# Mixing properties of a mechanical model of Brownian motion

PhD thesis in mathematics
by
Florian Friedrich Maximilian Voigtländer

advisor
Prof. Vladas Sidoravicius

IMPA - Instituto Nacional de Matemática Pura e Aplicada

# Contents

## Acknowledgements

I'm deeply indebted to my parents Anette and Fritz without their constant support during all time of my studies I could not have succeeded. I owe many thanks to my wife Gisele for her endless pacience, to my advisor Vladas Sidoravicius for innumerous fruitful discussions and explanations concerning the topic, as well to Maria Eulália Vares of CBPF for constructive suggestions. I also would like to mention the support of Departamento de Ensino for their administrative and organizational support during all my time at IMPA and CNPq for their financial grant. Finally many thanks to Steffi, Marie-Christine, Ilo, Regina, Leandro and Letícia, Wadinho, Rildo and Kátia, Waldomiro and Rosa, Julia and Steffen-Sebastian, Florian and Miwa, Amanda and Carlos Eduardo, Carina and Marcinho, Furlani and Mariana, Wanderley and Nelma Lobianco, Luis Eduardo and Dita.

# 1 Goals and results

In this work we study a one dimensional mechanical system of infinitely many point particles interacting through elastic collisions with a tracer particle, subject to a constant force. All point particles are field neutral and the mass of the tracer particle and field neutral particles are equal (the last condition can be removed, and one can consider the case of a heavy tracer particle which is more technically involved). A special feature of the model is that all neutral particles are equipped with a lifetime, which starts to discount after the first collision with the tracer particle. When the lifetime expires, the point particle is removed from the system, while the tracer particle has infinite lifetime and remains in the system forever. The principal question is to determine the long time behavior of the tracer particle. Our main goal is to generalize results (and extend techniques) of [11] to a as broad as possible class of distributions of lifetimes of the neutral particles in mechanical models of Brownian motion. It is believed that in the original model, i.e. with no lifetimes, during the evolution each neutral particle interacts with the tracer particle only finitely many times, and then flies away. However, the tail of the distribution of the last collision is expected to decay polynomially, thus producing long term memory in the system. Our motivation comes from the fact, that the understanding of the behaviour of models with a more general class of lifetime distribution might serve as another step forward in developing new (stochastic) tools which permit to analyze this long standing problem.

Applying an approach, which relies on line covering techniques by random intervals, proposed in collaboration with V. Beffara, V. Sidoravicius and M.E. Vares, we succeed to show that the strong law of large numbers (LLN) and the invariance principle (IP) for the rescaled position of the tracer particle holds as long as the lifetimes of the neutral particles are integrable random variables. Moreover, we are able to show that for the class of physically relevant distributions of lifetimes, such as inverse Gaussian, the mechanical system at low density of neutral particles still undergoes periods of clustering (against the predictions in physics literature), and, in fact, is a Bernoulli system. The key element of the proof is to show that under our assumptions, the different mechanical systems under consideration undergo the so-called clustering process, i.e. have infinitely many regeneration instants, when the system looses completely influence of its past on its future, and then to establish the tail asymptotic for the clustering event to occur. The control on tail decay determines the decay of correlations for the system. Once this is achieved, there is a number of available standard techniques which one applies in this case in a routine way to obtain the LLN and the IP. It is important to notice that we prove the LLN and the IP only for the discrete dynamics, obtained by observing our system at the times of the collisions of the tracer particle with freshly coming neutral particles (i.e. at the moment of the first collision between the tracer particle and each neutral particle). For future perspective, we believe that the idea to apply interval covering techniques is potentially very robust in the force driven systems, where one expects ballistic behaviour of a tracer particle at large time scales. Currently we are working on the extensions of these methods to the systems with neutral particles moving with Maxwellian velocities.

## 2 Introduction

We are concerned with the asymptotic behaviour of one dimensional mechanical systems, in particular with the motion of a tracer particle (t.p.) subject to a constant electric field in a random environment of neutral gas particles (n.p.s). This is one of fundamental questions in non-equilibrium statistical mechanics.
Our main model of interest, we call it Model 1 (M1), from a mechanical point of view is exactly the same as in [11], and informally can be described as follows. We consider the semi-infinite segment $\mathbb{R}_+ = [0, +\infty)$, with n.p.s initially located at positions $q_n > 0$, and the charged t.p. is located at the origin. All particles including the t.p. have equal mass one. The constant force $f > 0$ acts only on the charged particle, while the n.p.s do not feel the force. At the moment of collision the t.p. exchanges velocities with the n.p. elastically. N.p.s are initially standing, and interdistances between any two neighbouring particles are independent identically distributed (i.i.d.) exponential random variables with the parameter $\lambda > 0$. However differently from [11], where the lifetimes of particles were taken as i.i.d. exponential random variables with parameter 1, in the present work we will assume that lifetimes $\chi_n > 0$ of n.p.s are i.i.d. random variables which are integrable.
To obtain control on Model 1 we will consider the auxiliary Model 2 (M2). A one-dimensional particle system in $\mathbb{R}_+$ consisting of the t.p. interacting through elastic collisions with infinitely many point-like particles of an ideal gas and, as before, we suppose that all particles including the t.p. have equal mass one. Randomness enters through a measure under which the t.p. initially is at rest, located at the origin, and which governs injection of n.p.s into the system in the following way: the n.p.s collide with the t.p. for the first time at Poisson times, i.e. the times between consecutive first (fresh) collisions of the t.p. with the n.p.s are i.i.d. exponential random variables with intensity $\varrho > 0$. In other words, differently from M1, where n.p.s are initially assumed to be standing at exponential interdistances, in M2 we will assume that the n.p.s arrive (are injected into the system) at Poisson times at the position of the t.p. with zero incoming velocity. Then they remain in the system. Between collisions, a constant force $f > 0$ acts only on the t.p. while the n.p.s, as in M1, do not feel the force and do not interact among each other either. At collisions, the t.p. exchanges velocities with the n.p.s elastically. For convenience n.p.s are thought as undistinguishable pulses which only exchange velocities at collisions with each other and are relabeled afterwards, i.e. we may think they cross each other. Multiple arrivals at the same first collision time with equal velocities are excluded by our construction. In this way, the proof of the fact that the dynamics of the system seen from the

position of the t.p. is well-defined and governed by a uniform motion plus elastic collisions which obey the rules of classical mechanics follows the same lines as the proof of the main theorem in [13]. In the case of Model 1, the existence of the dynamics is proved in [13].
In the general situation (with no lifetimes) the system is expected to be asymptotically free and that the velocity of the t.p. does not approach to equilibrium with an exponential decay in general. The reason for this behaviour lies in the appearance of multiple recollisions between the t.p. and the same n.p.s of the environment. When the t.p. accelerates, it can collide with a n.p. many times which influences the friction force and affects the limiting velocity of the t.p. In particular, a n.p. which has collided earlier with the t.p., can recollide after an arbitrarily large time. This potentially can create a long tail memory which is responsible for a power law behaviour of correlations. So far, there is unfortunately no satisfactory way of treating fully Newtonian systems without any stochastic dynamics. One alternative approach proposed by [11] is the introduction of lifetimes for the n.p.s. The notion of lifetimes had already appeared indirectly in the models of [8], [9] or [5] for instance, where geometric restrictions and conditions on the velocity of the n.p.s lead (explicitly or not) to uniformly bounded lifetimes. Explicit exponential lifetimes of the n.p.s in this context were introduced for the first time in [11]. In this work we were concerned with relaxing the condition of lower uniform boundedness of the interdistances in [4]. M1 and M2 are some kind of asymptotic versions of the virtually one-dimensional model of [5], the so-called modified Rayleigh gas with a merely horizontally moving stick of height 1 subject to a constant force and collisions with n.p.s. In [5] the second dimension is only available to the n.p.s. Indeed, the horizontal initial velocity component $v_1$ and the vertical initial velocity component $v_2$ of the n.p.s in [5] determine their lifetimes $\chi = \frac{1}{v_2}$, due to the assumption that the vertical velocity components of the stick and the n.p.s do not change at collision times. In other words, $\chi$ is the time each of the n.p. remains inside the strip available to the stick, and once a n.p. leaves the strip, it can be considered extinct since it has become out of reach for the moving stick. In this way, our models can be interpreted as having zero horizontal velocity component and stick length going to zero, with the difference that M2 is initially a Poisson system in time and not in space, and secondly the n.p.s in [5] enter the strip available to the stick in a Poissonian manner and therefore do not necessarily collide for the first time at exponential interdistances with the stick as well. On the other hand, allowing both $v_1$ and $v_2$ to be normal distributed at the same time, $\chi$ becomes inverse Gaussian (one-sided $\frac{1}{2}$-stable), a heavy-tailed distribution, known to be the distribution of the first time a Brownian excursion hits some given level. This case is excluded

in [5] where a uniform lower bound of the vertical velocity distribution is imposed to exclude long living n.p.s and control recollisions. In view of this motivation, we suppose therefore that the t.p. has an infinite lifetime and the lifetimes of the n.p.s are i.i.d. with absolutely continuous distribution, independent of the first collision (arriving) times at which they start to be discounted.

In M1, the mechanical motion is delayed with respect to a Markovian evolution where n.p.s are annihilated immediately after collisions (and thus neglecting recollisions at all), i.e. $t_n(\omega) \geq \widetilde{t}_n(\omega)$ where $\widetilde{t}_n(\omega)$ (resp. $t_n(\omega)$) is the hitting time of the t.p. of the position of the $n$-th n.p. at first collision in the Markovian (resp. interaction) dynamics in some configuration $\omega$. In M2, it follows directly from the definition that the Markovian velocity is an upper bound for the velocity of the t.p. in the true dynamics for any time, since here $t_n(\omega) = t_n(\widetilde{\omega})$ for the suitable Markovian configuration $\widetilde{\omega}$, since in contrast to the above models, fresh n.p.s can arrive during the interaction of the t.p. with a block of already moving particles and the t.p. does not have to go through moving n.p.s in front of it first to reach the next fresh particle. In particular, the times when the n.p.s become extinct coincide in these two dynamics for M2. Observe that in Model 1, being specified initially in space, intercollision times of the t.p. with standing n.p.s in the Markovian evolution are proportional to the square root of the interdistances, making them Weibull distributed, in contrast to M2, where the intercollision times in both Markovian and true dynamics coincide and are exponential. By symmetry between these models, the interdistances in the Markovian evolution are Weibull with possibly different parameters. Still, since M2 is truly one-dimensional, the t.p. cannot overtake n.p.s during the evolution.

As all other models mentioned above, either directly or indirectly, our analysis relies essentially on the somewhat artificial notion of the so-called cluster times, i.e. first collision times at which the t.p. will not interact in the future with any n.p. it had collided with before, including the n.p. it collides with at this time. These times will then determine the mixing properties of the system. To construct a specific subset of cluster times, due to lack of mechanical arguments, we recurr first to the finding of conditions for the lifetime distribution which guarantee the existence of times of total extinction, i.e. stopping times with respect to the dynamics at which all previously moving n.p.s become extinct. The only memory of the past is then contained in the own velocity of the t.p. Upon this, since all particles have equal mass, cluster times are constructed by a simple mechanical argument and the general cluster times are then stochastically dominated by these special ones. This interpretation is indeed very close to the classical concepts of random covering problems in some different context and which allows to interpret the set of moments of total extinction of the n.p.s as the so-called uncovered set which is the closed image of an associated subordinator, i.e. an increasing Lévy process which represents the continuous time analogue of a renewal process. Cluster times are then stochastically bounded by the characteristics of this process. This is to some extent only of second interest, since in general the characteristics are hard to find explicitly for some given lifetime distribution. One natural way of generalization is to allow other distributions for the interarrival times of the n.p.s. One might think that if the interarrival time distribution were substituted by some heavier tailed distribution like Weibull, the most natural one in a Markovian (annihilation) version of the dynamics in M1 as already noted above, one may expect that such heavy-tailed interarrival distributions (still with finite mean) favour the non-covering of $\mathbb{R}_+$ more than the light-tailed exponential distribution, that is the heaviness of lifetime distributions which caused covering in the Poisson case might be weakend in the non-Poisson arrival case. Though we will not follow this direct approach, rather we establish some appropriate comparison principle between the mechanical models under consideration.

# 3 The mechanical models and results

For convenience, we begin first with the formal description of Model 2 (M2).

## 3.1 Model 2

The state space of this system seen from the position of the t.p. as described in the introduction is given by

$$\Omega = \mathbb{R}_+ \times X = \{\omega = (V, x) : V \in \mathbb{R}_+, x \in X\}$$

where for any bounded $A \in \mathcal{B}(\mathbb{R}_+)$,

$$X = \{x \subseteq \mathbb{M} \times (0, +\infty) : \ \mathrm{card}(x \cap (A \times \mathbb{R}_+) \times (0, +\infty)) < \infty \text{ and } \mathrm{card}(x \cap (q, v) \times (0, +\infty)) \leq 1\}$$

is the (marked) environment of the n.p.s and $\mathbb{M} = \mathbb{R}_+ \times \mathbb{R}_+$ is the one-particle state space consisting of the (relative w.r.t. the position of t.p.) position $q$ and (absolute) velocity $v$ of one n.p. Here $V$ stands for the velocity of the t.p. (the first particle) and $x$ is the point process of all locally finite subsets (in space) of $\mathbb{M}$, marked by the lifetimes, whose projection $x : \Omega \to X$ is given by

$$x(\omega) = x_m(\omega)$$

where $x_m$ are the moving n.p.s in the configuartion $\omega$. As for the main quantities, we write $(q_n(t))_{n \in \mathbb{N}}$ for the positions of the n.p.s relative to the t.p. at time $t$, $(v_n(t))_{n \in \mathbb{N}}$ for the absolute velocities of the n.p.s at time $t$, $(\sigma_n)_{n \in \mathbb{N}}$ denote the interarrival times of the n.p.s and $(\chi_n)_{n \in \mathbb{N}}$ the lifetimes of the n.p.s, with the convention that the t.p. has an infinite lifetime, $q_n \leq q_{n+1}$ and if $q_n = q_{n+1}$, then $v_n < v_{n+1}$ and $\chi_n < \chi_{n+1}$. The topology of $X$ is the one for which a fundamental system of neighbourhoods of a point $x \in X$ is given by

$$G_{A,B,C} = \{x' \in X : \mathrm{card}(x \cap (A \times B) \times C) = \mathrm{card}(x' \cap (A \times B) \times C)\}$$

with $A$, $B$ and $C$ open sets in $\mathbb{R}_+$ resp. $(0, +\infty)$ such that $A$ is bounded with $x \cap (\partial A \times B) \times C = \emptyset$ where $\partial A$ is the boundary of a set $A$. With this topology, $X$ is a Polish space and we denote by $\mathcal{B}(X)$ its Borel $\sigma$-algebra resp. $\mathcal{B}(\Omega) = \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(X)$ the Borel $\sigma$-algebra of $\Omega$. Initially, the t.p. is at rest and we endow $(\Omega, \mathcal{B}(\Omega))$ with the probability measure $\mu_0$, concentrated on the space of initial configurations

$$\Omega_0 = \{\omega \in \Omega : V(\omega) = 0, \ x(\omega) = \emptyset\},$$

under which the interarrival times $(\sigma_n)_{n\in\mathbb{N}}$ of the fresh n.p.s are i.i.d. exponential distributed with intensity $\varrho > 0$ and the lifetimes $(\chi_n)_{n\in\mathbb{N}}$ are i.i.d. and independent of $(\sigma_n)_{n\in\mathbb{N}}$, with common absolute continuous distribution function which we denote by $F_{\chi_1}(t) = \int_0^t f_{\chi_1}(y)dy$ for some density $f_{\chi_1}$. When we speak of arrivals of n.p.s, we mean the times at which n.p.s appear (are injected) at the position of the t.p. with incoming velocity zero. The lifetime of the $n$-th n.p. starts to be discounted at the $n$-th arrival time $t_n = \sum_{i=1}^n \sigma_i$ with $t_0 = 0$, whereas the t.p. has an infinite lifetime. Thus, the initial configuration can be described by the point process $\Xi = (t_n, \chi_n)_{n\in\mathbb{N}}$ on the upper right plane $\mathbb{H} = \mathbb{R}_+ \times (0, \infty)$ with intensity measure $n : \mathcal{B}(\mathbb{H}) \to \mathbb{R}_+$ given by

$$n(B) = \varrho \int_B F_{\chi_1}(ds)dq = \varrho \int_B f_{\chi_1}(s)dsdq$$

for $B \in \mathcal{B}(\mathbb{H})$. One then associates to each n.p. its lifetime interval, i.e. the interval $I_n = (t_n, t_n + \chi_n)$ for the $n$-th n.p. The dynamics, which we will denote by $(T^t)_{t\in\mathbb{R}_+}$, is then such that the t.p. is uniformly accelerated by the force $f > 0$ between consecutive collisions and at these collisions, it exchanges its velocity elastically with the n.p.s according to the mechanical rule

$$\Delta V = -\Delta v$$

where $\Delta V = V^+ - V^-$ and $\Delta v = v^+ - v^-$ are the velocity jumps and $V^+$ ($V^-$) and $v^+$ ($v^-$) are the outgoing (incoming) velocities of the t.p. and the n.p.s. The dynamics is right-continuous in the sense that at collision times the velocities are the outgoing ones, i.e. $V^+ = V$ and $v^+ = v$. In this way, the dynamics is $\mu_0$-a.s. well-defined on $\Omega$ (the same argument as in [4], Proposition A.1, works here as well). All statements about the dynamics will be understood such that they hold for those $\omega$ for which the dynamics is well-defined. If convenient, we may write as well for $\omega \in \Omega$ and $t > 0$,

$$\omega(t) = T^t\omega = (V_t(\omega), x(\omega(t)))$$

where $V_t = v_0(t) = v_0 \circ T^t$ is the velocity of the t.p. at time $t > 0$ given by

$$V_t = ft + \sum_{s\in J(V)\cap(0,t]} \Delta V_s$$

with $V_0 = V_{t_n} = 0$ and $J(V) = \{t > 0 : \Delta V_t = -ft\}$ is the set of jump times of the process $(V_t)_{t\in\mathbb{R}_+}$ where $\Delta V_t = V_t - V_{t-}$. The position $Q_t = q_0(t) = q_0 \circ T^t$ of the t.p. at time $t > 0$ is then

$$Q_t = \int_0^t V_s ds$$

with $Q_0 = 0$. The dynamics at the particular moments of first collisions is called the discrete dynamics and is (well-)defined by

$$T^n = T_{t_n}$$

for any $n \geq 1$ on the associated configuration space

$$X_1 = \{x \in X : V(x) = 0\}.$$

The evolution of the initial measure under the discrete dynamics $(T^n)_{n\in\mathbb{N}}$ is denoted by

$$\mu_n = \mu_0 \circ T^{-n}.$$

Later we will use a different notation with a hat on all quantities related to M2 in order to distinguish between M2 and M1 when we will compare them directly.

## 3.2 Model 1

As for M1, we make the suitable modifications and recall the description in the introduction and the notation for M2. The topology is analogous to the one of M2 with the difference that a configuration consists now of a sequence of positions, velocities and (residual) lifetimes instead of arrival times, velocities and (residual) lifetimes. We will maintain the notation of M2 and denote all related quantities of the dynamics of M1 by the same letters, otherwise introducing different notation at particular places to avoid confusion which model we are talking about. Formally, all particles are initially at rest, i.e. $V = v_n = 0$ for any $n \geq 1$ and the initial measure $\mu_0$ on $(\Omega, \mathcal{B}(\Omega))$ is such that the sequence of interparticle distances $\xi_n = q_n - q_{n-1}, n \geq 1$, is i.i.d. exponential with density $\lambda > 0$. The lifetimes $\chi_n, n \geq 1$, begin to be discounted at the times of first collisions $t_n, n \geq 1$, are i.i.d. with distribution function $F_{\chi_1}$ and independent of the whole sequence $(\xi_n)_{n\in\mathbb{N}}$. The initial configuration can then be described by the point process $\Xi = (q_n, \chi_n)_{n\in\mathbb{N}}$ on $\mathbb{H} = \mathbb{R}_+ \times (0, \infty)$ with intensity measure $n : \mathcal{B}(\mathbb{H}) \to \mathbb{R}_+$ given by

$$n(A) = \varrho \int_A F_{\chi_1}(dy)dt$$

for $A \in \mathcal{B}(\mathbb{H})$. Analogous to M2, the dynamics $(T_t)_{t\in\mathbb{R}_+}$ is such that the t.p. is uniformly accelerated by the force $f > 0$ between consecutive collisions and at these collisions, it exchanges its velocity elastically with the n.p.s according to $\Delta V = -\Delta v$. By the same conventions as in M2, the dynamics is well-defined by [11], Remark 2, since the proof is independent of the lifetime (distribution). At difference to M2, here we have a initial Poissonian system in space and the first hitting times $t_n$ depend heavily on possible recollisions, whereas in M2, the freshly arriving n.p.s rain down on the t.p. at exponential interarrival times independently of recollisions. In the sequel, we write generically $F_Y = \mu_0 \circ Y^{-1}$ for the distribution function under the initial measure $\mu_0$ (the same for M1 and M2) for some random element $Y$ on $\Omega$ and $\overline{F}_Y = 1 - F_Y$ for its tail. The expectation operator is denoted by $\mathbb{E}_\mu$ with respect to an arbitrary measure $\mu$. For a stochastic process we often also write the common abbreviation $Y = (Y_t)_{t\in\mathbb{R}_+}$.

We may state now our main results which concern M1, our main model of interest with initially standing n.p.s. Denote by $Q_t = q_0(t)$ the position of the t.p. at time $t > 0$ and $\mu_n$ the measure on the state space as seen from the t.p. at the moment of the first collision with the $n$-th n.p.

**Theorem 3.1.** (Law of Large Numbers) If the sequence $(\chi_n)_{n\in\mathbb{N}}$ of lifetimes of. n.p.s is i.i.d. with $\mathbb{E}_{\mu_0}\chi_1 < +\infty$, then there exists a positive constant $v_d > 0$ (the drift velocity) such that

$$\lim_{t\to+\infty} \frac{Q_t}{t} = v_d \qquad \mu_0\text{-a.s.}$$

**Theorem 3.2.** (Invariant Measure) If the sequence $(\chi_n)_{n\in\mathbb{N}}$ of lifetimes of. n.p.s is i.i.d. such that $\mathbb{E}_{\mu_0}\exp(a\chi_1) < +\infty$ for some $a > 0$, then there exists an invariant probability measure $\mu$ which is concentrated on $X_1$, such that

$$\lim_{n\to+\infty} \mu_n = \mu \quad \text{weakly.}$$

The proof of the above theorem is based on the construction of the so-called cluster index, and using this construction we in fact will get the following result.

**Corollary 3.3.** (Mixing and Invariance Principle) Under the assumptions of Theorem 3.1 and Theorem 3.2 the following holds:

1. Let $\mathcal{M}_m^n$ denote the $\sigma$-field generated by the variables $\{\tau_i : m \le i \le n\}$, where $\tau_i$ is the interarrival time between the $(i-1)$-th and $i$-th particle. Then there exist positive constants $c > 0$ and $c' > 0$ such that

$$\psi(n) = \sup_{A\in\mathcal{M}_0^k\,,\, B\in\mathcal{M}_{k+n}^{+\infty}} |\mu(B|A) - \mu(B)| \le c\exp(-c'n)$$

for all $k, n \geq 1$.

2. There exists a positive constant $\sigma > 0$ such that the process

$$\Big(\frac{Q_{ut} - v_d ut}{\sigma\sqrt{u}}\Big)_{t\in[0,1]}$$

converges in law as $u \to +\infty$, on the Skorokhod space $D([0,1],\mathbb{R})$, to standard Brownian motion.

As we just mentioned above, the proof of the above theorem is based on the construction of the cluster indices. This is the key part of the work, and is contained in section 5. We will achieve this by coupling three models, which in some stochastic sense dominate each other. First we will show that if the lifetimes of the n.p.s are integrable, then M2 has infinitely many regeneration times with any density of the injected n.p.s. Finally, using comparisons and couplings, we will show that this implies that the Markovian version of M1 with small enough density of particles inherits the same property from which follows the same for the original M1. Once this is achieved, we briefly outline consequences and give references to all necessary steps, which are at this point rather standard, to achieve the proof of Theorem 3.2 and Corollary 3.3.

# 4 Annihilation dynamics

From now on we will use different notations for M1 and M2 where we denote all quantities related to M2 with a hat. Due to possible recollisions, without enlarging the underlying probability space, the velocity process $V = (V_t)_{t\in\mathbb{R}_+}$ resp. $\widehat{V} = (\widehat{V}_t)_{t\in\mathbb{R}_+}$ is a non-Markovian càdlàg process which increases linearly in time proportional to the constant field $f > 0$ between successive collisions and at these collision times has negative jumps. An auxiliary Markovian dynamics can be achieved by annihilating the n.p.s immediately at each first collision with the t.p., which makes the corresponding velocity process Markovian due to the exclusion of recollisions. At each collision (which is then always a first one) the environment of the n.p.s is recreated according to the initial measure $\mu_0$ and the t.p. is the only moving particle in the system. The state space of this dynamics is denoted by $X^0$ resp. $\widehat{X}^0$ and all related quantities to this dynamics will carry superscript zero in both models. Since the velocity change in the interval $(t_{n-1}, t_n)$ resp. $(\widehat{t}_{n-1}, \widehat{t}_n)$ is only due to the constant field $f > 0$, the corresponding intercollisions times are given by

$$\tau_n^0 = t_n^0 - t_{n-1}^0 = \sqrt{\frac{2\xi_n}{f}}$$

where $\xi_n = q_n - q_{n-1}$ is exponential with parameter $\varrho > 0$ for M1, and

$$\widehat{\tau}_n^0 = \widehat{\tau}_n = \widehat{t}_n - \widehat{t}_{n-1}$$

which is exponential with parameter $\widehat{\varrho} > 0$ for M2. Recall that $(\xi_n)_{n\in\mathbb{N}}$ and $(\widehat{\tau}_n)_{n\in\mathbb{N}}$ are independent and the distribution of $\tau_1^0$ is known to be Weibull with scaling parameter $a = \frac{\varrho f}{2}$ and form parameter $b = 2$. This distribution is also known under the name Rayleigh distribution whose density and tail are given by

$$f_{\tau_1^0}(\tau) = \varrho f \tau \exp(-\frac{\varrho f}{2}\tau^2)$$

and

$$\overline{F}_{\tau_1^0}(\tau) = \exp(-\frac{\varrho f}{2}\tau^2).$$

For its mean and variance we have $\mathbb{E}_{\mu_0}\tau_1^0 = \frac{\sqrt{\pi}}{\sqrt{2\varrho f}}$ and $\Gamma = \mathbb{V}ar_{\mu_0}\tau_1^0 = \frac{4-\pi}{\sqrt{\varrho f}} > 0$.

If we denote by $\vartheta_t^0 = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{t_n^0 \le t\}}$ resp. $\widehat{\vartheta}_t = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{\widehat{t}_n \le t\}}$ the Weibull resp. Poisson process of the number of (first) collisions of n.p.s in the interval $[0, t]$, and by $t^0_{\vartheta_t^0} = \sum_{n=1}^{\vartheta_t^0} \tau_n^0$ resp. $\widehat{t}_{\widehat{\vartheta}_t} = \sum_{n=1}^{\widehat{\vartheta}_t} \widehat{\tau}_n$ the time of the last collision of a n.p. before time $t$, then the velocity process $V^0 = (V_t^0)_{t\in\mathbb{R}_+}$ resp. $\widehat{V}^0 = (\widehat{V}_t^0)_{t\in\mathbb{R}_+}$ becomes

$$V_t^0 = f(t - t_{\vartheta_t^0}^0) \quad \text{resp.} \quad \widehat{V}_t^0 = f(t - \widehat{t}_{\widehat{\vartheta}_t})$$

for any $t \geq 0$ with $V_0^0 = \widehat{V}_0^0 = 0$. Both processes are (strong) Markov with the same form of the infinitesimal generator, which for $\widehat{V}^0$ is

$$\mathcal{L}^{\widehat{V}^0}\varphi(v) = f\varphi'(v) + \widehat{\varrho}(\varphi(0) - \varphi(v))$$

with $v \in \mathbb{R}_+$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ bounded and continuous. Both processes hit the zero on the set of its jump times, namely $J(V^0) = \{t_n^0 : n \in \mathbb{N}\}$ resp. $J(\widehat{V}^0) = \{\widehat{t}_n : n \in \mathbb{N}\}$.

**Remark 4.1.**

1. By the renewal theorem (cf. [6]), the laws of the velocity processes converge as $t \to +\infty$ to the stationary distributions

   $$\nu^0(B) = \lambda\mathbb{E}_{\mu_0}\int_0^{\tau_1^0} \mathbb{1}_B(V_s^0)ds \quad \text{resp.} \quad \widehat{\nu}^0(B) = \widehat{\varrho}\mathbb{E}_{\mu_0}\int_0^{\widehat{\tau}_1} \mathbb{1}_B(\widehat{V}_s^0)ds$$

   for $B \in \mathcal{B}(\mathbb{R}_+)$, where we have set $\lambda = \frac{\sqrt{2\varrho f}}{\sqrt{\pi}}$. In particular, $\mathbb{E}_{\nu^0}V_t^0 = \frac{\lambda f}{2}\mathbb{E}_{\mu_0}(\tau_1^0)^2 = \frac{\lambda}{\varrho} = \frac{\sqrt{\frac{2f}{\varrho}}}{\sqrt{\pi}}$ and $\mathbb{E}_{\widehat{\nu}^0}\widehat{V}_t^0 = \frac{\widehat{\varrho}f}{2}\mathbb{E}_{\mu_0}\widehat{\tau}_1^2 = \frac{f}{\widehat{\varrho}}$ for any $t > 0$.

2. By symmetry, the interdistances in the annihilation dynamics of M2 become $\widehat{\xi}_n^0 = \frac{f}{2}\widehat{\tau}_n^2$ which are i.i.d. Weibull with parameters $a = \frac{f}{2\widehat{\varrho}^2}$ and $b = \frac{1}{2}$ whose density resp. tail take the form

   $$f_{\widehat{\xi}_1^0}(\xi) = \frac{\widehat{\varrho}}{\sqrt{2f\xi}}\exp(-\widehat{\varrho}\sqrt{\frac{2\xi}{f}})$$

   resp.

   $$\overline{F}_{\widehat{\xi}_1^0}(\xi) = \exp(-\widehat{\varrho}\sqrt{\frac{2\xi}{f}})$$

   with mean $\mathbb{E}_{\mu_0}\widehat{\xi}_1^0 = \frac{f}{\widehat{\varrho}^2}$ and variance $\widehat{\Gamma} = \mathbb{V}ar_{\mu_0}\widehat{\xi}_1^0 = \frac{5f^2}{\widehat{\varrho}^4} > 0$.

We have the almost immediate LLN for the annihilation dynamics of M1 and M2.

**Lemma 4.2.** (LLN for Markovian dynamics) There are constants $v_d^0 > 0$ and $\widehat{v}_d^0 > 0$ such that

$$\lim_{t\to+\infty} \frac{Q_t^0}{t} = v_d^0 \qquad \mu_0\text{-a.s. } (\nu^0\text{-a.s.})$$

and

$$\lim_{t\to+\infty} \frac{\widehat{Q}_t^0}{t} = \widehat{v}_d^0 \qquad \mu_0\text{-a.s. } (\widehat{\nu}^0\text{-a.s.})$$

where $v_d^0 = \frac{\sqrt{\frac{2f}{\varrho}}}{\sqrt{\pi}}$ and $\widehat{v}_d^0 = \frac{f}{\widehat{\varrho}}$.

*Proof.* It is sufficient to consider moments of first collisions only, since $t_{\vartheta_t} \le t < t_{\vartheta_t+1}$ and $Q_{t_{\vartheta_t}} \le Q_t \le Q_{t_{\vartheta_t+1}}$ imply

$$\frac{Q_{t_{\vartheta_t}}}{t_{\vartheta_t+1}} \le \frac{Q_t}{t} \le \frac{Q_{t_{\vartheta_t+1}}}{t_{\vartheta_t}}.$$

This holds for all dynamics related to the different models. Hence for the annihilation dynamics of M1, by the renewal theorem and the LLN for i.i.d. random variables,

$$\lim_{t\to+\infty} \frac{Q_t^0}{t} = \lim_{t\to+\infty} \frac{\vartheta_t^0}{t} \left( \frac{\sum_{k=1}^{\vartheta_t^0} \xi_k}{\vartheta_t^0} \right) = \frac{\mathbb{E}_{\mu_0}\xi_1}{\mathbb{E}_{\mu_0}\tau_1^0} \qquad \mu_0\text{-a.s. } (\nu^0\text{-a.s.})$$

where $\tau_1^0 = \sqrt{\frac{2\xi_1}{f}}$ is Rayleigh with $\mathbb{E}_{\mu_0}\tau_1^0 = \frac{\sqrt{\pi}}{\sqrt{2\varrho f}}$ and $\xi_1$ is exponential with $\mathbb{E}_{\mu_0}\xi_1 = \frac{1}{\varrho}$. As for the annihilation dynamics of M2, by the LLN for the Poisson process and for i.i.d. random variables,

$$\lim_{t\to+\infty} \frac{\widehat{Q}_t^0}{t} = \lim_{t\to+\infty} \frac{\widehat{\vartheta}_t}{t} \left( \frac{\sum_{k=1}^{\widehat{\vartheta}_t} \widehat{\xi}_k^0}{\widehat{\vartheta}_t} \right) = \frac{\mathbb{E}_{\mu_0}\widehat{\xi}_1^0}{\mathbb{E}_{\mu_0}\widehat{\tau}_1} \qquad \mu_0\text{-a.s. } (\widehat{\nu}^0\text{-a.s.})$$

where $\widehat{\xi}_1^0 = \frac{f}{2}\widehat{\tau}_1^2$ is Weibull with $\mathbb{E}_{\mu_0}\widehat{\xi}_1^0 = \frac{f}{\widehat{\varrho}^2}$ and $\widehat{\tau}_1$ is exponential with $\mathbb{E}_{\mu_0}\widehat{\tau}_1 = \frac{1}{\widehat{\varrho}}$. $\square$

**Remark 4.3.**

1. In M1, due to recollisions, we have $\tau_n^0 \le \tau_n$ for any $n \ge 1$ and $V_t|_{t\in(t_{n-1},t_n)} \le V_t^0|_{t\in(t_{n-1}^0,t_n^0)}$ for any $n \ge 1$ and initial configurations such that $x_0 = x_0^0$. The formal proof of the last statement follows the lines of Proposition 3.3. in [14]. In particular, $\vartheta_t \le \vartheta_t^0$ and consequently $\limsup_{t\to+\infty} \frac{Q_t}{t} \le v_d^0$ $\mu_0$-a.s.

2. In M2, due to recollisions, we have $\widehat{V}_t \leq \widehat{V}_t^0$ for any $t > 0$, but $\widehat{\tau}_n = \widehat{\tau}_n^0$ for any $n \geq 1$ and configurations such that $\widehat{x}_0 = \widehat{x}_0^0$. Consequently, $\widehat{\xi}_n \leq \widehat{\xi}_n^0$ for any $n \geq 1$ and $\limsup_{t\to+\infty} \frac{\widehat{Q}_t}{t} \leq \widehat{v}_d^0$ $\mu_0$-a.s.

# 5 Renewal structure of the dynamics

Coming back to the complete models M1 and M2, we make the following definitions.

**Definition 5.1.**

1. A moment $t > 0$ is called a time of (total) annihilation for the initial configuration $x \in X$ resp. $\widehat{x} \in \widehat{X}$ iff at $t > 0$, all n.p.s such that the t.p. had met before time $t$ are annihilated from the system (extinct) and the remaining memory of the past is contained only in the velocity of the t.p. at that time. We denote by $\mathbb{D}(x) \subseteq \mathbb{R}_+$ resp. $\mathbb{D}(\widehat{x}) \subseteq \mathbb{R}_+$ the (random) set of all times of annihilation for $x$ resp. $\widehat{x}$.

2. A first collision time $t_k > 0$ resp. $\widehat{t_k} > 0$ is called a cluster time for the initial configuration $x \in X$ resp. $\widehat{x} \in \widehat{X}$ iff the t.p. will never collide for any $t > t_k$ resp. $t > \widehat{t_k}$ with the n.p.s it had collided with for $t \leq t_k$ resp. $t \leq \widehat{t_k}$, including the n.p. it collides with at $t_k$ resp. $\widehat{t_k}$. If $t_k$ resp. $\widehat{t_k}$ is a cluster time for $x$ resp. $\widehat{x}$, we call the (random) integer $k$ a cluster index for $x$ resp. $\widehat{x}$.

3. A cluster time $t_k$ resp. $\widehat{t_k}$ is called double cluster time for $x$ resp. $\widehat{x}$ iff $t_{k-1}$ resp. $\widehat{t_{k-1}}$ is as well a cluster time for $x$ resp. $\widehat{x}$. The index $k$ is then called double cluster index for $x$ resp. $\widehat{x}$.

**Remark 5.2.**

1. In M1 (as well as in M2), the set $\mathbb{D}$ can be written as $\mathbb{D} = \mathbb{R}_+ \setminus \bigcup_{n \in \mathbb{N}} I_n$, the so-called uncovered set of $\mathbb{R}_+$ (cf. [7] and [12]), where $I_n = (t_n, t_n + \chi_n)$ are the lifetimes intervals. The zero is always contained in $\mathbb{D}$ if there are no initially moving n.p.s.

2. If in M1 (as well as in M2), there is an infinite sequence of cluster indices $k_1 < k_2 < ...$ for the configuration $x$, then $\mathbb{R}_+ = \bigcup_{n \in \mathbb{N}} J_{k_n}$ where on each of the intervals $J_{k_n} = [t_{k_n}, t_{k_{n+1}})$ with $J_{k_n} \cap J_{k_m} = \emptyset$ for any $n \neq m$, the t.p. can interact only with n.p.s which it collided with for the first time in such an $J_{k_n}$ and only with those. One then might say that the dynamics is regenerative or splits into independent clusters (blocks) $\mathcal{C}_n(x) = \{T_{k_n + t} x : 0 \leq t \leq t_{k_{n+1}}\}, n \geq 1$, since

$$\mu_0(T_{k_n} A \cap B) = \mu_0(A)\mu_0(B)$$

for any $A \in \mathcal{B}(X)$, $B \in \{T_{k_n}^{-1} A : A \in \mathcal{B}(X)\}$ and $n \geq 1$.

3. In M1, let $\mathcal{A}$ the $\sigma$-algebra generated by the cluster times $(t_{k_n})_{n\in\mathbb{N}}$. Then under the conditional measure $\mu_0^{\mathcal{A}} = \mu_0(.|\mathcal{A})$, the intercollision times $(\tau_{k_n+1})_{n\in\mathbb{N}}$ are i.i.d. Rayleigh with variance $\Gamma > 0$. Under $\mu_0^{\mathcal{A}}$, the vectors $(\tau_{k_n+2}, ..., \tau_{k_{n+1}})_{n\in\mathbb{N}}$ are independent and for any $m \neq n$, the $m$-vector $(\tau_{k_m+2}, ..., \tau_{k_{m+1}})$ is independent of $\tau_{k_n+1}$. As for M2, by symmetry, denoting by $\widehat{\mathcal{A}}$ the $\sigma$-algebra generated by the cluster times $(\widehat{t}_{k_n})_{n\in\mathbb{N}}$, then under the conditional measure $\mu_0^{\widehat{\mathcal{A}}} = \mu_0(.|\widehat{\mathcal{A}})$, the travelled distances $(\widehat{\xi}_{k_n+1})_{n\in\mathbb{N}}$ are i.i.d. Weibull with variance $\widehat{\Gamma} > 0$. Under $\mu_0^{\widehat{\mathcal{A}}}$, the vectors $(\widehat{\xi}_{k_n+2}, ..., \widehat{\xi}_{k_{n+1}})_{n\in\mathbb{N}}$ are independent and for any $m \neq n$, the $m$-vector $(\widehat{\xi}_{k_m+2}, ..., \widehat{\xi}_{k_{m+1}})$ is independent of $\widehat{\xi}_{k_n+1}$.

## 5.1 Random line covering

For future purposes, we make the following definiton.

**Definition 5.3.** Let $\Xi = (t_n, \chi_n)_{n\in\mathbb{N}}$ resp. $\widehat{\Xi} = (\widehat{t}_n, \chi_n)_{n\in\mathbb{N}}$ be the point process associated to M1 resp. M2 and denote by $\mathbb{D}_\Xi$ resp. $\mathbb{D}_{\widehat{\Xi}}$ the uncovered sets generated by them. We then say that $\Xi$ resp. $\widehat{\Xi}$ is non-covering iff $\mathbb{D}_\Xi$ resp. $\mathbb{D}_{\widehat{\Xi}}$ is unbounded $\mu_0$-a.s.

In the following, we will be concerned with M2 only and associate to $\widehat{\Xi} = (\widehat{t}_n, \chi_n)_{n\in\mathbb{N}}$ the auxiliary piecewise deterministic process $\widehat{R} = (\widehat{R}_t)_{t\in\mathbb{R}_+}$ which is formally defined as follows. If $\widehat{\eta}_t = t - t_{\widehat{\vartheta}_t}$ is the time spent since the last first collision, the linear part of $\widehat{R}$ in $\widehat{t}_{\widehat{\vartheta}_t} < t < \widehat{t}_{\widehat{\vartheta}_t+1}$ is given by

$$\widehat{R}_t = (\widehat{R}_{\widehat{t}_{\widehat{\vartheta}_t}} - \widehat{\eta}_t)^+ = (\widehat{R}_{\widehat{t}_{\widehat{\vartheta}_t}} - \widehat{\eta}_t) \vee 0 = (\widehat{R}_{\widehat{t}_{\widehat{\vartheta}_t}} - \widehat{\eta}_t)\mathbb{1}_{\{\widehat{R}_{\widehat{t}_{\widehat{\vartheta}_t}} > \widehat{\eta}_t\}}$$

which is decreasing linearly with slope one since $\widehat{\eta}$ is increasing in $(\widehat{t}_{\widehat{\vartheta}_t}, \widehat{t}_{\widehat{\vartheta}_t+1})$. The process possibly jumps at $t = \widehat{t}_n$ for some $n \geq 1$ by magnitude $\Delta\widehat{R}_{\widehat{t}_n} = (\chi_n - \widehat{R}_{\widehat{t}_n-})^+$, in particular

$$\widehat{R}_{\widehat{t}_n} = \chi_n \mathbb{1}_{\{\chi_n > \widehat{R}_{\widehat{t}_n-}\}} + \widehat{R}_{\widehat{t}_n-}\mathbb{1}_{\{\chi_n \leq \widehat{R}_{\widehat{t}_n-}\}}.$$

Assuming that $\widehat{R}_0 = r \geq 0$, the process has therefore the representation

$$\widehat{R}_t = (r - t + \sum_{k=1}^{\widehat{\vartheta}_t} (\chi_k - \widehat{R}_{\widehat{t}_k-})^+) \vee 0$$

for any $t \geq 0$. We refer to $\mathbb{P}_r$ as the distribution of $\widehat{R}$ starting at $r \geq 0$, and we consider the naturally filtered, complete probability space $(\widehat{X}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}_t})_{t\in\mathbb{R}_+}, \mu_0)$.

**Lemma 5.4.** The process $\widehat{R} = (\widehat{R}_t)_{t\in\mathbb{R}_+}$ is (strong) Markov with infinitesimal generator

$$\mathcal{L}^{\widehat{R}}\varphi(r) = \varrho \int_r^\infty \varphi'(s)\overline{F}_{\chi_1}(s)ds - \varphi'(r),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded and continuous with bounded derivatives.

*Proof.* The (strong) Markov property is standard and follows since the increment $\widehat{R}_{\widehat{H}+t} - \widehat{R}_{\widehat{H}} = (\sum_{\widehat{\vartheta}_{\widehat{H}}+1}^{\widehat{\vartheta}_{\widehat{H}+t}}(\chi_k - \widehat{R}_{\widehat{t}_k-})^+ - t) \vee 0$ starts at zero and is independent of $\widehat{\mathcal{F}}_{\widehat{H}}$ where $\widehat{H}$ is a finite $\widehat{\mathcal{F}}_t$-stopping time, due to the fact that $(\widehat{\vartheta}_{\widehat{H}+t} - \widehat{\vartheta}_{\widehat{H}})_{t\in\mathbb{R}_+}$ is a Poisson process, independent of $\widehat{\mathcal{F}}_{\widehat{H}}$ by the lack of memory of the exponential distribution. From the representation of $\widehat{R}$ one reads directly its inifinitesimal generator as

$$\mathcal{L}^{\widehat{R}}\varphi(r) \quad = \quad -\varphi'(r) + \widehat{\varrho}\mathbb{E}_{\mu_0}(\varphi(\chi_1 \vee r) - \varphi(r))$$

which is well-defined for $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ bounded and continuous with bounded derivatives. The last expression can be brought into the desired form using that

$$\begin{aligned}
\mathbb{E}_{\mu_0}\varphi(\chi_1 \vee r) &= \int_0^\infty \varphi(s \vee r)F_{\chi_1}(ds) \\
&= \varphi(r)F_{\chi_1}(r) + \int_r^\infty \varphi(s)F_{\chi_1}(ds) \\
&= \varphi(\infty) - \int_r^\infty \varphi'(s)F_{\chi_1}(s)ds \\
&= \int_r^\infty \varphi'(s)\overline{F}_{\chi_1}(s)ds + \varphi(r),
\end{aligned}$$

by virtue of integration by parts in the third equality. □

**Remark 5.5.** In the following we will use that by construction, $\mathbb{D}_{\widehat{\Xi}} = \{t \geq 0 : \widehat{R}_t = 0\}$ $\mu_0$-a.s. and that if the zero is a recurrent state, by the lack of memory, $\widehat{R}$ will hit the zero infinitely often eventually $\mu_0$-a.s. and hence $\mathbb{D}_{\widehat{\Xi}}$ is $\mu_0$-a.s. unbounded.

We use the following criterion for non-covering which is due to [1]. For completeness, we will give the proof which follows standard techniques.

**Lemma 5.6.** The Poisson point process $\widehat{\Xi}$ is non-covering iff $\widehat{\varrho} > 0$ and $F_{\chi_1}$ are such that

$$\int_0^\infty \exp(-\widehat{\varrho}\int_0^t \overline{F}_{\chi_1}(s)ds)dt = +\infty.$$

*Proof.* Define the survival probability

$$\pi(r) = \mu_0(\widehat{R}_t > 0 \text{ for any } t \geq 0 | \widehat{R}_0 = r)$$

and the first time of total annihilation

$$\widehat{H}(r) = \inf\{t > 0 : \widehat{R}_t = 0 | \widehat{R}_0 = r\}.$$

with $\pi(0) = 0$ and $\pi(+\infty) = 1$, and $\widehat{H}(+\infty) = +\infty$ $\mu_0$-a.s. Observe that by the Markov property of $\widehat{R}$, $\pi$ is a bounded invariant function, i.e. $\mathbb{E}_r\pi(\widehat{R}_t) = \pi(r)$ and hence $\mathcal{L}^{\widehat{R}}\pi(r) = 0$ if $\pi$ is in the domain of $\mathcal{L}^{\widehat{R}}$ (which will indeed be the case by the verification argument below). By standard arguments, $(\pi(\widehat{R}_t))_{t\in\mathbb{R}_+}$ is a martingale iff $\mathcal{L}^{\widehat{R}}\pi(r) = 0$. Plugging now in and differentiation to get rid of the integral term yields

$$\pi''(r) = -\widehat{\varrho}\overline{F}_{\chi_1}(r)\pi'(r).$$

Solving this equation and using integration, subject to the initial condition $\pi(0) = 0$, gives the expression for the survival probability

$$\pi(r) = \beta\int_0^r \exp(-\widehat{\varrho}\int_0^s \overline{F}_{\chi_1}(u)du)ds$$

for some norming constant $\beta \geq 0$. In particular $\pi$ is continuous and differentiable. Since $0 \leq \pi \leq 1$ and $\lim_{r\to+\infty}\pi(r) = \pi(+\infty) = 1$, if the right hand side above does not converge, it follows that one must have $\beta = 0$ and hence $\pi(r) = 0$ resp. $\widehat{H}(r) < +\infty$ $\mu_0$-a.s. for any $r > 0$. In other words, the zero is a recurrent state for $\widehat{R}$ and it follows from Remark 5.5. that $\mathbb{D}_{\widehat{\Xi}}$ is $\mu_0$-a.s. unbounded. On the other hand, if the above integral converges as $r \to +\infty$, again from $\lim_{r\to+\infty}\pi(r) = \pi(+\infty) = 1$ it follows that

$$\beta^{-1} = \int_0^\infty \exp(-\widehat{\varrho}\int_0^s \overline{F}_{\chi_1}(u)du)ds < +\infty$$

and $\pi(r) > 0$ for any $r > 0$. Since $\pi$ is a positive bounded invariant function, the process $(\widehat{M}_{t\wedge\widehat{H}(r)})_{t\in\mathbb{R}_+}$ is a (stopped) martingale which is defined by

$$\widehat{M}_t = \pi(\widehat{R}_t) = \mathbb{P}_{\widehat{R}_t}(\widehat{R}_s > 0 \text{ for any } s \geq 0)$$

with $\widehat{M}_0 = \pi(r) > 0$ for any $r > 0$ and $0 < \widehat{M}_t \leq 1$. By optimal stopping, $\mathbb{E}_{\mu_0}\widehat{M}_t = \mathbb{E}_{\mu_0}\widehat{M}_0 = \pi > 0$ for any $t \geq 0$. By continuity, $\pi(+\infty) = 1$ and the martingale convergence theorem, the limit $\widehat{M}_\infty = \lim_{t\to+\infty}\widehat{M}_t$ exists and equals to $\mathbb{1}_{\widehat{A}} = \widehat{M}_\infty = 1$ $\mu_0$-a.s. for the invariant set $\widehat{A} = \{\widehat{R}_t > 0$ for any $t \geq 0\}$. Taking expectation and bounded convergence yields $\pi(r) = \mathbb{E}_{\mu_0}\mathbb{1}_{\widehat{A}} = 1$ for any $r > 0$ which entails $\lim_{t\to+\infty}\widehat{R}_t = +\infty$ resp. $\widehat{H}(r) = +\infty$ $\mu_0$-a.s. for any $r > 0$. If $r = 0$, we set $\widehat{N}_t = \widehat{M}_{\widehat{\tau}_1\vee t}$ starting at $\widehat{N}_0 = \pi(\chi_1) > 0$ instead of $\widehat{M}_t$ and proceed as before. Hence if $r = 0$, then $\mathbb{D}_{\widehat{\Xi}} = \{0\}$ $\mu_0$-a.s. resp. if $r > 0$, then $\mathbb{D}_{\widehat{\Xi}} = \emptyset$ $\mu_0$-a.s. in this case. □

**Remark 5.7.** The lack of memory property of the exponential distribution is essential for the Markov property of $\widehat{R}$, since otherwise $\widehat{\vartheta} = (\widehat{\vartheta}_t)_{t\in\mathbb{R}_+}$ is not Markov and the future of $\widehat{R}$ depends through the distribution of the time to the next arrival on the past via the time spent since the last first collision of a particle, and hence the method as in the proof of Lemma 5.6 cannot be applied. If we denote by $\vartheta^0$ a renewal counting process with finite mean $\frac{1}{\varrho}$ and existing second moment of the i.i.d. interarrival times $(\tau_n^0)_{n\in\mathbb{N}}$ which have a absolute continuous distribution with density $f_{\tau_1^0}$ and distribution function $F_{\tau_1^0}$, which is independent of the lifetimes, then one can make the associated process $R^0 = (R^0)_{t\in\mathbb{R}_+}$ being Markovian without enlarging the underlying probability space by considering the bivariate process $(R^0, \alpha^0) = (R_t^0, \alpha_t^0)_{t\in\mathbb{R}_+}$, where $\alpha_t^0 = t - t^0_{\vartheta_t^0}$ is the spent time since the last first collision before time $t > 0$. The process $(R^0, \alpha^0)$ evolves deterministically for $s \in (t, t+\alpha_t^0)$ as $(R_s^0, \alpha_s^0) = (R_t^0 - (s-t), \alpha_t^0 - (s-t))$ and for $s = t^0_{\vartheta_t^0+1}$, $(R_s^0, \alpha_s^0) = (\chi_{\vartheta_t^0+1} \vee R^0_{t^0_{\vartheta_t^0+1}-}, 0)$. Since $\alpha^0$ is a (strong) Markov process, $(R^0, \alpha^0)$ becomes a strong Markov process too, whose infinitesimal generator can be calculated from the same arguments as the one in the Poisson case as

$$\mathcal{L}^{(R^0,\alpha^0)}\varphi(r,a) = \frac{f_{\tau_1^0}(a)}{\overline{F}_{\tau_1^0}(a)}\int_r^\infty \partial_y\varphi(y,a)\overline{F}_{\chi_1}(y)dy - \partial_r\varphi(r,a) + \partial_a\varphi(r,a)$$

for $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ in the domain of the generator, noting that $\alpha^0$ increases deterministically between two consecutive arrivals and if $\varrho(a)da + o(da)$ denotes the probability that there is an arrival in the interval $(a, a+da)$ conditionally that up to time $a > 0$ there is no arrival, then $\varrho(a) = \frac{f_{\tau_1^0}(a)}{\overline{F}_{\tau_1^0}(a)} = \lim_{h\to 0}\frac{1}{h}\mu_0(a < \tau_1^0 < a+h|\tau_1^0 > a)$ is the hazard (failure) rate for the distribution of the interarrival times. Like in the Poisson case, $(\varphi(R_t, \alpha_t))_{t\in\mathbb{R}_+}$

is a martingale iff

$$\partial_r \varphi(r,a) = \varrho(a) \int_r^\infty \partial_y \varphi(y,a) \overline{F}_{\chi_1}(y)dy + \partial_a \varphi(r,a).$$

One then could try to solve this PDE to get a criterion for non-covering as in the Poisson case. One example for such a renewal process in the light of the annihilation dynamics of M1 or M2 as in section 4 is where $\vartheta^0$ is a Weibull renewal counting process. Rather than following this strategy, we will take a different route in section 5.2.

**Remark 5.8.** Since $\widehat{\vartheta}$ is Poisson, the number $\widehat{\mathcal{N}}(A)$ of coordinates which fall into the set $A \in \mathcal{B}(\mathbb{H})$ is Poisson distributed with mean $\widehat{n}(A) = \widehat{\varrho} \int_A F_{\chi_1}(ds)dt$.

1. Hence for $A_t = \{(s,u) : 0 < t-s < u\}$ and $t > 0$ fixed, the probability that at time $t$ all n.p.s born before $t$ are annihilated (i.e. that $t$ is in the set $\mathbb{D}_{\Xi}$) is given by

   $$\mu_0(\widehat{R}_t = 0) = \mu_0(\widehat{\mathcal{N}}(A_t) = 0) = \exp(-\widehat{\varrho} \int_0^t \overline{F}_{\chi_1}(t-s)ds)$$

   resp. the mean number of alive n.p.s at time $t > 0$ is

   $$\widehat{n}(A_t) = \mathbb{E}_{\mu_0} \widehat{\mathcal{N}}(A_t) = \widehat{\varrho} \int_0^t \overline{F}_{\chi_1}(t-s)ds.$$

2. By symmetry and since $\overline{F}_{\chi_1}(t-s) = \overline{F}_{\chi_1}(s) - \mu_0(s < \chi_1 \le t-s)$ on $(0, \frac{t}{2}]$, one has

   $$\int_0^t \overline{F}_{\chi_1}(t-s)ds = 2\int_0^{t/2} \overline{F}_{\chi_1}(t-s)ds \le 2\int_0^t \overline{F}_{\chi_1}(s)ds.$$

   Thus if $\mathbb{E}_{\mu_0}\chi_1 < +\infty$ and using that $\mathbb{E}_{\mu_0}\chi_1 = \int_0^\infty \overline{F}_{\chi_1}(s)ds$, it follows that $\mu_0(\widehat{R}_t = 0) > 0$ and $\widehat{n}(A_t) \le 2\widehat{\varrho}\mathbb{E}_{\mu_0}\chi_1$ for any $t$, in particular $\int_0^\infty \exp(-\widehat{\varrho}\int_0^t \overline{F}_{\chi_1}(s)ds)dt = +\infty$.

We summarize the previous results in the following corollary.

**Corollary 5.9.** If $\widehat{\varrho} > 0$ and $F_{\chi_1}$ are such that

$$\int_0^\infty \exp(-\widehat{\varrho} \int_0^t \overline{F}_{\chi_1}(s)ds)dt = +\infty,$$

then there exists an infinite sequence $\widehat{H}_1 < \widehat{H}_2 < ...$ of times of total annihilation such that $\widehat{H}_k < +\infty$ $\mu_0$-a.s. and

$$\widehat{H}_k = \inf\{t > \widehat{H}_{k-1} : \widehat{R}_t = 0\}$$

with $\widehat{H}_0 = 0$ where

$$\widehat{\gamma}_k = \widehat{H}_k - \widehat{H}_{k-1}$$

are i.i.d. for any $k \geq 1$. The uncovered set generated by $\widehat{\Xi}$,

$$\mathbb{D}_{\widehat{\Xi}} = \bigcup_{k\in\mathbb{N}} (\widehat{H}_k, \widehat{t}_{\widehat{\vartheta}_{\widehat{H}_k}+1}),$$

is unbounded $\mu_0$-a.s. where

$$\widehat{\delta}_k = \widehat{t}_{\widehat{\vartheta}_{\widehat{H}_k}+1} - \widehat{H}_k$$

is i.i.d. exponential with intensity $\widehat{\varrho} > 0$ for any $k \geq 1$. $\square$

Concerning moments, we have the following results.

**Lemma 5.10.** If $\mathbb{E}_{\mu_0}\chi_1 < +\infty$, then $\mathbb{E}_{\mu_0}\widehat{\gamma}_1 < +\infty$.

*Proof.* If $\mathbb{E}_{\mu_0}\chi_1 < +\infty$, then $\int_0^\infty \exp(-\widehat{\varrho}\int_0^t \overline{F}_{\chi_1}(s)ds)dt = +\infty$ by Remark 5.7.2. On the other hand, if $\widehat{\mathcal{C}}_1$ is the first connected component, then

$$\mathbb{E}_{\mu_0}\widehat{\gamma}_1 \leq \mathbb{E}_{\mu_0}(\widehat{\tau}_1 + \sum_{k:\ \widehat{t}_k\in\widehat{\mathcal{C}}_1} \chi_k) < c_{\widehat{\varrho}}$$

for some constant $c_{\widehat{\varrho}} < +\infty$. $\square$

Using similar arguments we have the following corollary.

**Corollary 5.11.**

1. Suppose that there is a constant $a > 0$ such that $\mathbb{E}_{\mu_0}\exp(a\chi_1) < +\infty$. Then there is a constant $b > 0$ such that $\mathbb{E}_{\mu_0}\exp(b\widehat{\gamma}_1) < +\infty$.

2. If there is a constant $a > 0$ such that $\mathbb{E}_{\mu_0}\exp(a\chi_1) < +\infty$, then there are constants $c > 0$ and $c' > 0$ such that the i.i.d. sequence $(\widehat{\mathcal{N}}(\widehat{\mathcal{C}}_k))_{k\in\mathbb{N}}$ of the number of n.p.s in each connected component has tail

$$\mu_0(\widehat{\mathcal{N}}(\widehat{\mathcal{C}}_1) > n) \leq c\exp(-c'n).$$

## Excursion: The uncovered set as the image of an subordinator

The concrete distribution of $\widehat{\gamma}_1$ seems quite complicated. Using standard concepts from excursion theory, $\widehat{\gamma}_1$ equals to the first passage time of an associated subordinator above some fixed level and its distribution can be expressed in terms of the characteristics of the subordinator. We review shortly this construction. Suppose that $\widehat{R}_0 = 0$ and define the local time of $\widehat{R}$ at zero in $[0,t]$ as

$$\widehat{L}_t = \int_0^t \mathbb{1}_{\{\widehat{R}_s = 0\}} ds$$

which is constant on time intervals where $\widehat{R}$ is away from zero and increases on time intervals where $\widehat{R}$ hits the zero, hence $\mathbb{1}_{\{\widehat{R}_t=0\}} = 1$ iff $\widehat{L}_{t+s} > \widehat{L}_t$ for any $s > 0$. From the construction of the (strong) Markov process $\widehat{R}$ and the fact that the residual arrival time $\widehat{\kappa}_t = \widehat{t}_{\widehat{\vartheta}_t+1} - t$ at time $t \geq 0$ is exponential and has no atom at zero, one sees that $\mathbb{P}_0(\widehat{\gamma}_1 = 0) = \mu_0(\widehat{\kappa}_0 > 0) > 0$ which entails by Blumenthal's zero-one law that $\mathbb{P}_0(\widehat{\gamma}_1 = 0) = 1$, i.e. $\widehat{R}$ will hit the zero a.s. infinitely often during any initial time interval. It then follows from [3], Chapter V, that the right-continuous inverse of the local time of $\widehat{R}$ given by

$$\widehat{\sigma}_t = \widehat{L}_t^{-1} = \inf\{s \geq 0 : \widehat{L}_s > t\}$$

is a subordinator, i.e. an increasing Lévy process starting at zero whose Laplace exponent $\widehat{\Phi}(\theta) = -\log \mathbb{E}_{\mu_0} \exp(-\theta\widehat{\sigma}_1)$, $\theta \geq 0$, is given by the Lévy-Khintchin formula

$$\widehat{\Phi}(\theta) = \kappa\theta + \int_0^\infty (1 - \exp(-\theta s))\Pi(ds) = \theta(\kappa + \int_0^\infty \exp(-\theta s)\overline{\Pi}(s)ds)$$

where $\Pi$ is a Borel measure on $(0, +\infty)$ (Lévy measure) such that $\int_0^1 s\Pi(ds) = \int_0^1 \overline{\Pi}(s)ds < +\infty$ and $\kappa = \lim_{\theta\to+\infty} \frac{\Phi(\theta)}{\theta} > 0$ the drift. The range of $\widehat{\sigma}$ is the closure of the uncovered set $\mathbb{D}_{\widehat{\Xi}}$, whereas $\mathbb{D}_{\widehat{\Xi}}^c = \bigcup_{t\in J(\widehat{\sigma})} (\widehat{\sigma}_{t-}, \widehat{\sigma}_t)$ where $J(\widehat{\sigma})$ is the set of its jump times. Clearly, $\widehat{L}_t = \inf\{s \geq 0 : \widehat{\sigma}_s > t\}$ and the potential (renewal) measure is given by

$$\widehat{U}(B) = \int_B \mu_0(\widehat{R}_t = 0)dt = \int_0^\infty \mu_0(\widehat{\sigma}_t \in B)dt$$

for $B \in \mathcal{B}(\mathbb{R}_+)$ resp. $\widehat{U}([0,t]) = \mathbb{E}_{\mu_0}\widehat{L}_t$ for $t \geq 0$. As one sees from the Lévy-Khintchin formula, if $\widehat{\sigma}'$ is another subordinator with image the closure of $\mathbb{D}_{\widehat{\Xi}}$, there is a constant $c > 0$ such that $\widehat{\sigma}'_t = \widehat{\sigma}_{ct}$ $\mu_0$-a.s. for any $t \geq 0$. In this way, the subordinator is uniquely characterized up to a constant. Finally note that $\widehat{L}_\infty = +\infty$ $\mu_0$-a.s. iff the zero is eventually infinitely recurrent for $\widehat{R}$ $\mu_0$-a.s. Hence in terms of $\widehat{\sigma}$, if $\beta^{-1} = +\infty$, it follows as in Remark 5.8. that

$$\begin{aligned} \mathbb{E}_{\mu_0}\widehat{L}_\infty &= \int_0^\infty \mu_0(\widehat{R}_t = 0)dt \\ &\geq \int_0^\infty \exp(-2\widehat{\varrho}\int_0^t \overline{F}_{\chi_1}(s)ds)dt \\ &= c'\beta^{-1} = +\infty \end{aligned}$$

for some constant $c' > 0$, thus $\widehat{L}_\infty = +\infty$ $\mu_0$-a.s. and $\mathbb{D}_{\widehat{\Xi}}$ is unbounded. In particular, the renewal measure of $\widehat{\sigma}$ is a Radon measure with density

$$\widehat{u}(t) = \exp(-\widehat{\varrho}\int_0^t \overline{F}_{\chi_1}(t-s)ds).$$

To determine the distribution of $\widehat{\gamma}_1$, we can take the marginal distribution of the jump of $\widehat{\sigma}$ at its local time $\widehat{L}_t$ in Proposition 2, Chapter III, in [2], and obtain

$$F_{\widehat{\gamma}_1}(t) = \mu_0(\exists s > 0 : \widehat{\sigma}_s > t) = \mu_0(\widehat{\sigma}_{\widehat{L}_t} > t) = \int_0^t \overline{\Pi}(t-s)\widehat{u}(s)ds.$$

To make use of that expression, one has to recover the characteristics of $\widehat{\sigma}$. It is standard that the Laplace transform of the renewal measure can be expressed as

$$\Phi^{-1}(\theta) = \int_0^\infty \exp(-\theta t - \widehat{\varrho}\int_0^t \overline{F}_{\chi_1}(t-s)ds)dt,$$

whereas by the Lévy-Khintchin formula it follows that $\kappa = \lim_{\theta\to+\infty}\frac{\Phi(\theta)}{\theta}$ and $\int_0^\infty \exp(-\theta t)\overline{\Pi}(s)ds = \frac{\Phi(\theta)}{\theta} - \kappa$. Clearly, for some given lifetime distribution, in general except im some very special cases, it seems quite difficult to obtain concrete analytical expressions.

## 5.2 Comparison of line covering processes

The key difficulty is that in the original dynamics of M1 one can say very little about the distribution of the highly dependent intercollision times of the n.p.s. We therefore take a different approach and construct a suitable comparison between the dynamics of M2 and the random line covering problem for the annihilation dynamics in M1. For this purpose, we will introduce an auxiliary Poisson covering process, which we call $\epsilon$-reinforced covering process. As in the original covering process $\widehat{\Xi} = (\widehat{t}_n, \chi_n)_{n\in\mathbb{R}_+}$ associated with M2, we will assume that starting-points $\widehat{t}_n$ of the lifetime (covering) intervals $I_n = (t_n, t_n + \chi_n)$ are distributed according to the Poisson law with intensity $\widehat{\varrho} > 0$. However, from each such point now we will allow to start a multiple number of covering intervals, which are all going rightwards. The probability that out of a given point we start $k,\ k \geq 1$, additional intervals is geometric with parameter $\epsilon$, and equals to $\epsilon^k(1-\epsilon)$ for $0 < \epsilon < 1$, independently for each Poissonian point. The lengths of all intervals are chosen independently, and are distributed according to $F_{\chi_1}$ as before. The such obtained point process can thus be seen to be a Poisson covering process with a sequence of lifetimes which is constituted by the maximum lenghts of all lifetimes for every Poisson point. We thus have the following definition.

**Definition 5.12.** We say that $\widehat{\Xi}^\epsilon = (\widehat{t}_n, \chi_n^\epsilon)_{n\in\mathbb{N}}, 0 < \epsilon < 1$, is an $\epsilon$-reinforced covering process if it is a Poisson covering process with $\chi_n^\epsilon = \max_{1\leq k\leq \widehat{\mathcal{N}}_n^\epsilon} \chi_{nk}$ for $n \geq 1$, where $\{\chi_{nk} : n \in \mathbb{N}, 1 \leq k \leq \widehat{\mathcal{N}}_n^\epsilon\}$ is an independent array such that $(\widehat{\mathcal{N}}_n^\epsilon)_{n\in\mathbb{N}}$ is i.i.d. with $\mu_0(\widehat{\mathcal{N}}_1^\epsilon = k) = \epsilon^k(1-\epsilon)$ and independent of $(\chi_{nk})_{n\in\mathbb{N}}$ for any $k \geq 1$.

Recall that we denote by $\Xi = (t_n, \chi_n)_{n\in\mathbb{N}}$ the covering process associated to M1 and by $\Xi^0 = (t_n^0, \chi_n)_{n\in\mathbb{N}}$ the covering process associated to its annihilation dynamics where $\tau_n^0 = t_n^0 - t_{n-1}^0$ are i.i.d. Rayleigh distributed. The first comparison result below states that the annihilation dynamics dominates the covering of the complete dynamics of M1. The proof is based on the observation that while with larger intercollision times, uncovered components are shifted to the right, but they cannot shrink. This could be made formal, but since it is graphically clear, we will not give the more technically involved proof here.

**Proposition 5.13.** If $\Xi^0 = (t_n^0, \chi_n)_{n\in\mathbb{N}}$ is non-covering, then $\Xi = (t_n, \chi_n)_{n\in\mathbb{N}}$ is non-covering. □

The second comparison result states that the Poisson covering of M2 dominates the covering of its $\epsilon$-reinforced version. Though for the proof we will need the finite moment condition for the lifetime distribution.

**Proposition 5.14.** If $\mathbb{E}_{\mu_0}\chi_1 < +\infty$, then for any $0 < \epsilon < 1$, the $\epsilon$-reinforced

Poisson line covering process $\widehat{\Xi}^\epsilon$ is non-covering.

*Proof.* Note that by Remark 5.8.2, the condition $\mathbb{E}_{\mu_0}\chi_1 < +\infty$ entails that $\widehat{\Xi}$ is non-covering. Let $\widehat{\Delta}_k = \widehat{H}_k - \widehat{t}_{\vartheta_{\widehat{H}_{k-1}}+1}, k \geq 1$, be the length of the $k$-th connected component $\widehat{\mathcal{C}}_k$ where $\widehat{H}_k = \inf\{t > \widehat{H}_{k-1} : \widehat{R}_t = 0\}$ is the $k$-th time of total annihilation as in Corollary 5.9. If $\mathbb{E}\chi_1 < +\infty$, then $\mathbb{E}\widehat{\Delta}_1 < +\infty$ by Lemma 5.10 and since $\widehat{\Xi}$ is non-covering, the uncovered set splits into $\mathbb{D}_{\widehat{\Xi}} = \bigcup_{k\in\mathbb{N}}(\widehat{a}_k - \widehat{H}_{k-1})$ where we have set for short $\widehat{a}_k = t_{\vartheta_{\widehat{H}_{k-1}}+1}, k \geq 1$. Let $t \in (\widehat{a}_i - \widehat{H}_{i-1})$ for some large $i \geq 1$ and denote by $\widehat{U}_k(t)$ the event that the additional lifetime of at least one n.p. in some connected component $\widehat{\mathcal{C}}_k$, $k \leq i-1$, is covering $t$. If we denote by $\chi_{nj}$ the $j$-th lifetime at the $n$-th arrival time $t_n$ as in Definition 5.12, then we have

$$\widehat{U}_k(t) \subseteq \{ \sum_{j\geq 2,\ n:\ t_n \in \widehat{\mathcal{C}}_k} \chi_{nj} > \sum_{k+1\leq j\leq i-1} (\widehat{a}_j - \widehat{H}_{j-1})\}.$$

But $\sum_{j\geq 2,\ n:\ t_n\in\mathcal{C}_k} \chi_{nj}$ are i.i.d. random variables for any $k$ with

$$\mathbb{E}_{\mu_0}( \sum_{j\geq 2,\ n:\ t_n \in \widehat{\mathcal{C}}_k} \chi_{nj}) < c_\epsilon,$$

for some constant $c_\epsilon < +\infty$, while $\mathbb{E}_{\mu_0} \sum_{k+1\leq j\leq i-1} (\widehat{a}_j - \widehat{H}_{j-1}) = (k-i)\widehat{\varrho}$. Since $\mathbb{E}_{\mu_0}\chi_1 < +\infty$ (and as a consequence $\mathbb{E}_{\mu_0}\widehat{\Delta}_1 < +\infty$) we get by Borel-Cantelli that

$$\mu_0(\{ \sum_{j\geq 2,\ n:\ t_n \in \widehat{\mathcal{C}}_k} \chi_{nj} > \sum_{k+1\leq j\leq i-1} (\widehat{a}_j - \widehat{H}_{j-1})\} \text{ infinitely often}) = 0,$$

which implies that $\mu_0(\bigcap_{k\leq i-1} \widehat{U}_k^c(t)) > 0$ from which the assertion follows. □

We finally we come to the main result from which we deduce the strong cluster property of the dynamics in M1 in section 5.

**Lemma 5.15.** If $\mathbb{E}_{\mu_0}\chi_1 < +\infty$, then there is a $\varrho > 0$ with $0 < \varrho < \widehat{\varrho}$ such that the corresponding covering process $\Xi^0 = (t_n^0, \chi_n)_{n\in\mathbb{N}}$ is non-covering.

*Proof.* As before, $\mathbb{E}_{\mu_0}\chi_1 < +\infty$ implies that $\widehat{\Xi} = (\widehat{t}_n, \chi_n)_{n\in\mathbb{N}}$ is non-covering. Recall that $(\widehat{\tau}_n)_{n\in\mathbb{N}}$ and $(\tau_n^0)_{n\in\mathbb{N}}$ are independent and note that for $f > 0$ and $\widehat{\varrho} > 0$ fixed, there is $\varrho > 0$ such that $\mu_0(\tau_1^0 < \widehat{\tau}_1) \leq \epsilon$ for some given $0 < \epsilon < 1$. We will now perform the following surgery. Every segment $[\widehat{t}_{n-1}, \widehat{t}_n]$ such that

$\tau_n^0 < \widehat{\tau}_n$ will be subtracted by performing the left-shift of the semi-infinite configuration lying to the right of the Poisson point $\widehat{t}_n$ by the distance $\widehat{\tau}_n$ and identifying the point $\widehat{t}_{n-1}$ with $t_n^0$, while the point $\widehat{t}_{n-1}$ is endowed with the two lifetimes $\chi_n$ and $\chi_{n-1}$, and every segment $[\widehat{t}_{n-1}, \widehat{t}_n]$ such that $\tau_n^0 \geq \widehat{\tau}_n$ will be contracted by performing the left-shift of the semi-infinite configuration lying to the right of the Poisson point $\widehat{t}_n$ by the distance $\tau_n^0 - \widehat{\tau}_n \geq 0$ and identifying the point $\widehat{t}_n$ with the point $t_n^0$, while each of the points continues to have one lifetime, namely $\widehat{t}_n = t_n^0$ with $\chi_n$ and $\widehat{t}_{n-1}$ with $\chi_{n-1}$. The such obtained process $\widehat{\Xi}^\epsilon$ say, is the $\epsilon$-reinforced line covering process of $\widehat{\Xi}$, where for notational reasons we assume that intervals which were completely subtracted have lengths equal to zero. Since $\widehat{\Xi}$ is non-covering, according to Proposition 5.14 one can choose any $0 < \epsilon < 1$ such that $\widehat{\Xi}^\epsilon$ is non-covering. □

**Corollary 5.16.** If $\mathbb{E}\chi_1 < +\infty$, then $\Xi^0$ is non-covering for any $\varrho > 0$.

*Proof.* Note that an inspection of the proof of Lemma 5.15 implies that by inverting the roles of $\widehat{\Xi}$ and $\Xi^0$, for any given $\varrho > 0$ one can choose a sufficiently large $\widehat{\varrho} > 0$ such that $\Xi^0$ is non-covering. We then apply Proposition 5.13 and the assertion follows. □

## 5.3 Construction of cluster indices

By a simple mechanical argument we now show that the non-covering of $\Xi$ resp. $\widehat{\Xi}$ is a sufficient condition to guarantee the existence of cluster times in M1 resp. M2 as defined in Definition 5.1. Conversely though, from the opposite one cannot deduce in general the absence of cluster times since it does not exclude a more mechanical continuous loss of memory.

**Lemma 5.17.** Suppose that $\mathbb{E}_{\mu_0}\chi_1 < +\infty$. Then there exists an infinite sequence of cluster times for M1 and M2.

*Proof.* Recall that $\mathbb{E}_{\mu_0}\chi_1 < +\infty$ implies that there is an infinite sequence $\widehat{H}_1 < \widehat{H}_2 < ...$ of times of total annihilation for M2 by Corollary 5.9 where $(\widehat{\gamma}_k)_{k\in\mathbb{N}}$ is i.i.d. with distribution function $F_{\widehat{\gamma}_1}$ and $\widehat{H}_k < +\infty$ $\mu_0$-a.s. for any $k \geq 1$. The same is true for M1 by Corollary 5.16. We treat in the following only M1, since the case for M2 is analogous. Given $\gamma_1 = s$, fix a constant $a > 0$ and denote by $V_\star = V_s$ the velocity of the t.p. at that time. We can suppose that $V_\star = 0$. Then due to exclusion of recollisions, $\mu_0(\tau^0_{\vartheta_s+1} > a) = \exp(-\frac{\varrho f}{2}a^2)$ for some fixed constant $a > 0$. On the set

$\{\tau^0_{\vartheta_s+1} > a\}$, the outgoing velocity of the next fresh n.p. is the constant $v^+_{\vartheta_s+1} > fa$. Choose now the lifetime $\chi_{\vartheta_s+1}$ of the n.p. smaller than the time the t.p. would need to catch it up in the free dynamics, i.e. with no further n.p.s in $(t_{\vartheta_s+1}, +\infty]$, in particular smaller than $b = 2a$ determined by $bv^+_{\vartheta_s+1} > \int_0^b fsds = \frac{f}{2}b^2$. With the event $A_{\gamma_1} = \{\tau^0_{\vartheta_{\gamma_1}+1} > a\} \cap \{\chi_{\vartheta_{\gamma_1}+1} \leq 2a\}$ we thus have

$$\begin{aligned}
&\mu_0(\exists s \leq t : t_{\vartheta_s+1} \text{ is a cluster time}) \\
&\qquad \geq \mu_0(\{\gamma_1 \leq t\} \cap A_{\gamma_1}) = \int_0^t \mu_0(A_s | \gamma_1 = s) F_{\gamma_1}(ds) \\
&\qquad = \exp(-\frac{\varrho f}{2} a^2) F_{\chi_1}(2a) F_{\gamma_1}(t)
\end{aligned}$$

by the independence of $\tau^0_{\vartheta_{\gamma_1}+1}$ and $\chi_{\vartheta_{\gamma_1}+1}$. □

**Corollary 5.18.** The dynamics of M2 has infinitely many cluster indices and moreover, under the condition of Theorem 3.2 on the existence of exponential moment, the inter-cluster-indices distribution has exponentially decaying tails. □

**Remark 5.19.** Later we consider also double cluster times resp. indices as in Definition 5.1.3 which is indeed analogous to the notion of so-called good cluster indices in [4]. To produce them in our case, replace the event $A_H = \{\tau_H > a\} \cap \{\chi_{\vartheta_H+1} \leq b\}$ by $A'_H = A_H \cap B_H$ where $B_H = \{H_{\vartheta_H+1} > b\} \cap \{\chi_{\vartheta_H+2} \leq 2b\}$ and then one has as above

$$\mu_0(\exists s \leq t : t_{\vartheta_s+1} \text{ is a double cluster time}) \geq c' F_H(t)$$

for some constant $c' = c'(a, V_\star, f, \varrho)$ with $0 < c' < \widehat{c}$. If we set $Y'_k = 1_{\{t_{\vartheta_{H_k}+1} \text{ is a double cluster time}\}}$ for $k \geq 1$, then $\mu_0(Y'_1 = 1) \geq c' > 0$ and $\mu_0(Y'_k = 1 | Y'_1, ..., Y'_{k-1}) \geq c' > 0$ for $k \geq 2$. It follows that there is a $\gamma' > 0$ such that $\liminf_{n\to\infty} n^{-1} \sum_{k=1}^n Y'_k \geq \gamma' > 0$ $\mu_0$-a.s.

**Lemma 5.20.** If $\mathbb{E}_{\mu_0}\chi_1 < +\infty$, then sequence of measures $\{\mu_n : n \in \mathbb{N}\}$ is tight.

*Proof.* To show tightness we have for the mean number of alive n.p.s at time $t > 0$ in M2 by Remark 5.8 that

$$\mathbb{E}_{\mu_0}\widehat{\mathcal{N}}(A_t) = \widehat{\varrho} \int_0^t \overline{F}_{\chi_1}(t-s)ds \leq 2\widehat{\varrho} \int_0^t \overline{F}_{\chi_1}(s)ds \leq 2\widehat{\varrho}\mathbb{E}_{\mu_0}\chi_1.$$

By Markov's inequality $\mu_0(\widehat{\mathcal{N}}(A_t) > k) \leq k^{-1}\mathbb{E}_{\mu_0}\widehat{\mathcal{N}}(A_t) \leq k^{-1}2\widehat{\varrho}\mathbb{E}_{\mu_0}\chi_1$ for $k > 0$, denoting $A_n = A_{t_n}$, since $\lim_{n\to\infty} t_n = \infty$ and $\mathbb{E}_{\mu_0}\chi_1 < +\infty$, it follows that $\mu_0(\widehat{R}_t = 0) > 0$ for any $t > 0$ and

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu_0(\widehat{\mathcal{N}}(A_n) > k) = 0$$

which entails by [11], Lemma 4.5, the tightness of $\{\widehat{\mathcal{N}}(A_n) : n \in \mathbb{N}\}$. If $\widehat{\mu}_n = \mu_0 \circ \widehat{T}_n^{-1}$ is the law of the discrete dynamics as defined in section 3.1, the above result entails the tightness of the family $\{\widehat{\mu}_n : n \in \mathbb{N}\}$ and hence the convergence to a necessarily invariant measure $\widehat{\mu}$ say, i.e. $\widehat{\mu} = \widehat{\mu} \circ \widehat{T}_n^{-1}$ for any $n \geq 1$ such that $\widehat{\mu}(\widehat{X}_1) = 1$. For M1, in the light of section 5.2, the interarrival times (of standing n.p.s) of the Weibull line covering process associated to the Markovian dynamics of M1 dominate the (exponential) interarrival times in M2, hence by the previous considerations, this implies that the (mean) number of alive n.p.s in M1 at any time is smaller than the (mean) number of alive n.p.s in M2 and tightness for M1 follows. Thus we have Theorem 3.2. □

# 6 Mixing and law of large numbers

Existence of some limiting invariant measure $\widehat{\mu}$ under the discrete dynamics $(\widehat{T}_n)_{n\in\mathbb{N}}$ for M2 follows by tightness of the family $\{\widehat{\mu}_n : n \in \mathbb{N}\}$ as shown above under the condition of finite first moment of the lifetime distribution. Uniqueness follows by successfully coupling $\widehat{\mu}$ with the initial measure $\mu_0$. In the light of the construction of the cluster times in section 5.3 it is quite clear how to produce a successful coupling. For this, let $\widehat{\omega} \in \widehat{\Omega}$ be a configuration with decomposition $x(\widehat{\omega}) = x_0(\widehat{\omega}) \cup x_m(\widehat{\omega})$ and distributed according to the measure $\widehat{\mu}$ where we denote by $x_0(\widehat{\omega})$ the (freshly) arriving n.p.s at the Poisson times. Take two initial configurations $\widehat{\omega}'$ and $\widehat{\omega}''$ distributed according to $\mu_0$ resp. $\widehat{\mu}$ such that $x_0(\widehat{\omega}) = x_0(\widehat{\omega}') = x_0(\widehat{\omega}'')$, $x_m(\widehat{\omega}') = \emptyset$, $x_m(\widehat{\omega}) = x_m(\widehat{\omega}'')$, $\widehat{R}_0(\widehat{\omega}') = 0$ and $\widehat{R}_0(\widehat{\omega}'') = r_0$ for some $r_0 > 0$ where $\widehat{R}$ is the associated piecewiese deterministic process from section 5.1. Note also that under $\widehat{\mu}$, the freshly arriving n.p.s and the moving n.p.s are independent and the fresh n.p.s distributed according to the initial measure $\mu_0$, i.e. $\widehat{\mu} = \mu_0 \otimes \widehat{\mu}_m$ with $\mu_0(\widehat{\Omega}_0) = \widehat{\mu}_m(\widehat{\Omega}_0^c) = 1$. In words, the configuration $\widehat{\omega}'$ consists only of the t.p. and the freshly arriving n.p.s distributed according to $\mu_0$ and $\widehat{\omega}''$ has the same arriving n.p.s as $\widehat{\omega}'$ and its moving n.p.s are distributed according to $\widehat{\mu}_m$ with initial maximal residual lifetime $r_0$, i.e. their joint distribution is the measure $\widehat{\mathbb{Q}}$ on $\widehat{\Omega}_0 \times \widehat{\Omega}_0$ given by

$$\widehat{\mathbb{Q}}(d\widehat{\omega}' d\widehat{\omega}'') = \mathbb{1}_{\{x_m(\widehat{\omega}')=\emptyset\}} \mathbb{1}_{\{x_0(\widehat{\omega}')=x_0(\widehat{\omega}'')\}} \mu_0(d\widehat{\omega}')\widehat{\mu}(d\widehat{\omega}'').$$

If convenient, we write the related quantities to the two configurations with the corresponding superscripts like $\widehat{H}_1'$ or $\widehat{H}_1''$ for $\widehat{H}_1(\widehat{\omega}')$ resp. $\widehat{H}_1(\widehat{\omega}'')$, for instance. With the same notation as in the previous sections, letting $\widehat{H}_{r_0}(\widehat{\omega}') = \inf\{s > r_0 : \widehat{R}_s(\widehat{\omega}') = 0\}$, one has $H_{r_0}(\widehat{\omega}') = \widehat{\sigma}_{\widehat{L}_{r_0}}(\widehat{\omega}') = \widehat{H}(\widehat{\omega}'') = \widehat{H}(\widehat{\omega})$ where $\widehat{\sigma}$ is the associated subordinator and $\widehat{L}_{r_0}$ its local time at $r_0 > 0$ (see excursion at the end of section 5.1). Set $\widehat{H}_k(\widehat{\omega}') = \widehat{H}_k(\widehat{\omega}'')$ and $\chi_k(\widehat{\omega}') = \chi_k(\widehat{\omega}'')$ for any $1 \le k \le [\widehat{H}(\widehat{\omega})]$ where [.] denotes the integer part and let $\widehat{V}_\star' = \widehat{V}_{\widehat{H}}(\widehat{\omega}') > 0$ and $\widehat{V}_\star'' = \widehat{V}_{\widehat{H}}(\widehat{\omega}'') > 0$. Fix now some $a' > 0$, sample $\widehat{\tau}_{\widehat{H}}(\widehat{\omega}')$ according to the exponential distribution $\mu_0(\widehat{\tau}_{\widehat{H}}' \le s | \widehat{\tau}_{\widehat{H}}' > a')$ (again by lack of memory) and then set $\widehat{\tau}_{\widehat{H}}(\widehat{\omega}'') = \widehat{\tau}_{\widehat{H}}(\widehat{\omega}')$ in the configuration $\widehat{\omega}''$. Both n.p.s arriving at $\widehat{t}_{\vartheta_{\widehat{H}}+1}' (= \widehat{t}_{\vartheta_{\widehat{H}}+1}'')$ have some minimal outgoing velocity $\widehat{v}_{min}'$ resp. $\widehat{v}_{min}''$, depending on $\widehat{V}_\star'$ resp. $\widehat{V}_\star''$ (and of $f$ and $\widehat{a}'$). If $\chi_{\vartheta_{\widehat{H}}+1}'' \le \widehat{b}'$ and $\chi_{\vartheta_{\widehat{H}}+1}'' \le \widehat{b}''$ for the corresponding constants $\widehat{b}'$ and $\widehat{b}''$, determined as in section 5.3, then $\widehat{t}_{\vartheta_{\widehat{H}}+1}'$ is a cluster time for both configurations $\widehat{\omega}'$ and $\widehat{\omega}''$. Setting the interdistances as $\widehat{\xi}_{\vartheta_{\widehat{H}}+k}' = \widehat{\xi}_{\vartheta_{\widehat{H}}+k}''$ for any $k \ge 2$ concludes the coupling in this case. Otherwise, repeat the above procedure now with $\widehat{H}(\widehat{T}^{r_0}\widehat{\omega}')$ instead of $\widehat{H}(\widehat{\omega})$. Since $\widehat{\Xi}$ is non-covering and all other events involved have strictly positive probability, a successful coupling also in this case will be achieved in finite time. It then follows from Corollary 5.11.2 under exponential moment condition that there are constants $c > 0$ and $c' > 0$ such that

$$\widehat{\mathbb{Q}}((\widehat{\omega}', \widehat{\omega}'') : \exists i \leq n : \widehat{t}_{\widehat{\vartheta}_{\widehat{H}_i}+1} \text{ is an } (\widehat{\omega}', \widehat{\omega}'')\text{-cluster time}) \geq 1 - ce^{-c'n}$$

for any $n \geq 1$ and the mixing is exponentially decaying.
An analogous procedure applies for M1 by the following reasoning. The moments of total extinction are stopping times with respect to the dynamics of M1 (and M2) and hence future positions of standing n.p.s are independent of the past. However future hitting times and velocities of the t.p. with the n.p.s depend on the velocity of the t.p. at the times of total extinction. But since the velocity (and thus the time it takes to arrive to the next standing n.p.) is bounded from below by the time and velocity of the t.p. if it would have zero velocity at the time of total extinction, the velocity of the t.p. at collision with the next standing n.p. is bounded by an exponential random variable and the time by the square root of it. In the same way as in the construction of cluster indices for M2, this allows now to request that the lifetime of the first n.p. after moments of total extinction is short enough such that it dies before the t.p. catches it up, even if it flies with the lowest possible velocity determined by the lower bound. Thus we have Theorem 3.1 and Corollary 3.3.1.

# 7 Invariance principle

The proof of the IP for the displacement of the t.p. in M1 and M2 under exponential mixing is now classic. We will give the proof for M2 only, while for M1 one has to interchange the roles of interarrival times and interdistances. Define the random element on $(\widehat{\Omega}_1, \mathcal{B}(\widehat{\Omega}_1), \widehat{\mu})$ by $\widehat{S}_{[nt]} = n^{-1/2}\widehat{Z}_{[nt]}$ where $\widehat{Z}_n = \sum_{1\le k\le n}(\widehat{\xi}_k - \widehat{\varrho}^{-1}\widehat{v}_d)$ and $\widehat{v}_d = \widehat{\varrho}\mathbb{E}_{\widehat{\mu}}\widehat{\xi}_1$ is the drift. Note that $\mathbb{E}_{\widehat{\mu}}\widehat{\xi}_1^0 = \mathbb{E}_{\widehat{\mu}_0}\widehat{\xi}_1^0 = f\widehat{\varrho}^{-2}$, hence $\mathbb{E}_{\widehat{\mu}}\widehat{\xi}_1 \le f\widehat{\varrho}^{-2}$ and $\mathbb{E}_{\widehat{\mu}}\widehat{\xi}_1^2 \le 6f^2\widehat{\varrho}^{-4}$. Furthermore, we have $\sum_{n\in\mathbb{N}}\sqrt{\widehat{\psi}(n)} < \infty$ since the mixing coefficient $\widehat{\psi}(n)$ is exponentially decaying by the previous section. Then $\widehat{\sigma}^2 = \lim_{n\to\infty} n^{-1}\mathbb{E}_{\widehat{\mu}}\widehat{Z}_n^2 = \mathbb{E}_{\widehat{\mu}}(\widehat{\xi}_1 - \widehat{\varrho}^{-1}\widehat{v}_d) + 2\sum_{k\ge 2}\mathbb{E}_{\widehat{\mu}}(\widehat{\xi}_1 - \widehat{\varrho}^{-1}\widehat{v}_d)(\widehat{\xi}_k - \widehat{\varrho}^{-1}\widehat{v}_d) < \infty$ and if $\widehat{\sigma}^2 > 0$, $\widehat{S}_{[nt]}$ converges weakly on the Skorokhod space to $\widehat{\sigma}W$ where $W$ is standard one-dimensional Brownian motion. If $\widehat{\mathcal{C}}'$ is the $\sigma$-algebra generated by the double cluster indices, by the property of conditional variance, $\mathbb{E}_{\widehat{\mu}}\widehat{Z}_n^2 = \mathbb{E}_{\widehat{\mu}}(\mathbb{E}_{\widehat{\mu}}^{\widehat{\mathcal{C}}'}\widehat{Z}_n^2) + \mathbb{E}_{\widehat{\mu}}(\mathbb{E}_{\widehat{\mu}}^{\widehat{\mathcal{C}}'}\widehat{Z}_n)^2 \ge \mathbb{E}_{\widehat{\mu}}(\mathbb{E}_{\widehat{\mu}}^{\widehat{\mathcal{C}}'}\widehat{Z}_n^2)$ and by conditional independence,

$$n^{-1}\mathbb{E}_{\widehat{\mu}}^{\widehat{\mathcal{C}}'}\widehat{Z}_n^2 = n^{-1}\sum_{k=1}^{n}\mathbb{E}_{\widehat{\mu}}^{\widehat{\mathcal{C}}'}\widehat{\xi}_k^2 \ge \widehat{\Gamma}n^{-1}\sum_{k=1}^{n}\widehat{Y}_k'$$

with $\widehat{\Gamma} > 0$ and $\widehat{Y}_k{}'$ the indicator of the $k$-th double cluster time. Hence

$$\liminf_{n\to\infty} n^{-1}\mathbb{E}_{\widehat{\mu}}^{\widehat{\mathcal{C}}'}\widehat{Z}_n^2 \ge \widehat{\Gamma}\widehat{\gamma}' > 0 \quad \mu\text{-a.s.}$$

for some constant $\widehat{\gamma}' > 0$ and therefore $\widehat{\sigma}^2 > 0$ by integration. Replacing $n$ by $n^\kappa$ for $0 < \kappa < 1/2$ in the coupling of the previous section guarantees that $\widehat{\xi}_k(\widehat{\omega}') = \widehat{\xi}_k(\widehat{\omega}'')$ a.s. for $k \ge n^\kappa$ and $n$ large enough with the appropriate configurations $\widehat{\omega}'$ and $\widehat{\omega}''$. Using Minkowski's inequality and $\lim_{n\to\infty} n^{\kappa-1/2} = 0$, one sees that the IP for $\widehat{S}_{[nt]}$ is valid also on $(\widehat{\Omega}_1, \mathcal{B}(\widehat{\Omega}_1), \mu_0)$. By [1], Theorem 17.1, we have that $n^{-1/2}\widehat{Z}_{\widehat{\vartheta}_{nt}}$ converges weakly to $\widehat{\sigma}\widehat{\varrho}^{-1/2}W_t$. Noting that $\widehat{Q}_t = \sum_{k=1}^{\widehat{\vartheta}_t}\widehat{\xi}_k + \int_{t_{\widehat{\vartheta}_t}}^{t}\widehat{V}_s ds$ it follows that $|\widehat{S}_{[nt]} - \widehat{S}_{\widehat{\vartheta}_{nt}}| \le n^{-1/2}|\widehat{\xi}^0_{\widehat{\vartheta}_{nt}+1}|$ and by Chebyshev's inequality this converges in probability $\mu_0$ (resp. $\widehat{\mu}$) to zero as $n \to \infty$ uniformly in $t$ yielding the IP for $(\widehat{Q}_t)_{t\in\mathbb{R}_+}$, i.e. $n^{-1/2}\widehat{Q}_{[nt]}$ converges weakly on Skorokhod space to $\widetilde{\sigma}W_t$ with $\widetilde{\sigma} = \widehat{\sigma}\widehat{\varrho}^{-1/2} > 0$. Again as already mentioned at the beginning, the IP holds for M1 in the same way by changing from interdistances to interarrival times and Corollary 3.3.2 is proven.

## References


[1] Billingsley, P.: Convergence of probability measures. Wiley 1968.

[2] Bertoin, J.: Lévy processes. Cambridge Universiy Press 1996.

[3] Blumenthal, R. M., Getoor, R. K.: Markov processes and potential theory. Academic Press 1968.

[4] Boldrighini, C., Cosimi, G. C., Frigio, S., Nogueira, A.: Convergence to a stationary state and diffusion for a charged particle in a standing medium. Probability Theory and Related Fields 80, 481-500, 1989.

[5] Boldrighini, C., Soloveitchik, M.: Drift and diffusion for a mechanical system. Probability Theory and Related Fields 104, 349-379, 1997.

[6] Feller, W.: An introduction to probability theory and its applications. Volume 2. Wiley 1971.

[7] Fitzsimmons, P. J., Fristedt, B. E., Shepp, L. A.: The set of real numbers left uncovered by random covering intervals. Zeitschrift für Wahrscheinlichkeitstheorie und verwandt Gebiete 70, 175-189, 1985.

[8] Goldstein, S., Lebowitz, J. L., Ravishankar, K.: Ergodic properties of a system in contact with a heat bath. A one-dimensional model. Communications in Mathematical Physics 85, 419-427, 1982.

[9] Goldstein, S., Lebowitz, J. L., Ravishankar, K.: Approach to equilibrium in models of a system in contact with a heat bath. Journal of Statistical Physics 43, 303-315, 1986.

[10] Kallenberg, O.: Random measures. Akademie-Verlag Berlin 1983.

[11] Pellegrinotti, A., Sidoravicius, V., Vares, M. E.: Stationary state and diffusion for a charged particle in a one-dimensional medium with lifetimes. Probability Theory and Applications 44 (4), 697-721, 1998.

[12] Shepp, L.A.: Covering the line with random intervals. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 23, 163-170, 1972.

[13] Sidoravicius, V., Triolo, L., Vares, M. E.: On the forced motion of a heavy particle in random medium II. Existence of dynamics, Markov Processes and Related Fields, 4, 423-443, 1998.

[14] Sidoravicius, V., Triolo, L., Vares, M. E.: Mixing properties for a mechancial motion of a charged particle in a random medium. Communications in Mathematical Physics 219, 323-355, 2001.