\partial x^j}$, and

ii) $[e_a, e_b]_A = C_{ab}^c e_c$.

Now if $\{e^1, ..., e^r\}$ is a basis of local sections of $A^*$, dual to $\{e_1, ..., e_r\}$, we induce local coordinates $(x^i, \xi_a)$ on $A^*$. With respect to this local description of $A^*$, the linear Poisson bivector $\pi_{A^*} \in \mathfrak{X}^2(A^*)$ has the form

$$(\pi_{A^*})_{|(x,\xi)} = \rho_a^i(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial \xi_a} + \frac{1}{2} C_{ab}^c(x) \xi_c \frac{\partial}{\partial \xi_a} \wedge \frac{\partial}{\partial \xi_b}. \tag{2.3}$$

It can be easily verified that the linear Poisson structure on $A^*$ determines completely the Lie algebroid structure on $A$. See e.g. [13].

**Example 2.1.6.** Every finite dimensional Lie algebra $\mathfrak{g}$ can be seen as a Lie algebroid over a point.

**Example 2.1.7.** Let $M$ be a smooth manifold. The tangent bundle $TM$ has a natural Lie algebroid structure over $M$, with anchor map defined by $\mathrm{Id}_{TM}$ and Lie bracket on $\mathfrak{X}(M)$ given by the usual bracket of vector fields. We refer to $TM$ as the **canonical** Lie algebroid.

**Example 2.1.8.** Every regular distribution $F \subseteq TM$ which is involutive defines a Lie algebroid over $M$. The anchor map is given by the inclusion $F \longrightarrow TM$ and the Lie bracket on $\Gamma(F)$ is just the usual Lie bracket of vector fields.

**Example 2.1.9.** Let $\mathfrak{h}$ be a Lie algebra acting on a smooth manifold $M$. That is, there exists a Lie algebra morphism

$$\begin{aligned} \mathfrak{h} &\longrightarrow \mathfrak{X}(M) \\ u &\mapsto u_M. \end{aligned}$$

We endow the trivial bundle $A_{\mathfrak{h}} = \mathfrak{h} \times M$ with the structure of a Lie algebroid over $M$. The anchor map is defined by

$$\rho : \mathfrak{h} \times M \longrightarrow TM$$
$$(u, x) \mapsto u_M(x).$$

The Lie bracket $[\cdot,\cdot]_{A_{\mathfrak{h}}}$ on $\Gamma(A_{\mathfrak{h}}) \cong C^\infty(M) \bigotimes \mathfrak{h}$ is given by

$$[u, v]_{A_{\mathfrak{h}}} := [u, v],$$

for $u, v \in \mathfrak{h}$, and we extend it by requiring the Leibniz rule. The bundle $A_{\mathfrak{h}} \longrightarrow M$ with this Lie algebroid structure is referred to as the **transformation Lie algebroid**. See [50] for more details.

Given a Lie groupoid $G \rightrightarrows M$, we construct its Lie algebroid in the same way we do for the Lie algebra of a Lie group. For that, consider the distribution $T^sG$ tangent to the $s$-fibration of $G$, that is at every point $g \in G$ we have

$$T^sG = \ker(Ts : TG \longrightarrow TM).$$

**Definition 2.1.3.** A vector field $X$ on $G$ is called **right invariant** if it is tangent to the $s$-fibration, and for every composable pair $(g, h) \in G_{(2)}$ we have

$$T_g r_h(X_g) = X_{gh},$$

where $r_h : s^{-1}(t(h)) \longrightarrow s^{-1}(s(h))$ is the right multiplication by $h \in G$ and $T_g r_h$ denotes the derivative of $r_h$ at the point $g \in G$.

One can see that the space $\mathfrak{X}^r(G)$ of right invariant vector fields on $G$ is a Lie subalgebra of $\mathfrak{X}(G)$ with respect to the Lie bracket of vector fields. Furthermore, there is a one-to-one correspondence between $\mathfrak{X}^r(G)$ and the module of sections of the pull back vector bundle

$$AG := \epsilon^*(T^sG).$$

The **Lie algebroid** of $G$ is the vector bundle $AG \longrightarrow M$ equipped with the Lie bracket on $\Gamma(AG)$ induced by the identification $\mathfrak{X}^r(G) \cong \Gamma(AG)$, and anchor map defined by $Tt|_{AG} : AG \longrightarrow TM$. See [13].

**Example 2.1.10.** If $G$ is a Lie group, the construction of its Lie algebroid leads to the Lie algebra of $G$.

**Example 2.1.11.** The Lie algebroid of the fundamental groupoid $\Pi(M)$ is the tangent bundle $TM$ with the canonical Lie algebroid structure.

**Example 2.1.12.** The Lie algebroids of the Lie groupoids $M(\mathcal{F})$ and $H(\mathcal{F})$ coincide with the Lie algebroid associated to the distribution $F \subseteq TM$ tangent to the foliation $\mathcal{F}$.

Unlike finite dimensional Lie algebras, not every Lie algebroid is the Lie algebroid of a Lie groupoid. A Lie algebroid $A$ is called **integrable** if there exists a Lie groupoid $G$ with Lie algebroid isomorphic to $A$. It is easy to see that whenever $A$ integrates to a Lie groupoid $G$, then it admits an integration $\tilde{G}$ with simply connected $s$-fibers. For instance, the Lie algebroid associated to any integrable distribution $F \subseteq TM$ integrates to $H(\mathcal{F})$ and to $M(\mathcal{F})$, the latter being the source simply connected integration. Henceforth, we only consider source simply connected integrations. Explicit obstructions for the integrability of Lie algebroids can be found in [21].

**Definition 2.1.4.** ([41, 50]) Let $A_1 \longrightarrow M_1$ and $A_2 \longrightarrow M_2$ be Lie algebroids. A bundle map $\Psi : A_1 \longrightarrow A_2$ covering a map $\psi : M_1 \longrightarrow M_2$ is called a **morphism** of Lie algebroids if the following properties are fulfilled:

1. $\rho_{A_2} \circ \Psi = T\psi \circ \rho_{A_1}$
2. For every $u, v$ sections of $A_1$ with $\Psi(u) = f^i \phi^*(u_i)$ and $v = g^j \phi^*(v_j)$ where $u_i, v_j$ are sections of $A_2$ and $f^i, g^j$ are smooth functions on $M_1$, the following bracket preserving condition is satisfied

$$\Psi([u,v]_{A_1}) = f^i g^j \phi^*([u_i, v_j]_{A_2}) + \rho_{A_1}(u) g^j \phi^*(v_j) - \rho_{A_1}(v) f^i \phi^*(u_i)$$

Let $(\Phi, \varphi)$ be a Lie groupoid morphism between $G_1$ and $G_2$. The tangent functor applied to $\Phi$ gives rise to a bundle map $T\Phi : TG_1 \longrightarrow TG_2$ which sends the distribution $T^sG_1$ into $T^sG_2$. Since $(\Phi, \varphi)$ is compatible with the unit sections of $G_1$ and $G_2$, the bundle map $T\Phi$ restricts to a bundle map $A(\Phi) : AG_1 \longrightarrow AG_2$, which defines a morphism of Lie algebroids $(A(\Phi), \varphi)$ between $AG_1$ and $AG_2$. Let us denote by $\mathcal{LG}$ and $\mathcal{LA}$ the category of Lie groupoids and Lie algebroids, respectively.

**Definition 2.1.5.** There is a natural functor $A : \mathcal{LG} \longrightarrow \mathcal{LA}$, which maps each object $G \in \mathcal{LG}$ to the object $AG \in \mathcal{LA}$, and every morphism of groupoids $\Phi : G_1 \longrightarrow G_2$ is mapped to the Lie algebroid morphism $A(\Phi) : AG_1 \longrightarrow AG_2$. We refer to $A$ as the **Lie functor**.

We finish this subsection with the Lie's second fundamental theorem for morphisms of Lie algebroids. We will use this result several times along this thesis.

**Theorem 2.1.1.** *Let $\psi : A_1 \longrightarrow A_2$ be a morphism of integrable Lie algebroids, and let $G_1$ and $G_2$ be integrations of $A_1$ and $A_2$, respectively. If $G_1$ is source simply connected, then there exists a unique morphism of Lie groupoids $\Phi : G_1 \longrightarrow G_2$, such that $A(\Phi) = \psi$.*

A proof of this result can be found in [21, 48].

**Remark 2.1.1.** Assume that $G_1$ is a *source-connected* Lie groupoid. If $\Phi_1, \Phi_2 : G_1 \longrightarrow G_2$ are Lie groupoid morphisms inducing the same Lie algebroid morphism $\Psi : AG_1 \longrightarrow AG_2$, then necessarily $\Phi_1 = \Phi_2$. Indeed, we can consider the source simply connected Lie groupoid $\tilde{G}_1$ with $A\tilde{G}_1 = AG_1$ and integrate $\Psi : AG_1 \longrightarrow AG_2$ to a unique groupoid morphism $\tilde{\Phi} : \tilde{G}_1 \longrightarrow G_2$. The natural projection $pr : \tilde{G}_1 \longrightarrow G_1$ is a groupoid morphism, and we notice that $\Phi_1 \circ pr : \tilde{G}_1 \longrightarrow G_2$ is a groupoid morphism with $A(\Phi_1 \circ pr) = \Psi$. Similarly, the groupoid morphism $\Phi_2 \circ pr : \tilde{G}_1 \longrightarrow G_2$ satisfies $A(\Phi_2 \circ pr) = \Psi$. Therefore, the uniqueness of the integration $\tilde{\Phi} : \tilde{G}_1 \longrightarrow G_2$ implies that $\Phi_1 = \Phi_2$.

## 2.2 Basics on Dirac geometry

### 2.2.1 Dirac structures

Given a smooth manifold $M$ we consider $\mathbb{T}M := TM \oplus T^*M$. A **Dirac structure** ([17, 19]) is a subbundle $L \subset \mathbb{T}M$ satisfying the following properties:

1. It is maximal isotropic with respect to the non degenerate symmetric pairing on $\mathbb{T}M$,

$$\langle X \oplus \alpha, Y \oplus \beta \rangle = \alpha(Y) + \beta(X).$$

2. The space of smooth sections $\Gamma(L)$ is closed under the Courant bracket on $\Gamma(\mathbb{T}M)$,

$$[\![ X \oplus \alpha, Y \oplus \beta ]\!] = [X, Y] \oplus \mathcal{L}_X\beta - i_Y d\alpha.$$

It is whorthwhile to observe that, since the quadratic form $\langle \cdot, \cdot \rangle$ has split signature, condition 1. is equivalent to saying that $\langle \cdot, \cdot \rangle|_{L \times L} = 0$ and $\mathrm{rank}(L) = \dim(M)$. A maximal isotropic subbundle $L \subseteq \mathbb{T}M$ is referred to as a **Lagrangian** subbundle of $\mathbb{T}M$. We denote by $\mathrm{Dir}(M)$ the space of all Dirac structures on a smooth manifold $M$.

**Example 2.2.1.** (Closed 2-forms)

Let $\omega$ be a 2-form on $M$. The graph of $\omega$ is the subbundle of $\mathbb{T}M$ defined by

$$L_\omega = \{X \oplus \omega^\sharp(X) \mid X \in TM\}$$

where $\omega^\sharp : TM \longrightarrow T^*M$ is the natural bundle map induced by $\omega$. That is,

$$\omega^\sharp(X) = \omega(X, \cdot).$$

One easily checks that the skew symmetry of $\omega$ implies that $L_\omega$ is isotropic, and it is clear that $L_\omega$ has maximal dimension. The Courant integrability for $L_\omega$ is equivalent to $d\omega = 0$.

**Example 2.2.2.** (Poisson bivectors)

On the other extreme, the graph of a bivector $\pi \in \Gamma(\bigwedge^2 TM)$ is the subbundle of $\mathbb{T}M$ given by

$$L_\pi = \{\pi^\sharp(\alpha) \oplus \alpha \mid \alpha \in T^*M\}$$

Here $\pi^\sharp : T^*M \longrightarrow TM$ denotes the bundle map defined by $\pi^\sharp(\alpha) = \pi(\alpha, \cdot)$. Again the isotropy property of $L_\pi$ comes from the skew symmetry of $\pi$, and we observe that $L_\pi$ has maximal dimension. The Courant integrability for $L_\pi$ is equivalent to $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket of multivector fields.

The examples discussed previously show that Dirac structures interpolate presymplectic and Poisson structures. There is also another important class of Dirac structures, those given by regular foliations.

**Example 2.2.3.** (Regular foliations)

Let $F \subseteq TM$ be a regular distribution. Consider the graph

$$L_F = F \oplus F^\circ \subseteq TM \oplus T^*M$$

where $F^\circ$ denotes the annihilator of $F$. It is easy to see that $L_F$ defines a Dirac structure on $M$ if and only if $F$ is involutive in the sense of Frobenius.

**Example 2.2.4.** (Restriction to submanifolds)

Let $L$ be a Dirac structure on $M$, and let $Q \hookrightarrow M$ be a submanifold. For each $x \in Q$ define the Lagrangian subspace

$$(L_Q)_x := \frac{L_x \cap (T_xQ \oplus T_x^*M)}{L_x \cap (T_xQ)^\circ}. \tag{2.4}$$

The result of putting together the pointwise subspaces $(L_Q)_x \subseteq \mathbb{T}Q$ may not be a smooth vector bundle. The result will be a smooth bundle if for instance $L_x \cap (T_xQ)^\circ$ has constant dimension. When the family (2.4) defines a smooth bundle, we get a Dirac structure $L_Q$ on the submanifold $Q$ of $M$.

**Example 2.2.5.** (Moment level sets)

Let $(M, \pi)$ be a Poisson manifold, and let $H$ be a Lie group acting on $M$ in a Hamiltonian manner. Let $J : M \longrightarrow \mathfrak{h}^*$ be a momentum map for this action, and suppose that $\xi \in \mathfrak{h}^*$ is a regular value for $J$. Let $H_\xi$ denote the isotropy group of $\xi \in \mathfrak{h}^*$ with respect to the coadjoint action. The moment level set $Q := J^{-1}(\xi)$ is a submanifold of $M$, so we can consider the family of Lagrangian subspaces $(L_Q)_x \subseteq \mathbb{T}Q$ as in (2.4). If the isotropy groups of the $H_\xi$-action on $Q$ have constant dimension, e.g. if the action is free, then the result of putting together the subspaces $(L_Q)_x$ yields a smooth bundle over $Q$ which defines a Dirac structure on the moment level set $Q = J^{-1}(\xi)$. See [17] for more details.

As observed in [57], it is convenient to modify the Courant bracket by a closed 3-form $\phi$ on $M$. The **$\phi$-twisted Courant bracket** on $\Gamma(\mathbb{T}M) = \mathfrak{X}(M) \oplus \Omega^1(M)$ is defined by

$$[\![X \oplus \alpha, Y \oplus \beta]\!]_\phi = [X, Y] \oplus \mathcal{L}_X\beta - i_Y d\alpha + i_{X\wedge Y}\phi.$$

A **$\phi$-twisted Dirac structure** on $M$ is a Lagrangian subbundle $L \subseteq \mathbb{T}M$ whose space of sections $\Gamma(L)$ is closed under the $\phi$-twisted Courant bracket. If $\phi = 0$ we recover the usual bracket $[\![\cdot, \cdot]\!]$ introduced previously. It is important to observe that it is the Courant bracket that is twisted by the 3-form $\phi \in \Omega^3(M)$ and not the subbundle $L$. The addition of the *twist 3-form* is important since there are interesting examples of Dirac structures which turn out to be integrable up to a closed 3-form [57]. The study of twisted Dirac structures is motivated by the previous work of Klimčik and Strobl [36] on WZW-Poisson manifolds, where a 3-form background plays a role similar to the Wess-Zumino-Witten term in field theory. One observes easily that a 2-form $\omega$ on $M$ defines a $\phi$-twisted Dirac structure if and only if

$$d\omega + \phi = 0.$$

In this case we say that $\omega$ is closed with respect to $\phi \in \Omega^3(M)$. Similarly, a bivector $\pi \in \Gamma(\bigwedge^2 TM)$ defines a $\phi$-twisted Dirac structure if and only if

$$\frac{1}{2}[\pi, \pi] = (\wedge^3 \pi^\sharp)(\phi),$$

where $\wedge \pi^\sharp$ denotes the extension of the bundle map $\pi^\sharp : T^*M \longrightarrow TM$ to higher exterior powers.

**Example 2.2.6.** (*Cartan-Dirac structure*)

Let $G$ be a Lie group whose Lie algebra $\mathfrak{g}$ is equipped with a nondegenerate symmetric adjoint-invariant bilinear form $(\cdot, \cdot)_{\mathfrak{g}}$. We can use the bilinear form $(\cdot, \cdot)_{\mathfrak{g}}$ to identify $TG$ and $T^*G$. With respect to this identification, we define the Lagrangian subbundle

$$L_G := \{u^r - u^l \oplus \frac{1}{2}(u^r + u^l) \mid u \in \mathfrak{g}\},$$

where $u^r$ and $u^l$ denote the right and left invariant vector fields determined by $u \in \mathfrak{g}$. One can prove that $L_G$ is a $\phi_G$-twisted Dirac structure, where $\phi_G$ is the bi-invariant Cartan 3-form on $G$, defined at element in $\mathfrak{g}$ by

$$\phi_G(u, v, w) = \frac{1}{2}(u, [v, w])_{\mathfrak{g}}.$$

The $\phi_G$-twisted Dirac structure $L_G$ on $G$ is referred to as the **Cartan-Dirac** structure on $G$. The Cartan-Dirac structure on a Lie group is closely related to the theory of Lie group valued moment maps [1, 10], which arises in connection with the symplectic structure of the moduli space of flat connections on a compact Riemann surface [4].

Given a closed 3-form $\phi$ on $M$, we denote by $\mathrm{Dir}^\phi(M)$ the space of all $\phi$-twisted Dirac structures on $M$.

### 2.2.2 Properties

The involutivity of a Dirac subbundle with respect to the $\phi$-twisted Courant bracket may be thought of as a generalized Frobenius condition. It turns out that Dirac geometry has natural connections with foliation theory, in particular with the theory of Lie algebroids. More concretely, given a $\phi$-twisted Dirac structure $L$ on a smooth manifold $M$, the vector bundle $L \longrightarrow M$ inherits a canonical Lie algebroid structure with anchor map given by the restriction of the canonical projection $pr|_L : L \longrightarrow TM$, and Lie bracket on sections defined by the restriction $\phi$-twisted Courant bracket. Every Lie algebroid induces an integrable **singular** distribution given by the image of the anchor map, see e.g. [13]. In the case of a Lie algebroid induced by a Dirac structure $L$, this singular foliation comes with extra data. Actually, on each leaf $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow M$ there is a 2-form defined at each $x \in \mathcal{S}$ by

$$\Omega_{\mathcal{S}}(x)(X, Y) = \alpha(Y),$$

where $X, Y \in pr(L)_x$ and $\alpha \in T_x^*M$ satisfies $X \oplus \alpha \in L_x$. Observe that since $L \subseteq \mathbb{T}M$ is isotropic one concludes that $\Omega_{\mathcal{S}}$ is well defined, that is, it does not depend on the choice of $\alpha$. The integrability of $L$ with respect to the $\phi$-twisted Courant bracket implies that the leafwise 2-forms $\Omega_{\mathcal{S}}$ are closed up to $i_{\mathcal{S}}^*\phi$, that is

$$d\Omega_{\mathcal{S}} + i_{\mathcal{S}}^*\phi = 0.$$

We refer to this singular foliation with the leafwise 2-forms as the **presymplectic foliation** of $M$.

**Example 2.2.7.** Let $(M, \pi)$ be a Poisson manifold. The singular foliation on $M$ induced by the Dirac structure $L_\pi$ is the foliation tangent to $\pi^\sharp(T^*M) \subseteq TM$. The leafwise presymplectic forms recover the leafwise symplectic structure underlying the Poisson structure $\pi$.

The **kernel** of a Dirac structure $L$ on $M$ is defined by generally singular distribution $\ker(L) = L \cap TM$. It follows from the definition of the leafwise 2-forms that at each $x \in \mathcal{S}$ the fiber of $\ker(L)$ is given by $\ker(L)_x = \ker(\Omega_{\mathcal{S}}(x))$. As in the symplectic or Poisson case, we would like to define what the Hamiltonian vector field of a smooth function is. It turns out that on a Dirac manifold not every smooth function has a natural Hamiltonian vector field, and the leafwise presymplectic forms play an important role in this problem. An **admissible** function is a smooth function $f \in C^\infty(M)$ for which there exists a vector field $X_f \in \mathfrak{X}(M)$ such that $X_f \oplus df \in L$. By clear reasons, such a vector field is referred to as a **Hamiltonian** vector field of $f$. Notice that $X_f$ is well defined up to elements in $\ker(L)$, and whenever $\phi = 0$, the set $\mathcal{A}(M)$ of admissible functions inherits a Poisson algebra structure (see. e.g. [17]) defined by the bracket

$$\{f, g\} = dg(X_f).$$

Notice that, whenever $\ker(L) \subseteq TM$ has constant rank and defines a simple[1] foliation $\mathcal{K}$, then admissible functions are identified with smooth functions in the leaf space $M/\mathcal{K}$. Therefore, if $\mathcal{K}$ is a simple foliation, then the leaf space $M/\mathcal{K}$ inherits a Poisson structure denoted by $\pi_{red}$. The foliation $\mathcal{K}$ is called the **characteristic foliation** of $M$.

### 2.2.3 Dirac morphisms

Now we explain the notion of morphism of Dirac manifolds following [12]. A proper notion of morphism of Dirac manifolds should include pull backs of 2-forms and push forward of bivectors. In order to make a clear description of Dirac maps, we explain two extreme situations.

**Example 2.2.8.** (*Presymplectic maps*)

Let $(M, \omega_M)$ and $(N, \omega_N)$ presymplectic manifolds, that is $\omega_M$ and $\omega_N$ are closed 2-forms on $M$ and $N$, respectively. A presymplectic map is a smooth map $\varphi : M \longrightarrow N$ such that $\omega_M = \varphi^*\omega_N$. One observes that this is equivalent to the fact that the induced bundle maps $\omega_M^\sharp : TM \longrightarrow T^*M$ and $\omega_N^\sharp : TN \longrightarrow T^*N$ are related by

$$(\omega_M^\sharp)_x = (\omega_N^\sharp)_{\varphi(x)} \circ T_x\varphi,$$

for each $x \in M$. As in Example 2.2.1, we have Dirac structures $L_{\omega_M}$ and $L_{\omega_N}$ on $M$ and $N$, respectively. Therefore we conclude that a smooth map $\varphi : (M, \omega_M) \longrightarrow (N, \omega_N)$ is presymplectic if and only if

[1]A foliation $\mathcal{F}$ on a smooth manifold $M$ is said to be simple if the leaf space $M/\mathcal{F}$ is a smooth manifold such that the quotient map $M \longrightarrow M/\mathcal{F}$ is a surjective submersion.

$$(L_{\omega_M})_x = \{X \oplus (T_x\varphi)^*\beta \mid X \in T_xM, \beta \in T^*_{\varphi(x)}N, (T_x\varphi(X) \oplus \beta) \in (L_{\omega_N})_{\varphi(x)}\}.$$

**Example 2.2.9.** *(Poisson maps)*

Let $(M, \pi_M)$ and $(N, \pi_N)$ be Poisson manifolds. A smooth map $\varphi : M \longrightarrow N$ is a Poisson map if and only if the induced bundle maps $\pi^\sharp_M : T^*M \longrightarrow TM$ and $\pi^\sharp_N : T^*N \longrightarrow TN$ are related by

$$(\pi^\sharp_N)_{\varphi(x)} = T_x\varphi \circ (\pi^\sharp_M)_x \circ (T_x\varphi)^*.$$

As explained in Example 2.2.2, there are induced Dirac structures $L_{\pi_M}$ and $L_{\pi_N}$ on $M$ and $N$, respectively. The fact that $\varphi : (M, \pi_M) \longrightarrow (N, \pi_N)$ is a Poisson map is equivalent to

$$(L_{\pi_N})_{\varphi(x)} = \{T_x\varphi(X) \oplus \beta \mid X \in T_xM, \beta \in T^*_{\varphi(x)}N, (X \oplus (T_x\varphi)^*\beta) \in (L_{\pi_M})_x\}.$$

The examples discussed previously motivate the following definitions. Let $(M, L_M)$ and $(N, L_N)$ be Dirac manifolds. A map $\varphi : (M, L_M) \longrightarrow (N, L_N)$ is called a **backward Dirac map** if for every $x \in M$ we have

$$(L_M)_x = \{X \oplus (T_x\varphi)^*\beta \mid X \in T_xM, \beta \in T^*_{\varphi(x)}N, (T_x\varphi(X) \oplus \beta) \in (L_N)_{\varphi(x)}\}. \tag{2.5}$$

Similarly, we say that $\varphi$ is a **forward Dirac map** if for every $x \in M$,

$$(L_N)_{\varphi(x)} = \{T_x\varphi(X) \oplus \beta \mid X \in T_xM, \beta \in T^*_{\varphi(x)}N, X \oplus (T_x\varphi)^*\beta \in (L_M)_x\}. \tag{2.6}$$

Notice that a map between Poisson manifolds is a forward Dirac map if and only if it is a Poisson map. Similarly, a backward Dirac between presymplectic manifolds is the same that a presymplectic map. It is important to observe that even for symplectic manifolds, Poisson and symplectic maps may be different, thus forward and backward Dirac maps are different notions.

**Example 2.2.10.** Consider $\mathbb{R}^2$ with coordinates $(x^1, p_1)$ and symplectic form $\omega_2 = dx^1 \wedge$

$dp_1$. Assume also that on $\mathbb{R}^4$ we have coordinates $(x^1, p_1, x^2, p_2)$ and the symplectic form $\omega_4 = dx^1 \wedge dp_1 + dx^2 \wedge dp_2$. It is clear that the inclusion map

$$i : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$$
$$(x^1, p_1) \mapsto (x^1, p_1, 0, 0)$$

is a backward Dirac map, since it is symplectic. Notice that with respect to the Poisson brackets induced by $\omega_2$ and $\omega_4$, the map $i : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ is not a Poisson map[2], in particular it is not a forward Dirac map. Similary, the projection

$$\mathbb{R}^4 \longrightarrow \mathbb{R}^2$$
$$(x^1, p_1, x^2, p_2) \mapsto (x^1, p_1),$$

is a forward Dirac map, since it is a Poisson map. Clearly the projection is not symplectic[3], in particular it is not backward Dirac.

Denote the right hand side of (2.5) by $\varphi^* L_N$. This defines a natural way to pull Dirac structures back, though the result of putting together the pointwise subspaces of $\mathbb{T}M$ is not necessarily a smooth vector bundle. The result will be a Dirac structure if it defines a smooth bundle over $M$. For instance, the right hand side of (2.5) is smooth if $\varphi : M \longrightarrow N$ is a submersion. Therefore a smooth map $\varphi : (M, L_M) \longrightarrow (N, L_N)$ is a backward Dirac map if $L_M = \varphi^* L_N$. Also, we can write (2.6) as $L_N = \varphi_* L_M$, though $\varphi_* L_M$ may not be well defined. See [12] for more details.

**Example 2.2.11.** Let $L$ be a Dirac structure on $M$, and let $i : Q \hookrightarrow M$ be a smooth submanifold. Assume that the subspaces $(L_Q)_x \subseteq \mathbb{T}_x Q$ as in (2.4) define a smooth bundle over $Q$. Then we get a Dirac structure $L_Q$ on $Q$, and this Dirac structure is determined by the fact that the inclusion map $i : Q \hookrightarrow M$ is a backward Dirac map. That is $i^*(L) = L_Q$.

We finish our discussion about Dirac maps by illustrating two examples where the notions of backward and forward Dirac maps coincide.

**Example 2.2.12.** Let $(\mathcal{S}, \Omega_{\mathcal{S}})$ be a pre symplectic leaf of a Dirac manifold $(M, L)$. Then the inclusion map $i_{\mathcal{S}} : \mathcal{S} \longrightarrow M$ is both a forward and backward Dirac map.

[2]Recall that in this case the property of $i : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ being a Poisson map implies that $i$ has to be a submersion, which clearly it is not the case.

[3]Recall that if a map between symplectic manifolds is a symplectic map, then it has to be an immersion.

**Example 2.2.13.** Assume that the distribution $\ker(L) \subseteq TM$ is tangent to a simple foliation $\mathcal{K}$. Then the natural projection map $(M, L) \longrightarrow (M/\mathcal{K}, \pi_{red})$ is a backward and forward Dirac map.

**Example 2.2.14.** (Poisson reduction)

Let $(M, \pi)$ be a Poisson manifold with a Hamiltonian action of a Lie group $H$. Let $J : M \longrightarrow \mathfrak{h}^*$ be a moment map for this action and assume that $\xi \in \mathfrak{h}^*$ is a regular value of $J$. Assume that the $H_\xi$-action on $Q = J^{-1}(\xi)$ is free and proper. Then we conclude from example 2.2.5 that the restriction of $L_\pi$ to $Q$ defines a Dirac structure $L_Q$ on the level set $Q$. One can verify that the $H_\xi$-orbits of the action on $Q$ coincide with the characteristic leaves of the Dirac structure $L_Q$. Therefore, the reduced space $M_{red} := Q/H_\xi$ is the space of characteristic leaves of $L_Q$, so it inherits a canonical Poisson structure $\pi_{red}$ such that the projection map $Q \longrightarrow M_{red}$ is both a backward and forward Dirac map. See [17] for more details.

## 2.3 Tangent and cotangent structures

### 2.3.1 Tangent and cotangent groupoids

Let $G$ be a Lie groupoid over $M$ with Lie algebroid $AG$. The tangent bundle $TG$ has a natural Lie groupoid structure over $TM$. This structure is obtained by applying the tangent functor to each of the structure maps defining $G$ (source, target, multiplication, inversion and identity section). We refer to $TG$ with the groupoid structure over $TM$ as the **tangent groupoid** of $G$. Notice that the set of composable pairs $(TG)_{(2)} = T(G_{(2)})$, and for $(g, h) \in G_{(2)}$ and a tangent groupoid pair $(X_g, Y_h) \in (TG)_{(2)}$ the multiplication map on $TG$ is

$$X_g \bullet Y_h := Tm(X_g, Y_h)$$

**Example 2.3.1.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. The tangent bundle $TG$ is a Lie group as well. One can see that the multiplication on $TG$ is given by

$$X_g \bullet Y_h = T_g r_h(X_g) + T_h l_g(Y_h).$$

We can use right translations to trivialize $TG$ in such a way that $TG \cong G \times \mathfrak{g}$.

With respect to this identification, it is easy to see that the group structure on the tangent bundle corresponds to the semidirect group $G \ltimes \mathfrak{g}$ determined by the adjoint representation.

Consider now the cotangent bundle $T^*G$. It was shown in [16], that $T^*G$ is a Lie groupoid over $A^*G$. The source and target maps are defined by

$$\tilde{s}(\alpha_g)u = \alpha_g(Tl_g(u - Tt(u))) \quad \text{ and } \tilde{t}(\beta_g)v = \beta_g(Tr_g(v))$$

where $\alpha_g \in A^*_{s(g)}G$, $u \in A_{s(g)}G$ and $\beta_g \in A^*_{t(g)}G$, $v \in A_{t(g)}G$. The multiplication on $T^*G$ is defined by

$$(\alpha_g \circ \beta_h)(X_g \bullet Y_h) = \alpha_g(X_g) + \beta_h(Y_h)$$

for $(X_g, Y_h) \in T_{(g,h)}G_{(2)}$.

We refer to $T^*G$ with the groupoid structure over $A^*$ as the **cotangent groupoid** of $G$.

**Example 2.3.2.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Then the cotangent groupoid $T^*G$ has base manifold $\mathfrak{g}^*$. We can use right trivializations to identify $T^*G \cong G \times \mathfrak{g}^*$. In terms of this identification, the cotangent groupoid corresponds to the transformation groupoid $G \ltimes \mathfrak{g}^*$ with respect to the coadjoint action.

**Remark 2.3.1.** Notice that the tangent groupoid $TG \rightrightarrows TM$ and the cotangent groupoid $T^*G \rightrightarrows A^*G$ have an additional property. Namely, the space of arrows and objects are vector bundles and all the structure maps (source, target, multiplication, inversion and unit section) are morphisms of vector bundles. That is, they define Lie groupoid objects in the category of vector bundles. These are examples of a more general structure called a $\mathcal{VB}$-groupoid. The reader can find the definition and main properties of such structures in appendix A.

### 2.3.2 Tangent and cotangent algebroids

Let $M$ be a smooth manifold. The tangent bundle of $M$ is denoted by $p_M : TM \longrightarrow M$. We use $c_M : T^*M \longrightarrow M$ to indicate the cotangent bundle of a smooth manifold. Consider now $A \xrightarrow{q_A} M$ a vector bundle over $M$. The tangent bundle $TA$ has a natural structure of vector bundle over $TM$, defined by applying the tangent functor to each of the structure maps that define the vector bundle $A \xrightarrow{q_A} M$. This yields to a commutative diagram

$$\begin{CD} TA @>{Tq_A}>> TM \\ @V{p_A}VV @VV{p_M}V \\ A @>>{q_A}> M \end{CD} \tag{2.7}$$

In the terminology of [52, 41], this defines a double vector bundle[4]. Now we assume that $A \xrightarrow{q_A} M$ has a Lie algebroid structure with anchor map $\rho_A : A \longrightarrow TM$ and Lie bracket $[\cdot,\cdot]$ on $\Gamma_M(A)$. First note that any Poisson structure $\pi_M$ on a smooth manifold $M$ induces a Poisson structure on the tangent bundle $TM$. Indeed, since $T^*M$ is a Lie algebroid over $M$, then the dual bundle $TM$ has a linear Poisson structure $\pi_{TM}$ as in (2.3), which we call the **tangent Poisson structure**. Now, if $A$ is a Lie algebroid over $M$, then $A^*$ is a Poisson manifold. Consider the double vector bundle

$$\begin{CD} TA^* @>{Tq_{A^*}}>> TM \\ @V{p_{A^*}}VV @VV{p_M}V \\ A^* @>>{q_{A^*}}> M \end{CD} \tag{2.8}$$

The tangent Poisson structure on $TA^*$ is linear with respect to both vector bundle structures on $TA^*$. Therefore, the dual bundle $(TA^*)^* \longrightarrow TM$ inherits a Lie algebroid structure.

**Proposition 2.3.1.** *[46] There exists a canonical isomorphism of vector bundles $I : TA^* \longrightarrow (TA)^*$*

*Proof.* Consider the canonical pairing $A^* \times_M A \longrightarrow \mathbb{R}$. Applying the tangent functor and projecting onto the second component we get a nondegenerate pairing $TA^* \times_{TM} TA \longrightarrow \mathbb{R}$. We use this pairing to define an isomorphism of vector bundles $I : TA^* \longrightarrow (TA)^*$.

□

[4]The reader can find a review of the basics on double vector bundles in appendix A.

**Definition 2.3.1.** The **tangent Lie algebroid** of $A$ is the vector bundle $TA \longrightarrow TM$ equipped with the unique Lie algebroid structure that makes the canonical map $I^* : TA \longrightarrow (TA^*)^*$ into an isomorphism of Lie algebroids.

It will be useful to have an explicit description of the tangent anchor map, as well as the tangent Lie bracket on sections of $TA \longrightarrow TM$. First, recall that there exists a **canonical involution**

$$\begin{array}{ccc} TTM & \xrightarrow{J_M} & TTM \\ \downarrow{\scriptstyle p_{TM}} & & \downarrow{\scriptstyle Tp_M} \\ TM & \xrightarrow[\mathrm{Id}]{} & TM \end{array} \tag{2.9}$$

which in a local coordinates system $(x^i, \dot{x}^i, \delta x^i, \delta\dot{x}^i)$ on $TTM$ is given by

$$J_M((x^i, \dot{x}^i, \delta x^i, \delta\dot{x}^i)) = (x^i, \delta x^i, \dot{x}^i, \delta\dot{x}^i).$$

Now we can apply the tangent functor to the anchor map $\rho_A : A \longrightarrow TM$, and then compose with the canonical involution to obtain a bundle map $\rho_{TA} : TA \longrightarrow TTM$ defined by

$$\rho_{TA} = J_M \circ T\rho_A.$$

This defines the tangent anchor map. In order to define the tangent Lie bracket, we observe that every section $u \in \Gamma_M(A)$ induces two types of sections of $TA \longrightarrow TM$. The first type of section is $Tu : TM \longrightarrow TA$, which is given by applying the tangent functor to the section $u : M \longrightarrow A$. The second type of section is the *core* section $\hat{u} : TM \longrightarrow TA$, which is defined by

$$\hat{u}(X) = T(0^A)(X) + \overline{u(p_M(X))},$$

where $0^A : M \longrightarrow A$ denotes the zero section, and $\overline{u(p_M(X))} = \frac{d}{dt}(tu(p_M(X)))|_{t=0}$. As observed in [46], sections of the form $Tu$ and $\hat{u}$ generate the module of sections $\Gamma_{TM}(TA)$. Therefore, the tangent Lie bracket is determined by

$$[Tu, Tv] = T[u, v], \quad [Tu, \hat{v}] = \widehat{[u, v]}, \quad [\hat{u}, \hat{v}] = 0,$$

and we extend to other sections by requiring the Leibniz rule with respect to the tangent anchor $\rho_{TA}$.

**Example 2.3.3.** If $A = \mathfrak{g}$ is a Lie algebra, then $TA = \mathfrak{g} \times \mathfrak{g}$ is also a Lie algebra. Moreover, the tangent Lie algebra is the semidirect product Lie algebra $\mathfrak{g} \ltimes \mathfrak{g}$ with respect to the adjoint representation.

Now we explain how the cotangent bundle of a Lie algebroid inherits a Lie algebroid structure. For that, let us explain the vector bundle structure $T^*A \longrightarrow A^*$. If $(x^i, u^a)$ are local coordinates on $A$, we induce a local coordinates system $(x^i, u^a, p_i, \lambda_a)$ on $T^*A$, where $(p_i)$ determines a cotangent element in $T^*_x M$ and $(\lambda_a) \in A^*_x$ is a cotangent element with respect to the tangent direction to the fibers of $A$. Now the bundle projection $r : T^*A \longrightarrow A^*$ is described locally by $r(x^i, u^a, p_i, \lambda_a) = (x^i, \lambda_a)$. These vector bundle structures define a commutative diagram

$$\begin{array}{ccc} T^*A & \xrightarrow{\;r\;} & A^* \\ {\scriptstyle c_A}\big\downarrow & & \big\downarrow{\scriptstyle q_{A^*}} \\ A & \xrightarrow[\;q_A\;]{} & M \end{array} \tag{2.10}$$

This endows $T^*A$ with a double vector bundle structure. Suppose that $q_A : A \longrightarrow M$ carries a Lie algebroid structure. Then we can consider the dual bundle $A^*$ endowed with the linear Poisson structure induced by $A$. The cotangent bundle $T^*A^* \longrightarrow A^*$ has the Lie algebroid structure determined by the linear Poisson bivector on $A^*$. There exists a Legendre type map $R : T^*A^* \longrightarrow T^*A$ which is a anti-symplectomorphism with respect to the canonical symplectic structures, and it is locally defined by $R(x^i, \xi_a, p_i, u^a) = (x^i, u^a, -p_i, \xi_a)$. For an intrinsic definition see [46, 59].

**Definition 2.3.2.** The **cotangent algebroid** of $A$ is the vector bundle $T^*A \longrightarrow A^*$ equipped with the unique Lie algebroid structure that makes the Legendre type transform $R : T^*(A^*) \longrightarrow T^*A$ into an isomorphism of Lie algebroids.

**Example 2.3.4.** Let $\mathfrak{g}$ be a Lie algebra. Then the cotangent Lie algebroid $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^*$ is the transformation Lie algebroid $\mathfrak{g} \ltimes \mathfrak{g}^*$ with respect to the coadjoint representation.

**Remark 2.3.2.** The tangent and the cotangent algebroids have an additional property. They define Lie algebroid objects in the category of vector bundles. These are particular examples of a more general structure called a $\mathcal{VB}$-algebroid. In appendix A we have included the main properties and examples of such geometrical structures.

Notice that for a Lie group $G$ with Lie algebra $\mathfrak{g}$, the tangent and cotangent Lie algebroids of $\mathfrak{g}$ are exactly the Lie algebroids of the tangent and cotangent Lie groupoids of $G$. We will see that this is a general fact. For that, recall that the **Tulczyjew map** $\Theta_M : TT^*M \longrightarrow T^*TM$ is the isomorphism defined by

$$\Theta_M := J_M^* \circ I_M,$$

where $I_M : TT^*M \longrightarrow (TTM)^*$ is the map defined in Prop. 2.3.1 with $A = TM$. In a local coordinates system $(x^i, p_i, \dot{x}^i, \dot{p}_i)$ the Tulczyjew map is given by

$$\Theta_M(x^i, p_i, \dot{x}^i, \dot{p}_i) = (x^i, \dot{x}^i, \dot{p}_i, p_i).$$

Consider now a Lie groupoid $G$ over $M$ with Lie algebroid $AG$. There exists a natural injective bundle map

$$i_{AG} : AG \longrightarrow TG \tag{2.11}$$

The canonical involution $J_G : TTG \longrightarrow TTG$ restricts to an isomorphism of Lie algebroids $j_G : T(AG) \longrightarrow A(TG)$. More precisely, there exists a commutative diagram

$$\begin{array}{ccc} T(AG) & \xrightarrow{\ j_G\ } & A(TG) \\ {\scriptstyle T(i_{AG})}\downarrow & & \downarrow{\scriptstyle i_{A(TG)}} \\ TTG & \xrightarrow[\ J_G\ ]{} & TTG \end{array} \tag{2.12}$$

In particular, the Lie algebroid $A(TG)$ of the tangent groupoid is canonically isomorphic to the tangent Lie algebroid $T(AG)$ of $AG$. Similarly, the Lie algebroid of the cotangent groupoid $T^*G$ is isomorphic to the cotangent Lie algebroid $T^*(AG)$. For that, notice that the natural pairing $T^*G \oplus TG \longrightarrow \mathbb{R}$ defines a groupoid morphism, and the application of the Lie functor yields a symmetric pairing $\langle\langle\cdot,\cdot\rangle\rangle : A(T^*G) \oplus A(TG) \longrightarrow \mathbb{R}$, which is nondegenerate. See e.g. [46, 48]. In particular, we obtain an isomorphism $K_G : A(T^*G) \longrightarrow A(TG)^*$, where the target dual is with respect to the fibration $A(TG) \overset{A(p_G)}{\longrightarrow} AG$. Now we define a Lie algebroid isomorphism

$$j'_G : A(T^*G) \longrightarrow T^*(AG),$$

determined by the composition $j'_G = j^*_G \circ K_G$, where $j^*_G : A(TG)^* \longrightarrow T^*(AG)$ is the bundle map dual to the isomorphism $j_G : T(AG) \longrightarrow A(TG)$. As $j_G : T(AG) \longrightarrow A(TG)$ is a suitable restriction of the canonical involution $J_G : TTG \longrightarrow TTG$, the isomorphism $j'_G$ is related to the Tulczyjew map $\Theta_G : TT^*G \longrightarrow T^*TG$, via

$$j'_G = (Ti_{AG})^* \circ \Theta_G \circ i_{A(T^*G)}.$$

## 2.4 Examples of multiplicative structures

Now we present examples of geometrical structures defined on Lie groupoids which are compatible with the groupoid multiplication.

### 2.4.1 Poisson-Lie groups

Let $G$ be a Lie group and $\pi \in \Gamma(\bigwedge^2 TG)$ a Poisson bivector on $G$. One easily observes that the following statements are equivalent:

i) The multiplication map $m : G \times G \longrightarrow G$ is a Poisson map.

ii) The graph of the multiplication map defines a coisotropic[5] submanifold of $G \times G \times \overline{G}$.

iii) The bivector $\pi$ is multiplicative in the sense that

$$\pi_{gh} = (l_g)_* \pi_h + (r_h)_* \pi_g,$$

for every $g, h \in G$.

[5] Recall that a submanifold $Q \hookrightarrow M$ is said to be coisotropic with respect to a bivector $\pi$ on $M$ if $\pi^\sharp(N^*Q) \subseteq TQ$, where $N^*Q$ denotes the conormal bundle of $Q$.

A Poisson-Lie group [23, 40, 55] is a Lie group $G$ with a Poisson structure $\pi \in \Gamma(\Lambda^2 TG)$ satisfying one of the conditions above. Notice that condition iii) implies that a multiplicative bivector $\pi$ vanishes at the identity $e \in G$, and we conclude that Poisson-Lie groups are never symplectic. Since $\pi_e = 0$, there exists a canonical Lie algebra structure on the cotangent fiber $T_e^*G = \mathfrak{g}^*$, see [62]. The Lie bracket on $\mathfrak{g}^*$ will be denoted by $[\cdot,\cdot]_* : \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$. We would like to understand how the multiplicativity of $\pi$ is reflected in the Lie bracket $[\cdot,\cdot]_*$. For that, we dualize $[\cdot,\cdot]_*$ yielding a **cobracket**

$$F : \mathfrak{g} \longrightarrow \mathfrak{g} \times \mathfrak{g}.$$

On one hand the bivector $\pi$ is nothing else that a section $\pi : G \longrightarrow \wedge^2 TG$, and we use right translations to trivialize the vector bundle $\wedge^2 TG \cong G \times \wedge^2\mathfrak{g}$. With respect to this trivialization we induce a map

$$\begin{aligned} \tilde{\pi} : G &\longrightarrow \wedge^2\mathfrak{g} \\ g &\longmapsto (R_{g^{-1}})_* \pi_g \end{aligned}$$

Notice that the multiplicativity of $\pi$ implies that

$$\tilde{\pi}_{gh} = \tilde{\pi}_g + \mathrm{Ad}_g(\tilde{\pi}_h).$$

It turns out that $\tilde{\pi}$ defines a 1-cocycle on $G$ with valued in the $G$-module $\wedge^2\mathfrak{g}$, where the module structure is the one determined by the adjoint action extended to the second wedge product. See [39] for more details.

The linearization of $\tilde{\pi}$ at the identity coincides with the cobracket $F$ [23, 40]. Then $F$ defines a Lie algebra 1-cocycle with values in the $\mathfrak{g}$-module $\wedge^2\mathfrak{g}$, where the module structure is defined by

$$\mathrm{ad}_X(u \wedge v) = (\mathrm{ad}_X u) \wedge v + u \wedge (\mathrm{ad}_X v).$$

The 1-cocycle condition for the cobracket $F$ is

$$F([X,Y]) = \mathrm{ad}_X F(Y) - \mathrm{ad}_Y F(X).$$

**Definition 2.4.1.** A **Lie bialgebra** is a pair $(\mathfrak{g}, \mathfrak{g}^*)$ where $(\mathfrak{g}, [\cdot,\cdot])$ and $(\mathfrak{g}^*, [\cdot,\cdot]_*)$ are Lie algebras and the cobracket $F := [\cdot,\cdot]_*^*$ satisfies

$$F([X,Y]) = \mathrm{ad}_X F(Y) - \mathrm{ad}_Y F(X).$$

We have seen that every Poisson-Lie group $(G, \pi)$ induces a natural Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. Hence, Lie bialgebras may be regarded as the infinitesimal version of Poisson Lie groups. The converse result is true under the usual connectedness assumptions, establishing the so called Drinfeld's correspondence between Poisson-Lie groups and Lie bialgebras.

**Theorem 2.4.1.** *[23]*

*Let $G$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. There exists a one-to-one correspondence between*

1. *Lie bialgebra structures $(\mathfrak{g}, \mathfrak{g}^*)$, and*

2. *multiplicative Poisson structures on $G$.*

The proof of Drinfeld's correspondence is based on the correspondence between Lie group 1-cocycles and Lie algebra 1-cocycles. See [39] for a detailed discussion.

### 2.4.2 Symplectic groupoids

A symplectic groupoid [63, 16] is a symplectic manifold $(G, \omega_G)$ where $G$ is a Lie groupoid over $M$, and $\omega_G$ is a symplectic form compatible with the groupoid structure in the sense that the graph

$$\Lambda_m := \{(g, h, m(g,h)) \mid (g,h) \in G_{(2)}\},$$

is a Lagrangian submanifold of $G \times G \times \overline{G}$. Equivalently, the symplectic form $\omega_G$ is **multiplicative**, that is

$$m^*\omega_G = pr_1^*\omega_G + pr_2^*\omega_G,$$

where $pr_1, pr_2 : G_{(2)} \longrightarrow G$ are the natural projections. As observed in [63, 16], the base $M$ of a symplectic groupoid inherits a Poisson structure $\pi_M$, completely determined by the fact that the target map (resp. source) $t : G \longrightarrow M$ is a Poisson map (resp. anti-Poisson). Also, if $AG$ is the Lie algebroid of $G$, then there exists an isomorphism of Lie algebroids

$$\sigma : AG \longrightarrow T^*M \tag{2.13}$$

$$u \mapsto (i_u \omega_G)_{|TM} \tag{2.14}$$

where the Lie algebroid structure on $T^*M$ is the one induced by the Poisson bivector $\pi_M$ on $M$. It turns out that Poisson structures may be thought of as the infinitesimal counterpart of symplectic groupoids. To every symplectic groupoid one canonically associates a Poisson manifold. For this reason, symplectic groupoids are natural geometric objects that are useful for quantizing Poisson manifolds. Therefore, it seems that a suitable quantization of the symplectic groupoid $(G, \omega_G)$ should provide a natural way of quantizing the Poisson manifold $(M, \pi_M)$, see [65, 20] for more details about the prequantization of symplectic groupoids. See also Cattaneo and Felder's construction of symplectic groupoids as phase spaces of certain sigma models [14].

### 2.4.3 Poisson groupoids

In this subsection we study Lie groupoids endowed with a Poisson structure which satisfies an algebraic compatibility.

**Definition 2.4.2.** A **Poisson groupoid** is a pair $(G, \pi_G)$ where $G$ is a Lie groupoid over $M$ and $\pi_G$ is a Poisson structure on $G$ which is **multiplicative** in the sense that the graph of the multiplication map

$$\Lambda_m = \{(g, h, gh) \mid (g, h) \in G_2\}$$

is a coisotropic submanifold of $G \times G \times \bar{G}$.

Poisson groupoids were introduced by Alan Weinstein [64], providing a unified framework for the study of Poisson Lie groups [40] and symplectic groupoids [16]. A Poisson-Lie group is just a Poisson groupoid over a point, and a symplectic groupoid is nothing but a Poisson groupoid with nondegenerate Poisson bivector. In subsection 2.4.1 we observed that the infinitesimal invariant of a Poisson-Lie group is its Lie bialgebra. In order to find the infinitesimal counterpart of Poisson groupoids, one observes that the base $M$ of a Poisson groupoid $(G, \pi)$ is a coisotropic submanifold, in particular the conormal bundle $N^*(M) \cong A^*(G)$ inherits a Lie algebroid structure. Here $A^*(G)$ denotes the vector bundle dual to the Lie algebroid $A(G)$ of the Lie groupoid $G$. It turns out that for any Poisson groupoid there exists a pair of Lie algebroids $(A(G), A^*(G))$ in duality as vector bundles, which satisfies certain compatibility condition.

**Definition 2.4.3.** A **Lie bialgebroid** is a pair of Lie algebroids in duality $(A, A^*)$ satisfying

$$d_{A^*}([u,v]) = [d_{A*}(u), v] + [u, d_{A^*}(v)]$$

for every $u, v \in \Gamma(A)$.

Here $d_{A^*} : \Gamma(\bigwedge^k A) \longrightarrow \Gamma(\bigwedge^{k+1} A)$ denotes the Lie algebroid differential induced by $A^*$ and $[\cdot,\cdot]$ is the Schouten bracket on multisections of $A$.

**Example 2.4.1.** Any Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebroid.

**Example 2.4.2.** An interesting example coming from Poisson geometry is the following: given a Poisson manifold $(M, \pi)$, the cotangent bundle $T^*M$ inherits a canonical Lie algebroid structure with anchor map $\pi^\sharp : T^*M \longrightarrow TM$ and Lie bracket on $\Omega^1(M)$ given by

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d\pi(\alpha, \beta).$$

This Lie algebroid structure together with the trivial Lie algebroid structure on the tangent bundle of $M$ makes the pair $(T^*M, TM)$ into a Lie bialgebroid.

Just as Lie bialgebras arise as the infinitesimal counterpart of Poisson-Lie groups [23, 39], Lie bialgebroids are the infinitesimal version of Poisson groupoids according to the following result of K. Mackenzie and P. Xu.

**Theorem 2.4.2.** *[46]*

*Let $(G, \pi_G)$ be a Poisson groupoid with Lie algebroid $A(G)$. Then $(A(G), A^*(G))$ is a Lie bialgebroid.*

Let $(G, \omega_G)$ be a symplectic groupoid, viewed as a Poisson groupoid, then the Lie bialgebroid of $G$ is the one described in example 2.4.2, where $M$ has the Poisson structure induced by the symplectic groupoid $(G, \omega_G)$

The key point in Mackenzie-Xu's approach is based on the possibility of expressing the multiplicativity of a bivector in terms of Lie groupoid morphisms. Given a Lie groupoid $G \rightrightarrows M$, we consider the tangent groupoid $TG \rightrightarrows TM$ and the cotangent groupoid $T^*G \rightrightarrows A^*G$, as explained in subsection 2.3.1.

**Proposition 2.4.1.** *[46]*

*A bivector* $\pi_G \in \Gamma(\bigwedge^2 TG))$ *is a multiplicative bivector if and only if*

$$\begin{array}{ccc} T^*G & \xrightarrow{\pi_G^\sharp} & TG \\ \downarrow\downarrow & & \downarrow\downarrow \\ A^*(G) & \xrightarrow[\rho_{A^*G}]{} & TM \end{array} \tag{2.15}$$

*is a morphism of Lie groupoids covering some bundle map* $\rho_{A^*G}$.

This point of view is extremely useful since it provides a natural way for doing Lie theory for Poisson groupoids in terms of Lie's second theorem for morphisms of Lie algebroids 2.1.1. Now it is natural to expect that the property of $(A, A^*)$ being a Lie bialgebroid could be expressed in terms of suitable morphisms of Lie algebroids. First recall that as we explained in the first section of this chapter, the Lie algebroid $A^*$ induces a linear Poisson structure on $A$, given locally by

$$(\pi_A)_{|(x,u)} = \overline{\rho}^i_a(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial u^a} + \frac{1}{2} \overline{C}^c_{ab}(x) u^c \frac{\partial}{\partial u^a} \wedge \frac{\partial}{\partial u^b}.$$

where $\overline{\rho}^j_a$ and $\overline{C}^c_{ab}$ are the structure functions of the dual Lie algebroid $A^*$. Notice that the linearity of $\pi_A$ is reflected in the fact that the induced bundle map $\pi_A^\sharp : T^*A \longrightarrow TA$ is not only a morphism of vector bundles with respect to the usual bundle structures, but also it defines a morphism

$$\begin{CD} T^*A @>{\pi_A^\sharp}>> TA \\ @VVV @VVV \\ A^* @>>{\rho_{A^*}}> TM \end{CD} \tag{2.16}$$

with respect to the vector bundle structures $T^*A \longrightarrow A^*$ and $TA \longrightarrow TM$, explained in subsection 2.3.2. This can be seen directly from the local expression for the bivector $\pi_A$. Now, just as the multiplicativity of a bivector is translated to the language of morphisms of groupoids, the property of $(A, A^*)$ being a Lie bialgebroid is equivalent to saying that the double vector bundle morphism $\pi_A^\sharp$ is also a morphism of Lie algebroids. The proof of the following result can be found in [46].

**Theorem 2.4.3.** *[46]*

*Let $(A, A^*)$ be a pair of Lie algebroids in duality. Then $(A, A^*)$ is a Lie bialgebroid if and only if*

$$\begin{CD} T^*A @>{\pi_A^\sharp}>> TA \\ @VVV @VVV \\ A^* @>>{\rho_{A^*}}> TM \end{CD} \tag{2.17}$$

*is a morphism of Lie algebroids, where the top map is the linear Poisson bivector on $A$ and the bottom map is the anchor map of the dual algebroid.*

The transition from a Poisson groupoid to a Lie bialgebroid follows by applying the Lie functor to the morphism of groupoids (2.15), yielding a morphism of Lie algebroids

$$\begin{array}{ccc} A(T^*G) & \xrightarrow{A(\pi_G^\sharp)} & A(TG) \\ \Big\downarrow & & \Big\downarrow \\ A^*G & \xrightarrow[\rho_{A*}]{} & TM \end{array} \tag{2.18}$$

Consider the natural identifications $j_G : T(AG) \longrightarrow A(TG)$ and $j'_G : A(T^*G) \longrightarrow T^*(AG)$, as in subsection 2.3.2. It was proved in [46] that there is a commutative diagram

$$\begin{array}{ccc} A(T^*G) & \xrightarrow{A(\pi_G^\sharp)} & A(TG) \\ {\scriptstyle j'_G}\Big\downarrow & & \Big\uparrow{\scriptstyle j_G} \\ T^*(AG) & \xrightarrow[\pi_{AG}^\sharp]{} & T(AG) \end{array} \tag{2.19}$$

where $\pi_{AG}$ is the linear Poisson bivector on $AG$, induced by the dual Lie algebroid $A^*G$. In particular, it follows from Theorem 2.4.3 that $(AG, A^*G)$ is a Lie bialgebroid. The integration of Lie bialgebroids to Poisson groupoids is based on the same idea: under standard connectedness assumptions, Lie bialgebroids integrate to Poisson groupoids via Lie's second theorem.

**Theorem 2.4.4.** *[48]*

*Let $(A, A^*)$ a Lie bialgebroid. Assume that $A$ is the Lie algebroid of a source simply connected Lie groupoid $G$. There exists a unique Poisson structure $\pi_G$ on $G$ making the pair $(G, \pi_G)$ into a Poisson groupoid with Lie bialgebroid $(A, A^*)$.*

Since every Lie bialgebroid produces a morphism of Lie algebroids $\pi_A^\sharp : T^*A \longrightarrow TA$, we can integrate this morphism to a morphism of groupoids $\pi_G^\sharp : T^*G \longrightarrow TG$. It was shown in [48] that the morphism of groupoids $\pi_G^\sharp$ is linear with respect to the usual tangent and cotangent bundle structures and it is skew symmetric. Therefore, there is a well defined bivector $\pi_G$ on $G$, which it turns to be a Poisson bivector. This extends Drinfeld's correspondence 2.4.1 between Poisson-Lie groups and Lie bialgebras [23]. See [48] for details about the proof.

### 2.4.4 Multiplicative 2-forms

In this section we study Lie groupoids equipped with closed 2-forms which are compatible with the groupoid structure. Let $G$ be a Lie groupoid over $M$. A 2-form $\omega_G \in \Omega^2(G)$ is called **multiplicative** if

$$m^*\omega_G = pr_1^*\omega_G + pr_2^*\omega_G$$

where $m : G_{(2)} \longrightarrow G$ denotes the multiplication map and $pr_1, pr_2 : G_{(2)} \longrightarrow G$ are the natural projections. If $G$ is a Lie groupoid equipped with a multiplicative symplectic form $\omega_G$, we recover symplectic groupoids. If $\omega_G \in \Omega^2(G)$ is a closed multiplicative form, not necessarily symplectic, the bundle map $\sigma$ in (2.13) is no longer an isomorphism, and the bracket preserving property does not make sense at all, since the base manifold is not Poisson. In spite of this, the bundle map $\sigma$ has two interesting properties, as it was shown in [10].

**Proposition 2.4.2.** *Let $\phi$ be a closed 3-form on $M$. If $\omega_G \in \Omega^2(G)$ is a multiplicative form with $d\omega_G = s^*\phi - t^*\phi$, then the associated bundle map $\sigma : AG \longrightarrow T^*M$ satisfies the following conditions*

1. *for every $u, v \in \Gamma(AG)$ we have $\langle \sigma(u), \rho(v)\rangle = -\langle \sigma(v), \rho(u)\rangle$*
2. *$\sigma([u, v]) = \mathcal{L}_{\rho(u)}\sigma(v) - \mathcal{L}_{\rho(v)}(\sigma(u)) + d\langle \sigma(u), \rho(v)\rangle + i_{\rho(u)\wedge\rho(v)}\phi$,*

   *for every $u, v \in \Gamma(AG)$.*

A bundle map $\sigma : AG \longrightarrow T^*M$ satisfying properties *1.* and *2.* in Proposition 2.4.2 is called an **IM-2-form** with respect to $\phi \in \Omega^3(M)$. This terminology is due to the fact that an IM-2-form with respect to $\phi \in \Omega^3(M)$ may be thought of as an infinitesimal multiplicative 2-form. It turns out that under standard connectedness assumptions, a multiplicative $(s^*\phi - t^*\phi)$-twisted 2-form on a Lie groupoid is completely determined by its associated IM-2-form.

**Theorem 2.4.5.** *[10]*

*Let $G$ be a source simply connected Lie groupoid $G$ over $M$, with Lie algebroid $AG$. Consider a closed 3-form $\phi$ on $M$. There exists a one-to-one correspondence between*

*i) multiplicative 2-forms $\omega_G$ on $G$ with $d\omega_G = s^*\phi - t^*\phi$, and* *ii) IM-2-forms $\sigma : AG \longrightarrow T^*M$ with respect to $\phi$.*

Theorem 2.4.5 was proved in [10] using the path construction of Lie groupoids [21] and infinite dimensional reduction as in [14]. In chapter 3 we give an alternative proof of this result, avoiding infinite dimensional issues, which establishes a natural connection with Dirac groupoids, introduced in the end of this chapter.

We have seen that every $\phi$-twisted Dirac structure $L$ on $M$ gives rise to a canonical Lie algebroid. Now we explain how to construct twisted Dirac structures out of Lie algebroids. Let us consider a closed 3-form $\phi$ on $M$. The following definition was given in [10].

**Definition 2.4.4.** A **$\phi$-twisted presymplectic groupoid** over $M$ is a pair $(G, \omega_G)$ where $G$ is a Lie groupoid over $M$ and $\omega_G$ is a multiplicative 2-form on $G$ satisfying

1. $d\omega_G = s^*\phi - t^*\phi$
2. $\dim(G) = 2\dim(M)$
3. at every $x \in M$ the following nondegeneracy condition holds

$$\ker(T_x s) \cap \ker(T_x t) \cap \ker(\omega_G)_x = 0.$$

Consider the IM-2-form $\sigma : AG \longrightarrow T^*M$ associated to a presymplectic groupoid $(G, \omega_G)$. One easily checks that conditions 2. and 3. guarantee that the image $L_\sigma$ of the bundle map $\rho_{AG} \oplus \sigma : A(G) \longrightarrow TM \oplus T^*M$ defines a $\phi$-twisted Dirac structure on $M$. Moreover, the target map $t : (G, \omega_G) \longrightarrow (M, L_\sigma)$ is a forward Dirac map. Furthermore, the injective bundle map

$$\rho_{AG} \oplus \sigma : A(G) \longrightarrow TM \oplus T^*M,$$

establishes an isomorphism of Lie algebroids $AG \cong L_\sigma$ between the Lie algebroid of $G$ and the canonical Lie algebroid determined by the $\phi$-twisted Dirac structure $L_\sigma$. Hence, Dirac manifolds may be thought of as the infinitesimal data of presymplectic groupoids. In summary, the following result holds.

**Theorem 2.4.6.** *[10]*

*Let $(G, \omega_G, \phi)$ be a $\phi$-twisted presymplectic groupoid over $M$, then*

1. *There exists a canonical $\phi$-twisted Dirac structure $L_M$ on $M$, such that the target map $t : G \longrightarrow M$ is a forward Dirac map.*

2. *There is a canonical Lie algebroid isomorphism $AG \cong L_M$ between the Lie algebroid of $G$ and the Lie algebroid of the $\phi$-twisted Dirac structure $L_M$.*

A $\phi$-twisted presymplectic groupoid $(G, \omega_G, \phi)$ related to a $\phi$-twisted Dirac structure $L_M$ on the base $M$ as in Theorem 2.4.6 is referred to as an **integration** of $L_M$. The integration of twisted Dirac manifolds to presymplectic groupoids was also carried out in [10]. This follows as an immediate consequence of Theorem 2.4.5. More specifically, the following result holds.

**Theorem 2.4.7.** *[10]*

*Let $L_M$ be a $\phi$-twisted Dirac structure on $M$, whose associated Lie algebroid is integrable. Let $G$ be the source simply connected Lie groupoid integrating L, then there is a unique multiplicative 2-form $\omega_G$ on $G$ such that $(G, \omega_G, \phi)$ is an integration of $L_M$.*

The proof follows by applying Theorem 2.4.5 to the natural IM-2-form defined by the projection $L_M \subseteq TM \oplus T^*M \longrightarrow T^*M$.

In order to give a new proof of Theorem 2.4.5, avoinding path spaces, it is useful to notice that one has a characterization of multiplicative forms in terms of groupoid morphisms, in analogy with Theorem 2.4.1..

**Proposition 2.4.3.** *A 2-form $\omega_G$ on a Lie groupoid $G$ is multiplicative if and only if*

$$\begin{array}{ccc} TG & \xrightarrow{\ \omega_G^\sharp\ } & T^*G \\ \downarrow\downarrow & & \downarrow\downarrow \\ TM & \xrightarrow[\ -\sigma^t\ ]{} & A^*G \end{array} \tag{2.20}$$

*is a morphism of Lie groupoids, where $\sigma^t : TM \longrightarrow A^*G$ is the bundle map dual to the IM-form $\sigma : AG \longrightarrow T^*M$ induced by $\omega_G$.*

*Proof.* First we check that $\omega_G^\sharp$ preserves the target fibrations. Given $X_g \in T_gG$ we have a covector $\omega_G^\sharp(X_g) \in T_g^*G$. Applying the cotangent target map we obtain $\tilde{t}(\omega_G^\sharp(X_g)) \in A_{t(g)}^*G$, which at every $u_{t(g)} \in A_{t(g)}G$ acts via

$$\tilde{t}(\omega_G^\sharp(X_g))u_{t(g)} = \omega_G^\sharp(X_g)(T_{t(g)}r_g(u_{t(g)})).$$

We can write $X_g = Tt(g)X_g \bullet X_g$ and $T_{t(g)}r_g(u_{t(g)}) = u_{t(g)} \bullet 0_g$, then using the multiplicativity of $\omega_G$ one has the following identity

$$\begin{aligned}\tilde{t}(\omega_G^\sharp(X_g))u_{t(g)} =& \omega_G(Tt(g)X_g \bullet X_g, u_{t(g)} \bullet 0_g)\\ =& \omega_G(Tt(g)X_g, u_{t(g)})\\ =& -\sigma^t(Tt(g)X_g)u_{t(g)}.\end{aligned}$$

That is $\tilde{t}(\omega_G^\sharp(X_g)) = -\sigma^t(Tt(g)X_g)$ which is the compatibility of $\omega_G^\sharp$ with the target maps. A similar computation shows that $\omega_G^\sharp$ is compatible with the source maps. It remains to show that $\omega_G^\sharp$ preserves the groupoid multiplications. For that we consider composable groupoid pairs $(X_g, Y_h), (U_g, V_h) \in TG_{(2)}$, and we easily check that the multiplicativity of $\omega_G$ implies that

$$\begin{aligned}\omega_G^\sharp(X_g \bullet Y_h)(U_g \bullet V_h) =& \omega_G^\sharp(X_g)U_g + \omega_G^\sharp(Y_h)V_h\\ =& (\omega_G^\sharp(X_g) \circ \omega_G^\sharp(Y_h))(U_g \bullet V_h).\end{aligned}$$

That is, for every composable tangent pair $(X_g, Y_h)$ we have

$$\omega_G^\sharp(X_g \bullet Y_h) = \omega_G^\sharp(X_g) \circ \omega_G^\sharp(Y_h).$$

This shows that $\omega_G^\sharp$ is compatible with the groupoid multiplications, proving that $\omega_G^\sharp$ is a groupoid morphism.

□

This proposition suggests a different approach for the study of IM-2-forms. More precisely, since a multiplicative form induces a natural Lie groupoid morphism, it seems that the property of a bundle map $\sigma : A \longrightarrow T^*M$ being an IM-2-form could be translated into a suitable map $TA \longrightarrow T^*A$, constructed out of $\sigma$, being a morphism of Lie algebroids, the latter canonically related to the former via the Lie functor. This relation will be studied in detail in chapter 3.

## 2.5 Multiplicative Dirac structures

In this section we study Lie groupoids equipped with Dirac structures compatible with the groupoid multiplication. These new structures include both multiplicative Poisson and closed 2-forms as particular cases.

### 2.5.1 Definition and examples

Let $G$ be a Lie groupoid over $M$, with Lie algebroid $A(G)$. Consider the direct sum Lie groupoid $\mathbb{T}G = TG \oplus T^*G$ with base manifold $TM \oplus A^*G$.

**Definition 2.5.1.** Let $G$ be a Lie groupoid over $M$. A Dirac structure $L_G$ on $G$ is said to be **multiplicative** if $L_G \subseteq TG \oplus T^*G$ is a subgroupoid over some subbundle $E \subseteq TM \oplus A^*G$.

We refer to a pair $(G, L_G)$, made up of a Lie groupoid $G$ and a multiplicative Dirac structure $L_G$ on $G$, as a **Dirac groupoid**. We use the notation $\mathrm{Dir}_{mult}(G)$ to indicate the set consisting of all multiplicative Dirac structures on $G$.

Notice that a multiplicative Dirac structure $L_G$ on a Lie groupoid $G$ defines a $\mathcal{VB}$-subgroupoid $L_G \subseteq \mathbb{T}G$. See appendix A for this terminology.

**Example 2.5.1.** Let $\omega_G$ be a closed multiplicative 2-form on a Lie groupoid $G$. The multiplicativity property of $\omega_G$ is equivalent to saying that the bundle map $\omega_G^\sharp : TG \longrightarrow T^*G$ is a morphism of Lie groupoids. Hence, the corresponding Dirac structure $L_{\omega_G} = \mathrm{Graph}(\omega_G) \subseteq \mathbb{T}G$ is a multiplicative Dirac structure. In this case we have a groupoid $L_{\omega_G} \rightrightarrows E$ where $E \subseteq TM \oplus A^*G$ is the subbundle given by the graph of the bundle map $-\sigma^t$ determined by the IM-2-form $\sigma$ associated to $\omega_G$.

**Example 2.5.2.** Let $(G, \pi_G)$ be a Poisson groupoid. The multiplicativity of $\pi_G$ is equivalent to saying that $\pi_G^\sharp : T^*G \longrightarrow TG$ is a morphism of Lie groupoids. Therefore, the associated Dirac structure $L_{\pi_G} = \mathrm{Graph}(\pi_G) \subseteq \mathbb{T}G$ defines a multiplicative Dirac structure. In this case we have a groupoid $L_{\pi_G} \rightrightarrows E$ where $E \subseteq TM \oplus A^*G$ is the subbundle given by the graph of dual anchor map $\rho_{A^*G} : A^*G \longrightarrow TM$

**Example 2.5.3.** A regular distribution $F \subseteq TG$ is called **multiplicative** if it defines a Lie subgroupoid of the tangent groupoid $TG$. One checks that every involutive multiplicative distribution on $G$ defines a multiplicative Dirac structure on $G$. The foliation tangent to an involutive multiplicative distribution is called a **multiplicative foliation**. Multiplicative foliations which are simultaneously transversal to the $s$-fibration and to the $t$-fibration were studied in [58], providing interesting examples of noncommutative Poisson algebras.

The examples discussed previously show that Dirac groupoids lead to a natural generalization of Poisson groupoids and presymplectic groupoids. Our main aim is to describe Dirac groupoids infinitesimally, establishing in particular, a connection between such a infinitesimal description and Lie bialgebroids and IM-2-forms. This will be done in chapter 5.

We finish this section with an example of multiplicative Dirac structures given quotients of a Lie group action.

**Example 2.5.4.** Let $L_G$ be a multiplicative Dirac structure on a Lie groupoid $G \rightrightarrows M$, and let $H$ be a Lie group acting on $G$ by groupoid automorphisms. Assume that the $H$-action is free and proper and that the $H$-orbits coincide with the characteristic leaves of $L_G$. In this case the quotient space $G/H$ inherits the structure of a Lie groupoid over $M/H$. Moreover, since $G/H$ is the space of characteristic leaves of $L_G$, we conclude that there exists a Poisson structure $\pi_{red}$ on $G/H$, making the quotient map $G \longrightarrow G/H$ into both a backward and forward Dirac map. This fact together with the multiplicativity of $L_G$ imply that $\pi_{red}$ is a multiplicative Poisson bivector. In other words, the quotient space $G/H$ is a Poisson groupoid.

### 2.5.2 Functorial properties of multiplicative Dirac structures

This is the last section of this chapter. Here we are concerned with a functorial property of multiplicative Dirac structures that will be useful in the forthcoming chapters. Let $G_1 \rightrightarrows M_1$ and $G_2 \rightrightarrows M_2$ be Lie groupoids and $\Phi : G_1 \longrightarrow G_2$ a morphism of Lie groupoids. The tangent and cotangent Lie groupoids $TG_2$ and $T^*G_2$ are vector bundles over $G_2$, so we can consider the pull back vector bundles $\Phi^*(TG_2) \longrightarrow G_1$ and $\Phi^*(T^*G_2) \longrightarrow G_1$. The following property is natural.

**Proposition 2.5.1.** *Let $\Phi : G_1 \longrightarrow G_2$ be a morphism of Lie groupoids covering a map $\varphi : M_1 \longrightarrow M_2$. Assume that $\Phi$ is a surjective submersion. Then the following hold:*

1. *The pull back vector bundle* $\Phi^*(TG_2)$ *inherits a canonical Lie groupoid structure over* $\varphi^*(TM_2)$. *With respect to this groupoid structure the bundle map* $T\Phi : TG_1 \longrightarrow \Phi^*(TG_2)$ *is a morphism of Lie groupoids.*

2. *The pull back vector bundle* $\Phi^*(T^*G_2)$ *inherits a canonical Lie groupoid structure over* $\varphi^*(AG_2)$. *With respect to this groupoid structure the bundle map* $(T\Phi)^* : \Phi^*(T^*G_2) \longrightarrow T^*G_1$ *is a morphism of Lie groupoids.*

*Proof.* We begin with the proof of part *1.* For that we define the structure mappings for $\Phi^*(TG_2) \rightrightarrows \varphi^*(TM_2)$. For each arrow $Y_{\Phi(g)} \in \Phi^*(TG_2)$ we define the source and target maps by

$$s^\Phi(Y_{\Phi(g)}) = Ts_2(Y_{\Phi(g)}), \quad t^\Phi(Y_{\Phi(g)}) = Tt_2(Y_{\Phi(g)}).$$

At composable pairs $Y_{\Phi(g)}, \overline{Y}_{\Phi(h)} \in \Phi^*(TG_2)$ the multiplication map is defined by

$$m^\Phi(Y_{\Phi(g)}, \overline{Y}_{\Phi(h)}) = Y_{\Phi(g)} \bullet \overline{Y}_{\Phi(h)} \in T_{\Phi(gh)}G_2.$$

We also define the unit section $\epsilon^\Phi : \varphi^*(TM_2) \longrightarrow \Phi^*(TG_2)$ by the embedding

$$\epsilon^\Phi(U_{\varphi(x)}) = T\epsilon_2(U_{\varphi(x)}).$$

Finally, the inversion map is given by

$$i^\Phi(Y_{\Phi(g)}) = Ti_2(Y_{\Phi(g)}).$$

The fact that $\Phi : G_1 \longrightarrow G_2$ is a morphism of Lie groupoids implies that each of the mappings defined previously endows $\Phi^*(TG_2)$ with a Lie groupoid structure over $\varphi^*(TM_2)$. It remains to show that wth respect to this groupoid structure the map $T\Phi : TG_1 \longrightarrow \Phi^*(TG_2)$ is a Lie groupoid morphism. First we prove the compaibility with the source maps, which in this case reads

$$s^\Phi \circ T\Phi = T\varphi \circ Ts_1. \tag{2.21}$$

Since $\Phi$ is a groupoid morphism, we have that $s_2 \circ \Phi = \varphi \circ s_1$. Applying the tangent functor we get (2.21). The same argument shows that $T\Phi$ is compatible with the target and multiplication maps.

Now we prove part *2*. For every arrow $\beta_{\Phi(g)} \in \Phi^*(T^*G_2)$ we define the source and target maps by

$$\begin{aligned}\tilde{s}^{\Phi}(\beta_{\Phi(g)}) =& \tilde{s}_2(\beta_{\Phi(g)}) \in A^*_{\varphi(s_1(g))}G_2\\ \tilde{t}^{\Phi}(\beta_{\Phi(g)}) =& \tilde{t}_2(\beta_{\Phi(g)}) \in A^*_{\varphi(t_1(g))}G_2\end{aligned}$$

At every composable pair $\beta_{\Phi(g)}, \overline{\beta_{\Phi(h)}} \in \Phi^*(T^*G_2)$, the multiplication map is determined by

$$\tilde{m}^{\Phi}(\beta_{\Phi(g)}, \overline{\beta_{\Phi(h)}}) = \beta_{\Phi(g)} \circ \overline{\beta_{\Phi(h)}}.$$

Similarly, we can use the unit section and inversion map of $T^*G_2 \rightrightarrows A^*G_2$ to define the unit section and inversion map of $\Phi^*(T^*G_2) \rightrightarrows \varphi^*(A^*G_2)$. This defines the groupoid structure on $\Phi^*(T^*G_2)$. Finally, we show that with respect to this groupoid structure, the bundle map $(T\Phi)^* : \Phi^*(T^*G_2) \longrightarrow T^*G_1$ is a groupoid morphism over the bundle map $(A\Phi)^* : \varphi^*(A^*G_2) \longrightarrow A^*G_1$, which is dual to the map $A\Phi : AG_1 \longrightarrow AG_2$ obtained by applying the Lie functor to $\Phi : G_1 \longrightarrow G_2$. Let us check the compatibility of $(T\Phi)^*$ with the source maps, which in this case reads

$$\tilde{s}_1 \circ (T\Phi)^* = (A\Phi)^* \circ \tilde{s}^{\Phi}. \tag{2.22}$$

For that we consider an arrow $\beta_{\Phi(g)} \in \Phi^*(T^*G_2)$ with $\alpha_g := (T\Phi)^*\beta_{\Phi(g)}$. It follows from the definition of the cotangent source map explained in subsection 2.3.1, that for every $u \in A_{s_1(g)}G_1$ the following identity holds

$$\begin{aligned}\tilde{s}_1(\alpha_g)u =& \alpha_g(Tl_g(u - Tt_1(u))) \qquad (2.23)\\ =& \beta_{\Phi(g)}(T\Phi \circ Tl_g(u - Tt_1(u))), \qquad (2.24)\end{aligned}$$

where $l_g$ is the left multiplication by $g \in G_1$. The fact that $\Phi$ is a groupoid morphism implies $\Phi \circ l_g = l_{\Phi(g)} \circ \Phi$. Also, since the anchor map $\rho_{AG_1} = Tt_1|_{AG_1}$ and $A\Phi = T\Phi|_{AG_1}$ we see that (2.24) leads to

$$\tilde{s}_1((T\Phi)^*\beta_{\Phi(g)})u = \beta_{\Phi(g)}(Tl_{\Phi(g)}(A\Phi(u) - T\varphi \circ \rho_{AG_1}(u))) \tag{2.25}$$

On the other hand, using the definition of $\tilde{s}^\Phi$, we see that the right hand side of (2.22) is given by

$$(A\Phi)^*\tilde{s}^\Phi(\beta_{\Phi(g)})u = \tilde{s}^\Phi(\beta_{\Phi(g)})A\Phi(u) \tag{2.26}$$

$$= \beta_{\Phi(g)}(Tl_{\Phi(g)}(A\Phi(u) - Tt_2 \circ A\Phi(u))) \tag{2.27}$$

Recall that by definition $\rho_{AG_2} = Tt_2|_{AG_2}$. Also, the fact that $A\Phi : AG_1 \longrightarrow AG_2$ is a morphism of Lie algebroids implies that $\rho_{AG_2} \circ A\Phi = T\varphi \circ \rho_{AG_1}$. As a result (2.27) gives rise to

$$(A\Phi)^*\tilde{s}^\Phi(\beta_{\Phi(g)})u = \beta_{\Phi(g)}(Tl_{\Phi(g)}(A\Phi(u) - T\varphi \circ \rho_{AG_1}(u))) \tag{2.28}$$

Therefore, comparing (2.28) with (2.25) we conclude the compatibility (2.22) of $(T\Phi)^*$ with the source maps. A similar computation shows the compatibility of $(T\Phi)^*$ with target maps. That is,

$$\tilde{t}_1 \circ (T\Phi)^* = (A\Phi)^* \circ \tilde{t}^\Phi.$$

It remains to show that $(T\Phi)^*$ preserves multiplication. Indeed, assume that $(\beta_{\Phi(g)}, \overline{\beta_{\Phi(h)}})$ is a composable pair in $\Phi^*(T^*G_2)$, that is

$$\tilde{t}^\Phi(\overline{\beta}_{\Phi(h)}) = \tilde{s}^\Phi(\beta_{\Phi(g)}) \tag{2.29}$$

Define $\alpha_g = (T\Phi)^*\beta_{\Phi(g)}$ and $\overline{\alpha}_h = (T\Phi)^*\overline{\beta}_{\Phi(h)}$. Since $(T\Phi)^*$ is compatible with source and target maps, we see that $(\alpha_g, \overline{\alpha}_h)$ defines a composable pair in $T^*G_1$, so the product arrow $\alpha_g \circ \overline{\alpha}_h \in T^*G_1$ is well defined. Also, if $X_g \bullet Y_h \in T_{gh}G_1$ it follows from the definition of the cotangent multiplication that

$$\begin{aligned}(\alpha_g \circ \overline{\alpha}_h)(X_g \bullet Y_h) =& \alpha_g(X_g) + \overline{\alpha}_h(Y_h) && (2.30)\\ =& \beta_{\Phi(g)}(T\Phi(X_g)) + \overline{\beta}_{\Phi(h)}(T\Phi(Y_h)) && (2.31)\\ =& (\beta_{\Phi(g)} \circ \overline{\beta}_{\Phi(h)})(T\Phi(X_g) \bullet T\Phi(Y_h)) && (2.32)\\ =& (T\Phi)^*(\beta_{\Phi(g)} \circ \overline{\beta}_{\Phi(h)})(X_g \bullet Y_h), && (2.33)\end{aligned}$$

where in the last equality we have used the fact that $T\Phi : TG_1 \longrightarrow \Phi^*(TG_2)$ is a groupoid morphism. Thus we conclude that

$$(T\Phi)^*\beta_{\Phi(g)} \circ (T\Phi)^*\overline{\beta}_{\Phi(h)} = (T\Phi)^*(\beta_{\Phi(g)} \circ \overline{\beta}_{\Phi(h)}),$$

which is exactly the compatibility of $(T\Phi)^*$ with the multiplication.

□

Now we study how multiplicative Dirac structures change by groupoid morphisms which are Dirac maps as well. The following definition is general and it does not depend on groupoids.

**Definition 2.5.2.** Let $M, N$ be smooth manifolds and $\varphi : M \longrightarrow N$ a smooth map. We say that elements $a = X \oplus \alpha \in \mathbb{T}M_x$ and $b = Y \oplus \beta \in \mathbb{T}N_{\varphi(x)}$ are **$\varphi$-related** if $Y = T\varphi(X)$ and $\alpha = (T\varphi)^*\beta$.

Given a Lie groupoid $G \rightrightarrows M$ we consider the direct sum $\mathcal{VB}$-groupoid $\mathbb{T}G \rightrightarrows TM \oplus A^*G$; we denote the multiplication of a composable pair $(a_g, \overline{a}_h)$ in $(\mathbb{T}G)_{(2)}$ by $a_g * \overline{a}_h$.

**Proposition 2.5.2.** *Let $\Phi : G_1 \longrightarrow G_2$ be a morphism of groupoids over $\varphi : M_1 \longrightarrow M_2$, which is a surjective submersion. Assume that $a_g, \overline{a}_h \in \mathbb{T}G_1$ are $\Phi$-related to $b_{\Phi(g)}, \overline{b}_{\Phi(h)} \in \mathbb{T}G_2$. If $a_g, \overline{a}_h$ are composable then $b_{\Phi(g)}, \overline{b}_{\Phi(h)}$ are composable. In this case, $a_g * \overline{a}_h$ is $\Phi$-related to $b_{\Phi(g)} * \overline{b}_{\Phi(h)}$.*

*Proof.* We have seen that $\Phi^*(TG_2)$ and $\Phi^*(T^*G_2)$ have natural structures of Lie groupoids in such a way that $T\Phi : TG_1 \longrightarrow \Phi^*(TG_2)$ and $(T\Phi)^* : \Phi^*(T^*G_2) \longrightarrow T^*G_1$ are morphisms of groupoids.

Set $a_g = (X_g, \alpha_g)$, $\overline{a}_h = (\overline{X}_h, \overline{\alpha}_h)$, and similarly $b_{\Phi(g)} = (Y_{\Phi(g)}, \beta_{\Phi(g)})$, $\overline{b}_{\Phi(h)} = (\overline{Y}_{\Phi(h)}, \overline{\beta}_{\Phi(h)})$. The $\Phi$-relation between these elements reads

$$Y_{\Phi(g)} = T\Phi(X_g), \qquad \overline{Y}_{\Phi(h)} = T\Phi(\overline{X}_h) \tag{2.34}$$

$$\alpha_g = (T\Phi)^* \beta_{\Phi(g)}, \qquad \overline{\alpha}_h = (T\Phi)^* \overline{\beta}_{\Phi(h)}. \tag{2.35}$$

Since $\Phi : G_1 \longrightarrow G_2$ is a surjective submersion, we conclude that $A\Phi : AG_1 \longrightarrow AG_2$ is surjective. In particular, the dual map $(A\Phi)^* : \varphi^*(A^*G_2) \longrightarrow A^*G_1$ is injective. The fact that $a_g, \overline{a}_h$ are composable says that the corresponding tangent components $X_g, \overline{X}_h$ and the cotangent components $\alpha_g, \overline{\alpha}_h$ are composable. Due to the fact that $T\Phi$ is a groupoid morphism, we conclude from (2.34) that $Y_{\Phi(g)}, \overline{Y}_{\Phi(h)}$ are composable. Now we look at the cotangent components. Recall that $\alpha_g, \overline{\alpha}_h$ are composable if and only if

$$\tilde{s}_1(\alpha_g) = \tilde{t}_1(\overline{\alpha}_h). \tag{2.36}$$

The fact that $(T\Phi)^*$ is a groupoid morphism implies that the left hand side of (2.36) is

$$\tilde{s}_1(\alpha_g) = (A\Phi)^*(\tilde{s}_2(\beta_{\Phi(g)})). \tag{2.37}$$

Also, the same argument proves that the right hand side of (2.36) is

$$\tilde{t}_1(\overline{\alpha}_h) = (A\Phi)^*(\tilde{t}_2(\overline{\beta}_{\Phi(h)})). \tag{2.38}$$

Therefore (2.36) implies that

$$(A\Phi)^*(\tilde{s}_2(\beta_{\Phi(g)})) = (A\Phi)^*(\tilde{s}_2(\beta_{\Phi(g)})).$$

Using the injectivity of $(A\Phi)^*$ we conclude that $\tilde{s}_2(\beta_{\Phi(g)}) = \tilde{s}_2(\beta_{\Phi(g)})$, which says that $\beta_{\Phi(g)}, \overline{\beta_{\Phi(h)}}$ are composable. It remains to show that in this case, the product $a_g * \overline{a}_h$ is $\Phi$-related to the product $b_{\Phi(g)} * \overline{b}_{\Phi(h)}$. This is equivalent to the identities

$$Y_{\Phi(g)} \bullet \overline{Y}_{\Phi(h)} = (T\Phi)(X_g \bullet \overline{X}_h) \tag{2.39}$$

$$\alpha_g \circ \alpha_h = (T\Phi)^*(\beta_{\Phi(g)} \circ \overline{\beta}_{\Phi(h)}). \tag{2.40}$$

The fact that $T\Phi$ is a groupoid morphism together with (2.34) imply (2.39). Similarly, we use that $(T\Phi)^*$ is a groupoid morphism and (2.35) to conclude (2.40).

□

**Remark 2.5.1.** Notice that the converse of Proposition 2.5.2 holds only when the base map $\varphi : M_1 \longrightarrow M_2$ has injective derivative. In that case, the fact that $\varphi$ is also a submersion will imply that $\varphi : M_1 \longrightarrow M_2$ is a local diffeomorphism.

As a consequence of Proposition 2.5.2 we obtain a natural way of constructing multiplicative Dirac structures.

**Corollary 2.5.1.** *(Functoriality of multiplicative Dirac structures)*

*Let $\Phi : G_1 \longrightarrow G_2$ be a morphism of Lie groupoids, which is a surjective submersion. Assume that $L_1$ and $L_2$ are Dirac structures on $G_1$ and $G_2$, respectively. If $\Phi$ is a backward Dirac map and $L_2$ is multiplicative, then $L_1$ is multiplicative.*

*Proof.* Recall that $\Phi : (G_1, L_1) \longrightarrow (G_2, L_2)$ is a backward Dirac map if and only if at every $g \in G_1$ one has

$$(L_1)_g = \{X \oplus (T_g\Phi)^*\beta \mid X \in T_gG_1, \beta \in T^*_{\Phi(g)}G_2, \text{ and } T_g\Phi(X) \oplus \beta \in (L_2)_{\Phi(g)}\}.$$

That is, at every $g \in G_1$, the fiber $(L_1)_g$ consists of all elements $a_g$ which are $\Phi$-related to elements $b_{\Phi(g)} \in (L_2)_{\Phi(g)}$. In order to show that $L_1$ is multiplicative, we prove that $L_1 \subseteq \mathbb{T}G_1$ is closed by multiplication. For that, consider $a_g, \overline{a}_h \in L_1$ a composable pair. Since $\Phi$ is backward Dirac, there exist $b_{\Phi(g)}, \overline{b}_{\Phi(h)} \in L_2$, which are $\Phi$-related to $a_g$ and $\overline{a}_h$, respectively. Since $a_g, \overline{a}_h$ are composable, we use Proposition 2.5.2 to conclude that $b_{\Phi(g)}, \overline{b}_{\Phi(h)}$ are composable, and that the product $a_g * \overline{a}_h$ is $\Phi$-related to the product $b_{\Phi(g)} * \overline{b}_{\Phi(h)}$. The fact that $L_2$ is multiplicative implies that $b_{\Phi(g)} * \overline{b}_{\Phi(h)} \in (L_2)_{\Phi(gh)}$. Finally, since $a_g * \overline{a}_h$ is $\Phi$-related to $b_{\Phi(g)} * \overline{b}_{\Phi(h)} \in (L_2)_{\Phi(gh)}$ and the fiber $(L_1)_{gh}$ consists of all elements $\Phi$-related to elements of $(L_2)_{\Phi(gh)}$, we conclude that $a_g * \overline{a}_h \in (L_1)_{gh}$. This proves that $L_1$ is a multiplicative Dirac structure.

□

**Example 2.5.5.** (Reduction of Poisson groupoids)

Let $(G, \pi_G)$ be a Poisson groupoid, and let $J : G \longrightarrow \mathfrak{h}^*$ be a moment map for a Hamiltonian action of a Lie group $H$ on $G$. Assume that the $H$-action is by groupoid automorphisms and that the moment map is multiplicative in the sense that

$$J(g_1 g_2) = J(g_1) + J(g_2).$$

for every composable pair $(g_1, g_2) \in G_{(2)}$. In [49] the reader can find interesting situations where multiplicative moment maps arise. One observes that, whenever $0 \in \mathfrak{h}^*$ is a regular value for $J$, the moment level set $Q = J^{-1}(0)$ is a Lie subgroupoid of $G$. Consider the Dirac structure $L_Q$ as in example 2.2.14. Clearly this defines a multiplicative Dirac structure on the subgroupoid $Q \subseteq G$. Moreover, if the $H$-action is free and proper on the level set $Q$, we conclude from example 2.2.14 that the reduced space $G_{red} = Q/H$ inherits a canonical Poisson structure $\pi_{red}$ in such a way that the projection map $Q \longrightarrow G_{red}$ is both a backward and forward Dirac map. Now we use example 2.5.4 to conclude that $\pi_{red}$ is multiplicative. In other words, the reduced space $(G_{red}, \pi_{red})$ is a Poisson groupoid. See [25] for a detailed discussion about symmetries of Poisson groupoids.

**Example 2.5.6.** Given a Lie groupoid $G \rightrightarrows M$, we define the **isotropy group** at $x \in M$ as

$$G_x := s^{-1}(x) \cap t^{-1}(x).$$

It is clear that $G_x$ is a Lie group. Moreover, the inclusion map $i_{G_x} : G_x \hookrightarrow G$ is a groupoid morphism. Suppose now that $G$ is equipped with a multiplicative Dirac structure $L_G$, and that the restriction of $L_G$ to $G_x$ defines a smooth bundle $L_{G_x}$ over the isotropy group $G_x$. In this case, the bundle $L_{G_x}$ defines a Dirac structure on $G_x$. It follows from the functoriality of multiplicative Dirac structures that $L_{G_x}$ is a multiplicative Dirac structure on the Lie group $G_x$. This is what we call a **Dirac Lie group**. Dirac Lie groups are the main topic of chapter 4.

# Chapter 3

# Multiplicative 2-forms and their infinitesimal counterparts

This chapter is devoted to the study of multiplicative Dirac structures defined by graphs of multiplicative 2-forms. We show that the Lie functor acts naturally on multiplicative forms, establishing a correspondence between multiplicative 2-forms $\omega_G$ on a source simply connected Lie groupoid $G$ and linear 2-forms $\omega_A$ on the Lie algebroid $A$ of $G$ which also define a Lie algebroid morphism $\omega_A^\sharp : TA \longrightarrow T^*A$. The main result of this chapter is the characterization of IM-2-forms on a Lie algebroid $A$ in terms of suitable Lie algebroid morphisms $TA \longrightarrow T^*A$ between the tangent and the cotangent Lie algebroid. In particular, we use Lie's second theorem to give an alternative proof of the correspondence between multiplicative twisted 2-forms on a source simply connected Lie groupoid and IM-2-forms on its Lie algebroid, carried out in [10]. The results presented here may be thought of as dual versions of the results in [46, 48] where the integration of Lie bialgebroids is derived from a combination of Lie's second theorem and the characterization of Lie bialgebroids in terms of suitable *linear* bivectors $T^*A \longrightarrow TA$ which also define Lie algebroid morphisms. In order to understand what the dual version of a linear bivector should be, we recall the main properties and examples of linear forms on vector bundles. Along this chapter we will need some local computations, for that we begin by describing tangent and cotangent Lie algebroids locally. The results proved here are part of the preprint [7].

## 3.1 Tangent lifts of differential forms

This section discusses a natural way of constructing differential forms on a tangent bundle $TM \stackrel{p_M}{\longrightarrow} M$, out of differential forms on its base $M$. Most of the results exposed here can be found in [28, 60]. The direct sum over $M$ of $k$-copies of $TM$ will be denoted by $\prod_{p_M}^k TM$. Given a differential form $\alpha \in \Omega^k(M)$, we induce a canonical bundle map defined by

$$\alpha^\sharp : \prod_{p_M}^{k-1} TM \longrightarrow T^*M$$
$$(X_1, ...X_{k-1}) \mapsto \alpha(X_1, ..., X_{k-1}, \cdot)$$

Notice that the canonical involution $J_M : TTM \longrightarrow TTM$ extends to an isomorphism on higher products

$$J_M^{(k)} : \prod_{p_{TM}}^{k} TTM \longrightarrow \prod_{Tp_M}^{k} TTM.$$

We apply the tangent functor to the bundle map $\alpha^\sharp$, and using the extended canonical involution together with the Tulczyjew map, yields a bundle map $\alpha_T^\sharp : \prod_{p_{TM}}^{k-1} TTM \longrightarrow T^*TM$ defined by

$$\alpha_T^\sharp := \Theta_M \circ (T\alpha^\sharp) \circ J_M^{(k-1)}.$$

In this way, one defines an operation

$$\Omega^k(M) \longrightarrow \Omega^k(TM) \tag{3.1}$$
$$\alpha \mapsto \alpha_T \tag{3.2}$$

where $\alpha_T(V_1, ..., V_{k-1}, V_k) = \alpha_T^\sharp(V_1, ..., V_{k-1})(V_k)$. The $k$-form $\alpha_T$ is called the **tangent lift** of $\alpha$. For more details about tangent lifts of other tensors, see [28, 60]. Now we would like to understand how the de Rham differential acts on tangent lifts of differential forms. For that, let us consider the map $\tau : \Omega^k(M) \longrightarrow \Omega^{k-1}(TM)$ defined by

$$\tau(\alpha)_X = p_M^*(i_X\alpha).$$

The map in (3.1) is related to the map $\tau$ according to the following Cartan type formula

$$\alpha_T = d\tau(\alpha) + \tau(d\alpha). \tag{3.3}$$

See e.g. [7, 28]. In particular, if $\eta := d\alpha$, one has that $(d\alpha)_T = d(\alpha_T)$. That is, the tangent lift (3.1) commutes with the de Rham differential.

## 3.2 Linear forms on vector bundles

Let $A \xrightarrow{q_A} M$ be a vector bundle. The direct sum over $M$ of $k$-copies of $A$ will be denoted by $\prod_{q_A}^k A$.

**Definition 3.2.1.** A $k$-form $\omega_A$ on a vector bundle $A \xrightarrow{q_A} M$ is called **linear** if it defines a morphism of vector bundles

$$\begin{array}{ccc} \prod_{p_A}^{k-1} TA & \xrightarrow{\omega_A^\sharp} & T^*A \\ \downarrow & & \downarrow \\ \prod_{p_M}^{k-1} TM & \xrightarrow[\nu]{} & A^* \end{array} \tag{3.4}$$

where the bottom map $\nu$ is a vector bundle morphism, referred to as the **base bundle** map covered by $\omega_A$.

Henceforth, we will be mainly interested in linear forms of lower degree, namely 2-forms and 3-forms.

**Example 3.2.1.** The canonical symplectic form $\omega_{can}$ on the cotangent bundle $T^*M \longrightarrow M$ is a linear 2-form. The base bundle map $TM \longrightarrow TM$ is the identity map.

**Example 3.2.2.** Let $A, B$ be vector bundles over $M$. Consider a vector bundle morphism $\Psi : A \longrightarrow B$ covering the identity. If $\omega_B$ is a linear $k$-form on $B$, then the pull back form $\omega_A := \Psi^*\omega_B$ defines a linear $k$-form on $A$. Indeed, the induced bundle map

$$\omega_A^\sharp : \prod_{p_A}^{k-1} TA \longrightarrow T^*A,$$

is given, at every fiber over $u \in A$, by $(\omega_A^\sharp)_u = (T_u\Psi)^* \circ (\omega_B^\sharp)_{\Psi(u)} \circ (T_u\Psi)^{(k-1)}$, where $(T\Psi)^{(k-1)} : \prod_{p_A}^{k-1} TA \longrightarrow \prod_{p_B}^{k-1} TB$ denotes the natural extension of $T\Psi : TA \longrightarrow TB$. Thus $\omega_A^\sharp$ is a composition of vector bundle morphisms. The base bundle map covered by $\omega_A$ is given by the composition

$$\Psi^* \circ \nu : \prod_{p_M}^{k-1} TM \longrightarrow A^*.$$

**Example 3.2.3.** Let $\sigma : A \longrightarrow T^*M$ be a bundle map covering the identity. It follows from example 3.2.2 that there is a canonical linear 2-form on $A$, defined by

$$\omega_A := \sigma^*\omega_{can}.$$

Since the canonical form $\omega_{can}$ covers the identity $TM \longrightarrow TM$, we conclude from example 3.2.2 that the base map covered by $\omega_A = \sigma^*\omega_{can}$ is given by the bundle map

$$\sigma^t : TM \longrightarrow A^*,$$

dual to $\sigma$.

It turns out that all linear *closed* 2-forms on a vector bundle $A \longrightarrow M$ are included in example 3.2.3.

**Proposition 3.2.1.** *[37]*

*Every linear closed 2-form $\omega_A$ on a vector bundle $A \longrightarrow M$ is given by*

$$\omega_A = \sigma^*\omega_{can},$$

*where $\sigma : A \longrightarrow T^*M$ is the bundle map dual to the base bundle morphism in (3.4).*

### 3.2.1 Linear forms on Lie algebroids

Now we move to linear forms on a Lie algebroid. For that, assume that $A \xrightarrow{q_A} M$ is a Lie algebroid with anchor map $\rho : A \longrightarrow TM$. According to example 3.2.3, the pull back morphism $\rho^* : \Omega(TM) \longrightarrow \Omega(A)$ provides a natural way to produce linear forms on $A$ out of linear forms on $TM$. In subsection 3.1 we defined an operation

$$\tau : \Omega^k(M) \longrightarrow \Omega^{k-1}(TM),$$

with $\tau(\alpha)_X = p_M^*(i_X\alpha)$. One can see easily that for every $(k+1)$-form $\phi$ on $M$, the $k$-form $\tau(\phi) \in \Omega^k(TM)$ is a linear form, whose base bundle map $\prod_{p_M}^{k-1} TM \longrightarrow A^*$ is the fiberwise zero map. Combining this operation with the pull back morphism $\rho^* : \Omega^k(TM) \longrightarrow \Omega^k(A)$ we are led to a natural class of linear forms on Lie algebroids, those given by $\rho^*(\tau(\phi))$ for some differential form $\phi$ on $M$.

**Proposition 3.2.2.** *Let $A \longrightarrow M$ be a Lie algebroid with anchor map $\rho : A \longrightarrow TM$. Consider a closed 3-form $\phi$ on $M$. Assume that $\omega_A$ is a linear 2-form on A, whose exterior derivative satisfies $d\omega_A = d\rho^*\tau(\phi)$. Then*

$$\omega_A = \sigma^*\omega_{can} + \rho^*\tau(\phi),$$

*where $\sigma : A \longrightarrow T^*M$ is the base bundle map covered by $\omega_A$.*

*Proof.* The linear 2-form $\rho^*\tau(\phi)$ covers the bundle map $A \longrightarrow T^*M$ which is fiberwise zero. Therefore, the linear 2-form $\omega_A - \rho^*\tau(\phi)$ covers the same base bundle map $\sigma : A \longrightarrow T^*M$ covered by $\omega_A$. Since $\omega_A - \rho^*\tau(\phi)$ is closed, we use Proposition (3.2.1) to conclude the statement.

□

### 3.2.2 From multiplicative forms to linear forms

Now we explain another way of constructing linear forms on Lie algebroids. For that, let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid $AG$. Recall that a $k$-form $\omega_G$ on $G$ is called multiplicative if

$$m^*\omega_G = pr_1^*\omega_G + pr_2^*\omega_G,$$

where $m : G_{(2)} \longrightarrow G$ is the groupoid multiplication and $pr_1, pr_2 : G_{(2)} \longrightarrow G$ are the natural projections. We denote by $\Omega^k_{mult}(G)$ the set of all multiplicative $k$-forms on a Lie groupoid $G$. The $k$-degree version of the proof of Proposition 2.4.3 shows that the induced bundle map $\omega_G^\sharp : \prod_{p_G}^{k-1} TG \longrightarrow T^*G$ is a groupoid morphism, see e.g. [7]. The application of the Lie functor to $\omega^\sharp_G$, yields a Lie algebroid morphism

$$A(\omega^\sharp_G) : \prod_{A(p_G)}^{k-1} A(TG) \longrightarrow A(T^*G).$$

We can use the natural morphisms of Lie algebroids $j'_G : A(T^*G) \longrightarrow T^*(AG)$ and $j_G^{(k-1)} : \prod_{p_{AG}}^{k-1} T(AG) \longrightarrow \prod_{A(p_G)}^{k-1} A(TG)$, explained in section 2.3.2 of chapter 2, to define a Lie algebroid morphism

$$\omega^\sharp_{AG} : \prod_{p_{AG}}^{k-1} T(AG) \longrightarrow T^*(AG),$$

with $\omega^\sharp_{AG} := j'_G \circ A(\omega^\sharp_G) \circ j_G^{(k-1)}$. Notice the similarity of $\omega^\sharp_{AG}$ with the construction of the tangent lift of $k$-forms explained in subsection 3.1 of Appendix A. This similarity is clarified by the following proposition.

**Proposition 3.2.3.** *Let $(\omega_G)_T \in \Omega^k(TG)$ be the tangent lift of the multiplicative form $\omega_G \in \Omega^k(G)$. Consider the linear $k$-form $\Lambda$ on $AG$ defined by*

$$\Lambda = i^*_{AG}(\omega_G)_T,$$

*where $i_{AG} : AG \hookrightarrow TG$ is the natural bundle inclusion. Then $\Lambda^\sharp = \omega^\sharp_{AG}$.*

*Proof.* Recall that $j'_G = (Ti_{AG})^* \circ \Theta_G \circ i_{A(T^*G)}$ and $J_G \circ Ti_{AG} = i_{A(TG)} \circ j_G$. Thus extending to higher products we have that

$$J_G^{(k)} \circ (\prod^k Ti_{AG}) = (\prod^k i_{A(TG)}) \circ j_G^{(k)}.$$

On the other hand $i_{A(T^*G)} \circ A(\omega^\sharp_G) = T\omega^\sharp_G \circ \prod^{k-1} i_{A(TG)}$, thus we get

$$\begin{aligned} j'_G \circ A(\omega^\sharp_G) \circ j_G^{(k-1)} &= (Ti_{AG})^* \circ \Theta_G \circ T\omega^\sharp_G \circ \prod^{k-1} i_{A(TG)} \circ j_G^{(k-1)} \\ &= (Ti_{AG})^* \circ (\omega_G)^\sharp_T \circ (\prod^{k-1} Ti_{AG}) \\ &= (i^*_{AG}(\omega_G)_T)^\sharp \\ &= \omega^\sharp_{AG}, \end{aligned}$$

as desired. □

Due to the result above, we conclude that every multiplicative $k$-form $\omega_G$ on a Lie groupoid $G$ induces a linear $k$-form $\omega_{AG} := i^*_{AG}(\omega_G)_T$ on its Lie algebroid $AG$. Moreover, since

$$\omega^\sharp_{AG} = j'_G \circ A(\omega^\sharp_G) \circ j^{(k-1)}_G,$$

is the composition of morphism of Lie algebroids, we conclude that $\omega^\sharp_{AG} : T(AG) \longrightarrow T^*(AG)$ is a Lie algebroid morphism. In [47] the concept of morphic 1-form on a Lie algebroid was introduced. A 1-form $\alpha$ on a Lie algebroid $A$ is called morphic if $\alpha : A \longrightarrow T^*A$ is a Lie algebroid morphism. Moreover, they proved that the Lie functor applied to a multiplicative 1-form on a Lie groupoid gives rise to a morphic 1-form on its Lie algebroid. This motivates the following definition.

**Definition 3.2.2.** A linear $k$-form $\omega_A$ on a Lie algebroid $A \xrightarrow{q_A} M$ is called **morphic** if the induced bundle map (3.4) defines a morphism of Lie algebroids.

We denote by $\Omega^k_{mor}(A)$ the set of all morphic $k$-forms on a Lie algebroid $A$. Just as the Lie functor applied to multiplicative 1-forms on a Lie groupoid yields morphic 1-forms on its Lie algebroid [47], we see that the effect of the Lie functor on multiplicative $k$-forms on a Lie groupoid $G$ is determined by the map

$$\Omega^k_{mult}(G) \longrightarrow \Omega^k_{mor}(AG) \tag{3.5}$$

$$\omega_G \mapsto \omega_{AG}, \tag{3.6}$$

where $\omega_{AG} = i^*_{AG}(\omega_G)_T$.

**Remark 3.2.1.** Since the de Rham differential maps multiplicative forms into multiplicative forms, and it commutes with tangent lifts of differential forms (see formula 3.3 in Appendix A), we derive the following formula:

$$(d\omega_G)_{AG} = d\omega_{AG}. \tag{3.7}$$

In particular, (3.5) maps closed multiplicative forms into closed morphic forms.

Every closed 3-form $\phi$ on $M$ induces a multiplicative 3-form $\phi_G \in \Omega^3_{mult}(G)$, defined by

$$\phi_G = s^*\phi - t^*\phi.$$

Let us find the induced morphic 3-form $\phi_{AG}$ on $AG$.

**Proposition 3.2.4.** $\phi_{AG} = -d\rho^*(\tau(\phi))$.

*Proof.* By definition of the induced morphic form, we have

$$\phi_{AG} = i^*_{AG}(s^*\phi)_T - i^*_{AG}(t^*\phi)_T.$$

Combining the fact $d\phi = 0$ with the Cartan type formula (3.3) for the tangent lift of a differential form, we obtain

$$(s^*\phi)_T = d\tau(s^*\phi) \qquad \text{and } (t^*\phi)_T = d\tau(t^*\phi).$$

One easily observes that $\tau(s^*\phi) = (Ts)^*\tau(\phi)$ and $\tau(t^*\phi) = (Tt)^*\tau(\phi)$. Thus we get

$$\phi_{AG} = d(Ts \circ i_{AG})^*\tau(\phi) - d(Tt \circ i_{AG})^*\tau(\phi).$$

Since $AG = \ker(Ts)|_M$ and the anchor map is defined by $\rho = Tt \circ i_{AG}$, the statement follows.

□

Notice also that whenever $G$ has connected source fibers, the infinitesimal property $\phi_{AG} = -d\rho^*\tau(\phi)$ characterizes the multiplicative form $\phi_G = s^*\phi - t^*\phi$. See remark 2.1.1 in chapter 2.

Consider now a multiplicative 2-form $\omega_G$ on $G$ with

$$d\omega_G = s^*\phi - t^*\phi.$$

As in (2.13) we consider the associated bundle map

$$\sigma : AG \longrightarrow T^*M \tag{3.8}$$

$$u \mapsto (i_u\omega)_{|TM}. \tag{3.9}$$

One observes that the groupoid morphism $\omega_G^\sharp : TG \longrightarrow T^*G$ covers the bundle map $-\sigma^t : TM \longrightarrow A^*G$. See Proposition 2.4.3.

**Proposition 3.2.5.** *Let $\omega_G$ be a multiplicative 2-form on $G$. Let $\phi \in \Omega^3(M)$ be closed 3-form and assume that $d\omega_G = s^*\phi - t^*\phi$. Then the morphic 2-form on $AG$ associated to $\omega_G$ is*

$$\omega_{AG} = -(\sigma^*\omega_{can} + \rho^*\tau(\phi)),$$

*where $\omega_{can}$ is the canonical symplectic form on $T^*M$.*

*Proof.* The fact $d\omega_G = s^*\phi - t^*\phi$, combined with (3.7), imply that the morphic 2-form $\omega_{AG}$ satisfies

$$d\omega_{AG} = -d\rho^*\tau(\phi).$$

Thus, the hypothesis of Proposition 3.2.2 is fulfilled, and the statement follows.

□

The morphic 2-form $\omega_{AG} = -(\sigma^*\omega_{can} + \rho^*\tau(\phi))$, was constructed out of a global data. Namely, we applied the Lie functor to the multiplicative 2-form $\omega_G$ with $d\omega_G = s^*\phi - t^*\phi$. Conversely, assume that $A \longrightarrow M$ is a Lie algebroid with anchor map $\rho : A \longrightarrow TM$. Consider also a bundle map $\sigma : A \longrightarrow T^*M$, a closed 3-form $\phi$ on $M$, and look at the canonical linear 2-form $\Lambda$ on $A$ defined by

$$\Lambda = -\sigma^*\omega_{can} - \rho^*\tau(\phi).$$

We would like to find a *purely infinitesimal* condition on $\sigma$ and on $\phi$, in such a way that $\Lambda \in \Omega^2(A)$ be a morphic 2-form. This will be explained in the last section of this chapter.

## 3.3 Structure functions of tangent and cotangent Lie algebroids

Let $A \longrightarrow M$ be a Lie algebroid with anchor map $\rho : A \longrightarrow TM$ and Lie bracket $[\cdot,\cdot]$ on $\Gamma_M(A)$. As explained in chapter 2 section 2.3.2, there exist canonical Lie algebroid structures on the vector bundles $TA \longrightarrow TM$ and $T^*A \longrightarrow A^*$. Consider $(x^j)_{j=1,\ldots,\dim(M)}$ local coordinates on $M$, and $\{e_a\}$ a basis of local sections of $A$, which defines **structure functions** $\rho_a^j$ and $C_{ab}^c$ of the Lie algebroid $A$. These structure functions are determined by

$$\rho(e_a) = \rho_a^j \frac{\partial}{\partial x^j}, \qquad [e_a, e_b] = C_{ab}^c e_c.$$

According to subsection A.0.1 of appendix A, every section $u \in \Gamma_M(A)$ induces sections $Tu, \hat{u} \in \Gamma_{TM}(TA)$. According to the definition of the tangent anchor map $\rho_{TA} : TA \longrightarrow TTM$ and the tangent Lie bracket $[\cdot,\cdot]_{TA}$ on $\Gamma_{TM}(TA)$, we conclude that the structure functions of the tangent Lie algebroid $TA \longrightarrow TM$ are determined by

$$[\widehat{e}_a, \widehat{e}_b]_{TA} = 0, \quad [Te_a, \widehat{e}_b]_{TA} = C_{ab}^c \widehat{e}_c, \quad [Te_a, Te_b]_{TA} = C_{ab}^c Te_c + dC_{ab}^c \widehat{e}_c, \tag{3.10}$$

$$\rho_{TA}(Te_a) = \rho_a^j \frac{\partial}{\partial x^j} + d\rho_a^j \frac{\partial}{\partial \dot{x}^j}, \quad \rho_{TA}(\widehat{e}_a) = \rho_a^j \frac{\partial}{\partial \dot{x}^j}. \tag{3.11}$$

Consider $\{e^a\}$ the basis of local sections of $A^*$, dual to $\{e_a\}$. , we induce coordinates $(x^j, \xi_a)$ on $A^*$. With respect to $\{e_a\}$ we have coordinates $(x^j, u^a)$ on $A$. On the cotangent bundle $T^*A$ we use local coordinates of the form $(x^j, u^a, p_j, \lambda_a)$, where $(p_j)$ determines an element in $T_x^*M$ and $(\lambda_a)$ defines an element in $A_x^*$. As indicated in subsection A.0.1 of appendix A, every section $u \in \Gamma_M(A)$ induces a linear section $u^L \in \Gamma_{A^*}(T^*A)$, which is locally described by

$$u^L(x^i, \xi_a) = (x^i, u^a(x), 0, \xi_a),$$

where $u = u^a e_a$. Also, given a section $\alpha : M \longrightarrow T^*M$ of the core[1] of $T^*A \longrightarrow A^*$, we have the corresponding core section $\hat{\alpha} \in \Gamma_{A^*}(T^*A)$, which is locally given by

$$\hat{\alpha}(x^i, \xi_a) = (x^i, 0, \alpha_i(x), \xi_a),$$

[1]The definition of the core of a double vector bundle can be found in appendix A

where $\alpha = \alpha_i dx^i$. With respect to this local description, the structure functions of the cotangent algebroid $T^*A \longrightarrow A^*$ are determined by

$$[\widehat{dx^i}, \widehat{dx^j}]_{T^*A} = 0, \;\; [e^L_a, \widehat{dx^j}]_{T^*A} = \widehat{d\rho^j_a}, \;\; [e^L_a, e^L_b]_{T^*A}|_{(x,\xi)} = -\widehat{dC^c_{ab}}\xi_c + C^c_{ab}e^L_c, \tag{3.12}$$

$$\rho_{T^*A}(\widehat{dx^i}) = \rho^i_a \frac{\partial}{\partial \xi_a}, \quad \rho_{T^*A}(e^L_a)|_{(x,\xi)} = \rho^i_a \frac{\partial}{\partial x^i} + C^c_{ab}\xi_c \frac{\partial}{\partial \xi_b}. \tag{3.13}$$

## 3.4 Integration of IM-2-forms via Lie's second Theorem

Let $A \longrightarrow M$ be a Lie algebroid, with bracket $[\cdot, \cdot]$ and anchor $\rho$. Let $\sigma : A \longrightarrow T^*M$ be a vector bundle map and $\phi \in \Omega^3(M)$ a closed 3-form. Let us consider the linear 2-form $\Lambda \in \Omega^2(A)$ defined by

$$\Lambda = -(\sigma^*\omega_{can} + \rho^*\tau(\phi)), \tag{3.14}$$

covering $-\sigma^t : TM \longrightarrow T^*M$. We give a necessary and sufficient condition on $\sigma$ and $\phi$ in such a way that $\Lambda^\sharp : TA \longrightarrow T^*A$ defines a Lie algebroid morphism. Recall that the notion of morphism of Lie algebroids was presented in chapter 2 definition 2.1.4.

**Theorem 3.4.1.** *Let $\Lambda \in \Omega^2(A)$ be as in* (3.14)*. The following are equivalent:*

*(i) $\Lambda$ is a morphic 2-form on $A$.*

*(ii) The map $\sigma : A \longrightarrow T^*M$ is an IM-2-form with respect to $\phi$. That is*

$$\langle \sigma(u), \rho(v) \rangle = -\langle \sigma(v), \rho(u) \rangle$$
$$\sigma([u, v]) = \mathcal{L}_{\rho(u)}\sigma(v) - \mathcal{L}_{\rho(v)}\sigma(u) + i_{\rho(v)}i_{\rho(u)}\phi,$$

*for all $u, v \in \Gamma(A)$.*

In order to prove Theorem 3.4.1 it will be useful to make some local computations. For that we follow the local description of the tangent and cotangent algebroids, presented in the first section of this chapter.

*Proof.* A system of local coordinates $(x^j)$ on $M$ induces local coordinates $(x^j, \dot{x}^j)$ on the tangent bundle $TM$, and $(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j)$ on the double tangent bundle $TTM$. Let $\{e_a\}$

be a basis of local sections of $A$, and $\{e^a\}$ the basis of local sections of $A^*$, dual to $\{e_a\}$. We induce local coordinates $(x^j, u^a)$ on $A$ and $(x^j, \xi_a)$ on $A^*$. The tangent bundle $TA$ will be described by the local coordinates system $(x^j, u^a, \dot{x}^j, \dot{u}^a)$, and similary we have local coordinates $(x^j, u^a, p_j, \lambda_a)$ on the cotangent bundle $T^*A$. The bundle map $\sigma : A \longrightarrow T^*M$ can be locally written as

$$\sigma(x^j, u^a) = (x^j, u^a \sigma_{ja}(x)).$$

Thus the dual bundle map $\sigma^t : TM \longrightarrow A^*$ has the local form

$$\sigma^t(x^j, \dot{x}^j) = (x^j, \dot{x}^j \sigma_{ja}).$$

We also write the 3-form $\phi \in \Omega^3(M)$ locally, as

$$\phi = \frac{1}{6}\phi_{ijk} dx^i \wedge dx^j \wedge dx^k.$$

Consider the bundle map $\Lambda^\sharp : TA \longrightarrow T^*A$ induced by the linear 2-form $\Lambda$. Recall that $\Lambda^\sharp$ covers the bundle map $-\sigma^t : TM \longrightarrow A^*$. A straightforward computation shows that

$$\Lambda^\sharp(x^j, u^d, \dot{x}^j, \dot{u}^d) = (x^j, u^d, p_j, \lambda_d), \tag{3.15}$$

with coordinates $(p_j) \in T_x^*M$ and $(\lambda_d) \in A_x^*$ are determined by

$$\begin{aligned} p_j =& \dot{x}^l u^d \left( \frac{\partial \sigma_{jd}}{\partial x^l} - \frac{\partial \sigma_{ld}}{\partial x^j} \right) + \dot{u}^d \sigma_{jd} - \phi_{ijk} u^d \rho_d^k \dot{x}^i, \\ \lambda_d =& - \dot{x}^l \sigma_{ld}. \end{aligned}$$

We want to show that $(i)$ and $(ii)$ are equivalent. Recall that, by definition, $\Lambda$ is a morphic 2-form on $A$ if and only if $\Lambda^\sharp : TA \longrightarrow T^*A$ is a Lie algebroid morphism covering $-\sigma^t : TM \longrightarrow A^*$. Let us study the compatibility of $\Lambda^\sharp$ with the tangent and cotangent anchor maps, defined in (3.11) and (3.13), respectively. Recall that $\Gamma_{TM}(TA)$ is generated by sections of the form $Te_a, \hat{e}_a$, with $e_a \in \Gamma_M(A)$. Thefore, it suffices to show the compatibility of $\Lambda^\sharp$ with the anchors at linear and core sections $Te_a$ and $\hat{e}_a$, respectively.

For that, notice that the morphism of double vector bundles $\Lambda^\sharp : TA \longrightarrow T^*A$ maps core sections into core sections. We easily check that

$$\Lambda^\sharp(\hat{e}_b(x^j, \dot{x}^j)) = (x^j, 0, \sigma_{jb}, -\dot{x}^l\sigma_{ld}) \tag{3.16}$$

Similarly, the morphism of double vector bundles $\Lambda^\sharp : TA \longrightarrow T^*A$ maps linear sections into a combination of linear and core sections. Therefore, a direct computation using (3.15) gives

$$\Lambda^\sharp(Te_b(x^j, \dot{x}^j)) = (x^j, \delta_{bd}, p_j, -\dot{x}^l\sigma_{ld}), \tag{3.17}$$

with the coordinates $(p_j)$ determined by

$$p_j = \dot{x}^l \left( \frac{\partial \sigma_{jb}}{\partial x^l} - \frac{\partial \sigma_{lb}}{\partial x^j} \right) - \phi_{ijk}\rho_b^k \dot{x}^i.$$

Recall that the compatibility of $\Lambda^\sharp$ with the tangent and cotangent anchor maps means

$$\rho_{T^*A} \circ \Lambda^\sharp = T(-\sigma^t) \circ \rho_{TA}. \tag{3.18}$$

We will need an explicit formula for the derivative of the bundle map $-\sigma^t : TM \longrightarrow A^*$. Using the local description of $\sigma^t$, we conclude that

$$T(-\sigma^t)(x^j, \dot{x}^j, \delta x^j, \delta\dot{x}^j) = (x^j, -\dot{x}^l\sigma_{ld}, \delta x^j, \lambda_d) \in TA^*, \tag{3.19}$$

where the coordinates $(\lambda_d)$ are given by

$$\lambda_d = -\dot{x}^l \frac{\partial \sigma_{ld}}{\partial x^k}\delta x^k - \sigma_{ld}\delta\dot{x}^l.$$

Let us check (3.18) at a core section $\hat{e}_b$. One uses the definition of the cotangent anchor map (3.13) and the local descripition (3.16) for $\Lambda^\sharp$ at core sections to conclude that the left hand side of (3.18), at a core section $\hat{e}_b$, is determined by

$$\rho_{T^*A}(\Lambda^\sharp(\hat{e}_b(x, \dot{x}))) = (x^j, -\dot{x}^l\sigma_{ld}, 0, \rho_d^l\sigma_{lb}) \in TA^*$$

On the other hand, the tangent anchor applied to $\hat{e}_b$ is determined by (3.11). Thus we use (3.19) to conclude that the right hand side of (3.18) is given by

$$T(-\sigma^t)(\rho_{TA}(\hat{e}_b(x,\dot{x}))) = (x^j, -\dot{x}^l\sigma_{ld}, 0, -\sigma_{ld}\rho_b^l).$$

Thus we immediatly observe that, at a core section $\hat{e}_b$, the identity (3.18) holds if and only if

$$\rho_d^l\sigma_{lb} = -\sigma_{ld}\rho_b^l.$$

Or equivalently,

$$\langle\sigma(e_d), \rho(e_b)\rangle = -\langle\sigma(e_b), \rho(e_d)\rangle,$$

for every pair of sections $e_a, e_b$ of $A$. This is exactly the first property of an IM-2-form with respect to $\phi$. Now let us check that (3.18) holds at every linear section $Te_b$. We use the local description (3.17) of $\Lambda^\sharp$ at a linear section $Te_b$ and the definition of the cotangent anchor (3.13) to conclude that the left hand side of (3.18) is given by

$$\rho_{T^*A}(\Lambda^\sharp(Te_b(x,\dot{x}))) = (x^j, -\dot{x}^l\sigma_{ld}, \rho_b^j, \lambda_d) \in TA^*,$$

where the coordinates $(\lambda_d)$ are determined by

$$\begin{aligned}\lambda_d &= \dot{x}^l\rho_d^k\left(\frac{\partial\sigma_{kb}}{\partial x^l} - \frac{\partial\sigma_{lb}}{\partial x^k}\right) - \phi_{ijk}\rho_b^k\dot{x}^i\rho_d^j + C_{db}^c\dot{x}^l\sigma_{lc} \\ &= \big\langle -i_{\rho(e_d)}(d\sigma(e_b)) + i_{\rho(e_d)}i_{\rho(e_b)}\phi + \sigma([e_d, e_b]), \dot{x}\big\rangle. \end{aligned} \tag{3.20}$$

On the other hand, the tangent anchor applied to $Te_b$ is determined by (3.11). Thus we use (3.19) to conclude that the right hand side of (3.18) is given by

$$T(-\sigma^t)(\rho_{TA}(Te_b(x,\dot{x}))) = (x^j, -\dot{x}^l\sigma_{ld}, \rho_b^j, \lambda'_d) \in (-\sigma^t)^*TA^*,$$

where the coordinates $(\lambda'_d)$ are determined by,

$$\lambda'_d = -\dot{x}^l\left(\frac{\partial\sigma_{ld}}{\partial x^k}\rho_b^k + \sigma_{id}\frac{\partial\rho_b^i}{\partial x^l}\right) = -\big\langle\mathcal{L}_{\rho(e_b)}\sigma(e_d), \dot{x}\big\rangle \tag{3.21}$$

Thus the identity (3.18) holds at a linear section $Te_b$ if and only if (3.20) and (3.21) coincide.

Equivalently, if and only if

$$\sigma([u,v]) = \mathcal{L}_{\rho(u)}\sigma(v) - \mathcal{L}_{\rho(v)}\sigma(u) + i_{\rho(v)}i_{\rho(u)}\phi,$$

for every $u, v$ sections of $A$. This is exactly the second property of an IM-2-form with respect to $\phi$. This proves that (3.18) is fulfilled.

It remains to show that $\Lambda^\sharp : TA \longrightarrow T^*A$ is a bracket preserving map. Recall that, according to Definition 2.1.4 in chapter 2, the bracket preserving property for $\Lambda^\sharp$ means

$$\begin{aligned}\Lambda^\sharp([U,V]_{TA}) =& f_j g_i(-\sigma^T)^*[U_j, V_i]_{T^*A} + \mathcal{L}_{\rho_{TA}(U)} g_i(-\sigma^t)^* V_i \\ &- \mathcal{L}_{\rho_{TA}(V)} f_j(-\sigma^t)^* U_j,\end{aligned} \tag{3.22}$$

where $U, V \in \Gamma_{TM}(TA)$, and $f_j, g_i \in C^\infty(TM)$, $U_j, V_i \in \Gamma_{A^*}(T^*A)$ are such that

$$\Lambda^\sharp(U) = f_j(-\sigma^t)^* U_j \quad \text{and} \quad \Lambda^\sharp(V) = g_i(-\sigma^t)^* V_i.$$

Again, it suffices to check (3.22) when $U, V$ represent all the possible combinations of linear and core sections. We need to determine the functions $f_j, g_i \in C^\infty(TM)$ for each of these cases. Recall that every section $e_a$ of $A$ induces a linear section of $T^*A \longrightarrow A^*$, given locally by

$$e_a^L(x^j, \xi_d) = (x^j, \delta_{ad}, 0, \xi_d).$$

Similarly, as explained in appendix A, every section $\alpha : M \longrightarrow T^*M$ of the core of $T^*A$, induces a core section of $T^*A \longrightarrow A^*$, locally determined by

$$\alpha^L(x^j, \xi_d) = (x^j, 0, \alpha_j(x), \xi_d),$$

where $\alpha = \alpha_j dx^j$.

We use (3.16) to conclude that

$$\Lambda^\sharp(\hat{e}_a(x,\dot{x})) = \widehat{\sigma(e_a)}(-\sigma^t(x,\dot{x})) = g_i^a \widehat{dx^i}(-\sigma^t(x,\dot{x})), \tag{3.23}$$

where $g_i^a(x,\dot{x}) = \sigma_{ia}(x)$. Similarly, we use (3.17) to conclude that

$$\Lambda^\sharp(Te_a(x,\dot{x})) = e_a^L(-\sigma^t(x,\dot{x})) + f_j^a\widehat{dx^j}(-\sigma^t(x,\dot{x})), \tag{3.24}$$

where

$$f_j^a(x,\dot{x}) = \dot{x}^l\left(\frac{\partial\sigma_{ja}}{\partial x^l} - \frac{\partial\sigma_{la}}{\partial x^j}\right) - \phi_{ijk}\rho_a^k\dot{x}^i.$$

Let us observe that a 1-form on $TM$ of the type $\alpha_j(x,\dot{x})dx^j$ can be identified with a section of the pull back bundle $(-\sigma^t)^*(T^*A)$, via

$$\alpha_j(x,\dot{x})dx^j \mapsto \alpha_j(x,\dot{x})dx^j, \tag{3.25}$$

where, since there is no risk of confusion, in the right hand side of (3.25) we abuse notation writing $dx^j$ instead of $dx^j(-\sigma^t(x,\dot{x}))$. With respect to the identification (3.25) we have

$$f_j^a dx^j = i_{\dot{x}}d\sigma(e_a) - i_{\dot{x}}i_{\rho(e_a)}\phi, \tag{3.26}$$

with $\dot{x} = \dot{x}^l\frac{\partial}{\partial x^l} \in \mathfrak{X}(TM)$. Now we are ready to prove the bracket preserving property (3.22). As we said before, since we only need to consider all the possible combinations of linear and core sections, in order to check (3.22) we study three cases.

*Case 1: Core-Core sections*

Take $U = \hat{e}_a$ and $V = \hat{e}_b$ in (3.22). By definition of the tangent Lie bracket (3.10) we have $[\hat{e}_a,\hat{e}_b]_{TA} = 0$. Thus the left hand side of (3.22) vanishes. Similarly, the definition (3.12) of the cotangent bracket says $[\widehat{dx^i},\widehat{dx^j}]_{T^*A} = 0$. The tangent anchor at a core section gives a vertical vector field on $TM$, that is, a vector field tangent to the fibres. The right hand side of (3.22) is a combination of $[\widehat{dx^i},\widehat{dx^j}]_{T^*A} = 0$ and derivatives of $g_i^a(x,\dot{x}) = \sigma_{ia}(x)$ with respect to the variable $\dot{x}$. Since $g_i^a$ just depend on the variable $x$, we conclude that the right hand side of (3.22) vanishes as well. This shows that (3.22) holds at a pair of core sections.

*Case 2: Linear-Core sections*

Take $U = Te_a$ and $V = \hat{e}_b$ in (3.22). According to (3.10), the tangent bracket of linear and core sections is determined by $[Te_a, \hat{e}_b]_{TA} = C^c_{ab}\hat{e}_c$. Using (3.23) we see that the left hand side of (3.22) is given by

$$\Lambda^\sharp([Te_a, \hat{e}_b]) = \sigma([e_a, e_b]).$$

On the other hand, the right hand side of (3.22) is given by the sum of three terms $T_1 + T_2 + T_3$. A straightforward computation, based on the structure functions (3.11) and (3.12) for the tangent and cotangent algebroids, shows that

$$\begin{aligned}
T_1 =& \sigma_{ib} d\rho^i_a. \\
T_2 =& (\mathcal{L}_{\rho_{TA}(Te_a)}\sigma_{ib})dx^i \\
=& \mathcal{L}_{\rho^l_a \frac{\partial}{\partial x^l}}(\sigma_{ib}dx^i) - \sigma_{ib}\mathcal{L}_{\rho^l_a \frac{\partial}{\partial x^l}} dx^i \\
=& \mathcal{L}_{\rho(e_a)}\sigma(e_b) - \sigma_{ib}d\rho^i_a. \\
T_3 =& (\mathcal{L}_{\rho_{TA}(\widehat{e}_b)} f^a_j)dx^j \\
=& \Big(\rho^l_b \left(\frac{\partial \sigma_{ja}}{\partial x^l} - \frac{\partial \sigma_{la}}{\partial x^j}\right) - \phi_{ijk}\rho^k_a \rho^i_b\Big)dx^j \\
=& i_{\rho(e_b)} d\sigma(e_a) - i_{\rho(e_b)} i_{\rho(e_a)}\phi
\end{aligned}$$

Therefore, the bracket preserving property (3.22) in this case is equivalent to

$$\sigma([e_a, e_b]) = T_1 + T_2 + T_3 \tag{3.27}$$

$$= \sigma_{ib}d\rho^i_a + \mathcal{L}_{\rho(e_a)}\sigma(e_b) - \sigma_{ib}d\rho^i_a + \tag{3.28}$$

$$+ i_{\rho(e_b)}d\sigma(e_a) - i_{\rho(e_b)}i_{\rho(e_a)}\phi. \tag{3.29}$$

One easily observes that (3.29) is equivalent to

$$\sigma([u, v]) = \mathcal{L}_{\rho(u)}\sigma(v) - \mathcal{L}_{\rho(v)}\sigma(u) + i_{\rho(v)}i_{\rho(u)}\phi,$$

for every $u, v$ sections of $A$, which is exactly the second property of an IM-2-form with respect to $\phi$.

*Case 3: Linear-Linear sections*

This is the last case to be verified. Take $U = Te_a$ and $V = Te_b$ in (3.22). It follows from the definition of the tangent Lie bracket (3.10) and the formulas (3.24) and (3.23), that the left hand side of (3.22) is given by

$$\begin{aligned}\Lambda^\sharp([Te_a, Te_b]_{TA}) &= C^c_{ab} e^L_c|_{-\sigma^t(x,\dot{x})} + C^c_{ab} f^c_j dx^j + dC^c_{ab}(\dot{x})\sigma(e_c) \\ &= [e_a, e_b]^L|_{-\sigma^t(x,\dot{x})} + C^c_{ab}(i_{\dot{x}} d\sigma(e_c) - i_{\dot{x}} i_{\rho(e_c)}\phi) + dC^c_{ab}(\dot{x})\sigma(e_c) \\ &= [e_a, e_b]^L|_{-\sigma^t(x,\dot{x})} + i_{\dot{x}} d\sigma([e_a, e_b]) + dC^c_{ab}\langle\sigma(e_c), \dot{x}\rangle - i_{\dot{x}} i_{[\rho(e_a),\rho(e_b)]}\phi.\end{aligned}$$

Recall that (3.24) says that

$$\begin{aligned}\Lambda^\sharp(Te_a) =& e^L_a + f^a_j dx^j \\ \Lambda^\sharp(Te_b) =& e^L_b + f^b_i dx^i.\end{aligned}$$

The right hand side of (3.22) is given by the sum of three terms $S_1 + S_2 + S_3$. A direct computation, using the structure functions (3.11) and (3.12) for the tangent and cotangent algebroids, shows that

$$\begin{aligned}S_1 =& [e_a, e_b]^L|_{-\sigma^t(x,\dot{x})} + dC^c_{ab}\langle\sigma^t(\dot{x}), e_c\rangle - f^a_j d\rho^j_b + f^b_i d\rho^i_a. \\ S_2 =& \mathcal{L}_{\rho_{TA}(Te_a)}(f^b_i)dx^i. \\ S_3 =& \mathcal{L}_{\rho_{TA}(Te_b)}(f^a_j)dx^j.\end{aligned}$$

We can use the fact that the Lie derivative is a derivation of degree zero, to conclude that

$$S_2 = \mathcal{L}_{\rho_{TA}(Te_a)}(f^b_i dx^i) - f^b_i(\mathcal{L}_{\rho_{TA}(Te_a)} dx^i). \tag{3.30}$$

In the second term of the right hand side of (3.30) we can use Cartan's formula and the fact that the tangent anchor at $Te_a$ is given by

$$\rho_{TA}(Te_a) = \rho^j_a \frac{\partial}{\partial x^j} + d\rho^j_a \frac{\partial}{\partial \dot{x}^j}, \tag{3.31}$$

to conclude that $f^b_i(\mathcal{L}_{\rho_{TA}(Te_a)} dx^i) = f^b_i d\rho^i_a$. Recall also that

$$f_i^b dx^i = i_{\dot{x}} d\sigma(e_b) - i_{\dot{x}} i_{\rho(e_b)}\phi,$$

thus we derive the identity

$$S_2 = \mathcal{L}_{\rho_{TA}(Te_a)} i_{\dot{x}} d\sigma(e_b) - \mathcal{L}_{\rho_{TA}(Te_a)} i_{\dot{x}} i_{\rho(e_b)}\phi - f_i^b d\rho_a^i. \tag{3.32}$$

Notice that (3.31) can be written as

$$\rho_{TA}(Te_a) = \rho(e_a) + V_a^v,$$

with $V_a^v = d\rho_a^l(\dot{x})\frac{\partial}{\partial \dot{x}^l}$. Observe also that

$$[\rho(e_a), \dot{x}] = -V_a^h,$$

where $V_a^h = d\rho_a^l(\dot{x})\frac{\partial}{\partial x^l}$. It is easy to see, using local coordinates, that $\mathcal{L}_{V_a^v} i_{\dot{x}}\alpha = i_{V_a^h}\alpha$, for every 2-form $\alpha = \frac{1}{2}\alpha_{ij}(x)dx^i \wedge dx^j$. Therefore, using Cartan's calculus we see that the first term of the right hand side of (3.32) is given by

$$\begin{aligned}
\mathcal{L}_{\rho_{TA}(Te_a)} i_{\dot{x}} d\sigma(e_b) =& \mathcal{L}_{\rho(e_a)} i_{\dot{x}} d\sigma(e_b) + \mathcal{L}_{V_a^v} i_{\dot{x}} d\sigma(e_b) \\
=& - i_{V_a^h} d\sigma(e_b) + i_{\dot{x}}\mathcal{L}_{\rho(e_a)} d\sigma(e_b) + i_{V_a^h} d\sigma(e_b) \\
=& i_{\dot{x}} d i_{\rho(e_a)} d\sigma(e_b).
\end{aligned}$$

The second term of the right hand side of (3.32) is

$$\mathcal{L}_{\rho_{TA}(Te_a)} i_{\dot{x}} i_{\rho(e_b)}\phi = i_{\dot{x}}\mathcal{L}_{\rho(e_a)} i_{\rho(e_b)}\phi.$$

Therefore we conclude that

$$S_2 = i_{\dot{x}} d i_{\rho(e_a)} d\sigma(e_b) - i_{\dot{x}}\mathcal{L}_{\rho(e_a)} i_{\rho(e_b)}\phi - f_i^b d\rho_a^i.$$

The same argument applied to $S_3$ implies that

$$S_3 = i_{\dot{x}} d i_{\rho(e_b)} d\sigma(e_a) - i_{\dot{x}}\mathcal{L}_{\rho(e_b)} i_{\rho(e_a)}\phi - f_j^a d\rho_b^j.$$

Hence, the bracket preserving condition (3.22), which in this case, reduces to

$$\Lambda^{\sharp}([Te_a, Te_b]_{TA}) = S_1 + S_2 + S_3,$$

holds if and only if

$$\begin{aligned} i_{\dot{x}} d\sigma([e_a, e_b]) - i_{\dot{x}} i_{[\rho(e_a),\rho(e_b)]}\phi =& i_{\dot{x}} d i_{\rho(e_a)} d\sigma(e_b) - i_{\dot{x}} \mathcal{L}_{\rho(e_a)} i_{\rho(e_b)}\phi \\ & - i_{\dot{x}} d i_{\rho(e_b)} d\sigma(e_a) + i_{\dot{x}} \mathcal{L}_{\rho(e_b)} i_{\rho(e_a)}\phi. \end{aligned} \tag{3.33}$$

Now we use the formula $i_{[X,Y]} = [\mathcal{L}_X, i_Y]$ and the fact that $\phi$ is closed, to conclude that

$$i_{[\rho(e_a),\rho(e_b)]}\phi - \mathcal{L}_{\rho(e_a)} i_{\rho(e_b)}\phi + \mathcal{L}_{\rho(e_b)} i_{\rho(e_a)}\phi = d i_{\rho(e_b)} i_{\rho(e_a)}\phi.$$

Thus, the identity (3.33) is holds if and only if

$$\begin{aligned} d\sigma([e_a, e_b]) &= d(i_{\rho(e_a)} d\sigma(e_b) - i_{\rho(e_b)} d\sigma(e_a) + i_{\rho(e_b)} i_{\rho(e_a)}\phi) \\ &= d(\mathcal{L}_{\rho(e_a)}\sigma(e_b) - d i_{\rho(e_a)}\sigma(e_b) - i_{\rho(e_b)} d\sigma(e_a) + i_{\rho(e_b)} i_{\rho(e_a)}\phi) \\ &= d(\mathcal{L}_{\rho(e_a)}\sigma(e_b) - i_{\rho(e_b)} d\sigma(e_a) + i_{\rho(e_b)} i_{\rho(e_a)}\phi) \\ &= d(\mathcal{L}_{\rho(e_a)}\sigma(e_b) - \mathcal{L}_{\rho(e_b)}\sigma(e_a) + i_{\rho(e_b)} i_{\rho(e_b)}\phi), \end{aligned}$$

which can be derived by differentiating the second property of an IM-2-form with respect to $\phi$. This finishes the proof.

□

As an immediate consequence of Theorem 3.4.1 we obtain an alternative method to the one described in [10] for integrating IM-2-forms to multiplicative 2-forms.

**Corollary 3.4.1.** *Let $G$ be a source simply connected Lie groupoid $G$ over $M$, with Lie algebroid $AG$. Suppose that $\phi$ is a closed 3-form on $M$, and consider the 3-form $\phi_G$ on $G$ defined by $\phi_G = s^*\phi - t^*\phi$ There exists a one-to-one correspondence between*

*i) multiplicative 2-forms $\omega_G$ on $G$ with $d\omega_G = \phi_G$, and*

*ii) IM-2-forms $\sigma : AG \longrightarrow T^*M$ with respect to $\phi$.*

*Proof.* Let us consider a multiplicative 2-form $\omega_G$ on $G$ such that $d\omega_G = s^*\phi - t^*\phi$ for some closed 3-form $\phi$ on $M$. Consider also the bundle map $\sigma : AG \longrightarrow T^*M$ as in (2.13). The Lie functor applied to $\omega_G$ yields, in virtue of Proposition 3.2.5, a morphic 2-form $\omega_{AG}$ on $AG$ given by

$$\omega_{AG} = -\sigma^*\omega_{can} - \rho^*\tau(\phi).$$

Thus a direct application of Theorem 3.4.1 shows that $\sigma : AG \longrightarrow T^*M$ satisfies the axioms of an IM-2-form with respect to $\phi$. Conversely, given an IM-2-form with respect to $\phi$, we consider the induced linear 2-form on $AG$ given by

$$\Lambda = -\sigma^*\omega_{can} - \rho^*\tau(\phi).$$

It follows from Theorem 3.4.1 that $\Lambda$ is morphic, so the induced bundle map $\Lambda^\sharp : T(AG) \longrightarrow T^*(AG)$ is a Lie algebroid morphism. Since $G$ is a source simply connected Lie groupoid, the tangent Lie groupoid $TG \rightrightarrows TM$ is also source simply connected, and its Lie algebroid is $T(AG)$. Therefore, it follows from Lie's second theorem 2.1.1 that there exists a unique morphism of Lie groupoids

$$\omega_G^\sharp : TG \longrightarrow T^*G,$$

with $A(\omega_G^\sharp) = \Lambda^\sharp$. For every pair of tangent vectors $(X, Y) \in TG \oplus TG$, define

$$\omega_G(X, Y) := \omega_G^\sharp(X)(Y). \tag{3.34}$$

Notice that if $\omega_G$ was a 2-form on $G$, then it would be automatically a multiplicative form. In this case the morphic form induced by $d\omega_G$ is exactly $d\Lambda = -d\rho^*(\tau(\phi))$, and we conclude from Proposition 3.2.5 that $d\omega_G = s^*\phi - t^*\phi$, as required. Hence, we only need to check that (3.34) defines a 2-form on $G$. First let us check that $c_G \circ \omega_G^\sharp = p_G$. Notice that $p_G : TG \longrightarrow G$ and $c_G : T^*G \longrightarrow G$ are morphism of Lie groupoids, whose induced Lie algebroid morphisms are determined by $p_{AG} = j_G \circ A(p_G)$ and $c_{AG} = A(c_G) \circ (j'_G)^{-1}$, respectively. That is, up to canonical isomorphism of Lie algebroids, we have that $A(p_G) = p_{AG}$ and $A(c_G) = c_{AG}$. Since $c_{AG} \circ \Lambda^\sharp = p_{AG}$, we conclude from the uniqueness of the integration of a Lie algebroid morphism, that

$$c_G \circ \omega_G^\sharp = p_G.$$

Now we verify that $\omega^\sharp$ is linear with respect to the usual bundle structures $TG \longrightarrow G$ and $T^*G \longrightarrow G$. For that we observe that the fiberwise addition maps $+_{TG} : TG \oplus TG \longrightarrow TG$ and $+_{T^*G} : T^*G \oplus T^*G \longrightarrow T^*G$ are groupoid morphisms, whose induced Lie algebroid morphisms are, up to canonical identifications, given by the fiberwise addition maps $+_{TA} : TA \oplus TA \longrightarrow TA$ and $+_{T^*A} : T^*A \oplus T^*A \longrightarrow T^*A$, respectively. Since the bundle map $\Lambda^\sharp : TA \longrightarrow T^*A$ is linear with respect to the usual bundle structures $TA \longrightarrow A$ and $T^*A \longrightarrow A$, we conclude that

$$\Lambda^\sharp \circ +_{TA} = +_{T^*A} \circ \Lambda^\sharp. \tag{3.35}$$

Again by the uniqueness of the integration given by Lie's second theorem, we conclude that

$$\omega_G^\sharp \circ +_{TG} = +_{T^*G} \circ \omega_G^\sharp, \tag{3.36}$$

showing that $\omega_G^\sharp$ is additive. The same argument applied to the groupoid morphism given by scalar multiplication, shows that $\omega_G^\sharp(rX) = r\omega_G^\sharp(X)$ for every $X \in TG$ and $r \in \mathbb{R}$. Finally we prove that $\omega_G^\sharp : TG \longrightarrow T^*G$ is skew symmetric. This is equivalent to saying that the canonical pairing $T^*G \oplus TG \longrightarrow \mathbb{R}$ vanishes on the graph $L_{\omega_G}$ of $\omega_G^\sharp$. Observe that, since $\omega_G^\sharp$ is a groupoid morphism, the graph $L_{\omega_G}$ is a subgroupoid of $T^*G \oplus TG$, whose Lie algebroid coincides, up to canonical identifications, with the graph $L_\Lambda \subseteq T^*A \oplus TA$ of the Lie algebroid morphism $\Lambda^\sharp$. Also the skew symmetry of $\Lambda$ is equivalent to the fact that the canonical pairing $T^*A \oplus TA \longrightarrow \mathbb{R}$ vanishes on $L_\Lambda$. We observe also, that the canonical pairing $T^*G \oplus TG \longrightarrow \mathbb{R}$ is a groupoid morphism, whose induced morphism of Lie algebroids is, up to identifications, the canonical pairing $T^*A \oplus TA \longrightarrow \mathbb{R}$. Again, the uniqueness of Lie's second theorem implies that the canonical pairing $T^*G \oplus TG \longrightarrow \mathbb{R}$ vanishes on $L_{\omega_G}$, since this holds infinitesimally. This finishes the proof.

□

# Chapter 4

# The case of Lie groups

In this chapter we study multiplicative Dirac structures on Lie groups. We introduce Dirac-Lie groups as a natural generalization of Poisson-Lie groups in the category of Lie groups. The main results exposed in this chapter can be found in the author's work [51].

## 4.1 Dirac-Lie groups

A **Dirac-Lie group** is a pair $(G, L_G)$ where $G$ is a Lie group and $L_G \subseteq \mathbb{T}G$ is a multiplicative Dirac structure on $G$. We have seen that Dirac structures unify Poisson bivectors, closed 2-forms and regular foliations, therefore it is natural to study multiplicative versions of these three classes of Dirac structures. We will analyze them separately. First, we immediatly observe that a Dirac-Lie group $(G, L_G)$ defined by the graph of a Poisson bivector $\pi_G$ on $G$ is nothing but a Poisson-Lie group. On the other extreme, the following proposition says that there are no interesting Dirac-Lie groups defined by the graph of multiplicative 2-forms.

**Proposition 4.1.1.** *Let $G$ be a Lie group. The only multiplicative 2-form on $G$ is the zero 2-form.*

*Proof.* Let $\omega_G$ be a multiplicative 2-form on $G$. In virtue of Proposition 2.4.3, the multiplicativity of $\omega_G$ is equivalent to saying that the bundle map

$$\omega_G^\sharp : TG \longrightarrow T^*G \tag{4.1}$$

$$X \mapsto i_X\omega_G, \tag{4.2}$$

is a morphism of Lie groupoids. If $X_g \in T_gG$ is a tangent element, it follows from the definition of the cotangent target map $\tilde{t} : T^*G \longrightarrow \mathfrak{g}^*$ that $\tilde{t}(\omega_G^\sharp(X_g)) \in \mathfrak{g}^*$, which at every $u \in \mathfrak{g}$ is given by

$$\tilde{t}(\omega_G^\sharp(X_g))u = \omega_G(X_g, u^r),$$

where $u^r$ is the right invariant vector field on $G$ determined by $u \in \mathfrak{g}$. As explained in section 2.3.1 of chapter 2, the fact that $G$ is a Lie group implies that the tangent bundle $TG$ is also a Lie group. In particular, the tangent target map is the zero map $TG \longrightarrow \{0\}$. Thus, the fact that $\omega_G^\sharp$ is a groupoid morphism implies that

$$0 = \tilde{t}(\omega_G^\sharp(X_g))u = \omega_G(X_g, u^r).$$

Now, if $g \in G$ is fixed, then every tangent element $Y_g \in T_gG$ can be written as $Y_g = u^r(g)$ for some right invariant vector field $u^r$ on $G$. Thus we conclude that

$$\omega_G(X_g, Y_g) = 0,$$

for every $X_g, Y_g \in T_gG$, as desired.

□

Just as Poisson-Lie groups are Lie groups with a Poisson structure such that the multiplication map is a Poisson map, Dirac-Lie groups are Lie groups with a Dirac structure compatible with the multiplication in the sense that the multiplication map is a forward Dirac map. In order to explain this, we consider a Dirac structure $L_G$ on $G$. The direct product $G \times G$ is equipped with a Dirac structure defined by

$$(L_{G\times G})_{(g,h)} := \{(X_g, \overline{X}_h, \alpha_g, \overline{\alpha}_h) \mid X_g \oplus \alpha_g \in (L_G)_g, \overline{X}_h \oplus \overline{\alpha}_h \in (L_G)_h\}.$$

**Proposition 4.1.2.** *Let $G$ be a Lie group equipped with a Dirac structure $L_G$. Then $L_G$ is multiplicative if and only if the multiplication map $m : (G \times G, L_{G\times G}) \longrightarrow (G, L_G)$ is a forward Dirac map.*

*Proof.* Assume that $L_G$ is a multiplicative Dirac structure on $G$. Given $g, h \in G$ and $Y_{gh} \oplus \beta_{gh} \in (L_G)_{gh}$, we can write

$$Y_{gh} \oplus \beta_{gh} \in (L_G)_{gh} = X_g \bullet \overline{X}_h \oplus \alpha_g \circ \overline{\alpha}_h, \tag{4.3}$$

with $X_g \oplus \alpha_g \in (L_G)_g$ and $\overline{X}_h \oplus \overline{\alpha}_h \in (L_G)_h$. In order to show that the multiplication map is forward Dirac, it suffices to prove that

$$Y_{gh} \oplus \beta_{gh} = T_{(g,h)}m(X_g, \overline{X}_h) \oplus \beta_{gh}, \tag{4.4}$$

where $\beta_{gh} \in T^*_{gh}G$ and $(X_g, \overline{X}_h, (T_{(g,h)}m)^*\beta_{gh}) \in (L_{G\times G})_{(g,h)}$. Take $\beta_{gh} = \alpha_g \circ \overline{\alpha}_h$, then (4.3) implies (4.4), as desired. Conversely, if $m : G \times G \longrightarrow G$ is a forward Dirac map, then $L_G$ is multiplicative if and only if given $X_g \oplus \alpha_g \in (L_G)_g$ and $\overline{X}_h \oplus \overline{\alpha}_h \in (L_G)_h$, then

$$X_g \bullet \overline{X}_h \oplus \alpha_g \circ \overline{\alpha}_h \in (L_G)_{gh}. \tag{4.5}$$

Since $m$ is a forward Dirac map, every element in $(L_G)_{gh}$ has the form

$$T_{(g,h)}m(U_g, \overline{U}_h) \oplus \beta_{gh},$$

$(U_g, \overline{U}_h, (T_{(g,h)}m)^*\beta_{gh}) \in (L_{G\times G})_{(g,h)}$. Now (4.5) follows with $U_g = X_g$, $\overline{U}_h = \overline{X}_h$ and $\beta_{gh} = \alpha_g \circ \overline{\alpha}_h$. This finishes the proof.

□

## 4.2 Multiplicative foliations

In this section we give a detailed study of Dirac-Lie groups defined by regular foliations. Let us begin with the following observation.

**Proposition 4.2.1.** *Let $F \subseteq TG$ be a regular integrable distribution on a Lie group $G$. Then the corresponding Dirac structure $L_F = F \oplus F^\circ$ is multiplicative if and only if $F \subseteq TG$ is a Lie subgroup, where $TG$ has the natural Lie group structure induced from $G$.*

*Proof.* Assume that $F \subseteq TG$ is a Lie subgroup. Let $\alpha_g, \beta_h$ be composable elements in the annihilator $F^\circ$ of $F$. The cotangent product is defined by

$$(\alpha_g \circ \beta_h)(X_g \bullet Y_h) = \alpha_g(X_g) + \beta_h(Y_h).$$

where $X_g \in T_gG$ and $Y_h \in T_hG$. In particular, if $X_g, Y_h$ are composable elements of $F$, we conclude that $\alpha_g \circ \beta_h \in F^\circ$. This implies that $L_F = F \oplus F^\circ$ is a Lie subgroupoid of $\mathbb{T}G$, or equivalently, $L_F$ defines a multiplicative Dirac structure on $G$. Conversely, if $L_F$ is a multiplicative Dirac structure on $G$, we conclude that $F \subseteq TG$ is a Lie subgroup, since the groupoid structure on $L_F \subseteq \mathbb{T}G$ is defined out of the groupoid structures on $TG$ and $T^*G$, which are independent of each other.

□

A **multiplicative foliation** on a Lie group $G$ is a regular foliation $\mathcal{F}$ tangent to a Lie subgroup $F \subseteq TG$. The following proposition gives a natural way of constructing multiplicative foliations.

**Proposition 4.2.2.** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra. Consider the distribution $F \subseteq TG$ defined at every $g \in G$ by*

$$F_g := T_e l_g(\mathfrak{h}),$$

*where $l_g : G \longrightarrow G$ is the left multiplication by $g$ and $e \in G$ is the identity element. Then $F \subseteq TG$ is a Lie subgroup if and only if $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal.*

*Proof.* Assume that $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal. We will show that $F \subseteq TG$ is a Lie subgroup. In general, if $X_g \in T_gG$, then the tangent inverse is determined by

$$(X_g)^{-1} = -T_g(l_{g^{-1}} \circ r_{g^{-1}})X_g. \tag{4.6}$$

If $X_g \in F_g$, then there exists $u \in \mathfrak{h}$ with $X_g = T_e l_g(u)$. Using (4.6) we conclude that

$$(X_g)^{-1} = -T_e l_{g^{-1}}(\mathrm{Ad}_g(u)). \tag{4.7}$$

The fact that $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal is equivalent to the Ad-invariance of $\mathfrak{h}$. Thus (4.7) implies that $F \subseteq TG$ is closed by the inversion map in $TG$. It remains to show that $F \subseteq TG$ is closed by multiplication. For that, consider $X_g = T_e l_g(u)$ and $Y_h = T_e l_h(v)$, elements in $F$. If $m$ is the multiplication map of $G$, then the tangent multiplication gives

$$X_g \bullet Y_h = T_{(g,h)} m(X_g, Y_h) \tag{4.8}$$
$$= T_g r_h(T_e l_g(u)) + T_h l_g(T_e l_h(v)) \tag{4.9}$$
$$= T_g r_h(T_e l_g(u)) + T_e l_{gh}(v). \tag{4.10}$$

Notice that the second term of the right hand side of (4.10) belongs to $F_{gh}$. On the other hand, we claim that there exists a unique $u' \in \mathfrak{h}$ such that the first term of the right hand side of (4.10) is given by

$$T_g r_h(T_e l_g(u)) = T_e l_{gh}(u'). \tag{4.11}$$

Indeed, since $\mathfrak{h}$ is Ad-invariant, we see that $u' = \mathrm{Ad}_{h^{-1}} u \in \mathfrak{h}$ is the solution of (4.11). Thus, the right hand side of (4.10) defines an element of $F_{gh}$, showing that $F \subseteq TG$ is closed by multiplication. This proves that $F \subseteq TG$ is a subgroup. Conversely, if $F \subseteq TG$ is a subgroup, then $X_g \bullet Y_h \in F_{gh}$ for every $X_g = T_e l_g(u)$ and $Y_h = T_e l_h(v)$ elements in $F$. In particular (4.10) implies that $T_g r_h(T_e l_g(u)) \in F_{gh}$ for every $u \in \mathfrak{h}$. Now, $u' = \mathrm{Ad}_{h^{-1}} u$ defined by (4.11) necessarily defines an element in $\mathfrak{h}$, and we conclude that $\mathfrak{h}$ is Ad-invariant. That is $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal.

□

The distribution $F \subseteq TG$ defined in Proposition 4.2.2 is clearly an integrable distribution. Thus, the induced foliation $\mathcal{F}$ on $G$ is multiplicative. Notice that the leaf through the identity coincides with the connected normal Lie subgroup $H \subseteq G$ that integrates the ideal $\mathfrak{h} \subseteq \mathfrak{g}$. The other leaves of $\mathcal{F}$ are cosets of the normal Lie subgroup $H \subseteq G$. The following result says that this is the general picture of multiplicative foliations on a Lie group.

**Proposition 4.2.3.** *Let $\mathcal{F}$ be the foliation integrating a multiplicative distribution $F \subseteq TG$. The following holds:*

1. *The leaf through the identity $\mathcal{F}_e \subseteq G$ is a normal Lie subgroup.*
2. *The foliation $\mathcal{F}$ is given by cosets of $\mathcal{F}_e$.*

*Proof.* Since $F \subseteq TG$ is a subgroup, it is closed under multiplication in $TG$, that is $dm(g,h)(X_g, X_h) = dR_h(g)X_g + dL_g(h)X_h \in F_{gh}$ for every $X_g, X_h \in F$. In particular, for $X_h = 0$ we see that $F$ is right invariant, i.e. $dR_h(g)X_g \in F_{gh}$. Similarly we obtain left invariance of $F$: $dL_g(h)X_h \in F_{gh}$. This says that the distribution at each $g \in G$ is given by

$$F_g = dL_g(e)F_e = dR_g(e)F_e. \tag{4.12}$$

Consider now $\mathcal{F}_e$, the leaf of $F$ through the identity $e \in G$. For every $a, b \in \mathcal{F}_e$ there exist paths $a(t), b(t) \in G, t \in [0,1]$, tangent to the distribution $F$, joining the identity $e \in G$ to $a$ and $b$, respectively. We want to prove that $c = ab \in \mathcal{F}_e$. For this, take the path $c(t) = a(t)b(t)$, which joins the identity to $c = ab$. The path $c(t)$ is tangent to the distribution $F$: indeed, the bi-invariance of $F$ implies that

$$c'(t) = dR_b(t)(a(t))a'(t) + dL_a(t)(b(t))b'(t) \in F_{c(t)},$$

since $a'(t) \in F_{a(t)}$ and $b'(t) \in F_{b(t)}$. This shows that $c \in \mathcal{F}_e$. A similar computation shows that $\mathcal{F}_e$ is closed by the inversion map. Therefore the leaf through the identity is a subgroup of $G$. Moreover, it follows from (4.12) that the Lie algebra of $\mathcal{F}_e$ is Ad-invariant, which is equivalent to $\mathcal{F}_e$ being a normal subgroup. The assertion in 2. follows from the bi-invariance in (4.12).

□

## 4.3 The characteristic foliation of a Dirac-Lie group

In the previous section we discussed in detail three classes of Dirac-Lie groups. Another class of examples of Dirac-Lie groups is obtained as follows: Let $\Phi : G_1 \longrightarrow G_2$ be a homomorphism of Lie groups which is a surjective submersion. If $\pi$ is a multiplicative Poisson structure on $G_2$, then its pull back (in the sense of Dirac structures, see chapter 2 ) turns out to be a multiplicative Dirac structure on $G_1$, whose presymplectic leaves are the inverse images by $\Phi$ of the symplectic leaves of $G_2$, and whose characteristic foliation is given by the fibres of the submersion $\Phi$. Our main observation in this section is that, modulo a regularity condition, all multiplicative Dirac structures on Lie groups are of this form.

We observed in Proposition 2.5.2 that if $\Phi : G_1 \longrightarrow G_2$ is a morphism of Lie groupoids and a surjective submersion, and if $a_g, \overline{a}_h \in \mathbb{T}G_1$ are $\Phi$-related elements to $b_{\Phi(g)}, \overline{b}_{\Phi(h)} \in \mathbb{T}G_2$, we conclude that whenever $a_g, \overline{a}_h$ are composable elements, then $b_{\Phi(g)}, \overline{b}_{\Phi(h)}$ are composable as well. As a result we obtained a natural functorial property of multiplicative Dirac structures on Lie groupoids explained in Corollary 2.5.1. In the special case of multiplicative Dirac structures on Lie groups, we notice that the converse of Proposition 2.5.2 is true.

**Proposition 4.3.1.** *Let $\Phi : G_1 \longrightarrow G_2$ be a morphism of Lie groups, which is a surjective submersion. Assume that $a_g, \overline{a}_h \in \mathbb{T}G_1$ are $\Phi$-related to $b_{\Phi(g)}, \overline{b}_{\Phi(h)} \in \mathbb{T}G_2$. Then $a_g, \overline{a}_h$ are composable if and only if $b_{\Phi(g)}, \overline{b}_{\Phi(h)}$ are composable. In this case, $a_g * \overline{a}_h$ is $\Phi$-related to $b_{\Phi(g)} * \overline{b}_{\Phi(h)}$.*

*Proof.* It suffices to show that if $b_{\Phi(g)}, \overline{b}_{\Phi(h)}$ are composable, then $a_g, \overline{a}_h$ are composable, since the other direction was proved in Proposition 2.5.2. The cotangent parts of $b_{\Phi(g)}$ and $\overline{b}_{\Phi(h)}$ are composable, so the $\Phi$-relation assumption together with fact that $(T\Phi)^* : \Phi^*(T^*G_2) \longrightarrow T^*G_1$ is a groupoid morphism implies that the cotangent parts of $a_g$ and $\overline{a}_h$ are composable. Finally, notice that since $G_1$ is a Lie group, in particular $TG_1$ is a Lie group, then the tangent parts of $a_g, \overline{a}_h$ are always composable, and this fact does not depend on the composability of $b_{\Phi(g)}, \overline{b}_{\Phi(h)}$. This proves the statement.

□

Recall that Corollary 2.5.1 says that the multiplicativity property of a Dirac structure is preserved by groupoid morphisms which are surjective submersions and backward Dirac maps. In virtue of Proposition 4.3.1 we obtain a similar result for forward Dirac maps.

**Corollary 4.3.1.** *Let $\Phi : G_1 \longrightarrow G_2$ be a homomorphism of Lie groups, which is a surjective submersion. Assume that $L_1$, $L_2$ are Dirac structures on $G_1$, $G_2$ , respectively. If $\Phi$ is a forward Dirac map and $L_1$ is multiplicative, then $L_2$ is multiplicative. Also, if $\Phi$ is a backward Dirac map and $L_2$ is multiplicative, then $L_1$ is multiplicative.*

*Proof.* It suffices to show the forward case, since the backward case is a direct consequence of Corollary 2.5.1. Now, recall that $\Phi$ is a forward Dirac map if and only if $L_2$ is the bundle of all $\Phi$-related elements to elements in $L_1$. The statement follows from Proposition 4.3.1.

□

It turns out that such a functorial property is a useful tool for studying the space of characteristic leaves of Lie groups endowed with multiplicative Dirac structures. Consider now a Dirac Lie group $(G, L_G)$ and let $\mathcal{K}$ be the characteristic foliation of $L_G$, that is, the *generally singular* foliation of $G$ tangent to the distribution $\ker(L_G) = L_G \cap TG$. As explained in chapter 2, whenever $\mathcal{K}$ is a simple foliation, the space of characteristic leaves $G/\mathcal{K}$ inherits a Poisson structure denoted by $\pi_{red}$. In the special case of Dirac-Lie groups our main result is the following.

**Theorem 4.3.1.** *Let $G$ be a Lie group with a multiplicative Dirac structure $L_G \subseteq TG \oplus T^*G$. Then:*

1. *The kernel of $L_G$ is a multiplicative integrable distribution, and the leaves of the characteristic foliation $\mathcal{K}$ are cosets of the normal Lie subgroup $\mathcal{K}_e \subseteq G$.*
2. *If $\mathcal{K}_e$ is closed, then the leaf space $G/\mathcal{K}$ is smooth and the induced Poisson structure $\pi_{red}$ is multiplicative (i.e., $G/\mathcal{K}$ becomes a Poisson-Lie group). Moreover, $L_G$ is the pull back of $\pi_{red}$ by the quotient map $G \longrightarrow G/\mathcal{K}$.*

*Proof.* Since $L_G$ is multiplicative, we have that $\ker(L_G) = L_G \cap TG \subseteq TG$ is a subgroup, hence (4.12) implies that $\ker(L_G)$ has constant rank. In particular it defines an involutive distribution, whose leaves are given by cosets of the normal Lie subgroup $K = \mathcal{K}_e$ (the leaf through the identity) by Prop. 4.2.3. If $K$ is closed, then $G/K$ is a Lie group and the projection $G \longrightarrow G/K$ is a surjective submersion which is both a forward and backward Dirac map [11], where $G/K$ is equipped with the natural Poisson structure $\pi_{red}$ induced by $L_G$. The multiplicativity property of $\pi_{red}$ is a direct consequence of the functorial property of multiplicative Dirac structures.

□

## 4.4 Infinitesimal description

In this section we describe Dirac-Lie groups infinitesimally. We combine Theorem 4.3.1 and Drinfeld's correspondence between Poisson-Lie groups and Lie bialgebras [23], to obtain the infinitesimal counterpart of Dirac-Lie groups.

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$.

**Proposition 4.4.1.** *If $(G, L_G)$ is a Dirac-Lie group, then $\mathfrak{k} = \ker(L_G)_e$ is an ideal in $\mathfrak{g}$ and the quotient $\mathfrak{g}/\mathfrak{k}$ inherits the structure of a Lie bialgebra.*

*Proof.* The multiplicativity of the characteristic distribution implies that $\mathfrak{k} \subseteq \mathfrak{g}$ is an ideal. Now consider the connected and simply connected Lie group $T$ integrating the quotient Lie algebra $\mathfrak{g}/\mathfrak{k}$. The canonical projection $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{k}$ integrates to a homomorphism of Lie groups $\Phi : \tilde{G} \longrightarrow T$, where $\tilde{G}$ denotes the universal covering of $G$. The subgroup $H = \ker(\Phi)$ is closed and normal in $\tilde{G}$, therefore the connected component of the identity $H_0$ is closed and normal as well and the quotient group $\tilde{G}/H_0$ inherits a Poisson-Lie structure. Since $\tilde{G}/H$ is locally diffeomorphic to $\tilde{G}/H_0$, the Lie algebra $\mathfrak{g}/\mathfrak{k}$ inherits a Lie bialgebra structure. □

In the situation of Proposition 4.4.1 we say that $(G, L_G)$ is an **integration** of the infinitesimal data $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{k} \subseteq \mathfrak{g}$ is ideal and $\mathfrak{g}/\mathfrak{k}$ is a Lie bialgebra.

**Proposition 4.4.2.** *If $G$ is connected and simply connected and $\mathfrak{k} \subseteq \mathfrak{g}$ is an ideal such that $\mathfrak{g}/\mathfrak{k}$ is a Lie bialgebra, then there is a unique multiplicative Dirac structure on $G$ integrating $(\mathfrak{g}, \mathfrak{k})$.*

*Proof.* Let $T$ be the connected and simply connected Lie group integrating $\mathfrak{g}/\mathfrak{k}$. Consider the homomorphism $\Phi : G \longrightarrow T$ and $H \subseteq G$ as in the proof of Proposition 4.4.1. The quotient group $G/H \cong T$ has a multiplicative Poisson structure $\pi_T$ integrating the Lie bialgebra $\mathfrak{g}/\mathfrak{k}$. Since $\Phi$ is a surjective submersion, we induce a multiplicative Dirac structure $L_G$ on $G$ according to Corollary 4.3.1. This shows that $(G, L_G)$ is an integration of $(\mathfrak{g}, \mathfrak{k})$. □

# Chapter 5

# Natural functors on Dirac groupoids

In this chapter we study the effect of natural functors, such as the tangent functor and the Lie functor, on Lie groupoids equipped with multiplicative Dirac structures. On one direction, we extend a result of Grabowski-Urbanski [28] concerning tangent lifts of Poisson Lie groups. More precisely, we show that every Dirac groupoid $(G, L_G)$ can be lifted, in a natural manner, to a tangent Dirac groupoid $(TG, L_{TG})$. On the other direction, we show that any multiplicative Dirac structure $L_G \subseteq \mathbb{T}G$ is mapped, via the Lie functor, into a Lie subalgebroid $L_{AG} \subseteq \mathbb{T}(AG)$ which is also a *linear* Dirac subbundle. Conversely, if $A$ is an integrable Lie algebroid with source simply connected Lie groupoid $G$, then every Lie subalgebroid $L_A \subseteq \mathbb{T}A$ which also defines a Dirac structure integrates to a Lie subgroupoid $L_G \subseteq \mathbb{T}G$, making the pair $(G, L_G)$ into a Dirac groupoid. We also study multiplicative $B$-fields acting on Poisson groupoids and we explain the geometric structures obtained after applying the Lie functor.

## 5.1 The tangent functor

We start this section by motivating our construction of tangent Dirac structures. Recall that if $\pi_M$ is a Poisson bivector on $M$, then the cotangent bundle $T^*M$ carries a Lie algebroid structure over $M$, and we denote this Lie algebroid by $(T^*M)_{\pi_M}$. We can dualize this Lie algebroid structure, giving rise to a linear Poisson bivector $\pi_{TM}$ on the tangent bundle of $M$. This tangent Poisson structure coincides, up to canonical isomorphisms, with the derivative of $\pi_M$. More precisely, there exists a commutative diagram

$$\begin{array}{ccc} T(T^*M) & \xrightarrow{T(\pi_M^\sharp)} & T(TM) \\ \Theta_M \Big\downarrow & & \Big\downarrow J_M \\ T^*(TM) & \xrightarrow[\pi_{TM}^\sharp]{} & T(TM) \end{array} \tag{5.1}$$

where $J_M : TTM \longrightarrow TTM$ denotes the canonical involution and $\Theta_M : T(T^*M) \longrightarrow T^*(TM)$ is the Tulczyjew map. For a detailed discussion about this identifications see the original work [59] or section 2.3.2 in the second chapter of this work. Now we conclude that the tangent Poisson structure $\pi_{TM}$ induces a Lie algebroid structure on the cotangent bundle $T^*(TM) \longrightarrow TM$, which it turns to be isomorphic to the tangent Lie algebroid of $(T^*M)_{\pi_M}$. In terms of Dirac geometry, the Poisson bivector $\pi_M$ may be thought of as a Dirac structure $L_M \subseteq TM \oplus T^*M$ which, as a Lie algebroid, is isomorphic to the cotangent bundle $(T^*M)_{\pi_M}$. Similarly, the tangent Poisson bivector $\pi_{TM}$ induces a Dirac structure $L_{TM} \subseteq T(TM) \oplus T^*(TM)$ which, as a Lie algebroid, is isomorphic to $(T^*(TM))_{\pi_{TM}}$. Consequently, the canonical bundle map $J_M \oplus \Theta_M : T(TM) \oplus T(T^*M) \longrightarrow T(TM) \oplus T^*(TM)$ restricts to an isomorphism of Lie algebroids between the tangent prolongation Lie algebroid of $L_M$ and a Dirac subbundle $L_{TM} \subseteq T(TM) \oplus T^*(TM)$.

We generalize this tangent lifting procedure for an arbitrary Dirac structure. In order to make a clear exposition, we recall the canonical tangent lifts of multivector fields and differential forms, see [29, 60].

### 5.1.1 Tangent Dirac structures

We begin by summarizing some of the main properties of tangent lifts of vector fields and differential forms. Let $f \in C^\infty(M)$ be a smooth function. Then we have a pair of smooth functions on $TM$ defined by

$$f^v = f \circ p_M; \quad f^T = df.$$

We refer to $f^v$ and $f^T$ as the **vertical** lift and **tangent** lift of $f$, respectively. One can see easily that the algebra of functions $C^\infty(TM)$ is generated by functions of the form $f^v$ and

$f^T$. Now, given a vector field $X$ on $M$ we define the **vertical** lift of $X$ as the vector field $X^v$ on $TM$ which acts on vertical and tangent lifts of functions as

$$X^v(f^v) = 0, \quad X^v(f^T) = (Xf)^v.$$

The **tangent** lift of $X$ is the vector field $X^T$ on $TM$, which acts on vertical and tangent lifts of functions in the following manner:

$$X^T(f^v) = (Xf)^v, \quad X^T(f^T) = (Xf)^T.$$

It is easy to see that vertical and tangent lifts of vector fields generate the space of all vector fields on $TM$. Now let us consider a 1-form $\alpha$ on a smooth manifold $M$. We define the **vertical** lift of $\alpha$ as the 1-form $\alpha^v$ on $TM$, which is determined by its value at vertical and tangent lifts of vector fields,

$$\alpha^v(X^v) = 0, \quad \alpha^v(X^T) = (\alpha(X))^v.$$

The **tangent** lift of $\alpha$ is the 1-form $\alpha^T$ on $TM$ defined by

$$\alpha^T(X^v) = (\alpha(X))^v, \quad \alpha^T(X^T) = (\alpha(X))^T.$$

It is important to emphasize that vertical and tangent lifts of vector fields (resp. of 1-forms) are sections of the usual vector bundle structure $T(TM) \stackrel{p_{TM}}{\longrightarrow} TM$ (resp. sections of $T^*(TM) \stackrel{c_{TM}}{\longrightarrow} TM$), and they do not define sections of the tangent prolongation vector bundle $T(TM) \stackrel{Tp_M}{\longrightarrow} TM$ (resp. of the tangent prolongation $T(T^*M) \stackrel{Tc_M}{\longrightarrow} TM$). However, there exists a canonical relation between vector fields (resp. 1-forms) on $TM$ and sections of the tangent prolongation vector bundle $T(TM) \longrightarrow TM$ (resp. $T(T^*M) \longrightarrow TM$). Recall that for an arbitrary vector bundle $A \stackrel{q_A}{\longrightarrow} M$, every section $u \in \Gamma_M(A)$ induces two types of sections of $TA \longrightarrow TM$. The first type of section is $Tu : TM \longrightarrow TA$, which is given by applying the tangent functor to the section $u : M \longrightarrow A$. The second type of section is the *core* section $\hat{u} : TM \longrightarrow TA$, which is defined by

$$\hat{u}(X) = T(0^A)(X) + \overline{u(p_M(X))},$$

where $0^A : M \longrightarrow A$ denotes the zero section, and $\overline{u(p_M(X))} = \frac{d}{dt}(tu(p_M(X)))|_{t=0}$. Now, given a vector field $X$ and a 1-form $\alpha$ on $M$, we consider the linear sections $TX, T\alpha$ and the core sections $\hat{X}, \hat{\alpha}$ of the corresponding tangent prolongation vector bundles. It follows from the definition that

$$J_M(TX) = X^T, \quad J_M(\hat{X}) = X^v. \tag{5.2}$$

$$\Theta_M(T\alpha) = \alpha^T, \quad \Theta_M(\hat{\alpha}) = \alpha^v. \tag{5.3}$$

It turns out that many geometric properties of the direct sum vector bundle $T(TM) \oplus T^*(TM)$ can be understood in terms of tangent geometric properties of $T(TM) \oplus T(T^*M)$, using the canonical identification

$$J_M \oplus \Theta_M : T(TM) \oplus T(T^*M) \longrightarrow T(TM) \oplus T^*(TM).$$

Consider now a Dirac structure $L_M$ on $M$. Equivalently, we may think of $L_M$ as a Lie algebroid over $M$ with Lie bracket given by the Courant bracket on sections of $L_M$, and the anchor map $\rho_M$ is the natural projection from $L_M \subseteq TM \oplus T^*M$ onto $TM$. According to a construction of K. Mackenzie and P. Xu [46], we can consider the tangent prolongation Lie algebroid $TL_M \longrightarrow TM$, with anchor map

$$\rho_{TM} = J_M \circ T\rho_M,$$

and Lie bracket defined by

$$[\hat{a}_1, \hat{a}_2]_{TL_M} = 0, \quad [Ta_1, \hat{a}_2]_{TL_M} = \widehat{[a_1, a_2]}, \quad [Ta_1, Ta_2]_{TL_M} = T[a_1, a_2],$$

where $a_1, a_2$ are sections of $L_M \longrightarrow M$. We denote by $L_{TM}$ the image of $TL_M$ under the natural bundle map $J_M \oplus \Theta_M : TTM \oplus TT^*M \longrightarrow TTM \oplus T^*TM$.

**Proposition 5.1.1.** *The subbundle $L_{TM} \subseteq TTM \oplus T^*TM$ is isotropic with respect to the non degenerate symmetric pairing $\langle \cdot, \cdot \rangle_{TM}$ defined on $TTM \oplus T^*TM$.*

*Proof.* Consider the non degenerate symmetric pairing $\langle \cdot, \cdot \rangle_M$ defined on $TM \oplus T^*M$. The application of the tangent functor, followed by the projection onto de second factor, leads to a non degenerate symmetric pairing

$$\langle\langle\cdot,\cdot\rangle\rangle : TTM \times_{TM} TT^*M \longrightarrow \mathbb{R},$$

for which the subbundle $TL_M \subseteq TTM \oplus TT^*M$ is isotropic. Finally, for every $\dot{a}_1, \dot{a}_2 \in TL_M$ the well known identity

$$\langle\langle \dot{a}_1, \dot{a}_2\rangle\rangle = \langle (J_M \oplus \Theta_M)(\dot{a}_1), (J_M \oplus \Theta_M)(\dot{a}_2)\rangle_{TM},$$

says that the canonical map $J_M \oplus \Theta_M : T(TM) \oplus T(T^*M) \longrightarrow T(TM) \oplus T^*(TM)$ is a fiberwise isometry with respect to the pairings $\langle\langle\cdot,\cdot\rangle\rangle$ and $\langle\cdot,\cdot\rangle_{TM}$; see for instance [29, 46]. In particular, $L_{TM} = (J_M \oplus \Theta_M)(TL_M)$ is isotropic with respect to the canonical pairing on $TTM \oplus T^*TM$.

□

The tangent Lie algebroid $TL_M \longrightarrow TM$ induces a unique Lie algebroid structure on $L_{TM} \longrightarrow TM$ characterized by the property that $J_M \oplus \Theta_M : TL_M \longrightarrow L_{TM}$ is a Lie algebroid isomorphism. The space of sections $\Gamma(L_{TM})$ is generated by sections of the form $a^T := (J_M \oplus \Theta_M)(Ta)$ and $a^v := (J_M \oplus \Theta_M)\hat{a}$, where $a$ is a section of $L_M \longrightarrow M$. In particular the induced Lie bracket on sections of $L_{TM}$ is completely determined by identities

$$[a_1^v, a_2^v] = 0, \quad [a_1^T, a_2^v] = [\![a_1, a_2]\!]^v, \quad [a_1^T, a_2^T] = [\![a_1, a_2]\!]^T,$$

and the Leibniz rule with respect to the induced anchor map $pr_{TTM} : L_{TM} \longrightarrow TTM$.

**Proposition 5.1.2.** *The induced Lie bracket on sections $\Gamma(L_{TM})$ is a restriction of the Courant bracket $[\![\cdot,\cdot]\!]_{TM}$ on sections of $TTM \oplus T^*TM$.*

*Proof.* Due to the identities (5.2) and (5.3), we only need to check that the Courant bracket on sections of $L_{TM}$, naturally induced by $J_M \oplus \Theta_M$, satisfies the bracket identities that determine the induced Lie bracket on $\Gamma(L_{TM})$. One observes that vertical and tangent lifts are compatible with Lie derivatives in the sense that

1. $\mathcal{L}_{X^v}\alpha^v = 0$
2. $\mathcal{L}_{X^T}\alpha^v = (\mathcal{L}_X\alpha)^v$
3. $\mathcal{L}_{X^T}\alpha^T = (\mathcal{L}_X\alpha)^T$,

and we conclude that

1. $[\![X^v \oplus \alpha^v, Y^v \oplus \beta^v]\!] = 0$

2. $[\![X^T \oplus \alpha^T, Y^v \oplus \beta^v]\!] = [X,Y]^v \oplus (\mathcal{L}_X\beta - i_Y d\alpha)^v$

3. $[\![X^T \oplus \alpha^T, Y^T \oplus \beta^T]\!] = [X,Y]^T \oplus (\mathcal{L}_X\beta - i_Y d\alpha)^T$.

Thus the Lie bracket on $\Gamma_{TM}(L_{TM})$ induced by the tangent Lie bracket on $\Gamma_{TM}(TL_M)$ coincides with the Courant bracket.

□

We have shown the following.

**Proposition 5.1.3.** *Let $M$ be a smooth manifold. There exists a natural map*

$$\mathrm{Dir}(M) \longrightarrow \mathrm{Dir}(TM)$$
$$L_M \mapsto L_{TM},$$

*where $L_{TM} := (J_M \oplus \Theta_M)(TL_M)$.*

The Dirac structure $L_{TM} \in \mathrm{Dir}(TM)$ given by the proposition above is referred to as the **tangent Dirac structure** induced by $L_M \in \mathrm{Dir}(M)$.

**Example 5.1.1.** Let $\pi_M$ be a Poisson bivector on $M$ and consider the induced tangent Poisson bivector $\pi_{TM}$ on the tangent bundle of $M$. Let $L_M$ be the Dirac structure on $M$ defined by the graph of $\pi_M$. Then the tangent Dirac structure $L_{TM}$ induced by $L_M$ coincides with the graph of the tangent Poisson bivector $\pi_{TM}$.

**Example 5.1.2.** Let $\omega_M$ be a closed 2-form on $M$. The tangent lift of $\omega_M$ is a closed 2-form $\omega_{TM}$ on $TM$, determined by the commutative diagram

$$\begin{array}{ccc} T(TM) & \xrightarrow{T(\omega_M^\sharp)} & T(T^*M) \\ {\scriptstyle J_M}\downarrow & & \downarrow{\scriptstyle \Theta_M} \\ T(TM) & \xrightarrow[\omega_{TM}^\sharp]{} & T^*(TM) \end{array} \tag{5.4}$$

Let $L_M$ be the Dirac structure on $M$ given by the graph of $\omega_M$, then the tangent Dirac structure $L_{TM}$ induced by $L_M$ is exactly the graph of the tangent lift $\omega_{TM}$ of $\omega_M$.

**Remark 5.1.1.** The tangent lift of Dirac structures was originally studied by T. Courant [18], where tangent Dirac structures are described locally. In [61] I. Vaisman gives an intrinsic construction of tangent Dirac structures, where the tangent lift of a Dirac structure is described via the sheaf of local sections defining a Dirac subbundle of $TTM \oplus T^*TM$. Our construction is also intrinsic, and it provides an explicit description of the vector bundle $L_{TM}$ whose sheaf of sections coincides with the one described in [61]. Although we only give an alternative description of tangent Dirac structures, our construction is functorial and it has an important application to the study of multiplicative Dirac structures, namely, the Lie functor is just a restriction of the tangent functor.

Now we explain how the tangent functor acts on morphisms of Dirac manifolds. For every smooth map $\varphi : M \longrightarrow N$ between smooth manifolds, the tangent functor yields a bundle map $T\varphi : TM \longrightarrow TN$ between tangent bundles. When $M$ and $N$ carry Dirac structures, we are allowed to talk about Dirac maps. The following proposition explains the effect of the tangent functor on Dirac maps.

**Proposition 5.1.4.** *Let $\varphi : (M, L_M) \longrightarrow (N, L_N)$ be a backward Dirac map. Then $T\varphi : (TM, L_{TM}) \longrightarrow (TN, L_{TN})$ is a backward Dirac map with respect to the tangent Dirac structures induced by $L_M$ and $L_N$.*

*Proof.* The fact of $\varphi$ being a backward Dirac map is equivalent to saying that every $X \oplus \alpha \in L_M$ can be written as

$$X \oplus \alpha = X \oplus (T\varphi)^*\beta,$$

with $T\varphi(X) \oplus \beta \in \varphi^*(L_N)$. This implies that every element $\dot{X} \oplus \dot{\alpha} \in TL_M$ can be written as

$$\dot{X} \oplus \dot{\alpha} = \dot{X} \oplus T(T\varphi^*)\dot{\beta},$$

with $\dot{\beta} \in T(T^*N)$. We can apply the canonical map $J_M \oplus \Theta_M : TTM \oplus TT^*M \longrightarrow TTM \oplus T^*TM$, yielding

$$J_M(\dot{X}) \oplus \Theta_M(\dot{\alpha}) = J_M(\dot{X}) \oplus \Theta_M(T(T\varphi^*)\dot{\beta}).$$

Using the identity $\Theta_M \circ T(T\varphi^*) = (T(T\varphi))^* \circ \Theta_N$, one concludes that every element in $L_{TM}$ has the form

$$J_M(\dot{X}) \oplus \Theta_M(\dot{\alpha}) = J_M(\dot{X}) \oplus (T(T\varphi))^*\Theta_N(\dot{\beta}).$$

On the other hand, we can use the identity $T(T\varphi) \circ J_M = J_N \circ T(T\varphi)$ to conclude that $T(T\varphi)J_M(\dot{X}) = J_N(T(T\varphi)\dot{X})$. In particular, we have that $J_N(T(T\varphi)\dot{X}) \oplus \Theta_N(\dot{\beta}) \in L_{TN}$. This shows that for every $Y \in TM$

$$(L_{TM})_Y = \{V \oplus (T(T\varphi))^*\xi \mid V \in T_Y(TM), \xi \in T^*_{T\varphi(Y)}(TN), (T_Y(T\varphi)V \oplus \xi) \in (L_{TN})_{T\varphi(Y)}\}.$$

That is, the tangent map $T\varphi : (TM, L_{TM}) \longrightarrow (TN, L_{TN})$ is a backward Dirac map..

□

Consider now a Dirac manifold $(M, L_M)$ and let $(\mathcal{S}, \Omega_{\mathcal{S}})$ be a presymplectic leaf. The presymplectic structure $\Omega_{\mathcal{S}} \in \Omega^2(S)$ is characterized by the fact that the inclusion map $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow M$ is a backward Dirac map. As a consequence of Proposition 5.1.4 the presymplectic foliation of the tangent Dirac manifold $(TM, L_{TM})$ can be easily described.

**Corollary 5.1.1.** *Let $(M, L_M)$ be a Dirac manifold with presymplectic foliation $\{\mathcal{S}, \Omega_{\mathcal{S}}\}$. The presymplectic foliation of the tangent Dirac manifold $(TM, L_{TM})$ is given by $\{T\mathcal{S}, \Omega^T_{\mathcal{S}}\}$, where $\Omega^T_{\mathcal{S}} \in \Omega^2(T\mathcal{S})$ is the tangent lift of $\Omega_{\mathcal{S}} \in \Omega^2(\mathcal{S})$.*

*Proof.* It is clear that the foliation tangent to the generalized distribution $pr_{TTM}(L_{TM})$ has leaves given by $T\mathcal{S}$ where $\mathcal{S}$ is a leaf of the generalized foliation induced by $L_M$. On the other hand, since the inclusion map $i_{\mathcal{S}} : (\mathcal{S}, \Omega_{\mathcal{S}}) \longrightarrow (M, L_M)$ is a backward Dirac map, then the tangent functor applied to this map gives rise to the inclusion $(T\mathcal{S}, \Omega^T_{\mathcal{S}}) \longrightarrow (TM, L_{TM})$ which is, due to Proposition 5.1.4, a backward Dirac map as well. This characterizes the presymplectic foliation of $(TM, L_{TM})$, proving the statement.

□

**Remark 5.1.2.** We have constructed the tangent functor on Dirac structures. This is a map Dir($M$) $\longrightarrow$ Dir($TM$), which sends an object $L_M$ to the tangent object $L_{TM}$, and a Dirac morphism $\varphi : (M, L_M) \longrightarrow (N, L_N)$ to the tangent Dirac morphism $T\varphi : (TM, L_{TM}) \longrightarrow (TN, L_{TN})$. Since a Dirac structure is completely determined by its presymplectic foliation, we could define tangent Dirac structures by lifting the tangent presymplectic foliation, according to Corollary 5.1.1.

### 5.1.2 Tangent lift of a multiplicative Dirac structure

In this subsection we study tangent lifts of multiplicative Dirac structures. It was proved in [28] that whenever a Lie group $G$ carries a multiplicative Poisson bivector $\pi_G$, then the tangent Lie group $TG$ equipped with the tangent Poisson structure $\pi_{TG}$ becomes a Poisson Lie group. The next result extends the multiplicative Poisson case to abstract multiplicative Dirac structures. Assume that $G$ is a Lie groupoid over $M$ and consider the tangent groupoid $TG$ over $TM$ explained in section 2.3.1 of chapter 2.

**Proposition 5.1.5.** *The tangent Dirac structure $L_{TG} \subseteq TTG \oplus T^*TG$ induced by a multiplicative Dirac structure $L_G \subseteq TG \oplus T^*G$ is also a multiplicative Dirac structure.*

*Proof.* The bundle map $J_G : TTG \longrightarrow TTG$ is a groupoid isomorphism over $J_M : TTM \longrightarrow TTM$. Similarly, the bundle map $\Theta_G : TT^*G \longrightarrow T^*TG$ is a groupoid isomorphism over the canonical identification $I : T(A^*G) \longrightarrow (T(AG))^*$. Since $L_G$ is a Lie subgroupoid of $TG \oplus T^*G$, then the tangent functor yields a Lie subgroupoid $TL_G$ of $TTG \oplus TT^*G$. Due to the fact that $L_{TG}$ is the image of $TL_G$ via the groupoid isomorphism $J_G \oplus \Theta_G$, we see that $L_{TG}$ inherits a natural structure of Lie subgroupoid of $TTG \oplus T^*TG$. Hence we conclude that $L_{TG}$ defines a multiplicative Dirac structure on $TG$.

□

**Example 5.1.3.** Let $\omega_G$ be a multiplicative closed 2-form on $G$. Then the tangent Dirac structure $L_{TG}$ induced by the graph of $\omega_G$ coincides with the multiplicative Dirac structure on $TG$ given by the graph of the tangent lift 2-form

$$\omega_{TG} = (\omega_G^\sharp)^* \omega_{can},$$

where $\omega_{can}$ is the canonical symplectic form on $T^*G$. Notice that the multiplicativity of the Dirac structure $L_{TG}$ is also a consequence of the multiplicativity of $\omega_{can}$ and the functorial property of multiplicative Dirac structures (see Corollary 2.5.1 in chapter 2).

### 5.1.3 The Courant 3-tensor and integrability

In this section we are concerned with an alternative way of proving the integrability of tangent Dirac structures. First, notice that although we can check by hand that tangent lifts of closed 2-forms and Poisson bivectors are also closed 2-forms and Poisson bivectors, respectively, we can argue in a more direct way. In section 3.1 of chapter 3, we have seen that for every 2-form $\omega$ on $M$ we have

$$d(\omega^T) = (d\omega)^T,$$

where $(\cdot)^T$ denotes the tangent lift form on $TM$. In particular, the tangent lift of closed forms is a closed form as well. Similarly, in [28] the analogue formula for multivector fields was shown. More concretely, if $\pi$ is a multivector on $M$ and $\pi^T$ is the tangent lift multivector on $TM$, then

$$[\pi^T, \pi^T] = [\pi, \pi]^T,$$

where the bracket above is the Schouten bracket. In particular, the tangent lift $\pi^T$ of a Poisson bivector $\pi$ is also a Poisson bivector. We would like to find a direct argument that ensures the integrability of the tangent lift of a Dirac structure.

The Courant integrability of Lagrangian subbundles of $\mathbb{T}M$ is measured by a canonical tensorial object [17]. Given a Lagrangian sub bundle $L_M \subseteq \mathbb{T}M$, the **Courant 3-tensor** is the canonical section $\mu_M \in \Gamma_M(\bigwedge^3 L_M^*)$ defined by

$$\begin{aligned} \mu_M : \Gamma_M(L) \times \Gamma_M(L) \times \Gamma_M(L) &\longrightarrow C^\infty(M) \\ (a_1, a_2, a_3) &\mapsto \langle [\![ a_1, a_2 ]\!], a_3 \rangle_M \end{aligned}$$

Notice that a Lagrangian sub bundle $L_M \subseteq \mathbb{T}M$ defines a Dirac structure if and only if the Courant 3-tensor $\mu_M$ vanishes. Now let us observe that on the direct sum vector bundle

$$\prod_{p_M \oplus c_M}^{3} L_M := L_M \oplus_M L_M \oplus_M L_M,$$

we have a natural function, also denoted by $\mu_M$, defined by

$$\mu_M((a_1, a_2, a_3)_p) = \langle [\![\tilde{a}_1, \tilde{a}_2]\!], \tilde{a}_3 \rangle_M(p),$$

where $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ are sections of $L_M$ such that, at the point $p \in M$, satisfy $\tilde{a}_i(p) = a_i$ for $i = 1, 2, 3$. This is a well defined function due to the tensorial property of $\mu_M$. The application of the tangent functor to $\mu_M$ yields a function

$$T\mu_M : \prod_{Tp_M \oplus Tc_M}^{3} TL_M \longrightarrow \mathbb{R},$$

which is related to the function $\mu_{TM}$, induced by the Lagrangian sub bundle $L_{TM} \subseteq \mathbb{T}(TM)$ and the Courant 3-tensor $\mu_{TM}$, according to the following proposition.

**Proposition 5.1.6.** *For every* $(\dot{a}_1, \dot{a}_2, \dot{a}_3) \in TL_M$ *the following identity holds*

$$T\mu_M(\dot{a}_1, \dot{a}_2, \dot{a}_3) = \mu_{TM}((J_M \oplus \Theta_M)\dot{a}_1, (J_M \oplus \Theta_M)\dot{a}_2, (J_M \oplus \Theta_M)\dot{a}_3).$$

*Proof.* For every $a_1, a_2, a_3 \in \Gamma_M(L_M)$ one has $T\mu_M(Ta_1, Ta_2, Ta_3) = T(\mu_M(a_1, a_2, a_3))$. On the other hand, the canonical map $J_M \oplus \Theta_M$ applied to each of the sections $Ta_1, Ta_2, Ta_3$ gives $a_1^T, a_2^T, a_3^T \in \Gamma_{TM}(L_{TM})$. Thus we conclude that

$$\begin{aligned} \mu_{TM}(a_1^T, a_2^T, a_3^T) =& \langle [\![a_1^T, a_2^T]\!], a_3^T \rangle_{TM} \\ =& (\langle [\![a_1, a_2]\!], a_3 \rangle_M)^T, \end{aligned}$$

which is exactly the tangent functor applied to the function $\mu_M(a_1, a_2, a_3)$. Therefore, for every triple of sections $a_1, a_2, a_3$ of $L_M$ we get

$$T\mu_M(Ta_1, Ta_2, Ta_3) = \mu_{TM}(a_1^T, a_2^T, a_3^T). \tag{5.5}$$

Now we notice, using local coordinates, that for every point $\dot{a} \in TL_M$ above $\dot{x} \in TM$ there exists a section $a \in \Gamma_M(L_M)$ such that $Ta(\dot{x}) = \dot{a}$, where $Ta \in \Gamma_{TM}(TL_M)$ is the section obtained by applying the tangent functor to the section $a$ of $L_M$. This fact together with identity (5.5) prove the statement. □

As a consequence we obtain a direct proof of the Courant integrability of the tangent lift of a Dirac structure $L_M$ on $M$.

**Corollary 5.1.2.** *Let $L_M$ be an almost Dirac structure on $M$, and consider the induced almost Dirac structure $L_{TM}$ on $TM$. Then $L_{TM}$ is Courant integrable if $L_M$ is Courant integrable.*

*Proof.* An almost Dirac structure $L_M$ on $M$ is Courant integrable if and only if the associated Courant 3-tensor vanishes. The result follows by a direct application of the Proposition 5.1.6.

□

The identity $T\mu_M = \mu_{TM} \circ (J_M \oplus \Theta_M)^{(3)}$ will be extremely useful for finding the infinitesimal data of a Dirac groupoid. This will be done in the next section.

## 5.2 The Lie functor

### 5.2.1 From multiplicative to linear Dirac structures

Let $A \xrightarrow{q_A} M$ be a vector bundle. A Dirac structure $L_A \subseteq \mathbb{T}A$ is called **linear** if it defines a double vector sub bundle[1] $L_A \longrightarrow E$ of $\mathbb{T}A \longrightarrow TM \oplus A^*$. The set of all linear Dirac structures on $A$ will be denoted by $\mathrm{Dir}_{lin}(A)$.

**Example 5.2.1.** Consider a linear Poisson bivector $\pi_A$ on a vector bundle $A \xrightarrow{q_A} M$. The induced Dirac structure (see example 2.2.2 in chapter 2) $L_{\pi_A} \subseteq \mathbb{T}A$ is a linear Dirac structure on $A$.

**Example 5.2.2.** Let $\omega_A$ be a closed linear 2-form on a vector bundle $A \xrightarrow{q_A} M$. The Dirac structure $L_{\omega_A} \subseteq \mathbb{T}A$ determined by $\omega_A$ defines a linear Dirac structure on $A$.

We will be mainly interested in linear Dirac structures on Lie algebroids. In chapter 3 we discussed how multiplicative 2-forms on a Lie groupoid $G$ induce linear 2-forms on its Lie algebroid $AG$. In this section we extend this construction to the framework of multiplicative Dirac structures. For that, consider a Dirac groupoid $(G, L_G)$. We would like to answer the following question.

**Question 5.2.1.** How is the multiplicativity of $L_G \in \mathrm{Dir}_{mult}(G)$ reflected at the infinitesimal level?

[1] See appendix A for the definition and main examples of double vector bundles.

Given a Lie algebroid $A$ over $M$, we define the subset $\mathrm{Dir}_{alg}(A) \subseteq \mathrm{Dir}_{lin}(A)$ consisting of all linear Dirac structures $L_A$ on $A$, which also define a Lie subalgebroid of $\mathbb{T}A \longrightarrow TM \oplus A^*$, over some subbundle $E \subseteq TM \oplus A^*$. We will see that for any Lie groupoid $G$ with Lie algebroid $AG$ there exists a natural map

$$\mathrm{Dir}_{mult}(G) \longrightarrow \mathrm{Dir}_{alg}(AG)$$
$$L_G \mapsto L_{AG},$$

which up to canonical identifications, coincides with the Lie functor. The main idea for constructing linear Dirac structures out of multiplicative ones is based on the following observation. The canonical geometric objects associated to $\mathbb{T}G$ that are used to define Dirac structures (symmetric pairing and Courant bracket) are compatible with the groupoid structure of $\mathbb{T}G$. This observation suggests that $\mathbb{T}G$ is the prototype of a new geometric object that might be called a **$\mathcal{CA}$-groupoid**, that is, a Lie groupoid object in the category of Courant algebroids. See chapter 7 for more detailed discussion about such geometric structures.

Consider now the nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle_G$ on the direct sum Lie groupoid $\mathbb{T}G$.

**Proposition 5.2.1.** *The canonical pairing defines a morphism of Lie groupoids*

$$\langle \cdot, \cdot \rangle_G : \mathbb{T}G \oplus \mathbb{T}G \longrightarrow \mathbb{R},$$

*where* $\mathbb{R}$ *is equipped with the usual abelian group structure.*

*Proof.* Since $\mathbb{R}$ is a groupoid over a point, we only need to check the compatibility of $\langle \cdot, \cdot \rangle_G$ with the corresponding groupoid multiplications. For that, consider elements $(X_g \oplus \alpha_g), (Y_g \oplus \beta_g) \in \mathbb{T}_g G$ and $(X'_h \oplus \alpha'_h), (Y'_h \oplus \beta'_h) \in \mathbb{T}_h G$. Then by definition of the groupoid structure on $\mathbb{T}G \oplus \mathbb{T}G$, we have

$$((X_g \oplus \alpha_g) \oplus (Y_g \oplus \beta_g)) * ((X'_h \oplus \alpha'_h) \oplus (Y'_h \oplus \beta'_h)) = (X_g \bullet X'_h \oplus \alpha_g \circ \alpha'_h) \oplus (Y_g \bullet Y'_h \oplus \beta_g \circ \beta'_h),$$

therefore one gets

$$\begin{aligned}\langle (X_g \bullet X'_h \oplus \alpha_g \circ \alpha'_h), (Y_g \bullet Y'_h \oplus \beta_g \circ \beta'_h)\rangle_G =& (\alpha_g \circ \alpha'_h)(Y_g \bullet Y'_h) + (\beta_g \circ \beta'_h)(X_g \bullet X'_h)\\ =&\alpha_g(Y_g) + \alpha'_h(Y'_h) + \beta_g(X_g) + \beta'_h(X'_h)\\ =&\langle (X_g \oplus \alpha_g), (Y_g, \beta_g)\rangle_G + \langle (X'_h \oplus \alpha'_h), (Y'_h \oplus \beta'_h)\rangle_G\end{aligned}$$

This proves the statement. □

We can apply the Lie functor to the Lie groupoid morphism $\langle\cdot,\cdot\rangle_G$, yielding a nondegenerate symmetric pairing

$$A(\langle\cdot,\cdot\rangle_G) : (A(TG) \oplus A(T^*G)) \times_{AG} (A(TG) \oplus A(T^*G)) \longrightarrow \mathbb{R}.$$

Let $\langle\cdot,\cdot\rangle_{AG}$ denote the canonical non degenerate symmetric pairing on $\mathbb{T}(AG)$. Recall that there exist canonical isomorphisms of Lie algebroids $j_G : T(AG) \longrightarrow A(TG)$ and $j'_G : A(T^*G) \longrightarrow T^*(AG)$, as explained in section 2.3.2 of chapter 2. Since $\langle\cdot,\cdot\rangle_{AG}$ is just a suitable restriction of $T\langle\cdot,\cdot\rangle_G$, one concludes that the canonical map

$$j_G^{-1} \oplus j'_G : A(TG) \oplus A(T^*G) \longrightarrow T(AG) \oplus T^*(AG),$$

is a fiberwise isometry with respect to $A(\langle\cdot,\cdot\rangle_G)$ and $\langle\cdot,\cdot\rangle_{AG}$. This is a useful tool for transporting Lagrangian subbundles of $TG \oplus T^*G$ to Lagrangian subbundles of $T(AG) \oplus T^*(AG)$. For instance, given a $\mathcal{VB}$-subgroupoid $L_G$ of $TG \oplus T^*G$, we can apply the Lie functor to obtain a $\mathcal{VB}$-subalgebroid $A(L_G) \subseteq A(TG) \oplus A(T^*G)$. We mimic the construction of tangent Dirac structures, giving rise to a $\mathcal{VB}$-subalgebroid of $T(AG) \oplus T^*(AG)$ defined by

$$L_{AG} := (j_G^{-1} \oplus j'_G)(A(L_G)).$$

The following result is straightforward consequence of Proposition 5.2.1.

**Proposition 5.2.2.** *Let $L_G \subseteq TG \oplus T^*G$ be a source simply connected $\mathcal{VB}$-subgroupoid. Consider the associated $\mathcal{VB}$-subalgebroid $L_{AG} \subseteq T(AG) \oplus T^*(AG)$. Then $L_G$ is isotropic with respect to $\langle\cdot,\cdot\rangle_G$ if and only if $L_{AG}$ is isotropic with respect to $\langle\cdot,\cdot\rangle_{AG}$.*

In particular the Lie functor maps Lagrangian $\mathcal{VB}$-subgroupoids of $TG \oplus T^*G$ into

Lagrangian $\mathcal{VB}$-subalgebroids of $T(AG) \oplus T^*(AG)$.

**Corollary 5.2.1.** *Let $L_G \subseteq TG \oplus T^*G$ be a $\mathcal{VB}$-subgroupoid with associated $\mathcal{VB}$-subalgebroid $L_{AG} \subseteq T(AG) \oplus T^*(AG)$. Then $L_G$ is an almost Dirac structure on $G$ if and only if $L_{AG}$ is an almost Dirac structure on $AG$.*

This is just Proposition 5.2.2 rephrased in terms of Dirac structures. The main objective of this section is to show that the Lie functor, not only preserves almost Dirac structures, but also preserves the property of being integrable in the sense of Courant.

### 5.2.2 The Courant 3-tensor and integrability

This subsection is concerned with the integrability of linear Dirac structures obtained by the application of the Lie functor to multiplicative Dirac structures. In order to prove the integrability of the Lagrangian subbundle $L_{AG} \subseteq T(AG) \oplus T^*(AG)$, we extract, from the multiplicativity of $L_G$, a property that generalizes the fact that the de Rham differential leaves invariant the set of multiplicative forms. As explained in chapter 3, such a observation together with the compatibility of the exterior derivative with tangent lifts of differential forms, gave rise to the identity

$$(d\omega_G)_{AG} = d\omega_{AG}, \tag{5.6}$$

where $\omega_{AG}$ is the restriction to $AG$ of the tangent lift $\omega_G^T$ of $\omega_G$. In particular, we concluded immediatly that Lie$(\omega_G)$ is closed, whenever $\omega_G$ is a closed 2-form. As in the case of tangent Dirac structures, we would like to obtain an analogue of (5.6) that ensures the integrability of the subbundle $L_{AG} \subseteq \mathbb{T}(AG)$. As we did for tangent Dirac structures, we shall study the Courant 3-tensor $\mu_G \in \Gamma(\bigwedge^3 L_G^*)$ determined by the Lagrangian subbundle $L_G \subseteq \mathbb{T}G$. Since $\mu_G$ involves the Courant bracket, we need a compatibility between the Courant bracket and the groupoid structure of $\mathbb{T}G$.

In order to explain the relation between the Courant bracket and the Lie groupoid structure on the direct sum vector bundle $\mathbb{T}G = TG \oplus T^*G$, we consider the direct product vector bundle $\mathbb{T}G \times \mathbb{T}G \longrightarrow G \times G$. Every section $a^{(2)}$ of $\mathbb{T}G \times \mathbb{T}G$ can be written as

$$a^{(2)} = a_1 \circ pr_1 \oplus a_2 \circ pr_2,$$

where $a_1, a_2$ are sections of $\mathbb{T}G$, and $pr_1, pr_2 : \mathbb{T}G \times \mathbb{T}G \longrightarrow \mathbb{T}G$ denote the natural projections. The direct product bracket on sections of $\mathbb{T}G \times \mathbb{T}G$ is defined as usual

$$[a^{(2)}, \overline{a}^{(2)}] = [\![a_1, \overline{a}_1]\!] \circ pr_1 \oplus [\![a_2, \overline{a}_2]\!] \circ pr_2.$$

Since the Courant bracket in $\Gamma(\mathbb{T}G)$ does not satisfy Jacobi indentity, the direct product bracket is not a Lie bracket. In fact, the direct product bracket together with the componentwise projection map

$$\mathbb{T}G \times \mathbb{T}G \longrightarrow TG \times TG,$$

make the vector bundle $\mathbb{T}G \times \mathbb{T}G \longrightarrow G \times G$ into an almost Lie algebroid. Recall that an almost Lie algebroid is a vector bundle $A \longrightarrow M$ with a skew symmetric bilinear bracket $[\cdot, \cdot]_A$ on $\Gamma(A)$ and an anchor map $\rho_A : A \longrightarrow TM$ which are compatible in the sense that the usual Leibniz rule is fulfilled. On the other hand, the set of composable groupoid pairs $(\mathbb{T}G)_{(2)}$ is a vector bundle over $G_{(2)}$, and we consider the almost Lie algebroid structure on $(\mathbb{T}G)_{(2)}$ induced by the direct product $\mathbb{T}G \times \mathbb{T}G$. Now, the compatibility between the Courant bracket on $\Gamma(\mathbb{T}G)$ and the groupoid structure of $\mathbb{T}G$ becomes clear due to the following proposition.

**Proposition 5.2.3.** *Let $m_{\mathbb{T}} : (\mathbb{T}G)_{(2)} \longrightarrow \mathbb{T}G$ denote the groupoid multiplication of $\mathbb{T}G = TG \oplus T^*G$. Then the bundle map*

$$\begin{array}{ccc} (\mathbb{T}G)_{(2)} & \xrightarrow{m_{\mathbb{T}}} & \mathbb{T}G \\ \downarrow & & \downarrow \\ G_2 & \xrightarrow[m_G]{} & G \end{array} \tag{5.7}$$

*is a morphism of almost Lie algebroids.*

If $A_1 \longrightarrow M_1$ and $A_2 \longrightarrow M_2$ are almost Lie algebroids, then a bundle map $\Psi : A_1 \longrightarrow A_2$ covering $\psi : M_1 \longrightarrow M_2$ is a morphism of almost Lie algebroids if $\Psi$ satisfies the usual compatibility conditions with the anchor maps and the brackets on sections of $A_1$ and $A_2$. This definition makes sense since an almost Lie algebroid satisfies all the axioms of a Lie algebroid except the Jacobi identity. Now we proceed with the proof of Proposition 5.2.3.

*Proof.* We begin by checking the compatibility of $(m_{\mathbb{T}}, m_G)$ with the corresponding anchor maps. For that, consider a section $a^{(2)} = a_1 \circ pr_1 \oplus a_2 \circ pr_2$ of $(\mathbb{T}G)_{(2)}$ where $a_1 = X^1 \oplus \alpha^1$ and $a_2 = X^2 \oplus \alpha^2$ are sections of $\mathbb{T}G$. The multiplication on the Lie groupoid $\mathbb{T}G$ maps the section $a^{(2)}$ into

$$m_{\mathbb{T}}(a_1 \circ pr_1 \oplus a_2 \circ pr_2)(g,h) = X^1_g \bullet X^2_h \oplus \alpha^1_g \circ \alpha^2_h.$$

Applying the anchor map of $\mathbb{T}G$ we obtain

$$\rho_{\mathbb{T}G}(X^1_g \bullet X^2_h \oplus \alpha^1_g \circ \alpha^2_h) = X^1_g \bullet X^2_h.$$

On the other hand, the componentwise anchor map of $(\mathbb{T}G)_{(2)}$ applied to the section $a^{(2)}$ gives rise to

$$\rho_{(\mathbb{T}G)_{(2)}}(a_1 \circ pr_1 \oplus a_2 \circ pr_2)(g,h) = (X^1_g, X^2_h),$$

which followed by the derivative of $m_G : G_{(2)} \longrightarrow G$ yields

$$Tm_G(\rho_{(\mathbb{T}G)_{(2)}}(X^1_g \oplus \alpha^1_g, X^2_h \oplus \alpha^2_h)) = X^1_g \bullet X^2_h,$$

showing that $(m_{\mathbb{T}}, m_G)$ is compatible with the anchors. It remains to prove that $m_{\mathbb{T}}$ is bracket preserving. For that one observes that $m_{\mathbb{T}}$ is a fiberwise surjective map, so it suffices to check that, whenever

$$m_{\mathbb{T}} \circ a^{(2)} = a \circ m_G \quad (5.8)$$

$$m_{\mathbb{T}} \circ \overline{a}^{(2)} = \overline{a} \circ m_G, \quad (5.9)$$

where $a^{(2)}, \overline{a}^{(2)} \in \Gamma_{G_{(2)}}((\mathbb{T}G)_{(2)})$ and $a, \overline{a} \in \Gamma_G(\mathbb{T}G)$, then the following bracket preserving property is fulfilled

$$m_{\mathbb{T}} \circ [a^{(2)}, \overline{a}^{(2)}] = [\![a, \overline{a}]\!] \circ m_G.$$

See e.g. [33] Prop. 1.5. It will be convenient write down sections as

$$
\begin{aligned}
a^{(2)} &= (X^1 \oplus \alpha^1) \circ pr_1 \oplus (X^2 \oplus \alpha^2) \circ pr_2 \\
\overline{a}^{(2)} &= (\overline{X}^1 \oplus \overline{\alpha}^1) \circ pr_1 \oplus (\overline{X}^2 \oplus \overline{\alpha}^2) \circ pr_2 \\
a &= Y \oplus \beta \\
\overline{a} &= \overline{Y} \oplus \overline{\beta},
\end{aligned}
$$

then the identities (5.8), (5.9) become

$$X^1_g \bullet X^2_h \oplus \alpha^1_g \circ \alpha^2_h = Y_{gh} \oplus \beta_{gh} \tag{5.10}$$

$$\overline{X}^1_g \bullet \overline{X}^2_h \oplus \overline{\alpha}^1_g \circ \overline{\alpha}^2_h = \overline{Y}_{gh} \oplus \overline{\beta}_{gh}, \tag{5.11}$$

for any composable pair $(g, h) \in G \times G$. Now it follows directly from the definition of the direct product bracket that

$$[a^{(2)}, \overline{a}^{(2)}] = ([X^1, \overline{X}^1] \oplus \mathcal{L}_{X^1}\overline{\alpha}^1 - i_{\overline{X}^1} d\alpha^1) \circ pr_1 \oplus ([X^2, \overline{X}^2] \oplus \mathcal{L}_{X^2}\overline{\alpha}^2 - i_{\overline{X}^2} d\alpha^2) \circ pr_2.$$

Then, composing with the groupoid multiplication of $\mathbb{T}G$, we have

$$m_{\mathbb{T}} \circ [a^{(2)}, \overline{a}^{(2)}]_{(g,h)} = [X^1, \overline{X}^1]_g \bullet [X^2, \overline{X}^2]_h \oplus (\mathcal{L}_{X^1}\overline{\alpha}^1 - i_{\overline{X}^1} d\alpha^1)_g \circ (\mathcal{L}_{X^2}\overline{\alpha}^2 - i_{\overline{X}^2} d\alpha^2)_h.$$

On the other hand,

$$[\![a, \overline{a}]\!] \circ m_G(g, h) = [Y, \overline{Y}]_{gh} \oplus (\mathcal{L}_Y \overline{\beta} - i_{\overline{Y}} d\beta)_{gh},$$

and using the identities (5.10) and (5.11) one concludes that

$$[Y, \overline{Y}]_{gh} = [X^1, \overline{X}^1]_g \bullet [X^2, \overline{X}^2]_h.$$

Thus, the tangent component of $[\![a, \overline{a}]\!]_{gh}$ coincides with the tangent component of $m_{\mathbb{T}} \circ [a^{(2)}, \overline{a}^{(2)}]_{(g,h)}$. It remains to show that we also have the equality of the corresponding cotangent parts. This is equivalent to showing that

$$\begin{aligned}(\mathcal{L}_Y\overline{\beta} - \mathcal{L}_{\overline{Y}}\beta - d\langle\beta,\overline{Y}\rangle)_{gh} =&(\mathcal{L}_{X^1}\overline{\alpha}^1 - \mathcal{L}_{\overline{X}^1}\alpha^1 - d\langle\alpha^1,\overline{X}^1\rangle)_g\circ\\ &\circ(\mathcal{L}_{X^2}\overline{\alpha}^2 - \mathcal{L}_{\overline{X}^2}\alpha^2 - d\langle\alpha^2,\overline{X}^2\rangle)_h,\end{aligned}$$

for every composable pair $(g,h) \in G_{(2)}$. In order to prove this identity, we need to check that the left hand side ($LHS$), and the right hand side ($RHS$) above coincide at elements of the form $U_g \bullet V_h$. For that consider the 1-form on $G$ defined by $\gamma := \mathcal{L}_Y\overline{\beta} - \mathcal{L}_{\overline{Y}}\beta - d\langle\beta,\overline{Y}\rangle$. We can look at the pull back 1-form $m_G^*\gamma \in \Omega^1(G_{(2)})$, which at every tangent vector $(U_g, V_h) \in T_{(g,h)}G_{(2)}$ is given by

$$(m_G^*\gamma)_{(g,h)}(U_g,V_h) = \gamma_{gh}(U_g \bullet V_h) = (LHS)(U_g \bullet V_h).$$

The pull back form $m_G^*\gamma$ involves three terms. Let us analyze the first term $m_G^*(\mathcal{L}_Y\overline{\beta})$ of this pull back form. It follows from the relation $Y = (m_G)_*(X^1, X^2)$ that

$$m_G^*(\mathcal{L}_Y\overline{\beta}) = \mathcal{L}_{(X^1,X^2)}m_G^*\overline{\beta}.$$

Notice that (5.11) implies that

$$\begin{aligned}(m_G^*\overline{\beta})_{(g,h)}(U_g,V_H) =&\overline{\beta}_{gh}(U_g \bullet V_h)\\ =&(\overline{\alpha}_g^1 \circ \overline{\alpha}_h^2)(U_g \bullet V_h)\\ =&\overline{\alpha}_g^1(U_g) + \overline{\alpha}_h^2(V_h)\\ =&(\overline{\alpha}^1,\overline{\alpha}^2)_{(g,h)}(U_g,V_h).\end{aligned}$$

That is, $m_G^*(\mathcal{L}_Y\overline{\beta}) = \mathcal{L}_{X^1}\overline{\alpha}^1 \oplus \mathcal{L}_{X^2}\overline{\alpha}^2$. A similar argument can be applied to the other terms of the pull back form $m_G^*\gamma$, yielding

$$\begin{aligned}(LHS)(U_g \bullet V_h) =&(m_G^*\gamma)_{(g,h)}(U_g, V_h)\\ =&(\mathcal{L}_{X^1}\overline{\alpha}^1)_g(U_g) + (\mathcal{L}_{X^2}\overline{\alpha}^2)_h(V_h)+\\ &-(\mathcal{L}_{\overline{X}^1}\alpha^1)_g(U_g) - (\mathcal{L}_{\overline{X}^2}\alpha^2)_h(V_h)+\\ &-d\langle\alpha^1, \overline{X}^1\rangle_g(U_g) - d\langle\alpha^2, \overline{X}^2\rangle_h(V_h)\\ =&(RHS)(U_g \bullet V_h).\end{aligned}$$

Thus $RHS$ and $LHS$ coincide at elements of the form $U_g \bullet V_h$, and we conclude that $(m_{\mathbb{T}}, m_G)$ is bracket preserving.

□

Given a Lagrangian $\mathcal{VB}$-subgroupoid $L_G \rightrightarrows E_G$ of the direct sum $TG \oplus T^*G \rightrightarrows TM \oplus A^*G$, we induce a natural $\mathcal{VB}$-groupoid structure on the direct sum vector bundle

$$\prod_{p_G \oplus c_G}^{3} L_G \rightrightarrows \prod_{p_M \oplus q_{A^*G}}^{3} E_G.$$

Associated to $L_G$ is the natural function

$$\mu_G : \prod_{p_G \oplus c_G}^{3} L_G \longrightarrow \mathbb{R},$$

induced by the Courant 3-tensor $\mu_G \in \Gamma(\bigwedge^3 L_G^*)$. Since $L_G$ is multiplicative it is natural to expect that such a multiplicativity property could affect the nature of the function $\mu_G$.

**Proposition 5.2.4.** *Given a Lagrangian subgroupoid $L_G \subseteq TG \oplus T^*G$, the canonical function*

$$\mu_G : \prod_{p_G \oplus c_G}^{3} L_G \longrightarrow \mathbb{R},$$

*is a groupoid morphism. That is $\mu_G$ is a multiplicative function.*

*Proof.* Let us consider composable pairs $a_g^i, \overline{a}_h^i$ in $L_G$ with $i = 1, 2, 3$. Then,

$$\mu_G((a^1_g, a^2_g, a^3_g) * (\overline{a}^1_h, \overline{a}^2_h, \overline{a}^3_h)) = \langle [\![ a^1\overline{a}^1, a^2\overline{a}^2 ]\!]_{gh}, a^3_g\overline{a}^3_h\rangle_G$$
$$= \langle [\![ a^1, a^2 ]\!]_g [\![ \overline{a}^1, \overline{a}^2 ]\!]_h, a^3_g\overline{a}^3_h\rangle_G.$$

The last identity follows from the fact that $(m_{\mathbb{T}}, m_G)$ is bracket preserving. Now we use the fact that $\langle\cdot,\cdot\rangle_G$ is a groupoid morphism to conclude that

$$\mu_G((a^1_g, a^2_g, a^3_g) * (\overline{a}^1_h, \overline{a}^2_h, \overline{a}^3_h)) = \mu_G(a^1_g, a^2_g, a^3_g) + \mu_G(\overline{a}^1_h, \overline{a}^2_h, \overline{a}^3_h).$$

This proves that the function $\mu_G$ is multiplicative.

□

We can apply the Lie functor to the groupoid morphism $\mu_G$, yielding a Lie algebroid morphism

$$A(\mu_G) : \prod^3_{A(p_G\oplus c_G)} A(L_G) \longrightarrow \mathbb{R}.$$

Recall that $T\mu_G$ coincides, up to a canonical identification, with $\mu_{TG}$. Since $A(\mu_G)$ is a suitable restriction of $T\mu_G$, the following proposition follows directly.

**Proposition 5.2.5.** *Consider the Lagrangian subbundle $L_{AG} = (j_G^{-1} \oplus j'_G)A(L_G) \subseteq \mathbb{T}(AG)$. The following identity holds*

$$Lie(\mu_G) = \mu_{AG} \circ (j_G^{-1} \oplus j'_G)^{(3)},$$

*where $(j_G^{-1} \oplus j'_G)^{(3)} : \prod^3_{A(p_G\oplus c_G)} A(L_G) \longrightarrow \prod^3_{p_{AG}\oplus c_{AG}} L_{AG}$ denotes the natural extension of $(j_G^{-1} \oplus j'_G)$.*

Now we are ready to state the main theorem of this section.

**Theorem 5.2.1.** *Let $L_G \subseteq \mathbb{T}G$ be a multiplicative almost Dirac structure on $G$. Consider the associated linear almost Dirac structure $L_{AG} \subseteq \mathbb{T}(AG)$ on $AG$. If $L_G$ is a Dirac structure, then $L_{AG}$ is also a Dirac structure.*

*Proof.* The fact that $L_G$ is a Dirac structure is equivalent to saying that the Courant 3-tensor $\mu_G$ vanishes. Now the identity

$$\mathrm{Lie}(\mu_G) = \mu_{AG} \circ (j_G^{-1} \oplus j'_G)^{(3)}$$

implies that the Courant 3-tensor for the corresponding almost Dirac structure $L_{AG}$ on $AG$ vanishes as well. That is, $L_{AG}$ defines a Dirac structure on $AG$.

□

We notice that Theorem 5.2.1 explains the effect of the Lie functor on multiplicative Dirac structures. In particular, we are allowed to answer Question 5.2.1 proposed in subsection 5.2.1. Given a Lie groupoid $G$ with Lie algebroid $AG$, there is a natural map

$$\mathrm{Dir}_{mult}(G) \longrightarrow \mathrm{Dir}_{alg}(AG)$$

which sends every multiplicative Dirac structure $L_G$ on $G$ to a linear Dirac structure $L_{AG}$ on $AG$ which also defines a Lie subalgebroid of $\mathbb{T}(AG)$.

The Lie functor also can be applied on Dirac maps which are morphisms of Lie groupoids.

**Proposition 5.2.6.** *Let $\Phi : G \longrightarrow H$ be a morphism of Lie groupoids. Assume that $L_G$ and $L_H$ are multiplicative Dirac structures on $G$ and $H$, respectively. If $\Phi$ is a backward Dirac map then $A(\Phi) : (AG, L_{AG}) \longrightarrow (AH, L_{AH})$ is a backward Dirac map.*

*Proof.* This follows from the fact that $T\phi : (TG, L_{TG}) \longrightarrow (TH, L_{TH})$ is backward Dirac, and the fact that $A(\phi)$ is a suitable restriction of $T\phi$.

□

### 5.2.3 Examples

Now we discuss some familiar examples of Dirac structures on Lie algebroids.

**Example 5.2.3.** (*Linear Dirac structures induced by Poisson groupoids*)

Let $(G, \pi_G)$ be a Poisson groupoid. The Dirac structure $L_G$ on $G$ defined by the graph of $\pi_G$ is a multiplicative Dirac structure. The multiplicativity of this Dirac structure is equivalent to $\pi_G^\sharp : T^*G \longrightarrow TG$ being a morphism of Lie groupoids, and the associated

Lie algebroid morphism coincides, up to identifications, with $\pi^\sharp_{AG} : T^*(AG) \longrightarrow T(AG)$ where $\pi_{AG}$ denotes the linear Poisson bivector on $AG$ dual to the Lie algebroid $A^*G$. One concludes that the corresponding Dirac structure $L_{AG}$ on $AG$ is exactly the graph of $\pi_{AG}$.

**Example 5.2.4.** (*Linear Dirac structures induced by multiplicative forms*)

Let $\omega_G$ be a multiplicative closed 2-form on a Lie groupoid $G$. The graph of $\omega_G$ defines a multiplicative Dirac structure $L_G$ on $G$. Let $\sigma : AG \longrightarrow T^*M$ denote the IM-2-form determined by $\omega_G$. The multiplicativity of $\omega_G$ is equivalent to saying that $\omega^\sharp_G : TG \longrightarrow T^*G$ is a morphism of Lie groupoids, and the corresponding morphism of Lie algebroids is $\omega^\sharp_{AG} : T(AG) \longrightarrow T^*(AG)$ where $\omega_{AG}$ denotes the linear 2-form on $AG$ induced by the IM-2-form $\sigma$. Hence, the associated Dirac structure $L_{AG}$ on $AG$ is exactly the graph of the linear closed 2-form $\omega_{AG}$.

**Example 5.2.5.** (*Linear Dirac structures on Lie algebras*)

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $L_G \in \mathrm{Dir}_{mult}(G)$ be a multiplicatice Dirac structure such that the characteristic leaf $\mathcal{K}$ through the identity is closed. We have seen that $\mathcal{K}$ is a normal Lie subgroup of $G$, in particular its Lie algebra $\mathfrak{k}$ is an ideal of $\mathfrak{g}$. The canonical quotient map $q : G \longrightarrow G/\mathcal{K}$ is both a forward and a backward Dirac map, where $G/\mathcal{K}$ has the multiplicative Poisson structure $\pi_{red}$ induced by $L_G$. Applying the Lie functor to the group homomorphism $q$, we get a morphism of Lie algebras $\mathfrak{q} : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{k}$ which is a forward and backward Dirac map, with respect to the linear Dirac structures on $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{k}$ determined by $L_G$ and $\pi_{red}$, respectively. It follows from example 5.2.3 that the linear Dirac structure $L_{\pi_{\mathfrak{g}/\mathfrak{k}}}$ on $\mathfrak{g}/\mathfrak{k}$ is the graph of the linear Poisson bivector $\pi_{\mathfrak{g}/\mathfrak{k}}$ determined by the dual Lie algebra $(\mathfrak{g}/\mathfrak{k})^*$. One concludes that the linear Dirac structure on $\mathfrak{g}$ corresponds to the pull back Dirac structure $L_{\mathfrak{g}} := \mathfrak{q}^*(L_{\pi_{\mathfrak{g}/\mathfrak{k}}})$.

**Example 5.2.6.** (*Linear Dirac structures arising in Poisson reduction*)

Let $(G, \pi_G)$ be a Poisson groupoid with a Hamiltonian action of a Lie group $H$ as in example 2.5.5 of chapter 2. We have seen that the reduced space $(G_{red}, \pi_{red})$ is a Poisson groupoid. Let $A_{red}$ be the Lie algebroid of $G_{red}$. The induced Dirac structure on $A_{red}$ is the graph of the linear Poisson bivector $\pi_{A_{red}}$ on $A_{red}$, determined by the dual Lie algebroid $A^*_{red}$. See example 5.2.3.

## 5.3 Reconstructing multiplicative Dirac structures

In this chapter we extend the integration of Lie bialgebroids to Poisson groupoids and the integration of IM-2-forms to twisted multiplicative 2-forms, carried out in [48] and [10], respectively. Let $A \xrightarrow{q_A} M$ be an integrable Lie algebroid with source simply connected integration $G$. The fact that $G$ has simply connected $s$-fibers implies that the tangent groupoid $TG \rightrightarrows TM$ and the cotangent groupoid $T^*G \rightrightarrows A^*$ are source simply connected Lie groupoids. In particular, since $A(TG) \cong TA$ and $A(T^*G) \cong T^*A$ we conclude that the direct sum $\mathbb{T}G = TG \oplus T^*G$ is the source simply connected integration of $\mathbb{T}A = TA \oplus T^*A$. Consider now a Lie subalgebroid $L_A \subseteq \mathbb{T}A$ which has also a vector bundle structure over $A$. As explained in appendix A subsection A.0.2, $L_A \subseteq \mathbb{T}A$ integrates to a source simply connected Lie subgroupoid $L_G \subseteq \mathbb{T}G$ which inherits a vector bundle structure over $G$. That is,

$$\mathcal{VB}\text{-subalgebroid } L_A \subseteq \mathbb{T}A \mapsto \mathcal{VB}\text{-subgroupoid } L_G \subseteq \mathbb{T}G. \tag{5.12}$$

In the previous section, we explained the effect of the Lie functor on multiplicative Dirac structures in terms of the map

$$\mathrm{Dir}_{mult}(G) \longrightarrow \mathrm{Dir}_{alg}(AG) \tag{5.13}$$

$$L_G \mapsto L_{AG} \tag{5.14}$$

We will prove that, whenever $G$ has simply connected $s$-fibers, we can reconstruct a multiplicative Dirac structure out of elements in $\mathrm{Dir}_{alg}(AG)$.

**Theorem 5.3.1.** *Let $G \rightrightarrows M$ be a source simply connected Lie groupoid with Lie algebroid $A$. The map* (5.13) *is a bijection.*

*Proof.* We construct an inverse of (5.13). For that we take an element $L_A \in \mathrm{Dir}_{alg}(A)$, that is $L_A$ is linear Dirac structure on $A$ such that $L_A \subseteq \mathbb{T}A$ is a $\mathcal{VB}$-subalgebroid. Consider the integrating $\mathcal{VB}$-subgroupoid $L_G \subseteq \mathbb{T}G$ as explained in (5.12). We will prove that $L_G$ is a multiplicative Dirac structure on $G$. Since $L_A \subseteq \mathbb{T}A$ is Lagrangian with respect to the canonical symmetric pairing $\langle \cdot, \cdot \rangle_A$ on $\mathbb{T}A$, we conclude from Proposition 5.2.2 that $L_G$ is Lagrangian with respect to the canonical symmetric pairing $\langle \cdot, \cdot \rangle_G$ on $\mathbb{T}G$. It remains to show that $L_G \subseteq \mathbb{T}G$ is integrable with respect to the Courant bracket. Equivalently, we have to prove that the Courant 3-tensor $\mu_G \in \Gamma(\wedge^3 L_G^*)$ is zero. Recall that the fact that $L_A \subseteq \mathbb{T}A$ is a Dirac structure is equivalent to saying that the induced Courant 3-tensor $\mu_A \in \Gamma(\wedge^3 L_A^*)$ vanishes. Therefore, we use Proposition 5.2.5 and Lie's second theorem to conclude that $\mu_G \equiv 0$, as desired. This shows that $L_G$ is a Dirac structure on $G$, which by definition is multiplicative.

□

As a consequence of Theorem 5.3.1 we obtain the integration of Lie bialgebroids proved in [48].

**Corollary 5.3.1.** *Let $(A, A^*)$ be a Lie bialgebroid. Assume that $G$ is a source simply connected Lie groupoid with Lie algebroid $A$. Then there exists a unique Poisson bivector $\pi_G$ on $G$, making the pair $(G, \pi_G)$ into a Poisson groupoid with Lie algebroid $(A, A^*)$.*

*Proof.* The linear Poisson bivector $\pi_A$ on $A$ defines a Lie algebroid morphism $\pi_A^\sharp : T^*A \longrightarrow TA$. Let $L_{\pi_A}$ be the Dirac structure on $A$ determined by the graph of $\pi_A^\sharp$. Then $L_{\pi_A} \in \mathrm{Dir}_{alg}(A)$ and we can integrate $L_{\pi_A}$ to a unique multiplicative Dirac structure $L_G$ on $G$ according to Theorem 5.3.1. Since $L_{\pi_A}$ is the graph of a Lie algebroid morphism, we conclude that $L_G$ is the graph of a Lie groupoid morphism $\pi_G^\sharp : T^*G \longrightarrow TG$. The fact that $L_G$ is a vector bundle over $G$ says that there is a well defined bivector $\pi_G$ on $G$, given by

$$\pi_G(\alpha, \beta) = \pi_G^\sharp(\alpha)\beta.$$

The fact that $L_G$ is a Dirac structure over $G$ is equivalent to saying that $\pi_G$ is a Poisson bivector. Therefore, the pair $(G, \pi_G)$ is a Poisson groupoid with Lie bialgebroid $(A, A^*)$.

□

Finally, our construction of linear Dirac structures which are Lie subalgebroids of $\mathbb{T}A$, out of multiplicative Dirac structures on $G$ is strongly inspired on the tangent lift of arbitrary Dirac structures. Recall that the Courant integrability of the tangent lift of a Dirac structure came for free, since up to canonical identifications, a tangent Dirac structure is only a tangent prolongation Lie algebroid. The Courant bracket on $\mathfrak{X}(TM) \oplus \Omega^1(TM)$ is obtained via the Lie bracket of a tangent Lie algebroid and a suitable flip isomorphism

$J_M \oplus \Theta_M : TTM \oplus TT^*M \longrightarrow TTM \oplus T^*TM$. In our study of multiplicative and linear Dirac structures, we realized a linear Dirac structure $L_{AG}$ as the Lie functor applied to a multiplicative Dirac structure $L_G$. It seems that the Lie functor applied to the Lie algebroid $L_G \longrightarrow G$ associated to a multiplicative Dirac structure should lead to a Lie algebroid $A(L_G) \longrightarrow AG$, which up to canonical flip isomorphisms must coincide with the Lie algebroid $L_{AG} \longrightarrow AG$ associated to the linear Dirac structure $L_{AG}$. This approach is closely related to a second order geometry introduced by K. Mackenzie [42, 43], and it will be explained in chapter 6.

## 5.4 Multiplicative $B$-field transformations

A Dirac structure on $M$ is defined out of two objects canonically attached to the direct sum vector bundle $\mathbb{T}M = TM \oplus T^*M$, namely the symmetric pairing $\langle \cdot, \cdot \rangle$ and the Courant bracket $[\![\cdot, \cdot]\!]$. One can see easily that the there exists a natural extended action of the group $\mathrm{Diff}(M)$ on $\mathbb{T}M$, and this action preserves the symmetric pairing $\langle \cdot, \cdot \rangle$ and the Courant bracket. In this section we study extra symmetries of the geometric data $(\mathbb{T}M, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!])$. These symmetries are given by the so called $B$-field transformations. See e.g. [26, 32, 34] for the relation with generalized complex geometry.

Let $B \in \Omega^2(M)$ be a 2-form on $M$ and consider the Lagrangian subbundle $\tau_B(L) \subseteq \mathbb{T}M$ defined by

$$\tau_B(L) = \{X \oplus \alpha + i_X B \mid X \oplus \alpha \in L\}.$$

Now we see what condition on the 2-form $B$ implies that $\tau_B(L)$ defines a Dirac structure.

**Proposition 5.4.1.** *[26]*

*The subbundle $\tau_B(L)$ defines a Dirac structure on $M$ if and only if $B$ is a closed 2-form.*

*Proof.* Let $X \oplus \alpha$ and $Y \oplus \beta$ be sections of $L$. Then

$$[\![X \oplus \alpha + i_X B, Y \oplus \beta + i_Y B]\!] = [X, Y] \oplus \mathcal{L}_X \beta - i_X d\alpha + \mathcal{L}_X i_Y B - i_Y d i_X B,$$

and using the formula $i_{[X,Y]} = [\mathcal{L}_X, i_Y]$ one can see that $B$ is closed if and only if

$$[\![X \oplus \alpha + i_X B, Y \oplus \beta + i_Y B]\!] = [X,Y] \oplus \mathcal{L}_X \beta - i_X d\alpha + i_{[X,Y]} B,$$

which is equivalent to saying that $\tau_B(L)$ is a Dirac structure.

□

With this, the abelian group $\Omega^2_{cl}(M)$ of closed 2-forms on $M$ can be thought of as a group of symmetries of the space of Dirac structures on $M$. A Dirac structure $L'$ is **gauge equivalent** to $L$, if $L' = \tau_B(L)$ for some closed 2-form $B$ on $M$, see e.g. [5]. We also say that $L'$ is obtained out of $L$ by a $B$**-field** transformation. Notice that, it follows from the proposition above, that the injective bundle map

$$TM \oplus T^*M \xrightarrow{\tau_B} TM \oplus T^*M$$
$$X \oplus \alpha \mapsto X \oplus \alpha + i_X B,$$

preserves the Courant bracket. In particular, as observed in [5], gauge equivalent Dirac structures define isomorphic Lie algebroids

$$L \cong \tau_B(L).$$

One observes that gauge transformations may change the “relative position” of a Dirac subbundle $L$ inside $\mathbb{T}M$. For instance, if a Dirac sub bundle $L$ has null intersection with $TM$, that is $L$ is a Dirac structure induced by a Poisson bivector $\pi$ on $M$, then not necessarily $\tau_B(L)$ is the graph of a Poisson bivector.

**Definition 5.4.1.** [5]

A closed 2-form $B$ on $M$ is called $\pi$**-admissible** if $\tau_B(L) = L_{\tau_B(\pi)}$ for some Poisson bivector $\tau_B(\pi)$ on $M$.

As we have seen before, if $B$ is $\pi$-admissible then the Lie algebroid $L_\pi$ is isomorphic to $L_{\tau_B(\pi)}$ via the canonical map $\tau_B$. This induces a canonical isomorphism between the Lie algebroids $(T^*M)_\pi$ and $(T^*M)_B$ determined by the Poisson structures $\pi$ and $\tau_B(\pi)$, respectively. One can check easily, that the induced Lie algebroid isomorphism is given by

$$\varphi_B := \mathrm{Id} + B^\sharp \circ \pi^\sharp : (T^*M)_\pi \longrightarrow (T^*M)_{\tau_B(\pi)}.$$

Now we consider an integrable Poisson manifold $(M, \pi)$, with symplectic groupoid $(G, \omega_G)$. Assume that $B$ is a $\pi$-admissible 2-form on $M$, then the Lie algebroid associated to the Poisson manifold $(M, \tau_B(\pi))$ integrates to $G$. The natural question is what is the effect of a gauge transformation on the symplectic groupoid of $M$. Notice that the map $\mathrm{Id} + B^\sharp \circ \pi^\sharp : T^*M \longrightarrow T^*M$ is an IM-2-form, and since it is invertible, it corresponds to a symplectic form $\omega_B$ on the Lie groupoid $G$. Further, notice that $\mathrm{Id} : T^*M \longrightarrow T^*M$ is the IM-2-form associated to $\omega_G$, and $B^\sharp \circ \pi^\sharp : T^*M \longrightarrow T^*M$ is the IM-2-form associated to the multiplicative 2-form on $G$ defined by $B_G := t^*B - s^*B$. Now it is clear how the symplectic groupoid of an integrable Poisson manifold is modified under the action of a $B$-field.

**Theorem 5.4.1.** *[11]*

*Consider the multiplicative 2-form $\omega_B = \omega + B_G$. Then the pair $(G, \omega_B)$ is a symplectic groupoid integrating the Poisson manifold $(M, \tau_B(\pi))$.*

More generally, we can study gauge transformations of Poisson groupoids. In particular we are concerned with the effect of a gauge transformation on the Lie bialgebroid of a Poisson groupoid. Let $(G, \pi_G)$ be a Poisson groupoid with Lie bialgebroid $(A, A^*)$. Let $B_G \in \Omega^2(G)$ be a closed multiplicative form on $G$. Assume that $B_G$ is $\pi_G$-admissible and consider the Poisson bivector $\pi_G^B$ constructed via the $B_G$-field tansformation of $\pi_G$. One can check that

$$(\pi_G^B)^\sharp = \pi_G^\sharp \circ (\mathrm{Id} + B_G^\sharp \circ \pi_G^\sharp)^{-1}.$$

In particular the Poisson bivector $\pi_G^B$ is multiplicative, since $(\pi_G^B)^\sharp$ is the composition of groupoid morphisms. As explained in chapter 3, the multiplicative closed 2-form $B_G$ induces a linear closed 2-form $B_A$ on $A$, and it is easy to see that $B_A$ is $\pi_A$-admissible, where $\pi_A$ is the linear Poisson structure on $A$ induced by the dual Lie algebroid $A^*$. Thus, we obtain a new Poisson structure on $A$ which is determined by

$$(\pi_A^B)^\sharp = \pi_A^\sharp \circ (\mathrm{Id} + B_A^\sharp \circ \pi_A^\sharp)^{-1}.$$

One observes that $\pi_A^B$ is a linear bivector, since $(\pi_A^B)^\sharp$ is the composition of Lie algebroid morphisms. Therefore, in the presence of a multiplicative $B_G$-field, the Lie algebroid structure of $A$ is preserved. On the other hand, the Poisson structure on $A$ is modified by the linear $B_A$-field transformation, so the Lie algebroid structure on the dual bundle $A^*$ changes.

Now we see how the Lie algebroid $A^*$ changes under the action of a gauge transformation of $\pi_G$ by the multiplicative form $B_G = t^*B - s^*B$ where $B$ is a closed 2-form on the base manifold $M$. We denote by $A^*_B$ the Lie algebroid dual to the linear Poisson bivector $\pi^B_A$. Let us find the anchor $\rho^B_{A^*}$ and the Lie bracket $[\cdot,\cdot]^B_{A^*}$ of the Lie algebroid $A^*_B$. First we have a morphism of vector bundles

$$\begin{array}{ccc} T^*A & \xrightarrow{(\pi^B_A)^\sharp} & TA \\ \downarrow & & \downarrow \\ A^* & \xrightarrow[\rho_{A^*}\circ\psi_B^{-1}]{} & TM \end{array} \tag{5.15}$$

where $\psi_B = (\mathrm{Id} + \rho^*_A \circ B^\sharp \circ \rho_{A^*})$. On one hand, the linear bivector $\pi^B_A$ induces a morphism of Lie algebroids $(T^*A)_{B_A} \longrightarrow TA$, then it follows from Theorem 2.4.3 that the anchor of $A^*_B$ is given by

$$\rho^B_{A*} = \rho_{A^*} \circ \psi_B^{-1}.$$

Moreover the Lie bracket of the Lie algebroid $A^*_B$ is given by

$$[\xi_1, \xi_2]^B_{A^*} = \psi_B[\psi_B^{-1}(\xi_1), \psi_B^{-1}(\xi_2)]_{A^*}.$$

In summary, the action of $B_G = t^*B - s^*B$ on $(G, \pi_G)$ is reflected infinitesimally by the transition from the Lie bialgebroid $(A, A^*)$ to the Lie bialgebroid $(A, A^*_B)$. Notice that $(A, A^*_B)$ is actually a Lie bialgebroid due to the fact that (5.15) is a Lie algebroid morphism. See Theorem 2.4.3.

**Remark 5.4.1.** Recall that every Lie bialgebroid $(A, A^*)$ induces a Poisson structure $\pi$ on the base $M$, determined by

$$\pi^\sharp = \rho_A \circ \rho^*_{A^*},$$

where $\rho_A, \rho_{A^*}$ denote the anchor maps of $A, A^*$, respectively. See e.g. [5]. Notice that a closed 2-form $B$ on $M$ is $\pi$-admissible if and only if the map $\psi_B : A^* \longrightarrow A^*$ defined previously is invertible.

The notion of gauge transformation of a Lie bialgebroid was introduced in [5], and it becomes clear after the comments above. Let $(A, A^*)$ be a Lie bialgebroid with anchor maps $\rho_A : A \longrightarrow TM$, $\rho_{A^*} : A^* \longrightarrow TM$ and Lie brackets $[\cdot,\cdot]_A$, $[\cdot,\cdot]_{A^*}$. Let $\pi$ be the Poisson structure on $M$ induced by $(A, A^*)$. Suppose that $B$ is a closed 2-form on $M$ which is $\pi$-admissible.

**Definition 5.4.2.** The gauge transformation of the Lie bialgebroid $(A, A^*)$ by the closed 2-form $B$, is the Lie bialgebroid $(A, A^*_B)$ described before.

The following result was proved in [5].

**Theorem 5.4.2.** *Let $(G, \pi_G)$ be a Poisson groupoid over $M$, with Lie bialgebroid $(A, A^*)$, and induced Poisson structure $\pi$ on $M$. Let $B$ be a closed 2-form on $M$ and consider the 2-form $B_G = t^*B - s^*B$. Then $B$ is $\pi$-admissible if and only if $B_G$ is $\pi_G$-admissible. Moreover, the Poisson groupoid $(G, \tau_{B_G}(\pi_G))$ has Lie bialgebroid $(A, A^*_B)$.*

The following result describes the effect of a gauge transformation of a Poisson groupoid by a non admissible multiplicative 2-form. This is the original setting where multiplicative Dirac structures appeared.

**Theorem 5.4.3.** *Let $(G, \pi_G)$ be a Poisson groupoid over $M$ with Lie bialgebroid $(AG, A^*G)$. Let $B$ be a closed 2-form on $M$ and consider the multiplicative 2-form $B_G = t^*B - s^*B$. The following hold:*

1. *The Dirac structure $L_G^B := \tau_{B_G}(L_{\pi_G})$ is multiplicative.*

2. *The linear Dirac structure induced by $L_G^B$ is*

$$L_{AG}^B = \tau_{B_{AG}}(L_{\pi_{AG}}),$$

*where $B_{AG}$ is the morphic 2-form determined by $B_G$, and $\pi_{AG}$ is the linear Poisson structure on $AG$ induced by $A^*G$.*

*Proof.* Let us show the first statement. A straightforward computation shows that the multiplicativity of the form $B_G$ is equivalent to saying that

$$\tau_{B_G} : \mathbb{T}G \longrightarrow \mathbb{T}G$$

is a morphism of Lie groupoids. In particular, since $L_{\pi_G}$ is a Lie subgroupoid of $\mathbb{T}G$ we conclude that the image $L^B_{AG} = \tau_{B_G}(L_{\pi_G})$ is also a Lie subgroupoid of $\mathbb{T}G$, as required. In order to prove *2.* we observe that the application of the Lie functor to $\tau_{B_G}$ gives a morphism of Lie algebroids, which up to canonical identifications coincides with

$$\tau_{B_A} : \mathbb{T}A \longrightarrow \mathbb{T}A,$$

where $B_A$ is the morphic closed 2-form on $A$ induced by $B_G$. See chapter 3 to recall this construction. The isomorphism of Lie groupoids

$$L_{\pi_G} \stackrel{\tau_{B_G}}{\longrightarrow} L^B_G,$$

induces an isomorphism of Lie algebroids

$$L_{\pi_{AG}} \stackrel{\tau_{B_{AG}}}{\longrightarrow} L^B_{AG}.$$

Hence, the Dirac structure in $\mathrm{Dir}_{alg}(AG)$ induced by the multiplicative Dirac structure $L^B_G \in \mathrm{Dir}_{mult}(G)$ coincides with the subalgebroid $L^B_A \subseteq \mathbb{T}A$, as desired.

□

# Chapter 6

# Dirac groupoids and $\mathcal{LA}$-groupoids

This chapter is concerned with the second order geometry underlying multiplicative Dirac structures. The definition of a Dirac groupoid encompasses two geometric structures, namely a Dirac sub bundle $L_G \subseteq \mathbb{T}G$ which also defines a $\mathcal{VB}$-subgroupoid $L_G \rightrightarrows E$ of the natural $\mathcal{VB}$-groupoid structure on $\mathbb{T}G$. Double geometric structures have been vastly studied by Kirill Mackenzie [42, 43, 45, 44] providing a unified setting for several structures appearing in the theory of Poisson manifolds. The main observation of this chapter is that every multiplicative Dirac structure fits in Mackenzie's theory of double structures. More concretely, we show that every Dirac groupoid can be viewed as a double structure called $\mathcal{LA}$-groupoid, which roughly speaking is a Lie groupoid object in the category of Lie algebroids. A prolongation procedure, similar to the tangent prolongation of a Lie algebroid, gives rise to the infinitesimal data of an $\mathcal{LA}$-groupoid, in the terminology of [43] such a infinitesimal data is called a double Lie algebroid. If we think of a Dirac groupoid as a special type of $\mathcal{LA}$-groupoid, we are allowed to apply the Lie functor yielding the corresponding double Lie algebroid. It turns out that this double Lie algebroid encodes the linear Dirac structure associated to any multiplicative Dirac structure, as explained in chapter 5.

## 6.1 $\mathcal{LA}$-groupoids and double Lie algebroids

An $\mathcal{LA}$-groupoid is a Lie groupoid object in the category of Lie algebroids. More precisely, an **$\mathcal{LA}$-groupoid** is a square

$$
\begin{array}{ccc}
H & \xrightarrow{q_H} & G \\
\Downarrow & & \Downarrow \\
E & \xrightarrow[q_E]{} & M
\end{array}
\tag{6.1}
$$

where the single arrows denote Lie algebroids and the double arrows denote Lie groupoids. These structures are compatible in the sense that all the structure mappings (i.e. source, target, unit section, inversion and multiplication) defining the Lie groupoid $H$ are Lie algebroid morphisms over the corresponding structure mappings which define the Lie groupoid $G$. We also require that the anchor map $\rho_H : H \longrightarrow TG$ be a groupoid morphism over the anchor map $\rho_E : E \longrightarrow TM$. Here $TG$ is endowed with the tangent groupoid structure over $TM$. For describing the square given by an $\mathcal{LA}$-groupoid we use the notation $(H, G, E, M)$. It is worthwhile to explain how the groupoid multiplication defines a morphism of Lie algebroids. For that, let $m_H : H_{(2)} \subseteq H \times H \longrightarrow H$ denote the groupoid multiplication of $H$, and similarly let $m_G : G_{(2)} \subseteq G \times G \longrightarrow G$ denote the multiplication of $G$. The direct product vector bundle $H \times H \longrightarrow G \times G$ inherits a natural Lie algebroid structure, and we have a Lie subalgebroid $H_{(2)}$ over $G_{(2)}$ which is just a pull back algebroid, see e.g. [33] for details about the pull back operation in the category of Lie algebroids. With respect to this Lie algebroid structure, the multiplication map $m_H$ is required to be a Lie algebroid morphism covering $m_G$.

**Example 6.1.1.** Let $G$ be a Lie groupoid over $M$. The tangent functor leads to a canonical $\mathcal{LA}$-groupoid $(TG, G, TM, M)$, where the Lie groupoid structure on $TG$ is the tangent groupoid, and the Lie algebroid structure $TG \longrightarrow G$ corresponds to the trivial Lie algebroid. See example 2.1.7 in chapter 2.

The Lie functor applied to an $\mathcal{LA}$-groupoid (6.1) determines a square

$$\begin{array}{ccc} AH & \xrightarrow{A(q_H)} & AG \\ \big\downarrow & & \big\downarrow \\ E & \xrightarrow[q_E]{} & M \end{array} \tag{6.2}$$

where each of the arrows define Lie algebroids. The top Lie algebroid structure is non trivial, and it deserves a detailed explanation. The Lie algebroid structure $AH \longrightarrow AG$ was constructed in [43] as a prolongation procedure similar to the tangent prolongation of a Lie algebroid, except that we replace the tangent functor by the Lie functor.

**Remark 6.1.1.** The main ingredients for constructing the tangent Lie algebroid of $H \longrightarrow G$ are the tangent anchor map

$$\rho_{TH} = J_G \circ T\rho_H,$$

and the generators $Tu, \hat{u}$ of the space os sections $\Gamma_{TG}(TH)$, where $u$ is a section of the vector bundle $H \longrightarrow G$. In order to construct the prolonged Lie algebroid structure on $AH \longrightarrow AG$ we need to understand the analogue objects of the tangent anchor and the generating sections. More precisely, we would like to find conditions on the anchor map $\rho_H$ and a section $u \in \Gamma_G(H)$ in such a way that the tangent anchor, as well as the sections $Tu, \hat{u}$, restrict to an anchor map and sections of $AH \longrightarrow AG$ that generate $\Gamma_{AG}(AH)$.

Notice that the tangent anchor map is obtained by a direct application of the tangent functor to the anchor of $H$, and then twisting by a suitable morphism of double vector bundles. This suggests that the anchor map for $AH \longrightarrow AG$ should be defined by an application of the Lie functor to the Lie groupoid morphism $\rho_H : H \longrightarrow TG$ and then swap it in a proper manner.

**Definition 6.1.1.** The **prolonged anchor map** $AH \longrightarrow T(AG)$ is defined by

$$\tilde{\rho} := j_G^{-1} \circ A(\rho_H),$$

where $j_G : T(AG) \longrightarrow A(TG)$ is the canonical identification defined in appendix A.

Now we study the space of sections $\Gamma_{AG}(AH)$. First we notice that the induced section $Tu \in \Gamma_{TG}(TH)$ defines a section of $AH \longrightarrow AG$ if the section $u : G \longrightarrow H$ preserves the units and the source fibrations. This leads naturally to the following definition.

**Definition 6.1.2.** A section $u \in \Gamma_G(H)$ is called a **star section** if there exists a section $u_0 \in \Gamma_M(E)$ such that

1. $\epsilon_E \circ u_0 = u \circ \epsilon_M$,

2. $s_H \circ u = u_0 \circ s_G$.

Notice that since every star section $u : G \longrightarrow H$ preserves the units and the source fibrations, we are allowed to apply the Lie functor to $u$, yielding a section $A(u)$ of the vector bundle $AH \overset{A(q_H)}{\longrightarrow} AG$.

**Remark 6.1.2.** Recall that the core of the double vector bundle $(TH, TG, H, G)$ is the vector bundle $H \longrightarrow G$. Every section $u$ of the core $H \longrightarrow G$ gave rise to a core section $\hat{u} \in \Gamma_{TG}(TH)$ defined by

$$\hat{u}(X_g) = T(0^H)X_g + \overline{u(g)},$$

where $\overline{u(g)} = \frac{d}{dt}(tu(g))|_{t=0}$ is the core element induced by $u(g) \in H_g$. This suggests that in order to define sections of $AH \longrightarrow AG$ that play the role of $\hat{u}$, we need to find the core of the double vector bundle $(AH, AG, E, M)$.

**Definition 6.1.3.** Let $(H, G, E, M)$ be an $\mathcal{LA}$-groupoid. The **core** of $H$ is the vector bundle over $M$ defined by

$$K := \epsilon_M^* \ker(s_H).$$

**Example 6.1.2.** Let $G$ be a Lie groupoid and consider the canonical $\mathcal{LA}$-groupoid $(TG, G, TM, M)$. The core of $TG$ is nothing else that $K = AG$ the Lie algebroid of $G$.

Given a section $k \in \Gamma(K)$ we define a section $k_H \in \Gamma_G(H)$ in the following way

$$k_H(g) := k(t_G(g))0_g^H,$$

where $0_g^H$ is the zero element in the fiber $H_g$ above $g \in G$. Notice that for every section $k \in \Gamma(K)$ the induced section $k_H \in \Gamma_G(H)$ satisfies

$$k_H \circ \epsilon_M = k.$$

**Example 6.1.3.** For the canonical $\mathcal{LA}$-groupoid $(TG, G, TM, M)$ a section $k$ of the core $K$ is just a section of the Lie algebroid $AG$. The induced section $k_{TG} \in \Gamma_G(TG)$ is exactly the right invariant vector field on $G$ determined by the section $k \in \Gamma(AG)$. Indeed, the right invariant vector field determined by $k \in \Gamma(AG)$ is defined, at every $g \in G$ with $t(g) = x$, by

$$Tr_g(k(x)) = Tm_G(x, g)(k(x), 0_g^{TG}),$$

which is exactly the section $k_{TG}$.

It was proved in [43] that there exist an exact sequence of vector bundles over $E$,

$$q_E^*(K) \longrightarrow AH \longrightarrow q_E^*(AG),$$

and an exact sequence of vector bundles over $AG$

$$q_{AG}^*(K) \longrightarrow AH \longrightarrow q_{AG}^*(E).$$

In particular, the core of the double vector bundle $(AH, AG, E, M)$ is the vector bundle $K \longrightarrow M$.

Now let us see how a core element $k \in K$ induces a Lie algebroid element $\overline{k} \in AH$. For that, we observe that every element in $AH$ has the form

$$W = \frac{d}{dt}(h_t)|_{t=0},$$

where $h_t$ is a curve in $H$ sitting in a fixed source fiber $s_H^{-1}(e)$ with $h_0 = \epsilon_E(e)$. Thus, for every core element $k \in K$ above $x \in M$, that is $s_H(k) = 0_x^E$ and $q_H(k) = \epsilon_M(x)$, there exists a natural element $\overline{k} \in AH$, defined by

$$\overline{k} := \frac{d}{dt}(tk)|_{t=0}.$$

**Definition 6.1.4.** Given a section $k \in \Gamma(K)$, the **core** section induced by $k$ is the section $k^{\text{core}} \in \Gamma_{AG}(AH)$ defined by

$$k^{\text{core}}(u_x) := A(0^H)u_x + \overline{k(x)}.$$

Notice that every section $k \in \Gamma(K)$ induces a section of the tangent prolongation $TH \longrightarrow TG$. Indeed, first we consider the induced section $k_H \in \Gamma_G(H)$ and then we construct the core section $\hat{k}_H \in \Gamma_{TG}(TH)$ defined in the usual way

$$\hat{k}_H(X_g) = T(0^H)X_g + \overline{k_H(g)}.$$

For every $x \in \epsilon_M(M) \subseteq G$ one has $k_H(x) = k(x)$, and thus at any $u_x \in (AG)_x \subseteq T_xG$ we get

$$\hat{k}_H(u_x) = A(0^H)u_x + \overline{k(x)}.$$

Hence we conclude that the core section $\hat{u} \in \Gamma_{TG}(TH)$ restricts to a section of $AH \longrightarrow AG$ if $\hat{u} = \hat{k}_H$ comes from a section $k \in \Gamma(K)$ of the core of $(H, G, E, M)$. The following proposition was proved in [43].

**Proposition 6.1.1.** *The space of sections $\Gamma_{AG}(AH)$ is generated by sections of the form $A(u)$, where $u : G \longrightarrow H$ is a star section, and by sections of the form $k^{\text{core}}$, where $k : M \longrightarrow K$ is a section of the core of $H$.*

The Lie bracket on $\Gamma_{AG}(AH)$ is defined in terms of star sections and core sections. First we observe that whenever $u, v \in \Gamma_G(H)$ are star sections, then the Lie bracket $[u, v] \in \Gamma_G(H)$ is also a star section. Thus the Lie bracket between sections of the form $A(u), A(v)$ is defined by

$$[A(u), A(v)] = A([u, v]).$$

The bracket of a pair of core sections is defined by

$$[k_1^{\text{core}}, k_2^{\text{core}}] = 0.$$

In order to define the bracket of a star section and a core section we notice that every star section $u : G \longrightarrow H$ induces a covariant differential operator

$$\begin{aligned} D_u : \Gamma(K) &\longrightarrow \Gamma(K) \\ k &\mapsto [u, k_H] \circ \epsilon_M, \end{aligned}$$

now we define $[A(u), k^{\text{core}}] = (D_u(k))^{\text{core}}$.

The Lie bracket of other sections of $\Gamma_{AG}(AH)$ is defined by requiring the Leibniz rule

$$[w, fw'] = f[w, w'] + (\mathcal{L}_{\tilde{\rho}(w)} f)w'.$$

The vector bundle $AH \overset{A(q_H)}{\longrightarrow} AG$ endowed with the anchor map $\tilde{\rho} = j_G^{-1} \circ A(\rho)$ and the Lie bracket $[\cdot, \cdot]$ on $\Gamma_{AG}(AH)$ becomes a Lie algebroid called the **prolonged Lie algebroid** induced by $H \longrightarrow G$, see [43].

**Example 6.1.4.** Consider the canonical $\mathcal{LA}$-groupoid $(TG, G, TM, M)$ with the corresponding prolonged Lie algebroid $(A(TG), AG, TM, M)$. The canonical map $j_G : T(AG) \longrightarrow A(TG)$ is a morphism of double vector bundles, and whenever it is viewed as a morphism of vector bundles over $AG$, it becomes a Lie algebroid isomorphism between the trivial Lie algebroid $T(AG) \longrightarrow AG$ and the prologated Lie algebroid $A(TG) \longrightarrow AG$. In fact, the compatibility with the anchor maps follows directly from the definition of the prologated anchor map. On the other hand, for every star vector field $X \in \Gamma_G(TG)$, we have an induced vector field $\tilde{X} = j_G^{-1}(A(X))$ on $AG$, which is linear in the sense that the corresponding local 1-parameter family of diffeomorphisms of $AG$ is given by vector bundle isomorphisms. Similarly, for every section $k \in \Gamma(AG)$ of the core of $TG$, one has another vector field $k^{\uparrow}$ on $AG$, which is the core vector field induced by $k$, that is

$$k^{\uparrow}(a)F := \frac{d}{dt} F(a + tk(q_{AG}(a)))|_{t=0}.$$

The space of vector fields $\mathfrak{X}(AG)$ is generated by vector fields of the form $\tilde{X}$, where $X \in \mathfrak{X}(G)$ is a star vector field, and by vector fields of the form $k^{\uparrow}$, where $k \in \Gamma(AG)$. The Lie bracket of such a vector fields satisfies

$$[\tilde{X}, \tilde{Y}] = [X, \tilde{Y}]; \quad [\tilde{X}, k^{\uparrow}] = ([X, k^r] \circ \epsilon_M)^{\uparrow}; \quad [k_1^{\uparrow}, k_2^{\uparrow}] = 0.$$

In particular the prolonged Lie bracket on $\Gamma_{AG}(A(TG))$ is mapped, via $j_G^{-1} : A(TG) \longrightarrow T(AG)$, to the usual Lie bracket of vector fields on $AG$. See [47].

## 6.2 Dirac groupoids as $\mathcal{LA}$-groupoids

Let $L_G$ be a multiplicative Dirac structure on a Lie groupoid $G \rightrightarrows M$. This means that we have a $\mathcal{VB}$-subgroupoid $L_G \rightrightarrows E$ of $\mathbb{T}G \rightrightarrows TM \oplus A^*G$, such that $L_G \subseteq \mathbb{T}G$ is also a Dirac sub bundle. In particular there is a canonical Lie algebroid structure on $L_G \longrightarrow G$ with anchor map $L_G \longrightarrow TG$ the natural projection and Lie bracket $[\![\cdot,\cdot]\!]$ on $\Gamma_G(L_G)$. The dual of the Lie algebroid $AG$ can identified with the conormal bundle $N^*(M) \subseteq T^*G$, and we define a Courant like bracket on the space of sections of $E \subseteq TM \oplus A^*G$ by

$$[X_1 \oplus \xi_1, X_2 \oplus \xi_2] := [X_1, X_2] \oplus (\mathcal{L}_{X_1}\xi_2 - i_{X_2}d\xi_1).$$

With respect to this Lie bracket and the natural projection $E \longrightarrow TM$, the vector bundle $E \longrightarrow M$ becomes a Lie algebroid.

**Proposition 6.2.1.** *A multiplicative Dirac structure $L_G$ on $G$ gives rise to an $\mathcal{LA}$-groupoid*

$$\begin{array}{ccc} L_G & \xrightarrow{p_G \oplus c_G} & G \\ \downarrow\downarrow & & \downarrow\downarrow \\ E & \xrightarrow[q_E]{} & M \end{array} \tag{6.3}$$

*where $p_G$ and $c_G$ denote the tangent projection and the cotangent projection, respectively.*

*Proof.* Since the structure mappings defining the Lie groupoid $L_G \rightrightarrows E$ are restrictions of the structure mappings of the tangent and cotangent groupoids, a straightforward computation shows that these structure mappings are Lie algebroid morphisms over the structure mapping of $G$. In order to prove that the multiplication on $L_G$ is a Lie algebroid morphism over the multiplication on $G$, we reproduce the proof of Proposition 5.2.3 replacing $m_{\mathbb{T}}$ by $m_{L_G}$, where $m_{L_G}$ denotes the multiplication on $L_G$. An argument similar to the one used in the proof of Proposition 5.2.3 shows that the inversion map on $L_G$ is a Lie algebroid morphism. This proves the statement.

□

**Example 6.2.1.** Assume that $(G, \pi_G)$ is a Poisson groupoid. Then $\pi_G^\sharp : T^*G \longrightarrow TG$ is a groupoid morphism over the dual anchor map $\rho_{A^*G} : A^*G \longrightarrow TM$. The corresponding

$\mathcal{LA}$-groupoid associated with this structure is

$$\begin{array}{ccc} L_{\pi_G} & \xrightarrow{p_G \oplus c_G} & G \\ \downdownarrows & & \downdownarrows \\ E_\rho & \xrightarrow[q_{E_\rho}]{} & M \end{array} \tag{6.4}$$

where $L_{\pi_G}$ is the graph of the bivector $\pi_G$, and $E_\rho$ is the graph of the dual anchor map $\rho_{A^*G}$. The top Lie algebroid structure is the usual algebroid structure isomorphic to the cotangent bundle $T^*G \longrightarrow G$, and the Lie algebroid structure on $E_\rho$ is the one induced by the graph of the Lie algebroid morphism $\rho_{A^*G} : A^*G \longrightarrow TM$.

**Example 6.2.2.** Let $\omega_G$ be a multiplicative closed 2-form on a Lie groupoid $G$. Consider the corresponding IM-2-form $\sigma : AG \longrightarrow T^*M$. This determines an $\mathcal{LA}$-groupoid

$$\begin{array}{ccc} L_{\omega_G} & \xrightarrow{p_G \oplus c_G} & G \\ \downdownarrows & & \downdownarrows \\ E_\sigma & \xrightarrow[q_{E_\sigma}]{} & M \end{array} \tag{6.5}$$

where $L_{\omega_G}$ is the graph of the $\omega_G$ and $E_\sigma$ denotes the graph of the bundle map $-\sigma^t : TM \longrightarrow A^*G$.

Consider now a multiplicative Dirac structure $L_G \subseteq \mathbb{T}G$ with associated $\mathcal{LA}$-groupoid $(L_G, G, E, M)$. Applying the Lie functor we obtain the prolonged Lie algebroid structure on $A(L_G) \longrightarrow AG$, and we use the canonical map $j_G^{-1} \oplus j'_G : A(TG) \oplus A(T^*G) \longrightarrow T(AG) \oplus T^*(AG)$, to define a Lie algebroid $L_{AG} = (j_G^{-1} \oplus j'_G)(A(L_G))$ over $AG$, characterized by the fact that $j_G^{-1} \oplus j'_G : A(L_G) \longrightarrow L_{AG}$ is a Lie algebroid isomorphism. We have seen in chapter 5 that $L_{AG} \subseteq \mathbb{T}(AG)$ is a Lagrangian sub bundle with respect to the canonical pairing $\langle \cdot, \cdot \rangle_{AG}$ on $\mathbb{T}(AG)$. We claim that the Lie bracket on $\Gamma_{AG}(L_{AG})$ induced by the prolonged Lie bracket on $\Gamma_{AG}(A(L_G))$ coincides with the Courant bracket.

**Theorem 6.2.1.** *The Lie bracket on $\Gamma_{AG}(L_{AG})$ coincides with the Courant bracket $[\![\cdot, \cdot]\!]$ determined by the Courant algebroid $T(AG) \oplus T^*(AG)$. In particular, the Lie functor maps* *multiplicative Dirac structures on* $G$ *into linear Dirac structures on* $AG$ *which are Lie subalgebroids of* $\mathbb{T}(AG)$.

*Proof.* The space of sections $\Gamma_{AG}(L_{AG})$ is generated by sections of the form $\tilde{X} \oplus \tilde{\alpha} := j_G^{-1}(A(X)) \oplus j'_G(A(\alpha))$, where $X \oplus \alpha \in \Gamma_G(L_G)$ is a star section, and by sections of the form $k^{\uparrow} \oplus h^{\uparrow} = j_G^{-1}(k^{\text{core}}) \oplus j'_G(h^{\text{core}})$, where $k \oplus h \in \Gamma(K_G)$ is a section of the core $K_G \longrightarrow G$ of the $\mathcal{LA}$-groupoid $(L_G, G, E, M)$. The Lie bracket on $\Gamma_{AG}(L_{AG})$ is determined by the following identities

1. $[\tilde{X} \oplus \tilde{\alpha}, \tilde{Y} \oplus \tilde{\beta}] = (j_G^{-1} \oplus j'_G)A([\![X \oplus \alpha, Y \oplus \beta]\!]_G)$
2. $[\tilde{X} \oplus \tilde{\alpha}, k^{\uparrow} \oplus h^{\uparrow}] = (j_G^{-1} \oplus j'_G)([\![X \oplus \alpha, k_{L_G} \oplus h_{L_G}]\!]_G \circ \epsilon_M)^{\text{core}}$
3. $[k_1^{\uparrow} \oplus h_1^{\uparrow}, k_2^{\uparrow} \oplus h_2^{\uparrow}] = 0.$

Since $A(X \oplus \alpha)$ and $(k \oplus h)^{\text{core}}$ are suitable restrictions of $T(X \oplus \alpha)$ and $(k \oplus h)\hat{}_{L_G}$, respectively, and the Lie bracket on sections of $AH \longrightarrow AG$ comes from the Lie bracket on sections of $TH \longrightarrow TG$, we conclude that the isomorphism $A(L_G) \cong L_{AG}$, which is a suitable restriction of the isomorphism $(L_{TG}, [\![\cdot,\cdot]\!]) \cong (TL_G, [\cdot,\cdot])$ shown in Proposition 5.1.2, sends the prolonged Lie bracket to the Courant bracket.

□

**Example 6.2.3.** The prolonged Lie algebroid $A(L_{\pi_G}) \longrightarrow AG$ induced by the $\mathcal{LA}$-groupoid determined by a Poisson groupoid, is mapped via the canonical map $j_G^{-1} \oplus j'_G$ into the Lie algebroid $L_{\pi_{AG}} \longrightarrow AG$ given by the linear Dirac structure on $AG$ defined by the linear Poisson bivector $\pi_{AG}$ on $AG$.

**Example 6.2.4.** Consider the prolonged Lie algebroid $A(L_{\omega_G}) \longrightarrow AG$ induced by the $\mathcal{LA}$-groupoid determined by a multiplicative closed 2-form $\omega_G$ on $G$. The canonical map $j_G^{-1} \oplus j'_G$ sends the prolonged Lie algebroid to the Lie algebroid $L_{\omega_A} \longrightarrow AG$ defined by the linear Dirac structure on $AG$ which is the graph of the linear closed 2-form $\omega_A = \text{Lie}(\omega_G)$ on $AG$.

Although the $\mathcal{LA}$-groupoid approach to Dirac groupoids just explain the action of the Lie functor, we believe that it provides an explicit construction of the linear Dirac structure corresponding to a multiplicative Dirac structure, following the spirit of the construction of the linear Dirac structures associated to multiplicative Poisson bivectors and multiplicative closed 2-forms. However, the integration of double Lie algebroids to double Lie groupoids was perfomed by Kirill Mackenzie in some special cases. Our guess is that an intermediate integration step as

$$\{\text{double Lie algebroids}\} \longrightarrow \{\mathcal{LA}\text{-groupoids}\} \longrightarrow \{\text{double Lie groupoids}\},$$

would provide an explicit integration functor from linear Dirac structures to multiplicative Dirac structures. Furthermore, such intermediate step would be useful to find the presymplectic groupoid associated to a multiplicative Dirac structure. This will be treated in a future work.

# Chapter 7

# New research directions

This chapter contains a description of future work based on the main results exposed along this dissertation.

## 7.1 Lie's second theorem and the Van Est isomorphism

Let $A \xrightarrow{q_A} M$ be a Lie algebroid with anchor map $\rho : A \longrightarrow TM$ and Lie bracket $[\cdot, \cdot]$ on $\Gamma(A)$. Consider a closed $(k+1)$-form $\phi$ on $M$. An IM-$k$-form on $A$ with respect to $\phi$ is a bundle map $\sigma : A \longrightarrow \prod_{c_M}^{k-1} T^*M$, which satisfies the following conditions:

1. $i_{\rho(v)}\sigma(u) = -i_{\rho(u)}\sigma(v)$
2. $\sigma([u, v]) = \mathcal{L}_{\rho(u)}\sigma(v) - \mathcal{L}_{\rho(v)}\sigma(u) + di_{\rho(v)}\sigma(u) + i_{\rho(u)\wedge\rho(v)}\phi,$

   for every $u, v \in \Gamma(AG)$.

In [2, 3] was proved that for every source-simply connected Lie groupoid $G$ with Lie algebroid $A$, there exists a one-to-one correspondence between:

i) Multiplicative $k$-forms $\omega_G$ on $G$ with $d\omega_G = s^*\omega - t^*\omega_G$, and

ii) IM-$k$-forms on A with respect to $\phi$.

The method used in [2, 3] is based in the interpretation of multiplicative forms satisfying i) as cocycles in the so called Bott-Shulman complex of the Lie groupoid $G$. Similarly, IM-$k$-forms with respect to $\phi$ induces cocycles in the Weil algebra of the Lie algebroid $A$, see [2, 3]. The correspondence between multiplicative forms satisfying i) and IM-$k$-forms is constructed out of a Van Est type isomorphism between the cohomology of the Bott-Shulman complex of $G$ and the cohomology of the Weil algebra of $A$, see [2, 3]. We would like to find the relation between linear $k$-forms on a Lie algebroid $A$ and elements in the Weil algebra of $A$. In particular, morphic $k$-forms must induce cocycles in the Weil algebra. We can use Lie's second theorem to integrate morphic forms to multiplicative forms. This procedure must be related with the Van Est map approach for integrating IM-$k$-forms. We will discuss these topics in a future work.

## 7.2 Multiplicative Dirac structures and supergeometry

In this section we explain how the theory of graded supermanifolds can be used to study multiplicative Dirac structures.

**Definition 7.2.1.** [38, 53]

A **Courant algebroid** over $M$ is a vector bundle $E \longrightarrow M$ equipped with a nondegenerate symmetric fibrewise bilinear operation $\langle\cdot,\cdot\rangle$, a bilinear bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(E)$ and an **anchor** map $\rho : E \longrightarrow TM$, such that for every $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$, the following conditions are fulfilled:

1. $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$
2. $\rho([\![e_1, e_2]\!]) = [\rho(e_1), \rho(e_2)]$
3. $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\mathcal{L}_{\rho(e_1)}f)e_2$
4. $\langle e_1, [\![e_2, e_3]\!] + [\![e_3, e_2]\!]\rangle = \mathcal{L}_{\rho(e_1)}(\langle e_2, e_3\rangle)$
5. $\mathcal{L}_{\rho(e_1)}(\langle e_2, e_3\rangle) = \langle [\![e_{1,2}]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$

A Courant algebroid will be denoted by $(E, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle, \rho)$. The main example of a Courant algebroid is $TM \oplus T^*M$ with the canonical nondegenerate symmetric pairing $\langle\cdot,\cdot\rangle$ and the usual Courant bracket. As explained in [54], every Courant algebroid $(E, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle, \rho)$ has an interesting counterpart in supergeometry. Since Courant algebroids are the geometric structure where Dirac structures belong, it is useful to have such a supergeometric interpretation to study Dirac structures.

**Theorem 7.2.1.** *[54]*

*There exists a one-to-one correspondence between*

1. *Courant algebroids, and*
2. *Symplectic graded manifolds of degree* 2 *with a degree* 3 *function* $\theta$ *satisfying* $\{\theta, \theta\} = 0$.

*Moreover, this correspondence is constructed in such a way that the canonical Courant algebroid structure on* $TM \oplus T^*M$ *corresponds to the symplectic graded manifold* $T^*[2]T[1]M$.

We propose to study the inifinitesimal counterpart of multiplicative Dirac structures via Roytenberg's correspondence 7.2.1. For that, consider the usual Courant algebroid $TG \oplus T^*G$ over $G$, and let $(S(G), \theta)$ denote the graded symplectic supermanifold $T^*[2]T[1]G$ associated to $TG \oplus T^*G$ with the degree 3 function $\theta$ satisfying $\{\theta, \theta\} = 0$. Since the direct sum $TG \oplus T^*G$ is also a Lie groupoid over $TM \oplus A^*$, this property has to be reflected in the supermanifold $S(G) = T^*[2]T[1]G$. Indeed, $S(G)$ is a graded Lie groupoid over the graded manifold $S(P) := (T[1]A)^*[2]$. Moreover, the symplectic structure on $S(G)$ is the canonical symplectic structure on a cotangent groupoid, thus it defines a multiplicative symplectic structure on $S(G)$. The other data defining $S(G)$ is the degree 3-function $\theta$. The fact that $TG \oplus T^*G$ is a Lie groupoid must imply that $\theta$ is a multiplicative function on $S(G)$. Therefore, the supergeometric counterpart of the Courant algebroid $TG \oplus T^*G$ is a graded symplectic supergroupoid $S(G) \rightrightarrows S(P)$ equipped with a degree 3 multiplicative function $\theta$ satisfying $\{\theta, \theta\} = 0$. On the other hand, the base manifold $S(P)$ inherits the structure of a graded Poisson manifold, characterized by the fact that the target map $S(G) \longrightarrow S(P)$ is a morphism of graded Poisson manifolds.

Let us consider now a multiplicative Dirac structure $L$ on $G$. In the super side, a multiplicative Dirac structure corresponds to a graded Lagrangian *subgroupoid* $S(L) \rightrightarrows S(C)$ of the graded symplectic groupoid $S(G) \rightrightarrows S(P)$, where the degree 3 multiplicative function $\theta$ vanishes. Just as Lagrangian subgroupoids of symplectic groupoids have a coisotropic base [15, 66], the graded Lagrangian subgroupoid $S(L) \subseteq S(G)$, necessarily has a base $S(C)$ which is a *graded coisotropic* submanifold of the graded Poisson manifold $S(P)$. We can argue that the infinitesimal data of a graded Lagrangian subgroupoid $S(L)$ of $S(G)$ is the Lie subalgebroid $N^*(S(C))$ of $T^*(S(P))$[1]. However, we only need $S(C)$, since

[1]Recall that a submanifold $C$ of a Poisson manifold $P$ is coisotropic if and only if the conormal bundle $N^*(C) \subseteq T^*P$ is a Lie subalgebroid

we can go from $S(C)$ to $N^*(S(C))$ canonically. In terms of the algebra of functions, the infinitesimal version of a graded Lagrangian subgroupoid of $S(G)$ corresponds to a coisotropic ideal $\mathcal{I}$ of the graded Poisson algebra $\mathcal{C}^\infty(S(P))$.

The supergeometric approach to linear Dirac structures on a Lie algebroid should provide a finer infinitesimal invariant of a multiplicative Dirac structure. The fact that the ideal $\mathcal{I}$ is coisotropic means that

$$\{\mathcal{I},\mathcal{I}\} \subseteq \mathcal{I},$$

and such a relation might lead to a more natural description of Dirac structures $L_A \in \mathrm{Dir}_{alg}(A)$, such as Lie bialgebroids and IM-2-forms.

## 7.3 New higher structures: $\mathcal{CA}$-groupoids

This is the final section of this chapter. Along this thesis we have study multiplicative Dirac structures on Lie groupoids. It is the Courant algebroid $TG \oplus T^*G$ where multiplicative Dirac structures lie. In addition, the vector bundle $TG \oplus T^*G$ is a Lie groupoid over $TM \oplus A^*G$, and in chapter 5 we proved that all the structure data defining the Courant algebroid $TG \oplus T^*G$ is preserved by the structure mappings that define the Lie groupoid $TG \oplus T^*G$. In terms of double structures, we have a square

$$\begin{array}{ccc} TG \oplus T^*G & \longrightarrow & G \\ \downarrow\downarrow & & \downarrow\downarrow \\ TM \oplus A^*G & \longrightarrow & M \end{array} \tag{7.1}$$

where double arrows denote Lie groupoids, the top horizontal structure is a Courant algebroid and the bottom horizontal structure has a structure similar to that of a Courant groupoid, except that the natural pairing on $TM \oplus A^*G$ could be degenerate. The double structure (7.1) should be thought of as the model example of a new higher structure that might be called a **$\mathcal{CA}$-groupoid**. Roughly, a $\mathcal{CA}$-groupoid is a Lie groupoid object in the category of Courant algebroids. We believe that the techniques used along this work can be useful for the study $\mathcal{CA}$-groupoids and their infinitesimal versions. Also supergeometry can be used to understand what a $\mathcal{CA}$-groupoid is. This will be a future research project.

# Appendix A

# Double geometrical structures

We recall here some double geometric structures such as double vector bundles, $\mathcal{VB}$-groupoids and $\mathcal{VB}$-algebroids.

### A.0.1 Double vector bundles

The concept of double vector bundle was introduced by J. Pradines in [52]. Here we recall the main properties of these structures. We also recommend [41] for a detailed discussion about double structures. Roughly, a double vector bundle is a vector bundle object in the category of vector bundles. More specifically, a **double vector bundle** consists of square

$$\begin{array}{ccc} D & \xrightarrow{q_D^H} & B \\ {\scriptstyle q_D^V}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow[q_A]{} & M \end{array} \qquad \text{(A.1)}$$

where each of the arrows denote vector bundle structures. We require that all the structure mappings defining the horizontal vector bundle $D \xrightarrow{q_D^H} B$ be morphisms of vector bundles over the corresponding structure maps that define the vector bundle $A \xrightarrow{q_A} M$. We use the notation $(D, B, A, M)$ to indicate the double vector bundle (A.1).

**Example A.0.1.** (*Tangent double vector bundle*)

Given a vector bundle $A \xrightarrow{q_A} M$, there is a natural double vector bundle

$$\begin{array}{ccc} TA & \xrightarrow{Tq_A} & TM \\ {\scriptstyle p_A}\big\downarrow & & \big\downarrow{\scriptstyle p_M} \\ A & \xrightarrow[q_A]{} & M \end{array} \tag{A.2}$$

obtained by applying the tangent functor to all the structure mappings that define $A \longrightarrow M$.

**Example A.0.2.** (*Cotangent double vector bundle*)

Given a vector bundle $A \xrightarrow{q_A} M$, the cotangent bundle $T^*A$ gives rise to a double vector bundle

$$\begin{array}{ccc} T^*A & \xrightarrow{r} & A^* \\ {\scriptstyle c_A}\big\downarrow & & \big\downarrow{\scriptstyle q_{A^*}} \\ A & \xrightarrow[q_A]{} & M \end{array} \tag{A.3}$$

where the bundle projection $r : T^*A \longrightarrow A^*$ is described locally by $r(x^i, u^a, p_i, \lambda_a) = (\lambda_a)$.

Given a double vector bundle $(D, B, A, M)$, we define its **core** vector bundle as $C = \ker(q_D^H) \cap \ker(q_D^V)$. The core of a double vector bundle is canonically embedded in $D$, and it becomes a vector bundle $C \longrightarrow M$ in a natural way.

**Example A.0.3.** The core of the double vector bundle (A.2) is the vector bundle of vertical vectors tangent to the zero section $M \longrightarrow A$. Therefore the core of (A.2) identifies canonically with $A \longrightarrow M$.

**Example A.0.4.** The core of the double vector bundle (A.3) is described locally by elements $(x^i, u^a, p_i, \lambda_a)$ with $u^a = 0$ and $\lambda_a = 0$. Thus, the core of the double vector bundle (A.3) identifies with $T^*M \longrightarrow M$.

Let us consider a double vector bundle $(D, B, A, M)$ as in (A.1).

**Definition A.0.1.** A section $\tilde{u} \in \Gamma_B(D)$ is called **linear** if there exists a section $u \in \Gamma_M(A)$ such that $\tilde{u} : B \longrightarrow D$ is a vector bundle morphism over $u : M \longrightarrow A$.

**Example A.0.5.** Let $u$ be a section of a vector bundle $A \longrightarrow M$. The application of the tangent functor to $u$, yields a linear section $Tu : TM \longrightarrow TA$ of the double vector bundle (A.2).

**Example A.0.6.** A section $u$ of a vector bundle $A \longrightarrow M$ induces a linear section $u^L$ of the double vector bundle (A.3). If $\{e_a\}$ denotes a basis of local sections of $A$ such that $u = u^a e_a$, then the linear section $u^L$ is described locally by

$$u^L(x^i, \xi_a) = (x^i, u^a(x), 0, \xi_a).$$

Given a section $k$ of the core $C \longrightarrow M$ of a double vector bundle $(D, B, A, M)$, the **core** section induced by $k$ is a section $\hat{k}$ of $D \longrightarrow B$ defined by

$$\hat{k}(b) = 0^D(b) + \overline{k(q_B(b))},$$

here $0^D : B \hookrightarrow D$ is the zero section and $\overline{k(q_B(b))}$ denotes the image of $k(q_B(b))$ by the canonical embedding $C \hookrightarrow D$.

**Example A.0.7.** A section $u : M \longrightarrow A$ of the core of $(TA, TM, A, M)$ induces a core section $\hat{u} : TM \longrightarrow TA$ determined by

$$\hat{u}(X) = T(0^A)(X) + \overline{u(p_M(X))},$$

where $\overline{u(p_M(X))} = \frac{d}{dt}(tu(p_M(X)))|_{t=0}$.

**Example A.0.8.** A section $\alpha : M \longrightarrow T^*M$ of the core of $(T^*A, A^*, A, M)$ determines a core section $\hat{\alpha} : A^* \longrightarrow T^*A$, which is locally described by

$$\hat{\alpha}(x^i, \xi_a) = (x^i, 0, \alpha_i(x), \xi_a),$$

where $\alpha = \alpha_i dx^i$.

### A.0.2 The $\mathcal{VB}$-category

A $\mathcal{VB}$-groupoid is a Lie groupoid object in the category of vector bundles. This means that a **$\mathcal{VB}$-groupoid** is a square

$$\begin{array}{ccc} H & \xrightarrow{q_H} & G \\ \downdownarrows & & \downdownarrows \\ E & \xrightarrow[q_E]{} & M \end{array} \tag{A.4}$$

where double arrows denote Lie groupoid structures and single arrows denote vector bundles. We require that the structure mappings (source, target, multiplication, unit section and inversion) that define the Lie groupoid $H \rightrightarrows E$ be morphisms of vector bundles over the corresponding structure mappings defining the Lie groupoid $G \rightrightarrows M$.

**Example A.0.9.** Given a Lie groupoid $G \rightrightarrows M$ with Lie algebroid $A$, there are two canonical $\mathcal{VB}$-groupoids associated to it, namely, the tangent groupoid $TG \rightrightarrows TM$ and the cotangent groupoid $T^*G \rightrightarrows A^*$.

Now we want to understand what is the geometric object obtained by applying the Lie functor to the $\mathcal{VB}$-groupoid (A.4).

**Definition A.0.2.** An **$\mathcal{LA}$-vector bundle** is a double vector bundle

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} \tag{A.5}$$

where the vertical structures are Lie algebroids and the horizontal structures are vector bundles. These structures are compatible in the sense that all the structure mappings that define the vector bundle $A \longrightarrow E$ are morphisms of Lie algebroids over the corresponding mappings that define the vector bundle $B \longrightarrow M$.

As usual, one can say that an $\mathcal{LA}$-vector bundle is a vector bundle object in the category of Lie algebroid. There is also a symmetric version of an $\mathcal{LA}$-vector bundle, this double structure is called a $\mathcal{VB}$-algebroid. Recently, R. Mehta and A. Gracia-Saz [31] have shown that these symmetric notions of double structure coincide.

**Example A.0.10.** Given a Lie algebroid $A \longrightarrow M$, there are two canonical $\mathcal{LA}$-vector bundles associated to it, namely, the tangent Lie algebroid $(TA, TM, A, M)$ and the cotangent Lie algebroid $(T^*A, A^*, A, M)$.

It seems that the Lie functor maps $\mathcal{VB}$-groupoids into $\mathcal{LA}$-vector bundles. In fact, there exists a one-to-one correspondence between:

1. Source simply connected $\mathcal{VB}$-groupoids, and
2. Integrable $\mathcal{LA}$-vector bundles.

A $\mathcal{VB}$-groupoid (A.4) is called **source simply connected** if the Lie groupoid $H \rightrightarrows E$ is a source simply connected Lie groupoid. An $\mathcal{LA}$-vector bundle (A.5) is called integrable if the Lie algebroid $A \longrightarrow E$ is an integrable Lie algebroid. In order to understand the correspondence above, we briefly explain the main ideas of [6], where the determination of a vector bundle out of its fiberwise scalar multiplication, proved in [30], is used strongly.

**Definition A.0.3.** ([30]) A **homogeneous structure** on a smooth manifold $E$ is a smooth action $h : \mathbb{R}_+ \times E \longrightarrow E$ of the multiplicative monoid $\mathbb{R}_+$ which is non-singular in the sense that

$$\frac{d}{dt}(h(t,e))|_{t=0} = 0 \quad \text{if and only if} \quad e \in h_0(E)$$

In the terminology of Grabowski and Rotkiewicz [30], every smooth action $h : \mathbb{R}_+ \times E \longrightarrow E$ defines a projection $h_0 : E \longrightarrow E$, whose image is a closed subset $N = h_0(E)$. We can define the **vertical lift** of the action $\mathcal{V}_h : E \longrightarrow (TE)|_N$ by

$$\mathcal{V}_h(e) = \frac{d}{dt}(h(t,e))|_{t=0}.$$

The vertical lift of the action $h : \mathbb{R}_+ \times E \longrightarrow E$ may be thought of as an infinitesimal action on $E$. Notice that at each point $x \in N$, the vertical lift is given by $\mathcal{V}_h(x) = 0$, so a homogeneous structure is an action such that the set of singularities of the vertical lift is smallest as possible.

**Example A.0.11.** Let $E \longrightarrow M$ be a vector bundle. The action by homoteties

$$h : \mathbb{R}_+ \times E \longrightarrow E \tag{A.6}$$

$$(t, e) \mapsto te, \tag{A.7}$$

endows $E$ with a homogeneous structure.

It turns out that on a vector bundle $E$, the homogeneous structure given by homoteties, determines completely the vector bundle structure on $E$. See [30] for a proof of this result.

**Theorem A.0.1.** *[30]*

*If $h : \mathbb{R}_+ \times E \longrightarrow E$ is a homogeneous structure on a smooth manifold $E$, then there exists a unique vector bundle structure on $E$ such that $h$ coincides with the homoteties of $E$.*

Notice that a morphism of vector bundles $E_1 \longrightarrow E_2$ is just a map that commutes with the corresponding homogeneous structures on $E_1$ and $E_2$. Let us see how this characterization of vector bundle structures is useful to study Lie groupoids objects in the category of vector bundles.

**Proposition A.0.1.** *A $\mathcal{VB}$-groupoid structure $(H, G, E, M)$ is equivalent to homogeneous structures $h^H$ and $h^E$ on $H$ and $E$, respectively, which defines an action by groupoid endomorphisms*

$$\begin{array}{ccc} H & \xrightarrow{\;h^H\;} & H \\ \downdownarrows & & \downdownarrows \\ E & \xrightarrow[\;h^E\;]{} & E \end{array} \tag{A.8}$$

*Proof.* The compatibility of $(h^H, h^E)$ with each of the groupoid structure mappings is equivalent to saying that all the structure mappings defining the Lie groupoid $H \rightrightarrows E$ are vector bundle morphisms over the corresponding structure mappings that define the groupoid $G \rightrightarrows M$. This is exactly the definition of a $\mathcal{VB}$-groupoid.

□

A pair $(h^H, h^E)$ of homogeneous structures given by groupoid endomorphisms will be referred to as a **multiplicative homogeneous** structure. If we apply the Lie functor to a multiplicative homogeneous structure $(h^H, h^E)$ we obtain a homogeneous structure $h^{AH}$

on $AH$ given by Lie algebroid endomorphisms over $h^E$. Similarly, the following proposition is proved along the same idea.

**Proposition A.0.2.** *An $\mathcal{LA}$-vector bundle structure $(A, B, E, M)$ is equivalent to homogeneous structures $h^A$ and $h^E$ on $A$ and $E$, respectively, which define an action by Lie algebroid endomorphisms*

$$\begin{array}{ccc} A & \xrightarrow{h^A} & A \\ \downarrow & & \downarrow \\ E & \xrightarrow[h^E]{} & E \end{array} \tag{A.9}$$

Now, in order to show the correspondence between source simply connected $\mathcal{VB}$-groupoids and integrable $\mathcal{LA}$-vector bundles, we can resort to the correspondence between homogeneous structures given by groupoid endomorphisms and homogeneous structures given by algebroid endomorphisms. The latter are related to the former via the Lie functor. Also, as explained in [8], under standard connectedness assumptions it is possible to integrate morphisms of $\mathcal{VB}$-algebroids to morphisms of $\mathcal{VB}$-groupoids. As a result, sub objects in the category of integrable $\mathcal{VB}$-algebroids can be integrated to sub objects in the category of $\mathcal{VB}$-groupoids.

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