High Dimensional Nonlinear Dispersive Models

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Abstract

This work presents new results about several nonlinear equations. We investigate the (non)existence of solitary waves of the ZK equation and some of the their properties by using the variational methods. Next we study the initial value problem associated to some dissipative ZK equations (ZKB and Benney equations). We will also investigate the (non)existence and stability of solitary wave solutions of BO-ZK; and their properties. Furthermore, we are interested in studying the solitary wave solutions of the high dimensional Benjamin equations and their characteristics.

To My Parents
and My Wife

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Preliminaries

Let $S_L = \mathbb{R} \times (-L, L)$. We define a concentration function for nonnegative function $u \in L^1(S_L)$. Let $\xi \in \mathbb{R}$ and $r > 0$, then we define $S_r(\xi)$ as the rectangle of the form

$$[\xi - r, \xi + r] \times [-L, L].$$

We define the concentration function of u as

$$Q(r) = \sup_{\xi \in \mathbb{R}} \iint_{S_r(\xi)} d\mu, \tag{0.1}$$

for the measure on S_L given by $d\mu = u\; dxdy$.

The concentration compactness principle of Lions [52, 53] is a way of compensating for the well known failure of precompactness of bounded sets in infinite-dimensional Banach spaces (i.e. that bounded sequences need not have convergent subsequences). The principle, roughly speaking, asserts that given any bounded sequence, there exists a subsequence which resolves into the superposition of convergent sequences that have been shifted by an asymptotically orthogonal set of unitary group actions, plus an error term which goes to zero in certain coarse norms which are weaker than the original norm topology (but significantly stronger than the weak topology). It is useful in generating nonlinear profiles of solutions to nonlinear equations, and also combines well with the induction on energy method. For $r \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we denote the ball with radius r, centered at x, by $B_r(x)$.

LEMMA 0.0.1 (Concentration-Compactness) *Suppose that μ_m is a sequence of probability measures on $\mathbb{R}^n$ such that $\mu_m \geq 0$ and $\int_{\mathbb{R}^n} d\mu_m = 1$. There exists a subsequence $\{\mu_m\}$ such that one of the three following conditions holds:*

(i) (*Evanescence*) *For all $R > 0$ there holds*

$$\lim_{m \to \infty} \left(\sup_{x \in \mathbb{R}^n} \int_{B_R(x)} d\mu_m \right) = 0.$$

(ii) (*Compactness*) *There exists a sequence $\{x_m\} \subset \mathbb{R}^n$ such that for any $\varepsilon > 0$ there is a radius $R > 0$ with the property that*

$$\int_{B_R(x_m)} d\mu_m \geq 1 - \varepsilon$$

for all m.

(iii) (*Dichotomy*) *There exists a number* λ, $0 < \lambda < 1$, *such that for any* $\varepsilon > 0$ *there is a number* $R > 0$ *and a sequence* $\{x_m\} \subset \mathbb{R}^n$ *with the following property:*
Given $R' > R$, *there are non-negative measures* μ_m^1 *and* μ_m^2 *such that*

- $0 \leq \mu_m^1 + \mu_m^2 \leq \mu_m$,
- $\operatorname{supp}(\mu_m^1) \subset B_R(x_m)$ *and* $\operatorname{supp}(\mu_m^2) \subset \mathbb{R}^n \backslash B_{R'}(x_m)$,
- $\limsup_{n\to\infty} \left(\left| \lambda - \int_{\mathbb{R}^n} d\mu_n^1 \right| + \left| (1-\lambda) - \int_{\mathbb{R}^n} d\mu_n^2 \right| \right) \leq \varepsilon.$

LEMMA 0.0.2 (Periodic Concentration-Compactness) *Suppose that* $\{\mu_n\}$ *is a sequence of positive measures on* S_L *such that* $\lim_{n\to\infty} Q_n = L$, *where* Q_n *are defined by* (0.1) *corresponding to* μ_n. *Then there exists a subsequence of measures, denoted the same, such that one of the following three conditions holds:*

(i) (*Evanescence*) *For all* $r > 0$,

$$\lim_{n\to\infty} \sup_{\xi\in\mathbb{R}} \iint_{S_r(\xi)} d\mu_n = 0.$$

(ii) (*Compactness*) *There exists a sequence* ξ_n *in* $\mathbb{R}$ *such that for any* $\varepsilon > 0$, *there is a* $r > 0$ *such that*

$$\iint_{S_r(\xi_n)} d\mu_n \geq L - \varepsilon \qquad \text{for every } n \in \mathbb{N}.$$

(iii) (*Dichotomy*) *There exists* $l \in (0, L)$ *such that for any* $\varepsilon > 0$, *there exists a positive number* $r > 0$ *and a sequence* ξ_n *in* $\mathbb{R}$ *with the following property: Given* $r' > r$ *there exists nonnegative measures* μ_n^1 *and* μ_n^2 *such that*

- $0 \leq \mu_n^1 + \mu_n^2 \leq \mu_n$,
- $\operatorname{supp}(\mu_n^1) \subset S_r(\xi_n)$ *and* $\operatorname{supp}(\mu_n^2) \subset S_L \setminus S_{r'}(\xi_n)$,
- $\limsup_{n\to\infty} \left(\left| l - \iint_{S_L} d\mu_n^1 \right| + \left| (L-l) - \iint_{S_L} d\mu_n^2 \right| \right) \leq \varepsilon.$

We consider $L^p(\Omega)$ spaces of complex-valued functions. Ω being an open subset of $\mathbb{R}^n$, $L^p(\Omega)$ denotes the space of (classes) measurable functions $u : \Omega \to \mathbb{C}$ such that $\|u\|_{L^p(\Omega)} < \infty$ with

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p},$$

if $p \in [1, \infty)$, and

$$\|u\|_{L^p(\Omega)} = \operatorname*{ess\,sup}_{\Omega} |u|,$$

if $p = \infty$.

LEMMA 0.0.3 (Refined Fatou lemma) *Let* $p \in (0, \infty)$ *and* $\{u_j\}_j$ *be a bounded sequence in* $L^p(\mathbb{R}^n)$ *such that* $u_j \to u$ *a.e. in* $\mathbb{R}^n$. *Then*

$$\|u_j\|^p_{L^p(\mathbb{R}^n)} - \|u_j - u\|^p_{L^p(\mathbb{R}^n)} - \|u\|^p_{L^p(\mathbb{R}^n)} \longrightarrow 0.$$

The assumption $u_j \to u$ *a.e. can be removed if* $p = 2$.

To a proof, see [18].

We will use following variants of the Hörmander-Mikhlin multipliers theorem [56].

PROPOSITION 0.0.4 (Lizorkin) *Let $\Upsilon : \mathbb{R}^N \longrightarrow \mathbb{R}$ be C^N for $|\xi_j| > 0$, $j = 1, \ldots, N$. Assume that there exists $M > 0$ such that*

$$\left| \xi_1^{k_1} \ldots \xi_N^{k_N} \frac{\partial^k \Upsilon}{\partial \xi_1^{k_1} \ldots \partial \xi_n^{k_N}} (\xi) \right| \leq M,$$

with $k_i = 0$ or 1 and $k = k_1 + k_2 + \cdots + k_N = 0, 1, \ldots, N$. Then Υ is a Fourier multiplier on $L^r(\mathbb{R}^N)$, $1 < r < +\infty$.

PROPOSITION 0.0.5 *Let $1 \leq p \leq \infty$, T be a operator in $\mathcal{L}(L^p(\mathbb{R}^n))$ and Λ be a lattice in $\mathbb{R}^n$. Suppose Φ is the multiplier corresponding to T which is continuous at each point of Λ. Then there exists a unique periodized operator $\widetilde{T}$ defined by*

$$\widetilde{T} f = \sum_{m \in \Lambda} \lambda_m a_m e^{im.x},$$

where $f(x) = \sum_{m \in \Lambda} a_m e^{im.x}$ and $\lambda_m = \Phi(m)$ for every $m \in \Lambda$, such that $\widetilde{T}$ is in $\mathcal{L}(L^p(\mathbb{T}^n))$ and $\left\| \widetilde{T} \right\| \leq \|T\|$.

THEOREM 0.0.6 ([3]) *Let $\{A(t)\}$ be a family of bounded linear operators of a Hilbert space V into V^* (dual of V). Let also $\{B(t)\}$ be a family of operators belonging to $L^2([0,T]; L(V, V^*)$. Suppose that the following conditions hold:*

(i) *There is $A'(t) \in L^1([0,T]; L(V, V^*))$ such that for all $(u, v) \in V \times V$, one has $\frac{d}{dt}(A(t)u, v) = (A'(t)u, v)$,*

(ii) *for $t \in [0,T]$, the operator $A(t)$ is self-adjoint,*

(iii) *there is a real number $\alpha > 0$ such that $(A(t)u, u) \geq \alpha \|u\|^2$, for all $u \in V$ and $t \in [0,T]$.*

If $u \in L^2([0,T]; V)$ is a solution of $u' + A(t)u + B(t)u = 0$ verifying $u(T) = 0$, then $u \equiv 0$ on $[0,T]$.

We will use the standard multi-index notation. A multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a $n-$tuple of nonnegative integers. We define the symbols $|\alpha| := \sum_{i=1}^n \alpha_i$, $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}}.$$

Define for each $(\alpha, \beta) \in \mathbb{N}^{2n}$ the semi-norm, $\| \cdot \|_{\alpha,\beta}$, by

$$\|f\|_{\alpha,\beta} := \|x^\alpha D^\beta f\|_{L^\infty}.$$

DEFINITION 0.0.7 *We define the Schwartz space, $\mathscr{S}(\mathbb{R}^n)$, as*

$$\mathscr{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) \ : \ \|\varphi\|_{\alpha,\beta} < \infty \ \text{for any } (\alpha, \beta) \in \mathbb{N}^{2n}\}$$

The topology in $\mathscr{S}(\mathbb{R}^n)$ is that induced by the family of semi-norms $\{\|\cdot\|_{\alpha,\beta}\}_{(\alpha,\beta)\in\mathbb{N}^{2n}}$. We say that a linear functional $\Psi : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$, defines a *tempered distribution* if Ψ is continuous (and we denote $\mathscr{S}'(\mathbb{R}^n)$ as the set of all tempered distribution). Consider an open subset Ω of $\mathbb{R}^n$. For $m \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \ : \ D^\alpha u \in L^p(\Omega) \text{ for any } |\alpha| \le m\}.$$

$W^{m,p}(\Omega)$ is a Banach space when equipped with the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ defined by

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{0\le|\alpha|\le m} \|D^\alpha u\|_{L^p(\Omega)}.$$

When $p = 2$, set $W^{m,p}(\Omega) = H^m(\Omega)$.

THEOREM 0.0.8 (Gagliardo-Nirenberg's inequality) *Let $1 \le p, q, r \le \infty$ and let j, m be two integers such that $0 \le j < m$. If*

$$\frac{1}{p} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{(1-\theta)}{q}$$

for some $\theta \in \left[\frac{j}{m}, 1\right]$ ($\theta < 1$ if $r > 1$ and $m - j - \frac{n}{r} = 0$), then there exists $C = C(n, m, j, \theta, q, r)$ such that

$$\sum_{|\alpha|=j} \|D^\alpha u\|_{L^p} \le C \left(\sum_{|\alpha|=m} \|D^\alpha u\|_{L^r}\right)^\theta \|u\|_{L^q}^{1-\theta}$$

for every $u \in \mathscr{S}(\mathbb{R}^n)$.

DEFINITION 0.0.9 *Given $s \in \mathbb{R}$, one defines Sobolev spaces*

$$H^s(\mathbb{R}^n) = \left\{u \in \mathscr{S}'(\mathbb{R}^n) \ : \ \left[\left(1+|\xi|^2\right)^{\frac{s}{2}}\widehat{u}\right]^\vee \in L^2(\mathbb{R}^n)\right\}$$

and

$$\|u\|_{H^s(\mathbb{R}^n)} = \left\|\left[\left(1+|\xi|^2\right)^{\frac{s}{2}}\widehat{u}\right]^\vee\right\|_{L^2(\mathbb{R}^n)}.$$

Let Ω be an open set in $\mathbb{R}^n$ with smooth boundary. The Sobolev space $H^s(\Omega)$ is defined by the restriction of the elements of $H^s(\mathbb{R}^n)$ to Ω, and with the norm

$$\|f\|_{H^s(\Omega)} = \inf\{\|\varphi\|_{H^s(\mathbb{R}^n)} \ : \ \varphi \text{ coincides with } f \text{ in } \Omega\}.$$

DEFINITION 0.0.10 *Let $s_1, \cdots, s_n \in \mathbb{R}$. We define the anisotropic Sobolev spaces $H^{s_1,\cdots,s_n}(\mathbb{R}^n)$ endowed with the norm*

$$\|\varphi\|^2_{H^{s_1,\cdots,s_n}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi_1,\cdots,\xi_n)|^2 \prod_{i=1}^n \left(1+\xi_i^2\right)^{s_i} \, d\xi_1\cdots d\xi_n$$

for any $\varphi \in \mathscr{S}'(\mathbb{R}^n)$.

One can easily prove the following interpolation in the anisotropic spaces.

LEMMA 0.0.11 *If* $s_{1,i} \le \varrho_i \le s_{2,i},\ 1 \le i \le n$ *, with*

$$(\varrho_1, \cdots, \varrho_n) = \theta(s_{1,1}, \cdots, s_{1,n}) + (1-\theta)(s_{2,1}, \cdots, s_{2,n})$$

and $\theta \in [0,1]$*, then*

$$\|f\|_{H^{\varrho_1,\cdots,\varrho_n}(\mathbb{R}^n)} \le \|f\|^{\theta}_{H^{s_{1,1},\cdots,s_{1,n}}(\mathbb{R}^n)} \|f\|^{1-\theta}_{H^{s_{2,1},\cdots,s_{2,n}}(\mathbb{R}^n)}. \tag{0.2}$$

DEFINITION 0.0.12 *Let* $r_1, \cdots, r_n \in \mathbb{R}$*. We define the fractional Sobolev-Liouville space*

$$H^{(s_1,\cdots,s_n)}(\mathbb{R}^n) := H^{s_1,0,\cdots,0}(\mathbb{R}^n) \cap \cdots \cap H^{0,\cdots,0,s_n}(\mathbb{R}^n)$$

equipped with the norm $\|f\|_{H^{(s_1,\cdots,s_n)}(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n \left\|D^{s_i}_{x_i} f\right\|_{L^2(\mathbb{R}^n)}$*, where* $\widehat{D^{s_i}_{x_i} f}(\xi_1, \cdots, \xi_n) = |\xi_i|^{s_i} \widehat{f}(\xi_1, \cdots, \xi_n)$*.*

If for all $1 \le k \le n$, $r_k = r$, are integers, then $H^{(r_1,\cdots,r_n)}(\mathbb{R}^n)$ is the Sobolev space $W^{r,2}(\mathbb{R}^n)$.

REMARK 0.0.13 *If* $\frac{n}{2} - \frac{n}{p} \le s \le \min\{s_1, \cdots, s_n\}$ *and* $p \in [2, \infty)$*, then the following embedding are continuous*

$$H^{s_1+\cdots+s_n}(\mathbb{R}^n) \hookrightarrow H^{s_1,\cdots,s_n}(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$$

THEOREM 0.0.14 (Young's inequality) *Let* $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$*,* $1 \le p, q \le \infty$ *with* $\frac{1}{p} + \frac{1}{q} \ge 1$*. Then* $f * g \in L^r(\mathbb{R}^n)$ *where* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$*. Moreover*

$$\|f * g\|_{L^r(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

LEMMA 0.0.15 *Let* $1 \le p, q, p_1, q_1, p_2, q_2 \le \infty$ *with* $1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $1 + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$*. Then*

$$\|f * g\|_{L^p_x L^q_y(\mathbb{R}^2)} \le \|f\|_{L^{p_1}_x L^{q_1}_y(\mathbb{R}^2)} \|g\|_{L^{p_2}_x L^{q_2}_y(\mathbb{R}^2)},$$

LEMMA 0.0.16 *If* $s_i > \frac{1}{2}$ *, for all* $i = 1, \cdots, n$*, then* $H^{s_1,\cdots,s_n}(\mathbb{R}^n)$ *is an algebra.*

THEOREM 0.0.17 (Embedding) *If* $s > \frac{n}{2} + k$*, then* $H^s(\mathbb{R}^n)$ *is continuously embedded in* $C^k_\infty(\mathbb{R}^n)$*, the space of the functions with* k *continuous derivatives vanishing at infinity. In other words, if* $f \in H^s(\mathbb{R}^n)$ *and* $s > \frac{n}{2} + k$ *then (after a possible modification of* f *in a set of measure zero)* $f \in C^k_\infty(\mathbb{R}^n)$ *and*

$$\|f\|_{C^k(\mathbb{R}^n)} \le c_s \|f\|_{H^s(\mathbb{R}^n)}.$$

THEOREM 0.0.18 (Embedding) *The space* $H^s(\Omega)$ *is continuously embedded in* $L^p(\Omega)$*, if* $2 \le p < \infty$ *and* $\dfrac{1}{p} \ge \dfrac{1}{2} - \dfrac{s}{n}$*. Moreover,* $H^s(\Omega)$ *is continuously embedded in* $L^\infty(\Omega)$ *if* $s > n/2$*. This embedding is compact if* $\dfrac{1}{p} > \dfrac{1}{2} - \dfrac{s}{n}$*.*

LEMMA 0.0.19 *For any* $f \in H^{n/2+\epsilon}(\mathbb{R}^n)$ *and* $\epsilon \in (0, 1/2]$*, we have*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le c(n)\epsilon^{-1/2} \|f\|_{H^{n/2+\epsilon}(\mathbb{R}^n)}.$$

THEOREM 0.0.20 *If $s > \frac{n}{2}$, then $H^s(\mathbb{R}^n)$ is a algebra with respect to the product of the functions. That is, if $f, g \in H^s(\mathbb{R}^n)$, then $fg \in H^s(\mathbb{R}^n)$ with*

$$\|fg\|_{H^s(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)} \|g\|_{H^s(\mathbb{R}^n)}.$$

DEFINITION 0.0.21 *Let H be a Hilbert space and $I : H \to \mathbb{R}$ be a functional. We say I satisfies the* Palais-Smale *condition if $I \in C^1(H, \mathbb{R})$, and if every sequence $\{u_k\}_{k=1}^{\infty} \subset H$ such that:*

- *$\{I[u_k]\}_{k=1}^{\infty}$ is bounded, and*
- *$I'[u_k] \to 0$ in H,*

is precompact in H.

DEFINITION 0.0.22 *Let H be a Hilbert space and $I : H \to \mathbb{R}$ be a functional. We say $u \in H$ is a critical point if $I'[u] = 0$. Also the number c is a critical value if $K_c \neq \emptyset$, where*

$$K_c := \{u \in H \;\;|\;\; I[u] = c,\; I'[u] = 0\}.$$

The mountain pass theorem (see [19]) is an existence theorem from the calculus of variations. Given certain conditions on a function, the theorem demonstrates the existence of a saddle point. The theorem is unusual in that there are many other theorems regarding the existence of extremum, but few regarding saddle points.

THEOREM 0.0.23 (Mountain Pass) *Let H be a Hilbert space and $I : H \to \mathbb{R}$ be a functional. Assume that I satisfies the following conditions:*

- *$I \in C^1(H, \mathbb{R})$,*
- *I' is Lipschitz continuous on bounded subsets of H,*
- *I satisfies the Palais-Smale compactness condition,*
- *$I[0] = 0$,*
- *there exist positive constants r and a such that $I[u] \geq a$ if $\|u\| = r$, and*
- *there exists $v \in H$ with $\|v\| > r$ such that $I[v] \leq 0$.*

If we define:

$$\Gamma = \{\mathbf{g} \in C([0,1]; H) \,|\, \mathbf{g}(0) = 0, \mathbf{g}(1) = v\}.$$

Then

$$c = \inf_{\mathbf{g} \in \Gamma} \max_{0 \leq t \leq 1} I[\mathbf{g}(t)],$$

is a critical value of I.

THEOREM 0.0.24 (Mountain Pass) *Let X be a Banach space. Let M_0 be a closed subspace of the metric space M and $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma := \{\gamma \in C(M, X) \;\;:\;\; \gamma|_{M_0} \in \Gamma_0\}.$$

If $\varphi \in C^1(X,\mathbb{R})$ satisfies

$$\infty > c := \inf_{\gamma\in\Gamma} \sup_{u\in M} \varphi(\gamma(u)) > a := \sup_{\gamma_0\in\Gamma_0} \sup_{u\in M_0} \varphi(\gamma_0(u)),$$

then, for every $\varepsilon \in (0,(c-a)/2)$, $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_M \varphi\circ\gamma \le c+\varepsilon,$$

there exists $u \in X$ such that

- $c - 2\varepsilon \le \varphi(u) \le c + 2\varepsilon$,
- $\operatorname{dist}(u,\gamma(M)) \le 2\delta$,
- $\|\varphi'(u)\| \le 8\varepsilon/\delta$.

We recall the Hilbert Transform $\mathscr{H}$, defined by

$$\mathscr{H}(f)(x) = \frac{1}{\pi}\,\texttt{p.v.}\frac{1}{y} * f(x) = \frac{1}{\pi}\,\texttt{p.v.}\int \frac{f(x-y)}{y}\,dy,$$

is a unitary operator on $L^2(\mathbb{R})$; we remember that for any $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, the Hilbert transform $\mathscr{H}f(x)$ exits and is finite a.e.. Moreover, the Hilbert transform operator $\mathscr{H} : f \to \mathscr{H}f$ for $f \in L^p(\mathbb{R})$, $1 < p < \infty$, is bounded. Some properties of the Hilbert transform $\mathscr{H}$, for $f, g \in \mathscr{S}$:

$$\int g\mathscr{H}(f) = -\int f\mathscr{H}(g), \tag{0.3}$$

$$\widehat{\mathscr{H}(f)}(\xi) = -\texttt{i}\texttt{sgn}(\xi)\hat{f}, \tag{0.4}$$

$$\|\mathscr{H}(f)\|_{L^2} = \|f\|_{L^2}, \tag{0.5}$$

$$\partial_x\mathscr{H} = \mathscr{H}\partial_x, \tag{0.6}$$

$$\mathscr{H}(f(a\,\cdot))(x) = \texttt{sgn}(a)\mathscr{H}(f)(ax), \quad \text{for every } a\in\mathbb{R}, \tag{0.7}$$

$$\mathscr{H}(xf_x(x))(y) = y\mathscr{H}(f_x)(y), \tag{0.8}$$

$$\mathscr{H}(f(\cdot+a))(x) = \mathscr{H}(f)(x+a), \quad \text{for every } a\in\mathbb{R}, \tag{0.9}$$

$$\int xf_x\mathscr{H}f_x = 0, \tag{0.10}$$

$$\mathscr{H}(fg) = f\mathscr{H}(g) + g\mathscr{H}(f) + \mathscr{H}\left(\mathscr{H}(f)\mathscr{H}(g)\right). \tag{0.11}$$

We will explain now the notation of well-posedness that will be used. Let X and Y be Banach spaces such that $X \hookrightarrow Y$ and suppose that $f \in C(X,Y)$. Consider the initial value problem:

$$\frac{du}{dt} = f(u) \tag{0.12}$$

$$u(0) = \varphi. \tag{0.13}$$

DEFINITION 0.0.25 *We will say that (0.12)-(0.13) is locally well-posed in* X *if for any* $\varphi \in X$ *there exists* $T > 0$ *such that the following conditions hold,*

- *there exists a unique* $u \in C([0,T];X) \cap C^1([0,T];Y)$ *such that satisfies (0.12)-(0.13);*
- u *depends continuously on* φ *in the sense that if* $\varphi_n \to \varphi$ *in* X*, then for* n *large enough* $u_n \in C([0,T];X)$ *and* $u_n \to u$ *in* $C([0,T];X)$*.*

If T *can be taken arbitrary large, then the initial value problem is called globally well-posed.*
Observe that the first condition expresses the persistence of $u(t)$ *in the space* X*. The second condition says that the local flow defined by (0.12)-(0.13) is continuous.*
In what follows different constants may denoted by the same letter when their precise values are of no relevance to our arguments. When necessary, dependence on other quantities will be indicated.

Chapter 1

Solitary Waves of ZK equation

1.1 Introduction

The *Korteweg-de Vries* (KdV) equation depicts the evolution of the weakly nonlinear and weakly dispersive waves in such physical as plasma physics, ion-acoustic waves, stratified internal waves, and atmospheric waves [6]. Kakutani and Ono have shown that the modified KdV equation governs the propagation of Alfvén waves at a critical angle to the undisturbed magnetic field. The presence of the transverse dispersion has been physically attributed to the finite Larmor radius effects [1]. But, despite its overt fame, the KdV equation is restricted as a model by being spatially one-dimensional. On the basis of the great success in the soliton theory, a lot of works have recently been directed to thrive higher-dimensional models and investigations of soliton properties in multi-dimensional systems, particularly two and three spatial dimensions. There are several two-dimensional generalizations of the KdV equation, but the *Kadomtsev-Petviashvili* (KP) and *Zakharov-Kuznetsov* (ZK) [1] equations are the most well-known ones.

The ZK equation

$$u_t + \Delta u_x + u u_x = 0, \qquad (x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$$

was first derived in three dimensional form to describe nonlinear ion-acoustic waves in a magnetized Plasma. But a variety of physical phenomena, in the purely dispersive limit, are governed by this type of equation such as the long waves on a thin liquid film, the Rossby waves in rotating atmosphere, and the isolated vortex of the drift waves in three-dimensional plasma. Spatially localized solitary wave solutions decaying in all directions were also obtained analytically but the conclusion is restricted to specific situations. When the localized pulses decaying in all directions preserve their forms in the interaction with other pulses, they are called solitons in higher dimensional space. The multi-dimensional localized pulses are often actually observed in a variety of physical phenomena and some of them turn up to imply soliton-like properties; *magmons* in porous flow and vortex solitons in plasmas [1]. However, detailed investigation, either analytical or numerical, of those properties based on sound models of nonlinear differential equations are still defective.

A cylindrically symmetric solitary wave solution (bell-shaped pulse) of ZK equation was obtained numerically [40, 67]. The numerical study of the ZK equation shows that the cylindrically symmetric solitary wave solution are fundamental because these solutions arise from an arbitrary initial condition. In fact, the interesting issue is the existence of ground states of $-\Delta u + f(u) = 0$ which is well known, under suitable assumptions over the nonlinearity. It follows, for example, from the results of Berestycki and Lions [9].

The well-posedness of the ZK equation with the nonlinearity $u^p u_x$ (generalized ZK equation) can be seen using Kato's Theory for the general nonlinearity [42]. Faminskii showed the ZK equation is globally well-posed in $H^s(\mathbb{R}^2)$ for $s \geq 1$ [31]. Biagioni and Linares proved the modified ZK equation ($p = 2$) is globally well-posed in $H^s(\mathbb{R}^2)$ for $s \geq 1$ [12]. Recently, Linares and Saut obtained some results in the three dimensional case [51].

The present Chapter of this thesis is devoted to define a suitable space for traveling wave solutions and give a necessary condition for the existence related to the speed c. Then, we obtain periodic traveling wave solutions from a constrained minimization problem when the nonlinearity is $\frac{1}{p+1}u^{p+1}$ and $p = \frac{k}{\ell}$, where $k \in \mathbb{N}$ is even and k and ℓ are relatively prime. We use the *Steiner Symmetrization* to extend our result to the arbitrary p, also we will show that the found solution is, in fact, a smooth solitary wave solution with symmetry and decay (is called *bell-shaped pulse*). We also use the ideas of Pankov and Pflüger ([64, 65]) to show that the sequence of solitary wave solutions of ZK in cylinder tends to a traveling wave solution of ZK equation in $\mathbb{R}^n$. Next, we use other approach to obtain the solitary waves for a general nonlinearity of the ZK equation. Note that these results, in a appropriate context, can be repeated for other higher-dimensional models such as the generalization of the BBM equation, which one dimensional models the unidirectional propagation of long waves in a channel and is an alternative model for the Korteweg-de Vries equation. In Section 1.9, we will study instability of some of the minimizers of generalized KdV equation. We use the variational properties of our solutions and show that they are unstable for $p > 4$.

1.2 Nonexistence

We will study existence of traveling wave solutions of the equation of Zakharov-Kuznetsov of the form

$$u_t + \Delta u_x + (f(u))_x = 0 \tag{1.1}$$

in two dimensional cylinder and some of their properties. However, note that a similar procedure can be applied to the case of $\mathbb{R}^n \times \mathbb{T}^m$. One can easily show that the equation (1.1) inherits the following invariants:

$$\mathscr{E}_1(u) = \frac{1}{2}\iint \left(|\nabla u|^2 - F(u)\right) dxdy, \quad \mathscr{E}_2(u) = \frac{1}{2}\iint u^2 \, dxdy.$$

As far as we know 5 invariants for ZK equation, other three invariants are the following:

$$M_y(u) = \int u \, dx, \quad M(u) = \int\int u \, dxdy,$$
$$I(u) = \int\int (x\boldsymbol{i} + y\boldsymbol{j})u \, dxdy - t\boldsymbol{i}\int\int \frac{1}{2}u^2 \, dxdy,$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ denote the unit vectors in the x and y-directions; and the extent of integrals is over the total region under consideration. By a solitary wave solution we mean a solution of (1.1) of the form $u(x, y, t) = \varphi(x - ct, y)$, where $(\xi = x - ct, y) \in \mathbb{R} \times \mathbb{T}$, where $c \in \mathbb{R}$ represents the speed of the wave and

$$\varphi \longrightarrow 0, \tag{1.2}$$
$$\Delta\varphi \longrightarrow 0, \tag{1.3}$$
$$\nabla\varphi \longrightarrow 0, \tag{1.4}$$

as the variable $|\xi|$ approaches to $+\infty$, where $\Delta = \partial_\xi^2 + \partial_y^2$ and $\nabla = (\partial_\xi, \partial_y)$; and also

$$\varphi(\cdot, -L) = \varphi(\cdot, L), \quad \nabla\varphi(\cdot, -L) = \nabla\varphi(\cdot, L). \tag{1.5}$$

We consider $f(x) = \dfrac{x^{p+1}}{p+1}$. So putting this form of u in (1.1) and integrating once, we see that φ must satisfy the partial differential equation

$$-c\varphi + \Delta\varphi + \frac{\varphi^{p+1}}{p+1} = 0. \tag{1.6}$$

We will see that c has to be positive; in fact, there is no solution of (1.6) when $c < 0$. We start by defining the natural spaces needed to find a weak solution of equation (1.6). Hereafter we change the role of ξ by x.

DEFINITION 1.2.1 *Let $C_{per}^\infty(\mathbb{R}^2)$ be the space of smooth functions which are L-periodic in y and have compact support in x and define*

$$H_L^1(S_L) := \left\{\varphi|_{S_L} : \varphi \in C_{per}^\infty(\mathbb{R}^2)\right\}, \quad S_L = \mathbb{R} \times (-L, L). \tag{1.7}$$

Let H_L denote the closure of $H_L^1(S_L)$ with respect to the norm given by

$$\|\varphi\|_1^2 = \iint\limits_{S_L} \varphi^2 + \varphi_x^2 + \varphi_y^2 \, dxdy.$$

Similarly, we can define L^p, with the norm $\|\varphi\|_{L^p}^p = \iint\limits_{S_L} |\varphi|^p dxdy$.

THEOREM 1.2.2 *Let $c > 0$ and $p \in \mathbb{R}$. Then there exists no nontrivial solutions in H_L of*

$$c\varphi + \Delta\varphi + \frac{\varphi^{p+1}}{p+1} = 0, \tag{1.8}$$

such that (1.2 − 1.5) hold.

Proof. We will use *Pohozaev-type* identities to prove this result.
Let $\varphi \in H_L$. Multiplying (2.2) by φ and integrating over S_L, one gets

$$\iint\limits_{S_L} c\varphi^2 - |\nabla\varphi|^2 + \frac{\varphi^{p+2}}{p+1} \, dxdy = 0. \tag{1.9}$$

Similarly, multiplying (2.2) by $x\varphi_x$ and $y\varphi_y$ and integrating over S_L, one obtains

$$\iint\limits_{S_L} -c\varphi^2 - \varphi_x^2 + \varphi_y^2 - \frac{2\,\varphi^{p+2}}{(p+1)(p+2)} \, dxdy = 0, \tag{1.10}$$

and

$$\iint\limits_{S_L} -c\varphi^2 + \varphi_x^2 - \varphi_y^2 - \frac{2\,\varphi^{p+2}}{(p+1)(p+2)} \, dxdy = 0. \tag{1.11}$$

By summing (2.4) and (2.5), we have

$$\iint\limits_{S_L} c\varphi^2 + \frac{2\,\varphi^{p+2}}{(p+1)(p+2)}\,dxdy = 0. \tag{1.12}$$

From (2.3) and (2.6), we have

$$\iint\limits_{S_L} p\,c\,\varphi^2 + 2|\nabla\varphi|^2\,dxdy = 0.$$

So, $\varphi \equiv 0$. ■

Therefore, $c \in \mathbb{R}$ in the equation (1.6) must be positive.

1.3 Existence

We are going to make use of a variational method applied to a suitable minimization problem to prove the existence of the solitary wave solution of the ZK Equation in S_L. We consider the case that $p = \frac{k}{\ell}$, where $k \in \mathbb{N}$ is even and k and ℓ are relatively prime. We will prove there is a solution of (1.6) by variational methods. We start by defining the nonlinear continuous functional I on H_L

$$I(\varphi) = \frac{1}{2}\iint\limits_{S_L} |\nabla\varphi|^2 + c\varphi^2\,dxdy$$

and the following constrained minimization problem on H_L,

$$I_q = \inf\left\{ I(\varphi) : \varphi \in H_L\ ,\ J(\varphi) = \iint\limits_{S_L} \varphi^{p+2}\,dxdy = q > 0 \right\}. \tag{1.13}$$

Also, we consider the set of minimizers

$$G_q = \{\varphi \in H_L : I(\varphi) = I_q\ ,\ J(\varphi) = q\}\,. \tag{1.14}$$

We call a sequence $\{\varphi_n\} \subset H_L$ a minimizing sequence to G_q if

$$\lim_{n\to\infty} I(\varphi_n) = G_q, ,\quad J(\varphi_n) = q, \quad \text{for all } n \in \mathbb{N}.$$

We endeavor to show that $G_q \neq \emptyset$. If $g \in G_q$ then by the Lagrange Multiplier theorem, there exists $\theta \in \mathbb{R}$ such that $\delta I(g) + \theta\delta J(g) = 0$, where $\delta I(g)$ and $\delta J(g)$ are the Fréchet derivatives of I and J at g. Now δI and δJ are given (as distributions in H_L^{-1}) by

$$\delta I(g) = -\Delta g + cg, \qquad \delta J(g) = (p+2)g^{p+1}.$$

By the change of the scale $\varphi = sgn(\theta)(|\theta|(p+2)(p+1))^{1/p}g$, we obtain that φ satisfies in (1.6). Let $q > 0$, and $\{\varphi_n\}$ be a minimizing sequence to I_q. Therefore, since $I(\varphi_n) \to I_q$ and $I(\varphi)$ represents a equivalent norm to the $\|\varphi\|_1^2$, it follows that there exists $K > 0$ such that $\|\varphi_n\|_1^2 \leq K$. Also, it is clear that $I_q > 0$ for every $q > 0$. So,

LEMMA 1.3.1 *For all $q > 0$, one has $0 < I_q < \infty$.*

To each minimizing sequence $\{\varphi_n\}$, we associate a sequence of nondecreasing functions $Q_n : [0,\infty) \to [0,q]$ defined by

$$Q_n(r) = \sup_{\zeta\in\mathbb{R}} \int_{-L}^{L}\int_{\zeta-r}^{\zeta+r} \varphi_n^{p+2}\, dxdy.$$

An elementary argument shows that any uniformly bounded sequence of nondecreasing functions on $[0,\infty)$ must have a subsequence which converges pointwise to a nondecreasing limit function on $[0,\infty)$. Hence $\{Q_n\}$ has such a subsequence, which we denote again by $\{Q_n\}$. Let $Q : [0,\infty) \to [0,q]$ be the nondecreasing function to which Q_n converges, and define $\alpha = \lim_{r\to\infty} Q(r)$; then $0 \le \alpha \le q$. In fact, we are going to use the Lemma 0.0.2 and show that the evanescence and dichotomy do not occur for $Q_n(r)$. Suppose $\alpha = q$. Then there exists $r_0 > 0$ such that for all sufficiently large values of n we have

$$Q_n(r_0) = \sup_{\zeta} \int_{-L}^{L}\int_{\zeta-r_0}^{\zeta+r_0} \varphi_n^{p+2}\, dxdy\ > q/2.$$

Hence for each sufficiently large n we can find x_n such that

$$\int_{-L}^{L}\int_{x_n-r_0}^{x_n+r_0} \varphi_n^{p+2}\, dxdy\ > q/2.$$

Now, let $z > q/2$ be given. Since $\alpha = q$ then we can find $r_0(z)$ and $N(z)$ such that if $n \ge N(z)$ then

$$\int_{-L}^{L}\int_{x_n(z)-r_0(z)}^{x_n(z)+r_0(z)} \varphi_n^{p+2}\, dxdy\ > z$$

for some $x_n(z) \in \mathbb{R}$. Since $\int_{S_L} \varphi_n^{p+2}\, dxdy = q$, it follows that for large n the intervals $[x_n - r_0, x_n + r_0]$ and $[x_n(z) - r_0(z), x_n(z) + r_0(z)]$ must overlap. Thus, by defining $r = r(z) = 2r_0(z) + r_0$, we have that $[x_n - r, x_n + r]$ contains $[x_n(z) - r_0(z), x_n(z) + r_0(z)]$ and therefore

$$\int_{-L}^{L}\int_{x_n-r_0}^{x_n+r_0} \varphi_n^{p+2}\, dxdy\ > z,$$

for all sufficiently large n. Note that the case of $z \le q/2$ is clear. We define $\widetilde{\varphi}_n(x,y) = \varphi_n(x + x_n, y)$. Now if we put $z = 1 - \dfrac{1}{k}$ for every $k \in \mathbb{N}$, then there exists r_k such that for all sufficiently large $n \in \mathbb{N}$,

$$\int_{-L}^{L}\int_{-r_k}^{r_k} \widetilde{\varphi}_n^{p+2}\, dxdy\ > 1 - \frac{1}{k}.$$

We have the fact that $\{\widetilde{\varphi}_n\}$ is bounded in H_L, then compactness embedding H_L into L^{p+2} on bounded intervals, it follows that some subsequence of $\{\widetilde{\varphi}_n\}$ converges in $L^{p+2}([-r_k, r_k] \times [-L, L])$ norm to a limit function $g \in L^{p+2}([-r_k, r_k] \times [-L, L])$ satisfying

$$\int_{-L}^{L}\int_{-r_k}^{r_k} g^{p+2}\, dxdy\ > 1 - \frac{1}{k}.$$

By Cantor diagonalization argument, together with the fact that $J(\widetilde{\varphi}_n) = q$ for all n, we have then that some subsequence of $\{\widetilde{\varphi}_n\}$ converges in L^{p+2} norm to a function $g \in L^{p+2}$ satisfying $\iint\limits_{S_L} g^{p+2}\, dxdy = q$. By weak compactness and the weak lower semicontinuity of the norm in H_L, we know that $\widetilde{\varphi}_n$ converges weakly to g in H_L, and that $\|g\|_1 \le \liminf_{n\to\infty} \|\widetilde{\varphi}_n\|_1$. It follows that $I(g) \le \liminf_{n\to\infty} I(\widetilde{\varphi}_n) = I_q$, whence $I(g)$ and $g \in G_q$. Furthermore, $I(g) = \lim\limits_{n\to\infty} I(\widetilde{\varphi}_n)$, whence $\|g\|_1 = \lim\limits_{n\to\infty} \|\widetilde{\varphi}_n\|_1$ and $\widetilde{\varphi}_n$ converges to g in H_L norm. So, G_q is nonempty. In fact, we have proved

LEMMA 1.3.2 *Suppose $\alpha = q$. Thence there exists a sequence of real numbers $\{x_1, x_2, x_3, \cdots\}$ such that for every $z < q$ there exists $r = r(z)$ such that $\int_{-L}^{L}\int_{x_n-r}^{x_n+r} \varphi_n^{p+2}\, dxdy > z$ for all sufficiently large n. Moreover, the sequence $\{\widetilde{\varphi}_n\}$ defined by $\widetilde{\varphi}_n(x,y) = \varphi_n(x+x_n, y)$ has a subsequence which converges in H_L norm to a function $g \in G_q$.*

Now, we are going to show that the cases $\alpha = 0$ (*evanescence*) and $0 < \alpha < q$ (*dichotomy*) do not occur. First, we show the sub-additivity property of I_q. Let $q > 0$ and $\varphi \in H_L$. We define the function φ_θ by $\varphi_\theta(x,y) = \theta^{\frac{1}{p+2}}\varphi(x,y)$. Then $J(\varphi_\theta) = \theta J(\varphi)$, and $I(\varphi_\theta) = \theta^{\frac{2}{p+2}} I(\varphi)$. Hence

$$I_{\theta q} = \inf\{I(\varphi_\theta) :\ J(\varphi_\theta) = \theta q\} = \inf\left\{\theta^{\frac{2}{p+2}} I(\varphi)\ :\ J(\varphi) = q\right\} = \theta^{\frac{2}{p+2}} I_q.$$

Therefore $I_{q_1} = \left(\dfrac{q_1}{q_2}\right)^{\frac{2}{p+2}} I_{q_2}$ for all $q_1, q_2 > 0$. Now, if $\gamma \in (0,q)$ then there exists $\theta \in (0,1)$ such that $\gamma = \theta q$. Since the function $f(\theta) = \theta^{\frac{2}{p+2}} + (1-\theta)^{\frac{2}{p+2}}$ satisfies in $f(\theta) > 1$ for all $\theta \in (0,1)$ and $I_q > 0$, thereupon one has

$$I_\gamma + I_{q-\gamma} = \theta^{\frac{2}{p+2}} I_q + (1-\theta)^{\frac{2}{p+2}} I_q = \left(\theta^{\frac{2}{p+2}} + (1-\theta)^{\frac{2}{p+2}}\right) I_q >\ I_q.$$

Therefore,

LEMMA 1.3.3 *For all $\gamma \in (0,q)$, one has $I_q < I_{q-\gamma} + I_\gamma$.*

We choose a function $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi \equiv 1$ on $[-1,1]$, $\phi = 0$ for $x \notin [-2,2]$ and $|\phi'| \le K$ for some $K > 0$. Also we choose a function $\psi \in C_0^\infty(\mathbb{R})$ such that $\phi^2 + \psi^2 = 1$ on $\mathbb{R}$, and we define $\phi_r(x) = \phi\left(\frac{x}{r}\right)$ and $\psi_r(x) = \psi\left(\frac{x}{r}\right)$. Let $\epsilon > 0$. By the definition of α, there exists r_1 such that for every $r \ge r_1$, $\alpha - \epsilon < Q(r) \le Q(2r) \le \alpha$. Since Q_n converges pointwise to Q, there is $N \in \mathbb{N}$ such that for every $n \ge N$, $|Q_n(r) - Q(r)| < \epsilon/2$, and $|Q_n(2r) - Q(2r)| < \epsilon/2$. Thus $\alpha - \epsilon < Q_n(r) \le Q_n(2r) < \alpha + \epsilon$,for every $n \ge N$. Now by the definition of Q_n, for every $n \ge N$ there exists x_n such that

$$\int_{-L}^{L}\int_{x_n-r}^{x_n+r} \varphi_n^{p+2}\, dxdy\ >\ \alpha - \epsilon, \qquad \int_{-L}^{L}\int_{x_n-2r}^{x_n+2r} \varphi_n^{p+2}\, dxdy\ <\ \alpha + \epsilon. \tag{1.15}$$

Now we define $g_n(x,y) = \phi_r(x - x_n)\varphi_n(x,y)$ and $h_n(x,y) = \psi_r(x - x_n)\varphi_n(x,y)$. It is obvious that $g_n(x,y)$, $h_n(x,y)$ are in H_L. We have

$$\begin{aligned}\int_{-L}^{L}\int_{\mathbb{R}} g_n^{p+2}\, dxdy\ &= \int_{-L}^{L}\int_{-2r}^{2r} \phi^{p+2}\left(\frac{x}{r}\right)\varphi_n^{p+2}(x+x_n,y)\, dxdy \\ &\le\ \int_{-L}^{L}\int_{-2r}^{2r} \varphi_n^{p+2}(x+x_n,y)\, dxdy\ =\ \int_{-L}^{L}\int_{x_n-2r}^{x_n+2r} \varphi_n^{p+2}(x,y)\, dxdy\ <\ \alpha + \epsilon\end{aligned}$$

and

$$\int_{-L}^{L}\int_{\mathbb{R}} h_n^{p+2}\,dxdy \;=\int_{-L}^{L}\int_{-2r}^{2r}\phi_r^{p+2}(x)\varphi_n^{p+2}(x+x_n,y)\,dxdy$$
$$\geq\int_{-L}^{L}\int_{-2r}^{2r}\varphi_n^{p+2}(x+x_n,y)\,dxdy \;=\; \int_{-L}^{L}\int_{x_n-2r}^{x_n+2r}\varphi_n^{p+2}(x,y)\,dxdy \;>\; \alpha-\epsilon.$$

Consequently,

$$|J(g_n)-\alpha|<\epsilon \quad for\ every\ n\geq N, \tag{1.16}$$

since $J(\varphi_n)=q$. Likewise we have,

$$|J(h_n)-(q-\alpha)|<\epsilon \quad for\ every\ n\geq N \tag{1.17}$$

from the inequalities of (1.15). Now, we have

$$\begin{aligned}I(g_n)+I(h_n) &= \frac{1}{2}\iint_{S_L}|\nabla g_n|^2+c\,g_n^2\;+\;|\nabla h_n|^2\;+\;c\,h_n^2\,dxdy\\ &= \frac{1}{2}\iint_{S_L}{\phi_r'}^2\varphi_n^2\;+\;\phi_r^2(\varphi_n)_x^2+2\phi_r\phi_r'\varphi_n(\varphi_n)_x\\ &\quad+\;\phi_r^2(\varphi_n)_y^2+c\,\phi_r^2\varphi_n^2\,dxdy\;+\;\frac{1}{2}\iint_{S_L}{\psi_r'}^2\varphi_n^2\\ &\quad+\;\psi_r^2(\varphi_n)_x^2\;+\;2\psi_r\psi_r'\varphi_n(\varphi_n)_x\;+\;\psi_r^2(\varphi_n)_y^2\\ &\quad+c\psi_r^2\varphi_n^2\,dxdy=\frac{1}{2}\iint_{S_L}|\nabla\varphi_n|^2+c\,\varphi_n^2\,dxdxy\;+\;O\left(\frac{1}{r}\right),\end{aligned}$$

since $\phi^2+\psi^2=1$ and $|{\phi'}_r|_\infty=|\phi'|_\infty/r,\ \ |{\psi'}_r|_\infty=|\psi'|_\infty/r$. We make the choice r so large such that the $O(1/r)$ term in the preceding paragraph is less than ϵ in absolute value. In consequence, $|I(\varphi_n)-I(g_n)-I(h_n)|<\epsilon$ for all $n\geq N(r)$. Therefore,

LEMMA 1.3.4 *For every $\epsilon>0$, there exists a number $N\in\mathbb{N}$, sequences $\{g_n\}$ and $\{h_n\}$ of H_L functions such that for every $n\geq N$,*

- $|J(g_n)-\alpha|<\epsilon$
- $|J(h_n)-(q-\alpha)|<\epsilon$
- $I(\varphi_n)\geq I(g_n)+I(h_n)-\epsilon.$

Now, suppose the case $0<\alpha<q$ occurs. Let $\epsilon>0$ be given. We consider N and $\{g_n\}_{n\geq N}$, $\{h_n\}_{n\geq N}$ in H_L as above.
For $n\geq N$, we set

$$\widetilde{g}_n=\frac{\alpha^{\frac{1}{p+2}}}{\|g_n\|_{p+2}}\,g_n \quad , \quad \widetilde{h}_n=\frac{(q-\alpha)^{\frac{1}{p+2}}}{\|h_n\|_{p+2}}\,h_n.$$

Then $J(\widetilde{g}_n) = \alpha$ and $J(\widetilde{h}_n) = q - \alpha$. So, it follows that $I(\widetilde{g}_n) \geq I_\alpha$ and $I(\widetilde{h}_n) \geq I_{q-\alpha}$. Accordingly,

$$I(g_n) \geq \frac{\|g_n\|_{p+2}^2}{\alpha^{\frac{2}{p+2}}}\, I_q \quad , \quad I(h_n) \geq \frac{\|h_n\|_{p+2}^2}{(q-\alpha)^{\frac{2}{p+2}}}\, I_{q-\alpha}.$$

From (1.16) and (1.17), it follows

$$\begin{aligned} I(\varphi_n) \geq I(g_n) + I(h_n) - \epsilon &\geq \frac{\|g_n\|_{p+2}^2}{\alpha^{\frac{2}{p+2}}}\, I_q \;+\; \frac{\|h_n\|_{p+2}^2}{(q-\alpha)^{\frac{2}{p+2}}}\, I_{q-\alpha} - \epsilon \\ &\geq \frac{(\alpha-\epsilon)^{\frac{2}{p+2}}}{\alpha^{\frac{2}{p+2}}}\, I_q \;+\; \frac{(q-\alpha-\epsilon)^{\frac{2}{p+2}}}{(q-\alpha)^{\frac{2}{p+2}}}\, I_{q-\alpha} - \epsilon. \end{aligned}$$

For a fixed ϵ, we take $n \to \infty$, and then as $\epsilon \to \infty$, we obtain

THEOREM 1.3.5 (Ruling out Dichotomy) *If* $0 < \alpha < q$ *then* $I_q \geq I_\alpha + I_{q-\alpha}$.

This theorem contradicts the sub-additivity property of I_q if we consider $\alpha \in (0, q)$. So, we have ruled out the case $0 < \alpha < q$. Now, we will use ideas of Albert and Lied to rule out the evanescence case [2, 50]. Let $\xi \in \mathscr{S}(\mathbb{R})$ (Schwartz space) such that *supp* $\xi(x) = [-2, 2]$ and ξ is positive in $[-2\ ,\ 2]$. We define $F(x) = \sum_{n\in\mathbb{Z}} \xi(x-n)$. Then for every x, the sum defining $F(x)$ has not more than four of the terms in the sum being nonzero. So, $F(x) > 0$ for every $x \in \mathbb{R}$. We define $\omega(x) = \dfrac{\xi(x)}{F(x)}$. Therefrom $\omega \in C^\infty(\mathbb{R})$ and *supp* $\omega \subseteq [-2\ ,\ 2]$. We have

$$\sum_{n\in\mathbb{Z}} \omega(x-n) = \sum_{n\in\mathbb{Z}} \frac{\xi(x-n)}{F(x-n)} = \frac{1}{F(x)} \sum_{n\in\mathbb{Z}} \xi(x-n) = 1$$

by the equalities

$$F(x-m) = \sum_{n\in\mathbb{Z}} \xi(x-n-m) = \sum_{n\in\mathbb{Z}} \xi(x-n) = F(x).$$

Also, since $\sum_{n\in\mathbb{Z}} \xi(x-n) \geq \xi(x)$, it follows that $\omega(x) \leq 1$ for every $x \in \mathbb{R}$.

LEMMA 1.3.6 *Let* $\omega \in C^\infty(\mathbb{R})$ *be given such that* $0 \leq \omega \leq 1$, $\omega(x) = 0$ *for* $x \notin [-2\ ,\ 2]$, *and* $\sum_{n\in\mathbb{Z}} \omega(x-n) = 1$ *for all* $x \in \mathbb{R}$. *Then there exists a positive constant* k *such that for all* $\varphi \in H_L$,

$$\sum_{n\in\,\mathbb{Z}} \|\omega(x-n)\varphi\|_1^2 \leq k\, \|\varphi\|_1^2. \tag{1.18}$$

Proof. We define $\omega_n(x) = \omega(x-n)$ for $n \in \mathbb{Z}$. Also, let $l^2(H_L)$ denote the Hilbert space of all sequences $\{f_n\}_{n\in\,\mathbb{Z}}$ such that $f_n \in H_L$ for each $n \in \mathbb{Z}$ and $\sum_{n\in\,\mathbb{Z}} \|f_n\|_1^2 < \infty$. So we have

$$\begin{aligned} \|\{\omega_n\varphi\}_n\|_{l^2(H_L)}^2 &= \sum_{n\in\mathbb{Z}} \|\omega_n\varphi\|_{H_L}^2 = \sum_{n\in\mathbb{Z}} \left(\|\omega_n\varphi\|_{L^2}^2 + \|(\omega_n\varphi)_x)\|_{L^2}^2 + \|(\omega_n\varphi)_y\|_{L^2}^2\right) \\ &= \sum_{n\in\mathbb{Z}} \iint_{S_L} \left(|\omega_n\varphi|^2 + |(\omega_n\varphi)_x|^2 + |(\omega_n\varphi)_y|^2\, dxdy\right). \end{aligned} \tag{1.19}$$

By the definition of ω_n, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\|(\omega_n\varphi)_y\|_{L^2}^2 &= \sum_{n\in\mathbb{Z}}\iint_{S_L}\omega^2(x-n)|\varphi_y|^2\,dxdy = \iint_{S_L}|\varphi_y|^2\,dxdy\sum_{n\in\mathbb{Z}}\omega^2(x-n)\\ &\le \iint_{S_L}|\varphi_y|^2\,dxdy\sum_{n\in\mathbb{Z}}\omega(x-n) \;=\; \iint_{S_L}|\varphi_y|^2\,dxdy\end{aligned}\tag{1.20}$$

and

$$\sum_{n\in\mathbb{Z}}\|(\omega_n\varphi)_x\|_{L^2}^2 = \sum_{n\in\mathbb{Z}}\|\omega_n'\varphi+\omega_n\varphi_x\|_{L^2}^2 \lesssim \sum_{n\in\mathbb{Z}}\|\omega_n'\varphi\|_{L^2}^2+\|\omega_n\varphi_x\|_{L^2}^2.\tag{1.21}$$

Also, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\|\omega_n'\varphi\|_{L^2}^2 &\le \sum_{n\in\mathbb{Z}}\int_{-L}^{L}\int_{-2}^{2}(\omega')^2|\varphi(x+n,y)|^2\,dxdy \le k(\omega)\sum_{n\in\mathbb{Z}}\int_{-L}^{L}\int_{n-2}^{n+2}|\varphi(x,y)|^2\,dxdy\\ &\le 4k(\omega)\int_{-L}^{L}\int_{\mathbb{R}}|\varphi|^2\,dxdy = 4k(\omega)\,\|\varphi\|_{L^2}^2.\end{aligned}\tag{1.22}$$

Therefore from (1.19), (1.20), (1.21) and (1.22), we obtain (1.18). ■

LEMMA 1.3.7 *Suppose $B>0$ and $\delta>0$ are given. Then there exists $\eta=\eta(B,\delta)$ such that if $f\in H_L$ with $\|f\|_1\le B$ and $\|f\|_{L^{p+2}}\ge\delta$, then*

$$\sup_r\int_{-L}^{L}\int_{r-2}^{r+2}|f|^{p+2}\,dxdy \;\ge\; \eta.$$

Proof. Let ω be as in the preceding lemma. Since $\sum_{n\in\mathbb{Z}}\omega(x-n)=1$, it implies that no more than four of the terms in the sum are nonzero at any given value of x, it follows that there exists a constant $k_1>0$ such that $\sum_{n\in\mathbb{Z}}\omega^{p+2}(x-n)\ge\; k_1$ for all $x\in\mathbb{R}$. Suppose there exists f (which is not identically zero) such that $\|f\|_1\le B$ and

$$\|\omega_nf\|_1^2 \ge \left(1+k_2\|f\|_{L^{p+2}}^{-p-2}\right)\|\omega_nf\|_{L^{p+2}}^{p+2}\tag{1.23}$$

for every $n\in\mathbb{Z}$ where $k_2=\dfrac{k\,B^2}{k_1}$. By summing over n and using Lemma 1.3.6, we have

$$\begin{aligned}kB^2\ge k\|f\|_1^2 &\ge \left(1+k_2\|f\|_{L^{p+2}}^{-p-2}\right)\sum_{n\in\mathbb{Z}}\|\omega_nf\|_{L^{p+2}}^{p+2} = \left(1+k_2\|f\|_{L^{p+2}}^{-p-2}\right)\sum_{n\in\mathbb{Z}}\iint_{S_L}\omega_n^{p+2}|f|^{p+2}\\ &\ge \left(1+k_2\|f\|_{L^{p+2}}^{-p-2}\right)k_1\|f\|_{L^{p+2}}^{p+2} = k_1\|f\|_{L^{p+2}}^{p+2}+k_1k_2 = k_1\|f\|_{L^{p+2}}^{p+2}+kB^2.\end{aligned}$$

But it is impossible, since f is not identically zero. So, there exists $n_0\in\mathbb{Z}$ such that

$$\|\omega_{n_0}f\|_1^2 \le \left(1+k_2\|f\|_{L^{p+2}}^{-p-2}\right)\|\omega_{n_0}f\|_{L^{p+2}}^{p+2},\tag{1.24}$$

where $k_2=\dfrac{k\,B^2}{k_1}$. Therefore, from (1.24) and Sobolev embedding, we have

$$\|\omega_{n_0}f\|_{L^{p+2}}^2 \le k_3^2\|\omega_{n_0}f\|_1^2 \le k_3^2\left(1+\frac{k_2}{\delta^{p+2}}\right)\|\omega_{n_0}f\|_{L^{p+2}}^{p+2}.$$

Then,

$$\int_{-L}^{L}\int_{n_0-2}^{n_0+2} |f|^{p+2}\,dxdy \geq \int_{-L}^{L}\int_{\mathbb{R}} |\omega_{n_0} f|^{p+2} \geq \left[k_3^2\left(1+\frac{k_2}{\delta^{p+2}}\right)\right]^{\frac{-p-2}{p}} \equiv \eta.$$

■

Now, for every minimizing sequence $\{\varphi_n\}$ of I_q, we know that $\|\varphi_n\|_{L^{p+2}}^{p+2} \geq q$ and $\|\varphi_n\|_1 \leq B$ for every $n \in \mathbb{Z}$. Thus by preceding lemma, there exists $\eta > 0$ such that $Q_n(2) \geq \eta$ for all n. Therefore,

$$\alpha = \lim_{r\to\infty} Q(r) \geq Q(2) = \lim_{n\to\infty} Q_n(2) \geq \eta > 0.$$

Therefore, we have showed

THEOREM 1.3.8 (Ruling out Evanescence) *For every minimizing sequence of I_q , $\alpha > 0$.*

Thus, we have ruled out the evanescence case; hence the *Compactness* case occurs, i.e. $\alpha = q$.

1.4 Regularity of the Solitary Waves

In this section, we prove that any solitary wave of (1.6) is a C^∞ function, for all $p \in \mathbb{N}$. More precisely we have

THEOREM 1.4.1 *Any solitary wave solution of (1.6) belongs to $H_L^\infty := H_L^\infty(S_L)$.*

Proof. There may exist many ways to prove this, but we will proceed by bootstrapping argument, using Lemmata 0.0.4 and 0.0.5. Setting $\psi \equiv \frac{-1}{p+1}\varphi^{p+1}$, (1.6) yields

$$\mathcal{F}_x\mathcal{F}_y(\varphi_{yy}) = -n^2\ \mathcal{F}_x\mathcal{F}_y(\varphi) =\ q(\xi,n)\ \mathcal{F}_x\mathcal{F}_y(\psi),$$

where $q(\xi,n) = \dfrac{n^2}{c+\xi^2+n^2}$, $n \in \frac{\pi}{L}\mathbb{Z}$, $\xi \in \mathbb{R}$ and $\mathcal{F}_x$, $\mathcal{F}_y$ are the Fourier transforms with respect to x and y (respectively). It can be rewrited as follows

$$\mathcal{F}_y(\varphi_{yy}) = \mathcal{F}_x^{-1}\left[q(\xi,n)\ \mathcal{F}_x\mathcal{F}_y\psi\right] = \widetilde{q}(n)\ \psi,$$

where $\widetilde{q}(n)$ is the operator $\mathcal{F}_x^{-1}\ q(\cdot,n)\ \mathcal{F}_x$ for any fixed n. It is easy to verify that $\widetilde{q}(n) \in \mathcal{L}\left(L^2(\mathbb{R}_x)\right)$, the space of bounded linear operators in $L^2(\mathbb{R}_x)$. As well, it is easily checked out that $q(\xi,n)$ satisfies the assumption of Proposition 0.0.4, if we take $n \in \mathbb{R}$, so $q(\xi,n)$ is a multiplier in $L^2(\mathbb{R}^2)$. Thusly, so is it for $\widetilde{q}(n)$ in the space $L^2\left(\mathbb{R}_y, L^2(\mathbb{R}_x)\right) = L^2\left(\mathbb{R}^2\right)$. Additionally, we have that $\widetilde{q}(n)$ depends continuously on n with respect to the norm in $\mathcal{L}\left(L^2(\mathbb{R}_x)\right)$ at any point $n \neq 0$. Using the Proposition 0.0.5, for every $x \in \mathbb{R}$ fixed, it follows that $\widetilde{q}(n)$ is a multiplier in the space $L^2\left((-L,L), L^2(\mathbb{R}_x)\right)^{\curlyvee} = L^2(S_L)^{\curlyvee}$ considered as the space of L-periodic functions in y, where superscript $\curlyvee$ means that for functions from this space $\mathcal{F}_y\ \varphi$ vanishes at $n = 0$. Since $q(\xi,0) = 0$, the corresponding multiplier vanishes on $\left\{\varphi \in L^2(S_L)\ :\ \mathcal{F}_y\varphi = 0\ \text{ if }\ n = 0\right\}$ and, hence, is a bounded operator on the entire space $L^2(S_L)$. In fact, here, we need an extension of that theorem for operator-valued multipliers which may be discontinuous at the point zero; nevertheless, the proof presented in [68] of Proposition 0.0.5 brings about without any change; although the last argument does not need for φ_{xx}. So, since $\varphi^{p+1} \in L^2(S_L)$, we obtain that $\varphi_{yy} \in L^2(S_L)$, and analogously $\varphi_{xx} \in L^2(S_L)$. So $\varphi \in H^2(S_L)$. By differentiating of (1.6) in the sense of distribution with respect to x, y and reiteration of the process leads to proof of Theorem 1.4.1. ■

REMARK 1.4.2 *Theorem 1.4.1 implies that we do not initially need to put the conditions* (1.2) − (1.4) *on* φ.

THEOREM 1.4.3 *Suppose* $p \in \mathbb{N}$ *and* $1 \leq q \leq \infty$ *and* φ_c *is a solution of (1.6) which we obtained by minimization. Then* $\varphi_c \in W^{1,q}(S_L)$.

Proof. By (1.44), we have $\mathcal{F}_x\mathcal{F}_y\left(\frac{\partial\varphi_c}{\partial x}\right) = i\xi\ \mathcal{F}_x\mathcal{F}_y(\varphi_c) = i\xi\widehat{K_c}(\xi,n)\mathcal{F}_x\mathcal{F}_y(\psi)$, where $\psi = \frac{1}{p+1}\widehat{\varphi_c{}^{p+1}}$. By a similar argument as the one used obtaining the regularity, we obtain that $\frac{\partial\varphi_c}{\partial x}$ is in $L^q(S_L)$. Similarly $\frac{\partial\varphi_c}{\partial y} \in L^q(S_L)$. Combining these estimates, we have

$$\|\varphi_c\|_{W^{1,q}(S_L)} = \|\varphi_c\|_{L^q(S_L)} + \left\|\frac{\partial\varphi_c}{\partial x}\right\|_{L^q(S_L)} + \left\|\frac{\partial\varphi_c}{\partial y}\right\|_{L^q(S_L)} \leq C\|\varphi_c^{p+1}\|_{L^q(S_L)}.$$

So $\varphi_c \in W^{1,q}(S_L)$. ■

REMARK 1.4.4 *In fact, one can show that* $\|\varphi_c\|_{W^{2,q}(S_L)}$ *is equivalent to* $\|\varphi_c\|^{p+1}_{L^{q(p+1)}(S_L)}$. *Furthermore, it can be shown that* $\varphi_c \in W^{k,p}(S_L)$ *for* $1 \leq k, p \leq \infty$.

1.5 Traveling Wave Solution of ZK in the Plane

We proved the existence of the traveling wave solution u_k of the Zakharov-Kuznetsov equation of period k in y-direction, for every $k \in \mathbb{Z}$. In this section, we are going to demonstrate the sequence $\{u_k\}_k$ converges to a solitary wave solution of the Zakharov-Kuznetsov in $\mathbb{R}^2$ as $k \to \infty$. Let $k \in \mathbb{N}$ and $S_k = \mathbb{R} \times (-k, k)$ and H_k be as in the Definition 1.2.1. As we saw before, we obtained, u_k, a solitary wave solution of the ZK equation in S_k. By a simple calculation we see that u_k is a critical point of $\mathcal{J}_k$, where

$$\mathcal{J}_k(u) = \frac{1}{2}\int_{S_k} cu^2 + |\nabla u|^2 - \frac{2}{(p+1)(p+2)}u^{p+2}\ dxdy.$$

We define the functional

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^2} c\ u^2 + |\nabla u|^2 - \frac{2}{(p+1)(p+2)}u^{p+2}\ dxdy,$$

We will show that there exists a minimizer of I_q in $\mathbb{R}^2$. We denotes $S_r(\xi)$ the cube with the side length r, centered at the point $\xi \in \mathbb{R}^2$.

LEMMA 1.5.1 *If* $u_n \in H_k$, $n = 1, 2, \dots$ *is a bounded sequence and there exists a* $r > 0$ *such that*

$$\lim_{n\to\infty}\sup_{\xi\in S_k}\int_{S_r(\xi)} |u_n|^2\ dxdy = 0,$$

then $\|u_n\|_{L^p(S_k)} \longrightarrow 0$, *for all* $2 \leq p < \infty$.

Proof. By Hölder's inequality we have

$$\int_{S_r(\xi)} |u_n|^p\ dxdy \leq \left(\int_{S_r(\xi)} |u_n|^2\ dxdy\right)^{1/2}\left(\int_{S_r(\xi)} |u_n|^{2(p-1)}\ dxdy\right)^{1/2} \tag{1.25}$$

To proceed, we use the embedding theorem, so by (6.1) we have

$$\int_{S_r(\xi)} |u_n|^p \, dxdy \le \left(\int_{S_r(\xi)} |u_n|^2 \, dxdy \right)^{1/2} \|u_n\|^{p-1}_{H_k(S_r(\xi))} \le A^{p-1} \|u_n\|_{L^2(S_r(\xi))}.$$

On the other hand, for $r > 0$, we can choose a fixed number $m = m(r) \in \mathbb{N}$ such that the countable set

$$\mathscr{U} = \left\{ S_r(\xi) \ : \ \xi = (a,b) \in \frac{1}{2r}\mathbb{Z}^2 \ , \ |b| \le k \right\}$$

can cover S_k and any point of S_k is contained in at most m of the cubes of $\mathscr{U}$. Summing up, using the inequality above, we find that

$$\int_{S_k} |u_n|^p \, dxdy \le \widetilde{A} \left(\sup_{\xi \in S_k} \int_{S_r(\xi)} |u_n|^2 \, dxdy \right)^{1/2}.$$

By the assumption we conclude that $\|u_n\|_{L^p(S_k)} \to 0$, for $2 \le p < \infty$. ■

LEMMA 1.5.2 *Let $\{u_n\} \subset H_k$ be a sequence of y-periodic functions of period k such that it is uniformly bounded in the H_k-norm and satisfies $\mathcal{J}_k'(u_k) \to 0$, then the following alternative holds: Either*

(i) $\|u_k\|_{H_k} \to 0$ *as* $k \to \infty$, *or*

(ii) *There exist $r, \eta > 0$ and a sequence of the points $\xi_k \in \mathbb{R}^2$ such that, up to a subsequence,*

$$\lim_{k \to \infty} \left(\int_{S_r(\xi)} |u_k|^2 \, dxdy \right) > \eta.$$

Proof. Assume that the case (ii) does not hold, then from Lemma 1.5.1, we get $\|u_k\|_{L^p(S_k)} \to 0$ for all $2 \le p < \infty$. By the definition of $\mathcal{J}$, we have

$$\mathcal{J}_k(u_k) - \frac{1}{2}\langle \mathcal{J}_k'(u_k), u_k \rangle = \frac{p}{2(p+1)(p+2)} \int_{S_k} u_k^{p+2} \, dxdy. \tag{1.26}$$

So, we obtain

$$\frac{\min\{1,c\}}{2} \|u_k\|^2_{H_k} \le \int_{S_k} \frac{1}{2} c u_k^2 + \frac{1}{2} |\nabla u_k|^2 \, dxdy = \mathcal{J}_k(u_k) + \frac{1}{(p+1)(p+2)} \|u_k\|^{p+2}_{L^{p+2}(S_k)} =$$
$$= \frac{1}{2}\langle \mathcal{J}_k'(u_k), u_k \rangle + \frac{1}{2(p+1)} \|u_k\|^{p+2}_{L^{p+2}(S_k)} \le \frac{1}{2} \|\mathcal{J}'(u_k)\| \|u_k\| + \frac{1}{2(p+1)} \|u_k\|^{p+2}_{L^{p+2}(S_k)}.$$

So, the right hand side tends to zero, which implies the condition (i) holds. ■

LEMMA 1.5.3 *Any critical point u_k of $\mathcal{J}_k$ satisfies the estimate $\|u_k\|_{H_k} \le C_k$ with a constant $C_k > 0$ only depending on the critical value.*

Proof. Since $c_k = \mathcal{J}_k(u_k)$ and $\mathcal{J}_k'(u_k) = 0$, then we obtain

$$c_k = \mathcal{J}_k(u_k) - \frac{1}{2}\langle \mathcal{J}_k'(u_k), u_k\rangle = \frac{p}{2(p+1)(p+2)} \int_{S_k} u_k^{p+2}\, dxdy.$$

This implies

$$\begin{aligned}\min\{1,c\}\|u_k\|_{H_k}^2 &\leq \int_{S_k} cu_k^2 + |\nabla u_k|^2\, dxdy = 2\mathcal{J}_k(u_k) + \frac{2}{(p+1)(p+2)}\int_{S_k} u_k^{p+2}\, dxdy\\ &= 2c_k + \frac{2}{(p+1)(p+2)}\int_{S_k} u_k^{p+2}\, dxdy = \left(2+\frac{4}{p}\right)c_k.\end{aligned}$$

Therefore $\|u_k\|_{H_k}^2 \leq \frac{(2+\frac{4}{p})}{\min\{1,c\}} c_k$. ■

LEMMA 1.5.4 *Let $u_k \in H_k$ and $u \in H^1(\mathbb{R}^2)$ be nontrivial solutions of y-periodic and non-periodic equations which satisfy $\langle \mathcal{J}_k'(u_k), u_k\rangle = 0$ and $\langle \mathcal{J}'(u), u\rangle = 0$, respectively. Then there exist $\epsilon_1 > 0, \epsilon_2 > 0$ not depending on k such that $\|u_k\|_{H_k} \geq \epsilon_1$, $\|u\|_{H^1} \geq \epsilon_1$, $\mathcal{J}_k(u_k) \geq \epsilon_2$ and $\mathcal{J}(u) \geq \epsilon_2$.*

Proof. Since $\langle \mathcal{J}_k'(u_k), u_k\rangle = 0$ then we get

$$\min\{1,c\}\|u_k\|_{H_k}^2 \leq \int_{S_k} cu_k^2 + |\nabla u_k|^2\, dxdy = \frac{1}{p+1}\int_{S_k} u_k^{p+2}\, dxdy \leq \frac{C}{p+1}\|u_k\|_{H_k}^{p+2},$$

where $C > 0$ depends on the embedding constants.
So, this shows that

$$\|u_k\|_{H_k} \geq \left(\frac{\min\{1,c\}(p+1)}{C}\right)^{\frac{1}{p}}.$$

On the other hand, we have

$$\mathcal{J}_k(u_k) = \int_{S_k} cu_k^2 + |\nabla u_k|^2 - \frac{1}{(p+1)(p+2)}u_k^{p+2}\, dxdy \geq \min\{1,c\}\|u_k\|_{H_k}^2 - \frac{1}{(p+1)(p+2)}u_k^{p+2}\, dxdy.$$

By the assumptions, we obtain

$$\mathcal{J}_k(u_k) \geq \min\{1,c\}\|u_k\|_{H_k}^2 - \frac{2}{p}\mathcal{J}_k(u_k).$$

Therefore, we get

$$\mathcal{J}_k(u_k) \geq \min\{1,c\}\frac{p}{p+2}\|u_k\|_{H_k}^2.$$

This together with the first estimate, gives the desired lower bound. Clearly, the arguments for $u \in H^1(\mathbb{R}^2)$ are similar. ■

In the following constructions, we need a operator from H_k to $H^1(\mathbb{R}^2)$. Let χ_k be a $C_0^\infty(\mathbb{R})$ cut-off function satisfying $\chi_k(s) = 1$ for $|s| \leq k$, $\chi_k(s) = 0$ for $|s| \geq k + \frac{1}{2}$ and $|\chi_k'|, |\chi_k''| \leq c_0$. We define the cut-off operator $\mathcal{P}_k : H_k \longrightarrow H^1(\mathbb{R}^2)$ by $\mathcal{P}_k u(x,y) = \partial_y\left(\chi_k(y)\partial_{y,k}^{-1}u(x,y)\right)$, where $\partial_{y,k}^{-1}u(x,y) = \int_{-k}^{y} u(x,s)\, ds$. Then we have the following lemma.

LEMMA 1.5.5 *$\mathcal{P}_k$ is a uniformly bounded (with respect to k) linear operator from H_k into $H^1(\mathbb{R}^2)$ and $\mathcal{P}_k u(x,y) = u(x,y)$ for $(x,y) \in S_k$.*

Proof. For $u \in H_k$ we have

$$\iint\limits_{\mathbb{R}^2} |\mathcal{P}_k u(x,y)|^2 \, dxdy \le 2 \iint\limits_{\mathbb{R}^2} \left(|\chi_k(y) u(x,y)|^2 + \left| \chi_k'(y) \partial_{y,k}^{-1} u(x,y) \right|^2 \right) dxdy. \tag{1.27}$$

The first integral on the right hand side can easily estimated by $2\|u\|^2_{L^2(S_k)}$. To estimate the second one, we denote $\mathscr{N}_1 = [-k - \frac{1}{2}, -k]$ and $\mathscr{N}_2 = [k, k + \frac{1}{2}]$. For $y \in \mathscr{N}_1$, by the Cauchy-Schwarz inequality, we have

$$\left| \partial_{y,k}^{-1} u(x,y) \right|^2 \le \left| \int_{-k}^{y} |u(x,s)| \, ds \right|^2 \lesssim \int\limits_{\mathscr{N}_1} |u(x,s)|^2 \, ds,$$

and for $y \in \mathscr{N}_2$, by the Cauchy-Schwarz inequality, we have

$$\left| \partial_{y,k}^{-1} u(x,y) \right|^2 \le \left| \int_{k}^{y} |u(x,s)| \, ds \right|^2 \lesssim \int\limits_{\mathscr{N}_2} |u(x,s)|^2 \, ds.$$

The second integral in (1.27) can now estimated by

$$\begin{aligned} \iint\limits_{\mathbb{R}^2} \left| \chi_k'(y) \partial_{y,k}^{-1} u(x,y) \right|^2 dxdy &= \int\limits_{\mathscr{N}_1 \cup \mathscr{N}_2} \int\limits_{\mathbb{R}} \left| \chi_k'(y) \partial_{y,k}^{-1} u(x,y) \right|^2 dxdy \\ &\le \int\limits_{\mathscr{N}_1 \cup \mathscr{N}_2} \int\limits_{\mathbb{R}} |\chi_k'(y)|^2 \int\limits_{-k}^{k} |u(x,s)|^2 \, ds \, dxdy \\ &\le C_0^2 \int\limits_{\mathscr{N}_1 \cup \mathscr{N}_2} \|u\|^2_{L^2(S_k)} \, dy \le C_0^2 \, \|u\|^2_{L^2(S_k)}. \end{aligned}$$

This shows that

$$\iint\limits_{\mathbb{R}^2} |\mathcal{P}_k u(x,y)|^2 \, dxdy \le C_1 \, \|u\|^2_{L^2(S_k)}.$$

Similarly, we can estimate

$$\begin{aligned} \iint\limits_{\mathbb{R}^2} |(\mathcal{P}_k u(x,y))_y|^2 \, dxdy &\le 4 \iint\limits_{\mathbb{R}^2} \left[|\chi_k u_y|^2 + 2 \left| \chi_k' u \right|^2 + \left| \chi_k'' \partial_{y,k} u \right|^2 \right] dxdy \\ &\le C_2 \left(\|u_y\|^2_{L^2(S_k)} + \|u\|^2_{L^2(S_k)} \right). \end{aligned}$$

Finally, we get $\iint\limits_{\mathbb{R}^2} |(\mathcal{P}_k u(x,y))_x|^2 \, dxdy \le \iint\limits_{\mathbb{R}^2} |u_x(x,y)|^2 \, dxdy$. This shows that $\mathcal{P}_k : H_k \longrightarrow H^1(\mathbb{R}^2)$ is uniformly bounded. The second statement of the lemma is obvious. ∎

THEOREM 1.5.6 *Let $u_k \in H_k$ be a minimizer for $\mathcal{J}_k$. Then there exists a sequence $\xi_k \in \mathbb{R}^2$ and a function $u \in H^1(\mathbb{R}^2)$ such that $\mathcal{P}_k u_k(\cdot + \xi_k)$ converges weakly to u along a subsequence. Moreover, u is a nontrivial solution of the Zakharov-Kuznetsov equation, a minimizer of I_q in $\mathbb{R}^2$ and*

$$\lim_{k\to\infty} \|u_k - u(\cdot + \xi_k)\|_{H_k} = 0.$$

Proof. For $k \geq 1$ we can choose $\varrho \in C_0^\infty(S_1)$ having $\int_{S_1} \varrho^{p+2}\, dxdy > 0$. Since supp $\varrho \subset S_1 \subset S_k$, we define a periodic extension of ϱ such that $\varrho_k(x,y) = \varrho(x,y)$ when $(x,y) \in S_1$ and $\varrho_k(x,y) = 0$ when $(x,y) \in S_k \setminus S_1$. We take $\beta \in \mathbb{R}$ in such way that $e_k = \beta\varrho_k$ satisfies that $e_k \in H_k$. Then, this leads us to conclude that $\mathcal{J}_k(e_k) = \mathcal{J}_1(e_1)$. Therefore from Lemma 1.5.3 and Lemma 1.5.4 we have $\|u_k\|_{H_k}$ is uniformly bounded from below and above. Thus, case (i) of Lemma 1.5.2 is not possible and from case (ii), we obtain a sequence $\xi_k \in \mathbb{R}^2$ such that the shifted functions $\widetilde{u}_k = u_k(\cdot + \xi_k)$ satisfy (for large k)

$$\int_{S_r(0)} |\widetilde{u}_k|^2\, dxdy > \eta/2,$$

with appropriate $r, \eta > 0$. Clearly, $\widetilde{u}_k$ is also critical points of $\mathcal{J}_k$. Since the sequence $\mathcal{P}_k\widetilde{u}_k$ is bounded in $H^1(\mathbb{R}^2)$, there exists a subsequence which converges weakly in $H^1(\mathbb{R}^2)$ to a nontrivial function $u \in H^1(\mathbb{R}^2)$. We claim that u is a nontrivial solution of Zakharov-Kuznetsov equation. Since the embedding $H^1(\Omega) \hookrightarrow L^{p+2}(\Omega)$ is compact for any bounded domain Ω in $\mathbb{R}^2$, we claim $\widetilde{u}_k \to u$ strongly in $L^{p+2}(\Omega)$. Let $\vartheta \in C_0^\infty(\mathbb{R}^2)$. Then for sufficiently large k we have that supp $\vartheta \subset S_k$ and so, ϑ can be considered as an element of H_k for k large just by defining its periodic extension.
Now, we have

$$\langle \mathcal{P}_k\widetilde{u}_k, \vartheta\rangle_{H^1} = \int_\Omega \mathcal{P}_k\widetilde{u}_k\vartheta + \nabla(\mathcal{P}_k\widetilde{u}_k)\cdot\nabla\vartheta\, dxdy = \int_{S_k} \mathcal{P}_k\widetilde{u}_k\vartheta + \nabla(\mathcal{P}_k\widetilde{u}_k)\cdot\nabla\vartheta\, dxdy = \langle \widetilde{u}_k, \vartheta\rangle_{H_k}.$$

This clearly implies $\langle \widetilde{u}_k, \vartheta\rangle_{H_k} \to \langle u, \vartheta\rangle_{H^1}$. On the other hand, since ϑ is also a member of H^1, for large k, then we see

$$\langle \mathcal{J}(u), \vartheta\rangle = \langle u, \vartheta\rangle_{H^1} - \frac{1}{p+1}\int_\Omega u^{p+1}\vartheta\, dxdy = \lim_{k\to\infty}\left(\langle \widetilde{u}_k, \vartheta\rangle_{H_k} - \frac{1}{p+1}\int_\Omega \widetilde{u}^{p+1}\vartheta\, dxdy\right) = \lim_{k\to\infty}\langle \mathcal{J}_k'(u_k), \vartheta\rangle = 0;$$

that means u is a nontrivial weak solution of ZK. We want to show that $u \in G_q$, where

$$G_q = \left\{u \in H^1(\mathbb{R}^2) \;:\; I(u) = I_q\;,\; J(u) = q\right\}, \tag{1.28}$$

$$I_q = \inf\left\{I(u) \;:\; u \in\; H^1(\mathbb{R}^2)\;,\; J(u) = \iint_{\mathbb{R}^2} u^{p+2}\, dxdy = q\right\}, \tag{1.29}$$

$$I(u) = \frac{1}{2}\iint_{\mathbb{R}^2} c\, u^2 + |\nabla u|^2\, dxdy, \tag{1.30}$$

$$G_q^k = \left\{u \in H_k \;:\; I^k(u) = I_q^k\;,\; J^k(u) = q\right\}, \tag{1.31}$$

$$I_q^k = \inf\left\{I^k(u) \;:\; u \in\; H_k\;,\; J^k(u) = \iint_{S_k} u^{p+2}\, dxdy = q\right\}, \tag{1.32}$$

$$I^k(u) = \frac{1}{2}\iint_{S_k} c\, u^2 + |\nabla u|^2\, dxdy. \tag{1.33}$$

First note that for given $w \in H^1(\mathbb{R}^2)$ such that $J(w) = q$, there exists a sequence $w_k \in C_0^\infty(S_k)$ such that $\|w_k - w\|_{H^1(\mathbb{R}^2)}$, as $k \to \infty$. By the Sobolev embedding, we have $\|w_k - w\|_{L^{p+2}(\mathbb{R}^2)} \to 0$ implying $\big|\|w_k\|_{L^{p+2}(\mathbb{R}^2)} - \|w\|_{L^{p+2}(\mathbb{R}^2)}\big| \to 0$. So, we obtain $J(w_k) = q$, hence $J^k(w_k) = q$. Moreover $J(u) = q$ because u is a critical point of $\mathcal{J}$ and a nontrivial weak solution of the ZK equation, and $u_k \in G_q^k$. From continuity for the functional I, we conclude that $I(w_k) \to I(w)$, as $k \to \infty$. Thus for given $\epsilon > 0$, there exists k_ϵ such that $I(w_k) \leq I(w) + \epsilon$, which implies that $\limsup I_q^k \leq I(w) + \epsilon$ for any $w \in H^1(\mathbb{R}^2)$ with $J(w) = q$ and for any $\epsilon > 0$. Moreover $\limsup I_q^k \leq I_q$. Now, note that for a given bounded $\mathcal{D} \subset \mathbb{R}^2$, we have that $\mathcal{D} \subset S_k$ for k large enough, and so

$$I_k^q = I^k(\widetilde{u}_k) \geq \frac{1}{2} \iint_{\mathcal{D}} c\ \widetilde{u}_k^2 + |\nabla \widetilde{u}_k|^2\ dxdy.$$

Taking $\liminf$, we get that

$$\liminf I_q^k = I^k(\widetilde{u}_k) = \liminf \frac{1}{2} \iint_{\mathcal{D}} c\ \widetilde{u}_k^2 + |\nabla \widetilde{u}_k|^2\ dxdy \geq \frac{1}{2} \iint_{\mathcal{D}} c\ u^2 + |\nabla u|^2\ dxdy,$$

due to the local compactness result. In other words, we have shown that $\liminf I_q^k \geq I(u)$, since $\mathcal{D}$ is arbitrary. But u is a nontrivial weak solution of the ZK equation, then $\liminf I_q^k \geq I_q$. In other words, $\lim_{k\to\infty} I_q^k = I_q = I(u)$, which is equivalent to say that u is a nontrivial solution of ZK. Now, let $w_k \in C_0^\infty(S_k)$ such that $w_k \to u$ in $H^1(\mathbb{R}^2)$. Then, a direct computation shows that

$$\lim_{k\to\infty} \|u_k - u(\cdot + \xi_k)\|_{H_k} = \lim_{k\to\infty} \|\widetilde{u}_k - u\|_{H_k} = 0 \Leftrightarrow \lim_{k\to\infty} \|\widetilde{u}_k - w_k\|_{H_k} = 0.$$

On the other hand, we have that

$$\begin{aligned} I^k(\widetilde{u}_k - w_k) &= I^k(\widetilde{u}_k) + I^k(w_k) - 2 \iint_{S_k} \widetilde{u}_k\ w_k + \nabla \widetilde{u}_k \cdot w_k\ dxdy \\ &= I^k(\widetilde{u}_k) + I^k(w_k) - 2 \iint_{S_k} \widetilde{u}_k\ u + \nabla \widetilde{u}_k \cdot \nabla u\ dxdy \\ &- 2 \iint_{S_k} \widetilde{u}_k\ (w_k - u) + \nabla \widetilde{u}_k \cdot \nabla (w_k - u)\ dxdy \\ &= I^k(\widetilde{u}_k) + I^k(w_k) - 2\langle \widetilde{u}_k, u\rangle_1 - 2\langle \widetilde{u}_k, w_k - u\rangle_1. \end{aligned}$$

Since w_k converges strongly to u in $H^1(\mathbb{R}^2)$ and $\|\widetilde{u}_k\|_{H_k}$ is bounded, we conclude that $|\langle \widetilde{u}_k, w_k - u\rangle| = o(1)$. But we proved that $\langle \widetilde{u}_k, u\rangle \to I(u)$. So, taking limit as $k \to \infty$ and using $I^k(\cdot) \sim \|\cdot\|_{H_k}$, $I(\cdot) \sim \|\cdot\|_{H^1(\mathbb{R}^2)}$,

$$\|\widetilde{u}_k - w_k\|_{H_k}^2 \sim I^k(\widetilde{u}_k - w_k) = o(1).$$

So, $\lim_{k\to\infty} \|u_k - u(\cdot + \xi_k)\|_{H_k}^2 = 0$. ■

1.6 Extension

We studied the equation (1.6) in the case that $p = \frac{k}{\ell}$, where $k \in \mathbb{N}$ is even and k and ℓ are relatively prime. The evenness of k was necessary to define the concentration functions $Q_n(\cdot)$. Now, we want extend our results to the case p is arbitrary. Let $q > 0$ and $S_L = \mathbb{R} \times (-L, L)$. We define the functional I on H_L as before,

$$I(\varphi) = \frac{1}{2} \iint\limits_{S_L} |\nabla\varphi|^2 + c\varphi^2 dxdy$$

and the following constrained minimization problem on H_L,

$$I_q = \inf \left\{ I(\varphi) : \varphi \in H_L \ , \ J(\varphi) = \iint\limits_{S_L} \varphi^{p+2}(x,y)dxdy = q > 0 \right\}. \tag{1.34}$$

Also, we consider the set of minimizers $G_q = \{\varphi \in H_L : I(\varphi) = I_q \ , \ J(\varphi) = q\}$. But, here, we cannot define the concentration functions, because we do not know the sign of $J(\cdot)$. So, we consider the following constrained minimization on H_L,

$$I_q^\star := \inf \left\{ I(\varphi) : \varphi \in H_L \ , \ J^\star(\varphi) := \iint\limits_{S_L} |\varphi(x,y)|^{p+2} \, dxdy = q > 0 \right\}. \tag{1.35}$$

Also, associated to this minimization problem we consider the set of minimizers $G_q^\star$, defined by

$$G_q^\star := \left\{ \varphi \in H_L : I(\varphi) = I_q^\star \ , \ J^\star(\varphi) = q \right\}.$$

To each minimizing sequence $\{\varphi_n\}$ of $I_q^\star$, we can define the concentration functions $Q_n^\star : [0,\infty) \to [0,q]$ defined by

$$Q_n^\star(r) = \sup_\zeta \int_{-L}^{L} \int_{\zeta-r}^{\zeta+r} |\varphi(x,y)|^{p+2} \, dxdy.$$

Now, the method of Section 1.3 works analogously, and we have

THEOREM 1.6.1 *Let $c > 0$, and let $\{\varphi_n\}$ be a minimizing sequence to $I_q^\star$. Then there is a subsequence $\{\varphi_{n_k}\}$ and a sequence of numbers $\{x_{n_k}\} \subset \mathbb{R}^2$ such that $\varphi_{n_k}(\cdot + x_{n_k})$ converges strongly in H_L to some $\varphi \in H_L$. The limit φ is a minimizer for $I_q^\star$; i.e., $G_q^\star \neq \emptyset$.*

In the rest, we use the ideas of [8].

Rearrangement

In this section, we rearrange the region in a suitable way that we are able to use the results of Section 1.3.

Monotone Decreasing Rearrangement

Let Ω be a bounded open subset of $\mathbb{R}^n$ and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_0^+ = [0,\infty)$ be a nonnegative measurable function. We define level sets $\Omega_s := \{x \ : \ u(x) \geq s\}$, $s \in \mathbb{R}$ of u. We denote a point $x \in \mathbb{R}^n$ by (x', y) with $x' \in \mathbb{R}^{n-1}$. Furthermore, we introduce the notation $\Omega(x') = \Omega \cap \{(x', y) \in \mathbb{R}^n \ ; \ y \in \mathbb{R}\}$, for fixed $x' \in \mathbb{R}^{n-1}$.

The length of this one-dimensional set $\Omega(x')$ can be calculated as follows $m(\Omega(x')) = \int_{\mathbb{R}} \chi_\Omega(x', y)\, dy$, where χ_Ω is the characteristic function of Ω and m denotes the *Lebesgue* measure. Now, we define

$$\Omega^\star(x') := \begin{cases} \{(x', y) \in \mathbb{R}^n \ : \ 0 \leq y \leq m(\Omega(x'))\} & \text{if} \quad \Omega(x') \neq \emptyset \\ \\ \emptyset & \text{if} \quad \Omega(x') = \emptyset \end{cases}$$

and $\Omega^\star := \bigcup_{x' \in \Omega'} \Omega^\star(x')$, where $\Omega' \subseteq \mathbb{R}^{n-1}$ is the set of those $x' \in \mathbb{R}^{n-1}$ for which $\Omega(x')$ is not empty. Then the set $\Omega^\star$ is a bounded open subset of $\mathbb{R}^n$ whose measure is equal to $m(\Omega)$, and is connected if and only if the projection of Ω on hyperplane $\{y = 0\}$ is connected; this will be called the monotone decreasing rearrangement of Ω relative to the direction y. It is included in $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n,\ x = (x', y),\ x' \in \mathbb{R}^{n-1}, y \in \mathbb{R}^+_0\}$.

DEFINITION 1.6.2 *We define the monotone decreasing rearrangement in the direction y of u by*

$$u^\star(x) := \sup\{s \in \mathbb{R} \ ; \ x \in \Omega^\star_s\}.$$

$u^\star$ is the unique function defined on $\Omega^\star$, which is decreasing in the y-direction (that is, for any $x' \in \mathbb{R}^{n-1}$, $a > b > 0$, $u^\star(x', a) \leq u^\star(x', b)$) and y-equimeasurable with u, i.e., satisfies

$$m\left(\{\zeta \in \Omega(x'),\ u(x', \zeta) > \tau\}\right) = m\left(\{\zeta \in \Omega^\star(x'),\ u^\star(x', \zeta) > \tau\}\right) \tag{1.36}$$

for all $\tau \geq 0$, and all $x' \in \mathbb{R}^{n-1}$.

REMARK 1.6.3 *If u is a function with compact support, we can choose for Ω either the support of u or any open set containing it, without changing $u^\star$; therefore, we shall not always indicate Ω in the following, for the rearrangement of functions with compact support.*

REMARK 1.6.4 *In particular, it is clear that $\Omega = \Omega^\star$ when Ω is the cylinder $S_L = \mathbb{R} \times (-L, L)$, and u and $u^\star$ have the same domain of definition.*

Steiner Symmetrization

Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^+_0$ be a nonnegative measurable function.
We define

$$\Omega^\star(x') := \begin{cases} \left\{(x', y) \in \mathbb{R}^n \ ; \ 0 \leq |y| \leq \frac{1}{2} m(\Omega(x'))\right\} & \text{if} \quad \Omega(x') \neq \emptyset \\ \\ \emptyset & \text{if} \quad \Omega(x') = \emptyset \end{cases}$$

and $\Omega^\star := \bigcup\limits_{x' \in \Omega'} \Omega^\star(x')$. Then we define *Steiner symmetrization* to $\{y = 0\}$ of u by $u^\#(x) := \sup\{s \in \mathbb{R} \ ; \ x \in \Omega^\star_s\}$.

REMARK 1.6.5 *The monotone decreasing rearrangement just described is equivalent to Steiner symmetrization applied to y-even functions (those verifying $u(x', y) = u(x', -y)$). To see that, let u be a function with compact support, and define $u^\ddagger$ by $u^\ddagger(x', y) = u(x', -y)$, and $U = u + u^\ddagger$. If $U^\#$ is the Steiner symmetrization of U relative to the y, then $u^\star$ and $U^\#$ coincide on $\mathbb{R}^n_+$. Therefore all properties of Steiner symmetrization are possessed by the monotone decreasing rearrangement.*

REMARK 1.6.6 *The monotone decreasing rearrangement is order-preserving, for sets (i.e. $\Omega_1 \subseteq \Omega_2$ imply $\Omega_1^\star \subseteq \Omega_2^\star$) and for functions ($u \leq v$ imply $u^\star \leq v^\star$).*

REMARK 1.6.7 *One of the very useful properties of any equimeasurable rearrangement is that if F is any continuous function defined on $\mathbb{R}^{n-1} \times \mathbb{R}$, then*

$$\int_\Omega F(x', u(x))\, dx = \int_{\Omega^\star} F(x', u^\star(x))\, dx. \tag{1.37}$$

This is a clear consequence of (1.36). As a particular case, we have

$$\|u\|_{L^p(\Omega)} = \|u^\star\|_{L^p(\Omega^\star)} \tag{1.38}$$

for all $p \in [1, \infty]$.

REMARK 1.6.8 *It will be easy to see that for all $s > 0$, $(u^s)^\star = (u^\star)^s$ pointwise.*

Inequalities

There are some inequalities between u and $u^\star$ that are important and useful.

LEMMA 1.6.9 (Hardy-Littlewood) *Let $\Omega = S_L$ and u, v be two nonnegative functions of $L^2(\Omega)$. Then we have*

$$\int_\Omega u(x)v(x)\, dx \leq \int_\Omega u^\star(x)v^\star(x)\, dx. \tag{1.39}$$

This property is proved for the Steiner symmetrization in one dimension in [36], and for all equimeasurable and order-preserving rearrangement in [45]. Also a similar property is valid for functions with arbitrary sign:

$$\left| \int uv \right| \leq \int |u^\star||v^\star|.$$

REMARK 1.6.10 *This inequality and (7.5) imply that the monotone decreasing rearrangement is continuous and 1-Lipschitz in the L^2 norm:*

$$\int_\Omega |u^\star(x) - v^\star(x)|^2\, dx \leq \int_\Omega |u(x) - v(x)|\, dx.$$

LEMMA 1.6.11 (Riesz) *Let $u_1, \ldots, u_k$ be nonnegative measurable functions on Ω such satisfying $m(\{x\,; u_i(x) \geq s\}) < \infty$ for all $s > 0$ and all $1 \leq i \leq k$. Then $|(u_1 * u_2 * \cdots * u_k)(0)| \leq (u_1^\star * u_2^\star * \cdots * u_k^\star)(0)$, in the sense that if the right hand side is finite, then the left hand side exists and the inequality holds.*

THEOREM 1.6.12 (Riesz-Sobolev) *Let u and v be two nonnegative functions in $L^2(\Omega)$, and w a function with support in $\mathbb{R}^n_+$, positive and nonincreasing in the y-direction in $\mathbb{R}^n_+$, verifying $w(x', a) \geq w(x', b) > 0$, for all $x' \in \mathbb{R}^{n-1}$, and for all $a > b > 0$. Then we have:*

$$\iint\limits_{\Omega\times\Omega} u(x_1)v(x_2)w(x_1' - x_2', |y_1 - y_2|)\, dx_1dx_2 \leq \iint\limits_{\Omega\times\Omega} u^\star(x_1)v^\star(x_2)w(x_1' - x_2', |y_1 - y_2|)\, dx_1dx_2,$$

and

$$\iint\limits_{\Omega\times\Omega} u(x_1)v(x_2)w(x_1 + x_2)\, dx_1dx_2 \leq \iint\limits_{\Omega\times\Omega} u^\star(x_1)v^\star(x_2)w(x_1 + x_2)\, dx_1dx_2,$$

where $x_1 = (x_1', y_1)$ and $x_2 = (x_2', y_2) \in \mathbb{R}^n$.

For the monotone decreasing rearrangement or Steiner symmetrization we have

THEOREM 1.6.13 *Let $\Omega = S_L$ and u be a function in $H_L := H^1_L(S_L)$. Then $u^\star$ belongs to H_L, and*

$$\int_\Omega |\nabla u^\star|^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx. \tag{1.40}$$

Proof. Let $t > 0$, and $\mathcal{K}_t(x,y)$ be the *Heat* kernel in Ω. We know that $\mathcal{K}_t$ is in all the $L^p(\Omega)$ spaces, so

$$\mathcal{B}_t(u) = t^{-1}\left(\int_{\Omega\times\Omega} |u|^2 \, dxdy - \int_{\Omega\times\Omega} u(x)\mathcal{K}_t(x,y)u(y)\, dxdy\right)$$

is well-defined. Due to Theorem 1.6.12, we have $\mathcal{B}_t(u) \geq \mathcal{B}_t(u^\star)$, since $\mathcal{B}_t(u)$ is symmetric decreasing. We have $u^\star \in L^2(\Omega)$, since $u \in L^2(\Omega)$. To complete the proof, by a similar argument in [45, Lemma 2.6] in the case of cylinder Ω, $\lim_{t\to 0} \mathcal{B}_t(f) = \|f\|^2_{L^2}$ if $f \in H_L$ and $\lim_{t\to 0}\mathcal{B}_t(f) = \infty$ if $f \notin H_L$. ■

It is easy to generalize (1.40) with a separation of y and the other coordinates:

LEMMA 1.6.14 *Under the assumptions of Theorem 1.6.13, we have*

$$\int_\Omega |\nabla_{x'} u^\star|^2 \, dx \leq \int_\Omega |\nabla_{x'} u|^2 \, dx \quad \text{and} \quad \int_\Omega |\partial_y u^\star|^2 \, dx \leq \int_\Omega |\partial_y u|^2 \, dx. \tag{1.41}$$

DEFINITION 1.6.15 *If $u : \Omega \longrightarrow \mathbb{C}$, we define $u^\star = |u|^\star$ and $u^\# = |u|^\#$.*

REMARK 1.6.16 *Note that all the results above hold for this definition.*

Now, we come back to our problem.

LEMMA 1.6.17 *If $\varphi \in G_q^\star$ then $|\varphi|^\star \in G_q$. moreover, $I_q \leq I_q^\star$.*

Proof. By (1.38), we know that the rearrangement preserves L^p-norm, so it follows that $J^\star(|\varphi|^\star) = J^\star(\varphi) = q$. On the other hand, from [35, Lemma 7.6], we know that if $\varphi \in H_L$ then $|\varphi| \in H_L$. Therefore by Theorem 1.6.13, we have $I_q^\star = I(\varphi) \geq I(|\varphi|^\star) \geq I_q^\star$. Hence, we have $|\varphi|^\star \in G_q^\star$. Since $J(|\varphi|^\star) = J^\star(\varphi) = q$, it follows $I_q \leq I_q^\star$. Now, suppose that $|\varphi|^\star \neq G_q$. Then there exists $\psi \in H_L$ such that $J(\psi) = q$ and $I(\psi) < I(|\psi|^\star)$. By defining

$$\zeta = \frac{q^{\frac{1}{p+2}}}{\|\psi\|_{L^{p+2}}}\psi,$$

we have $J^\star(\zeta) = q$ and $I(\zeta) = \dfrac{q^{\frac{2}{p+2}}}{\|\psi\|_{L^{p+2}}} I(\psi) \leq I(\psi)$, since $q = J(\psi) \leq J^\star(\psi)$. Therefore,

$$I(|\varphi|^\star) = I_q^\star \leq I(\zeta) \leq I(\psi) < I(|\varphi|^\star).$$

which is a contradiction. So $|\varphi|^\star \in G_q$. ■

PROPOSITION 1.6.18 *If $\varphi \in G_q$ then $\varphi^\star \in G_q$. Moreover $I_q = I_q^\star$ and $G_q \subseteq G_q^\star$.*

Proof. Let $\varphi \in G_q$. Since φ^p, $\varphi^2 \in L^2(S_L)$, it follows from the properties of the decreasing monotone rearrangement that

$$0 < q = \int_{S_L} \varphi^p \varphi^2 \, dxdy \leq \int_{S_L} (\varphi^p)^\star (\varphi^2)^\star \, dxdy = \int_{S_L} (\varphi^\star)^{p+2} \, dxdy =: \tau. \tag{1.42}$$

We want to show that $q = \tau$ and $I(\varphi) = I(\varphi^\star)$. Indeed, considering $\upsilon \in G_\tau$ and defining $\beta = (\frac{q}{\tau})^{\frac{1}{p+2}}$, we have $J(\beta^{-1}\varphi) = \tau$ and $J(\beta\upsilon) = q$. Whence, we obtain that $I(\upsilon) \leq \beta^{-2} I(\varphi) = I(\beta^{-1}\varphi)$ and $I(\varphi) \leq \beta^2 I(\upsilon) = I(\beta\upsilon)$. Thusly, we have $I(\upsilon) \leq \beta^{-2} I(\varphi) \leq I(\upsilon)$ implying $I(\varphi) = \beta^2 I(\upsilon)$. By (1.42) and the properties of the decreasing monotone rearrangement we have that $I(\varphi^\star) \leq I(\varphi) = \beta^2 I(\upsilon) \leq I(\upsilon) \leq I(\varphi^\star)$, so it follows that $I(\varphi^\star) = I(\varphi)$ and the equality $\beta^2 I(\upsilon) = I(\upsilon)$ implies that $q = \tau$.

For the second part, if $\varphi \in G_q$ thence by the first part of proposition, we obtain that $J^\star(|\varphi|^\star) = J(|\varphi|^\star) = q$, since $\varphi^\star \equiv |\varphi|^\star \geq 0$. So, $I_q^\star \leq I(|\varphi|^\star) = I(\varphi) = I_q$. Now, by Lemma 1.6.17, we obtain that

$$I_q = I_q^\star. \tag{1.43}$$

It remains to prove that $G_q \subseteq G_q^\star$. Let $\varphi \in G_q$. Then by the first part and the properties of the decreasing monotone rearrangement we have that $q = J(\varphi) = J(\varphi^\star) = J^\star(\varphi^\star) = J^\star(\varphi)$. Now, by (1.43), we get $I(\varphi) = I_q = I_q^\star$. Thusly, $\varphi \in G_q^\star$. ■

Now, by the Lemma 1.6.17, we have that $G_q \neq \emptyset$. So if $\psi \in G_q$ then by the Lagrange multiplier theorem, there exists $\theta \in \mathbb{R}$ such that $\delta I(\psi) + \theta \delta J(\psi) = 0$. By a scaling change, we obtain that ψ satisfies in equation (1.6) in the pointwise sense, similar to Section 1.4. Thus, by choosing $\varphi = |\psi|^\star$ with $\psi \in G_q^\star$, we have that φ is a solitary wave solution of the ZK equation, which is a nonnegative smooth function and is even decreasing in y-direction. We see that φ is strictly positive. In fact, we have

$$\varphi(x,y) = \frac{1}{p+1} K_c * \varphi^{p+1}(x,y), \tag{1.44}$$

where $K_c(x,y)$ is the inverse Fourier transform of $\widehat{K_c}(\xi, n) = \frac{1}{c+\xi^2+n^2}$, for all $\xi \in \mathbb{R}$ and $n \in (\pi/L)\mathbb{Z}$; and $*$ is the convolution operator in S_L defined by

$$f * g(x,y) = \int_{\mathbb{R}} \int_{-L}^{L} f(x-\sigma, y-\rho) g(\sigma, \rho) \, d\sigma d\rho.$$

By an integral approximation, one can show that K_c is positive, so if $\varphi(x_0, y_0) = 0$, then $\varphi \equiv 0$, which is contradiction. In fact

THEOREM 1.6.19 *For every $c > 0$ and $p \in \mathbb{N}$, the ZK equation (1.6) has a solitary wave solution $\varphi \in H_L^\infty$ which is cylindrically symmetric, strictly positive and decreasing in y-direction (bell-shaped pulse).*

REMARK 1.6.20 *Note that the Fourier transform of Kernel of generalized BBM is in the form of*

$$\widehat{\mathcal{K}_c}(\xi, n) = \frac{1}{c' + c(\xi^2 + n^2)}.$$

1.7 A General Nonlinearity

In this section we want to study the existence of the solitary wave solution, $u(x,y)=\varphi(x-ct,y)$, of the equation (1.1) with a general nonlinearity such that $(x,y)\in\mathbb{R}\times\mathbb{T}$, with periodic L. Indeed, we are looking for a solution of

$$-c\varphi+\Delta\varphi+f(\varphi)=0 \tag{1.45}$$

with the conditions (1.2)-(1.5) where $\Delta=\partial_x^2+\partial_y^2$. We assume that f is a differentiable real-valued function on $\mathbb{R}$ such that $f(0)=0$; and denote $F(x)=\int_0^x f(s)\,ds$ as the primitive function of f. We define the energy functional

$$\mathbb{E}(\varphi)=\int_{S_L}\frac{1}{2}(c\varphi^2+|\nabla\varphi|^2)-F(\varphi)\,dxdy.$$

LEMMA 1.7.1 *Suppose that f satisfies the following assumption:*

$f(x)\le p(x)$, where $p(x)$ is polynomial in the form of

$$p(x)=a_1x+a_2x^2+\cdots+a_mx^m$$

for some $m\in\mathbb{N}$, where $a_i\in\mathbb{R}$, for each $1\le i\le m$, and $\min\{1,c\}>|a_1|$(remember that c is velocity of solitary wave solution.

In consequence, there exists $\rho>0$ and $\delta>0$, independent on L, such that $\mathbb{E}(\varphi)\ge\delta>0$, if $\varphi\in H^1(S_L)$ and $\|\varphi\|_1=\rho$.

Proof. From the assumptions of the lemma and the Sobolev embedding, we have

$$\begin{aligned}\mathbb{E}(\varphi)&\ge\frac{1}{2}\min\{1,c\}\|\varphi\|_1^2-\int_{S_L}F(\varphi)\,dxdy\\&\ge\frac{1}{2}\min\{1,c\}\|\varphi\|_1^2-\frac{a_1}{2}\int_{S_L}\varphi^2\,dxdy\;-\cdots-\;\frac{a_m}{m+1}\int_{S_L}\varphi^{m+1}\,dxdy\\&\ge\frac{1}{2}\min\{1,c\}\|\varphi\|_1^2-\frac{|a_1|}{2}\|\varphi\|^2_{L^2(S_L)}\;-\cdots-\;\frac{|a_m|}{m+1}\|\varphi\|^{m+1}_{L^{m+1}(S_L)}\,dxdy\\&\ge\frac{1}{2}(\min\{1,c\}-|a_1|)\|\varphi\|_1^2-\frac{|a_2|}{3}\|\varphi\|_1^3-\cdots-\frac{|a_m|}{m+1}\|\varphi\|_1^{m+1}.\end{aligned}$$

Now, we can choose $\rho>0$ small enough such that $\mathbb{E}(\varphi)\ge\delta>0$, if $\|\varphi\|_1=\rho$. ■

LEMMA 1.7.2 *Suppose that F satisfies the following assumption:*

There exists $\upsilon\in C^\infty_{per}(\mathbb{R}^2)$ such that $\lambda^{-2}\int_{\mathbb{R}^2}F(\lambda\upsilon)\,dxdy$ is sufficiently large as $\lambda>0$ tends to infinity.

Then there exists $e\in H^1(S_L)$ such that $\mathbb{E}(e)<0$ and $\|e\|_1>\rho$, where $\rho>0$ is the same constant in the Lemma 1.7.1.

Proof. Without loss of generality, we may assume that υ has compact support in S_1. Therefore

$$\mathbb{E}(\lambda\upsilon) = \int_{S_1} \frac{\lambda^2}{2}(c\upsilon^2 + |\nabla\upsilon|) - F(\lambda\upsilon)\, dxdy$$

is negative and small enough, as λ tends to infinity. So there exists $\lambda_0 > 0$ such that $\mathbb{E}(\lambda\upsilon) < 0$ and $\|\lambda_0\upsilon\|_1 > \rho$. We set $e_1 = \lambda_0\upsilon$. For $L > 1$, we define

$$\upsilon_L := \begin{cases} \upsilon & \text{if} \quad (x,y) \in S_1 \\ 0 & \text{if} \quad (x,y) \in S_L \setminus S_1. \end{cases}$$

By extending υ_L periodically, and setting $e_L = \lambda_0\upsilon_L$, we have $e_L \in H^1(S_L)$, $\|e_L\|_{H^1(S_L)} = \|e_1\|_{H^1(S_1)} > \rho$ and $\mathbb{E}(e_L) = \mathbb{E}(e_1) < 0$. ∎

THEOREM 1.7.3 *Suppose f satisfies the assumptions of Lemma 1.7.1 and Lemma 1.7.2. Also assume that one of the following conditions holds:*

- *There exists $\mu > 2$ such that $\mu F(x) \leq xf(x)$.*
- *There exists $\mu < 1$ such that $\mu F(x) \geq xf(x)$.*

Then there exists a nontrivial solution of (1.45) in $H^1(S_L)$.

Proof. We define

$$\mathrm{d} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathbb{E}(\gamma(t))$$

where $\Gamma = \{\gamma \in \mathrm{C}([0,1], H_L)\ ;\ \gamma(0) = 0\ , \gamma(1) = e\}$, with e obtained in Lemma 1.7.2. Note that, according to our choice of e, the set $\{t\ e\ ;\ t \in [0,1]\}$ belongs to Γ and

$$\max_{0\leq t\leq 1} \mathbb{E}(te) \geq \mathrm{d} \geq \delta > 0,$$

which shows that d is uniformly bounded (from below and above), independent of L. Now, by using Lemmata 1.7.1, 1.7.2 and Theorem 0.0.24 , we obtain that there exists a sequence $\varphi_n \in H_L$ such that $\mathbb{E}(\varphi_n) \longrightarrow \mathrm{d}$ and $\|\mathbb{E}'(\varphi_n)\|_{H_L^\star} \longrightarrow 0$ as $n \to \infty$, where $H_L^\star = H^{-1}(S_L)$. Note that the functional $\mathbb{E}$ does not satisfy the *Palais-Smale* condition. For instance, if $\varphi_0 \neq 0$ is a critical point of $\mathbb{E}$, then $\varphi(\cdot + j, \cdot)$ is also a critical point of $\mathbb{E}$, for each $j \in \mathbb{Z}$; but the sequence $\{\varphi(\cdot + j, \cdot)\}_j$ does not have any convergent subsequence in H_L.

We prove that the sequence $\{\varphi_n\}$ is bounded in H_L. Indeed, by the hypotheses we have

$$\min\{1, c\}(1-\mu)\|\varphi_n\|_1^2 \leq \langle \mathbb{E}'(\varphi_n), \varphi_n\rangle - \mu\mathbb{E}(\varphi_n) \leq \|\mathbb{E}'(\varphi_n)\|_{H_L^\star}\|\varphi_n\|_1 - \mu \mathrm{d},$$

or

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\min\{1, c\}\|\varphi_n\|_1^2 \leq \mathbb{E}(\varphi_n) - \frac{1}{\mu}\langle \mathbb{E}'(\varphi_n), \varphi_n\rangle \leq \mathrm{d} + \frac{1}{\mu}\|\mathbb{E}'(\varphi_n)\|_{H_L^\star}\|\varphi_n\|_1.$$

Hence the sequence $\{\varphi_n\}$ is bounded in H_L.

Now, we use from Lemma 1.5.1 to show that this sequence has a nontrivial limit. Assume, on contrary, there exists $r > 0$ such that $\sup_{\xi \in S_L} \int_{S_r(\xi)} |u_n|^2 \, dxdy \longrightarrow 0$, then $\|u_n\|_{L^p(S_L)}$ tends to zero for all $2 \leq p < \infty$. So we can choose a sequence ε_n such that $\varepsilon_n \to 0$, and

$$\begin{aligned} \mathrm{d} = \mathbb{E}(\varphi_n) - \frac{1}{2}\langle \mathbb{E}'(\varphi_n), \varphi_n \rangle + \varepsilon_n &\leq \int_{S_L} \frac{1}{2}|f(\varphi_n)\varphi_n| + |F(\varphi_n)| \, dxdy \; + \; \varepsilon_n \\ &\leq b_1 \|\varphi_n\|^2_{L^2(S_L)} + \cdots + b_m \|\varphi_n\|^{m+1}_{L^{m+1}(S_L)} + \varepsilon_n \end{aligned}$$

where $b_1, \cdots, b_m$ are positive constants. Since $\mathrm{d} > 0$ and the right hand side can be made arbitrary small as $n \to \infty$, this arises a contradiction. Consequently, there exists a sequence $\{(x_n, y_n)\}$ in $\mathbb{R}^2$ and $r > 0$ such that, along a subsequence,

$$\int\limits_{S_r(0)} |\widetilde{\varphi}_n|^2 \, dxdy \geq \mathrm{d} > 0,$$

for all n, where $\widetilde{\varphi}_n(x, y) = \varphi_n(x + x_n, y + y_n)$; and there exists φ in H_L such that $\widetilde{\varphi}_n$ converges weakly in H_L and strongly in $L^p_{loc}(S_L)$ to φ. It is obvious that $\varphi \neq 0$, also for every $v \in C^\infty_{per}(\mathbb{R}^2)$

$$\langle \mathbb{E}'(\varphi), v \rangle = \lim_{n \to \infty} \langle \mathbb{E}'(\widetilde{\varphi}_n), v \rangle = 0,$$

which implies that φ is a nontrivial solution of (1.45). ■

REMARK 1.7.4 *Similarly, one can show that the argument discussed in Section 1.5 holds in the general nonlinearity case.*

1.8 Asymptotic Properties

In this section we are going to study some asymptotic properties and the behavior of solitary wave solutions. These investigations may be important in instability theory. To study the demeanors of the solutions of (1.6), it is natural to peruse the behavior of K_c, the Kernel of the equation ZK, where

$$\widehat{K_c}(\xi, n) := \mathcal{F}_y \mathcal{F}_x(K_c)(\xi, n) = \frac{1}{c + \xi^2 + n^2},$$

where $\xi \in \mathbb{R}$ and $n \in (\pi/L)\mathbb{Z}$. First we will try to represent K_c in the forms which may be more convenient to deal with them. To do this, we will need the following propositions.

PROPOSITION 1.8.1 (Poisson Summation Formula) *Let f be a function on $\mathbb{R}^N$ such that for some $\delta > 0$ and $A > 0$*

$$|f(x)| \leq \frac{A}{(1 + |x|)^{N+\delta}} \quad \text{and} \quad |\hat{f}(\xi)| \leq \frac{A}{(1 + |\xi|)^{N+\delta}},$$

then

$$\sum_{m \in \mathbb{Z}^N} f(x + m) = \sum_{m \in \mathbb{Z}^N} \hat{f}(m) \, e^{2\pi i x \cdot m}.$$

The proof of the following proposition is elementary.

PROPOSITION 1.8.2 *Let $c > 0$ and*

$$\hat{B}(\xi) = \frac{1}{c + |\xi|^2}$$

where $\xi \in \mathbb{R}^N$. Then B is an even real-valued function in $L^1(\mathbb{R}^N)$. Moreover, B decays to zero when $|x|$ tends to infinity.

Now, by the definition, it is easy to check that $K_c(x, y)$ is a real-valued function for every $(x, y) \in S_L$. Also by definition,

$$K_c(x,y) = \int_{\mathbb{R}} \sum_{n \in \frac{\pi}{L}\mathbb{Z}} \frac{e^{ix\xi} e^{iny}}{c + \xi^2 + n^2} = \int_{\mathbb{R}} \sum_{n \in \frac{\pi}{L}\mathbb{Z}} \frac{\cos(x\xi + ny)}{c + \xi^2 + n^2} = \int_{\mathbb{R}} \sum_{n \in \frac{\pi}{L}\mathbb{Z}} \frac{\cos(x\xi)\cos(ny)}{c + \xi^2 + n^2}.$$

By using Proposition 1.8.2, we have that $\int_{\mathbb{R}} \frac{e^{ix\xi}}{c + \xi^2}\, d\xi = \frac{\pi}{\sqrt{c}} e^{-\sqrt{c}|x|}$. Therefore K_c can be written by

$$K_c(x,y) = \sqrt{\pi} \int_0^\infty \sum_{n \in \frac{\pi}{L}\mathbb{Z}} \frac{\cos(ny)}{\sqrt{t(c+n^2)}}\, e^{-t-(c+n^2)\frac{x^2}{4t}}\, dt = \sum_{n \in \frac{\pi}{L}\mathbb{Z}} \frac{\pi \cos(ny)}{\sqrt{c+n^2}}\, e^{-\sqrt{c+n^2}|x|}. \tag{1.46}$$

So we see that $K_c(x, y)$ will increase when (x, y) moves to $(0, 0)$. On the other hand,

$$K_c(x,y) = \int_0^{+\infty} e^{-ct} \int_{\mathbb{R}} \sum_{n \in \frac{\pi}{L}\mathbb{Z}} e^{iny + i\xi x} e^{-t(\xi^2 + n^2)}.$$

Therefore, by Propositions 1.8.1 and 1.8.2, we obtain that

$$K_c(x,y) = 2L \int_0^\infty \sum_{n \in \mathbb{Z}} \frac{e^{-ct}}{t}\, e^{-\frac{1}{4t}\left(x^2 + |y+2nL|^2\right)}\, dt, \tag{1.47}$$

for every $(x, y) \in S_L$ and $(x, y) \neq (0, 0)$. Also we can write K_c in the following form

$$K_c(x,y) = \int_{\mathbb{R}} e^{ix\xi} \sum_{n \in \frac{\pi}{L}\mathbb{Z}} \hat{h}(n)\, e^{iny}\, d\xi, \tag{1.48}$$

where $\hat{h}(n) = \dfrac{1}{c + \xi^2 + n^2}$. So, for $y \neq 0$, by using the Poisson Summation Formula, we have

$$K_c(x,y) = 2L \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \cos(x\xi) \frac{e^{-\sqrt{c+\xi^2}|y+2nL|}}{\sqrt{c+\xi^2}}\, d\xi = 4L \sum_{n \in \mathbb{Z}} \mathscr{K}_0\left(\sqrt{c}(x^2 + |y+2nL|^2)^{1/2}\right), \tag{1.49}$$

where $\mathscr{K}_0$ is known as the *modified Bessel function of third order*, or a *Macdonald function*; and can be represented by

$$\mathscr{K}_0(r) = \frac{(\pi/2)^{1/2}}{\Gamma(\frac{1}{2})}\, r^{-1/2} e^{-r} \int_0^\infty e^{-t}\, t^{-1/2} \left(1 + \frac{t}{2r}\right)^{-1/2} dt.$$

Note that $\mathscr{K}_0(r) \sim \log(1/r)$, as $r \to 0$ and $\mathscr{K}_0(r) \sim (\frac{\pi}{r})^{1/2}\, e^{-r}$ as $r \to \infty$. (See Figure 1.1 - 1.3).

Figure 1.1: Kernel of equation (1.6) in $\mathbb{R} \times \mathbb{T}$.

LEMMA 1.8.3 *$K_c \in L^p$ for $p \in [1, +\infty)$, $K_c \in H^s$ for $s < 1$ and $\widehat{K_c} \in H^s$ for $s \geq 0$.*

Proof. It is easy to see that $\widehat{K_c} \in L^p$ for any $p \in (1, +\infty]$, $K_c \in L^p$ for any $p \in [1, +\infty)$ and $K_c \in H^s$ for $s < 1$. However note that $K_c \in L^\infty(S_L \setminus \{\mathbf{0}\})$. It is also easy to see that $|\nabla \widehat{K_c}| \in L^p$, for any $p \in (3/4, +\infty]$. So $\widehat{K_c} \in \dot{H}^1_p$. Therefore $\widehat{K_c} \in H^s$ for any $s \leq 2$, since $\dot{H}^1_p \subset \dot{H}^s_2$, for any $s = 2\left(1 - \frac{1}{p}\right)$. On the other hand, it is easy to see that $\widehat{K_c} \in H^s$ for $s \geq 1$. This completes the proof. ■

LEMMA 1.8.4 *$r^\alpha |\nabla \varphi| \in L^2$, for any $\alpha > 0$, where $r = (1 + x^2 + y^2)^{1/2}$. Moreover, $\varphi \in H^1(r^\alpha\, dxdy)$.*

Proof. Multiply (1.6) by $\chi_j(x)|x|^\alpha \varphi$ and $\chi_j(y)|y|^\alpha \varphi$, respectively, where $\chi_j(t) = \chi_0\left(\dfrac{t^2}{j^2}\right)$ and $\chi_0 \in C_0^\infty(\mathbb{R})$, $0 \leq \chi_0 \leq 1$, $\chi_0(t) = 1$ if $-L/4 \leq t \leq L/4$ and $\chi_0(t) = 0$ if $|t| \geq L/2$. The proof follows by using several integrations by parts, the properties of χ_j, Theorem 1.4.1 and Lebesgue's theorem. ■

LEMMA 1.8.5 *$rh \in L^\infty$, where $\widehat{h} = \dfrac{\widehat{K_c}}{r}$. Furthermore, $r^\alpha \varphi \in L^\infty$, for any $\alpha \geq 0$.*

Proof. It is easy to see that $\widehat{h} \in L^p$, for any $p \in (1/2, +\infty]$ and $h \in L^p$, for any $p \in [1, +\infty]$. Since

$$|r(f * g)| \leq C|(rf) * g| + C|f * (rg)|,$$

then $|rh| \leq C \left\|rK_c\right\|_{L^\infty} \left\|K_c\right\|_{L^1} < \infty$. The second part comes from $\varphi = h * \left((1 - \Delta)^{1/2} \varphi^{p+1}\right)$, for $\alpha = 1$. For general $\alpha > 0$, the proof is similar. ■

THEOREM 1.8.6 *Let φ_c be a solution of (1.6) which we obtained by minimization. Then there exists $\sigma_0, \sigma'_L > 0$ such that*

$$|\varphi_c(x, y)| \leq C_1 e^{-\sigma|x|} \quad \text{and} \quad |\varphi_c(x, y)| \leq C_2 e^{-\sigma'_L|y|},$$

for any $\sigma < \sigma_0$ and $\sigma_L < \sigma'_L$. Moreover φ_c, $e^{\sigma'_L|y|}\varphi_c$ and $e^{\sigma|x|}\varphi_c$ are in $L^1(S_L)$.

Proof. The proof is strongly related to the kernel K_c. By (1.47), we have $e^{\sigma|x|} K_c(x, y) \in L^2(S_L)$, for any $\sigma < \sigma_0 = \sqrt{c}$. Then by (1.44) we see φ_c decays exponentially in the x-direction. So $K_c(x, y) = O\left(e^{-\sigma|x|}\right)$. Similarly by (1.49), we obtain that the solution decays exponentially in y-direction. By (1.46)-(1.49), an application of Fubini's theorem and Young's inequality in $L^1(S_L)$, we obtain the second part, since $\varphi_c^{p+1} \in L^1(S_L)$. ■

So that the solitary wave solution φ_c is rapidly decreasing.

Figure 1.2: Kernel of equation (1.6) in $\mathbb{R} \times \mathbb{R}$.

Figure 1.3: (left) Projection of Kernel on yz-plane, (right) Projection on xz-plane

1.9 Instability

In this section, we consider a solitary wave solution $\varphi(x,y)$ of ZK equation obtained from a minimization problem such as of type of Section 1.3 such that φ does not depend on y; or indeed the minimizers of the KdV equation with a suitable constraint value. We are going to show that this type of solutions are unstable in H_L for some special p. We will use the ideas of [16]. First we state a well-posedness result for ZK equation.

THEOREM 1.9.1 *Let $s>2$. Then for any $u_0 \in H^s(S_L)$, there exists $T = T(\|u_0\|_{H^s}) > 0$ and there exists a unique solution $u \in C([0,T];H^s(S_L))$ of ZK equation with $u(0) = u_0$ and $u(t)$ depending on u_0 continuously in the H^s-norm.*

Proof. The proof can be obtained via Kato's Theory ([42]). ■

For any $X \subseteq H_L$ and $\varepsilon > 0$, we denote the set $\mathcal{V}(X,\varepsilon) = \{g \in H_L;\ \inf_{v\in X} \|v-g\|_{H_L} < \varepsilon\}$, the $\varepsilon-$neighborhood of X in H_L. Also for $Y \subseteq L^p$, we denote $\Omega_Y = \{\tau_\alpha v\ ;\ \alpha \in \mathbb{R}^2, v \in Y\}$, where τ_α denotes the translation operator by α.

DEFINITION 1.9.2 *We say $X \subseteq H_L$ is stable by the flow of ZK iff for any ε there exists δ such that for any $u_0 \in \mathcal{V}(X,\delta)$, the solution of the ZK equation with initial data $u(0) = u_0$ is in $\mathcal{V}(X,\varepsilon)$ for all $t \geq 0$. Otherwise we say that X is unstable.*

A direct consequence of this definition is the following.

LEMMA 1.9.3 *Let $X \subseteq H_L$ and $\epsilon > 0$. Then $\mathcal{V}(\Omega_X,\epsilon) = \Omega_{\mathcal{V}(X,\epsilon)}$.*

Throughout this section, φ is a solitary wave as described at the beginning.

LEMMA 1.9.4 *Consider φ which is not necessarily $\varphi_y = 0$. Then*

(I) *There exists $\epsilon_0 > 0$ such that for any $v \in \mathcal{V}(\Omega_\varphi,\epsilon_0)$, there exists a unique $\mathcal{N}(v) \in \Omega_\varphi$ such that $\|v - \mathcal{N}(v)\| \leq \|v - w\|$, for all $w \in \Omega_\varphi$. Moreover $\mathcal{N} : \mathcal{V}(\Omega_\varphi,\epsilon_0) \to \Omega_\varphi$ is C^2.*

(II) *There exists a unique C^2 functional $\Lambda : \mathcal{V}(\Omega_\varphi,\epsilon_0) \to \mathbb{R}^2$ which satisfies the following for $\alpha \in \mathbb{R}^2$ and $v \in \mathcal{V}(\Omega_\varphi,\epsilon_0)$:*

   (i) *$\Lambda(\tau_\alpha v) = \Lambda(v) + \alpha$, modulus L in the second component of α,*

   (ii) *$\langle v, \tau_{\Lambda(v)}\varphi_x\rangle = 0$,*

   (iii) *$\langle \Lambda'(v), v\rangle = 0$, if $v \in \Omega_\varphi$,*

   (iv) *for any $v \in \mathcal{V}(\Omega_\varphi,\epsilon_0)$, if $D = BC - A^2 \neq 0$, then*

$$\Lambda_1'(v) = \frac{1}{D}\left(\varphi_y(\cdot + \Lambda(v))A - \varphi_x(\cdot + \Lambda(v))C\right), \tag{1.50}$$

$$\Lambda_2'(v) = \frac{1}{D}\left(\varphi_x(\cdot + \Lambda(v))A - \varphi_y(\cdot + \Lambda(v))B\right), \tag{1.51}$$

where

$$B = \int v(x,y)\varphi_{xx}((x,y)+\Lambda(v))\, dxdy, \tag{1.52}$$

$$C = \int v(x,y)\varphi_{yy}((x,y)+\Lambda(v))\, dxdy, \tag{1.53}$$

$$A = \int v(x,y)\varphi_{xy}((x,y)+\Lambda(v))\, dxdy, \tag{1.54}$$

(v) *if v is a function such that $v(x,y) = v(x,-y)$, then $\Lambda(v) = (\Lambda_1(v), 0)$; and*

$$\Lambda_1'(v) = -\frac{\tau_{\Lambda(v)}\varphi_x}{\langle v, \tau_{\Lambda(v)}\varphi_{xx}\rangle}. \tag{1.55}$$

Proof. Let $\epsilon > 0$. Define $\mathcal{G} : \mathcal{V}(\varphi,\epsilon)\times\mathbb{R}^2 \to \mathbb{R}^2$, given by $\mathcal{G}(v,\alpha) = \frac{1}{2}\int_{S_L} |\tau_\alpha\varphi(x,y) - v(x,y)|^2\, dxdy$. Then from regularity of φ, we obtain $\mathcal{G} \in C^3$ and $\nabla_\alpha\mathcal{G}(v,\alpha) : \mathcal{V}(\varphi,\epsilon)\times\mathbb{R}^2 \to \mathbb{R}^2$ by $\nabla_\alpha\mathcal{G}(v,\alpha) = -\langle\tau_\alpha\nabla\varphi, v\rangle$. So we have $\nabla_\alpha\mathcal{G}(\varphi,0) = 0$. Notice that Jacobian matrix of $\nabla_\alpha\mathcal{G}$ at $(\varphi,0)$ is invertible and the determinant is positive, because $\langle\varphi_x,\varphi_y\rangle = 0$, since φ is cylindrically symmetric. Hence from the Implicit Function Theorem, we get that there exists $\epsilon_0 > 0$ and a unique C^2 function $\Lambda : \mathcal{V}(\varphi,\epsilon_0) \to \mathbb{R}^2$, such that for every $v \in \mathcal{V}(\varphi,\epsilon_0)$, $\nabla_\alpha\mathcal{G}(v,\Lambda(v)) = 0$. We define $\mathcal{N}(v) = \tau_{\Lambda(v)}\varphi$, for $v \in \mathcal{V}(\varphi,\epsilon_0)$. For every $v \in \mathcal{V}(\varphi,\epsilon_0)$, $\mathcal{N}(v)$ is the unique element of $\mathcal{V}(\varphi,\epsilon_0)$ satisfying $\|v - \mathcal{N}(v)\| \le \|v - w\|$, for all $w \in \mathcal{V}(\varphi,\epsilon_0)\cap\Omega_\alpha$. But we know that $\mathcal{G}(v,\alpha) = \mathcal{G}(\tau_\beta v, \alpha+\beta)$, for all $\beta \in \mathbb{R}^2$. By Lemma 1.9.3, Λ can be extensible to $\mathcal{V}(\Omega_\varphi,\epsilon_0)$ in such way that for all $v \in \mathcal{V}(\varphi,\epsilon_0)$, $\Lambda(\tau_\alpha v) = \Lambda(v) + \alpha$, modulus the second component of α. The derivatives (1.50) and (1.51) are obtained by differentiating the relation

$$\langle\tau_{\Lambda(v)}\nabla\varphi, v\rangle = 0,$$

with respect to v. Now if $v(x,y) = v(x,-y)$, analogously, by using the Implicit Function Theorem, we can find $\Lambda_1(v)$ for $v \in \mathcal{V}(\varphi,\epsilon_0)$ such that $\langle\tau_{(\Lambda_1(v),0)}\varphi_x, v\rangle = 0$. Note that $\langle\tau_{(\Lambda_1(v),0)}\varphi_y, v\rangle = 0$, because of having cylindrically symmetry of φ and v. Hence the uniqueness provided by the Implicit Function Theorem gives $\Lambda(v) = (\Lambda_1(v), 0)$, taking ϵ_0 smaller if necessary. Also, (1.55) follows by differentiating the relation

$$\langle\tau_{(\Lambda_1(v),0)}\varphi_x, v\rangle = 0,$$

with respect to v.

■

Now, suppose that ψ is a function such that ψ_x and ψ_{xx} is $\in H_L$. Then we define

$$\mathcal{B}_\psi(u) \equiv \tau_{\Lambda(u)}\psi_x - \frac{\langle u, \tau_{\Lambda(u)}\psi_x\rangle}{\langle u, \tau_{\Lambda(u)}\varphi_{xx}\rangle}\tau_{\Lambda(u)}\varphi_{xx}. \tag{1.56}$$

LEMMA 1.9.5 *$\mathcal{B}_\psi$ is a C^1 function with bounded derivatives from $\mathcal{V}(\Omega_\varphi,\epsilon_0)$ into H_L. Moreover*

(i) *$\mathcal{B}_\psi$ commutes with translation in the x-variable,*

(ii) $\langle \mathcal{B}_\psi(u), u\rangle = 0$ *for all* $u \in \mathcal{V}(\Omega_\varphi, \epsilon_0)$,

(iii) $\mathcal{B}_\psi(\varphi) = \psi_x$ *if* $\langle \varphi, \psi_x\rangle = 0$,

(iv) *if* $\langle \varphi_x, \psi_x\rangle = 0$ *then* $\langle \mathcal{B}_\psi(u), u_x\rangle = 0$ *for all* $u \in \Omega_\varphi$.

Proof. The proof follows from the previous lemma and differentiation. ■

REMARK 1.9.6 *Note that in the preceding lemma we used the geometry of* $\mathbb{R}\times\mathbb{T}$*, which does not hold in* $\mathbb{R}^2$*.*

Now we will consider the following situation

$$\langle \mathcal{S}''(\varphi)\mathcal{B}_\psi(\varphi), \mathcal{B}_\psi(\varphi)\rangle < 0, \tag{1.57}$$

where $\mathcal{S} = \mathscr{E}_1 + c\mathscr{E}_2$ and $\psi(x) = \psi(x,y) = \int_{-\infty}^x \varphi(z,y) + 2z\varphi_z(z,y)\,dz$. Note that φ satisfies $-c\varphi + \varphi_{xx} + \frac{1}{p+1}\varphi^{p+1} = 0$. Now for $v_0 \in \mathcal{V}(\Omega_\varphi, \epsilon_0)$, we consider the initial value problem

$$\frac{d}{ds}v(s) = \mathcal{B}_\psi(v(s)), \quad v(0) = v_0. \tag{1.58}$$

From Lemma 1.9.5, we have that for each $v_0 \in \mathcal{V}(\Omega_\varphi, \epsilon_0)$, this system admits a unique maximal solution $v \in C^2((-\sigma,\sigma); \mathcal{V}(\Omega_\varphi, \epsilon_0))$ where $v(0) = v_0$ and $\sigma = \sigma(v_0) \in (0, +\infty]$. Moreover, for each $\epsilon_1 < \epsilon_0$ there exists $\sigma_1 > 0$ such that $\sigma(v_0) \geq \sigma_1$ for each $v_0 \in \mathcal{V}(\Omega_\varphi, \epsilon_1)$. Now for fixed ϵ_1, σ_1, we consider the flow of (1.58)

$$\begin{aligned} \mathscr{U} : (-\sigma_1, \sigma_1) \times \mathcal{V}(\Omega_\varphi, \epsilon_1) &\to \mathcal{V}(\Omega_\varphi, \epsilon_0) \\ (s, v_0) &\to \mathscr{U}(s)v_0, \end{aligned}$$

where $s \to \mathscr{U}(s)v_0$ is the maximal solution of (1.58) with initial data v_0. From Lemma 1.9.5, we have that $\mathscr{U}$ is C^1 and for each $v_0 \in \mathcal{V}(\Omega_\varphi, \epsilon_1)$, $s \in (-\sigma_1, \sigma_1) \to \mathscr{U}(s)v_0$ is C^2. Also the flow commutes with the translations with respect to the x-variable (and then commutes with τ_α for each $\alpha \in \mathbb{R}^2$). Also from the relation

$$\mathscr{U}(s)\varphi = \varphi + \int_0^s \tau_{\Lambda(\mathscr{U}(t)\varphi)}\psi_x\,dt - \int_0^s \mathcal{F}(t)\tau_{\Lambda(\mathscr{U}(t)\varphi)}\varphi_{xx}\,dt,$$

and the properties of φ, we have that $\mathscr{U}(s) \in W^{2,1}(S_L)$, *for* $s \in (-\sigma, \sigma)$, where $s \in (-\sigma,\sigma) \to \mathcal{F}(s)$ is a continuous function. Now for every $v_0 \in \mathcal{V}(\Omega_\varphi, \epsilon_1)$, we get from Taylor's theorem that there exists $\theta \in (0,1)$ such that

$$\mathcal{S}(\mathscr{U}(s)v_0) = \mathcal{S}(v_0) + \mathcal{P}(v_0)s + \frac{1}{2}\mathcal{R}(\mathscr{U}(\theta s)v_0)s^2, \tag{1.59}$$

where

$$\mathcal{P}(v) = \langle \mathcal{S}'(v), \mathcal{B}_\psi(v)\rangle, \tag{1.60}$$

$$\mathcal{R}(v) = \langle \mathcal{S}''(v)\mathcal{B}_\psi(v), \mathcal{B}_\psi(v)\rangle + \langle \mathcal{S}'(v), \mathcal{B}'_\psi(v)\mathcal{B}_\psi(v)\rangle, \tag{1.61}$$

are functionals defined on $\mathcal{V}(\Omega_\varphi, \epsilon_1)$. Since $\mathcal{R}$ and $\mathcal{U}$ are continuous, $\mathcal{S}'(\varphi) = 0$. We are going to show that the condition (1.57) implies the instability. In fact, by (1.57) we have $\mathcal{R}(\varphi) < 0$. Therefore there exists $\epsilon_2 \in (0, \epsilon_1]$ and $\sigma_2 \in (0, \sigma_1]$ such that

$$\mathcal{S}(\mathcal{U}(s)v_0) \leq \mathcal{S}(v_0) + \mathcal{P}(v_0)s, \tag{1.62}$$

for $v_0 \in \mathcal{V}(\varphi, \epsilon_2)$ and $s \in (-\sigma_2, \sigma_2)$. We can extend the inequalities (1.62) to $v_0 \in V(\Omega_\varphi, \epsilon_2)$ by Lemma 1.9.3 and the commutation between $\mathcal{U}(s)v_0$ and the translations. Now we put $v_0 = \mathcal{U}(\rho)\varphi$ with $\rho \neq 0$ small enough. Then we obtain

$$\mathcal{S}(\mathcal{U}(s)\mathcal{U}(\rho)\varphi) \leq \mathcal{S}(\mathcal{U}(\rho)\varphi) + \mathcal{P}(\mathcal{U}(\rho)\varphi)s. \tag{1.63}$$

Hence for $s = -\rho < 0$, we have

$$\mathcal{S}(\varphi) \leq \mathcal{S}(\mathcal{U}(\rho)\varphi) - \mathcal{P}(\mathcal{U}(\rho)\varphi)\rho. \tag{1.64}$$

Also from (1.57) we have that the function $\rho \to \mathcal{S}(\mathcal{U}(\rho)\varphi)$ has a strict local maximum at zero, so

$$\mathcal{S}(\mathcal{U}(\rho)\varphi) < \mathcal{S}(\varphi), \tag{1.65}$$

for $\rho \in (-\sigma, \sigma_2)$ and $\rho \neq 0$. From (1.64) and (1.65), we have that for some $\sigma_3 \leq \sigma_2$,

$$\mathcal{P}(\mathcal{U}(\rho)\varphi) < 0, \tag{1.66}$$

for $\rho \in (0, \sigma_3)$. But since φ a minimizer of $\mathcal{S}$ under the constraint $J(u) = q$, then we have from (1.57) that

$$\langle J'(\varphi), \mathcal{B}_\psi(\varphi) \rangle \neq 0, \tag{1.67}$$

where $J(\varphi) = \int_{S_L} \varphi^{p+2}$. Now we consider the function $(v_0, s) \in V(\Omega_\varphi, \epsilon_1) \times (-\sigma_1, \sigma_1) \to J(\mathcal{U}(s)v_0)$. This function is C^1 and $(\varphi, 0) \to q$. From (1.67), we have

$$\frac{d}{ds} J(\mathcal{U}(s)v_0)|_{(\varphi,0)} = \langle J'(\varphi), \mathcal{B}_\psi(v)(\varphi) \rangle \neq 0. \tag{1.68}$$

Thusly by the Implicit Function Theorem, there exists $\epsilon_3 \in (0, \epsilon_2)$ and $\sigma_3 \in (0, \sigma_2)$ such that for each $v_0 \in V(\varphi, \epsilon_3)$, there exists a unique $s = s(v_0) \in (-\sigma_3, \sigma_3)$ such that $J(\mathcal{U}(s)v_0) = q$. Now by using (1.62) for $(v_0, s(v_0)) \in V(\varphi, \epsilon_3) \times (-\sigma_3, \sigma_3)$ and since φ is a minimizer of $\mathcal{S}$ under the constraint $J(u) = q$, we have that for $v_0 \in V(\varphi, \epsilon_3)$ there exists $s \in (-\sigma_3, \sigma_3)$ such that

$$\mathcal{S}(\varphi) \leq \mathcal{S}(v_0) + \mathcal{P}(v_0)s. \tag{1.69}$$

Therefore from Lemma 1.9.3 and the commutation between $\mathcal{U}(s)v_0$ and translations we can extend (1.69) to $V(\Omega_\varphi, \epsilon_3)$. Note that since $\mathcal{B}_{-\psi}(\varphi) = -\mathcal{B}_\psi(\varphi)$, we assume that $\langle J'(\varphi), \mathcal{B}_\psi(\varphi) \rangle < 0$, by using (1.67). So for $\tau > 0$ small enough we can get some δ small such that

$$J(\mathcal{U}(\tau)\varphi) = J(\varphi) + \int_0^\tau \langle J'(\mathcal{U}(\varpi)\varphi), \mathcal{B}_\psi(\mathcal{U}(\varpi)\varphi) \rangle \, d\varpi = q - \delta < q. \tag{1.70}$$

Note that if u_0 is a function satisfying (1.65), (1.66) and (1.70) (by substituting u_0 instead of $\mathcal{U}(\rho)\varphi$), then since φ is a minimizer of $\mathcal{S}$ under the constraint $J(u) = q$, the ZK solution $u(t)$ corresponding to initial data u_0 satisfies (1.65), (1.66) and (1.70).

In the rest, we need the following lemma.

LEMMA 1.9.7 *Assume that $u_0 \in \mathcal{V}(\Omega_\varphi, \epsilon_0)$ and satisfies in Lemma 1.9.9. If $u(t)$ is a solution of (1.1) corresponding to u_0 as initial data and $u(t) \in \mathcal{V}(\Omega_\varphi, \epsilon_0)$ for $t \in [0,T]$, then for $\psi(x) = \psi(x,y) = \int_{-\infty}^{x} \varphi(z,y) + x\varphi_x(z,y)\, dz$, we have*

$$\mathcal{A}_\psi(u(t)) \equiv \int_{S_L} \psi(x - \Lambda_1(u(t)), y) u(t)\, dxdy < +\infty,$$

for all $t \in [0,T]$.

The functional $\mathcal{A}_\psi$ is called the *Lyapunov* functional along ψ.
Proof. Put $\omega(t) = \Lambda_1(u(t))$. Therefore

$$\mathcal{A}_\psi(u(t)) = \int_{S_L} \psi(x - \omega(t), y) u(t)\, dxdy =$$
$$\int_{S_L} \left[\psi(x - \omega(t), y) - \nu \mathcal{H}(x - \omega(t))\right] u(t)\, dxdy + \int_{-L}^{L} \int_{\omega(t)}^{+\infty} \nu u(t)\, dxdy,$$

where $\nu = \int_{\mathbb{R}} \varphi + 2x\varphi_x\, dx$ and $\mathcal{H}$ is the Heaviside function. Thus we obtain

$$|\mathcal{A}_\psi(u(t))| \lesssim \|\psi - \nu\mathcal{H}\|_{L^2} \|u\|_{L^2} + |\nu| \left| \int_{-L}^{L} \int_{\omega(t)}^{+\infty} u(t)\, dxdy \right|. \tag{1.71}$$

By the decaying at infinity of ψ in x, we have that $\|\psi - \nu\mathcal{H}\|_{L^2}$ is finite. Thusly

$$|\mathcal{A}_\psi(u(t))| \lesssim \|\psi - \nu\mathcal{H}\|_{L^2} \|u_0\|_{L^2} + |\nu| \left\| \int_{\omega(t)}^{+\infty} u(x,\cdot,t)\, dx \right\|_{L^\infty(-L,L)}.$$

To estimate $\left\| \int_{\omega(t)}^{+\infty} u(x,\cdot,t)\, dx \right\|_{L^\infty}$, we use the following lemmata.

LEMMA 1.9.8 *Let $s > 2$ and $u_0 \in \mathcal{F}^{s,2}_{\frac{r}{4},0}$, for some $r \le 2$. Then the solution of the ZK equation corresponding to initial data $u_0 \in C([0,T); H^s) \cap L^\infty\left([0,T); L^2\left(\left(1 + |x|^{r/2}\right) dx\right)\right)$ satisfies*

$$\|u(t)\|_{L^2((1+|x|^{r/2})dx)} \le C(1+t)^{1/2}$$

for any t such that $0 \le t \le T_1 < T$, where

$$C = C\left(\sup_{t\in[0,T_1]} \|u(t)\|_{H_L}, \|u_0\|_{L^2\left(\left(1+|x|^{r/2}\right)dx\right)} \right).$$

Proof. Denote $w(x) = w(x,y) = (1 + |x|^r)^{1/4}$, so we have $(wu)_t + wu\Delta u_x + wu^p u_x = 0$. By taking L^2-inner product in the last equation with wu, we obtain $\|wu\|^2_{L^2} \le C(1+t)$, where we used the fact that $\left(w^2\right)_x$ and $\left(w^2\right)_{xx}$ are bounded on S_L. ■

LEMMA 1.9.9 *Let $s > 3$ and $u_0 \in H^s(S_L)$. Also, suppose that $u_0 \in \mathcal{F}^{s,1}_{\frac{5}{8},0}(S_L) \cap \mathcal{F}^{s,2}_{\frac{1}{2},0}(S_L)$. Then*

$$\left\| \int_x^{+\infty} u(s, \cdot, t) \right\|_{L^\infty} \leq C \left(t^{-3/4}(1 + |x|)^{5/4} + t^{1/4}(1 + t + |x|)^{1/4} \right), \tag{1.72}$$

where $u(t)$ is the solution of the ZK equation corresponding to initial data $u(0) = u_0$ and

$$C = C \left(\sup_{0 \leq t \leq T} \|u(t)\|_{H_L}, \|u_0\|_{\mathcal{F}^{s,1}_{\frac{5}{8},0}}, \|u_0\|_{\mathcal{F}^{s,2}_{\frac{1}{2},0}} \right).$$

Proof. We denote $\mathcal{W}(y) = \int_{\mathbb{R}} u_0(x, y)\, dx$, then $\mathcal{M}(x, y, t) = \int_x^{+\infty} u(s, y, t)\, ds$ is the solution of

$$\mathcal{M}_t + \Delta \mathcal{M}_x - \frac{1}{p+1} u^{p+1} = 0$$

with initial data $\mathcal{M}(0) = \int_x^{+\infty} u_0(s, y)\, ds$. We also have

$$\begin{aligned}
\mathcal{M}(x, y, t) &= U(t) * \mathcal{M}(0)(x, y) - \frac{1}{p+1} \int_0^t U(t - \tau) * u^{p+1}(\tau)\, d\tau \\
&= U(t) * f_1(x, y) + U(t) * f_2(x, y) - \frac{1}{p+1} \int_0^t U(t - \tau) * u^{p+1}(\tau)\, d\tau,
\end{aligned}$$

where $U(t)=\int_{S_L} e^{i\left(t\xi^3+t\xi|\eta|^2+x\xi+y\cdot\eta\right)}\,d\xi d\eta$, $f_1(x,y)=\mathcal{H}(-x)\mathcal{W}(y)$, $f_2(x,y)=\mathcal{M}(x,0)-\mathcal{H}(-x)\mathcal{W}(y)$ and $\mathcal{H}(x)$ is the Heaviside function. By Lemma 2.3.4, we have

$$\begin{aligned}
\|\mathcal{M}(x,\cdot,t)\|_{L^\infty_y} &\leq \|U(t)*_y \mathcal{W}\|_{L^\infty_y} *_x \mathcal{H}(-x)+\|U(t)\|_{L^\infty_y} *_x \|\mathcal{M}(0)-\mathcal{W}\mathcal{H}(-\cdot)\|_{L^1_y}\\
&\quad+\frac{1}{p+1}\int_0^t\int_{\mathbb{R}}\left\|U(x-s,\cdot,t-\tau)*_y u^{p+1}(s,\cdot,\tau)\right\|_{L^\infty_y}\,dsd\tau\\
&\leq \int_{-\infty}^0 \|\mathcal{W}\|_{L^1_y}\|U(x-s,\cdot,t)\|_{L^\infty_y}\,ds+Ct^{-2/3}\,\|\mathcal{M}-\mathcal{W}\mathcal{H}(-\cdot)\|_{L^1}\\
&\quad+Ct^{-3/4}\int_{S_L}|x-s|^{1/4}\,|\mathcal{M}(s,y,0)-\mathcal{W}(y)\mathcal{H}(-s)|\,\,dyds\\
&\quad+\frac{1}{p+1}\int_0^t\|U(\cdot,\cdot,t-\tau)\|_{L^\infty_y} *_x \left\|u^{p+1}(\cdot,\cdot,\tau)\right\|_{L^1_y}\\
&\leq C\|u_0\|_{L^1}\int_{-\infty}^0 t^{-2/3}e^{-\frac{2}{3}(x-s)^{\frac{3}{2}}t^{-1/2}}\mathcal{H}(x-s)\,ds\\
&\quad+C\|u_0\|_{L^1}\int_{-\infty}^0\left(t^{-2/3}+t^{-3/4}|s-x|^{\frac{1}{4}}\right)\mathcal{H}(s-x)\,ds\\
&\quad+Ct^{-2/3}\,\|\mathcal{M}-\mathcal{W}\mathcal{H}(-\cdot)\|_{L^1}+Ct^{-3/4}\int_{S_L}|x|^{\frac{1}{4}}\,|\mathcal{M}(s,y,0)-\mathcal{W}(y)\mathcal{H}(-s)|\,\,dyds\\
&\quad+Ct^{-3/4}\int_{S_L}(1+|s|)^{\frac{1}{4}}\,|\mathcal{M}(s,y,0)-\mathcal{W}(y)\mathcal{H}(-s)|\,\,dyds\\
&\quad+\frac{1}{p+1}\|U(\cdot,\cdot,t-\tau)\|_{L^\infty_y} *_x \|u(\cdot,\cdot,\tau)\|^{p+1}_{L^{p+1}_y}\\
&=\mathcal{Y}_1+\mathcal{Y}_2+\mathcal{Y}_3+\mathcal{Y}_4+t^{3/4}\mathcal{Y}_5+\frac{1}{p+1}\mathcal{Y}_6.
\end{aligned}$$

On the other hand, we have

$$\mathcal{Y}_1\leq\int_{\max\{0,x\}}^{+\infty} e^{-\frac{2}{3}\eta^{3/2}t^{-1/2}}\,d\eta,\qquad \mathcal{Y}_2\leq\int_{\min\{0,x\}}^{+\infty}\left(t^{-2/3}+t^{-3/4}|\eta|^{1/4}\right)\,d\eta.$$

and

$$\begin{aligned}
\mathcal{Y}_5 &\leq C\int_{-L}^{L}\int_0^{+\infty}\int_0^{\eta}(1+|s|)^{1/4}|u_0(\eta,y)|\,dsd\eta dy\\
&\quad+C\int_{-L}^{L}\int_{-\infty}^{0}\int_{\eta}^{0}(1+|s|)^{1/4}|u_0(\eta,y)|\,dsd\eta dy\leq C\int_{S_L}\left(1+|s|^{5/4}\right)|u_0(\eta,y)|\,d\eta ds.
\end{aligned}$$

Also

$$\begin{aligned}
\mathcal{Y}_6 &\leq C\int_0^t\left((t-\tau)^{-2/3}+\int_{\mathbb{R}}|x-s|^{\frac{1}{4}}(t-\tau)^{-\frac{3}{4}}\right)\|u(\tau)\|^{p+1}_{L^{p+1}}\,d\tau\\
&\leq C\left(t^{1/3}+|x|^{1/4}t^{1/4}\right)+C\int_0^t\int_{S_L}(1+|s|^2)^{1/8}(t-\tau)^{-3/4}|u(s,y,\tau)|^{p+1}\,dsdyd\tau.
\end{aligned}$$

But

$$\int_{S_L}(1+|s|^2)^{1/8}|u(s,y,\tau)|^{p+1}\,dsdy \le \left(\int_{S_L}(1+|s|^2)^{1/2}|u(s,y)|^2\right)^{1/4}\|u\|_{L^{\gamma_1}}^{\gamma_2},$$

where $\gamma_1=\frac{2}{3}(2p+1)$ and $\gamma_2=\frac{9}{8(p+1)}$. By Lemma 1.9.8, we obtain that

$$\int_{S_L}(1+|s|^2)^{1/8}|u(s,y,\tau)|^{p+1}\,dsdy \le C(1+\tau)^{1/4},$$

and then

$$\int_0^t(t-\tau)^{-3/4}\int_{S_L}(1+|s|^2)^{1/8}|u(s,y,\tau)|^{p+1}\,dsdyd\tau \le Ct^{1/4}(1+t^{1/4}),$$

and consequently

$$\mathcal{Y}_6 \le C(t^{1/3}+t^{1/4}(1+t^{1/4}+|x|^{1/4})).$$

This completes the proof.

■

LEMMA 1.9.10 *Let $s>3$ and $u_0\in H^s(S_L)\cap\mathcal{V}(\Omega_\varphi,\epsilon_0)$. We also assume that $u(t)$ is the solution of the ZK equation corresponding to initial data u_0 and $\epsilon_0<\|\varphi_x\|_{L^2}^2\|\varphi_{xx}\|_{L^2}^{-1}$. Then $|\Lambda_1(u(t))|\le|\Lambda_1(u_0)|+C|t|$, where C does only depend on ϵ.*

Proof. Notice that from the ZK equation we have that u(x,y,t)=u(x,-y,t). So by differentiating the relation $\langle\tau_{(\Lambda_1(u),0)}\varphi_x,u\rangle=0$ in t (see Lemma 1.9.4), we obtain

$$\Lambda_1'(u(t))=\frac{\langle\Delta u_x+u^pu_x,\tau_{(\Lambda_1(u),0)}\varphi_x\rangle}{\langle u(t),\tau_{(\Lambda_1(u),0)}\varphi_{xx}\rangle}.$$

Let $u(x,y,t)=\tau_{(\Lambda_1(u),0)}\varphi(x,y)+h(x,y,t)$ with $\|h(t)\|_{H_L}\le\epsilon_0$. Denote $\mathscr{b}=(\Lambda_1(u),0)$. Since φ satisfies the ZK equation, we have that

$$\langle\Delta u_x+u^pu_x,\tau_{\mathscr{b}}\varphi_x\rangle=\langle\Delta h_x+c\tau_{\mathscr{b}}\varphi_x,\tau_{\mathscr{b}}\varphi_x\rangle+\frac{1}{p+1}\langle u^{p+1}(t),\tau_{\mathscr{b}}\varphi_x\rangle-\frac{1}{p+1}\left\langle(\tau_{\mathscr{b}}\varphi^{p+1})_x,\tau_{\mathscr{b}}\varphi_x\right\rangle=c\|\varphi_x\|_{L^2}^2+\mathcal{Z}(t),$$

where

$$\mathcal{Z}(t)=-\langle h(t),\tau_{\mathscr{b}}\Delta\varphi_{xx}\rangle+\frac{1}{p+1}\langle\tau_{\mathscr{b}}\varphi_{xx},(\tau_{\mathscr{b}}\varphi+h)^{p+1}-\tau_{\mathscr{b}}\varphi_{xx}\rangle.$$

So we obtain

$$|\mathcal{Z}(t)|\le C_1\|h\|_{L^2}+C_2\|h\|_{L^{p+1}}^{p+1}+C_0\le C_1\epsilon_0^{p+1}+C_0. \tag{1.73}$$

On the other hand, we have

$$\langle u(t),\tau_{\mathscr{b}}\varphi_{xx}\rangle=-\|\varphi_x\|_{L^2}^2+\langle h(t),\tau_{\mathscr{b}}\varphi_{xx}\rangle\le-\|\varphi_x\|_{L^2}^2+\epsilon_0\|\varphi_{xx}\|_{L^2}<0. \tag{1.74}$$

Therefore $|\Lambda_1'(u(t))|\le C_{\epsilon_0}$, and the proof is complete. ■

By Lemmata 1.9.8, 1.9.9 and 1.9.10, we conclude that for $\varrho = -3/4$ and $\zeta = 1/2$,

$$|\mathcal{A}_\psi(u(t))| \leq C(t^{-\varrho} + t^{\zeta}), \tag{1.75}$$

where C does not depend on time. ■

So, we have

$$\begin{aligned}\partial_t \mathcal{A}_\psi(u(t)) &= \int_{S_L} \big(\langle \tau_{\omega(t)}\psi_x, u(t)\rangle \omega'(t) + \tau_{\omega(t)}\psi\big)\, u(t)\; dxdy = -\left\langle \langle \tau_{\omega(t)}\psi_x, u\rangle \frac{d}{dx}\Lambda_1'(u(t)) + \tau_{\omega(t)}\psi_x, \mathscr{E}_1'(u(t))\right\rangle \\ &= -\langle \mathcal{B}_\psi(u(t)), \mathcal{S}'(u(t))\rangle + c\langle \mathcal{B}_\psi(u(t)), u(t)\rangle = -\mathcal{P}(u(t)).\end{aligned} \tag{1.76}$$

Furthermore, $\int_{S_L} c|u(t)|^2 + |\nabla u(t)|^2\; dxdy = 2\mathcal{S}(u(t)) + \frac{2}{p+1}J(u(t)) < 2\mathcal{S}(\varphi) + \frac{2q}{p+1} < +\infty$. On the other hand, by using (1.65), (1.66) and (1.70), we obtain that $\mathscr{U}(\tau)$ satisfies these inequalities for $\tau \in (0, \sigma_3)$. Now we take a sequence $\{\tau_j\} \subset (0, \sigma_3)$ such that $\tau_j \to 0$ as $j \to +\infty$; and consider $u_j(t)$ as the ZK solutions corresponding to initial datum $u_{0,j} = \mathscr{U}(\tau_j)\varphi$. Note that $u_{0,j}$ tends to φ in $H^s(S_L)$ (and then in H_L) as $j \to +\infty$. We are going to show that $u_j(t)$ do not stay in $\mathcal{V}(\Omega_\varphi, \epsilon_3)$ for all $j \in \mathbb{N}$. We define the maximum time which each u_j stays near to the orbit of φ:

$$T_j = \sup\{\tau > 0 \quad ; \quad u_j(t) \in \mathcal{V}(\Omega_\varphi, \epsilon_3), \text{ for all } t \in (0, \tau)\}. \tag{1.77}$$

Then it follows from (1.69) that for each $j \in \mathbb{N}$ and $t \in (0, T_j)$, there exists $s = s_j(t) \in (-\sigma_3, \sigma_3)$ such that

$$\mathcal{S}(\varphi) \leq \mathcal{S}(u_j(t)) + \mathcal{P}(u_j(t))s = \mathcal{S}(u_{0,j}) + \mathcal{P}(u_j(t))s. \tag{1.78}$$

As mentioned above, $\mathcal{P}(u_j(t)) < 0$, for $t \in (0, T_j)$; so we have that

$$-\mathcal{P}(u_j(t)) \geq \frac{\mathcal{S}(\varphi) - \mathcal{S}(u_{0,j})}{\sigma_3} = \ell_j > 0, \tag{1.79}$$

for all $t \in (0, T_j)$. We will show that $T_j < +\infty$ which implies the instability. Suppose that for some j, we have $T_j = +\infty$. Then from (1.76), (1.79) and the properties of the flow, we have that

$$\mathcal{A}_\psi(u_j(t)) \geq \mathcal{A}_\psi(u_{0,j}) + t\ell_j, \tag{1.80}$$

for all $t \in (0, +\infty)$. Then from (1.75), we obtain that

$$\frac{\mathcal{A}_\psi(u_{0,j}) + t\ell_j}{t^{-\varrho} + t^{\zeta}} \leq +\infty, \tag{1.81}$$

for all $t \in (0, +\infty)$, where $\varrho + \zeta < 1$. Consequently, it implies that $T_j < +\infty$.

Now we want to investigate when the condition (1.57) occurs. Note that $\langle \varphi, \psi_x\rangle = 0$, so $\mathcal{B}_\psi(\varphi) = \psi_x$. Also, by the definition, we have $\mathcal{S}''(\varphi) = -\Delta + c - \varphi^p$. Therefore, we obtain that

$$\begin{aligned}\langle \mathcal{S}''(\varphi)\psi_x, \psi_x\rangle &= \langle \mathcal{S}''(\varphi)\varphi, \varphi\rangle + 4\langle \mathcal{S}''(\varphi)\varphi, x\varphi_x\rangle + 4\langle \mathcal{S}''(\varphi)(x\varphi_x), x\varphi_x\rangle \\ &= \frac{-p}{p+1}\int_{S_L} \varphi^{p+2}\; dxdy + \frac{4p}{(p+1)(p+2)}\int_{S_L} \varphi^{p+2}\; dxdy + 4\int_{S_L} \varphi_x^2\; dxdy \\ &= \frac{2p - p^2}{(p+1)(p+2)}\int_{S_L} \varphi^{p+2}\; dxdy + 4\int_{S_L} \varphi_x^2\; dxdy.\end{aligned}$$

On the other hand, we know that

$$\int_{S_L} \varphi_x^2 \, dxdy = \frac{p}{2(p+1)(p+2)} \int_{S_L} \varphi^{p+2} \, dxdy,$$

by Pohozaev-type identities. Therefore

$$\langle \mathcal{S}''(\varphi)\psi_x, \psi_x \rangle = \frac{4p - p^2}{(p+1)(p+2)} \int_{S_L} \varphi^{p+2} \, dxdy. \tag{1.82}$$

Thusly, we have proved

THEOREM 1.9.11 *The orbit Ω_φ is unstable by the flow of ZK equation, if $p > 4$.*

Chapter 2

ZK with Dissipation

2.1 Introduction

The dispersion terms in the ZK equation appear in the *Zakharov-Kuznetsov-Burgers* (ZKB) equation, with directional dispersion:

$$u_t + \left(\Delta u - \alpha u_x + \frac{1}{2}u^2\right)_x = 0, \quad (x,y) \in \mathbb{R}^2, \quad t \in \mathbb{R}^+;$$

and the ZK equation with higher order dissipation which is known as a 2D version of the *Benney* equation :

$$u_t + uu_x + \alpha u_{xx} + \Delta u_x + \beta \Delta^2 u = 0, \quad (x,y) \in \mathbb{R}^2, \quad t \in \mathbb{R}^+$$

where $\alpha > 0$, $\beta > 0$ are real constants, u is a real-valued function. The ZKB equation describes the propagations, of nonlinear dust acoustic waves in a nonuniform magnetized dusty plasma [26, 61]. The Benney equation describes a variety of physical phenomena in two dimensions (mainly, of hydrodynamic origin), for example, long waves on a thin liquid film, the Rossby waves in rotating atmosphere and the drift waves in plasma [7, 39, 58]. 2D pulses in the Benney equation were numerically identified in the limiting case of zero dispersion [66].

In Sections 2.2 and 2.6, we will investigate the Cauchy problem associated to the generalized ZKB equation and the Benney equation in Sobolev spaces $H^s\left(\mathbb{R}^2\right)$. The Benney equation can be considered as a high dimensional generalization of the KdV-KS equation

$$u_t + \delta u_{xxx} + uu_x + \mu(u_{xx} + u_{xxxx}) = 0, \tag{2.1}$$

where $\delta, \mu \in \mathbb{R}$ are constants. In [11], using the dissipative effect of the linear part, Biagioni, Bona, Iorio and Scialom showed that the Cauchy problem associated to (2.1) is globally well-posed in $H^s(\mathbb{R})$ for $s \geq 1$. We use purely dissipative methods applied by Dix to study the initial value problem for the KdV-Burgers equation [25] (see also Giga in [34]). The main argument consists in applying a fixed point theorem to the integral equation associated to the Benney equation in time weighted spaces. Indeed, we can observe that the structure of the Benney equation possesses a dissipation *stronger* (in some sense) than the dispersion. So that we do not need to use Bourgain's-type spaces as in [59, 60]. But in the ZKB equation, the dissipation is *weaker* than the dispersion so we need to use the effects of the dispersion. On the other hand, the directional dissipation u_{xx} does not permit to use the Sobolev spaces directly by linear properties of the ZKB equation. Therefore we have to apply the techniques of Molinet and Ribaud in the Bourgain spaces which is strongly related to the results of the ZK equation. Unfortunately, the Cauchy problem of the (generalized) ZK equation in Bourgain's type spaces seems not to work.

Our strategy is to use the regularization by applying more dissipative terms to the equation [37]; in fact, we will study the following regularized ZKB problem:

$$u_t + (\Delta u + f(u) - \alpha u_x)_x - \beta \Delta_\perp u = 0;$$

where $\beta \in \mathbb{R}$ is nonnegative.

2.2 ZKB Equation

In this section, we will study the Cauchy problem of the Zakharov-Kuznetsov-Burgers (ZKB) equation:

$$u_t + (\Delta u + f(u) - \alpha u_x)_x = 0, \tag{2.2}$$

where $\alpha \in \mathbb{R}$ is nonnegative and f is a differentiable real-valued function on $\mathbb{R}$ such that $f(0) = 0$ and $f'(0) = 0$. We also assume that $f(x) = O(x^{p+1})$, for $p \in \mathbb{N}$. We denote $F(x) = \int_0^x f(s)\,ds$ as the primitive function of f. We are going to investigate the local well-posedness of initial value problem of (2.2) in Sobolev spaces H^s and some weighted spaces. Our strategy is to use the regularization by applying more dissipative terms to the equation; in fact, we will study the following regularized ZKB problem:

$$u_t + (\Delta u + f(u) - \alpha u_x)_x - \beta \Delta_\perp u = 0, \tag{2.3}$$

where $\beta \in \mathbb{R}$ is nonnegative; and then by using the properties of (2.2) we obtain the solution of (2.3). Here, we consider $u(x, y)$ such that $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $\triangle = \partial_x^2 + \triangle_\perp$. The directional dissipation term u_{xx} just provides the mass conservation. But it is worth knowing the behavior of the ZKB equation under invariants of the ZK equation. We suppose that $u(t)$ is sufficiently regular, then one can see that:

PROPOSITION 2.2.1 *For any $t \in [0, T]$, we have*

- $$\int_{\mathbb{R}^n} u(t)^2\,dxdy + 2\alpha \int_0^t \int_{\mathbb{R}^n} u_x^2(t')\,dxdydt' = \int_{\mathbb{R}^n} u_0^2\,dxdy.$$

  Consequently, $\|u(t)\|_{L^2}$ is a non-increasing function of t; and

  $$\sup_{t\in[0,T]} \|u(t)\|_{L^2} \le \|u_0\|_{L^2}.$$

- $$E(u(t)) + \alpha \int_0^t \int_{\mathbb{R}^n} \left[u_{xy_1}^2(t') + \cdots + u_{xy_{n-1}}^2(t')\right]\,dxdydt' = \alpha \int_0^t \int_{\mathbb{R}^n} u_x^2(t') f'(u)\,dxdydt' + E(u_0),$$

  where $E(u) = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla u|^2 - F(u)\,dxdy$.

2.3 Linear Properties

Consider the initial value problem

$$\begin{cases} u_t + (\Delta u - \alpha u_x)_x - \beta \Delta_\perp u = 0 \\ u(x,y,0) = u_0(x,y) \qquad (x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} \end{cases} \tag{2.4}$$

where $u_0 \in H^s$, $s \in \mathbb{R}$. Solutions of (2.4) are described by the semigroup $\{U_{\alpha,\beta}(t)\}_{t\geq 0}$, that is,

$$u_{\alpha,\beta}(t) = U_{\alpha,\beta}(t)u_0 = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} e^{t\left(i\xi^3 + i\xi|\eta|^2 - \alpha\xi^2 - \beta|\eta|^2\right) + ix\xi + iy\cdot\eta}\, \widehat{u}_0(\xi,\eta)\, d\xi d\eta.$$

In fact, $\widehat{U_{\alpha,\beta}(t)u_0} = K_{\alpha,\beta}(\xi,\eta,t)\hat{u}_0$, where

$$K_{\alpha,\beta}(\xi,\eta,t) = e^{it\left(\xi^3 + \xi|\eta|^2 + \alpha i\xi^2 + \beta i|\eta|^2\right)}.$$

By an elementary calculation, one has that

$$\left\| \left(\xi^2 + |\eta|^2\right)^s \; K_{\alpha,\beta}(\xi,\eta,t) \right\|_\infty \precsim t^{-s} \max\{\alpha^{-s}, \beta^{-s}\}, \tag{2.5}$$

where $\precsim$ means the inequality needs to a harmless positive constant (which in fact only depends on s). So, we can obtain a useful property of the solutions of (2.4).

LEMMA 2.3.1 *Let $\alpha, \beta > 0$ and $s \in \mathbb{R}$, then for every $\delta \geq 0$ and all $t > 0$, $U_{\alpha,\beta}(t) \in \mathcal{L}(H^s, H^{s+\delta})$; moreover*

$$\|u_{\alpha,\beta}(t)\|_{H^{s+\delta}} \precsim \left(1 + t^{-s}\max\{\alpha^{-s}, \beta^{-s}\}\right)^{1/2} \|u_0\|_{H^s}, \tag{2.6}$$

for any $u_0 \in H^s$.

However, it is straightforward that $U_{\alpha,\beta}$ is a contraction semigroup in H^s and is extensible to a strongly continuous unitary group whenever $\alpha = \beta = 0$. Next we study the L^p estimates of $U_{\alpha,\beta}(t)\delta_0$ which may be useful. Denote $K_{\alpha,\beta}(\xi,\eta,t) = g(\xi,\eta,t)\; f_{\alpha,\beta}(\xi,\eta,t)$ where $g(\xi,\eta,t) = e^{it(\xi^3+\xi|\eta|^2)}$. By induction it is easy to see that for any $k \in \mathbb{N}$, $\frac{\partial^k}{\partial\xi^k} f_{\alpha,\beta} = \; p_k(\xi,\eta,t)\; f_{\alpha,\beta}$, where p_k is a polynomial of degree k in each of the variables t and ξ with its higher order term of the form $(\alpha t\xi)^k$. Also $\frac{\partial^k}{\partial\eta^k} f_{\alpha,\beta}$ can be analogously obtained for the components of η. On the other hand,

$$\partial_\xi^k g = g \sum_{j=[\frac{k}{3}]}^{k} t^j g_{k,j}(\xi,\eta),$$

where $g_{k,j}(\xi,\eta)$ is a polynomial of degree $2j$ with respect to ξ and $|\eta|$ and also $g_{k,j}(0,0) = 0$. Also

$$\partial_\eta^k g = g_k(\xi,\eta,t)g,$$

where p_k is a polynomial of degree $k+1$ with its higher order term of the form $(\beta t\xi\eta)^k$. Then by Leibniz's rule we have

$$\partial_{\eta_i}^k K_{\alpha,\beta} = K_{\alpha,\beta} \sum_{j=0}^{[\frac{k}{2}]} t^{k-j}|\eta|^{k-2j} h_{k,j}(\xi,\beta) \tag{2.7}$$

for each component of $\eta = (\eta_1, \cdots, \eta_{n-1})$. So we can obtain the estimates of the Green function $U^0_{\alpha,\beta}(t) = U_{\alpha,\beta}(t)\delta_0$:

LEMMA 2.3.2 *Let* $m = (m_1, \cdots, m_n), k = (k_1, \cdots, k_n) \in (\mathbb{Z}^+)^n$, $x \in \mathbb{R}^n$ *and* $t > 0$.

(i) *If* $2 \le p \le \infty$, *then*

$$\left\| x^k D^m U^0_{\alpha,\beta}(t) \right\|_{L^p} \le C(\alpha,\beta) \langle t \rangle^{\frac{1}{2}|k|}\ t^{-\frac{1}{2}|m| - n\left(1-\frac{1}{p}\right)}. \tag{2.8}$$

(ii) *If* $1 \le p \le 2$, *then*

$$\left\| x^k D^m U^0_{\alpha,\beta}(t) \right\|_{L^p} \le C(\alpha,\beta) \langle t \rangle^{\frac{1}{2}\left(|k| - \frac{n}{2}\right)}\ t^{-n\left(1-\frac{1}{p}\right) - \frac{|m|}{2}}, \tag{2.9}$$

where $|k| = k_1 + \cdots + k_n$, $|m| = m_1 + \cdots + m_n$ *and* $\langle t \rangle = \left(1 + t^2\right)^{1/2}$.

Proof. Suppose that $2 \le p \le \infty$. We have by the Plancherel theorem and the preceding lemma

$$\begin{aligned} \left\| x^k D^m U^0_{\alpha,\beta}(t) \right\|_{L^p} &\le C \left\| x^m D^k K_{\alpha,\beta} \right\|_{L^{\frac{p}{p-1}}} \le C(\alpha,\beta) \langle t \rangle^{\frac{|k|}{2}} \left\| x^m K_{\alpha,\beta} \right\|_{L^{\frac{p}{p-1}}} \\ &\le C(\alpha,\beta) \langle t \rangle^{\frac{1}{2}\left(|k| - \frac{n}{2}\right)}\ t^{-n\left(1-\frac{1}{p}\right) - \frac{|m|}{2}}. \end{aligned}$$

The second estimate, for $1 \le p \le 2$, follows from the last estimate, interpolation and the following inequality:

$$\begin{aligned} \left\| x^k D^m U^0_{\alpha,\beta}(t) \right\|_{L^1} &= \int_{S_r(0)} \left| x^k D^m U^0_{\alpha,\beta}(t) \right|\, dx\ +\ \int_{\mathbb{R}^n \setminus S_r(0)} \left| x^{-\mathbf{1}} \right| \left| x^{\hat{k}} D^m U^0_{\alpha,\beta}(t) \right|\, dx \\ &\le \langle t \rangle^{\frac{n}{4}} \left\| x^k D^m U^0_{\alpha,\beta}(t) \right\|_{L^2} + \langle t \rangle^{-\frac{n}{4}} \left\| x^{\hat{k}} D^m U^0_{\alpha,\beta}(t) \right\|_{L^2} \le \langle t \rangle^{\frac{1}{2}\left(|k| - \frac{n}{2}\right)} t^{-\frac{|m|}{2}}, \end{aligned}$$

where $\hat{k} = k + \mathbf{1} = (k_1 + 1, \cdots, k_n + 1)$, $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^n$ and $S_r(0)$ denotes the cube centered at $0 \in \mathbb{R}^n$ with the side length $r = \sqrt{\langle t \rangle}$. ∎

A direct corollary of preceding lemma is L^p-estimates of the solutions $u_{\alpha,\beta}(t)$ of (2.4):

PROPOSITION 2.3.3 *Let* $u_0 \in L^2$. *Then* $u_{\alpha,\beta}(t) \in L^p$ *for any* $2 \le p \le \infty$; *moreover* $\|u_{\alpha,\beta}(t)\|_{L^p} \le Ct^{-\theta}\|u_0\|$, *where* $\theta = \theta(p) = 1 - \frac{2}{p}$.

Proof. By using of the Plancherel theorem and the Hölder Inequality, we have

$$\|u_{\alpha,\beta}(t)\|_{L^p} \le C \|K_{\alpha,\beta} \hat{u}_0\|_{L^{\frac{p}{p-1}}} \le C \|u_0\|_{L^2} \left\| U^0_{\alpha,\beta}(t) \right\|_{L^{\frac{2p}{p-2}}}.$$

But from last lemma, we have $\|U_{\alpha,\beta}(t)\|_{L^{\frac{2p}{p-2}}} \le Ct^{\frac{2}{p}-1}$. Thus $\|u_{\alpha,\beta}(t)\|_{L^p} \le Ct^{-\theta}\|u_0\|_{L^2}$. ∎

Now we will obtain some $y-$directional estimates of $U^0_{\alpha,\beta}(t)$ which may be useful in the instability analysis.

LEMMA 2.3.4 *Let* $2 \le p \le \infty$. *Then for any* $t > 0$,

(i) *if $x \geq 0$ then*

$$\|U^0_{\alpha,\beta}(t)\|_{L^p_y(\mathbb{R}^{n-1})} \leq C\, t^{-\frac{1}{3}\left(n-\frac{n-1}{p}\right)}\; e^{-\frac{2}{3}x^{3/2}t^{-1/2}}. \tag{2.10}$$

(ii) *if $x \leq 0$ then*

$$\|U^0_{\alpha,\beta}(t)\|_{L^p_y(\mathbb{R}^{n-1})} \leq C\; \left(t^{-\frac{1}{3}\left(1+\frac{n-1}{p'}\right)} + t^{\frac{1}{3}\left(\frac{1-2n}{p'}\right)-\frac{1}{4}+\frac{1}{2p'}}\; |x|^{\frac{n}{p'}-\frac{3}{2p'}-\frac{1}{4}}\right), \tag{2.11}$$

where $\frac{1}{p'} = 1 - \frac{1}{p}$.

Proof. By a change of variable we have

$$\begin{aligned} U^0_{\alpha,\beta}(x,y,t) &= \int_{\mathbb{R}^n} e^{t(i\xi^3+i\xi|\eta|^2-\alpha\xi^2-\beta|\eta|^2)+ix\xi+iy\cdot\eta}\; d\xi d\eta \\ &= C\; t^{-1/3}\int_{\mathbb{R}^n} e^{i\xi^3+it^{2/3}\xi|\eta|^2-\alpha t^{1/3}\xi^2-\beta t|\eta|^2+it^{-1/3}x\xi+iy\cdot\eta}\; d\xi d\eta \\ &= C\; t^{-1/3}\int_{\mathbb{R}^{n-1}} \mathtt{Ai}\left(t^{-1/3}x+t^{2/3}|\eta|^2\right)\; e^{iy\cdot\eta-\alpha t^{1/3}\xi^2-\beta t|\eta|^2}\; d\eta, \end{aligned}$$

where $\mathtt{Ai}$ is the *Airy* function, defined by $\mathtt{Ai}(x) = \int_{\mathbb{R}} e^{i\xi^3+ix\xi}\; d\xi$. By using Plancherel theorem, we obtain

$$\left\|U^0_{\alpha,\beta}(x,.,t)\right\|_{L^p(\mathbb{R}^{n-1})} \leq C\; t^{-1/3}\left\|\mathtt{Ai}\left(t^{-1/3}x+t^{2/3}|\cdot|^2\right)\right\|_{L^q(\mathbb{R}^{n-1})}.$$

Now, if $x \geq 0$ then we know that $|\mathtt{Ai}(x)| \leq e^{-\frac{2}{3}x^{3/2}}$, (see[32]), this leads us to the inequality (2.10). If $x \leq 0$ then we divide $\mathbb{R}^{n-1}$ to $\left\{\eta \in \mathbb{R}^{n-1}\; ;\; |\eta|^2 + t^{-1}x \geq 0\right\}$ and its complement, but by using the preceding bound on the Airy function, it is clear that

$$\int_{|\eta|^2+t^{-1}x\geq 0} \left|\mathtt{Ai}\left(t^{-1/3}x+t^{2/3}|\eta|^2\right)\right|^q\; d\eta \tag{2.12}$$

is bounded independently of x and $t > 0$. On the other hand, if we consider the other region, then by a change of variable, we have

$$\begin{aligned} I &= \int_{|\eta|^2+t^{-1}x<0} \left|\mathtt{Ai}\left(t^{-1/3}x+t^{2/3}|\eta|^2\right)\right|^q\; d\eta = C\; t^{\frac{1-n}{3}}\int_{|\eta|^2+t^{-1/3}x\leq 0}\left|\mathtt{Ai}\left(t^{-1/3}x+|\eta|^2\right)\right|^q\; d\eta \\ &= C\; t^{\frac{1-n}{3}}\int_{t^{-1/3}x}^0 |\mathtt{Ai}(w)|^q\left(w-t^{-1/3}x\right)^{n-\frac{5}{2}}\; dw, \end{aligned}$$

where C depends on $\left|\mathbb{S}^{n-1}\right|$. Since $w \leq 0$ then $|Ai(w)| \leq |w|^{-1/4}$, thusly

$$\begin{aligned} I &\leq C\; t^{\frac{1-n}{3}}\int_{t^{-1/3}x}^0 |w|^{-q/4}\left(w-t^{-1/3}x\right)^{n-\frac{5}{2}}\; dw; \\ &\leq C\; t^{-\frac{2}{3}n+\frac{7}{6}}\; |x|^{n-\frac{5}{2}}\int_{t^{-1/3}x}^0 |w|^{-q/4}dw; \leq C\; t^{\frac{1-2n}{3}+\frac{q}{12}+\frac{1}{2}}\; |x|^{n-\frac{q}{4}-\frac{3}{2}}. \end{aligned} \tag{2.13}$$

By (2.10), (2.11) and the bound on (2.12), we obtain (b). ■

Now we want to study the initial value problem (2.4) in *weighted* spaces.

DEFINITION 2.3.5 *Let $s \in \mathbb{R}$ be nonnegative, $p \in \mathbb{N}$ and $r = (r_1, \cdots, r_n) \in (\mathbb{R}^+)^n$. We denote $L^p\left(\left(1+|x^{2r}|\right)dx\right)$ the space of all real valued functions f such that*

$$\|f\|^p_{L^p((1+|x^{2r}|)dx)} = \int f^p(x)\left(1+|x^{2r}|\right)\, dx < \infty,$$

where $|x^{2r}| = \sum_{i=1}^{n} x_i^{2r_i}$. Also we denote $\mathcal{F}_r^{s,p}$ the space of all real valued measurable functions f such that

$$\|f\|_{\mathcal{F}_r^{s,p}} = \|f\|_{H^s} + \|f\|_{L^p((1+|x^{2r}|)dx)} < \infty.$$

The following lemma is useful in the weighted spaces.

LEMMA 2.3.6 *Let $n=2$, $p, m \in \mathbb{N}$, $\beta > 0$, $t > 0$ and $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$. Then for any $f \in \mathcal{F}_{0,m}^{0,p}$,*

$$\|D^\omega U_{\alpha,\beta}(t) f\|_{\mathcal{F}_{0,m}^{0,p}} \le C(m, \beta, |\omega|)\left(1 + t^{-\frac{|\omega|}{2}} + t^{\frac{m-|\omega|}{2}}\right)\|f\|_{\mathcal{F}_{0,m}^{0,p}}.$$

Proof. The proof follows from Plancherel theorem, Leibniz's rule, (2.6) and (2.7). In fact, we have

$$\begin{aligned}
\|y^m D^\omega U_{\alpha,\beta}(t) f\|_{L^p} &= \left\|\partial_\eta^m\left(\xi^{\omega_1}\eta^{\omega_2}K_{\alpha,\beta}(\xi,\eta,t)\hat{f}\right)\right\|_{L^p} \\
&\le C \sum_{k=0}^{\min\{m,\omega_2\}} \left\|\xi^{\omega_1}\eta^{\omega_2-k}\partial_\eta^{m-k}K_{\alpha,\beta}(\xi,\eta,t)\hat{f}\right\|_{L^p} \\
&\le C\sum_k \left\|\xi^{\omega_1}\eta^{\omega_2-k}\sum_{j=0}^{m-k}\sum_{\ell=0}^{[\frac{j}{2}]} t^{j-\ell}\eta^{j-2\ell}(1+|\xi|)^{j-2\ell}K_{\alpha,\beta}(\xi,\eta,t)\,\partial_\eta^{m-k-j}\hat{f}\right\|_{L^p} \\
&\le C\sum_k\sum_j\sum_\ell t^{j-\ell}\left\|\xi^{\omega_1}(1+|\xi|)^{j-2\ell}\eta^{j+\omega_2-k-2\ell}e^{-t(\alpha\xi^2+\beta\eta^2)}\right\|_\infty \left\|\partial_\eta^{m-k-j}\hat{f}\right\|_{L^p} \\
&\le C\sum_k\sum_j\sum_\ell\left(t^{\frac{j+k-|\omega|}{2}} + t^{\frac{2\ell+k-|\omega|}{2}}\right)\left\|\partial_\eta^{m-k-j}\hat{f}\right\|_{L^p} \\
&\le C\left(t^{-\frac{|\omega|}{2}} + t^{\frac{m-|\omega|}{2}}\right)\left\|(1+y^{2k})^{1/2}f\right\|_{L^p},
\end{aligned}$$

where C may change from line to line and depends on β, m, $|\omega|$ and the maximum of the coefficients of $\partial_\eta^i\widehat{f}$. This completes the proof. ∎

By using the definition of the $U_{\alpha,\beta}$, Lemma 2.3.1 and L^p interpolation theorem (see [68]), we can easily obtain the persistence and boundedness and regularity of the solutions in our weighted spaces.

LEMMA 2.3.7 *Let $n=2$, $s \ge 2$, $r \in [0,1]$ and $t \ge 0$. Then if $f \in \mathcal{F}_1^{s,2}$ and $\beta \ge 0$, then $U_{\alpha,\beta}(t) \in \mathcal{L}\left(\mathcal{F}_1^{s,2}\right)$ and*

$$\|U_{\alpha,\beta}(t)f\|_{\mathcal{F}_1^{s,2}} \le C\|f\|_{\mathcal{F}_1^{s,2}},$$

where C is a polynomial of first degree in t with positive coefficients depending on α and β. Moreover $U_{\alpha,\beta}(\cdot)f \in C\left([0,+\infty); \mathcal{F}_1^{s,2}\right)$.

Furthermore, if $f \in \mathcal{F}_r^{s,2}$, $\delta \ge 0$ and $\beta > 0$, then $U_{\alpha,\beta}(t) \in \mathcal{L}\left(\mathcal{F}_r^{s,2}, \mathcal{F}_r^{s+\delta,2}\right)$ and

$$\|U_{\alpha,\beta}(t)f\|_{{}_r^{s+\delta,2}} \le C\|f\|_{\mathcal{F}_r^{s,2}}.$$

Moreover $U_{\alpha,\beta}(\cdot)f \in C\left([0,+\infty); \mathcal{F}_r^{s+\delta,2}\right)$.

The following properties are some direct consequences of Lemmata 2.3.1, 2.3.6 and 2.3.7.

LEMMA 2.3.8 *Let* $s \geq 0$, $\omega \in \mathbb{R}^2$, $r\mathbb{N}$, $\delta \geq 0$, $\beta > 0$ *and* $t > 0$. *Then, if* $f \in \mathcal{F}_{1,r}^{s,2}$, *then* $U_{\alpha,\beta}(t) \in \mathcal{L}\left(\mathcal{F}_{1,r}^{s,2}, \mathcal{F}_{1,r}^{s+\delta,2}\right)$ *and*

$$\|U_{\alpha,\beta}(t)f\|_{\mathcal{F}_{1,r}^{s+\delta,2}} \leq C\|f\|_{\mathcal{F}_{1,r}^{s,2}},$$

where $C = C(t,\alpha,\beta,\delta)$ *is a continuous function such that* $C \sim O\left(t^{-\frac{\delta}{2}}\right)$ *as* $t \to 0^+$. *Moreover* $U_{\alpha,\beta}(\cdot)f \in C\left([0,+\infty); \mathcal{F}_{1,r}^{s+\delta,2}\right)$.

Furthermore, $D^\omega U_{\alpha,\beta}(t) \in \mathcal{L}\left(\mathcal{F}_{1,r}^{s,2}\right)$ *and*

$$\|D^\omega U_{\alpha,\beta}(t)f\|_{\mathcal{F}_{1,r}^{s,2}} \leq C\|f\|_{\mathcal{F}_{1,r}^{s,2}},$$

where $C = C(t,\alpha,\beta,|\omega|)$ *is a continuous function such that* $C \sim O\left(t^{-\frac{|\omega|}{2}}\right)$ *as* $t \to 0^+$. *Moreover* $D^\omega U_{\alpha,\beta}(\cdot)f \in C\left([0,+\infty); \mathcal{F}_{1,r}^{s,2}\right)$.

2.4 Local Existence

Now we are going to study the Cauchy problem (2.3). We use the obtained properties of linear problem and a Poincare argument to get the existence in a suitable space. Without loss of generality we assume that $f(x) = \frac{1}{p+1}x^{p+1}$ in (2.3). The main theorem is the following:

THEOREM 2.4.1 *Let* $\alpha, \beta > 0$ *and* $s > \frac{n}{2}+1$. *Then for any* $u_0 \in H^s$, *there exists* $T_{\alpha,\beta}^s = T\left(\alpha,\beta,\|u_0\|_{H^s}\right)$ *and a unique solution of initial value problem (2.3),* $u_{\alpha,\beta}(\cdot)$ *defined in the interval* $\left[0, T_{\alpha,\beta}^s\right]$ *satisfying*

$$u_{\alpha,\beta} \in C\left(\left[0,T_{\alpha,\beta}^s\right]; H^s\right) \cap C^1\left(\left[0,T_{\alpha,\beta}^s\right]; H^{s-2}\right), \tag{2.14}$$

$$\|u_{\alpha,\beta}(t)\|_{H^s} \leq \|u_0\|_{H^s} \exp\left\{c\int_0^t \|u_{\alpha,\beta}(\tau)\|_{L^\infty}^{p-1}\|\nabla u_{\alpha,\beta}(\tau)\|_{L^\infty}\, d\tau\right\}, \tag{2.15}$$

for all $t \in \left[0, T_{\alpha,\beta}^s\right]$. *Moreover* $u_{\alpha,\beta} \in C\left(\left(0,T_{\alpha,\beta}^s\right]; H^\infty\right)$.

Proof. As usual we consider the integral equation associated to the initial value problem (2.3), that is,

$$u_{\alpha,\beta}(t) = U_{\alpha,\beta}(t)u_0 + \int_0^t U_{\alpha,\beta}(t-\tau)(\partial_x u_{\alpha,\beta}u_{\alpha,\beta}^p)(\tau)\, d\tau. \tag{2.16}$$

We define the operator

$$\Phi(v(t)) = U_{\alpha,\beta}(t)u_0 + \int_0^t U_{\alpha,\beta}(t-\tau)(v^p v_x)(\tau)\, d\tau, \tag{2.17}$$

and the metric space

$$E\left(T_{\alpha,\beta}^s\right) = \left\{v \in C\left(\left[0,T_{\alpha,\beta}^s\right]; H^s\right)\ :\ |v|_E \leq \|u_0\|_{H^s}\right\},$$

where $|v|_E = \sup_{t\in\left[0,T^s_{\alpha,\beta}\right]} \|v(t) - U_{\alpha,\beta}(t)u_0\|_{H^s}$. First we will show that $\Phi(v) \in E\left(T^s_{\alpha,\beta}\right)$ if $v \in E\left(T^s_{\alpha,\beta}\right)$ and $T^s_{\alpha,\beta}$ is suitable . In fact, by Hölder inequality and (2.6), we have

$$\begin{aligned}\|\Phi(v(t)) - U_{\alpha,\beta}(t)u_0\|_{H^s} &\le c\int_0^t \left\|U_{\alpha,\beta}(t-\tau)v^{p+1}(\tau)\right\|_{H^{s+1}}\, d\tau \le c\int_0^t (1+\tau^{-s}\max\{\alpha^{-s},\beta^{-s}\})^{1/2}\|v\|^{p+1}_{H^s}\, d\tau\\ &\le c\|u_0\|^{p+1}_{H^s}\int_0^t (1+\tau^{-s}\max\{\alpha^{-s},\beta^{-s}\})^{1/2}\, d\tau.\end{aligned}$$

Therefore $\Phi(v) \in E\left(T^s_{\alpha,\beta}\right)$ for $T^s_{\alpha,\beta}$ small enough. A similar computation shows that Φ is a contraction (by choosing $T^s_{\alpha,\beta}$ smaller if necessary). So the obtained fixed point via contraction is a solution of equation (2.3) with initial data u_0. Note that the obtained solution $u_{\alpha,\beta}(t)$ with initial data u_0 is more regular for $t > 0$ and is in $C\left(\left(0, T^s_{\alpha,\beta}\right]; H^\infty\right)$. In fact, for any $\lambda > 0$ and $T > 0$, we have that $U_{\alpha,\beta}(t)u_0 \in H^{s+\lambda}$, by (2.6). But for any $\epsilon \in (0,1)$, we have

$$\begin{aligned}\left\|\int_0^t U_{\alpha,\beta}(t-\tau)u^p_{\alpha,\beta}(\tau)\partial_x u_{\alpha,\beta}(\tau)\, d\tau\right\|_{H^{s+\epsilon}} &\le c\int_0^t \left\|U_{\alpha,\beta}(t-\tau)u^{p+1}_{\alpha,\beta}(\tau)\right\|_{H^{s+1+\epsilon}}\, d\tau\\ &\le c\sup_\tau \|u_{\alpha,\beta}(\tau)\|^{p+1}_{H^s}\int_0^t \left(1+\tau^{-s}\max\{\alpha^{-s},\beta^{-s}\}\right)^{1/2}\, d\tau,\end{aligned}$$

where c depends on ϵ. This implies that $u_{\alpha,\beta}(t) \in H^{s+\epsilon}$ for all $t \in \left(0, T^s_{\alpha,\beta}\right]$. By reiterating this procedure, one gets that $u_{\alpha,\beta}(t) \in H^\infty$ for all $t \in \left(0, T^s_{\alpha,\beta}\right]$. Now suppose that $t \in \left(0, T^s_{\alpha,\beta}\right]$, so we have $u_{\alpha,\beta}(t)$ and $\frac{d}{dt}u_{\alpha,\beta}(t)$ are in H^∞. Define $J^s = (1-\Delta)^{s/2}$. We know that $J^s \in \mathcal{L}(H^r, H^{r-s})$ for every $s, r \in \mathbb{R}$. Thus $J^s u_{\alpha,\beta}(t) \in H^\infty$. By applying J^s to (2.3) and taking the inner product with $J^s u_{\alpha,\beta}(t)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_{\alpha,\beta}(t)\|^2_{H^s} + \langle J^s\Delta\partial_x u_{\alpha,\beta}(t), J^s u_{\alpha,\beta}(t)\rangle + \frac{1}{p+1}\left\langle J^s\partial_x u^{p+1}_{\alpha,\beta}(t), J^s u_{\alpha,\beta}(t)\right\rangle&\\ - \alpha\langle J^s\partial_x^2 u_{\alpha,\beta}(t), J^s u_{\alpha,\beta}(t)\rangle - \beta\langle J^s\Delta_\perp u_{\alpha,\beta}(t), J^s u_{\alpha,\beta}(t)\rangle = 0.&\end{aligned}$$

Since J^s commutes with the derivative operators, then from integrating by parts, we have

$$\begin{aligned}0 = \frac{1}{2}\frac{d}{dt}\|u_{\alpha,\beta}(t)\|^2_{H^s} + \alpha\|J^s\partial_x u_{\alpha,\beta}(t)\|^2_{L^2} + \beta\|J^s\nabla_\perp u_{\alpha,\beta}(t)\|^2_{L^2}.&\\ + \left\langle u^p_{\alpha,\beta}(t)J^s\partial_x u_{\alpha,\beta}(t), J^s u_{\alpha,\beta}(t)\right\rangle + \left\langle [J^s, u^p_{\alpha,\beta}(t)]\partial_x u_{\alpha,\beta}(t), J^s u_{\alpha,\beta}(t)\right\rangle&\end{aligned}$$

Now, we use the Kato-Ponce commutator ([44]) :

LEMMA 2.4.2 *If $f, g \in \mathscr{S}(\mathbb{R}^n)$, $s > 0$ and $p \in (1, +\infty)$, then*

$$\|[J^s, M_f]g\|_{L^p} \le C\left(\|\nabla f\|_{L^{p_1}}\left\|J^{s-1}g\right\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}\right), \tag{2.18}$$
$$\|fg\|_{L^p} \le C\left(\|f\|_{L^{p_1}}\|J^s g\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}\right), \tag{2.19}$$

where $p_2, p_3 \in (1, +\infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

By using the preceding lemma, we obtain that

$$\begin{aligned}\frac{d}{dt}\|u_{\alpha,\beta}(t)\|_{H^s}^2 &\leq C\left|\left\langle u_{\alpha,\beta}^{p-1}(t)\partial_x u_{\alpha,\beta}(t),(J^s u_{\alpha,\beta}(t))^2\right\rangle\right| + C\left|\left\langle [J^s,u_{\alpha,\beta}^p(t)],J^s u_{\alpha,\beta}(t)\right\rangle\right|\\ &\leq C\left|\left\langle u_{\alpha,\beta}^{p-1}(t)\partial_x u_{\alpha,\beta}(t),(J^s u_{\alpha,\beta}(t))^2\right\rangle\right| +\\ &\quad + C\|\nabla(u_{\alpha,\beta}^p(t))\|_{L^\infty}\|J^{s-1}\partial_x u_{\alpha,\beta}(t)\|_{L^2}\|u_{\alpha,\beta}(t)\|_{H^s}+\\ &\quad + C\|\partial_x u_{\alpha,\beta}(t)\|_{L^\infty}\|J^s u_{\alpha,\beta}^p(t)\|_{L^2}\|u_{\alpha,\beta}(t)\|_{H^s}\\ &\leq C\left(\|u_{\alpha,\beta}(t)\|_{L^\infty}^{p-1}\|\partial_x u_{\alpha,\beta}(t)\|_{L^\infty}\|u_{\alpha,\beta}(t)\|_{H^s}^2\right)+\\ &\quad + C\left(\|u_{\alpha,\beta}(t)\|_{L^\infty}^{p-1}\|\nabla u_{\alpha,\beta}(t)\|_{L^\infty}\|u_{\alpha,\beta}(t)\|_{H^s}^2\right)\\ &\leq C\|u_{\alpha,\beta}(t)\|_{L^\infty}^{p-1}\|\nabla u_{\alpha,\beta}(t)\|_{L^\infty}\|u_{\alpha,\beta}(t)\|_{H^s}^2,\end{aligned}$$

where we used the fact that $\|fg\|_{H^s}\leq C(\|f\|_{L^\infty}\|g\|_{H^s}+\|f\|_{H^s}\|g\|_{L^\infty})$, for every $s\geq 0$. The Gronwall inequality leads to (2.15). ■

Now we are going to find a time interval of existence of solutions of (2.3) which is independent of parameters α and β. This can be easily obtained by considering the solution of $\frac{d}{dt}\mathcal{X}=c_s\mathcal{X}^q(t)$ for $t\in[0,T^s)$, with initial data $\mathcal{X}(0)=\|u_0\|_{H^s}^2$, where $q=\frac{p+2}{2}$ and c_s is in the inequality

$$\frac{d}{dt}\|u_{\alpha,\beta}(t)\|_{H^s}^2\leq c_s\|u_{\alpha,\beta}(t)\|_{H^s}^p+2,$$

for $t\in\left(0,T_{\alpha,\beta}^s\right)$, by using the Sobolev embedding. Now we take $T\in(0,T^s)$. Thus $\|u_{\alpha,\beta}(t)\|_{H^s}\leq\mathcal{X}^{1/2}(t)$, for $t\in[0,T^*]$ where $T^*=\min\{T,T_{\alpha,\beta}^s\}$. So all solutions can be extended to $[0,T]$ and then to $[0,T^s)$ and also for any $T\in(0,T^s)$, there is $\mathscr{A}_T$ such that $\|u_{\alpha,\beta}(t)\|_{H^s}\leq\mathscr{A}_T$ for all $\alpha,\beta>0$ and $0\leq t\leq T$. To get the solutions of (2.2), we need to study the behavior of the solutions of (2.3) when the parameter β varies. In fact, we will investigate more general case where α and β vary. Let $\alpha_1,\alpha_2,\beta_1,\beta_2>0$ and $u_0,v_0\in H^s$ where $s>\frac{n}{2}+1$. Also let $u_{\alpha_1,\beta_1}(t),u_{\alpha_2,\beta_2}(t)\in C([0,T];H^s)$ be the solutions of (2.3) corresponding to the initial data and the parameters u_0,α_1,β_1 and v_0,α_2,β_2 respectively. Note that T does not depend on the parameters. Denote $w(t)=u_{\alpha_1,\beta_1}(t)-u_{\alpha_2,\beta_2}(t)$, then we have

$$\begin{aligned}\frac{d}{dt}\|w(t)\|_{L^2}^2 &= \frac{2}{p+1}\left\langle u_{\alpha_1,\beta_1}^{p+1}(t)-u_{\alpha_2,\beta_2}^{p+1}(t),w_x(t)\right\rangle+2\alpha_1\langle w_{xx}(t),w(t)\rangle+2\beta_1\langle\Delta_\perp w(t),w(t)\rangle\\ &\quad+2(\alpha_1-\alpha_2)\langle u_{\alpha_1,\beta_1}(t)_{xx},w(t)\rangle+2(\beta_1-\beta_2)\langle\Delta_\perp u_{\alpha_1,\beta_1}(t),w(t)\rangle\\ &= \frac{1}{p+1}\left\langle g(w)(t),(w^2(t))_x\right\rangle-2\alpha_1\|w_x(t)\|_{L^2}^2-2\beta_1\|\nabla_\perp w(t)\|_{L^2}^2+\\ &\quad+2(\alpha_1-\alpha_2)\langle\partial_x^2 u_{\alpha_1,\beta_1}(t),w(t)\rangle+2(\beta_1-\beta_2)\langle\Delta_\perp u_{\alpha_1,\beta_1}(t),w(t)\rangle\\ &\leq C\mathscr{K}^p\|w(t)\|_{L^2}^2+2\left(|\alpha_1-\alpha_2|+|\beta_1-\beta_2|\right)\mathscr{K}^2,\end{aligned}$$

where $g(w)=u_{\alpha_1,\beta_1}^p+u_{\alpha_1,\beta_1}^{p-1}u_{\alpha_2,\beta_2}+\cdots+u_{\alpha_1,\beta_1}u_{\alpha_2,\beta_2}^{p-1}+u_{\alpha_2,\beta_2}^p$, and C does not depend on the parameters. Therefore by Gronwall's inequality, we obtain that

$$\|w(t)\|_{L^2}^2\leq C\left(\|w_0\|_{L^2}^2+|\alpha_1-\beta_1|+|\beta_1-\beta_2|\right), \tag{2.20}$$

for all $t \in [0,T]$, where $w_0 = u_0 - v_0$ and $C = C(s,p,\mathscr{K},T)$. In particular, for $u_0 = v_0$, we have

$$\|u_{\alpha_1,\beta_1}(t) - u_{\alpha_2,\beta_2}(t)\|_{L^2} \le C(|\alpha_1 - \beta_1| + |\beta_1 - \beta_2|), \tag{2.21}$$

for all $t \in [0,T]$.

Now, let $\{u_{\alpha,\beta}(t)\}_{\beta>0}$ be the solutions of (2.3) corresponding to initial data u_0 and the parameters α (fixed) and β. Then $\{u_{\alpha,\beta}(t)\}_{\beta>0}$ is a Cauchy sequence in $C([0,T];L^2)$, by (2.21). Denote $u_\alpha(t) = \lim_{\beta\to 0} u_{\alpha,\beta}(t)$ and $w_\alpha(t) = u_{\alpha,\beta_1}(t) - u_{\alpha,\beta_2}(t)$. Then we have

$$|\langle w_\alpha(t),\phi\rangle_{H^s}| \le |\langle w_\alpha(t),\phi-\psi\rangle_{H^s}| + |\langle w_\alpha(t),\psi\rangle_{H^s}| \le 2\mathscr{A}_T\|\phi-\psi\|_{H^s} + \|w_\alpha(t)\|_{L^2}\|\psi\|_{H^s}$$

for any $\psi \in \mathscr{S}$, $\phi \in H^s$ and $t \in [0,T]$. Thus $u_{\alpha,\beta}(t) \to u_\alpha(t)$ in $C_w([0,T];H^s)$ as $\beta \to 0$ and $u_\alpha \in C_w([0,T];H^s)$. Also by interpolation, we have $u_{\alpha,\beta}(t) \to u_\alpha(t)$ in $C([0,T];H^r)$ for any $r \in [0,s)$. So

$$\Delta\partial_x u_{\alpha,\beta}(t) \to \Delta\partial_x u_\alpha(t), \tag{2.22}$$
$$\Delta_\perp u_{\alpha,\beta}(t) \to \Delta_\perp u_\alpha(t), \tag{2.23}$$
$$\partial_x^2 u_{\alpha,\beta}(t) \to \partial_x^2 u_\alpha(t), \tag{2.24}$$

in $C_w([0,T];H^{s-2}) \cap C([0,T];H^{r-2})$, and

$$\partial_x\left(u_{\alpha,\beta}^{p+1}(t)\right) \to \partial_x\left(u_\alpha^{p+1}(t)\right), \tag{2.25}$$

in $C_w([0,T];H^{s-1}) \cap C([0,T];H^{r-1})$ as $\beta \to +0$. But we know that

$$u_{\alpha,\beta}(t) = u_0 - \int_0^t \left[\Delta\partial_x u_{\alpha,\beta}(\tau) + \frac{1}{p+1}\partial_x\left(u_{\alpha,\beta}^{p+1}(\tau)\right) - \alpha\partial_x^2 u_{\alpha,\beta}(\tau) - \beta\Delta_\perp u_{\alpha,\beta}(\tau)\right] d\tau.$$

By taking the limit from the last identity and using the (2.22)-(2.25), we conclude that u_α is a solution of (2.2). The uniqueness implies that there is no other solution different from u_α. Also the inequality of (2.15) can be obtained by the weak lower semicontinuity. Note that u_α has right continuity at zero. In fact,

$$\|u_0\|_{H^s} \le \liminf_{t\to 0^+}\|u_\alpha(t)\|_{H^s} \le \limsup_{t\to 0^+} X^{1/2}(t) = \|u_0\|_{H^s}.$$

Thus $\lim_{t\to 0^+} u_\alpha(t) = u_0$. Analogously, one can obtain that as $\alpha = \beta \to 0$, there is a unique solution $u(t)$ of (1.1) in $C([0,T];H^s)$ for $s > \frac{n}{2} + 1$. Notice that in this case we have also the left continuity of the solution $u(t)$ at T (and in fact any $t \in [0,T]$), by the uniqueness and the invariance of (1.1) by $(x,y,t) \to (-x,y,a-t)$ for all $a \in \mathbb{R}$. So Theorem 2.4.1 holds for $\beta = 0$ (in the weak sense) and $\alpha = \beta = 0$.

Now we state the continuous dependence of the solutions to the initial data.

THEOREM 2.4.3 *For $R > 0$, the correspondence $u_0 \to u_{\alpha,\beta}$ that associates to $u_0 \in \mathscr{B}_R$ the solution $u_{\alpha,\beta}$ of (2.3) with initial data u_0, is continuous mapping from $\mathscr{B}_R$ into $E^s_{\alpha,\beta}$, where $\mathscr{B}_R$ is the ball of radius R centered at the origin of H^s.*

Proof. To prove this, one may use the Bona-Smith approximation. The following lemma (with slight modifications) appears in [17].

LEMMA 2.4.4 *Let $\varphi \in H^s$ and $s \in \mathbb{R}$. Then we have*

(a) $\|\varphi^\varepsilon - \varphi\|_{L^2} \to 0$, *as* $\varepsilon \to 0$,

(b) $\|\varphi^\varepsilon - \varphi^\ell\|_{L^2} \leq |\varepsilon - \ell| \|\varphi\|_{L^2}$, *for any* $\varepsilon, \ell \geq 0$,

(c) $\|\varphi^\varepsilon\|_{L^2} \leq \|\varphi^\ell\|_{L^2}$ *iff* $\varepsilon \geq \ell \geq 0$,

(d) $\|\varphi^\varepsilon\|_{L^2} \leq \left(\frac{r}{s\varepsilon}\right)^{r/s} e^{-r/s} \|\varphi\|_{L^2}$, *for any* $r \in \mathbb{R}^+$,

where $\widehat{\varphi^\varepsilon}(\xi) = e^{-\varepsilon(1+|\xi|^2)^{s/2}} \widehat{\varphi}(\xi)$.

Suppose that $u_{0,n} \to u_0$ in H^s and also $u_{\alpha,\beta,n}$ and $u_{\alpha,\beta}$ are the corresponding solutions of (2.3) with $u_{\alpha,\beta,n}(0) = u_{0,n}$ and $u_{\alpha,\beta}(0) = u_0$. Take $T \in (0, T^s)$ and denote $\varphi_n = u_{0,n}$ and $\varphi = u_0$. By using the preceding lemma and its notation, we have $\|\varphi_n^\varepsilon - \varphi\|_{H^s} \to 0$ as $\varepsilon \to 0$ and $n \to +\infty$. So there is $\varepsilon_0 > 0$ and $N \in \mathbb{N}$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $n \geq N$, $u_{\alpha,\beta,n}$, $u^\varepsilon_{\alpha,\beta,n}$ and $u^\varepsilon_{\alpha,\beta}$ are defined in $[0,T]$. For simplicity we remove α and β. Therefore we have

$$\|u_n(t) - u(t)\|_{H^s} \leq \|u_n(t) - u_n^\varepsilon(t)\|_{H^s} + \|u_n^\varepsilon(t) - u^\varepsilon(t)\|_{H^s} + \|u^\varepsilon(t) - u(t)\|_{H^s}.$$

Denote $v = u^\varepsilon$, $v_n = u_n^\varepsilon$, $w = u - v$ and $w_n = u_n - v_n$. By the preceding lemma and Lemma 2.4.2, we have

$$\begin{aligned}
\frac{d}{dt}\|w\|_{H^s}^2 &\lesssim \langle u^p u_x - v^p v_x, w\rangle_{H^s} \lesssim (\langle u^p w_x, w\rangle_{H^s} + \langle (u^p - v^p) v_x, w\rangle_{H^s}) \\
&\lesssim |\langle [J^s, u^p] w_x, J^s w\rangle| + |\langle u^p J^s w_x, J^s w\rangle| + \|(u^p - v^p) v_x\|_{H^s} \|w\|_{H^s} \\
&\lesssim \|w\|_{H^s}^2 + \|g(w)\|_{H^s} \|w v_x\|_{H^s} \|w\|_{H^s} \lesssim \|w\|_{H^s}^2 + \|w v_x\|_{H^s} \|w\|_{H^s} \\
&\lesssim \|w\|_{H^s}^2 + \|w\|_{H^s} \left(\|[J^s, w] v_x\|_{L^2} + \|w J^s v_x\|_{L^2}\right) \\
&\lesssim \|w\|_{H^s}^2 + \|w\|_{H^s} \left(\|w\|_{H^r} \|v\|_{H^{s+1}} + \|w\|_{H^s} \|v\|_{L^2}\right),
\end{aligned}$$

for some $r \in (1, s-1)$ and where $\lesssim$ means the inequalities need to a positive constant depending on $p, \mathscr{A}_T$. By interpolation and the preceding lemma, we have that

$$\|w\|_{H^r} \leq C \|w\|_{L^2}^{1-\frac{r}{s}} \|w\|_{H^s}^{\frac{r}{s}} \leq C \|\varphi^\varepsilon - \varphi\|_{L^2}^{1-\frac{r}{s}} \leq C \varepsilon^{1-\frac{r}{s}},$$

and $\|v\|_{H^{s+1}} \leq C \|\varphi^\varepsilon\|_{H^{S+1}} \leq C_{(\|\varphi\|_{H^s}, T)} \varepsilon^{-1/2}$. Then

$$\frac{d}{dt}\|w\|_{H^s} \leq C\left(\|w\|_{H^s} + \varepsilon^\theta\right),$$

where $C = C(\mathscr{A}_T, \|\varphi\|_{H^s}, r, p)$ and $\theta = 1 - \frac{r+1}{s}$. By Gronwall's inequality, we obtain that

$$\|w\|_{H^s}^2 \leq C(\|\varphi^\varepsilon - \varphi\|_{H^s}^2 + \varepsilon^\theta).$$

Thusly

$$\|w\|_{H^s}^2 + \|w_n\|_{H^s}^2 \leq C(\|\varphi^\varepsilon - \varphi\|_{H^s}^2 + \|\varphi_n^\varepsilon - \varphi_n\|_{H^s}^2 + \varepsilon^\theta), \tag{2.26}$$

and by interpolation we obtain

$$\|v - v_n\|_{H^s}^2 \leq C \|v - v_n\|_{H^{2s}} \|v - v_n\|_{L^2} \leq C \|\varphi_n^\varepsilon - \varphi^\varepsilon\|_{L^2} \leq C \|\varphi_n - \varphi\|_{L^2}, \tag{2.27}$$

where $C = C(\mathscr{A}_T, \|\varphi\|_{H^{2s}})$. The proof follows from (2.26) and (2.27). ■

Corollary 2.4.5 *The same result holds when $\beta = 0$ (and for $\alpha = \beta = 0$) in $C_w([0,T];H^s)$ (in $C([0,T];H^s)$), by weak lower semicontinuity.*

THEOREM 2.4.6 *If $p = 1$, $s > \frac{n}{2} + 1$ and $\alpha, \beta > 0$ then the correspondence $u_0 \to u_{\alpha,\beta}$ is analytic.*

Proof. We define the mapping

$$\Lambda : H^s \times E^s_{\alpha,\beta} \to E^s_{\alpha,\beta}$$

given by

$$\Lambda(u_0, v(t)) = v(t) - U_{\alpha,\beta}(t)u_0 - \int_0^t U_{\alpha,\beta}(t-\tau)(vv_x)(\tau)\, d\tau.$$

Due to (2.6), Λ is smooth for $s > \frac{n}{2} + 1$ and $\alpha, \beta > 0$. Let $\Lambda(u_0, u(t)) = 0$, which is to say, suppose $u(t)$ is a solution of (2.3) with $u(0) = u_0$. Then taking the Fréchet derivative with respect to the second variable, we have

$$\Lambda'_u(u_0, u(t))\phi = \phi - \int_0^t U_{\alpha,\beta}(t-\tau)(v\phi)_x(\tau)\, d\tau.$$

But we know that

$$\|U_{\alpha,\beta}(t-\tau)(u\phi)_x(\tau)\, d\tau\|_{H^s} \le C_{\left(\alpha,\beta,s,T^s_{\alpha,\beta}\right)}\|u\|_{H^s}\|\phi\|_{H^s}.$$

It is deduced that for $T^s_{\alpha,\beta}$ small enough $\Lambda'_u(u_0, u(t))$ is invertible since it is of the form of $I + \Theta$ such that $\|\Theta\|_X < 1$, where $X = \mathcal{L}\left(E^s_{\alpha,\beta}, E^s_{\alpha,\beta}\right)$. The proof is complete by the Implicit Function Theorem. ∎

2.5 Weighted Spaces

Now we are going to obtain some result in weighted spaces. We state our results in two dimensional case. By some easy calculation, one can obtain the following useful lemma.

LEMMA 2.5.1 *Let $\mathcal{W}$ be a function with all its first and second derivatives bounded and such that*

$$|\mathcal{W}(x,y)| \le C_\varepsilon e^{\varepsilon\left(x^2+y^2\right)}$$

for all $(x,y) \in \mathbb{R}^2$ and any $\varepsilon \in (0, \widetilde{\varepsilon})$, for some $\widetilde{\varepsilon} > 0$ and $C_\varepsilon > 0$. Then there exist the constants $C_1, \cdots, C_5 > 0$, independent of ε such that

$$\|\nabla \mathcal{W}_\varepsilon\|_{L^\infty} \le C_1 \|\nabla \mathcal{W}\|_{L^\infty} + C_2, \tag{2.28}$$

$$\|D^\omega \mathcal{W}_\varepsilon\|_{L^\infty} \le C_3 \|\nabla \mathcal{W}\|_{L^\infty} + C_4 \|D^\omega \mathcal{W}\|_{L^\infty} + C_5, \tag{2.29}$$

where $\mathcal{W}_\varepsilon(x,y) = \mathcal{W}(x,y)\exp\left(-\varepsilon\left(x^2+y^2\right)\right)$ and $\omega \in \mathbb{R}^2$ such that $|\omega| = 2$.

THEOREM 2.5.2 *Let $u_0 \in H^s\left(\mathcal{W}^2\right)$, $s > 2$ and $\mathcal{W}$ be a weight function as in Lemma 2.5.1. Then the solution $u_{\alpha,\beta}$ of the equation (2.3) corresponding to the initial data u_0 is in $C\left(\left[0, T^s_{\alpha,\beta}\right); H^s\left(\mathcal{W}^2\right)\right)$. Moreover, the continuous dependence of solutions of the equation (2.3) holds in $H^s\left(\mathcal{W}^2\right)$.*

Proof. By Theorem 2.4.1, it suffices to prove that $\|\mathcal{W}u(t)\|_{L^2}$ remains bounded as long as $t \in [0,T]$ for any $T \in \left(0, T^s_{\alpha,\beta}\right)$. By the hypothesis, $\|\mathcal{W}_\varepsilon u_{\alpha,\beta}\|_{L^2} < \infty$ and $\|\mathcal{W}_\varepsilon \partial_t u_{\alpha,\beta}\|_{L^2} < \infty$, for all $t \in (0,T]$. By using the equation (2.3), we obtain that

$$\frac{d}{dt}\|\mathcal{W}_\varepsilon u_{\alpha,\beta}\|^2_{L^2} = 2\left\langle \mathcal{W}_\varepsilon u_{\alpha,\beta}, \mathcal{W}_\varepsilon \partial_x \Delta u_{\alpha,\beta} - \mathcal{W}_\varepsilon u^p u_x + \alpha \mathcal{W}_\varepsilon \partial_x^2 u_{\alpha,\beta} + \beta \mathcal{W}_\varepsilon \Delta_\perp u_{\alpha,\beta}\right\rangle.$$

On the other hand, it is simple to see that

$$\langle \mathcal{W}_\varepsilon u_{\alpha,\beta}, \mathcal{W}_\varepsilon \partial_x \Delta u_{\alpha,\beta}\rangle = \langle \mathcal{W}_\varepsilon u_{\alpha,\beta}, [\mathcal{W}_\varepsilon, \partial_x]\Delta u_{\alpha,\beta}\rangle + \langle \mathcal{W}_\varepsilon u_{\alpha,\beta}, \partial_x[\mathcal{W}_\varepsilon, \Delta] u_{\alpha,\beta}\rangle, \tag{2.30}$$

$$\left\langle \mathcal{W}_\varepsilon u_{\alpha,\beta}, \mathcal{W}_\varepsilon \partial_x^2 u_{\alpha,\beta}\right\rangle + \|\partial_x(\mathcal{W}_\varepsilon u_{\alpha,\beta})\|^2_{L^2} = \left\langle \mathcal{W}_\varepsilon u_{\alpha,\beta}, \left[\mathcal{W}_\varepsilon, \partial_x^2\right] u_{\alpha,\beta}\right\rangle, \tag{2.31}$$

$$\langle \mathcal{W}_\varepsilon u_{\alpha,\beta}, \mathcal{W}_\varepsilon \Delta_\perp u_{\alpha,\beta}\rangle + \|\nabla_\perp(\mathcal{W}_\varepsilon u_{\alpha,\beta})\|^2_{L^2} = \langle \mathcal{W}_\varepsilon u_{\alpha,\beta}, [\mathcal{W}_\varepsilon, \triangle_\perp]\, u_{\alpha,\beta}\rangle, \tag{2.32}$$

$$\left\|\mathcal{W}_\varepsilon u^p_{\alpha,\beta}\partial_x u_{\alpha,\beta}\right\|_{L^2} \le \left\|u^{p-1}\partial_x u_{\alpha,\beta}\right\|_{L^\infty} \|\mathcal{W}_\varepsilon u_{\alpha,\beta}\|_{L^2} \le \mathscr{A}_T^p \|\mathcal{W}_\varepsilon u_{\alpha,\beta}\|_{L^2}. \tag{2.33}$$

By using the integration by parts and Lemma 2.5.1, one obtain that

$$\begin{aligned}\frac{d}{dt}\|\mathcal{W}_\varepsilon u_{\alpha,\beta}\|^2_{L^2} &\le 2\left(C\|u_{\alpha,\beta}\|_{H^s}\|\mathcal{W}_\varepsilon u_{\alpha,\beta}\|_{L^2} + \mathscr{A}_T^p\|\mathcal{W}_\varepsilon u_{\alpha,\beta}\|^2_{L^2}\right)\\ &\le C\mathscr{A}_T^2 + (1+\mathscr{A}_T^p)\|\mathcal{W}_\varepsilon u_{\alpha,\beta}\|^2_{L^2};\end{aligned}$$

and by Gronwall's inequality, it follows that

$$\|\mathcal{W}_\varepsilon u_{\alpha,\beta}\|^2_{L^2} \le e^{1+\mathscr{A}_T^p}\left(\|\mathcal{W}_\varepsilon u_0\|^2_{L^2} + T\,(C\mathscr{A}_T)^2\right).$$

Applying now the Monotone Convergence Theorem yields that

$$\|\mathcal{W} u_{\alpha,\beta}\|^2_{L^2} \le e^{1+\mathscr{A}_T^p}\left(\|\mathcal{W}_\varepsilon u_0\|^2_{L^2} + T\,(C\mathscr{A}_T)^2\right).$$

Thus $u_{\alpha,\beta} \in H^s\left(\mathcal{W}^2\right)$ for all $t \in \left(0, T^s_{\alpha,\beta}\right)$. The continuity and dependence continuity can be derived from analogous estimates, similar to Theorem 2.4.1. ∎

The following theorem shows the persistence of the solutions of the equation 2.3 in the weighted spaces $\mathcal{F}^{s,2}_{1,s}$.

THEOREM 2.5.3 *Let $s \in \mathbb{N}$, $s \ge 3$ and $\beta \ge 0$. Also suppose that $u_{\alpha,\beta} \in C\left(\left[0, T^s_{\alpha,\beta}\right); H^s\right)$ is maximal solution of the equation (1.6) corresponding to the initial data $\mathcal{F}^{s,2}_{1,s}$. Then $u_{\alpha,\beta} \in C\left(\left[0, T^s_{\alpha,\beta}\right); \mathcal{F}^{s,2}_{1,s}\right)$.*

Proof. Take $\alpha, \beta > 0$. Note that $\mathcal{F}^{s,2}_{r_1,r_2}$ is a Banach algebra, for $s > 1$ and $r_1, r_2 > 0$. Thus, as in the proof of the Theorem 2.4.1 and using the Lemma 2.3.8, one obtain a local solution of the equation (2.3) in the complete space

$$\mathcal{E}\left(\widetilde{T}\right) = \left\{v \in C\left(\left[0, \widetilde{T}\right]; \mathcal{F}^{s,2}_{1,s}\right) \;:\; |v|_{\mathcal{E}} \le \|u_0\|_{\mathcal{F}^{s,2}_{1,s}}\right\},$$

where $|v|_{\mathcal{E}} = \sup\limits_{t\in[0,\widetilde{T}]} \|v(t) - U_{\alpha,\beta}(t)u_0\|_{\mathcal{F}^{s,2}_{1,s}}$, for some $\widetilde{T} > 0$. By the uniqueness of the solution in H^s, this solution must be $u_{\alpha,\beta}$. Therefore, it only remains to prove that it belongs to $\mathcal{F}^{0,2}_{0,s}$ for $t \in \left[0, T^s_{\alpha,\beta}\right)$, by using the Theorem 2.5.2. So it is enough to estimate $\|y^s u_{\alpha,\beta}\|_{L^2}$ for all $t \in \left[0, \widetilde{T}\right]$. Note that for $t \in \left[0, \widetilde{T}\right]$, $y^s u_{\alpha,\beta}^p \partial_x u_{\alpha,\beta}(t) \in L^2$, since

$$\left\| y^s u_{\alpha,\beta}^p \partial_x u_{\alpha,\beta}(t) \right\|_{L^2} \le C(s) \mathscr{A}_{\widetilde{T}}^p \left\| y^s u_{\alpha,\beta}(t) \right\|_{L^2}.$$

By using (2.16) and Lemma 2.5.1, one can easily see that $y^s D^\omega u_{\alpha,\beta}(t)$ and $y^s \partial_t u_{\alpha,\beta}(t)$ are in L^2 for all $t \in \left(0, \widetilde{T}\right]$ and $|\omega| = 2$. Therefore, using some integration by parts, one obtains that

$$\partial_t \left\| y^s u_{\alpha,\beta}(t) \right\|_{L^2}^2 \le C \left\| y^s u_{\alpha,\beta}(t) \right\|_{L^2}^2,$$

where $C = C(s, \alpha, \beta)$ is increasing in α and β. Thusly

$$\left\| y^s u_{\alpha,\beta}(t) \right\|_{L^2}^2 \le e^{C\widetilde{T}} \left(\left\| y^s u_0 \right\|_{L^2}^2 + C\widetilde{T} \mathscr{A}_{\widetilde{T}}^2 \right). \tag{2.34}$$

This allows us to extend the solution $u_{\alpha,\beta}$ to its interval of existence in H^s.
For the case $\beta = 0$, observe that by (2.34) and the Theorem 2.4.1, there exists $T > 0$ such that

$$\left\| y^s u_{\alpha,\beta}(t) \right\|_{L^2}^2 \le C \left(T, s, \widetilde{\beta}, \|u_0\|_{\mathcal{I}_{1,s}^{s,2}} \right),$$

for any $\beta \in \left(0, \widetilde{\beta}\right)$, $\widetilde{\beta} > 0$ arbitrary and for any $t \in [0, T]$. Then for any $t \in [0, T]$, there exists a sequence β_n such that $u_{\alpha,\beta_n}(t) \rightharpoonup v(t)$ in $L^2\left(\left(1 + y^{2s}\right) dxdy\right)$, for some $v(t) \in L^2\left(\left(1 + y^{2s}\right) dxdy\right)$. But since $L^2\left(\left(1 + y^{2s}\right) dxdy\right) \hookrightarrow L^2$, $v = u_{\alpha,0}$. The inequality (2.34) follows also for $\beta = 0$ by the weak lower semicontinuity of the norm $L^2\left(\left(1 + y^{2s}\right) dxdy\right)$. Finally, an extension argument yields the result. ■

REMARK 2.5.4 *Note that we are able to prove that the ill-posedness of the ZKB equation the anisotropic spaces; indeed , in [27], we proved that, when $p = 1$, the ZKB equation (2.3) is ill-posed (in some sense) in $H^{s,0}\left(\mathbb{R}^2\right)$ for $s < -\frac{3}{4}$. Further more we obtained some explicit traveling wave solutions of (2.3), by using the improved tanh method.*

2.6 Equation with Higher Order Dissipation

In this section, We are going to investigate the Cauchy problem of the Benney equation :

$$u_t + uu_x + \alpha u_{xx} + \Delta u_x + \beta \Delta^2 u = 0, \quad (x, y) \in \mathbb{R}^2, \tag{2.35}$$

where $\alpha > 0$, $\beta > 0$, and $\Delta^2 = \left(\partial_x^2 + \partial_y^2\right)^2$. In fact we wish to obtain local and global well posedness of initial value problem of (2.35) in Sobolev spaces. If we assume that $u(t)$ is sufficiently regular in $[0, T]$, then

PROPOSITION 2.6.1 *For any $t \in [0, T]$,*

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \beta \int_0^t \int_{\mathbb{R}^2} u_{xx}^2(t') + 2u_{xy}^2(t') + u_{yy}^2(t')\, dxdydt' = \alpha \int_0^t \int_{\mathbb{R}^2} u_x^2(t')\, dxdydt' + \frac{1}{2}\|u_0\|_{L^2}^2.$$

The linear problem associated to equation (2.35) is :

$$\begin{cases} u_t + \Delta u_x + \alpha u_{xx} + \beta \Delta^2 u = 0, & (x,y) \in \mathbb{R}^2, \ \ t \in \mathbb{R}, \\ \\ u(x,y,0) = u_0(x,y) \end{cases} \tag{2.36}$$

where $u_0 \in H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$. The unique solution of (2.36) is given by the semigroup $\{U(t)\}_{t\geq 0}$, that is,

$$u(t) = U(t)u_0 = \int_{\mathbb{R}^2} e^{t(i\xi^3 + i\xi\eta^2 + \alpha\xi^2 - \beta(\xi^2+\eta^2)^2)} e^{i(x\xi + y\eta)} \ \hat{u}_0(\xi,\eta) \ d\xi d\eta.$$

It is convenient to define :

$$K(\xi,\eta,t) = e^{it\left(\xi^3 + \xi\eta^2 - i\alpha\xi^2 + i\beta\left(\xi^2+\eta^2\right)^2\right)}.$$

In fact, $\widehat{U(t)u_0} = K(\xi,\eta,t)\widehat{u}_0$. The following lemma provides a very useful inequality.

LEMMA 2.6.2 *For any $\alpha, \beta > 0$ and $a > 0$, we have for all $\xi, \eta \in \mathbb{R}$,*

$$F(\xi,\eta) = \left(\xi^2 + \eta^2\right)^a \ e^{\alpha\xi^2 - \beta\left(\xi^2+\eta^2\right)^2} \leq \varrho^a \ e^{\frac{1}{2}(\alpha\varrho - a)}, \tag{2.37}$$

where $\varrho = \dfrac{\alpha + \sqrt{\alpha^2 + 8a\beta}}{4\beta}$.

Proof. By calculating the gradient of the function F and its Hessian matrix we obtain that $(\xi,\eta)_{\max} = (\sqrt{\varrho}, 0)$. ■

Now, the following properties of $U(t)$ can be derived.

LEMMA 2.6.3 *Let $s, \lambda, \alpha \geq 0$ and $\beta > 0$. Then $U(t) \in \mathcal{L}\left(H^s\left(\mathbb{R}^2\right), H^{s+\lambda}\left(\mathbb{R}^2\right)\right)$ for all $t > 0$ and satisfies the estimate*

$$\|U(t)f\|_{H^{s+\lambda}} \lesssim \left(e^{\frac{\alpha^2 t}{4\beta}} + \left(\frac{\alpha}{4\beta}\right)^{\lambda/2} \left[1 + \sqrt{1 + \frac{4\lambda\beta}{\alpha^2 t}}\right]^{\lambda/2} e^{\frac{\alpha^2 t}{4\beta}\left(1 + \sqrt{1 + \frac{4\lambda\beta}{\alpha^2 t}}\right)} \right) \|f\|_{H^s}, \tag{2.38}$$

for all $f \in H^s(\mathbb{R}^2)$, where $\lesssim$ means the inequality needs to a constant depending on λ. Moreover, the map $t \in (0,\infty) \longmapsto U(t)f$ is continuous with respect to the topology of $H^{s+\lambda}(\mathbb{R}^2)$.

Proof. The first part and (2.38) is a direct consequent of Lemma 2.6.2. For the continuity result, assume, without loss of generality, that $t > t'$ and apply the dominated convergence theorem to deduce that

$$\begin{aligned} &\|U(t)f - U(t')f\|^2_{H^{s+\lambda}} \\ &= \int_{\mathbb{R}^2} \left(1 + \xi^2 + \eta^2\right)^{s+\lambda} \left[e^{t\left(\alpha\xi^2 - \beta\left(\xi^2+\eta^2\right)^2\right)} - e^{t'\left(\alpha\xi^2 - \beta\left(\xi^2+\eta^2\right)^2\right)}\right]^2 \left|\widehat{f}(\xi,\eta)\right|^2 \ d\xi d\eta \\ &= \int_{\mathbb{R}^2} \left(1 + \xi^2 + \eta^2\right)^{s+\lambda} \ e^{t\left(\alpha\xi^2 - \beta\left(\xi^2+\eta^2\right)^2\right)} \left[1 - e^{(t-t')\left(\alpha\xi^2 - \beta\left(\xi^2+\eta^2\right)^2\right)}\right]^2 \left|\widehat{f}(\xi,\eta)\right|^2 \ d\xi d\eta \end{aligned}$$

tens to zero as $t \to t'$. ■

In fact, Lemma 2.6.3 expresses a regularizing property of the semigroup $U(t)$. Now we state our result on local well-posedness in $H^s(\mathbb{R}^2)$.

THEOREM 2.6.4 *Let $\alpha > 0$, $\beta > 0$ be fixed and suppose $u_0 \in H^s(\mathbb{R}^2)$ to be given, where $s > 1$. Then there exists $T_s > 0$ depending on s, $\|u_0\|_{H^s(\mathbb{R}^2)}$, α and β; and a unique solution $u(t)$ of the equation (2.35) such that $u(0) = u_0$ and*

$$u(t) \in C\left([0,T_s]; H^s\left(\mathbb{R}^2\right)\right).$$

Proof. We define the operator

$$\Phi(u(t)) = U(t)u_0 + \int_0^t U(t-t')u(t')u_x(t')\,dt' \tag{2.39}$$

and the complete metric space

$$E(T_s) = \left\{u \in C\left([0,T_s]; H^s\left(\mathbb{R}^2\right)\right) \;:\; |u|_E \leq \|u_0\|_{H^s}\right\}, \tag{2.40}$$

where $|u|_E = \sup\limits_{t\in[0,T_s]} \|u(t) - U(t)u_0\|_{H^s}$. Let $u \in E(T_s)$. By the Hölder inequality and (2.38)(Lemma 2.6.3), we have

$$\begin{aligned}
\|\Phi(u(t)) - U(t)u_0\|_{H^s} &\leq c\int_0^t \|U(t-t')u^2(t')\|_{H^{s+1}}\,dt' \\
&\leq c\int_0^t \left[e^{\frac{\alpha^2 t'}{4\beta}} + \sqrt{\frac{\alpha}{4\beta}}\left(1+\sqrt{1+\frac{4\beta}{\alpha^2 t'}}\right)^{1/2} e^{\frac{\alpha^2 t'}{4\beta}\left(1+\sqrt{1+\frac{4\beta}{\alpha^2 t'}}\right)}\right]\|u\|_{H^s}^2\,dt' \\
&\leq c\|u_0\|_{H^s(\mathbb{R}^2)}^2 \int_0^t \left[e^{\frac{\alpha^2 t'}{4\beta}} + \sqrt{\frac{\alpha}{4\beta}}\left(1+\sqrt{1+\frac{4\beta}{\alpha^2 t'}}\right)^{1/2} e^{\frac{\alpha^2 t'}{4\beta}\left(1+\sqrt{1+\frac{4\beta}{\alpha^2 t'}}\right)}\right]\,dt' \\
&= c\|u_0\|_{H^s(\mathbb{R}^2)}^2 \Upsilon(t).
\end{aligned}$$

Therefore $\Phi(u) \in E(T_s)$ for T_s small enough; in fact for this T_s, $\Phi(E(T_s)) \subset E(T_s)$. One can also see that $\Upsilon(T_s)$ tends to zero as T_s tends to zero. A similar computation shows that Φ is a contraction (by choosing T_s smaller if necessary). In fact, for $t \in [0,T_s]$, one has

$$\begin{aligned}
&\|\Phi(u(t)) - \Phi(v(t))\|_{H^s} \leq c\sup_{t\in[0,T_s]} \left\|u^2(t) - v^2(t)\right\|_{H^s}\Upsilon(T_s) \\
&\leq c\sup_{t\in[0,T_s]} \|u(t)-v(t)\|_{H^s}(\|u(t)\|_{H^s} + \|v(t)\|_{H^s})\Upsilon(T_s) \leq c\sup_{t\in[0,T_s]} \|u(t)-v(t)\|_{H^s}\left(e^{\frac{\alpha^2 T_s}{2\beta}}+1\right)\|u_0\|_{H^s}\Upsilon(T_s),
\end{aligned}$$

where we used this fact that $\|u(t)\|_{H^s} \leq \left(e^{\frac{\alpha^2 T_s}{4\beta}}+1\right)\|u_0\|_{H^s}$, by (2.38). Choosing T_s small enough gives us the contraction. So there exists a unique solution $u(t)$ of the equation (2.35) with initial data u_0. ■

PROPOSITION 2.6.5 *Let $u \in C([0,T_s]; H^s(\mathbb{R}^2))$ be the solution of equation (2.35) with initial condition u_0 in $H^s(\mathbb{R}^2)$, where $s > 1$, $\alpha > 0$ and $\beta > 0$. Then $u \in C([0,T_s]; H^\infty(\mathbb{R}^2))$.*

Proof. The proof follows from an easy bootstrapping argument similar what we mentioned in the equation (2.3), by using Lemma 2.6.3. ■

Now we are going to obtain a global a priori estimates that enable the local solutions to be extended to temporal half line $[0, \infty)$.

LEMMA 2.6.6 *Consider the initial value problem (2.35) with initial data $u_0 \in H^k\left(\mathbb{R}^2\right)$ for some integer $k \geq 0$. Let u be a solution of (2.35) in $C([0,T]; H^k\left(\mathbb{R}^2\right))$ for some $T > 0$. Then we have*

$$\|u\|_{L^2(\mathbb{R}^2)} \leq e^{cT}\|u_0\|_{L^2(\mathbb{R}^2)}, \tag{2.41}$$

if there exists a constant $\varepsilon > 0$ such that $\varepsilon \leq \beta$. Furthermore, we have

$$\|u\|_{H^j(\mathbb{R}^2)} \leq g\left(\|u_0\|_{H^{j-1}(\mathbb{R}^2)}\right)\|u_0\|_{H^j(\mathbb{R}^2)}, \tag{2.42}$$

for $j = 1, \cdots$, where $g(\|u_0\|_{H^{j-1}(\mathbb{R}^2)})$ is a nondecreasing function of $\|u_0\|_{H^{j-1}(\mathbb{R}^2)}$, α, β and T.

Proof. We begin by proving the lemma in L^2-norm. Multiplying the equation (2.35) by u and integrating over $\mathbb{R}^2$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \alpha\langle u, u_{xx}\rangle + \beta\langle u, \Delta^2 u\rangle = 0, \tag{2.43}$$

where the inner product is that of L^2. Integration by parts and the Cauchy-Schwarz inequality then imply

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \leq \alpha\|u\|_{L^2}\|u_{xx}\|_{L^2} - \beta\|\Delta u\|_{L^2}^2.$$

By using Young's inequality, for any $\varepsilon > 0$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \leq \frac{\alpha^2}{4\varepsilon}\|u\|_{L^2}^2 + (\varepsilon - \beta)\|u_{xx}\|_{L^2}^2.$$

By using the hypothesis and applying Gronwall's lemma, we conclude

$$\|u\|_{L^2} \leq e^{\frac{\alpha^2 T}{4\varepsilon}}\|u_0\|_{L^2}. \tag{2.44}$$

In H^1 space, by differentiating the equation (2.35) with respect to x and y; and multiplying them by u_x and u_y respectively, we obtain that

$$\frac{1}{2}\frac{d}{dt}\left(\|u_x\|_{L^2}^2 + \|u_y\|_{L^2}^2\right) + \langle u_x, (uu_x)_x\rangle + \langle u_y, (uu_x)_y\rangle + \alpha(\langle u_x, u_{xxx}\rangle + \langle u_y, u_{xxy}\rangle) + \beta\langle\nabla u, \nabla\Delta^2 u\rangle = 0.$$

Integration by parts and the Cauchy-Schwarz inequality then imply

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|u_x\|_{L^2}^2 + \|u_y\|_{L^2}^2\right) &\leq |\langle u^2, u_{xxx}\rangle| + |\langle u^2, u_{xyy}\rangle| + \alpha\left(\|u_x\|_{L^2}\|u_{xxx}\|_{L^2} + \|u_x\|_{L^2}\|u_{xyy}\|_{L^2}\right)\\ &\quad - \beta\left(\|u_{xxx}\|_{L^2}^2 + 3\|u_{xxy}\|_{L^2}^2 + 3\|u_{xyy}\|_{L^2}^2 + \|u_{yyy}\|_{L^2}^2\right).\end{aligned}$$

By using the Cauchy-Schwarz inequality, Young's inequality and the Gagliardo-Nirenberg inequality, we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|u_x\|_{L^2}^2+\|u_y\|_{L^2}^2\right) &\le \|u\|_{L^4}^2\|u_{xxx}\|_{L^2}+\|u\|_{L^4}^2\|u_{xyy}\|_{L^2}\\ &\quad+\varepsilon_1\alpha\|u_x\|_{L^2}^2+c_{\varepsilon_1}\alpha\|u_{xxx}\|_{L^2}^2+\varepsilon_2\alpha\|u_x\|_{L^2}^2+c_{\varepsilon_2}\alpha\|u_{xyy}\|_{L^2}^2-\beta\|u_{xxx}\|_{L^2}^2-3\beta\|u_{xyy}\|_{L^2}^2\\ &\le \varepsilon_3\|u\|_{L^4}^4+c_{\varepsilon_3}\|u_{xxx}\|_{L^2}^2+\varepsilon_4\|u\|_{L^4}^4+c_{\varepsilon_4}\|u\|_{L^2}^2\\ &\quad+(\varepsilon_1+\varepsilon_2)\alpha\|u_x\|_{L^2}^2+c_{\varepsilon_1}\alpha\|u_{xxx}\|_{L^2}^2+c_{\varepsilon_2}\alpha\|u_{xyy}\|_{L^2}^2-\beta\|u_{xxx}\|_{L^2}^2-3\beta\|u_{xyy}\|_{L^2}^2\\ &\le(\varepsilon_3+\varepsilon_4)\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2+(c_{\varepsilon_1}+c_{\varepsilon_3}-\beta)\|u_{xxx}\|_{L^2}^2\\ &\quad+(c_{\varepsilon_4}+c_{\varepsilon_2}\alpha-3\beta)\|u_{xyy}\|_{L^2}^2+(\varepsilon_1+\varepsilon_2)\alpha\|u_x\|_{L^2}^2\\ &=\kappa_1\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2+\kappa_2\|u_{xxx}\|_{L^2}^2+\kappa_3\|u_{xyy}\|_{L^2}^2+\kappa_4\|u_x\|_{L^2}^2.\end{aligned}$$

By choosing ε_1, ε_2, ε_3 and ε_4 suitably such that $\kappa_2,\kappa_3\le 0$, applying Gronwall's lemma and using (2.44) we obtain (2.42). Now by applying the operators ∂_x^2, ∂_y^2 and $\partial_x\partial_y$ on the equation (2.35) and multiplying them by u_{xx}, u_{yy} and u_{xy} respectively, we obtain that

$$\frac{1}{2}\frac{d}{dt}\sum_{|j|=2}\|D^ju\|_{L^2}^2+\langle uu_x,h(u)\rangle+\alpha\langle u_{xx},h(u)\rangle+\beta\langle\Delta^2u,h(u)\rangle=0, \tag{2.45}$$

where $h(u)=u_{xxxx}+u_{xxyy}+u_{yyyy}$. By integration by parts, the Cauchy-Schwarz inequality and the Gagliardo-Nirenberg inequality, we have that

$$\begin{aligned}|\langle uu_x,u_{xxxx}\rangle|=\frac{5}{2}|\langle u_x,u_{xx}^2\rangle|&\le\|u_x\|_{L^2}\|u_{xx}\|_{L^4}^2\le\|u_x\|_{L^2}^2\|u_{xx}\|_{L^2}^2\|\nabla u_{xx}\|_{L^2}^2\\ &\le\|u_x\|_{L^2}^2\|u_{xx}\|_{L^2}^2\|u_{xxx}\|_{L^2}^2+\|u_x\|_{L^2}^2\|u_{xx}\|_{L^2}^2\|u_{xxy}\|_{L^2}^2.\end{aligned} \tag{2.46}$$

On the other hand, by using Young's inequality and the Gagliardo-Nirenberg inequality, we have

$$\|u_x\|_{L^2}^2\|u_{xx}\|_{L^2}^2\|u_{xxx}\|_{L^2}^2\lesssim\varepsilon_1\|u_{xx}\|_{L^2}^2+c_{\varepsilon_1}g^{3/2}(u)\|u\|_{L^2}^{1/2}\|u_x\|_{L^2}^2\lesssim\varepsilon_1\|u_{xx}\|_{L^2}^2+c_{\varepsilon_1}g^2(u)\|u\|_{L^2}^2, \tag{2.47}$$

where $g(u)=\sum_{|j|=4}\|D^ju\|_{L^2}$. Similarly, we obtain

$$\|u_x\|_{L^2}^2\|u_{xx}\|_{L^2}^2\|u_{xxy}\|_{L^2}^2\lesssim\varepsilon_2\|u_{xx}\|_{L^2}^2+c_{\varepsilon_2}g^2(u)\|u\|_{L^2}^2, \tag{2.48}$$

for any $\varepsilon_2>0$. Therefore,

$$|\langle uu_x,u_{xxxx}\rangle|\lesssim(\varepsilon_1+\varepsilon_2)\|u_{xx}\|_{L^2}^2+(c_{\varepsilon_1}+c_{\varepsilon_2})g^2(u)\|u\|_{L^2}^2. \tag{2.49}$$

Also we can analogously obtain

$$|\langle uu_x,u_{yyyy}\rangle|\lesssim(\varepsilon_3+\varepsilon_4)\|u_{xx}\|_{L^2}^2+(c_{\varepsilon_3}+c_{\varepsilon_4})g^2(u)\|u\|_{L^2}^2, \tag{2.50}$$

$$|\langle uu_x,u_{xxyy}\rangle|\lesssim(\varepsilon_5+\varepsilon_6)\|u_{xx}\|_{L^2}^2+(c_{\varepsilon_5}+c_{\varepsilon_6})g^2(u)\|u\|_{L^2}^2. \tag{2.51}$$

Thusly, by using Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{|j|=2}\|D^ju\|_{L^2}^2&\le\alpha((\varepsilon_7+\varepsilon_8+\varepsilon_9)\|u_{xx}\|_{L^2}^2+c_{\varepsilon_7}\|u_{xxxx}\|_{L^2}^2\\ &\quad+c_{\varepsilon_8}\|u_{yyyy}\|_{L^2}^2+c_{\varepsilon_9}\|u_{xxyy}\|_{L^2}^2)-\beta(\|u_{xxxx}\|_{L^2}^2+3\|u_{xxxy}\|_{L^2}^2\\ &\quad+4\|u_{xxyy}\|_{L^2}^2+\|u_{yyyy}\|_{L^2}^2+3\|u_{xyyy}\|_{L^2}^2)-\langle uu_x,h(u)\rangle,\end{aligned}$$

for any $\varepsilon_1, \cdots, \varepsilon_9 > 0$. Now by (2.49), (2.50), (2.51) and the above inequality and choosing $\varepsilon_1, \cdots, \varepsilon_9 > 0$ suitably, we obtain (2.44).
In general, for $j \geq 2$ in $\mathbb{N}$, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^j}^2 + \langle u, uu_x\rangle_{H^j} + \langle u, \Delta u_x\rangle_{H^j} + \alpha\langle u, u_{xx}\rangle_{H^j} + \beta\langle u, \Delta^2 u\rangle_{H^j} = 0, \tag{2.52}$$

where $\langle u, v\rangle_{H^j} = \sum_{|\ell|\leq j}\left\langle D^\ell u, D^\ell v\right\rangle_{L^2}$. Therefore, by integration by parts, the Cauchy-Schwarz inequality, the Gagliardo-Nirenberg inequality and Young's inequality we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{H^j}^2 &\leq |\langle u, uu_x\rangle| + \sum_{\substack{|\ell|\leq j\\ |\ell'|=|\ell|+2}}\left(\alpha\|D^\ell u\|_{L^2}^2\|D^{\ell'}u\|_{L^2}^2 - \beta\|D^{\ell'}u\|_{L^2}^2\right)\\ &\lesssim \|u\|_{L^2}\|u\|_{H^j}^2 + \sum_{|\ell|\leq j}\|D^\ell u\|_{L^2}^2 = (\|u\|_{L^2}^2 + 1)\|u\|_{H^j}^2.\end{aligned} \tag{2.53}$$

The Gronwall inequality implies (2.44).

■

The Global well posedness follows from the local theory and the a priori estimates obtained in the previous lemma, when $s \geq 2$ is integer. For non-integer values of s, nonlinear interpolation theory is applied. We will use the following theorems [15].

THEOREM 2.6.7 *Let B_0^j, B_1^j be Banach spaces such that $B_0^j \supset B_1^j$ with inclusion mapping, $j = 1, 2$. Let λ, q lie in the ranges $0 < \lambda < 1$, $1 \leq q \leq \infty$. Suppose that $A : B_{\lambda,q}^1 \longrightarrow B_0^2$ and for $f, g \in B_{\lambda,q}^1$,*

$$\|Af - Ag\|_{B_0^2} \leq c_0\left(\|f\|_{B_{\lambda,q}^1} + \|g\|_{B_{\lambda,q}^1}\right)\|f - g\|_{B_0^1},$$

and $A : B_1^1 \longrightarrow B_1^2$ and for $h \in B_1^1$,

$$\|Ah\|_{B_1^2} \leq c_1\left(\|h\|_{B_{\lambda,q}^1}\right)\|h\|_{B_1^1},$$

where $c_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous nondecreasing functions, $i = 0, 1$. Then if $(\theta, p) \geq (\lambda, q)$, A maps $B_{\theta,p}^1$ into $B_{\theta,p}^2$ and for $f \in B_{\theta,p}^1$, $\|Af\|_{B_{\theta,p}^2} \leq c\left(\|f\|_{B_{\lambda,q}^1}\right)\|f\|_{B_{\theta,p}^1}$, where for $\gamma > 0$, $c(\gamma) = 4c_0(4\gamma)^{1-\theta}c_1(3\gamma)^\theta$.

THEOREM 2.6.8 *Let B_i^j, λ,q and A, $i = 0, 1$, $j = 1, 2$ be as in Theorem 2.6.7. Assume that the pair B_0^1, B_1^1 has a (θ, p) approximate identity for some $(\theta, p) \geq (\lambda, q)$ and A is continuous as a map of B_1^1 to B_1^2. Then A is a continuous map from $B_{\theta,p}^1$ to $B_{\theta,p}^2$.*

THEOREM 2.6.9 *Let $\alpha > 0$, $\beta > 0$ and $s \geq 2$. Then the equation (2.35) is globally well posed for initial data in $H^s(\mathbb{R}^2)$.*

Proof. Let $k-1 \leq s \leq k$. To use Theorems 2.6.7 and 2.6.8, we put $B_0^1 = L^2(\mathbb{R}^2)$, $B_0^2 = C([0, T]; L^2(\mathbb{R}^2))$, $B_1^1 = H^k(\mathbb{R}^2)$, $B_1^2 = C([0, T]; H^k(\mathbb{R}^2))$, $\lambda = \frac{k-1}{k}$ and $\theta = \frac{s}{k}$. Thus $B_{\lambda,2}^1 = [B_0, H^k(\mathbb{R}^2)]_{\lambda,2} = H^{k-1}(\mathbb{R}^2)$, $B_{\lambda,2}^1 = [B_0, H^k(\mathbb{R}^2)]_{\theta,2} = H^s(\mathbb{R}^2)$, $B_{\theta,2}^1 = C([0, T]; H^{k-1}(\mathbb{R}^2))$ and $B_{\theta,2}^2 = C([0, T]; H^s(\mathbb{R}^2))$. Let Φ be the map which takes the initial data $u_0 \in H^k(\mathbb{R}^2)$ into the unique solution $u \in C([0,T]; H^k(\mathbb{R}^2))$ of the equation (2.35) obtained in Theorem 2.6.4. It follows from Lemma 2.6.6 that

$$\|\Phi(u_0)\|_{H^k} \leq c_1 \left(\|u_0\|_{H^{k-1}}\right) \|u_0\|_{H^k}, \tag{2.54}$$

for all $u_0 \in H^k(\mathbb{R}^2)$, where $c_1 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous, nondecreasing function. Now let $u_0, v_0 \in H^{k-1}(\mathbb{R}^2)$, $u = \Phi(u_0)$, $v = \Phi(v_0)$, $w = u - v$ and $w_0 = u_0 - v_0$. It is easy (see (2.44)) to obtain that $\|w\|_{L^2} \leq c\|w_0\|_{L^2}$, where c depends on α, β and T. Thus Φ is continuous; in fact

$$\|\Phi(u_0) - \Phi(v_0)\|_{C([0,T];L^2(\mathbb{R}^2))} \leq c_0 \|u_0 - v_0\|_{L^2}. \tag{2.55}$$

On the other hand, we have

$$w(t) = U(t)w_0 + \int_0^t U(t-\tau)(u(\tau)u_x(\tau) - v(\tau)v_x(\tau))\, d\tau. \tag{2.56}$$

By using Lemma 2.6.3, it follows that

$$\begin{aligned}
&\|w\|_{H^k} \leq e^{\frac{\alpha^2 t}{4\beta}} \|w_0\|_{H^k} + \int_0^t \|U(t-\tau)\partial_x(u^2(\tau) - v^2(\tau))\|_{H^k}\, d\tau \\
&\leq e^{\frac{\alpha^2 t}{4\beta}} \|w_0\|_{H^k} + \\
&\int_0^t \left(e^{\frac{\alpha^2 (t-\tau)}{4\beta}} + \mu \left[1 + \sqrt{1 + \frac{1}{\mu(t-\tau)}}\,\right]^{1/2} e^{-\frac{1}{2} + \alpha\mu(t-\tau)\left[1+\sqrt{1+\frac{1}{\mu(t-\tau)}}\,\right]} \right) \|u^2 - v^2\|_{H^k}\, d\tau \\
&\lesssim e^{\frac{\alpha^2 t}{4\beta}} \|w_0\|_{H^k} + \\
&\sup_t \left(\|u\|_{H^k} + \|v\|_{H^k}\right) \left[e^{\frac{\alpha^2 T}{4\beta}} \left(1 + \left(\frac{T}{\mu}\right)^{1/4}\right) e^{-\frac{1}{2} + \alpha\mu T + \mu\sqrt{\alpha^2 T^2 + 4\alpha\beta T}} \right] \int_0^t \frac{\|w(\tau)\|_{H^k}}{(t-\tau)^{1/4}}\, d\tau,
\end{aligned}$$

where $\mu = \frac{\alpha^2}{\beta}$. Now we use a generalization of Gronwall's inequality (see for example [71]):

LEMMA 2.6.10 *Let $\mathscr{b} > 0$ be a real constant and $\mathscr{f}(t)$ be a nonnegative function locally integrable on $[0, \mathscr{T}]$ (some $\mathscr{T} \leq \infty$) and let $\mathscr{g}(t)$ be a nonnegative, nondecreasing continuous function defined on $[0, \mathscr{T}]$, $\mathscr{g}(t) \leq \mathscr{M}$ (constant). Suppose that $\mathscr{u}(t)$ is nonnegative and locally integrable on $[0, \mathscr{T}]$ with*

$$\mathscr{u}(t) \leq \mathscr{f}(t) + \mathscr{g}(t) \int_0^t (t-\tau)^{\mathscr{b}-1} \mathscr{u}(\tau)\, d\tau$$

on this interval. Then

$$\mathscr{u}(t) \leq \mathscr{f}(t) + \int_0^t \left[\sum_{\ell=1}^{\infty} \frac{(\mathscr{g}(t)\Gamma(\mathscr{b}))^{\ell}}{\Gamma(\ell\mathscr{b})} (t-\tau)^{\ell\mathscr{b}-1} \mathscr{f}(\tau) \right] d\tau \quad 0 \leq t \leq \mathscr{T}.$$

By using the above lemma with $\mathscr{b} = \frac{3}{4}$, there obtains $\|w(t)\|_{H^k} \lesssim e^{\frac{\alpha^2 T}{4\beta}} \|w_0\|_{H^k} \mathcal{Z}_{\frac{3}{4}}(\mathscr{k}t)$, where

$$\mathscr{k} = \left(\sup_{0\le t\le T} (\|u(t)\|_{H^k} + \|v(t)\|_{H^k}) C(\alpha,\beta,T)\Gamma\left(\frac{3}{4}\right)\right)^{\frac{4}{3}} \tag{2.57}$$

$$C(\alpha,\beta,T) = e^{\frac{\alpha^2 T}{4\beta}} \left(1 + \left(\frac{T}{\mu}\right)^{1/4}\right) e^{-\frac{1}{2}+\alpha\mu T+\mu\sqrt{\alpha^2T^2+4\alpha\beta T}} \tag{2.58}$$

$$\mathcal{Z}_\rho(z) = \sum_{\ell=1}^{\infty} \frac{z^{\rho\ell-1}}{\Gamma(\rho\ell)}. \tag{2.59}$$

The proof is completed by using Theorems 2.6.7 and 2.6.8. ■

2.7 Negative Sobolev Indices

In this section, we are going to extend our well-posedness results for the equation (2.35) to the Sobolev spaces with lower indices. In fact, we state the local well-posedness in the following.

THEOREM 2.7.1 *Let $\alpha,\beta > 0$ be fixed and $s > -2$, then for all $u_0 \in H^s(\mathbb{R}^2)$, there exists $T = T\left(\|u_0\|_{H^s(\mathbb{R}^2)}\right) > 0$, a space*

$$\mathcal{X}_T^s \hookrightarrow C\left([0,T]; H^s(\mathbb{R}^2)\right)$$

and a unique solution $u(t)$ of (2.35) such that $u(0) = u_0$. Moreover, u satisfies $u \in C((0,T); H^\infty(\mathbb{R}^2))$ and the map solution

$$\mathbb{F} : H^s(\mathbb{R}^2) \longrightarrow \mathcal{X}_T^s \cap C\left([0,T]; H^s(\mathbb{R}^2)\right), \quad u_0 \mapsto u,$$

is smooth.

In order to prove Theorem 2.7.1, we will make the assumption $-2 < s \le 0$, since the case $0 < s < 2$ follows by similar arguments. Our strategy is again to use a contraction argument on the integral equation associated to (2.35)

$$u(t) = \Phi(u(t)) := U(t)u_0 + \int_0^t U(t-t')u(t')u_x(t')\, dt', \tag{2.60}$$

in some well-adapted function space, where as before $U(t)$ is the semigroup associated to the linear part of (2.35). In order to do this, we will adapt the spaces like ones used by Dix [25] for the dissipative Burgers equation. For $0 < T \le 1$, we define

$$\mathcal{X}_T^s = \left\{ u \in C\left([0,T]; H^s(\mathbb{R}^2)\right) \ : \ \|u\|_{\mathcal{X}_T^s} < \infty \right\},$$

where

$$\|u\|_{\mathcal{X}_T^s} = \sup_{t\in[0,T]} \left(\|u(t)\|_{H^s(\mathbb{R}^2)} + t^{\frac{|s|}{4}} \|u(t)\|_{L^2(\mathbb{R}^2)} \right).$$

First, we will turn our attention to estimate the linear part.

PROPOSITION 2.7.2 *Let $\alpha, \beta > 0$, $0 < T \leq 1$, $s \leq 0$ and $u_0 \in H^s(\mathbb{R}^2)$, then*

$$\sup_{t\in[0,T]} \|U(t)u_0\|_{H^s(\mathbb{R}^2)} \leq e^{\frac{\alpha^2 T}{4\beta}} \|u_0\|_{H^s(\mathbb{R}^2)}, \tag{2.61}$$

and

$$\sup_{t\in[0,T]} t^{\frac{|s|}{4}} \|U(t)u_0\|_{H^s(\mathbb{R}^2)} \lesssim \Upsilon^s_{\alpha,\beta}(T) \|u_0\|_{H^s(\mathbb{R}^2)}, \tag{2.62}$$

where

$$\Upsilon^s_{\alpha,\beta}(t) = e^{\mu t} + \left[\left(\frac{\mu t}{\alpha}\right)^{\frac{|s|}{4}} + \left(\frac{\alpha}{4\beta\mu t}\right)^{\frac{|s|}{4}} \right] e^{\mu t \sqrt{1+\frac{|s|}{2\mu t}}},$$

is continuous nondecreasing function on $[0,1]$, $\mu = \frac{\alpha^2}{4\beta}$ and $\lesssim$ means $\leq$ with a constant depending on s.

Proof. The inequality (2.61) follows immediately from Lemma 2.6.3 with $\lambda = 0$. To prove the inequality (2.62), we first observe , since $0 < T \leq 1$, that

$$t^{\frac{|s|}{4}} \leq \frac{\left(1 + t^{\frac{1}{2}}\xi^2 + t^{\frac{1}{2}}\eta^2\right)^{\frac{|s|}{2}}}{(1+\xi^2+\eta^2)^{\frac{|s|}{2}}},$$

for all $t \in [0, T]$. Hence, by using the Plancherel theorem and the definition of $U(t)$, we deduce that

$$\begin{aligned}
&t^{\frac{|s|}{2}} \|U(t)u_0\|_{L^2(\mathbb{R}^2)} \\
&\leq \left\| \left(1 + t^{\frac{1}{2}}\xi^2 + t^{\frac{1}{2}}\eta^2\right)^{\frac{|s|}{2}} e^{\alpha t\xi^2 - \beta t(\xi^2+\eta^2)^2} (1+\xi^2+\eta^2)^{\frac{s}{2}} |\widehat{u_0}(\xi,\eta)|^2 \right\|_{L^2(\mathbb{R}^2)} \\
&\lesssim \left(\left\| e^{\alpha t\xi^2 - \beta t(\xi^2+\eta^2)^2} \right\|_{L^\infty(\mathbb{R}^2)} + t^{\frac{|s|}{4}} \left\| (\xi^2+\eta^2)^{\frac{|s|}{2}} e^{\alpha t\xi^2 - \beta t(\xi^2+\eta^2)^2} \right\|_{L^\infty(\mathbb{R}^2)} \right) \|u_0\|_{H^s(\mathbb{R}^2)}.
\end{aligned}$$

Lemma 2.6.3 implies the inequality (2.62). ■

Next step is to derive the bilinear estimates.

PROPOSITION 2.7.3 *Let $\alpha, \beta > 0$, $0 \leq t \leq T \leq 1$ and $s \in (-2, 0]$, then*

$$\left\| \int_0^t U(t-t')\partial_x(uv)\, dt' \right\|_{X^s_T} \lesssim e^{\frac{\alpha^2 T}{2\beta}} T^{\frac{1}{2}\left(1+\frac{s}{2}\right)} \|u\|_{X^s_T} \|v\|_{X^s_T} \tag{2.63}$$

for all $u, v \in X^s_T$, where $\lesssim$ means $\leq$ with a constant depending on s.

Proof. Let $0 \leq t \leq T$. We have $(1+\xi^2+\eta^2)^{s/2} \leq (\xi^2+\eta^2)^{s/2}$, since $s \leq 0$. So by using the Minkowski inequality and the definition of $U(t)$, we obtain that

$$\begin{aligned}
\left\| \int_0^t U(t-t')\partial_x(uv) \right\|_{H^s(\mathbb{R}^2)} &\leq \int_0^t \left\| \xi(1+\xi^2+\eta^2)^{\frac{s}{2}} e^{(t-t')\left(\alpha\xi^2 - \beta(\xi^2+\eta^2)^2\right)} (u(t')v(t'))^\wedge(\xi,\eta) \right\|_{L^2(\mathbb{R}^2)} dt' \\
&\leq \int_0^t \left\| \xi(\xi^2+\eta^2)^{\frac{s}{2}} e^{(t-t')\left(\alpha\xi^2 - \beta(\xi^2+\eta^2)^2\right)} \right\|_{L^2(\mathbb{R}^2)} \left\| \widehat{u(t')} * \widehat{v(t')}(\xi,\eta) \right\|_{L^\infty(\mathbb{R}^2)} dt'.
\end{aligned}$$

The Young inequality implies that

$$\left\|\widehat{u(t')}*\widehat{v(t')}(\xi,\eta)\right\|_{L^\infty(\mathbb{R}^2)} \leq \frac{\|u\|_{X_T^s}\|v\|_{X_T^s}}{|t'|^{\frac{|s|}{2}}}.$$

Therefore we obtain

$$\left\|\int_0^t U(t-t')\partial_x(uv)\right\|_{H^s(\mathbb{R}^2)} \leq \left(\int_0^t \left\|\xi\left(\xi^2+\eta^2\right)^{\frac{|s|}{2}} e^{\alpha t'\xi^2-\beta t'\left(\xi^2+\eta^2\right)^2}\right\|_{L^2(\mathbb{R}^2)} \frac{dt'}{|t-t'|^{\frac{|s|}{2}}}\right)\|u\|_{X_T^s}\|v\|_{X_T^s}. \tag{2.64}$$

To estimate the integral on the right-hand side of (2.64), we use a change of variable to deduce that

$$\begin{aligned}&\left\|\xi\left(\xi^2+\eta^2\right)^{\frac{|s|}{2}} e^{\alpha t'\xi^2-\beta t'\left(\xi^2+\eta^2\right)^2}\right\|_{L^2(\mathbb{R}^2)}\\ &\leq |t'|^{-\frac{1}{4}(2+s)}\left\|e^{\alpha\sqrt{t'}\xi^2-\frac{\beta}{2}\left(\xi^2+\eta^2\right)^2}\right\|_{L^\infty(\mathbb{R}^2)}\left\|\left(\xi^2+\eta^2\right)^{\frac{|s|}{2}}|\xi|e^{-\frac{\beta}{2}\left(\xi^2+\eta^2\right)^2}\right\|_{L^2(\mathbb{R}^2)} \lesssim e^{\frac{\alpha^2 t'}{2\beta}}|t'|^{-\frac{1}{4}(2+s)}.\end{aligned} \tag{2.65}$$

Therefore, we get from (2.64) and (2.65) that

$$\begin{aligned}&\left\|\int_0^t U(t-t')\partial_x(uv)\right\|_{H^s(\mathbb{R}^2)} \lesssim e^{\frac{\alpha^2 T}{2\beta}}T^{\frac{1}{2}\left(1+\frac{s}{2}\right)}\left(\int_0^1 |t'|^{-\frac{1}{4}(2+s)}|1-t'|^{-\frac{|s|}{2}}\,dt'\right)\|u\|_{X_T^s}\|v\|_{X_T^s}\\ &\lesssim e^{\frac{\alpha^2 T}{2\beta}}T^{\frac{1}{2}\left(1+\frac{s}{2}\right)}\|u\|_{X_T^s}\|v\|_{X_T^s},\end{aligned} \tag{2.66}$$

for all $0\leq t\leq T$. On the other hand, by a similar argument, we deduce that for all $0\leq t\leq T$,

$$\begin{aligned}&|t|^{\frac{|s|}{4}}\left\|\int_0^t U(t-t')\partial_x(uv)\right\|_{L^2(\mathbb{R}^2)} \leq T^{\frac{|s|}{4}}\int_0^t\left\|\xi\, e^{(t-t')\left(\alpha\xi^2-\beta(\xi^2+\eta^2)^2\right)}\right\|_{L^2(\mathbb{R}^2)}\left\|\widehat{u(t')}*\widehat{v(t')}(\xi,\eta)\right\|_{L^\infty(\mathbb{R}^2)}\,dt'\\ &\leq\left(\int_0^t |t'|^{-\frac{1}{2}}\left\||\xi|\, e^{\alpha\sqrt{t'}\xi^2-\beta\left(\xi^2+\eta^2\right)^2}\right\|_{L^2(\mathbb{R}^2)}\frac{dt'}{|t-t'|^{\frac{|s|}{2}}}\right)\|u\|_{X_T^s}\|v\|_{X_T^s}\\ &\lesssim e^{\frac{\alpha^2 T}{2\beta}}T^{\frac{1}{2}\left(1+\frac{s}{2}\right)}\left(\int_0^1 |t'|^{-\frac{1}{2}}|1-t'|^{-\frac{|s|}{2}}\,dt'\right)\|u\|_{X_T^s}\|v\|_{X_T^s}\lesssim e^{\frac{\alpha^2 T}{2\beta}}T^{\frac{1}{2}\left(1+\frac{s}{2}\right)}\|u\|_{X_T^s}\|v\|_{X_T^s}.\end{aligned}$$

This completes the proof. ■

Next, we derive a regularity property.

PROPOSITION 2.7.4 *Let $\alpha,\beta>0$, $0\leq t\leq T\leq 1$, $s\in(-2,0]$ and $\kappa\in[0,s+2)$, then*

$$\mathbb{V}: t\longmapsto \int_0^t U(t-t')\partial_x(u^2)(t')\,dt'$$

is in $C\left([0,T];H^{s+\kappa}(\mathbb{R}^2)\right)$, for all $u\in X_T^s$.

Proof. Consider $t_0, t_1 \in [0,T]$ be fixed such that $t_0 < t_1$. Then by the Minkowski inequality, we have

$$\|\mathbb{V}(t_1) - \mathbb{V}(t_0)\|_{H^{s+\kappa}(\mathbb{R}^2)} \le \mathbb{V}_1(t_0,t_1) + \mathbb{V}_2(t_0,t_1),$$

where $\mathbb{V}_1(t_0,t_1) = \int_{t_0}^{t_1} \left\|U(t_1-t')\partial_x(u^2)\right\|_{H^{s+\kappa}(\mathbb{R}^2)}\, dt'$, and

$$\mathbb{V}_2(t_0,t_1) = \int_0^{t_0} \left\|\left(U(t_1-t') - U(t_0-t')\right)\partial_x(u^2)\right\|_{H^{s+\kappa}(\mathbb{R}^2)}\, dt'.$$

By performing a change of variable, we obtain

$$\begin{aligned}\mathbb{V}_1(t_0,t_1) &\le \left(\int_0^t \left\|\xi(\xi^2+\eta^2)^{\frac{s+\kappa}{2}}\, e^{(t_1-t')\left(\alpha\xi^2-\beta(\xi^2+\eta^2)^2\right)}\right\|_{L^2(\mathbb{R}^2)} |t'|^{-\frac{|s|}{2}}\, dt'\right)\|u\|_{\mathcal{X}_T^s}^2\\ &\lesssim \left(\int_{t_0}^{t_1} |t_1-t'|^{-\frac{1}{4}(2+s+\kappa)} e^{\frac{\alpha^2(t_1-t')}{2\beta}} |t'-t_0|^{-\frac{|s|}{2}}\, dt'\right)\|u\|_{\mathcal{X}_T^s}^2\\ &\lesssim e^{\frac{\alpha^2 T}{2\beta}}(t_1-t_0)^{\frac{1}{2}\left(1+\frac{s-\kappa}{2}\right)}\left[\int_0^1 |1-t'|^{-\frac{1}{4}(2+s+\kappa)}|t'|^{-\frac{|s|}{2}}\ t'\right]\|u\|_{\mathcal{X}_T^s}^2.\end{aligned}$$

Now, by using the hypotheses, we get that $\lim_{t_1\to t_0}\mathbb{V}_1(t_0,t_1) = 0$. On the other hand, we have

$$\mathbb{V}_2(t_0,t_1) \le \left(\int_0^{t_0} \|f(t_0,t_1,t',\xi,\eta)\|_{L^2(\mathbb{R}^2)}|t'|^{-\frac{|s|}{2}}\, dt'\right)\|u\|_{\mathcal{X}_T^s}^2,$$

where

$$\begin{aligned}f(t_0,t_1,t',\xi,\eta) &= \left(\xi^2+\eta^2\right)^{\frac{s+\kappa}{2}}|\xi|\left[e^{(t_1-t')\left(\alpha\xi^2-\beta(\xi^2+\eta+2)^2\right)}\, e^{i(t_1-t')(\xi^3+\xi\eta^2)}\right]\\ &\quad - \left(\xi^2+\eta^2\right)^{\frac{s+\kappa}{2}}|\xi|\left[e^{(t_0-t')\left(\alpha\xi^2-\beta(\xi^2+\eta+2)^2\right)}\, e^{i(t_0-t')(\xi^3+\xi\eta^2)}\right].\end{aligned}$$

It is clear that $f(t_0,t_1,t',\xi,\eta)$ tends to zero pointwise for almost every $(\xi,\eta)\in\mathbb{R}^2$ and $t'\in[0,t_0]$ when $|t_1-t_0|\to 0$. So

$$|f(t_0,t_1,t',\xi,\eta)| \lesssim \chi_{\left\{|\xi|\le\sqrt{\frac{\alpha}{\beta}}\right\}}(\xi)e^{\frac{\alpha^2 T}{2\beta}} + \left(\xi^2+\eta^2\right)^{\frac{s+\kappa}{2}}|\xi|e^{(t_0-t')\left(\alpha\xi^2-\beta(\xi^2+\eta+2)^2\right)}.$$

Thusly, we deduce from the Lebesgue dominated convergence theorem that $\|f(t_0,t_1,t',\xi,\eta)\|_{L^2(\mathbb{R}^2)} \longrightarrow 0$, as $t_1\to t_0$. Using again the Lebesgue dominated convergence theorem, we conclude that $\lim_{t_1\to t_0}\mathbb{V}_2(t_0,t_1) = 0$. This completes the proof. ■

Now we are in the position to give the proof of Theorem 2.7.1.

Proof of Theorem 2.7.1. Let $\alpha,\beta>0$, $s\in(-2,0]$ and $u_0\in H^s(\mathbb{R}^2)$. We are going to show that the operator Φ defined in (2.60) is a contraction in some closed ball of $\mathcal{X}_T^s$. By Propositions 2.7.2 and 2.7.3, there exists two positive constants $C = C(s,\alpha,\beta)$ and $\theta=\theta(s)$ such that

$$\|\Phi(u)\|_{\mathcal{X}_T^s} \le C\left(\|u_0\|_{H^s(\mathbb{R}^2)} + T^\theta\|u\|_{\mathcal{X}_T^s}^2\right), \tag{2.67}$$

and

$$\|\Phi(u) - \Phi(v)\|_{\mathcal{X}_T^s} \leq CT^\theta \|u - v\|_{\mathcal{X}_T^s} \|u + v\|_{\mathcal{X}_T^s}, \tag{2.68}$$

for all $u, v \in \mathcal{X}_T^s$ and $0 < T \leq 1$. Now we define $\mathcal{X}_T^s(a) = \left\{u \in \mathcal{X}_T^s \ : \ \|u\|_{\mathcal{X}_T^s} \leq a\right\}$ with $a = 2C\|u_0\|_{H^s(\mathbb{R}^2)}$; and we choose $0 < T < \min\left\{1, (2Ca)^{-\frac{1}{\theta}}\right\}$. The estimates (2.67) and (2.68) imply that Φ is a contraction on the Banach space $\mathcal{X}_T^s(a)$; so that we deduce by the fixed point theorem, the existence of a unique solution u of the integral equation (2.35) in $\mathcal{X}_T^s(a)$ with the initial data $u(0) = u_0$. Note that the Proposition 2.7.4 assures that $\Phi(u) \in C\left([0, T]; H^s(\mathbb{R}^2)\right)$.

The uniqueness of the solution of (2.35) on the whole space $\mathcal{X}_T^s$ and the smoothness of the flow map solution follow by the standard arguments as we did before.

Note that a similar contraction argument shows that the existence result holds for any $s' > s > -2$, in the time interval $[0, T]$ with $T = T(\|u_0\|_{H^s(\mathbb{R}^2)})$. Finally, we know that the map $t \longmapsto U(t)u_0$ is continuous in the time interval $(0, T]$ with respect to the topology of $H^\infty(\mathbb{R}^2)$. Since our solution u belongs to $\mathcal{X}_T^s$, we deduce from the Proposition 2.7.4 that there exists $\kappa > 0$ such that the map $\mathbb{V}$ belongs to $C\left([0, T]; H^{s+\kappa}(\mathbb{R}^2)\right)$, so that

$$u \in C\left((0, T]; H^{s+\kappa}(\mathbb{R}^2)\right).$$

Therefore, by a standard bootstrapping argument, using the uniqueness result and the fact that the time interval of the existence of the solutions only depends on the $H^s(\mathbb{R}^2)$-norm of the initial data, we deduce that

$$u \in C\left([0, T]; H^\infty(\mathbb{R}^2)\right).$$

■

Similarly, as before, by using the global a priori estimates, we can extend the local solutions to be extended to temporal half line $[0, \infty)$.

THEOREM 2.7.5 *Let $\alpha > 0$, $\beta > 0$ and $s \geq 0$. Then the equation (2.35) is globally well posed for initial data in $H^s(\mathbb{R}^2)$, provided that there exists a constant $\varepsilon > 0$ such that $\varepsilon \leq \beta$.*

REMARK 2.7.6 *By a similar argument, one can obtain the global well-posedness in n dimensional case of the equation (2.35):*

$$u_t + uu_x + \alpha u_{xx} + \Delta u_x + \beta \Delta^2 u = 0, \quad (x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}, \quad t \in \mathbb{R}^+.$$

In fact, one can show that the associated initial value problem is globally well-posed in Sobolev spaces $H^s(\mathbb{R}^n)$ for $s > \frac{n}{2} - 3$.

REMARK 2.7.7 *One can see that for $s > 2$, there exists $T_s > 0$ and a positive function $\mathscr{X} \in C([0, T_s))$ independent of α and β such that solution $u_{\alpha,\beta}$ of (2.35), associated to $u_0 \in H^s$, α and β, is defined in $[0, T_s)$ (possibly extended) and $\|u_{\alpha,\beta}(t)\|_{H^s} \leq \mathscr{X}^{1/2}(t)$, for all $t \in [0, T_s)$.*

REMARK 2.7.8 *One can see that for $u_0 \in H^s(\mathbb{R}^2)$, $s > -2$, the time existence of the solution $u_{\varepsilon,\alpha,\beta}$ of the equation*

$$u_t + uu_x + \varepsilon \Delta u_x + \alpha u_{xx} + \beta \Delta^2 u = 0, \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}^+$$

in Theorem 2.7.1 is independent of $\varepsilon > 0$, therefore the limit $u^0 = \lim_{\varepsilon \to 0} u_{\varepsilon,\alpha,\beta}$ exists in $C\left([0, T]; H^s\left(\mathbb{R}^2\right)\right)$ and is the unique solution of the biharmonic equation

$$u_t + uu_x + \alpha u_{xx} + \beta \Delta^2 u = 0, \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}^+;$$

with continuity of the map $u_0 \in H^s(\mathbb{R}^2) \mapsto u^0 \in C([0, T]; H^s(\mathbb{R}^2))$.

REMARK 2.7.9 *We are able to show that our results are sharp [29]. We establish that the flow map of the Benney equation fails to be C^2 in $H^s\left(\mathbb{R}^2\right)$ for $s < -2$. This means that a Picard iteration cannot be used to obtain solution of (2.35). We proved that solutions of the Benney equation tend to solutions of the ZK equation in the $C([0,T]; H^s\left(\mathbb{R}^2\right))$ topology when α and β tend to zero and $s > 2$. Furthermore, we used the improved tanh method to obtain some explicit traveling wave solution s of Benney equation.*

Chapter 3

BO-ZK Equation

3.1 Introduction

This chapter is concerned with (non)-existence, stability and properties of solitary wave solutions for the two dimensional BO-ZK equation:

$$u_t + u^p u_x + \alpha \mathscr{H} u_{xx} + \varepsilon u_{xyy} = 0, \quad (x, y) \in \mathbb{R}^2, \;\; t \in \mathbb{R}^+, \tag{3.1}$$

where the constant ϵ measures the transverse dispersion effects and is normalized to ± 1 and the constant α is real. When $p = 1$, the equation (3.1) appears in electromigration and the interaction of the nanoconductor with the surrounding medium [41, 49], by considering Benjamin-Ono dispersive term with the anisotropic effects included via weak dispersion of ZK-type. The instability of solitary waves, the well-posedness and the unique continuation property of the equation (3.1) and the generalized higher dimensional BO-ZK have been studied in [28].

In fact, the equations (3.1) is generalizations of the one dimensional Benjamin-Ono equation (see also [30]). The questions of existence, asymptotic and stability of solitary wave solutions of the Benjamin-Ono type equations were studies by Benjamin in [4, 5]. The initial value problem associated to the Benjamin-Ono equation has been studied by several authors [20, 46, 69].

We will investigate the existence of solitary wave solutions of (3.1) and their properties.

It can be seen the flow associated to (3.1) satisfies the conservation quantities $\mathscr{F}(\cdot) = \frac{1}{2}\|\cdot\|_{L^2}$ and E, where

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\varepsilon u_y^2 - \alpha u \mathscr{H} u_x - \frac{2}{(p+1)(p+2)} u^{p+2} \right) \, dxdy.$$

Indeed, we are looking for a solution of (3.1) of the form $u = \varphi(x - ct, y)$ decaying to zero at infinity; so, substituting this form of u in (3.1) and integrating once, we see that φ must satisfy

$$-c\varphi + \frac{1}{p+1}\varphi^{p+1} + \alpha \mathscr{H}\varphi_x + \varepsilon \varphi_{yy} = 0. \tag{3.2}$$

REMARK 3.1.1 *Note that we can assume that $|c| = 1$, since the scale change*

$$\psi(x,y) = |c|^{-\frac{1}{p}} \varphi\left(\frac{x}{|c|}, \frac{y}{\sqrt{|c|}}\right),$$

transforms (3.2) in φ, into the same in ψ, but with $|c| = 1$.

REMARK 3.1.2 *The scale-invariant spaces for the BO-ZK equations (3.1) are $\dot{H}^{s_1,s_2}(\mathbb{R}^2)$, $2s_1 + s_2 = \frac{3}{2} - \frac{2}{p}$. Hence a reasonable framework for studying the local well-posedness of the BO-ZK equations (3.1) is the family of spaces $\dot{H}^{s_1,s_2}(\mathbb{R}^2)$, $2s_1 + s_2 \geq \frac{3}{2} - \frac{2}{p}$ (see [28]).*

We shall denote, $\mathscr{Z} = H^{\left(\frac{1}{2},1\right)}\left(\mathbb{R}^2\right)$. By Theorem 1 in [48] (see also [62, 63]) and Remark 0.0.13 imply the following embedding $\mathscr{Z}$ in $L^p\left(\mathbb{R}^2\right)$ spaces:

$$\mathscr{Z} \hookrightarrow L^p\left(\mathbb{R}^2\right), \text{ for all } p \in [2,6]. \tag{3.3}$$

3.2 (Non)existence

THEOREM 3.2.1 *The equations (3.2) do not admit any nontrivial solitary wave solution $\varphi \in \mathscr{Z}$ if none of the following cases occurs:*

(i) $\varepsilon = 1$, $c > 0$, $\alpha < 0$, $p < 4$,

(ii) $\varepsilon = -1$, $c < 0$, $\alpha > 0$, $p < 4$,

(iii) $\varepsilon = 1$, $c < 0$, $\alpha < 0$, $p > 4$,

(iv) $\varepsilon = -1$, $c > 0$, $\alpha > 0$, $p > 4$.

Sketch of the proof. To prove, we apply a truncation argument to gain the regularity we need (see Chapter 4), then by using the Lebesgue dominated convergence theorem, we obtain some useful identities. In fact, by multiplying the equation (3.2) by φ, $x\varphi_x$ and $y\varphi_y$, respectively, integrating over $\mathbb{R}^2$ and using (0.3)-(0.11), we obtain the following relations:

$$\int_{\mathbb{R}^2} \left(-c\varphi^2 + \alpha\varphi\mathscr{H}\varphi_x - \varepsilon\varphi_y^2 + \frac{1}{p+1}\varphi^{p+2}\right) dxdy = 0, \tag{3.4}$$

$$\int_{\mathbb{R}^2} \left(c\varphi^2 + \varepsilon\varphi_y^2 - \frac{2}{(p+1)(p+2)}\varphi^{p+2}\right) dxdy = 0, \tag{3.5}$$

$$\int_{\mathbb{R}^2} \left(c\varphi^2 - \alpha\varphi\mathscr{H}\varphi_x - \varepsilon\varphi_y^2 - \frac{2}{(p+1)(p+2)}\varphi^{p+2}\right) dxdy = 0. \tag{3.6}$$

By adding (3.4) and (3.5), we get

$$\int_{\mathbb{R}^2} \left(\alpha\varphi\mathscr{H}\varphi_x + \frac{p}{(p+1)(p+2)}\varphi^{p+2}\right) dxdy = 0. \tag{3.7}$$

Also by adding (3.5) and (3.6) yields

$$\int_{\mathbb{R}^2}\left(c\varphi^2-\frac{\alpha}{2}\varphi\mathscr{H}\varphi_x-\frac{2}{p+1}\varphi^{p+2}\right)\,dxdy=0. \tag{3.8}$$

Eliminating φ^{p+2} from (3.7) and (3.8) leads to

$$\int_{\mathbb{R}^2}\left(2pc\varphi^2+\alpha(4-p)\varphi\mathscr{H}\varphi_x\right)\,dxdy=0. \tag{3.9}$$

On the other hand, adding (3.4) and (3.6) yields

$$\int_{\mathbb{R}^2}\left(2\varepsilon\varphi_y^2-\frac{p}{(p+1)(p+2)}\varphi^{p+2}\right)\,dxdy=0. \tag{3.10}$$

Plugging (3.5) in (3.10) we obtain

$$\int_{\mathbb{R}^2}\left(pc\varphi^2+\varepsilon(p-4)\varphi_y^2\right)\,dxdy=0. \tag{3.11}$$

The proof follows from (3.9) and (3.11). ■

THEOREM 3.2.2 *Let $\alpha\varepsilon, c\alpha<0$ and $p=\frac{k}{m}<4$, where $m\in\mathbb{N}$ is odd and m and k are relatively prime. Then the equation (3.2) admits a nontrivial solution $\varphi\in\mathscr{Z}$.*

Sketch of the proof. The proof is based on Lemma 0.0.1. We suppose that $\alpha<0$. The proof for $\alpha>0$ is similar. Without loss of generality we assume that $\alpha=-1$ and $c=1$. We consider the minimization problem

$$I_\lambda=\inf\left\{I(\varphi)=\frac{1}{2}\int_{\mathbb{R}^2}\left(\varphi^2+\varphi\mathscr{H}\varphi_x+\varphi_y^2\right)\,dxdy\ ;\ \varphi\in\mathscr{Z}\ ,\ J(\varphi)=\|\varphi\|_{L^{p+2}}^{p+2}=\lambda\right\}, \tag{3.12}$$

where $\lambda>0$. Let $\{\varphi_n\}\subset\mathscr{Z}$ be a minimizing sequence of I_λ. By using (3.3), we obtain that

$$\lambda=\left|\int_{\mathbb{R}^2}\varphi^{p+2}\,dxdy\right|\le C\|\varphi\|_{\mathscr{Z}}^{p+2}\le CI_\lambda^{\frac{p+2}{2}},$$

for any $\varphi\in\mathscr{Z}$ and $p<4$. Hence $I_\lambda<\infty$ and $I_\lambda>0$ for any positive λ. Also, since $I(\varphi)\sim\|\varphi\|_{\mathscr{Z}}^2$, so $\|\varphi_n\|_{\mathscr{Z}}<\infty$. Now we define the concentration functions

$$Q_n(r)=\sup_{(\tilde{x},\tilde{y})\in\mathbb{R}^2}\int_{B_r(\tilde{x},\tilde{y})}\rho_n\,dxdy,$$

where $\rho_n=|\varphi_n|^2+\left|D_x^{1/2}\varphi_n\right|^2+|\partial_y\varphi_n|^2$. If the evanescence occurs, i.e., that for any $R>0$,

$$\lim_{n\to+\infty}\sup_{(\tilde{x},\tilde{y})\in\mathbb{R}^2}\int_{B_r(\tilde{x},\tilde{y})}\rho_n\,dxdy=0,$$

then by using (3.3), we obtain that $\lim_{n\in\infty}\|\varphi_n\|_{L^{p+2}}=0$, which would contradict the constraint of the minimization problem. Now suppose that $\gamma\in(0,I_\lambda)$, where

$$\gamma=\lim_{r\to+\infty}\lim_{n\to+\infty}\sup_{(\widetilde{x},\widetilde{y})\in\mathbb{R}^2}\int_{B_r(\widetilde{x},\widetilde{y})}\rho_n\,dxdy.$$

By the definition of γ, for $\epsilon>0$, there exist $r_1\in\mathbb{R}$ and $N\in\mathbb{N}$ such that $\gamma-\epsilon<Q_n(r)\le Q_n(2r)\le\gamma<\gamma+\epsilon$, for any $r\ge r_1$ and $n\ge N$. Hence, there exists a sequence $\{(\widetilde{x}_n,\widetilde{y}_n)\}\subset\mathbb{N}$ such that

$$\int_{B_r(\widetilde{x}_n,\widetilde{y}_n)}\rho_n\,dxdy>\gamma-\epsilon,\qquad\int_{B_{2r}(\widetilde{x}_n,\widetilde{y}_n)}\rho_n\,dxdy<\gamma+\epsilon.$$

Let $(\phi,\psi)\in(C_0^\infty(\mathbb{R}^2))^2$ satisfy

- supp $\phi\subset B_2(0)$, $\phi\equiv1$ on $B_1(0)$ and $0\le\phi\le1$,
- supp $\psi\subset\mathbb{R}^2\setminus B_2(0)$, $\psi\equiv1$ on $\mathbb{R}^2\setminus B_1(0)$ and $0\le\psi\le1$.

Now we define

$$g_n(x,y)=\phi_r((x,y)-(\widetilde{x}_n,\widetilde{y}_n))\varphi_n\quad\text{and}\quad h_n(x,y)=\psi_r((x,y)-(\widetilde{x}_n,\widetilde{y}_n))\varphi_n,$$

where

$$\phi_r(x,y)=\phi\left(\frac{(x,y)}{r}\right)\quad\text{and}\quad\psi_r(x,y)=\psi\left(\frac{(x,y)}{r}\right).$$

It is easy to see that $g_n,h_n\in\mathscr{X}$. The following splitting lemma is proved similar to Lemma 4.2.5, by using Lemma 4.2.7.

LEMMA 3.2.3 *For every $\epsilon>0$, there exists a $\delta(\epsilon)$ with $\lim_{\epsilon\to0}\delta(\epsilon)=0$, $\varrho\in(0,I_\lambda)$, $\rho\in(0,\lambda)$ and two sequences $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ in $\mathscr{X}$ with satisfying the following for $n\ge n_0$.*

$$|I(\varphi_n)-I(g_n)-I(h_n)|\le\delta(\epsilon),\tag{3.13}$$
$$|I(g_n)-\varrho|\le\delta(\epsilon),\quad|I(h_n)-I_\lambda+\varrho|\le\delta(\epsilon),\tag{3.14}$$
$$|J(\varphi_n)-J(g_n)-J(h_n)|\le\delta(\epsilon),\tag{3.15}$$
$$|J(g_n)-\rho|\le\delta(\epsilon),\quad|J(h_n)-\lambda+\rho|\le\delta(\epsilon).\tag{3.16}$$

The previous lemma imply that $I_\lambda\ge I_\rho+I_{\lambda-\rho}$. This inequality contradicts the subadditivity condition of I_λ coming from $I_\lambda=\lambda^{2/(p+2)}I_1$. Therefore the remaining case in the Lemma 0.0.1 is locally compactness. There exist a sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}}\subset\mathbb{R}^2$, such that for all $\epsilon>0$, there exists a finite $R>0$ and $n_0>0$, with

$$\int_{B_R(x_n,y_n)}\rho_n\,dxdy\ge\iota_\lambda-\epsilon,$$

for $n\ge n_0$, where $\iota_\lambda=\lim_{n\to+\infty}\int_{\mathbb{R}^2}\rho_n\,dxdy$. This implies that for n large enough

$$\int_{B_R(x_n,y_n)}|\varphi_n|^2\,dxdy\ge\int_{\mathbb{R}^2}|\varphi_n|^2\,dxdy-2\epsilon.$$

Since φ_n is bounded in the Hilbert space $\mathscr{X}$, there exists $\varphi \in \mathscr{X}$ such that a subsequence of $\{\varphi_n(\cdot - (x_n, y_n))\}_{n\in\mathbb{N}}$ (denoted by the same) converges weakly in $\mathscr{X}$. We then have

$$\int_{\mathbb{R}^2} |\varphi|^2 \, dxdy \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |\varphi_n|^2 \, dxdy \leq \liminf_{n\to+\infty} \int_{B_R(x_n,y_n)} |\varphi_n|^2 \, dxdy + 2\epsilon.$$

But we know the compactness embedding $\mathscr{X}$ into L^2 on bounded intervals. Consequently $\{u_n(\cdot - (x_n, y_n))\}_{n\in\mathbb{N}}$ converges strongly in $L^2_{loc}(\mathbb{R}^2)$. But the last inequality above implies that this strong convergence also takes place in $L^2(\mathbb{R}^2)$. Thus by (3.3), $\{\varphi_n(\cdot - (x_n, y_n))\}_{n\in\mathbb{N}}$ also converges to φ strongly in $L^{p+2}(\mathbb{R}^2)$ so that $J(\varphi) = \lambda$ and $I_\lambda = \lim_{n\to+\infty} I(\varphi_n) = I(\varphi)$, that is, φ is a solution of I_λ.

Now by using the preceding theorem and the Lagrange multiplier theorem, there exists $\theta \in \mathbb{R}$ such that

$$\varphi - \alpha\mathscr{H}\varphi_x - \varphi_{yy} = \theta(p+2)\varphi^{p+1}, \tag{3.17}$$

in $\mathscr{X}'$. By a scale change, φ satisfies (3.2). ■

We are also able to prove that our solitary wave solutions are the ground state solutions of BO-ZK equation [28]

3.3 Stability

The following Theorem is a consequence of Theorem 3.2.2 and it will be main key to obtain our stability results of the solutions of BO-ZK equations. Hereafter, without loss of generality we assume that $\alpha = -1$.

THEOREM 3.3.1 *Let $\lambda > 0$. Then*

(i) *every minimizing sequence to I_λ converges, up to a translation, in $\mathscr{X}$ to an element of the minimizers set*

$$M_\lambda = \{\varphi \in \mathscr{X};\ I(\varphi) = I_\lambda,\ J(\varphi) = \lambda\}.$$

(ii) *Let $\{\varphi_n\}$ be a minimizing sequence for I_λ. Then we have*

$$\lim_{n\to+\infty} \inf_{\psi\in M_\lambda,\, z\in\mathbb{R}^2} \|\varphi_n(\cdot + z) - \psi\|_{\mathscr{X}} = 0, \tag{3.18}$$

$$\lim_{n\to+\infty} \inf_{\psi\in M_\lambda} \|\varphi_n - \psi\|_{\mathscr{X}} = 0. \tag{3.19}$$

The following lemma easily show that there exists a $\lambda > 0$ such that every element in the set of minimizers satisfies (3.2).

LEMMA 3.3.2 *For $\lambda = (2(p+1)I_1)^{\frac{p+2}{p}}$ in our minimization problem, we have that if $\varphi \in M_\lambda$, then φ is a solitary wave solution for the BO-ZK equation (3.2).*

Now for λ in the above lemma, we define the set $\mathscr{N}_c = \{\varphi \in \mathscr{X};\ J(\varphi) = 2(p+1)I(\varphi) = \lambda\}$. It is easy to see that $M_\lambda = \mathscr{N}_c$. Now for any $c > 0$ and any $\varphi \in \mathscr{N}_c$, we define the function $d(c) = E(\varphi) + c\mathscr{F}(\varphi)$.

LEMMA 3.3.3 *$d(c)$ is constant on $\mathscr{N}_c$ and is differentiable and strictly increasing for $c > 0$ and $p < \frac{4}{3}$. Moreover, $d''(c) > 0$ if and only if $p < \frac{4}{3}$.*

Proof. It is easy to see that

$$d(c) = I(\varphi) - \frac{1}{(p+1)(p+2)}J(\varphi) = \frac{p}{2(p+1)(p+2)}J(\varphi) = \frac{p(2(p+1))^{\frac{2}{p}}}{p+2}I_1^{\frac{p+2}{p}}.$$

Therefore, $d(c) = \dfrac{p}{2(p+1)(p+2)}c^{\frac{2}{p}-\frac{1}{2}}J(\psi)$, where $\psi(x,y) = c^{-\frac{1}{p}}\varphi\left(\dfrac{x}{c}, \dfrac{y}{\sqrt{c}}\right)$. Note that ψ satisfies (3.2), with $c = 1$. But we know that

$$\frac{1}{(p+1)(p+2)}J(\varphi) = \frac{4c}{4-p}\mathscr{F}(\varphi).$$

Thusly, we obtain that $d''(c) = \left(\dfrac{2}{p} - \dfrac{3}{2}\right)c^{\frac{2}{p}-\frac{5}{2}}\mathscr{F}(\psi)$. ■

Now we are going to study the behavior of d in a neighborhood of the set $\mathscr{N}_c$.

LEMMA 3.3.4 *Let $c > 0$. Then there exists a small positive number ϵ and a C^1-map $v : \mathscr{B}_\epsilon(\mathscr{N}_c) \to (0, +\infty)$ defined by*

$$v(u) = d^{-1}\left(\frac{p}{2(p+1)(p+2)}J(\varphi)\right),$$

such that $v(\varphi) = c$ for every $\varphi \in \mathscr{N}_c$, where $\mathscr{B}_\epsilon(\mathscr{N}_c) = \{\varphi \in \mathscr{Z} \; ; \; \inf_{\psi\in\mathscr{N}_c} \|\varphi - \psi\|_{\mathscr{Z}} < \epsilon\}$.

Proof. Without loss of generality we assume that $c = 1$. It is easy to see that $\mathscr{N}_c$ is a bounded set in $\mathscr{Z}$. Moreover

$$\mathscr{N}_c \subset B(0,r) \subset \mathscr{Z},$$

where $r = (2(p+1))^{\frac{2}{p}}I_1^{\frac{p+2}{p}}$ and $B(0,r)$ is the ball of radius $r > 0$ centered at the origin in $\mathscr{Z}$. Let $\rho > 0$ be sufficiently large such that $\mathscr{N}_c \subset B(0,\rho) \subset \mathscr{Z}$. Since the function $u \to J(u)$ is uniformly continuous on bounded sets, then there exists $\epsilon > 0$ such that if $u, v \in B(0,\rho)$ and $\|u - v\|_{\mathscr{Z}} < 2\epsilon$ then $|J(u) - J(v)| < \rho$. Considering the neighborhoods $\mathscr{I} = (d(c) - \rho, d(c) + \rho)$ and $\mathscr{B}_\epsilon(\mathscr{N}_c)$ of $d(c)$ and $\mathscr{N}_c$, respectively, we have that if $u \in \mathscr{B}_\epsilon(\mathscr{N}_c)$ then $J(u) \in \mathscr{I}$. Therefore v is well defined on $\mathscr{B}_\epsilon(\mathscr{N}_c)$ and satisfies $v(\varphi) = c$, for all $\varphi \in \mathscr{N}_c$. ■

Next, we establish the main inequality in our study of stability.

LEMMA 3.3.5 *Let $c > 0$ and suppose that $d''(c) > 0$. Then for all $u \in \mathscr{B}_\epsilon(\mathscr{N}_c)$ and any $\varphi \in \mathscr{N}_c$,*

$$E(u) - E(\varphi) + v(u)\left(\mathscr{F}(u) - \mathscr{F}(\varphi)\right) \geq \frac{1}{4}d''(c)|v(u) - c|^2.$$

Proof. Denote the functional $I_\omega(\varphi) = \frac{1}{2}\int_{\mathbb{R}^2}\left(\omega\varphi^2 + \varphi\mathscr{H}\varphi_x + \varphi_y^2\right)\,dxdy$, and φ_ω any element of $\mathscr{N}_\omega$. Then we have

$$E(u) + v(u)\mathscr{F}(u) = I_{v(u)}(u) - \frac{1}{(p+1)(p+2)}.$$

On the other hand, we have $J(u) = J\left(\varphi_{v(u)}\right)$, since $d(v(u)) = \frac{p}{2(p+1)(p+2)}J(u)$ for $u \in \mathscr{B}_\epsilon(\mathscr{N}_c)$ and $d(v(u)) = \frac{p}{2(p+1)(p+2)}J\left(\varphi_{v(u)}\right)$. Thusly $I_{v(u)}(u) \geq I_{v(u)}\left(\varphi_{v(u)}\right)$. Therefore by using the Taylor expansion of d at c, we obtain that

$$\begin{aligned} E(u)+v(u)\mathscr{F}(u) &\geq I_{v(u)}\left(\varphi_{v(u)}\right)-\frac{1}{(p+1)(p+2)}J\left(\varphi_{v(u)}\right) \\ &= d(v(u)) \geq d(c)+\mathscr{F}(\varphi)(v(u)-c)+\frac{1}{4}d''(c)|v(u)-c|^2 \\ &= E(u)+v(\varphi)\mathscr{F}(\varphi)+\frac{1}{4}d''(c)|v(u)-c|^2 \end{aligned}$$

■

First we state a well-posedness result for (3.1); which can be proved by using an argument similar to Section 2.2.

THEOREM 3.3.6 *Let $s>2$. Then for any $u_0\in H^s(\mathbb{R}^2)$, there exists $T=T(\|u_0\|_{H^s})>0$ and there exists a unique solution $u\in C([0,T];H^s(\mathbb{R}^2))$ of the equation (3.1) with $u(0)=u_0$ and $u(t)$ depends on u_0 continuously in the H^s-norm. In addition, $u(t)$ satisfies $E(u(t))=E(u_0)$, $\mathscr{F}(u(t))=\mathscr{F}(u_0)$, for all $t\in[0,T)$.*

Now we will prove our nonlinear stability result of the set $\mathscr{N}_c$ in $\mathscr{X}$.

THEOREM 3.3.7 *Let $c>0$ and $\lambda=(2(p+1)I_1)^{\frac{p+2}{p}}$. Then the set $\mathscr{N}_c=M_\lambda$ is $\mathscr{X}$-stable with regard to the flow of the BO-ZK equation if $p<\dfrac{4}{3}$.*

Proof. Assume that $\mathscr{N}_c$ is $\mathscr{X}$-unstable with regard to the flow of the BO-ZK equation. Then there is a sequence of initial data $u_k(0)\in\mathscr{B}_{\frac{1}{k}}(\mathscr{N}_c)\cap H^s\left(\mathbb{R}^2\right)$, $s>2$, such that

$$\sup_{t\in[0,+\infty)}\inf_{\varphi\in\mathscr{N}_c}\|u_k(t)-\varphi\|_{\mathscr{X}}\geq\epsilon, \tag{3.20}$$

where $u_k(t)$ is the solution of (3.1) with initial data $u_k(0)$. So we can find, for k large enough, a time t_k such that

$$\inf_{\varphi\in\mathscr{N}_c}\|u_k(t_k)-\varphi\|_{\mathscr{X}}=\frac{\epsilon}{2}, \tag{3.21}$$

by continuity in t. Now since E and $\mathscr{F}$ are conserved, we can find $\varphi_k\in\mathscr{N}_c$ such that

$$|E(u_k(t_k))-E(\varphi_k)|=|E(u_k(0))-E(\varphi_k)|\to 0, \tag{3.22}$$

$$|\mathscr{F}(u_k(t_k))-\mathscr{F}(\varphi_k)|=|\mathscr{F}(u_k(0))-\mathscr{F}(\varphi_k)|\to 0, \tag{3.23}$$

as $k\to+\infty$. By using Lemma 3.3.5, we have

$$E(u_k(t_k))-E(\varphi_k)+v(u_k(t_k))\left(\mathscr{F}(u_k(t_k))-\mathscr{F}(\varphi_k)\right)\geq\frac{1}{4}d''(c)|v(u_k(t_k))-c|^2,$$

by choosing k large enough. This implies that $v(u_k(t_k))\to c$ as $k\to+\infty$, since $u_k(t_k)$ is uniformly bounded for k. Hence, by the definition of v and continuity of d, we have

$$\lim_{k\to+\infty}J(u_k(t_k))=\frac{2(p+1)(p+2)}{p}\,d(c). \tag{3.24}$$

On the other hand, by Lemma 3.3.3, we have

$$\begin{aligned} I(u_k(t_k)) &= E(u_k(t_k)) + c\mathscr{F}(u_k(t_k)) + \frac{1}{(p+1)(p+2)} J(u_k(t_k)) \\ &= d(c) + E(u_k(t_k)) - E(\varphi_k) + c\left(\mathscr{F}(u_k(t_k)) - \mathscr{F}(\varphi_k)\right) + \frac{1}{(p+1)(p+2)} J(u_k(t_k)). \end{aligned}$$

Then by (3.24), we obtain that

$$\lim_{k\to+\infty} I(u_k(t_k)) = \frac{p+2}{p} d(c) = (2(p+1))^{\frac{2}{p}}\, I_1^{\frac{p+2}{p}}. \tag{3.25}$$

By defining $\vartheta_k(t_k) = (J(u_k(t_k)))^{-\frac{1}{p+2}} u_k(t_k)$, in $\mathscr{Z}$, we obtain that $J(\vartheta_k(t_k)) = 1$. Therefore by using (3.24), (3.25) and Lemma 3.3.3, we obtain that

$$\lim_{k\to+\infty} I(\vartheta_k(t_k)) = I_1. \tag{3.26}$$

Hence $\{\vartheta_k(t_k)\}$ is a minimizing sequence of I_1, so, from Theorem 3.3.1, there exists a sequence $\psi_k \subset M_1$ such that

$$\lim_{k\to+\infty} \|\vartheta_k(t_k) - \psi_k\|_{\mathscr{Z}} = 0. \tag{3.27}$$

On the other hand, from the Lagrange multiplier theorem, there exist $\theta_k \in \mathbb{R}$ such that

$$\mathscr{H}\,(\psi_k)_x + c\psi_k - (\psi_k)_{yy} = \theta_k(p+2)\psi_k^{p+1} \tag{3.28}$$

so $2I_1 = \theta_k(p+2)$, which implies $\theta_k = \theta$ for all k. By scaling $\varphi_k = \mu\psi_k$ with $\mu^p = \theta(p+1)(p+2) = 2(p+1)I_1$, we obtain that φ_k satisfy (3.2) and $2(p+1)I(\varphi_k) = J(\varphi_k) = \mu^{p+2}$, which implies that $\varphi_k \in \mathscr{N}_c$ for every k. Also, by (3.24)-(3.27) and Lemma 3.3.3, we have

$$\begin{aligned} \|u_k(t_k) - \varphi_k\|_{\mathscr{Z}} &= (J(u_k(t_k)))^{\frac{1}{p+2}} \left\| (J(u_k(t_k)))^{-\frac{1}{p+2}} (u_k(t_k) - \varphi_k) \right\|_{\mathscr{Z}} \\ &\le (J(u_k(t_k)))^{\frac{1}{p+2}} \left(\left\|\vartheta_k(t_k) - \mu^{-1}\varphi_k\right\|_{\mathscr{Z}} + \mu^{-1}\|\varphi_k\|_{\mathscr{Z}} - (J(u_k(t_k)))^{-\frac{1}{p+2}} \right). \end{aligned}$$

This implies that $\lim_{k\to+\infty} \|u_k(t_k) - \varphi_k\|_{\mathscr{Z}} = 0$, as $k \to +\infty$; which contradicts (3.21); and the proof is complete. ■

3.4 Decay and Regularity

In order to investigate the regularity and the decaying properties of the solitary wave solutions of (3.1), we need to study the kernel of (3.2).

REMARK 3.4.1 *Note that by using the Residue theorem, the kernel of the solution of (3.2) can be written in the following form*

$$K(x,y) = C \int_0^{+\infty} \frac{|\alpha|\sqrt{t}}{t^2 + \alpha^2 x^2}\, e^{-\left(ct + \frac{y^2}{4t}\right)}\, dt,$$

where $C > 0$, *independent of* α, x *and* y, *and* $\widehat{K}(\xi,\eta) = \dfrac{1}{c - \alpha|\xi| + \eta^2}$. *Also by Fubini's theorem, we obtain that*

$$\|K\|_{L^1} = C \int_0^{+\infty} \int_{\mathbb{R}^2} \frac{|\alpha|\sqrt{t}}{t^2 + \alpha^2 x^2}\, e^{-\left(ct + \frac{y^2}{4t}\right)}\, dxdydt = C(\alpha) \int_0^{+\infty} e^{-ct}\, dt.$$

Therefore,

LEMMA 3.4.2 *K is an even (in x and y), positive and decreasing function and $K \in L^\infty\left(\mathbb{R}^2 \setminus \{\boldsymbol{0}\}\right) \cap C^\infty\left(\mathbb{R}^2 \setminus \{\boldsymbol{0}\}\right)$. Furthermore, $\widehat{K} \in L^p(\mathbb{R}^2)$, for any $p \in (3/2, +\infty]$ and $K \in L^p(\mathbb{R}^2)$, for any $p \in [1, +\infty)$.*

THEOREM 3.4.3 *Any solitary wave solution φ of (3.2), with $p \in \mathbb{N}$, belongs to $W^{k,r}(\mathbb{R}^2)$, for all $k \in \mathbb{N}$ and all $r \in [1, +\infty]$. Furthermore, any solitary wave solution φ is continuous and in $L^\infty(\mathbb{R}^2)$; and tends to zero at infinity, for any $0 < p < 4$.*

Proof. Setting $g \equiv -\dfrac{\varphi^{p+1}}{p+1}$, (3.2) yields

$$\widehat{\varphi} = \frac{\widehat{g}}{c - \alpha|\xi| + \varepsilon\eta^2}. \tag{3.29}$$

This implies that $\varphi \in H^{\frac{1}{2},1}(\mathbb{R}^2) \cap H^{0,2}(\mathbb{R}^2) \cap H^{1,0}(\mathbb{R}^2)$. By using Lemma 0.0.11 and the embedding (3.3), we obtain that $\varphi \in H^{s,2(1-s)}(\mathbb{R}^2)$, for any $s \in [0,1]$. By a bootstrapping argument and using Lemma 0.0.11 and Lemma 0.0.16, the proof of first part will be complete. The second part follows from the embedding (3.3), the Young inequality and the properties of K in Lemma 3.4.2. ■

REMARK 3.4.4 *Note that $H^{(s,2s)}\left(\mathbb{R}^2\right)$ is an algebra if $s > \frac{3}{2}$ (see Proposition 2.5 in [33]).*

Now, we are going to study the symmetry properties of the solitary wave solutions of (3.1). Here, for $u : \mathbb{R}^2 \to \mathbb{R}$, $u^\sharp$ will represent the Steiner symmetrization of u with respect to $\{x = 0\}$ and $u^\star$ the Steiner symmetrization of u with respect to $\{y = 0\}$ (see Section 1.6).

LEMMA 3.4.5 *If $f \in \mathscr{X}$, then $f^\star, f^\sharp, |f| \in \mathscr{X}$.*

Proof. By setting $g = |f|$, then we have

$$\langle f, K * f \rangle \le \langle g, K * g \rangle,$$

for every $c > 0$. Therefore

$$\int_{\mathbb{R}^2} \widehat{K}(\xi,\eta)\, \left|\widehat{f}(\xi,\eta)\right|^2\, d\xi d\eta = \langle f, K * f \rangle \le \langle g, K * g \rangle = \int_{\mathbb{R}^2} \widehat{K}(\xi,\eta)\, |\widehat{g}(\xi,\eta)|^2\, d\xi d\eta.$$

So, we have

$$\int_{\mathbb{R}^2} c\left(1 - c\widehat{K}\right)\, |\widehat{g}(\xi,\eta)|^2\, d\xi d\eta \le \int_{\mathbb{R}^2} c\left(1 - c\widehat{K}\right)\, \left|\widehat{f}(\xi,\eta)\right|^2\, d\xi d\eta, \tag{3.30}$$

since $\left\|\widehat{f}\right\|_{L^2(\mathbb{R}^2)} = \|\widehat{g}\|_{L^2(\mathbb{R}^2)}$. By taking the limit as $c \to +\infty$ on both sides of (3.30) and using the Monotone Convergence Theorem, we obtain that

$$\int_{\mathbb{R}^2} \left(|\xi| + \eta^2\right) \, |\widehat{g}(\xi,\eta)|^2 \, d\xi d\eta \leq \int_{\mathbb{R}^2} \left(|\xi| + \eta^2\right) \, \left|\widehat{f}(\xi,\eta)\right|^2 \, d\xi d\eta,$$

which shows that $|f| \in \mathscr{X}$.

Let us prove that $f^\sharp \in \mathscr{X}$. One can see that $K^\sharp = K = K^\star$. Then Theorem 1.6.12 implies that

$$\int_{\mathbb{R}^4} f(x,y)f(s,t)K(x-s,y-t) \, dsdt \, dxdy \leq \int_{\mathbb{R}^4} f^\sharp(x,y)f^\sharp(s,t)K(x-s,y-t) \, dsdt \, dxdy.$$

Then it follows that

$$\int_{\mathbb{R}^2} \widehat{K}(\xi,\eta) \left|\widehat{f}(\xi,\eta)\right|^2 \leq \int_{\mathbb{R}^2} \widehat{K}(\xi,\eta) \left|\widehat{f^\sharp}(\xi,\eta)\right|^2.$$

On the other hand, by using the fact that

$$\left\|\widehat{f}\right\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R}^2)} = \left\|f^\sharp\right\|_{L^2(\mathbb{R}^2)} = \left\|\widehat{f^\sharp}\right\|_{L^2(\mathbb{R}^2)},$$

a similar analysis as in the preceding proof shows that $f^\sharp \in \mathscr{X}$. Analogously, one can prove that $f^\star \in \mathscr{X}$. ■

LEMMA 3.4.6 *If $\varphi \in M_\lambda$, then $\varphi^\sharp, \varphi^\star \in M_\lambda$.*

Proof. Since Steiner symmetrization preserves the L^{p+2}−norm, it follows that $J(\varphi) = J(\varphi^\sharp)$. So, by using Lemma 3.4.5, we get

$$M_\lambda \leq I\left(\varphi^\sharp\right) \leq I(\varphi) = M_\lambda.$$

Therefore, we have that $\varphi^\sharp \in M_\lambda$. Similarly, $\varphi^\star \in M_\lambda$. ■

Now, we prove our theorem concerning the symmetry properties of the solitary wave solutions of the equation (3.1).

THEOREM 3.4.7 *The solitary wave solutions of the equation (3.1) are radially symmetric with respect to the transverse direction and the propagation direction.*

Proof. By Theorems 3.2.2 and 3.4.3, there is the function ψ satisfying (3.2). By choosing $\varphi = \psi^{\sharp\star} \equiv \psi^{\star\sharp}$, we have that φ is a solitary wave solution of the equation (3.1) which is symmetric with respect to $\{x=0\}$ and $\{y=0\}$. ■

THEOREM 3.4.8 *The solitary wave solution φ obtain in Theorem 3.2.2 is positive.*

Proof. The proof follows from the proof of Theorem 3.2.2, Lemma 3.4.2, Lemma 3.4.5 and the following identity

$$\varphi(x,y) = \frac{1}{p+1} K * \varphi^{p+1}(x,y). \tag{3.31}$$

■

Regarding on the decay properties of the solitary wave solutions of (3.1), one can prove the following properties of the kernel K.

LEMMA 3.4.9 *$K \in H^{s_1,0}(\mathbb{R}^2) \cap H^{0,s_2}(\mathbb{R}^2)$, for any $s_1 < \frac{1}{4}$ and $s_2 < \frac{1}{2}$. Moreover $K \in H^{r,s}(\mathbb{R}^2) \cap H^{(s_1,s_2)}(\mathbb{R}^2)$, where $rs_2 + ss_1 = s_1 s_2$ and $r \in [0,1]$.*

LEMMA 3.4.10 (i) *$\widehat{K} \in H^{s_1,0}(\mathbb{R}^2) \cap H^{0,s_2}(\mathbb{R}^2)$, for any $s_1 < \frac{3}{2}$ and $s_2 \in \mathbb{R}$. Moreover $\widehat{K} \in H^{r,s}(\mathbb{R}^2) \cap H^{(s_1,s_2)}(\mathbb{R}^2)$, where $rs_2 + ss_1 = s_1 s_2$ and $r \in [0,1]$.*

(ii) *$\widehat{K} \in H_p^{(s_1,s_2)}(\mathbb{R}^2)$, for any $s_1 < 1 + \dfrac{1}{p}$, $p \geq 2$ and $s_2 \in \mathbb{R}$. Moreover $|x|^{s_1}|y|^{s_2}K \in L^p(\mathbb{R}^2)$, for any $s_1 < 2 - \dfrac{1}{p}$, $2s_1 + s_2 \geq 3\left(1 - \dfrac{1}{p}\right)$ and $p \geq 1$.*

LEMMA 3.4.11 *Let ℓ and m be two constants satisfying $0 < \ell < m - n$. Then there exists $C > 0$, depending only on ℓ, m and n, such that for all $\epsilon > 0$, we have*

$$\int_{\mathbb{R}^n} \frac{|y|^\ell}{(1+\epsilon|y|)^m (1+|x-y|)^m}\, dy \leq \frac{\mathcal{C}\,|x|^\ell}{(1+\epsilon|x|)^m}, \qquad \forall |x| \geq 1, \tag{3.32}$$

$$\int_{\mathbb{R}^n} \frac{1}{(1+\epsilon|y|)^m (1+|x-y|)^m}\, dy \leq \frac{\mathcal{C}}{(1+\epsilon|x|)^m}, \qquad \forall x \in \mathbb{R}^n. \tag{3.33}$$

The proof of Lemma 3.4.11 is elementary and is essentially the same as the proof of Lemma 3.1.1 in [13].

THEOREM 3.4.12 *For any solitary wave solution of (3.2), we have*

(i) *$|x|^\ell |y|^\varrho \varphi(x,y) \in L^p(\mathbb{R}^2)$ for all $p \in (1,+\infty)$, any $\ell \in [0,1)$ and any $\varrho \geq 0$,*

(ii) *$|(x,y)|^\theta \varphi(x,y) \in L^p(\mathbb{R}^2)$ for all $p \in (1,+\infty)$ and any $\theta \in [0,1)$,*

(iii) *$\varphi \in L^1(\mathbb{R}^2)$.*

Proof. (i) Choose $\ell \in \left[0, s_1 - 1 + \dfrac{1}{p}\right)$ and $p > 1$, where $s_1 < 2 - \dfrac{1}{p}$. For $0 < \epsilon < 1$, we denote

$$\hbar_\epsilon(x,y) = \mathcal{A}(x,y)\,\varphi(x,y),$$

where $\mathcal{A}(x,y) = \dfrac{|x|^\ell |y|^\varrho}{(1+\epsilon|x|)^{s_1}(1+\epsilon|y|)^{s_2}}$ and $s_2 \geq 3\left(1 - \dfrac{1}{p}\right)$. Then $\hbar_\epsilon \in L^{p'}(\mathbb{R}^2)$. Using Hölder's inequality, we obtain that

$$|f(x,y)| \leq C(s_1,s_2,p)\left(\int_{\mathbb{R}^2} |\mathcal{G}(z,w)|^{p'}\, dzdw\right)^{\frac{1}{p'}},$$

where $g(t) = \dfrac{t^{p+1}}{p+1}$,

$$\mathcal{G}(z,w) = \frac{g(\varphi)(z,w)}{(1+|x-z|)^{s_1}(1+|y-w|)^{s_2}}$$

and $C(s_1,s_2,p) = \|(1+|x|)^{s_1}(1+|y|)^{s_2}K\|_{L^p(\mathbb{R}^2)}$. Note that the fact that $\varphi \to 0$ as $|(x,y)| \to +\infty$ implies that for every $\delta > 0$, there exists $R_\delta > 1$ such that if $|(x,y)| \geq R_\delta$, we have $\left|g(\varphi)(x,y)\right| \leq \delta|\varphi(x,y)|$. By using Hölder's inequality, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'}\,dxdy &= \int_{\mathbb{R}^2\setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'-r}\mathcal{A}^r(x,y)|\varphi(x,y)|^r\,dxdy\\ &\le C(s_1,s_2,p)^r\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'-r}\mathcal{A}^r(x,y)\,\|\mathcal{G}\|^r_{L^{p'}(\mathbb{R}^2)}\,dxdy\\ &\le C(s_1,s_2,p)^r\|\hbar_\epsilon\|^{p'-r}_{L^{p'}(\mathbb{R}^2\setminus B(0,R_\delta))}\left\|\,\mathcal{A}\,\|\mathcal{G}\|_{L^{p'}(\mathbb{R}^2)}\right\|^r_{L^{p'}(\mathbb{R}^2\setminus B(0,R_\delta))}\end{aligned}$$

Thusly,

$$\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'}\,dxdy \le C(s_1,s_2,p)^{p'}\int_{\mathbb{R}^2\setminus B(0,R_\delta)} \mathcal{A}^{p'}(x,y)\|\mathcal{G}\|^{p'}_{L^{p'}(\mathbb{R}^2)}\,dxdy.$$

Using Fubini's theorem and Lemma 3.4.11, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2\setminus B(0,R_\delta)} \mathcal{A}^{p'}(x,y)\|\mathcal{G}\|^{p'}_{L^{p'}(\mathbb{R}^2)}\,dxdy\\ &= \int_{\mathbb{R}^2}\left|g(\varphi)(z,w)\right|^{p'}\left(\int_{\mathbb{R}^2\setminus B(0,R_\delta)}\frac{\mathcal{A}^{p'}(x,y)}{(1+|x-z|)^{p's_1}(1+|y-w|)^{p's_2}}\,dxdy\right)dzdw\\ &\le \mathcal{C}\int_{\mathbb{R}^2\setminus B(0,R_\delta)}\left|g(\varphi)(z,w)\right|^{p'}\mathcal{A}^{p'}(z,w)\,dzdw\\ &+\int_{B(0,R_\delta)}\left|g(\varphi)(z,w)\right|^{p'}\left(\int_{\mathbb{R}^2\setminus B(0,R_\delta)}\frac{\mathcal{A}^{p'}(x,y)}{(1+|x-z|)^{p's_1}(1+|y-w|)^{p's_2}}\,dxdy\right)dzdw.\end{aligned}$$

The last integral is bounded by a constant $\mathcal{C}'$ depending on φ and R_δ and independent of ϵ. Therefore, by using the fact that $\left|g(\varphi)(x,y)\right| \le \delta|\varphi(x,y)|$ on $\mathbb{R}^2\setminus B(0,R_\delta)$, we get

$$\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'}\,dxdy \le C(s_1,s_2,p)^{p'}\left(\mathcal{C}\delta^{p'}\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'}\,dxdy + \mathcal{C}'\right).$$

Choosing δ such that $C(s_1,s_2,p)\delta\mathcal{C}^{\frac{1}{p'}} < 1$, from the last inequality, we deduce that

$$\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'}\,dxdy \le \mathcal{C}, \tag{3.34}$$

where $\mathcal{C}$ is a constant independent of ϵ. Now, we let $\epsilon\to 0$ in (3.34) and apply Fatou's lemma to obtain that

$$\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |x|^{\ell p'}|y|^{\varrho p'}|\varphi(x,y)|^{p'}\,dxdy \le \mathcal{C}.$$

Hence $|x|^\ell|y|^\varrho\varphi(x,y)\in L^{p'}\left(\mathbb{R}^2\right)$, for $p' = \dfrac{p}{p-1}$.

Now by taking the limits $p\to 1$ and $p\to+\infty$, we obtain that $\ell\to 1$ and $p'\in(1,+\infty)$. This proves the theorem.

(ii) The proof follows from (i).

(iii) Let $s > 1$ and $\mathcal{g}$, δ and R_δ be the same in (i). Define $\mathcal{A}_\epsilon(x,y) = \dfrac{1}{(1+\epsilon|(x,y)|)^s}$. Therefore, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |\varphi(x,y)|\mathcal{A}_\epsilon(x,y)\, dxdy \\
&\quad \le \int_{\mathbb{R}^2} \left|\mathcal{g}(\varphi)(z,w)\right| \left(\int_{\mathbb{R}^2\setminus B(0,R_\delta)} \mathcal{A}_\epsilon(x,y) K(x-z,y-w)\, dxdy \right) dzdw \\
&\quad \le \int_{\mathbb{R}^2} \left|\mathcal{g}(\varphi)(z,w)\right| \left(\int_{\mathbb{R}^2\setminus B(0,R_\delta)} \mathcal{A}_1^{-2}(x-z,y-w) K^2(x-z,y-w)\, dxdy \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_{\mathbb{R}^2\setminus B(0,R_\delta)} \mathcal{A}_1^2(x-z,y-w)\mathcal{A}_\epsilon^2(x,y)\, dxdy \right)^{\frac{1}{2}} dzdw \\
&\quad \le C(s)\mathcal{C}^{\frac{1}{2}} \int_{\mathbb{R}^2} \left|\mathcal{g}(\varphi)(z,w)\right| \mathcal{A}_\epsilon(z,w)\, dzdw \\
&\quad \le C(s)\mathcal{C}^{\frac{1}{2}}\delta \int_{\mathbb{R}^2\setminus B(0,R_\delta)} |\varphi(z,w)|\mathcal{A}_\epsilon(z,w)\, dzdw + C(s)\mathcal{C}^{\frac{1}{2}} \int_{B(0,R_\delta)} \left|\mathcal{g}(\varphi)(z,w)\right|\, dzdw,
\end{aligned}
$$

by using Fubini's theorem, Lemma 3.4.11 and this fact that φ, $\mathcal{A}_\epsilon \in L^2(\mathbb{R}^2)$ and $\varphi\mathcal{A}_\epsilon \in L^1(\mathbb{R}^2)$. Hence by the restriction on δ, and using Fatou's lemma as $\epsilon \to 0$, it concludes that $\varphi \in L^1(\mathbb{R}^2)$. ■

The following corollary is an immediate consequence of (3.31), the Theorem 3.4.12 and the inequality

$$|t|^\theta \le C\left(|t-s|^\theta + |s|^\theta\right). \tag{3.35}$$

Corollary 3.4.13 *Suppose that $\varphi \in L^\infty(\mathbb{R}^2)$ satisfies (3.2) and $\varphi \to 0$ at infinity. Then*

(i) $|x|^\ell |y|^\varrho \varphi(x,y) \in L^\infty(\mathbb{R}^2)$, *for all $\ell \in [0,1)$ and any $\varrho \ge 0$,*

(ii) $|(x,y)|^\theta \varphi(x,y) \in L^\infty(\mathbb{R}^2)$, *for all $\theta \in [0,1)$.*

LEMMA 3.4.14 $|x|^\ell |y|^\varrho K \in L^\infty(\mathbb{R}^2)$, *for any $\ell \le 2$ and any $\varrho \ge 0$.*

Proof. Suppose that $|x| \ge 1$, so we have

$$K(x,y) < C\int_0^{+\infty} \frac{e^{-ct}\sqrt{t}}{x^2} e^{-\frac{y^2}{4t}}\, dt \le C\int_0^{+\infty} \frac{e^{-ct}\sqrt{t}}{x^2}\left(\frac{4t}{y^2}\right)^\nu dt = \frac{C}{x^2|y|^{2\nu}},$$

for any $\nu \ge 0$. On the other hand, for $0 < |x| \le 1$, by a change of variables, we have that

$$K(x,y) \le \frac{C}{\sqrt{|x|}} \int_0^{+\infty} \frac{e^{-t|x|}\sqrt{t}}{1+t^2} e^{-\frac{y^2}{4|x|t}}\, dt = C\frac{|x|^{\nu-\frac{1}{2}}}{|y|^{2\nu}},$$

for any $\nu \ge 0$. This completes the proof. ■

Corollary 3.4.15 *$|x|^\ell |y|^\varrho \varphi(x,y) \in L^\infty(\mathbb{R}^2)$, for any $\ell \leq 2$ and any $\varrho \geq 0$.*

Proof. Without loss of generality, we assume that $\varrho = 0$. Let $\ell < 1$ and $\gamma = \min\{2, (p+1)\ell\}$. Because

$$|x|^\gamma \left|\varphi^{p+1}(x,y)\right| \leq \left(|\varphi(x,y)||x|^{\frac{\gamma}{p+1}}\right)\left(|\varphi(x,y)||x|^{\frac{\gamma}{p+1}}\right)^p,$$

then by using (3.35), Corollary 3.4.13, Lemma 3.4.14 and Theorem 3.4.12, we obtain that $|x|^\gamma\varphi(x,y) \in L^\infty(\mathbb{R}^2)$. If $\gamma = (p+1)\ell$, one may use the above argument to show that $|x|^\gamma\varphi(x,y) \in L^\infty(\mathbb{R}^2)$ for $\gamma = \min\{2, (p+1)^2\ell\}$. Then repeating this argument at most finitely many times leads to the conclusion. ■

The following corollary follows from (3.35), Corollary 3.4.13 and Theorem 3.4.12.

Corollary 3.4.16 (i) *$|x|^\ell|y|^\varrho\varphi(x,y) \in L^1(\mathbb{R}^2)$, for all $\ell \in [0,1)$ and any $\varrho \geq 0$,*

(ii) *$|(x,y)|^\theta\varphi(x,y) \in L^1(\mathbb{R}^2)$, for all $\theta \in [0,1)$.*

LEMMA 3.4.17 *There exists $\sigma_0 > 0$ such that for any $\sigma < \sigma_0$ and any $s < \frac{3}{2}$, we have*

(i) *$|x|^s e^{\sigma|y|}K \in L^2(\mathbb{R}^2)$,*

(ii) *$K \in L^p_y L^1_x(\mathbb{R}^2)$; for any $1 \leq p \leq \infty$,*

(iii) *$|x|^s e^{\sigma|y|}K \in L^2_y L^1_x(\mathbb{R}^2)$; where $\|\cdot\|_{L^q_y L^p_x(\mathbb{R}^2)} = \left\|\|\cdot\|_{L^p_x}\right\|_{L^q_y}$.*

Proof.

By a change of variables, K can be written in the following form

$$K(x,y) = |\alpha| \int_0^{+\infty} \frac{e^{-c|x|t}}{\alpha^2+t^2}\left(\frac{t}{|x|}\right)^{\frac{1}{2}} e^{-\frac{y^2}{4|x|t}}\,dt.$$

Hence,

$$\left\||x|^s e^{\sigma|y|}K\right\|_{L^2(\mathbb{R}^2)} \leq \alpha\int_0^{+\infty}\frac{\sqrt{t}}{\alpha^2+t^2}\left(\int_{\mathbb{R}}|x|^{2s-1}\,e^{-2c|x|t}\int_{\mathbb{R}} e^{2\sigma|y|}e^{-\frac{y^2}{2|x|t}}\,dydx\right)^{\frac{1}{2}}dt$$

$$\leq C(\alpha)\int_0^{+\infty}\frac{t^{\frac{3}{4}}}{\alpha^2+t^2}\left(\int_{\mathbb{R}}|x|^{2s-\frac{1}{2}}\,e^{-2|x|t\left(c-\sigma^2\right)}dx\right)^{\frac{1}{2}}dt = C(\alpha)\frac{\sqrt{\Gamma\left(2s+\frac{1}{2}\right)}}{(2(c-\sigma^2))^{s+\frac{1}{4}}}\int_0^{+\infty}\frac{t^{\frac{1}{2}-s}}{\alpha^2+t^2}\,dt,$$

which is finite for any $\sigma < \sqrt{c}$ and any $s < \frac{3}{2}$. The proof of (ii) follows from the identity

$$\|K\|_{L^1_x} = C(\alpha)\int_0^\infty\frac{e^{-ct-\frac{y^2}{4t}}}{\sqrt{t}}\,dt = \frac{C(\alpha)}{\sqrt{c}}e^{-\sqrt{c}|y|}.$$

The proof of (iii) is similar. ■

The following corollary is a consequence of Lemma 0.0.15.

Corollary 3.4.18 *$\varphi \in L^p_y L^1_x\left(\mathbb{R}^2\right)$, for any $1 \le p \le \infty$.*

Now we state our main result of decaying of the solitary wave solutions.

THEOREM 3.4.19 *There exists $\sigma_0 > 0$ such that for any $\sigma \in [0, \sigma_0)$ and any $s < \frac{3}{2}$, we have that $|x|^s e^{\sigma|y|}\varphi(x,y) \in L^1\left(\mathbb{R}^2\right) \cap L^\infty\left(\mathbb{R}^2\right)$.*

Proof. Without loss of generality we assume that $s = 0$. By using Lemma 3.4.17 and the proof of Corollary 3.14 in [13], with natural modifications, there exists $\widetilde{\sigma} \ge \sigma_0$ such that $e^{\sigma|y|}\varphi(x,y) \in L^1\left(\mathbb{R}^2\right)$, for any $\sigma < \widetilde{\sigma}$. Now by using the following inequality:

$$|\varphi(x,y)|e^{\sigma|y|} \le \int_{\mathbb{R}^2} |K(x-z, y-w)|e^{\sigma|y-w|}|\varphi(z,w)|e^{\sigma|w|}|\varphi(z,w)|^p \, dz dw, \tag{3.36}$$

and the facts $\varphi(x,y)e^{\sigma|y|} \in L^1\left(\mathbb{R}^2\right)$, $\varphi \in L^\infty\left(\mathbb{R}^2\right)$ and $K(x,y)e^{\sigma|y|} \in L^2\left(\mathbb{R}^2\right)$, for any $\sigma < \sigma_0$, we obtain that $\varphi(x,y)e^{\sigma|y|} \in L^\infty\left(\mathbb{R}^2\right)$, for any $\sigma < \sigma_0$. ■

Finally, the following theorem shows that analyticity of our solitary wave solutions.

THEOREM 3.4.20 *There exists $\sigma > 0$ and an holomorphic function f of two variables z_1 and z_2, defined in the domain $\mathscr{H}_\sigma = \left\{(z_1, z_2) \in \mathbb{C}^2 \,;\, |\mathrm{Im}(z_1)| < \sigma,\ |\mathrm{Im}(z_2)| < \sigma\right\}$ such that $f(x,y) = \varphi(x,y)$ for all $(x,y) \in \mathbb{R}^2$.*

Proof. By the Cauchy-Schwarz inequality, we have that $\widehat{\varphi} \in L^1\left(\mathbb{R}^2\right)$. Equation (3.2) implies that

$$|\xi|\,|\widehat{\varphi}|\,(\xi,\eta) \le \overbrace{|\widehat{\varphi}| * \cdots * |\widehat{\varphi}|}^{p+1}(\xi,\eta), \tag{3.37}$$

$$|\eta|\,|\widehat{\varphi}|\,(\xi,\eta) \le \underbrace{|\widehat{\varphi}| * \cdots * |\widehat{\varphi}|}_{p+1}(\xi,\eta). \tag{3.38}$$

We denote $\mathscr{T}_1(|\widehat{\varphi}|) = |\widehat{\varphi}|$ and for $m \ge 1$, $\mathscr{T}_{m+1}(|\widehat{\varphi}|) = \mathscr{T}_m(|\widehat{\varphi}|) * |\widehat{\varphi}|$. It can be seen by induction that

$$r^m|\widehat{\varphi}|(\xi,\eta) \le (m-1)!\,(p+1)^{m-1}\mathscr{T}_{mp+1}(|\widehat{\varphi}|)(\xi,\eta), \tag{3.39}$$

where $r = |(\xi,\eta)|$. Then we have

$$\begin{aligned} r^m|\widehat{\varphi}|(\xi,\eta) &\le (m-1)!\,(m+1)^{m-1}\left\|\mathscr{T}_{mp+1}(|\widehat{\varphi}|)\right\|_{L^\infty(\mathbb{R}^2)} \le (m-1)!\,(m+1)^{m-1}\left\|\mathscr{T}_{mp}(|\widehat{\varphi}|)\right\|_{L^2(\mathbb{R}^2)}\|\widehat{\varphi}\|_{L^2(\mathbb{R}^2)} \\ &\le (m-1)!\,(m+1)^{m-1}\|\widehat{\varphi}\|^{mp}_{L^1(\mathbb{R}^2)}\|\widehat{\varphi}\|^2_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Let $a_m = \dfrac{(m-1)!\,(m+1)^{m-1}\|\widehat{\varphi}\|^{mp}_{L^1(\mathbb{R}^2)}\|\widehat{\varphi}\|^2_{L^2(\mathbb{R}^2)}}{m!}$, then it is clear that

$$\frac{a_{m+1}}{a_m} \longrightarrow (p+1)\|\widehat{\varphi}\|^p_{L^1(\mathbb{R}^2)},$$

as $m \to +\infty$. Therefore the series $\sum_{m=0}^\infty t^m r^m|\widehat{\varphi}|(\xi,\eta)/m!$ converges uniformly in $L^\infty(\mathbb{R}^2)$, if $0 < t < \sigma = \frac{1}{p+1}\|\widehat{\varphi}\|^{-p}_{L^1(\mathbb{R}^2)}$. Hence $e^{tr}\widehat{\varphi}(\xi,\eta) \in L^\infty(\mathbb{R}^2)$, for $t < \sigma$.

We define the function

$$f(z_1, z_2) = \int_{\mathbb{R}^2} e^{i(\xi z_1 + \eta z_2)}\widehat{\varphi}(\xi,\eta)\, d\xi d\eta.$$

By the Paley-Wiener Theorem, f is well defined and analytic in $\mathcal{H}_\sigma$; and by Plancherel's Theorem, we have $f(x, y) = \varphi(x, y)$ for all $(x, y) \in \mathbb{R}^2$. ■

Chapter 4

High Dimensional Benjamin Equations

The Benjamin equation

$$u_t + 2uu_x + \rho u_{xxx} + \varrho \mathscr{H} u_{xx} = 0,$$

where $\mathscr{H} = \mathscr{H}^{(x)}$ is the Hilbert transform with respect to x variable, governs the propagation of straight-crested unidirectional weakly nonlinear long waves on the interface of this two-fluid system, ignoring the effects of viscosity and assuming that interfacial tension is large and the fluid densities are nearly equal. Under these flow conditions, the two dimensional Benjamin equation

$$u_t + \alpha u_{xxx} + \beta \mathscr{H} u_{xx} + \epsilon v_y + u^p u_x = 0, \tag{4.1}$$

$$u_y = v_x, \quad (x, y) \in \mathbb{R}^2, \quad t \geq 0, \tag{4.2}$$

and three dimensional Benjamin equation

$$u_t + \alpha u_{xxx} + \beta \mathscr{H} u_{xx} + a v_y + b w_z + u^p u_x = 0, \tag{4.3}$$

$$u_y = v_x, \tag{4.4}$$

$$u_z = w_x, \quad (x, y, z) \in \mathbb{R}^3, \quad t \geq 0, \tag{4.5}$$

are some extensions of the Benjamin equation that allows for weak spatial variations transverse to the propagation direction, and can be derived by a standard weakly nonlinear long-wave expansion, where the constants ϵ, a, b measure the transverse dispersion effects and are normalized to ± 1 and the constants α and β are real. These equations are some models for interfacial gravity-capillary waves; namely, we consider an extension to two spatial dimensions of the evolution equation derived by Benjamin ([4]) for weakly nonlinear long waves on the interface of a two-fluid system, in the case that the upper layer is bounded by a rigid lid and lies on top of an infinitely deep fluid. The usual 2-DB equation and 3-DB equation correspond to the nonlinearity uu_x (see [47] and references therein). When $\beta = 0$, the 2-DB and 3-DB equations are known as two and three dimensional KP equations, respectively.

This Chapter is devoted to nonexistence, existence of the solitary wave solutions of generalized n-dimensional Benjamin equation. We also use the variational properties of the problem to obtain the symmetry and blow-up results. Furthermore, we will show that some regularity and decay of the solitary wave solutions.

4.1 (Non)Existence

In order to give a precise definition of our needed spaces, we use the following spaces. We shall denote, $\mathscr{X}$ the closure of $\partial_x(C_0^\infty(\mathbb{R}^n))$ for the norm

$$\|\partial_x\varphi\|^2_{\mathscr{X}} = \|\nabla\varphi\|^2_{L^2(\mathbb{R}^n)} + \|\partial_x^2\varphi\|^2_{L^2(\mathbb{R}^n)} \tag{4.6}$$

where $n = 2, 3$ and $\partial_x(C_0^\infty(\mathbb{R}^n))$ denotes the space of functions of the form $\partial_x\varphi$ with $\varphi \in \partial_x(C_0^\infty(\mathbb{R}^n))$, that is, the space of functions $\psi \in \partial_x(C_0^\infty(\mathbb{R}^n))$ in $\partial_x(C_0^\infty(\mathbb{R}^n))$ such that $\int_{\mathbb{R}} \psi(x, y)\, dx = 0$, for every $y \in \mathbb{R}^{n-1}$.

By a solitary wave solution of 2-DB equation (respectively 3-DB equation), we mean a solution of (4.1)-(4.2) (respectively (4.3)-(4.5)) of the type $u(x - c_1t, y - c_2t)$ (respectively $u(x - c_1t, y - c_2t, z - c_3t)$), where $u \in \mathscr{X}$, $c_1, c_2, c_3 \in \mathbb{R}$ are the speeds of propagation of the wave along each direction. So we are looking for *localized* solutions of the systems

$$\begin{cases} -c_1u_x - c_2u_y + \alpha u_{xxx} + \beta\mathscr{H}u_{xx} + \epsilon v_y + u^pu_x = 0 \\ u_y = v_x \end{cases} \tag{4.7}$$

and

$$\begin{cases} -c_1u_x - c_2u_y - c_3u_z + \alpha u_{xxx} + \beta\mathscr{H}u_{xx} + a v_y v_y + b w_z + u^pu_x = 0 \\ u_y = v_x \\ u_z = w_x, \end{cases} \tag{4.8}$$

respectively. By a change of variables $\widetilde{x} = x$, $\widetilde{y} = y - \frac{1}{2}\epsilon c_2 x$ in the two dimensional case, and $\widetilde{x} = x$, $\widetilde{y} = y - \frac{1}{2}a c_2 x$, $\widetilde{z} = z - \frac{1}{2}b c_3 x$ in the three dimensional case, after dropping the tilde, we obtain the new systems

$$\begin{cases} -cu_x + \alpha u_{xxx} + \beta\mathscr{H}u_{xx} + \epsilon v_y + u^pu_x = 0 \\ u_y = v_x \end{cases} \tag{4.9}$$

with $c = c_1 + \frac{1}{4}\epsilon c_2^2$ and

$$\begin{cases} -cu_x + \alpha u_{xxx} + \beta\mathscr{H}u_{xx} + a v_y v_y + b w_z + u^pu_x = 0 \\ u_y = v_x \\ u_z = w_x \end{cases} \tag{4.10}$$

with $c = c_1 + \frac{1}{4}a c_2^2 + \frac{1}{4}b c_3^2$. By using the Pohozaev type identities, we obtain the situations of the nonexistence of the solitary wave solutions. We apply the following truncation argument to gain the regularity we need.

THEOREM 4.1.1 *Let $|\alpha| + |\beta| > 0$. The equation (4.9) does not admit any nontrivial solitary wave satisfying*

$$u = \partial_x\varphi \in \mathscr{X}, \quad u \in H^1(\mathbb{R}^2) \cap L^\infty_{loc}(\mathbb{R}^2),$$
$$\partial_y^2\varphi \in L^2_{loc}(\mathbb{R}^2), \quad \partial_x^2\varphi \in L^2_{loc}(\mathbb{R}^2),$$

(I) *if* $\epsilon = 1$ *and one of the following cases occurs:*

(i) $\alpha, \beta \geq 0$, $c < 0$ *and* $p \geq 4/3$,

(ii) $\alpha \leq 0$, $\beta \geq 0$, $c > 0$ *and* $p \leq 4/3$,

(iii) $\alpha \leq 0$, $\beta \geq 0$, $c < 0$ *and* $p \geq 4$,

(iv) $\alpha \geq 0$, $\beta \leq 0$, $c \in \mathbb{R}^*$ *and* p *is arbitrary,*

(v) $\alpha, \beta \leq 0$, $c > 0$ *and* $p \leq 4$,

or

(II) *if* $\epsilon = -1$ *and one of the following cases occurs:*

(i) $\alpha, \beta \geq 0$, $c < 0$ *and* $p \leq 4$,

(ii) $\alpha \leq 0$, $\beta \geq 0$, $c \in \mathbb{R}^*$ *and* p *is arbitrary,*

(iii) $\alpha \geq 0$, $\beta \leq 0$, $c > 0$ *and* $p \geq 4$,

(iv) $\alpha \geq 0$, $\beta \leq 0$, $c < 0$ *and* $p \leq 4/3$,

(v) $\alpha, \beta \leq 0$, $c > 0$ *and* $p \geq 4/3$.

THEOREM 4.1.2 *Let* $|\alpha| + |\beta| > 0$. *The equation (4.10) does not admit any nontrivial solitary wave satisfying*

$$u = \partial_x \varphi \in \mathscr{X}, \quad u \in H^1(\mathbb{R}^3) \cap L^\infty_{loc}(\mathbb{R}^3) \cap L^{2(p+1)}(\mathbb{R}^3),$$
$$\partial_y^2 \varphi \in L^2_{loc}(\mathbb{R}^3), \quad \partial_z^2 \varphi \in L^2_{loc}(\mathbb{R}^3), \quad \partial_x^2 \varphi \in L^2_{loc}(\mathbb{R}^3),$$

(I) *if* $\mathscr{a}\mathscr{b} = -1$, *or*

(II) *if* $\mathscr{a} = \mathscr{b} = 1$ *and one of the following cases occurs:*

(i) $\alpha, \beta \geq 0$, $c < 0$ *and* $p \geq 2/3$,

(ii) $\alpha \leq 0$, $\beta \geq 0$, $c > 0$ *and* $p \leq 2/3$,

(iii) $\alpha \leq 0$, $\beta \geq 0$, $c < 0$ *and* $p \geq 4/3$,

(iv) $\alpha \geq 0$, $\beta \leq 0$, $c \in \mathbb{R}^*$ *and* p *is arbitrary,*

(v) $\alpha, \beta \leq 0$, $c > 0$ *and* $p \leq 4/3$,

or

(III) *if* $\mathscr{a} = \mathscr{b} = -1$ *and one of the following cases occurs:*

(i) $\alpha, \beta \geq 0$, $c < 0$ *and* $p \leq 4/3$,

(ii) $\alpha \leq 0$, $\beta \geq 0$, $c \in \mathbb{R}^*$ *and* p *is arbitrary,*

(iii) $\alpha \geq 0$, $\beta \leq 0$, $c > 0$ *and* $p \geq 4/3$,

(iv) $\alpha \geq 0$, $\beta \leq 0$, $c < 0$ *and* $p \leq 2/3$,

(v) $\alpha, \beta \leq 0$, $c > 0$ *and* $p \geq 2/3$.

Proof of Theorem 4.1.1. Let $\chi_0 \in C_0^\infty(\mathbb{R})$, such that $0 \leq \chi_0 \leq 1$ and $\chi_0(s) = 1$ if $0 \leq |s| \leq 1$, $\chi_0(s) = 0$ if $|s| \geq 2$. We set $\chi_j(x, y) = \chi_0\left(\frac{r^2}{j^2}\right)$, where $r^2 = x^2 + y^2$, $j \in \mathbb{N}$ and $y \in \mathbb{R}^{n-1}$ for $n = 2, 3$.
First, we consider the two dimensional case.
We multiply the first equation of the system (4.9) by $xu\chi_j$ and we integrate over $\mathbb{R}^2$ to get

$$-c \int x\chi_j uu_x \, dxdy + \int x\chi_j u^{p+1}u_x \, dxdy + \alpha \int x\chi_j uu_{xxx} \, dxdy + \beta \int x\chi_j u\mathscr{H}u_{xx} \, dxdy + \epsilon \int x\chi_j uv_y \, dxdy = 0.$$

By using the integration by parts we obtain

$$\begin{aligned}
0 = &\frac{c}{2} \int \chi_j u^2 \, dxdy + \frac{c}{j^2} \int x^2\chi_0'\left(\frac{r^2}{j^2}\right) u^2 \, dxdy - \frac{3\alpha}{j^2} \int \chi_0'\left(\frac{r^2}{j^2}\right) u^2 \, dxdy - \frac{2\beta}{j^2} \int \chi_0'\left(\frac{r^2}{j^2}\right) u\mathscr{H}u_x \, dxdy \\
&- \frac{6\alpha}{j^4} \int \chi_0''\left(\frac{r^2}{j^2}\right) u^2 \, dxdy - \frac{6\alpha}{j^4} \int x\chi_0''\left(\frac{r^2}{j^2}\right) u^2 - \frac{4\alpha}{j^6} \int x^3\chi_0'''\left(\frac{r^2}{j^2}\right) u^2 \, dxdy + \frac{3\alpha}{j^2} \int x\chi_0'\left(\frac{r^2}{j^2}\right) u_x^2 \, dxdy \\
&+ \frac{3\alpha}{2} \int \chi_j u_x^2 \, dxdy + \frac{\epsilon}{2} \int \chi_j v^2 \, dxdy - \beta \int \chi_j u\mathscr{H}u_x \, dxdy - \frac{2\epsilon}{j^2} \int xy\chi_0'\left(\frac{r^2}{j^2}\right) uv \, dxdy \\
&- \frac{1}{p+2} \int \chi_j u^{p+2} \, dxdy + \frac{1}{j^2} \int x\chi_0'\left(\frac{r^2}{j^2}\right) v^2 \, dxdy - \frac{2}{j^2(p+2)} \int x^2\chi_0'\left(\frac{r^2}{j^2}\right) u^{p+2} \, dxdy - \beta \int x\chi_j u_x\mathscr{H}u_x \, dxdy.
\end{aligned}$$

Now by using Lebesgue dominated convergence theorem, we get

$$\int_{\mathbb{R}^2} \left(cu^2 + 3\alpha u_x^2 - 2\beta u\mathscr{H}u_x + \epsilon v^2 - \frac{2}{p+2} u^{p+2} \right) \, dxdy = 0. \tag{4.11}$$

Next we multiply the first equation of the system (4.9) by $yv\chi_j$ and we integrate over $\mathbb{R}^2$; similar to above, by using the integration by parts and Lebesgue dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^2} \left(-cu^2 - \alpha u_x^2 + \beta u\mathscr{H}u_x - \epsilon v^2 - \frac{2}{(p+1)(p+2)} u^{p+2} \right) \, dxdy = 0. \tag{4.12}$$

Now we multiply the first equation of the system (4.9) by $u\chi_j$ and we integrate over $\mathbb{R}^2$; similar to above, by using the integration by parts and Lebesgue dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^2} \left(-cu^2 - \alpha u_x^2 + \beta u\mathscr{H}u_x + \epsilon v^2 + \frac{1}{p+1} u^{p+2} \right) \, dxdy = 0. \tag{4.13}$$

By adding (4.11) and (4.12) we get

$$\int_{\mathbb{R}^2} \left(2\alpha u_x^2 - \beta u\mathscr{H}u_x - \frac{2p}{(p+1)(p+2)} u^{p+2} \right) \, dxdy = 0. \tag{4.14}$$

By subtracting (4.11) from (4.12) we obtain

$$\int_{\mathbb{R}^2} \left(cu^2 + 2\alpha u_x^2 - \frac{3}{2}\beta u\mathscr{H}u_x - \frac{1}{p+1} u^{p+2} \right) \, dxdy = 0. \tag{4.15}$$

Adding (4.14) and (4.13) yields

$$\int_{\mathbb{R}^2} \left(-cu^2 + \frac{\beta}{2} u\mathscr{H}u_x + \epsilon v^2 + \frac{2}{(p+1)(p+2)} u^{p+2} \right) \, dxdy = 0. \tag{4.16}$$

By adding (4.15) and (4.13) we have

$$\int_{\mathbb{R}^2} \left(2\alpha u_x^2 - \beta u\mathscr{H}u_x + 4\epsilon v^2 \right) \, dxdy = 0, \tag{4.17}$$

which rules out (I)(iv) and (II)(ii).
Subtracting (4.17) from (4.14) yields

$$\int_{\mathbb{R}^2} \left(2\epsilon v^2 + \frac{p}{(p+1)(p+2)} u^{p+2} \right) \, dxdy = 0. \tag{4.18}$$

Eliminating u^{p+2} by (4.18) and (4.16) leads to

$$\int_{\mathbb{R}^2} \left[-cu^2 + \frac{\beta}{2} u\mathscr{H}u_x + \epsilon \left(\frac{p-4}{p} \right) v^2 \right] \, dxdy = 0, \tag{4.19}$$

which rules out (I)(iii), (I)(v), (II)(i) and (II)(iii).
Adding (4.11) and 2 times (4.12), and using (4.18), we obtain

$$\int_{\mathbb{R}^2} \left[-cu^2 + \alpha u_x^2 + \epsilon \left(\frac{3p-4}{p} \right) v^2 \right] \, dxdy = 0, \tag{4.20}$$

which rules out (I)(i), (I)(ii), (II)(iv) and (II)(v). ∎

Proof of Theorem 4.1.2. In dimension three, by the aforementioned truncation process, by multiplying the first equation of the system (4.10) by $xu\chi_j$, $yv\chi_j$, $zw\chi_j$ and $u\chi_j$ respectively; integration by parts, and Lebesgue dominated convergence theorem, we obtain the following relations:

$$\int_{\mathbb{R}^3} \left(cu^2 + 3\alpha u_x^2 - 2\beta u\mathscr{H}u_x + \mathfrak{a} v^2 + \mathfrak{b} w^2 - \frac{2}{p+2} u^{p+2} \right) \, dxdydz = 0, \tag{4.21}$$

$$\int_{\mathbb{R}^3} \left(-cu^2 - \alpha u_x^2 + \beta u\mathscr{H}u_x - \mathfrak{a} v^2 + \mathfrak{b} w^2 + \frac{2}{(p+1)(p+2)} u^{p+2} \right) \, dxdydz = 0, \tag{4.22}$$

$$\int_{\mathbb{R}^3} \left(-cu^2 - \alpha u_x^2 + \beta u\mathscr{H}u_x + \mathfrak{a} v^2 - \mathfrak{b} w^2 + \frac{2}{(p+1)(p+2)} u^{p+2} \right) \, dxdydz = 0, \tag{4.23}$$

$$\int_{\mathbb{R}^3} \left(-cu^2 - \alpha u_x^2 + \beta u\mathscr{H}u_x + \mathfrak{a} v^2 + \mathfrak{b} w^2 + \frac{1}{p+1} u^{p+2} \right) \, dxdydz = 0. \tag{4.24}$$

By subtracting (4.22) from (4.23) yields

$$\int_{\mathbb{R}^3} \left(\mathfrak{a} v^2 - \mathfrak{b} w^2 \right) \, dxdydz = 0, \tag{4.25}$$

which rules out (I).
By adding (4.22) and (4.23) we obtain

$$\int_{\mathbb{R}^3} \left(-cu^2 - \alpha u_x^2 + \beta u \mathscr{H} u_x + \frac{2}{(p+1)(p+2)} u^{p+2} \right) dxdydz = 0. \tag{4.26}$$

Subtracting (4.24) from (4.26), using (4.25), we infer

$$\int_{\mathbb{R}^3} \left(2\mathfrak{a} v^2 + \frac{p}{(p+1)(p+2)} u^{p+2} \right) dxdydz = 0. \tag{4.27}$$

Adding 3 times (4.24) and 2 times (4.21) yields

$$\int_{\mathbb{R}^3} \left(-2cu^2 + \beta u \mathscr{H} u_x + 8\mathfrak{a} v^2 + \frac{p+4}{(p+1)(p+2)} u^{p+2} \right) dxdydz = 0. \tag{4.28}$$

Eliminating u^{p+2} by (4.28) and (4.27) leads to

$$\int_{\mathbb{R}^3} \left[-2cu^2 + \beta u \mathscr{H} u_x + \left(\frac{6p-8}{p} \right) \mathfrak{a} v^2 \right] dxdydz = 0, \tag{4.29}$$

which rules out (II)(iii), (II)(v), (III)(i) and (III)(iii).

Adding twice (4.26) and (4.11), using (4.27), we obtain

$$\int_{\mathbb{R}^3} \left[-\frac{c}{2} u^2 + \frac{\alpha}{2} u_x^2 + \mathfrak{a} \left(\frac{3p-2}{p} \right) v^2 \right] dxdydz = 0, \tag{4.30}$$

which rules out (II)(i), (II)(ii), (III)(iv) and (III)(v).
Adding (4.21) and (4.24), plugging the identity (4.25) and (4.27), yields

$$\int_{\mathbb{R}^3} \left(2\alpha u_x^2 - \beta u \mathscr{H} u_x + 6\mathfrak{a} v^2 \right) dxdydz = 0, \tag{4.31}$$

which rules out (II)(iv) and (III)(ii). ■

4.2 The Existence

In this section, we prove the existence of solitary wave solutions of equations (4.9) and (4.10) by using the minimization problem as before. The main theorems are stated in the following.

THEOREM 4.2.1 *Let $\alpha, c > 0$, $\beta \in \mathbb{R}$ and $p = \frac{k}{m}$, where $m \in \mathbb{N}$ is odd and m and k are relatively prime. We also suppose that if $\beta > 0$, then $\beta \in (0, 2\sqrt{\alpha c}\,)$.*

(I) *Let $n = 2$, $0 < p < 4$ and $\epsilon = -1$, Then the system (4.9) admits a nontrivial solution $u \in \mathscr{X}$.*

(II) *Let $n = 3$, $0 < p < 4/3$ and $\mathfrak{a} = \mathfrak{b} = -1$. Then the system (4.10) admits a nontrivial solution $u \in \mathscr{X}$.*

The proof will be done by using Lemma 0.0.1. The help of an embedding theorem for anisotropic Sobolev spaces due to Besov et al. [10] is needed to encounter the fact that $\mathscr{X}$ is not embedded in $H^2(\mathbb{R}^n)$. In particular, our minimizing sequence is not bounded in $H^1(\mathbb{R}^n)$ and we have to apply a compactness lemma in $L^2(\mathbb{R}^n)$ of bounded sequences of $\mathscr{X}$, a minimizing sequence u_n in $\mathscr{X}$ leads to a minimum u.

THEOREM 4.2.2 *Let $n,m\in\mathbb{N}$, $p\in(1,+\infty)^m$, $q\in(1,+\infty)$, $\varrho\in\mathbb{N}_0^n$, $\{\kappa_i\}_{i=1}^m\subset\mathbb{N}^n$ and $\mu\in(0,1)^m$ such that $|\mu|=1$ and satisfying*

$$\frac{1}{q}\leq\sum_{j=1}^m\frac{\mu_j}{p_j}\quad \textit{and}\quad \varrho_i-\frac{1}{q}=\sum_{j=1}^m\mu_j\left(\kappa_{ji}-\frac{1}{p_j}\right),$$

for each $1\leq i\leq n$. Then there exists constant $C>0$ such that $\|D^\varrho f\|_{L^q(\mathbb{R}^n)}\leq C\prod_{j=1}^m\|D^{\kappa_j}f\|_{L^{p_j}(\mathbb{R}^n)}$, for all $f\in C_0^\infty(\mathbb{R}^n)$.

Using this theorem, one can induce embedding of $\mathscr{X}$ into the $L^q(\mathbb{R}^n)$ spaces.

LEMMA 4.2.3 *Let $p_n=\frac{4n-2}{2n-3}$, $n=2,3$. Then for any $p\in[2,p_n]$, there exists $C>0$ such that*

$$\|u\|_{L^q(\mathbb{R}^n)}\leq C\|u\|_{\mathscr{X}}. \tag{4.32}$$

As a consequence of this lemma we have, the following, if $u\in\mathscr{X}$ and $n=3$, then $u=\partial_x\varphi$ where $\varphi\in L^6_{loc}(\mathbb{R}^3)$; and if $n=2$ and $u\in\mathscr{X}$, then $u=\partial_x\varphi$ where $\varphi\in L^q_{loc}(\mathbb{R}^2)$, for any $q\in[2,+\infty)$.
In fact, the proof follows from the previous theorem:

$$\|u\|^q_{L^q(\mathbb{R}^2)}\leq C\|u\|^{\frac{6-q}{2}}_{L^2(\mathbb{R}^2)}\|u_x\|^{q-2}_{L^2(\mathbb{R}^2)}\|v\|^{\frac{q-2}{2}}_{L^2(\mathbb{R}^2)},\quad q\in[2,6] \tag{4.33}$$

$$\|u\|^q_{L^q(\mathbb{R}^3)}\leq C\|u\|^{\frac{10-3q}{2}}_{L^2(\mathbb{R}^3)}\|u_x\|^{\frac{3(q-2)}{2}}_{L^2(\mathbb{R}^3)}\|v\|^{\frac{q-2}{2}}_{L^2(\mathbb{R}^3)}\|w\|^{\frac{q-2}{2}}_{L^2(\mathbb{R}^3)},\quad q\in[2,10/3]. \tag{4.34}$$

Proof of Theorem 4.2.1(I). We consider the minimization problem

$$I_\lambda=\inf\left\{\mathfrak{H}(u)\mid u\in\mathscr{X},\ J(u)=\int_{\mathbb{R}^2}u^{p+2}(x,y)\,dxdy=\lambda\right\}, \tag{4.35}$$

where $u=\varphi_x$, $\lambda>0$ and $\mathfrak{H}(u)=c\|\varphi_x\|^2_{L^2(\mathbb{R}^2)}+\|\varphi_y\|^2_{L^2(\mathbb{R}^2)}+\alpha\|\varphi_{xx}\|^2_{L^2(\mathbb{R}^2)}-\beta\left\|D_x^{1/2}\varphi_x\right\|^2_{L^2(\mathbb{R}^2)}$.

REMARK 4.2.4 *Note that according to the assumptions of the Theorem 4.2.1, $\mathfrak{H}(u)>0$, for all $0\neq u\in\mathscr{X}$. Also if $\beta\in\mathbb{R}\setminus[2\sqrt{\alpha c},+\infty)$, then $\mathfrak{H}(u)\sim\|u\|^2_{\mathscr{X}}$. However $\mathfrak{H}(\cdot)$ defines an equivalent norm to $\|\cdot\|_{\mathscr{X}}$ whenever $\beta\leq 0$.*

Let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence of I_λ in $\mathscr{X}$. Then there exists a sequence of functions $\varphi_n\in L^6(\mathbb{R}^3)$ satisfying $u_n=\partial_x\varphi_n$. We set $v_n=\partial_y\varphi_n=D_x^{-1}\partial_y u_n$ and

$$\rho_n=|u_n|^2+|v_n|^2+|\partial_x u_n|^2.$$

By using (4.32), we can not have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\rho_n\,dxdy=0.$$

By using again of (4.32), we obtain that

$$\lambda = \left| \int_{\mathbb{R}^3} u^{p+2}\, dxdy \right| \leq C\|u\|_{\mathscr{X}}^{p+2} \leq C I_\lambda^{p+2},$$

for any $u \in \mathscr{X}$. Hence $I_\lambda > 0$ for any positive λ. Since $\mathfrak{H}(u_n) \longrightarrow I_\lambda$ as $n \to \infty$ and by Remark 4.2.4, there exists $C > 0$ such that $\mathfrak{H}(u_n) < C, \quad \rho_n < C$ and $\|u_n\|_{\mathscr{X}} < C$, for all $n \in \mathbb{N}$.

Now suppose the evanescence occurs, i.e. that for any $R > 0$,

$$\lim_{n\to+\infty} \sup_{(\widetilde{x},\widetilde{y})\in\mathbb{R}^2} \int_{(\widetilde{x},\widetilde{y})+B_R} \rho_n\, dxdy = 0, \tag{4.36}$$

where B_R is the ball of radius R centered at *zero*. By (4.32), for $q \in (2,6)$, there exists a constant $C > 0$, independent of $(x,y) \in \mathbb{R}^2$ such that,

$$\int_{(\widetilde{x},\widetilde{y})+B_1} |\partial_x \varphi_n|^q\, dxdy \leq C \left(\int_{(\widetilde{x},\widetilde{y})+B_1} |u_n|^2 + |v_n|^2 + |\partial_x u_n|^2\, dxdy \right)^{q/2}$$
$$\leq C \left(\int_{(\widetilde{x},\widetilde{y})+B_1} \rho_n\, dxdy \right)^{q/2} \leq C \left(\sup_{(\widetilde{x},\widetilde{y})\in\mathbb{R}^2} \int_{(\widetilde{x},\widetilde{y})+B_1} \rho_n\, dxdy \right)^{(q-2)/2} \left(\int_{(\widetilde{x},\widetilde{y})+B_1} \rho_n\, dxdy \right).$$

Since $\mathbb{R}^2$ can be covered by balls of radius one in such a way that each point is at most included in three balls, one infers

$$\int_{\mathbb{R}^2} |\partial_x \varphi_n|^q\, dxdy \leq C \left(\sup_{(\widetilde{x},\widetilde{y})\in\mathbb{R}^2} \int_{(\widetilde{x},\widetilde{y})+B_1} \rho_n\, dxdy \right)^{(q-2)/2} \left(\int_{\mathbb{R}^2} \rho_n \right).$$

Now this inequality and the vanishing assumptions imply that $\lim_{n\to+\infty} \|u_n\|_{L^q(\mathbb{R}^2)} = 0$ for any $q \in (2,6)$, which contradicts that the constraint in I_λ.

Assume now that dichotomy occurs, i.e. that there exists $\gamma \in (0, I_\lambda)$ such that $\lim_{t\to+\infty} \mathscr{M}(t) = \gamma$, where

$$\mathscr{M}(t) = \lim_{n\to+\infty} \sup_{(\widetilde{x},\widetilde{y})\in\mathbb{R}^2} \int_{(\widetilde{x},\widetilde{y})+B_t} \rho_n\, dxdy, \quad \text{for all } t \geq 0.$$

Dichotomy implies the splitting of u_n in two sequences $u_{1,n}$ and $u_{2,n}$ with disjoint support. To keep $u_n, u_{1,n}, u_{2,n} \in \mathscr{X}$, we focus on φ_n and localize it. This splitting lemma is stated as follows.

LEMMA 4.2.5 *For every $\varepsilon > 0$, there exists a $\delta(\varepsilon)$ with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$, $\varrho \in (0, I_\lambda)$, $\rho \in (0, \lambda)$ and two sequences $\{u_{1,n}\}_{n\in\mathbb{N}}$ and $\{u_{2,n}\}_{n\in\mathbb{N}}$ in $\mathscr{X}$ with satisfying the following for $n \geq n_0$.*

$$\|u_n - u_{1,n} - u_{2,n}\|_{\mathscr{X}} \leq \delta(\varepsilon), \tag{4.37}$$
$$|\mathfrak{H}(u_n) - \mathfrak{H}(u_{1,n}) - \mathfrak{H}(u_{2,n})| \leq \delta(\varepsilon), \tag{4.38}$$
$$|\mathfrak{H}(u_{1,n}) - \varrho| \leq \delta(\varepsilon), \quad |\mathfrak{H}(u_{2,n}) - I_\lambda + \varrho| \leq \delta(\varepsilon), \tag{4.39}$$
$$|J(u_n) - J(u_{1,n}) - J(u_{2,n})| \leq \delta(\varepsilon), \tag{4.40}$$
$$|J(u_{1,n}) - \rho| \leq \delta(\varepsilon), \quad |J(u_{2,n}) - \lambda + \rho| \leq \delta(\varepsilon). \tag{4.41}$$

Proof. According to the definition $\mathscr{M}$, it is an increasing function and then fix $\varepsilon > 0$. Then there exist $R_0 > 1/\varepsilon$ and a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ and given $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$\gamma - \varepsilon \leq \int_{(x_n,y_n)+B_{R_0}} \rho_n \, dxdy \leq \gamma \leq \gamma + \varepsilon.$$

One defines

$$\mathscr{M}_n(t) = \sup_{(\widetilde{x},\widetilde{y})\in\mathbb{R}^2} \int_{(\widetilde{x},\widetilde{y})+B_t} \rho_n \, dxdy,$$

for all $t > 0$. Hence we can find a sequence $\{R_m > 0\}_{m\in\mathbb{N}}$ (taking a subsequence if necessary) such that $\lim_{n\to+\infty} R_n = +\infty$ and $\mathscr{M}_n(2R_n) \leq \gamma + \varepsilon$, for $n \geq n_0$. It follows that

$$\int\limits_{R_0\leq|(x,y)-(x_n,y_n)|\leq 2R_n} \left(|u_n|^2 + |\partial_x u_n|^2 + |v_n|^2\right) \, dxdy \leq 2\varepsilon. \tag{4.42}$$

Let $(\zeta, \eta) \in C_0^\infty(\mathbb{R}^2)^2$ satisfy

- supp $\zeta \subset B_2(0)$, $\zeta \equiv 1$ on $B_1(0)$ and $0 \leq \zeta \leq 1$;
- supp $\eta \subset \mathbb{R}^2 \setminus B_2(0)$, $\eta \equiv 1$ on $\mathbb{R}^2 \setminus B_1(0)$ and $0 \leq \eta \leq 1$.

For $n \in \mathbb{N}$, we set $\zeta_n = \zeta\left(\dfrac{\cdot - (x_n, y_n)}{R_1}\right)$, $\eta_n = \eta\left(\dfrac{\cdot - (x_n, y_n)}{R_n}\right)$, and we define

$$\begin{aligned} u_{1,n} &= \partial_x(\zeta_n(\varphi_n - a_n)), \quad u_{2,n} = \partial_x(\eta_n(\varphi_n - b_n)),\\ v_{1,n} = D_x^{-1}(u_{1,n})_y &= \partial_y(\zeta_n(\varphi_n - a_n)), \quad v_{2,n} = D_x^{-1}(u_{2,n})_y = \partial_y(\eta_n(\varphi_n - b_n)), \end{aligned}$$

where $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are real sequences that will be fixed later. Obviously $u_{1,n}$ and $u_{2,n}$ are in $\mathscr{X}$ and supp $u_{1,n} \cap$ supp $u_{2,n} = \emptyset$.
If u_n is written as $u_n = u_{1,n} + u_{2,n} + \hbar_n$, then

$$\begin{aligned} \|\hbar_n\|_{\mathscr{X}} &= \|\partial_x \left(\varphi_n - \zeta_n(\varphi_n - a_n) - \eta_n(\varphi_n - b_n)\right)\|_{\mathscr{X}}\\ &\leq \|(1 - \zeta_n - \eta_n)u_n\|_{L^2(\mathbb{R}^2)} + \|(1 - \zeta_n - \eta_n)v_n\|_{L^2(\mathbb{R}^2)} + \|\partial_x \hbar_n\|_{L^2(\mathbb{R}^2)}\\ &+ \|(\varphi_n - a_n)\partial_x\zeta_n\|_{L^2(\mathbb{R}^2)} + \|(\varphi_n - a_n)\partial_y\zeta_n\|_{L^2(\mathbb{R}^2)}\\ &+ \|(\varphi_n - b_n)\partial_x\eta_n\|_{L^2(\mathbb{R}^2)} + \|(\varphi_n - b_n)\partial_y\eta_n\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

By using (4.42), we have

$$\|(1 - \zeta_n - \eta_n)u_n\|^2_{L^2(\mathbb{R}^2)} \leq \int_{R_1\leq|(x,y)-(x_n,y_n)|\leq R_n} |u_n|^2 \, dxdy \leq 2\varepsilon.$$

Analogously, $\|(1 - \zeta_n - \eta_n)v_n\|_{L^2(\mathbb{R}^2)} \leq \sqrt{2\varepsilon}$.
To estimates the rest, we need a Poincaré-type lemma.

LEMMA 4.2.6 *For $q \in [2, +\infty]$, there exists $C > 0$ such that, for all $f \in L^2_{loc}(\mathbb{R}^2)$ satisfying $\nabla f \in L^2_{loc}(\mathbb{R}^2)$, and for every $R > 0$ and for all $(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2$,*

$$\left(\int\limits_{R\le|(x,y)-(\widetilde{x},\widetilde{y})|\le 2R} |f(x,y)-m_R(f)|^q\,dxdy\right)^{1/q} \le CR^{2/q}\left(\int\limits_{R\le|(x,y)-(\widetilde{x},\widetilde{y})|\le 2R} |\nabla f|^2\,dxdy\right)^{1/2}$$

where

$$m_R(f)=\frac{1}{|\mathscr{B}(\widetilde{x},\widetilde{y};R)|}\int\limits_{R\le|(x,y)-(\widetilde{x},\widetilde{y})|\le 2R} f(x,y)\,dxdy$$

and $\mathscr{B}(\widetilde{x},\widetilde{y};R)$ *denotes the annulus* $\{(x,y)\in\mathbb{R}^2, \quad R\le|(x,y)-(\widetilde{x},\widetilde{y})|\le 2R\}$.

For a proof of this lemma, see [23, Lemma 3.1].

We choose the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$:

$$a_n=m_{R_1}(\varphi_n)=\frac{1}{|\mathscr{B}(\widetilde{x},\widetilde{y};R_1)|}\int_{R_1\le|(x,y)-(\widetilde{x},\widetilde{y})|\le 2R_1}\varphi_n\,dxdy,$$

and

$$b_n=m_{R_n}(\varphi_n)=\frac{1}{|\mathscr{B}(\widetilde{x},\widetilde{y};R_n)|}\int_{R_n\le|(x,y)-(\widetilde{x},\widetilde{y})|\le 2R_n}\varphi_n\,dxdy.$$

Therefore, we have

$$\begin{aligned}\|(\varphi_n-a_n)\partial_x\zeta_n\|^2_{L^2(\mathbb{R}^2)} &\le \frac{1}{R_1^2}\|\partial_x\zeta\|^2_{L^\infty(\mathbb{R}^2)}\int_{\mathscr{B}(x_n,y_n;R_1)}|\varphi_n-a_n|^2\,dxdy\\ &\le \|\zeta_x\|^2_{L^\infty(\mathbb{R}^2)}\left[\int_{\mathscr{B}(x_n,y_n;R_1)}|u_n|^2+|v_n|^2\,dxdy\right]\le C\varepsilon.\end{aligned}$$

Similarly, one can estimate $\|(\varphi_n-b_n)\partial_x\eta_x\|^2_{L^2(\mathbb{R}^2)}\le\delta(\varepsilon)$, $\|(\varphi_n-a_n)\partial_y\zeta_x\|^2_{L^2(\mathbb{R}^2)}\le\delta(\varepsilon)$ and $\|(\varphi_n-b_n)\partial_y\eta_x\|^2_{L^2(\mathbb{R}^2)}\le\delta(\varepsilon)$. On the other hand,

$$\partial_x\hbar_n=(1-\zeta_n-\eta_n)\partial_x u_n+2u_n\partial_x\zeta_n+2u_n\partial_x\eta_n+(\varphi_n-a_n)\partial_x^2\zeta_n+(\varphi_n-b_n)\partial_x^2\eta_n.$$

By using (4.42) and the Lemma 4.2.6, one can obtain $\|(1-\zeta_n-\eta_n)\partial_x u_n\|_{L^2(\mathbb{R}^2)}\le\delta(\varepsilon)$, $\|(\varphi_n-a_n)\partial_x^2\zeta_n\|_{L^2(\mathbb{R}^2)}\le\delta(\varepsilon)$ and $\|(\varphi_n-b_n)\partial_x^2\eta_n\|_{L^2(\mathbb{R}^2)}\le\delta(\varepsilon)$.
But by using (4.32), we obtain

$$\|u_n\partial_x\zeta_n\|_{L^2(\mathbb{R}^2)}\le C\|\zeta_x\|_{L^\infty(\mathbb{R}^2)}\left(\int_{\mathscr{B}(x_n,y_n;R_1)}|u_n|^2\,dxdy\right)^{1/2}\le C\sqrt{\varepsilon}.$$

Similarly, $\|u_n\partial_x\eta_n\|_{L^2(\mathbb{R}^2)}\le\delta(\varepsilon)$. From the above inequalities one obtains

$$\|\hbar_n\|_{\mathscr{X}}=\|u_n-u_{1,n}-u_{2,n}\|_{\mathscr{X}}\le\delta(\varepsilon).$$

We are going to estimate $|\mathfrak{H}(u_n)-\mathfrak{H}(u_{1,n})-\mathfrak{H}(u_{2,n})|$. Note that

$$|\mathfrak{H}(u_n)-\mathfrak{H}(u_{1,n})-\mathfrak{H}(u_{2,n})|\le\|u_n-u_{1,n}-u_{2,n}\|^2_{\mathscr{X}}+\left\|D_x^{1/2}\hbar_n\right\|^2_{L^2(\mathbb{R}^2)}.$$

So it is enough to estimate the second term on the right hand side of the above inequality. We have

$$\begin{aligned}
&\left\| D_x^{1/2} u_{1,n} \right\|_{L^2(\mathbb{R}^2)}^2 \\
&= \int_{\mathbb{R}^2} ((\varphi_n - a_n)\partial_x \zeta_n + \zeta_n u_n) \mathscr{H} \left((\varphi_n - a_n)\partial_x^2 \zeta_n + 2u_n \partial_x \zeta_n + \zeta_n \partial_x u_n \right) \, dxdy \\
&= \int_{\mathbb{R}^2} (\varphi_n - a_n)\partial_x \zeta_n \mathscr{H} \left((\varphi_n - a_n)\partial_x^2 \zeta_n \right) \, dxdy + 2 \int_{\mathbb{R}^2} \zeta_n u_n \mathscr{H} \left(u_n \partial_x \zeta_n \right) \, dxdy \\
&\quad + 2 \int_{\mathbb{R}^2} (\varphi_n - a_n)\partial_x \zeta_n \mathscr{H} \left(u_n \partial_x \zeta_n \right) \, dxdy + \int_{\mathbb{R}^2} \zeta_n u_n \mathscr{H} \left((\varphi_n - a_n)\partial_x^2 \zeta_n \right) \, dxdy \\
&\quad + \int_{\mathbb{R}^2} (\varphi_n - a_n)\partial_x \zeta_n \mathscr{H}(\zeta_n \partial_x u_n) \, dxdy + \int_{\mathbb{R}^2} \zeta_n u_n [\mathscr{H}, \zeta_n] \partial_x u_n \, dxdy \\
&\quad + \int_{\mathbb{R}^2} \zeta_n^2 u_n \mathscr{H} \partial_x u_n \, dxdy.
\end{aligned} \tag{4.43}$$

Before estimating we need the following classical Calderón Commutator Theorem (see [21, 22]).

LEMMA 4.2.7 *Let $g \in C^\infty(\mathbb{R})$ with $g' \in L^\infty(\mathbb{R})$. Then $[\mathscr{H}, g]\partial_x \in \mathcal{L}\left(L^2(\mathbb{R}), L^2(\mathbb{R})\right)$ with*

$$\|[\mathscr{H}, g]\partial_x f\|_{L^2(\mathbb{R})} \leq C \|g'\|_{L^\infty(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.$$

By using the fact that $\|\mathscr{H} u\|_{L^2(\mathbb{R}^2)} = \|u\|_{L^2(\mathbb{R}^2)}$ and other properties of $\mathscr{H}$, the first five terms on the right hand side of (4.43) are bounded as the preceding estimate. By using the Cauchy-Schwarz inequality and the previous lemma, we have

$$\int_{\mathbb{R}^2} \zeta_n u_n [\mathscr{H}, \zeta_n] \partial_x u_n \, dxdy \leq \|\zeta_n u_n\|_{L^2(\mathbb{R}^2)} \|[\mathscr{H}, \zeta_n]\partial_x u_n\|_{L^2(\mathbb{R}^2)} \leq C \|\partial_x \zeta_n\|_{L^\infty(\mathbb{R}^2)} \|u_n\|_{L^2(\mathscr{B}(x_n, y_n; R_1))}^2 \leq C\varepsilon.$$

Therefore, by similar estimates for $\left\| D_x^{1/2} u_{2,n} \right\|_{L^2(\mathbb{R}^2)}$, we obtain

$$\left\| D_x^{1/2} \hbar_n \right\|_{L^2(\mathbb{R}^2)} \leq \left\| \left(1 - \zeta_n^2 - \eta_n^2\right) u_n \right\|_{L^2(\mathbb{R}^2)} + C\sqrt{\varepsilon} \leq \delta(\varepsilon).$$

Since $\|u_{1,n}\|_{\mathscr{X}}$ and $\|u_{2,n}\|_{\mathscr{X}}$ are bounded, then $\mathfrak{H}(u_{1,n})$ and $\mathfrak{H}(u_{2,n})$ are bounded. From the above inequality, one infers that there exists $\varrho(\varepsilon) \in [0, I_\lambda]$ (and taking subsequences if necessary) such that $\lim_{n\to\infty} \mathfrak{H}(u_{1,n}) = \varrho(\varepsilon)$, and thus $|\mathfrak{H}(u_{2,n}) - I_\lambda + \varrho| \leq \delta(\varepsilon)$. Analogously, one can obtain $|J(u_n) - J(u_{1,n}) - J(u_{2,n})| \leq \delta(\varepsilon)$. Therefore we assume that $\lim_{n\to+\infty} J(u_{1,n}) = \rho(\varepsilon)$ and $\lim_{n\to+\infty} J(u_{2,n}) = \widetilde{\rho}(\varepsilon)$, with $|\lambda - \rho(\varepsilon) - \widetilde{\rho}(\varepsilon)| \leq \delta(\varepsilon)$. If $\lim_{\varepsilon\to 0} \rho(\varepsilon) = 0$, then choosing ε sufficiently small, we have for n large enough $J(u_{2,n}) > 0$. Hence by considering $(\widetilde{\rho}(\varepsilon) J(u_{2,n}))^{\frac{1}{p+2}} u_{2,n}$, we obtain that (note that $J\left((\widetilde{\rho}(\varepsilon) J(u_{2,n}))^{\frac{1}{p+2}} u_{2,n} \right) = \widetilde{\rho}(\varepsilon)$)

$$I_{\widetilde{\rho}(\varepsilon)} \leq \liminf_{n\to+\infty} \mathfrak{H}(u_{2,n}) \leq I_\lambda - \gamma + \delta(\varepsilon),$$

which leads to a contradiction since $\lim_{\varepsilon\to 0} \widetilde{\rho}(\varepsilon) = \lambda$. Thus $\rho = \lim_{\varepsilon\to 0} \rho(\varepsilon) > 0$. Necessarily $\rho < \lambda$, because the case $\rho = \lambda$ is ruled out in the same manner with $u_{2,n}$ instead of $u_{1,n}$. Since $\rho \in (0, \lambda)$, one infers that necessarily $\varrho = \lim_{\varepsilon\to+\infty} \varrho(\varepsilon) \in (0, I_\lambda)$. This completes the proof of lemma. ∎

Let us continue the proof of the theorem and show that the dichotomy cannot occur. The previous lemma imply that $I_{\lambda-\rho} + I_\rho \le I_\lambda$. This inequality contradicts the subadditivity condition of I_λ coming from $I_\lambda = \lambda^{2/(p+2)} I_1$.

Therefore the remaining case in Lemma 0.0.1 is locally compactness. There exist a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$, such that for all $\varepsilon > 0$, there exists a finite $R > 0$ and $n_0 > 0$, with

$$\int_{(x_n,y_n)+B_R} \rho_n \, dxdy \ge \iota_\lambda - \varepsilon,$$

for $n \ge n_0$, where $\iota_\lambda = \lim_{n\to+\infty} \int_{\mathbb{R}^2} \rho_n \, dxdy$. This implies that for n large enough

$$\int_{(x_n,y_n)+B_R} |u_n|^2 \, dxdy \ge \int_{\mathbb{R}^2} |u_n|^2 \, dxdy - 2\varepsilon.$$

Since u_n is bounded in the Hilbert space $\mathscr{X}$, there exists $u \in \mathscr{X}$ such that a subsequence of $\{u_n(\cdot - (x_n, y_n))\}_{n\in\mathbb{N}}$ (denoted by the same) converges weakly in $\mathscr{X}$. We then have

$$\int_{\mathbb{R}^2} |u|^2 \, dxdy \le \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |u_n|^2 \, dxdy \le \liminf_{n\to+\infty} \int_{(x_n,y_n)+B_R} |u_n|^2 \, dxdy + 2\varepsilon.$$

But we know there is an injection of $\mathscr{X}$ into $L^2_{loc}(\mathbb{R}^2)$ (due to [23]). Consequently $\{u_n(\cdot - (x_n, y_n))\}_{n\in\mathbb{N}}$ converges strongly in $L^2_{loc}(\mathbb{R}^2)$. But the last inequality above implies that this strong convergence also takes place in $L^2(\mathbb{R}^2)$. Thus by the (4.32), $\{u_n(\cdot - (x_n, y_n))\}_{n\in\mathbb{N}}$ also converges to u strongly in $L^{p+2}(\mathbb{R}^2)$ so that $J(u) = \lambda$ and $I_\lambda = \lim_{n\to+\infty} \mathfrak{H}(u_n) = \mathfrak{H}(u)$, that is, u is a solution of I_λ.

(II) The proof for three dimensional case is basically the same. ■

Now by using the preceding theorem and the Lagrange multiplier theorem, there exists $\theta \in \mathbb{R}$ such that

$$-\alpha u_{xx} - \beta \mathscr{H} u_x + cu + D_x^{-1} v_y = \frac{\theta}{p+1} u^{p+1},$$

in $\mathscr{X}'$. Using the scale change $u = \operatorname{sgn}(\theta)|\theta|^{-1/p}\widetilde{u}$, one can easily see that $\widetilde{u}$ satisfies (4.9) (and similarly for three dimensional case (4.10)).

It is easy to see, by multiplying the previous equation by u and integrating by parts over $\mathbb{R}^2$, that

$$(p+1)\mathfrak{H}(u) = \theta\lambda,$$

so $\theta > 0$. θ is a continuous function of λ. Furthermore,

PROPOSITION 4.2.8 *There exists $\lambda_0 > 0$ such that $\theta(\lambda_0) = 1$. Moreover*

$$\lim_{\lambda\to 0^+} \theta(\lambda) = +\infty, \quad \textit{and} \quad \lim_{\lambda\to+\infty} \theta(\lambda) = 0.$$

Proof. We give a proof in two dimensional case; proof of three dimensional case is similar and we will omit it.

Let $\chi \in C_0^\infty(\mathbb{R}^2)$. For any $\lambda > 0$, we assume that u_λ is a minimizer of I_λ. By defining

$$\chi_\lambda = \left(\frac{\lambda}{\int_{\mathbb{R}^2} \chi^{p+2} \, dxdy} \right)^{1/(p+2)} \chi,$$

we obtain that
$$0 < \theta(\lambda)\lambda = (p+1)\mathfrak{H}(u_\lambda) \leq (p+1)\mathfrak{H}(\chi_\lambda) = (p+1)\lambda^{2/(p+2)}\mathfrak{H}(\chi).$$
Therefore $\lim_{\lambda\to+\infty}\theta(\lambda) = 0$. On the other hand, by using (4.32) and Remark 4.2.4, we have
$$\theta(\lambda)\lambda = (p+1)\mathfrak{H}(u_\lambda) \geq (p+1)\|u\|^2_{\mathscr{X}} \geq (p+1)\lambda^{2/(p+2)}.$$
Thusly $\lim_{\lambda\to 0^+}\theta(\lambda) = +\infty$. ■

4.3 Variational Characterizations

Throughout in section we assume that $\beta \leq 0$.

DEFINITION 4.3.1 *Let $\mathscr{F}(\cdot) = \|\cdot\|^2_{L^2(\mathbb{R}^n)}$. A ground state is a solitary wave which minimizes the action*
$$S(u) = E(u) + \frac{c}{2}\mathscr{F}(u), \quad n = 2, 3, \quad (x, y) \in \mathbb{R}\times\mathbb{R}^{n-1}$$
among all the nonzero solutions of (4.9)(resp. (4.10)), where c is the velocity of the solitary wave and E is the energy defined by
$$E(u) = \frac{1}{2}\int_{\mathbb{R}^2}\left[\alpha u_x^2 - \epsilon v^2 - \beta u\mathscr{H}u_x - \frac{2}{(p+1)(p+2)}u^{p+2}\right]\,dxdy,$$
in two dimensional case and
$$E(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left[\alpha u_x^2 - a v^2 - b w^2 - \beta u\mathscr{H}u_x - \frac{2}{(p+1)(p+2)}u^{p+2}\right]\,dxdydz,$$
in three dimensional case.

We will consider two dimensional case. See [30], for the three dimensional case.

We are going to show that the solutions u obtained from (4.35) are exactly the ground states of equation (4.9) and also gives us some interesting characterizations of those solutions, which may appear to be useful to demonstrate the symmetry property and instability.

Let us define the following functionals[1]:

$$\mathscr{K}(u) = \frac{c}{2}\mathscr{F}(u) + \frac{1}{2}\int_{\mathbb{R}^2}\left((D_x^{-1}u_y)^2 - \frac{\beta}{2}u\mathscr{H}u_x\right)\,dxdy - \frac{1}{(p+1)(p+2)}J(u), \tag{4.44}$$

$$\mathscr{G}(u) = \int_{\mathbb{R}^2}\left(\alpha u_x^2 - \frac{\beta}{2}u\mathscr{H}u_x\right)\,dxdy, \quad \mathscr{I}(u) = \mathfrak{H}(u) - \frac{1}{p+1}J(u) \tag{4.45}$$

$$\mathscr{Q}(u) = 2\mathscr{K}(u) - c\mathscr{F}(u) + \mathscr{G}(u) + \frac{4-3p}{2(p+1)(p+2)}J(u) + \frac{\beta}{2}\int_{\mathbb{R}^2}u\mathscr{H}u_x\,dxdy, \tag{4.46}$$

$$\widetilde{S}(u) = \frac{c}{2}\mathscr{F}(u) + \frac{2-3p}{6p}\beta\int_{\mathbb{R}^2}u\mathscr{H}u_x\,dxdy + \frac{3p-4}{6p}\int_{\mathbb{R}^2}\left[\alpha u_x^2 + \left(D_x^{-1}u_y\right)^2\right]\,dxdy, \tag{4.47}$$

$$\mathscr{P}(u) = \mathscr{G}(u) - \frac{p}{(p+1)(p+2)}J(u), \tag{4.48}$$

$$\widetilde{\mathscr{P}}(u) = \frac{c}{2}\mathscr{F}(u) + \frac{1}{2}\int_{\mathbb{R}^2}\left[\left(D_x^{-1}u_y\right)^2 - \alpha u_x^2\right]\,dxdy + \frac{p-1}{p}\mathscr{G}(u). \tag{4.49}$$

LEMMA 4.3.2 *Let $\beta \le 0$; then there exists a real positive number λ^* such that for $u^* \in \mathscr{X}$, the following assertions are equivalent:*

(i) *u^* is a ground state,*

(ii) *$J(u^*) = \lambda^*$ and u^* is a minimum of I_{λ^*},*

(iii) *$\mathscr{K}(u^*) = 0$ and $\mathscr{G}(u^*) = \inf\{\mathscr{G}(u),\ u \in \mathscr{X},\ u \ne 0,\ \mathscr{K}(u) = 0\}$,*

(iv) *$\mathscr{K}(u^*) = \inf\{\mathscr{K}(u),\ u \in \mathscr{X},\ \mathscr{G}(u) = \mathscr{G}(u^*)\} = 0$,*

(v) *$\mathscr{I}(u^*) = 0$ and $J(u^*) = \inf\{J(u),\ u \in \mathscr{X},\ u \ne 0,\ \mathscr{I}(u) = 0\} = \mathfrak{m}$,*

(vi) *$\mathscr{I}(u^*) = \inf\{\mathscr{I}(u),\ u \in \mathscr{X},\ J(u) = \mathfrak{m}\} = 0$,*

where $\mathfrak{m} = \inf\{J(u),\ u \in \mathscr{N}\}$ and $\mathscr{N}$ is the Nehari manifold $\{u \in \mathscr{X},\ u \ne 0,\ \mathscr{I}(u) = 0\}$.
If $p \in \left(\frac{4}{3}, 4\right)$, then the assertions (i)-(vi) are equivalent to the following ones:

(vii) $\mathscr{Q}(u^*) = 0$ and $\ell = S(u^*) = \inf\{S(u),\ u \in \mathscr{X},\ u \ne 0,\ \mathscr{Q}(u) = 0\}$,

(viii) $\mathscr{Q}(u^*) = 0$ and $\ell' = \widetilde{S}(u^*) = \inf\left\{\widetilde{S}(u),\ u \in \mathscr{X},\ u \ne 0,\ \mathscr{Q}(u) \le 0\right\}$.

If $p \in (2, 4)$, then the assertions (i)-(viii) are equivalent to the following ones:

(ix) $\mathscr{P}(u^*) = 0$ and $\wp = S(u^*) = \inf\{S(u),\ u \in \mathscr{X},\ u \ne 0,\ \mathscr{P}(u) = 0\}$,

(x) $\mathscr{P}(u^*) = 0$ and $\wp' = \widetilde{S}(u^*) = \inf\left\{\widetilde{S}(u),\ u \in \mathscr{X},\ u \ne 0,\ \mathscr{P}(u) \le 0\right\}$.

[1] The functional $\mathscr{I}$ is so-called Nehari functional.

Proof. As we mentioned before, if u is a minimum of I_λ; then there is a positive Lagrange parameter θ_λ such that $(p+1)I_\lambda = \lambda\theta_\lambda$ for each positive λ. Since $I_\lambda = \lambda^{2/(p+2)}I_1$, we get $\theta_\lambda = 1$, by choosing $\lambda = \lambda^* = [(p+1)I_1]^{(p+2)/p}$.

Let us now prove that the lemma holds with this choice of λ.

`(iii)` $\Rrightarrow$ `(iv)` : Assume that `(iii)` holds; let $u \in \mathscr{X}$ with $\mathscr{G}(u) = \mathscr{G}(u^*)$. We have $\mathscr{K}(\tau u) > 0$ for $\tau > 0$ sufficiently small, so that if $\mathscr{K}(u) < 0$, then there is a $\tau_0 \in (0,1)$ such that $\mathscr{K}(\tau_0 u) = 0$; then by setting $\widetilde{u} = \tau_0 u$, one has $\widetilde{u} \in \mathscr{X}$, $\mathscr{K}(\widetilde{u}) = 0$ and $\mathscr{G}(\widetilde{u}) < \mathscr{G}(u) = \mathscr{G}(u^*)$, which contradicts `(iii)`, and shows that u^* satisfies `(iv)` since $\mathscr{K}(u^*) = 0$.

`(iv)` $\Rrightarrow$ `(iii)` : Assume that u^* satisfies `(iv)` and let $u \in \mathscr{X}$ with $\mathscr{K}(u) = 0$ and $u \neq 0$. Then $\mathscr{K}(\tau u) < 0$ for some $\tau > 1$, so that if $\mathscr{G}(u) < \mathscr{G}(u^*)$, one can find $\tau_0 > 1$ with $\mathscr{G}(\tau_0 u) = \mathscr{G}(u^*)$ and $\mathscr{K}(\tau_0 u) < 0$. This leads us a contradiction with `(iv)`.

`(iii)` $\Rrightarrow$ `(i)` : If u^* satisfies `(iii)`, then there exists a Lagrange parameter θ such that u^* solves the Euler-Lagrange equation

$$cu + D_x^{-2}u_{yy} - \frac{u^{p+1}}{p+1} = \theta\left(\alpha u_{xx} + \frac{\beta}{2}\mathscr{H}u_x\right).$$

It is easily seen, by multiplying this equation by u^*, integrating by parts, and using $\mathscr{K}(u^*) = 0$, that $\theta > 0$. We set $u^\diamond(x,y) = \theta^{1/p}u^*\left(x, \theta^{2/p}y\right)$. Suppose that $\theta > 1$. By a simple calculation, we can see that $\mathscr{G}(u^\diamond) = \mathscr{G}(u^*)$; and also $\mathscr{K}(u^\diamond) < 0$. This contradicts $0 \le \mathscr{K}(u^*) \le \mathscr{K}(u^\diamond)$. Therefore $\theta \le 1$. On the other hand, by setting $u_\diamond(x,y) = \theta^{-1/p}u^*(x, \theta^{-2/p}y)$, it can be easily seen that $\mathscr{G}(u_\diamond) = \mathscr{G}(u^*)$; and $\mathscr{K}(u_\diamond) < 0$ if $\theta < 1$. But this is contradicts $\mathscr{K}(u_\diamond) \ge 0$. Thusly $\theta \ge 1$, hence $\theta = 1$. Now identity $S(u) = \mathscr{K}(u) + \frac{1}{2}\mathscr{G}(u)$ shows that if u is a solution of (4.9), then $S(u) = \frac{1}{2}\mathscr{G}(u) \ge \frac{1}{2}\mathscr{G}(u^*) = S(u^*)$; thus u^* is a ground state.

`(ii)` $\Rrightarrow$ `(iii)` : Assume that u^* satisfies `(i)`. Let $u \in \mathscr{X}$ with $u \neq 0$ and $\mathscr{K}(u) = 0$. Since $\mathscr{K}(u) = 0$, so $J(u) > 0$ unless $u = 0$. Thus we set

$$u_\mu = u\left(\frac{\cdot}{\mu}\right), \quad \text{with} \quad \mu = \left(\frac{J(u^*)}{J(u)}\right)^{1/2}.$$

We obtain $J(u_\mu) = J(u^*)$ and $\mathscr{K}(u_\mu) = \frac{\beta}{4}\mu(\mu-1)\int_{\mathbb{R}^2} u\mathscr{H}u_x\,dxdy$. We have $\mathscr{K}(u^*) = 0$, since u^* is a minimum of I_{λ^*}. On the other hand,

$$\mathscr{K}(u^*) + \frac{1}{2}\mathscr{G}(u^*) + \frac{1}{(p+1)(p+2)}J(u^*) \le \mathscr{K}(u_\mu) + \frac{1}{2}\mathscr{G}(u_\mu) + \frac{1}{(p+1)(p+2)}J(u_\mu);$$

which implies

$$\mathscr{G}(u^*) \le \beta\mu\left(\frac{\mu}{2} - 1\right)\int_{\mathbb{R}^2} u\mathscr{H}u_x\,dxdy + \alpha\int_{\mathbb{R}^2} u_x^2\,dxdy \le \mathscr{G}(u),$$

and `(iii)` holds.

`(i)` $\Rrightarrow$ `(ii)` : By using the identities of the proof of Theorem 4.2.1, one has, for any solution u of (4.9), we have $\mathscr{K}(u) = 0$ and

$$\mathfrak{H}(u) = \int_{\mathbb{R}^2}\left(\alpha u_x^2 + cu^2 - \beta u\mathscr{H}u_x + (D_x^{-1}u_y)^2\right)\,dxdy = \left(1 + \frac{2}{p}\right)\mathscr{G}(u).$$

Hence if u^* is a ground state, u^* minimizes both $\mathscr{G}(u)$ and $\mathfrak{H}(u)$ among all the solutions of (4.9). Let $J(u) = \lambda$ and $u^\diamond$ be a minimum of I_λ. Then $I_\lambda = \mathfrak{H}(u^\diamond) \le \mathfrak{H}(u^*)$ and there is a positive θ such that

$$cu^\diamond + D_x^{-2}u^\diamond_{yy} - \alpha u^\diamond_{xx} - \beta \mathscr{H} u^\diamond_x = \frac{\theta}{p+1}(u^\diamond)^{p+1}.$$

Using the equations satisfied by $u^\diamond$ and u^*, the preceding inequality is written as $I_\lambda = \dfrac{\theta\lambda}{p+1} \le \dfrac{\lambda}{p+1}$; hence $\theta \le 1$. On the other hand, by setting $u_\diamond = \theta^p u^\diamond$, we obtain that $u_\diamond$ satisfies (4.9), and since u^* is a ground state, $\mathfrak{H}(u^*) \le \mathfrak{H}(u_\diamond) \le \theta^{2p}\mathfrak{H}(u^\diamond)$; so that $\theta \ge 1$.
Hence $u^* = u^\diamond$ is a minimum of I_λ with $\lambda = \lambda^*$.

`(v)` $\Rrightarrow$ `(i)` : Assume that u^* satisfies `(v)`. Then there exists a Lagrange parameter θ such that $J'(u^*) = \theta \mathscr{I}'(u^*)$. It is easily seen, by multiplying this equation by u^*, integrating by parts , and using $\mathscr{I}(u^*) = 0$, that $\theta = 0$. Therefore u^* is a ground state.

`(v)` $\Rrightarrow$ `(vi)` : For u^* as in `(v)`, $\mathscr{I}(u^*) = 0$. Assume that there is $u \in \mathscr{X}$ such that $J(u) = \mathfrak{m}$ and $\mathscr{I}(u) < 0$. Then $J(u) > 0$ and there exists a $\tau_0 \in (0,1)$ such that $\mathscr{I}(\tau_0 u) = 0$. But $J(\tau_0 u) < J(u) = \mathfrak{m}$, which is impossible.

`(v)` $\Lleftarrow$ `(vi)` : Let u^* satisfy `(vi)`. Then $J(u^*) \ge \mathfrak{m}$. Assume that $J(u^*) > \mathfrak{m}$. Again we have $J(u^*) > 0$. So there exists a $\tau_0 \in (0,1)$ such that $J(\tau_0 u) = \mathfrak{m}$. However $\mathscr{I}(u^*) > 0$ and this contradicts `(vi)`.

`(i)` $\Rrightarrow$ `(v)` : See [30].

`(vii)` $\Rrightarrow$ `(viii)` : It is trivial from the definition of ℓ and ℓ' that $\ell' \le \ell$.

`(vii)` $\Lleftarrow$ `(viii)` : Note that $S(u) = \widetilde{S}(u) + \frac{2}{3p}\mathscr{Q}(u)$ and that $\widetilde{S}(u) > 0$ for $p > \frac{4}{3}$. Let u be in $\mathscr{X}$ such that $\mathscr{Q}(u) < 0$. Since $\mathscr{Q}(\tau u) > 0$ for some sufficiently small $\tau > 0$, there exists a $\tau_0 \in (0,1)$ such that $\mathscr{Q}(\tau_0 u) = 0$; hence we have $\ell \le S(\tau_0 u) = \widetilde{S}(\tau_0 u) = \tau_0^2 \widetilde{S}(u) < \widetilde{S}(u) = S(u)$. Therefore $\ell = \ell'$.

`(i)` $\Rrightarrow$ `(viii)` : Let u^* be a ground state. It is easy to see that $\mathscr{Q}(u^*) = 0$. Hence, there exists a minimizing sequence u_j such that $\widetilde{S}(u_j) \to \ell'$, $\mathscr{Q}(u_j) \le 0$ and the sequence $\{u_j\}$ is bounded in $\mathscr{X}$. By using (4.32), we can obtain a subsequence of $\{u_j\}$, denoting by the same $\{u_j\}$, and $u_0 \in \mathscr{X} \cap L^{p+2}(\mathbb{R}^2)$ such that $u_j \rightharpoonup u_0$ in $\mathscr{X}$ and $L^{p+2}(\mathbb{R}^2)$ for $p \in (0,4)$. Using the injection $\mathscr{X} \hookrightarrow L^2_{loc}(\mathbb{R}^2)$, we obtain that $u_j \to u_0$ a.e. in $\mathbb{R}^2$. We show that $u_0 = u^*$. Note that if there is a subsequence of $\{u_j\}$ such that $\|u_j\|^{p+2}_{L^{p+2}(\mathbb{R}^2)} \to 0$, then

$$\left\| D_x^{1/2} u_j \right\|^2_{L^2(\mathbb{R}^2)} + \|\partial_x u_j\|^2_{L^2(\mathbb{R}^2)} + \left\| D_x^{-1} u_j \right\|^2_{L^2(\mathbb{R}^2)} \longrightarrow 0,$$

since $\mathscr{Q}(u_j) \le 0$ and $p > \frac{4}{3}$. By using (4.33) and the conservation under L^2-norm, it follows that

$$\begin{aligned}\|\partial_x u_j\|^2_{L^2(\mathbb{R}^2)} + \left\| D_x^{-1} u_j \right\|^2_{L^2(\mathbb{R}^2)} &\lesssim \|u_j\|^{(4-p)/2}_{L^2(\mathbb{R}^2)} \left(\|\partial_x u_j\|^2_{L^2(\mathbb{R}^2)} + \left\| D_x^{-1} u_j \right\|^2_{L^2(\mathbb{R}^2)} \right)^{3p/4} \\ &\lesssim \left(\|\partial_x u_j\|^2_{L^2(\mathbb{R}^2)} + \left\| D_x^{-1} u_j \right\|^2_{L^2(\mathbb{R}^2)} \right)^{3p/4}.\end{aligned}$$

Therefore

$$1 \lesssim \left(\|\partial_x u_j\|^2_{L^2(\mathbb{R}^2)} + \left\| D_x^{-1} u_j \right\|^2_{L^2(\mathbb{R}^2)} \right)^{(3p-4)/4};$$

which is impossible. Consequently $\inf_j \|u_j\|^{p+2}_{L^{p+2}(\mathbb{R}^2)} = \varpi > 0$. On the other hand,

$$\begin{aligned} \varpi \le \|u_j\|^{p+2}_{L^{p+2}(\mathbb{R}^2)} &= \|u_j\|^{p+2}_{L^{p+2}(\mathscr{A}_1)} + \|u_j\|^{p+2}_{L^{p+2}(\mathscr{A}_2)} + \|u_j\|^{p+2}_{L^{p+2}(\mathscr{A}_3)} \\ &\le \|u_j\|^{p+2}_{L^{p+2}(\mathscr{A}_1)} + \varepsilon^p \|u_j\|^2_{L^2(\mathscr{A}_2)} + \varepsilon^{-(p+2)} |\mathscr{A}_4| \le \varepsilon^\gamma \|u_j\|^{p+2+\gamma}_{L^{p+2+\gamma}(\mathscr{A}_1)} + \varepsilon^p \|u_j\|^2_{L^2(\mathscr{A}_2)} + C_\varepsilon |\mathscr{A}_4|, \end{aligned}$$

where $\mathscr{A}_1 = \left\{ |u_j| \ge \varepsilon^{-1} \right\}$, $\mathscr{A}_2 = \{|u_j| \le \varepsilon\}$, $\mathscr{A}_3 = \left\{ \varepsilon < |u_j| < \varepsilon^{-1} \right\}$, $\mathscr{A}_4 = \{|u_j| > \varepsilon\}$ and $\gamma \in (0, 4-p)$. Thus, by using (4.32), we obtain that $\|u_j\|^{p+2+\gamma}_{L^{p+2+\gamma}(\mathscr{A}_1)} \le C_1$, and $\|u_j\|^2_{L^2(\mathscr{A}_1)} \le C_2$. Choosing $\varepsilon > 0$ sufficiently small, we obtain that

$$|\mathscr{A}_4| \ge \frac{\varpi - \varepsilon^\gamma C_1 - \varepsilon^p C_2}{C_\varepsilon} > 0.$$

To continue, we need the following lemma. The proof of the following lemma is similar to [54], with the natural modifications.

LEMMA 4.3.3 *Let $u \in \mathscr{X}$ such that $\{|u| > \varepsilon\} \ge \sigma > 0$. Then there exit $r, s \in \mathbb{R}$ such that for some constant $\delta = \delta(\|u\|_{\mathscr{X}}, \sigma, \varepsilon)$,*

$$\left| \mathbb{D}^2 \cap \left\{ |\tau_{r,s} u| > \frac{\varepsilon}{2} \right\} \right| > \delta,$$

where $\tau_{r,s} u(x, y) = u(x + r, y + s)$ and $\mathbb{D}^2$ is the unit ball in $\mathbb{R}^2$.

It follows from Lemma 4.3.3 that $\left|\mathbb{D}^2 \cap \left\{ |u_0| > \frac{\varepsilon}{2} \right\}\right| > \delta$ because $u_j \to u_0$ a.e. in $\mathbb{R}^2$. Consequently $u_0 \ne 0$ a.e. in $\mathbb{R}^2$. Using Lemma 0.0.3, we have that $\mathscr{Q}(u_j) - \mathscr{Q}(u_j - u_0) - \mathscr{Q}(u_0)$ and $\widetilde{S}(u_j) - \widetilde{S}(u_j - u_0) - \widetilde{S}(u_0)$ tend to zero as $j \to +\infty$. Now we show that $\mathscr{Q}(u_0) = 0$. If $\mathscr{Q}(u_0) > 0$, then $\mathscr{Q}(u_j - u_0) \le 0$ as $j \to +\infty$ since $\mathscr{Q}(u_j) \le 0$. It follows from $\widetilde{S}(u_j) \to \ell'$ and $\widetilde{S}(u_j - u_0) \ge \ell'$ that $\widetilde{S}(u_0) \le 0$; which is contradiction. Therefore $\ell' \le \widetilde{S}(u_0) \le \liminf_{j \to +\infty} \widetilde{S}(u_j) = \ell'$; and thusly $\ell' = \widetilde{S}(u_0)$ and $\mathscr{Q}(u_0) \le 0$.

Now, suppose that $\mathscr{Q}(u_0) < 0$. Choosing a small $\tau > 0$ we obtain that $\mathscr{Q}(\tau u_0) > 0$. Therefore there exists a $\tau_0 \in (0, 1)$ such that $\mathscr{Q}(\tau_0 u_0) = 0$. But $\ell' \le \widetilde{S}(\tau_0 u_0) = \tau_0^2 \widetilde{S}(u_0) < \widetilde{S}(u_0) = \ell'$ leads us to a contradiction; consequently $\mathscr{Q}(u_0) = 0$. On the other hand, there exists a Lagrange parameter θ such that

$$S'(u_0) + \theta \mathscr{Q}'(u_0) = 0. \tag{4.50}$$

If $\theta = 0$, then $S'(u_0) = 0$. But it can be easily seen that $\mathscr{Q}(u) = 0$ for any $u \in \mathscr{X}$ such that $S'(u) = 0$. Hence $S(u_0) \le S(u)$ and $u_0 = u^*$ in the sequence, since u^* is a ground state.
So suppose that $\theta \ne 0$. Let $u_0^\diamond(x, y) = \mu^{\frac{3}{2}} u_0(\mu x, \mu^2 y)$. It is easy to see that

$$\begin{aligned} 0 = \left\langle \left(S' + \theta \mathscr{Q}'\right)(u_0), \frac{\partial u_0^\diamond}{\partial \mu}\Big|_{\mu=1} \right\rangle &= \mathscr{Q}(u_0) - \frac{9\theta p^2}{4(p+1)(p+2)} J(u_0) \\ &+ \theta \left[2 \int_{\mathbb{R}^2} \left(\alpha (\partial_x u_0)^2 + \left(D_x^{-1} \partial_y u_0\right)^2 - \frac{\beta}{4} u_0 \mathscr{H} \partial_x u_0 \right) dxdy \right]. \end{aligned}$$

Using this fact that $\mathscr{Q}(u_0) = 0$ and $p \in \left(\frac{4}{3}, 4\right)$, it follows that

$$0 = \frac{4 - 3p}{2} \int_{\mathbb{R}^2} \left[\alpha (\partial_x u_0)^2 + \left(D_x^{-1} \partial_y u_0\right)^2 - \frac{\beta}{2} u_0 \mathscr{H} \partial_x u_0 \right] dxdy + \frac{\beta}{2} \int_{\mathbb{R}^2} u_0 \mathscr{H} \partial_x u_0 \, dxdy < 0,$$

which is impossible; hence $\theta = 0$.

The proofs of (viii) $\Rrightarrow$ (i) and (ix) $\Rrightarrow$ (i) follow from the definition of the ground state.

(x) $\Rrightarrow$ (ix) : It is trivial from the definition of $\wp$ and $\wp'$ that $\wp' \leq \wp$.

(x) $\Lleftarrow$ (ix) : Note that $S(\cdot) = \widetilde{\mathscr{P}}(\cdot) + \frac{1}{p}\mathscr{P}(\cdot)$ and that $\widetilde{\mathscr{P}}(\cdot) > 0$ for $p > 2$. Suppose that $u \in \mathscr{X}$ such that $\mathscr{P}(u) < 0$. Then there exists a $\tau \in (0,1)$ such that $\mathscr{P}(\tau u) = 0$, so $\wp \leq S(\tau u) = \widetilde{\mathscr{P}}(\tau u) = \tau^2 \widetilde{\mathscr{P}}(u) < \widetilde{\mathscr{P}}(u)$. Consequently $\wp = \wp'$.

(i) $\Rrightarrow$ (x) : The proof is basically similar to the proof of (i) $\Rrightarrow$ (viii) with some natural modifications. ■

Corollary 4.3.4 *Let $p \in [1,4)$ and u^* be a ground state, then $S(u^*) = \jmath = \jmath'$, where*

$$\jmath = \inf\{S(u),\ u \in \mathscr{X},\ u \neq 0,\ \mathscr{G}(u) = \mathscr{G}(u^*)\}$$

and $\jmath' = \inf\{S(u),\ u \in \mathscr{X},\ u \neq 0,\ J(u) = J(u^*)\}$.

Proof. Since $S(\cdot) = \mathscr{K}(\cdot) + \frac{1}{2}\mathscr{G}(\cdot)$, then by using the preceding lemma, we have

$$\jmath = \frac{1}{2}\mathscr{G}(u^*) + \inf\{\mathscr{K}(u),\ u \in \mathscr{X},\ u \neq 0,\ \mathscr{G}(u) = \mathscr{G}(u^*)\} = \mathscr{G}(u^*) = S(u^*).$$

On the other hand, using again the preceding lemma, we obtain that

$$\begin{aligned} \jmath' &= \frac{1}{2}\inf\{\mathfrak{H}(u),\ u \in \mathscr{X},\ u \neq 0,\ J(u) = J(u^*)\} - \frac{1}{(p+1)(p+2)}J(u^*) \\ &= \frac{1}{2}\mathfrak{H}(u^*) - \frac{1}{(p+1)(p+2)}J(u^*) = S(u^*). \end{aligned}$$

■

Let u^* be a ground state obtained above.

Corollary 4.3.5 *Let $p \in [1,4)$ and u_0 be the initial data such that the corresponding solution $u(t)$ of equations (4.1)-(4.2) is in $C([0,T);\mathscr{X})$ for some $T > 0$ and satisfies $E(u(t)) = E(u_0)$ and $\mathscr{F}(u(t)) = \mathscr{F}(u_0)$ for $t \in [0,T)$. Then we have the following assertions.*

(i) *If $p \in \left(\frac{4}{3}, 4\right)$ and $u_0 \in \mathscr{M}_i$ then $u(t) \in \mathscr{M}_i$ for $t \in [0,T)$ and $i = 1,2$;*

(ii) *If $u_0 \in \mathscr{Y}_i$ then $u(t) \in \mathscr{Y}_i$ for $t \in [0,T)$ and $i = 1,2$;*

(iii) *If $u_0 \in \mathscr{W}_i$ then $u(t) \in \mathscr{W}_i$ for $t \in [0,T)$ and $i = 1,2$;*

(iv) *If $u_0 \in \mathscr{N}_i$ then $u(t) \in \mathscr{N}_i$ for $t \in [0,T)$ and $i = 1,2$,* *where*

$\mathscr{M}_1 = \{u \in \mathscr{X},\ u \neq 0,\ \ S(u) < S(u^*),\ \mathscr{Q}(u) \geq 0\}$, $\mathscr{M}_2 = \{u \in \mathscr{X},\ u \neq 0,\ \ S(u) < S(u^*),\ \mathscr{Q}(u) < 0\}$,
$\mathscr{Y}_1 = \{u \in \mathscr{X},\ u \neq 0,\ \ S(u) < S(u^*),\ \mathscr{G}(u) \leq \mathscr{G}(u^*)\}$, $\mathscr{Y}_2 = \{u \in \mathscr{X},\ u \neq 0,\ \ S(u) < S(u^*),\ \mathscr{G}(u) > \mathscr{G}(u^*)\}$,
$\mathscr{W}_1 = \{u \in \mathscr{X},\ u \neq 0,\ \ S(u) < S(u^*),\ J(u) \geq J(u^*)\}$, $\mathscr{W}_2 = \{u \in \mathscr{X},\ u \neq 0,\ \ S(u) < S(u^*),\ J(u) < J(u^*)\}$,
$\mathscr{N}_1 = \{u \in \mathscr{X},\ u \neq 0,\ \ S(u) < S(u^*),\ \mathscr{P}(u) > 0\}$, $\mathscr{N}_2 = \{u \in \mathscr{X},\ u \neq 0,\ \ S(u) < S(u^*),\ \mathscr{P}(u) \leq 0\}$.

Furthermore, If $u_0 \in \mathscr{M}_2$, then $\mathscr{Q}(u(t)) < \frac{3p}{2}(S(u_0) - S(u^*))$, for $t \in [0,T)$.

Proof. (i) Let $u_0 \in \mathscr{M}_2$. Because of the invariance of E and $\mathscr{F}$, we have $S(u(t)) = S(u_0) < S(u^*)$. Suppose that $u(t_0) \notin \mathscr{M}_2$ for some $t_0 \in (0,T)$, so $\mathscr{Q}(u(t_0)) \geq 0$. By using $\mathscr{Q}(u_0) < 0$ and the continuity of $\mathscr{Q}(u(t))$ with respect to t, there exists a $t_1 \in (0,t_0]$ such that $\mathscr{Q}(u(t_1)) = 0$. Then by using the preceding lemma we obtain that

$$S(u^*) > S(u(t_1)) \geq \inf\{S(u),\ u \in \mathscr{X},\ u \neq 0,\ \mathscr{Q}(u) = 0\} = S(u^*),$$

which is contradiction.
Now suppose that $u_0 \in \mathscr{M}_2$, then $u(t) \in \mathscr{M}_2$; so $S(u(t)) < S(u^*)$ and $\mathscr{Q}(u(t)) < 0$ for $t \in [0,T)$. On the other hand, it is easy to see that $\mathscr{Q}(\tau u) > 0$ for some sufficiently small $\tau > 0$. Therefore there exists a $\tau_0 \in (0,1)$ such that $\tau = 0$. Thus $S(u^*) \leq S(\tau_0 u(t)) = \tau_0^2 \widetilde{S}(u(t)) < \widetilde{S}(u(t))$. Consequently

$$\mathscr{Q}(u(t)) < \frac{3p}{2}(S(u_0) - S(u^*)).$$

The other cases in (i), (ii), (iii) and (iv) can be proved analogously. ■

The solution of the Cauchy problem associated to (4.1)-(4.2) (see [30]) can be extend globally by using the conservation laws E and $\mathscr{F}$, if $u_0 \in \mathscr{M}_2 \cap \mathscr{W}_2$ (see [70]). Now we are able to extend our blow-up results in the last section to the case $p > 4/3$.

THEOREM 4.3.6 *Let u be the solution of the equations (4.1)-(4.2) in $C([0,T);\mathscr{X})$ with $u(0) = u_0$ and is conserved under E and $\mathscr{F}$. Then there exists a finite time T^* such that*

$$\lim_{t \to T^*} \|u_y(t)\|_{L^2(\mathbb{R}^2)} = +\infty,$$

if one of the following cases occurs.

(i) $p \in \left(\frac{4}{3}, 4\right)$ *and* $u_0 \in \mathscr{M}_2 \cap \mathscr{W}_2 \cap \mathscr{Y}_2 \cap L^2\left(y^2 dxdy\right)$.

(ii) $p \in (2,4)$ *and* $u_0 \in \mathscr{M}_2 \cap \mathscr{N}_1 \cap L^2\left(y^2 dxdy\right)$.

(iii) $p \in \left[\frac{4}{3}, 4\right)$, $E(u_0) < 0$ *and* $u_0 \in \mathscr{W}_2 \cap \mathscr{Y}_2 \cap L^2\left(y^2 dxdy\right)$.

Proof. Suppose that $u(t)$ stays in $\mathscr{X}$. In [30], we proved the following Viriel-type identity

$$\frac{d^2}{dt^2}\mathcal{I}(t) = 8\left(Q(u(t)) - \mathscr{G}(u(t)) + \frac{p}{(p+1)(p+2)}J(u(t))\right),$$

where $\mathcal{I}(t) = \int_{\mathbb{R}^2} y^2 u^2(x,y,t)\, dxdy$. So It follows from Corollary 4.3.5 and $\mathscr{P}(u^*) = 0$ that

$$\frac{d^2}{dt^2}\mathcal{I}(t) < 8\left(\frac{3p}{2}[S(u_0) - S(u^*)] - \mathscr{G}(u^*) + \frac{p}{(p+1)(p+2)}J(u^*)\right) = -\varrho < 0.$$

Therefore $\lim_{t\to T^*}\mathcal{I}(t) = 0$ for a finite time T^*. Using Weyl-Heisenberg's inequality, we obtain the blow-up result immediately. The case (`ii`) is similar. It can be easily checked that if φ_c is a solitary wave solution of (4.9), then $\mathscr{G}(\varphi_c) - \frac{p}{(p+1)(p+2)}J(\varphi_c) = 0$. On the other hand, for every $p \in \left[\frac{4}{3}, 4\right)$ there exists $s \in (0,2]$ such that $ps = 4 - p$, so by using the viriel identity and Corollary 4.3.5,

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{I}(t) &\leq 16E(u(t)) - 4s\,\mathscr{G}(u(t)) - \frac{4(4-p)}{(p+1)(p+2)}J(u(t)) \\ &< 16E(u_0) - 4s\,\mathscr{G}(\varphi_c) - \frac{4(4-p)}{(p+1)(p+2)}J(\varphi_c) = 16E(u_0).\end{aligned}$$

This completes the proof of (`iii`). ■

The preceding theorem implies the instability of solitary wave solutions of (4.1)-(4.2).

THEOREM 4.3.7 *Let $p \in [2,4]$. Suppose that φ is the solitary wave solution of (4.1)-(4.2) with $c > 0$. For any $\delta > 0$, there is an initial data $u_0 \in \mathfrak{X}_s(\mathbb{R}^2)$, $s > 2$ with $\|u_0 - \varphi\|_{\mathscr{X}} < \delta$, such that the solution u of (4.1)-(4.2) with $u(0) = u_0$ blows up in finite time, where $\mathfrak{X}_s(\mathbb{R}^n) = \left\{f \in H^s(\mathbb{R}^n), \left(\frac{\widehat{f}(\xi,\eta)}{\xi}\right)^{\vee} \in H^s(\mathbb{R}^n)\right\}$, equipped with the norm*

$$\|f\|_{\mathfrak{X}_s} = \|f\|_{H^s} + \left\|\left(\frac{\widehat{f}(\xi,\eta)}{\xi}\right)^{\vee}\right\|_{H^s}.$$

Proof. Consider the initial data $u_0(x,y) = \varrho\varphi(x,\rho y)$, for any $\varrho, \rho > 0$. By Theorem 4.3.6, it suffices to show that u_0 is close to the solitary wave φ, for small ρ and ϱ; and $u_0 \in \mathscr{M}_2 \cap \mathscr{N}_1$. One can easily check that for $\rho^2 = 1 - \tau$, with $\tau > 0$, ρ and ϱ sufficiently small $u_0 \in \mathscr{M}_2 \cap \mathscr{N}_1$. ■

Now, we use the Lemma 4.3.2 to obtain the symmetry properties of the ground state solutions of the equation (4.9).

THEOREM 4.3.8 *Let $\beta \leq 0$. Any ground state $u^*(x,y)$ of the equation (4.9) is radial in y (cylindrically symmetric), up to a translation of the origin of the coordinates in y.*

Proof. Choose $r \in \mathbb{R}$, in order that if

$$\Upsilon = \{(x,y) \in \mathbb{R}^2 \ : \ y = r\},$$

then

$$\mathscr{G}^+(u^*) = \mathscr{G}^-(u^*) = \frac{1}{2}\mathscr{G}(u^*),$$

where $\mathscr{G}^+$ and $\mathscr{G}^-$ are the same $\mathscr{G}$ with Υ^+ and Υ^- as the domains of integral, respectively; and Υ^+ and Υ^- are the half-planes delimited by Υ. Let u^+ be defined by $u^+ = u^*$ in Υ^+ and u^+ be symmetric with respect to Υ. Then $u^+ \in \mathscr{X}$; indeed, if $\varphi \in L^2_{loc}\left(\mathbb{R}^2\right)$ is such that $\varphi_x = u^*$ and $\varphi_y = D_x^{-1}u_y^*$, then $\varphi_x^+ = u^+$ and $\left\|\varphi_y^+\right\|_{L^2(\mathbb{R}^2)} = 2\left\|\varphi_y\right\|_{L^2(\Upsilon^+)} < +\infty$, where $\varphi^+(x,y) = \varphi(x,y)$, if $y \geq r$ and $\varphi^+(x,y) = \varphi(x, 2r-y)$, if $y \leq r$. Since there is a sequence $\varphi_n \in C_0^\infty\left(\mathbb{R}^2\right)$ such that $\partial_x\varphi_n$ converges to $\varphi_x = u^*$ in $\mathscr{X}$, $D_x^{-1}u_y^+ = \varphi_y^+$. Moreover, $\mathscr{G}(u^*) = \mathscr{G}(u^+)$. In the same way, if $u^- = u^*$ in Υ^- and u^- is symmetric with respect to Υ, then $u^- \in \mathscr{X}$ and $\mathscr{G}(u^*) = \mathscr{G}(u^-)$. Hence it follows from Lemma 4.3.2 (iv) that $\mathscr{K}(u^+) \geq 0$ and $\mathscr{K}(u^-) \geq 0$. Therefore, u^+ and u^- satisfy (iv) of Lemma 4.3.2, since $\mathscr{K}(u^+) + \mathscr{K}(u^-) = 2\mathscr{K}(u^*) = 0$. Thusly, u^+ and u^- are ground states of the equation (4.9). On the other hand, since $u^+ = u^*$ in Υ^+ and $u^- = u^*$ in Υ^-, by using Theorem 4.3.9 applied to $u^+ - u^*$ and $u^- - u^*$, we conclude that $u^+ = u^- = u$ and u^* is symmetric with respect to Υ.

THEOREM 4.3.9 *Let $\alpha > 0$, $\beta \leq 0$, $a, b, c \in L^\infty(\mathbb{R}^2)$ and $u, u_y, u_{xx}, u_{xy}, u_{xxx} \in L^2(\mathbb{R}^2)$ and*

$$u_{yy} - \alpha u_{xxxx} - \beta\mathscr{H}u_{xxx} = a(x,y)u + b(x,y)u_x + c(x,y)u_{xx} \quad \text{in } \mathbb{R}^2. \tag{4.51}$$

Then, if u vanishes on a half-plane Λ in $\mathbb{R}^2$, it vanishes everywhere in $\mathbb{R}^2$.

Proof. Without loss of generality we assume that Λ is parallel to the x-axis. It suffices obviously to prove that if u satisfying the hypotheses of Theorem 0.0.6 is such that $u \equiv 0$ on $\{(x,y)\ ,\ y \leq 0\}$ then it vanishes on $\Lambda_T = \{(x,y)\ ,\ y \in [0,T]\}$ for any $T > 0$. We can rewrite (4.51) as

$$\vartheta_y - \mathfrak{A}\vartheta = -\vartheta + au + bu_x + cu_{xx},$$

where

$$\vartheta = u_y + Au, \quad \mathfrak{A} = I + A,$$

A being the operator, defined by

$$\widehat{Au}(\xi) = \left(\alpha\xi^4 - \beta|\xi|^3\right)^{1/2}\widehat{u}.$$

Obviously, A is a self adjoint operator, continuous from $H^1(\mathbb{R})$ to $H^{-1}(\mathbb{R})$ which satisfies the hypotheses of Theorem 0.0.6, yielding $\vartheta \equiv 0$ on Λ_T and therefore $u \equiv 0$ on Λ_T. ∎

∎

4.4 Regularity and Decay

First, we are going to show that any solitary wave solution of (4.9) and (4.10) is analytic. Indeed,

THEOREM 4.4.1 *Let $\alpha, c > 0$ and $\beta \in \mathbb{R}^2$. We also suppose that $\beta < 2\sqrt{\alpha c}$, if $\beta > 0$. Then any solitary wave u of (4.9) and (4.10) is continuous and tends to zero at infinity. Moreover, u is a real analytic function, provided p is an integer.*

Proof. We prove the theorem for the two dimensional case (see [30] for the three dimensional case). (4.9) implies that u satisfies

$$-cu_{xx} - u_{yy} + \alpha u_{xxxx} + \beta\mathscr{H}u_{xxx} + \frac{1}{p+1}\left(u^{p+1}\right)_{xx} = 0. \tag{4.52}$$

We get from (4.52)

$$\widehat{u}(\xi,\eta)=\frac{\xi^2}{c\xi^2-\beta|\xi|^3+\alpha\xi^4+\eta^2}\,\widehat{g}(\xi,\eta),$$

where $g=-\frac{1}{p+1}u^{p+1}$. The proof is essentially the same as in [24] with natural modifications by using the Proposition 0.0.4 and bootstrapping argument; which imply that the solution is C^∞. The analyticity of the solitary wave solution follows from the Taylor formula, the regularity of the solution and using the following lemma ([43]).

LEMMA 4.4.2 *There exist two constants $C>0$ and $A>0$ such that for all $\omega\in\mathbb{N}^n$,*

$$\|\partial_\omega u\|_{L^2(\mathbb{R}^n)}\le CA^{|\omega|-1}\left(|\omega|-2\right)!.$$

■

Now, we present some results regarding the decay of the solitary wave solutions of (4.9).

LEMMA 4.4.3 *Let $\alpha,c>0$ and $\beta\in(-\infty,2\sqrt{\alpha c}\,)$. Then any solitary wave solution of (4.9) satisfies*

$$\int_{\mathbb{R}^2} r^2\left(|\nabla u|^2+u_{xx}^2\right)\,dxdy<+\infty, \tag{4.53}$$

where $r^2=x^2+y^2$. Furthermore,

$$\int_{\mathbb{R}^2} r^2\left(|\nabla u|^2+\left|D_x^{\frac{3}{2}}u\right|^2+u_{xx}^2\right)\,dxdy<+\infty. \tag{4.54}$$

Proof. Let χ_0 be the same function in the proof of Theorem 4.1.1. We set $\chi_j(x)=\chi_0\left(\frac{x^2}{j^2}\right)$, $j\in\mathbb{N}$. We multiply (4.52) by $\chi_j(x)x^2u$ and $\chi_j(y)y^2u$, separately, and integrate over $\mathbb{R}^2$. Using several integrations by parts, the Plancherel theorem and the terms in (4.52) are computed as follows.

$$-\int_{\mathbb{R}^2} u_{xx}\chi_j(x)x^2u\,dxdy=\int_{\mathbb{R}^2}\left[x^2\chi_j(x)u_x^2-\chi_j(x)u^2-2x\chi_j'(x)u^2-x^2\chi_j''(x)u^2\right]\,dxdy, \tag{4.55}$$

$$-\int_{\mathbb{R}^2} u_{yy}\chi_j(x)x^2u\,dxdy=\int_{\mathbb{R}^2} x^2\chi_j(x)u_y^2\,dxdy, \tag{4.56}$$

$$\begin{aligned}\int_{\mathbb{R}^2} u_{xxxx}\chi_j(x)x^2u\,dxdy=&\int_{\mathbb{R}^2}\left[x^2\chi_j(x)u_{xx}^2-4\chi_j(x)u_x^2-8x\chi_j'(x)u_x^2-2x^2\chi_j''(x)u_x^2\right]\,dxdy\\&+\int_{\mathbb{R}^2}\left[6\chi_j''(x)4x\chi_j'''(x)+\frac{1}{2}x^2\chi_j^{(4)}(x)\right]\,dxdy,\end{aligned} \tag{4.57}$$

$$\begin{aligned}\int_{\mathbb{R}^2} x^2u\chi_j(x)\mathscr{H}u_{xxx}\,dxdy&=-\int_{\mathbb{R}^2}\left[3xu\chi_j(x)\mathscr{H}u_{xx}-xu\chi_j(x)D_x^3(xu)\right]\,dxdy\\&=\int_{\mathbb{R}^2}\left[3\left(u\chi_j(x)+xu\chi_j'(x)+x\chi_j(x)u_x\right)\mathscr{H}u_x-\left(u\chi_j(x)+xu\chi_j'(x)+x\chi_j(x)u_x\right)D_x(u+xu_x)\right]\,dxdy,\end{aligned} \tag{4.58}$$

$$\int_{\mathbb{R}^2} x^2 u\chi_j(x)\left(u^{p+1}\right)_{xx}\ dxdy = 2\frac{p+1}{p+2}\int_{\mathbb{R}^2}\chi_j(x)u^{p+2}\ dxdy - (p+1)\int_{\mathbb{R}^2} x^2\chi_j(x)u^p u_x^2\ dxdy + \frac{p+1}{p+2}\int_{\mathbb{R}^2}\left[x^2\chi_j''(x)+4x\chi_j'(x)\right]u^{p+2}\ dxdy, \tag{4.59}$$

$$-\int_{\mathbb{R}^2} y^2\chi_j(y)uu_{yy}\ dxdy = \int_{\mathbb{R}^2}\left[y^2\chi_j(y)u_y^2 - \chi_j(y)u^2\right]\ dxdy - \int_{\mathbb{R}^2}\left[2y\chi_j'(y)+\frac{1}{2}y^2\chi_j''(y)\right]u^2\ dxdy, \tag{4.60}$$

$$-\int_{\mathbb{R}^2} y^2 u\chi_j(y)u_{xx}\ dxdy = \int_{\mathbb{R}^2} y^2\chi_j(y)u_x^2\ dxdy, \tag{4.61}$$

$$\int_{\mathbb{R}^2} y^2 u\chi_j(y)u_{xxxx}\ dxdy = \int_{\mathbb{R}^2} y^2\chi_j(y)u_{xx}^2\ dxdy, \tag{4.62}$$

$$\int_{\mathbb{R}^2} y^2 u\chi_j(y)\mathscr{H}u_{xxx}\ dxdy = -\int_{\mathbb{R}^2} y^2\chi_j(y)\left|D_x^{\frac{3}{2}}u\right|^2\ dxdy, \tag{4.63}$$

$$\int_{\mathbb{R}^2} y^2 u\chi_j(y)\left(u^{p+1}\right)_{xx}\ dxdy = -(p+1)\int_{\mathbb{R}^2} y^2\chi_j(y)u_x^2u^p\ dxdy. \tag{4.64}$$

By using the properties of χ_j, Lebesgue's theorem, the following equality

$$\left\|D_x^{\frac{1}{2}}(u+xu_x)\right\|_{L^2(\mathbb{R}^2)}^2 = \left\|xD_x^{\frac{3}{2}}u\right\|_{L^2(\mathbb{R}^2)}^2 - C_1\left\|D_x^{\frac{1}{2}}u\right\|_{L^2(\mathbb{R}^2)}^2 + C_2\left\|D_x^{\frac{3}{4}}u\right\|_{L^2(\mathbb{R}^2)}^2$$

as $j\to+\infty$, for some $C_1, C_2 > 0$ and the fact that u tends to zero as $r\to+\infty$, (4.55)-(4.64) imply that

$$\int_{\mathbb{R}^2} r^2\left(cu_x^2+u_y^2+\beta\left|D_x^{\frac{3}{2}}u\right|^2+\alpha u_{xx}^2\right)\ dxdy < +\infty.$$

■

Now by using the analysis of the decay of the solitary wave solution, based on the convolution equations

$$u = ih_1 * (u^p u_x) = -\frac{1}{p+1}h_2 * u^{p+1},$$

where

$$\widehat{h_1}(\xi,\eta) = \frac{\xi}{c\xi^2-\beta|\xi|^3+\alpha\xi^4+\eta^2} \quad \text{and} \quad \widehat{h_2}(\xi,\eta) = \frac{\xi^2}{c\xi^2-\beta|\xi|^3+\alpha\xi^4+\eta^2},$$

and the Lemmata 3.2-3.6 in [24], with some modifications, we obtain that

THEOREM 4.4.4 *Let $\alpha, c > 0$ and $\beta\in(-\infty, 2\sqrt{\alpha c}\,)$. Then any solitary wave solution of (4.9) satisfies*

$$r^2 u \in L^\infty\left(\mathbb{R}^2\right).$$

Furthermore, $r^s u,\ \ r^{1+s}\nabla u,\ \ r^{1+s}u_{xx}\in L^2\left(\mathbb{R}^2\right)$, for any $s\in[0,1)$.

See also [30] for the three dimensional case.

REMARK 4.4.5 *Note that when $\beta\le 0$, then by using the Residue theorem, h_2 can be written in the following form*

$$h_2(x,y) = C\int_0^{+\infty}\int_0^{+\infty}\frac{e^{-(\tau+ct)}}{\sqrt{t\nu\tau}}\mathscr{H}\partial_x\left(e^{-\frac{x^2}{4\nu}}\right)\ dt\ d\tau,$$

where $\nu = \alpha t + \frac{1}{4}\left(\frac{\beta^2t^2}{\tau}+\frac{y^2}{t}\right)$ and $C > 0$.

4.5 (Generalized)Benjamin-Ono-KP

In this section we will looking for solitary wave solutions of the generalized high dimensional BO-KP equation (nBOKP)

$$(u_t + u^p u_x - \beta D_x^\alpha u_x)_x + \sum_{i=1}^{n-1} \epsilon_i \partial_{y_i}^2 u = 0, \tag{4.65}$$

where $D_x = \left(-\partial_x^2\right)^{1/2}$, $n \geq 2$, $\alpha \geq 1$, $\epsilon_i = \pm 1$ and the constant β is real. In fact, as before, we are looking for a solution of (4.65) of the form $u\,(x - c_0 t, y_1 - c_1 t, \cdots, y_{n-1} - c_{n-1} t)$. By a change of variables $\widetilde{x} = x$, $\widetilde{y_i} = y_i - \frac{1}{2}\epsilon_i c_i x$, after dropping the tilde, we obtain the systems

$$(-cu_x + u^p u_x - \beta D_x^\alpha u_x)_x + \sum_{i=1}^{n-1} \epsilon_i \partial_{y_i}^2 u = 0, \tag{4.66}$$

with $c = c_0 + \frac{1}{4}\sum_{i=1}^{n-1} \epsilon_i c_i^2$.

REMARK 4.5.1 *Note that we can assume that $|c| = 1$, since the scale change*

$$\upsilon(x, y) = |c|^{-\frac{1}{p}} u\left(|c|^{-\frac{1}{\alpha}}, |c|^{-\frac{\alpha+2}{2\alpha}} y\right),$$

where $y = (y_1, \cdots, y_{n-1})$, transforms (4.66) in u, into the same in υ, but with $|c| = 1$.

DEFINITION 4.5.2 *We shall denote, $\mathscr{X}$ the closure of $\partial_x(C_0^\infty(\mathbb{R}^n))$ for the norm*

$$\|\partial_x \varphi\|_{\mathscr{X}}^2 = \|\nabla \varphi\|_{L^2(\mathbb{R}^n)}^2 + \left\|D_x^{\alpha/2} \varphi_x\right\|_{L^2(\mathbb{R}^n)}^2. \tag{4.67}$$

REMARK 4.5.3 *Equation (4.65) admits the conservation quantities*

$$\mathscr{F}(u(t)) = \frac{1}{2}\int_{\mathbb{R}^n} u^2(t)\; dxdy, \tag{4.68}$$

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^n} \left[\beta\left(D_x^{\alpha/2} u(t)\right)^2 - \sum_{i=1}^{n-1} \epsilon_i u_{y_i}^2(t) - \frac{2}{(p+1)(p+2)} u^{p+2}(t)\right] dxdy \tag{4.69}$$

LEMMA 4.5.4 *Let $\alpha \geq 2$ be given and let $p \leq p_{n,\alpha} = \dfrac{4\alpha}{2n + (n-3)\alpha}$. Then there exists $C > 0$, depending on α, n and p, such that for any $\varphi \in \mathscr{X}$,*

$$\|\varphi\|_{L^{p+2}(\mathbb{R}^n)}^{p+2} \leq C \|\varphi\|_{L^2(\mathbb{R}^n)}^{q} \|\varphi\|_{H^{\alpha/2,\boldsymbol{0}}(\mathbb{R}^n)}^{\frac{pn}{\alpha}} \left\|\partial_x^{-1} \varphi_y\right\|_{L^2(\mathbb{R}^n)}^{\frac{p(n-1)}{2}}, \tag{4.70}$$

where $\boldsymbol{0} \in \mathbb{R}^{n-1}$ and $q = 2 - \dfrac{pn}{\alpha} - \dfrac{p(n-3)}{2}$. Furthermore, when $n = 2$, $p \leq p_{2,\alpha} = \frac{4\alpha}{4-\alpha}$ and $\alpha \in [1, 2]$, there exists $C > 0$, depending on α and p, such that for any $\varphi \in \mathscr{X}$,

$$\|\varphi\|_{L^{p+2}(\mathbb{R}^2)}^{p+2} \leq C \|\varphi\|_{L^2(\mathbb{R}^2)}^{\frac{2}{q'}} \|\varphi\|_{H^{\alpha/2,0}(\mathbb{R}^2)}^{\frac{p}{2}+\frac{2}{q}} \left\|\partial_x^{-1} \varphi_y\right\|_{L^2(\mathbb{R}^2)}^{\frac{p}{2}}, \tag{4.71}$$

where $q = \dfrac{2(\alpha+2)}{2p + (2-\alpha)(p+2)}$, $\quad q' = \dfrac{2(\alpha+2)}{4\alpha - (4-\alpha)p}$.

REMARK 4.5.5 *Note that by an argument similar to one in [14], one can see that a solution u of (4.65) that starts in $\mathscr{X}$ will remain in this space throughout its period of existence, regardless of the sign of ϵ_i.*

THEOREM 4.5.6 (I) *The equation (4.66) does not admit any nontrivial solitary wave satisfying*

$$u = \partial_x \varphi \in \mathscr{X}, \quad u \in L^\infty_{loc}(\mathbb{R}^n),$$
$$\partial^2_{y_i} \varphi \in L^2_{loc}(\mathbb{R}^n), \quad D_x^{\alpha/2}\partial_x \varphi \in L^2_{loc}(\mathbb{R}^n), \quad 1 \le i \le n-1,$$

if one of the following cases occurs:

(i) $\epsilon_i \epsilon_j < 0$, *for some* $i \neq j$,

(ii) $\epsilon_i \beta < 0$, *for some* i,

(iii) $p \ge \dfrac{4\alpha}{2n + (n-3)\alpha}$, $c\beta > 0$ *and* $c\epsilon_i > 0$, *for all* $1 \le i \le n-1$,

(iv) $p \le \dfrac{4\alpha}{2n + (n-3)\alpha}$, $c\beta < 0$ *and* $c\epsilon_i < 0$, *for all* $1 \le i \le n-1$.

(II) *Let $c > 0$ and $p = \frac{k}{m}$, where $m \in \mathbb{N}$ is odd and m and k are relatively prime. Then the equation (4.66) admits a nontrivial solution $u \in \mathscr{X}$, if $\epsilon_i = -1$ for all $1 \le i \le n-1$, $p < p_{n,\alpha}$, $\alpha \ge 2$ for $n \ge 2$ and $\alpha \ge 1$ for $n = 2$. Furthermore, there is a positive number λ^* such that the minimization problem*

$$\widetilde{I}_{\lambda^*} = \inf \{E(u)\ ;\ u \in \mathscr{X},\ \mathscr{F}(u) = \lambda^*\}$$

has at least one solution.

Idea of the proof. The proof of (I) follows from the following identity:

$$(\alpha(3p+4) - pn(\alpha+2))\,\beta \int_{\mathbb{R}^n} \left(D_x^{\alpha/2} u\right)^2\, dxdy = 4\alpha pc \int_{\mathbb{R}^n} u^2\, dxdy. \tag{4.72}$$

For (II), similar to Section 4.2, one can prove that the existence of solitary wave solutions, when $c > 0$, $\epsilon_i = -1$, for all $i = 1, \cdots, n-1$ and

$$p < \frac{4\alpha}{2n + \alpha(n-3)},$$

by the following minimization problem

$$I_\lambda = \inf \left\{ \|u\|_{\mathscr{X}}\ ;\ u \in \mathscr{X},\ J(u) = \int_{\mathbb{R}^n} u^{p+2}(x,y)\, dxdy = \lambda \right\}, \tag{4.73}$$

by using Lemma 0.0.1. ■

REMARK 4.5.7 *One can also see that the classical function $d(c) = E(u) + c\mathscr{F}(u)$ is strictly increasing for $c > 0$ and*

$$p < p^c_{n,\alpha} = \frac{4\alpha}{2n + \alpha(n-1)}.$$

Moreover, we have $d(c) = c^{\omega} J(v)$*, where* v *satisfies (4.66) with* $c = 1$ *and*

$$\omega = \frac{4\alpha + p((3-n)\alpha) - 2n}{2\alpha p}.$$

Therefore $d''(c) > 0$ *if and only if* $p < p^c_{n,\alpha}$.

Let $\mathbb{G}_c$ be the set of the ground state solutions of the equation (4.66). The stability is an immediate corollary of Theorem 4.5.6 by using classical arguments and the fact that $\widetilde{I}_\lambda = \lambda^{\frac{4\alpha - p(2n+\alpha(n-3))}{4\alpha - p(2n+\alpha(n-1))}} \widetilde{I}_1$, by setting

$$u_\lambda(x,y) = \lambda^{\frac{2\alpha}{4\alpha - p(2n+\alpha(n-1))}} u\left(\lambda^{\frac{2p}{4\alpha - p(2n+\alpha(n-1))}} x, \lambda^{\frac{p(\alpha+2)}{4\alpha - p(2n+\alpha(n-1))}} y\right).$$

Corollary 4.5.8 *Let* $p < p^c_{n,\alpha}$, $c > 0$ *and* $\varphi_c \in \mathbb{G}_c$*; then for all positive* $\varepsilon > 0$*, there is a positive* $\delta > 0$ *such that if* $u_0 \in \left\{ u \in H^s(\mathbb{R}^n)\ ;\ \left(\frac{\widehat{u}(\xi,\eta)}{\xi}\right)^\vee \in H^s(\mathbb{R}^n) \right\}$, $s \geq 3$ *with* $\|u_0 - \varphi_c\|_{\mathscr{X}} < \delta$*, then the solution* $u(t)$ *of (4.65) with* $u(0) = u_0$ *satisfies*

$$\sup_{t\geq 0} \inf_{\psi \in \mathbb{G}_c} \|u(t) - \psi\| \leq \varepsilon.$$

Now we are going to study instability by using the mechanism of blow-up. Let us denote functionals similar to ones in Section 4.3.

$$\mathscr{G}(u) = \frac{\beta}{2} \int_{\mathbb{R}^n} \left(D_x^{\alpha/2} u\right)^2 \, dxdy, \quad \mathscr{I}(u) = \|u\|^2_{\mathscr{X}} - \frac{1}{p+1} J(u), \tag{4.74}$$

$$\mathscr{Q}(u) = \beta \mathscr{G}(u) - \frac{np(2+\alpha) - p\alpha}{2\alpha(p+1)(p+2)} J(u) + \sum_{i=1}^{n-1} \int_{\mathbb{R}^n} v_j^2(x,y)\, dxdy, \quad \mathscr{K}(u) = S(u) - \frac{\alpha}{n} \mathscr{G}(u), \tag{4.75}$$

$$S(u) = E(u) + c\mathscr{F}(u), \quad \widetilde{S}(u) = S(u) - \frac{2\alpha}{2np + p\alpha(n-1)} \mathscr{Q}(u), \tag{4.76}$$

$$\mathscr{P}(u) = \mathscr{G}(u) - \frac{p(1-\alpha+n) + 2(1-\alpha)}{2(p+1)(p+2)} J(u), \quad \widetilde{\mathscr{P}}(u) = S(u) - \frac{2}{p(1+n-\alpha) + 2(1-\alpha)} \mathscr{P}(u), \tag{4.77}$$

where $v_j = \partial_x^{-1} u_{y_i}$. Considering the above functionals, we have a lemma similar to Lemma 4.3.2.

LEMMA 4.5.9 *Let* $\mathscr{G}$, $\mathscr{I}$, $\mathscr{Q}$, S, $\widetilde{S}$, $\mathscr{P}$, $\widetilde{\mathscr{P}}$ *and* $\mathscr{K}$ *be as above. There exists a real positive number* λ^* *such that for* $u^* \in \mathscr{X}$*, we have as in Lemma 4.3.2,*

(I) (i)-(vi) are equivalent,

(II) (i)-(vi) are equivalent to (vii)-(viii), if $p > \dfrac{2\alpha}{1-\alpha+n}$,

(III) (i)-(viii) are equivalent to (ix)-(x) , if $p > p^c_{n,\alpha}$.

Corollary 4.5.10 *Let* $\mathscr{G}$, $\mathscr{I}$, $\mathscr{Q}$, S, $\widetilde{S}$, $\mathscr{P}$, $\widetilde{\mathscr{P}}$ *and* $\mathscr{K}$ *be as Lemma 4.5.9. Then* (i)-(iii) in Corollary 4.3.5 hold. Also (i) in Corollary 4.3.5 holds if $p > p^c_{n,\alpha}$. Moreover, if $u_0 \in \mathscr{M}_2$, then

$$\mathscr{Q}(u(t)) < \frac{1}{p^c_{n,\alpha}} (S(u_0) - S(u^*)), \tag{4.78}$$

for $t \in [0,T)$.

REMARK 4.5.11 *Similar to Section 4.3, the quantity*

$$\mathcal{I}(t) = \int_{\mathbb{R}^n} |y|^2 u^2(x,y)\, dxdy$$

plays an important role in our blow-up and instability results. We can similarly obtain the following Viriel-type identity. In fact, we have

$$\frac{d}{dt}\mathcal{I}(t) = 4\sum_{i=1}^{n-1} \epsilon_i \int_{\mathbb{R}^n} u y_i v_i\, dxdy, \tag{4.79}$$

and

$$\begin{aligned} \frac{d^2}{dt^2}\mathcal{I}(t) &= 8\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \epsilon_i\epsilon_j \int_{\mathbb{R}^n} v_j^2\, dxdy + \frac{4p}{(p+1)(p+2)}\sum_{i=1}^{n-1}\epsilon_i \int_{\mathbb{R}^n} u^{p+2}\, dxdy \\ &= (8-2p)\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\epsilon_i\epsilon_j \int_{\mathbb{R}^n} v_j^2\, dxdy + 2p\sum_{i=1}^{n-1}\epsilon_i\left[\beta\int_{\mathbb{R}^n}\left(D_x^{\alpha/2}u\right)^2 dxdy - 2E(u)\right]. \end{aligned} \tag{4.80}$$

The proof of the following theorem is similar to Theorem 4.3.6, by using (4.80).

THEOREM 4.5.12 *Let u be the solution of the equations (4.65) in $C([0,T);H^s(\mathbb{R}^n))$ with $u(0)=u_0$ and is conserved under E and $\mathscr{F}$. Then there exists a finite time T^* such that*

$$\lim_{t\to T^*}\sum_{i=1}^{n-1}\|u_{y_i}(t)\|_{L^2(\mathbb{R}^n)} = +\infty,$$

if one of the following cases occurs.

(i) $p\in(p_{n,\alpha}^c, p_{n,\alpha})$ *and* $u_0\in\mathscr{M}_2\cap\mathscr{W}_2\cap\mathscr{Y}_2\cap L^2\left(|y|^2dxdy\right)$,

(ii) $p\in\left(\frac{2\alpha}{1-\alpha+n}, p_{n,\alpha}\right)$ *and* $u_0\in\mathscr{M}_2\cap\mathscr{N}_1\cap L^2\left(|y|^2dxdy\right)$,

(iii) $p\in\left[p_{n,\alpha}^c, p_{n,\alpha}\right)$, $E(u_0)<0$ *and* $u_0\in\mathscr{W}_2\cap\mathscr{Y}_2\cap L^2\left(|y|^2dxdy\right)$.

Theorem 4.5.12 enables us to obtain strong instability of solitary wave solutions of (4.65).

THEOREM 4.5.13 *Let $p\geq p_{n,\alpha}^c$. Suppose φ is the solitary wave solution of the nBOKP equation (4.66) with $c>0$. Then for any $\delta>0$, there is an initial data $u_0\in\left\{u\in H^s(\mathbb{R}^n)\,;\,\left(\frac{\widehat{u}(\xi,\eta)}{\xi}\right)^\vee\in H^s(\mathbb{R}^n)\right\}$, $s>2$ with $\|u_0-\varphi\|_{\mathscr{X}}<\delta$, such that the solution u of (4.65) with $u(0)=u_0$ blows up in finite time. More precisely,*

$$\lim_{t\to T^*}\sum_{i=1}^{n-1}\|u_{y_i}(t)\|_{L^2(\mathbb{R}^n)} = +\infty.$$

Proof. We define $u_0(x,y) = \sigma\varphi(\kappa x, \rho y)$, where $\rho^2 = (1-\varepsilon)\kappa^\alpha + 2$ with a sufficiently small $\varepsilon > 0$. By Theorem 4.5.12, it suffices to show that u_0 is close to the solitary wave φ for small $\varepsilon > 0$ and $u_0 \in \mathscr{M}_2 \cap \mathscr{W}_2 \cap \mathscr{Y}_2$; in fact, by using the facts

$$c\,\mathscr{F}(\varphi) = B_1\mathscr{G}(\varphi), \qquad J(\varphi) = \frac{2\alpha(p+1)(p+2)}{pn}\mathscr{G}(\varphi),$$
$$\|v_j\|_{L^2(\mathbb{R}^n)} = \frac{\alpha}{n}\mathscr{G}(\varphi), \quad j = 1,\cdots,n-1 \quad \text{and} \quad S(\varphi) = \frac{\alpha}{n}\mathscr{G}(\varphi),$$

we should show that ρ, σ and κ satisfy the following conditions

$$B_1\kappa^{-1}\rho^{1-n} + B_2\kappa^{\alpha-1}\rho^{1-n} + B_3\kappa^{-3}\rho^{3-n} < \frac{\alpha}{n}\sigma^{-2}, \tag{4.81}$$
$$B_4\kappa^{-2} + \kappa^\alpha\rho^{-2} < B_5\rho^{-2}\sigma^p, \tag{4.82}$$
$$\sigma^2\kappa^{\alpha-1}\rho^{1-n} > 1, \tag{4.83}$$
$$\sigma^{p+2}\rho^{1-n} < \kappa, \tag{4.84}$$

where $B_1 = \alpha\left(\frac{3}{2n} + \frac{2}{np} - \frac{1}{2}\right) - 1$, $B_2 = 1 - \frac{2\alpha}{B_5}$, $B_4 = \frac{\alpha}{2n}(n-1)$, $B_3 = B_4B_2$ and $B_5 = 1 + B_4$. By a simple computation, one can show that all conditions (4.81)-(4.84) are satisfied if $\kappa \in (A_1, A_2)$, where

$$A_1 = \left(\frac{nB_1}{(1+\varepsilon)^{-2\theta} - nB_2 - (1-\varepsilon)B_3}\right)^{1/\alpha}, \quad \sigma = (1+\varepsilon)^\theta(1-\varepsilon)^{\frac{1}{4}(\alpha+2)(n-1)}\kappa^{\frac{1}{4}((n-3)\alpha+2n)},$$

$$A_2 = \left(\frac{B_5(1-\varepsilon)^{\frac{1}{4}(\alpha+2)(n-1)}}{B_4(1-\varepsilon)+1}\right)^{1/\tau}, \quad \tau = \frac{1}{4}\left[\alpha(4+3p) - np(2+\alpha)\right],$$

with sufficiently small $\varepsilon > 0$ and $\theta > 0$. It is easy to verify that $\kappa \to 1$, $\sigma \to 1$, $\rho \to 1$ and $u_0 \to \varphi$ in $\mathscr{X}$ as $\varepsilon \to 0$. This completes the proof. ■

REMARK 4.5.14 *Note that in [30], we could obtain some local well-posedness of two and three dimensional Benjamin equations in the anisotropic spaces with negative indices.*

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