# SIMPLICITY OF THE LYAPUNOV SPECTRUM OF MULTIDIMENSIONAL CONTINUED FRACTION ALGORITHMS

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ABSTRACT. We prove that the Lyapunov spectrum of the Selmer Multidimensional Continued Fractions Algorithm is simple. The proof is based on the simplicity criterium used by Avila and Viana for proving the Zorich-Kontsevich conjecture. But our approach for checking the pinching and twisting conditions of the criterium is different, with a flavor from algebraic geometry. We expect this approach to apply in great generality for continued fraction algorithms.

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## 1. Introduction

In a recent paper [2], Avila and Viana prove that the Lyapunov spectra of all Rauzy-Veech-Zorich linear cocycles [20, 26, 29, 30] are simple. Their methods suggest that simplicity may actually be a very general feature for multidimensional continued fraction algorithms, and the present work may be viewed as a contribution towards establishing this fact. We prove here that *Selmer's Continued Fractions Algorithm has simple Lyapunov spectrum in any dimension.* The definition of the Selmer algorithm will be recalled in a while; see also Lagarias [15] and Schweiger [22]. Our approach for checking the assumptions of the Avila-Viana criterium is different, with a flavor from algebraic geometry, and we expect it can be applied in great generality for continued fraction algorithms and specially to the algorithm of Brun [7, 8]. Beforehand, let us recall some important background material.

### 1.1. One-dimensional continued fractions.

Recall that the *classical continued fraction expansion* of a real number $x \in (0,1)$ is defined by

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{1}$$

where $a_{n+1} = \left[1/T^n(x)\right]$ for $n \geq 1$ and

$$T : (0,1) \to [0,1], \qquad T(y) = \frac{1}{y} - \left[\frac{1}{y}\right]$$

is the *Gauss transformation*. This is a powerful tool for studying the arithmetic properties of real numbers, and exhibits several important properties. For one thing, $x \in (0,1)$ is rational if and only if the algorithm stops after finitely many steps, that is, $T^k(x) = 0$ for some $k \geq 1$. Moreover, as observed by Lagrange, $x$ is an algebraic number of degree 2 if and only if its continued fraction expansion is periodic. Most important, the *convergent*

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} \tag{2}$$

provide the best approximations of each (irrational) $x \in (0,1)$, in the sense that

$$|x - \frac{p_n}{q_n}| \leq |x - \frac{p}{q}| \quad \text{for all} \quad q \geq q_n.$$

Moreover, these approximations are uniformly good, in the sense that

$$|x - \frac{p_n}{q_n}| < \frac{1}{q_n^2} \leq \frac{1}{\xi^{2n}}$$

where $\xi = (1+\sqrt{5})/2$ is the *golden mean*.

As one interesting application, let us mention Lagrange's solution of *Pell's Diophantine equation*

$$x^2 - cy^2 = 1, \quad \text{where } c \text{ is a square-free integer.}$$

A pair $(x_1, y_1)$ of positive integers is called *fundamental solution* if it satisfies the equation and $x_1$ is minimal among all (positive) solutions. Lagrange showed that the fundamental solution may be found by considering the continued fraction expansion of $\sqrt{c}$ and testing each successive convergent $p_n/q_n$ until a solution to Pell's equation is found in the form $(x_1, y_1) = (p_n, q_n)$. Then all the other solutions $(x_i, y_i)$ may be calculated algebraically through

$$x_i + y_i\sqrt{c} = (x_1 + y_1\sqrt{c})^i.$$

### 1.2. **Multidimensional algorithms.**

The use of multidimensional analogs of this algorithm goes back to C. G. J. Jacobi, whose goal was to extend to the cubic case Lagrange's characterization of quadratic algebraic numbers that was mentioned previously. To this end, Jacobi proposed a continued fraction algorithm for pairs $(x_1, x_2)$ of real numbers which he hoped would yield a periodic expansion if and only if $x_1$ and $x_2$ belong to the same cubic field. It remains an open problem whether such an algorithm does exist.

Multidimensional continued fractions arise naturally in many other areas, particularly in connection to renormalization. One important class of applications in Dynamics is to KAM theory, where the properties of invariant tori are very closely linked to the arithmetics of the corresponding rotation vectors. See Khanin, Sinai [13], Kosygin [14], MacKay [16], Khanin, Lopes Dias, Marklof [12], and references therein for some of the applications in this direction.

In general terms, a *continued fraction algorithm in dimension* $d \geq 1$ assigns to each $\theta = (\theta_1, \ldots, \theta_d) \in (0, 1]^d$ a sequence $(L^n_\theta)_n$ in $GL(d+1, \mathbb{Z})$ such that the vector $\Theta = (\theta_1, \ldots, \theta_d, 1)$ belongs to the positive quadrant $L^n_\theta \cdot \mathbb{R}^{d+1}_+$ for every $n$. Let $\{e_1, \ldots, e_d, e_{d+1}\}$ denote the canonical basis of $\mathbb{R}^{d+1}$, so that $\ell^n_j = L^n_\theta \cdot e_j$, $j = 1, \ldots, d, d+1$ are the column vectors of each $L^n_\theta$. The algorithm is *weakly convergent* if for (almost) every $\theta$, the sequence of positive quadrants converges to the direction of $\Theta$. In other words, weak convergence means that the directions of all $\ell^n_j$, $j = 1, \ldots, d, d+1$ converge to the direction of $\Theta$ as $n \to \infty$. Then

$$\left(\frac{\ell^n_{j,1}}{\ell^n_{j,d+1}}, \ldots, \frac{\ell^n_{j,d}}{\ell_{j,d+1}}\right) \tag{3}$$

provide rational approximations to the vector $\theta \in \mathbb{R}^d$. One speaks of *strong convergence* if the base vectors $\ell^n_j$, $j = 1, \ldots, d, d+1$ themselves, not just their directions, converge to the radius $\mathbb{R}_+\Theta$ as $n \to \infty$.

The theory of multidimensional continued fractions is currently much less satisfactory than its one-dimensional counterpart. Several different algorithms have been proposed, usually with properties particularly suited to some specific goal, but an algorithm combining all the nice properties of the classical construction can not exist in dimension larger than one (see Szekeres [24] for a discussion). For the vast majority of models in the literature, strong convergence either fails or is unknown (see [12] for a discussion and references) and, in some cases, even weak convergence fails (see Nogueira [17]). Another important issue that is largely open, and which is of even more direct concern for our work, is the quantitative analysis of the convergence for rather general families of expansion algorithms. To discuss this issue we focus on a class of dynamically defined continued fraction algorithms that contains many of the most important models.

### 1.3. **Some motivation to Selmer's algorithm.**

Consider the question: Are the components of a given vector the base of some algebraic field?. It was historically the first motivation for the introduction of multidimensional continued fractions algorithms and the Jacobi-Perron algorithm was the first attempt in this direction. As a consequence this algorithm has been the earliest and most extensively studied. A lot of research, without success, in the question of periodicity for the Jacobi-Perron algorithm has been conducted by many notable mathematicians. Some partial results on periodicity has been obtained and it was also proved that the algorithm is weakly convergent in every dimension.

Nevertheless some skepticism have been expressed about the performance of the Jacobi-Perron algorithm. The difficulty of the problem of periodicity led some mathematicians to introduce variations on the algorithm or even to propose very different algorithms and modifications of them.

Poincaré [19] proposed an algorithm for two dimensions motivated by a geometric idea. But his algorithm is not convergent for some particular examples. Attempting to remove this defect Brun [7, 8] proposed in 1919 a rather simple algorithm with very interesting periodicity properties first for $n = 2$, later for general $n$. In 1977 Greiter [11] brought the state of knowledge about Brun's algorithm to the level of that of Jacobi-Perron when he proved that Brun's algorithm is weakly convergent in any dimension. This together with some other arguments led him to enounce that Brun's algorithm was a more natural generalization of classical continued fractions than that of Jacobi-Perron. But Brun's algorithm has some defect, it is not strongly convergent as some counterexamples for its 2-dimensional version show.

Selmer [23] introduced a variation in the algorithm of Brun which apparently could give more approximations than Brun's. A very remarkable property of this algorithm being the fact that the expansions can essentially be codified by two symbols. So the algorithm is simpler than that of Brun and maybe more suitable to investigate periodicity properties. Other good properties like weak convergence has been established for this algorithm too making it subject of many recent investigations.

### 1.4. **Lyapunov exponents.**

Many continued fraction algorithms can be defined in terms of a transformation $T : X \to X$, defined on some positive, or even full measure subset $X$ of $(0, 1]^d$, and a function $A : X \to GL(d + 1, \mathbb{Z})$, in the sense that the sequence $(L^n_\theta)_n$ of linear operators is given by

$$L^n_\theta = A(\theta)A(T(\theta)) \cdots A(T^{n-1}(\theta)), \quad \text{for each } n. \tag{4}$$

See Lagarias [15] for the closely related notion of *Markovian Multidimensional Continued Fractions Algorithms*. Notice the one-dimensional continued fraction fits in this class:

**Example 1.1.** Let $T : (0, 1] \to [0, 1]$ be the Gauss map, $T(\theta) = (1/\theta) - [1/\theta]$ and $A : (0, 1] \to SL(2, \mathbb{Z})$ be defined by

$$A(\theta) = \begin{pmatrix} 0 & 1 \\ 1 & [1/\theta] \end{pmatrix}.$$

It follows directly from the definitions that, for every $\theta$,

$$(\theta, 1) \quad \text{is collinear to} \quad A(\theta) \cdot (T(\theta), 1). \tag{5}$$

By induction, this ensures that $(\theta, 1)$ is in the positive quadrant of $L_\theta^n$ for every $n$. Also by induction,

$$\ell_1^n = L_\theta^n \cdot (1, 0) = (p_{n-1}, q_{n-1}) \quad \text{and} \quad \ell_2^n = L_\theta^n \cdot (0, 1) = (p_n, q_n),$$

where the integer sequences $(p_n)_n$ and $(q_n)_n$ are as in (2).

Suppose that the map $T$ admits some interesting (e.g. absolutely continuous) invariant probability measure $\mu$. This is a rather general property: existence of an absolutely continuous invariant probability was first observed by Gauss himself for the one-dimensional continued fraction algorithm and it is now been verified for many models in higher dimensions as well. In particular, it holds for Selmer's multidimensional continued fractions algorithm (Lagarias [15]) which is of particular relevance for our purposes here.

Then, assuming the function $\log \|A^{-1}\|$ is $\mu$-integrable, for $\mu$-almost every $\theta$ there exists a stratification

$$\{0\} = E_\theta^0 < E_\theta^1 < E_\theta^2 < \cdots < E_\theta^\kappa = \mathbb{R}^{d+1} \tag{6}$$

into vector subspaces such that $A^{-1}(\theta) \cdot E_\theta^j = E_{T(\theta)}^j$ and the *Lyapunov exponents*

$$\lambda_j(\theta) = -\lim_n \frac{1}{n} \log \|(L_\theta^n)^{-1} \cdot v\|$$

are well-defined for every $v \in E_\theta^j \setminus E_\theta^{j-1}$, with $\lambda_1(\theta) > \lambda_2(\theta) > \cdots > \lambda_\kappa(\theta)$. This follows directly from applying the multiplicative ergodic theorem of Oseledets [18] to the linear cocycle $(T, A^{-1})$: notice that $(L_\theta^n)^{-1}$ coincides with $A^{-n}(\theta)$, where the latter is defined by the cocycle chain rule

$$A^{-n}(\theta) = A(T^{n-1}(\theta))^{-1} \cdots A(T(\theta))^{-1} A(\theta)^{-1}.$$

These objects provide important information on the convergence properties of the continued fraction algorithm, as we are going to explain.

To begin with, weak convergence (for almost every $\theta$) can often be deduced from knowing that the largest Lyapunov exponent $\lambda_1$ is simple, that is,

$$\dim E^1 = 1 \quad \text{almost everywhere.}$$

Indeed, it is easy to see that for typical cones $C \subset \mathbb{R}^{d+1}$ their iterates under $L_\theta^n$ converge exponentially fast to the direction of $E_\theta^1$, with exponent given by $\lambda_1 - \lambda_2$. In many cases, one can check this holds for $C = \mathbb{R}_+^{d+1}$ and the Oseledets subspace $E_\theta^1$ coincides with the direction of $\Theta = (\theta, 1)$, and then weak convergence follows.

**Example 1.2.** Suppose $T$ and $A$ are such such that $A^{-1}(\theta) \cdot (\theta, 1)$ is collinear to $(T(\theta), 1)$ almost everywhere. This holds for the one-dimensional continued fraction algorithm, as we have seen in (5). More generally, it holds for the linear simplex-splitting algorithms (Lagarias [15]), where the map $T$ corresponds to the projectivization of $A^{-1}$. Then the line bundle

$$\theta \mapsto \mathbb{R}\Theta = \mathbb{R}(\theta, 1) \tag{7}$$

is invariant under the cocycle. Suppose, in addition, that there exists $n \geq 1$ such that $L_\theta^n > 0$ (all entries positive) almost everywhere. Then a Perron-Frobenius type argument proves that (7) is the unique invariant line bundle inside $\mathbb{R}_+^d$ and it coincides with the Oseledets subbundle $E_\theta^1$. See [27, Section 5] for a detailed presentation of this argument in a related situation.

Similarly, strong convergence is closely related to knowing that $\lambda_1$ is the only non-negative Lyapunov exponent. Few multidimensional continued fraction algorithms are known to be strongly convergent (see for instance the discussions in Baladi, Nogueira [3], Khanin, Lopes Dias, Markloff [12], and Tourigny, Smart [25]).

### 1.5. **Simplicity of the Lyapunov spectrum.**

A much finer property has been proved recently by Avila, Viana [2] for the Rauzy-Veech-Zorich multidimensional continued fraction algorithms: their Lyapunov spectra are *simple*, meaning that the stratifications (6) have

$$\dim E^j = j \quad \text{for every } j \qquad \text{(almost everywhere).} \tag{8}$$

This leads to a particularly detailed description of the convergence for this algorithm: each approximating vector $\ell_j^n$ may be written as a sum

$$\ell_j^n = v_{j,1}^n + v_{j,2}^n + \cdots + v_{j,d+1}^n, \quad \text{where } v_{j,1}^n \in \mathbb{R}\Theta$$

and all the terms have well defined and distinct rates of growth:

$$\lim \frac{1}{n} \log \|v_{j,i}\| = \lambda_i \quad \text{for } i = 1, 2, \ldots, d+1$$

(possibly, restricted to an infinite subset of values of $n$).

The argument of Avila, Viana is based on an abstract simplicity criterium that they improve [1, 2] from Bonatti, Viana [4]: for a broad class of linear cocycles, *if the monoid generated by the cocycle is pinching and twisting then the Lyapunov spectrum is simple. Pinching* and *twisting monoid* are transversality type conditions whose definitions will be recalled later. For the time being let us just mention that they both measure how "rich" the monoid is: in particular, any monoid that has some pinching submonoid is also pinching, and the same is true for twisting . Then the bulk of the work in [2] is devoted to proving that the monoid generated by every Rauzy-Veech-Zorich cocycle satisfies these conditions. This the authors do by induction on the complexity of the corresponding base dynamics or, more precisely, on the genus and the number of singularities of the stratum of Teichmüller space associated to each Rauzy-Veech-Zorich cocycle.

Here we develop a very different and perhaps more direct approach to proving the pinching and twisting properties, and we apply it to the Selmer continued fraction algorithm in any dimension. It is worth pointing out that the monoid generated by Selmer's algorithm is relatively "poor", as it admits a generating set with only two elements. Yet, our arguments show that it is indeed pinching and twisting.

For proving the pinching property, we find in the Selmer monoid in any dimension an infinite family of *pinching matrices*, that is, such that all their eigenvalues are real numbers with different absolute values. The proof that the eigenvalues of our matrices are indeed real and distinct in absolute values uses ideas from the theory of orthogonal polynomials, taking advantage of the fact that certain sequences of orthogonal polynomials arise among the characteristic polynomials of those matrices. For proving the twisting property we find an algebraic curve containing the eigenvectors of all our pinching matrices and which exhibits a beautiful property of linear independence of all non-trivial sets of vectors over the curve. These arguments apply directly to any monoid with pinching matrices having only one row depending on a parameter and, presumably, can be extended to even greater generality.

In particular, this approach should apply without major modifications to Brun's algorithm.

Before we outline the structure of the paper let us mention one more implication of simplicity of the Lyapunov spectrum on the convergence of the algorithm.

Lagarias [15] defines the *uniform approximation exponent* $\eta^*(\cdot)$ of a multidimensional continued fraction algorithm by

$$\eta^*(\theta) = \liminf_{n\to\infty}[\min_{1\le j\le d+1} \eta(l_j^n, \theta)]$$

where

$$\eta(l,\theta) = -\frac{1}{\log || l_{d+1} ||} \log || (l_1, \cdots, l_d) - l_{d+1}\theta ||$$

He proves that $\eta^*(\theta) \le 1 + \frac{1}{d}$ and calls the algorithm *optimal* if the equality holds for almost every $\theta$. For a class of Markovian (dynamically defined) algorithms he obtains

$$\eta^*(\theta) = 1 - \frac{\lambda_2}{\lambda_1}$$

almost everywhere. Using that the sum of all the exponents is zero, one easily concludes that in this case the algorithm is not optimal unless $\lambda_2 = \cdots = \lambda_{d+1}$. So if simplicity of the Lyapunov spectrum is as general as we believe, them optimal algorithms might actually not exist in this dynamically defined class. A large part of the information given by the fact of simplicity of Lyapunov Spectrum is obviously not been used here.

1.6. **Outline of the work.**

This work is organized as follows:
In section 2 we introduce a class of Multidimensional Continued Fractions Algorithms containing almost all of the classical algorithms. Selmers algorithm pertains to this class, so we present it in subsection 2.3, thereafter in subsection 2.4 some of its already known properties. Preparing the ground for the inducing argument in section 4 we give in subsection 2.5 a conjugation of the base application for Selmer's algorithm, restricted to some domain containing essentially all the dynamics, to some map on the projective plane.

The theoretical tools we need in the proof of simplicity for Selmer's algorithm are given in section 3. Here we introduce some important definitions involved in the proof and the basic simplicity criterium. In the last three subsections we touch superficially the fundamentals steps (inducing, pinching, twisting) to be verified.

These fundamentals steps we formalize in sections 4, 5 and 6. The argument used in section 4 is standard and there is no novelty here. Ideas used in sections 5 and 6 are rather elementary, and we believe a rigorous reader could follow them letting calculations and details for the Appendices.

## 2. Selmer's MCFA

In this section we introduce Selmer's Multidimensional Continued Fractions Algorithm and state some of its known properties. This algorithm is a particular case of a class of algorithms which Lagarias [15] called *Markovian multidimensional continued fractions algorithms* and, more specifically, of the subclass of *linear simplex-splitting multidimensional continued fractions algorithms*. In our proof of simplicity of the Lyapunov spectrum for Selmer's algorithm we first apply a standard inductive procedure to reduce the problem to the question of simplicity of the Lyapunov spectrum of a suitable projective map then we just have to prove that the associated monoid is pinching and twisting. We will be a little more specific on the three final subsections.

### 2.1. **Markovian MCF algorithms.**

Brentjes [6] and Szekeres [24] developed a general concept of multidimensional continued fractions (briefly MCF). A MCF algorithm associates to each $\theta \in [0,1]^d$ a sequence $\{A^n(\theta) : n \geq 1\}$ of matrices of partial quotients $A^n(\theta) \in GL(d+1, Z)$ with the convergent matrices associated defined by:

$$C^n(\theta) = A^n(\theta) \cdot A^{n-1}(\theta) \cdots A^1(\theta)$$

Simultaneous Diofantine approximations for $\theta$ will be obtained from the rows of $C^n(\theta)$ by:

$$\hat{w}_i^n = (\frac{C^n_{i,1}}{C^n_{i,d+1}}, \cdots, \frac{C^n_{i,d}}{C^n_{i,d+1}})$$

Where the denominator will be the last element $C^n_{i,d+1}$ from each row of $C^n_\theta$. A MCF algorithm is called weakly convergent if for any $\theta \in [0,1]^d$ the approximating vectors satisfy:

$$\hat{w} \xrightarrow[n\to\infty]{} \theta = (\theta_1, \cdots, \theta_d) \quad \text{for} \quad 1 \leq i \leq d+1$$

Lagarias [15] introduced the concepts of Markovian MCF algorithm and Simplex Splitting MCF Algorithm . A Markovian MCF Algorithm is determined by a pair $(T, A)$ where $T$ and $A$ are two functions

$$T : [0,1]^d \to [0,1]^d$$
$$A : [0,1]^d \to GL(d+1, Z)$$

continued by parts. The $n$-th matrix of the partial quotients of $\theta \in [0,1]^d$ is:

$$A^n(\theta) = A(T^{n-1}(\theta))$$

and $C^n(\theta) = A^n(\theta) \cdot A^{n-1}(\theta) \cdots A^1(\theta)$. Observe that $A^n(\theta)$ is a function of $T^{n-1}(\theta)$ so for determining $A^n(\theta)$ we don't need to know $\theta, T^1(\theta), \cdots, T^{n-2}(\theta)$. Hence the algorithm "forgets" the initial orbit motivating the name of Markovian for these algorithms. For arbitrary MCF Algorithms given by a pair (T,A) like before the rows of the matrices $C^n(\theta)$ may have not approximation properties to $\theta$. Therefore we call *Markovian MCF* those with the weak convergence property.

2.2. **Linear simplex-splitting multidimensional continued fractions algorithms.**

Must of the Markovian MCF Algorithms constructed until now are of a special class called *linear simplex-splitting multidimensional continued fractions algorithms*. They arise from some piecewise linear map $\widetilde{T}$ on the cone:

$$\mathbb{R}^{d+1}_{++} = \{(y_1, \cdots, y_{d+1}) \in \mathbb{R}^{d+1} : y_i \geq 0, 1 \leq i \leq d+1, \quad y_{d+1} = \max_{1 \leq i \leq d+1} y_i\}$$

where $d \in \{1, 2, 3, \cdots\}$. Hence

$$(y_1, \cdots, y_{d+1}) \in \mathbb{R}^{d+1}_{++} \setminus \{0\} \Rightarrow (\frac{y_1}{y_{d+1}}, \cdots, \frac{y_d}{y_{d+1}}, 1) \in [0,1]^d \times \{1\}$$

The cone $\mathbb{R}^{d+1}_{++}$ is partitioned into a (finite or infinite) family of subcones, and on each subcone the map $\widetilde{T}$ is linear, with

$$\widetilde{T}(y) = \widetilde{A}(y).y \tag{9}$$

where $\widetilde{A}(y) \in GL(d+1, Z)$ is constant on each subcone. $\widetilde{T}$ respect the rays, thus the set of rays can be identified with $[0,1]^d$ choosing as representative of the ray $[y] = \{t.(y_1, \cdots, y_{d+1}) : t > 0\}$ the unique vector with $y_{d+1} = 1$. This permits us, given the projection

$$p : \mathbb{R}^{d+1}_{++} \to [0,1]^d$$
$$(y_1, \cdots, y_{d+1}) \mapsto (\frac{y_1}{y_{d+1}}, \cdots, \frac{y_d}{y_{d+1}})$$

to define the map $T : [0,1]^d \to [0,1]^d$ making commutative the following diagram:

$$\begin{array}{ccc} \mathbb{R}^{d+1}_{++} & \xrightarrow{\widetilde{T}} & \mathbb{R}^{d+1}_{++} \\ p\downarrow & & \downarrow p \\ [0,1]^d & \xrightarrow{T} & [0,1]^d \end{array}$$

We can also induce the map:

$$A : [0,1]^d \to GL[d+1Z]$$
$$(x_1, \cdots, x_d) \mapsto ((\widetilde{A}(x_1, \cdots, x_d, 1))^T)^{-1}$$

So that the pair $(T, A)$ constitutes a Markovian MCF algorithm whenever it has the weak convergence property.

2.3. **Selmer's algorithm.**

As explained before, Selmer's algorithm was originally introduced [23] in connection with Brun's algorithm [7, 8], one main advantage being that the base dynamics can be codified (eventually) with only two symbols. Let us give the precise definition of this algorithm.
First of all define $\triangle^{d+1}_L$ and $B^d_L$ by

$$\triangle^{d+1}_L = \{y \in \mathbb{R}^{d+1} : 0 \leq y_1 \leq \cdots \leq y_{d+1}\}$$
$$B^d_L = \{x = (x_1, \ldots, x_d) : 0 \leq x_1 \leq \cdots \leq x_d \leq 1\} \subset \mathbb{R}^d$$

$\triangle_L^{d+1}$ is a cone with $\triangle_L^{d+1} \subset \mathbb{R}_{++}^{d+1}$ and $B_L^d \subset \triangle_L^{d-1+1} = \triangle_L^d$. Consider now the partition $\{B_L^d(i) : 0 \leq i \leq d\}$ of $B_L^d$ where the $B_L^d(i)$ are given by:

$$\begin{aligned}
B_L^d(0) &= B_L^d \cap \{x = (x_1, \ldots, x_d) : 0 \leq x_1 \leq 1 - x_d\} \\
&= \{x \in B_L^d : x_d + x_1 \leq 1 \leq 1 + x_1\} \\
B_L^d(1) &= B_L^d \cap \{x = (x_1, \ldots, x_d) : 1 - x_d < x_1 \leq 1 - x_{d-1}\} \\
&= \{x \in B_L^d : x_{d-1} + x_1 \leq 1 < x_d + x_1\} \\
B_L^d(2) &= B_L^d \cap \{x = (x_1, \ldots, x_d) : 1 - x_{d-1} < x_1 \leq 1 - x_{d-2}\} \\
&= \{x \in B_L^d : x_{d-2} + x_1 \leq 1 < x_{d-1} + x_1\} \\
\vdots \quad & \quad \vdots \\
B_L^d(d-1) &= B_L^d \cap \{(x_1, \ldots, x_d) : 1 - x_2 < x_1 \leq 1 - x_1\} \\
&= \{x \in B_L^d : x_1 + x_1 \leq 1 < x_2 + x_1\} \\
B_L^d(d) &= B_L^d \cap \{(x_1, \ldots, x_d) : 1 - x_1 < x_1 \leq 1\} \\
&= \{x \in B_L^d : x_1 \leq 1 < x_1 + x_1\}
\end{aligned}$$

Then $B_L^d(i) = \{x \in B_L^d : x_{d+1-i} + x_1 \geq 1 \geq x_{d-i} + x_1\}; \quad 0 \leq i \leq d$ where $x_{d+1} = 0$ and $x_0 = 0$.

*Selmer's multidimensional continued fractions algorithm* is a Linear Simplex-Splitting MCF algorithm. The corresponding map

$$\widetilde{T}_{\mathscr{S}} : \mathbb{R}_{++}^{d+1} \to \mathbb{R}_{++}^{d+1} \tag{10}$$

being defined by:

-If $y \in \mathbb{R}_{++}^{d+1} \setminus \triangle_L^{d+1}$ then $\widetilde{T}_{\mathscr{S}}(y)$ is the vector obtained by permuting the coordinates of $y$ in a way that the resulting vector has its coordinates in increasing order. So $\widetilde{T}_{\mathscr{S}}(\mathbb{R}_{++}^{d+1} \setminus \triangle_L^{d+1}) \subset \triangle_L^{d+1}$.

-If $y \in \triangle_L^{d+1}$ then $\widetilde{T}_{\mathscr{S}}(y)$ is obtained from $(y_1, y_2, \cdots, y_d, y_{d+1} - y_1)$ making the necessary permutations on its coordinates for getting a vector in $\triangle_L^{d+1}$.

Then $\widetilde{T}_{\mathscr{S}}(\mathbb{R}_{++}^{d+1}) \subset \triangle_L^{d+1}$ and it suffice to describe the action of $\widetilde{T}_{\mathscr{S}}$ on $\triangle_L^{d+1}$. Let us describe the regions of $\triangle_L^{d+1}$ corresponding to the different expressions defining $\widetilde{T}_{\mathscr{S}}$. First observe that

$$0 \leq 0 + y_1 \leq y_1 + y_1 \leq y_2 + y_1 \leq \cdots \leq y_{d-1} + y_1 \leq y_d + y_1 \leq y_{d+1} + y_1 \tag{11}$$

And the implications:

$$\boxed{0}\quad \widetilde{T}_{\mathscr{S}}(y_1,y_2,\dots,y_{d+1}) = (y_1,y_2,\dots,y_d,y_{d+1}-y_1)$$
$$\Leftrightarrow 0 \le y_1 \le y_2 \le \cdots \le y_d \le y_{d+1}-y_1$$
$$\Rightarrow y_d + y_1 \le y_{d+1}$$

$$\boxed{1}\quad \widetilde{T}_{\mathscr{S}}(y_1,y_2,\dots,y_{d+1}) = (y_1,y_2,\dots,y_{d+1}-y_1,y_d)$$
$$\Leftrightarrow 0 \le y_1 \le y_2 \le \cdots \le y_{d-1} \le y_{d+1}-y_1 \le y_d$$
$$\Rightarrow y_{d-1} + y_1 \le y_{d+1} \le y_d + y_1$$

$$\boxed{2}\quad \widetilde{T}_{\mathscr{S}}(y_1,y_2,\dots,y_{d+1}) = (y_1,y_2,\dots,y_{d-2},y_{d+1}-y_1,y_{d-1},y_d)$$
$$\Leftrightarrow 0 \le y_1 \le y_2 \le \cdots \le y_{d-2} \le y_{d+1}-y_1 \le y_{d-1} \le y_d$$
$$\Rightarrow y_{d-2} + y_1 \le y_{d+1} \le y_{d-1} + y_1$$

$$\vdots \qquad \vdots$$

$$\boxed{d-1}\quad \widetilde{T}_{\mathscr{S}}(y_1,y_2,\dots,y_{d+1}) = (y_1,y_{d+1}-y_1,y_2,\dots,y_d)$$
$$\Leftrightarrow 0 \le y_1 \le y_{d+1}-y_1 \le y_2 \le \cdots \le y_d$$
$$\Rightarrow y_1 + y_1 \le y_{d+1} \le y_2 + y_1$$

$$\boxed{d}\quad \widetilde{T}_{\mathscr{S}}(y_1,y_2,\dots,y_{d+1}) = (y_{d+1}-y_1,y_1,y_2,\dots,y_d)$$
$$\Leftrightarrow 0 \le y_{d+1}-y_1 \le y_1 \le y_2 \le \cdots \le y_d$$
$$\Rightarrow 0 + y_1 \le y_{d+1} \le y_1 + y_1$$

So we can decompose $\triangle_L^{d+1}$ in some mutually disjoint regions $\triangle_L^{d+1}(0),\cdots,\triangle_L^{d+1}(d)$ such that

$$\triangle_L^{d+1} = \triangle_L^{d+1}(0) \cup \cdots \cup \triangle_L^{d+1}(d) \tag{12}$$

Where

$$\begin{aligned}
\triangle_L^{d+1}(0) &= \triangle_L^{d+1} \cap \{y_d + y_1 \le y_{d+1} \le y_{d+1} + y_1\} \\
&= \triangle_L^{d+1} \cap \{y : 0 \le x_1 \le 1 - x_d\} \\
\triangle_L^{d+1}(1) &= \triangle_L^{d+1} \cap \{y_{d-1} + y_1 \le \ y_{d+1} < y_d + y_1\} \\
&= \triangle_L^{d+1} \cap \{y : 1 - x_d < x_1 \le 1 - x_{d-1}\} \\
\triangle_L^{d+1}(2) &= \triangle_L^{d+1} \cap \{y_{d-2} + y_1 \le \ y_{d+1} < y_{d-1} + y_1\} \\
&= \triangle_L^{d+1} \cap \{y : 1 - x_{d-1} < x_1 \le 1 - x_{d-2}\} \\
\vdots \quad & \quad \vdots \\
\triangle_L^{d+1}(d-1) &= \triangle_L^{d+1} \cap \{y_1 + y_1 \le \ y_{d+1} < y_2 + y_1\} \\
&= \triangle_L^{d+1} \cap \{y : 1 - x_2 < x_1 \le 1 - x_1\} \\
\triangle_L^{d+1}(d) &= \triangle_L^{d+1} \cap \{0 + y_1 \le \ y_{d+1} < y_1 + y_1\} \\
&= \triangle_L^{d+1} \cap \{y : 1 - x_1 < x_1 \le 1\}
\end{aligned}$$

and $(x_1,\cdots,x_d) = (\frac{y_1}{y_{d+1}},\cdots,\frac{y_d}{y_{d+1}})$ satisfy

$$0 \le x_1 \le x_2 \le \cdots \le x_d \le 1$$

So $(x_1, \cdots, x_d) \in B^d_L$. The projection $p$ verifies $p(\triangle^{d+1}_L(k)) = B^d_L(k); \quad 0 \leq k \leq d$.

Then we get a pair $(T_{\mathscr{S}}, A)$ like at the end of subsection 2.2. And we can recover the expressions for $T_{\mathscr{S}}$ and the matrices defining the map $A$ as follows:

$\boxed{i=0}$

$$\begin{array}{ccc} (y_1,\dots,y_{d+1}) \in \triangle^{d+1}_L(0) & \xrightarrow{\tilde{T}_{\mathscr{S}}} & \triangle^{d+1}_L \ni (y_1,\dots,y_d,y_{d+1}-y_1) \\ p\big\downarrow & & \big\downarrow p \\ (\frac{y_1}{y_{d+1}},\dots,\frac{y_d}{y_{d+1}}) \in B^d_L(0) & \xrightarrow{T_{\mathscr{S}}} & B^d_L \ni (\frac{y_1}{y_{d+1}-y_1},\dots,\frac{y_d}{y_{d+1}-y_1}) \end{array}$$

$$\Rightarrow ((\frac{y_1}{y_{d+1}},\dots,\frac{y_d}{y_{d+1}}) \mapsto (\frac{\frac{y_1}{y_{d+1}}}{1-\frac{y_1}{y_{d+1}}},\dots,\frac{\frac{y_d}{y_{d+1}}}{1-\frac{y_1}{y_{d+1}}}))$$

Then

$$T_{\mathscr{S}}(x_1,\cdots,x_d) = (\frac{x_1}{1-x_1},\cdots,\frac{x_d}{1-x_1}) \quad \text{if} \quad (x_1,\cdots,x_d) \in B^d_L(0)$$

$$\tilde{A}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$((\tilde{A}_0)^{-1})^T = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\boxed{i=1}$

$$\begin{array}{ccc} (y_1,\dots,y_{d+1}) \in \triangle^{d+1}_L(1) & \xrightarrow{\tilde{T}_{\mathscr{S}}} & \triangle^{d+1}_L \ni (y_1,\dots,y_{d+1}-y_1,y_d) \\ p\big\downarrow & & \big\downarrow p \\ (\frac{y_1}{y_{d+1}},\dots,\frac{y_d}{y_{d+1}}) \in B^d_L(1) & \xrightarrow{T_{\mathscr{S}}} & B^d_L \ni (\frac{y_1}{y_d},\dots,\frac{y_{d-1}}{y_d},\frac{y_{d+1}-y_1}{y_d}) \end{array}$$

$$\Rightarrow ((\frac{y_1}{y_{d+1}},\dots,\frac{y_d}{y_{d+1}}) \mapsto (\frac{\frac{y_1}{y_{d+1}}}{\frac{y_d}{y_{d+1}}},\dots,\frac{\frac{y_{d-1}}{y_{d+1}}}{\frac{y_d}{y_{d+1}}},\frac{1-\frac{y_1}{y_{d+1}}}{\frac{y_d}{y_{d+1}}}))$$

Then

$$T_{\mathscr{S}}(x_1,\cdots,x_d)=(\frac{x_1}{x_d},\cdots,\frac{x_{d-1}}{x_d},\frac{1-x_1}{x_d})\quad\text{if}\quad(x_1,\cdots,x_d)\in B^d_L(1))$$

$$\tilde{A}_1=\begin{pmatrix}1&0&0&0&\dots&0&0&0&0\\0&1&0&0&\dots&0&0&0&0\\0&0&1&0&\dots&0&0&0&0\\0&0&0&1&\dots&0&0&0&0\\\dots&\dots&\dots&\dots&\dots&\dots&\dots&\dots&\dots\\0&0&0&0&\dots&1&0&0&0\\0&0&0&0&\dots&0&1&0&0\\-1&0&0&0&\dots&0&0&0&1\\0&0&0&0&\dots&0&0&1&0\end{pmatrix}$$

and

$$((\tilde{A}_1)^{-1})^T=\begin{pmatrix}1&0&0&0&\dots&0&0&0&1\\0&1&0&0&\dots&0&0&0&0\\0&0&1&0&\dots&0&0&0&0\\0&0&0&1&\dots&0&0&0&0\\\dots&\dots&\dots&\dots&\dots&\dots&\dots&\dots&\dots\\0&0&0&0&\dots&1&0&0&0\\0&0&0&0&\dots&0&1&0&0\\0&0&0&0&\dots&0&0&0&1\\0&0&0&0&\dots&0&0&1&0\end{pmatrix}$$

$$\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots$$

$\boxed{i=d-1}$

$$\begin{array}{ccc}(y_1,\dots,y_{d+1})\in\triangle^{d+1}_L(d-1) & \xrightarrow{\widetilde{T}_{\mathscr{S}}} & \triangle^{d+1}_L\ni(y_1,y_{d+1}-y_1,y_2,\dots,y_{d-1},y_d)\\ \downarrow p & & \downarrow p\\ (\frac{y_1}{y_{d+1}},\dots,\frac{y_d}{y_{d+1}})\in B^d_L(d-1) & \xrightarrow{T_{\mathscr{S}}} & B^d_L\ni(\frac{y_1}{y_d},\frac{y_{d+1}-y_1}{y_d},\frac{y_2}{y_d},\cdots,\frac{y_{d-1}}{y_d})\end{array}$$

$$\Rightarrow((\frac{y_1}{y_{d+1}},\dots,\frac{y_d}{y_{d+1}})\mapsto(\frac{\frac{y_1}{y_{d+1}}}{\frac{y_d}{y_{d+1}}},\frac{1-\frac{y_1}{y_{d+1}}}{\frac{y_d}{y_{d+1}}},\frac{\frac{y_2}{y_{d+1}}}{\frac{y_d}{y_{d+1}}},\dots,\frac{\frac{y_{d-1}}{y_{d+1}}}{\frac{y_d}{y_{d+1}}}))$$

Then

$$T_{\mathscr{S}}(x_1,\cdots,x_d)=(\frac{x_1}{x_d},\frac{1-x_1}{x_d},\frac{x_2}{x_d},\cdots,\frac{x_{d-1}}{x_d})\quad\text{if}\quad(x_1,\cdots,x_d)\in B^d_L(d-1)$$

$$\tilde{A}_{d-1}=\begin{pmatrix}1&0&0&0&\dots&0&0&0&0\\-1&0&0&0&\dots&0&0&0&1\\0&1&0&0&\dots&0&0&0&0\\0&0&1&0&\dots&0&0&0&0\\\dots&\dots&\dots&\dots&\dots&\dots&\dots&\dots&\dots\\0&0&0&0&\dots&0&0&0&0\\0&0&0&0&\dots&1&0&0&0\\0&0&0&0&\dots&0&1&0&0\\0&0&0&0&\dots&0&0&1&0\end{pmatrix}$$

and

$$((\tilde{A}_{d-1})^{-1})^T = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix}$$

$\boxed{i = d}$

$$\begin{array}{ccc} (y_1, \dots, y_{d+1}) \in \triangle_L^{d+1}(d) & \xrightarrow{\widetilde{T}_{\mathscr{S}}} & \triangle_L^{d+1} \ni (y_{d+1} - y_1, y_1, y_2, \dots, y_d) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ (\frac{y_1}{y_{d+1}}, \dots, \frac{y_d}{y_{d+1}}) \in B_L^d(d) & \xrightarrow{T_{\mathscr{S}}} & B_L^d \ni (\frac{y_{d+1}-y_1}{y_d}, \frac{y_1}{y_d}, \cdots, \frac{y_{d-1}}{y_d}) \end{array}$$

$$\Rightarrow ((\frac{y_1}{y_{d+1}}, \dots, \frac{y_d}{y_{d+1}}) \mapsto (\frac{1 - \frac{y_1}{y_{d+1}}}{\frac{y_d}{y_{d+1}}}, \frac{\frac{y_1}{y_{d+1}}}{\frac{y_d}{y_{d+1}}}, \cdots, \frac{\frac{y_{d-1}}{y_{d+1}}}{\frac{y_d}{y_{d+1}}}))$$

Then

$$T_{\mathscr{S}}(x_1, \cdots, x_d) = (\frac{1 - x_1}{x_d}, \frac{x_1}{x_d}, \cdots, \frac{x_{d-1}}{x_d}) \quad \text{if} \quad (x_1, \cdots, x_d) \in B_L^d(d)$$

$$\tilde{A}_d = \begin{pmatrix} -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$((\tilde{A}_d)^{-1})^T = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix}$$

From now on we will denote the matrices $(\tilde{A}_{d-1}^T)^{-1}$ and $(\tilde{A}_d^T)^{-1}$ by $A_1$ and $A_2$ respectively.

## 2.4. **Some known properties.**

This presentation of Selmer's algorithm we have just given looks different from the one in [22] but, in fact, the two versions are conjugated by the diffeomorphism $f$ defined as

$$\begin{array}{ccc} f : B^d & \longrightarrow & B^d_L \\ (x_1, \cdots, x_d) & \longrightarrow & (x_d, x_{d-1}, \cdots, x_1) \end{array}$$

So the following diagram conmutes

$$\begin{array}{ccc} B^d & \xrightarrow{T} & B^d \\ f\downarrow & & \downarrow f \\ B^d_L & \xrightarrow{T_{\mathscr{S}}} & B^d_L \end{array}$$

We are going to state some of the nice properties proven for $T$ in [22] which can be rewritten as analogous assertions for $T_{\mathscr{S}}$. Observe that not all of them are necessary for us.

For example Lemma 16, page 55 in [22] becomes

$$\begin{aligned} f(B^d(i)) &= f(\{x \in B^d : x_i + x_d \geq 1 \geq \ x_{i+1} + x_d)\} \\ &= \{(x_d, \cdots, x_1) : x_i + x_d \geq 1 \geq x_{i+1} + x_d\} \\ &= \{(y_1, \cdots, y_d) : y_{d+1-i} + y_1 \geq 1 \geq y_{d-i} + y_1\} = B^d_L \end{aligned}$$

Then taking $D_L = B^d_L(d-1) \cup B^d_L(d)$ we have

$$\begin{aligned} D_L &= f(B^d(d-1)) \cup f(B^d(d)) = f(B^d(d-1) \cup B^d(d)) \\ &= f(\{x \in B^d : x_{d-1} + x_d \geq 1\}) = \{x \in B^d_L : x_2 + x_1 \geq 1\} \end{aligned}$$

And Theorem 22 page 55, Lemma 16 page 55 in [22] can be rewritten as

**Theorem 2.1.** *$D_L$ is an absorbing set, which means:*

*(i) $T_{\mathscr{S}} D_L = D_L$*

*(ii)For almost all $x \in B^d_L$ there exists $N = N(x)$ such that $T^N_{\mathscr{S}}(x) \in D_L$.*

**Lemma 2.1.** *For $0 \leq i \leq d$ we have $B^d_L(i) = \{x \in B^d_L : x_{d+1-i} + x_1 \geq 1 \geq x_{d-i} + x_1\}$ which implies*

*$T_{\mathscr{S}} B^d_L(i) = \{x \in B^d_L : x_{d+1-i} + x_1 \geq 1\}$ for $0 \leq i \leq d-1$. So*

$$T_{\mathscr{S}} B^d_L(i) = \bigcup_{j \geq 1} B^d_L(j)$$

So that $D_L$ contains essentially all the dynamics of $T_{\mathscr{S}}$ and we call it the *fundamental domain*. The following Lemma shows that $\mathcal{T} = T\mid_{D_L}$ have a nice Markov structure.

**Lemma 2.2.** *$\mathcal{T} : B^{d-1}_L \to D_L$ and $\mathcal{T} : B^d_L \to D_L$ are bijections.*

*Proof.*

$\mathcal{T}\mid_{B^d(d-1)}$ is a bijection:
From Lemma 1

$$T_{\mathscr{S}} B^d_{d-1} = B^d_{d-1} \cup B^d_d = D_L$$

and $\mathcal{T}\mid_{B^d_{d-1}}$ is surjective.
Suppose now that $(x_1, \cdots, x_d) \in B^d(d-1)$ is such that

$$T(x_1, \cdots, x_d) = (y_1, \cdots, y_d) \in D$$

Then:

$$(\frac{x_1}{x_d}, \frac{1-x_1}{x_d}, \frac{x_2}{x_d}, \cdots, \frac{x_{d-1}}{x_d}) = (y_1, \cdots, y_d)$$

which implies that

$$(x_1, \cdots, x_d) = (\frac{y_1}{y_2+y_1}, \frac{y_3}{y_2+y_1}, \cdots, \frac{y_d}{y_2+y_1}, \frac{1}{y_2+y_1})$$

and $\mathcal{T}\mid_{B^d(d-1)}$ is injective.

$\mathcal{T}\mid_{B^d(d)}$ is a bijection:
Observe that $(x_1, \cdots, x_d) \in B^d(d)$ is equivalent to

$$0 \le x_1 \le \cdots \le x_d \le 1; \quad 1 - x_1 \le x_1 \le 1$$

which implies

$$\frac{1-x_1}{x_d} \le \frac{x_1}{x_d} \le \frac{x_2}{x_d} \le \cdots \le \frac{x_{d-1}}{x_d}; \quad \frac{x_1}{x_d} + \frac{1-x_1}{x_d} = \frac{1}{x_d} \ge 1$$

so that

$$T(x_1, \cdots, x_d) = (\frac{1-x_1}{x_d}, \frac{x_1}{x_d}, \cdots, \frac{x_{d-1}}{x_d}) \in D_L$$

and $\mathcal{T}(B^d(d)) \subset D_L$
Let $(y_1, \cdots, y_d)$ and $(x_1, \cdots, x_d)$ be such that:

$$(\frac{1-x_1}{x_d}, \frac{x_1}{x_d}, \cdots, \frac{x_{d-1}}{x_d}) = (y_1, \cdots, y_d) \tag{13}$$

Then:

$$(x_1, \cdots, x_d) = (\frac{y_2}{y_2+y_1}, \frac{y_3}{y_2+y_1}, \cdots, \frac{y_d}{y_2+y_1}, \frac{1}{y_2+y_1})$$

Now observe that

$$(y_1, \cdots, y_d) \in D_L$$

is equivalent to

$$0 \le y_1 \le y_2 \le \cdots \le y_d \le 1; \quad y_2 + y_1 \ge 1$$

which implies

$$0 \le x_1 \le \cdots \le x_d \le 1 \quad \text{and} \quad x_1 = \frac{y_2}{y_2+y_1} \ge \frac{1}{2}$$

so that $(x_1, \cdots, x_d) \in B_L(d)$

By the above $\mathcal{T}\mid_{B_L(d)}$ is a bijection. □

Now Lemma 19 page 58 and Theorem 23 page 60 in [22] translates to:

**Lemma 2.3.** $\bigcap_{s=1}^{\infty} B_L^d(k_1(x), \cdots, k_s(x)) = \{x\}$ *for any* $x \in B_L^d$ *where*

$$B_L^d(k_1(x), \cdots, k_s(x)) = \{x \in B_L^d : T_{\mathscr{S}}^j(x) \in B_L^d(k_{j+1}(x)), \quad 0 \leq j \leq s-1\}$$

**Theorem 2.2.** *Selmer's algorithm is ergodic in relation to the Lebesgue measure and admits an absolutely continous invariant measure* $\mu_{T_{\mathscr{S}}}$ *(which is finite for* $d \geq 2$*) with density:*

$$h_{T_{\mathscr{S}}}(x) = \frac{1}{x_1 x_2 \cdots x_d}$$

The cylinders for the application $\mathcal{T}$ coincide with those for $T_{\mathscr{S}}$ in the form

$$B_L^d(k_1, \cdots, k_s), \quad \text{where} \quad k_1, \cdots, k_s \in \{d-1, d\}$$

We then conclude from Lemma 2.2 and Theorem 2.2 that all the cylinders $B_L^d(k_1, \cdots, k_s)$ are non empty and that $\mathcal{T}_s(B_L^d(k_1, \cdots, k_s)) = D_L$. Also from Lemma 2.3 and Theorem 2.2 we get the

**Proposition 2.1.** $\mathcal{T}$ *is ergodic and admits an absolutely continuous invariant measure* $\mu(A) = \frac{1}{\mu_{T_{\mathscr{S}}}(D)} \cdot \mu_{T_{\mathscr{S}}}(A)$ *with density* $h(x) = \frac{1}{\mu_{T_{\mathscr{S}}}(D) x_1 \cdots x_d}$ *and*

$$\lim_{s \to \infty} diam B_L^d(k_1, \cdots, k_s) = 0$$

Observe that if $(x_1, \cdots, x_d) \in D_L$ is such that $x_i = 0$, then from $i > 1$ we deduce that $x_1 = x_2 = \cdots = x_i = 0$ and $x_2 + x_1 = 0$ which is a contradiction. So $i = 1$ and $x_1 = 0$ which implies $x_2 = x_2 + x_1 \geq 1$, $x_2 = 1$ and finally $x_2 = x_3 = \cdots = x_d = 1$. We thus get

*Remark* 2.1. The density $h(x)$ is infinite in and only in the point

$$(0, 1, \cdots, 1) \in B_L^d(d-1)$$

## 2.5. **Projectivization of the fundamental domain.**

For understanding the inducing procedure to be applied on section 4 it is useful to see Selmer's cocycle as a cocycle over some map defined on a convenient subset of the projective plane. This is our next objective.
Let now $\tilde{D}^d$ be defined by

$$\tilde{D}^d = \{(x_1, \cdots, x_d, 1) : (x_1, \cdots, x_d) \in D_L\} \subset \mathbb{R}^{d+1}$$

and $\mathbb{D}^d = \triangle_L^{d+1}(d-1) \cup \triangle_L^{d+1}(d)$. It is true that

$$\mathbb{D}^d = \{(y_1, \cdots, y_{d+1}) : 0 \leq y_1 \leq \cdots \leq y_{d+1}, y_2 + y_1 \geq y_{d+1}\}$$

because

$$\left.\begin{array}{l} (y_1, \ldots, y_{d+1}) \in \triangle_L^{d+1}(d-1) \Rightarrow y_2 + y_1 \geq y_{d+1} \\ (y_1, \ldots, y_{d+1}) \in \triangle_L^{d+1}(d) \Rightarrow y_2 + y_1 \geq y_1 + y_1 \geq y_{d+1} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \mathbb{D}^d \subset \{(y_1, \cdots, y_{d+1}) : 0 \leq y_1 \leq \cdots \leq y_{d+1}, \quad y_2 + y_1 \geq y_{d+1}\}$$

We also have that if $(y_1, \cdots, y_{d+1})$ is such that $0 \le y_1 \le \cdots \le y_{d+1}, \quad y_2+y_1 \ge y_{d+1}$ there are two possibilities:

1)$y_2 + y_1 \le y_{d+1} \Rightarrow y_1 + y_1 \le y_{d+1} \le y_2 + y_1 \Rightarrow (y_1, \cdots, y_{d+1}) \in \triangle_L^{d+1}(d-1)$

2)$y_{d+1} \le y_2 + y_1 \Rightarrow 0 + y_1 \le y_{d+1} \le y_1 + y_1 \Rightarrow (y_1, \cdots, y_{d+1}) \in \triangle_L^{d+1}(d)$

Observe that if $(y_1, \cdots, y_{d+1}) \in \mathbb{D}$ then $(y_1, y_3, \cdots, y_{d+1}, y_2+y_1) \in \triangle_L^{d+1}(d-1)$ and

$$\widetilde{T}_{\mathscr{S}}(y_1, y_3, \cdots, y_{d+1}, y_2 + y_1) = (y_1, y_2, y_3, \cdots, y_{d+1}) \tag{14}$$

Analogously $(y_2, y_3, \cdots, y_{d+1}, y_2 + y_1) \in \triangle_L^{d+1}(d)$ and

$$\widetilde{T}_{\mathscr{S}}(y_2, y_3, \cdots, y_{d+1}, y_2 + y_1) = (y_1, y_2, y_3, \cdots, y_{d+1}) \tag{15}$$

It follows that $\widetilde{T}_{\mathscr{S}}(\triangle_L^{d+1}(d-1)) = \mathbb{D} = \widetilde{T}_{\mathscr{S}}(\triangle_L^{d+1}(d))$. By construction the maps $\widetilde{T}_{\mathscr{S}} \mid_{\triangle_L^{d+1}(d-1)}$ and $\widetilde{T}_{\mathscr{S}} \mid_{\triangle_L^{d+1}(d)}$ are injective so we deduce that the maps $\widetilde{T}_{\mathscr{S}} : \triangle_L^{d+1}(d-1) \to \mathbb{D}$ and $\widetilde{T}_{\mathscr{S}} : \triangle_L^{d+1}(d) \to \mathbb{D}$ are bijections. Also

$$(\widetilde{T}_{\mathscr{S}} \mid_{\triangle_L^{d+1}(d-1)})^{-1} : \mathbb{D} \to \triangle_L^{d+1}(d-1) \quad \text{and} \quad (\widetilde{T}_{\mathscr{S}} \mid_{\triangle_L^{d+1}(d)})^{-1} : \mathbb{D} \to \triangle_L^{d+1}(d)$$

are restrictions of the linear applications given by the non negative matrices $(\tilde{A}_{d-1})^{-1}$ and $(\tilde{A}_d)^{-1}$ from $\mathbb{R}_+^{d+1}$ to $\mathbb{R}_+^{d+1}$.

Let $\Lambda_{d+1} = \{(z_1, \cdots, z_{d+1}) \in \mathbb{R}_+^{d+1} : z_1 + \cdots + z_{d+1} = 1\}$ and define the maps:

$$\begin{aligned} j : \bar{D}_L^d &\to \bar{\bar{D}}^d \\ (x_1, \cdots, x_d) &\mapsto (x_1, \cdots, x_d, 1) \end{aligned}$$

$$\begin{aligned} p : \bar{\bar{D}} &\to \Lambda_{d+1} \\ (y_1, \cdots, y_{d+1}) &\mapsto (\frac{y_1}{y_1 + \cdots + y_{d+1}}, \cdots, \frac{y_{d+1}}{y_1 + \cdots + y_{d+1}}) \end{aligned}$$

$$\begin{aligned} h : \bar{D}_L &\to \Lambda_{d+1} \\ (x_1, \cdots, x_d) &\mapsto p \circ j(x_1, \cdots, x_d) \\ &= (\frac{x_1}{x_1 + \cdots + x_d + 1}, \cdots, \frac{x_d}{x_1 + \cdots + x_d + 1}, \frac{1}{x_1 + \cdots + x_d + 1}) \end{aligned}$$

Where $\bar{A}$ denotes the closure of A.

Being $\bar{D}_L$ a compact set, $\Lambda_{d+1}$ Hausdorff, $h \in C^\infty$ injective it follows that $h$ is a homeomorphism onto its image. Observing that $(x_1, \cdots, x_d) \in \bar{D}_L \Rightarrow \frac{1}{x_1 + \cdots + x_d + 1} \neq 0 \Rightarrow y_{d+1} \neq 0$ if $(y_1, \cdots, y_{d+1}) \in h(\bar{D}_L)$ and that $h^{-1}(y_1, \cdots, y_{d+1}) = (\frac{y_1}{y_{d+1}}, \cdots, \frac{y_d}{y_{d+1}})$ results that $h$ is a diffeomorphism $C^\infty$ between $D_L$ and $h(D_L)$. And it is easy to prove that

$$h(B_L^d(d-1)) = \Lambda_{d+1} \cap \Lambda_L^{d+1}(d-1), \quad h(B_L^d(d)) = \Lambda_{d+1} \cap \Lambda_L^{d+1}(d)$$

Let $\widetilde{T}_p : h(D_L) \to \Lambda_{d+1}$ be the projectivization of $\widetilde{T}_{\mathscr{S}}$. Since

$$\widetilde{T}_{\mathscr{S}}(\triangle_L^{d+1}(d-1)) = \mathbb{D} = \triangle_L^{d+1}(d-1) \cup \triangle_L^{d+1}(d) = \mathbb{D} = \widetilde{T}_{\mathscr{S}}(\triangle_L^{d+1}(d))$$

it's clear that:

$$\begin{aligned}\widetilde{T}_p(h(B_L^d(d-1))) &= \widetilde{T}_p(\Lambda_{d+1}\cap\triangle_L^{d+1}(d-1)) = \Lambda_{d+1}\cap(\triangle_L^{d+1}(d-1)\cup\triangle_L^{d+1}(d)) \\ &= \Lambda_{d+1}\cap\mathbb{D} \\ \widetilde{T}_p(h(B_L^d(d))) &= \widetilde{T}_p(\Lambda_{d+1}\cap\triangle_L^{d+1}(d)) = \Lambda_{d+1}\cap(\triangle_L^{d+1}(d-1)\cup\triangle_L^{d+1}(d)) \\ &= \Lambda_{d+1}\cap\mathbb{D}\end{aligned}$$

Let us now define $\beta = h(D_L)$, $\beta(1) = h(B_L^d(d-1))$, $\beta(2) = h(B_L^d(d))$. Then:

$$\begin{aligned}\widetilde{T}_p\mid_{\beta(1)}(y_1,\cdots,y_{d+1}) &= \frac{(y_1, y_{d+1}-y_1, y_2,\cdots,y_{d-1},y_d)}{y_2+y_3+\cdots+y_{d+1}} \\ \widetilde{T}_p\mid_{\beta(2)}(y_1,\cdots,y_{d+1}) &= \frac{(y_{d+1}-y_1, y_1, y_2,\cdots,y_d)}{y_2+y_3+\cdots+y_{d+1}}\end{aligned}$$

So we can easily prove that

$$\mathcal{T}(x_1,\cdots,x_d) = h^{-1}\circ\widetilde{T}_p\circ h(x_1,\cdots,x_d) \tag{16}$$

and therefore $\mathcal{T},\widetilde{T}_p$ are conjugated by the diffeomorphism $C^\infty, h$.

*Remark* 2.2. $\beta(2) = h(B_L^d(d))$ is compactly contained in $\Lambda_{d+1}$.

*Proof.* Observe that if $(x_1,\cdots,x_d)\in B_L^d(d)$ then $\frac{1}{2}\le x_1\le\cdots\le x_d\le 1$ so that $\frac{d}{2}+1\le x_1+\cdots+x_d+1\le d+1$ and, consequently

$$\begin{aligned}\frac{1}{d+1} \le \frac{1}{x_1+\cdots+x_d+1} \le \frac{1}{\frac{d}{2}+1} \\ \frac{\frac{1}{2}}{d+1} \le \frac{x_i}{x_1+\cdots+x_d+1} \le \frac{1}{\frac{d}{2}+1}, \quad 1\le i\le d\end{aligned}$$

□

By the conjugation (16) we can substitute the cocycle $(D_L,\mathcal{T},\mu,((\tilde{A})^T)^{-1})$ by the cocycle $(\beta,\widetilde{T}_p,h_*\mu,(\tilde{A}^T)^{-1}(h^{-1}))$. The Markov structure for the new cocycle given by $\beta=\beta(1)\cup\beta(2)$ with $\widetilde{T}_p(\beta(1))=\beta=\widetilde{T}_p(\beta(2)))$.

*Remark* 2.3. Simplicity of the Lyapunov Spectrum for the cocycle $(\beta,\widetilde{T}_p,\mu_p,(\tilde{A}^T)^{-1})(h^{-1}))$ imply simplicity of the Lyapunov Spectrum for the cocycle $(D_L,\mathcal{T},\mu,(\tilde{A}^T)^{-1})$.

## 3. The simplicity criterium and outline of the proof

Sufficient conditions for Simplicity of Lyapunov Spectrum given by Avila-Viana in [1], and [2] will permit us to prove simplicity for Selmer's Multidimensional Continued Fractions Algorithm. For our purposes its enough Theorem 7.1 from [2]. The following closely resembles this work.

### 3.1. **Simplicity criterium.**

Let $(\triangle, \mu_0)$ be a probability space, and let

$$T : \bigcup_{(l)\in\Lambda} \triangle^{(l)} \to \triangle$$

be a transformation where $\Lambda$ is finite or countable, $\triangle^{(l)} \subset \triangle$ for all $(l) \in \Lambda$,

$$\mu_0(\bigcup_{(l)\in\Lambda}) = 1, \quad \mu_0(\triangle^{(l)}) > 0, \quad \forall (l) \in \Lambda$$

$T : \triangle^{(l)} \to \triangle$ is an invertible transformation and $T_*(\mu_0 \mid_{\triangle^{(l)}})$ is equivalent to $\mu_0$. Let $\Omega$ be the set of finite sequences of elements of $\Lambda$, including the empty sequence. If $\underline{l} = (l_1, \cdots, l_m) \in \Lambda$, let us define $\triangle^{\underline{l}}$ by

$$\triangle^{\underline{l}} = \{x \in \triangle : T^k(x) \in \triangle^{(l_{k+1})} \quad \text{for} \quad 0 \le k \le m\}$$

and $T^{\underline{l}} \overset{def.}{=} T^m : \triangle^{\underline{l}} \to \triangle$. Note that $T^{\underline{l}}$ is an invertivel and measuravel transformation.

**Definition 3.1.** We say that $(T, \mu_0)$ has *approximate product structure* if there exists $C > 0$ such that

$$\frac{1}{C} \le \frac{1}{\mu_0(\triangle^{\underline{l}})} \frac{dT_*^{\underline{l}}(\mu_0 \mid_{\triangle^{\underline{l}}})}{d\mu_0} \le C \quad \text{for all} \quad \underline{l} \in \Omega$$

Under those conditions there exists one and only one probability measure $\mu$ on the cylinder $\sigma$-algebra which is invariant under $T$ and is absolutely continuous with respect to $\mu_0$; $(T, \mu)$ has approximate product structure as well.

**Definition 3.2.** Let $(T, \mu)$ have approximate product structure and $H$ be some finite-dimensional vector space. Let $A^{(l)} \in SL(H), \quad l \in \Lambda$, and define $A : \triangle \to SL(H)$ by $A(x) = A^{(l)}$ if $x \in \triangle^{(l)}$. We say that $(T, A)$ is a *locally constant cocycle*. The *supporting monoid* of $(T, A)$ is the monoid generated by the $A^{(l)}$'s $l \in \Lambda$.

If $T$ is ergodic in relation to $\mu$ and there exists Lyapunov exponents for almost any $x \in \bigcup_{(l)\in\Lambda} \triangle^{(l)}$ then it is well defined the Lyapunov spectrum for the cocycle.

Now for $p \ge 2$ we call $P\mathbb{R}_+^p$ the *standard simplex*. A *projective contraction* is a projective transformation taking the standard simplex into itself or, in other words, it is the projectivization of some matrix $B \in GL(p, \mathbb{R})$ with non-negative entries. The image of the standard simplex by a projective contraction is called a *simplex*.

A *projective expanding map* $T$ is a map $T : \cup\triangle^{(l)} \to \triangle$, where $\triangle$ is a simplex compactly contained in the standard simplex, the $\triangle^{(l)}$'s form a finite or countable family of pairwise disjoint simplexes contained in $\triangle$ and covering almost all of $\triangle$, and $T^{(l)} \overset{def.}{=} T \mid_{\triangle^{(l)}}: \triangle^{(l)} \to \triangle$ is a bijection such that $(T^{(l)})^{-1}$ is the restriction of a projective contraction.

**Lemma 3.1.** *If $T : \cup\triangle^{(l)} \to \triangle$ is a projective expanding map then it has approximate product structure with respect to Lebesgue measure.*

Suppose now that we have an inner product on the vector space $H$. Then we can speak of the *singular values* of a linear isomorphism acting on $H$ which are the square roots of the eigenvalues (counted with multiplicity) of the positive self-adjoint operator $A^*A$. We always order them

$$\sigma_1(A) \geq \cdots \geq \sigma_{\dim H}(A) > 0$$

A different inner product gives singular values differing from the $\sigma_i$'s by bounded factors, where the bound is independent of $A$.
Lets denote the supporting monoid by $\mathfrak{B}$.

**Definition 3.3.** The monoid $\mathfrak{B}$ is *pinching* if for every $C > 0$ there exists $A \in \mathfrak{B}$ such that

$$\sigma_i(A) > C\sigma_{i+1}(A), \quad \text{for all} \quad 1 \leq i \leq \dim H - 1$$

**Definition 3.4.** We will say that the operator $A$ is *pinching* if all its eigenvalues are real and with different modules. In case $H = \mathbb{R}^{d+1}$ we identify $A$ with its matrix in the canonic base and say that $A$ is a *pinching matrix*.

Let $M^{d+1}$ denote the set of all $(d+1) \times (d+1)$ matrices over the field of complex numbers. Given a matrix $A \in M^{d+1}$, we denote the eigenvalues of $A$ by $\lambda_1(A), \lambda_2(A), \cdots, \lambda_{d+1}(A)$, with the convention that multiple eigenvalues are repeated according to the multiplicities and indexed so that $\mid \lambda_i(A) \mid \leq \mid \lambda_{i+1}(A) \mid$. The singular values of $A$ are denoted as before by $\sigma_1(A), \sigma_2(A), \cdots, \sigma_{d+1}(A)$, where $\sigma_i(A)$ is the nonnegative square root of $\lambda_i(A^*A)$.

Pinching operators are useful for us because an asymptotic relation between the eigenvalues of an operator $A$ and the singular values of $A^n$ given by the following theorem first proved by Yamamoto in [28]:

**Theorem 3.1.** *Let $A \in M^{d+1}$. For each $i = 1, 2, \cdots, d+1$*

$$\lim_{n\to\infty} (\sigma_i(A^n))^{\frac{1}{n}} = \mid \lambda_i(A) \mid$$

Therefore the numbers $\sigma_i(A^n)$ behaves like the numbers $\mid \lambda_i(A) \mid^n$ for big values of $n$ giving us a simple way for proving pinching:

**Lemma 3.2.** *If there exists some pinching operator in $\mathfrak{B}$ then the monoid is pinching.*

**Definition 3.5.** The monoid $\mathfrak{B}$ is *twisting* if for any $k$-dimensional subspace $F \subset H$ for any $1 \leq k \leq \dim H - 1$ and for every finite subset $\{F_i\}_{i=1}^m$ of $(\dim H - k)$-dimensional subspaces of $H$ there exists $A \in \mathfrak{B}$ such that

$$A(F) \cap F_i = \{0\}, \quad 1 \leq i \leq m$$

**Definition 3.6.** The monoid $\mathfrak{B}$ is *simple* if it is pinching and twisting.

**Definition 3.7.** Let $B_1$ and $B_2$ be two $(d+1)$-dimensional pinching matrices such that any set $C$, constituted by $d+1$ eigenvectors from $B_1$ or $B_2$ ($C$ could have eigenvectors from $B_1$ and eigenvectors from $B_2$), is linearly independent. Then we say that the pair $(B_1, B_2)$ is in *general position*.

We will see (Lemma 6.2) that any monoid which contains a pair of pinching matrices in general position is twisting.
The fact of a monoid being pinching and twisting imply the same properties for some kinds of submonoids.

**Lemma 3.3.** *Let $\mathfrak{B}_0 \subset \mathfrak{B}$ be a large submonoid in the sense that there exists a finite subset $Y \subset \mathfrak{B}$ and $z \in \mathfrak{B}$ such that for every $x \in \mathfrak{B}$ there is some $y \in Y$ such that $yxz \in \mathfrak{B}_0$. If $\mathfrak{B}$ is twisting or pinching then $\mathfrak{B}_0$ also is.*

This Lemma is important for us because we can reduce the study of simplicity of Lyapunov spectrum, applying an inducing process in some suitable region of the base dynamics.

Finally the following theorem in [2], which is an adaptation of the main result in [1], give us the promised sufficient condition to prove simplicity of the Lyapunov spectrum in our setting.

**Theorem 3.2** (Sufficient condition)**.** *Let $(T, A)$ be a locally constant measurable cocycle. If the supporting monoid is pinching and twisting then the Lyapunov spectrum is simple.*

3.2. **Inducing.**

First of all by an inducing process explained in section 4 the problem is reduced to the question of simplicity of the Lyapunov spectrum for a first return map whose dynamics is codified by an infinite shift. This map satisfies the conditions on Lemma 3.1 so that we have a locally constant cocycle and by Theorem 3.2 it suffice to prove that the supporting monoid is simple. Because on Lemma 3.3 for the simplicity of the supporting monoid we just need to prove that the monoid generated by the two matrices $A_1$ and $A_2$ introduced at the end of subsection 2.3 is simple.

3.3. **Checking the pinching condition.**

For the pinching condition we construct pinching matrices $B_m$ in the form $B_m = A_1^{md-1} A_2^{2d+2}$, where $d$ is the algorithm's dimension. In section 10 is obtained the characteristic polynomial $p_m$ of these matrices which can be written in the form:

$$\begin{aligned} p_m(x) &= m\{\frac{1}{m} r_{d+1}(x) + q_d(x)\} \\ &= m\{\frac{1}{m} \tilde{r}_{d+1}(y) + \tilde{q}_d(y)\} \quad \text{where} \quad y = 2 - x \end{aligned}$$

In subsection 5.1 we see that for big values of $m$ the polynomials $p_m$ have a root with absolute value on the interval $(2m, 4m)$. There we prove also that if all the roots of the polynomial $q_d$ are real and different then all the remaining $d$ roots of the polynomials $p_m$ are real and different for any $m$ big enough and converge to the roots of $q_d$. As a consequence we conclude that all the roots of $p_m$ are real for any $m$ sufficiently big.

The fact that all the roots of the polynomial $q_d$ are real and with different modules is verified in subsection 5.2. There we exploit the fact that the polynomials $\tilde{q}_d$ can be decomposed into the product of three more elementary polynomials

$$\begin{aligned} \tilde{q}_{2n-1}(x) &= (y-2) s_{n-1}(y) o_{n-1}(y) \\ \tilde{q}_{2n}(x) &= (y-2) p_n(y) g_{n-1}(y) \end{aligned}$$

The polynomials $s_{n-1}$, $o_{n-1}$, $p_n$, $g_{n-1}$ pertain to some sequences of orthogonal polynomials so their roots are all real and distinct and different from 2 as we see in section 7. In subsection 5.2 we check that the polynomials $s_{n-1}$ and $o_{n-1}$ do not have roots in common and the same is true for the polynomials $p_n$ and $g_{n-1}$.

### 3.4. **Checking the twisting condition.**

After establishing the existence of an infinite sequence of pinching matrices we check in subsection 6.3 that there exist a pair of such matrices in general position. The crucial fact is that their last $d$ rows are the same. For this king of matrices we can express the components of their eigenvectors by means of some fixed polynomials on the eigenvalues (see 6.2) depending on the last $d$ rows. Then we just have to establish using the properties of their characteristic polynomials that we can find a pair of them without common eigenvalues. By means of a generalized Vandermonde's determinant 6.1 we conclude that this pair is in general position. Them by the Twisting Lemma 6.2 the monoid is twisting.

## 4. Inducing and simplicity

Let us denote $\triangle = \beta(2)$. As $\widetilde{T}_p : \beta \to \beta$ preserves the probability $\mu_p$, Poincare's theorem guarantees that the set $P$ of points from $\triangle$ which return infinitely many times to $\triangle$ while iterating $\widetilde{T}_p$ is of total measure in $\triangle$. Let $\beta_R$ be the set of points which return to $\triangle$ at least one time while iterating $\widetilde{T}_p$. Let $\mathfrak{F}$ be the countable family of subsets from $\triangle$ given by:

$$\mathfrak{F} = \{\triangle^{(l)}, \quad l \geq 0\} \quad \text{where} \quad \triangle^{(l)} = \beta(2, \overbrace{1, \cdots, 1}^{l}, 2)$$

It is easy to see that $\beta_R$ is measurable and that $\mathfrak{F}$ constitutes a partition of $\beta_R$. As $P \subset \beta_R \subset \triangle$ we have that $\beta_R$ is of total measure in $\triangle$.
The *time of the first return* is defined on each point of $\beta_R$ by

$$m(x) = \min\{m \geq 1 : \widetilde{T}_p^m(x) \in \triangle\}$$

and the *application of the first return* $F : \beta_R \to \triangle$ is then well defined on each point from $\beta_R$ by

$$F(x) = \widetilde{T}_p^{m(x)}(x)$$

Obviously $P$ is invariant under $F$ and (see [21] for example) we know that $F\mid_P$ preserves the probability

$$\mu_F = \frac{1}{\mu_p(\triangle)} \mu_p \mid_\triangle$$

Let us define the skew product $(y, v) \mapsto (F(y), A_F(y) \cdot v)$ on $F : \beta_R \to \triangle$ making $A_F(y) = A_P(\widetilde{T}_p^{m(y)-1}(y)) \cdots A_P(y)$. This skew-product induces a function

$$\begin{aligned} A : \beta_R &\to SL(H) \\ y &\mapsto A_F(y) \end{aligned}$$

Observe that $A$ is constant on each $\triangle^{(l)}$, so that if we denote $A(x) = A^{(l)}$ for $x \in \triangle^{(l)}$, results:

$$A(x) = A^{(l)} = A_1^l A_2 \quad \text{for} \quad x \in \triangle^{(l)}, \quad l \geq 0$$

Clearly $F\mid_{\triangle^{(l)}} : \triangle^{(l)} \to \triangle$ is a bijection for any $l \in \mathbb{N}$ and

$$(F\mid_{\triangle^{(l)}})^{-1}$$

is the restriction of a projective contraction (see section 9). By Remark 9.1 we conclude that $F$ is a projective expanding map and by Lemma 3.1 it have approximate product structure in relation to the Lebesgue measure. Then

*Remark* 4.1. $(F, A_F)$ is a locally constant cocycle and is a known fact that the Lyapunov exponents of $(F, A_F)$ can be obtained multiplying the Lyapunov exponents of the cocycle $(\beta, \widetilde{T}_p, \mu_P, A_P)$ by $\frac{1}{\mu_p(\triangle)}$.

Let $M_F$ be the monoid generated by the $A^{(l)}, l \in \mathbb{N}$. If $A_{i_1} \cdots A_{i_n}$ is an arbitrary element from the monoid $M_{\widetilde{T}_p}$ associated to the cocycle $(\beta, \widetilde{T}_P, \mu_P, A_P)$ it is easy to see that $A_2 A_{i_1} \cdots A_{i_n} A_2 \in M_F$, so that by Lemma 3.3 we obtain

$$M_{\widetilde{T}_p} \text{twisting and pinching} \Rightarrow M_F \quad \text{twisting and pinching} \tag{17}$$

From Proposition 5.5 and Corollary 6.2 we get that $M_{\widetilde{T}_p}$ is pinching and twisting. Therefore using 17 and Theorem 3.2 we get that the spectrum of $F$ is simple. By Remark 4.1 it follows that the spectrum of $\widetilde{T}_p$ is simple so by Remark 2.3 the spectrum of $\mathcal{T}$ is simple. And we have proved that:

**Theorem 4.1.** *The Lyapunov spectrum for Selmer's algorithm is simple.*

## 5. Pinching

In this section we check the pinching condition for Selmer's algorithm. We will require a fine understanding of the characteristic polynomial of our candidate pinching matrices and this will be very important for our proof of twisting too.

### 5.1. A first incursion on the characteristic polynomial.

Define $d_d(m, x)$ by

$$d_d(m, x) \stackrel{\text{def.}}{=} \text{Det}[A_1^{md-1} A_2^{2d+2} - x\text{Id}]$$

In section 5.2 we use the following assertion proved in section 10:

$$d_d(m, x) = r_{d+1}(x) + m \cdot q_d(x) = m \cdot \left\{ \frac{1}{m} r_{d+1}(x) + q_d(x) \right\}$$

Where $r_{d+1}(x)$ and $q_d(x)$ are two polynomials of degrees $d+1$ and $d$ respectively. We can feel that for $m$ big the localization of the zeroes of such $d_d(m, x)$ depends strongly on the zeroes of $q_d(x)$. In fact there will be one zero far from the origin and $d$ zeroes converging to the zeroes of $q_d$. The following two Lemmas will permit us to formalize this affirmation.

**Lemma 5.1.** *There exists $m_0$ such that if $m \geq m_0$ the characteristic polynomial of the matrix $A_1^{md-1} A_2^{2d+2}$ have one root with absolute value contained in the open interval $(2m, 4m)$.*

*Proof.* We know that

$$\text{Spectrum}(\frac{M}{m}) = \frac{1}{m} \text{Spectrum(M)}$$

for any matrix $M$. Let the matrices $V_{d,m}$ and $C$ be defined by:

$$V_{d,m} = \frac{A_1^{md-1} A_2^{2d+2}}{m} =$$

$$= \begin{pmatrix} 3-\frac{1}{m} & 4-\frac{1}{m} & 4 & 4 & \dots & 4 & 4 & 4 & 4 & 5-\frac{1}{m} \\ \frac{1}{m} & \frac{2}{m} & \frac{1}{m} & 0 & \dots & 0 & 0 & 0 & 0 & \frac{1}{m} \\ 0 & \frac{1}{m} & \frac{2}{m} & \frac{1}{m} & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m} & \frac{2}{m} & \dots & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{m} & \frac{2}{m} & \frac{1}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{m} & \frac{2}{m} & \frac{1}{m} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{m} & \frac{2}{m} & \frac{1}{m} \\ \frac{2}{m} & \frac{1}{m} & 0 & 0 & \dots & 0 & 0 & 0 & \frac{1}{m} & \frac{3}{m} \end{pmatrix} \boxed{d+1}$$

$$C = \begin{pmatrix} 3 & 4 & 4 & 4 & \dots & 4 & 4 & 4 & 4 & 5 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The caracteristic polynomial of $C$ is $(3-x)x^d$, so that Spectrum$(C) = \{3, 0\}$ where 3 is a simple zero. Clearly $V_{d,m} \underset{m}{\longrightarrow} C$. Thus by continued dependence of the eigenvalues it follows that if $m$ is big enough $V_{d,m}$ have an eigenvalue $\lambda_m$ converging to 3. That is $\lambda_m = 3 + \theta_m$ where $\theta_m \underset{m}{\longrightarrow} 0$. This is why the matrix $A_1^{md-1} A_2^{2d+2} = m \cdot V_{d,m}$ have an eigenvalue $\mu_m = m\lambda_m$. Let $m_0$ be big enough such that $\mid \theta_m \mid < 1$ for $m \geq m_0$. Then:

$$\mu_m = 3m + \theta_m \cdot m$$

so

$$\mid \mu_m - 3m \mid = \mid \theta_m \mid \cdot m < m$$

and $2m < \mid \mu_m \mid < 4m$ for $m \geq m_0$. □

**Lemma 5.2.** *Let $q(x)$ and $r(x)$ be two polynomials with degrees $d$ and $d+1$ respectively and such that $q$ have $d$ real and different zeroes $x_1 < x_2 < \cdots < x_d$ and $\theta = \min_{1 \leq i \leq d-1} \frac{1}{2}(x_{i+1} - x_i)$. Then for all $\epsilon < \theta$ there exists $N \in \mathbb{N}$ such that if $m \geq N$ the polynomial $p_m(x) = \frac{1}{m} r(x) + q(x)$ with degree $d+1$ have $d$ real zeroes $y_1^m, \cdots, y_d^m$ satisfying $\mid y_i^m - x_i \mid < \epsilon$ for $1 \leq i \leq d$.*

*Proof.*
Observation: If $\epsilon \leq \theta$ then $q$ have no zero different from $x_i$ in $[x_i - \epsilon, x_i + \epsilon]$ and $\text{sign}[q(x_i - \epsilon)] = -\text{sign}[q(x_i + \epsilon)]$.

Take $A > 0$ such that:

$$[x_1 - \theta, \cdots, x_d + \theta] \subset [-A, A]$$

and $M$ defined by

$$M = \min_{1 \leq i \leq d} \{\min\{\mid q(x_i - \epsilon) \mid, \mid q(x_i + \epsilon) \mid\}\}$$

Because on the continuity of $r$ there exist $N$ such that $\mid \frac{1}{m}r(x) \mid \leq \frac{1}{2}M$ for $m \geq N$ and $-A \leq x \leq A$ so we get:

$$\begin{aligned}\operatorname{sign}[p_m(x_i \pm \epsilon)] &= \operatorname{sign}[\frac{1}{m}r(x_i \pm \epsilon) + q(x_i \pm \epsilon)] \\ &= \operatorname{sign}[q(x_i \pm \epsilon)] \quad \text{for,} \quad m \geq N, \quad 1 \leq i \leq d\end{aligned}$$

Now using this and the initial Observation we deduce that $p_m$ have at least one real zero on the open interval $(x_i - \epsilon, x_i + \epsilon)$ for, $m \geq N$, $1 \leq i \leq d$. □

5.2. **The polynomials involved.**

Now we define some sequences of polynomials appearing in the course of our proof of simplicity. These polynomials satisfy an infinitude of relations and identities from which we will need a few. So we are concentrating some of them in Proposition 5.1 and other important relations in Proposition 5.2. From now on a super-index inside a box for a matrix specifies de order of the matrix.
Let then the sequences of polynomials $(g_n), (t_n), (k_n), (h_t), (u_t), (\tilde{r}_d), (\tilde{q}_d)$ be defined by the following equalities:

$$g_{-1}(y) \equiv 0, \quad g_0(y) \equiv 1, \quad g_n(y) = \begin{vmatrix} y & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & y & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & y & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & y & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & y & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & y \end{vmatrix}^{\boxed{n}}, \quad \text{for} \quad n \geq 1$$

$$t_0(y) \equiv 0, \quad t_n(y) = \begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & y & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & y & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & y & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & y & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & y \end{vmatrix}^{\boxed{n}}, \quad \text{for} \quad n \geq 1$$

$$k_n(y) = \begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ y & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & y & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & y & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & y & 1 \end{vmatrix}^{\boxed{n}}, \quad \text{for} \quad n \geq 1$$

$$\begin{aligned}
&h_t(y) = g_t(y)(g_t(y) - g_{t-1}(y)) \quad \text{for} \quad t \geq 0 \\
&u_t(y) = g_{t-1}(y)(g_t(y) - g_{t-1}(y)) \quad \text{for} \quad t \geq 0 \\
&\tilde{r}_d(y) = yg_{d-1}(y) - 3g_{d-1}(y) + yg_{d-2}(y) - yg_{d-3}(y) + g_{d-3}(y) + \\
&\qquad + (-3 + 2y)(-1)^d \quad \text{for} \quad d \geq 2 \\
&\tilde{q}_d(y) = 3g_d(y) - 11g_{d-1}(y) + 6g_{d-2}(y) + (-8 + 12y)t_{d-2}(y) - 12t_{d-3}(y) + \\
&\qquad + 7(-1)^{d+2} \quad \text{for} \quad d \geq 3
\end{aligned}$$

On each item of the following Proposition assume the argument is $y$ in the functions involved. This is a part of the larger Proposition 10.1.

**Proposition 5.1.**

$$\begin{aligned}
&1)\tilde{q}_{2n-1} = (y-2)(3g_{n-1} - 2g_{n-2})(g_{n-1} - g_{n-2}) \quad \textit{for} \quad n \geq 2 \\
&2)\tilde{q}_{2n} = (y-2)(3g_n + 2g_{n-2} - 5g_{n-1})g_{n-1} \quad \textit{for} \quad n \geq 1 \\
&3)\tilde{r}_{2n} = (y+2)(y-2)(g_{n-1} - g_{n-2})^2 + 1 \quad \textit{for} \quad n \geq 1 \\
&4)\tilde{r}_{2n+1} = (y+2)(g_n - 2g_{n-1} + g_{n-2})^2 - 1 \quad \textit{for} \quad n \geq 1 \\
&5)(y+2)(y-2)g_{t-1}^2 + 4 = (g_t - g_{t-2})^2 \quad \textit{for} \quad n \geq 1 \\
&6)(y+2)(g_{n-1} - g_{n-2})^2 = (y-2)(g_{n-1} + g_{n-2})^2 + 4 \quad \textit{for} \quad n \geq 1
\end{aligned}$$

We also postpone the tedious but easy calculations for proving next Proposition until section 10.

**Proposition 5.2.** *The expression* $Det[A_1^{md-1} \cdot A_2^{2d+2} - xId]$ *can be written as:*

$$1)m \cdot \left\{ \frac{1}{m}\Big(x^2 - 4x + 1\Big) + (-3x) \right\}, \quad \textit{for} \quad d = 1$$

$$2)m \cdot \left\{ \frac{1}{m}\Big(-x^3 + 4x^2 - 1\Big) + x(3x - 1) \right\}, \quad \textit{for} \quad d = 2$$

$$\begin{aligned}
&3)m \cdot \left\{ \frac{1}{m}\Big[(2-x)g_d(2-x) - 3g_d(2-x) + (2-x)g_{d-1}(2-x) - (2-x)g_{d-2}(2-x) + \right. \\
&+ g_{d-2}(2-x) + (-3 + 2(2-x))(-1)^{d+1}\Big] + 3g_d(2-x) - 11g_{d-1}(2-x) + 6g_{d-2}(2-x) + \\
&\left. + (-8 + 12(2-x))t_{d-2}(2-x) - 12t_{d-3}(2-x) + 7(-1)^{d+2} \right\}, \quad \textit{for} \quad d \geq 3
\end{aligned}$$

For clarifying our argumentations let us define the polynomials

$$\begin{aligned}
r_{d+1}(x) &= -(1+x)g_d(2-x) + (2-x)g_{d-1}(2-x) + \\
&\quad + (-1+x)g_{d-2}(2-x) + (1-2x)(-1)^{d+1} \\
q_d(x) &= 3g_d(2-x) - 11g_{d-1}(2-x) + 6g_{d-2}(2-x) + \\
&\quad + 4(4-3x)t_{d-2}(2-x) - 12t_{d-3}(2-x) + 7(-1)^{d+2}
\end{aligned}$$

Observe that

*Remark* 5.1.

$$\begin{aligned}
r_{d+1}(x) &= \tilde{r}_{d+1}(2-x) \\
q_d(x) &= \tilde{q}_d(2-x)
\end{aligned}$$

So making $y = 2 - x$ we have the following

*Remark* 5.2.

$$\begin{aligned}\mathrm{Det}[A_1^{md-1}A_2^{2d+2} - xId] &= r_{d+1}(x) + m \cdot q_d(x) = \\ &= \tilde{r}_{d+1}(y) + m \cdot \tilde{q}_d(y), \quad \text{for} \quad d \geq 3\end{aligned}$$

**Proposition 5.3.** *The zeroes of $\tilde{q}_d$ are all real and different and contained in the semi-open interval* $(-2, 2]$.

*Proof.* Let us first observe that by Proposition 5.1.1) and Proposition 5.1.2) we have:

$$\tilde{q}_{2n-1}(y) = (y-2)s_{n-1}(y)o_{n-1}(y)$$
$$\tilde{q}_{2n}(y) = (y-2)p_n(y)g_{n-1}(y)$$

where $s_{n-1}, o_{n-1}, p_n, g_{n-1}$ are like in section 7. By Proposition 7.3 each one of these polynomials have all their zeroes real and distinct and contained on the open interval $(-2, 2)$. Then it suffice to prove that $s_{n-1}$ and $o_{n-1}$ do not have any common zeroes and that $p_n$ and $g_{n-1}$ do not have any common zeroes.

Case 1

$s_{n-1}$ and $o_{n-1}$ do not have any common zeroes because if we suppose that on the contrary there exists $y_0$ such that $s_{n-1}(y_0) = o_{n-1}(y_0) = 0$ then

$$\begin{aligned}&3g_{n-1}(y_0) - 2g_{n-2}(y_0) = 0 \\ &\quad g_{n-1}(y_0) - g_{n-2}(y_0) = 0 / \cdot 2 \qquad \Rightarrow 2g_{n-1}(y_0) - 2g_{n-2}(y_0) = 0\end{aligned}$$

$\Rightarrow g_{n-1}(y_0) = 0 \Rightarrow g_{n-2}(y_0) = 0 \Rightarrow y_0$ is a common zero of $g_{n-1}$ and $g_{n-2}$ which contradicts the fact that the zeroes of $g_{n-1}$ and $g_{n-2}$ strictly interlace (Proposition 7.3).

Case 2

$p_n$ and $g_{n-1}$ do not have any common zeroes because if we suppose that on the contrary there exists $y_0$ such that $p_n(y_0) = g_{n-1}(y_0) = 0$ then

$$\begin{aligned}&3g_n(y_0) + 2g_{n-2}(y_0) - 5g_{n-1}(y_0) = 0 \\ &\qquad\qquad\qquad\qquad g_{n-1}(y_0) = 0 \qquad \Rightarrow 3g_n(y_0) + 2g_{n-2}(y_0) = 0\end{aligned}$$

$\Rightarrow 3[y_0 \cdot g_{n-1}(y_0) - g_{n-2}(y_0)] + 2g_{n-2}(y_0) = 0 \Rightarrow g_{n-2}(y_0) = 0 \Rightarrow g_{n-1}(y_0) = g_{n-2}(y_0) = 0$ arriving again to a contradiction because the zeroes of $g_{n-1}$ and $g_{n-2}$ strictly interlace (Proposition 7.3). □

**Proposition 5.4.** *If $\lambda$ is an eigenvalue of $A_1^{md-1}A_2^{2d+2}$ then $q_d(\lambda) \neq 0$.*

*Proof.* Suppose that on the contrary $\lambda$ is an eigenvalue of $A_1^{md-1}A_2^{2d+2}$ and that $q_d(\lambda) = 0$. Let $y_0 = 2 - \lambda$, then by Remark 5.1 and Remark 5.2:

$$\tilde{q}_d(y_0) = \tilde{q}_d(2-\lambda) = q_d(\lambda) = 0$$
$$\tilde{r}_{d+1}(y_0) = \tilde{r}_{d+1}(y_0) + m \cdot \tilde{q}_d(y_0) = \mathrm{Det}[A_1^{md-1}A_2^{2d+2} - \lambda \mathrm{Id}] = 0$$

Thus $y_0$ satisfy the system:

$$\begin{Bmatrix} \tilde{r}_{d+1}(y_0) = 0 \\ \tilde{q}_d(y_0) = 0 \end{Bmatrix} \tag{18}$$

And we have two cases:

Case 1 $\boxed{d = 2n}$
Using Proposition 5.1.4) and Proposition 5.1.2), the system 18 writes:

$$\left\{ \begin{array}{r} (y_0+2)[g_n(y_0) - 2g_{n-1}(y_0) + g_{n-2}(y_0)]^2 - 1 = 0 \\ (y_0-2)[3g_n(y_0) + 2g_{n-2}(y_0) - 5g_{n-1}(y_0)]g_{n-1}(y_0) = 0 \end{array} \right\} \tag{19}$$

From the second equation of 19 we have 3 cases:
Case 1.1 $\boxed{y_0 = 2}$

Observe that $g_2(2) - 2g_1(2) + g_0(2) = 0 = g_3(2) - 2g_2(2) + g_1(2)$ and that:

$$\begin{aligned} g_k - 2g_{k-1} + g_{k-2} &= (yg_{k-1} - g_{k-2}) - 2(yg_{k-2} - g_{k-3}) + (yg_{k-3} - g_{k-4}) \\ &= y(g_{k-1} - 2g_{k-2} + g_{k-3}) - (g_{k-2} - 2g_{k-3} + g_{k-4}) \end{aligned}$$

So by induction we obtain:

$$g_n(2) - 2g_{n-1}(2) + g_{n-2}(2) = 0, \quad \forall n \geq 2$$

Now substituting in the first equation of 19 we get $-1 = 0$ which is an absurd.

Case 1.2 $\boxed{g_{n-1}(y_0) = 0}$
In this case:

$$\begin{aligned} g_n(y_0) - 2g_{n-1}(y_0) + g_{n-2}(y_0) &= y_0 g_{n-1}(y_0) - g_{n-2}(y_0) - 2g_{n-1}(y_0) + g_{n-2}(y_0) \\ &= g_{n-1}(y_0)(y_0 - 2) = 0 \end{aligned}$$

And substituting in the first equation of 19 we arrive at $-1 = 0$ a contradiction.

Case 1.3 $\boxed{3g_n(y_0) + 2g_{n-2}(y_0) - 5g_{n-1}(y_0) = 0}$
By Proposition 7.3 we know that $y_0 \in (-2, 2)$. We also have the equivalences:

$$\begin{aligned} & 3g_n(y_0) + 2g_{n-2}(y_0) - 5g_{n-1}(y_0) = 0 \\ & \Leftrightarrow 3(g_n(y_0) - 2g_{n-1}(y_0) + g_{n-2}(y_0)) + (g_{n-1}(y_0) - g_{n-2}(y_0)) = 0 \\ & \Leftrightarrow g_n(y_0) - 2g_{n-1}(y_0) + g_{n-2}(y_0) = \frac{-(g_{n-1}(y_0) - g_{n-2}(y_0))}{3} \end{aligned}$$

Substituting in the first equation of 19 results:

$$(y_0+2)(g_{n-1}(y_0) - g_{n-2}(y_0))^2 - 9 = 0 \tag{20}$$

Now we use Proposition 5.1.6) and the fact that $y_0 \in (-2, 2)$ to obtain:

$$\begin{aligned} 0 &= (y_0+2)(g_{n-1}(y_0) - g_{n-2}(y_0))^2 - 9 = \\ &= (y_0-2)(g_{n-1}(y_0) + g_{n-2}(y_0))^2 - 5 < -5 < 0 \\ &\Rightarrow 0 < 0 \end{aligned}$$

A contradiction.

Case 2 $\boxed{d = 2n - 1}$
Using Proposition 5.1.3) and Proposition 5.1.1) the system 18 is written:

$$\left\{ \begin{array}{r} (y_0+2)(y_0-2)[g_{n-1}(y_0) - g_{n-2}(y_0)]^2 + 1 = 0 \\ (y_0-2)[3g_{n-1}(y_0) - 2g_{n-2}(y_0)][g_{n-1}(y_0) - g_{n-2}(y_0)] = 0 \end{array} \right\} \tag{21}$$

from the second equation of 21 we get 3 cases:

Case 2.1 $\boxed{y_0 = 2}$
Substitute in the first equation of 21 to obtain $1 = 0$, a contradiction.

Case 2.2 $\boxed{g_{n-1}(y_0) - g_{n-2}(y_0) = 0}$
Substitute in the first equation of 21 to obtain $1 = 0$, a contradiction.

Case 2.3 $\boxed{3g_{n-1}(y_0) - 2g_{n-2}(y_0) = 0}$
Observe the equivalence:

$$3g_{n-1}(y_0) - 2g_{n-2}(y_0) = 0$$
$$\Leftrightarrow g_{n-1}(y_0) - g_{n-2}(y_0) = \frac{-g_{n-1}(y_0)}{2}$$

Now substitute in the first equation of 21 to obtain:

$$(y_0 + 2)(y_0 - 2)\frac{g_{n-1}(y_0)^2}{4} + 1 = 0 \tag{22}$$

And use Proposition 5.1.5), 22 and the fact that $3g_{n-1}(y_0) - 2g_{n-2}(y_0) = 0$ to deduce that $y_0$ is a solution of the system:

$$\begin{Bmatrix} g_n(y_0) - g_{n-2}(y_0) = 0 \\ 3g_{n-1}(y_0) - 2g_{n-2}(y_0) = 0 \end{Bmatrix} \tag{23}$$

From the first equation we get that:

$$y_0 g_{n-1}(y_0) - 2g_{n-2}(y_0) = 0$$

combining with the second equation results:

$$(3 - y_0)g_{n-1}(y_0) = 0 \tag{24}$$

But $y_0$ being a solution of $s_{n-1}(y_0) = 3g_{n-1}(y_0) - 2g_{n-2}(y_0) = 0$ Proposition 7.3 guarantees that $y_0 \in (-2, 2)$. This is why from 24 we deduce that $g_{n-1}(y_0) = 0$. Now if we substitute in 23 again results that $g_{n-2}(y_0) = g_{n-1}(y_0) = 0$ a contradiction because the zeroes of $g_{n-1}$ and $g_{n-2}$ strictly interlace (Proposition 7.3). □

### 5.3. **Selmer's algorithm is pinching.**

As we explain in Lemma 3.2 for proving that Selmer's algorithm is pinching its enough to prove the existence of some pinching matrix in the associated monoid. This is the content of the following Proposition.

**Proposition 5.5.** *For any $d \geq 1$ there exists $N_0(d)$ such that the matrices $A_1^{md-1}A_2^{2d+2}$ are pinching for $m \geq N_0(d)$. Consequently Selmer's algorithm is pinching.*

*Proof.* Remember that from Proposition 5.2:
$\boxed{d = 1}$
1)$\text{Det}[A_1^{md-1}A_2^{2d+2} - x\text{Id}] = m \cdot \left\{\frac{1}{m}(x^2 - 4x + 1) + (-3x)\right\}$

$\boxed{d = 2}$
2)$\text{Det}[A_1^{md-1}A_2^{2d+2} - x\text{Id}] = m \cdot \left\{\frac{1}{m}(-x^3 + 4x^2 - 1) + x(3x - 1)\right\}$

$\boxed{d \geq 3}$

3)$\text{Det}[A_1^{md-1}A_2^{2d+2} - x\text{Id}] = m \cdot \left\{\frac{1}{m}r_{d+1}(x) + q_d(x)\right\}$

We have $q_d(x) = \tilde{q}_d(2-x)$ and Proposition 5.3 guarantees that the zeroes of $\tilde{q}_d$ are all real and distinct and contained in the semi-open interval $(-2, 2]$ for $d \geq 2$. Therefore the zeroes of $q_d$ are all real and distinct and contained in the semi-open interval $[0, 4)$ for $d \geq 3$. Thus the zeroes of $q_d$ are all real and with different modules. The same affirmation holds obviously for the polynomials $(-3x)$ and $x(3x-1)$ appearing in 1) and 2).

Now using this last fact, the expressions $1), 2), 3)$ and Lemma 5.2 we deduce that the characteristic polynomial of $A_1^{md-1}A_2^{2d+2}$ has $d$ real zeroes converging to the $d$ real, distinct and of different modules zeroes of $q_d$ contained in the interval $(0, 4]$ for $d \geq 1$. Consequently all its $d+1$ zeroes are real. Lemma 5.1 guarantees the existence of a zero with absolute module greater than $2m$ for $m \geq m_0$. Now the Proposition follows easily. □

## 6. Twisting

Now we can check the twisting condition for Selmer's algorithm. Our first objective is a Twisting Lemma giving a sufficient condition for a monoid to be twisting, we just need to find pairs of matrices in general position. Then we convince ourselves that our pinching matrices are appropriate for applying this Lemma. Our previous study of the characteristic polynomials make it easier to conclude the existence of such pairs of matrices.

### 6.1. **A Twisting Lemma.**

The following Lemma constitutes a generalization of the formula for *Vandermonde's determinant*.

**Lemma 6.1.** *Let* $p_i(x) = \sum_{j=0}^{d} a_j^i x^j, \quad 1 \leq i \leq d+1$ *be polynomials with degrees least than or equal to d. Then there exist* $C \in \mathbb{R}$ *such that:*

$$Det[(p_i(x_j))] = \begin{vmatrix} p_1(x_1) & p_2(x_1) & \dots & p_{d+1}(x_1) \\ p_1(x_2) & p_2(x_2) & \dots & p_{d+1}(x_2) \\ \dots & \dots & \dots & \dots \\ p_1(x_{d+1}) & p_2(x_{d+1}) & \dots & p_{d+1}(x_{d+1}) \end{vmatrix} = C \cdot \prod_{1 \leq i < j \leq d+1} (x_i - x_j)$$

*Proof.* Splitting each $p_i(x)$ as the sum of its monomials, we can write the determinant as a linear combination of

$$\begin{vmatrix} x_1^{m_1} & x_1^{m_2} & \dots & x_1^{m_{d+1}} \\ x_2^{m_1} & x_2^{m_2} & \dots & x_2^{m_{d+1}} \\ \dots & \dots & \dots & \dots \\ x_{d+1}^{m_1} & x_{d+1}^{m_2} & \dots & x_{d+1}^{m_{d+1}} \end{vmatrix} \tag{25}$$

with $0 \leq m_1, m_2, \cdots, m_{d+1} \leq d$. If $m_i = m_j$ for some pair $(i, j)$ then, clearly, 25= 0. In all other cases we have a classical Vandermonde determinant

$$\prod_{1 \leq i < j \leq d+1} (x_i - x_j)$$

□

Last Lemma gives as a Corollary that if there are $d+1$ linearly independent vectors over the curve $c(t) = (p_1(t), \cdots, p_{d+1}(t))$ then it is injective and have the beautiful property that any subset of $d+1$ different vectors over the curve is also linearly independent. This constitutes the heart of our argument for proving that Selmer's MCFA associated monoid is twisting.

**Corollary 6.1.** *Suppose there exist $x_1^0, \cdots, x_{d+1}^0$ such that $Det[(p_i(x_j^0))] \neq 0$, then for any set $\{x_1, \cdots, x_{d+1}\}$ satisfying that $x_i \neq x_j \quad if \quad i \neq j$ we have that*

$$Det[(p_i(x_j))] \neq 0$$

*and the curve $c(t) = (p_1(t), \cdots, p_{d+1}(t))$ is injective.*

The pinching matrices we found on Proposition 5.5 are of a very special kind giving us the possibility of finding pairs of such matrices satisfying the conditions on the following Proposition:

**Proposition 6.1.** *Let $M_1$ and $M_2$ be two pinching matrices of order $d+1$ with eigenvalues $\lambda_1^1, \cdots, \lambda_{d+1}^1$ and $\lambda_1^2, \cdots, \lambda_{d+1}^2$ respectively and such that $\lambda_i^1 \neq \lambda_j^2$ for $1 \leq i, j \leq d+1$. Suppose that there exist some polynomials $p_1, \cdots, p_{d+1}$ of order least than or equal to $d$ such that for each pair $(i,j)$ with $1 \leq i \leq 2, \quad 1 \leq j \leq d+1$ exists one non trivial vector $\vec{v}_j^{\,i}$ satisfying the following equalities:*

$$M_i \cdot \vec{v}_j^{\,i} = \lambda_j^i \cdot \vec{v}_j^{\,i}, \quad \vec{v}_j^{\,i} = c(\lambda_j^i) \quad where \quad c(t) = (p_1(t), \cdots, p_{d+1}(t))$$

*Then any set of vectors $\left\{\vec{v}_{i_1}^{\,1}, \cdots, \vec{v}_{i_k}^{\,1}, \vec{v}_{i_{k+1}}^{\,2}, \cdots, \vec{v}_{i_{d+1}}^{\,2}\right\}$ with $i_m \neq i_n$ for $m \neq n, \quad 1 \leq m, n \leq k \quad$ and $i_r \neq i_s$ for $r \neq s, \quad k+1 \leq r, s \leq d+1$ is linearly independent for any $0 \leq k \leq d+1$. That is the pair $(M_1, M_2)$ is in general position.*

*Proof.* Observe that the vectors $\vec{v}_1^{\,1}, \cdots, \vec{v}_{d+1}^{\,1}$ constitute a linear independent system. This is why if we take $x_1^0 = \lambda_1^1, \cdots, x_{d+1}^0 = \lambda_{d+1}^1$ we obtain:

$$\text{Det}[(p_i(x_j^0))] = \begin{vmatrix} p_1(\lambda_1^1) & \ldots & p_{d+1}(\lambda_1^1) \\ \ldots & \ldots & \ldots \\ p_1(\lambda_{d+1}^1) & \ldots & p_{d+1}(\lambda_{d+1}^1) \end{vmatrix} \neq 0$$

because

$$(p_1(\lambda_1^1), \cdots, p_{d+1}(\lambda_1^1)) = c(\lambda_1^1) = \vec{v_1^1}$$
$$\vdots \qquad\qquad \vdots$$
$$(p_1(\lambda_{d+1}^1), \cdots, p_{d+1}(\lambda_{d+1}^1)) = c(\lambda_{d+1}^1) = \vec{v_{d+1}^1}$$

Therefore applying Corollary 6.1 we conclude that $\text{Det}[(p_i(x_j))] \neq 0$ for any set of numbers $\{x_1, \cdots, x_{d+1}\}$ such that $x_i \neq x_j$ if $i \neq j$.

As $M_1$ and $M_2$ are pinching we know that $\lambda_i^1 \neq \lambda_j^1$ and $\lambda_i^2 \neq \lambda_j^2$ for $i \neq j$. Also the hypotheses of the Proposition 6.2 guarantee that $\lambda_i^1 \neq \lambda_j^2$ for $1 \leq i, j \leq d+1$.

So we deduce that if $z_1 = \lambda^1_{i_1}, \cdots, z_k = \lambda^1_{i_k}, z_{k+1} = \lambda^2_{i_{k+1}}, \cdots, z_{d+1} = \lambda^2_{i_{d+1}}$ then $z_i \neq z_j$ for $i \neq j$ and so:

$$\begin{vmatrix} p_1(\lambda^1_{i_1}) & p_2(\lambda^1_{i_1})\dots & p_{d+1}(\lambda^1_{i_1}) \\ p_1(\lambda^1_{i_2}) & p_2(\lambda^1_{i_2})\dots & p_{d+1}(\lambda^1_{i_2}) \\ \dots & \dots & \dots \\ p_1(\lambda^1_{i_k}) & p_2(\lambda^1_{i_k})\dots & p_{d+1}(\lambda^1_{i_k}) \\ p_1(\lambda^2_{i_{k+1}}) & p_2(\lambda^2_{i_{k+1}})\dots & p_{d+1}(\lambda^2_{i_{k+1}}) \\ \dots & \dots & \dots \\ p_1(\lambda^2_{i_{d+1}}) & p_2(\lambda^2_{i_{d+1}})\dots & p_{d+1}(\lambda^2_{i_{d+1}}) \end{vmatrix} = \mathrm{Det}[(p_i(x_j))] \neq 0$$

Therefore the set $\left\{ \overset{\rightarrow}{v}{}^1_{i_1}, \cdots, \overset{\rightarrow}{v}{}^1_{i_k}, \overset{\rightarrow}{v}{}^2_{i_{k+1}}, \cdots, \overset{\rightarrow}{v}{}^2_{i_{d+1}} \right\}$ is linearly independent.

□

Proposition 6.1 give us an elegant way to prove the twisting condition by means of the following Lemma:

**Lemma 6.2** (Twisting Lemma)**.** *Let $\mathfrak{B}$ be a monoid containing a pair of twisting matrices in general position. Then the monoid is twisting.*

*Proof.* Let $(B_1, B_2)$ a pair of twisting matrices in general position on the monoid $\mathfrak{B}$, $\{\overset{\rightarrow}{v}{}^1_1, \cdots, \overset{\rightarrow}{v}{}^1_{d+1}\}$ and $\{\overset{\rightarrow}{v}{}^2_1, \cdots, \overset{\rightarrow}{v}{}^2_{d+1}\}$ basis of eigenvectors of $B_1$ and $B_2$ respectively.

Observe that if $V, W$ are subspaces of $\mathbb{R}^{d+1}$ with complementary dimensions being sums of eigenspaces belonging to $B_1$ there exist $1 \leq k \leq d$ and some indices $i_1, \cdots, i_k, i_{k+1}, \cdots, i_{d+1}$ such that

$$V = < \overset{\rightarrow}{v}{}^1_{i_{k+1}}, \cdots, \overset{\rightarrow}{v}{}^1_{i_{d+1}} >, \quad W = < \overset{\rightarrow}{v}{}^1_{i_1}, \cdots, \overset{\rightarrow}{v}{}^1_{i_k} >$$

Then by Proposition 11.2 we know that $B_2^n(V)$ converge to some subspace

$$< \overset{\rightarrow}{v}{}^2_{i_{k+1}}, \cdots, \overset{\rightarrow}{v}{}^2_{i_{d+1}} >$$

with dimension $d+1-k$ of $\mathbb{R}^{d+1}$ which is the sum of some eigenspaces belonging to $B_2$. Now from the fact that the pair $(B_1, B_2)$ is general position we deduce that $B_2^n(V) \cap W = \{0\}$ if $n$ is big enough.

Let now $F, G_1, \cdots, G_r$ be subspaces with $\mathrm{dim}F + \mathrm{dim}G_i = d+1, \quad 1 \leq i \leq r$. We know from Proposition 11.2 that $B_1^s(F), B_1^{-s}(G_i), \cdots, B_1^{-s}(G_r)$ converge to some subspaces $V, W_1, \cdots, W_r$ of $\mathbb{R}^{d+1}$ which are sums of eigenspaces of $B_1$ and $\mathrm{dim}V = \mathrm{dim}F$, $\mathrm{dim}W_i = \mathrm{dim}G_i, \cdots, \mathrm{dim}W_r = \mathrm{dim}G_r$. As we observed before for $n$ big enough we have

$$B_2^n(V) \cap W_i = \{0\}, \quad 1 \leq i \leq r$$

Therefore for $s$ big enough we have

$$\begin{aligned} & B_2^n(B_1^s(F)) \cap B_1^{-s}(G_i) = \{0\}, \quad 1 \leq i \leq r \\ & \Rightarrow B_1^s B_2^n B_1^s(F) \cap G_i = \{0\}, \quad 1 \leq i \leq r \\ & \Rightarrow L(F) \cap G_i = \{0\}, \quad 1 \leq i \leq r \end{aligned}$$

where

$$L = B_1^s B_2^n B_1^s = (A_1^{m_1 d-1} A_2^{2d+2})^s (A_1^{m_2 d-1} A_2^{2d+2})^n (A_1^{m_1 d-1} A_2^{2d+2})^s$$

for some naturals $m_1$ and $m_2$ in case $B_1$ and $B_2$ are taken from our pinching matrices. So the monoid $\mathfrak{B}$ is twisting. □

### 6.2. **We can apply the Twisting Lemma.**

For applying the Twisting Lemma we have just to prove that our pinching matrices satisfy the conditions on Proposition 6.1. The reason is contained in the following Proposition combined with the fact that our pinching matrices coincide in the last $d$ rows.

**Proposition 6.2.** *Let $M = (m_{ij})$ be a pinching matrix of order $d+1$. There exist some polynomials $p_1, \cdots, p_{d+1}$ with degrees least than or equal to $d$ whose coefficients are determined by the last $d$ rows of $M$ and such that the vectors:*

$$x = (p_1(\lambda), \cdots, p_{d+1}(\lambda))$$

*are non zero for any eigenvalue $\lambda$ of $M$ and satisfy:*

$$M \cdot x = \lambda \cdot x$$

We comment the proof for $d = 2$, the general case is analogous. A detailed proof is given in section 12.

*Proof.* Let $v = (v_1, v_2, v_3)$ be a non-zero eigenvector corresponding to the eigenvalue $\lambda$. There exist an index $i$ such that $v_i = 0$. Suppose without loss of generality that $i = 3$. Obviously there exist one and only one vector $w = (w_1, w_2, 1)$ such that $(M - \lambda \mathrm{Id}) \cdot w = 0$. So we have the following system:

$$m_{21}w_1 + (m_{22} - \lambda)w_2 = -m_{23}$$
$$m_{31}w_1 + m_{32}w_2 = -(m_{33} - \lambda)$$

As $w_1$ and $w_2$ are uniquely determined we conclude that the last system have a unique solution given by:

$$(w_1, w_2) = \left( \frac{\begin{vmatrix} m_{22}-\lambda & m_{23} \\ m_{32} & m_{33}-\lambda \end{vmatrix}}{\begin{vmatrix} m_{21} & m_{22}-\lambda \\ m_{31} & m_{32} \end{vmatrix}}, \frac{-\begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33}-\lambda \end{vmatrix}}{\begin{vmatrix} m_{21} & m_{22}-\lambda \\ m_{31} & m_{32} \end{vmatrix}} \right)$$

So the vector

$$u = \left( \begin{vmatrix} m_{22}-\lambda & m_{23} \\ m_{32} & m_{33}-\lambda \end{vmatrix}, -\begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33}-\lambda \end{vmatrix}, \begin{vmatrix} m_{21} & m_{22}-\lambda \\ m_{31} & m_{32} \end{vmatrix} \right)$$

satisfies $M \cdot u = \lambda \cdot u$. Now observe that the components of $u$ are the minors of order 2 of the matrix resulting by dropping the first row of $M$, and so polynomials of degree least than or equal to 2.

□

### 6.3. **Selmer's algorithm is twisting.**

Now we are in condition to use some of our pinching matrices for proving the twisting condition.

**Proposition 6.3.** *There exist $m_1$ and $m_2$ naturals and different such that the following matrices are pinching*

$$B_1 = A_1^{m_1 d-1} A_2^{2d+2}, \qquad B_2 = A_1^{m_2 d-1} A_2^{2d+2}$$

*And the pair $(B_1, B_2)$ is in general position.*

*Proof.* Observe that if $N_0$ is like in Proposition 5.5 then for any integer $m$ greater than $N_0$ the matrix $A_1^{md-1}A_2^{2d+2}$ is pinching. Let $\lambda_1^m, \cdots, \lambda_{d+1}^m$ be its eigenvalues in increasing order. As we observed during the Proof of Proposition 5.5 the first $d$ eigenvalues converge to the zeroes of $q_d$ and are different from them by Proposition 5.4. Then by Lemma 5.1 the last eigenvalue is real and contained in the open interval $(2m, 4m)$ for $m$ big enough. So we can easily deduce that there exist $m_1$ and $m_2$ greater than or equal to $N_0$ with $m_1 \neq m_2$ and such that $\lambda_i^{m_1} \neq \lambda_j^{m_2} \quad \forall i, j \in \{1, \cdots, d+1\}$.

If we take $B_1 = A_1^{m_1 d-1}A_2^{2d+2}$, $B_2 = A_1^{m_2 d-1}A_2^{2d+2}$ from equation 35 in section 8 we conclude that the last $d$ rows of $B_1$ and $B_2$ coincide and so Proposition 6.2 guarantees the existence of some polynomials $p_1, \cdots, p_{d+1}$ of orders least than or equal to $d$ such that the vectors

$$\vec{v}^{\,i}_j = (p_1(\lambda_j^i), \cdots, p_{d+1}(\lambda_j^i)), \quad 1 \leq i \leq 2, \quad 1 \leq j \leq d+1$$

are non trivial and solve the equation

$$B_i \cdot \vec{v}^{\,i}_j = \lambda_j^i \cdot \vec{v}^{\,i}_j$$

where $\lambda_j^i, \quad 1 \leq i \leq 2, \quad 1 \leq j \leq d+1$ are the eigenvalues of $B_i, \quad 1 \leq i \leq 2$. Now Proposition 6.1 permit us to conclude that any set of vectors in the form

$$\left\{ \vec{v}^{\,1}_{i_1}, \cdots, \vec{v}^{\,1}_{i_k}, \vec{v}^{\,2}_{i_{k+1}}, \cdots, \vec{v}^{\,2}_{i_{d+1}} \right\} \tag{26}$$

is linearly independent for $0 \leq k \leq d+1$ and the pair $(B_1, B_2)$ is in general position. □

Now as a consequence of the Twisting Lemma we get

**Corollary 6.2.** *Selmer's algorithm is twisting.*

## 7. Appendix A: Orthogonal polynomials

In general, a nondecreasing bounded function $\alpha$ defined on $\mathbb{R}$ is called an $m$-distribution if it takes infinitely many distinct values, and the improper integrals

$$\int_{-\infty}^{+\infty} x^n d\alpha(x) = \lim_{\substack{w_1 \to -\infty \\ w_2 \to +\infty}} \int_{w_1}^{w_2} x^n d\alpha(x)$$

exist and are finite for $n = 0, 1, \ldots$.
Using a distribution $\alpha$ we can define in a natural way an internal product in the vector space $\mathcal{P}$ of the polynomials

$$(p, q) := \int_R pq d\alpha$$

The following theorem (see [5]) guarantees the existence and unicity of a *sequence of orthogonal polynomials* in relation to this internal product.

**Theorem 7.1.** *For any $m$-distribution $\alpha$ there exists a unique sequence of polynomials $(p_n)_{n=0}^{\infty}$ with the following properties:*

*(i)*$p_n(x) = \gamma_n(x)x^n + r_{n-1}(x), \gamma_n > 0, r_{n-1} \in \mathcal{P}_{n-1}$

*(ii)* $\int_R p_n(x)p_m(x)d\alpha(x) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$

This polynomials also satisfy a very special recurrent relation

**Theorem 7.2.** *Suppose that* $(p_n)_{n=0}^{\infty}$ *is a sequence of orthogonal polynomials in relation to an* $m$*-distribution* $\alpha$*. Then:*

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1}p_{n-1}(x); \quad n = 0, 1, \dots$$

*where:* $p_{-1} = 0, \quad a_{-1} = 0, \quad a_n = \frac{\gamma_n}{\gamma_{n+1}} > 0, \quad b_n \in \mathbb{R}, \quad n = 0, 1, \dots$

($\gamma_n$ *is the maximal coefficient of* $p_n$).

In fact this is a characterization for sequences of orthogonal polynomials because on the following converse to this theorem due to Favard (see [9]).

**Theorem 7.3.** *Given* $(a_n)_{n=0}^{\infty} \subset (0, \infty)$ *and* $(b_n)_{n=0}^{\infty} \subset \mathbb{R}$*, the polynomials* $p_n \in \mathcal{P}$ *are defined by:*
$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1}p_{n-1}(x)$,
$p_{-1} = 0, \quad p_0 = \gamma_0 > 0$
*Then there exists an* $m$*-distribution* $\alpha$ *such that:*

$$\int_{\mathbb{R}} p_n(x)p_m(x)d\alpha(x) = 0$$

*for any two non negative integers* $n$ *and* $m$*.*

Orthogonal polynomials satisfy some amazing properties a couple of them will be very useful for us. So we are going to enunciate them in the following Proposition :

**Proposition 7.1.** *Let* $(p_n)_{n=0}^{\infty}$ *be a sequence of orthogonal polynomials in relation to an* $m$*-distribution* $\alpha$*. Then:*

-(Simple Real Zeros) $p_n$ *has exactly* $n$ *simple real zeros lying in the interior of the smallest interval containing supp*$\alpha$*.*

-(Interlacing of Zeros) *The zeros of* $p_n$ *and* $p_{n+1}$ *strictly interlace. That is, there is exactly one zero of* $p_n$ *strictly between any two consecutive zeroes of* $p_{n+1}$*.*

7.1. **Some interesting sequences of orthogonal polynomials.**
Let us study for example the following sequences of polynomials $(g_k)_{k=1}^{\infty}, (s_k)_{k=1}^{\infty}, (o_k)_{k=1}^{\infty}, (p_k)_{k=1}^{\infty}$ defined by:

$$\begin{aligned} &g_{-1} \equiv 0, \quad g_0 \equiv 1, \quad g_k(y) = yg_{k-1}(y) - g_{k-2}(y) \quad \forall k \geq 1 \\ &s_{-1} \equiv 0, \quad s_0 \equiv 3, \quad s_k(y) = 3g_k(y) - 2g_{k-1}(y) \quad \forall k \geq 0 \\ &o_{-1} \equiv 0, \quad o_0 \equiv 1, \quad o_k(y) = g_k(y) - g_{k-1}(y) \quad \forall k \geq 0 \\ &p_{-1} \equiv 0, \quad p_0 \equiv \sqrt{3}, \quad p_k(y) = 3g_k(y) + 2g_{k-2}(y) - 5g_{k-1}(y) \quad \forall k \geq 1 \end{aligned}$$

These polynomials appear inside the characteristic polynomials of some matrices of the monoid associated to Selmer's MCFA and will play a central role in our proof if pinching. So we will take some time to study them. First observe that:

$$
\begin{aligned}
&g_1(y) = y, \quad g_2(y) = y^2 - 1, \quad g_3(y) = y^3 - 2y, \quad g_4(y) = y^4 - 3y^2 + 1 \\
&s_1(y) = 3y - 2, \quad s_2(y) = 3y^2 - 2y - 3, \quad s_3(y) = 3y^3 - 2y^2 - 6y + 2 \\
&o_1(y) = y - 1, \quad o_2(y) = y^2 - y - 1, \quad o_3(y) = y^3 - y^2 - 2y + 1 \\
&p_1(y) = 3y - 5, \quad p_2(y) = 3y^2 - 5y - 1, \quad p_3(y) = 3y^3 - 5y^2 - 4y + 5 \\
&p_4(y) = 3y^4 - 5y^3 - 7y^2 + 10y + 1
\end{aligned}
$$

Using the recurrent relation which define the sequence $(g_k)_{k=1}^{\infty}$ it is easy to verify that the other three sequences satisfy the same recurrent relation, that is:

$$
\begin{aligned}
s_k(y) &= y s_{k-1}(y) - s_{k-2}(y) \quad \forall k \geq 2 \\
o_k(y) &= y o_{k-1}(y) - o_{k-2}(y) \quad \forall k \geq 2 \\
p_k(y) &= y p_{k-1}(y) - p_{k-2}(y) \quad \forall k \geq 3
\end{aligned}
$$

And these are sequences of orthogonal polynomials because they verify the conditions in Favard's theorem:

$$
\left.\begin{array}{l}
y \cdot g_0(y) = 1 \cdot g_{0+1}(y) + 0 \cdot g_0(y) + 0 \cdot g_{-1}(y) \\
y \cdot g_n(y) = 1 \cdot g_{n+1}(y) + 0 \cdot g_n(y) + 1 \cdot g_{n-1}(y) \\
\qquad \forall n \geq 1 \\
\qquad g_{-1} = 0, \quad g_0 = 1 > 0
\end{array}\right\} \Rightarrow \left\{\begin{array}{l}
1 = a_0 = a_1 = \cdots \\
\quad \Rightarrow (a_n)_{n=0}^{\infty} \subset (0, \infty) \\
0 = b_0 = b_1 = \cdots \\
\quad \Rightarrow (b_n)_{n=0}^{\infty} \subset \mathbb{R}
\end{array}\right.
$$

$$
\left.\begin{array}{l}
y \cdot s_0(y) = 1 \cdot s_{0+1}(y) + \frac{2}{3} \cdot s_0(y) + 0 \cdot s_{-1}(y) \\
y \cdot s_n(y) = 1 \cdot s_{n+1}(y) + 0 \cdot s_n(y) + 1 \cdot s_{n-1}(y) \\
\qquad \forall n \geq 1 \\
\qquad s_{-1} = 0, \quad s_0 = 3 > 0
\end{array}\right\} \Rightarrow \left\{\begin{array}{l}
1 = a_0 = a_1 = \cdots \\
\quad \Rightarrow (a_n)_{n=0}^{\infty} \subset (0, \infty) \\
\frac{2}{3} = b_0, \quad 0 = b_1 = b_2 = \cdots \\
\quad \Rightarrow (b_n)_{n=0}^{\infty} \subset \mathbb{R}
\end{array}\right.
$$

$$
\left.\begin{array}{l}
y \cdot o_0(y) = 1 \cdot o_{0+1}(y) + 1 \cdot o_0(y) + 0 \cdot o_{-1}(y) \\
y \cdot o_n(y) = 1 \cdot o_{n+1}(y) + 0 \cdot o_n(y) + 1 \cdot o_{n-1}(y) \\
\qquad \forall n \geq 1 \\
\qquad o_{-1} = 0, \quad o_0 = 1 > 0
\end{array}\right\} \Rightarrow \left\{\begin{array}{l}
1 = a_0 = a_1 = \cdots \\
\quad \Rightarrow (a_n)_{n=0}^{\infty} \subset (0, \infty) \\
1 = b_0, \quad 0 = b_1 = b_2 = \cdots \\
\quad \Rightarrow (b_n)_{n=0}^{\infty} \subset \mathbb{R}
\end{array}\right.
$$

$$
\left.\begin{array}{l}
y \cdot p_0(y) = \frac{\sqrt{3}}{3} \cdot p_{0+1}(y) + \frac{5}{3} \cdot p_0(y) + 0 \cdot p_{-1}(y) \\
y \cdot p_1(y) = 1 \cdot p_{1+1}(y) + 0 \cdot p_1(y) + \frac{\sqrt{3}}{3} \cdot p_{1-1}(y) \\
y \cdot p_n(y) = 1 \cdot p_{n+1}(y) + 0 \cdot p_n(y) + 1 \cdot p_{n-1}(y) \\
\qquad \forall n \geq 2 \\
\qquad p_{-1} = 0, \quad p_0 = \sqrt{3} > 0
\end{array}\right\} \Rightarrow \left\{\begin{array}{l}
\frac{\sqrt{3}}{3} = a_0, \quad 1 = a_1 = a_2 = \cdots \\
\quad \Rightarrow (a_n)_{n=0}^{\infty} \subset (0, \infty) \\
\frac{5}{3} = b_0, \quad 0 = b_1 = b_2 = \cdots \\
\quad \Rightarrow (b_n)_{n=0}^{\infty} \subset \mathbb{R}
\end{array}\right.
$$

It is easy to prove that the polynomials $g_k, s_k, o_k, p_k$ have all of their zeroes on the open interval $(-2,2)$. To see this let us first observe that if we define:

$$\begin{aligned}
g_p(y) &= g_2(y) - g_0(y) = y^2 - 2\\
g_i(y) &= g_3(y) - g_1(y) = y^3 - 3y\\
s_p(y) &= s_2(y) - s_0(y) = 3y^2 - 2y - 6\\
s_i(y) &= s_3(y) - s_1(y) = 3y^3 - 2y^2 - 9y + 4\\
o_p(y) &= o_2(y) - o_1(y) = y^2 - y - 2\\
o_i(y) &= o_3(y) - o_1(y) = y^3 - y^2 - 3y + 2\\
p_p(y) &= p_4(y) - p_2(y) = 3y^4 - 5y^3 - 10y^2 + 15y + 2\\
p_i(y) &= p_3(y) - p_1(y) = 3y^3 - 5y^2 - 7y + 10
\end{aligned}$$

Then:

$g_p(-2) = 2, g_p(0) = -2, g_p(2) = 2$ $\quad$ $g_i(-2) = -2, g_i(-1) = 2, g_i(1) = -2, g_i(2) = 2$

$s_p(-2) = 10, s_p(0) = -6, s_p(2) = 2$ $\quad$ $s_i(-2) = -10, s_i(-1) = 8, s_i(1) = -4, s_i(2) = 2$

$o_p(-2) = 4, o_p(0) = -2, o_p(2) = 0$ $\quad$ $o_i(-2) = -4, o_i(0) = 2, o_i(1) = -1, o_i(2) = 0$

$p_p(-2) = 20, p_p(-1) = -15, p_p(0) = 2$ $\quad$ $p_i(-2) = -20, p_i(0) = 10, p_i(1.2) = -0.416$

$p_p(1.6) = -0.4192, p_p(2) = 0$ $\quad$ $p_i(2) = 0$

From there we deduce that:

1)zeroes of $g_p$ are in $(-2,2)$ and $g_p(x) > 0$ if $x \leq -2$, $g_p(x) > 0$ if $x \geq 2$

2)zeroes of $g_i$ are in $(-2,2)$ and $g_i(x) < 0$ if $x \leq -2$, $g_i(x) > 0$ if $x \geq 2$

3)zeroes of $s_p$ are in $(-2,2)$ and $s_p(x) > 0$ if $\mid x \mid \geq 2$

4)zeroes of $s_i$ are in $(-2,2)$ and $s_i(x) < 0$ if $x \leq -2$, $s_i(x) > 0$ if $x \geq 2$

5)zeroes of $o_p$ are in $(-2,2]$ and $o_p(x) > 0$ if $x \leq -2$, $o_p(x) \geq 0$ if $x \geq 2$

6)zeroes of $o_i$ are in $(-2,2]$ and $o_i(x) < 0$ if $x \leq -2$, $o_i(x) \geq 0$ if $x \geq 2$

7)zeroes of $p_p$ are in $(-2,2]$ and $p_p(x) > 0$ if $x \leq -2$, $p_p(x) \geq 0$ if $x \geq 2$

8)zeroes of $p_i$ are in $(-2,2]$ and $p_i(x) < 0$ if $x \leq -2$, $p_i(x) \geq 0$ if $x \geq 2$

We also have:

$g_2(-2) = 3, g_2(0) = -1, g_2(2) = 3, g_3(-2) = -4, g_3(-1) = 1, g_3(1) = -1, g_3(2) = 4$

$s_2(-2) = 13, s_2(0) = -3, s_2(2) = 5, s_3(-2) = -18, s_3(-1) = 3, s_3(1) = -3, s_3(2) = 6$

$o_2(-2) = 5, o_2(0) = -1, o_2(2) = 1, o_3(-2) = -7, o_3(0) = 1, o_3(1) = -1, o_3(2) = 1$

$p_2(-2) = 21, p_2(0) = -1, p_2(2) = 1, p_3(-2) = -31, p_3(0) = 5, p_3(1) = -1, p_3(2) = 1$

So the polynomials $g_2, g_3, s_2, s_3, o_2, o_3, p_2, p_3$ have all their zeroes on the open interval $(-2,2)$ and:
$g_2(x) > 0, s_2(x) > 0, o_2(x) > 0, p_2(x) > 0$ for $\mid x \mid \geq 2$;

$g_{2\cdot1+1}(x) = g_3(x) < 0, s_{2\cdot1+1}(x) = s_3(x) < 0, o_{2\cdot1+1}(x) = o_3(x) < 0, p_{2\cdot1+1}(x) = p_3(x) < 0$ for $x \leq -2$;

$g_{2\cdot1+1}(x) = g_3(x) > 0, s_{2\cdot1+1}(x) = s_3(x) > 0, o_{2\cdot1+1}(x) = o_3(x) > 0, p_{2\cdot1+1}(x) = p_3(x) > 0$ for $x \geq 2$.

Combining all of these observations with the following Proposition it is easy to deduce that all the zeroes of the polynomials $g_k, s_k, o_k, p_k$ are contained in the open interval $(-2, 2)$.

**Proposition 7.2.** *Let $\mathfrak{p}$ be a sequence of polynomials satisfying the recurrent relation $\mathfrak{p}_n(x) = x\mathfrak{p}_{n-1}(x) - \mathfrak{p}_{n-2}(x)$. And suppose that there exist $m_0$ and $n_0$ such that $\mathfrak{p}_{2m_0}(x) > 0$ if $\mid x \mid \geq 2$, $\mathfrak{p}_{2n_0+1}(x) < 0$ if $x \leq -2$, $\mathfrak{p}_{2n_0+1}(x) > 0$ if $x \geq 2$ and $\mathfrak{p}_{2m_0+2}(x) - \mathfrak{p}_{2m_0}(x) \geq 0$ if $\mid x \mid \geq 2$, $\mathfrak{p}_{2n_0+3}(x) - \mathfrak{p}_{2n_0+1}(x) \leq 0$ if $x \leq -2$, $\mathfrak{p}_{2n_0+3}(x) - \mathfrak{p}_{2n_0+1}(x) \geq 0$ if $x \geq 2$. Then the following chains of inequalities are true for any $k$:*

$$1) p_{2k}(x) \geq p_{2k-2}(x) \geq \cdots \geq p_{2m_0}(x) > 0 \quad \textit{if} \quad \mid x \mid \geq 2$$
$$2) p_{2k+1}(x) \leq p_{2k-1}(x) \leq \cdots \leq p_{2n_0+1}(x) < 0 \quad \textit{if} \quad x \leq -2$$
$$3) p_{2k+1}(x) \geq p_{2k-1}(x) \geq \cdots \geq p_{2n_0+1}(x) > 0 \quad \textit{if} \quad x \geq 2$$

*Proof.* First of all observe that :

$$\left.\begin{aligned} \mathfrak{p}_{n+2}(x) - \mathfrak{p}_n(x) &= x\mathfrak{p}_{n+1}(x) - \mathfrak{p}_n(x) - \mathfrak{p}_n(x) = \\ &= x[x\mathfrak{p}_n(x) - \mathfrak{p}_{n-1}(x)] - 2p_n(x) = \\ &= (x^2-2)\mathfrak{p}_n(x) - x\mathfrak{p}_{n-1}(x) \end{aligned}\right\} \tag{27}$$

1)We will proceed by induction. For $m_0$ its true that $\mathfrak{p}_{2m_0}(x) > 0$ if $\mid x \mid \geq 2$. Now suppose that:
$p_{2k}(x) \geq p_{2k-2}(x) \geq \cdots \geq p_{2m_0}(x) > 0$ if $\mid x \mid \geq 2$
Using (2) and the fact that $\mathfrak{p}_{2k}(x) > 0$ results:

$$\begin{aligned} \mathfrak{p}_{2k+2}(x) - \mathfrak{p}_{2k}(x) &= (x^2-2)\mathfrak{p}_{2k}(x) - x\mathfrak{p}_{2k-1}(x) \geq \\ &\geq 2\mathfrak{p}_{2k}(x) - x\mathfrak{p}_{2k-1}(x) = \\ &= \mathfrak{p}_{2k}(x) - \mathfrak{p}_{2k-2}(x) \quad \text{if} \quad \mid x \mid \geq 2 \end{aligned}$$

But $\mathfrak{p}_{2k}(x) \geq \mathfrak{p}_{2k-2}(x)$ if $\mid x \mid \geq 2$ so we obtain:
$\mathfrak{p}_{2k+2}(x) \geq \mathfrak{p}_{2k}(x)$ if $\mid x \mid \geq 2$

2)We prove by induction too. For $n_0$ it is true that $\mathfrak{p}_{2n_0+1}(x) < 0$ if $x \leq -2$. Suppose now that:
$p_{2k+1}(x) \leq p_{2k-1}(x) \leq \cdots \leq p_{2n_0+1}(x) < 0$ if $x \leq -2$
Using (2) and the fact that $\mathfrak{p}_{2k+1}(x) < 0$ we obtain:

$$\begin{aligned} \mathfrak{p}_{2k+3}(x) - \mathfrak{p}_{2k+1}(x) &= (x^2-2)\mathfrak{p}_{2k+1}(x) - x\mathfrak{p}_{2k}(x) \leq \\ &\leq 2\mathfrak{p}_{2k+1}(x) - x\mathfrak{p}_{2k}(x) = \\ &= \mathfrak{p}_{2k+1}(x) - \mathfrak{p}_{2k-1}(x) \quad \text{for} \quad x \leq -2 \end{aligned}$$

But $\mathfrak{p}_{2k+1}(x) \leq \mathfrak{p}_{2k-1}(x)$ if $x \leq -2$ so we get:
$\mathfrak{p}_{2k+3}(x) \leq \mathfrak{p}_{2k+1}(x)$ for $x \leq -2$

3)By induction again. For $n_0$ we have $\mathfrak{p}_{2n_0+1}(x) > 0$ if $x \geq 2$. Suppose that :
$p_{2k+1}(x) \geq p_{2k-1}(x) \geq \cdots \geq p_{2n_0+1}(x) > 0$ if $x \geq 2$

Using (2) and the fact that $\mathfrak{p}_{2k+1}(x) > 0 \quad \text{for} \quad x \geq 2$ we derive:

$$\begin{aligned} \mathfrak{p}_{2k+3}(x) - \mathfrak{p}_{2k+1}(x) &= (x^2-2)\mathfrak{p}_{2k+1}(x) - x\mathfrak{p}_{2k}(x) \geq \\ &\geq 2\mathfrak{p}_{2k+1}(x) - x\mathfrak{p}_{2k}(x) = \\ &= \mathfrak{p}_{2k+1}(x) - \mathfrak{p}_{2k-1}(x) \quad \text{if} \quad x \geq 2 \end{aligned}$$

But $\mathfrak{p}_{2k+1}(x) \geq \mathfrak{p}_{2k-1}(x) \quad \text{if} \quad x \geq 2$ therefore:
$\mathfrak{p}_{2k+3}(x) \geq \mathfrak{p}_{2k+1}(x) \quad \text{for} \quad x \geq 2$ □

Collecting all the previous observations we get the following

**Proposition 7.3.** *The four sequences $(g_k)_k$, $(s_k)_k$, $(o_k)_k$, $(p_k)_k$ are sequences of orthogonal polynomials. The zeroes of each one of the polynomials are real, simple and contained in the open interval $(-2, 2)$. And the roots of two consecutive polynomials in the same sequence strictly interlace.*

## 8. Appendix B: The matrices involved

Let us first study the potences of $A_2$. In order to do this define the recurrent sequence $x_n$ given by:
$x_1 = x_2 = \cdots = x_{d-1} = x_d = 0, \quad x_{d+1} = 1, \quad x_{n+d+1} = x_n + x_{n+1}, \forall n \geq 1$

*Remark* 8.1.

$$\begin{aligned} x_{d+1+i} &= x_{i+d+1} = x_i + x_{i+1} = 0, \quad \text{for} \quad 1 \leq i \leq d-1 \\ x_{2d+1} &= x_{d+d+1} = x_d + x_{d+1} = 1 \end{aligned}$$

*Remark* 8.2. There exists $n_0$ such that $x_n > 0, \forall n \geq n_0$

<u>Proof</u>

It is easy to prove that if in the sequence $(x_n)_n$ there are $k$ consecutive elements all of them greater than 0,
$x_{n+1} > 0, \ldots, x_{m+k} > 0$ then there are $k+1$ consecutive elements all of them greater than 0 because
$x_{m+d+1} = x_m + x_{m+1} > 0, x_{m+1+d+1} = x_{m+1} + x_{m+2} > 0, \cdots, x_{m+k+d+1} = x_{m+k} + x_{m+k+1} > 0$
Being $x_{d+1} = 1$ we obtain by induction that for any $k \geq 1$ there are $k$ consecutive elements from the sequence greater than 0. Taking $k = d+1$ we conclude that there exists $N$ such that
$x_{N+1} > 0, x_{N+2} > 0, \cdots, x_{N+d+1} > 0$
Now we can proof by induction that $x_{N+i} > 0, \forall i \geq 1$. Observe that $x_{N+i} > 0$ for $1 \leq i \leq d+1$ then its enough to proof that for all $k \geq d+1$ its true that $x_{N+i} > 0$ for $1 \leq i \leq k$. Obviously the Affirmation is true for $k = d+1$. Suppose its also true for $k = q > d+1$ i.e $x_{N+i} > 0$ for $1 \leq i \leq q$. Then:

$$x_{N+(q+1)} = x_{(N+q-d)+d+1} = x_{N+q-d} + x_{N+q-d+1} > 0$$

because

$$N+1 \leq N+q-d \leq N+q-1 \leq N+q, \quad N+2 \leq N+q-d+1 \leq N+q$$

and $x_{N+i} > 0$ for $1 \leq i \leq q+1$, thus concluding the induction step and
$x_{N+i} > 0, \quad \forall i \geq 1$.

*Remark* 8.3.

$$A_2^n = \begin{pmatrix} x_n & x_{n+1} & \dots & x_{n+d-1} & x_{n+d} \\ x_{n+d} & x_{n+d+1} & \dots & x_{n+2d-1} & x_{n+2d} \\ x_{n+d-1} & x_{n+d} & \dots & x_{n+2d-2} & x_{n+2d-1} \\ \multicolumn{5}{c}{\dotfill} \\ x_{n+1} & x_{n+2} & \dots & x_{n+d} & x_{n+d+1} \end{pmatrix}^{\boxed{d+1}}$$

Just look at:

$$A_2^1 = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 1 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}^{\boxed{d+1}} = \begin{pmatrix} x_1 & x_2 & \dots & x_d & x_{d+1} \\ x_{d+1} & x_{d+2} & \dots & x_{2d} & x_{2d+1} \\ x_d & x_{d+1} & \dots & x_{2d-1} & x_{2d} \\ \multicolumn{5}{c}{\dotfill} \\ x_2 & x_3 & \dots & x_{d+1} & x_{d+2} \end{pmatrix}$$

And them proceeding by induction suppose that:

$$A_2^k = \begin{pmatrix} x_k & x_{k+1} & \dots & x_{k+d-1} & x_{k+d} \\ x_{k+d} & x_{k+d+1} & \dots & x_{k+2d-1} & x_{k+2d} \\ x_{k+d-1} & x_{k+d} & \dots & x_{k+2d-2} & x_{k+2d-1} \\ \multicolumn{5}{c}{\dotfill} \\ x_{k+1} & x_{k+2} & \dots & x_{k+d} & x_{k+d+1} \end{pmatrix}$$

So we have the following equalities:

$$A_2^{k+1} = A_2^k A_2^1 = \begin{pmatrix} x_k & x_{k+1} & \dots & x_{k+d-1} & x_{k+d} \\ x_{k+d} & x_{k+d+1} & \dots & x_{k+2d-1} & x_{k+2d} \\ x_{k+d-1} & x_{k+d} & \dots & x_{k+2d-2} & x_{k+2d-1} \\ \multicolumn{5}{c}{\dotfill} \\ x_{k+1} & x_{k+2} & \dots & x_{k+d} & x_{k+d+1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 0 & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_{k+1} & x_{k+2} & \dots & x_{k+d} & x_k + x_{k+1} \\ x_{k+d+1} & x_{k+d+2} & \dots & x_{k+2d} & x_{k+d} + x_{k+d+1} \\ x_{k+d} & x_{k+d+1} & \dots & x_{k+2d-1} & x_{k+d-1} + x_{k+d} \\ \multicolumn{5}{c}{\dotfill} \\ x_{k+2} & x_{k+3} & \dots & x_{k+d+1} & x_{k+1} + x_{k+2} \end{pmatrix}$$

$$= \begin{pmatrix} x_{k+1} & x_{k+2} & \dots & x_{k+d} & x_{k+d+1} \\ x_{k+d+1} & x_{k+d+2} & \dots & x_{k+2d} & x_{k+2d+1} \\ x_{k+d} & x_{k+d+1} & \dots & x_{k+2d-1} & x_{k+2d} \\ \multicolumn{5}{c}{\dotfill} \\ x_{k+2} & x_{k+3} & \dots & x_{k+d+1} & x_{k+d+2} \end{pmatrix}$$

$$= \begin{pmatrix} x_{(k+1)} & x_{(k+1)+1} & \dots & x_{(k+1)+d-1} & x_{(k+1)+d} \\ x_{(k+1)+d} & x_{(k+1)+d+1} & \dots & x_{(k+1)+2d-1} & x_{(k+1)+2d} \\ x_{(k+1)+d-1} & x_{(k+1)+d} & \dots & x_{(k+1)+2d-2} & x_{(k+1)+2d-1} \\ \multicolumn{5}{c}{\dotfill} \\ x_{(k+1)+1} & x_{(k+1)+2} & \dots & x_{(k+1)+d} & x_{(k+1)+d+1} \end{pmatrix}$$

From Remark 8.3 we obtain:

$$A_2^{2d+2} = \begin{pmatrix} x_{2d+2} & x_{2d+3} & x_{2d+4} & x_{2d+5} & \dots & x_{3d} & x_{3d+1} & x_{3d+2} \\ x_{3d+2} & x_{3d+3} & x_{3d+4} & x_{3d+5} & \dots & x_{4d} & x_{4d+1} & x_{4d+2} \\ x_{3d+1} & x_{3d+2} & x_{3d+3} & x_{3d+4} & \dots & x_{4d-1} & x_{4d} & x_{4d+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_{2d+3} & x_{2d+4} & x_{2d+5} & x_{2d+6} & \dots & x_{3d+1} & x_{3d+2} & x_{3d+3} \end{pmatrix}$$

Now if observe that:
$x_1 = x_2 = \cdots = x_d = 0, x_{d+1} = 1, x_{d+2} = x_{d+3} = \cdots = x_{2d} = 0$ by Remark 8.1, and from the recurrence relation between the consecutive terms of the sequence we have:
$x_{2d+1} = 1, x_{2d+2} = 1, x_{2d+3} = x_{2d+4} = \cdots = x_{3d} = 0, x_{3d+1} = 1, x_{3d+2} = 2, x_{3d+3} = 1, x_{3d+4} = x_{3d+5} = \cdots = x_{4d} = 0, x_{4d+1} = 1, x_{4d+2} = 3$
Therefore

$$A_2^{2d+2} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & \dots & 0 & 0 & 1 & 3 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 & 1 \end{pmatrix} \tag{28}$$

It is also easy to proof inductively that:

$$A_1^m = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad 1 \le m \le d \tag{29}$$

Where in the first row there appear $m+1$ numbers 1. Therefore:

$$A_1^{d-1} = \begin{pmatrix} 1 & 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{30}$$

Now multiplying by $A_1$:

$$A_1^d = A_1^{d-1} \cdot A_1 \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{31}$$

Again by induction:

$$A_1^{md} = \begin{pmatrix} 1 & m & m & \dots & m & m \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{32}$$

And using the fact that:

$$A_1^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{33}$$

We easily arrive at:

$$A_1^{md-1} = \begin{pmatrix} 1 & m-1 & m & m & \dots & m & m \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{34}$$

Finally combining (9) and (15) we get:

$$A_1^{md-1} A_2^{2d+2} = \begin{pmatrix} 3m-1 & 4m-1 & 4m & 4m & \dots & 4m & 5m-1 \\ 1 & 2 & 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 2 & 1 & 0 & 0 & \dots & 1 & 3 \end{pmatrix} \tag{35}$$

*Remark* 8.4. Combining Remark 8.1 and Remark 8.2 we can assert that there exists $n_0$ such that $A_2^n$ is positive for all $n \geq n_0$.

## 9. Appendix C: Cylinders are simplices

The region $D_L$ containing the dynamics of $T_{\mathscr{S}}$ have a simple geometric structure because it is the convex hull of certain finite set of points.

**Proposition 9.1.**

$$\{x \in \mathbb{R}^d : 0 \leq x_1 \leq x_2 \cdots \leq x_{d-1} \leq x_d \leq 1, x_2 + x_1 \geq 1\}$$
$$= \textit{Convex hull of}\{(0,1,\cdots,1),(1,\cdots,1),(\frac{1}{2},\frac{1}{2},1,\cdots,1),$$
$$(\frac{1}{2},\frac{1}{2},\frac{1}{2},1,\cdots,1),\cdots,(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2})\}$$

*Proof.* $\boxed{\subset}$
Let $x$ be such that $0 \leq x_1 \leq x_2 \cdots \leq x_{d-1} \leq x_d \leq 1, x_2 + x_1 \geq 1$. Then $x_2 \geq \frac{1}{2}$

and $x_3 \geq \frac{1}{2}, \cdots, x_d \geq \frac{1}{2}$, so that we have the following inequalities:

$$\left.\begin{array}{l} 0 \leq -x_1 + x_2 \leq x_2 \leq 1 \\ 0 \leq x_1 + x_2 - 1 \leq x_2 \leq 1 \\ 0 \leq -2x_2 + 2x_3 = 2(x_3 - x_2) \leq 2 \cdot \dfrac{1}{2} = 1 \\ 0 \leq -2x_3 + 2x_4 = 2(x_4 - x_3) \leq 2 \cdot \dfrac{1}{2} = 1 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ 0 \leq -2x_{d-1} + 2x_d = 2(x_d - x_{d-1}) \leq 2 \cdot \dfrac{1}{2} = 1 \end{array}\right\} \mathrm{I}$$

$$0 \leq 2x_d - 1 \leq 1 \quad \}\,\mathrm{II}$$

Observing that
$(-x_1+x_2)+(x_1+x_2-1)+(-2x_2+2x_3)+(-2x_3+2x_4)+\cdots+(-2x_{d-1}+2x_d) = 2x_d-1$
and using I and II results that the vector:

$$(t_1, t_2, t_3, t_4, \cdots, t_{d+1}) =$$
$$= (-x_1 + x_2, x_1 + x_2 - 1, -2x_2 + 2x_3, -2x_3 + 2x_4, \cdots, -2x_{d-1} + 2x_d,$$
$$1 - (-x_1 + x_2) + (x_1 + x_2 - 1) + (-2x_2 + 2x_3) + \cdots + (-2x_{d-1} + 2x_d))$$

satisfy $0 \leq t_i \leq 1, 1 \leq i \leq d+1$. We also have that:

$$(x_1, x_2, \cdots, x_d) =$$
$$= (t_2 + \frac{1}{2}t_3 + \cdots + \frac{1}{2}t_{d+1}, t_1 + t_2 + \frac{1}{2}t_3 + \cdots + \frac{1}{2}t_{d+1}, t_1 + t_2 + t_3 +$$
$$\frac{1}{2}t_4 + \cdots + \frac{1}{2}t_{d+1}, \cdots, t_1 + t_2 + \cdots + t_d + \frac{1}{2}t_{d+1})$$
$$= t_1(0, 1, \cdots, 1) + t_2(1, \cdots, 1) + t_3(\frac{1}{2}, \frac{1}{2}, 1, \cdots, 1) + t_4(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \cdots, 1) +$$
$$+ t_5(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \cdots, 1) + \cdots + t_{d+1}(\frac{1}{2}, \frac{1}{2})$$
$$\Rightarrow (x_1, \cdots, x_d) \in \text{Convex hull of}\{(0, 1, \cdots, 1), (1, \cdots, 1), (\frac{1}{2}, \frac{1}{2}, 1, \cdots, 1),$$
$$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \cdots, 1), \cdots, (\frac{1}{2}, \frac{1}{2}), \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})\}$$

$\boxed{\supset}$
Take

$$(x_1, \cdots, x_d) \in \text{Convex hull of}\{(0, 1, \cdots, 1), (1, \cdots, 1), (\frac{1}{2}, \frac{1}{2}, 1, \cdots, 1),$$
$$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \cdots, 1), \cdots, (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})\}$$

Then:

$$(x_1, \cdots, x_d) = (t_2 + \frac{1}{2}t_3 + \cdots + \frac{1}{2}t_{d+1}, t_1 + t_2 + \frac{1}{2}t_3 + \cdots + \frac{1}{2}t_{d+1}, t_1 + t_2 + t_3 +$$
$$\frac{1}{2}t_4 + \cdots + \frac{1}{2}t_{d+1}, \cdots, t_1 + t_2 + \cdots + t_d + \frac{1}{2}t_{d+1})$$

for some $0 \le t_i \le 1, \quad 1 \le i \le d+1, \quad t_1 + \cdots + t_d + t_{d+1} = 1$
Obviously:

$$0 \le x_1 \le x_2 \le x_3 \le \cdots \le x_d \le 1$$

and

$$x_2 + x_1 = t_1 + 2t_2 + t_3 + \cdots + t_{d+1} = 1 + t_2 \ge 1$$

So

$$(x_1, \cdots, x_d) \in \{x : 0 \le x_1 \le \cdots \le x_d \le 1, \quad x_2 + x_1 \ge 1\}$$

□

Lemma 3.1 requires $T$ to be a projective expanding map. So the fact that $\beta$ and the cylinders $\beta(i_1, \cdots, i_k)$ are simplexes is relevant to conclude in section 4 that some application has approximate product structure. In the following we proof this fact and that some of the cylinders $\beta(i_1, \cdots, i_k)$ are simplexes compactly contained in the standard simplex.

Now observe that as the vertices of the simplex $D_L$ are the points

$$(0,1,\cdots,1),(1,1,\cdots,1),(\frac{1}{2},\frac{1}{2},1,\cdots,1),(\frac{1}{2},\frac{1}{2},\frac{1}{2},1,\cdots,1),\cdots,(\frac{1}{2},\cdots,\frac{1}{2})$$

and $j(D_L) = \tilde{D}$ then the vertices $P_1, \cdots, P_{d+1}$ of $\tilde{D}$ are given by:

$$\left.\begin{array}{rl} P_1 = j(0,1,\cdots,1) = & (0,1,\cdots,1,1) \\ P_2 = j(1,1,\cdots,1) = & (1,1,\cdots,1,1) \\ P_3 = j(\frac{1}{2},\frac{1}{2},1,\cdots,1) = & (\frac{1}{2},\frac{1}{2},1,\cdots,1,1) \\ \vdots \qquad \vdots & \qquad \vdots \\ P_{d+1} = j(\frac{1}{2},\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2}) = & (\frac{1}{2},\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2},1) \end{array}\right\}$$

Let us define the linear application $l$ by:

$$\begin{aligned} l : \mathbb{R}^{d+1} &\to \mathbb{R}^{d+1} \\ x &\mapsto M \cdot x \end{aligned}$$

Where

$$M = \begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ 1 & 1 & 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ 1 & 1 & 1 & 1 & \cdots & \frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & \cdots & \frac{1}{2} \end{pmatrix} \boxed{d+1}$$

From:

$$\mathrm{Det(M)} = \begin{vmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ 1 & 0 & 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ 1 & 0 & 0 & 0 & \cdots & -\frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & \cdots & -\frac{1}{2} \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{vmatrix}\boxed{d+1}$$

$$= (-1)^{d+1+1} \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} \end{vmatrix}\boxed{d}$$

$$= (-1)^{d+1+1}(-\frac{1}{2})^{d-1} = -(\frac{1}{2})^{d-1} \neq 0$$

results that $l$ is an isomorphism. Also $l(\overrightarrow{e_i}) = P_i, 1 \leq i \leq d+1$ and

$$\begin{aligned} l(t_1\overrightarrow{e_1} + \cdots + t_{d+1}\overrightarrow{e_{d+1}}) &= t_1 \cdot l(\overrightarrow{e_1}) + \cdots + t_{d+1} \cdot l(\overrightarrow{e_{d+1}}) \\ &= t_1 \cdot P_1 + \cdots + t_{d+1} \cdot P_{d+1} \end{aligned}$$

for $0 \leq t_1 \leq \cdots \leq t_{d+1} \leq 1$ with $t_1 + \cdots + t_{d+1} = 1$. Therefore $l(P\mathbb{R}^{d+1}) = \tilde{D}$, but $\beta$ is the projection of $\tilde{D}$ in $P\mathbb{R}^{d+1}$ so we obtain that $L(P\mathbb{R}^{d+1}) = \beta$ for some linear application given by a non negative matrix $L$.
Remember that

$$\beta(i_1, \cdots, i_k) = (\tilde{T}_p^{(i_1, \cdots, i_k)})^{-1}(\beta) = (\tilde{T}_p^{(i_1, \cdots, i_k)})^{-1} \circ L(P\mathbb{R}^{d+1})$$

Hence the $\beta(i_1, \cdots, i_k)$ are simplexes. And by Remark 2.2 we get:

*Remark* 9.1. $\beta(2)$ is a simplex compactly contained in $P\mathbb{R}^{d+1}_+$.

We have another easy way to conclude the existence of a compactly contained simplex in $P\mathbb{R}^{d+1}_+$. Let $n_0$ be like in Remark 8.4 then $(\tilde{T}_p^{(\overbrace{2, \cdots, 2}^{n_0})})^{-1}$ is the projectivization of a positive matrix. Also

$$\beta(\overbrace{2, \cdots, 2}^{n_0}) = (\tilde{T}_p^{(\overbrace{2, \cdots, 2}^{n_0})})^{-1}(\beta) = (\tilde{T}_p^{(\overbrace{2, \cdots, 2}^{n_0})})^{-1} \circ L(\mathbb{R}^{d+1}_+) = N(\mathbb{R}^{d+1}_+)$$

where $N$ is a positive matrix because $L$ is non negative and $(\tilde{T}_p^{(\overbrace{2, \cdots, 2}^{n_0})})^{-1}$ is positive. Then $\beta(\overbrace{2, \cdots, 2}^{n_0})$ is a simplex compactly contained in $P\mathbb{R}^{d+1}_+$.

## 10. Appendix D: Some polynomials and identities

**Proposition 10.1.**

1) $g_n = yg_{n-1} - g_{n-2} = \sum_{i=0}^{[\frac{n}{2}]}(-1)^i \binom{n-i}{i} y^{n-2i} \quad for \quad n \geq 1$

2) $t_n = g_{n-1} - t_{n-1} \quad for \quad n \geq 0$

3) $t_n = (-1)^{n+1}k_n \quad for \quad n \geq 1$

4) $t_n = (-1)^2 g_{n-1} + (-1)^3 g_{n-2} + (-1)^4 g_{n-3} + \cdots + (-1)^n g_n + (-1)^{n+1} g_0 \quad for \quad n \geq 0$

5) $t_n = (-1)^n g_n t_{n-1} + (-1)^{n+1} g_{n-1} t_n \quad for \quad n \geq 0$

6) $t_n = (-1)^n g_{n-1} t_{n+1} + (-1)^{n+1} g_n t_n \quad for \quad n \geq 0$

7) $h_n = t_{2n+1} \quad for \quad n \geq 0$

8) $u_n = t_{2n} \quad for \quad n \geq 0$

9) $g_n(g_n - g_{n-1}) - g_{n-1}(g_{n-1} - g_{n-2}) = g_{2n} - g_{2n-1} \quad for \quad n \geq 0$

10) $g_{n-2}(g_{n-1} - g_{n-2}) - g_{n-1}(g_n - g_{n-1}) = g_{2n-2} - g_{2n-1} \quad for \quad n \geq 0$

11) $(3g_n - 2g_{n-1})(g_n - g_{n-1}) - (3g_{n-1} - 2g_{n-2})(g_{n-1} - g_{n-2}) =$
$= 3g_{2n} - 5g_{2n-1} + 2g_{2n-2} \quad for \quad n \geq 0$

12) $(3g_{n+1} + 2g_{n-1} - 5g_n)g_n - (3g_n + 2g_{n-2} - 5g_{n-1})g_{n-1} =$
$= 3g_{2n+1} - 5g_{2n} + 2g_{2n-1} \quad for \quad n \geq 0$

13) $\tilde{q}_{2n+1} - \tilde{q}_{2n-1} = (y-2)(3g_{2n} - 5g_{2n-1} + 2g_{2n-2}) \quad for \quad n \geq 2$

14) $\tilde{q}_{2n+2} - \tilde{q}_{2n} = (y-2)(3g_{2n+1} - 5g_{2n} + 2g_{2n-1}) \quad for \quad n \geq 1$

15) $\tilde{q}_{2n-1} = (y-2)(3g_{n-1} - 2g_{n-2})(g_{n-1} - g_{n-2}) \quad for \quad n \geq 2$

16) $\tilde{q}_{2n} = (y-2)(3g_n + 2g_{n-2} - 5g_{n-1})g_{n-1} \quad for \quad n \geq 1$

17) $(g_n - g_{n-1})^2 = g_{2n} - 2t_{2n} \quad for \quad n \geq 0$

18) $(g_n - g_{n-1})^2 - (g_{n-1} - g_{n-2})^2 = g_{2n} - 2g_{2n-1} + g_{2n-2} \quad for \quad n \geq 0$

19) $[(g_n - g_{n-1}) - (g_{n-1} - g_{n-2})]^2 = g_{2n} + g_{2n-2} - 2t_{2n} + 2t_{2n-1} - 2yt_{2n-1} \quad for \quad n \geq 0$

20) $[(g_{n+1} - g_n) - (g_n - g_{n-1})]^2 - [(g_n - g_{n-1}) - (g_{n-1} - g_{n-2})]^2 =$
$= g_{2n+2} - 4g_{2n+1} + 6g_{2n} - 4g_{2n-1} + g_{2n-2} \quad for \quad n \geq 1$

21) $\tilde{r}_{2n+2} - \tilde{r}_{2n} = (y-2)(y+2)(g_{2n} - 2g_{2n-1} + g_{2n-2}) \quad for \quad n \geq 1$

22) $\tilde{r}_{2n+3} - \tilde{r}_{2n+1} = (y+2)(g_{2n+2} - 4g_{2n+1} + 6g_{2n} - 4g_{2n-1} + g_{2n-2}) \quad for \quad n \geq 1$

23) $\tilde{r}_{2n} = (y+2)(y-2)(g_{n-1} - g_{n-2})^2 + 1 \quad for \quad n \geq 1$

24) $\tilde{r}_{2n+1} = (y+2)(g_n - 2g_{n-1} + g_{n-2})^2 - 1 \quad for \quad n \geq 1$

25) $g_n^2 - g_{n-1}g_{n+1} = 1 \quad for \quad n \geq 0$

26) $(y+2)(y-2)g_{t-1}^2 + 4 = (g_t - g_{t-2})^2 \quad for \quad n \geq 1$

27) $(y+2)(g_{n-1} - g_{n-2})^2 = (y-2)(g_{n-1} + g_{n-2})^2 + 4 \quad for \quad n \geq 1$

*Proof.*
1) The first equality comes from developing the determinant defining $g_n(y)$ by the first column. The second equality is easily proved by induction on $n$.

2)Its enough to develop the determinant defining $t_n(y)$ by the first column.

3) We will proceed by induction. So observe that:

$$t_1(y) = 1 = (-1)^{1+1} \cdot k_1(y)$$
$$t_2(y) = \begin{vmatrix} 1 & 1 \\ 1 & y \end{vmatrix} = - \begin{vmatrix} 1 & 1 \\ y & 1 \end{vmatrix} = (-1)^{2+1} k_2(y)$$

Now if we suppose that 3) is true for $n = k$ we have:

$$t_{k+1}(y) = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & y & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & y & 1 & \cdots & 0 & 0 & 0 \\ \cdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & y & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & y & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & y \end{vmatrix} \boxed{k+1}$$

$$= (-1)^k \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \overset{\downarrow}{1} \\ y & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & y & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & y & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & y & 0 \end{vmatrix} \boxed{k+1}$$

$$= (-1)^k [(-1)^{1+k+1} g_k + (-1)^{2+k+1} t_k]$$
$$= (-1)^{k+2} (-1)^{k+2} (g_k - t_k)$$
$$= (-1)^{k+2} [(-1)^{k+2} g_k + (-1)^{k+1} t_k]$$
$$= (-1)^{k+2} [(-1)^{1+k+1} g_k + k_k]$$

$$= (-1)^{k+2} \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ y & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & y & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & y & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & y & 0 \end{vmatrix} \boxed{k+1}$$

$$= (-1)^{k+2} k_{k+1}(y)$$

4) Its enough to observe that $t_1 = (-1)^{1+1} g_0$, $\quad t_2 = \begin{vmatrix} 1 & 1 \\ 1 & y \end{vmatrix} = g_1 + (-1)^{2+1} g_0$ and then proceed by induction using 2).

5)For $n = 2, n = 3$ the identity is satisfied trivially:

$$
\begin{aligned}
t_2 &= \begin{vmatrix} 1 & 1 \\ 1 & y \end{vmatrix} = y - 1 = (y^2 - 1) \cdot 1 - y(y-1) = g_2 \cdot t_1 - g_1 \cdot t_2 \\
t_3 &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & y & 1 \\ 0 & 1 & y \end{vmatrix} = y^2 - y = -(y^3 - 2y)(y-1) + (y^2-1)(y^2-y) \\
&= -g_3 \cdot t_2 + g_2 \cdot t_3
\end{aligned}
$$

Now we can prove by induction differentiating the cases $n$ even and $n$ odd.
Case $n$ even
Suppose that 5) is true for $n = 2k$, that is:

$$t_{2k} = g_{2k} \cdot t_{2k-1} - g_{2k-1} \cdot t_{2k}$$

Observe the equivalences:

$$
\begin{aligned}
&1 + g_1 - g_0 = yg_0 \\
&\Leftrightarrow 1 - g_{2k-2} + g_{2k-3} - \cdots + g_3 - g_2 + g_1 - g_0 = \\
&= -g_{2k-2} + g_{2k-3} - \cdots + g_3 - g_2 + yg_0 \\
&\Leftrightarrow 1 - t_{2k-1} \\
&= -g_{2k} + g_{2k-1} - g_{2k-2} + g_{2k-3} - \cdots + g_3 - g_2 + yg_0 + g_{2k} - g_{2k-1} \\
&\Leftrightarrow 1 - t_{2k-1} = \\
&= (-yg_{2k-1} + yg_{2k-2} - yg_{2k-3} + yg_{2k-4} - \cdots + yg_2 - yg_1 + yg_0) + \\
&+ (g_{2k-2} - g_{2k-3} + \cdots + g_2 - g_1 + g_0) + (g_{2k} - g_{2k-1}) \\
&\Leftrightarrow 1 - t_{2k-1} = -yt_{2k} + (g_{2k} - g_{2k-1}) + t_{2k-1} \\
&\Leftrightarrow g_{2k} - g_{2k}t_{2k-1} + g_{2k-1}t_{2k} = -yg_{2k}t_{2k} + g_{2k}(g_{2k} - g_{2k-1}) + \\
&+ g_{2k-1}t_{2k} + g_{2k}t_{2k-1} \\
&\Leftrightarrow g_{2k} - t_{2k} = -(yg_{2k} - g_{2k-1})t_{2k} + g_{2k}(g_{2k} - g_{2k-1} + t_{2k-1}) \\
&\Leftrightarrow t_{2k+1} = -g_{2k+1}t_{2k} + g_{2k}t_{2k+1}
\end{aligned}
$$

and 5) holds for $n = 2k + 1$.
Case $n$ odd
Now suppose that 5) is true for $n = 2k + 1$, that is:

$$t_{2k+1} = -g_{2k+1} \cdot t_{2k} + g_{2k} \cdot t_{2k+1}$$

Analogously we have the equivalences:

$$
\begin{aligned}
&1 + g_1 - g_0 = yg_0 \\
&\Leftrightarrow 1 + g_{2k-1} - g_{2k-2} + \cdots + g_3 - g_2 + g_1 - g_0 = \\
&= g_{2k-1} - g_{2k-2} + \cdots + g_3 - g_2 + yg_0 \\
&\Leftrightarrow 1 + t_{2k} \\
&= g_{2k+1} - g_{2k} + g_{2k-1} - g_{2k-2} + \cdots + g_3 - g_2 + yg_0 - (g_{2k+1} - g_{2k})
\end{aligned}
$$

$$\Leftrightarrow 1 + t_{2k} =$$
$$= (yg_{2k} - yg_{2k-1} + \cdots + yg_2 - yg_1 + yg_0) -$$
$$- (g_{2k-1} - g_{2k-2} + \cdots + g_1 - g_0) - (g_{2k+1} - g_{2k})$$
$$\Leftrightarrow 1 + t_{2k} = yt_{2k+1} - t_{2k} - (g_{2k+1} - g_{2k})$$
$$\Leftrightarrow g_{2k+1} + g_{2k+1}t_{2k} - g_{2k}t_{2k+1} = yg_{2k+1}t_{2k+1} - g_{2k+1}(g_{2k+1} - g_{2k}) -$$
$$- g_{2k}t_{2k+1} - g_{2k+1}t_{2k}$$
$$\Leftrightarrow g_{2k+1} - t_{2k+1} = (yg_{2k+1} - g_{2k})t_{2k+1} - g_{2k+1}(g_{2k+1} - g_{2k} + t_{2k})$$
$$\Leftrightarrow t_{2k+2} = g_{2k+2}t_{2k+1} - g_{2k+1}t_{2k+2}$$

and 5) holds for $n = 2k + 2$.

6)For $n = 2, n = 3$ the identity is satisfied trivially:

$$t_2 = \text{Det}\begin{pmatrix} 1 & 1 \\ 1 & y \end{pmatrix} = y - 1 = y(y^2 - y) - (y^2 - 1)(y - 1) = g_1 \cdot t_3 - g_2 \cdot t_2$$

$$t_3 = \text{Det}\begin{pmatrix} 1 & 1 & 1 \\ 1 & y & 1 \\ 0 & 1 & y \end{pmatrix} = y^2 - y = (-1)^3(y^2 - 1)(y^3 - y^2 - y) +$$
$$+ (-1)^4(y^3 - 2y)(y^2 - y) = (-1)^3 g_2 \cdot t_4 + (-1)^4 g_3 \cdot t_3$$

Like in 5) we will prove by induction and making distinction for the cases $n$ even and $n$ odd. <u>Case $n$ even</u>
Suppose that 6) is valid for $n = 2k$, that is:

$$t_{2k} = g_{2k-1} \cdot t_{2k+1} - g_{2k} \cdot t_{2k}$$

Its enough to observe the following equivalences:

$$1 + g_1 - g_0 = yg_0$$
$$\Leftrightarrow 1 + g_{2k-1} - g_{2k-2} + \cdots + g_3 - g_2 + g_1 - g_0 =$$
$$= g_{2k-1} - g_{2k-2} + \cdots + g_3 - g_2 + yg_0$$
$$\Leftrightarrow 1 + t_{2k}$$
$$= -g_{2k+1} + g_{2k} + (g_{2k+1} - g_{2k} + g_{2k-1} - g_{2k-2} + \cdots + g_3 - g_2 + yg_0)$$
$$\Leftrightarrow 1 + t_{2k} =$$
$$= -g_{2k+1} + g_{2k} + (yg_{2k} - yg_{2k-1} + \cdots + yg_2 - yg_1 + yg_0) -$$
$$- (g_{2k-1} - g_{2k-2} + \cdots + g_3 - g_2 + g_1 - g_0)$$
$$\Leftrightarrow 1 + t_{2k} = -g_{2k+1} + g_{2k} + yt_{2k+1} - t_{2k}$$
$$\Leftrightarrow g_{2k} - g_{2k-1}t_{2k+1} + g_{2k}t_{2k} = -g_{2k}(g_{2k+1} - g_{2k}) + yg_{2k}t_{2k+1} - g_{2k}t_{2k} - g_{2k-1}t_{2k+1}$$
$$\Leftrightarrow g_{2k} - t_{2k} = -g_{2k}(g_{2k+1} - g_{2k} + t_{2k}) + (yg_{2k} - g_{2k-1})t_{2k+1}$$
$$\Leftrightarrow t_{2k+1} = -g_{2k}t_{2k+2} + g_{2k+1}t_{2k+1}$$

and 6) holds for $n = 2k + 1$.

<u>Case $n$ odd</u>
Suppose that 6) is true for $n = 2k + 1$, that is:

$$t_{2k+1} = -g_{2k} \cdot t_{2k+2} + g_{2k+1} \cdot t_{2k+1}$$

Again we have some equivalences:

$1 + g_1 - g_0 = yg_0$
$\Leftrightarrow 1 - g_{2k} + g_{2k-1} - \cdots + g_3 - g_2 + g_1 - g_0 =$
$= -g_{2k} + g_{2k-1} - \cdots + g_3 - g_2 + yg_0$
$\Leftrightarrow 1 - t_{2k+1} =$
$= g_{2k+2} - g_{2k+1} + (-g_{2k+2} + g_{2k+1} - g_{2k} + g_{2k-1} - \cdots + g_3 - g_2 + yg_0)$
$\Leftrightarrow 1 - t_{2k+1} =$
$= g_{2k+2} - g_{2k+1} + (-yg_{2k+1} + yg_{2k} - \cdots - yg_3 + yg_2 - yg_1 + yg_0)+$
$+ (g_{2k} - g_{2k-1} + \cdots + g_2 - g_1 + g_0)$
$\Leftrightarrow 1 - t_{2k+1} = g_{2k+2} - g_{2k+1} - yt_{2k+2} + t_{2k+1}$
$\Leftrightarrow g_{2k+1} + g_{2k}t_{2k+2} - g_{2k+1}t_{2k+1} = g_{2k+1}(g_{2k+2} - g_{2k+1}) - yg_{2k+1}t_{2k+2} + g_{2k+1}t_{2k+1} + g_{2k}t_{2k+2}$
$\Leftrightarrow g_{2k+1} - t_{2k+1} = g_{2k+1}(g_{2k+2} - g_{2k+1} + t_{2k+1}) - (yg_{2k+1} - g_{2k})t_{2k+2}$
$\Leftrightarrow t_{2k+2} = g_{2k+1}t_{2k+3} - g_{2k+2}t_{2k+2}$

and 6) holds for $n = 2k + 1$.

7)Expand the determinant defining $t_{2n+1}$ by its last $n$ columns, using Laplace's formulae, and then apply 5). We are marking the numbers of rows and columns for convenience:

$$\left.\begin{array}{l} J = \{n+2, n+3, \cdots, 2n, 2n+1\} \\ J^{'} = \{1, 2, \cdots, n+1\} \end{array}\right\} \Rightarrow \varepsilon(J) = (-1)^{n(n+1)} = 1$$

$$\left.\begin{array}{l} K = \{n+2, n+3, \cdots, 2n, 2n+1\} \\ K^{'} = \{1, 2, \cdots, n+1\} \end{array}\right\} \Rightarrow \varepsilon(K) = (-1)^{n(n+1)} = 1$$

$$\left.\begin{array}{l} K = \{n+1, n+3, \cdots, 2n, 2n+1\} \\ K^{'} = \{1, 2, \cdots, n, n+2\} \end{array}\right\} \Rightarrow \varepsilon(K) = (-1)^{n \cdot n + n - 1} = -1$$

$$\left.\begin{array}{l} K = \{1, n+3, \cdots, 2n, 2n+1\} \\ K^{'} = \{2, 3, \cdots, n+1, n+2\} \end{array}\right\} \Rightarrow \varepsilon(K) = (-1)^{(n-1)(n+1)} = (-1)^{n+1}$$

$t_{2n+1} =$

$$
= \begin{vmatrix}
1 & \dots & 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 & \underline{\mathit{1}} \\
1 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \underline{\mathit{2}} \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \underline{\mathit{3}} \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \underline{\mathit{4}} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & 1 & y & 1 & 0 & 0 & \dots & 0 & 0 & \underline{n+\mathit{1}} \\
0 & \dots & 0 & 1 & y & 1 & 0 & \dots & 0 & 0 & \underline{n+\mathit{2}} \\
0 & \dots & 0 & 0 & 1 & y & 1 & \dots & 0 & 0 & \underline{n+\mathit{3}} \\
0 & \dots & 0 & 0 & 0 & 1 & y & \dots & 0 & 0 & \underline{n+\mathit{4}} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & y & 1 & \underline{\mathit{2}n} \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 1 & y & \underline{\mathit{2}n+\mathit{1}} \\
\underline{\mathit{1}} & \dots & \underline{n} & \underline{n+\mathit{1}} & \underline{n+\mathit{2}} & \underline{n+\mathit{3}} & \underline{n+\mathit{4}} & \dots & \underline{\mathit{2}n} & \underline{\mathit{2}n+\mathit{1}} &
\end{vmatrix}^{\boxed{2n+1}}
$$

$$
= \begin{vmatrix}
y & 1 & 0 & \dots & 0 & 0 \\
1 & y & 1 & \dots & 0 & 0 \\
0 & 1 & y & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & y & 1 \\
0 & 0 & 0 & \dots & 1 & y
\end{vmatrix}^{\boxed{n}}
\cdot
\begin{vmatrix}
1 & 1 & 1 & 1 & \dots & 1 & 1 \\
1 & y & 1 & 0 & \dots & 0 & 0 \\
0 & 1 & y & 1 & \dots & 0 & 0 \\
0 & 0 & 1 & y & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & \dots & 1 & y
\end{vmatrix}^{\boxed{n+1}}
-
$$

$$
- \begin{vmatrix}
1 & 0 & 0 & \dots & 0 & 0 \\
1 & y & 1 & \dots & 0 & 0 \\
0 & 1 & y & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & y & 1 \\
0 & 0 & 0 & \dots & 1 & y
\end{vmatrix}^{\boxed{n}}
\cdot
\begin{vmatrix}
1 & 1 & 1 & 1 & \dots & 1 & 1 \\
1 & y & 1 & 0 & \dots & 0 & 0 \\
0 & 1 & y & 1 & \dots & 0 & 0 \\
0 & 0 & 1 & y & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & \dots & y & 1 \\
0 & 0 & 0 & 0 & \dots & 0 & 1
\end{vmatrix}^{\boxed{n+1}}
+
$$

$$
+ (-1)^{n+1} \begin{vmatrix}
1 & 1 & 1 & \dots & 1 & 1 \\
1 & y & 1 & \dots & 0 & 0 \\
0 & 1 & y & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & y & 1 \\
0 & 0 & 0 & \dots & 1 & y
\end{vmatrix}^{\boxed{n}}
\cdot
\begin{vmatrix}
1 & y & 1 & 0 & \dots & 0 & 0 \\
0 & 1 & y & 1 & \dots & 0 & 0 \\
0 & 0 & 1 & y & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & \dots & 1 & y \\
0 & 0 & 0 & 0 & \dots & 0 & 1
\end{vmatrix}^{\boxed{n+1}}
$$

$$
\begin{aligned}
&= g_n t_{n+1} - g_{n-1} t_n + (-1)^{n+1} t_n \\
&= g_n(g_n - g_{n-1} + t_{n-1}) - g_{n-1} t_n + (-1)^{n+1} t_n \\
&\overset{5)}{=} g_n(g_n - g_{n-1}) + g_n t_{n-1} - g_{n-1} t_n + (-1)^{n+1}[(-1)^n g_n t_{n-1} + (-1)^{n+1} g_{n-1} t_n] \\
&= g_n(g_n - g_{n-1}) \\
&= h_n
\end{aligned}
$$

8)Expand the determinant defining $t_{2n}$ by its last $n$ columns, by means of Laplace's formulae and then apply 6):

$$\left.\begin{aligned} J &= \{n+1, n+2, \cdots, 2n-1, 2n\} \\ J^{'} &= \{1, 2, \cdots, n\} \end{aligned}\right\} \Rightarrow \varepsilon(J) = (-1)^{n\cdot n} = (-1)^n$$

$$\left.\begin{aligned} K &= \{n+1, n+2, \cdots, 2n-1, 2n\} \\ K^{'} &= \{1, 2, \cdots, n\} \end{aligned}\right\} \Rightarrow \varepsilon(K) = (-1)^{n\cdot n} = (-1)^n$$

$$\left.\begin{aligned} K &= \{n, n+2, \cdots, 2n-1, 2n\} \\ K^{'} &= \{1, 2, \cdots, n-1, n+1\} \end{aligned}\right\} \Rightarrow \varepsilon(K) = (-1)^{n-1+n(n-1)} = (-1)^{n+1}$$

$$\left.\begin{aligned} K &= \{1, n+2, \cdots, 2n-1, 2n\} \\ K^{'} &= \{2, 3, \cdots, n, n+1\} \end{aligned}\right\} \Rightarrow \varepsilon(K) = (-1)^{n(n-1)} = 1$$

$t_{2n} =$

$$= \begin{vmatrix} 1 & \dots & 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 & \underline{\mathit{1}} \\ 1 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \underline{\mathit{2}} \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \underline{\mathit{3}} \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \underline{\mathit{4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & y & 1 & 0 & 0 & 0 & \dots & 0 & 0 & \underline{\mathit{n}} \\ 0 & \dots & 0 & 1 & y & 1 & 0 & 0 & \dots & 0 & 0 & \underline{\mathit{n+1}} \\ 0 & \dots & 0 & 0 & 1 & y & 1 & 0 & \dots & 0 & 0 & \underline{\mathit{n+2}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & y & 1 & \underline{\mathit{2n-1}} \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & y & \underline{\mathit{2n}} \\ \underline{\mathit{1}} & \dots & \underline{\mathit{n-1}} & \underline{\mathit{n}} & \underline{\mathit{n+1}} & \underline{\mathit{n+2}} & \underline{\mathit{n+3}} & \underline{\mathit{n+4}} & \dots & \underline{\mathit{2n-1}} & \underline{\mathit{2n}} & \end{vmatrix}^{\boxed{2n}}$$

$$= (-1)^n(-1)^n \begin{vmatrix} y & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & y & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & y & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & y \end{vmatrix}^{\boxed{n}} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & y & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & y & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & y & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & y \end{vmatrix}^{\boxed{n}} +$$

$$+ (-1)^n(-1)^{n+1} \begin{vmatrix} \rightarrow 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & y & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & y & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & y \end{vmatrix}^{\boxed{n}} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & y & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & y & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & y & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & y & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \leftarrow \end{vmatrix}^{\boxed{n+1}} +$$

$$+\,(-1)^n \begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & y & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & y & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & y \end{vmatrix}^{\boxed{n}} \cdot \begin{vmatrix} 1 & y & 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & y & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & y & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & y \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix}^{\boxed{n}}$$

$$\begin{aligned}
&= g_n t_n - g_{n-1} t_{n-1} + (-1)^n t_n \\
&\overset{6)}{=} g_n t_n - g_{n-1} t_{n-1} + (-1)^n [(-1)^n g_{n-1} t_{n+1} + (-1)^{n+1} g_n t_n] \\
&= -g_{n-1} t_{n-1} + g_{n-1} t_{n+1} \\
&= g_{n-1}(t_{n+1} - t_{n-1}) \\
&= g_{n-1}(g_n - g_{n-1} + t_{n-1} - t_{n-1}) \\
&= g_{n-1}(g_n - g_{n-1}) \\
&= u_n
\end{aligned}$$

9)By 7) we have

$$\begin{aligned}
&h_n = t_{2n+1} = g_{2n} - g_{2n-1} + t_{2n-1} = g_{2n} - g_{2n-1} + h_{n-1} \\
&\Rightarrow h_n - h_{n-1} = g_{2n} - g_{2n-1} \\
&\Rightarrow g_n(g_n - g_{n-1}) - g_{n-1}(g_{n-1} - g_{n-2}) = g_{2n} - g_{2n-1}
\end{aligned}$$

10)By 8) we have

$$\begin{aligned}
&u_{n-1} - u_n = t_{2n-2} - t_{2n} = t_{2n-2} - (g_{2n-1} - g_{2n-2} + t_{2n-2}) = \\
&= g_{2n-2} - g_{2n-1} \\
&\Rightarrow g_{n-2}(g_{n-1} - g_{n-2}) - g_{n-1}(g_n - g_{n-1}) = g_{2n-2} - g_{2n-1}
\end{aligned}$$

11)By 9) and 10) we have

$$\begin{aligned}
3[g_n(g_n - g_{n-1}) - g_{n-1}(g_{n-1} - g_{n-2})] &= 3(g_{2n} - g_{2n-1}) \\
2[g_{n-2}(g_{n-1} - g_{n-2}) - g_{n-1}(g_n - g_{n-1})] &= 2(g_{2n-2} - g_{2n-1})
\end{aligned}$$

summing

$$\begin{aligned}
&3[g_n(g_n - g_{n-1}) - g_{n-1}(g_{n-1} - g_{n-2})] + 2[g_{n-2}(g_{n-1} - g_{n-2}) - g_{n-1}(g_n - g_{n-1})] = \\
&= 3(g_{2n} - g_{2n-1}) + 2(g_{2n-2} - g_{2n-1}) \\
&\Leftrightarrow (3g_n - 2g_{n-1})(g_n - g_{n-1}) - (3g_{n-1} - 2g_{n-2})(g_{n-1} - g_{n-2}) = \\
&= 3g_{2n} - 5g_{2n-1} + 2g_{2n-2}
\end{aligned}$$

12)

$$\begin{aligned}
&(3g_{n+1}+2g_{n-1}-5g_n)g_n-(3g_n+2g_{n-2}-5g_{n-1})g_{n-1}=\\
&=3(g_{n+1}g_n-g_ng_{n-1}-g_ng_n+g_{n-1}g_{n-1})+2(g_{n-1}g_n-g_{n-2}g_{n-1}-g_ng_n+g_{n-1}g_{n-1})\\
&=3[g_n(g_{n+1}-g_n)-g_{n-1}(g_n-g_{n-1})]+2[-g_n(g_n-g_{n-1})+g_{n-1}(g_{n-1}-g_{n-2})]\\
&=3[u_{n+1}-u_n]+2[-h_n+h_{n-1}]\\
&\overset{7),8)}{=}3[t_{2n+2}-t_{2n}]+2[-t_{2n+1}+t_{2n-1}]\\
&=3[g_{2n+1}-g_{2n}+t_{2n}-t_{2n}]+2[-g_{2n}+g_{2n-1}-t_{2n-1}+t_{2n-1}]\\
&=3g_{2n+1}-5g_{2n}+2g_{2n-1}
\end{aligned}$$

13)Observe that:

$$\begin{aligned}
\tilde{q}_{2n-1}&=3g_{2n-1}-11g_{2n-2}+6g_{2n-3}+(-8+12y)t_{2n-3}-12t_{2n-4}+7(-1)^{2n-1+2}\\
&=3g_{2n-1}-11g_{2n-2}+6g_{2n-3}+(-8+12y)(g_{2n-4}-t_{2n-4})-12t_{2n-4}-7\\
&=3g_{2n-1}-11g_{2n-2}+6g_{2n-3}-8g_{2n-4}-4t_{2n-4}+(12yg_{2n-4}-12yt_{2n-4}-7)
\end{aligned}$$

Analogously

$$\begin{aligned}
\tilde{q}_{2n+1}&=3g_{2n+1}-11g_{2n}+6g_{2n-1}-8g_{2n-2}-4t_{2n-2}+\\
&\quad+(12yg_{2n-2}-12yt_{2n-2}-7)\\
&=3g_{2n+1}-11g_{2n}+6g_{2n-1}-8g_{2n-2}-4(g_{2n-3}-g_{2n-4}+t_{2n-4})+\\
&\quad+(12yg_{2n-2}-12yg_{2n-3}+12yg_{2n-4}-12yt_{2n-4}-7)\\
&=(3g_{2n-1}-11g_{2n-2}+6g_{2n-3}-8g_{2n-4})-4t_{2n-4}+\\
&\quad+(12yg_{2n-4}-12yt_{2n-4}-7)-(3g_{2n-1}-11g_{2n-2}+6g_{2n-3}-8g_{2n-4})+\\
&\quad+(3g_{2n+1}-11g_{2n}+6g_{2n-1}-8g_{2n-2}-4g_{2n-3}+4g_{2n-4})+\\
&\quad+(12yg_{2n-2}-12yg_{2n-3})\\
&\quad=\tilde{q}_{2n-1}+3g_{2n+1}-11g_{2n}+3g_{2n-1}+12(yg_{2n-2}-g_{2n-3})+\\
&\quad+3g_{2n-2}+2g_{2n-3}-12(yg_{2n-3}-g_{2n-4})\\
\Rightarrow \tilde{q}_{2n+1}-\tilde{q}_{2n-1}&=3g_{2n+1}-11g_{2n}+3g_{2n-1}+12g_{2n-1}+3g_{2n-2}+2g_{2n-3}-12g_{2n-2}\\
&=3(yg_{2n}-g_{2n-1})-6g_{2n}-5g_{2n}+15g_{2n-1}-9g_{2n-2}+2g_{2n-3}\\
&=3(y-2)g_{2n}-5g_{2n}+12g_{2n-1}-9g_{2n-2}+2g_{2n-3}\\
&=3(y-2)g_{2n}-5(yg_{2n-1}-g_{2n-2})+5\cdot 2g_{2n-1}+2g_{2n-1}-9g_{2n-2}+2g_{2n-3}\\
&=(y-2)(3g_{2n})-5(y-2)g_{2n-1}+5g_{2n-2}+2g_{2n-1}-9g_{2n-2}+2g_{2n-3}\\
&=(y-2)(3g_{2n}-5g_{2n-1})+2(yg_{2n-2}-g_{2n-3})-4g_{2n-2}+2g_{2n-3}\\
&=(y-2)(3g_{2n}-5g_{2n-1})+2(y-2)g_{2n-2}\\
&=(y-2)(3g_{2n}-5g_{2n-1}+2g_{2n-2})
\end{aligned}$$

14)Observe that:

$$\begin{aligned}
\tilde{q}_{2n}&=3g_{2n}-11g_{2n-1}+6g_{2n-2}+(-8+12y)t_{2n-2}-12t_{2n-3}+7(-1)^{2n+1+1}\\
&=3g_{2n}-11g_{2n-1}+6g_{2n-2}+(-8+12y)(g_{2n-3}-t_{2n-3})-12t_{2n-3}+7\\
&=3g_{2n}-11g_{2n-1}+6g_{2n-2}-8g_{2n-3}-4t_{2n-3}+(12yg_{2n-3}-12yt_{2n-3}+7)
\end{aligned}$$

Analogously:

$$\begin{aligned}
\tilde{q}_{2n+2} &= 3g_{2n+2} - 11g_{2n+1} + 6g_{2n} - 8g_{2n-1} - 4t_{2n-1} + (12yg_{2n-1} - 12yt_{2n-1} + 7)\\
&= 3g_{2n+2} - 11g_{2n+1} + 6g_{2n} - 8g_{2n-1} - 4(g_{2n-2} - g_{2n-3} + t_{2n-3}) +\\
&+ (12yg_{2n-1} - 12yg_{2n-2} + 12yg_{2n-3} - 12yt_{2n-3} + 7)\\
&= 3g_{2n} - 11g_{2n-1} + 6g_{2n-2} - 8g_{2n-3} - 4t_{2n-3} +\\
&+ (12yg_{2n-3} - 12yt_{2n-3} + 7) - (3g_{2n} - 11g_{2n-1} + 6g_{2n-2} - 8g_{2n-3}) +\\
&+ (3g_{2n+2} - 11g_{2n+1} + 6g_{2n} - 8g_{2n-1} - 4g_{2n-2} + 4g_{2n-3}) +\\
&+ (12yg_{2n-1} - 12yg_{2n-2})\\
&= \tilde{q}_{2n} + 3g_{2n+2} - 11g_{2n+1} + 3g_{2n} + 3(4y+1)g_{2n-1} -\\
&- 2(6y+5)g_{2n-2} + 12g_{2n-3}\\
\Rightarrow \tilde{q}_{2n+2} - \tilde{q}_{2n} &= 3g_{2n+2} - 11g_{2n+1} + 3g_{2n} + 12(yg_{2n-1} - g_{2n-2}) +\\
&+ 3g_{2n-1} + 2g_{2n-2} - 12(yg_{2n-2} - g_{2n-3})\\
&= 3g_{2n+2} - 11g_{2n+1} + 3g_{2n} + 12g_{2n} + 3g_{2n-1} + 2g_{2n-2} - 12g_{2n-1}\\
&= 3g_{2n+2} - 11g_{2n+1} + 15g_{2n} - 9g_{2n-1} + 2g_{2n-2}\\
&= 3(yg_{2n+1} - g_{2n}) - 6g_{2n+1} - 5g_{2n+1} + 15g_{2n} - 9g_{2n-1} + 2g_{2n-1}\\
&= 3(y-2)g_{2n+1} - 5g_{2n+1} + 12g_{2n} - 9g_{2n-1} + 2g_{2n-2}\\
&= 3(y-2)g_{2n+1} - 5(yg_{2n} - g_{2n-1}) + 5\cdot 2g_{2n} + 2g_{2n} - 9g_{2n-1} + 2g_{2n-2}\\
&= 3(y-2)g_{2n+1} - 5(y-2)g_{2n} + 5g_{2n-1} + 2g_{2n} - 9g_{2n-1} + 2g_{2n-2}\\
&= (y-2)(3g_{2n+1} - 5g_{2n}) + 2(yg_{2n-1} - g_{2n-2}) - 4g_{2n-1} + 2g_{2n-2}\\
&= (y-2)(3g_{2n+1} - 5g_{2n} + 2g_{2n-1})
\end{aligned}$$

15)For $n = 2$ we have:

$$\begin{aligned}
\tilde{q}_{2\cdot 2-1} = \tilde{q}_3 &= 3g_3 - 11g_2 + 6g_1 + (-8+12y)t_1 - 12t_0 + 7(-1)^{2\cdot 2-1+2}\\
&= 3(y^3 - 2y) - 11(y^2 - 1) + 6\cdot y + (-8+12y)\cdot 1 - 12\cdot 0 + 7(-1)\\
&= 3y^3 - 11y^2 + 12y - 7\\
&= (y-2)(3y-2)(y-1)\\
&= (y-2)(3g_{2-1} - 2g_{2-2})(g_{2-1} - g_{2-2})
\end{aligned}$$

Suppose now the validity of 15) for $n = k$, that is:

$$\tilde{q}_{2k-1} = (y-2)(3g_{k-1} - 2g_{k-2})(g_{k-1} - g_{k-2}) \tag{36}$$

From 13) we have:

$$\begin{aligned}
\tilde{q}_{2k+1} &= \tilde{q}_{2k-1} + (y-2)(3g_{2k} - 5g_{2k-1} + 2g_{2k-2})\\
&\overset{11)}{=} \tilde{q}_{2k-1} + (y-2)[(3g_k - 2g_{k-1})(g_k - g_{k-1}) - (3g_{k-1} - 2g_{k-2})(g_{k-1} - g_{k-2})]\\
&\overset{(17)}{=} (y-2)(3g_{k-1} - 2g_{k-2})(g_{k-1} - g_{k-2}) +\\
&\quad + (y-2)[(3g_k - 2g_{k-1})(g_k - g_{k-1}) - (3g_{k-1} - 2g_{k-2})(gk - 1 - g_{k-2})]\\
&= (y-2)(3g_k - 2g_{k-1})(g_k - g_{k-1})
\end{aligned}$$

and (17) holds for $n = 2k + 1$.

16)For $n = 2$ we have:

$$\begin{aligned}\tilde{q}_{2\cdot 2} = \tilde{q}_4 &= 3g_4 - 11g_3 + 6g_2 + (-8+12y)t_2 - 12t_1 + 7(-1)^{4+2}\\ &= 3(y^4 - 3y^2 + 1) - 11(y^3 - 2y) + 6(y^2 - 1) +\\ &\quad + (-8+12y)(y-1) - 12\cdot 1 + 7\\ &= 3y^4 - 11y^3 + 9y^2 + 2y\\ &= (y-2)(3y^2 - 3 + 2 - 5y)y\\ &= (y-2)(3g_2 + 2g_0 - 5g_1)g_1\end{aligned}$$

Suppose now the validity of 16) for $n = k$,that is:

$$\tilde{q}_{2k} = (y-2)(3g_k + 2g_{k-2} - 5g_{k-1})g_{k-1} \tag{37}$$

From 14) we have:

$$\begin{aligned}\tilde{q}_{2k+2} &= \tilde{q}_{2k} + (y-2)(3g_{2k+1} - 5g_{2k} + 2g_{2k-1})\\ &\overset{12)}{=} \tilde{q}_{2k} + (y-2)[(3g_{k+1} + 2g_{k-1} - 5g_k)g_k - (3g_k + 2g_{k-2} - 5g_{k-1})g_{k-1}]\\ &\overset{(18)}{=} (y-2)(3g_k + 2g_{k-2} - 5g_{k-1})g_{k-1} +\\ &\quad + (y-2)[(3g_{k+1} + 2g_{k-1} - 5g_k)g_k - (3g_k + 2g_{k-2} - 5g_{k-1})g_{k-1}]\\ &= (y-2)(3g_{k+1} + 2g_{k-1} - 5g_k)g_k\end{aligned}$$

and (18) holds for $n = 2k$.

17)

$$\begin{aligned}(g_n - g_{n-1})^2 &= g_n(g_n - g_{n-1}) - g_{n-1}(g_n - g_{n-1})\\ &= h_n - u_n\\ &\overset{7),8)}{=} t_{2n+1} - t_{2n}\\ &= g_{2n} - t_{2n} - t_{2n}\\ &= g_{2n} - 2t_{2n}\end{aligned}$$

18)

$$\begin{aligned}(g_n - g_{n-1})^2 - (g_{n-1} - g_{n-2})^2 &\overset{17)}{=} (g_{2n} - 2t_{2n}) - (g_{2n-2} - 2t_{2n-2})\\ &= g_{2n} - 2(g_{2n-1} - g_{2n-2} + t_{2n-2}) - g_{2n-2} + 2t_{2n-2}\\ &= g_{2n} - 2g_{2n-1} + g_{2n-2}\end{aligned}$$

19)

$$\begin{aligned}&[(g_n - g_{n-1}) - (g_{n-1} - g_{n-2})]^2 =\\ &= (g_n - g_{n-1})^2 + (g_{n-1} - g_{n-2})^2 - 2g_n(g_{n-1} - g_{n-2}) + 2g_{n-1}(g_{n-1} - g_{n-2})\\ &= g_{2n} - 2t_{2n} + g_{2n-2} - 2t_{2n-2} + 2h_{n-1} - 2(yg_{n-1} - g_{n-2})(g_{n-1} - g_{n-2})\\ &\overset{7)}{=} g_{2n} + g_{2n-2} - 2t_{2n} + 2t_{2n-1} - 2t_{2n-2} - 2yg_{n-1}(g_{n-1} - g_{n-2}) + 2g_{n-2}(g_{n-1} - g_{n-2})\\ &= g_{2n} + g_{2n-2} - 2t_{2n} + 2t_{2n-1} - 2t_{2n-2} - 2yh_{n-1} + 2u_{n-1}\\ &\overset{7),8)}{=} g_{2n} + g_{2n-2} - 2t_{2n} - 2t_{2n-2} - 2yt_{2n-1} + 2t_{2n-2}\\ &= g_{2n} + g_{2n-2} - 2t_{2n} + 2t_{2n-1} - 2yt_{2n-1}\end{aligned}$$

20)

$$\begin{aligned}
&[(g_{n+1}-g_n)-(g_n-g_{n-1})]^2-[(g_n-g_{n-1})-(g_{n-1}-g_{n-2})]^2=\\
&\overset{19)}{=}(g_{2n+2}+g_{2n}-2t_{2n+2}+2t_{2n+1}-2yt_{2n+1})-\\
&\quad-(g_{2n}+g_{2n-2}-2t_{2n}+2t_{2n-1}-2yt_{2n-1})\\
&=g_{2n+2}+g_{2n}-2(g_{2n+1}-g_{2n}+t_{2n})+2(g_{2n}-t_{2n})-2y(g_{2n}-t_{2n})-\\
&\quad-g_{2n}-g_{2n-2}+2t_{2n}-2t_{2n-1}+2yt_{2n-1}\\
&=g_{2n+2}-2g_{2n+1}+4g_{2n}-2yg_{2n}-g_{2n-2}-2(g_{2n-1}-t_{2n-1})+\\
&\quad+2y(g_{2n-1}-t_{2n-1})-2t_{2n-1}+2yt_{2n-1}\\
&=g_{2n+2}-2g_{2n+1}-2yg_{2n}+4g_{2n}+2yg_{2n-1}-2g_{2n-1}-g_{2n-2}\\
&=g_{2n+2}-2g_{2n+1}-2(yg_{2n}-g_{2n-1})+4g_{2n}+2yg_{2n-1}-4g_{2n-1}-g_{2n-2}\\
&=g_{2n+2}-2g_{2n+1}-2g_{2n+1}+2(yg_{2n-1}-g_{2n-2})+4g_{2n}-4g_{2n-1}+g_{2n-2}\\
&=g_{2n+2}-4g_{2n+1}+2g_{2n}+4g_{2n}-4g_{2n-1}+g_{2n-2}\\
&=g_{2n+2}-4g_{2n+1}+6g_{2n}-4g_{2n-1}+g_{2n-2}
\end{aligned}$$

21)

$$\begin{aligned}
&\tilde r_{2n+2}-\tilde r_{2n}=\\
&=(yg_{2n+1}-3g_{2n+1}+yg_{2n}-yg_{2n-1}+g_{2n-1}+2y-3)-\\
&\quad-(yg_{2n-1}-3g_{2n-1}+yg_{2n-2}-yg_{2n-3}+g_{2n-3}+2y-3)\\
&=yg_{2n+1}-3g_{2n+1}+yg_{2n}-2yg_{2n-1}+4g_{2n-1}-yg_{2n-2}+yg_{2n-3}-g_{2n-3}\\
&=(y-2)g_{2n+1}-(yg_{2n}-g_{2n-1})+yg_{2n}-2yg_{2n-1}+4g_{2n-1}-yg_{2n-2}+yg_{2n-3}-g_{2n-3}\\
&=(y-2)g_{2n+1}-2(y-2)g_{2n-1}+g_{2n-1}-yg_{2n-2}+yg_{2n-3}-g_{2n-3}\\
&=(y-2)(g_{2n+1}-2g_{2n-1})+(yg_{2n-2}-g_{2n-3})-yg_{2n-2}+yg_{2n-3}-g_{2n-3}\\
&=(y-2)(g_{2n+1}-2g_{2n-1}+g_{2n-3})\\
&=(y-2)(yg_{2n}-g_{2n-1}-2g_{2n-1}+g_{2n-3})\\
&=(y-2)((y+2)g_{2n}-2g_{2n}-3g_{2n-1}+g_{2n-3})\\
&=(y-2)((y+2)g_{2n}-2(yg_{2n-1}-g_{2n-2})-3g_{2n-1}+g_{2n-3})\\
&=(y-2)((y+2)g_{2n}-2(y+2)g_{2n-1}+g_{2n-1}+2g_{2n-2}+g_{2n-3})\\
&=(y-2)((y+2)(g_{2n}-2g_{2n-1})+yg_{2n-2}-g_{2n-3}+2g_{2n-2}+g_{2n-3})\\
&=(y-2)(y+2)(g_{2n}-2g_{2n-1}+g_{2n-2})
\end{aligned}$$

22)

$$\begin{aligned}
&\tilde r_{2n+3}-\tilde r_{2n+1}=\\
&=(yg_{2n+2}-3g_{2n+2}+yg_{2n+1}-yg_{2n}+g_{2n}+2y-3)-\\
&\quad-(yg_{2n}-3g_{2n}+yg_{2n-1}-yg_{2n-2}+g_{2n-2}-2y+3)\\
&=yg_{2n+2}-3g_{2n+2}+yg_{2n+1}-2yg_{2n}+4g_{2n}-yg_{2n-1}+yg_{2n-2}-g_{2n-2}\\
&=(y+2)g_{2n+2}-5g_{2n+2}+yg_{2n+1}-2yg_{2n}+4g_{2n}-yg_{2n-1}+yg_{2n-2}-g_{2n-2}\\
&=(y+2)g_{2n+2}-5(yg_{2n+1}-g_{2n})+yg_{2n+1}-2yg_{2n}+4g_{2n}-yg_{2n-1}+yg_{2n-2}-g_{2n-2}
\end{aligned}$$

$= (y+2)g_{2n+2} - 4(y+2)g_{2n+1} + 8g_{2n+1} - 2yg_{2n} + 9g_{2n} - yg_{2n-1} + yg_{2n-2} - g_{2n-2}$
$= (y+2)(g_{2n+2} - 4g_{2n+1}) + 8(yg_{2n} - g_{2n-1}) - 2yg_{2n} + 9g_{2n} - yg_{2n-1} + yg_{2n-2} - g_{2n-2}$
$= (y+2)(g_{2n+2} - 4g_{2n+1}) + 6(y+2)g_{2n} - 3g_{2n} - yg_{2n-1} - 8g_{2n-1} + yg_{2n-2} - g_{2n-2}$
$= (y+2)(g_{2n+2} - 4g_{2n+1} + 6g_{2n}) - 3(yg_{2n-1} - g_{2n-2}) - yg_{2n-1} - 8g_{2n-1} + yg_{2n-2} - g_{2n-2}$
$= (y+2)(g_{2n+2} - 4g_{2n+1} + 6g_{2n}) - 4(y+2)g_{2n-1} + yg_{2n-2} + 2g_{2n-2}$
$= (y+2)(g_{2n+2} - 4g_{2n+1} + 6g_{2n} - 4g_{2n-1} + g_{2n-2})$

23)For $n = 1$ we have:

$$\begin{aligned}
\tilde{r}_{2\cdot 1} &= yg_{2\cdot 1-1} - 3g_{2\cdot 1-1} + yg_{2\cdot 1-2} - yg_{2\cdot 1-3} + g_{2\cdot 1-3} + (-3+2y)(-1)^{2\cdot 1} \\
&= y\cdot y - 3y + y\cdot 1 - y\cdot 0 + 0 - 3 + 2y \\
&= y^2 - 3 \\
&= (y-2)(y+2)(1-0)^2 + 1 \\
&= (y-2)(y+2)(g_{1-1} - g_{1-2})^2 + 1
\end{aligned}$$

Now if we suppose that 23) holds for $n = k$, that is:

$$\tilde{r}_{2k} = (y-2)(y+2)(g_{k-1} - g_{k-2})^2 + 1 \tag{38}$$

Results:

$$\begin{aligned}
\tilde{r}_{2k+2} &\overset{21)}{=} \tilde{r}_{2k} + (y-2)(y+2)(g_{2k} - 2g_{2k-1} + g_{2k-2}) \\
&\overset{18),(19)}{=} (y-2)(y+2)(g_{k-1} - g_{k-2})^2 + 1 + (y-2)(y+2)[(g_k - g_{k-1})^2 - (g_{k-1} - g_{k-2})^2] \\
&= (y-2)(y+2)(g_k - g_{k-1})^2 + 1
\end{aligned}$$

and (19) holds for n=k+1.

24) For $n = 1$ we have:

$$\begin{aligned}
\tilde{r}_{2\cdot 1+1} &= yg_{2\cdot 1} - 3g_{2\cdot 1} + yg_{2\cdot 1-1} - yg_{2\cdot 1-2} + g_{2\cdot 1-2} + (-3+2y)(-1)^{2\cdot 1+1} \\
&= y\cdot(y^2-1) - 3\cdot(y^2-1) + y\cdot y - y\cdot 1 + 1 + 3 - 2y \\
&= y^3 - 2y^2 - 4y + 7 \\
&= (y+2)(y-2)^2 - 1 \\
&= (y+2)(y-2)(g_1 - 2g_0 + g_{-1})^2 - 1
\end{aligned}$$

Now suppose that 24) holds for $n = k$:

$$\tilde{r}_{2k+1} = (y+2)(g_k - 2g_{k-1} + g_{k-2})^2 - 1 \tag{39}$$

Then:

$$\begin{aligned}
\tilde{r}_{2k+3} &\overset{22)}{=} \tilde{r}_{2k+1} + (y+2)(g_{2k+2} - 4g_{2k+1} + 6g_{2k} - 4g_{2k-1} + g_{2k-2}) \\
&\overset{20),(20)}{=} (y+2)(g_k - 2g_{k-1} + g_{k-2})^2 - 1 + \\
&\quad + (y+2)\{[(g_{k+1} - g_k) - (g_k - g_{k-1})]^2 - [(g_k - g_{k-1}) - (g_{k-1} - g_{k-2})]^2\} \\
&= (y+2)(g_k - 2g_{k-1} + g_{k-2})^2 - 1 + \\
&\quad + (y+2)[(g_{k+1} - 2g_k + g_{k-1})^2 - (g_k - 2g_{k-1} + g_{k-2})^2] \\
&= (y+2)(g_{k+1} - 2g_k + g_{k-1})^2 - 1
\end{aligned}$$

and (20) holds for $n = k + 1$.

25) For $n = 0$ it is trivial.
Suppose that 25) holds for $n = k$, the induction step follows from:

$$\begin{aligned}
&g_k^2 - g_{k-1}g_{k+1} = 1\\
&\Rightarrow g_{k+1}(-g_{k+1} + g_{k+1} - g_{k-1}) + g_k^2 = 1\\
&\Rightarrow g_{k+1}(-yg_k + g_{k-1} + g_{k+1} - g_{k-1}) + g_k^2 = 1\\
&\Rightarrow g_{k+1}(g_{k+1} - yg_k) + g_k^2 = 1\\
&\Rightarrow g_{k+1}^2 + g_k^2 - yg_kg_{k+1} = 1\\
&\Rightarrow g_{k+1}^2 - g_k(yg_{k+1} - g_k) = 1\\
&\Rightarrow g_{k+1}^2 - g_kg_{k+2} = 1
\end{aligned}$$

and 25) holds for $n = k + 1$.

26) Using 25) we get:

$$\begin{aligned}
&g_n \cdot g_{n-2} = g_{n-1}^2 - 1\\
&\Leftrightarrow g_n^2 + g_{n-2}^2 - 4g_{n-1}^2 + 2g_ng_{n-2} + 4 = g_n^2 + g_{n-2}^2 - 2g_ng_{n-2}\\
&\Leftrightarrow (g_n + 2g_{n-1} + g_{n-2})(g_n - 2g_{n-1} + g_{n-2}) + 4 = (g_n - g_{n-2})^2\\
&\Leftrightarrow (yg_{n-1} - g_{n-2} + g_{n-2} + 2g_{n-1})(yg_{n-1} - g_{n-2} + g_{n-2} - 2g_{n-1}) + 4 = (g_n - g_{n-2})^2\\
&\Leftrightarrow (yg_{n-1} + 2g_{n-1})(yg_{n-1} - 2g_{n-1}) + 4 = (g_n - g_{n-2})^2\\
&\Leftrightarrow (y + 2)(y - 2)g_{n-1}^2 + 4 = (g_n - g_{n-2})^2
\end{aligned}$$

27) Using 25) we get:

$$\begin{aligned}
&g_{n-1}^2 = g_ng_{n-2} + 1\\
&\Leftrightarrow g_{n-1}^2 + g_{n-2}^2 = g_ng_{n-2} + g_{n-2}^2 + 1\\
&\Leftrightarrow g_{n-1}^2 + g_{n-2}^2 = (yg_{n-1} - g_{n-2} + g_{n-2})g_{n-2} + 1\\
&\Leftrightarrow g_{n-1}^2 + g_{n-2}^2 = yg_{n-1}g_{n-2} + 1\\
&\Leftrightarrow yg_{n-1}^2 + yg_{n-2}^2 - 2yg_{n-1}g_{n-2} + 2g_{n-1}^2 + 2g_{n-2}^2 - 4g_{n-1}g_{n-2}\\
&= yg_{n-1}^2 + yg_{n-2}^2 + 2yg_{n-1}g_{n-2} - 2g_{n-1}^2 - 2g_{n-2}^2 - 4g_{n-1}g_{n-2} + 4\\
&\Leftrightarrow (y + 2)(g_{n-1} - g_{n-2})^2 = (y - 2)(g_{n-1} + g_{n-2})^2 + 4
\end{aligned}$$

□

Now we prove Proposition 5.2

*Proof.* :
1) $\boxed{d = 1}$
In this case

$$A_1^d = A_1^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_1^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

By induction

$$A_1^{md} = A_1^{m\cdot 1} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \Rightarrow A_1^{md-1} = \begin{pmatrix} 1 & m-1 \\ 0 & 1 \end{pmatrix}$$

As

$$A_2^{2\cdot d+2} = A_2^4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^4 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$$

we get

$$A_1^{md-1} \cdot A_2^{2d+2} = \begin{pmatrix} 3m-1 & 5m-2 \\ 3 & 5 \end{pmatrix}$$

So

$$\text{Det}[A_1^{md-1} \cdot A_2^{2d+2} - xId] = \begin{vmatrix} 3m-1-x & 5m-2 \\ 3 & 5-x \end{vmatrix} = m \cdot \{\frac{1}{m}(x^2-4x+1) + (-3x)\}$$

2) $\boxed{d = 2}$
Now

$$A_1^d = A_1^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

And by induction

$$A_1^{md} = A_1^{m\cdot 2} = \begin{pmatrix} 1 & m & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow A_1^{md-1} = \begin{pmatrix} 1 & m-1 & m \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

But

$$A_2^{2\cdot d+2} = A_2^6 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^6 = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$$

So we get

$$A_1^{md-1} \cdot A_2^{2d+2} = \begin{pmatrix} 3m-1 & 4m-1 & 5m-1 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

And finally

$$\text{Det}[A_1^{md-1} \cdot A_2^{2d+2} - xId] = \begin{vmatrix} 3m-1-x & 4m-1 & 5m-1 \\ 1 & 2-x & 2 \\ 2 & 2 & 3-x \end{vmatrix} =$$
$$= m \cdot \left\{ \frac{1}{m}(-x^3 + 4x^2 - 1) + x(3x-1) \right\}$$

3) $\boxed{d \geq 3}$
Let us first prove some preliminary identities marked by $i)$, $ii)$,and $iii)$:

*i*)

$$\left|\begin{array}{cccccccccc}
\overset{\downarrow}{y} & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1\\
1 & y & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 1 & y & 1 & \dots & 0 & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & \dots & 1 & y & 1 & 0 & 0\\
0 & 0 & 0 & 0 & \dots & 0 & 1 & y & 1 & 0\\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & y & 1\\
1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 1+y
\end{array}\right|^{\boxed{d}} =$$

$$= y\cdot\left|\begin{array}{cccccccc}
y & 1 & 0 & \dots & 0 & 0 & 0 & \overset{\downarrow}{0}\\
1 & y & 1 & \dots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \dots & y & 1 & 0 & 0\\
0 & 0 & 0 & \dots & 1 & y & 1 & 0\\
0 & 0 & 0 & \dots & 0 & 1 & y & 1\\
0 & 0 & 0 & \dots & 0 & 0 & 1 & 1+y
\end{array}\right|^{\boxed{d-1}} - \left|\begin{array}{cccccccc}
\rightarrow 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1\\
1 & y & 1 & \dots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \dots & y & 1 & 0 & 0\\
0 & 0 & 0 & \dots & 1 & y & 1 & 0\\
0 & 0 & 0 & \dots & 0 & 1 & y & 1\\
0 & 0 & 0 & \dots & 0 & 0 & 1 & 1+y
\end{array}\right|^{\boxed{d-1}} +$$

$$+\,(1)^{d+1}\left|\begin{array}{cccccccc}
\rightarrow 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1\\
y & 1 & 0 & \dots & 0 & 0 & 0 & 0\\
1 & y & 1 & \dots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \dots & y & 1 & 0 & 0\\
0 & 0 & 0 & \dots & 1 & y & 1 & 0\\
0 & 0 & 0 & \dots & 0 & 1 & y & 1
\end{array}\right|^{\boxed{d-1}} =$$

$$= y\left\{\left|\begin{array}{cccccccc}
y & 1 & 0 & \dots & 0 & 0 & 0 & \overset{\downarrow}{0}\\
1 & y & 1 & \dots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \dots & 1 & y & 1 & 0\\
0 & 0 & 0 & \dots & 0 & 1 & y & 1\\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 1
\end{array}\right|^{\boxed{d-2}} + (1+y)\left|\begin{array}{cccccccc}
y & 1 & 0 & \dots & 0 & 0 & 0 & 0\\
1 & y & 1 & \dots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \dots & 1 & y & 1 & 0\\
0 & 0 & 0 & \dots & 0 & 1 & y & 1\\
0 & 0 & 0 & \dots & 0 & 0 & 1 & y
\end{array}\right|^{\boxed{d-2}}\right\} -$$

$$-\left\{\left|\begin{array}{cccccccc}
y & 1 & \dots & 0 & 0 & 0 & 0 & \overset{\downarrow}{0}\\
1 & y & \dots & 0 & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & \dots & 1 & y & 1 & 0 & 0\\
0 & 0 & \dots & 0 & 1 & y & 1 & 0\\
0 & 0 & \dots & 0 & 0 & 1 & y & 1\\
0 & 0 & \dots & 0 & 0 & 0 & 1 & 1+y
\end{array}\right|^{\boxed{d-2}} + (-1)^{d}\left|\begin{array}{cccccccc}
1 & y & 1 & \dots & 0 & 0 & 0 & 0\\
0 & 1 & y & \dots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \dots & 1 & y & 1 & 0\\
0 & 0 & 0 & \dots & 0 & 1 & y & 1\\
0 & 0 & 0 & \dots & 0 & 0 & 1 & y\\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 1
\end{array}\right|^{\boxed{d-2}}\right\} +$$

$$+(-1)^{d+1}\left\{\begin{vmatrix}1 & 0 & \dots & 0 & 0 & 0\\ y & 1 & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & y & 1 & 0\\ 0 & 0 & \dots & 1 & y & 1\end{vmatrix}^{\boxed{d-2}} + (-1)^d\begin{vmatrix}y & 1 & 0 & \dots & 0 & 0 & 0\\ 1 & y & 1 & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1 & y & 1\\ 0 & 0 & 0 & \dots & 0 & 1 & y\end{vmatrix}^{\boxed{d-2}}\right\} =$$

$$= y\left\{-\begin{vmatrix}y & 1 & 0 & \dots & 0 & 0 & 0\\ 1 & y & 1 & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1 & y & 1\\ 0 & 0 & 0 & \dots & 0 & 1 & y\end{vmatrix}^{\boxed{d-3}} + (1+y)g_{d-2}(y)\right\} -$$

$$-\left\{-\begin{vmatrix}y & 1 & \dots & 0 & 0 & 0 & \overset{\downarrow}{0}\\ 1 & y & \dots & 0 & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1 & y & 1 & 0\\ 0 & 0 & \dots & 0 & 1 & y & 1\\ 0 & 0 & \dots & 0 & 0 & 0 & 1\end{vmatrix}^{\boxed{d-3}} + (1+y)\begin{vmatrix}y & 1 & \dots & 0 & 0 & 0 & 0\\ 1 & y & \dots & 0 & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1 & y & 1 & 0\\ 0 & 0 & \dots & 0 & 1 & y & 1\\ 0 & 0 & \dots & 0 & 0 & 1 & y\end{vmatrix}^{\boxed{d-3}} + (-1)^d\right\} +$$

$$+(-1)^{d+1}\left\{1+(-1)^d g_{d-2}(y)\right\} =$$

$$= y\left\{-g_{d-3}(y)+(1+y)g_{d-2}(y)\right\} - \left\{-g_{d-4}(y)+(1+y)g_{d-3}(y)+(-1)^d\right\} +$$

$$+(-1)^{d+1}\left\{1+(-1)^d g_{d-2}(y)\right\} \quad \text{for} \quad d \geq 3$$

*ii*)

$$\begin{vmatrix} 4\overset{\downarrow}{m}-1 & 4m & 4m & 4m & \dots & 4m & 4m & 4m & 4m & 5m-1 \\ 1 & y & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & y & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \hdotsfor{10} \\ 0 & 0 & 0 & 0 & \dots & 1 & y & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & y & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & y & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 1+y \end{vmatrix}^{\boxed{d}} =$$

$$= (4m-1) \begin{vmatrix} y & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \overset{\downarrow}{0} \\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \hdotsfor{9} \\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & y & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 1+y \end{vmatrix}^{\boxed{d-1}} -$$

$$- \begin{vmatrix} 4m & 4m & 4m & \dots & 4m & 4m & 4m & 4m & 5\overset{\downarrow}{m}-1 \\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \hdotsfor{9} \\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & y & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 1+y \end{vmatrix}^{\boxed{d-1}} +$$

$$+ (-1)^{d+1} \begin{vmatrix} 4m & 4m & 4m & \dots & 4m & 4m & 4m & 5\overset{\downarrow}{m}-1 \\ y & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0 \\ \hdotsfor{8} \\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1 \end{vmatrix}^{\boxed{d-1}} =$$

$$= (4m-1) \left\{ - \begin{vmatrix} y & 1 & 0 & \dots & 0 & 0 & 0 & \overset{\downarrow}{0} \\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0 \\ \hdotsfor{8} \\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{vmatrix}^{\boxed{d-2}} + (1+y) \begin{vmatrix} y & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0 \\ \hdotsfor{8} \\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & y \end{vmatrix}^{\boxed{d-2}} \right\} -$$

$$
-\left\{(-1)^d(5m-1)\begin{vmatrix}1 & y & 1 & \dots & 0 & 0 & 0 & 0\\ 0 & 1 & y & \dots & 0 & 0 & 0 & 0\\ \hdotsfor{8}\\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0\\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1\\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & y\\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1\end{vmatrix}^{\boxed{d-2}} - \right.
$$

$$
-\begin{vmatrix}4m & 4m & 4m & \dots & 4m & 4m & 4m & \overset{\downarrow}{4m}\\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0\\ \hdotsfor{8}\\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0\\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1\\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1\end{vmatrix}^{\boxed{d-2}} +
$$

$$
\left. +(1+y)\begin{vmatrix}4m & 4m & 4m & \dots & 4m & 4m & 4m & 4m\\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0\\ \hdotsfor{8}\\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0\\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1\\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & y\end{vmatrix}^{\boxed{d-2}} \right\} +
$$

$$
+(-1)^{d+1}\left\{(-1)^d(5m-1)\begin{vmatrix}y & 1 & 0 & \dots & 0 & 0 & 0 & 0\\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0\\ \hdotsfor{8}\\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0\\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1\\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & y\end{vmatrix}^{\boxed{d-2}} + \right.
$$

$$
\left. +\begin{vmatrix}4m & 4m & 4m & \dots & 4m & 4m & 4m & 4m\\ y & 1 & 0 & \dots & 0 & 0 & 0 & 0\\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0\\ \hdotsfor{8}\\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0\\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1\end{vmatrix}^{\boxed{d-2}} \right\} =
$$

$$
\begin{aligned}
&= (4m-1)\left\{-g_{d-3}(y) + (1+y)g_{d-2}(y)\right\} - \\
&- \left\{(-1)^d(5m-1) - 4m\cdot t_{d-3}(y) + (1+y)(4m)t_{d-2}(y)\right\} + \\
&+ (-1)^{d+1}\left\{(-1)^d(5m-1)g_{d-2}(y) + (4m)k_{d-2}(y)\right\} \quad \text{for} \quad d \geq 3
\end{aligned}
$$

*iii)*

$$
\begin{vmatrix}
4m-1 & 4m & 4m & 4m & \dots & 4m & 4m & 4m & 4m & \overset{\downarrow}{5m}-1 \\
y & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \\
1 & y & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & y & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\
\hdotsfor{10} \\
0 & 0 & 0 & 0 & \dots & 1 & y & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 1 & y & 1 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & y & 1
\end{vmatrix}^{\boxed{d}} =
$$

$$
= (-1)^{d+1}(5m-1)
\begin{vmatrix}
y & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
1 & y & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 1 & y & 1 & \dots & 0 & 0 & 0 & 0 \\
\hdotsfor{9} \\
0 & 0 & 0 & 0 & \dots & 1 & y & 1 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 1 & y & 1 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & y
\end{vmatrix}^{\boxed{d-1}} +
$$

$$
+ (-1)^{d+2}
\begin{vmatrix}
\overset{\downarrow}{4m}-1 & 4m & 4m & 4m & \dots & 4m & 4m & 4m & 4m \\
1 & y & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 1 & y & 1 & \dots & 0 & 0 & 0 & 0 \\
\hdotsfor{9} \\
0 & 0 & 0 & 0 & \dots & 1 & y & 1 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 1 & y & 1 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & y
\end{vmatrix}^{\boxed{d-1}} +
$$

$$
+
\begin{vmatrix}
4m-1 & 4m & 4m & 4m & \dots & 4m & 4m & 4m & 4m \\
y & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
1 & y & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 1 & y & 1 & \dots & 0 & 0 & 0 & 0 \\
\hdotsfor{9} \\
0 & 0 & 0 & 0 & \dots & 1 & y & 1 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 1 & y & 1
\end{vmatrix}^{\boxed{d-1}} =
$$

$$= (-1)^{d+1}(5m-1)g_{d-1}(y) + (-1)^{d+2}\left\{ (4m-1)\begin{vmatrix} y & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & y \end{vmatrix}^{\boxed{d-2}} - \right.$$

$$\left. - \begin{vmatrix} 4m & 4m & 4m & \dots & 4m & 4m & 4m & 4m \\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & y \end{vmatrix}^{\boxed{d-2}} \right\} +$$

$$+ \left\{ (4m-1)\begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ y & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1 \end{vmatrix}^{\boxed{d-2}} - \right.$$

$$- y \begin{vmatrix} 4m & 4m & 4m & \dots & 4m & 4m & 4m & 4m \\ y & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1 \end{vmatrix}^{\boxed{d-2}} +$$

$$\left. + \begin{vmatrix} \overset{\downarrow}{4m} & 4m & 4m & \dots & 4m & 4m & 4m & 4m \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & y & 1 & \dots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & 1 & y & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & y & 1 \end{vmatrix}^{\boxed{d-2}} \right\} =$$

$$= (-1)^{d+1}(5m-1)g_{d-1}(y) + (-1)^{d+2}\{(4m-1)g_{d-2}(y) - (4m)t_{d-2}(y)\} +$$
$$+ \{(4m-1) - y(4m)k_{d-2}(y) - (4m)k_{d-3}(y)\} \quad \text{for} \quad d \geq 3$$

4)Now combining $i), ii), iii)$ and making $2 - x = y$ we get:

$\mathrm{Det}[A_1^{md-1}A_2^{2d+2} - xId] =$

$$= \begin{vmatrix} \overset{\downarrow}{3m-1-x} & 4m-1 & 4m & 4m & 4m & \dots & 4m & 4m & 4m & 4m & 5m-1 \\ 1 & 2-x & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2-x & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2-x & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 2-x & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2-x & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 2-x & 1 \\ 2 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 3-x \end{vmatrix}^{\boxed{d+1}}$$

$$= (3m-1-x)\begin{vmatrix} y & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \\ 1 & y & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & y & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \dots & 1 & y & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & y & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & y & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 1+y \end{vmatrix}^{\boxed{d}} -$$

$$- \begin{vmatrix} 4m-1 & 4m & 4m & 4m & \dots & 4m & 4m & 4m & 4m & 5m-1 \\ 1 & y & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & y & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \dots & 1 & y & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & y & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & y & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 1+y \end{vmatrix}^{\boxed{d}} +$$

$$+ (-1)^{d+2}\cdot 2\begin{vmatrix} 4m-1 & 4m & 4m & 4m & \dots & 4m & 4m & 4m & 4m & 5m-1 \\ y & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \\ 1 & y & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & y & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \dots & 1 & y & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & y & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & y & 1 \end{vmatrix}^{\boxed{d}} =$$

$$
\begin{aligned}
&= (3m-1-x)\left\{y[-g_{d-3}(y)+(1+y)g_{d-2}(y)]-[-g_{d-4}(y)+(1+y)g_{d-3}(y)+(-1)^d]+\right.\\
&+\left.(-1)^{d+1}[1+(-1)^d g_{d-2}(y)]\right\}-\\
&-\{(4m-1)[-g_{d-3}(y)+(1+y)g_{d-2}(y)]-\\
&-[(-1)^d(5m-1)-(4m)t_{d-3}(y)+(4m)(1+y)t_{d-2}(y)]+\\
&+(-1)^{d+1}[(-1)^d\,(5m-1)g_{d-2}(y)+(4m)k_{d-2}(y)]\}+\\
&+(-1)^{d+2}\cdot 2\left\{(-1)^{d+1}(5m-1)g_{d-1}(y)+(-1)^{d+2}[(4m-1)g_{d-2}(y)-(4m)t_{d-2}(y)]+\right.\\
&+\left.[(4m-1)-y(4m)k_{d-2}(y)-(4m)k_{d-3}(y)]\right\}=\\
&= (3m-1-x)\left\{y[g_{d-1}(y)+g_{d-2}(y)]-[g_{d-3}(y)+g_{d-2}(y)+(-1)^d]+\right.\\
&+\left.(-1)^{d+1}[1+(-1)^d g_{d-2}(y)]\right\}-\\
&-\left\{(4m-1)[g_{d-1}(y)+g_{d-2}(y)]-[(-1)^d(5m-1)-(4m)t_{d-3}(y)+4m(1+y)t_{d-2}(y)]+\right.\\
&+\left.(-1)^{d+1}[(-1)^d(5m-1)g_{d-2}(y)+(4m)k_{d-2}(y)]\right\}+\\
&+(-1)^{d+2}\cdot 2\left\{\,(-1)^{d+1}(5m-1)g_{d-1}(y)+(-1)^{d+2}[(4m-1)g_{d-2}(y)-(4m)t_{d-2}(y)]+\right.\\
&+\left.[(4m-1)-y(4m)k_{d-2}(y)-(4m)k_{d-3}(y)]\right\}=\\
&= (3m-1-x)\left\{g_d(y)+g_{d-1}(y)-g_{d-2}(y)+2(-1)^{d+1}\right\}-\\
&-\left\{(4m-1)g_{d-1}(y)+(4m-1)g_{d-2}(y)+(-1)^{d+1}(5m-1)+(4m)t_{d-3}(y)+\right.\\
&+\left.4m(-1-y)t_{d-2}(y)-(5m-1)g_{d-2}(y)+(4m)(-1)^{d+1}(-1)^{d-2+1}t_{d-2}(y)\right\}+\\
&+(-10m+2)g_{d-1}(y)+(8m-2)g_{d-2}(y)-(8m)t_{d-2}(y)+(-8m+2)(-1)^{d+1}+\\
&+2(-1)^{d+1}y(4m)(-1)^{d-2+1}t_{d-2}(y)+2(-1)^{d+1}(4m)(-1)^{d-3+1}t_{d-3}(y)=\\
&= (3m-1-x)g_d(y)+(3m-1-x-4m+1-10m+2)g_{d-1}(y)+\\
&+(-3m+1+x-4m+1+5m-1+8m-2)g_{d-2}(y)+\\
&+(4m+4my-4m-8m+8my)t_{d-2}(y)+(-4m-4m\cdot 2)t_{d-3}(y)+\\
&+(6m-5m-8m)(-1)^{d+1}+(-2-2x+1+2)(-1)^{d+1}=\\
&= -(1+x)g_d(y)+(2-x)g_{d-1}(y)+(-1+x)g_{d-2}(y)+(1-2x)(-1)^{d+1}+\\
&+m\cdot[3g_d(y)-11g_{d-1}(y)+6g_{d-2}(y)+(-8+12y)t_{d-2}(y)-12t_{d-3}(y)+7(-1)^{d+2}]\\
&= -(1+x)g_d(2-x)+(2-x)g_{d-1}(2-x)+(-1+x)g_{d-2}(2-x)+(1-2x)(-1)^{d+1}+\\
&+m\cdot[3g_d(2-x)-11g_{d-1}(2-x)+6g_{d-2}(2-x)+4(4-3x)t_{d-2}(2-x)-\\
&-12t_{d-3}(2-x)+7(-1)^{d+2}]\\
&= (2-x)g_d(2-x)-3g_d(2-x)+(2-x)g_{d-1}(2-x)-(2-x)g_{d-2}(2-x)+\\
&+g_{d-2}(2-x)+(-3+2(2-x))(-1)^{d+1}+m\cdot\left[3g_d(2-x)-11g_{d-1}(2-x)+\right.\\
&+\left.6g_{d-2}(2-x)+(-8+12(2-x))t_{d-2}(2-x)-12t_{d-3}(2-x)+7(-1)^{d+2}\right],\quad \text{for}\quad d\geq 3
\end{aligned}
$$

□

## 11. Appendix E: Subspaces convergence

Consider $\mathbb{R}^n$ endowed with the standard scalar product and the corresponding norm $\| x \|= (\sum_{i=1}^n | x_i |^2)^{\frac{1}{2}}$. The norm of an $n \times n$ matrix id defined accordingly:

$$\| A \|= \max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\| Ax \|}{\| x \|}$$

Now we introduce a concept that serves as a measure of distance between subspaces (see [10]).

**Definition 11.1.** The *gap between subspaces* $\mathscr{L}$ and $\mathscr{M}$ in $\mathbb{R}^n$ is defined as

$$\theta(\mathscr{L},\mathscr{M}) =\| P_{\mathscr{M}} - P_{\mathscr{L}} \|$$

where $P_{\mathscr{L}}$ and $P_{\mathscr{M}}$ are the orthogonal projectors on $\mathscr{L}$ and $\mathscr{M}$, respectively.

It is easy from the definition to prove that $\theta(\mathscr{L},\mathscr{M})$ is a metric. Note that $\theta(\mathscr{L},\mathscr{M}) \leq 1$.

In the following paragraphs we denote by $S_{\mathscr{L}}$ the unit sphere in a subspace $\mathscr{L} \subset \mathbb{R}^n$. And the distance from $x \in \mathbb{R}^n$ to a set $Z \subset \mathbb{R}^n$ is defined as usual by

$$d(x, Z) = \inf_{t\in Z} \| \ x - t \|$$

**Theorem 11.1.** *Let $\mathscr{M}$, $\mathscr{L}$ be subspaces in $\mathbb{R}^n$. Then*

$$\theta(\mathscr{L},\mathscr{U}) = max\left\{ \underset{x\in S_{\mathscr{U}}}{sup}\ d(x,\mathscr{L}),\ \underset{x\in S_{\mathscr{L}}}{sup}\ d(x,\mathscr{U})\right\}$$

*If exactly one of the subspaces $\mathscr{L}$ and $\mathscr{M}$ is the zero subspace, then the right hand side is interpreted as* 1*; if $\mathscr{L} = \mathscr{M} = 0$ then the right hand side is interpreted as* 0.

**Proposition 11.1.** *Suppose that the sequences of vectors $\{v_1^k\}_{k\in\mathbb{N}}, \cdots, \{v_n^k\}_{k\in\mathbb{N}}$ and $w_1, \cdots, w_n$ are such that:*

$$(i) v_1^k, \cdots, v_n^k \quad \textit{is a linearly independent system} \quad \forall k \in \mathbb{N} \quad \textit{and}$$
$$\| v_1^k \|= \cdots =\| v_n^k \|= 1$$
$$(ii) w_1, \cdots, w_n \quad \textit{is a linearly independent system} \quad \textit{and}$$
$$\| w_1 \|= \cdots =\| w_n \|= 1$$
$$(iii) v_1^k \underset{k}{\rightarrow} w_1, \cdots, v_n^k \underset{k}{\rightarrow} w_n$$

*Let $V^k$ and $W$ be the subspaces defined by $V^k =< v_1^k, \cdots, v_n^k >$, $W =< w_1, \cdots, w_n >$*
*Then*

$$\theta(V^k, W) \xrightarrow[k\to\infty]{} 0$$

*Proof.* Observe that

$$\theta(V^k, W) = \max\left\{ \sup_{x\in S_{V^k}} d(x, S_W), \sup_{x\in S_W} d(x, S_{V^k})\right\}$$

Choose $x \in S_{V^k}$ and $\bar{x} \in S_W$ with:

$$x = \alpha_1 \cdot v_1^k + \cdots + \alpha_n \cdot v_n^k$$
$$\bar{x} = \alpha_1 \cdot w_1 + \cdots + \alpha_n \cdot w_n$$

Then:

$$\left\{\begin{array}{l} d(x,S_W)\leq d(x,\bar{x})\leq\parallel \alpha_1(v_1^k-w_1)+\cdots+\alpha_n(v_n^k-w_n)\parallel\leq \\ \qquad\leq\mid\alpha_1\mid\parallel v_1^k-w_1\parallel+\cdots+\mid\alpha_n\mid\parallel v_n^k-w_n\parallel \end{array}\right. \tag{40}$$

Observing that $\parallel v_1^k-w_1\parallel\underset{k}{\rightarrow}0,\cdots,\parallel v_n^k\underset{k}{-}w_n\parallel\rightarrow 0$ we perceive it suffice to prove that $\alpha_1\cdots,\alpha_n$ are limited. And we have the hypotheses that

$$\parallel\alpha_1\cdot v_1^k+\cdots+\alpha_n\cdot v_n^k\parallel=1$$

Using these fact and(33) will be easy to prove that $d(x,S_W)\rightarrow 0$ and also that $\sup\limits_{x\in S_{V^k}} d(x,S_W)\rightarrow 0$.
Now we prove the convergence using just the definition of $\theta(V^k,W)$.
Let $w_1,\cdots,w_n$ be a base of $W$ and $w_{n+1},\cdots,w_{n+m}$ a base of $<w_1,\cdots,w_n>^{\perp}$.
We know that

$$\left\{\begin{array}{c} v_1^k\rightarrow w_1 \\ \vdots \\ v_n^k\rightarrow w_n \end{array}\right\}\Leftrightarrow \begin{array}{c} \parallel v_1^k-w_1\parallel\rightarrow 0 \\ \vdots \\ \parallel v_n^k-w_n\parallel\rightarrow 0 \end{array}$$

There exist some scalars $\lambda_{ij}^k,\quad k=1,2\cdots,\quad,1\leq i\leq n,\quad,1\leq j\leq n+m$ such that:

$$\left\{\begin{array}{rl} v_1^k &= \lambda_{11}^k w_1+\cdots+\lambda_{1n}^k w_n+\lambda_{1n+1}^k w_{n+1}+\cdots+\lambda_{1n+m}^k w_{n+m} \\ \vdots & \vdots \\ v_n^k &= \lambda_{n1}^k w_1+\cdots+\lambda_{nn}^k w_n+\lambda_{nn+1}^k w_{n+1}+\cdots+\lambda_{nn+m}^k w_{n+m} \\ w_{n+1} &= 0\cdot w_1+\cdots+0\cdot w_n+1\cdot w_{n+1}+\cdots+0\cdot w_{n+m} \\ \vdots & \vdots \\ w_{n+m} &= 0\cdot w_1+\cdots+0\cdot w_n+0\cdot w_{n+1}+\cdots+1\cdot w_{n+m} \end{array}\right\} \tag{41}$$

Observe that:

$$\parallel\lambda_{i1}^k w_1+\cdots+(\lambda_{ii}^k-1)w_i+\cdots+\lambda_{in}^k w_n+\lambda_{in+1}^k w_{n+1}+\cdots+\lambda_{in+m}^k w_{n+m}\parallel=$$
$$=\parallel v_i^k-w_i\parallel\underset{k}{\rightarrow}0$$

So we deduce that:

$$\begin{aligned}
\lambda_{i1}^k &\xrightarrow{k} 0\\
\vdots\ &\ \vdots\\
\lambda_{ii-1}^k &\xrightarrow{k} 0\\
\lambda_{ii}^k &\xrightarrow{k} 1\\
\lambda_{ii+1}^k &\xrightarrow{k} 0\\
\vdots\ &\ \vdots\\
\lambda_{in}^k &\xrightarrow{k} 0\\
\lambda_{in+1}^k &\xrightarrow{k} 0\\
\vdots\ &\ \vdots\\
\lambda_{in+m}^k &\xrightarrow{k} 0
\end{aligned}$$

Thus

$$\begin{vmatrix}
\lambda_{11}^k & \dots & \lambda_{n1}^k & 0 & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots\\
\lambda_{1n}^k & \dots & \lambda_{nn}^k & 0 & \dots & 0\\
\lambda_{1n+1}^k & \dots & \lambda_{nn+1}^k & 1 & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots\\
\lambda_{1n+m}^k & \dots & \lambda_{nn+m}^k & 0 & \dots & 1
\end{vmatrix}
\xrightarrow{k}
\begin{vmatrix}
1 & \dots & 0 & 0 & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots\\
0 & \dots & 1 & 0 & \dots & 0\\
0 & \dots & 0 & 1 & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots\\
0 & \dots & 0 & 0 & \dots & 1
\end{vmatrix} = 1$$

$\Rightarrow$ The system (34) is linearly independent for $k$ big.
$\Rightarrow$ The system $v_1^k, \cdots, v_n^k, w_{n+1}, \cdots, w_{n+m}$ is linearly independent for $k$ big.

Let us apply the Gram-Smidt process to the systems of vectors:

$$\{w_1, \cdots, w_n, w_{n+1}, \cdots, w_{n+m}\} \xrightarrow{\text{G-S}} \{\bar{w}_1, \cdots, \bar{w}_n, \bar{w}_{n+1}, \cdots, \bar{w}_{n+m}\} \quad \text{orthonormal}$$

$$\{v_1^k, \cdots, v_n^k, w_{n+1}, \cdots, w_{n+m}\} \xrightarrow{\text{G-S}} \{\bar{v}_1^k, \cdots, \bar{v}_n^k, \bar{w}_{n+1}^k, \cdots, \bar{w}_{n+m}^k\} \quad \text{orthonormal}$$

Observation(1): As $v_1^k \xrightarrow{k} w_1, \cdots, v_n^k \xrightarrow{k} w_n$ and the Gram-Smidt process is continuous we obtain that:

$$\| \bar{v}_1^k - \bar{w}_1 \| \xrightarrow{k} 0, \cdots, \| \bar{v}_n^k - \bar{w}_n \| \xrightarrow{k} 0,\ \| \bar{w}_{n+1}^k - \bar{w}_{n+1} \| \xrightarrow{k} 0, \cdots, \| \bar{w}_{n+m}^k - \bar{w}_{n+m} \| \xrightarrow{k} 0$$

Observation (2):

$$\begin{aligned}
< w_1, \cdots, w_n > &= < \bar{w}_1, \cdots, \bar{w}_n > \Rightarrow\\
\Rightarrow W^\perp &= < w_1, \cdots, w_n >^\perp = < \bar{w}_1, \cdots, \bar{w}_n >^\perp\\
&= < \bar{w}_{n+1}, \cdots, \bar{w}_{n+m} >\\
< v_1^k, \cdots, v_n^k > &= < \bar{v}_1^k, \cdots, \bar{v}_n^k > \Rightarrow\\
\Rightarrow (V^k)^\perp &= < v_1^k, \cdots, v_n^k >^\perp = < \bar{v}_1^k, \cdots, \bar{v}_n^k >^\perp\\
&= < \bar{w}_{n+1}^k, \cdots, \bar{w}_{n+m}^k >
\end{aligned}$$

Let now $v$ be an unitary vector $\| v \|= 1$ such that:

$$\begin{cases} v = x_1\bar{w}_1 + \cdots + x_n\bar{w}_n + x_{n+1}\bar{w}_{n+1} + \cdots + x_{n+m}\bar{w}_{n+m} = \\ \quad = x_1^k\bar{v}_1^k + \cdots + x_n^k\bar{v}_n^k + x_{n+1}^k\bar{w}_{n+1}^k + \cdots + x_{n+m}^k\bar{w}_{n+m}^k \end{cases} \tag{42}$$

Then by the orthonormality of the involved systems we have that:

$$\begin{aligned} &| x_1^k |^2 + \cdots + | x_n^k |^2 + | x_{n+1}^k |^2 + \cdots + | x_{n+m}^k |^2 = \\ &= | x_1 |^2 + \cdots + | x_n |^2 + | x_{n+1} |^2 + \cdots + | x_{n+m} |^2 = \| v \|^2 = 1 \end{aligned}$$

And from Cauchy-Schwartz's inequality:

$$\begin{cases} | x_1^k | + \cdots + | x_n^k | + | x_{n+1}^k | + \cdots + | x_{n+m}^k | \leq \\ \leq \sqrt{n+m} \cdot \sqrt{| x_1^k |^2 + \cdots + | x_n^k |^2 + | x_{n+1}^k |^2 + \cdots + | x_{n+m}^k |^2} \\ = \sqrt{n+m} \end{cases} \tag{43}$$

Let now the norm $||| \cdot |||$ be defined by

$$||| \alpha_1 \cdot \bar{w}_1 + \cdots + \alpha_{n+m} \cdot \bar{w}_{n+m} ||| = \max\{| \alpha_1 |, \cdots, | \alpha_{n+m} |\}$$

Then by the equivalence of norms in spaces of finite dimension we have that there exists a constant $\triangle$ such that:

$$\| v \| \geq \triangle ||| v ||| \quad \forall v \tag{44}$$

Let for each $k$, be $\epsilon_k$ defined by

$$\epsilon_k = \max\{\| \bar{v}_1^k - \bar{w}_1 \|, \cdots, \| \bar{v}_n^k - \bar{w}_n \|, \| \bar{w}_{n+1}^k - \bar{w}_{n+1} \|, \cdots, \| \bar{w}_{n+m}^k - \bar{w}_{n+m} \|\}$$

Then by Observation (2) we obtain that:

$$\epsilon_k \xrightarrow{k} 0 \tag{45}$$

Now from (35) we deduce that

$$\begin{cases} [x_1^k(\bar{v}_1^k - \bar{w}_1) + \cdots + x_{n+m}^k(\bar{w}_{n+m}^k - \bar{w}_{n+m})] + \\ + [(x_1^k - x_1)\bar{w}_1 + \cdots + (x_{n+m}^k - x_{n+m})\bar{w}_{n+m}] = \\ = (x_1^k\bar{v}_1^k - x_1^k\bar{w}_1 + x_1^k\bar{w}_1 - x_1\bar{w}_1) + \cdots + \\ + (x_{n+m}^k\bar{w}_{n+m}^k - x_{n+m}^k\bar{w}_{n+m} + x_{n+m}^k\bar{w}_{n+m} - x_{n+m}\bar{w}_{n+m}) = \\ = (x_1^k\bar{v}_1^k - x_1\bar{w}_1) + \cdots + (x_{n+m}^k\bar{w}_{n+m}^k - x_{n+m}\bar{w}_{n+m}) = v - v = 0 \\ \Rightarrow \| (x_1^k - x_1)\bar{w}_1 + \cdots + (x_{n+m}^k - x_{n+m})\bar{w}_{n+m} \| = \\ = \| x_1^k(\bar{v}_1^k - \bar{w}_1) + \cdots + x_{n+m}^k(\bar{w}_{n+m}^k - \bar{w}_{n+m}) \| \leq \\ \leq | x_1^k | \| \bar{v}_1^k - \bar{w}_1 \| + \cdots + | x_{n+m}^k | \| \bar{w}_{n+m}^k - \bar{w}_{n+m} \| \leq \\ \leq (| x_1^k | + \cdots + | x_{n+m}^k |) \cdot \epsilon_k \leq \\ \leq \sqrt{n+m} \cdot \epsilon_k \end{cases} \tag{46}$$

And from(37) we have

$$
(47) \qquad \begin{cases}
\triangle \max\{| x_1^k - x_1 |, \cdots, | x_{n+m}^k - x_{n+m} |\} = \\
= \triangle ||| (x_1^k - x_1)\bar{w}_1 + \cdots + (x_{n+m}^k - x_{n+m})\bar{w}_{n+m} ||| \leq \\
\leq \| (x_1^k - x_1)\bar{w}_1 + \cdots + (x_{n+m}^k - x_{n+m})\bar{w}_{n+m} \| \\
\overset{(39)}{\Rightarrow} \triangle \max\{| x_1^k - x_1 |, \cdots, | x_{n+m}^k - x_{n+m} |\} \leq \sqrt{n+m} \cdot \epsilon_k \\
\Rightarrow \max\{| x_1^k - x_1 |, \cdots, | x_{n+m}^k - x_{n+m} |\} \leq \dfrac{\sqrt{n+m}}{\triangle} \cdot \epsilon_k
\end{cases}
$$

So

$$
\begin{aligned}
\| P_{V^k}(v) - P_W(v) \| &= \| (x_1^k \bar{v}_1^k + \cdots + x_n^k \bar{v}_n^k) - (x_1 \bar{w}_1 + \cdots + x_n \bar{w}_n) \| = \\
&= \| (x_1^k \bar{v}_1^k - x_1^k \bar{w}_1 + x_1^k \bar{w}_1 - x_1 \bar{w}_1) + \cdots + \\
&+ (x_n^k \bar{v}_n^k - x_n^k \bar{w}_n + x_n^k \bar{w}_n - x_n \bar{w}_n) \| \leq \\
&\leq (| x_1^k | \| \bar{v}_1^k - \bar{w}_1 \| + \cdots + | x_n^k | \| \bar{v}_n^k - \bar{w}_n \|) + \\
&+ (| x_1^k - x_1 | \| \bar{w}_1 \| + \cdots + | x_n^k - x_n | \| \bar{w}_n \|) \leq \\
&\leq (| x_1^k | + \cdots + | x_n^k |) \cdot \epsilon_k + \\
&+ (| x_1^k - x_1 | \| \bar{w}_1 \| + \cdots + | x_n^k - x_n | \| \bar{w}_n \|) \leq \\
&\leq (| x_1^k | + \cdots + | x_n^k | + | x_{n+1}^k | + \cdots + | x_{n+m}^k |) \cdot \epsilon_k + \\
&+ (| x_1^k - x_1 | \| \bar{w}_1 \| + \cdots + | x_n^k - x_n | \| \bar{w}_n \|) \leq \\
&\overset{(36)}{\leq} \sqrt{n+m} \cdot \epsilon_k + (| x_1^k - x_1 | \| \bar{w}_1 \| + \cdots + | x_n^k - x_n | \| \bar{w}_n \|) \leq \\
&\leq \sqrt{n+m} \cdot \epsilon_k + \frac{\sqrt{n+m}}{\triangle} \cdot \epsilon_k (\| \bar{w}_1 \| + \cdots + \| \bar{w}_n \|) = \\
&= \sqrt{n+m}(1 + \frac{1}{\triangle}(\| \bar{w}_1 \| + \cdots + \| \bar{w}_n \|)) \cdot \epsilon_k \quad \forall \quad v \quad \text{with} \quad \| v \| = 1 \\
&\Rightarrow \| P_{V^k} - P_W \| \overset{k}{\to} 0 \\
&\Rightarrow \theta(P_{V^k}, P_W) \overset{k}{\to} 0
\end{aligned}
$$

□

**Proposition 11.2.** *Let $\lambda_1, \cdots, \lambda_d$ with $| \lambda_d | < | \lambda_{d-1} | < \cdots < | \lambda_1 |$ be the eigenvalues of a pinching matrix $M$ and $\vec{v}_1, \cdots, \vec{v}_d$ the corresponding eigenvectors with*

$$\| \vec{v}_1 \| = \cdots = \| \vec{v}_d \| = 1$$

*And let $\vec{u}_1, \cdots, \vec{u}_k$ be a system of linearly independent of vectors. Then there exists some $(m_1, \cdots, m_k)$ such that*

$$\theta(M^n(W), < \vec{v}_{m_1}, \cdots, \vec{v}_{m_k} >) \underset{n}{\to} 0$$

*where $W = < \vec{u}_1, \cdots, \vec{u}_k >$*

Given $\vec{w} = \beta^1 \vec{v}_1 + \cdots + \beta^d \vec{v}_d$ let us define $d(w)$ by $d(w) \overset{\text{def}}{=} \min\{i : \beta^i \neq 0\}$, that is $d(w)$ is the index corresponding to the eigenvalue with greater modular value such that $\vec{w}$ have a non zero component in the direction of the eigenvector corresponding when it is expressed in the base $\vec{v}_1, \cdots, \vec{v}_d$. Then when we apply the potences method to $\vec{w}$, that is when we construct the sequence $\vec{w}_n = \frac{M^n(\vec{w})}{\|M^n(\vec{w})\|}$ results $\vec{w} \underset{n}{\to} \vec{v}_{d(\vec{w})}$.

Let us suppose that:

$$\begin{aligned}
\vec{u}_1 &= \alpha_1^1 \cdot \vec{v}_1 + \cdots + \alpha_1^d \cdot \vec{v}_d \\
\vdots \quad & \quad \vdots \\
\vec{u}_k &= \alpha_k^1 \cdot \vec{v}_1 + \cdots + \alpha_k^d \cdot \vec{v}_d, \quad 1 \le k \le d
\end{aligned}$$

Applying the Gauss process to the matrix

$$\begin{bmatrix} \alpha_1^1 & \dots & \alpha_1^d \\ \dots & \dots & \dots \\ \alpha_k^1 & \dots & \alpha_k^d \end{bmatrix}$$

we get a matrix in the form

$$\begin{bmatrix}
0 & \dots & 0 & \bar{\alpha}_1^{m_1} \cdot \vec{v}_{m_1} & \dots & \bar{\alpha}_1^{m_2} \cdot \vec{v}_{m_2} & \dots & \bar{\alpha}_1^{m_k} \cdot \vec{v}_{m_k} & \dots & \bar{\alpha}_1^{d} \cdot \vec{v}_{d} \\
0 & \dots & 0 & 0 & \dots & \bar{\alpha}_2^{m_2} \cdot \vec{v}_{m_2} & \dots & \bar{\alpha}_2^{m_k} \cdot \vec{v}_{m_k} & \dots & \bar{\alpha}_2^{d} \cdot \vec{v}_{d} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & 0 & 0 & \dots & 0 & \dots & \bar{\alpha}_k^{m_k} \cdot \vec{v}_{m_k} & \dots & \bar{\alpha}_k^{d} \cdot \vec{v}_{d}
\end{bmatrix}$$

Remember that maybe it was necessary to permute the rows sometimes during the Gauss process of elimination to get these final distribution of zeroes. Then the vectors:

$$\begin{aligned}
\vec{\bar{u}}_1 &= \bar{\alpha}_1^{m_1} \cdot \vec{v}_{m_1} + \cdots + \bar{\alpha}_1^d \cdot \vec{v}_d \\
\vec{\bar{u}}_2 &= \bar{\alpha}_2^{m_2} \cdot \vec{v}_{m_2} + \cdots + \bar{\alpha}_2^d \cdot \vec{v}_d \\
\vdots \quad & \quad \vdots \\
\vec{\bar{u}}_k &= \bar{\alpha}_k^{m_k} \cdot \vec{v}_{m_k} + \cdots + \bar{\alpha}_k^d \cdot \vec{v}_d
\end{aligned}$$

Also constitute a base of $W$ and:

$$\begin{aligned}
\frac{M^n(\vec{\bar{u}}_1)}{\| M^n(\vec{\bar{u}}_1) \|} &\underset{n}{\to} \vec{v}_{m_1} \\
\vdots \quad & \quad \vdots \\
\frac{M^n(\vec{\bar{u}}_k)}{\| M^n(\vec{\bar{u}}_k) \|} &\underset{n}{\to} \vec{v}_{m_k}
\end{aligned}$$

So using Proposition 11.1 we obtain :

$$\theta(< M^n(\vec{\bar{u}}_1), \cdots, M^n(\vec{\bar{u}}_k) >, < \vec{v}_{m_1}, \cdots, \vec{v}_{m_k} >) \underset{n}{\to} 0$$

that is

$$\theta(M^n(W), < \vec{v}_{m_1}, \cdots, \vec{v}_{m_k} >) \underset{n}{\to} 0$$

## 12. Appendix F: We can apply the Twisting Lemma

Now we prove rigorously Proposition 6.2.

*Proof.* Take $n = d+1$. The eigenspaces of $M$ are unidimensional and if $(v_1, \cdots, v_n)$ is a non trivial eigenvalue associated to $\lambda$ there exists one index $k$ such that $v_k \neq 0$. Then there exists one and only one vector $y = (y_1, \cdots, y_{k-1}, 1, y_{k+1}, \cdots, y_n)$ such that:

$$My = \lambda y$$

Which is equivalent to the following system of equations:

$$\begin{aligned}
(m_{11}-\lambda)y_1 + m_{12}y_2 + \cdots + m_{1k-1}y_{k-1} + m_{1k}\cdot 1 + m_{1k+1}y_{k+1} + \cdots + m_{1n}y_n &= 0\\
m_{21}y_1 + (m_{22}-\lambda)y_2 + \cdots + m_{2k-1}y_{k-1} + m_{2k}\cdot 1 + m_{2k+1}y_{k+1} + \cdots + m_{2n}y_n &= 0\\
\vdots \qquad\qquad &\ \ \vdots\\
m_{k-11}y_1 + \cdots + (m_{k-1k-1}-\lambda)y_{k-1} + m_{k-1k}\cdot 1 + m_{k-1k+1}y_{k+1} + \cdots + m_{k-1n}y_n &= 0\\
m_{k1}y_1 + \cdots + m_{kk-1}y_{k-1} + (m_{kk}-\lambda)\cdot 1 + m_{kk+1}y_{k+1} + \cdots + m_{kn}y_n &= 0\\
m_{k+11}y_1 + \cdots + m_{k+1k-1}y_{k-1} + m_{k+1k}\cdot 1 + (m_{k+1k+1}-\lambda)y_{k+1} + \cdots + m_{k+1n}y_n &= 0\\
\vdots \qquad\qquad &\ \ \vdots\\
m_{n1}y_1 + m_{n2}y_2 + \cdots + m_{nk-1}y_{k-1} + m_{nk}\cdot 1 + m_{nk+1}y_{k+1} + \cdots + (m_{nn}-\lambda)y_n &= 0
\end{aligned}$$

Therefore the following system have one and only one solution $y = (y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_n)$:

$$Ay = b \tag{48}$$

where

$$A = \begin{bmatrix}
m_{21} & (m_{22}-\lambda) & \dots & m_{2k-1} & m_{2k+1} & \dots & m_{2n}\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots\\
m_{k-11} & m_{k-12} & \dots & (m_{k-1k-1}-\lambda) & m_{k-1k+1} & \dots & m_{k-1n}\\
m_{k1} & m_{k2} & \dots & m_{kk-1} & m_{kk+1} & \dots & m_{kn}\\
m_{k+11} & m_{k+12} & \dots & m_{k+1k-1} & (m_{k+1k+1}-\lambda) & \dots & m_{k+1n}\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots\\
m_{n1} & m_{n2} & \dots & m_{nk-1} & m_{nk+1} & \dots & (m_{nn}-\lambda)
\end{bmatrix}$$

$$b = \begin{bmatrix}
-m_{2k}\\
\dots\dots\\
-m_{k-1k}\\
-(m_{kk}-\lambda)\\
-m_{k+1k}\\
\dots\dots\\
-m_{nk}
\end{bmatrix}$$

Now observe that $\mathrm{Det}A \neq 0$ because if $\mathrm{Det}A = 0$ we can take a non trivial solution $x_0$ of the homogeneous system $Ax_0 = 0$, and then the vector $z = y + x_0$ satisfy:

$$Az = Ay + Ax_0 = Ay = b$$

and $z$ is a solution of (28) different from $y$ which is a contradiction.
As $\mathrm{Det}A \neq 0$ we can use Cramer's rule for obtaining the unique solution $y$ of (28)

which is then given by the following formulae:

$$y_1 = \\ = \frac{\begin{vmatrix} -m_{2k} & (m_{22}-\lambda) & \dots & m_{2k-1} & m_{2k+1} & \dots & m_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -m_{k-1k} & m_{k-12} & \dots & (m_{k-1k-1}-\lambda) & m_{k-1k+1} & \dots & m_{k-1n} \\ -(m_{kk}-\lambda) & m_{k2} & \dots & m_{kk-1} & m_{kk+1} & \dots & m_{kn} \\ -m_{k+1k} & m_{k+12} & \dots & m_{k+1k-1} & (m_{k+1k+1}-\lambda) & \dots & m_{k+1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -m_{nk} & m_{n2} & \dots & m_{nk-1} & m_{nk+1} & \dots & (m_{nn}-\lambda) \end{vmatrix}}{\mathrm{Det}A} = \\ = (-1)^{k-1} \frac{\begin{vmatrix} (m_{22}-\lambda) & \dots & m_{2k-1} & m_{2k} & m_{2k+1} & \dots & m_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{k-12} & \dots & (m_{k-1k-1-\lambda}) & m_{k-1k} & m_{k-1k+1} & \dots & m_{k-1n} \\ m_{k2} & \dots & m_{kk-1} & (m_{kk}-\lambda) & m_{kk+1} & \dots & m_{kn} \\ m_{k+12} & \dots & m_{k+1k-1} & m_{k+1k} & (m_{k+1k+1}-\lambda) & \dots & m_{k+1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{n2} & \dots & m_{nk-1} & m_{nk} & m_{nk+1} & \dots & (m_{nn}-\lambda) \end{vmatrix}}{\mathrm{Det}A}$$

$$y_2 = \\ = \frac{\begin{vmatrix} m_{21} & -m_{2k} & m_{23} & \dots & m_{2k-1} & m_{2k+1} & \dots & m_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{k-11} & -m_{k-1k} & m_{k-13} & \dots & (m_{k-1k-1}-\lambda) & m_{k-1k+1} & \dots & m_{k-1n} \\ m_{k1} & -(m_{kk}-\lambda) & m_{k3} & \dots & m_{kk-1} & m_{kk+1} & \dots & m_{kn} \\ m_{k+11} & -m_{k+1k} & m_{k+13} & \dots & m_{k+1k-1} & (m_{k+1k+1}-\lambda) & \dots & m_{k+1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{n1} & -m_{nk} & m_{n3} & \dots & m_{nk-1} & m_{nk+1} & \dots & (m_{nn}-\lambda) \end{vmatrix}}{\mathrm{Det}A} = \\ = (-1)^{k-2} \frac{\begin{vmatrix} m_{21} & m_{23} & \dots & m_{2k-1} & m_{2k} & m_{2k+1} & \dots & m_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{k-11} & m_{k-13} & \dots & (m_{k-1k-1-\lambda}) & m_{k-1k} & m_{k-1k+1} & \dots & m_{k-1n} \\ m_{k1} & m_{k3} & \dots & m_{kk-1} & (m_{kk}-\lambda) & m_{kk+1} & \dots & m_{kn} \\ m_{k+11} & m_{k+13} & \dots & m_{k+1k-1} & m_{k+1k} & (m_{k+1k+1}-\lambda) & \dots & m_{k+1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{n1} & m_{n3} & \dots & m_{nk-1} & m_{nk} & m_{nk+1} & \dots & (m_{nn}-\lambda) \end{vmatrix}}{\mathrm{Det}A}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$y_{k-1} =$

$$= \frac{\begin{vmatrix} m_{21} & \dots & m_{2k-2} & -m_{2k} & m_{2k+1} & \dots & m_{2n} \\ \multicolumn{7}{c}{\dotfill} \\ m_{k-11} & \dots & m_{k-1k-2} & -m_{k-1k} & m_{k-1k+1} & \dots & m_{k-1n} \\ m_{k1} & \dots & m_{kk-2} & -(m_{kk}-\lambda) & m_{kk+1} & \dots & m_{kn} \\ m_{k+11} & \dots & m_{k+1k-2} & -m_{k+1k} & (m_{k+1k+1}-\lambda) & \dots & m_{k+1n} \\ \multicolumn{7}{c}{\dotfill} \\ m_{n1} & \dots & m_{nk-2} & -m_{nk} & m_{nk+1} & \dots & (m_{nn}-\lambda) \end{vmatrix}}{\mathrm{Det}A} =$$

$$= (-1)^1 \frac{\begin{vmatrix} m_{21} & \dots & m_{2k-2} & m_{2k} & m_{2k+1} & \dots & m_{2n} \\ \multicolumn{7}{c}{\dotfill} \\ m_{k-11} & \dots & m_{k-1k-2} & m_{k-1k} & m_{k-1k+1} & \dots & m_{k-1n} \\ m_{k1} & \dots & m_{kk-2} & (m_{kk}-\lambda) & m_{kk+1} & \dots & m_{kn} \\ m_{k+11} & \dots & m_{k+1k-2} & m_{k+1k} & (m_{k+1k+1}-\lambda) & \dots & m_{k+1n} \\ \multicolumn{7}{c}{\dotfill} \\ m_{n1} & \dots & m_{nk-2} & m_{nk} & m_{nk+1} & \dots & (m_{nn}-\lambda) \end{vmatrix}}{\mathrm{Det}A}$$

$y_{k+1} =$

$$= \frac{\begin{vmatrix} m_{21} & \dots & m_{2k-1} & -m_{2k} & m_{2k+2} & \dots & m_{2n} \\ \multicolumn{7}{c}{\dotfill} \\ m_{k-11} & \dots & (m_{k-1k-1}-\lambda) & -m_{k-1k} & m_{k-1k+2} & \dots & m_{k-1n} \\ m_{k1} & \dots & m_{kk-1} & -(m_{kk}-\lambda) & m_{kk+2} & \dots & m_{kn} \\ m_{k+11} & \dots & m_{k+1k-1} & -m_{k+1k} & m_{k+1k+2} & \dots & m_{k+1n} \\ \multicolumn{7}{c}{\dotfill} \\ m_{n1} & \dots & m_{nk-1} & -m_{nk} & m_{nk+2} & \dots & (m_{nn}-\lambda) \end{vmatrix}}{\mathrm{Det}A} =$$

$$= (-1)^1 \frac{\begin{vmatrix} m_{21} & \dots & m_{2k-1} & m_{2k} & m_{2k+2} & \dots & m_{2n} \\ \multicolumn{7}{c}{\dotfill} \\ m_{k-11} & \dots & (m_{k-1k-1}-\lambda) & m_{k-1k} & m_{k-1k+2} & \dots & m_{k-1n} \\ m_{k1} & \dots & m_{kk-1} & (m_{kk}-\lambda) & m_{kk+2} & \dots & m_{kn} \\ m_{k+11} & \dots & m_{k+1k-1} & m_{k+1k} & m_{k+1k+2} & \dots & m_{k+1n} \\ \multicolumn{7}{c}{\dotfill} \\ m_{n1} & \dots & m_{nk-1} & m_{nk} & m_{nk+2} & \dots & (m_{nn}-\lambda) \end{vmatrix}}{\mathrm{Det}A}$$

$$y_{k+2} =$$

$$= \frac{\begin{vmatrix} m_{21} & \dots & m_{2k+1} & -m_{2k} & m_{2k+3} & \dots & m_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{k-11} & \dots & m_{k-1k+1} & -m_{k-1k} & m_{k-1k+3} & \dots & m_{k-1n} \\ m_{k1} & \dots & m_{kk+1} & -(m_{kk}-\lambda) & m_{kk+3} & \dots & m_{kn} \\ m_{k+11} & \dots & (m_{k+1k+1}-\lambda) & -m_{k+1k} & m_{k+1k+3} & \dots & m_{k+1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{n1} & \dots & m_{nk+1} & -m_{nk} & m_{nk+3} & \dots & (m_{nn}-\lambda) \end{vmatrix}}{\text{Det}A} =$$

$$= (-1)^2 \frac{\begin{vmatrix} m_{21} & \dots & m_{2k-1} & m_{2k} & m_{2k+1} & m_{2k+3} & \dots & m_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{k-11} & \dots & (m_{k-1k-1}-\lambda) & m_{k-1k} & m_{k-1k+1} & m_{k-1k+3} & \dots & m_{k-1n} \\ m_{k1} & \dots & m_{kk-1} & (m_{kk}-\lambda) & m_{kk+1} & m_{kk+3} & \dots & m_{kn} \\ m_{k+11} & \dots & m_{k+1k-1} & m_{k+1k} & (m_{k+1k+1}-\lambda) & m_{k+1k+3} & \dots & m_{k+1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{n1} & \dots & m_{nk-1} & m_{nk} & m_{nk+1} & m_{nk+3} & \dots & (m_{nn}-\lambda) \end{vmatrix}}{\text{Det}A}$$

$$y_n =$$

$$= \frac{\begin{vmatrix} m_{21} & \dots & m_{2k-1} & m_{2k+1} & \dots & m_{2n-1} & -m_{2k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{k-11} & \dots & (m_{k-1k-1}-\lambda) & m_{k-1k+1} & \dots & m_{k-1n-1} & -m_{k-1k} \\ m_{k1} & \dots & m_{kk-1} & m_{kk+1} & \dots & m_{kn-1} & -(m_{kk}-\lambda) \\ m_{k+11} & \dots & m_{k+1k-1} & (m_{k+1k+1}-\lambda) & \dots & m_{k+1n-1} & -m_{k+1k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{n1} & \dots & m_{nk-1} & m_{nk+1} & \dots & m_{nn-1} & -m_{nk} \end{vmatrix}}{\text{Det}A} =$$

$$= (-1)^{n-k} \frac{\begin{vmatrix} m_{21} & \dots & m_{2k-1} & m_{2k} & m_{2k+1} & \dots & m_{2n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{k-11} & \dots & (m_{k-1k-1}-\lambda) & m_{k-1k} & m_{k-1k+1} & \dots & m_{k-1n-1} \\ m_{k1} & \dots & m_{kk-1} & (m_{kk}-\lambda) & m_{kk+1} & \dots & m_{kn-1} \\ m_{k+11} & \dots & m_{k+1k-1} & m_{k+1k} & (m_{k+1k+1}-\lambda) & \dots & m_{k+1n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{n1} & \dots & m_{nk-1} & m_{nk} & m_{nk+1} & \dots & m_{nn-1} \end{vmatrix}}{\text{Det}A}$$

Let us define $x = (x_1, \cdots, x_n) = (-1)^{k+1}\text{Det}A \cdot (y_1, \cdots, y_{k-1}, 1, y_{k+1}, \cdots, y_n)$. Being $y$ a solution of $Ay = b$ and $\text{Det}A \neq 0$ we deduce that $x$ is a non trivial solution of the system:

$$(49)\quad \begin{cases} m_{21}x_1 + (m_{22}-\lambda)x_2 + \cdots + m_{2k-1}x_{k-1} + m_{2k}x_k + \cdots + m_{2n}x_n = 0 \\ \vdots \qquad\qquad \vdots \\ m_{k-11}x_1 + \cdots + (m_{k-1k-1}-\lambda)x_{k-1} + m_{k-1k}x_k + \cdots + m_{k-1n}x_n = 0 \\ m_{k1}x_1 + m_{k2}x_2 + \cdots + (m_{kk}-\lambda)x_k + m_{kk+1}x_{k+1} + \cdots + m_{kn}x_n = 0 \\ m_{k+11}x_1 + \cdots + m_{k+1k}x_k + (m_{k+1k+1}-\lambda)x_{k+1} + \cdots + m_{k+1n}x_n = 0 \\ \vdots \qquad\qquad \vdots \\ m_{n1}x_1 + m_{n2}x_2 + \cdots + m_{nk-1}x_{k-1} + m_{nk}x_k + \cdots + (m_{nn}-\lambda)x_n = 0 \end{cases}$$

Define now the polynomials $p_1, \cdots, p_n$ by:

$p_1(t) =$

$$= (-1)^2 \begin{vmatrix} m_{22}-t & \dots & m_{2k-1} & m_{2k} & m_{2k+1} & \dots & m_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{k-12} & \dots & (m_{k-1k-1}-t) & m_{k-1k} & m_{k-1k+1} & \dots & m_{k-1n} \\ m_{k2} & \dots & m_{kk-1} & (m_{kk}-t) & m_{kk+1} & \dots & m_{kn} \\ m_{k+12} & \dots & m_{k+1k-1} & m_{k+1k} & (m_{k+1k+1}-t) & \dots & m_{k+1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{n2} & \dots & m_{nk-1} & m_{nk} & m_{nk+1} & \dots & (m_{nn}-t) \end{vmatrix}$$

$p_2(t) =$

$$= (-1)^3 \begin{vmatrix} m_{21} & m_{23} & \dots & m_{2k-1} & m_{2k} & m_{2k+1} & \dots & m_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{k-11} & m_{k-13} & \dots & (m_{k-1k-1}-t) & m_{k-1k} & m_{k-1k+1} & \dots & m_{k-1n} \\ m_{k1} & m_{k3} & \dots & m_{kk-1} & (m_{kk}-t) & m_{kk+1} & \dots & m_{kn} \\ m_{k+11} & m_{k+13} & \dots & m_{k+1k-1} & m_{k+1k} & (m_{k+1k+1}-t) & \dots & m_{k+1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{n1} & m_{n3} & \dots & m_{nk-1} & m_{nk} & m_{nk+1} & \dots & (m_{nn}-t) \end{vmatrix}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$p_n(t) =$

$$= (-1)^{n+1} \begin{vmatrix} m_{21} & (m_{22}-t) & \dots & m_{2k-1} & m_{2k} & m_{2k+1} & \dots & m_{2n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{k-11} & m_{k-12} & \dots & (m_{k-1k-1}-t) & m_{k-1k} & m_{k-1k+1} & \dots & m_{k-1n-1} \\ m_{k1} & m_{k2} & \dots & m_{kk-1} & (m_{kk}-t) & m_{kk+1} & \dots & m_{kn-1} \\ m_{k+11} & m_{k+12} & \dots & m_{k+1k-1} & m_{k+1k} & (m_{k+1k+1}-t) & \dots & m_{k+1n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{n1} & m_{n2} & \dots & m_{nk-1} & m_{nk} & m_{nk+1} & \dots & (m_{nn}-t) \end{vmatrix}$$

Let $\bar{M}_t$ be the $(n-1)\times(n-1)$ matrix obtained by removing the first row from the matrix $M - t\cdot \mathrm{Id}$. Observe that $p_i(t), 1 \leq i \leq n$ is the determinant of the matrix obtained by dropping the $i$-th column from $\bar{M}_t$. Obviously the polynomials defined in this way have degree least than or equal to $n-1$ and:

$$(50)\qquad x = (p_1(\lambda), \cdots, p_n(\lambda))\}$$

As $\lambda$ is an eigenvalue of $M$ we have that:

$$\begin{aligned}&\mathrm{Det}(M-\lambda\mathrm{Id})=0\\ \Leftrightarrow\ &(m_{11}-\lambda)\cdot p_1(\lambda)+m_{12}\cdot p_2(\lambda)+\cdots+m_{1n}\cdot p_n(\lambda)=0\\ \Leftrightarrow\quad &\{(m_{11}-\lambda)\cdot x_1+m_{12}\cdot x_2+\cdots+m_{1n}\cdot x_n=0\}\end{aligned}$$

From this and from 49 we deduce that $x$ is a non trivial solution of

$$Mx=\lambda x$$

given by the expression 50 where the polynomials $p_1,\cdots,p_n$ have degree least than or equal to $n-1$ and by definition its coefficients depend only on the last $n-1$ rows of $M$. □

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