Instituto Nacional de Matemática Pura e Aplicada

Hydrostatics, Statical and Dynamical Large Deviations of Boundary Driven Gradient Symmetric Exclusion Processes

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Resumo

Nos últimos anos princípios de grandes desvios estáticos e dinâmicos de sistemas de partículas interagentes com bordos estocásticos têm sido muito estudados como um primeiro passo para o entendimento de estados estacionários fora do equilíbrio.

Neste trabalho consideramos Processos de exclusão simêtrico gradiente com bordos estocásticos em cualquer dimensão e estudamos neste contexto os seguintes problemas.

Primeiro apresentamos uma prova da hidrostática baseada no limite hidrodinâmico e o fato que o perfil estacionario é um atrator global da equação hidrodinâmica. Também sâo provados o limite hidrodinâmico e a lei de Fick.

Depois apresentamos uma prova do principio dos grandes desvios dinâmico para a medida empirica. A prova apresentada aqui é mais simples do que a usual ja que ao invez de aproximarmos trajetorias com funçã custo finita por trajetorias suaves, aproximamos o campo externo associado a ele com campos externos suaves e provamos que as soluçoes fracas da equação hidrodinâmica com estes campos externos aproximam a trajetoria original. Isto simplifica consideravelmente a prova dos grandes desvios dinâmicos.

Por último, apresentamos uma prova do principio de grandes desvios para a medida estacionaria. Mais precisamente, seguindo a estratégia de Freidlin e Wentzell provamos que a medida estacionária de nosso sistema satisfaz um princípio de grandes desvios com função custo dada pelo quase potencial da função custo dinâmica.

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Contents

Introduction

Statical and dynamical large deviations principles of boundary driven interacting particles systems has attracted attention recently as a first step in the understanding of nonequilibrium thermodynamics (cf. [5, 7, 9] and references therein). One of the main dificulties is that in general the stationary measure is not explicitly known and moreover it presents long range correlations (cf. [25]).

This work has three purposes. First, inspired by the dynamical approach to stationary large deviations, introduced by Bertini et al. in the context of boundary driven interacting particles systems [3], we present a proof of the hydrostatics[1] based on the hydrodynamic behaviour of the system and on the fact that the stationary profile is a global attractor of the hydrodynamic equation.

More precisely, if $\bar{\rho}$ represents the stationary density profile and π^N the empirical measure, to prove that π^N converges to $\bar{\rho}$ under the stationary state μ_{ss}^N, we first prove the hydrodynamic limit stated as follows. If we start from an initial configuration which has a density profile γ, on the diffusive scale the empirical measure converges to an absolutely continuous measure, $\pi(t, du) = \rho(t, u)du$, whose density ρ is the solution of the parabolic equation

$$\begin{cases} \partial_t \rho = (1/2)\nabla \cdot D(\rho)\nabla\rho \, , \\ \rho(0, \cdot) = \gamma(\cdot) \, , \\ \rho(t, \cdot) = b(\cdot) \text{ on } \Gamma \, , \end{cases}$$

where D is the diffusivity of the system, ∇ the gradient, b is the boundary condition imposed by the stochastic dynamics and Γ is the boundary of the domain in which the particles evolve. Since for all initial profile $0 \le \gamma \le 1$, the solution ρ_t is bounded above, resp. below, by the solution with initial condition equal to 1, resp. 0, and since these solutions converge, as $t \uparrow \infty$, to the stationary profile $\bar{\rho}$, hydrostatics follows from the hydrodynamics and the weak compactness of the space of measures.

The second contribution of this work is a simplification of the proof of the dynamical large deviations[2] of the empirical measure. The original proof [18, 11, 16] relies on the convexity of the rate functional, a very special property only fulfilled by very few interacting particle systems as the symmetric simple exclusion process. The extension to general processes [22, 23, 6] is relatively technical. The main difficulty appears in the proof of the lower bound where one needs to show that any trajectory λ_t, $0 \le t \le T$, with finite rate function, $I_T(\lambda) < \infty$, can be approximated by a sequence of smooth trajectories $\{\lambda^n : n \ge 1\}$ such that

$$\lambda^n \longrightarrow \lambda \quad \text{and} \quad I_T(\lambda^n) \longrightarrow I_T(\lambda) \, . \tag{0.0.1}$$

[1] This is part of one work in partnership with Claudio Landim and Mustapha Mourragui.
[2] This result is also part of the same work mentioned above

This property is proved by approximating in several steps a general trajectory λ by a sequence of profiles, smoother at each step, the main ingredient being the regularizing effect of the hydrodynamic equation. This part of the proof is quite elaborate and relies on properties of the Green kernel associated to the second order differential operator.

We propose here a simpler proof. It is well known that a path λ with finite rate function may be obtained from the hydrodynamical path through an external field. More precisely, if $I_T(\lambda) < \infty$, there exists H such that

$$I_T(\lambda) \;=\; \frac{1}{2}\int_0^T dt \int \sigma(\lambda_t)\,[\nabla H_t]^2\,dx\;,$$

where σ is the mobility of the system and H is related to λ by the equation

$$\begin{cases} \partial_t\lambda - (1/2)\nabla\cdot D(\lambda)\nabla\lambda \;=\; -\nabla\cdot[\sigma(\lambda)\nabla H_t] \\ H(t,\cdot) = 0 \text{ at the boundary }. \end{cases} \tag{0.0.2}$$

This is an elliptic equation for the unknown function H for each $t \geq 0$. Note that the left hand side of the first equation is the hydrodynamical equation. Instead of approximating λ by a sequence of smooth trajectories, we show that approximating H by a sequence of smooth functions, the corresponding smooth solutions of (0.0.2) converge in the sense (0.0.1) to λ. This approach, closer to the original one, simplifies considerably the proof of the hydrodynamical large deviations.

The third contribuition of this work is the proof of a large deviation principle of the empirical density under the invariant measure. More precisely, we prove that the quasi potential of the dynamical rate function is the large deviation functional of the stationary state.

We follow closely the approach given in [8]. In fact, the arguments presented in there can be adapted modulo technical dificulties to our context. However there is a case not considered in the proof of the upper bound in [8], which we describe in detail in the following.

For a fixed closed set $\mathcal{C}$ in the weak topology not containing the stationary density $\bar\rho$, small neighborhoods $\mathcal{V}_\delta$ (which depends on a parameter $\delta > 0$) of $\bar\rho$ are considered. By following the Freidlin and Wentzell strategy, the proof of the upper bound is reduced to prove that the minimal quasi-potential of densities in $\mathcal{C}$ can be estimated from above by the minimal dynamical rate function of trajectories which start at $\mathcal{V}_\delta$ and touch $\mathcal{C}$ before a time $T = T_\delta$.

At this moment, in [8] it is supposed that the time $T = T_\delta$ is fixed and then, by a direct aplication of the dynamical large deviation upper bound, the desired result is obtained. The same argument still works if we assume the existence of a sequence of parameters $\delta_n \downarrow 0$ with the sequence of times T_{δ_n} bounded. The problem here, is that such bounded sequence doesn't necessarily exist. Moreover, by the construction of such times T^δ, it is expected that $T_\delta \to \infty$ as $\delta \downarrow 0$.

In our context, to solve this missing case, we first prove that long trajectories which have their dynamical rate functions uniformly bounded has to be close in some moment to the stationary density $\bar\rho$ in the L^2 metric, and then we prove that the quasi potential is continuous at the stationary density $\bar\rho$ in the L^2 topology.

In this way, we fullfill the gap in [8] described above and extend their result for a broader class of models. Finally, as a consequence of these facts we obtain a direct proof of the lower semicontinuity of the quasi potential. In the context of one dimensional boundary driven SSEP, the lower semicontinuity of the quasi potential was obtained indirectly by using its exact formulation given in [10, 4].

The organization of this thesis is as follows.

1. Notations and Results

Here we introduce the boundary driven gradient symmetric exclusion processes (Section 1.1) and establish some notations in order to describe in detail the three results mentionated above: hydrostatics (Section 1.2), dynamical large deviations (Section 1.3), and statical large deviations (Section 1.4).

2. Hydrodynamics and Hydrostatics

In this chapter we prove hydrostatics based on the hydrodynamic behaviour of the system and on the fact that the stationary profile is a global attractor of the hydrodynamic equation. More presicely, in Section 2.1 we present a proof for the hydrostatics and Fick's law by supposing the hydrodynamic behavior. A proof for the hydrodynamic behavior can be found in [14] but for the sake of completeness we present in Section 2.2 a detailed proof based on the entropy method.

3. Dynamical Large Deviations

Here we obtain a dynamical large deviation principle for the empirical measure. We start by investigating the dynamical rate function $I_T(\cdot|\gamma)$ in Section 3.1. The main result obtained here is the fact that the dynamical rate function has compact level sets.

Then, in Section 3.2, we prove $I_T(\cdot|\gamma)$ density, which means that any trajectory λ_t, $0 \le t \le T$, with finite rate function, $I_T(\lambda|\gamma) < \infty$, can be approximated by a sequence of smooth trajectories $\{\lambda^n : n \ge 1\}$ such that

$$\lambda^n \longrightarrow \lambda \quad \text{and} \quad I_T(\lambda^n|\gamma) \longrightarrow I_T(\lambda|\gamma) \; .$$

This is fundamental for the obtention of the lower bound.

In Section 3.3, we prove large deviation upper and lower bound. The last one is obtained by the usual arguments (cf. Chapter 10 in [16]) and the I_T density proved in the last section. To prove the upper bound,we have to take care of some additional technical dificulties, the first one is the fact that the invariant measure is not explicitly known which difficulties the obtention of superexponential estimates, the second one is the necessity of energy estimates in order to prove that trajectories with infinite energy are negligible in the context of large deviations, and the last one is that we are working with the empirical measure instead of (as usual) the empirical density.

4. Statical Large Deviations

In this chapter we prove a large deviation principle for the stationary measure. More precisely, Following the Freidling and Wentzell [15] strategy and more closely the article [8], we prove that the large deviation functional for the stationary measure is given by the quasi potential of the dynamical rate function.

In Section 4.1 we introduce the functional I_T closely related to the dynamical rate function $I_T(\cdot|\gamma)$ and prove that trajectories which stays a long time far away from the stationary state $\bar{\rho}$ pays a nonnegligible cost.

In Section 4.2 we study some properties of the quasi potential. The first main result obtained here is the continuity of the quasi potential at the stationary state $\bar{\rho}$ in the L^2 topology. The second one is a direct proof of the lower semicontinuity of the quasi potential.

Finally, in Section 4.3 we prove large deviations lower and upper bounds. The first one is an inmediate consequence of the hydrostatic result and the dynamical large deviation lower bound. To prove the upper bound, we proceed as in [8] and fulfill the gap in there mentioned above by using the results developing in the previous sections.

5. Weak Solutions

Finally, we study weak solutions of the hydrodynamic equation, of its stationary solutions and of the equation with external field (0.0.2). These results are essential in the derivation of many of the results of the previous chapters. However, we have postponed their proofs until here because they are naturally expected for weak solutions of quasilinear parabolic equations.

In Section 5.1 we establish existence and uniqueness of weak solutions as well as monotonicity and uniformly infinitely propagation of speed.

In Section 5.2 we establish energy estimates for weak solutions, which is one of the mai ingredients in the proof of the results in Chapter 4.

Chapter 1

Notations and Results

1.1 Boundary Driven Exclusion Process

Fix a positive integer $d \geq 2$. Denote by Ω the open set $(-1,1)\times\mathbb{T}^{d-1}$, where $\mathbb{T}^k$ is the k-dimensional torus $[0,1)^k$, and by Γ the boundary of Ω: $\Gamma = \{(u_1,\dots,u_d) \in [-1,1]\times\mathbb{T}^{d-1} : u_1 = \pm 1\}$.

For an open subset Λ of $\mathbb{R}\times\mathbb{T}^{d-1}$, $\mathcal{C}^m(\Lambda)$, $1 \leq m \leq +\infty$, stands for the space of m-continuously differentiable real functions defined on Λ.

Fix a positive function $b : \Gamma \to \mathbb{R}_+$. Assume that there exists a neighbourhood V of Ω and a smooth function $\beta : V \to (0,1)$ in $\mathcal{C}^2(V)$ such that β is bounded below by a strictly positive constant, bounded above by a constant smaller than 1 and such that the restriction of β to Γ is equal to b.

For an integer $N \geq 1$, denote by $\mathbb{T}_N^{d-1} = \{0,\dots,N-1\}^{d-1}$, the discrete $(d-1)$-dimensional torus of length N. Let $\Omega_N = \{-N+1,\dots,N-1\}\times\mathbb{T}_N^{d-1}$ be the cylinder in $\mathbb{Z}^d$ of length $2N-1$ and basis $\mathbb{T}_N^{d-1}$ and let $\Gamma_N = \{(x_1,\dots,x_d) \in \mathbb{Z}\times\mathbb{T}_N^{d-1} \,|\, x_1 = \pm(N-1)\}$ be the boundary of Ω_N. The elements of Ω_N are denoted by letters x, y and the elements of Ω by the letters u, v.

We consider boundary driven symmetric exclusion processes on Ω_N. A configuration is described as an element η in $X_N = \{0,1\}^{\Omega_N}$, where $\eta(x) = 1$ (resp. $\eta(x) = 0$) if site x is occupied (resp. vacant) for the configuration η. At the boundary, particles are created and removed in order for the local density to agree with the given density profile b.

The infinitesimal generator of this Markov process can be decomposed in two pieces:

$$\mathcal{L}_N = \mathcal{L}_{N,0} + \mathcal{L}_{N,b} \ ,$$

where $\mathcal{L}_{N,0}$ corresponds to the bulk dynamics and $\mathcal{L}_{N,b}$ to the boundary dynamics. The action of the generator $\mathcal{L}_{N,0}$ on functions $f : X_N \to \mathbb{R}$ is given by

$$\big(\mathcal{L}_{N,0} f\big)(\eta) = \sum_{i=1}^{d} \sum_{x} r_{x,x+e_i}(\eta) \left[f(\eta^{x,x+e_i}) - f(\eta) \right] ,$$

where $(e_1,\dots,e_d)$ stands for the canonical basis of $\mathbb{R}^d$ and where the second sum is performed over all $x \in \mathbb{Z}^d$ such that $x, x+e_i \in \Omega_N$. For $x, y \in \Omega_N$, $\eta^{x,y}$ is the configuration obtained from η by exchanging the occupations variables

$\eta(x)$ and $\eta(y)$:

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x\,, \\ \eta(x) & \text{if } z = y\,, \\ \eta(z) & \text{if } z \neq x, y\,. \end{cases}$$

For $a > -1/2$, the rate functions $r_{x,x+e_i}(\eta)$ are given by

$$r_{x,x+e_i}(\eta) \;=\; 1 \;+\; a\big\{\eta(x-e_i) + \eta(x+2e_i)\big\}$$

if $x - e_i$, $x + 2e_i$ belongs to Ω_N. At the boundary, the rates are defined as follows. Let $\check{x} = (x_2, \cdots, x_d) \in \mathbb{T}_N^{d-1}$. Then,

$$\begin{aligned} r_{(-N+1,\check{x}),(-N+2,\check{x})}(\eta) \;&=\; 1 \;+\; a\big\{\eta(-N+3,\check{x}) + b(-1,\check{x}/N)\big\}\,, \\ r_{(N-2,\check{x}),(N-1,\check{x})}(\eta) \;&=\; 1 \;+\; a\big\{\eta(N-3,\check{x}) + b(1,\check{x}/N)\big\}\,. \end{aligned}$$

The non-conservative boundary dynamics can be described as follows. For any function $f : X_N \to \mathbb{R}$,

$$(\mathcal{L}_{N,b} f)(\eta) = \sum_{x \in \Gamma_N} C^b(x,\eta)\big[f(\eta^x) - f(\eta)\big]\,,$$

where η^x is the configuration obtained from η by flipping the occupation variable at site x:

$$\eta^x(z) = \begin{cases} \eta(z) & \text{if } z \neq x \\ 1 - \eta(x) & \text{if } z = x \end{cases}$$

and the rates $C^b(x,\cdot)$ are chosen in order for the Bernoulli measure with density $b(\cdot)$ to be reversible for the flipping dynamics restricted to this site:

$$\begin{aligned} C^b\big((-N+1,\check{x}),\eta\big) \;&=\; \eta(-N+1,\check{x})\big[1 - b(-1,\check{x}/N)\big] \\ &\qquad + \big[1 - \eta(-N+1,\check{x})\big] b(-1,\check{x}/N)\,, \\ C^b\big((N-1,\check{x}),\eta\big) \;&=\; \eta(N-1,\check{x})\big[1 - b(1,\check{x}/N)\big] \\ &\qquad + \big[1 - \eta(N-1,\check{x})\big] b(1,\check{x}/N)\,, \end{aligned}$$

where $\check{x} = (x_2, \cdots, x_d) \in \mathbb{T}_N^{d-1}$, as above.

Denote by $\{\eta_t = \eta_t^N : t \geq 0\}$ the Markov process associated to the generator $\mathcal{L}_N$ *speeded up* by N^2. For a smooth function $\rho : \Omega \to (0,1)$, let $\nu^N_{\rho(\cdot)}$ be the Bernoulli product measure on X_N with marginals given by

$$\nu^N_{\rho(\cdot)}(\eta(x) = 1) = \rho(x/N)\,.$$

It is easy to see that the Bernoulli product measure associated to any constant function is invariant for the process with generator $\mathcal{L}_{N,0}$. Moreover, if $b(\cdot) \equiv b$ for some constant b then the Bernoulli product measure associated to the constant density b is reversible for the full dynamics $\mathcal{L}_N$.

1.2 Hydrostatics

Denote by μ^N_{ss} the unique stationary state of the irreducible Markov process $\{\eta_t : t \geq 0\}$. We examine in Section 2.1 the asymptotic behavior of the empirical measure under the stationary state μ^N_{ss}.

Let $\mathcal{M} = \mathcal{M}(\Omega)$ be the space of positive measures on Ω with total mass bounded by 2 endowed with the weak topology. For each configuration η, denote by $\pi^N = \pi^N(\eta)$ the positive measure obtained by assigning mass N^{-d} to each particle of η :

$$\pi^N \;=\; N^{-d} \sum_{x\in\Omega_N} \eta(x)\, \delta_{x/N} \;,$$

where δ_u is the Dirac measure concentrated on u. For a measure ϑ in $\mathcal{M}$ and a continuous function $G:\Omega\to\mathbb{R}$, denote by $\langle \vartheta, G\rangle$ the integral of G with respect to ϑ:

$$\langle \vartheta, G\rangle \;=\; \int_\Omega G(u)\, \vartheta(du)\;.$$

To define rigorously the quasi-linear elliptic problem the empirical measure is expected to solve, we need to introduce some Sobolev spaces. Let $L^2(\Omega)$ be the Hilbert space of functions $G:\Omega\to\mathbb{C}$ such that $\int_\Omega |G(u)|^2 du < \infty$ equipped with the inner product

$$\langle G, J\rangle_2 = \int_\Omega G(u)\, \bar{J}(u)\, du\;,$$

where, for $z\in\mathbb{C}$, $\bar{z}$ is the complex conjugate of z and $|z|^2 = z\bar{z}$. The norm of $L^2(\Omega)$ is denoted by $\|\cdot\|_2$.

Let $H^1(\Omega)$ be the Sobolev space of functions G with generalized derivatives $\partial_{u_1}G,\dots,\partial_{u_d}G$ in $L^2(\Omega)$. $H^1(\Omega)$ endowed with the scalar product $\langle\cdot,\cdot\rangle_{1,2}$, defined by

$$\langle G, J\rangle_{1,2} = \langle G, J\rangle_2 + \sum_{j=1}^d \langle \partial_{u_j} G\,,\, \partial_{u_j} J\rangle_2\;,$$

is a Hilbert space. The corresponding norm is denoted by $\|\cdot\|_{1,2}$. For each G in $H^1(\Omega)$ we denote by ∇G its generalized gradient: $\nabla G = (\partial_{u_1}G,\dots,\partial_{u_d}G)$.

Let $\overline{\Omega} = [-1,1]\times\mathbb{T}^{d-1}$ and denote by $\mathcal{C}_0^m(\overline{\Omega})$ (resp. $\mathcal{C}_c^m(\Omega)$), $1\le m\le+\infty$, the space of m-continuously differentiable real functions defined on $\overline{\Omega}$ which vanish at the boundary Γ (resp. with compact support in Ω). Let $\varphi:[0,1]\to\mathbb{R}_+$ be given by $\varphi(r) = r(1{+}ar)$ and let $\|\cdot\|$ be the Euclidean norm: $\|(v_1,\dots,v_d)\|^2 = \sum_{1\le i\le d} v_i^2$. A function $\rho:\Omega\to[0,1]$ is said to be a weak solution of the elliptic boundary value problem

$$\left\{ \begin{array}{ll} \Delta\varphi(\rho) \;=\; 0 & \text{on } \Omega\,, \\ \rho \;=\; b & \text{on } \Gamma\,, \end{array}\right. \tag{1.2.1}$$

if

(S1) ρ belongs to $H^1(\Omega)$:

$$\int_\Omega \|\, \nabla\rho(u)\, \|^2 du \;<\; \infty\;.$$

(S2) For every function G in $\mathcal{C}_0^2\left(\overline{\Omega}\right)$,

$$\int_\Omega \big(\Delta G\big)(u)\, \varphi\big(\rho(u)\big)\, du = \int_\Gamma \varphi(b(u))\, \mathbf{n}_1(u)\, (\partial_{u_1}G)(u) \mathrm{dS}\;,$$

where $\mathbf{n}{=}(\mathbf{n}_1,\dots,\mathbf{n}_d)$ stands for the outward unit normal vector to the boundary surface Γ and dS for an element of surface on Γ.

We prove in Section 5.1 existence and uniqueness of weak solutions of (1.2.1). The first main result of this work establishes a law of large number for the empirical measure under $\mu^N_{\rm ss}$. Denote by E^μ the expectation with respect to a probability measure μ.

Theorem 1.2.1. *For any continuous function $G : \overline{\Omega} \to \mathbb{R}$,*

$$\lim_{N\to\infty} E^{\mu^N_{ss}} \Big[\Big| \langle \pi^N, G\rangle - \int_\Omega G(u)\bar\rho(u) du \Big| \Big] \;=\; 0\,,$$

where $\bar\rho(u)$ is the unique weak solution of (1.2.1).

Denote by Γ_-, Γ_+ the left and right boundary of Ω:

$$\Gamma_\pm = \{(u_1, \dots, u_d) \in \Omega \mid u_1 = \pm 1\}$$

and denote by $W_{x,x+e_i}$, x, $x+e_i \in \Omega_N$, the instantaneous current over the bond $(x, x+e_i)$. This is the rate at which a particle jumps from x to $x+e_i$ minus the rate at which a particle jumps from $x+e_i$ to x. A simple computation shows that

$$W_{x,x+e_i}(\eta) = \tau_{x+e_i} h_i(\eta) - \tau_x h_i(\eta)\,,$$

provided $x - e_i$ and $x + 2e_i$ belong to Ω_N. Here,

$$h_i(\eta) = \eta(0) + a\{\eta(0)[\eta(-e_i) + \eta(e_i)] - \eta(-e_i)\eta(e_i)\}\,.$$

Furthermore, if $x_1 = N - 1$,

$$W_{x-e_1,x} = \big\{\eta(x-e_1) - \eta(x)\big\}\big\{1 + a\,\eta(x-2e_1) + a\,b((x+e_1)/N)\big\}$$

and if $x_1 = -N + 1$,

$$W_{x,x+e_1} = \big\{\eta(x) - \eta(x+e_1)\big\}\big\{1 + a\,\eta(x+2e_1) + a\,b((x-e_1)/N)\big\}\,.$$

Theorem 1.2.2. (Fick's law) *Fix $-1 < u < 1$. Then,*

$$\lim_{N\to\infty} E^{\mu^N_{ss}} \Big[\frac{2N}{N^{d-1}} \sum_{y\in\mathbb{T}^{d-1}_N} W_{([uN],y),([uN]+1,y)} \Big]$$
$$= \int_{\Gamma_-} \varphi(b(v))\, \mathrm{S}(dv) - \int_{\Gamma_+} \varphi(b(v))\, \mathrm{S}(dv)\,.$$

Remark 1.2.3. *We could have considered different bulk dynamics. The important feature used here to avoid painful arguments is that the process is gradient, which means that the currents can be written as the difference of a local function and its translation.*

1.3 Dynamical Large Deviations

Fix $T > 0$. Let $\mathcal{M}^0$ be the subset of $\mathcal{M}$ of all absolutely continuous measures with respect to the Lebesgue measure with positive density bounded by 1:

$$\mathcal{M}^0 = \big\{\vartheta \in \mathcal{M} : \vartheta(du) = \rho(u)du \quad \text{and} \quad 0 \le \rho(u) \le 1 \;\text{ a.e.}\big\}\,,$$

and let $D([0,T],\mathcal{M})$ be the set of right continuous with left limits trajectories $\pi : [0,T] \to \mathcal{M}$, endowed with the Skorohod topology. $\mathcal{M}^0$ is a closed subset of $\mathcal{M}$ and $D([0,T],\mathcal{M}^0)$ is a closed subset of $D([0,T],\mathcal{M})$.

Let $\Omega_T = (0,T) \times \Omega$ and $\overline{\Omega_T} = [0,T] \times \overline{\Omega}$. For $1 \le m,n \le +\infty$, denote by $\mathcal{C}^{m,n}(\overline{\Omega_T})$ the space of functions $G = G_t(u) : \overline{\Omega_T} \to \mathbb{R}$ with m continuous derivatives in time and n continuous derivatives in space. We also denote by $\mathcal{C}_0^{m,n}(\overline{\Omega_T})$ (resp. $\mathcal{C}_c^{\infty}(\Omega_T)$) the set of functions in $\mathcal{C}^{m,n}(\overline{\Omega_T})$ (resp. $\mathcal{C}^{\infty,\infty}(\overline{\Omega_T})$) which vanish at $[0,T] \times \Gamma$ (resp. with compact support in Ω_T).

Let the energy $\mathcal{Q}_T : D([0,T],\mathcal{M}) \to [0,+\infty]$ be given by

$$\mathcal{Q}_T(\pi) = \sum_{i=1}^{d} \sup_{G \in \mathcal{C}_c^{\infty}(\Omega_T)} \left\{ 2\int_0^T \langle \pi_t, \partial_{u_i} G_t \rangle dt - \int_0^T dt \int_\Omega G(t,u)^2 \, du \right\} .$$

For each $G \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$ and each measurable function $\gamma : \overline{\Omega} \to [0,1]$, let $\hat{J}_G = \hat{J}_{G,\gamma,T} : D([0,T],\mathcal{M}^0) \to \mathbb{R}$ be the functional given by

$$\begin{aligned} \hat{J}_G(\pi) \quad &= \quad \langle \pi_T, G_T \rangle - \langle \gamma, G_0 \rangle - \int_0^T \langle \pi_t, \partial_t G_t \rangle \, dt \\ &\quad - \int_0^T \langle \varphi(\rho_t), \Delta G_t \rangle \, dt \; + \; \int_0^T dt \int_{\Gamma^+} \varphi(b) \, \partial_{u_1} G \, dS \\ &\quad - \int_0^T dt \int_{\Gamma^-} \varphi(b) \, \partial_{u_1} G \, dS \; - \; \frac{1}{2} \int_0^T \langle \sigma(\rho_t), \|\nabla G_t\|^2 \rangle \, dt \; , \end{aligned}$$

where $\sigma(r) = 2r(1-r)(1+2ar)$ is the mobility and $\pi_t(du) = \rho_t(u)du$. Define $J_G = J_{G,\gamma,T} : D([0,T],\mathcal{M}) \to \mathbb{R}$ by

$$J_G(\pi) = \begin{cases} \hat{J}_G(\pi) & \text{if } \pi \in D([0,T],\mathcal{M}^0), \\ +\infty & \text{otherwise .} \end{cases}$$

We define the rate functional $I_T(\cdot|\gamma) : D([0,T],\mathcal{M}) \to [0,+\infty]$ as

$$I_T(\pi|\gamma) = \begin{cases} \sup\limits_{G \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})} \big\{ J_G(\pi) \big\} & \text{if } \mathcal{Q}_T(\pi) < \infty \, , \\ +\infty & \text{otherwise .} \end{cases}$$

We are no ready to state our second main result.

Theorem 1.3.1. *Fix $T > 0$ and a measurable function $\rho_0 : \Omega \to [0,1]$. Consider a sequence η^N of configurations in X_N associated to ρ_0 in the sense that:*

$$\lim_{N\to\infty} \langle \pi^N(\eta^N), G \rangle \; = \; \int_\Omega G(u) \rho_0(u) \, du$$

for every continuous function $G : \overline{\Omega} \to \mathbb{R}$. Then, the measure $\mathbf{Q}_{\eta^N} = \mathbb{P}_{\eta^N}(\pi^N)^{-1}$ on $D([0,T],\mathcal{M})$ satisfies a large deviation principle with speed N^d and rate function $I_T(\cdot|\rho_0)$. Namely, for each closed set $\mathcal{C} \subset D([0,T],\mathcal{M})$,

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbf{Q}_{\eta^N}(\mathcal{C}) \le - \inf_{\pi \in \mathcal{C}} I_T(\pi|\rho_0)$$

and for each open set $\mathcal{O} \subset D([0,T],\mathcal{M})$,

$$\varliminf_{N\to\infty} \frac{1}{N^d} \log \mathbf{Q}_{\eta^N}(\mathcal{O}) \geq - \inf_{\pi\in\mathcal{O}} I_T(\pi|\rho_0) \ .$$

Moreover, the rate function $I_T(\cdot|\rho_0)$ is lower semicontinuous and has compact level sets.

1.4 Statical Large Deviations

Let us introduce $\mathcal{P}_N = \mu_{ss}^N \circ (\pi^N)^{-1}$, which is a probability measure on $\mathcal{M}$ and describes the behavior of the empirical measure under the invariant measure.

Let $\bar{\rho} : \overline{\Omega} \to [0,1]$ be the weak solution of (1.2.1) . Following [3], [15], we define $V : \mathcal{M} \to [0,+\infty]$ as the quasi potential for the dynamical rate function $I_T(\cdot|\bar{\rho})$.

$$V(\vartheta) = \inf \left\{ I_T(\pi|\bar{\rho}) : T > 0,\ \pi \in D([0,T],\mathcal{M}) \ \text{ and } \ \pi_T = \vartheta \right\} \ .$$

It is clear that for the measure $\overline{\vartheta}(du) = \bar{\rho}(u)du$ we have that $V(\overline{\vartheta}) = 0$.

We will prove in Section 3.1 that if $I_T(\pi|\bar{\rho})$ is finite then π belongs to $C([0,T],\mathcal{M}^0)$. Therefore we may restrict the infimum in the definition of $V(\vartheta)$ to paths in $C([0,T],\mathcal{M}^0)$ and if $V(\vartheta)$ is finite, ϑ belongs to $\mathcal{M}^0$. Reciprocally, we will see in Section 4.2 that V is bounded on $\mathcal{M}^0$.

The last main result of this work establishes a large deviation principle for the invariant measure.

Theorem 1.4.1. *The measure $\mathcal{P}_N$ satisfies a large deviation principle on $\mathcal{M}$ with speed N^d and lower semicontinuous rate function V. Namely, for each closed set $\mathcal{C} \subset \mathcal{M}$ and each open set $\mathcal{O} \subset \mathcal{M}$,*

$$\varlimsup_{N\to\infty} \frac{1}{N^d} \log \mathcal{P}_N(\mathcal{C}) \leq - \inf_{\vartheta\in\mathcal{C}} V(\vartheta)\,,$$

$$\varliminf_{N\to\infty} \frac{1}{N^d} \log \mathcal{P}_N(\mathcal{O}) \geq - \inf_{\vartheta\in\mathcal{O}} V(\vartheta)\,.$$

Chapter 2

Hydrodynamics and Hydrostatics

2.1 Hydrodynamics, Hydrostatics and Fick's Law

We prove in this section Theorem 1.2.1. The idea is to couple three copies of the process, the first one starting from the configuration with all sites empty, the second one starting from the stationary state and the third one from the configuration with all sites occupied. The hydrodynamic limit states that the empirical measure of the first and third copies converge to the solution of the initial boundary value problem (2.1.1) with initial condition equal to 0 and 1. Denote these solutions by ρ_t^0, ρ_t^1, respectively. In turn, the empirical measure of the second copy converges to the solution of the same boundary value problem, denoted by ρ_t, with an unknown initial condition. Since all solutions are bounded below by ρ^0 and bounded above by ρ^1, and since ρ^j converges to a profile $\bar{\rho}$ as $t \uparrow \infty$, ρ_t also converges to this profile. However, since the second copy starts from the stationary state, the distribution of its empirical measure is independent of time. Hence, as ρ_t converges to $\bar{\rho}$, $\rho_0 = \bar{\rho}$. As we shall see in the proof, this argument does not require attractiveness of the underlying interacting particle system. This approach has been followed in [21] to prove hydrostatics for interacting particles systems with Kac interaction and random potential.

We first describe the hydrodynamic behavior. Fix $T > 0$ and a profile $\rho_0 \colon \overline{\Omega} \to [0, 1]$. A measurable function $\rho : \overline{\Omega_T} \to [0, 1]$ is said to be a weak solution of the initial boundary value problem

$$\begin{cases} \partial_t \rho = \Delta \varphi\big(\rho\big) \, , \\ \rho(0, \cdot) = \rho_0(\cdot) \, , \\ \rho(t, \cdot)\big|_{\Gamma} = b(\cdot) \quad \text{for } 0 \le t \le T \, , \end{cases} \tag{2.1.1}$$

in the layer $[0, T] \times \Omega$ if

(H1) ρ belongs to $L^2\big([0, T], H^1(\Omega)\big)$:

$$\int_0^T ds \, \left(\int_\Omega \| \, \nabla \rho(s, u) \, \|^2 du \right) < \infty \, ;$$

(**H2**) For every function $G = G_t(u)$ in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\int_\Omega \big\{G_T(u)\rho(T,u) - G_0(u)\rho_0(u)\big\}du - \int_0^T ds \int_\Omega du\,(\partial_s G_s)(u)\rho(s,u)$$
$$= \int_0^T ds \int_\Omega du\,(\Delta G_s)(u)\varphi\big(\rho(s,u)\big) - \int_0^T ds \int_\Gamma \varphi(b(u))\mathbf{n}_1(u)(\partial_{u_1} G_s(u))\mathrm{dS}\ ,$$

where $\mathbf{n}=(\mathbf{n}_1,\dots,\mathbf{n}_d)$ stands for the outward unit normal vector to the boundary surface Γ and dS for an element of surface on Γ.

We prove in Section 5.1 existence and uniqueness of weak solutions of (2.1.1).

For a measure μ on X_N, denote by $\mathbb{P}_\mu = \mathbb{P}_\mu^N$ the probability measure on the path space $D(\mathbb{R}_+, X_N)$ corresponding to the Markov process $\{\eta_t : t \geq 0\}$ with generator $N^2\mathcal{L}_N$ starting from μ, and by $\mathbb{E}_\mu$ expectation with respect to $\mathbb{P}_\mu$. Recall the definition of the empirical measure π^N and let $\pi_t^N = \pi^N(\eta_t)$:

$$\pi_t^N \;=\; N^{-d} \sum_{x\in\Omega_N} \eta_t(x)\,\delta_{x/N}\ .$$

Theorem 2.1.1. *Fix a profile $\rho_0 : \Omega \to (0,1)$. Let μ^N be a sequence of measures on X_N associated to ρ_0 in the sense that :*

$$\lim_{N\to\infty} \mu^N \left\{ \left| \langle \pi^N, G\rangle - \int_\Omega G(u)\rho_0(u)\,du \right| > \delta \right\} \;=\; 0\ , \qquad (2.1.2)$$

for every continuous function $G : \Omega \to \mathbb{R}$ and every $\delta > 0$. Then, for every $t > 0$,

$$\lim_{N\to\infty} \mathbb{P}_\mu^N \left\{ \left| \langle \pi_t^N, G\rangle - \int_\Omega G(u)\rho(t,u)\,du \right| > \delta \right\} \;=\; 0\ ,$$

where $\rho(t,u)$ is the unique weak solution of (2.1.1).

A proof of this result can be found in [14]. Denote by $\mathbf{Q}_{\mathrm{ss}}^N$ the probability measure on the Skorohod space $D([0,T],\mathcal{M})$ induced by the stationary measure μ_{ss}^N and the process $\big\{\pi^N(\eta_t) : 0 \le t \le T\big\}$. Note that, in contrast with the usual set-up of hydrodynamics, we do not know that the empirical measure at time 0 converges. We can not prove, in particular, that the sequence $\mathbf{Q}_{\mathrm{ss}}^N$ converges, but only that this sequence is tight and that all limit points are concentrated on weak solution of the hydrodynamic equation for some unknown initial profile.

We first show that the sequence of probability measures $\big\{\mathbf{Q}_{\mathrm{ss}}^N : N \ge 1\big\}$ is weakly relatively compact:

Proposition 2.1.2. *The sequence $\{\mathbf{Q}_{ss}^N,\, N \ge 1\}$ is tight and all its limit points $\mathbf{Q}_{ss}^*$ are concentrated on absolutely continuous paths $\pi(t,du) = \rho(t,u)du$ whose density ρ is positive and bounded above by* 1 *:*

$$\mathbf{Q}_{ss}^*\Big\{\pi \,:\, \pi(t,du) = \rho(t,u)du\,,\ \textit{for } 0 \le t \le T\Big\} = 1\ ,$$
$$\mathbf{Q}_{ss}^*\Big\{\pi \,:\, 0 \le \rho(t,u) \le 1\,,\ \textit{for } (t,u) \in \overline{\Omega_T}\Big\} = 1\ .$$

The proof of this statement is similar to the one of Proposition 3.2 in [19]. Actually, the proof is even simpler because the model considered here is gradient.

The next two propositions show that all limit points of the sequence $\big\{\mathbf{Q}_{\rm ss}^N : N \geq 1\big\}$ are concentrated on absolutely continuous measures $\pi(t,du) = \rho(t,u)du$ whose density ρ are weak solution of (2.1.1) in the layer $[0,T]\times\Omega$. Denote by $\mathcal{A}_T \subset D\big([0,T],\mathcal{M}^0\big)$ the set of trajectories $\{\pi(t,du) = \rho(t,u)du : 0 \leq t \leq T\}$ whose density ρ satisfies condition (**H2**).

Proposition 2.1.3. *All limit points $\mathbf{Q}_{ss}^*$ of the sequence $\{\mathbf{Q}_{ss}^N, N > 1\}$ are concentrated on paths $\pi(t,du) = \rho(t,u)du$ in $\mathcal{A}_T$:*

$$\mathbf{Q}_{ss}^*\{\mathcal{A}_T\} = 1\ .$$

The proof of this proposition is similar to the one of Proposition 3.3 in [19]. Next result implies that every limit point $\mathbf{Q}_{\rm ss}^*$ of the sequence $\{\mathbf{Q}_{\rm ss}^N, N > 1\}$ is concentrated on paths whose density ρ belongs to $L^2([0,T],H^1(\Omega))$:

Proposition 2.1.4. *Let $\mathbf{Q}_{ss}^*$ be a limit point of the sequence $\{\mathbf{Q}_{ss}^N, N > 1\}$. Then,*

$$E_{\mathbf{Q}_{ss}^*}\left[\int_0^T ds\left(\int_\Omega \parallel \nabla\rho(s,u)\parallel^2 du\right)\right] < \infty\ .$$

The proof of this proposition is similar to the one of Lemma A.1.1 in [17]. We are now ready to prove the first main result of this work.

Proof of Theorem 1.2.1. Fix a continuous function $G : \overline{\Omega} \to \mathbb{R}$. We claim that

$$\lim_{N\to\infty} E^{\mu_{\rm ss}^N}\Big[\,\big|\langle\pi,G\rangle - \langle\bar\rho(u)du,G\rangle\big|\,\Big] \;=\; 0\ .$$

Note that the expectations are bounded. Consider a subsequence N_k along which the left hand side converges. It is enough to prove that the limit vanishes. Fix $T>0$. Since $\mu_{\rm ss}^N$ is stationary, by definition of $\mathbf{Q}_{\rm ss}^{N_k}$,

$$E^{\mu_{\rm ss}^{N_k}}\Big[\,\big|\langle\pi,G\rangle - \langle\bar\rho(u)du,G\rangle\big|\,\Big] \;=\; \mathbf{Q}_{\rm ss}^{N_k}\Big[\,\big|\langle\pi_T,G\rangle - \langle\bar\rho(u)du,G\rangle\big|\,\Big]\ .$$

By Proposition 2.1.2, there is a limit point $\mathbf{Q}_{\rm ss}^*$ of $\big\{\mathbf{Q}_{\rm ss}^{N_k} : k \geq 1\big\}$. Since the expression inside the expectation is bounded, by Propositions 2.1.3 and 2.1.4,

$$\begin{aligned}\lim_{k\to\infty}\mathbf{Q}_{\rm ss}^{N_k}\Big[\big|\langle\pi_T,G\rangle - \langle\bar\rho(u)du,G\rangle\big|\Big] \;&=\; \mathbf{Q}_{\rm ss}^*\Big[\big|\langle\pi_T,G\rangle - \langle\bar\rho(u)du,G\rangle\big|\,\mathbf{1}\{\mathcal{S}_T\}\Big]\\ &\leq\; \|G\|_\infty\mathbf{Q}_{\rm ss}^*\Big[\big\|\rho(T,\cdot) - \bar\rho(\cdot)\big\|_1\,\mathbf{1}\{\mathcal{S}_T\}\Big]\ ,\end{aligned}$$

where $\|\cdot\|_1$ stands for the $L^1(\Omega)$ norm and where $\mathcal{S}_T$ stands for the subset of $D([0,T],\mathcal{M}^0)$ consisting on those trajectories $\{\pi(t,du) = \rho(t,u)du : 0 \leq t \leq T\}$ whose density ρ is a weak solution of (2.1.1). Denote by $\rho^0(\cdot,\cdot)$ (resp. $\rho^1(\cdot,\cdot)$) the weak solution of the boundary value problem (2.1.1) with initial condition $\rho(0,\cdot)\equiv 0$ (resp. $\rho(0,\cdot)\equiv 1$). By Lemma 5.1.4, each profile ρ in $\mathcal{A}_T$, including the stationary profile $\bar\rho$, is bounded below by ρ^0 and above by ρ^1. Therefore

$$\lim_{k\to\infty} E^{\mu_{\rm ss}^{N_k}}\Big[\,\big|\langle\pi,G\rangle - \langle\bar\rho(u)du,G\rangle\big|\,\Big] \;\leq\; \|G\|_\infty\,\big\|\rho^0(T,\cdot) - \rho^1(T,\cdot)\big\|_1\ .$$

Note that the left hand side does not depend on T. To conclude the proof it remains to let $T\uparrow\infty$ and to apply Lemma 5.1.7.

□

Fick's law, announced in Theorem 1.2.2, follows from the hydrostatics and elementary computations presented in the Proof of Theorem 2.2 in [17]. The arguments here are even simpler and explicit since the process is gradient.

In the next section we will show that Propositions 2.1.2, 2.1.3 and 2.1.4 holds for any sequence of probability measures μ^N on X_N in the place of the stationary ones μ^N_{ss}. Furthermore, if the sequence μ^N satisfies (2.1.2) with some profile $\rho_0 : \Omega \to [0,1]$ then all limit points $\mathbf{Q}^*$ of $\mathbf{Q}_{\mu^N}$ are concentrated on paths π with $\pi(0,du) = \rho_0(u)du$:

$$\mathbf{Q}^*\big\{\pi : \pi(0,du) = \rho_0(u)du\big\} = 1\,.$$

From these facts and the uniqueness of weak solutions of (2.1.1) we may obtain the next result.

Theorem 2.1.5. *Under the conditions in Theorem 2.1.1, the sequence of probability measures $\mathbf{Q}_{\mu^N}$ converges weakly to the measure $\mathbf{Q}^*$ that is concentrated on the absolutely continuous path $\pi(t,du) = \rho(t,u)du$ whose density $\rho(\cdot,\cdot)$ is the unique weak solution of the hydrodynamic equation* (2.1.1)*.*

Theorem 2.1.1 follows from this last result by standard arguments (cf. Section 4.2 in [16]).

2.2 Proofs of Propositions 2.1.2, 2.1.3 and 2.1.4

Fix $T > 0$ and a sequence μ^N of measures on X_N. Denote by $\mathbf{Q}^N$ the probability measure on the path space $D([0,T],\mathcal{M})$ induced by the process $\{\pi^N(\eta_t) : 0 \le t \le T\}$ and with initial distribution μ^N. Fix a limit point $\mathbf{Q}^*$ of the sequence $\mathbf{Q}^N$ and assume, without loss of generality, that $\mathbf{Q}^N$ converges to $\mathbf{Q}^*$.

For a function G in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$, consider the martingales $M_t^G = M_t^{G,N}$, $N_t^G = N_t^{G,N}$ defined by

$$\begin{aligned} M_t^G &= \big\langle \pi_t^N, G_t\big\rangle - \big\langle \pi_0^N, G_0\big\rangle - \int_0^t ds\ (\partial_s + N^2\mathcal{L}_N)\big\langle \pi_s^N, G_s\big\rangle\,, \\ N_t^G &= \big(M_t^G\big)^2 - \int_0^t ds\ A_s^{G,N}\,, \end{aligned}$$

where

$$A_s^{G,N} = N^2\mathcal{L}_N\big\langle \pi_s^N, G_s\big\rangle^2 - 2\big\langle \pi_s^N, G_s\big\rangle N^2\mathcal{L}_N\big\langle \pi_s^N, G_s\big\rangle\,.$$

A simple computation give us that $A_s^{G,N}$ is bounded above by $C(G)N^{-d}$. Therefore, by Doob's and Chebychev's inequalities, for every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N}\Big\{ \sup_{0\le t\le T} \big|M_t^G\big| > \delta \Big\} = 0 \tag{2.2.1}$$

Denote by Γ_N^-, resp. Γ_N^+, the left, resp. right, boundary of Ω_N:

$$\Gamma_N^{\pm} = \{(x_1,\cdots,x_d) \in \Gamma_N : x_1 = \pm(N-1)\}\,.$$

For each x in $\Gamma_N^\pm$, let $\hat{x} = x \pm e_1$. After two summations by parts we may rewrite the part inside the integral term of the martingale M_t^G as

$$
\begin{aligned}
&\langle \pi_s^N, \partial_s G_s \rangle + \frac{1}{N^d} \sum_{i=1}^d \sum_x \big(\Delta_i^N G_s\big)(x/N) \tau_x h_i(\eta_s) \\
&+ \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} \big(\partial_1^N G_s\big)(x/N)[\varphi(b(\hat{x}/N)) + aV^-(x, \eta_s)] \\
&- \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^+} \big(\partial_1^N G_s\big)(x/N)[\varphi(b(\hat{x}/N)) + aV^+(x, \eta_s)] \\
&+ O_G(N^{-1}),
\end{aligned}
\tag{2.2.2}
$$

where $\Delta_i^N G$ stands for the discrete second partial derivative in the i-th direction,

$$
\big(\Delta_i^N G\big)(x/N) = N^2[G(x + e_i/N) + G(x - e_i/N) - 2G(x/N)],
$$

and

$$
V^\pm(x, \eta) = [\eta(x) + b(\hat{x}/N)][\eta(x \mp e_1) - b(\hat{x}/N)].
$$

Proof of Proposition 2.1.2. In order to prove tightness for the sequence $\mathbf{Q}^N$, we just need to prove tightness of the real process $\langle \pi_t^N, G \rangle$ for any function G in $\mathcal{C}^2(\overline{\Omega})$. Moreover, by approximations of G in $L^1(\Omega)$ and since there is at most one particle per site, we may assume that G belongs $\mathcal{C}_0^2(\overline{\Omega})$. In that case, tightness for $\langle \pi_t^N, G \rangle$ follows from (2.2.1), (2.2.2) and the fact that the total mass of the empirical measure π_t^N is bounded by 2.

The other two statements follows from the fact that there is at most one particle per site (cf. Section 4.2 in [16]). $\square$

Fix here and throughout the rest of the section a real number α in $(0, 1)$ and a function β as in the beginning of Section 1.1 and such that there is a $\theta > 0$ such that for all $\check{u}$ in $\mathbb{T}^{d-1}$:

$$
\begin{aligned}
&\beta(u_1, \check{u}) = b(-1, \check{u}) \qquad \text{if } \; -1 \le u_1 \le -1 + \theta, \\
&\beta(u_1, \check{u}) = b(1, \check{u}) \qquad \text{if } \; 1 - \theta \le u_1 \le 1.
\end{aligned}
\tag{2.2.3}
$$

Notice that, for N large enough, $\nu_{\beta(\cdot)}^N$ is reversible with respect to the generator $\mathcal{L}_{N,b}$.

For a cylinder function Ψ, denote the expectation of Ψ with respect to the Bernoulli product measure ν_α^N by $\tilde{\Psi}(\alpha)$:

$$
\tilde{\Psi}(\alpha) = E^{\nu_\alpha^N}[\Psi]
$$

For each integer $l > 0$ and each site x in Ω_N, denote the empirical mean density on a box of size $2l + 1$ centered at x by $\eta^l(x)$:

$$
\eta^l(x) = \frac{1}{|\Lambda_l(x)|} \sum_{y \in \Lambda_l(x)} \eta(y),
$$

where

$$
\Lambda_l(x) = \Lambda_{N,l}(x) = \{y \in \Omega_N : |y - x| \le l\}.
$$

For each cylinder function Ψ and each $\varepsilon > 0$, let

$$V^{\Psi}_{N,\varepsilon}(\eta) = \frac{1}{N^d}\sum_x \left| \frac{1}{\Lambda_{\varepsilon N}(x)} \sum_{y\in\Lambda_{\varepsilon N}(x)} \tau_{x+y}\Psi(\eta) - \widetilde{\Psi}(\eta^{\varepsilon N}(x)) \right| ,$$

where the sum is carried over all x for which the support of $\tau_{x+y}\Psi$ is contained in Ω_N for every y in $\Lambda_{\varepsilon N}(x)$.

For a continuous function $H : [0,T] \times \Gamma \to \mathbb{R}$, let

$$V^{\pm}_{N,H} = \int_0^T ds\ \frac{1}{N^d} \sum_{x\in\Gamma_N^{\pm}} V^{\pm}(x,\eta_s) H(s,\hat{x}/N) .$$

Proposition 2.1.3 follows in the ususal way from (2.2.2) and the next replacement Lemma (cf. Section 5.1 in [16]).

Lemma 2.2.1. *Let Ψ be a cylinder function and $H : [0,T]\times\Gamma \to \mathbb{R}$ a continuous function. For every $\delta > 0$,*

$$\overline{\lim_{\varepsilon\to\infty}}\ \overline{\lim_{N\to\infty}}\ \mathbb{P}_{\mu^N}\left[\int_0^T ds\ V^{\Psi}_{N,\varepsilon}(\eta_s) > \delta\right] = 0 \tag{2.2.4}$$

$$\overline{\lim_{N\to\infty}}\ \mathbb{P}_{\mu^N}[|V^{\pm}_{N,H}| > \delta] = 0 \tag{2.2.5}$$

For probability measures μ,ν in X_N, denote by $H(\mu|\nu)$ the entropy of μ with respect to ν. Since there are at most one particle per site, there exists a constant $C = C(\beta) > 0$ such that

$$H\left(\mu|\nu^N_{\beta(\cdot)}\right) \le CN^d \tag{2.2.6}$$

for any probability measure μ on X_N (cf. comments following Remark 5.5.6 in [16]).

For the proof of (2.2.4) we need to establish an estimate on the entropy production. Denote by S^N_t the semigroup associated with the infinitesimal generator $N^2\mathcal{L}_N$ and let $\mu^N_t = \mu^N S^N_t$. Let also f^N_t, resp. g^N_t, be the density of μ^N_t with respect to $\nu^N_{\beta(\cdot)}$, resp. ν^N_α). Notice that

$$\partial_t f^N_t = N^2\mathcal{L}^*_N f^N_t , \tag{2.2.7}$$

where $\mathcal{L}^*_N$ is the adjoint of $\mathcal{L}_N$ in $L^2\big(\nu^N_{\beta(\cdot)}\big)$.

For a density f with respect to a probability measure μ on X_N, let

$$D^N_0(f,\mu) = \sum_{i=1}^d \sum_x D^N_{x,x+e_i}(f,\mu) ,$$

where the second sum is performed over all x such that $x, x+e_i$ belong to Ω_N and

$$D^N_{x,x+e_i}(f,\mu) = \frac{1}{2}\int r_{x,x+e_i}(\eta)\big(\sqrt{f(\eta^{x,x+e_i})} - \sqrt{f(\eta)}\big)^2 \mu(d\eta) .$$

Denote by $D^N_\beta(\cdot)$ the Dirichlet form of the generator $\mathcal{L}_{N,b}$ with respect to its reversible probabilty measure $\nu^N_{\beta(\cdot)}$ and let $H_N(t) = H\big(\mu^N_t|\nu^N_{\beta(\cdot)}\big)$.

Lemma 2.2.2. *There exists a positive constant $C = C(\beta)$ such that*

$$\partial_t H_N(t) \le -N^2 D_0^N\big(g_t^N, \nu_\alpha^N\big) + CN^d$$

Proof. By (2.2.7) and the explicit formula for the entropy,

$$\partial_t H_N(t) = N^2 \int f_t^N \mathcal{L}_N \log f_t^N \nu_{\beta(\cdot)}^N(d\eta)$$

Since $\nu_{\beta(\cdot)}^N$ is reversible with respect to $\mathcal{L}_{N,b}$, standard estimates gives that the piece of the right-hand side of the last equation corresponding to $\mathcal{L}_{N,b}$ is bounded above by $-2N^2 D_\beta^N\big(f_t^N\big)$. Hence, in order to conclude the proof, we just need to show that there exists a constant $C = C(\beta) > 0$ such that, for any sites $x, y = x + e_i$ in Ω_N,

$$\int f_t^N \mathcal{L}_{x,y} \log f_t^N \nu_{\beta(\cdot)}^N(d\eta) \le -D_{x,y}^N(g_t^N, \nu_\alpha^N) + CN^{-2}\,, \tag{2.2.8}$$

where $\mathcal{L}_{x,y}$ is the piece of the generator $\mathcal{L}_{N,0}$ that corresponds to jumps between x and y.

Fix then $x, y = x + e_i$ in Ω_N. By the definitions of f_t^N and g_t^N,

$$\begin{aligned} \int f_t^N \mathcal{L}_{x,y} \log f_t^N \nu_{\beta(\cdot)}^N(d\eta) \;&=\; \int g_t^N \mathcal{L}_{x,y} \log g_t^N \nu_\alpha^N(d\eta) \\ &\quad + \int g_t^N \mathcal{L}_{x,y} \log\left(\frac{\nu_\alpha^N(\eta)}{\nu_{\beta(\cdot)}^N(\eta)}\right)\nu_\alpha^N(d\eta)\,. \end{aligned} \tag{2.2.9}$$

Since the product measure ν_α^N is invariant for the generator $\mathcal{L}_{x,y}$, by standard estimates, the first term of the right-hand side of (2.2.9) is bounded above by $-2D_{x,y}^N\big(g_t^N, \nu_\alpha^N\big)$.

On the other side, since ν_α^N and $\nu_{\beta(\cdot)}^N$ are product measures, we may compute the second term on the right-hand side of (2.2.9). It is equal to

$$\begin{aligned} &\big[\Phi(y/N) - \Phi(x/N)\big] \int [\eta(y) - \eta(x)] r_{x,y}(\eta) g_t^N(\eta) \nu_\alpha^N(d\eta) \\ &= \big[\Phi(y/N) - \Phi(x/N)\big] \int \eta(x) r_{x,y}(\eta)[g_t^N(\eta^{x,y}) - g_t^N(\eta)]\nu_\alpha^N(d\eta)\,, \end{aligned}$$

where $\Phi = \log(\frac{\beta}{1-\beta})$. By the elementary inequality $2ab \le Aa^2 + A^{-1}b^2$, the previous expression is bounded above by

$$\begin{aligned} &\frac{1}{2}\big[\Phi(y/N) - \Phi(x/N)\big]^2 \int \eta(x) r_{x,y}(\eta)\Big(\sqrt{g_t^N(\eta^{x,y})} + \sqrt{g_t^N(\eta)}\Big)^2 \nu_\alpha^N(d\eta) \\ &+ D_{x,y}^N(g_t^N, \nu_\alpha^N). \end{aligned}$$

This and the fact that g_t^N is a density with respect to ν_α^N permit us to deduce (2.2.8).

□

The proof of (2.2.5) requires the following estimate.

Lemma 2.2.3. *There exists a positive constant $C = C(\beta)$ such that if f is a density with respect to $\nu^N_{\beta(\cdot)}$, then*

$$\langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu^N_{\beta(\cdot)}} \le -\frac{1}{2} D^N_0\big(f, \nu^N_{\beta(\cdot)}\big) - D^N_\beta(f) + CN^{d-2} .$$

Proof. It is enough to show that there is a constant $C = C(\beta)$ such that

$$\langle \mathcal{L}_{x,y} \sqrt{f}, \sqrt{f} \rangle_{\nu^N_{\beta(\cdot)}} \le -\frac{1}{2} D_{x,y}\big(f, \nu^N_{\beta(\cdot)}\big) + CN^{-2} , \tag{2.2.10}$$

for any $x, y = x + e_i$ in Ω_N.

Fix then $x, y = x + e_i$ in Ω_N.

$$\begin{aligned} \langle \mathcal{L}_{x,y} \sqrt{f}, \sqrt{f} \rangle_{\nu^N_{\beta(\cdot)}} &= -\frac{1}{2} \int r_{x,y}(\eta)\big(\sqrt{f(\eta^{x,y})} - \sqrt{f(\eta)}\big)^2 \nu^N_\beta(d\eta) \\ &\quad + \frac{1}{2} \int r_{x,y}(\eta) f(\eta) \left[\frac{\nu^N_\beta(\eta^{x,y})}{\nu^N_{\beta(\cdot)}(\eta)} - 1 \right] \nu^N_\beta(d\eta) \\ &= -D^N_{x,y}\big(f, \nu^N_\beta\big) \\ &\quad + \frac{1}{4} \int r_{x,y}(\eta)[f(\eta^{x,y}) - f(\eta)] \left[1 - \frac{\nu^N_\beta(\eta^{x,y})}{\nu^N_\beta(\eta)} \right] \nu^N_\beta(d\eta) . \end{aligned}$$

Notice that, for some constant $C_1 = C_1(\beta)$,

$$\left| 1 - \frac{\nu^N_\beta(\eta^{x,y})}{\nu^N_\beta(\eta)} \right| \le C_1 |B(y/N) - B(x/N)| , \tag{2.2.11}$$

where $B = \frac{1-\beta}{\beta}$. Hence, by the elementary inequality $2ab \le Aa^2 + A^{-1}b^2$, the left hand side in (2.2.10) is bounded above by

$$\begin{aligned} &\frac{{C_1}^2}{16} \int r_{x,y}(\eta)\big(\sqrt{f(\eta^{x,y})} + \sqrt{f(\eta)}\big)^2 [B(x/N) - B(y/N)]^2 \nu^N_\beta(d\eta) \\ &\qquad - \frac{1}{2} D^N_{x,y}\big(f, \nu^N_{\beta(\cdot)}\big) . \end{aligned}$$

From this fact and since f is a density with respect to $\nu^N_{\beta(\cdot)}$ we obtain (2.2.10). □

Proof of Lemma 2.2.1. By (2.2.6) and Lemma 2.2.2, the proof of (2.2.4) may be reduced (cf. Section 5.3 in [16]) to show that for every positive constant C_0,

$$\varlimsup_{\varepsilon \to 0} \varlimsup_{N \to \infty} \sup_g \int V^\Psi_{N,\varepsilon}(\eta) g(\eta) \nu^N_\alpha(d\eta) = 0 , \tag{2.2.12}$$

where the supremum is carried over all densities g with respect to ν^N_α such that $D^N_0\big(g, \nu^N_\alpha\big) \le C_0 N^{d-2}$. Moreover, since $V^\Psi_{N,\varepsilon}$ is bounded, we may replace g by its conditional expectation g_ε given $\{\eta(x) : x \in \Omega_{N-2\varepsilon N}\}$ in the left hand side of (2.2.12) . In that case, this limit may be estimated by the one of the periodic case. Hence, (2.2.12) follows from Lemma 5.5.7 in [16].

We turn now to the proof of (2.2.5). Fix $A > 0$. By the entropy inequality and (2.2.6),

$$\mathbb{E}_{\mu^N}\big[|V^-_{N,H}|\big] \le \frac{C}{A} + \frac{1}{AN^d}\big[\log \mathbb{E}_{\nu^N_{\beta(\cdot)}}\big[\exp\big\{AN^d|V^-_{N,H}|\big\}\big]\big]\,.$$

Thus we just need to show that for some constant $C = C(\beta) > 0$,

$$\overline{\lim_{N\to\infty}}\, \frac{1}{N^d} \log \mathbb{E}_{\nu^N_{\beta(\cdot)}}\big[\exp\big\{AN^d|V^-_{N,H}|\big\}\big] \le C\,. \tag{2.2.13}$$

Since $e^{|x|} \le e^x + e^{-x}$ and

$$\overline{\lim_{N\to\infty}}\, \frac{1}{N^d}\log(a_N + b_N) \le \max\left\{\overline{\lim_{N\to\infty}}\, \frac{1}{N^d}\log a_N\,,\ \overline{\lim_{N\to\infty}}\, \frac{1}{N^d}\log b_N\right\}\,,$$

we may remove the absolute value in (2.2.13), provided our estimates remain in force if we replace H with $-H$.

Let

$$V^{\pm}_H(x,\eta,s) = V^{\pm}(x,\eta)H(s,\hat{x}/N)\,.$$

By the Feynman-Kac formula, the left hand side of (2.2.13), without the absolute value, is bounded by

$$\frac{1}{N^d}\int_0^T \lambda^N_s\, ds\,,$$

where λ^N_s stands for the largest eigenvalue of the $\nu^N_{\beta(\cdot)}$-reversible operator $N^2\mathcal{L}^{sym}_N + AN\sum_{x\in\Gamma^-} V^-_H(x,\eta,s)$ and $\mathcal{L}^{sym}_N$ is the symmetric part of the operator $\mathcal{L}_N$ in $L^2\big(\nu^N_{\beta(\cdot)}\big)$. By the variational formula for the largest eigenvalue, for each $s \in [0,T]$, $N^{-d}\lambda^N_s$ is equal to

$$\sup_f\left\{\frac{A}{N^{d-1}}\left\langle \sum_{x\in\Gamma^-_N} V^-_H(x,\eta,s), f\right\rangle_{\nu^N_{\beta(\cdot)}} + \frac{1}{N^{d-2}}\big\langle L_N\sqrt{f}, \sqrt{f}\big\rangle_{\nu^N_{\beta(\cdot)}}\right\},$$

where the supremum is carried over all densities f with respect to $\nu^N_{\beta(\cdot)}$.

By Lemma 2.2.3, for a constant $C_1 = C_1(\beta) > 0$, the expression inside braces is less than or equal to

$$C_1 + \frac{A}{N^{d-1}}\left\{\sum_{x\in\Gamma^-_N}\langle V^-_H(x,\eta,s), f\rangle_{\nu^N_{\beta(\cdot)}} - \frac{N}{A}\Big(\frac{1}{2}D^N_0\big(f,\nu^N_{\beta(\cdot)}\big) + D^N_b\big(f,\nu^N_{\beta(\cdot)}\big)\Big)\right\}.$$

In this last expression, for some positive constant $C_2 = C_2(b)$, the part inside braces is bounded above by

$$\sum_{x\in\Gamma^-_N}\left\{\langle V^-_H(x,\eta,s), f\rangle_{\nu^N_{\beta(\cdot)}} - \frac{NC_2}{A}\left[\int\big(\sqrt{f(\eta^{x,x+e_1})} - \sqrt{f(\eta)}\big)^2 \right.\right.$$
$$\left.\left. + \big(\sqrt{f(\eta^x)} - \sqrt{f(\eta)}\big)^2\nu^N_\beta(d\eta)\right]\right\}$$

Hence, in order to prove (2.2.5), it is enough to show that the part inside braces in the last expression is bounded above by some positive constant c_N, not depending on s, f or x, which converges to 0 as $N \uparrow \infty$.

Fix such s, f and x, and denote by f_x the conditional expectation of f given $\{\eta(x), \eta(x+e_1)\}$. Since $V_{H,x,s}(\eta) = V_H^-(x, \eta, s)$ depends on the configuration η only through $\{\eta(x), \eta(x+e_1)\}$, the part inside braces in the last expression is bounded above by

$$\langle V_{H,x,s}, f_x \rangle_{\nu^N_{\beta(\cdot)}} - \frac{NC_2}{A}\Big[\int \big(\sqrt{f_x(\eta^{x,x+e_1})} - \sqrt{f_x(\eta)}\big)^2 + \big(\sqrt{f_x(\eta^x)} - \sqrt{f_x(\eta)}\big)^2 \nu^N_\beta(d\eta)\Big] .$$

Let $\Lambda_x = \{0,1\}^{\{x,x+e_1\}}$ and denote by $\hat{f}_x$ the restriction of f_x to Λ_x. Note that, for N large enough, the restriction of $\nu^N_{\beta(\cdot)}$ to Λ_x is the Bernoulli product measure associated to the constant function $b_x = b(\hat{x}/N)$. Hence, for a constant $C_3 = C_3(b) > 0$, the last expression is bounded above by

$$\langle V_{H,x,s}, \hat{f}_x \rangle_{\nu^N_{b_x}} - \frac{NC_3}{A} Var_{\nu^N_{b_x}}\Big(\sqrt{\hat{f}_x}\Big) .$$

which, by the elementary inequality $2ab \le Aa^2 + A^{-1}b^2$ and since $E_{\nu^N_{b_x}}(V_{H,x,s}) = 0$, is bounded by

$$\frac{A}{4NC_3} E_{\nu^N_{b_x}}\Big[(V_{H,x,s})^2 \Big(\sqrt{\hat{f}_x} + E\sqrt{\hat{f}_x}\Big)^2\Big] .$$

Since $\hat{f}_x$ is a density with respect to ν^N_b and $|V_{H,x,s}| \le 2\|H\|_\infty$, the previous expression is bounded by

$$c_N = \frac{4A\|H\|^2_\infty}{NC_3} ,$$

which concludes the proof.

□

For each function G in $\mathcal{C}^\infty_c(\Omega_T)$, each integer $1 \le i \le d$ and $C > 0$, let $\mathcal{Q}^{G,i,C}_T : D([0,T], \mathcal{M}) \to \mathbb{R}$ be the functional given by

$$\mathcal{Q}^{G,i,C}_T(\pi) = \int_0^T \big\langle \pi_s, \partial_{u_i} G_s \big\rangle \, ds - C \int_0^T ds \int_\Omega du \; G(s,u)^2 .$$

Recall from Section 1.3 that the energy $\mathcal{Q}_T(\pi)$ was defined as

$$\mathcal{Q}_T(\pi) = \sum_{i=1}^d \mathcal{Q}^i_T(\pi) ,$$

where

$$\mathcal{Q}^i_T(\pi) = \sup_{G \in \mathcal{C}^\infty_c(\Omega_T)} \Big\{ 2\int_0^T \langle \pi_t, \partial_{u_i} G_t \rangle dt - \int_0^T dt \int_\Omega G(t,u)^2 \, du \Big\} .$$

Notice that

$$\sup_{G\in\mathcal{C}_c^\infty(\Omega_T)} \left\{\mathcal{Q}_T^{G,i,C}(\pi)\right\} = \frac{\mathcal{Q}_T^i(\pi)}{4C}\,. \tag{2.2.14}$$

The next result is the key ingredient in the proof of Proposition (2.1.4).

Lemma 2.2.4. *There exists a constant $C_0 = C_0(\beta) > 0$ such that for every integer $1 \le i \le d$ and every function G in $\mathcal{C}_c^\infty(\Omega_T)$,*

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbb{E}_{\nu^N_{\beta(\cdot)}} \left[\exp\left\{N^d \mathcal{Q}_T^{G,i,C_0}(\pi^N)\right\}\right] \le C_0$$

Proof. By the Feynman-Kac formula,

$$\frac{1}{N^d}\log \mathbb{E}_{\nu^N_{\beta(\cdot)}}\left[\exp\left\{N\int_0^T ds \sum_{x\in\Omega_N}(\eta_s(x)-\eta_s(x-e_i))G(s,x/N)\right\}\right]$$

is bounded above by

$$\frac{1}{N^d}\int_0^T \lambda_s^N\, ds\,,$$

where λ_s^N stands for the largest eigenvalue of the $\nu^N_{\beta(\cdot)}$-reversible operator $N^2\mathcal{L}_N^{sym} + N\sum_{x\in\Omega_N}[\eta(x)-\eta(x-e_i)]G(s,x/N)$. By the variational formula for the largest eigenvalue, for each $s\in[0,t]$, λ_s^N is equal to

$$\sup_f\left\{\left\langle N\sum_{x\in\Omega_N}(\eta(x)-\eta(x-e_i))G(s,x/N), f\right\rangle_{\nu^N_{\beta(\cdot)}} + N^2\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\beta(\cdot)}}\right\},$$

where the supremum is carried over all densities f with respect to $\nu^N_{\beta(\cdot)}$. By Lemma 2.2.3, for a constant $C = C(\beta) > 0$, the expression inside braces is bounded above by

$$CN^d - \frac{N^2}{2}D_0^N(f,\nu^N_{\beta(\cdot)}) + \sum_{x\in\Omega_N}\left\{NG(s,x/N)\int[\eta(x)-\eta(x-e_i)]f(\eta)\nu^N_\beta(d\eta)\right\},$$

By the elementary inequality $2ab \le Aa^2 + A^{-1}b^2$, the part inside braces in the last expression is bounded above by

$$\begin{aligned}
& G(s,x/N)^2\int f(\eta^{x-e_i,x})\nu^N_\beta(d\eta)\\
+\;& \frac{1}{4}\int \eta(x)f(\eta^{x-e_i,x})\left[N\left(1-\frac{\nu^N_\beta(\eta^{x-e_i,x})}{\nu^N_\beta(\eta)}\right)\right]^2\nu^N_\beta(d\eta)\\
+\;& G(s,x/N)^2\int\frac{\eta(x)}{r_{x-e_i,x}(\eta)}\left(\sqrt{f(\eta^{x-e_i,x})}+\sqrt{f(\eta)}\right)^2\nu^N_\beta(d\eta)\,,
\end{aligned}$$

which is bounded above by $C_1G(s,x/N)^2 + C_1$, by some positive constant $C_1 = C_1(\beta)$, because of (2.2.11) and the fact that f is a density with respect to $\nu^N_{\beta(\cdot)}$. Thus, $C_0 = C + C_1$ satisfies the statement of the Lemma.

□

It is well known that a trajectory $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}^0)$ has finite energy, $\mathcal{Q}_T(\pi) < \infty$, if and only if its density ρ belongs to $L^2([0,T],H^1(\Omega))$, in which case,

$$\mathcal{Q}_T(\pi) = \int_0^T dt \int_\Omega du\, \|\nabla \rho_t(u)\|^2 < \infty\,.$$

Proof of Proposition 2.1.4. Fix a constant $C_0 > 0$ satisfying the statement of Lemma 2.2.4. Let $\{G_k : k \geq 1\}$ be a sequence of smooth functions dense in $L^2([0,T],H^1(\Omega))$ and $1 \leq i \leq d$ an integer. By the entropy inequality and (2.2.6), there is a constant $C = C(\beta) > 0$ such that

$$\mathbb{E}_{\mu^N}\left[\max_{1\leq k\leq r}\left\{\mathcal{Q}_T^{G_k,i,C_0}(\pi^N)\right\}\right]$$

is bounded above by

$$C + \frac{1}{N^d}\log \mathbb{E}_{\nu^N_{\beta(\cdot)}}\left[\exp\left\{N^d \max_{1\leq k\leq r}\left\{\mathcal{Q}_T^{G_k,i,C_0}(\pi^N)\right\}\right\}\right]\,.$$

Hence, Lemma 2.2.4 together with the facts that $e^{\max\{x_1,\dots,x_n\}} \leq e^{x_1} + \cdots + e^{x_n}$ and that

$$\varlimsup_{N\to\infty} \frac{1}{N^d}\log(a_N + b_N) \leq \max\left\{\varlimsup_{N\to\infty}\frac{1}{N^d}\log a_N\,,\ \varlimsup_{N\to\infty}\frac{1}{N^d}\log b_N\right\}\,,$$

imply

$$\begin{aligned} E_{\mathbf{Q}^*}\left[\max_{1\leq k\leq r}\left\{\mathcal{Q}_{i,C_0}^{G_k}\right\}\right] &= \lim_{N\to\infty}\mathbb{E}_{\mu^N}\left[\max_{1\leq k\leq r}\left\{\mathcal{Q}_{i,C_0}^{G_k}(\pi^N)\right\}\right] \\ &\leq C + C_0\,. \end{aligned}$$

This together with (2.2.14) and the monotone convergence theorem prove the desired result.

□

Chapter 3

Dynamical Large Deviations

In this chapter, we investigate the large deviations from the hydrodynamic limit.

3.1 The Dynamical Rate Function

We examine in this section the rate function $I_T(\cdot|\gamma)$. The main result, presented in Theorem 3.1.7 below, states that $I_T(\cdot|\gamma)$ has compact level sets. The proof relies on two ingredients. The first one, stated in Lemma 3.1.2, is an estimate of the energy and of the H_{-1} norm of the time derivative of the density of a trajectory in terms of the rate function. The second one, stated in Lemma 3.1.6, establishes that sequences of trajectories, with rate function uniformly bounded, whose densities converges weakly in L^2 converge in fact strongly.

We start by introducing some Sobolev spaces. Recall that we denote by $\mathcal{C}_c^\infty(\Omega)$ the set of infinitely differentiable functions $G : \Omega \to \mathbb{R}$, with compact support in Ω. Recall from Section 1.2 the definition of the Sobolev space $H^1(\Omega)$ and of the norm $\| \cdot \|_{1,2}$. Denote by $H_0^1(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. Since Ω is bounded, by Poincaré's inequality, there exists a finite constant C_1 such that for all $G \in H_0^1(\Omega)$

$$\|G\|_2^2 \;\le\; C_1 \|\partial_{u_1} G\|_2^2 \;\le\; C_1 \sum_{j=1}^{d} \langle \partial_{u_j} G \,,\, \partial_{u_j} G \rangle_2 \; .$$

This implies that, in $H_0^1(\Omega)$

$$\|G\|_{1,2,0} \;=\; \left\{ \sum_{j=1}^{d} \langle \partial_{u_j} G \,,\, \partial_{u_j} G \rangle_2 \right\}^{1/2}$$

is a norm equivalent to the norm $\| \cdot \|_{1,2}$. Moreover, $H_0^1(\Omega)$ is a Hilbert space with inner product given by

$$\langle G \,,\, J \rangle_{1,2,0} \;=\; \sum_{j=1}^{d} \langle \partial_{u_j} G \,,\, \partial_{u_j} J \rangle_2 \; .$$

To assign boundary values along the boundary Γ of Ω to any function G in $H^1(\Omega)$, recall, from the trace Theorem ([26], Theorem 21.A.(e)), that there exists a continuous linear operator $B : H^1(\Omega) \to L^2(\Gamma)$, called trace, such that $BG = G\big|_\Gamma$ if $G \in H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Moreover, the space $H_0^1(\Omega)$ is the space of functions G in $H^1(\Omega)$ with zero trace ([26], Appendix (48b)):

$$H_0^1(\Omega) = \left\{ G \in H^1(\Omega) : \; BG = 0 \right\} .$$

Since $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$ ([26], Corollary 21.15.(a)), for functions F, G in $H^1(\Omega)$, the product FG has generalized derivatives $\partial_{u_i}(FG) = F\partial_{u_i} G + G\partial_{u_i} F$ in $L^1(\Omega)$ and

$$\begin{aligned} &\int_\Omega F(u)\, \partial_{u_1} G(u)\, du \; + \int_\Omega G(u)\, \partial_{u_1} F(u)\, du \\ &\quad = \int_{\Gamma_+} BF(u)\, BG(u)\, du \; - \int_{\Gamma_-} BF(u)\, BG(u)\, du \, . \end{aligned} \tag{3.1.1}$$

Moreover, if $G \in H^1(\Omega)$ and $f \in \mathcal{C}^1(\mathbb{R})$ is such that f' is bounded then $f \circ G$ belongs to $H^1(\Omega)$ with generalized derivatives $\partial_{u_i}(f \circ G) = (f' \circ G)\partial_{u_i} G$ and trace $B(f \circ G) = f \circ (BG)$.

Finally, denote by $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$. $H^{-1}(\Omega)$ is a Banach space with norm $\| \cdot \|_{-1}$ given by

$$\|v\|_{-1}^2 = \sup_{G \in \mathcal{C}_c^\infty(\Omega)} \left\{ 2\langle v, G \rangle_{-1,1} - \int_\Omega \|\nabla G(u)\|^2 du \right\} ,$$

where $\langle v, G \rangle_{-1,1}$ stands for the values of the linear form v at G.

For each function G in $\mathcal{C}_c^\infty(\Omega_T)$ and each integer $1 \le i \le d$, let $\mathcal{Q}_T^{G,i} : D([0,T], \mathcal{M}^0) \to \mathbb{R}$ be the functional given by

$$\mathcal{Q}_T^{G,i}(\pi) = 2 \int_0^T \langle \pi_t, \partial_{u_i} G_t \rangle \, dt - \int_0^T dt \int_\Omega du \; G(t,u)^2 \, ,$$

and recall, from Section 1.3, that the energy $\mathcal{Q}(\pi)$ was defined as

$$\mathcal{Q}_T(\pi) = \sum_{i=1}^d \mathcal{Q}_T^i(\pi) \quad \text{with} \quad \mathcal{Q}_T^i(\pi) = \sup_{G \in \mathcal{C}_c^\infty(\Omega_T)} \mathcal{Q}_T^{G,i}(\pi) \, .$$

The functional $\mathcal{Q}_T^{G,i}$ is convex and continuous in the Skorohod topology. Therefore $\mathcal{Q}_T^i$ and $\mathcal{Q}_T$ are convex and lower semicontinuous. Furthermore, it is well known that a trajectory $\pi(t,du) = \rho(t,u)du$ in $D([0,T], \mathcal{M}^0)$ has finite energy, $\mathcal{Q}_T(\pi) < \infty$, if and only if its density ρ belongs to $L^2([0,T], H^1(\Omega))$, in which case,

$$\mathcal{Q}_T(\pi) = \int_0^T dt \int_\Omega du \; \|\nabla \rho_t(u)\|^2 < \infty \, .$$

Let $D_\gamma = D_{\gamma,b}$ be the subset of $C([0,T], \mathcal{M}^0)$ consisting of all paths $\pi(t,du) = \rho(t,u)du$ with initial profile $\rho(0,\cdot) = \gamma(\cdot)$, finite energy $\mathcal{Q}_T(\pi)$ (in which case ρ_t belongs to $H^1(\Omega)$ for almost all $0 \le t \le T$ and so $B(\rho_t)$ is well defined for those t) and such that $B(\rho_t) = b$ for almost all t in $[0,T]$.

Lemma 3.1.1. *Let π be a trajectory in $D([0,T], \mathcal{M})$ such that $I_T(\pi|\gamma) < \infty$. Then π belongs to D_γ.*

Proof. Fix a path π in $D([0,T],\mathcal{M})$ with finite rate function, $I_T(\pi|\gamma)<\infty$. By definition of I_T, π belongs to $D([0,T],\mathcal{M}^0)$. Denote its density by ρ: $\pi(t,du)=\rho(t,u)du$.

The proof that $\rho(0,\cdot)=\gamma(\cdot)$ is similar to the one of Lemma 3.5 in [4] and is therefore omitted. To prove that $B(\rho_t)=b$ for almost all $t\in[0,T]$, since the function $\varphi:[0,1]\to[0,1+a]$ is a $\mathcal{C}^1$ diffeomorphism and since $B(\varphi\circ\rho_t)=\varphi(B\rho_t)$ (for those t such that ρ_t belongs to $H^1(\Omega)$), it is enough to show that $B(\varphi\circ\rho_t)=\varphi(b)$ for almost all $t\in[0,T]$. To this end, we just need to show that, for any function $H_\pm\in\mathcal{C}^{1,2}([0,T]\times\Gamma_\pm)$,

$$\int_0^T dt\int_{\Gamma_\pm} du\,\big\{B(\varphi(\rho_t))(u)-\varphi(b(u))\big\}\,H_\pm(t,u)\;=\;0\,. \tag{3.1.2}$$

Fix a function $H\in\mathcal{C}^{1,2}([0,T]\times\Gamma_-)$. For each $0<\theta<1$, let $h_\theta:[-1,1]\to\mathbb{R}$ be the function given by

$$h_\theta(r)=\begin{cases} r+1 & \text{if } -1\le r\le -1+\theta\,,\\ \frac{-\theta r}{1-\theta} & \text{if } -1+\theta\le r\le 0\,,\\ 0 & \text{if } 0\le r\le 1\,,\end{cases}$$

and define the function $G_\theta:\overline{\Omega_T}\to\mathbb{R}$ as $G(t,(u_1,\check u))=h_\theta(u_1)H(t,(-1,\check u))$ for all $\check u\in\mathbb{T}^{d-1}$. Of course, G_θ can be approximated by functions in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$. From the integration by parts formula (3.1.1) and the definition of J_{G_θ}, we obtain that

$$\lim_{\theta\to0}J_{G_\theta}(\pi)=\int_0^T dt\int_{\Gamma_-}du\,\big\{B(\varphi(\rho_t))(u)-\varphi(b(u))\big\}\,H(t,u)\,,$$

which proves (3.1.2) because $I_T(\pi|\gamma)<\infty$.

We deal now with the continuity of π. We claim that there exists a positive constant C_0 such that, for any $g\in\mathcal{C}_c^\infty(\Omega)$, and any $0\le s<r<T$,

$$\begin{aligned}|\langle\pi_r,g\rangle-\langle\pi_s,g\rangle|\;\le\;C_0(r-s)^{1/2}\Big\{&I_T(\pi|\gamma)+\|g\|_{1,2,0}^2\\ &+(r-s)^{1/2}\|\Delta g\|_1\Big\}\,.\end{aligned} \tag{3.1.3}$$

Indeed, for each $\delta>0$, let $\psi^\delta:[0,T]\to\mathbb{R}$ be the function given by

$$(r-s)^{1/2}\psi^\delta(t)=\begin{cases}0 & \text{if } 0\le t\le s \ \text{ or }\ r+\delta\le t\le T\,,\\ \frac{t-s}{\delta} & \text{if } s\le t\le s+\delta\,,\\ 1 & \text{if } s+\delta\le t\le r\,,\\ 1-\frac{t-r}{\delta} & \text{if } r\le t\le r+\delta\,,\end{cases}$$

and let $G^\delta(t,u)=\psi^\delta(t)g(u)$. Of course, G^δ can be approximated by functions in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$ and then

$$\begin{aligned}(r-s)^{1/2}\lim_{\delta\to0}J_{G^\delta}(\pi)\;&=\;\langle\pi_r,g\rangle-\langle\pi_s,g\rangle-\int_s^r dt\,\langle\varphi(\rho_t),\Delta g\rangle\\ &\quad-\frac{1}{2(r-s)^{1/2}}\int_s^r dt\,\langle\sigma(\rho_t),\|\nabla g\|^2\rangle\,.\end{aligned}$$

To conclude the proof, it remains to observe that the left hand side is bounded by $(r-s)^{1/2} I_T(\pi|\gamma)$, and to note that φ, σ are positive and bounded above on $[0,1]$ by some positive constant. $\square$

Denote by $L^2([0,T],H_0^1(\Omega))^*$ the dual of $L^2([0,T],H_0^1(\Omega))$. By Proposition 23.7 in [26], $L^2([0,T],H_0^1(\Omega))^*$ corresponds to $L^2([0,T],H^{-1}(\Omega))$, i.e., for each v in $L^2([0,T],H_0^1(\Omega))^*$, there exists a unique $\{v_t : 0 \le t \le T\}$ in $L^2([0,T],H^{-1}(\Omega))$ such that for any G in $L^2([0,T],H_0^1(\Omega))$,

$$\langle\!\langle v, G\rangle\!\rangle_{-1,1} \;=\; \int_0^T \langle v_t, G_t\rangle_{-1,1}\, dt\;, \tag{3.1.4}$$

where the left hand side stands for the value of the linear functional v at G. Moreover, if we denote by $|\!|\!| v |\!|\!|_{-1}$ the norm of v,

$$|\!|\!| v |\!|\!|_{-1}^2 \;=\; \int_0^T \|v_t\|_{-1}^2\, dt\;.$$

Fix a path $\pi(t,du) = \rho(t,u)du$ in D_γ and suppose that

$$\sup_{H\in\mathcal{C}_c^\infty(\Omega_T)} \Big\{ 2\int_0^T \langle \rho_t, \partial_t H_t\rangle\, dt - \int_0^T dt \int_\Omega du\, \|\nabla H_t\|^2 \Big\} \;<\; \infty\;. \tag{3.1.5}$$

In this case $\partial_t\rho : C_c^\infty(\Omega_T) \to \mathbb{R}$ defined by

$$\partial_t\rho(H) \;=\; -\int_0^T \langle \rho_t, \partial_t H_t\rangle\, dt$$

can be extended to a bounded linear operator $\partial_t\rho : L^2([0,T],H_0^1(\Omega)) \to \mathbb{R}$. It belongs therefore to $L^2([0,T],H_0^1(\Omega))^* = L^2([0,T],H^{-1}(\Omega))$. In particular, there exists $\{v_t : 0 \le t \le T\}$ in $L^2([0,T],H^{-1}(\Omega))$, which we denote by $v_t = \partial_t\rho_t$, such that for any H in $L^2([0,T],H_0^1(\Omega))$,

$$\langle\!\langle \partial_t\rho, H\rangle\!\rangle_{-1,1} \;=\; \int_0^T \langle \partial_t\rho_t, H_t\rangle_{-1,1}\, dt\;.$$

Moreover,

$$\begin{aligned} |\!|\!| \partial_t\rho |\!|\!|_{-1}^2 \;&=\; \int_0^T \|\partial_t\rho_t\|_{-1}^2\, dt \\ &=\; \sup_{H\in\mathcal{C}_c^\infty(\Omega_T)} \Big\{ 2\int_0^T \langle \rho_t, \partial_t H_t\rangle\, dt - \int_0^T dt \int_\Omega du\, \|\nabla H_t\|^2 \Big\}\;. \end{aligned}$$

Let W be the set of paths $\pi(t,du) = \rho(t,u)du$ in D_γ such that (3.1.5) holds, i.e., such that $\partial_t\rho$ belongs to $L^2\big([0,T],H^{-1}(\Omega)\big)$. For G in $L^2\big([0,T],H_0^1(\Omega)\big)$, let $\mathbb{J}_G : W \to \mathbb{R}$ be the functional given by

$$\begin{aligned} \mathbb{J}_G(\pi) \;&=\; \langle\!\langle \partial_t\rho, G\rangle\!\rangle_{-1,1} + \int_0^T dt \int_\Omega du\, \nabla G_t(u)\cdot \nabla(\varphi(\rho_t(u))) \\ &-\; \frac{1}{2}\int_0^T dt \int_\Omega du\, \sigma(\rho_t(u))\, \|\nabla G_t(u)\|^2\,. \end{aligned}$$

By Proposition 23.23 in [26], if $\pi(t,du) = \rho(t,u)du$ belongs to W and G belongs to $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\langle \rho_T, G_T\rangle - \langle \rho_0, G_0\rangle - \int_0^T \langle \rho_t, \partial_t G_t\rangle \, dt = \int_0^T \langle \partial_t \rho_t, G_t\rangle_{-1,1} \, dt \,,$$

which together with Lemma 3.1.1 and the integration by parts formula (3.1.1) implies that

$$\mathbb{J}_G(\pi) = J_G(\pi)\,. \tag{3.1.6}$$

Then, since $\mathbb{J}_\cdot(\pi)$ is continuous in $L^2\big([0,T], H_0^1\big)$ and since $\mathcal{C}_c^\infty(\Omega_T) \subset \mathcal{C}_0^{1,2}(\overline{\Omega_T})$ are dense in $L^2\big([0,T], H_0^1(\Omega)\big)$, for every π in W,

$$I_T(\pi|\gamma) = \sup_{G\in L^2([0,T],H_0^1)} \mathbb{J}_G(\pi) \;=\; \sup_{G\in\mathcal{C}_c^\infty(\Omega_T)} J_G(\pi)\,. \tag{3.1.7}$$

Lemma 3.1.2. *There exists a constant $C_0 > 0$ such that if the density ρ of some path $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}^0)$ has a generalized gradient, $\nabla\rho$, then*

$$\begin{aligned} \int_0^T dt \; \|\partial_t \rho_t\|_{-1}^2 \quad &\le \quad C_0 \left\{ I_T(\pi|\gamma) + \mathcal{Q}_T(\pi) \right\} \,, & (3.1.8)\\ \int_0^T dt \int_\Omega du \; \frac{\|\nabla \rho_t(u)\|^2}{\chi(\rho_t(u))} \quad &\le \quad C_0 \left\{ I_T(\pi|\gamma) + 1 \right\} \,, & (3.1.9)\end{aligned}$$

where $\chi(r) = r(1-r)$ is the static compressibility.

Proof. Fix a path $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}^0)$. In view of the discussion presented before the lemma, we need to show that the left hand side of (3.1.5) is bounded by the right hand side of (3.1.8). Such an estimate follows from the definition of the rate function $I_T(\cdot|\gamma)$ and from the elementary inequality $2ab \le Aa^2 + A^{-1}b^2$.

We turn now to the proof of (3.1.9). We may of course assume that $I_T(\pi|\gamma) < \infty$, in which case $\mathcal{Q}_T(\pi) < \infty$. Fix a function β as in the beginning of Section 1.1. For each $\delta > 0$, let $h^\delta : [0,1]^2 \to \mathbb{R}$ be the function given by

$$h^\delta(x,y) = (x+\delta)\log\left(\frac{x+\delta}{y+\delta}\right) + (1-x+\delta)\log\left(\frac{1-x+\delta}{1-y+\delta}\right)\,.$$

By (3.1.8), $\partial_t\rho$ belongs to $L^2([0,T], H^{-1}(\Omega))$. We claim that

$$\begin{aligned} \int_0^T \langle \partial_t \rho_t, \partial_x h^\delta(\rho_t,\beta)\rangle_{-1,1} \, dt \quad &= \quad \int_\Omega h^\delta(\rho_T(u),\beta(u))du \\ &\quad - \int_\Omega h^\delta(\rho_0(u),\beta(u))du\,. \qquad (3.1.10)\end{aligned}$$

Indeed, By Lemma 3.1.1 and (3.1.8), $\rho-\beta$ belongs to $L^2\big([0,T], H_0^1(\Omega)\big)$ and $\partial_t(\rho-\beta) = \partial_t\rho$ belongs to $L^2([0,T],H^{-1}(\Omega))$. Then, there exists a sequence $\{\widetilde{G}^n : n \ge 1\}$ of smooth functions $\widetilde{G}^n : \overline{\Omega_T} \to \mathbb{R}$ such that $\widetilde{G}^n_t$ belongs to $\mathcal{C}_c^\infty(\Omega)$ for every t in $[0,T]$, $\widetilde{G}^n$ converges to $\rho-\beta$ in $L^2([0,T],H_0^1(\Omega))$ and $\partial_t\widetilde{G}^n$ converges to $\partial_t(\rho-\beta)$ in $L^2([0,T],H^{-1}(\Omega))$ (cf. [26], Proposition 23.23(ii)).

For each positive integer n, let $G^n = \widetilde{G}^n + \beta$ and for each $\delta > 0$, fix a smooth function $\tilde{h}^\delta : \mathbb{R}^2 \to \mathbb{R}$ with compact support and such that its restriction to $[0,1]^2$ is h^δ. It is clear that

$$\begin{aligned}\int_0^T \langle \partial_t G_t^n, \partial_x \tilde{h}^\delta(G_t^n, \beta)\rangle \, dt \; &= \int_\Omega \tilde{h}^\delta(G_T^n(u), \beta(u)) du \\ &- \int_\Omega \tilde{h}^\delta(G_0^n(u), \beta(u)) du \, .\end{aligned} \tag{3.1.11}$$

On the one hand, $\partial_x h^\delta : [0,1]^2 \to \mathbb{R}$ is given by

$$\partial_x h^\delta(x,y) = \log\left(\frac{x+\delta}{1-x+\delta}\right) - \log\left(\frac{y+\delta}{1-y+\delta}\right) .$$

Hence, $\partial_x h^\delta(\rho,\beta)$ and $\partial_x \tilde{h}^\delta(G^n,\beta)$ belongs to $L^2\left([0,T], H_0^1(\Omega)\right)$. Moreover, since $\partial_x \tilde{h}^\delta$ is smooth with compact support and G^n converges to ρ in $L^2([0,T], H^1(\Omega))$, $\partial_x \tilde{h}^\delta(G^n,\beta)$ converges to $\partial_x h^\delta(\rho,\beta)$ in $L^2([0,T], H_0^1(\Omega))$. From this fact and since $\partial_t G^n$ converges to $\partial_t \rho$ in $L^2([0,T], H^{-1}(\Omega))$, if we let $n \to \infty$, the left hand side in (3.1.11) converges to

$$\int_0^T \langle \partial_t \rho_t, \partial_x h^\delta(\rho_t, \beta)\rangle_{-1,1} \, dt \, .$$

On the other hand, by Proposition 23.23(ii) in [26], G_0^n, resp. G_T^n, converges to ρ_0, resp. ρ_T, in $L^2(\Omega)$. Then, if we let $n \to \infty$, the right hand side in (3.1.11) goes to

$$\int_\Omega h^\delta(\rho_T(u), \beta(u)) du - \int_\Omega h^\delta(\rho_0(u), \beta(u)) du \, ,$$

which proves claim (3.1.10).

Notice that, since β is bounded away from 0 and 1, there exists a positive constant $C = C(\beta)$ such that for δ small enough,

$$h^\delta(\rho(t,u), \beta(u)) \le C \;\text{ for all }\; (t,u) \;\text{ in }\; \overline{\Omega_T} \, . \tag{3.1.12}$$

For each $\delta > 0$, let $H^\delta : \overline{\Omega_T} \to \mathbb{R}$ be the function given by

$$H^\delta(t,u) = \frac{\partial_x h^\delta(\rho(t,u), \beta(u))}{2(1+2\delta)} \, .$$

A simple computation shows that

$$\begin{aligned}\mathbb{J}_{H^\delta}(\pi) \;\; \ge \;\; & \int_0^T dt \, \langle \partial_t \rho_t, H_t^\delta \rangle_{-1,1} + \frac{1}{4}\int_0^T dt \int_\Omega du \, \varphi'(\rho_t(u)) \frac{\|\nabla \rho_t(u)\|^2}{\chi_\delta(\rho_t(u))} \\ & -\frac{1}{8}\int_0^T dt \int_\Omega du \, \sigma_\delta(\rho_t(u)) \frac{\|\nabla \beta(u)\|^2}{\chi_\delta(\beta(u))^2} \, ,\end{aligned}$$

where $\chi_\delta(r) = (r+\delta)(1-r+\delta)$ and $\sigma_\delta(r) = 2\chi_\delta(r)\varphi'(r)$. This last inequality together with (3.1.10), (3.1.7) and (3.1.12) show that there exists a positive constant $C_0 = C_0(\beta)$ such that for δ small enough

$$C_0 \left\{ I_T(\pi|\gamma) + 1 \right\} \ge \int_0^T dt \int_\Omega du \, \frac{\|\nabla \rho(t,u)\|^2}{\chi_\delta(\rho(t,u))} \, .$$

We conclude the proof by letting $\delta \downarrow 0$ and by using Fatou's lemma. $\square$

Corollary 3.1.3. *The density ρ of a path $\pi(t, du) = \rho(t, u)du$ in $D([0, T], \mathcal{M}^0)$ is the weak solution of the equation (2.1.1) with initial profile γ if and only if the rate function $I_T(\pi|\gamma)$ vanishes. Moreover, in that case*

$$\int_0^T dt \int_\Omega du \, \frac{\|\nabla \rho_t(u)\|^2}{\chi(\rho_t(u))} \; < \; \infty \, .$$

Proof. On the one hand, if the density ρ of a path $\pi(t, du) = \rho(t, u)du$ in $D([0, T], \mathcal{M}^0)$ is the weak solution of equation (2.1.1), by assumption (**H1**), the energy $\mathcal{Q}_T(\pi)$ is finite. Moreover, since the initial condition is γ, in the formula of $\hat{J}_G(\pi)$, the linear part in G vanishes which proves that the rate functional $I_T(\pi|\gamma)$ vanishes. On the other hand, if the rate functional vanishes, the path ρ belongs to $L^2([0, T], H^1(\Omega))$ and the linear part in G of $J_G(\pi)$ has to vanish for all functions G. In particular, ρ is a weak solution of (2.1.1). Moreover, in that case, by the previous lemma, the bound claimed holds. □

For each $q > 0$, let E_q be the level set of $I_T(\pi|\gamma)$ defined by

$$E_q = \{\pi \in D([0, T], \mathcal{M}) : I_T(\pi|\gamma) \leq q\} \, .$$

By Lemma 3.1.1, E_q is a subset of $C([0, T], \mathcal{M}^0)$. Thus, from the previous lemma, it is easy to deduce the next result.

Corollary 3.1.4. *For every $q \geq 0$, there exists a finite constant $C(q)$ such that*

$$\sup_{\pi \in E_q} \left\{ \int_0^T \|\partial_t \rho_t\|_{-1}^2 \; dt \; + \; \int_0^T dt \int_\Omega du \, \frac{\|\nabla \rho(t, u)\|^2}{\chi(\rho(t, u))} \right\} \; \leq \; C(q) \, .$$

Next result together with the previous estimates provide the compactness needed in the proof of the lower semicontinuity of the rate function.

Lemma 3.1.5. *Let $\{\rho^n : n \geq 1\}$ be a sequence of functions in $L^2(\Omega_T)$ such that uniformly on n,*

$$\int_0^T \|\rho_t^n\|_{1,2}^2 \; dt + \int_0^T \|\partial_t \rho_t^n\|_{-1}^2 \; dt < C$$

for some positive constant C. Suppose that there exists a function $\rho \in L^2(\Omega_T)$ such that ρ^n converges to ρ weakly in $L^2(\Omega_T)$. Then ρ_n converges to ρ strongly in $L^2(\Omega_T)$.

Proof. Since $H^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ with compact embedding $H^1(\Omega) \to L^2(\Omega)$, from Corollary 8.4, [24], the sequence $\{\rho_n\}$ is relatively compact in $L^2\big([0, T], L^2(\Omega)\big)$. Therefore the weak convergence implies the strong convergence in $L^2\big([0, T], L^2(\Omega)\big)$. □

Next result is a straightforward consequence of Corollary 3.1.4 and Lemma 3.1.5.

Lemma 3.1.6. *Let $\{\pi^n(t, du) = \rho^n(t, u)du : n \geq 1\}$ be a sequence of trajectories in $D([0, T], \mathcal{M}^0)$ such that, for some positive constant C,*

$$\sup_{n \geq 1} \{I_T(\pi^n|\gamma)\} \; \leq \; C.$$

If ρ^n converges to ρ weakly in $L^2(\Omega_T)$ then ρ^n converges to ρ strongly in $L^2(\Omega_T)$.

Theorem 3.1.7. *The functional $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets.*

Proof. We have to show that, for all $q \geq 0$, E_q is compact in $D([0,T],\mathcal{M})$. Since $E_q \subset C([0,T],\mathcal{M}^0)$ and $C([0,T],\mathcal{M}^0)$ is a closed subset of $D([0,T],\mathcal{M})$, we just need to show that E_q is compact in $C([0,T],\mathcal{M}^0)$.

We will show first that E_q is closed in $C([0,T],\mathcal{M}^0)$. Fix $q \in \mathbb{R}$ and let $\{\pi^n(t,du) = \rho^n(t,u)du : n \geq 1\}$ be a sequence in E_q converging to some $\pi(t,du) = \rho(t,u)du$ in $C([0,T],\mathcal{M}^0)$. Then, for all $G \in \mathcal{C}(\overline{\Omega_T})$,

$$\lim_{n\to\infty} \int_0^T \langle \rho_t^n, G_t \rangle \, dt = \int_0^T \langle \rho_t, G_t \rangle \, dt \, .$$

Notice that this means that $\rho^n \to \rho$ weakly in $L^2(\Omega_T)$, which together with Lemma 3.1.6 imply that $\rho^n \to \rho$ strongly in $L^2(\Omega_T)$. From this fact and the definition of J_G it is easy to see that, for all G in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\lim_{n\to\infty} J_G(\pi^n) = J_G(\pi) \, .$$

This limit, Corollary 3.1.4 and the lower semicontinuity of $\mathcal{Q}_T$ permit us to conclude that $\mathcal{Q}_T(\pi) \leq C(q)$ and that $I_T(\pi|\gamma) \leq q$.

We prove now that E_q is relatively compact. To this end, it is enough to prove that for every continuous function $G : \overline{\Omega} \to \mathbb{R}$,

$$\lim_{\delta\to 0} \sup_{\pi \in E_q} \sup_{\substack{0\leq s,r\leq T \\ |r-s|<\delta}} |\langle \pi_r, G\rangle - \langle \pi_s, G\rangle| = 0 \, . \tag{3.1.13}$$

Since $E_q \subset C([0,T],\mathcal{M}^0)$, we may assume by approximations of G in $L^1(\Omega)$ that $G \in \mathcal{C}_c^\infty(\Omega)$. In which case, (3.1.13) follows from (3.1.3). □

We conclude this section with an explicit formula for the rate function $I_T(\cdot|\gamma)$. For each $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}^0)$, denote by $H_0^1(\sigma(\rho))$ the Hilbert space induced by $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$ endowed with the inner product $\langle \cdot,\cdot\rangle_{\sigma(\rho)}$ defined by

$$\langle H, G\rangle_{\sigma(\rho)} = \int_0^T dt \, \langle \sigma(\rho_t), \nabla H_t \cdot \nabla G_t \rangle \, .$$

Induced means that we first declare two functions F, G in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$ to be equivalent if $\langle F-G, F-G\rangle_{\sigma(\rho)} = 0$ and then we complete the quotient space with respect to the inner product $\langle \cdot,\cdot\rangle_{\sigma(\rho)}$. The norm of $H_0^1(\sigma(\rho))$ is denoted by $\|\cdot\|_{\sigma(\rho)}$.

Fix a path ρ in $D([0,T],\mathcal{M}^0)$ and a function H in $H_0^1(\sigma(\rho))$. A measurable function $\lambda : \overline{\Omega_T} \to [0,1]$ is said to be a weak solution of the nonlinear boundary value parabolic equation

$$\begin{cases} \partial_t \lambda & = \quad \Delta\varphi(\lambda) - \sum_{i=1}^d \partial_{u_i} \left(\sigma(\lambda)\partial_{u_i} H\right) \, , \\ \lambda(0,\cdot) & = \quad \gamma \, , \\ \lambda(t,\cdot)|_\Gamma & = \quad b \quad \text{for } \; 0 \leq t \leq T \, . \end{cases} \tag{3.1.14}$$

if it satisfies the following two conditions.

(H1') λ belongs to $L^2\big([0,T], H^1(\Omega)\big)$:

$$\int_0^T ds \left(\int_\Omega \| \nabla\lambda(s,u) \|^2 du\right) < \infty \,;$$

(H2') For every function $G(t,u) = G_t(u)$ in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\begin{aligned}
&\int_\Omega \big\{G_T(u)\rho(T,u) - G_0(u)\gamma(u)\big\}\, du - \int_0^T ds \int_\Omega du\, (\partial_s G_s)(u)\lambda(s,u) \\
&= \int_0^T \! ds \int_\Omega du\, (\Delta G_s)(u)\varphi\big(\lambda(s,u)\big) - \int_0^T \! ds \int_\Gamma \varphi(b(u))\mathbf{n}_1(u)(\partial_{u_1} G_s(u))\mathrm{dS} \\
&\quad + \int_0^T ds \int_\Omega du\, \sigma(\lambda(s,u))\nabla H_s(u)\cdot\nabla G_s(u)\,.
\end{aligned}$$

In Section 5.1 we prove uniqueness of weak solutions of equation (3.1.14) when $\|\nabla H\|$ belongs to $L^2(\Omega_T)$, i.e., provided

$$\int_0^T dt \int_\Omega du\, \|\nabla H_t(u)\|^2 < \infty\,.$$

Lemma 3.1.8. *Assume that $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}^0)$ has finite rate function: $I_T(\pi|\gamma) < \infty$. Then, there exists a function H in $H_0^1(\sigma(\rho))$ such that ρ is a weak solution to* (3.1.14). *Moreover,*

$$I_T(\pi|\gamma) = \frac{1}{2}\|H\|^2_{\sigma(\rho)}\,. \tag{3.1.15}$$

The proof of this lemma is similar to the one of Lemma 5.3 in [16] and is therefore omitted.

3.2 $I_T(\cdot|\gamma)$-Density

The main result of this section, stated in Theorem 3.2.3, asserts that any trajectory λ_t, $0 \le t \le T$, with finite rate function, $I_T(\lambda|\gamma) < \infty$, can be approximated by a sequence of smooth trajectories $\{\lambda^n : n \ge 1\}$ such that

$$\lambda^n \longrightarrow \lambda \quad \text{and} \quad I_T(\lambda^n|\gamma) \longrightarrow I_T(\lambda|\gamma)\,.$$

This is one of the main steps in the proof of the lower bound of the large deviations principle for the empirical measure. The proof reposes mainly on the regularizing effects of the hydrodynamic equation and is one of the main contributions of this article, since it simplifies considerably the existing methods.

A subset A of $D([0,T],\mathcal{M})$ is said to be $I_T(\cdot|\gamma)$-dense if for every π in $D([0,T],\mathcal{M})$ such that $I_T(\pi|\gamma) < \infty$, there exists a sequence $\{\pi^n : n \ge 1\}$ in A such that π^n converges to π and $I_T(\pi^n|\gamma)$ converges to $I_T(\pi|\gamma)$.

Let Π_1 be the subset of $D([0,T],\mathcal{M}^0)$ consisting of paths $\pi(t,du) = \rho(t,u)du$ whose density ρ is a weak solution of the hydrodynamic equation (2.1.1) in the time interval $[0,\delta]$ for some $\delta > 0$.

Lemma 3.2.1. *The set Π_1 is $I_T(\cdot|\gamma)$-dense.*

Proof. Fix π in $D([0,T],\mathcal{M})$ such that $I_T(\pi|\gamma) < \infty$. By Lemma 3.1.1, π belongs to $C([0,T],\mathcal{M}^0)$. For each $\delta > 0$, let ρ^δ be the path defined as

$$\rho^\delta(t,u) = \begin{cases} \lambda(t,u) & \text{if } 0 \le t \le \delta\,, \\ \lambda(2\delta - t, u) & \text{if } \delta \le t \le 2\delta\,, \\ \rho(t-2\delta, u) & \text{if } 2\delta \le t \le T\,, \end{cases}$$

where $\pi(t,du) = \rho(t,u)du$ and where λ is the weak solution of the hydrodynamic equation (2.1.1) starting at γ. It is clear that $\pi^\delta(t,du) = \rho^\delta(t,u)du$ belongs to D_γ, because so do π and λ and that $\mathcal{Q}_T(\pi^\delta) \le \mathcal{Q}_T(\pi) + 2\mathcal{Q}_T(\lambda) < \infty$. Moreover, π^δ converges to π as $\delta \downarrow 0$ because π belongs to $\mathcal{C}([0,T],\mathcal{M})$. By the lower semicontinuity of $I_T(\cdot|\gamma)$, $I_T(\pi|\gamma) \le \underline{\lim}_{\delta\to 0} I_T(\pi^\delta|\gamma)$. Then, in order to prove the lemma, it is enough to prove that $I_T(\pi|\gamma) \ge \overline{\lim}_{\delta\to 0} I_T(\pi^\delta|\gamma)$. To this end, decompose the rate function $I_T(\pi^\delta|\gamma)$ as the sum of the contributions on each time interval $[0,\delta]$, $[\delta, 2\delta]$ and $[2\delta, T]$. The first contribution vanishes because π^δ solves the hydrodynamic equation in this interval. On the time interval $[\delta, 2\delta]$, $\partial_t \rho^\delta_t = -\partial_t \lambda_{2\delta - t} = -\Delta\varphi(\lambda_{2\delta-t}) = -\Delta\varphi(\rho^\delta_t)$. In particular, the second contribution is equal to

$$\sup_{G\in\mathcal{C}_0^{1,2}(\overline{\Omega_T})} \left\{ 2\int_0^\delta ds \int_\Omega du\; \nabla\varphi(\lambda)\cdot\nabla G - \frac{1}{2}\int_0^\delta \langle \sigma(\lambda_t), \|\nabla G_t\|^2\rangle\, ds \right\}$$

which, by Schwarz inequality, is bounded above by

$$\int_0^\delta ds \int_\Omega du\; \varphi'(\lambda)\frac{\|\nabla\lambda\|^2}{\chi(\lambda)}\,.$$

By Corollary 3.1.3, this last expression converges to zero as $\delta \downarrow 0$. Finally, the third contribution is bounded by $I_T(\pi|\gamma)$ because π^δ in this interval is just a time translation of the path π. □

Let Π_2 be the set of all paths π in Π_1 with the property that for every $\delta > 0$ there exists $\epsilon > 0$ such that $\epsilon \le \pi_t(\cdot) \le 1-\epsilon$ for all $t \in [\delta, T]$.

Lemma 3.2.2. *The set Π_2 is $I_T(\cdot|\gamma)$-dense.*

Proof. By the previous lemma, it is enough to show that each path $\pi(t,du) = \rho(t,u)du$ in Π_1 can be approximated by paths in Π_2. Fix π in Π_1 and let λ be as in the proof of the previous lemma. For each $0 < \varepsilon < 1$, let $\pi^\varepsilon = (1-\varepsilon)\pi + \varepsilon\lambda$. Note that $\mathcal{Q}_T(\pi^\varepsilon) < \infty$ because $\mathcal{Q}_T$ is convex and both $\mathcal{Q}_T(\pi)$ and $\mathcal{Q}_T(\lambda)$ are finite. Hence, π^ε belongs to D_γ since both π and λ satisfy the boundary conditions. Moreover, It is clear that π^ε converges to π as $\varepsilon \downarrow 0$. By the lower semicontinuity of $I_T(\cdot|\gamma)$, in order to conclude the proof, it is enough to show that

$$\overline{\lim_{\varepsilon\to 0}}\, I_T(\pi^\varepsilon|\gamma) \le I_T(\pi|\gamma)\,. \tag{3.2.1}$$

By Lemma 3.1.8, there exists $H \in H_0^1(\sigma(\rho))$ such that ρ solves the equation (3.1.14). Let $\mathbf{P} = \sigma(\rho)\nabla H - \nabla\varphi(\rho)$ and $\mathbf{P}^\lambda = -\nabla\varphi(\lambda)$. For each $0 < \varepsilon < 1$, let $\mathbf{P}^\varepsilon = (1-\varepsilon)\mathbf{P} + \varepsilon\mathbf{P}^\lambda$. Since ρ solves the equation (3.1.14), for every $G \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\int_0^T \langle \mathbf{P}^\varepsilon_t, \nabla G_t\rangle\, dt = \langle \pi^\varepsilon_T, G_T\rangle - \langle \pi^\varepsilon_0, G_0\rangle - \int_0^T \langle \pi^\varepsilon_t, \partial_t G_t\rangle\, dt\,.$$

Hence, by (3.1.7), $I_T(\pi^\varepsilon|\gamma)$ is equal to

$$\sup_{G\in\mathcal{C}_0^{1,2}(\overline{\Omega_T})} \left\{ \int_0^T dt \int_\Omega du \left\{\mathbf{P}^\varepsilon + \nabla\varphi(\rho^\varepsilon)\right\} \cdot \nabla G \;-\; \frac{1}{2}\int_0^T dt \int_\Omega du\, \sigma(\rho^\varepsilon)\|\nabla G\|^2 \right\} .$$

This expression can be rewritten as

$$\begin{aligned} &\frac{1}{2}\int_0^T dt \int_\Omega du\, \frac{\|\mathbf{P}^\varepsilon + \nabla\varphi(\rho^\varepsilon)\|^2}{\sigma(\rho^\varepsilon)} \\ &\qquad - \frac{1}{2}\inf_G \Big\{ \int_0^T dt \int_\Omega \frac{\|\mathbf{P}^\varepsilon + \nabla\varphi(\rho^\varepsilon) - \sigma(\rho^\varepsilon)\nabla G\|^2}{\sigma(\rho^\varepsilon)}\, du \Big\} \end{aligned}$$

Hence,

$$I_T(\pi^\varepsilon|\gamma) \;\le\; \frac{1}{2}\int_0^T dt \int_\Omega \frac{\|\mathbf{P}^\varepsilon + \nabla\varphi(\rho^\varepsilon)\|^2}{\sigma(\rho^\varepsilon)}\, du \cdot$$

In view of this inequality and (3.1.15), in order to prove (3.2.1), it is enough to show that

$$\lim_{\varepsilon\to 0}\int_0^T dt \int_\Omega du\, \frac{\|\mathbf{P}^\varepsilon + \nabla\varphi(\rho^\varepsilon)\|^2}{\sigma(\rho^\varepsilon)}\, du \;=\; \int_0^T dt \int_\Omega \frac{\|\mathbf{P} + \nabla\varphi(\rho)\|^2}{\sigma(\rho)}\, du \cdot$$

By the continuity of φ', σ and from the definition of $\mathbf{P}^\varepsilon$,

$$\lim_{\varepsilon\to 0}\frac{\|\mathbf{P}^\varepsilon + \nabla\varphi(\rho^\varepsilon)\|^2}{\sigma(\rho^\varepsilon)} \;=\; \frac{\|\mathbf{P} + \nabla\varphi(\rho)\|^2}{\sigma(\rho)}$$

almost everywhere. Therefore, to prove (3.2.1), it remains to show the uniform integrability of the families

$$\left\{ \frac{\|\mathbf{P}^\varepsilon\|^2}{\chi(\rho^\varepsilon)} \;:\; \varepsilon > 0 \right\} \qquad \text{and} \qquad \left\{ \frac{\|\nabla\rho^\varepsilon\|^2}{\chi(\rho^\varepsilon)} \;:\; \varepsilon > 0 \right\} .$$

Since $I_T(\pi|\gamma) < \infty$, by (3.1.9), (3.1.15) and Corollary 3.1.3, the functions $\frac{\|\mathbf{P}\|^2}{\chi(\rho)}$, $\frac{\|\mathbf{P}_\lambda\|^2}{\chi(\lambda)}$, $\frac{\|\nabla\rho\|^2}{\chi(\rho)}$ and $\frac{\|\nabla\lambda\|^2}{\chi(\lambda)}$ belong to $L^1(\Omega_T)$. In particular, the function

$$g = \max\left\{ \frac{\|\mathbf{P}\|^2}{\chi(\rho)}, \frac{\|\mathbf{P}_\lambda\|^2}{\chi(\lambda)}, \frac{\|\nabla\rho\|^2}{\chi(\rho)}, \frac{\|\nabla\lambda\|^2}{\chi(\lambda)} \right\} ,$$

also belongs to $L^1(\Omega_T)$. By the convexity of $\|\cdot\|^2$ an the concavity of $\chi(\cdot)$,

$$\frac{\|\mathbf{P}^\varepsilon\|^2}{\chi(\rho^\varepsilon)} \le \frac{(1-\varepsilon)\|\mathbf{P}\|^2 + \varepsilon\|\mathbf{P}_\lambda\|^2}{(1-\varepsilon)\chi(\rho) + \varepsilon\chi(\lambda)} \le g\,,$$

which proves the uniform integrability of the family $\frac{\|\mathbf{P}^\varepsilon\|^2}{\chi(\rho^\varepsilon)}$. The uniform integrability of the family $\frac{\|\nabla\rho_\varepsilon\|^2}{\chi(\rho_\varepsilon)}$ follows from the same estimate with $\nabla\rho_\varepsilon$, $\nabla\rho$ and $\nabla\lambda$ in the place of $\mathbf{P}_\varepsilon$, $\mathbf{P}$ and $\mathbf{P}_\lambda$, respectively. □

Let Π be the subset of Π_2 consisting of all those paths π which are solutions of the equation (3.1.14) for some $H \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$.

Theorem 3.2.3. *The set* Π *is* $I_T(\cdot|\gamma)$*-dense.*

Proof. By the previous lemma, it is enough to show that each path π in Π_2 can be approximated by paths in Π. Fix $\pi(t,du) = \rho(t,u)du$ in Π_2. By Lemma 3.1.8, there exists $H \in H^1_0(\sigma(\rho))$ such that ρ solves the equation (3.1.14). Since π belongs to $\Pi_2 \subset \Pi_1$, ρ is the weak solution of (2.1.1) in some time interval $[0, 2\delta]$ for some $\delta > 0$. In particular, $\nabla H = 0$ a.e in $[0, 2\delta] \times \Omega$. On the other hand, since π belongs to Π_1, there exists $\epsilon > 0$ such that $\epsilon \le \pi_t(\cdot) \le 1-\epsilon$ for $\delta \le t \le T$. Therefore,

$$\int_0^T dt \int_\Omega du \, \|\nabla H_t(u)\|^2 \; < \; \infty \, . \tag{3.2.2}$$

Since H belongs to $H^1_0(\sigma(\rho))$, there exists a sequence of functions $\{H^n : n \ge 1\}$ in $\mathcal{C}^{1,2}_0(\overline{\Omega_T})$ converging to H in $H^1_0(\sigma(\rho))$. We may assume of course that $\nabla H^n_t \equiv 0$ in the time interval $[0, \delta]$. In particular,

$$\lim_{n\to\infty} \int_0^T dt \int_\Omega du \, \|\nabla H^n_t(u) - \nabla H_t(u)\|^2 = 0 \, . \tag{3.2.3}$$

For each integer $n > 0$, let ρ^n be the weak solution of (3.1.14) with H^n in place of H and set $\pi^n(t,du) = \rho^n(t,u)du$. By (3.1.15) and since σ is bounded above in $[0,1]$ by a finite constant,

$$I_T(\pi^n|\gamma) = \frac{1}{2}\int_0^T \langle \sigma(\rho^n_t), \|\nabla H^n_t\|^2\rangle \, dt \le C_0 \int_0^T dt \int_\Omega du \, \|\nabla H^n_t(u)\|^2 \, .$$

In particular, by (3.2.2) and (3.2.3), $I_T(\pi^n|\gamma)$ is uniformly bounded on n. Thus, by Theorem 3.1.7, the sequence π^n is relatively compact in $D([0,T], \mathcal{M})$.

Let $\{\pi^{n_k} : k \ge 1\}$ be a subsequence of π^n converging to some π^0 in $D([0,T], \mathcal{M}^0)$. For every G in $\mathcal{C}^{1,2}_0(\overline{\Omega_T})$,

$$\begin{aligned}\langle \pi^{n_k}_T, G_T\rangle - \langle \gamma, G_0\rangle - \int_0^T \langle \pi^{n_k}_t, \partial_t G_t\rangle \, dt = \int_0^T \langle \varphi(\rho^{n_k}_t), \Delta G_t\rangle \, dt \\ - \int_0^T dt \int_\Gamma \varphi(b)\mathbf{n_1}(\partial_{u_1} G) dS - \int_0^T \langle \sigma(\rho^n_t), \nabla H^{n_k}_t \cdot \nabla G_t\rangle \, dt \, .\end{aligned}$$

Letting $k \to \infty$ in this equation, we obtain the same equation with π^0 and H in place of π^{n_k} and H^{n_k}, respectively, if

$$\begin{gathered}\lim_{k\to\infty} \int_0^T dt \, \langle \varphi(\rho^{n_k}_t), \Delta G_t\rangle \; = \; \int_0^T dt \, \langle \varphi(\rho^0_t), \Delta G_t\rangle \, , \\ \lim_{k\to\infty} \int_0^T dt \, \langle \sigma(\rho^{n_k}_t), \nabla H^{n_k}_t \cdot \nabla G_t\rangle = \int_0^T dt \, \langle \sigma(\rho^0_t), \nabla H_t \cdot \nabla G_t\rangle \, .\end{gathered} \tag{3.2.4}$$

We prove the second claim, the first one being simpler. Note first that we can replace H^{n_k} by H in the previous limit, because σ is bounded in $[0,1]$ by some positive constant and (3.2.3) holds. Now, ρ^{n_k} converges to ρ^0 weakly in $L^2(\Omega_T)$ because π^{n_k} converges to π^0 in $D([0,T], \mathcal{M}^0)$. Since $I_T(\pi^n|\gamma)$ is uniformly bounded, by Lemma 3.1.6, ρ^{n_k} converges to ρ^0 strongly in $L^2(\Omega_T)$ which implies (3.2.4). In particular, since (3.2.2) holds, by uniqueness of weak solutions of equation (3.1.14), $\pi^0 = \pi$ and we are done. □

3.3 Large Deviations

We prove in this section the dynamical large deviation principle for the empirical measure of boundary driven symmetric exclusion processes in dimension $d \geq 1$. The proof relies on the results presented in the previous sections and is quite similar to the original one presented in [18, 11]. There are just three additional difficulties. On the one hand, the lack of explicitly known stationary states hinders the derivation of the usual estimates of the entropy and the Dirichlet form, so important in the proof of the hydrodynamic behaviour. On the other hand, due to the definition of the rate function, we have to show that trajectories with infinite energy can be neglected in the large deviations regime. Finally, since we are working with the empirical measure, instead of the empirical density, we need to show that trajectories which are not absolutely continuous with respect to the Lebesgue measure and whose density is not bounded by one can also be neglected. The first two problems have already been faced and solved. The first one in [20, 4] and the second in [22, 6]. The approach here is quite similar, we thus only sketch the main steps in sake of completeness.

3.3.1 Superexponential estimates

It is well known that one of the main steps in the derivation of the upper bound is a super-exponential estimate which allows the replacement of local functions by functionals of the empirical density in the large deviations regime. Essentially, the problem consists in bounding expressions such as $\langle V, f^2\rangle_{\mu_{ss}^N}$ in terms of the Dirichlet form $\langle -N^2\mathcal{L}_N f, f\rangle_{\mu_{ss}^N}$. Here V is a local function and $\langle \cdot,\cdot\rangle_{\mu_{ss}^N}$ indicates the inner product with respect to the invariant state μ_{ss}^N. In our context, the fact that the invariant state is not known explicitly introduces a technical difficulty.

Let β be as in the beginning of Section 1.1. Following [20], [4], we use $\nu_{\beta(\cdot)}^N$ as reference measure and estimate everything with respect to $\nu_{\beta(\cdot)}^N$. However, since $\nu_{\beta(\cdot)}^N$ is not the invariant state, there are no reasons for $\langle -N^2\mathcal{L}_N f, f\rangle_{\nu_{\beta(\cdot)}^N}$ to be positive. The next statement shows that this expression is almost positive.

We may suppose that β satisfies (2.2.3), in which case, for every N large enough, $\nu_{\beta(\cdot)}^N$ is reversible for the process with generator $\mathcal{L}_{N,b}$ and then $\langle -N^2\mathcal{L}_{N,b} f, f\rangle_{\nu_{\beta(\cdot)}^N}$ is positive.

Recall from Section 2.2 the definition of $D_0^N(\cdot,\cdot)$ and recall also that we denote by $D_\beta^N(\cdot)$ the Dirichlet form of the generator $\mathcal{L}_{N,b}$ with respect to its reversible probability measure $\nu_{\beta(\cdot)}^N$.

Lemma 3.3.1. *There exists a constant C depending only on β such that if f is density with respect to $\nu_{\beta(\cdot)}^N$ and $g = f\frac{d\nu_{\beta(\cdot)}^N}{d\nu_\alpha^N}$ then*

$$\big\langle \mathcal{L}_N \sqrt{f}, \sqrt{f}\big\rangle_{\nu_{\beta(\cdot)}^N} \leq -\frac{1}{4} D_0^N(g, \nu_\alpha^N) - D_\beta^N(f) + CN^{d-2}\,.$$

Proof. By Lemma (2.2.3), it is enough to show that there is a constant $C_1 = C_1(\beta) > 0$ such that

$$D_{x,y}^N(f, \nu_{\beta(\cdot)}^N) \leq \frac{1}{2} D_{x,y}^N(g, \nu_\alpha^N) - C_1 N^{-2}\,, \tag{3.3.1}$$

for any $x, y = x + e_i \in \Omega_N$.

Fix then x, $x + e_i$ in Ω_N. By the inequality $2ab \le Aa^2 + A^{-1}b^2$,

$$\begin{aligned}
D^N_{x,y}(f, \nu^N_\gamma) &= \frac{1}{2}\int r_{x,y}(\eta)\Big[\sqrt{g(\eta^{x,y})}\Big(\sqrt{\frac{\nu^N_\gamma(\eta)}{\nu^N_\gamma(\eta^{x,y})}} - 1\Big) \\
&\qquad\qquad + \sqrt{g(\eta^{x,y})} - \sqrt{g(\eta)}\Big]^2 \nu^N_\alpha(d\eta) \\
&\ge \frac{1}{4}\int r_{x,y}(\eta)\big(\sqrt{g(\eta^{x,y})} - \sqrt{g(\eta)}\big)^2 \nu^N_\alpha(d\eta) \\
&\quad -\frac{1}{2}\int r_{x,y}(\eta) g(\eta^{x,y})\Big[\sqrt{\frac{\nu^N_\gamma(\eta)}{\nu^N_\gamma(\eta^{x,y})}} - 1\Big]^2 \nu^N_\alpha(d\eta)\,.
\end{aligned}$$

Hence, (3.3.1) follows from (2.2.11).

□

This lemma together with the computation presented in [2], p. 78, for non-reversible processes, permits to prove the super-exponential estimate. Recall from Section 2.2 that, for a cylinder function Ψ we denote the expectation of Ψ with respect to the Bernoulli product measure ν^N_α by $\widetilde{\Psi}(\alpha)$:

$$\widetilde{\Psi}(\alpha) = E^{\nu^N_\alpha}[\Psi]\,.$$

recall also that, for a positive integer l and $x \in \Omega_N$, we denote the empirical mean density on a box of size $2l+1$ centered at x by $\eta^l(x)$:

$$\eta^l(x) = \frac{1}{|\Lambda_l(x)|}\sum_{y\in\Lambda_l(x)} \eta(y)\,,$$

where

$$\Lambda_l(x) = \Lambda_{N,l}(x) = \{y \in \Omega_N : |y - x| \le l\}\,.$$

For each $G \in \mathcal{C}(\overline{\Omega_T})$, each cylinder function Ψ and each $\varepsilon > 0$, let

$$V^{G,\Psi}_{N,\varepsilon}(s,\eta) = \frac{1}{N^d}\sum_x G(s, x/N)\left[\tau_x\Psi(\eta) - \widetilde{\Psi}(\eta^{\varepsilon N}(x))\right]\,,$$

where the sum is carried over all x such that the support of $\tau_x\Psi$ belongs to Ω_N.

For a continuous function $H : [0,T]\times\Gamma \to \mathbb{R}$, let

$$V^{\pm}_{N,H} = \int_0^T ds\, \frac{1}{N^{d-1}}\sum_{x\in\Gamma^\pm_N} V^\pm(x,\eta_s) H\left(s, \frac{x\pm e_1}{N}\right)\,,$$

where

$$V^\pm(x,\eta) = \left[\eta(x) + b\left(\frac{x\pm e_1}{N}\right)\right]\left[\eta(x\mp e_1) - b\left(\frac{x\pm e_1}{N}\right)\right]\,.$$

Proposition 3.3.2. *Let $G : \overline{\Omega_T} \to \mathbb{R}$ and $H : [0,T]\times\Gamma \to \mathbb{R}$ be continuous functions. Fix a cylinder function Ψ and a sequence $\{\eta^N : N \ge 1\}$ of configurations with η^N in X_N. For every $\delta > 0$,*

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{N\to\infty}\frac{1}{N^d}\log\mathbb{P}_{\eta^N}\left[\left|\int_0^T V^{G,\Psi}_{N,\varepsilon}(s,\eta_s)ds\right| > \delta\right] = -\infty\,, \qquad (3.3.2)$$

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbb{P}_{\eta^N} \Big[|V^{\pm}_{N,H}| > \delta \Big] = -\infty \, . \tag{3.3.3}$$

Proof. Recall that, for some constant $C = C(\beta) > 0$,

$$\frac{d\delta\eta^N}{d\nu^N_{\beta(\cdot)}} \le C^{2N^d} \, . \tag{3.3.4}$$

Hence, we just need to show the Proposition with $\mathbb{P}_{\nu^N_{\beta(\cdot)}}$ in the place of $\mathbb{P}_{\eta^N}$.

The proof of (3.3.2) is almost the same as the one of Theorem 10.3.1 in [16]. It follows from (2.2.12), we just need in addition Lemma 3.3.1 for estimates on $\big\langle -N^2 \mathcal{L}_N \sqrt{f}, \sqrt{f} \big\rangle_{\nu^N_{\beta(\cdot)}}$.

We turn now to the proof of (3.3.3). By the exponential Chevychev inequality, for every $A > 0$, the left han side of (3.3.3) is bounded above by

$$-A\delta + \overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbb{E}_{\nu^N_{\beta(\cdot)}} [\exp\{AN^d |V^{\pm}_{N,H}|\}] \, ,$$

which, by (2.2.13), is bounded by $-A\delta + C_1$ for some positive constant $C_1 = C_1(\beta) > 0$. Hence, (3.3.3) follows from the arbitrariness of A.

□

For each $\varepsilon > 0$ and π in $\mathcal{M}$, denote by $\Xi_\varepsilon(\pi) = \pi^\varepsilon$ the absolutely continuous measure obtained by smoothing the measure π:

$$\Xi_\varepsilon(\pi)(dx) \;=\; \pi^\varepsilon(dx) \;=\; \frac{1}{U_\varepsilon} \frac{\pi(\mathbf{\Lambda}_\varepsilon(x))}{|\mathbf{\Lambda}_\varepsilon(x)|} \, dx \, ,$$

where $\mathbf{\Lambda}_\varepsilon(x) = \{y \in \Omega : |y - x| \le \varepsilon\}$, $|A|$ stands for the Lebesgue measure of the set A, and $\{U_\varepsilon : \varepsilon > 0\}$ is a strictly decreasing sequence converging to 1: $U_\varepsilon > 1$, $U_\varepsilon > U_{\varepsilon'}$ for $\varepsilon > \varepsilon'$, $\lim_{\varepsilon\downarrow 0} U_\varepsilon = 1$. Let

$$\pi^{N,\varepsilon} \;=\; \Xi_\varepsilon(\pi^N) \, .$$

A simple computation shows that $\pi^{N,\varepsilon}$ belongs to $\mathcal{M}^0$ for N sufficiently large because $U_\varepsilon > 1$, and that for each continuous function $H : \Omega \to \mathbb{R}$,

$$\langle \pi^{N,\varepsilon}, H \rangle \;=\; \frac{1}{N^d} \sum_{x\in\Omega_N} H(x/N) \eta^{\varepsilon N}(x) \;+\; O(N,\varepsilon) \, ,$$

where $O(N,\varepsilon)$ is absolutely bounded by $C_0\{N^{-1} + \varepsilon\}$ for some finite constant C_0 depending only on H.

For each H in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$ consider the exponential martingale M^H_t defined by

$$\begin{aligned} M^H_t \;\; = \;\; \exp\Big\{ N^d \Big[& \big\langle \pi^N_t, H_t \big\rangle - \big\langle \pi^N_0, H_0 \big\rangle \\ & - \frac{1}{N^d} \int_0^t e^{-N^d \langle \pi^N_s, H_s \rangle} \big(\partial_s + N^2 \mathcal{L}_N \big) \, e^{N^d \langle \pi^N_s, H_s \rangle} \, ds \Big] \Big\} \, . \end{aligned}$$

Recall from Section 1.3 the definition of the functional $\hat{J}_H$. An elementary computation shows that

$$M^H_T = \exp\Big\{ N^d \Big[\hat{J}_H(\pi^{N,\varepsilon}) + \mathbb{V}^H_{N,\varepsilon} + c^1_H(\varepsilon) + c^2_H(N^{-1}) \Big] \Big\} \, . \tag{3.3.5}$$

In this formula,

$$
\begin{aligned}
\mathbb{V}^H_{N,\varepsilon} \;&=\; -\sum_{i=1}^d \int_0^T V^{\partial^2_{u_i}H,h_i}_{N,\varepsilon}(s,\eta_s)\,ds - \frac{1}{2}\sum_{i=1}^d \int_0^T V^{(\partial_{u_i}H)^2,g_i}_{N,\varepsilon}(s,\eta_s)\,ds\\
&\quad +\; a\,V^+_{N,\partial_{u_1}H} \;-\; a\,V^-_{N,\partial_{u_1}H} \;+\; \langle \pi^N_0, H_0\rangle - \langle \gamma, H_0\rangle\,;
\end{aligned}
$$

the cylinder functions h_i, g_i are given by

$$
\begin{aligned}
h_i(\eta) \;&=\; \eta(0) \;+\; a\Big\{\eta(0)[\eta(-e_i)+\eta(e_i)] - \eta(-e_i)\eta(e_i)\Big\}\,,\\
g_i(\eta) \;&=\; r_{0,e_i}(\eta)\,[\eta(e_i)-\eta(0)]^2\;;
\end{aligned}
$$

and $c^j_H : \mathbb{R}_+ \to \mathbb{R}$, $j = 1, 2$, are functions depending only on H such that $c^j_H(\delta)$ converges to 0 as $\delta \downarrow 0$. In particular, the martingale M^H_T is bounded by $\exp\big\{C(H,T)N^d\big\}$ for some finite constant $C(H,T)$ depending only on H and T. Therefore, Proposition 3.3.2 holds for $\mathbb{P}^H_{\eta^N} = \mathbb{P}_{\eta^N} M^H_T$ in place of $\mathbb{P}_{\eta^N}$.

3.3.2 Energy estimates

To exclude paths with infinite energy in the large deviations regime, we need an energy estimate.

Fix a constant C_0 satisfying the statement of Lemma 2.2.4. Recall from Section 2.2 the definition of the functional $\mathcal{Q}^{G,i,C_0}_T$ and that

$$
\sup_{G\in\mathcal{C}^\infty_c(\Omega_T)} \Big\{\mathcal{Q}^{G,i,C_0}_T(\pi)\Big\} = \frac{\mathcal{Q}^i_T(\pi)}{4C_0}\,. \tag{3.3.6}
$$

Fix a sequence $\{G_k : k \geq 1\}$ of smooth functions dense in $L^2([0,T], H^1(\Omega))$. For any positive integers r, l, let

$$
B_{r,l} = \Big\{\pi \in D([0,T],\mathcal{M}) : \max_{\substack{1\le k\le r\\ 1\le i\le d}} \mathcal{Q}^{G_k,i,C_0}_T(\pi) \le l\Big\}\,.
$$

Since, for fixed G in $\mathcal{C}^\infty_c(\Omega_T)$ and $1 \le i \le d$ integer, the function $\mathcal{Q}^{G_k,i,C_0}_T$ is continuous, $B_{r,l}$ is a closed subset of $D([0,T],\mathcal{M})$.

Lemma 3.3.3. *There is a positive constant $\widetilde{C}$ such that, for any positive integers r, l,*

$$
\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbf{Q}_{\eta^N}\,[{B_{r,l}}^c] \le -l + \widetilde{C}\,.
$$

Proof. Let $C > 0$ be a constant satisfying (3.3.4). For integers $1 \le k \le r$ and $1 \le i \le d$, by the exponential Chevychev inequality and Lemma 2.2.4,

$$
\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbb{P}_{\eta^N}\Big[\mathcal{Q}^{G_k,i,C_0}_T > l\Big] \le -l + C_0 + C\,.
$$

Therefore, since

$$
\overline{\lim_{N\to\infty}} \frac{1}{N^d}\log(a_N+b_N) \le \max\Big\{\overline{\lim_{N\to\infty}}\frac{1}{N^d}\log a_N\,,\ \overline{\lim_{N\to\infty}}\frac{1}{N^d}\log b_N\Big\}\,, \tag{3.3.7}
$$

the desired inequality is obtained with $\widetilde{C} = C_0 + C$.

□

3.3.3 Upper bound

Fix a sequence $\{F_k : k \geq 1\}$ of smooth nonnegative functions dense in $\mathcal{C}(\overline{\Omega})$ for the uniform topology. For $k \geq 1$ and $\delta > 0$, let

$$D_{k,\delta} = \Big\{ \pi \in D([0,T], \mathcal{M}) : \, 0 \leq \langle \pi_t, F_k \rangle \leq \int_\Omega F_k(x)\, dx \, + \, C_k \delta \ , \, 0 \leq t \leq T \Big\} \, ,$$

where $C_k = \|\nabla F_k\|_\infty$ and ∇F is the gradient of F. Clearly, the set $D_{k,\delta}$, $k \geq 1$, $\delta > 0$, is a closed subset of $D([0,T], \mathcal{M})$. Moreover, if

$$E_{m,\delta} \;=\; \bigcap_{k=1}^{m} D_{k,\delta} \ ,$$

we have that $D([0,T], \mathcal{M}^0) = \cap_{n\geq 1} \cap_{m \geq 1} E_{m,1/n}$. Note, finally, that for all $m \geq 1$, $\delta > 0$,

$$\pi^{N,\varepsilon} \text{ belongs to } E_{m,\delta} \text{ for } N \text{ sufficiently large.} \tag{3.3.8}$$

Fix a sequence of configurations $\{\eta^N : N \geq 1\}$ with η^N in X_N and such that $\pi^N(\eta^N)$ converges to $\gamma(u)du$ in $\mathcal{M}$. Let A be a subset of $D([0,T], \mathcal{M})$,

$$\frac{1}{N^d} \log \mathbb{P}_{\eta^N} \left[\pi^N \in A \right] = \frac{1}{N^d} \log \mathbb{E}_{\eta^N} \left[M_T^H \, (M_T^H)^{-1} \, \mathbf{1}\{\pi^N \in A\} \right] \, .$$

Maximizing over π^N in A, we get from (3.3.5) that the last term is bounded above by

$$- \inf_{\pi \in A} \hat{J}_H(\pi^\varepsilon) + \frac{1}{N^d} \log \mathbb{E}_{\eta^N} \left[M_T^H \, e^{-N^d \mathbb{V}^H_{N,\varepsilon}} \right] - c^1_H(\varepsilon) - c^2_H(N^{-1}) \, .$$

Since $\pi^N(\eta^N)$ converges to $\gamma(u)du$ in $\mathcal{M}$ and since Proposition 3.3.2 holds for $\mathbb{P}^H_{\eta^N} = \mathbb{P}_{\eta^N} M_T^H$ in place of $\mathbb{P}_{\eta^N}$, the second term of the previous expression is bounded above by some $C_H(\varepsilon, N)$ such that

$$\overline{\lim_{\varepsilon \to 0}} \, \overline{\lim_{N \to \infty}} \, C_H(\varepsilon, N) = 0 \, .$$

Hence, for every $\varepsilon > 0$, and every H in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\overline{\lim_{N \to \infty}} \, \frac{1}{N^d} \log \mathbb{P}_{\eta^N}[A] \leq - \inf_{\pi \in A} \hat{J}_H(\pi^\varepsilon) + C'_H(\varepsilon) \, , \tag{3.3.9}$$

where $\lim\limits_{\varepsilon \to 0} C'_H(\varepsilon) = 0$.

For each $H \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$, each $\varepsilon > 0$ and any $r, l, m, n \in \mathbb{Z}_+$, let $J^{r,l,m,n}_{H,\varepsilon} : D([0,T], \mathcal{M}) \to \mathbb{R} \cup \{\infty\}$ be the functional given by

$$J^{r,l,m,n}_{H,\varepsilon}(\pi) = \begin{cases} \hat{J}_H(\pi^\varepsilon) & \text{if } \pi \in B_{r,l} \cap E_{m,1/n} \, , \\ +\infty & \text{otherwise} \, . \end{cases}$$

This functional is lower semicontinuous because so is $\hat{J}_H \circ \Xi_\varepsilon$ and because $B_{r,l}$, $E_{m,1/n}$ are closed subsets of $D([0,T], \mathcal{M})$.

Let $\mathcal{O}$ be an open subset of $D([0,T],\mathcal{M})$. By Lemma 3.3.3, (3.3.7), (3.3.8) and (3.3.9),

$$\begin{aligned}
\varlimsup_{N\to\infty} \frac{1}{N^d} \log \mathbf{Q}_{\eta^N}[\mathcal{O}] \;&\le\; \max\Big\{ \varlimsup_{N\to\infty} \frac{1}{N^d} \log \mathbf{Q}_{\eta^N}[\mathcal{O}\cap B_{r,l}\cap E_{m,1/n}]\,, \\
&\qquad\qquad \varlimsup_{N\to\infty} \frac{1}{N^d} \log \mathbf{Q}_{\eta^N}[(B_{r,l})^c]\Big\} \\
&\le\; \max\Big\{ -\inf_{\pi\in\mathcal{O}\cap B_{r,l}\cap E_{m,1/n}} \hat{J}_H(\pi^\varepsilon) + C'_H(\varepsilon)\,,\, -l+\widetilde{C}\Big\} \\
&=\; -\inf_{\pi\in\mathcal{O}} L^{r,l,m,n}_{H,\varepsilon}(\pi)\,,
\end{aligned}$$

where

$$L^{r,l,m,n}_{H,\varepsilon}(\pi) = \min\Big\{ J^{r,l,m,n}_{H,\varepsilon}(\pi) - C'_H(\varepsilon)\,,\, l-\widetilde{C}\Big\}\;.$$

In particular,

$$\varlimsup_{N\to\infty} \frac{1}{N^d} \log \mathbf{Q}_{\eta^N}[\mathcal{O}] \le -\sup_{H,\varepsilon,r,l,m,n}\;\inf_{\pi\in\mathcal{O}} L^{r,l,m,n}_{H,\varepsilon}(\pi)\,.$$

Note that, for each $H\in\mathcal{C}^{1,2}_0(\overline{\Omega_T})$, each $\varepsilon>0$ and $r,l,m,n\in\mathbb{Z}_+$, the functional $L^{r,l,m,n}_{H,\varepsilon}$ is lower semicontinuous. Then, by Lemma A2.3.3 in [16], for each compact subset $\mathcal{K}$ of $D([0,T],\mathcal{M})$,

$$\varlimsup_{N\to\infty} \frac{1}{N^d} \log \mathbf{Q}_{\eta^N}[\mathcal{K}] \;\le\; -\inf_{\pi\in\mathcal{K}}\;\sup_{H,\varepsilon,r,l,m,n} L^{r,l,m,n}_{H,\varepsilon}(\pi)\,.$$

By (3.3.6) and since $D([0,T],\mathcal{M}^0)=\cap_{n\ge1}\cap_{m\ge1}E_{m,1/n}$,

$$\varlimsup_{\varepsilon\to0}\varlimsup_{l\to\infty}\varlimsup_{r\to\infty}\varlimsup_{m\to\infty}\varlimsup_{n\to\infty} L^{r,l,m,n}_{H,\varepsilon}(\pi) \;=\; \begin{cases} \hat{J}_H(\pi) & \text{if } \mathcal{Q}_T(\pi)<\infty \text{ and } \pi\in D([0,T],\mathcal{M}^0)\,,\\ +\infty & \text{otherwise}\,. \end{cases}$$

This result and the last inequality imply the upper bound for compact sets because $\hat{J}_H$ and J_H coincide on $D([0,T],\mathcal{M}^0)$. To pass from compact sets to closed sets, we have to obtain exponential tightness for the sequence $\{\mathbf{Q}_{\eta^N}\}$. This means that there exists a sequence of compact sets $\{\mathcal{K}_n : n\ge1\}$ in $D([0,T],\mathcal{M})$ such that

$$\varlimsup_{N\to\infty} \frac{1}{N^d} \log \mathbf{Q}_{\eta^N}({\mathcal{K}_n}^c) \le -n\,.$$

The proof presented in [1] for the non interacting zero range process is easily adapted to our context.

3.3.4 Lower bound

The proof of the lower bound is similar to the one in the convex periodic case. We just sketch it and refer to [16], section 10.5. Fix a path π in Π and let $H\in\mathcal{C}^{1,2}_0(\overline{\Omega_T})$ be such that π is the weak solution of equation (3.1.14). Recall from the previous section the definition of the martingale M^H_t and denote by $\mathbb{P}^H_{\eta^N}$

the probability measure on $D([0,T],X_N)$ given by $\mathbb{P}^H_{\eta^N}[A] = \mathbb{E}_{\eta^N}[M^H_T \mathbf{1}\{A\}]$. Under $\mathbb{P}^H_{\eta^N}$ and for each $0 \le t \le T$, the empirical measure π^N_t converges in probability to π_t. Further,

$$\lim_{N\to\infty} \frac{1}{N^d} H\left(\mathbb{P}^H_{\eta^N} \middle| \mathbb{P}_{\eta^N}\right) = I_T(\pi|\gamma)\,,$$

where $H(\mu|\nu)$ stands for the relative entropy of μ with respect to ν. From these two results we can obtain that for every open set $\mathcal{O} \subset D([0,T],\mathcal{M})$ which contains π,

$$\varliminf_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^N}\left[\mathcal{O}\right] \ge -I_T(\pi|\gamma)\,.$$

The lower bound follows from this and the $I_T(\cdot|\gamma)$-density of Π established in Theorem 3.2.3.

Chapter 4

Statical Large Deviations

We prove here that the quasi potential is the large deviation functional of the stationary measure. Throughout this chapter, we denote by $\bar{\rho}$ the weak solution of (1.2.1), and by $\overline{\vartheta}$ the measure in $\mathcal{M}^0$ with density $\bar{\rho}$, i.e., $\overline{\vartheta}(du) = \bar{\rho}(u)du$.

4.1 The Functional I_T

Fix $T > 0$. For each $G \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$, let $\check{J}_G = \check{J}_{G,T} : D([0,T], \mathcal{M}^0) \to \mathbb{R}$ be the functional given by

$$\begin{aligned}\check{J}_G(\pi) \quad &= \quad \langle \pi_T, G_T \rangle - \langle \pi_0, G_0 \rangle - \int_0^T \langle \pi_t, \partial_t G_t \rangle \, dt \\ &\quad - \int_0^T \langle \varphi(\rho_t), \Delta G_t \rangle \, dt \; + \; \int_0^T dt \int_{\Gamma^+} \varphi(b) \, \partial_{u_1} G \, dS \\ &\quad - \int_0^T dt \int_{\Gamma^-} \varphi(b) \, \partial_{u_1} G \, dS \; - \; \frac{1}{2} \int_0^T \langle \sigma(\rho_t), \|\nabla G_t\|^2 \rangle \, dt \, ,\end{aligned}$$

where $\pi_t(du) = \rho_t(u)du$. Define $\widetilde{J}_G = \widetilde{J}_{G,T} : D([0,T], \mathcal{M}) \to \mathbb{R}$ by

$$\widetilde{J}_G(\pi) = \begin{cases} \check{J}_G(\pi) & \text{if } \pi \in D([0,T], \mathcal{M}^0), \\ +\infty & \text{otherwise} \, . \end{cases}$$

We define the functional $I_T : D([0,T], \mathcal{M}) \to [0, +\infty]$ as

$$I_T(\pi) = \begin{cases} \sup\limits_{G \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})} \big\{ \widetilde{J}_G(\pi) \big\} & \text{if } \mathcal{Q}_T(\pi) < \infty \, , \\ +\infty & \text{otherwise} \, . \end{cases}$$

Notice that, by Lemma 3.1.1, for any measurable function $\gamma : \Omega \to [0, 1]$,

$$I_T(\pi|\gamma) = \begin{cases} I_T(\pi) & \text{if } \pi_0(du) = \gamma(u)du \, , \\ +\infty & \text{otherwise} \, . \end{cases}$$

By this reason, it is easy to see that most of the results stated in Section 3.1 which holds for the dynamical rate function $I_T(\cdot|\gamma)$ also holds for the functional

I_T. Thus when we refer to a result stated in Section 3.1 and concerning the dynamical rate function $I_T(\cdot|\gamma)$, we mean the same result with I_T in the place of $I_T(\cdot|\gamma)$.

By Lemma 3.1.1, if a trajectory π in $D([0,T],\mathcal{M})$ satisfies $I_T(\pi) < \infty$ then π belongs to $C([0,T],\mathcal{M}^0)$. Thus when we let a trajectory π with $I_T(\pi) < \infty$, we will assume automatically that it belongs to $C([0,T],\mathcal{M}^0)$.

Let $\mathcal{C}_1(\overline{\Omega})$ be the set of continuous functions $f : \overline{\Omega} \to \mathbb{R}$ such that

$$\sup_{u\in\Omega} |f(u)| = 1\,.$$

Recall that we may define a metric on $\mathcal{M}$ by introducing a dense countable family $\{f_k \,:\, k \geq 1\}$ of functions in $\mathcal{C}_1(\Omega)$, with $f_1 = 1$, and by defining the distance

$$d(\vartheta_1,\vartheta_2) = \sum_{k=1}^{\infty} \frac{1}{2^k} |\langle \vartheta_1, f_k\rangle - \langle \vartheta_2, f_k\rangle|\,.$$

Let $\mathbb{D}$ be the space of measurable functions bounded below by 0 and bounded above by 1 endowed with the $L^2(\Omega)$ topology.

For $\vartheta \in \mathcal{M}$, $\rho \in \mathbb{D}$ and $\varepsilon > 0$, let us denote by $\mathcal{B}_\varepsilon(\vartheta)$ the open ε-ball in $\mathcal{M}$ with centre ϑ in the d-metric,

$$\mathcal{B}_\varepsilon(\vartheta) = \Big\{\widetilde{\vartheta} \in \mathcal{M} : d\big(\widetilde{\vartheta},\vartheta\big) < \varepsilon\Big\}\,,$$

and by $\mathbb{B}_\varepsilon(\rho)$ the open ε-ball in $\mathbb{D}$ with centre ρ in the $L^2(\Omega)$ norm,

$$\mathbb{B}_\varepsilon(\rho) = \{\tilde\rho \in \mathbb{D} : \|\tilde\rho - \rho\|_2 < \varepsilon\}\,.$$

Next result states that any trajectory whose density stays a long time far away from $\bar\rho$ in the $L^2(\Omega)$ norm pays a nonnegligible cost.

For each $\delta > 0$ and each $T > 0$ denote by $D([0,T],\mathcal{M}^0\backslash\mathbb{B}_\delta(\bar\rho))$ the set of trajectories $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}^0)$ such that $\rho_t \notin \mathbb{B}_\delta(\bar\rho)$ for all $0 \leq t \leq T$.

Lemma 4.1.1. *For every $\delta > 0$, there exists $T > 0$ such that*

$$\inf\{I_T(\pi) :\ \pi \in D([0,T],\mathcal{M}^0\backslash\mathbb{B}_\delta(\bar\rho))\} > 0\,.$$

Proof. By Corollary 5.1.8, there exists $T_0 = T_0(\delta) > 0$ such that for any weak solution λ of (2.1.1),

$$\|\lambda_t - \bar\rho\|_2 < \delta/2 \qquad \text{for all } t \geq T_0\,. \tag{4.1.1}$$

We assert that the statement of the lemma holds with $T = 2T_0$. If this is not the case, there exists a sequence of trajectories $\{\pi^k_t(du) = \rho^k(t,u)du : k \geq 1\}$ in $D\left([0,T],\mathcal{M}^0\backslash\mathbb{B}_\delta(\bar\rho)\right)$ such that $I_T(\pi^k) \leq 1/k$. Since I_T has compact level sets, by passing to a subsequence if necessary, we may assume that π^k converges to some $\pi_t(du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}^0)$. Moreover, by Lemma 3.1.6, ρ^k converges to ρ strongly in $L^2(\Omega_T)$.

On the other hand, the lower semicontinuity of I_T implies that $I_T(\pi) = 0$ or equivalently, by Corollary 3.1.3, that ρ is a weak solution of (2.1.1). Hence, by (4.1.1) and since $\left\|\rho^k_t - \bar\rho\right\|_2 \geq \delta$ for all $t \in [0,T]$ and for all positive integer k,

$$\int_0^T \left\|\rho^k_t - \rho_t\right\|_2^2 dt \geq \int_{T_0}^{2T_0} \left\|\rho^k_t - \rho_t\right\|_2^2 dt \geq \delta^2 T_0/4\,,$$

which contradicts the strong convergence of ρ^k to ρ in $L^2(\Omega_T)$ and we are done. $\square$

The same ideas permit us to establish an analogous result for the weak topology as follows.

Corollary 4.1.2. *For every $\varepsilon > 0$, there exists $T > 0$ such that*

$$\inf\left\{ I_T(\pi) : \pi \in D([0,T],\mathcal{M}) \ \text{ and } \ \pi_T \notin \mathcal{B}_\varepsilon(\overline{\vartheta})\right\} > 0\,.$$

Proof. Let $\delta = \varepsilon/\sqrt{2}$ and consider $T_0 > 0$ satisfying (4.1.1). Set $T = T_0$ and assume that the statement of the corollary does not hold. In that case, since I_T has compact level sets, by Lemma 3.1.6 and Corollary 3.1.3, there exists a sequence of trajectories $\{\pi^k : \ k \geq 1\}$ in $\mathcal{C}([0,T],\mathcal{M}^0)$, with $\pi^k_T \notin \mathcal{B}_\varepsilon(\overline{\vartheta})$, converging to some π whose density is a weak solution of (2.1.1). By (4.1.1) and since $\mathbb{B}_{\delta/2}(\bar{\rho}) \subset \mathcal{B}_{\varepsilon/2}(\overline{\vartheta})$, π_T belongs to $\mathcal{B}_{\varepsilon/2}(\bar{\vartheta})$. Hence, for every integer $k > 0$,

$$d\big(\pi^k_T, \pi_T\big) > \varepsilon/2\,,$$

which contradicts the convergence of π^k to π in $C([0,T],\mathcal{M}^0)$. $\square$

Fix a weak solution ρ of (2.1.1). By Corollary 3.1.3,

$$\mathcal{E}_T(\rho) = \int_0^T dt \int_\Omega du\, \frac{\|\nabla\rho_t(u)\|^2}{\chi(\rho_t(u))} < \infty. \tag{4.1.2}$$

Recall from Section 3.1 the definition of the functional $\partial_t\rho : C_c^\infty(\Omega_T) \to \mathbb{R}$. By Lemma 3.1.1, by the integration by parts formula (3.1.1) and since ρ is a weak solution of (2.1.1),

$$\partial_t\rho(H) \ = \ -\int_0^T \langle \rho_t, \partial_t H_t\rangle\, dt \ = \ -\int_0^T \int_\Omega \nabla\varphi(\rho_t(u)) \cdot \nabla H_t(u)\, du$$

for every $H \in C_c^\infty(\Omega_T)$. Then, $\partial_t\rho$ can be extended to a bounded linear operator in $L^2([0,T], H_0^1(\Omega))^*$ which corresponds to the path $\{\partial_t\rho_t : 0 \leq t \leq T\}$ in $L^2([0,T], H^{-1}(\Omega))$ with $\partial_t\rho_t : H_0^1(\Omega) \to \mathbb{R}$, $0 \leq t \leq T$, given by

$$\langle \partial_t\rho_t, G\rangle_{-1,1} = \int_\Omega \nabla\varphi(\rho_t(u)) \cdot \nabla G_t(u) du\,. \tag{4.1.3}$$

We conclude this section with an estimate on the cost paying by backwards solutions of the hydrodynamic equation (2.1.1). Let $\pi(t,du) = \tilde{\rho}(t,u)du$ be the path in $C([0,T],\mathcal{M}^0)$ with density given by $\tilde{\rho}(t,du) = \rho(T-t,u)du$. It is clear that $\mathcal{Q}_T(\pi) = \mathcal{Q}_T(\rho(u)du) < \infty$ and that $\partial_t\tilde{\rho}_t = -\partial_t\rho_{T-t}$. By (3.1.6) and since

ρ is a weak solution of (2.1.1), for each G in $\mathcal{C}_0^{1,2}(\Omega_T)$,

$$\begin{aligned}
\widetilde{J}_G(\pi) &= \int_0^T \langle \partial_t \tilde{\rho}_t, G_t\rangle_{-1,1}\, dt + \int_0^T dt \int_\Omega du\, \nabla\varphi(\tilde{\rho}_t(u)) \cdot \nabla G_t(u) \\
&\quad -\frac{1}{2}\int_0^T dt \int_\Omega du\, \sigma(\tilde{\rho}_t(u))\|\nabla G_t(u)\|^2 \\
&= -\int_0^T \langle \partial_t \rho_t, \widehat{G}_t\rangle_{-1,1}\, dt + \int_0^T dt \int_\Omega du\, \nabla\varphi(\rho_t(u)) \cdot \nabla \widehat{G}_t(u) \\
&\quad -\frac{1}{2}\int_0^T dt \int_\Omega du\, \sigma(\rho_t(u))\|\nabla \widehat{G}_t(u)\|^2 \\
&= 2\int_0^T dt \int_\Omega du\, \nabla\varphi(\rho_t(u)) \cdot \nabla \widehat{G}_t(u) - \frac{1}{2}\int_0^T dt \int_\Omega du\, \sigma(\rho_t(u))\|\nabla \widehat{G}_t(u)\|^2 \\
&\le \int_0^T dt \int_\Omega du\, \varphi'(\rho_t(u)) \frac{\|\nabla \rho_t(u)\|^2}{\chi(\rho_t(u))}\,,
\end{aligned}$$

where $\widehat{G}(t,u) = G(T-t,u)$ and where the last inequality follows from the elementary inequality $2ab \le Aa^2 + A^{-1}b^2$. In particular, since φ' is bounded above on $[0,1]$ by some constant $C_0 > 0$,

$$I_T(\pi) \le C_0 \mathcal{E}_T(\lambda)\,. \tag{4.1.4}$$

4.2 The Statical Rate Function

In this section we study some properties of the quasi potential V. The first main result, presented in Theorem 4.2.2, states that V is continuous at $\bar{\rho}$ in the $L^2(\Omega)$ topology. The second one, presented in Theorem 4.2.4, states that V is lower semicontinuous.

We start with an estimate on $V(\vartheta)$. Let $\mathbb{V} : \mathbb{D} \to [0,+\infty]$ be the functional given by $\mathbb{V}(\rho) = V(\rho(u)du)$. For each $h > 0$ and each $\delta > 0$, let $\mathbb{D}_\delta^h$ be the subset of $\mathbb{D}$ consisting of those profiles ρ satisfying the following conditions:

i) ρ belongs to $\in H^1(\Omega)$ and $B\rho = b$.

ii) $\int_\Omega \|\nabla\rho(u)\|^2 du \le h$.

iii) $\delta \le \rho(u) \le 1-\delta$ a.e. in Ω.

Lemma 4.2.1. *For every $h > 0$ and every $\delta > 0$, there exists a constant $C > 0$ such that*

$$\mathbb{V}(\rho) \le C \left\{ \|\rho - \bar{\rho}\|_2^2 \int_0^1 \alpha'(t)^2 dt + \int_0^1 \alpha(t)^2 dt \right\}$$

for any ρ in $\mathbb{D}_\delta^h$ and any increasing $\mathcal{C}^1$-diffeomorphism $\alpha : [0,1] \to [0,1]$.

Proof. Fix $h > 0$ and $\delta > 0$. Let $\rho \in \mathbb{D}_\delta^h$ and let $\alpha : [0,1] \to [0,1]$ be an increasing $\mathcal{C}^1$-diffeomorphism. Consider the path $\pi^\alpha(t,du) = \rho^\alpha(t,u)du$ in $\mathcal{C}([0,T],\mathcal{M}^0)$ with density given by $\rho_t^\alpha = (1-\alpha(t))\bar{\rho} + \alpha(t)\rho$. From condition *i)*, it is clear that ρ^α belongs to $L^2([0,1],H^1(\Omega))$ (which implies that $\mathcal{Q}_1(\pi^\alpha) < \infty$) and that

$B\rho_t^\alpha = b$ for every t in $[0,1]$. Further, since $\bar\rho$ solves (1.2.1), for every function G in $C_0^{1,2}(\overline{\Omega_1})$ and every t in $[0,1]$,

$$\begin{aligned} \int_\Omega \nabla\varphi(\rho_t^\alpha(u)) \cdot \nabla G_t(u) du &= \int_\Omega \nabla[\varphi(\rho_t^\alpha(u)) - \varphi(\bar\rho(u))] \cdot \nabla G_t(u) du \\ &= \int_\Omega \Psi_t^\alpha(u) \cdot \nabla G_t(u) du\,, \end{aligned}$$

where $\Psi_t^\alpha = \alpha(t)\varphi'(\rho_t^\alpha)\nabla(\rho - \bar\rho) + [\varphi'(\rho_t^\alpha) - \varphi'(\bar\rho)]\nabla\bar\rho$. From the definition of ρ^α it is easy to see that $\partial_t \rho^\alpha(t,u) = \alpha'(t)(\rho(u) - \bar\rho(u))$. Hence, by (3.1.6),

$$\begin{aligned} \widetilde{J}_G(\pi^\alpha) \;=\; & \int_0^1 \alpha'(t)\langle \rho - \bar\rho, G_t\rangle \, dt + \int_0^1 dt \int_\Omega du \, \Psi_t^\alpha(u) \cdot \nabla G_t(u) \\ & - \frac{1}{2}\int_0^1 \langle \sigma(\rho_t^\alpha), \|\nabla G_t\|^2 \rangle \, dt\,. \end{aligned} \tag{4.2.1}$$

Recall that $\bar\rho$ is bounded away from 0 and 1. Therefore, from condition *iii*), there exists a constant $C_1 = C_1(\delta) > 0$ such that the third term on the right hand side of (4.2.1) is bounded above by

$$-C_1 \int_0^1 dt \int_\Omega du \, \|\nabla G_t(u)\|^2\,.$$

On the other hand, by the inequality $2ab \le Aa^2 + A^{-1}b^2$ and by Poincaré's inequality, there exists a constant $C_2 > 0$ such that the first term on the right hand side of (4.2.1) is bounded by

$$C_2 \|\rho - \bar\rho\|_2^2 \int_0^1 \alpha'(t)^2 dt + \frac{C_1}{2}\int_0^1 dt \int_\Omega du \, \|\nabla G_t(u)\|^2\,.$$

Finally, from condition *ii*) and since φ' is bounded and Lipschitz on $[0,1]$, there is a constant $C' = C'(h) > 0$ such that $\int_\Omega \|\Psi_t^\alpha(u)\|^2 du \le C'\alpha(t)^2$ for every t in $[0,1]$. Hence, by the inequality $2ab \le Aa^2 + A^{-1}b^2$ and by Schwarz inequality, there exists a constant $C_3 = C_3(h,\delta) > 0$ such that the second term on the right hand side of (4.2.1) is bounded by

$$C_3 \int_0^1 \alpha(t)^2 dt + \frac{C_1}{2}\int_0^1 dt \int_\Omega du \, \|\nabla G_t(u)\|^2\,.$$

Adding these three bounds, we obtain that

$$\widetilde{J}_G(\pi^\alpha) \le C_2 \|\rho - \bar\rho\|_2^2 \int_0^1 \alpha'(t)^2 dt + C_3 \int_0^1 \alpha(t)^2 dt$$

for any function G in $\mathcal{C}_0^{1,2}(\overline{\Omega_1})$, which implies the desired result with $C = \max\{C_2, C_3\}$ □

Theorem 4.2.2. *$\mathbb{V}$ is continuous at $\bar\rho$.*

Proof. We will prove first that the restriction of $\mathbb{V}$ to the sets $\mathbb{D}_\delta^h$ is continuous at $\bar\rho$. Fix then $h > 0$ and $\delta > 0$. Let $\{\rho^n : n \ge 1\}$ be a sequence in $\mathbb{D}_\delta^h$ converging to $\bar\rho$. By Lemma 4.2.1, there is a constant $C = C(h,\delta)$ such that

$$\mathbb{V}(\rho^n) \le C \left\{ \|\rho^n - \bar\rho\|_2^2 \int_0^1 \alpha'(t)^2 dt + \int_0^1 \alpha(t)^2 dt \right\}$$

for any integer $n > 0$ and any increasing $\mathcal{C}^1$-diffeomorphism $\alpha : [0,1] \to [0,1]$. Thus, by letting $n \uparrow \infty$ and then taking the infimum over all the increasing $\mathcal{C}^1$-diffeomorphisms $\alpha : [0,1] \to [0,1]$, we conclude that

$$\overline{\lim_{n\to\infty}} \mathbb{V}(\rho^n) \le C \inf_\alpha \left\{ \int_0^1 \alpha(t)^2 dt \right\} = 0 \,.$$

We deal now with the general case. Let $\{\rho^n : n \ge 1\}$ be a sequence in $\mathbb{D}$ converging to $\bar\rho$. Fix $\varepsilon > 0$. For each integer $n > 0$, let λ^n be the weak solution of (2.1.1) starting at ρ^n. By Lemma 5.2.1, there exist $T = T(\varepsilon) > 0$ and $N_0 = N_0(\varepsilon) > 0$ such that, for all integer $n > N_0$,

$$\mathcal{E}_T(\lambda^n) \le \varepsilon \,. \tag{4.2.2}$$

In particular, there exists $T' \le T_n \le 2T' = T$ such that

$$\int_\Omega \|\nabla \lambda^n_{T_n}(u)\|^2 du \le \varepsilon / T' \,.$$

Moreover, by Lemma 5.1.6, there exists $\delta = \delta(T') > 0$ such that $\delta \le \lambda^n_{T_n}(u) \le 1 - \delta$ for every integer $n > N_0$ and for every u in Ω. Hence, $\lambda^n_{T_n}$ belongs to $\mathbb{D}_\delta^{\varepsilon/T'}$. Further, by Lemma 5.1.2,

$$\sqrt{2}\|\rho^n - \bar\rho\|_2 \ge \|\rho^n - \bar\rho\|_1 \ge \|\lambda^n_{T_n} - \bar\rho\|_1 \ge \|\lambda^n_{T_n} - \bar\rho\|_2^2 \,,$$

which implies that $\lambda^n_{T_n}$ also converges to $\bar\rho$ in $L^2(\Omega)$. Therefore, by the first part of the proof,

$$\lim_{n\to\infty} \mathbb{V}(\lambda^n_{T_n}) = 0 \,.$$

For each integer $n > 0$, let π^n be the path in $C([0,T_n], \mathcal{M})$ given by $\pi^n_t(du) = \lambda^n(T_n - t, u) du$. By (4.1.4) and (4.2.2), for every integer $n > N_0$,

$$I_{T_n}(\pi^n) \le C_0 \mathcal{E}_{T_n}(\lambda^n) \le C_0 \varepsilon \,.$$

In particular,

$$\overline{\lim_{n\to\infty}} \mathbb{V}(\rho^n) \le \lim_{n\to\infty} \mathbb{V}(\lambda^n_{T_n}) + \overline{\lim_{n\to\infty}} I_{T_n}(\pi^n) \le C_0 \varepsilon \,,$$

which, by the arbitrariness of ε, implies the desired result. $\square$

Similar arguments permit us to show that the quasi potentials of measures in $\mathcal{M}^0$ are uniformly bounded.

Proposition 4.2.3. *$V(\vartheta)$ is finite if and only if ϑ belongs to $\mathcal{M}^0$. Moreover,*

$$\sup_{\vartheta \in \mathcal{M}^0} V(\vartheta) < \infty \,.$$

Proof. By Lemmas 5.1.6 and 5.2.1, there exist constants $\delta > 0$ and $h > 0$ such that, for every weak solution λ of the equation (2.1.1),

$$\delta \le \lambda(t,u) \le 1-\delta \ \ \forall (t,u) \in [1,\infty)\times\Omega \quad \text{and} \quad \mathcal{E}_2(\lambda) \le h\,. \tag{4.2.3}$$

Fix $\vartheta(du) = \rho(u)du$ in $\mathcal{M}^0$ and let λ be the weak solution of (2.1.1) starting at ρ. By (4.2.3), there exists a time $1 \le T \le 2$ such that λ_T belongs to $\mathbb{D}^h_\delta$. Moreover, if we denote by π the path in $C([0,T],\mathcal{M}^0)$ given by $\pi_t(du) = \lambda(T-t,u)du$, by (4.1.4), there exists a constant $C_1 = C_1(h) > 0$ such that $I_T(\pi) \le C_1$. Hence, by Lemma 4.2.1, there exists a constant $C_2 = C_2(h,\delta) > 0$ such that

$$\mathbb{V}(\rho) \le \mathbb{V}(\lambda_T) + I_T(\pi) \le C_2 + C_1\,.$$

□

For any real numbers $r < s$ and any trajectory π in $D([\tilde r,\tilde s],\mathcal{M})$ with $\tilde r \le r \le s \le \tilde s$, let $\pi^{[r,s]}$ be the trajectory in $D([0,s-r],\mathcal{M})$ given by $\pi^{[r,s]}_t = \pi_{t+r}$, and let

$$I_{[r,s]}(\pi) = I_{s-r}\big(\pi^{[r,s]}\big)$$

For each π in $D((-\infty,0],\mathcal{M})$, let $I_\pi : (-\infty,0] \to [0,+\infty]$ be the function given by

$$I_\pi(t) = I_{[t,0]}(\pi)\,.$$

It is clear that this is a nonincreasing function and then

$$I(\pi) = \lim_{t\downarrow -\infty} I_\pi(t) \in [0,+\infty]$$

is well defined. We claim that, for every path π in $D((-\infty,0],\mathcal{M})$,

$$I(\pi) \ge V(\pi_0)\,. \tag{4.2.4}$$

Moreover, if $I(\pi) < \infty$ then, as $t \downarrow -\infty$, π_t converges to $\overline{\vartheta}$ in $\mathcal{M}^0$.

Indeed, the last assertion is an immediate consequence of Corollary 4.1.2. To prove (4.2.4), we may assume of course that $I(\pi) < \infty$. In that case, $\pi(t,du) = \rho(t,u)du$ belongs to $\mathcal{C}((-\infty,0],\mathcal{M}^0)$ and, by Lemma 4.1.1, there exists a sequence of nonpositive times $\{t_n : n \ge 1\}$ such that, for each integer $n > 0$, ρ_{t_n} belongs to $\mathbb{B}_{1/n}(\bar\rho)$. Hence, for all integer $n > 0$,

$$V(\pi_0) \le \mathbb{V}(\rho_{t_n}) + I_\pi(t_n) \le \mathbb{V}(\rho_{t_n}) + I(\pi)\,.$$

To conclude the proof of (4.2.4) it remains to let $n \uparrow \infty$ and to apply Theorem 4.2.2.

As a consequence of these facts we recover the definition for the quasi potential given in [3], in which the infimum appearing in the definition of $V(\vartheta)$ is carried over all paths π in $D([-\infty,0],\mathcal{M})$ with $\pi_{-\infty} = \overline{\vartheta}$ and $\pi_0 = \vartheta$.

Theorem 4.2.4. *The functional V is lower semicontinuous.*

Proof. Since $V(\vartheta)$ is finite only for measures ϑ in $\mathcal{M}^0$, which is a closed subset of $\mathcal{M}$, we just need to prove that, for all $q \in \mathbb{R}_+$, the set

$$V_q = \{\vartheta \in \mathcal{M}^0 : V(\vartheta) \le q\}\,,$$

is closed in $\mathcal{M}^0$. Fix then $q \in \mathbb{R}_+$ and let $\{\vartheta^n(du) = \rho(u)du : n \geq 1\}$ be a sequence of measures in V_q converging to some $\vartheta(du) = \rho(u)du$ in $\mathcal{M}^0$.

By definition of V, for each integer $n > 0$, there exists a path π^n in $C([0, T_n], \mathcal{M}^0)$ with $\pi^n_0 = \overline{\vartheta}$, $\pi^n_{T_n} = \vartheta^n$ and such that

$$I_{T_n}(\pi^n) \leq V(\vartheta^n) + 1/n \leq q + 1/n \tag{4.2.5}$$

Let us assume first that the sequence $\{T_n : n \geq 1\}$ is bounded above by some $T > 0$. In that case, for each integer $n > 0$, let $\hat{\pi}^n$ be the path in $C([0, T], \mathcal{M}^0)$ obtained from π^n by staying a time $T - T_n$ at $\overline{\vartheta}$,

$$\hat{\pi}^n_t \;=\; \begin{cases} \overline{\vartheta} & \text{if } 0 \leq t \leq T - T_n\,, \\ \pi^n_{t-T+T_n} & \text{if } T - T_n \leq t \leq T\,. \end{cases}$$

It is clear that $I_T(\hat{\pi}^n|\bar{\rho}) = I_{T_n}(\pi^n|\bar{\rho})$. Moreover, from (4.2.5) and since $I_T(\cdot|\bar{\rho})$ has compact level sets, there exists a subsequence of $\hat{\pi}^n$ converging to some π in $C([0, T], \mathcal{M}^0)$ such that $\pi_T(du) = \rho(u)du$ and $I_T(\pi|\bar{\rho}) \leq q$. In particular, ρ belongs to V_q and we are done.

To complete the proof, let us now assume that T_n has a subsequence which converges to ∞. We may suppose, without loss of generality that this subsequence is the sequence T_n itself. For each integer $n > 0$, let $\tilde{\pi}^n$ be the path in $C([-T_n, 0], \mathcal{M}^0)$ given by $\tilde{\pi}^n_t = \pi^n_{t+T_n}$.

Since I_T is lower semicontinuous with compact level sets, for any integer $l > 0$ and for any subsequence $\{\tilde{\pi}^{n_r} : r \geq 1\}$ of $\tilde{\pi}^n$, there exists a subsequence of $\tilde{\pi}^{n_r}$ converging to some $\breve{\pi}^l$ in $C([-l, 0], \mathcal{M}^0)$ with $\breve{\pi}^l_0 = \rho$ and $I_{[-k,0]}(\breve{\pi}^l) \leq q$. Then, by a Cantor's diagonal argument, we may obtain a path $\breve{\pi}$ in $C((-\infty, 0], \mathcal{M}^0)$ with $\breve{\pi}_0 = \rho$ and $I(\breve{\pi}) \leq q$. This together with (4.2.4) conclude the proof of the theorem. □

4.3 Large Deviations

4.3.1 Lower bound

The proof of the lower bound is essentially the same as the one in [8] but for the sake of completeness we present here the detailed proof. In fact, it is a simple consequence of Theorem 1.2.1 (hydrostatics) and the dynamical large deviation lower bound.

Fix an open subset $\mathcal{O}$ of $\mathcal{M}$. We have to prove that

$$\varliminf_{N\to\infty} \frac{1}{N^d} \log \mathcal{P}_N(\mathcal{O}) \geq - \inf_{\vartheta \in \mathcal{O}} V(\vartheta)\,.$$

To this end, it is enough to show that for any measure ϑ in $\mathcal{O} \cap \mathcal{M}^0$ and any trajectory $\tilde{\pi}$ in $C([0, T], \mathcal{M}^0)$ with $\tilde{\pi}_T = \vartheta$,

$$\varliminf_{N\to\infty} \frac{1}{N^d} \log \mathcal{P}_N(\mathcal{O}) \geq -I_T(\tilde{\pi}|\bar{\rho}) \tag{4.3.1}$$

holds. Since μ^N_{ss} is stationary,

$$\mathcal{P}_N(\mathcal{O}) = \mathbb{E}_{\mu_N}\left[\mathbb{P}_{\eta_0}\left(\pi^N_T \in \mathcal{O}\right)\right]\,.$$

Theorem 1.2.1 guarantees that for any fixed $\delta > 0$,

$$\lim_{N\to\infty} \mathcal{P}_N(\mathcal{B}_\delta(\overline{\vartheta})) = 1,$$

which is equivalent to the existence of a sequence of positive numbers $\{\varepsilon_N : N \geq 1\}$ converging to 0 and such that $\mathcal{P}_N(\mathcal{B}_{\varepsilon_N}(\overline{\vartheta}))$ converges to 1. Hence, for N large enough,

$$\mathcal{P}_N(\mathcal{O}) \geq \frac{1}{2} \inf_{\eta\in\mathcal{B}_N} \left\{\mathbb{P}_\eta \left[\pi_T^N \in \mathcal{O}\right]\right\},$$

where $\mathcal{B}_N = \left(\pi^N\right)^{-1} (\mathcal{B}_{\varepsilon_N}(\overline{\vartheta}))$. For each integer $N > 0$, consider a configuration η^N in $\mathcal{B}_N$ satisfying

$$\mathbb{P}_{\eta^N} \left[\pi_T^N \in \mathcal{O}\right] = \inf_{\eta\in\mathcal{B}_N} \left\{\mathbb{P}_\eta \left[\pi_T^N \in \mathcal{O}\right]\right\} .$$

Since $\pi^N \left(\eta^N\right)$ converges to $\bar{\rho}$ in $\mathcal{M}$ and since $\mathcal{O}_T = {\pi_T}^{-1}(\mathcal{O})$ is an open subset of $D([0,T],\mathcal{M})$, by the dynamical large deviations lower bound,

$$\begin{aligned}
\underline{\lim}_{N\to\infty} \frac{1}{N^d} \log \mathcal{P}_N(\mathcal{O}) &\geq \underline{\lim}_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^N} \left[\pi_T^N \in \mathcal{O}\right] \\
&= \underline{\lim}_{N\to\infty} \frac{1}{N^d} \log \mathbf{Q}_{\eta^N}(\mathcal{O}_T) \\
&\geq -\inf_{\pi\in\mathcal{O}_T} I_T(\pi|\bar{\rho}) \geq -I_T(\tilde{\pi}|\bar{\rho}),
\end{aligned}$$

which proves (4.3.1) and we are done.

4.3.2 Upper bound

In this subsection we prove the upper bound. We follow closely the approach given in [8] and solve the missing case mentioned in the introduction. Fix a closed subset $\mathcal{C}$ of $\mathcal{M}$. We have to show that

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathcal{P}_N(\mathcal{C}) \leq -V(\mathcal{C}), \tag{4.3.2}$$

where $V(\mathcal{C}) = \inf_{\vartheta\in\mathcal{C}} V(\vartheta)$.

Notice that if $\overline{\vartheta}$ belongs to $\mathcal{C}$, $V(\mathcal{C}) = 0$ and the upper bound is trivially verified. Thus, we may assume that $\overline{\vartheta} \notin \mathcal{C}$.

We may assume of course that the left hand side of (4.3.2) is finite, which implies that $\mathcal{C} \cap X_N \neq \emptyset$ for infinitely many integers N. By the compactness of $\mathcal{M}$ and since $\mathcal{C}$ is a closed subset of $\mathcal{M}$, there exists a sequence of configurations $\{\eta^{N_k} : k \geq 1\}$ with $\pi^N(\eta^{N_k})$ in $\mathcal{C} \cap X_{N_k}$ converging to some ϑ in $\mathcal{C}$. Moreover, since each configuration in X_N has at most one particle per site, ϑ belongs to $\mathcal{M}^0$. In particular, by Proposition 4.2.3, $V(\mathcal{C}) < \infty$.

Fix $\delta > 0$ such that $\mathcal{B}_{3\delta}(\overline{\vartheta}) \cap \mathcal{C} = \emptyset$. Let $B = B_\delta$ be the open δ-ball with centre $\overline{\vartheta}$ in the d-metric,

$$B = \mathcal{B}_\delta(\overline{\vartheta}),$$

and let $R = R_\delta$ be the subset of $\mathcal{M}$ defined by

$$R = \{\vartheta \in \mathcal{M} : 2\delta \leq d(\vartheta, \overline{\vartheta}) \leq 3\delta\}.$$

For each integer $N > 0$ and each subset A of $\mathcal{M}$, let $A^N = (\pi^N)^{-1}(A)$ and let $H_A^N : D(\mathbb{R}_+, X_N) \to [0, \infty]$ be the entry time in A^N:

$$H_A^N = \inf\left\{t \geq 0 : \eta_t \in A^N\right\} .$$

Let $\partial B^N = \partial B_\delta^N$ be the set of configurations η in X_N for which there exists a finite sequence of configurations $\{\eta^i : 0 \leq i \leq k\}$ in X_N with η^0 in R^N, $\eta^k = \eta$ and such that

i) For every $1 \leq i \leq k$, the configuration η^i can be obtained from η^{i-1} by a jump of the dynamics.

ii) The unique configuration of the sequence that can enter into B^N after a jump is η^k.

Let also $\tau_1 = \tau_1^N : D(\mathbb{R}_+, X_N) \to [0, \infty]$ be the stopping time given by

$$\tau_1 = \inf\left\{t > 0 : \text{ there exists } s < t \text{ such that } \eta_s \in R^N \text{ and } \eta_t \in \partial B^N\right\} .$$

The sequence of stopping times obtained by iterating τ_1 is denoted by τ_k. This sequence generates a Markov chain X_k on ∂B^N by setting $X_k = \eta_{\tau_k}$.

Notice that this Markov chain is irreducible. In fact, let ζ, η be configurations in ∂B^N. By definition of the set ∂B^N, there exist a sequence $\{\eta^i : 0 \leq i \leq k\}$ in X_N which satisfies $\eta^0 \in R^N$, $\eta^k = \eta$, *i)* and *ii)*. Further, it is clear that there exists a sequence $\{\zeta^i : 0 \leq i \leq l\}$ in X_N which satisfies $\zeta^0 = \zeta$, $\zeta^l = \eta^0$ and *i)*. Consider then the sequence $\{\tilde{\eta}^j : 0 \leq j \leq l + k\}$ in X_N given by

$$\tilde{\eta}^j = \begin{cases} \zeta^j & \text{if } \ 0 \leq j \leq l \,, \\ \eta^{j-l} & \text{if } \ l < j \leq l + k \,. \end{cases}$$

Let $j_0 = 0$ and for $i \geq 1$ let

$$j_{2i-1} = \min_{\substack{j > j_{2i-2} \\ \eta^j \in R^N}} \{j\} \qquad \text{and} \qquad j_{2i} = \min_{\substack{j > j_{2i-1} \\ \eta^j \in \partial B^N}} \{j\} \,.$$

Thus, by setting $\xi^i = \tilde{\eta}^{j_{2i}}$, we obtain a sequence $\{\xi^i : 0 \leq i \leq r\}$ in ∂B^N starting at $\xi^0 = \zeta$, ending at $\xi^r = \eta$ and such that

$$\mathbb{P}_{\xi_{i-1}}[\eta_{\tau_1} = \xi_i] > 0 \,,$$

for every $1 \leq i \leq r$. This implies the irreducibility of X_k.

Hence, since the state space ∂B^N is finite, this Markov chain has a unique stationary measure ν_N. Following [15], we represent the stationary measure μ_N of a subset A of X_N as

$$\mu_{ss}^N(A) = \frac{1}{C_N} \int_{\partial B^N} \mathbb{E}_\eta \left(\int_0^{\tau_1} \mathbf{1}_{\{\eta_s \in A\}} ds \right) d\nu_N(\eta) \,,$$

where

$$C_N = \int_{\partial B^N} \mathbb{E}_\eta \left(\tau_1\right) d\nu_N(\eta) \,.$$

In particular, by this representation an by the strong Markov property,

$$\mathcal{P}_N(\mathcal{C}) \le \frac{1}{C_N} \sup_{\eta\in\partial B^N} \left\{\mathbb{P}_\eta\left[H_{\mathcal{C}}^N < \tau_1\right]\right\} \sup_{\eta\in\mathcal{C}^N} \left\{\mathbb{E}_\eta\left(\tau_1\right)\right\} .$$

Recall that a configuration in X_N can jump by the dynamics to less than other $2dN^d$ configurations and that the jump rates are of order N^2. Hence, since any trajectory in $D(\mathbb{R}_+, X_N)$ has to perform at least a jump before the stopping time τ_1, $C_N \ge 1/CN^{d+2}$ for some constant $C > 0$.

Notice that the jumps of the process $d(\pi^N(\eta_t), \bar\rho)$ are of order N^{-d}. Thus, for N large enough, any trajectory in $D(\mathbb{R}_+, X_N)$ starting at some configuration in ∂B^N, resp. $\mathcal{C}^N$, satisfies $H_R^N \le H_{\mathcal{C}}^N$, resp. $\tau_1 \le H_B^N$. Hence, by the strong Markov property,

$$\mathcal{P}_N(\mathcal{C}) \le CN^{d+2} \sup_{\eta\in R^N} \left\{\mathbb{P}_\eta\left[H_{\mathcal{C}}^N < H_B^N\right]\right\} \sup_{\eta\in\mathcal{C}^N} \left\{\mathbb{E}_\eta\left(H_B^N\right)\right\} .$$

Therefore, in order to prove (4.3.2), it is enough to show the next lemma.

Lemma 4.3.1. *For every $\delta > 0$,*

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \sup_{\eta\in X_N} \left\{\mathbb{E}_\eta\left(H_{B_\delta}^N\right)\right\} \le 0 . \tag{4.3.3}$$

For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \sup_{\eta\in R_\delta^N} \left\{\mathbb{P}_\eta\left[H_{\mathcal{C}}^N < H_{B_\delta}^N\right]\right\} \le -V(\mathcal{C}) + \varepsilon . \tag{4.3.4}$$

To prove this lemma, we will need the following technical result.

Lemma 4.3.2. *For every $\delta > 0$, there exists $T_0, C_0, N_0 > 0$ such that*

$$\sup_{\eta\in X_N} \left\{\mathbb{P}_\eta\left[H_{B_\delta}^N \ge kT_0\right]\right\} \le \exp\left\{-kC_0N^d\right\} ,$$

for any integers $N > N_0$ and $k > 0$.

Proof. Fix $\delta > 0$. By Corollary 4.1.2, there exists $T_0 > 0$ and $C_0 > 0$ such that

$$\inf_{\pi\in\mathcal{D}} I_{T_0}(\pi) > C_0 ,$$

where $\mathcal{D} = D([0,T_0], \mathcal{M}\backslash B)$. For each integer $N > 0$, consider a configuration η^N in X_N such that

$$\mathbb{P}_{\eta^N}\left[H_B^N \ge T_0\right] = \sup_{\eta\in X_N} \left\{\mathbb{P}_\eta\left[H_B^N \ge T_0\right]\right\} .$$

By the compactness of $\mathcal{M}$, every subsequence of $\pi^N(\eta^N)$ contains a subsequence converging to some ϑ in $\mathcal{M}$. Moreover, since each configuration in X_N has at most one particle per site, ϑ belongs to $\mathcal{M}^0$. From this and since $\mathcal{D}$ is a closed subset of $D([0,T_0],\mathcal{M})$, by the dynamical large deviations lower bound, there exists a measure $\vartheta(du)=\gamma(u)du$ in $\mathcal{M}^0$ such that

$$\begin{aligned}\overline{\lim_{N\to\infty}}\frac{1}{N^d}\log\mathbb{P}_{\eta^N}\left[H_B^N\geq T_0\right] &= \overline{\lim_{N\to\infty}}\frac{1}{N^d}\log\mathbf{Q}_{\eta^N}(\mathcal{D})\\ &\leq -\inf_{\pi\in\mathcal{D}} I_{T_0}(\pi|\gamma)\\ &< -C_0\,.\end{aligned}$$

In particular, there exists $N_0>0$ such that for every integer $N>N_0$,

$$\mathbb{P}_{\eta^N}\left[H_B^N\geq T_0\right]\leq\exp\{-C_0N^d\}\,.$$

To complete the proof, we proceed by induction. Suppose then that the statement of the lemma is true until an integer $k-1>0$. Let $N>N_0$ and let $\hat\eta$ be a configuration in X_N. By the strong Markov property,

$$\begin{aligned}\mathbb{P}_{\hat\eta}\left[H_B^N\geq kT_0\right] &= \mathbb{E}_{\hat\eta}\left[\mathbf{1}_{\{H_B^N\geq T_0\}}\mathbb{P}_{\eta_{T_0}}\left[H_B^N\geq(k-1)T_0\right]\right]\\ &\leq \mathbb{P}_{\hat\eta}\left[H_B^N\geq T_0\right]\sup_{\eta\in X_N}\left\{\mathbb{P}_\eta\left[H_B^N\geq(k-1)T_0\right]\right\}\\ &\leq \exp\left\{-kC_0N^d\right\}\,,\end{aligned}$$

which concludes the proof.

□

Proof of Lemma 4.3.1. Let $\delta>0$ and consider $T_0,C_0,N_0>0$ satisfying the statement of Lemma 4.3.2. For every integer $N>N_0$ and every configuration η in X_N,

$$\mathbb{E}_\eta\left(H_B^N\right)\leq T_0\sum_{k=0}^\infty\mathbb{P}_\eta\left(H_B^N\geq kT_0\right)\leq T_0\sum_{k=0}^\infty\exp\left\{-kC_0N^d\right\}\leq\frac{T_0}{1-e^{-C_0}}\,,$$

which proves (4.3.3).

We turn now to the proof of (4.3.4). Fix $\varepsilon>0$. By Lemma 4.3.2 and since $V(\mathcal{C})<\infty$, for every $\delta>0$, there exists $T_\delta>0$ such that

$$\overline{\lim_{N\to\infty}}\frac{1}{N^d}\log\sup_{\eta\in X_N}\left\{\mathbb{P}_\eta\left[T_\delta\leq H_{B_\delta}^N\right]\right\}\leq -V(\mathcal{C})\,.$$

For each integer $N>0$, consider a configuration η^N in R_δ^N such that

$$\mathbb{P}_{\eta^N}\left[H_{\mathcal{C}}^N\leq T_\delta\right]=\sup_{\eta\in R_\delta^N}\left\{\mathbb{P}_\eta\left[H_{\mathcal{C}}^N\leq T_\delta\right]\right\}\,.$$

Let $\mathcal{C}_\delta$ be the subset of $D([0,T_\delta],\mathcal{M})$ consisting of all those paths π for which there exists t in $[0,T_\delta]$ such that $\pi(t)$ or $\pi(t-)$ belongs to $\mathcal{C}$. Notice that $\mathcal{C}_\delta$ is the closure of $\pi^N(\{H_{\mathcal{C}}^N\leq T_\delta\})$ in $D([0,T_\delta],\mathcal{M})$.

Recall that every subsequence of $\pi^N(\eta^N)$ contains a subsequence converging in $\mathcal{M}$ to some ϑ that belongs to $\mathcal{M}^0$. Hence, by the dynamical large deviations upper bound, there exists a measure $\vartheta_\delta(du)=\gamma_\delta(u)du$ in $R_\delta\cap\mathcal{M}^0$ such that

$$\overline{\lim_{N\to\infty}}\frac{1}{N^d}\log\mathbb{P}_{\eta^N}\left(H_{\mathcal{C}}^N\leq T_\delta\right)\leq\overline{\lim_{N\to\infty}}\frac{1}{N^d}\log\mathbf{Q}_{\eta^N}(\mathcal{C}_\delta)\leq-\inf_{\pi\in\mathcal{C}_\delta}I_{T_\delta}(\pi|\gamma_\delta)\,.$$

Therefore, since

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log\{a_N + b_N\} \le \max\left\{ \overline{\lim_{N\to\infty}} \frac{1}{N^d} \log a_N , \overline{\lim_{N\to\infty}} \frac{1}{N^d} \log b_N \right\} ,$$

the left hand side in (3.3.9) is bounded above by

$$\max\left\{ -V(\mathcal{C}), -\inf_{\pi\in\mathcal{C}_\delta} I_{T_\delta}(\pi|\gamma_\delta) \right\}$$

for every $\delta > 0$. Thus, in order to conclude the proof, it is enough to check that there exists $\delta > 0$ such that

$$\inf_{\pi\in\mathcal{C}_\delta} I_{T_\delta}(\pi|\gamma_\delta) \ge V(\mathcal{C}) - \varepsilon .$$

Assume that this is not true. In that case, for every integer $n > 0$ large enough, there exists a path π^n in $\mathcal{C}_{1/n} \cap C([0, T_{1/n}], \mathcal{M}^0)$ such that

$$I_{T_{1/n}}(\pi^n|\gamma_{1/n}) < V(\mathcal{C}) - \varepsilon .$$

Moreover, since π^n belongs to $\mathcal{C}_{1/n} \cap C([0, T_{1/n}], \mathcal{M}^0)$, there exists $0 < \widetilde{T}_n \le T_{1/n}$ such that $\pi^n_{\widetilde{T}_n}$ belongs to $\mathcal{C}$.

Let us assume first that the sequence of times $\{\widetilde{T}_n : n \ge 1\}$ is bounded above by some $T > 0$. For each integer $n > 0$, let $\tilde{\pi}^n$ be the path in $C([0,T], \mathcal{M}^0)$ given by

$$\tilde{\pi}^n_t = \begin{cases} \pi^n_t & \text{if } 0 \le t \le \widetilde{T}_n , \\ \pi^n_{\widetilde{T}_n} & \text{if } \widetilde{T}_n \le t \le T . \end{cases}$$

Since I_T has compact level sets and since $\pi^n_0(du) = \gamma_{1/n}(u)du$ belongs to $R_{1/n} \cap \mathcal{M}^0$ for every integer $n > 0$, we may obtain a subsequence of $\tilde{\pi}^n$ converging to some π in $C([0,T], \mathcal{M}^0)$ such that $\pi_0 = \overline{\vartheta}$, $\pi(T) \in \mathcal{C}$ and $I_T(\pi) \le V(\mathcal{C}) - \varepsilon$, which contradicts the definition of $V(\mathcal{C})$ and we are done.

To complete the proof, let us assume now that there exists a subsequence $\{\widetilde{T}_{n_k} : k \ge 1\}$ of $\widetilde{T}_n$ converging to ∞. By Theorem 4.2.2, there exists $\delta > 0$ such that $\mathbb{V}(\rho) < \varepsilon$ for every ρ in $\mathbb{B}_\delta(\bar{\rho})$. Moreover, if $\pi^{n_k}_t(du) = \rho^{n_k}(t,u)du$, by Lemma 4.1.1, for any integer k large enough, there exists $0 \le t_k \le \widetilde{T}_{n_k}$ such that $\rho^{n_k}_{t_k}$ belongs to $\mathbb{B}_\delta(\bar{\rho})$. Then,

$$\begin{aligned} V(\pi^{n_k}(\widetilde{T}_{n_k})) &\le \mathbb{V}(\rho^{n_k}_{t_k}) + I_{[t_k, \widetilde{T}_{n_k}]}(\pi^{n_k}) \\ &< \varepsilon + V(\mathcal{C}) - \varepsilon = V(\mathcal{C}) , \end{aligned}$$

which also contradicts the definition of $V(\mathcal{C})$ and we are done. □

Chapter 5

Weak Solutions

We establish in this chapter some properties of the weak solutions of the boundary value problems (1.2.1) and (2.1.1) (3.1.14).

5.1 Existence and Uniqueness

We prove in this section existence and uniqueness of weak solutions of the boundary value problems (1.2.1), (2.1.1) and (3.1.14). We start with the parabolic differential equation (2.1.1).

Proposition 5.1.1. *Let $\rho_0 : \overline{\Omega} \to [0,1]$ be a measurable function. There exists a unique weak solution of* (2.1.1)*.*

Proof. Existence of weak solutions of (2.1.1) is warranted by the tightness of the sequence $\mathbf{Q}^N$ proved in Section 2.1. Indeed, fix a profile $\rho_0 : \Omega \to [0,1]$ and consider a sequence $\{\mu^N : N \ge 1\}$ of probability measures in $\mathcal{M}$ associated to ρ_0 in the sense (2.1.2). Fix $T > 0$ and denote by $\mathbf{Q}^N$ the probability measure on $D([0,T],\mathcal{M})$ induced by the measure μ^N and the process π^N_t. In Section 2.1, we proved that the sequence $\{\mathbf{Q}^N : N \ge 1\}$ is tight and that any limit point of $\{\mathbf{Q}^N : N \ge 1\}$ is concentrated on weak solutions of (2.1.1). This proves existence. Uniqueness follows from Lemma 5.1.2 below. $\square$

Next lemma states that the $L^1(\Omega)$-norm of the difference of two weak solutions of the boundary value problem (2.1.1) decreases in time:

Lemma 5.1.2. *Fix two profiles ρ^1_0, $\rho^2_0 : \Omega \to [0,1]$. Let ρ^j, $j = 1$, 2, be weak solutions of* (2.1.1) *with initial condition ρ^j_0. Then, $\|\rho^1_t - \rho^2_t\|_1$ decreases in time. In particular, there is at most one weak solution of* (2.1.1)*.*

Proof. Fix two profiles ρ^1_0, $\rho^2_0 : \Omega \to [0,1]$. Let ρ^j, $j = 1$, 2, be weak solutions of (2.1.1) with initial condition ρ^j_0. Fix $0 \le s < t$. For $\delta > 0$ small, denote by R_δ the function defined by

$$R_\delta(u) \;=\; \frac{u^2}{2\delta}\mathbf{1}\{|u| \le \delta\} \;+\; \big(|u| - \delta/2\big)\mathbf{1}\{|u| > \delta\}\;.$$

Let $\psi : \mathbb{R}^d \to \mathbb{R}_+$ be a smooth approximation of the identity:

$$\psi(u) \ge 0\;, \quad \text{supp}\;\psi \subset [-1,1]^d\;, \quad \int \psi(u)\,du = 1\;.$$

For each positive ϵ, define ψ_ϵ as

$$\psi_\epsilon(u) = \epsilon^{-d}\psi(u\epsilon^{-1}) \; .$$

Taking the time derivative of the convolution of ρ^j_t with ψ_ϵ, after some elementary computations based on properties (**H1**), (**H2**) of weak solutions of (2.1.1), one can show that

$$\begin{aligned} &\int_\Omega du\, R_\delta\big(\rho^1(t,u) - \rho^2(t,u)\big) - \int_\Omega du\, R_\delta\big(\rho^1(s,u) - \rho^2(s,u)\big) \\ &\quad = -\delta^{-1} \int_s^t d\tau \int_{A_\delta} du\, \nabla(\rho^1 - \rho^2) \cdot \big\{\varphi'(\rho^1)\nabla\rho^1 - \varphi'(\rho^2)\nabla\rho^2\big\} \; , \end{aligned}$$

where A_δ stands for the subset of $[0,T] \times \Omega$ where $|\rho^1(t,u) - \rho^2(t,u)| \le \delta$. We may rewrite the previous expression as

$$\begin{aligned} &- \delta^{-1} \int_s^t d\tau \int_{A_\delta} du\, \varphi'(\rho^1) \|\nabla(\rho^1 - \rho^2)\|^2 \\ &\quad - \delta^{-1} \int_s^t d\tau \int_{A_\delta} du\, \big\{\varphi'(\rho^1) - \varphi'(\rho^2)\big\} \nabla(\rho^1 - \rho^2) \cdot \nabla\rho^2 \; . \end{aligned}$$

Since ρ^1, ρ^2 are positive and bounded by 1, there exists a positive constant c_0 such that $c_0 \le \varphi'(\rho^j(\tau,u))$. The first line in the previous formula is then bounded above by

$$-c_0\delta^{-1} \int_s^t d\tau \int_{A_\delta} du\, \|\nabla(\rho^1 - \rho^2)\|^2 \; .$$

On the other hand, since φ' is Lipschitz, on the set A_δ, $|\varphi'(\rho^1) - \varphi'(\rho^2)| \le M|\rho^1 - \rho^2| \le M\delta$ for some positive constant M. In particular, by Schwarz inequality, the second line of the previous formula is bounded by

$$\delta^{-1} M A \int_s^t d\tau \int_{A_\delta} du\, \|\nabla(\rho^1 - \rho^2)\|^2 \; + \; \delta M A^{-1} \int_s^t d\tau \int_{A_\delta} du\, \|\nabla\rho^2\|^2$$

for every $A > 0$. Choose $A = M^{-1}c_0$ to obtain that

$$\begin{aligned} &\int_\Omega du\, R_\delta\big(\rho^1(t,u) - \rho^2(t,u)\big) - \int_\Omega du\, R_\delta\big(\rho^1(s,u) - \rho^2(s,u)\big) \\ &\quad \le \; \delta c_0^{-1} M^2 \int_0^t d\tau \int du\, \|\nabla\rho^2\|^2 \; . \end{aligned}$$

Letting $\delta \downarrow 0$, we conclude the proof of the lemma because $R_\delta(\cdot)$ converges to the absolute value function as $\delta \downarrow 0$. □

Lemma 5.1.3. *Fix two profiles* ρ^1_0, $\rho^2_0 : \Omega \to [0,1]$. *Let* ρ^j, $j = 1, 2$, *be weak solutions of* (3.1.14) *for the same* H *satisfying* (3.2.2) *and with initial condition* ρ^j_0. *Then,* $\|\rho^1_t - \rho^2_t\|_1$ *decreases in time. In particular, there is at most one weak solution of* (3.1.14) *when* H *satisfies* (3.2.2).

Proof. Following the same procedure of the proof of the previous lemma, we get first

$$\begin{aligned}
&\int_\Omega du\, R_\delta\big(\rho^1(t,u)-\rho^2(t,u)\big) - \int_\Omega du\, R_\delta\big(\rho^1(s,u)-\rho^2(s,u)\big)\\
&\quad = -\delta^{-1}\int_s^t d\tau \int_{A_\delta} du\, \nabla(\rho^1-\rho^2)\cdot\big\{\varphi'(\rho^1)\nabla\rho^1-\varphi'(\rho^2)\nabla\rho^2\big\}\\
&\qquad -\delta^{-1}\int_s^t d\tau \int_{A_\delta} du\, \big\{\sigma(\rho^1)-\sigma(\rho^2)\big\}\nabla(\rho^1-\rho^2)\cdot\nabla H\ ,
\end{aligned}$$

and then

$$\begin{aligned}
&\int_\Omega du\, R_\delta\big(\rho^1(t,u)-\rho^2(t,u)\big) - \int_\Omega du\, R_\delta\big(\rho^1(s,u)-\rho^2(s,u)\big)\\
&\quad \le\ \delta C_1\int_0^t d\tau\int du\, \|\nabla\rho^2\|^2 + \delta C_2\int_0^t d\tau\int du\,\|\nabla H\|^2\ ,
\end{aligned}$$

for some positive constants C_1 and C_2. Hence, letting $\delta \downarrow 0$ we conclude the proof of the lemma. □

The same ideas permit to show the monotonicity of weak solutions of (2.1.1). This is the content of the next result which plays a fundamental role in proving existence and uniqueness of weak solutions of (1.2.1).

Lemma 5.1.4. *Fix two profiles ρ_0^1, $\rho_0^2 : \Omega \to [0,1]$. Let ρ^j, $j = 1, 2$, be the weak solutions of (2.1.1) with initial condition ρ_0^j. Assume that there exists $s \ge 0$ such that*

$$\lambda\big\{u\in\Omega\ :\ \rho^1(s,u)\le\rho^2(s,u)\big\} = 1\ ,$$

where λ is the Lebesgue measure on Ω. Then, for all $t \ge s$

$$\lambda\big\{u\in\Omega\ :\ \rho^1(t,u)\le\rho^2(t,u)\big\} = 1\ .$$

Proof. We just need to repeat the same proof of the Lemma 5.1.2 by considering the function $R_\delta^+(u) = R_\delta(u)\mathbf{1}\{u \ge 0\}$ instead of R_δ. □

Corollary 5.1.5. *Denote by ρ^0 (resp. ρ^1) the weak solution of* (2.1.1) *associated to the initial profile constant equal to* 0 *(resp.* 1*). Then, for $0 \le s \le t$, $\rho_t^1(\cdot) \le \rho_s^1(\cdot)$ and $\rho_s^0(\cdot) \le \rho_t^0(\cdot)$ a.e.*

Proof. Fix $s \ge 0$. Note that $\hat\rho(r,u)$ defined by $\hat\rho(r,u) = \rho^1(s+r,u)$ is a weak solution of (2.1.1) with initial condition $\rho^1(s,u)$. Since $\rho^1(s,u) \le 1 = \rho^1(0,u)$, by the previous lemma, for all $r \ge 0$, $\rho^1(r+s,u) \le \rho^1(r,u)$ for almost all u. □

Corollary 5.1.6. *For every $\delta > 0$, there exists $\epsilon > 0$ such that for all weak solution ρ of* (2.1.1) *with any initial profile ρ_0,*

$$\epsilon \le \rho(t,u) \le 1-\epsilon \qquad \textit{for almost all } (t,u) \textit{ in } [\delta,+\infty)\times\overline{\Omega}\,.$$

Proof. Let ρ^0 and ρ^1 be as in the statement of the previous corollary. For fixed $\delta > 0$, there exists $\epsilon > 0$ such that

$$\epsilon \le \rho^0(t,u)\ \text{ and }\ \rho^1(t,u) \le 1-\epsilon\ \text{ for almost all } (t,u) \text{ in } [\delta,\infty)\times\overline{\Omega}\,.$$

This and Lemma 5.1.4 permit us to conclude the proof. □

We now turn to existence and uniqueness of the boundary value problem (1.2.1). Recall the notation introduced in the beginning of Section 3.1. Consider the following classical boundary-eigenvalue problem for the Laplacian:

$$\begin{cases} -\Delta U = \alpha U\,, \\ U \in H_0^1(\Omega)\,. \end{cases} \tag{5.1.1}$$

By the Sturm–Liouville theorem (cf. [12], Subsection 9.12.3), problem (5.1.1) has a countable system $\{U_n, \alpha_n : n \geq 1\}$ of eigensolutions which contains all possible eigenvalues. The set $\{U_n : n \geq 1\}$ of eigenfunctions forms a complete orthonormal system in the Hilbert space $L^2(\Omega)$, each U_n belong to $H_0^1(\Omega)$, all the eigenvalues α_n, have finite multiplicity and

$$0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots \to \infty\,.$$

The set $\{U_n/\alpha_n^{1/2} : n \geq 1\}$ is a complete orthonormal system in the Hilbert space $H_0^1(\Omega)$. Hence, a function V belongs to $L^2(\Omega)$ if and only if

$$V \;=\; \lim_{n\to\infty} \sum_{k=1}^{n} \langle V, U_k\rangle_2\, U_k$$

in $L^2(\Omega)$. In this case,

$$\langle V, W\rangle_2 \;=\; \sum_{k=1}^{\infty} \langle V, U_k\rangle_2\, \overline{\langle W, U_k\rangle_2}$$

for all W in $L^2(\Omega)$. Moreover, a function V belongs to $H_0^1(\Omega)$ if and only if

$$V \;=\; \lim_{n\to\infty} \sum_{k=1}^{n} \langle V, U_k\rangle_2\, U_k$$

in $H_0^1(\Omega)$. In this case,

$$\langle V, W\rangle_{1,2,0} \;=\; \sum_{k=1}^{\infty} \alpha_k \langle V, U_k\rangle_2\, \overline{\langle W, U_k\rangle_2} \tag{5.1.2}$$

for all W in $H_0^1(\Omega)$.

Lemma 5.1.7. *Fix two profiles ρ_0^1, $\rho_0^2 : \Omega \to [0,1]$. Let ρ^j, $j = 1$, 2, be the weak solutions of (2.1.1) with initial condition ρ_0^j. Then,*

$$\int_0^\infty \|\rho_t^1 - \rho_t^2\|_1^2\, dt \;<\; \infty\,.$$

In particular,

$$\lim_{t\to\infty} \|\rho_t^1 - \rho_t^2\|_1 \;= 0\;\;.$$

Proof. Fix two profiles ρ_0^1, $\rho_0^2 : \Omega \to [0,1]$ and let ρ^j, $j = 1$, 2, be the weak solutions of (2.1.1) with initial condition ρ_0^j. Let $\rho_t^j(\cdot) = \rho^j(t,\cdot)$. For $n \geq 1$ let $F_n : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by

$$F_n(t) = \sum_{k=1}^{n} \frac{1}{\alpha_k} \big|\langle \rho_t^1 - \rho_t^2\,,\, U_k\rangle_2\big|^2\,.$$

Since ρ^1, ρ^2 are weak solutions, F_n is time differentiable. Since $\Delta U_k = -\alpha_k U_k$ and since $\alpha_k > 0$, for $t > 0$,

$$\begin{aligned} \frac{d}{dt} F_n(t) = - \sum_{k=1}^{n} \Big\{ & \langle \rho_t^1 - \rho_t^2 \,,\, U_k \rangle_2 \, \overline{\langle \varphi(\rho_t^1) - \varphi(\rho_t^2) \,,\, U_k \rangle_2} \\ & + \langle \varphi(\rho_t^1) - \varphi(\rho_t^2) \,,\, U_k \rangle_2 \, \overline{\langle \rho_t^1 - \rho_t^2 \,,\, U_k \rangle_2} \Big\} \,. \end{aligned} \tag{5.1.3}$$

Fix $t_0 > 0$. Integrating (5.1.3) in time, applying identity (5.1.2), and letting $n \uparrow \infty$, we get

$$\begin{aligned} \int_{t_0}^{T} dt \int_{\Omega} du \, \big[\varphi(\rho_t^1(u)) - \varphi(\rho_t^2(u)) \big] \big[\rho_t^1(u) - \rho_t^2(u) \big] &= \lim_{n \to \infty} \frac{1}{2} \Big\{ F_n(t_0) - F_n(T) \Big\} \\ &\le \frac{1}{2\alpha_1} \| \rho_{t_0}^1 - \rho_{t_0}^2 \|_2^2 \end{aligned}$$

for all $T > t_0$. Since $\rho_{t_0}^1 - \rho_{t_0}^2$ belongs to $L^2(\Omega)$,

$$\int_{t_0}^{\infty} dt \int_{\Omega} du \, \big[\varphi(\rho_t^1(u)) - \varphi(\rho_t^2(u)) \big] \big[\rho_t^1(u) - \rho_t^2(u) \big] \; < \; \infty \,.$$

There exists a positive constant C_2 such that, for all $a, b \in [0, 1]$

$$C_2 (b - a)^2 \le \big(\varphi(b) - \varphi(a) \big) (b - a) \,.$$

On the other hand, by Schwarz inequality, for all $t \ge t_0$,

$$\| \rho_t^1 - \rho_t^2 \|_1^2 \le 2 \| \rho_t^1 - \rho_t^2 \|_2^2 \,.$$

Therefore

$$\int_{t_0}^{\infty} \| \rho_t^1 - \rho_t^2 \|_1^2 \, dt < \infty \,.$$

and the first statement of the lemma is proved because the integral between $[0, t_0]$ is bounded by $4t_0$. The second statement of the lemma follows from the first one and from Lemma 5.1.2. $\square$

Corollary 5.1.8. *There is a nonnegative function Ψ in $L^2(\mathbb{R}_+)$ such that for any profiles ρ_0^1, $\rho_0^2 : \Omega \to [0, 1]$, the weak solutions ρ^j, $j = 1, 2$ of* (2.1.1) *with initial conditions ρ_0^j satisfy*

$$\| \rho_t^1 - \rho_t^2 \|_1 \; \le \; \Psi(t) \,,$$

for every $t \ge 0$.

Proof. Let ρ^0, resp. ρ^1, be the weak solution of the hydrodynamic equation (2.1.1) with initial condition $\rho^0(0, \cdot) \equiv 0$, resp. $\rho^1(0, \cdot) \equiv 1$, and set $\Psi(t) = \| \rho_t^1 - \rho_t^0 \|_1$. By the previous lemma, Ψ belongs to $L^2(\mathbb{R}_+)$. The last statement of the corollary follows from the monotonicity of weak solutions established in Lemma 5.1.4. $\square$

Proposition 5.1.9. *There exists a unique weak solution of the boundary value problem* (1.2.1).

Proof. We start with existence. Let $\rho^1(t,u)$ (resp. $\rho^0(t,u)$) be the weak solution of the boundary value problem (2.1.1) with initial profile constant equal to 1 (resp. 0). By Lemma (5.1.4), the sequence of profiles $\{\rho^1(n,\cdot) : n \geq 1\}$ (resp. $\{\rho^0(n,\cdot) : n \geq 1\}$) decreases (resp. increases) to a limit denoted by $\rho^+(\cdot)$ (resp. $\rho^-(\cdot)$). In view of Lemma 5.1.7, $\rho^+ = \rho^-$ almost surely. Denote this profile by $\bar\rho$ and by $\bar\rho(t,\cdot)$ the solution of (2.1.1) with initial condition $\bar\rho$. Since $\rho^0(t,\cdot) \leq \bar\rho(\cdot) \leq \rho^1(t,\cdot)$ for all $t \geq 0$, by Lemma 5.1.4, $\rho^0(t+s,\cdot) \leq \bar\rho(s,\cdot) \leq \rho^1(t+s,\cdot)$ a.e. for all s, $t \geq 0$. Letting $t \uparrow \infty$, we obtain that $\bar\rho(s,\cdot) = \bar\rho(\cdot)$ a.e. for all s. In particular, $\bar\rho$ is a solution of (1.2.1).

Uniqueness is simpler. Assume that ρ^1, $\rho^2 : \Omega \to [0,1]$ are two weak solution of (1.2.1). Then, $\rho^j(t,u) = \rho^j(u)$, $j = 1, 2$, are two stationary weak solutions of (2.1.1). By Lemma 5.1.7, $\rho^1 = \rho^2$ almost surely. □

5.2 Energy Estimates

We establish here an energy estimate for weak solutions in terms of the time T and the L^1 distance between its initial profile and the stationary density $\bar\rho$.

Fix $T > 0$ and let ρ be a weak solution of (2.1.1). Recall from (4.1.2) the definiton of $\mathcal{E}_T(\rho)$

Lemma 5.2.1. *There exists a positive constant C such that for any $T > 0$ and any weak solution ρ of (2.1.1) with initial profile $\rho_0 : \Omega \to [0,1]$,*

$$\mathcal{E}_T(\rho) \leq C\,\{T + \|\rho_0 - \bar\rho\|_1\}\ .$$

Proof. Fix $T > \delta > 0$, a weak solution ρ of (2.1.1) and a function $\beta : \overline{\Omega} \to (0,1)$ of class $\mathcal{C}^2$ such that $\beta\big|_\Gamma = b$. Let $\epsilon > 0$ such that

$$1 - \epsilon \ \leq\ \beta\ ,\ \rho_t\ \leq\ \epsilon \qquad \text{for every } t \geq \delta.$$

Let $F, U : [\delta, T] \times \overline{\Omega} \to \mathbb{R}$ be the functions given by

$$F(t,u) = \rho(t,u)\log\left(\frac{\rho(t,u)}{\beta(u)}\right) + (1-\rho(t,u))\log\left(\frac{1-\rho(t,u)}{1-\beta(u)}\right)\ ,$$

and

$$U(t,u) = \log\left(\frac{\rho(t,u)}{1-\rho(t,u)}\right) - \log\left(\frac{\beta(u)}{1-\beta(u)}\right)\ .$$

We claim that

$$\int_\Omega [F(T,u) - F(\delta,u)]\,du \quad = \quad \int_\delta^T \langle \partial_t \rho_t, U_t\rangle_{-1,1}\,dt\,. \tag{5.2.1}$$

Indeed, let $h : [\epsilon, 1-\epsilon]^2 \to \mathbb{R}$ be the smooth function given by

$$h(x,y) = x\log\left(\frac{x}{y}\right) + (1-x)\log\left(\frac{1-x}{1-y}\right)\ .$$

By proceeding as in the proof of Lemma 3.1.2 with h in the place of h^δ we may show that

$$\int_\delta^T \langle \partial_t \rho_t, \partial_x h(\rho_t,\beta)\rangle_{-1,1}\, dt \;=\; \int_\Omega h(\rho_{_T}(u),\beta(u))du - \int_\Omega h(\rho_\delta(u),\beta(u))du\,,$$

which is equivalent to the claim (5.2.1). By this claim, (4.1.3) and since ρ is a weak solution of (2.1.1),

$$\begin{aligned}
\int_\Omega [F(T,u)-F(\delta,u)]du \;&=\; -\int_\delta^T dt \int_\Omega du\, \nabla\varphi(\rho_t(u))\cdot\nabla U_t(u)\\
&= \int_\delta^T dt \int_\Omega du\, \frac{\varphi'(\rho_t(u))}{\chi(\beta(u))}\nabla\beta(u)\cdot\nabla\rho_t(u) \\
&\quad -\int_\delta^T dt \int_\Omega du\, \varphi'(\rho_t(u))\frac{\|\nabla\rho_t(u)\|^2}{\chi(\rho_t(u))}\,.
\end{aligned} \tag{5.2.2}$$

Let

$$\mathcal{E}_{[\delta,T]}(\rho) = \int_\delta^T dt \int_\Omega du\, \frac{\|\nabla\rho_t(u)\|^2}{\chi(\rho_t(u))}\,.$$

Since φ' is bounded bellow on $[0,1]$ by some positive constant C_1, by (5.2.2) and the elementary inequality $2ab \le A^{-1}a^2 + Ab^2$,

$$\begin{aligned}
2\mathcal{E}_{[\delta,T]}(\lambda) \;&\le\; \frac{2}{C_1}\int_\delta^T dt \int_\Omega du\, \varphi'(\rho_t(u))\frac{\|\nabla\rho_t(u)\|^2}{\chi(\rho_t(u))}\\
&\le\; \mathcal{E}_{[\delta,T]}(\rho) + \frac{1}{C_1^2}\int_\delta^T dt \int_\Omega du\, \frac{\varphi'(\rho_t(u))^2\chi(\rho_t(u))}{\chi(\beta(u))^2}\|\nabla\beta(u)\|^2\\
&\quad + \frac{2}{C_1}\|F_T - F_\delta\|_1\,.
\end{aligned}$$

Therefore, since φ', χ are bounded above on $[0,1]$ by some positive constant and since β is a function in $\mathcal{C}^2(\overline{\Omega})$ bounded away from 0 and 1, there exists a constant $C_2 = C_2(\beta)$ such that

$$\mathcal{E}_{[\delta,T]}(\rho) \le C_2(T-\delta) + \frac{2}{C_1}\|F_T - F_\delta\|_1\,.$$

Thus, in order to conclude the proof, we just need to show that there is a constant $C' > 0$ such that

$$\|F_T - F_\delta\|_1 \le C'\|\rho_0 - \bar\rho\|_1\,, \tag{5.2.3}$$

and then let $\delta \downarrow 0$. From the definition of F and since β is bounded away from 0 and 1 it is easy to see that $\|F_T - F_\delta\|_1$ is bounded above by

$$\begin{aligned}
&\int_\Omega \{|f(\rho_T(u)) - f(\rho_\delta(u))| + |f(1-\rho_T(u)) - f(1-\rho_\delta(u))|\}\, du\\
&+ C_3\|\rho_T - \rho_\delta\|_1\,,
\end{aligned} \tag{5.2.4}$$

where $f(r) = r\log r$ and $C_3 = C_3(\beta)$ is a positive constant.

Fix $\delta_0 > 0$ such that $2\delta_0 \le \bar{\rho}(u) \le 1 - 2\delta_0$ for all u in Ω. Let A_δ be the subset of Ω defined by

$$A_\delta = \{u \in \Omega : \, |\rho_\delta(u) - \bar{\rho}(u)| > \delta_0 \ \text{ or } \ |\rho_T(u) - \bar{\rho}(u)| > \delta_0\} \, .$$

Decompose the integral term in (5.2.4) as the sum of two integral terms $\int_{A_\delta} + \int_{A_\delta^c}$.

On the one hand, it is clear that $m(A_\delta) \le \delta_0^{-1}(\|\rho_\delta - \bar{\rho}\|_1 + \|\rho_T - \bar{\rho}\|_1)$ and then, since $-e^{-1} \le f(r) \le 0$ for all $r \in (0,1]$, the first integral term is bounded above by

$$\frac{2}{e\delta_0} \left\{ \|\rho_\delta - \bar{\rho}\|_1 + \|\rho_T - \bar{\rho}\|_1 \right\} \, .$$

On the other hand, $A_\delta^c \subset \{u \in \Omega : \, \delta_0 \le \rho_\delta(u), \rho_T(u) \le 1 - \delta_0\}$ and there exists a constant $C_4 = C_4(\delta_0) > 0$ such that $|f(r) - f(s)| \le C_4 |r - s|$ for all $r, s \in [\delta_0, 1]$. Hence, the second integral term is bounded by

$$2C_4 \|\rho_T - \rho_\delta\|_1 \, .$$

These bounds together with (5.2.4) and Lemma 5.1.2 prove (5.2.3) and we are done.

$\square$

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