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On Moving Singularities in Fibrations by Algebraic Curves

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Abstract

Bertini's theorem on variable singular points may fail in positive characteristic, as was discovered by Zariski in 1944. In fact, he found fibrations by nonsmooth curves. In this work we continue to classify this phenomenon in characteristic three by constructing a two-dimensional algebraic fibration by non-smooth plane projective quartic curves, that is universal in the sense that the data about some fibrations by nonsmooth plane projective quartics are condensed in it. Our approach was motivated by the close relation between this phenomenon and the theory of regular but nonsmooth curves, or equivalently, nonconservative function fields in one variable. Actually, it also provided to understand the interesting effect of the relative Frobenius morphism in fibrations by nonsmooth curves. In analogy to the Kodaira-Nron classification of special fibers of minimal fibrations by elliptic curves, we also construct the minimal proper regular model of some fibrations by non-smooth projective plane quartic curves, determine the structure of the bad fibers, and study the global geometry of the total spaces.

Keywords

Bertini's theorem. Fibrations by nonsmooth curves. Relative Frobenius morphism. Nonconservative function fields. Regular but nonsmooth curves. Minimal models.

Contents

Chapter 1

Introduction

Eugenio Bertini in his 1882 paper [B] published the two theorems that now bear his name: Bertini's theorem on variable singular points, and Bertini's theorem on reducible linear systems. We are particularly interested in the first statement, that essentially claims: if a linear system, on a projective space over the field of complex numbers, has no fixed components, then a general member has no singular points outside the base locus. For various versions of Bertini's theorems and his life story, which is a fascinating drama, we refer to Kleiman's article [K].

Bertini's theorems soon became widely used tools in algebraic geometry. In one of his works Guido Castelnuovo wrote that both theorems "come into play at every step in all the work". Later on, until 1950, the statements were approached in more general contexts by Bertini himself, Frederigo Enriques, Bartel van der Waerden and Oscar Zariski.

Bertini's theorem on variable singular points can be stated today as: almost all fibers of a dominant morphism between smooth algebraic varieties over an algebraically closed field of characteristic zero are smooth (see [Sh2], ch II 6.2). In other words, it just means the generic smoothness of morphisms between smooth algebraic varieties over an algebraically closed field of characteristic zero. For many authors, this theorem is also called Sard's theorem, by analogy with the theory of differentiable manifolds.

In his 1944 paper [Z1] Zariski discovered that this theorem is false in positive characteristic. In other words, he found fibrations by nonsmooth varieties $f : X \to Y$ between varieties over an algebraically closed field of positive characteristic. It motivated him, according to Mumford [M], to discover the existence of two different concepts of simple point on varieties over imperfect fields: regular in the sense of having a regular local ring, and smooth in the sense that the usual Jacobian criterion is satisfied. Now we present an interesting example constructed in this work.

Example 1.1. Let us consider an algebraically closed field k of characteristic three and the irreducible surface

$$S \subset \mathbb{P}^2(k) \times \mathbb{P}^1(k)$$

given by the pairs $((x : y : z), (s : t))$ satisfying the bihomogeneous polynomial equation

$$sy^3z - tz^4 - sx^4 = 0.$$

We also consider the surjective morphism

$$\eta : S \to \mathbb{P}^1(k)$$

obtained by restricting the second projection $\mathbb{P}^2(k) \times \mathbb{P}^1(k) \to \mathbb{P}^1(k)$ to S.

By the Jacobian criterion, though each fiber of η is singular the total space S has only one singular point, namely $((0 : 1 : 0), (0 : 1))$. Therefore, by restricting the base of η to the open subset $\mathbb{P}^1(k) \setminus \{(0 : 1)\}$ of $\mathbb{P}^1(k)$, we obtain that η is a fibration by nonsmooth curves.

We notice that the rational map of $\mathbb{P}^2(k) \times \mathbb{P}^1(k)$, given by the assignment

$$((x : y : z), (s : t)) \mapsto ((x^2 : yz : z^2), (s : t))$$

induces a rational double cover from S onto the surface

$$S' \subset \mathbb{P}^2(k) \times \mathbb{P}^1(k)$$

given by the bihomogeneous polynomial $sY^3 - tZ^3 - sZX^2$.

Moreover, η factors into this rational map followed by the fibration by cuspidal cubics

$$\eta' : S' \longrightarrow \mathbb{P}^1(k),$$

induced by the second projection of $\mathbb{P}^2(k) \times \mathbb{P}^1(k)$ onto $\mathbb{P}^1(k)$.

Fibrations by cusps arose in the extension of Enriques' classification of surfaces to positive characteristic, due to Bombieri and Mumford [BM], in order to characterize quasi-hyperelliptic surfaces. In particular, they showed that each fibration by cusps is locally formally obtained from the fibration η' in the previous example. It provides a first example that fibrations by nonsmooth varieties represent a rich phenomenon in algebraic geometry, and not just a simple pathology. In addition, it enables us to find other geometrical constructions that never occur in characteristic zero, as we can see in the following example.

Example 1.2. Let us consider the rational map

$$\mathbb{P}^2(k) \dashrightarrow \mathbb{P}^1(k)$$

defined by the assignment $(x : y : z) \mapsto (z^4 : y^3z - x^4)$, where k in an algebraically closed field of characteristic three. By looking at its fibers, we may see that $\mathbb{P}^2(k)$ can be covered by a family of singular curves, whose smooth points are always inflection points.

Although this example suggests wild properties of points of the general fiber of such fibrations, this is not necessarily true for their singularities, since they are at least Gorenstein.

By a fibration by curves, we mean a proper dominant morphism between algebraic varieties, over an algebraically closed field k, such that almost all of its fibers are algebraic curves and the total space is smooth after restricting the base to a dense open subset. Algebraic varieties are required to be integral. In this work we wish to investigate the classification of fibrations by nonsmooth curves. Hence, k must be a field of positive characteristic, according to Bertini's theorem on variable singular points. Two fibrations are said to be birational equivalent if there is a birational map between the total spaces and another one between the bases, such that the corresponding diagram commutes.

In the second chapter we begin with a brief discussion on Zariski's point of view concerning singularities appearing on the general fiber of a dominant morphism of algebraic varieties $f : X \to Y$. We also assume that the general fiber of f is an algebraic curve and f is a proper morphism. Hence, its general fiber is a projective algebraic curve. More precisely, by using scheme-theoretic language, we observe the correspondence between horizontal prime divisors of the total space X and closed points of the generic fiber of f, which is a complete algebraic curve over the function field $k(Y)$ of Y.

Moreover, we also assume that the generic fiber of f is geometrically integral, or equivalently, almost all fibers of f are integral projective curves. In this case, we observe the beautiful correspondence between nonsmooth closed points of the generic fiber of f and horizontal prime divisors contained in the nonsmooth locus of f, that is, horizontal prime divisors of X, containing singularities of almost all fibers. The morphism f is called not generically smooth when its nonsmooth locus contains horizontal prime divisors of X.

In this way, it is equivalent to classify fibrations by nonsmooth curves and regular but nonsmooth curves over isomorphic fields. However, this problem will be approached in a subtle point of view. In fact, we will classify regular but nonsmooth curves over a fixed field of positive characteristic.

Also in the second chapter, we investigate the Frobenius pullback $X^{(p)}$ of the Y-scheme X, together with the relative Frobenius morphism $F_{X/Y}$, that factors into f

followed by other proper dominant morphism $f^{(p)} : X^{(p)} \to Y$, with geometrically integral algebraic curve as generic fiber. By adopting classical ideas from the theory of function fields in one variable, we compare the generic fibers of f and $f^{(p)}$, in order to obtain another beautiful characterization of not generically smooth morphisms, displayed in Proposition 2.8.

Proposition 1.3. *Let $f : X \to Y$ be a proper dominant morphism between algebraic varieties, with a geometrically integral curve as generic fiber. Then the horizontal prime divisors contained in the singular locus of $X^{(p)}$ are precisely the images, by $F_{X/Y}$, of the horizontal prime divisors contained in the nonsmooth locus of f.*

Chapter three is devoted to classifying regular but nonsmooth complete and geometrically integral algebraic curves over a fixed field K. Since the arithmetic genus is invariant under base change we have that a regular complete and geometrically integral algebraic curve C is nonsmooth if and only if its geometric genus decreases on extending its base field to the algebraic closure $\overline{K}$. By looking at the function field $K(C)|K$ in one variable, this means that the genus decreases when we pass from $K(C)|K$ to $K(C)\overline{K}|\overline{K}$. According to Artin [Ar], function fields admitting genus drop are called nonconservative. Therefore, it is equivalent to classify regular but nonsmooth complete and geometrically integral algebraic curves, over a fixed field, and nonconservative separably generated function fields, whose field of coefficients is algebraically closed in it. Thus, in order to obtain a geometric background, we adopt the most important objects from the classical approach of nonconservative function fields.

By a theorem of Tate [T], we obtain an upper bound for the characteristic p of the base field of a regular but nonsmooth curve C, in terms of its geometric genus g, as follows.

$$p \leq 2g + 1$$

nonconservative function fields of genus one were classified by Queen [Q], and of genus two by Borges Neto [BN]. Considering genus three, the above inequality implies that seven, five, three and two are the possible characteristics. Stichtenoth [St] has treated the case of characteristic seven inside the general case of genus $(p-1)/2$ with $p > 2$, while Stöhr and Villela [SVi] have treated the case of characteristic five in the general case of genus $(p+1)/2$ with $p \geq 5$. In this work we begin to classify the case of genus three and characteristic three.

Our method is to obtain an affine plane curve, birationally equivalent to C, by studying the smoothness of the normalization $\widetilde{C^{(p^n)}}$ of the iterated Frobenius pullback $C^{(p^n)}$ of C, for some positive integer n. However, the affine curve does not reflect precisely information about nonsmooth points of C, although the Jacobian criterion provides the possibilities (see Corollary 3.25). On the other hand, it can be obtained by extending the analysis provided by Bedoya and Stöhr [BS] between local properties at $P \in C$ and $\widetilde{F^n_{C/K}}(P) \in \widetilde{C^{(p^n)}}$, where $\widetilde{F^n_{C/K}} : C \to \widetilde{C^{(p^n)}}$ is constructed by lifting the iterated Frobenius morphism $F^n_{C/K} : C \to C^{(p^n)}$. Roughly speaking, they compare degrees, semigroups and conductors of these points, or equivalently, their geometric singularity degrees, that measure their smoothness.

If the characteristic of the base field K is seven or five, then a regular but nonsmooth curve of genus three admits only one nonsmooth point. But the number of nonsmooth points may grow if we assume characteristic three. By making a systematic study we present all possibilities (see Table 3.1 and Table 3.2) and we describe one of them, as stated in Theorem 3.22.

Theorem 1.4. *Let C be a regular and complete algebraic curve over a field K of characteristic three. Then C is geometrically integral of genus three and admits a nondecomposed nonsmooth point of geometric singularity degree three, with rational image under $\widetilde{F_{C/K}}$, if and only if it is birationally equivalent to an affine plane curve given by the polynomial*

$$Y^3 - a_6X^6 - a_3X^3 - X^2 \in K[X, Y]$$

where $a_6 \in K \setminus K^3$ and $a_3 \in K$. Moreover, a second regular and complete curve over K, birational to an affine plane curve, given by $Y^3 - a'_6X^6 - a'_3X^3 - X^2$ with $a'_6 \in K \setminus K^3$ and $a'_3 \in K$, is isomorphic to C if and only if there are $c_1, c_2, d \in K$, with $d \neq 0$, satisfying

$$a'_6 = \frac{c_2^3 + d^6a_6}{d^{18}} \quad \textit{and} \quad a'_3 = \frac{c_1^3 + d^6a_3}{d^9}.$$

In order to obtain fibrations by nonsmooth curves and investigate their geometric properties, we finish this chapter studying, as in [S2] and [S4], when these curves are canonically embedded into the projective plane. In fact, all of these curves admit a such embedding, as stated in Theorem 3.29.

In Chapter 4 we construct a two dimensional fibration by nonsmooth projective quartics $\pi : T \to \mathbb{A}^2(k)$, that is universal in the sense that the data about all fibrations by nonsmooth plane quartics, whose generic fiber satisfies the hypothesis in the above theorem, are condensed in it. The total space of π is the rational threefold

$$T \subset \mathbb{P}^2(k) \times \mathbb{A}^2(k)$$

given by the polynomial $ZY^3 - aZ^4 - bZX^3 - X^4$. Thus, as stated in Theorem 4.1 and Corollary 4.2 we summarize the geometric questions studied in Chapter 3 and information about fibration by nonsmooth curves, as follows.

Theorem 1.5. *1. Let us consider k be an algebraically closed field of characteristic three. If C is an integral affine plane curve defined by the irreducible polynomial $\varphi(x,y) \in k[x,y]$, then the restricted projection morphism $\pi^{-1}(C) \to C$ is a fibration by nonsmooth curves if and only if*

$$\varphi(x,y) \notin k[x,y^3].$$

Also, the restricted fibration $\pi^{-1}(C) \to C$ admits a factorization by a rational double cover followed by a fibration by plane projective cuspidal cubics if and only if $\varphi \in k[x^3,y]$.

2. Each fibration by nonsmooth plane projective quartic curves with a point of singularity degree three, whose generic fiber satisfies the hypothesis of Theorem 3.22, is up to birational equivalence obtained by a base extension either from the two-dimensional fibration $\pi : T \to \mathbb{A}^2$ or from an one-dimensional fibration $\pi^{-1}(C) \to C$, obtained by restricting the base of π to an irreducible curve C on $\mathbb{A}^2$.

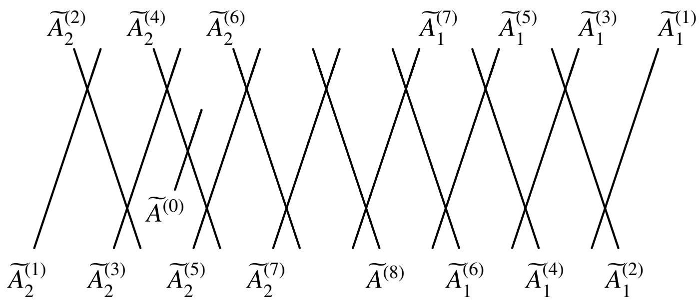
Corollary 1.6. *Almost all fibers of a fibration by nonsmooth plane projective quartic curves with a point of singularity degree three, whose generic fiber satisfies the hypothesis of Theorem 3.22, are non-classical curves and admit a unique Weierstrass point.*

In analogy to the Kodaira–Néron classification of special fibers of minimal fibrations by elliptic curves (cf. [Ko] and [N]), we finish Chapter 4 by describing the minimal proper regular model of some fibrations by nonsmooth curves over the projective line, and determine the structure of their bad fibers. For instance if we consider the fibration $\eta : S \to \mathbb{P}^1(k)$ in the first example, we may obtain its minimal model as follows. First of all, by resolving the singularity of S we obtain a fibration by curves $\widetilde{\eta} : \widetilde{S} \to \mathbb{P}^1(k)$ birationally equivalent to η, where $\widetilde{S}$ is a smooth surface. By the Castelnuovo's contractibility criterion on smooth surfaces we may contract the rational fiber components of self-intersection -1 and obtain a regular minimal model of η. Since the arithmetic genus of almost all fibers of η is different from zero we may obtain the uniqueness of the minimal model, up to isomorphism over the base curve $\mathbb{P}^1(k)$, by a variant of Enrique's theorem on minimal models of algebraic surfaces (see Lichtenbaum [Li] Theorem 4.4 or Shafarevich [Sh1] p.155). Therefore, it is uniquely determined by the function field in one variable $k(S)|k(\mathbb{P}^1(k))$. As displayed in Theorem 4.4, we present the minimal model of η, as follows.

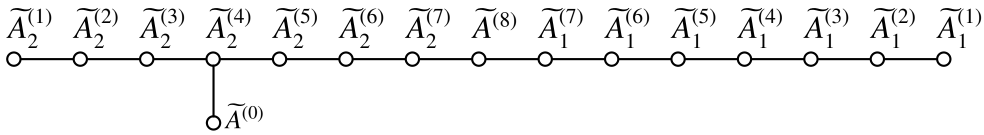
Theorem 1.7. *The fibration $\widetilde{\eta} : \widetilde{S} \to \mathbb{P}^1(k)$ is the minimal proper regular model of the fibration $\eta : S \to \mathbb{P}^1(k)$ by nonsmooth curves. Its fiber over $(1:t)$ coincides with the integral rational fiber $\eta^*(1:t)$ of the original fibration η, for each t in $\mathbb{A}^1(k)$, and over $(0:1)$ is a linear combination of smooth rational curves*

$$\widetilde{\eta}^*(0:1) = 4\widetilde{A}^{(0)} + \widetilde{A}_1^{(1)} + 3\widetilde{A}_2^{(1)} + 2\widetilde{A}_1^{(2)} + 6\widetilde{A}_2^{(2)} + 3\widetilde{A}_1^{(3)} + 9\widetilde{A}_2^{(3)} + 4\widetilde{A}_1^{(4)} +$$
$$12\widetilde{A}_2^{(4)} + 5\widetilde{A}_1^{(5)} + 11\widetilde{A}_2^{(5)} + 6\widetilde{A}_1^{(6)} + 10\widetilde{A}_2^{(6)} + 7\widetilde{A}_1^{(7)} + 9\widetilde{A}_2^{(7)} + 8\widetilde{A}^{(8)}$$

whose intersection configurations are obtained from the diagram:



or equivalently from the Coxeter–Dynkin diagram:



Chapter 2

Preliminaries

2.1 Not Generically Smooth Morphisms

In order to study the behavior of moving singularities on families of curves, we relate in this section, the existence of this phenomenon with the existence of singular points of a certain curve, called geometric generic fiber.

Throughout the present chapter we will assume that k is an algebraically closed field. Algebraic varieties are always required to be integral schemes of finite type, over the field k, unless otherwise stated.

Let us consider a dominant morphism between two algebraic varieties

$$f : X \to Y.$$

In this situation, we can identify the function field $k(Y)$ of Y as a subfield of the function field $k(X)$ of X. In the classical situation, this can be done by identifying rational functions on Y with rational functions on X that are constant along the fibers of the morphism f.

Let us assume that $k(Y)$ is algebraically closed in $k(X)$ and the field extension $k(X)|k(Y)$ is separably generated (i.e., there are indeterminates $x_1, \dots, x_n \in k(X)$ over $k(Y)$ such that the field extension $k(X)|k(Y)(x_1, \dots, x_n)$ is algebraic and separable). Matsusaka [Ma] has shown that these assumptions are equivalent to that almost all fibers of f be integral, where the fiber of f over each point $y \in Y$ is the scheme

$$X_y := X \times_Y \operatorname{Spec} k(y)$$

over $k(y)$, where $k(y)$ is the residual field of y, that is, the quotient of the local ring $\mathcal{O}_{Y,y}$ of Y at y by its maximal ideal $\mathfrak{m}_{Y,y}$.

By a regular point of a scheme we mean a point with regular local ring. Otherwise, we say that the point is nonregular or singular. On the other hand, given a scheme V over a not necessarily algebraically closed field K and $P \in V$, we say that P is a smooth point of V if all points of

$$V \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}$$

lying over P are regular points, where $\overline{K}$ is the algebraic closure of K. Moreover, V is called smooth if all of its points are smooth. It is possible to find, in the literature, others equivalent definitions for smooth point. To check these equivalences we refer to [Liu]. Obviously, if K is an algebraically closed field, then the definitions of smooth and regular point coincide.

We also say that the morphism f is smooth at $x \in X$ if it is flat at x and x is a smooth point of the fiber $X_{f(x)}$, as a curve over $k(f(x))$. We also say that f is smooth if it is smooth at all points of X.

As we are intend to study moving singularities on families of curves, we will assume that f is a proper morphism and

$$\dim X = \dim Y + 1.$$

It follows from these assumptions that almost all fibers of f are integral projective curves.

It can be proved that the subset of X of smooth points of the fibers of f, called smooth locus of f, is an open subset of X. We wish to investigate the existence of irreducible closed subvarieties of X, covering Y by f, whose points are nonsmooth points of the fibers of f. In other words, we wish to investigate not generically smooth morphisms, in the sense that the complementary set of the smooth locus, also called nonsmooth locus of f, contains prime divisors whose image surject onto Y. A prime divisor of X, whose image surjects onto Y, will be called horizontal.

Let us consider the generic point η of Y and

$$X_\eta := X \times_Y \operatorname{Spec} k(Y)$$

the generic fiber of the morphism f. Observe that we have the following bijective correspondence.

$$\left\{\begin{matrix}\text{subvarieties} \\ \text{of } X \text{ covering} \\ Y \text{ by } f\end{matrix}\right\} \longleftrightarrow X_\eta$$

Hence, a natural question is to identify the properties that characterize the points of X_η corresponding to the horizontal irreducible components of the nonsmooth locus of f. However, it is necessary to consider a preceding question. In fact, we have to identify which points of X_η correspond to the horizontal proper subvarieties of X. To analyze these questions we point out an important local property that X_η satisfies.

Let x be a point of X_η. By looking at smaller affine neighborhoods of η and x, we can easily see the following isomorphism of local rings:

$$\mathcal{O}_{X_\eta,x} \simeq \mathcal{O}_{X,x} \tag{2.1}$$

Its first consequence is that the finite type scheme X_η over $k(Y)$ is integral. Hence, by restricting the bijective correspondence, previously obtained, we have the following correspondence:

$$\left\{ \begin{array}{c} \text{horizontal prime} \\ \text{divisors of } X \end{array} \right\} \longleftrightarrow \{\text{closed points of } X_\eta\} \tag{2.2}$$

Before characterizing the desired horizontal proper subvarieties of X, we will make some remarks.

Since f is a finite type morphism we have that f is generically flat, that is, there exists a nonempty open subset U of Y, such that the restricted morphism $f|_{f^{-1}(U)}$ is flat. Hence we conclude that

$$\dim X_\eta = \dim X - \dim Y = 1.$$

Therefore X_η is an integral algebraic curve over $k(Y)$.

To analyze the horizontal prime divisors contained in the nonsmooth locus of f, we are allowed to restrict ourselves to a dense open subset of Y. In this way, let us assume Y regular and f flat.

Let x be a closed point of X_η and $Z = \overline{\{x\}}$ its correspondent horizontal prime divisor of X. Since x is the generic point of Z and regularity is an open property, it follows from (2.1) that x is a nonregular point of X_η if and only if the points of an open dense subset of Z are nonregular points of X, and hence nonregular points of the fibers over its images by f. Therefore the singular locus of X is contained in the nonsmooth locus of f.

We summarize this in the following bijective correspondence, obtained by restricting (2.2).

$$\left\{ \begin{array}{c} \text{horizontal prime divisors} \\ \text{contained in the singular} \\ \text{locus of } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{nonregular closed} \\ \text{points of } X_\eta \end{array} \right\} \tag{2.3}$$

Example 2.1. Let us consider the restriction of the base of the morphism in the first example given in the introduction, that is,

$$X = V(zy^3 - tz^4 - x^4) \subset \mathbb{P}^2(k) \times \mathbb{A}^1(k)$$

and

$$Y = \mathbb{A}^1(k) = \operatorname{Spec} k[t]$$

where k is an algebraically closed field of characteristic three and f is given by the restriction of the second projection

$$\mathbb{P}^2(k) \times \mathbb{A}^1(k) \longrightarrow \mathbb{A}^1(k)$$

to X.

There exists just one horizontal prime divisor contained in the nonsmooth locus of f, which is given by the ideal

$$\frac{(x, y^3 - t)}{(y^3 - t - x^4)}$$

in the affine open subset $\operatorname{Spec} \frac{k[x,y,t]}{(y^3-t-x^4)}$ of X.

As we can see in this example, there exist horizontal prime divisors of X, containing nonsmooth points of the fibers of f, that contain only regular points of X.

In general, a second consequence of the local ring isomorphism (2.1) is the regularity of X_η if X is assumed to be regular. In the case that f is a not generically smooth morphism between regular algebraic varieties, the last conclusion seems to be somewhat contradictory, because in characteristic zero almost all fibers inherit the properties of the generic fiber. To understand this phenomenon we present the following example.

Example 2.2. Let us consider P be the point of $X_\eta = \operatorname{Spec} \frac{k(t)[x,y]}{(y^3-t-x^4)}$ given by the ideal

$$P = \frac{(x, y^3 - t)}{(y^3 - t - x^4)}.$$

It is a regular point, because the relation $y^3 - t = x^4$, in the local ring $\mathcal{O}_{X_\eta,P}$, implies that the quotient $\mathfrak{m}_{X_\eta,P}/\mathfrak{m}^2_{X_\eta,P}$ is an one-dimensional vector space generated by x. On the other hand, by the Jacobian criterion, the closed point

$$\frac{(x, y - t^{1/3})}{(y^3 - t - x^4)} \in X_\eta \times_{\operatorname{Spec} k(t)} \operatorname{Spec} \overline{k(t)},$$

which lies over P, is the unique nonregular point of $X_\eta \times_{\operatorname{Spec} k(t)} \operatorname{Spec} \overline{k(t)}$.

Hence, we can formulate this phenomenon in the general case as follows.

Proposition 2.3. *Let $f : X \to Y$ be a proper dominant morphism between algebraic varieties and X_η be its generic fiber. Then we have the following bijective correspondence.*

$$\left\{\begin{array}{c}\textit{horizontal prime divisors}\\ \textit{contained in the}\\ \textit{nonsmooth locus of } f\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{nonsmooth closed}\\ \textit{points of } X_\eta\end{array}\right\} \tag{2.4}$$

Proof. We start the proof, by denoting

$$\overline{X_\eta} := X_\eta \times_{\operatorname{Spec} k(Y)} \operatorname{Spec} \overline{k(Y)}.$$

Let P and Q be closed points of X_η and $\overline{X_\eta}$, respectively, with Q lying over P. Given a nonempty affine open subset V of Y, there exists a nonempty affine open subset U of $f^{-1}(V)$ such that $U \cap \overline{\{P\}}$ is nonempty. Since f is of finite type, we have that $\mathcal{O}_X(U)$ is a finitely generated $\mathcal{O}_Y(V)$-algebra.

Hence, by restricting X and Y to these open subsets we may suppose that

$$Y = \operatorname{Spec} \frac{k[T_1, \ldots, T_m]}{(F_1, \ldots, F_r)}.$$

and

$$X = \operatorname{Spec} \left(\frac{\frac{k[T_1,\ldots,T_m]}{(F_1,\ldots,F_r)}[S_1, \ldots, S_n]}{(g_1, \ldots, g_s)} \right)$$

Since f is a dominant morphism, we have the inclusion of rings $\mathcal{O}_Y(Y) \subseteq \mathcal{O}_X(X)$. On the other hand,

$$X \simeq \operatorname{Spec} \left(\frac{k[T_1, \ldots, T_m, S_1, \ldots, S_n]}{(F_1, \ldots, F_r, G_1, \ldots, G_s)} \right)$$

where G_i is a polynomial in these $m + n$ variables T_j and S_l, whose image under the surjective homomorphism of polynomial rings

$$k[T_1, \ldots, T_m, S_1, \ldots, S_n] \to \frac{k[T_1, \ldots, T_m]}{(F_1, \ldots, F_r)}[S_1, \ldots, S_n]$$

is g_i, for each $i = 1, \ldots, s$. Therefore, we may view f as a restriction of the natural projection $\mathbb{A}^m(k) \times \mathbb{A}^n(k) \to \mathbb{A}^m(k)$ to X. Moreover, these assumptions yield

$$X_\eta = \operatorname{Spec} \left(\frac{k(Y)[S_1, \ldots, S_n]}{(g_1, \ldots, g_s)} \right), \quad \overline{X_\eta} = \operatorname{Spec} \left(\frac{\overline{k(Y)}[S_1, \ldots, S_n]}{(g_1, \ldots, g_s)} \right)$$

where $\overline{k(Y)}$ is the algebraic closure of $k(Y)$ and

$$Q = \frac{(S_1 - q_1, \ldots, S_n - q_n)}{(g_1, \ldots, g_s)}, \quad P = \frac{(S_1 - q_1, \ldots, S_n - q_n)}{(g_1, \ldots, g_s)} \bigcap \frac{k(Y)[S_1, \ldots, S_n]}{(g_1, \ldots, g_s)}$$

with $q_1, \ldots, q_n$ belonging to $\overline{k(Y)}$.

We notice that the regularity at points of fibers of a finite type morphism is stable under base change (see [EGA IV] 6.8.2). Hence, by passing from Y to $Y' := Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[q_1, \ldots, q_n]$ we may assume that $q_1, \ldots, q_n$ belong to the field $k(Y)$, that is, P is rational.

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

If necessary, after doing the same argument as above, we can assume that all points of X_η that are nonsmooth or correspond to the horizontal prime divisors contained in the nonsmooth locus of f are rational. Hence let us suppose that P is one of these points.

With the above notations we can write

$$q_1 = u_1/h_1, \ldots, q_n = u_n/h_n$$

where $u_1, \ldots, u_n, h_1, \ldots, h_n$ are regular functions on Y(i.e. they belong to $\mathcal{O}_Y(Y)$) and $h_1, \ldots, h_n$ are different from zero. Therefore,

$$P = \frac{(h_1 \cdot S_1 - u_1, \ldots, h_n \cdot S_n - u_n)}{(g_1, \ldots, g_s)}$$

and the horizontal proper subvariety Z of X associated to P is the closed subset of X

$$V(h_1 \cdot S_1 - u_1, \ldots, h_n \cdot S_n - u_n).$$

Since Y is irreducible and $h_1, \ldots, h_n$ are different from zero, we conclude that $\widetilde{V} = Y \setminus V(h_1, \ldots, h_n)$ is a nonempty open subset of Y. Moreover, if we denote by $c = (c_1, \ldots, c_m)$ a closed point of $\widetilde{V}$ given by the ideal $(T_1 - c_1, \ldots, T_m - c_m)$, then the intersection $Z \cap f^{-1}(c)$ is exactly the closed point

$$P_c := \left(\frac{u_1(c)}{h_1(c)}, \ldots, \frac{u_n(c)}{h_n(c)} \right)$$

and the fiber of f over c is

$$X_c = \operatorname{Spec} \left(\frac{k[S_1, \ldots, S_n]}{(\widetilde{g_1}, \ldots, \widetilde{g_s})} \right),$$

where $\widetilde{g_i} := g_i(c, S_1, \ldots, S_n)$, for each $i = 1, \ldots, s$.

We notice that the Jacobian matrix of X_c at P_c,

$$J(P_c) = \left(\frac{\partial \widetilde{g_i}}{\partial S_j}(P_c)\right)_{1 \le i \le s, 1 \le j \le n}$$

can be obtained from the Jacobian matrix of $\overline{X_\eta}$ at Q,

$$J(Q) = \left(\frac{\partial g_i}{\partial S_j}(q_1, \ldots, q_n)\right)_{1 \le i \le s, 1 \le j \le n},$$

by specializing the quotients of polynomials $q_1, \ldots, q_n$ in $c \in \widetilde{V}$.

Thus, we may compare the rank of these matrices, as follows

$$\operatorname{rank} J(P_c) \le \operatorname{rank} J(Q),$$

and obtain the equality on a nonempty open subset of $\widetilde{V}$. Therefore, the points that Q determines on the fibers over each point of this nonempty open subset of $\widetilde{V}$ are nonregular if and only if Q is a nonregular point of $\overline{X_\eta}$. □

As we can see in the proof of the previous proposition, almost all fiber of the morphism f inherit the properties of the curve

$$X_\eta \times_{\operatorname{Spec} k(Y)} \operatorname{Spec} \overline{k(Y)}.$$

Also, as it was mentioned in the begin of this chapter, another inherited property is the integrality of almost all fibers. Indeed, by the assumptions on the field extension $k(X)|k(Y)$, given in the first page of this section, we may conclude that the tensor product $k(X) \otimes_{k(Y)} \overline{k(Y)}$ is a field. In this case X_η is called geometrically integral. By these strong relations, this curve over $\overline{k(Y)}$ is called geometric generic fiber of f.

In view of such connection, we can point out another interesting property, in the following example.

Example 2.4. Let us consider the same morphism and the same points of the last example. We can verify, by the formula which relates the singularity degree at a given point and the multiplicities at the points obtained from its successive blow-ups (cf. [An], Korollar II 1.8), that the point

$$(0, t^{1/3}) \in \overline{X_\eta}$$

has singularity degree three, while for each $t \in \mathbb{A}^1(k)$ the point $(0, t^{1/3})$ has the same singularity degree, as a point of the fiber X_t.

Remark 2.5. By using the same arguments given in the end of the proof of the above proposition and in the previous example, we are able to derive another property of almost all fibers that is inherited from the generic fiber. Indeed, let $f : X \to Y$ be a proper dominant morphism between algebraic varieties and X_η be its generic fiber. In additional, let us assume that X_η is locally an affine plane curve. Then the singularity degree at each point of the geometric generic fiber $\overline{X_\eta}$ is the same at the unique point determined by it in almost all fiber of f. However, it seems that we don't need the assumption that the geometric generic fiber is plane.

2.2 Frobenius Pullback

In this section we shall give a nice interpretation of not generically smooth morphisms, by looking at its special factorization given by the relative Frobenius morphism. Before obtaining this relation it is necessary to develop some basic notations and highlight the importance of Frobenius pullback, in order to provide a different criterion for smoothness of curves.

2.2.1 Relative Frobenius Map

In this subsection we shall always consider schemes over $\mathbb{F}_p$, where p is a fixed positive prime number. For a more detailed approach we refer to [Liu] page 94.

Let S be a scheme and

$$F_S : S \to S$$

be the absolute Frobenius morphism of S, that is, the map induced by the following ring homomorphism.

$$\begin{array}{ccc} O_S & \to & O_S \\ a & \mapsto & a^p \end{array}$$

In addition, if we consider a scheme X over S, that is, a scheme together a morphism $\pi : X \to S$, then the Frobenius pullback of the S-scheme X is the S-scheme

$$X^{(p)}$$

obtained by pulling back of π via F_S. In other words, $X^{(p)}$ is the fibered product $X \times_S S$, where the second factor S is endowed with structure of a S-scheme via F_S.

We have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ S \arrow[r, "F_S"] & S \end{tikzcd}$$

and, by the universal property of the fibered product, there exists a unique morphism

$$F_{X/S} : X \to X^{(p)}$$

making the following diagram commutative:

$$\begin{tikzcd}
& X \arrow[dr, "\pi"] & \\
X \arrow[ur, "F_X"] \arrow[r, "F_{X/S}"'] \arrow[dr, "\pi"'] & X^{(p)} \arrow[u, "\pi_1"'] \arrow[r] \arrow[d, "\pi_2"] & S \\
& S \arrow[ur, "F_S"'] &
\end{tikzcd}$$

where π_1 and π_2 are the first and the second projections of $X \times_S S$, respectively.

The morphism $F_{X/S}$ is called the relative Frobenius morphism. In the case that S is the spectrum of a field K, we also denote $F_{X/S}$ by

$$F_{X/K}.$$

By using the last commutative diagrams, we can see that $F_{X/S}$ is in fact a homeomorphism.

Remark 2.6. For instance, if we assume that $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$, then $X^{(p)} = \operatorname{Spec}(B \otimes_A A)$, where the second factor in the tensor product is endowed with structure of A-algebra via $a \mapsto a^p$. Moreover, the morphisms F_X and $F_{X/S}$ correspond to the ring homomorphisms $b \mapsto b^p$ and $b \otimes a \mapsto b^p a$, respectively. Hence, in the particular case that A is an $\mathbb{F}_p$-algebra and

$$X = \operatorname{Spec} \frac{A[T_1, \ldots, T_n]}{I}$$

we can see that

$$X^{(p)} = \operatorname{Spec} \frac{A[T_1, \ldots, T_n]}{I^{(p)}}$$

where $I^{(p)}$ is the ideal generated by the elements of the form

$$f^{(p)} := \sum a^p_{i_1 \ldots i_n} T_1^{i_1} \cdots T_n^{i_n}$$

with $f = \sum a_{i_1 \ldots i_n} T_1^{i_1} \cdots T_n^{i_n} \in I$.

2.2.2 Relation With Not Generically Smooth Morphisms

In this subsection we relate not generically smooth morphisms with properties of the Frobenius pullback, by discussing the image under the relative Frobenius morphism at closed points of affine varieties over a fixed algebraically closed field k of positive characteristic p.

Let $X = \operatorname{Spec} k[T_1, \ldots, T_n]/I$ be an affine variety over k. The set of closed points of X is identified with

$$Z(I) := \{(a_1, \ldots, a_n) \in k^n \mid f((a_1, \ldots, a_n)) = 0 \ \ \forall f \in I\},$$

while the set $Z(I^{(p)})$ of closed points of $X^{(p)}$ can be identified in the same way. Thus, we can easily see that the restriction of the absolute Frobenius morphism $F_{X/k}$ to these sets, is given by the restriction of the, well known, endomorphism of $\mathbb{A}^n(k)$

$$\begin{array}{ccc} k^n & \longrightarrow & k^n \\ (a_1, \ldots, a_n) & \longmapsto & (a_1^p, \ldots, a_n^p) \end{array}$$

to $Z(I)$.

Another interesting case to be observed, is the morphism $f : X \to Y$ between algebraic varieties over k, considered in the previous subsection. In this case, X can be viewed as a variety over Y and the relative Frobenius morphism

$$F_{X/Y} : X \to X^{(p)}$$

factors into f followed by the second projection π_2, also denoted by $f^{(p)}$.

$$\begin{tikzcd} X \arrow[dd, "f"'] \arrow[dr, "F_{X/Y}"] & \\ & X^{(p)} \arrow[dl, "f^{(p)}"] \\ Y & \end{tikzcd}$$

We notice that $f^{(p)}$ inherits the required properties of f, that is, it is a flat and dominant morphism with geometrically integral algebraic curve, over $k(Y)$, as generic fiber.

By the local characterization, obtained in the begin of the proof of Proposition 2.3, we will assume that

$$Y = \operatorname{Spec} \frac{k[T_1, \ldots, T_m]}{(F_1, \ldots, F_r)}\ , \ \ X = \operatorname{Spec} \frac{k[T_1, \ldots, T_m, S_1 \ldots, S_n]}{(F_1, \ldots, F_r, G_1, \ldots, G_s)}$$

and f is the restriction of the second projection $\mathbb{A}^m(k) \times \mathbb{A}^n(k) \to \mathbb{A}^m(k)$ to X.

By looking to the previous remark we may see that

$$X^{(p)} = \operatorname{Spec} \frac{k[T_1, \dots, T_m, S_1 \dots, S_n]}{(F_1, \dots, F_r, G_1^{(p)}, \dots, G_s^{(p)})}$$

where

$$G^{(p)} := \sum a_{i_1 \dots i_n}(S_1, \dots, S_n)^p T_1^{i_1} \cdots T_n^{i_n}$$

if we consider $G = \sum a_{i_1 \dots i_n}(S_1, \dots, S_n) T_1^{i_1} \cdots T_n^{i_n} \in k[T_1, \dots, T_m, S_1 \dots, S_n]$.

Moreover, the restriction of the relative Frobenius morphism $F_{X/Y}$ sends the set

$$Z(F_1, \dots, F_r, G_1, \dots, G_s) \subseteq k^m \times k^n,$$

of closed points of X, onto the set

$$Z(F_1, \dots, F_r, G_1^{(p)}, \dots, G_s^{(p)}) \subseteq k^m \times k^n,$$

of closed points of $X^{(p)}$, by the following map.

$$\begin{array}{ccc} k^m \times k^n & \longrightarrow & k^m \times k^n \\ (y_1, \dots, y_m, x_1, \dots, x_n) & \longmapsto & (y_1, \dots, y_m, x_1^p, \dots, x_n^p) \end{array}$$

Example 2.7. Let us consider the restriction of the morphism f used in the previous examples, that is, the restriction of the natural projection of $\mathbb{A}^2(k) \times \mathbb{A}^1(k)$ over $\mathbb{A}^1(k)$ to the surface

$$X = V(y^3 - t - x^4) \subset \mathbb{A}^2(k) \times \mathbb{A}^1(k)$$

where the characteristic of k is three.

In this situation, the Frobenius pullback of the Y-scheme X is equal to the surface

$$X^{(p)} = V(y^3 - t^3 - x^4) \subset \mathbb{A}^2(k) \times \mathbb{A}^1(k).$$

Let us consider the horizontal prime divisor $V(x, y^3 - t)$ contained in the non-smooth locus of f. We notice that the image of its closed points

$$\{((0, t^{1/3}), t) \mid t \in \mathbb{A}^1(k)\}$$

under the relative Frobenius morphism $F_{X/Y}$ is equal to the set

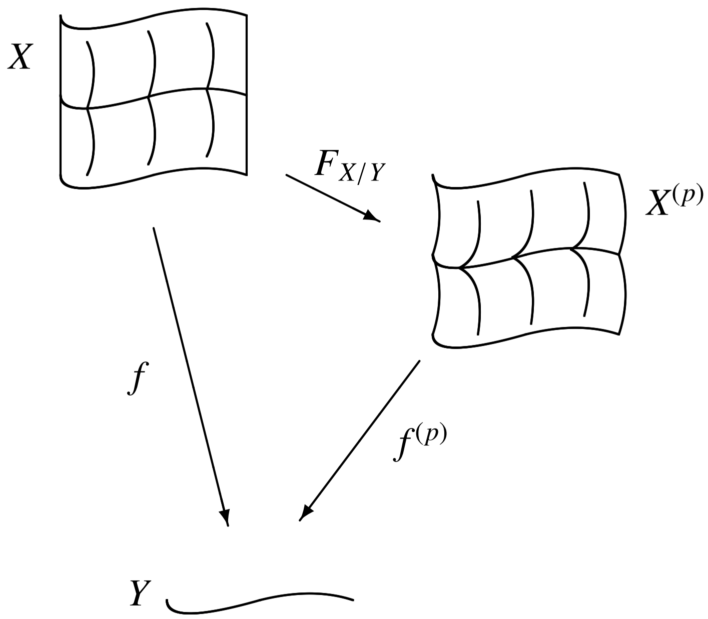
$$\{((0, t), t) \mid t \in \mathbb{A}^1(k)\}$$

of closed points of the horizontal prime divisor $V(x, y - t)$ contained in the nonsmooth locus of $f^{(p)}$.

On the other hand, by the Jacobian criterion, the closed points of $V(x, y - t)$ are exactly the singular closed points of $X^{(p)}$.

This raises the interesting question: how general is this phenomenon? We answer this question with the following result.

Proposition 2.8. *Let $f : X \to Y$ be a proper dominant morphism between algebraic varieties, with geometrically integral curve as generic fiber. Then, the horizontal prime divisors contained in the singular locus of $X^{(p)}$ are precisely the images, by $F_{X/Y}$, of the horizontal prime divisors contained in the nonsmooth locus of f.*



To show this proposition we recall the correspondence between horizontal prime divisors of $X^{(p)}$, contained in its singular locus, and singular points of the generic fiber $(X^{(p)})_\eta$ of the morphism $f^{(p)} : X^{(p)} \to Y$, as in (2.3). Since the generic fiber $(X^{(p)})_\eta$ is exactly the Frobenius pullback of X_η, viewed as a curve over $k(Y)$, that is,

$$(X^{(p)})_\eta = X_\eta^{(p)},$$

we just need to understand the singularities of $X_\eta^{(p)}$.

To do this, we notice that f is a proper morphism and hence the geometrically integral algebraic curve X_η, over $k(Y)$, is complete. Therefore, it remains to analyze the singularities of the Frobenius pullback of a complete and geometrically integral algebraic curve over a non necessarily algebraically closed field of positive characteristic, that we leave for the next subsection.

2.2.3 Curve Smoothness

In what follows, C will be a complete and geometrically integral algebraic curve over a non necessarily algebraically closed field K, of positive characteristic p. For convenience, the extended curve $C \times_{\operatorname{Spec} K} \operatorname{Spec} K'$ will be denoted by

$$C \otimes_K K'$$

where K' is a field containing K.

Lemma 2.9. *The nonregular points of the Frobenius pullback $C^{(p)}$ are precisely the images of the nonsmooth points of C, under the relative Frobenius morphism $F_{C/K}$.*

Proof. Firstly, we point out the isomorphism of curves

$$C^{(p)}|K \simeq (C \otimes_K K^{1/p})|K^{1/p}$$

obtained by the product of morphisms $id_C \times F : C^{(p)}|K \to (C \otimes_K K^{1/p})|K^{1/p}$, where id_C is the identity on C and F is induced by the field isomorphism $K^{1/p} \to K$, $a \mapsto a^p$. Moreover, its inverse morphism can be seen as part of the following commutative diagram of homeomorphisms

$$\begin{tikzcd}
 & C \otimes_K K^{1/p} \arrow[dl, "\pi_1"'] \arrow[dd, "id_C \times F^{-1}"] \\
C \arrow[dr, "F_{C/K}"'] & \\
 & C^{(p)}
\end{tikzcd}$$

where π_1 is the first projection. Therefore, it just remains to analyze the nonregular points of $C \otimes_K K^{1/p}$.

Since the field extension $K^{1/p}|K$ is purely inseparable we have that π_1 is an homeomorphism. We still have that the local ring at the point of the extended curve, lying over a point $P \in C$, is identified with $\mathcal{O}_{P,C}K^{1/p}$. Hence the points of $C \otimes_K K^{1/p}$ lying over nonregular points of C are also nonregular.

On the other hand, as a immediate consequence of the local version of Kimura's theorem (cf. [S1] Corollary 3.2 or Proposition 3.3), the nonregular points of $C \otimes_K K^{1/p}$, lying over regular points of C, correspond exactly to the regular but nonsmooth points of C. ◻

2.3 Fibrations by nonsmooth Schemes

Throughout this section we start to treat the main purpose of this work. In fact, we want to study *fibrations by nonsmooth schemes* between algebraic varieties

$$f : X \to Y$$

in the sense that all fibers are nonsmooth though the total space X is smooth after eventually restricting the base to a dense open subset of Y. Additionally, we always assume fibrations satisfying the required properties of the first section, that is, f will be a dominant proper morphism between algebraic varieties, with geometrically integral curve as generic fiber. In this case f is also called a *fibration by nonsmooth curves*.

Since the total space X is smooth after eventually restricting the base to a dense open subset of Y, the local rings isomorphism (2.1) provides that the generic fiber X_η is a regular algebraic curve over the field $k(Y)$. Thus it is the regular projective model of the algebraic function field $k(X)|k(Y)$. Since fibration by nonsmooth curves must be not generically smooth morphisms, we conclude by Proposition 2.3, that:

$$f \ \text{ is a fibration by nonsmooth curves} \quad \Longleftrightarrow X_\eta \ \text{ is regular but nonsmooth.}$$

In what follows we wish to investigate the birational classes of fibration by nonsmooth curves. We mean that two fibrations $f : X \to Y$, $f' : X' \to Y'$, of varieties over the same algebraically closed field k, are birational equivalent if there is a birational map between the total spaces

$$g : X \dashrightarrow X'$$

and another one between the bases

$$h : Y \dashrightarrow Y',$$

such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, dashed, "h"] & Y' \end{tikzcd}$$

Therefore, two fibrations $f : X \to Y$, $f' : X' \to Y'$ are birational equivalent if and only if there exist an isomorphism between the generic fibers

$$X_\eta|k(Y) \simeq X'_{\eta'}|k(Y')$$

together with an isomorphism between their base fields

$$k(Y) \simeq k(Y'),$$

where η and η' are the generic points of Y and Y', respectively.

In the present work we will treat the birational classification of fibrations by nonsmooth curves, but with a subtle point of view. In fact, we fix the base and allow only birational maps between the total spaces. Equivalently, we wish to classify regular but nonsmooth curves, over a fixed field.

Chapter 3

Classification of Regular but nonsmooth Curves

3.1 Regular but nonsmooth Curves

As we have seen in the last chapter, there is a close relation between nonsmoothness of an algebraic curve and nonregularity of its image under the relative Frobenius morphism. To improve the knowledge on regular but nonsmooth curves we shall investigate the local properties of their Frobenius pullbacks normalizations. This approach will be determinant to obtain the classification of regular but nonsmooth complete and geometrically integral algebraic curves.

In what follows, C will denote a regular complete and geometrically integral algebraic curve, over a non algebraically closed field K, of positive characteristic p. In this way, C is the non-singular projective model of the function field $K(C)|K$, that satisfies the following two important properties: K is algebraically closed in $K(C)$ and $K(C)|K$ is separably generated.

Observe that C is nonsmooth if and only if $C \otimes_K \overline{K}$ is a nonregular algebraic curve, that is, its arithmetic genus is bigger than its geometric genus. Since the arithmetic genus is invariant under base field extensions (see [R] p. 182) and C is a regular curve, this means that the geometric genus of C is bigger than the geometric genus of $C \otimes_K \overline{K}$, that is,

$$g > \overline{g}$$

where g and $\overline{g}$ are the geometric genera of C and $C \otimes_K \overline{K}$, respectively.

In the language of algebraic function field theory, g and $\overline{g}$ are the genera of $K(C)|K$ and $K(C) \otimes_K \overline{K}|\overline{K}$ (or simply, of $K(C)\overline{K}|\overline{K}$), respectively. A function field admitting genus drop was called nonconservative by Artin [Ar]. Thus, classify isomorphism classes of regular but nonsmooth complete and geometrically integral algebraic curves, over a fixed field, is equivalent to classify isomorphism classes of nonconservative and separably generated function fields whose field of coefficients is algebraically closed in the ground field.

Tate [T] has shown that the genus drop $g - \overline{g}$ is a multiple of $(p-1)/2$. Hence it provides an upper bound for the characteristic of the base field of a regular but nonsmooth curve C, in terms of its geometric genus, as follows.

$$p \leq 2g + 1$$

nonconservative function fields of genus one were classified, up to isomorphisms, by Queen [Q] and of genus two by Borges Neto [BN]. Considering genus three, the above inequality implies that seven, five, three and two are the characteristics that can be occur. Stichtenoth [St] has treated the case of characteristic seven inside the general case of genus $(p-1)/2$ with $p > 2$, while Stöhr and Villela [SVi] have treated the case of characteristic five in the general case of genus $(p+1)/2$ with $p \geq 5$.

In the last two cases, the analysis of properties of the Frobenius pullback normalization had exercised a significant role in order to obtain normal forms for regular but nonsmooth algebraic curves. To provide a simple example of this kind of influence, we recall that the iterated Frobenius pullback

$$C^{(p^n)}$$

can be defined, together with the iterated relative Frobenius morphism

$$F^n_{C/K} : C \to C^{(p^n)}$$

by making the iterated Frobenius maps. According to [St] Lemma 5, the rationality of the iterated Frobenius pullback normalization

$$\widetilde{C^{(p^n)}}$$

implies the existence of a normal form for C. However, it is not sufficient because these normal forms do not reflect informations about nonsmooth points. In this way, Bedoya and Stöhr [BS] have made a strong analysis of local properties at points in C, by looking to their images under the lifted Frobenius morphism

$$\widetilde{F^n_{C/K}}$$

that is, the unique morphism that commutes the following diagram.

$$\begin{tikzcd} C \arrow[r, "F^n_{C/K}"] \arrow[d, "\widetilde{F^n_{C/K}}"'] & C^{(p^n)} \\ \widetilde{C^{(p^n)}} \arrow[ur] & \end{tikzcd}$$

In order to continue the classification to the case of characteristic three, we will investigate the behavior of the iterated Frobenius pullback normalization, concerning its smoothness.

In the same manner, as observed in the proof of Lemma 2.9, we still have the isomorphisms

$$C^{(p^n)}|K \simeq (C \otimes_K K^{1/p^n})|K^{1/p^n}$$

for all positive integer n. Hence, by definition, it can be easily extended to its normalizations

$$\widetilde{C^{(p^n)}}|K \simeq (C \widetilde{\otimes_K K^{1/p^n}})|K^{1/p^n}$$

for all positive integer n. Thus, the behavior study of both curves are equivalent. The first property to be observed is the following global result.

Lemma 3.1. *There is a positive integer N such that $\widetilde{C^{(p^n)}}$ is smooth, for all $n \geq N$.*

Indeed, it is a simple consequence of the existence of a purely inseparable finite extension L of K, satisfying the geometric genera equality $p_g(C \otimes_K L) = p_g(C \otimes_K \overline{K})$.

Now we want to improve the knowledge of this phenomenon by finding conditions that determine exactly the mentioned N, as above. To give positive answers, we adopt the local study of function field theory, as we did implicitly in Lemma 2.9.

The local description provided in Remark 2.6 guarantees the identification, via the relative Frobenius morphism $F_{C/K}$, between $K(C^{(p)})$ and the compositum of fields $K(C)^pK$ contained in $K(C)$. On the other hand, $K'(C \otimes_K K') = K(C) \otimes_K K'$ can be also identified with the compositum $K(C)K'$, where $K' := K^{1/p}$.

$$\begin{tikzcd}
 & & K(C)K^{1/p} \arrow[ddd, dash] \\
K(C) \arrow[urr, dash] \arrow[dr, dash] \arrow[ddd, dash] & & \\
 & K(C)^pK \arrow[ddl, dash] & \\
 & & K^{1/p} \\
K \arrow[urr, dash] & &
\end{tikzcd}$$

By the above identifications, the isomorphism between $C \otimes_K K'$ and $C^{(p)}$, or between $C \widetilde{\otimes_K K'}$ and $\widetilde{C^{(p)}}$, reflects the following well known Frobenius isomorphism of function fields.

$$\begin{array}{ccc} K(C)K'|K' & \to & K(C)^pK|K \\ x & \mapsto & x^p \end{array}$$

Since the field extension $K'|K$ is purely inseparable, we may conclude that the following natural morphisms are in fact homeomorphisms.

$$C \longleftarrow C \otimes_K K' \longleftarrow C\widetilde{\otimes_K} K'$$

Indeed, if $P \in C$ then there is a unique discrete valuation on $K(C)K'|K'$, over the discrete valuation v_P on $K(C)|K$, determined by P. Thus, the unique point $P' \in C\widetilde{\otimes_K} K'$ over P is exactly the point determining this discrete valuation.

In addition, if P is a point of C then the local ring at its image by $\widetilde{F_{C/K}}$ (or $F_{C/K}$) on $\widetilde{C^{(p)}}$ (or $C^{(p)}$) is identified as follows.

$$\mathcal{O}_{\widetilde{F_{C/K}}(P),\widetilde{C^{(p)}}} \simeq \mathcal{O}^p_{P',C\widetilde{\otimes_K}K'} \quad \text{and} \quad \mathcal{O}_{F_{C/K}(P),C^{(p)}} \simeq \mathcal{O}^p_{P,C}K$$

Also, if we denote by P_1 the image at P by $\widetilde{F_{C/K}}$, then the discrete valuation v_{P_1} on $K(C)^pK|K$, determined by P_1, is the unique valuation under v_P.

In what follows we will use the concise notation

$$\mathcal{O}_P := \mathcal{O}_{P,C}, \quad \mathcal{O}_{P_1} := \mathcal{O}_{P_1,\widetilde{C^{(p)}}} \quad \text{and} \quad \mathcal{O}_{P'} := \mathcal{O}_{P',C\widetilde{\otimes_K}K'}.$$

To measure how a point P on C is smooth, we consider the *L-singularity degree* at $P \in C$, defined by

$$\dim_L \frac{\widetilde{L\mathcal{O}_{P,C}}}{L\mathcal{O}_{P,C}}$$

where L is an algebraic extension over K and $\widetilde{L\mathcal{O}_{P,C}}$ is the integral closure of the semilocal ring $L\mathcal{O}_{P,C}$ in $K(C)L$. When L is the algebraic closure $\overline{K}$ of K we simply call it by *geometric singularity degree*. In the same manner, this definition can be done for any normal curve.

The geometric meaning of the geometric singularity degree is obtained from its equality with the sum of singularity degrees at all points of $C \otimes_K \overline{K}$ lying above P. Indeed, it can be seen by the Chinese Remainder theorem. Therefore, a point $P \in C$ is smooth if and only if the semilocal noetherian domain $\overline{K}\mathcal{O}_P$ is integrally closed.

In the literature, a nonsmooth point of C is called singular prime of the function field $K(C)|K$.

One way to relate singularity degrees and global informations, of the extended curve, can be done as follows. Since the arithmetic genus is invariant under base field extension and C is a regular curve, we may write Rosenlicht's formula (see [R] p. 182) involving L-singularity degrees, arithmetic and geometric genera of $C \otimes_K L$, by

$$g - g_L = \sum_{P \in C} \dim_L \frac{\widetilde{L\mathcal{O}_P}}{L\mathcal{O}_P}, \tag{3.1}$$

where g and g_L are the geometric genera of C and $C \otimes_K L$, respectively.

To understand the smoothness behavior of $\widetilde{C \otimes_K K'}$ (or equivalently of $\widetilde{C^{(p)}}$), we will study how the geometric singularity degree at $P \in C$ can affect the geometric singularity degree at the unique point $P' \in \widetilde{C \otimes_K K'}$ (or $P_1 \in \widetilde{C^{(p)}}$) lying over (under) P. However, before stating the desired study, we need to compare the geometric singularity degrees at P, P_1 and P' in different ways.

First of all, we observe that the isomorphism of local rings $\mathcal{O}_{P_1} \simeq \mathcal{O}_{P'}^p$ implies the following equality between geometric singularity degrees.

$$\dim_{\overline{K}} \frac{\widetilde{\overline{K}\mathcal{O}_{P'}}}{\overline{K}\mathcal{O}_{P'}} = \dim_{\overline{K}} \frac{\widetilde{\overline{K}\mathcal{O}_{P_1}}}{\overline{K}\mathcal{O}_{P_1}}$$

The second relation to be observed is the following result.

Lemma 3.2. $\dim_{\overline{K}} \frac{\widetilde{\overline{K}\mathcal{O}_P}}{\overline{K}\mathcal{O}_P} = \dim_{K'} \frac{\widetilde{K'\mathcal{O}_P}}{K'\mathcal{O}_P} + \dim_{\overline{K}} \frac{\widetilde{\overline{K}\mathcal{O}_{P'}}}{\overline{K}\mathcal{O}_{P'}}$

Proof. Since $\mathcal{O}_{P'} = \widetilde{K'\mathcal{O}_P}$ we conclude, by the transitivity of integral closure, that $\widetilde{\overline{K}\mathcal{O}_P} = \widetilde{\overline{K}\mathcal{O}_{P'}}$. Hence the inclusion of rings $\overline{K}\mathcal{O}_P \subseteq \overline{K}\mathcal{O}_{P'} \subseteq \widetilde{\overline{K}\mathcal{O}_{P'}}$ provides the following equality of dimensions.

$$\dim_{\overline{K}} \frac{\widetilde{\overline{K}\mathcal{O}_P}}{\overline{K}\mathcal{O}_P} = \dim_{\overline{K}} \frac{\overline{K}\mathcal{O}_{P'}}{\overline{K}\mathcal{O}_P} + \dim_{\overline{K}} \frac{\widetilde{\overline{K}\mathcal{O}_{P'}}}{\overline{K}\mathcal{O}_{P'}}$$

To conclude the proof, we must use the equality in the first line and the invariance of the dimension of vector spaces under base field extension. □

As an immediate consequence of this lemma, we observe that a point of C is nonsmooth if its K'-singularity degree is different from zero. However, the converse seems to be more difficult to imagine. It was proved by Stöhr [S1], Corollary 3.2. We state it, for convenience.

Proposition 3.3. *A point of C is nonsmooth if and only if its $K^{1/p}$-singularity degree is different from zero.*

Other consequence of the previous lemma is the smoothness of the unique point of $\widetilde{C \otimes_K K^{1/p^n}}$ lying over P, for each sufficiently large positive integer n. Actually, in [S1] Corollary 3.6, Stöhr also shows that it happens for n larger than

$$\frac{\log\left(\frac{2\dim_{\overline{K}} \frac{\widetilde{\overline{K}\mathcal{O}_P}}{\overline{K}\mathcal{O}_P}}{p-1}\right)}{\log(p)}.$$

Remark 3.4. In fact, this makes sense because each L-singularity degree is a multiple of $(p-1)/2$, as follows from Corollary 1.2 and Corollary 2.4 in [S1].

The above lower bound implies the smoothness of the unique point of $\widetilde{C \otimes_K K'}$, lying over a point $P \in C$ of geometric singularity degree less than $p(p-1)/2$. But, when P has geometric singularity degree equal to $p(p-1)/2$, the above lower bound just provides the smoothness of the unique point of $\widetilde{C \otimes_K K^{1/p^2}}$ lying over it. In order to understand the behavior of the point of $\widetilde{C \otimes_K K'}$, lying over P, we present the last relation, obtained by Stöhr [S1] Corollary 2.5, that can be written as follows.

$$\dim_L \frac{\widetilde{L\mathcal{O}_P}}{L\mathcal{O}_P} = p \cdot \dim_L \frac{\widetilde{L\mathcal{O}_{P_1}}}{L\mathcal{O}_{P_1}} + \frac{p-1}{2} \cdot \sum_Q m_Q \deg Q \qquad (3.2)$$

where L is an algebraic extension of K, Q runs the points on $\widetilde{C \otimes_K L}$ lying over P and m_Q are non negative integers. In fact, the non negative integers can be expressed as coefficients of a divisor on $\widetilde{C \otimes_K L}$ whose points of its support lie over the points on C of L-singularity degree different from zero. However we prefer to state in this way, because it is enough for our purpose.

Now we are able to state and prove, in a suitable case, the exercised influence of geometric singularity degrees on smoothness of curves obtained by base field extension.

Theorem 3.5. *If the geometric singularity degree at $P \in C$ is equal to $p(p-1)/2$, then the unique point of $\widetilde{C \otimes_K K^{1/p}}$ lying over P is smooth, that is, its geometric singularity degree is equal to zero. Equivalently, the same happens to the image at P on $\widetilde{C^{(p)}}$, under $\widetilde{F_{C/K}}$.*

Proof. As previously denoted, let us consider, $K' := K^{1/p}$ and P' the unique point of $\widetilde{C \otimes_K K'}$ lying over P. We start writing the geometric singularity degree at P as $n(p-1)/2$, where $n \leq p$.

By applying (3.2), with L being the algebraic closure of K, we conclude that

$$\dim_{\overline{K}} \frac{\widetilde{\overline{K}\mathcal{O}_P}}{\overline{K}\mathcal{O}_P} = p \cdot \dim_{\overline{K}} \frac{\widetilde{\overline{K}\mathcal{O}_{P'}}}{\overline{K}\mathcal{O}_{P'}} + m \cdot \frac{(p-1)}{2}$$

where m is a non negative integer.

Hence, the previous remark implies the vanishing of the geometric singularity degree at P', if we suppose $n < p$.

Now let us assume the equality $n = p$ and the non-vanishing of the geometric singularity degree at P'. Hence the previous equality implies that $\dim_{\overline{K}} \frac{\widetilde{\overline{K}O_{P'}}}{\overline{K}O_{P'}} = (p-1)/2$. Moreover, the previous Lemma, says that

$$\dim_{K'} \frac{\widetilde{K'O_P}}{K'O_P} = \frac{(p-1)(p-1)}{2}.$$

Also by the previous lemma, but applied for the curve $\widetilde{C^{(p)}}$ instead of C, we obtain $\frac{(p-1)}{2} = \dim_{K'} \frac{\widetilde{K'O_{P_1}}}{K'O_{P_1}} + \dim_{\overline{K}} \frac{\widetilde{\overline{K}O_{P'_1}}}{\overline{K}O_{P'_1}}$, where P'_1 is the unique point in the normalization of $\widetilde{C^{(p)}} \otimes_K K'$, lying over P_1. Since P_1 is a nonsmooth point, we have by Proposition 3.3 that

$$\dim_{K'} \frac{\widetilde{K'O_{P_1}}}{K'O_{P_1}} = \frac{(p-1)}{2}.$$

By replacing the K'-singularity degrees evidenced above on the equality (3.2), applied to K', we obtain the absurd inequality $p - 1 \geq p$. $\square$

As an immediate consequence of the this theorem and of the lower bound, given by logarithmic expressions, we may present conditions to improve the first lemma of this chapter.

Corollary 3.6. *If the geometric genus drop $g - \overline{g}$ does not exceed $p(p-1)/2$, then the Frobenius pullback normalization $\widetilde{C^{(p)}}$ is smooth. In particular, the geometric genera of $C^{(p)}$ and $C \otimes_K \overline{K}$ coincide.*

To offer support for our method of classification, we present how the singularity degree affects the degree of a point in the normalization of the Frobenius pullback. Previously, we just recall some useful notation.

A point is called rational if its degree is equal to one. If L is an algebraic extension of K and Q is a point of $\widetilde{C \otimes_K L}$, lying over $P \in C$, then its ramification and inertia indexes are denoted by $e(Q|P)$ and $f(Q|P)$, respectively. Its definitions are inherited from the definitions over the induced valuations on the algebraic function fields $K(C)L|L$ and $K(C)|K$. In the same manner, these definitions can be done for any normal, complete and geometrically integral algebraic curve.

Now we present in a suitable case how the geometric singularity degree at a point on C can affect the degree at the unique point on $\widetilde{C^{(p^n)}}$ lying under it.

Corollary 3.7. *Let P be a point on C of geometric singularity degree $p(p-1)/2$. Then the unique point $P_2 \in \widetilde{C^{(p^2)}}$, under P, is rational.*

Proof. If P_1 is rational, then P_2 is also rational, because $\deg P_1 = f(P_1|P_2)\deg P_2$. Now let us assume that P_1 is non-rational. The last theorem implies that P_1 is a smooth point of $\widetilde{C^{(p)}}$, which is equivalent to

$$\dim_{K^{1/p}} \frac{\widetilde{K^{1/p}\mathcal{O}_{P_1}}}{K^{1/p}\mathcal{O}_{P_1}} = 0$$

by Proposition 3.3.

On the other hand, from Lemma 3.2 and the smoothness at P_1 we conclude that

$$\dim_{K^{1/p}} \frac{\widetilde{K^{1/p}\mathcal{O}_P}}{K^{1/p}\mathcal{O}_P} = p(p-1)/2.$$

By the isomorphism between the normalizations of $C^{(p)}$ and $C \otimes_K K^{1/p}$, we obtain the equality between the degrees at P' and P_1, besides the equality between their geometric singularities degrees, remarked previously. Therefore (3.2), applied to $L = K^{1/p}$, says that the degree at P_1 divides p, that is,

$$\deg P_1 = p$$

from the initial assumption.

Since $p = f(P_1|P_2) \cdot \deg P_2$, it just remains to prove that the valuation v_{P_1} is non-ramified over $K(C)^{p^2}K$. Otherwise, since the residual field $K(P_1)$ at P_1 is a purely inseparable extension of K, we would have the nonsmoothness of P_1 (see [St], Satz 2), which contradicts the previous theorem. □

From now we want to describe the possible values for the degree at a point $P \in C$ of geometric singularity degree $p(p-1)/2$. To do this we need to recall some useful objects and facts that were studied in [BS] in a bit special case. In fact, they initially assume that the base field of C is separably closed. However the proofs remain the same, with the assumptions that will be made here. In this way, we will just state the results and leave the proofs for the reader to consult [BS].

Let us consider P be a nondecomposed point of C, that is, there is a unique point $\overline{P} \in \widetilde{C \otimes_K \overline{K}}$, lying over P. Let $\mathcal{C}_P$ be the conductor of the local ring $\overline{K}\mathcal{O}_P$, that is, the largest ideal of

$$\mathcal{O}_{\overline{P}} := \mathcal{O}_{\overline{P},\widetilde{C \otimes_K \overline{K}}} = \widetilde{\overline{K}\mathcal{O}_P}$$

contained in $\overline{K}\mathcal{O}_P$. Since the geometric singularity degree at P if finite, the conductor of P is a non-zero ideal of $\mathcal{O}_{\overline{P}}$, and hence, there is a non-negative integer c_P such that

$$\mathcal{C}_P = \{z \in \mathcal{O}_{\overline{P}} \mid v_{\overline{P}}(z) \geq c_P\}.$$

The semigroup associated to the point P is defined by

$$H_P := v_{\overline{P}}(\overline{K}O_P \setminus 0),$$

and the non-negative integers that do not belong to H_P are called gaps of H_P.

We can relate the integer c_P and the geometric singularity degree at P with the semigroup H_P, as follows.

Proposition 3.8. *The integer c_P is the conductor of the semigroup H_P, that is, $c_P - 1$ is the largest gap of H_P. Moreover, the geometric singularity degree at P is exactly the number of gaps of H_P.*

Proof. See [BS], Proposition 1.1. □

Corollary 3.9. *A nondecomposed point P of C is smooth if and only if H_P is the semigroup of all non-negative integers $\mathbb{N}$.*

Corollary 3.10. *The geometric singularity degree at a nondecomposed point P is equal to $c_P/2$ and the semigroup H_P is symmetric, that is, an integer i belongs to H_P if and only if $c_P - 1 - i$ does not belong to H_P.*

Proof. See [BS] Corollary 1.4. □

In order to compute the degree at a point, as previously required, we present the following result.

Lemma 3.11. *The degree at a nondecomposed point $P \in C$ belongs to the semigroup H_P.*

Proof. See [BS], Lemma 1.5. □

Now we are able to present the required computation.

Corollary 3.12. *Let P be a nondecomposed and nonsmooth point on C, of geometric singularity degree $p(p-1)/2$. We have one of the following situations:*

1. *If P_1 is rational, then* $\deg P = p$*;*
2. *If P_2 is rational, then* $\deg P = p$ *or* p^2.

Proof. If P_1 is rational, then the fundamental equality says that the degree at P is equal to 1 or p. On the other hand, P is a nonsmooth point of C, and hence, its semigroup of values is different from the semigroup of all non-negative integers (cf. Corollary 3.9). But the degree at P belongs to its semigroup (cf. Lemma 3.11). Hence it must be p.

In the case that P_1 is non rational and P_2 is rational, the fundamental equality says that $\deg P_1 = p$ and hence $\deg P$ is equal to p or p^2. □

3.2 The Case of Genus Three

Throughout this section we apply the general results on local properties of points on algebraic curves, previously obtained, in order to approach the classification of regular but nonsmooth algebraic curves.

For instance, let us fix C of geometric genus three. If the characteristic of the field K is equal to seven, then (3.1) and Remark 3.4 say that C admits just one nonsmooth point, which has geometric singularity degree three. In addition, they still provide the vanishing of the geometric genus of the extended curve $C \otimes_K \overline{K}$. In the case of characteristic five, C admits only one nonsmooth point, which has geometric singularity degree two, and the geometric genus of the extended curve is equal to one.

On the other hand, in the case of characteristic and geometric genus being equal to three the number of possibilities may grows significantly. In fact, all of the possibilities are represented in the following table.

$\overline{g}$	Number of Non-Smooth Points	Geometric Singularity Degree of Each Point
o	1	3
	2	1 and 2
	3	1
1	1	2
	2	1
2	1	1

Table 3.1: *Possibilities for nonsmooth points and their geometric singularity degrees.*

On the other hand, we are not considering the existence of more than one point in the extended curve $C \otimes_K \overline{K}$, or in its normalization $\widetilde{C \otimes_K \overline{K}}$, lying over a nonsmooth point of C. However, in the spirit of Galois cohomology, we are allowed to make finite separable extensions L of the base field of C, in order to work with $C \otimes_K L$ instead of C. Indeed, since $L|K$ is a separable field extension we obtain that $C \otimes_K L$ remains regular with the same geometric genus of C. In this way we present the followings statements, in order to improve local properties at nonsmooth points of C.

Remark 3.13. Let L be an algebraic extension of K and P be a point of C. Then there is a finite extension L' of K, contained in L, such that each point of $\widetilde{C \otimes_K L'}$, lying over P, is nondecomposed in $\widetilde{C \otimes_K L}$, that is, only one point of $\widetilde{C \otimes_K L}$, lies over it.

By taking L being the separably closure of K, in the previous remark, we may suppose that each nonsmooth point of C is nondecomposed. Moreover, we may assume that all of their degrees are powers of p. Indeed, for each nonsmooth point we just extend the base to its separably closure in the residual field.

After these assumptions we may generalize Lemma 2.1 in [BS] that offers support for their local descriptions on nonsmooth points.

Lemma 3.14. *Let P be a point of C. Then, there exists a finite purely inseparable extension L of K such that the unique point of $\widetilde{C \otimes_K L}$, lying over P, is rational.*

Proof. By considering a purely inseparable finite extension L' of K, in which the geometric genera of $\widetilde{C \otimes_K L'}$ and $\widetilde{C \otimes_K \overline{K}}$ coincide, we may assume that P is smooth.

Let us consider L be the residual field $K(P)$ at P and Q be the point of $\widetilde{C \otimes_K L}$ lying over P. Since P is smooth we obtain $\mathcal{O}_{Q,\widetilde{C \otimes_K L}} = L\mathcal{O}_P$, and hence, the residual field at Q is equal to $LK(P) = L$. □

Corollary 3.15. *The image P_n at $P \in C$, under $\widetilde{F^n_{C/K}}$, is rational for some positive integer n.*

Proof. Indeed the finite purely inseparable extension of K, obtained in the above lemma, is contained in K^{1/p^n}, for some positive integer n. □

Remark 3.16. As was observed above, we proceed with base change in order to obtain the last corollary. In fact, it is important to use the local descriptions on nonsmooth points, provided in [BS], since its principal results can be obtained just with the two hypotheses: P is nondecomposed and P_n is rational, for some positive integer n. In particular, for nondecomposed points of geometric singularity degree $p(p-1)/2$, as we can see in Corollary 3.7.

Let us consider n as in the above corollary and y be a separating variable of $F|K$. According to Algorithm 2.2 in [BS], it is possible to construct, from y, an element $z \in \mathcal{O}_P$ that generates the free $\mathcal{O}_{P_1}$-modulo of rank p

$$\mathcal{O}_P = \mathcal{O}_{P_1}[z] = \bigoplus_{i=0}^{p-1} \mathcal{O}_{P_1} z^i.$$

Moreover, by induction it is possible to construct a base of $\mathcal{O}_P$ as a free $\mathcal{O}_{P_n}$-modulo of rank p^n, where $\mathcal{O}_{P_n} := \mathcal{O}_{P_n, \widetilde{C^{(p^n)}}}$. Then it is possible to obtain the conductor c_P, by induction, as follows.

Theorem 3.17. *If z is an element that generates $\mathcal{O}_P$ as a free $\mathcal{O}_{P_1}$-modulo, then*

$$c_P = pc_{P_1} + (p-1)v_{P_n}(dz^{p^n})$$

where dz^{p^n} is a differential of the function field $K(C)^{p^n}K|K$.

Proof. Follows from Proposition 2.3, Theorem 1.1.a and Lemma 2.1 in [S1]. □

Since z is an element of $K(C) \setminus K(C)^p K$ it is also a separating variable of $K(C)|K$. We notice that z remains a separating variable of $K^{1/p^n}K(C)|K^{1/p^n}$, or equivalently, z^{p^n} is a separating variable of $K(C)^{p^n}K|K$. Therefore the differential dz^{p^n} is different from zero, and hence, we obtain the following restrictions for the conductor of a point.

Corollary 3.18.

$$c_P \equiv 0 \bmod p-1$$

and

$$c_P \not\equiv 1 \bmod p$$

Then, up to a finite separable base extension, each nonsmooth point of a regular algebraic curve of geometric genus 3, over a non-algebraically closed field of characteristic three, has geometric singularity degree different from two. Hence the previous table can be simplified in Table 3.2, as follows, where each nonsmooth point is nondecomposed.

$\overline{g}$	Number of Non-Smooth Points	Geometric Singularity Degree of Each Point
o	1	3
	3	1
1	2	1
2	1	1

Table 3.2: *Possibilities for nonsmooth points and their geometric singularity degrees, up to a finite separable base change.*

Before start to discuss one of the above cases, we present, in a suitable situation, how the conductor or equivalently the geometric singularity degree at a nondecomposed point $P \in C$ can be obtained in terms of some special generators of the function field $K(C)|K$.

Let us consider n, as in Corollary 3.15 and F be the field of rational functions of C. If we take $x \in F_n := F^{p^n}K$ be a local parameter at v_{P_n} and y be a separating variable of $F|K$, then $F = F_n(y)$. We notice that y remains a separating variable of $K^{1/p^n}F|K^{1/p^n}$, or equivalently, y^{p^n} is a separating variable of $F_n|K$. Therefore the differential dy^{p^n} of the function field $F_n|K$ is different from zero. By the rationality at P_n we may consider the representation at y^{p^n}, as a Laurent series in the local parameter x, namely

$$y^{p^n} = \sum a_i x^i \in K((x)) \setminus K((x^p)).$$

Hence

$$dy^{p^n} = \sum ia_i x^{i-1}$$

and we can define:

$$\gamma := \min\{i \,|\, a_1 \neq 0\} = v_{P_n}(y^{p^n})$$

$$\mu := \min\{i \,|\, i \not\equiv 0 \bmod p,\, a_1 \neq 0\} = v_{P_n}(dy^{p^n}) + 1.$$

From the fundamental equality, $v_P(y) = v_{P_n}(y^{p^n})/\deg P$. But the orders of the differentials can be related as follows.

Theorem 3.19. $v_P(dy) = (c_P + v_{P_n}(y^{p^n}))/\deg P$.

Proof. See [BS] Theorem 2.7. □

According to [BS] Algorithm 3.1, it is possible to compute the semigroup H_P, by using the separating variable y. In the particular case where $n = 1$, that is, P_1 is a rational point of $\widetilde{C^{(p)}}$, it is summarized in the following statement.

Proposition 3.20. *If P_1 is a rational point, then P is non-rational if and only if there is an integer τ smaller than μ such that $a_\tau \notin K^p$. If τ is minimal with this property, then $K(P) = K(a_\tau^{1/p})$, $H_P = p\mathbb{N} + (\mu - \tau)\mathbb{N}$ and in particular*

$$c_P = (p-1)(\mu - \tau - 1).$$

Proof. See [BS] Proposition 4.1. □

We notice, in the case that P is non-rational, that $\tau \equiv 0 \bmod p$, since τ is smaller than μ and $a_\tau \neq 0$.

Remark 3.21. As was observed in Remark 3.16, we don't need to extend the base field, in order to prove this proposition, if we assume P a nondecomposed point. In particular, it follows for a nondecomposed point of geometric singularity degree $p(p-1)/2$.

3.3 Main Theorem

In this section we are going to discuss one of the cases presented in the last table. Indeed, we present the case: $\overline{g} = 0$ with one nondecomposed and nonsmooth point of geometric singularity degree three.

From Corollary 3.7 we have two cases to be considered: P_1 rational or P_1 non-rational and P_2 rational. We are going to analyze the first case, leaving the second for a next work.

Theorem 3.22. *Let C be a regular and complete algebraic curve over a field K of characteristic three. Then C is geometrically integral of genus three and admits a nondecomposed nonsmooth point of geometric singularity degree three, with rational image under $\widetilde{F_{C/K}}$, if and only if it is birational equivalent to an affine plane curve given by the polynomial*

$$Y^3 - aX^6 - bX^3 - X^2 \in K[X, Y]$$

where $a \in K \setminus K^3$ and $b \in K$. Moreover, a second regular and complete curve over K, birational equivalent to an affine plane curve given by $Y^3 - a'X^6 - b'X^3 - X^2$ with $a' \in K \setminus K^3$ and $b' \in K$, is isomorphic to C if and only if there are c_1, c_2, d in K, with $d \neq 0$, satisfying

$$a' = \frac{c_2^3 + d^6 a}{d^{18}} \quad \textit{and} \quad b' = \frac{c_1^3 + d^6 b}{d^9}.$$

In the remainder of this section, we are going to prove this theorem. Firstly, we need to recall some useful objects. It will be made separately, in the following remark.

Remark 3.23. Let C be a complete and integral algebraic curve over a field K. By a divisor on C we mean a coherent fractional ideal sheaf, that can be represented by a formal product of its stalks

$$\mathfrak{a} = \prod_{P \in C} \mathfrak{a}_P$$

where $\mathfrak{a}_P$ is a non-zero fractional ideal of $\mathcal{O}_P := \mathcal{O}_{C,P}$ for each $P \in C$ and $\mathfrak{a}_P = \mathcal{O}_P$ for almost all $P \in C$. The product of two divisors is defined by taking the products of their stalks. The locally principal divisors or Cartier divisors, which are the divisors whose stalks are principal ideals, form a multiplicative abelian group whose identity is the structure sheaf

$$\mathcal{O} := \prod_{P \in C} \mathcal{O}_P.$$

The divisor of a non-zero rational function $z \in K(C)$ is defined by the following formal product of principal ideals.

$$\operatorname{div}(z) := \prod_{P \in C} z^{-1} \mathcal{O}_P$$

Moreover, given a regular point $Q \in C$ we associate a divisor

$$\mathfrak{q} = \prod_{P \in C} \mathfrak{q}_P$$

by taking for $\mathfrak{q}_Q$ the inverse of the maximal ideal of O_Q and $\mathfrak{q}_P = O_P$, if P is different from Q.

The degree of a divisor is defined by the properties $\deg O = 0$ and

$$\deg(\mathfrak{a}) - \deg(\mathfrak{b}) = \sum_{P \in C} \dim_K \mathfrak{a}_P/\mathfrak{b}_P,$$

whenever $\mathfrak{a} \geq \mathfrak{b}$, that is, $\mathfrak{a}_P \supseteq \mathfrak{b}_P$ for each P on C. If Q is a regular point of C, then its degree and the degree of its associated divisor $\mathfrak{q}$ coincide.

The global sections of a divisor $\mathfrak{a}$ form a finite dimensional vector space

$$H^0(C, \mathfrak{a}) := \bigcap_{P \in C} \mathfrak{a}_P = \{z \in K(C)^* \mid \operatorname{div}(z) \cdot \mathfrak{a} \geq O\} \cup \{0\}$$

over the field K.

It can be verified, in the case where C is regular, that these definitions coincide with the classical definitions of divisors, degree and global sections over function fields, as we can find in [C]. For a more detailed presentation and more informations we refer to [S3], [Ha] or [Se].

Let us start the proof by establishing two concise notations.

$$F := K(C) \;\text{ and }\; F_1 := K(C)^p K \simeq K(C^{(p)}).$$

We also fix P be the nonsmooth point of C and P_1 be its image by $\widetilde{F_{C/K}}$ in $\widetilde{C^{(p)}}$.

As we can see in Corollary 3.6, the geometric genera of $\widetilde{C^{(p)}}$ and $C \otimes_K \overline{K}$ coincide. Hence $\widetilde{C^{(p)}}$ has geometric genus zero. Since P_1 is assumed to be rational, it follows that

$$F_1 = K(x)$$

where $x \in H^0(\widetilde{C^{(p)}}, \mathfrak{p}_1) \setminus K = \{z \in F_1^* \mid v_{P_1}(z) \geq -1 \text{ and } v_Q(z) \geq 0 \text{ for } Q \neq P\} \setminus K$ and $\mathfrak{p}_1$ is the divisor of $\widetilde{C^{(p)}}$ associated to P_1.

As we can see in Corollary 3.12, P is a point of degree three. Hence, by Riemann's theorem the K-vector space $H^0(C, \mathfrak{p}^2)$, of rational functions $y \in F$ with $v_P(y) \geq -2$ and $v_Q(y) \geq 0$ for $Q \neq P$, has dimension equal to $2 \deg P + 1 - g = 4$, where $g = 3$ is the geometric genus of C and $\mathfrak{p}$ is the divisor of C associated to P. In addition, the same corollary and fundamental equality imply that the ramification index $e(P|P_1)$ is equal to one. Therefore, for all positive integer n

$$H^0(C, \mathfrak{p}^n) \cap F_1 = H^0(\widetilde{C^{(p)}}, \mathfrak{p}_1^n) = K \oplus Kx \oplus \cdots \oplus Kx^n. \tag{3.3}$$

Hence, there exists $y \in H^0(C, \mathfrak{p}^2) \setminus H^0(\widetilde{C^{(p)}}, \mathfrak{p}_1^2)$. Since y belongs to $F \setminus F_1$, it follows that it is a separating variable of $F|K$, and thus

$$F = K(x, y).$$

On the other hand, $y^3 \in H^0(C, \mathfrak{p}^6) \cap F_1 = H^0(\widetilde{C^{(p)}}, \mathfrak{p}_1^6) = Kx^0 \oplus \cdots \oplus Kx^6$. Hence there are $a_0, \ldots, a_6 \in K$ such that

$$y^3 = a_0 + a_1 x + \cdots + a_6 x^6. \tag{3.4}$$

Since x^{-1} is a local parameter at v_{P_1}, we have that (3.4) express the representation at y^3 as a Laurent series in this local parameter. On the other hand, since P is non-rational we may conclude, from Proposition 3.20(see also Remark 3.21), that $\mu - \tau = 4$, and hence

$$a_4 = a_5 = 0, \quad a_2 \neq 0 \text{ and } a_6 \in K \setminus K^3.$$

From now, we will work systematically in order to obtain isomorphisms classes of function fields $F|K$, satisfying the required properties, together with possible normalizations for (3.4). Equivalently, let us consider C' be another curve over K, satisfying the hypothesis of Theorem 3.22, P' be its unique nondecomposed nonsmooth point and $F' = K(C')$. Then $F' = K(x', y')$ with $y'^3 = a'_0 + a'_1 x' + \cdots + a'_6 x'^6$, $a'_4 = a'_5 = 0$, $a'_2 \neq 0$ and $a'_6 \in K \setminus K^3$.

Let us take $\sigma : F' \to F$ be a K-isomorphism of fields. Since P and P' are the unique nonsmooth points of C and C', respectively, it follows that P' must be sent on P, via the associated isomorphism between C and C', that is, $\sigma(\mathcal{O}_{P'}) = \mathcal{O}_P$. Thus, σ induces an isomorphism of K-vector spaces

$$H^0(C', \mathfrak{p}'^n) \simeq H^0(C, \mathfrak{p}^n)$$

for all positive integer n. Moreover, since F'_1 (or F_1) is the unique subfield of F' (or F), containing K, such that $F'|F'_1$ (or $F|F_1$) is purely inseparable of degree 3, we have by (3.3) that σ also induces an isomorphism of K-vector spaces

$$H^0(\widetilde{C'^{(p)}}, \mathfrak{p}'^n) \simeq H^0(\widetilde{C^{(p)}}, \mathfrak{p}^n)$$

for all positive integer n.

It follows that $\sigma(x') \in H^0(C, \mathfrak{p}) \setminus K$, or equivalently,

$$\sigma(x') = ax + b$$

for some $a, b \in K$ with $a \neq 0$. Indeed, by Clifford's theorem we have the inequality $\dim_K H^0(C, \mathfrak{p}) \leq \frac{\deg \mathfrak{p}}{2} + 1 = \frac{5}{2}$, and hence, by (3.3) we have $H^0(C, \mathfrak{p}) = H^0(\widetilde{C^{(p)}}, \mathfrak{p}_1) = K \oplus Kx$. Moreover, we also obtain $\sigma(y') \in H^0(C, \mathfrak{p}^2) \setminus H^0(\widetilde{C^{(p)}}, \mathfrak{p}_1^2)$, that is,

$$\sigma(y') = c_0 + c_1 x + c_2 x^2 + cy$$

for some $c_0, c_1, c_2, c \in K$ with $c \neq 0$. Indeed, $y' \in H^0(C', \mathfrak{p}'^2) \setminus H^0(\widetilde{C'^{(p)}}, \mathfrak{p}_1'^2)$ and $H^0(C, \mathfrak{p}^2) = H^0(\widetilde{C^{(p)}}, \mathfrak{p}_1^2) \oplus Ky$.

Applying σ to the equation (3.4), for y' and x', and substituting the expression of y^3, as in (3.4), on the resulting equation, we obtain a polynomial equation on x. By comparing its coefficients we must have

$$\begin{aligned} a'_6 &= (c_2^3 + c^3 a_6)/a^6; \\ a'_3 &= (c_1^3 + a_3 c^3 - a'_6 2a^3 b^3)/a^3; \\ a'_2 &= c^3 a_2/a^2; \\ a'_1 &= (c^3 a_1 - a'_2 2ab)/a; \\ a'_0 &= c_0^3 + c^3 a_0 - a'_6 b^6 - a'_2 b^2 - a_1 b. \end{aligned} \tag{3.5}$$

Now we are able to perform normalizations in (3.4). Firstly, by taking the isomorphism defined by

$$\sigma(x') = x - a_1/a_2$$
$$\sigma(y') = y$$

we can assume $a_1 = 0$. Hence, by comparing the equality involving a_1 and a'_1 in (3.5), with $a_1 = a'_1 = 0$, we conclude that the isomorphisms of $F|K$ are of the form

$$\sigma(x') = ax$$
$$\sigma(y') = c_0 + c_1 x + c_2 x^2 + cy$$

with $a, c_0, c_1, c_2, c \in K$ and $a \cdot c \neq 0$.

By taking the isomorphism of $F|K$ defined by

$$\sigma(x') = x/a_2$$
$$\sigma(y') = y/a_2$$

we may normalize $a_2 = 1$. Hence, by comparing the equality involving a_2 and a'_2 in (3.5), with $a_2 = a'_2 = 1$, we conclude that the isomorphisms of $F|K$ are of the form

$$\sigma(x') = ax$$
$$\sigma(y') = c_0 + c_1 x + c_2 x^2 + cy$$

with $a, c_0, c_1, c_2, c \in K$, $a \cdot c \neq 0$ and $a^2 = c^3$. On the other hand, the K-rational points of the algebraic curve given by the polynomial $X^2 - Y^3$ over K, are identified with the points (t^3, t^2) where $t \in K$. Thus the isomorphisms of $F|K$ are of the form

$$\sigma(x') = d^3 x$$
$$\sigma(y') = c_0 + c_1 x + c_2 x^2 + d^2 y$$

with $c_0, c_1, c_2, d \in K$ and $d \neq 0$. By reformulating (3.5) we obtain:

$$\begin{aligned} a'_6 &= (c_2^3 + d^6 a_6)/d^{18}; \\ a'_3 &= (c_1^3 + a_3 d^6)/d^9; \\ a'_0 &= c_0^3 + d^6 a_0. \end{aligned} \tag{3.6}$$

After these normalizations, we are able to understand the meaning of the conditions on C given by the hypothesis of Theorem 3.22. Firstly, we present some general facts that provide support for this purpose.

Lemma 3.24. *Let C be a geometrically integral algebraic curve over a non-algebraically closed field K and $\widetilde{C}$ be its normalization. If a point P of $\widetilde{C}$ is nonsmooth, then the unique point Q of C, lying under P, is also nonsmooth.*

Proof. By extending the normalization map $\widetilde{C} \to C$ we obtain a commutative diagram

$$\begin{tikzcd}
 & \widetilde{C} \otimes_K \overline{K} \arrow[dl] \arrow[dd] \\
\widetilde{C} \arrow[dd] & \\
 & C \otimes_K \overline{K} \arrow[dl] \\
C &
\end{tikzcd}$$

Let us consider P' be a nonregular point of $\widetilde{C} \otimes_K \overline{K}$, lying over P. By the commutativity of the previous diagram we may see that the image $Q' \in C \otimes_K \overline{K}$ of P' lies over Q. But the morphism $\widetilde{C} \otimes_K \overline{K} \to C \otimes_K \overline{K}$ remains surjective. Hence we obtain the following inclusion of local rings.

$$\mathcal{O}_{Q', C\otimes_K \overline{K}} \subseteq \mathcal{O}_{P', \widetilde{C}\otimes_K \overline{K}}$$

On the other hand, if $\mathcal{O}_{Q', C\otimes_K \overline{K}}$ is a discrete valuation ring of the function field in one variable $K(C) \otimes_K \overline{K}|\overline{K}$, then the same happens for $\mathcal{O}_{P', \widetilde{C}\otimes_K \overline{K}}$, which contradicts the assumption on P'. $\square$

Corollary 3.25. *Let C be a geometrically integral affine plane curve, over a non-algebraically closed field K, given by the absolutely irreducible polynomial $F(X, Y) \in K[X, Y]$. We consider P be a point of $\widetilde{C}$ and a point of $C \otimes_K \overline{K}$, which we* *identify with $(a,b) \in \overline{K}^2$, lying over the point of C under P. If P is a nonsmooth point, then*

$$\frac{\partial F}{\partial X}(a,b) = \frac{\partial F}{\partial Y}(a,b) = 0.$$

Proof. By the previous lemma, the point of C, lying under P is nonsmooth. Since all points of $C \otimes_K \overline{K}$, lying over it, are nonregular (see [Liu] Remark 3.31 p.142) the assertion follows from the Jacobian criterion. □

Now we use the above corollary, in order to analyze a certain example, that was also studied by Stichtenoth [St].

Remark 3.26. Let C be a geometrically integral affine plane curve, over a non-algebraically closed field K of positive characteristic p, given by the absolutely irreducible polynomial $F(X,Y) = Y^{p^n} - f(X) \in K[X,Y]$, with $f'(X) \neq 0$ (see [St] Lemma 4).

The function field of C is equal to $K(x,y)$, where x and y correspond to the residual classes of X and Y in $K[X,Y]/(F(X,Y))$, respectively. Hence they satisfy the following relation.

$$y^{p^n} = f(x)$$

Since $K(C)|K(x)$ is a purely inseparable field extension, we obtain a bijection between points of $\widetilde{C}$ and discrete valuations of the function field $K(x)|K$, different from the pole of x, that is, the discrete valuation of $K(X)|K$ that sends x on a negative integer.

Let P be a point of $\widetilde{C}$ and P' be a point of $\widetilde{C \otimes_K \overline{K}}$, over P. Since $v_P(x) \geq 0$, it follows that $v_{P'}(x) \geq 0$, and hence, $v_{P'}(y) \geq 0$. Then there are $x(P'), y(P') \in \overline{K}$, uniquely determined by the property

$$v_{P'}(x - x(P')) > 0 \text{ and } v_{P'}(y - y(P')) > 0.$$

We notice that $(x(P'), y(P')) \in \overline{K}^2$ is identified with a point of $C \otimes_K \overline{K}$, lying over the point of C under P. If P is nonsmooth, we have by the above corollary, that

$$\frac{\partial F}{\partial X}(x(P'), y(P')) = \frac{\partial F}{\partial Y}(x(P'), y(P')) = 0,$$

or equivalently,

$$f'(x(P')) = 0,$$

that is,

$$v_{P'}(f'(x)) > 0,$$

that is,

$$v_P(f'(x)) > 0.$$

In this case we say that v_P is a zero of $f'(x)$.

Now let us return to the proof of the theorem. By the previous remark, the nonsmooth points of C, different from P, are exactly the points $Q \in C$ whose corresponding discrete valuation v_Q are zero of $f'(x)$, where $f(x) = a_6x^6 + a_3x^3 + x^2 + a_0$; that is, the zeros of x. On the other hand, since the field extension $F|F_1$ is purely inseparable, the unique zero of x is the discrete valuation of $F|K$ extending the unique zero of x in $F_1|K$. Let Q be its corresponding point in C. Then

$$\mathrm{div}(x) = \mathfrak{p}^{-1} \cdot \mathfrak{q}^{v_Q(x)},$$

where $\mathfrak{q}$ is the divisor of C associated to Q. We notice that Q is a nondecomposed point with rational image Q_1, under $\widetilde{F_{C/K}}$. We also observe that x is a local parameter of the discrete valuation associated to Q_1 and we recall that

$$y^3 = a_6x^6 + a_3x^3 + x^2 + a_0.$$

We claim that Q is a rational point. Otherwise, by Proposition 3.20 (see also Remark 3.21) applied to Q, there exists $\tau < \mu = 2$ such that $a_\tau \notin K^3$. Thus $\tau = 0$ and hence the conductor C_Q must be equal to two, which contradicts the smoothness at Q (see Corollary 3.10). In this way, Proposition 3.20 still provides that $a_0 \in K^3$.

Finally, this allows us to make the last normalization, by taking the isomorphism of $F|K$ defined by

$$\sigma(x') = x$$
$$\sigma(y') = -a_0^{1/3} + y.$$

Therefore we may assume $a_0 = 0$ and the conditions, as in (3.6), are exactly the conditions stated in Theorem 3.22, where $a := a_6$, $a' := a_6'$, $b := a_3$ and $b' := a_3'$.

Now it remains to prove the sufficiency of the given conditions in Theorem 3.22. Firstly, we notice that $F_1 := F^3K = K(x^3, y^3) = K(x)$, since the relation between x and y implies that $x = x^3/(y^3 - ax^6 - bx^3) \in F^3K$. In this way, $F = F_1(y)$ and hence $F|K$ is a separably generated function field. On the other hand, since $aX^6 + bX^3 + X^2$ is not a cube in $\overline{K}[X]$, we have the irreducibility of the polynomial $Y^3 - aX^6 - bX^3 - X^2$ in $\overline{K}[X, Y]$, that is, K is algebraically close in F. Therefore, C is geometrically integral.

Let us consider P be the point of C, whose corresponding discrete valuation v_P on $F|K$ extends the pole of x in $K(x)|K$. By the conditions on the coefficients of the equation involving x and y, Proposition 3.20 and Corollary 3.10, we may conclude that the geometric singularity degree at P is equal to three.

Since the genus of the function field $F_1|K$ is equal to zero, it remains to prove that P is the unique nonsmooth point of C, to conclude that its geometric genus is equal to three. To show this, we mention again Remark 3.26 together with Proposition 3.20, to conclude that the unique point of C, different from P, which could be nonsmooth, is rational. Hence it is smooth. □

3.4 Canonical Embedding

In order to obtain fibrations by nonsmooth curves from the regular but nonsmooth curves, studied in the previous section, we are going to find the canonical embedding of the curves obtained by their base extension to the algebraic closure. In addition, this allows us to investigate intrinsic geometric properties of the extended curve, by studying the extrinsic geometric properties of the canonical embedding.

For this purpose we need to study holomorphic differentials of such curves. We start recalling some of the most important definitions and facts about it. For a more detailed approach we refer [S3].

Let C be a complete and geometrically integral algebraic curve over a field K, of arithmetic genus g.

If $\mathfrak{a} = \prod \mathfrak{a}_P$ is a divisor on C, then the paralleletop of $\mathfrak{a}$ is the cartesian product

$$\Lambda(\mathfrak{a}) = \prod_{P \in C} \hat{\mathfrak{a}}_P$$

where $\hat{\mathfrak{a}}_P$ is the completion of the stalk $\mathfrak{a}_P$. Each paralleletop is contained in the K-algebra A_C of adeles of the function field $K(C)|K$, defined to be the restricted product of local fields $\hat{K}_Q$ of the brunches $Q \in \widetilde{C}$.

By a differential on C we mean a K-linear functional $A_C \to K$ vanishing on $\Lambda(\mathfrak{a}) + K$ for some divisor $\mathfrak{a}$ of C. Let Ω_C denote the space of differentials on C and λ be a non-zero differential on C. It can be proven that there exists a largest paralleletope $\Lambda(\mathfrak{c})$ vanishing λ. The divisor

$$\mathrm{div}(\lambda) := \mathfrak{c}$$

is called canonical divisor.

We say that C is a Gorenstein curve when each stalk $\mathfrak{c}_P$ of the canonical divisor is a principal O_P-ideal.

Let $\Omega_C(O)$ denote the set of holomorphic differentials on C, that is, the set of differentials vanishing $\Lambda(O)$. We may deduce that

$$\Omega_C(O) = H^0(C, \mathfrak{c}) \cdot \lambda$$

is a K-vector space of dimension g.

For instance, let us consider K be an algebraically closed field. A basis

$$\omega_1 = x_1 \cdot \lambda, \dots, \omega_g = x_g \cdot \lambda$$

of the K-vector space $\Omega_C(O)$ induces a morphism

$$(x_1 : \cdots : x_g) : \widetilde{C} \longrightarrow \mathbb{P}^{g-1}(K)$$

usually denoted by $(\omega_1 : \cdots : \omega_g)$, where $\widetilde{C}$ is the normalization of C. As we can see in [R] Theorem 15 it is birational if and only if C is a non-hyperelliptic curve, that is, it does not admit a morphism $C \to \mathbb{P}^1(K)$ of degree 2. On the other hand, this morphism induces a morphism

$$(\omega_1 : \cdots : \omega_g) : C \longrightarrow \mathbb{P}^{g-1}(K)$$

if and only if C is a Gorenstein curve (see [S3], Theorem 3.2). In this case it is an isomorphism, called canonical embedding, if and only if C is non-hyperelliptic (see [R], Theorem 17).

Now we return to the case where K is non-necessarily algebraically closed. If we assume that C is regular, then C is a Gorenstein curve (see [R], Theorem 10). Therefore the extended curve, $C \otimes_K \overline{K}$, remains Gorenstein (see [S1] Theorem 1.1 or [WITO]).

Thus, a natural question arise, in order to investigate the existence of a canonical embedding for the extended curve $C \otimes_K \overline{K}$. What property of C is related with the non-hyperellipticity of $C \otimes_K \overline{K}$?

To answer this question we notice that the differentials of $C \otimes_K \overline{K}$ are completely determined by the differentials of C, and the same occurs for the holomorphic differentials of $C \otimes_K \overline{K}$. Indeed,

$$\Omega_{C\otimes_K\overline{K}} \simeq \Omega_C \otimes_K \overline{K} \quad \text{and} \quad \Omega_{C\otimes_K\overline{K}}(O_{C\otimes_K\overline{K}}) \simeq \Omega_C(O_C) \otimes_K \overline{K}.$$

On the other hand, the non-hyperellipticity of $C \otimes_K \overline{K}$ is equivalent to the birationality of the map $(\omega_1 : \cdots : \omega_g) : \widetilde{C \otimes_K \overline{K}} \to \mathbb{P}^{g-1}(\overline{K})$, where $\omega_1, \ldots, \omega_g$ form a basis of the $\overline{K}$-vector space of holomorphic differentials of $C \otimes_K \overline{K}$. This in turn occurs if and only if the field $\overline{K}(C \otimes_K \overline{K})$ is generated by the quotients of holomorphic differentials ω_i/ω_j for all i and j in $\{1, \ldots, g\}$, that is, the field $K(C)$ is generated by all quotients of a basis of the K-vector space of holomorphic differentials of C. Furthermore, when the arithmetic genus of C is $g \geq 2$, the last condition means that C does not admit a morphism

$$C \longrightarrow C'$$

of degree 2, where C' is a regular complete genus zero curve over K. In this case, we say that C is a non-hyperelliptic curve.

In what follows, we will analyze a curve, as in Theorem 3.22, with respect to its non-hyperellipticity. When this occurs, we will find its basis of holomorphic differentials, together with the image of the canonical embedding of the extended curve. So let C be such a curve. Firstly, we notice that the non-hyperellipticity of C is equivalent to the existence of a quadratic genus zero subfield of the function field $K(C)|K$. But $K(C)|K$ admits a quadratic subfield if and only if it admits an automorphism of order two.

As in Theorem 3.22, $F := K(C) = K(x,y)$ where $y^3 = ax^6 + bx^3 + x^2$, with $a \in K \setminus K^3$ and $b \in K$. Also by Theorem 3.22, the K-automorphisms $\sigma : F \to F$ are of the form

$$\sigma(x) = d^3 x$$
$$\sigma(y) = c_0 + c_1 x + c_2 x^2 + d^2 y$$

with $c_0, c_1, c_2, d \in K$, $d \neq 0$, $a_6 = (c_2^3 + d^6 a_6)/d^{18}$ and $a_3 = (c_1^3 + a_3 d^6)/d^9$.

Since $a \in K \setminus K^3$, we must have that $c_2 = 0$ and $d^4 = 1$. In addition, if we assume $b \in K \setminus K^3$, then the group of automorphisms of $F|K$ is trivial.

If $b \in K^3$, then we can normalize $b = 0$, by taking the isomorphism

$$\sigma(x') = x$$
$$\sigma(y') = 2b^{1/3} + y.$$

In this case, $F = K(x,y)$ where $y^3 = ax^6 + x^2$, $a \in K \setminus K^3$ and its automorphisms are of the form

$$\sigma(x) = d^3 x$$
$$\sigma(y) = d^2 y$$

with $d \in K \setminus \{0\}$ and $d^4 = 1$. Moreover, we may clearly see that its unique automorphism of order two is given by $d = -1$. Thus the subfield of F whose elements are fixed by the automorphism of order two is equal to $E := K(x^2, y)$.

In order to investigate its genus, if we take $z := x^2$, then we can see that E is the function field of the regular complete and geometrically integral algebraic curve C' over K, defined by the homogeneous polynomial

$$Y^3 - aZ^3 - ZX^2 \text{ with } a \in K \setminus K^3.$$

Since it is a nonsmooth curve, it follows that its geometric genus must be different from zero. Thus there is no quadratic subfield of $F|K$ of genus zero.

Actually, we are able to compute the geometric genus of C'. To do this, let us notice that

$$E^3 K = K(x^6, y^3) = K(z)$$

and hence $E^3K|K$ has genus zero. Let us consider Q be the unique point of C' whose associated discrete valuation v_Q on $E|K$ lies over the discrete valuation on $K(z)|K$, given by the pole of z. Since Q is nondecomposed with rational image under $\widetilde{F_{C'/K}}$, we can apply Proposition 3.20 and Corollary 3.10 to conclude that the geometric singularity degree at Q is equal to one.

On the other hand, in the same manner as in the proof of Theorem 3.22, we can use Remark 3.26 to conclude that C' does not admit other nonsmooth point beyond

Q. Therefore, (3.1) together Corollary 3.6 implies that the geometric genus of C' is equal to one.

To summarize, we have the following result.

Theorem 3.27. *A curve C as in Theorem 3.22 is non-hyperelliptic. Furthermore, $b \in K^3$ if and only if C is a double cover of the regular but nonsmooth complete and geometrically integral algebraic curve of genus one defined by the equation*

$$Y^3 - aZ^3 - ZX^2, \text{ with } a \in K \setminus K^3.$$

In this case, its group of automorphisms is cyclic of order four, if K contains a non-trivial fourth root of unity, and cyclic of order two, otherwise. In the case where $b \in K \setminus K^3$, its group of automorphisms is trivial.

In the remainder of this section we will investigate the holomorphic differentials of C. Our strategy is to find at least one of these, by using the differentials of Frobenius pullback normalization (see Theorem 3.19), and after we use a result of adjoint plane curves, due to Gorenstein [G], in order to obtain all of its holomorphic differentials.

Let us consider $f(x) := ax^6+bx^3+x^2 \in F := K(C)$, where a and b are obtained by the affine curve birational equivalent to C, as in Theorem 3.22. Let us consider the differential dy^3 of $\widetilde{C^{(p)}}$. Since $y^3 = f(x)$ in $F_1 := K(\widetilde{C^{(p)}}) = F^pK$ we have

$$dy^3 = 2xdx.$$

As in the proof of Theorem 3.22, if we consider P be the unique nonsmooth point of C and P_1 be its image in $\widetilde{C^{(p)}}$, then the discrete valuation v_{P_1} on $F_1|K$ is the pole of x. So, $v_{P_1}(dy^3) = -3$ and hence the order of the differential dy at P is given by

$$v_P(dy) = \frac{2\dim_{\overline{K}} \frac{\widetilde{\overline{K}\mathcal{O}_P}}{\overline{K}\mathcal{O}_P} + v_{P_1}(dy^3)}{\deg P} = \frac{2 \cdot 3 - 3}{3} = 1$$

(see Theorem 3.19).

Let us consider Q be a point of C, different from P, and Q_1 be its image in $\widetilde{C^{(p)}}$. In this case, v_{Q_1} is the discrete valuation on $F_1|K$ associated to some irreducible polynomial π in $K[x]$.

Since $dy^3 = \frac{2x}{\pi'}d\pi$, we have that $v_{Q_1}(dy^3)$ is equal to one, if $\pi = x$, and zero otherwise. On the other hand, Q is a smooth point of C and $v_Q(dy) = v_{Q_1}(dy^3)/\deg Q$ is an integer. Then

$$v_Q(dy) = \begin{cases} 1 & \text{if } \pi = x \\ 0 & \text{if } \pi \neq x \end{cases}$$

and $\deg Q = 1$ if $\pi = x$. Thus,

$$\operatorname{div}(dy) = \mathfrak{p} \cdot \mathfrak{q}$$

where v_Q is the unique extension to $F|K$ of the zero of x in $F_1 = K(x)|K$ and $\mathfrak{p}$ and $\mathfrak{q}$ are the divisors of C associated to P and Q, respectively. Therefore, dy is a holomorphic differential on C.

From now, let us find a basis of the three dimensional K-vector space of holomorphic differentials on C. By a result of Gorenstein [G], the holomorphic differentials on C are of the form

$$\omega = \frac{h \cdot dy}{f'(x)}$$

with $h \in K[x, y]$ and $\deg h = \deg f - 3 = 3$. So, $\operatorname{div}(\omega) = \operatorname{div}(h) \cdot \operatorname{div}(x^{-1}) \cdot \operatorname{div}(dy)$.

Since $\deg P = 3$, $\deg P_1 = 1$ and $\deg Q = 1$, we have that $e(P|P_1) = 1$ and $e(Q|Q_1) = 3$. Thus $\operatorname{div}(x) = \mathfrak{p}^{-1} \cdot \mathfrak{q}^3$ and therefore

$$\operatorname{div}(\omega) = \operatorname{div}(h) \cdot \mathfrak{p}^2 \cdot \mathfrak{q}^{-2}.$$

In this way, ω is holomorphic if and only if $h \in H^0(C, \mathfrak{p}^2 \cdot \mathfrak{q}^{-2})$. On the other hand, since $v_P(h) = v_{P_1}(h^3)/3$ and $v_Q(h) = v_{Q_1}(h^3)$ we have that

$$\omega \text{ is holomorphic } \iff v_{P_1}(h^3) \geq -6 \text{ and } v_{Q_1}(h^3) \geq 2.$$

By writing

$$h = c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{30}x^3 + c_{21}x^2y + c_{12}xy^2 + c_{03}y^3,$$

where $c_{ij} \in K$ for every i, j in $\{0, \ldots, 3\}$, and substituting $y^3 = ax^6 + bx^3 + x^2$ in h^3, the last equivalence means that the coefficients $c_{03}, c_{12}, c_{02}, c_{21}, c_{11}, c_{30}$ and c_{00} must be zero. Therefore, every holomorphic differential on C is of the type

$$\omega = c_{10}dy + c_{01}\frac{y}{x}dy + c_{20}xdy$$

with $c_{10}, c_{01}, c_{20} \in K$.

We summarize it in the following theorem.

Theorem 3.28. *The set $\{dy, \frac{y}{x}dy, xdy\}$ provides a basis of the K-vector space of holomorphic differentials on a curve C, as in Theorem 3.22.*

In the remainder of this section, we will analyze the image of the canonical embedding

$$(\omega_1 : \omega_2 : \omega_3) : C \otimes_K \overline{K} \hookrightarrow \mathbb{P}^2(\overline{K}),$$

where $\omega_1 := dy$, $\omega_2 := \frac{y}{x}dy$, $\omega_3 := xdy$ and C is a curve as in Theorem 3.22.

By taking the third power of ω_2 and multiplying it by ω_3 we may easily obtain that the homogeneous polynomial $ZY^3 - aZ^4 - bZX^3 - X^4 \in K[X, Y, Z]$ vanishes the image of the canonical morphism of $C \otimes_K \overline{K}$. On the other hand, we can conclude that this polynomial is irreducible if we observe that the image of the canonical embedding has degree four. Therefore we have the following result.

Theorem 3.29. *The non-singular projective model of a curve C, as in Theorem 3.22, is isomorphic to a plane quartic curve over K defined by the homogeneous polynomial*

$$F(X, Y, Z) = ZY^3 - aZ^4 - bZX^3 - X^4$$

where $a \in K \setminus K^3$ and $b \in K$. Its nonsmooth point corresponds to the point lying under the point $(0 : a^{1/3} : 1)$ of the projective plane $\mathbb{P}^2(\overline{K})$.

Remark 3.30. We notice that a plane affine curve over $\overline{K}$, given by the polynomial

$$Y^3 - aX^6 - bX^3 - X^2$$

with $a \in K \setminus K^3$ and $b \in K$, admits $(0, 0)$ as unique nonregular point. By blowing up this point we obtain a plane affine quartic curve over $\overline{K}$ given by the polynomial

$$XY^3 - aX^4 - bX - 1.$$

Therefore, we rediscover the last theorem, in a very simple way. However, we prefer to present the first proof for two reasons. Firstly, this approach is closer to solve a general case. Secondly, in order to investigate the non-hyperellipticity of a curve, we need to study the existence of a quadratic subfield of its function field. Thus we can find amazing geometric properties, as we can see in the next section.

3.5 Geometry of the Canonical Embedding

In this section we describe some interesting intrinsic properties of the base extension of a curve, as in Theorem 3.22, by identifying their corresponding extrinsic properties, in the canonical embedding.

Let us consider P be the nonsmooth point of C and $\overline{P}$ be the point of the base extension $C \otimes_K \overline{K}$, lying above P. In the canonical embedding of $C \otimes_K \overline{K}$, that is, the projective plane quartic curve over $\overline{K}$ given by the homogeneous polynomial

$$ZY^3 - aZ^4 - bZX^3 - X^4,$$

$\overline{P}$ corresponds to the point $(0 : a^{1/3} : 1)$. This is the unique singularity of $C \otimes_K \overline{K}$. Moreover, its multiplicity and its singularity degree are both equal to three.

On the other hand, with the aim to finish the classification of regular curves of genus three with one nondecomposed and nonsmooth point of geometric singularity degree three, in a forthcoming work, we present the behavior at the singularity of the extended curve, with respect to its multiplicity.

Proposition 3.31. *Let C be a regular complete and geometrically integral algebraic curve of genus three, with one nondecomposed and nonsmooth point of singularity degree three. Then the multiplicity at the singularity of the extended curve $C \otimes_K \overline{K}$ is three.*

Proof. By the formula relating the singularity degree and the multiplicity at the singular branch (cf. e.g. [An] Korollar II 1.8), the possibilities for the multiplicity of the singular point of $C \otimes_K \overline{K}$ are three and two. If we assume that the singularity has multiplicity two, then its semi-group of values is $2\mathbb{N} + 7\mathbb{N}$. Indeed, its multiplicity belongs to its semigroup and its singularity degree is exactly the numbers of gaps of its semigroup (cf. Proposition 3.8). But it is a contradiction, since the analysis in the end of page 321 in [BS] provides $3\mathbb{N} + 4\mathbb{N}$ as unique possibility for this semigroup. □

The last geometric property is amazing in the sense that intrinsic properties can be reflected by extrinsic properties. First of all we recall some basic definitions in the following remark. For more details we refer [SVo].

Remark 3.32. Let C be a complete and integral algebraic curve of arithmetic genus g, over an algebraically closed field of characteristic p, and $\widetilde{C}$ be its normalization. Additionally, we assume that C is Gorenstein and non-hyperelliptic.

As previously observed, the base of holomorphic differentials of C defines a morphism

$$f : \widetilde{C} \longrightarrow \mathbb{P}^{g-1}$$

and consequently a base point free linear system of hyperplane sections

$$\mathscr{D} = \{f^*(H) \mid H \text{ hyperplane in } \mathbb{P}^{g-1}\}.$$

Let Q be a point of $\widetilde{C}$. An integer j is called hermitian Q-invariant if there exists a hyperplane intersecting the branch Q with multiplicity j. Equivalently, if there exists a divisor D in $\mathscr{D}$ such that

$$v_Q(D) = j$$

where v_Q is the discrete valuation associated to Q on the function field of C.

There are exactly g hermitian Q-invariants denoted by

$$j_0(Q) < j_1(Q) < \cdots < j_{g-1}(Q)$$

where $j_0(Q) = 0$, $j_1(Q)$ is the multiplicity of Q and $j_{g-1}(Q) < 2g - 2$. It can be proven the existence of a generic sequence $\varepsilon_0, \ldots, \varepsilon_{g-1}$ in the following sense:

$$j_i(Q) = \varepsilon_i \ (i = 0, \ldots, g-1) \text{ for all but many branches } Q \in \widetilde{C}$$

and

$$j_i(Q) \geq \varepsilon_i \ (i = 0, \ldots, g-1) \text{ for all branches } Q \in \widetilde{C}.$$

A point $Q \in \widetilde{C}$ whose sequence of hermitian invariants $j_0(Q), j_1(Q), \ldots, j_{g-1}(Q)$ differs from the generic sequence is called Weierstrass branch. A smooth point of C is called Weierstrass point if its branch is a Weierstrass branch.

It is possible to define the weight of each branch $Q \in \widetilde{C}$, denoted by $w(Q)$, satisfying the inequality

$$w(Q) \geq \sum (j_i(Q) - \varepsilon_i).$$

The number of Weierstrass branches, counted according to their weights, is given by the total weight formula

$$\sum_{Q \in \widetilde{C}} w(Q) = \left(\sum \varepsilon_i\right)(2\widetilde{g} - 2) + g(2g - 2)$$

where $\widetilde{g}$ is the geometric genus of C.

The curve C is called classical when the generic sequence is the classical sequence

$$0, 1, \ldots, g - 1.$$

For instance if the arithmetic genus of C is equal to three, then C is non-classical if and only if all of its smooth points are inflection points. Indeed, by looking for smooth points of C we can see that $\varepsilon_1 = 1$ and hence non-classicality means that

$$j_2(Q) \geq 3$$

for each branch Q lying over a smooth point of C. Equivalently, for each smooth point of C, there exists a line in the projective plane that intersects C with multiplicity at least three in this point.

To finish this remark we just notice that the classification of non-classical Gorenstein curves of arithmetic genus three and four can be found in [FS].

Proposition 3.33. *Let C be a curve as in Theorem 3.22. Then the extended curve $C \otimes_K \overline{K}$ is non-classical and admits a unique smooth Weierstrass point.*

Proof. As we can see in Theorem 3.29, the canonical form of $C \otimes_K \overline{K}$ is the plane projective curve over $\overline{K}$ given by the homogeneous polynomial

$$F(X, Y, Z) = ZY^3 - aZ^4 - bZX^3 - X^4,$$

where $a, b \in K$.

Since all second partial differentials of $F(X, Y, Z)$ vanish, we obtain, by the Hessian criterion, that all smooth points of the curve in $\mathbb{P}^2(\overline{K})$ given by this polynomial are inflection points. Thus, by the previous remark we may conclude that $C \otimes_K \overline{K}$ is non-classical.

On the other hand, the sequence of hermitian invariants of the branches of the points corresponding to

$$(0 : a^{1/3} : 1) \quad \text{and} \quad (0 : 1 : 0)$$

are

$$0, 3, 4 \quad \text{and} \quad 0, 1, 4$$

respectively. Hence, by the total weight formula, the unique smooth Weierstrass point of $C \otimes_K \overline{K}$ corresponds to $(0 : 1 : 0)$. □

Chapter 4

Fibrations by nonsmooth Curves

In this chapter we use the canonical embedding of the regular but nonsmooth curves, previously studied, in order to list fibrations by nonsmooth curves within a universal fibration. Moreover, in analogy to the Kodaira-Néron classification of special fibers of minimal fibrations by elliptic curves, we describe the minimal proper regular models of some fibrations by nonsmooth curves over the projective line and determine the structure of their bad fibers.

4.1 A Universal Two-Dimensional Fibration by nonsmooth Curves

Throughout this chapter k will denote an algebraically closed field of characteristic three. We consider the rational threefold

$$T \subset \mathbb{P}^2(k) \times \mathbb{A}^2(k)$$

given by the polynomial $ZY^3 - aZ^4 - bZX^3 - X^4$, which is smooth by a simple consequence of the Jacobian criterion. We also consider the morphism

$$\pi : T \longrightarrow \mathbb{A}^2(k)$$

induced by the natural projection of $\mathbb{P}^2(k) \times \mathbb{A}^2(k)$ over $\mathbb{A}^2(k)$.

Let us fix $P = (a, b)$ be a point in $\mathbb{A}^2(k)$. The fiber of π over P is a plane projective quartic curve. By the Jacobian criterion it admits just one singular point, namely

$$(0 : a^{1/3} : 1).$$

Since the singularity is unibranch of singularity degree three we have that the fiber over P is a rational and integral plane projective quartic curve. Hence π is a fibration by nonsmooth curves. According to Theorem 3.22 and Theorem 3.29 the image at the nonsmooth point of the generic fiber Z of π, under the lifted Frobenius morphism $\widetilde{F_{Z/K}}$, is rational. Actually, the above fibration by nonsmooth curves is universal in the sense that the data about all fibrations by nonsmooth plane quartics, whose generic fiber satisfies this property, are condensed in it. Indeed it can be seen in the following result.

Theorem 4.1. *1. Let C be an integral affine plane curve and $\varphi(x, y)$ be the irreducible polynomial in $k[x, y]$ defining it. Then the restricted projection morphism $\pi^{-1}(C) \to C$ is a fibration by nonsmooth curves if and only if*

$$\varphi(x, y) \notin k[x, y^3].$$

Also, the restricted fibration $\pi^{-1}(C) \to C$ admits a factorization by a rational double cover followed by a fibration by plane projective cuspidal cubics if and only if $\varphi \in k[x^3, y]$.

2. Each fibration by nonsmooth plane projective quartic curves with a point of singularity degree three, whose generic fiber satisfies the hypothesis of Theorem 3.22, is up to birational equivalence obtained by a base extension either from the two-dimensional fibration $\pi : T \to \mathbb{A}^2$ or from an one-dimensional fibration $\pi^{-1}(C) \to C$, obtained by restricting the base of π to an irreducible curve C on $\mathbb{A}^2$.

Before we present the proof of this theorem we notice an interesting property of such fibrations, that follows immediately from Proposition 3.33.

Corollary 4.2. *Almost all fiber of a fibration by nonsmooth plane projective quartic curves with a point of singularity degree three, whose generic fiber satisfies the hypothesis of Theorem 3.22, are non-classical curves and admit a unique Weierstrass point.*

In the remainder of this section we present the proof of the previously stated theorem.

To prove the first assertion of the first item we look for the generic fiber of $\pi^{-1}(C) \to C$, that is, the projective plane curve over $K := k(a, b)$ given by the homogeneous polynomial

$$ZY^3 - aZ^4 - bZX^3 - X^4$$

where a and b are the residual classes at x and y in $k[x, y]/\varphi(x, y)k[x, y]$, respectively.

The restriction $\pi^{-1}(C) \to C$ is a fibration by nonsmooth curves if and only if a does not belong to K^3. Since k is algebraically closed, the last condition means that a is a separating variable in the function field $K|k$, that is, the differential da of a in the function field $K|k$ is different from zero. Since $\varphi(a,b) = 0$ we have the equality

$$\frac{\partial\varphi}{\partial x}(a,b)\,da + \frac{\partial\varphi}{\partial y}(a,b)\,db = 0$$

where db is the differential of b in $K|k$. Moreover, since k is algebraically closed it follows that the function field $K|k$ has a separating variable, that is,

$$da \neq 0 \text{ or } db \neq 0\,.$$

On the other hand,

$$\frac{\partial\varphi}{\partial y}(a,b) \neq 0 \text{ or } \frac{\partial\varphi}{\partial x}(a,b) \neq 0\,.$$

Indeed, otherwise we would have $\frac{\partial\varphi}{\partial x}(x,y) = \frac{\partial\varphi}{\partial y}(x,y) = 0$ from Nullstellensatz and from the fact that the degree of $\frac{\partial\varphi}{\partial x}(x,y)$ (or $\frac{\partial\varphi}{\partial y}(x,y)$) in x (or y) is strictly less than the degree of $\varphi(x,y)$ in x (or y). But it implies that $\varphi(x,y)$ belongs to $k[x,y]^3$, that contradicts the irreducibility of $\varphi(x,y)$. Therefore, we can conclude from the three last evidenced sentences that $da \neq 0$ if and only if $\frac{\partial\varphi}{\partial y}(a,b) \neq 0$, which in turn is equivalent to $\frac{\partial\varphi}{\partial y}(x,y) \neq 0$, that proves the first assertion of the theorem.

From now we analyze the existence of a factorization by a rational double cover followed by a fibration by plane projective cuspidal cubics. From Theorem 3.27 it is equivalent to find conditions from that b belongs to the field K^3. It can be done by observing that $b \in K^3$ is equivalent to $db = 0$, which in turn is equivalent to $\frac{\partial\varphi}{\partial x}(a,b) = 0$. Indeed the necessity in the last equivalence follows from the equality evidenced above and from the inequality $da \neq 0$. The sufficiency follows from the last sentence evidenced above and the equivalence between the non vanishing of $\frac{\partial\varphi}{\partial x}(a,b)$ and $\frac{\partial\varphi}{\partial x}(x,y)$.

To prove the assertion in the second item of Theorem 4.1, it is enough to prove that the function field of the generic fiber of such a fibration by nonsmooth curves is obtained, up to isomorphism, by a base extension either from the function field of the generic fiber of $\pi : T \to \mathbb{A}^2$ or from the function field of the generic fiber of $\pi^{-1}(C) \to C$, for some irreducible affine plane curve. For this purpose we simply take a and b as in Theorem 3.29. Thus the conclusion follows by observing that the field extension $k(a,b)|k$ may have transcendent degree two or one.

4.2 Regular Minimal Models

In order to investigate the special fibers of fibrations by nonsmooth curves we present, in this section, the behavior study of two non-birationally equivalent fibrations by nonsmooth plane projective quartics over the projective line and their relative minimal models. Also, in the case that such a fibration admits a factorization, as in the previous theorem, we present the relative minimal model of the fibration by plane projective cuspidal cubics over the projective line.

The first fibration is obtained by considering the surface

$$S \subset \mathbb{P}^2(k) \times \mathbb{P}^1(k)$$

given by the bihomogeneous polynomial

$$sZY^3 - tZ^4 - sX^4$$

and the morphism

$$\eta : S \longrightarrow \mathbb{P}^1(k)$$

induced by the second projection $((x : y : z), (s : t)) \mapsto (s : t)$.

Over each point of the form $(1 : t)$, in the projective line $\mathbb{P}^1(k)$, the fiber of η is identified with the plane projective quartic curve given by the homogeneous polynomial

$$ZY^3 - tZ^4 - X^4.$$

It is rational, integral and admits just one singular point $(0 : t^{1/3} : 1)$, which is unibranch of multiplicity and singularity degree three. Indeed it can be seen with direct computations or by observing that the restricted morphism

$$\pi^{-1}(C) \to C,$$

where π is the two dimensional fibration by nonsmooth curves constructed in the previous section and C is the affine plane curve given by the irreducible polynomial $y \in k[x, y]$, coincides with the fibration obtained by restricting η to the open subset of $\mathbb{P}^1(k)$, whose points are of the form $(1 : t)$ with $t \in k$.

On the other hand the fiber over $(0 : 1)$ is identified with a non-reduced projective plane curve given by the polynomial Z^4.

By the Jacobian criterion, though each special fiber is singular the total space S has only one singular point, namely $((0 : 1 : 0), (0 : 1))$.

According to Theorem 4.1, the rational map from $\mathbb{P}^2(k) \times \mathbb{P}^1(k)$ onto itself, given by the assignment

$$((x : y : z), (s : t)) \mapsto ((x^2 : yz : z^2), (s : t))$$

induces a rational double cover from S onto the surface

$$S' \subset \mathbb{P}^2(k) \times \mathbb{P}^1(k)$$

given by the bihomogeneous polynomial

$$sY^3 - tZ^3 - sZX^2.$$

Let us consider
$$\eta' : S' \longrightarrow \mathbb{P}^1(k)$$
be the morphism induced by the natural projection of $\mathbb{P}^2(k) \times \mathbb{P}^1(k)$ onto $\mathbb{P}^1(k)$. Its fiber over each point of the form $(1 : t)$ is a rational integral projective cubic that admits just one singularity, namely

$$(0 : t^{1/3} : 1),$$

which is unibranch of multiplicity two and singularity degree one.

We notice that the fibrations by cusps was studied by Bombieri and Mumford [BM] in order to extend the Enriques' classification of surfaces in characteristic positive. In fact, they arise as an Albanese mapping of an Albanese variety associated to an quasi-hyperelliptic surface.

The rational double cover factors into η, outside of its indeterminacy locus, followed by η', as in the following diagram.

$$\begin{tikzcd} S \setminus (\{(0:1:0)\} \times \mathbb{P}^1(k)) \arrow[d, "\eta"'] \arrow[dr] & \\ \mathbb{P}^1(k) & S' \arrow[l, "\eta'"] \end{tikzcd}$$

Therefore we can see that almost all fiber of η admits a rational double cover over a plane projective cuspidal cubic. In this way, we notice the next characterization of such fibrations that follows immediately from Theorem 3.27 and Theorem 3.29.

Corollary 4.3. *Each fibration by nonsmooth plane projective quartics curves, as in the second item of Theorem 4.1, whose general fiber admits a rational double cover over an integral projective plane cuspidal cubic is up to birational equivalence obtained by a base extension from* $\eta : S \longrightarrow \mathbb{P}^1(k)$.

From now we present the study of the relative minimal model of the fibrations $\eta : S \to \mathbb{P}^1(k)$ and $\eta' : S' \to \mathbb{P}^1(k)$.

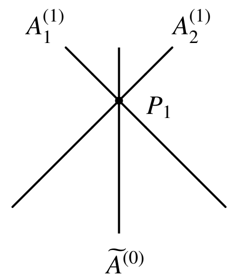
Firstly we present the study of η. Thus it is necessary to describe the desingularization morphism $\widetilde{S} \to S$, obtained by blowups, and determine the fibers of the composed morphism
$$\widetilde{\eta} : \widetilde{S} \longrightarrow S \longrightarrow \mathbb{P}^1(k).$$
In order to obtain the relative minimal model of η it remains to blow down on $\widetilde{S}$ the smooth rational fibral components of self-intersection -1. Since the arithmetic

genus of almost all fiber is positive, it follows by a variant of Enriques's theorem on minimal models of algebraic surfaces (see [Sh1] p. 155 or [Li] Theorem 4.4), that the minimal model is unique up to isomorphisms over the base curve $\mathbb{P}^1$.

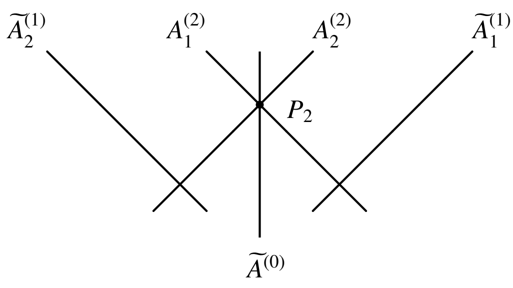
For each $t \in k$ the fiber of η over $(1 : t)$ is a rational and integral algebraic curve not containing the singularity of the total space S. Thus the fiber of $\widetilde{\eta}$ over $(1 : t)$ remains rational and integral. Since a fiber meets its components with intersection number zero, the self-intersection of $\widetilde{\eta}^*(1 : t)$ is different from -1. Therefore we just need to analyze the fiber $\widetilde{\eta}^*(0 : 1)$.

Let us fix $A^{(0)} := \eta^*(0 : 1)$. It is a curve on S, containing its singularity $P_0 := ((0 : 1 : 0), (0 : 1))$, and is given by the polynomial equation "$z = 0$".

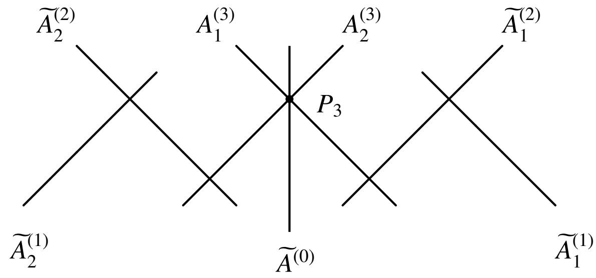
By blowing up P_0 we obtain the exceptional divisor as union of two rational smooth curves $A_1^{(1)}$ and $A_2^{(1)}$ containing a common point P_1, which is the unique singularity of the total space and belongs to the strict transform $\widetilde{A}^{(0)}$ of $A^{(0)}$. This configuration can be seen in the following figure, where the intersections are always transversal.



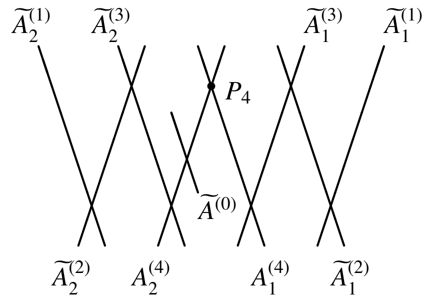
By blowing up P_1 we obtain the exceptional divisor as union of two rational smooth curves $A_1^{(2)}$ and $A_2^{(2)}$ containing a common point P_2, which is the unique singularity of the total space and still belongs to $\widetilde{A}^{(0)}$. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{A}_1^{(1)}$ and $\widetilde{A}_2^{(1)}$ are the strict transforms of $A_1^{(1)}$ and $A_2^{(1)}$, respectively.



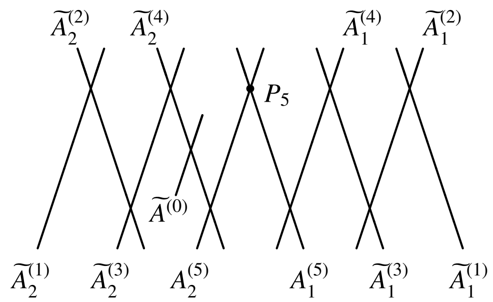
By blowing up P_2 we obtain the exceptional divisor as union of two rational smooth curves $A_1^{(3)}$ and $A_2^{(3)}$ containing a common point P_3, which is the unique singularity of the total space and still belongs to $\widetilde{A}^{(0)}$. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{A}_1^{(2)}$ and $\widetilde{A}_2^{(2)}$ are the strict transforms of $A_1^{(2)}$ and $A_2^{(2)}$, respectively.



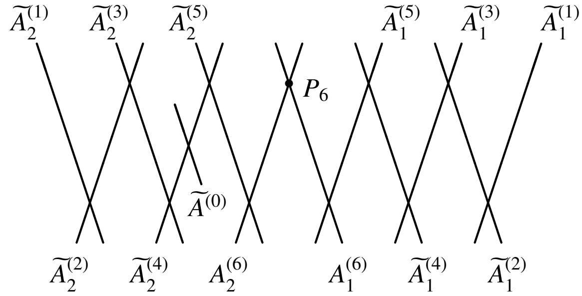
By blowing up P_3 we obtain the exceptional divisor as union of two rational smooth curves $A_1^{(4)}$ and $A_2^{(4)}$ containing a common point P_4, which is the unique singularity of the total space. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{A}_1^{(3)}$ and $\widetilde{A}_2^{(3)}$ are the strict transforms of $A_1^{(3)}$ and $A_2^{(3)}$, respectively.



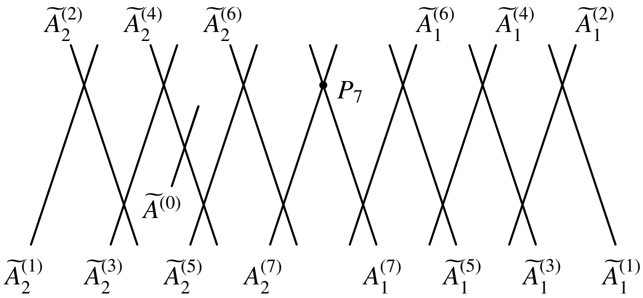
By blowing up P_4 we obtain the exceptional divisor as union of two rational smooth curves $A_1^{(5)}$ and $A_2^{(5)}$ containing a common point P_5, which is the unique singularity of the total space. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{A}_1^{(4)}$ and $\widetilde{A}_2^{(4)}$ are the strict transforms of $A_1^{(4)}$ and $A_2^{(4)}$, respectively.



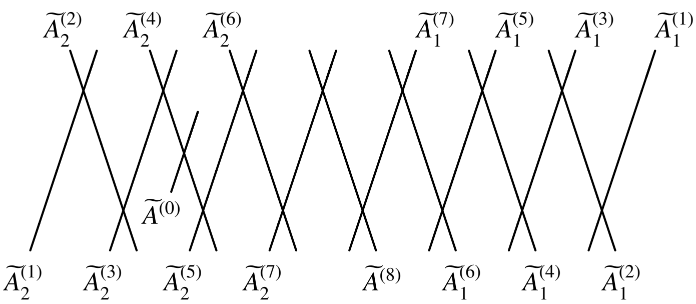
By blowing up P_5 we obtain the exceptional divisor as union of two rational smooth curves $A_1^{(6)}$ and $A_2^{(6)}$ containing a common point P_6, which is the unique singularity of the total space. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{A}_1^{(5)}$ and $\widetilde{A}_2^{(5)}$ are the strict transforms of $A_1^{(5)}$ and $A_2^{(5)}$, respectively.



By blowing up P_6 we obtain the exceptional divisor as union of two rational smooth curves $A_1^{(7)}$ and $A_2^{(7)}$ containing a common point P_7, which is the unique singularity of the total space. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{A}_1^{(6)}$ and $\widetilde{A}_2^{(6)}$ are the strict transforms of $A_1^{(6)}$ and $A_2^{(6)}$, respectively.



By blowing up P_7 we obtain an irreducible rational smooth curve $\widetilde{A}^{(8)}$ as exceptional divisor, not containing singularities of the total space. The final configuration can be seen in the following figure, where $\widetilde{A}_1^{(7)}$ and $\widetilde{A}_2^{(7)}$ are the strict transforms of $A_1^{(7)}$ and $A_2^{(7)}$, respectively. The intersections are always transversal.



By looking to the charts of each blowup we can compute the fiber of $\widetilde{\eta}$ over $(0:1)$ as follows.

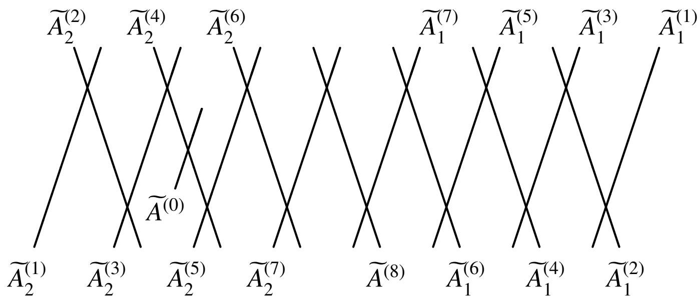
$$\begin{aligned}\widetilde{\eta}^*(0:1) = {} & 4\,\widetilde{A}^{(0)} + \widetilde{A}_1^{(1)} + 3\,\widetilde{A}_2^{(1)} + 2\,\widetilde{A}_1^{(2)} + 6\,\widetilde{A}_2^{(2)} + 3\,\widetilde{A}_1^{(3)} + 9\,\widetilde{A}_2^{(3)} + 4\,\widetilde{A}_1^{(4)} + \\ & 12\,\widetilde{A}_2^{(4)} + 5\,\widetilde{A}_1^{(5)} + 11\,\widetilde{A}_2^{(5)} + 6\,\widetilde{A}_1^{(6)} + 10\,\widetilde{A}_2^{(6)} + 7\,\widetilde{A}_1^{(7)} + 9\,\widetilde{A}_2^{(7)} + 8\,\widetilde{A}^{(8)}\end{aligned}$$

Since a fiber meets its components with intersection number zero, we can calculate the self-intersection numbers of the components from the intersection numbers of pairs of different components. Indeed we obtain the self-intersection of $\widetilde{A}_j^{(i)}$ and $\widetilde{A}^{(8)}$ equal to -2, for all $i = 1, \ldots, 7$ and $j = 1, 2$, and the self-intersection of $\widetilde{A}^{(0)}$ equal to -3. Thus, there are no curves in the fibers of $\widetilde{\eta}$ with self-intersection -1. We summarize:

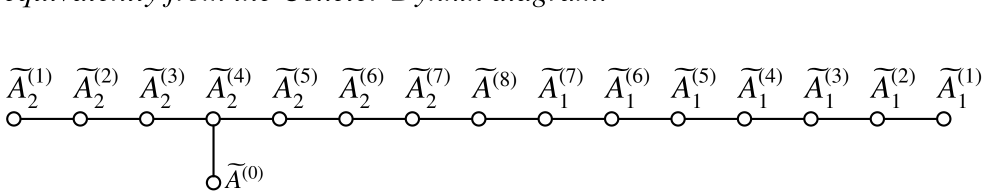
Theorem 4.4. *The fibration $\widetilde{\eta} : \widetilde{S} \to \mathbb{P}^1(k)$ is the minimal proper regular model of the fibration by nonsmooth curves $\eta : S \to \mathbb{P}^1(k)$. Its fiber over $(1:t)$ coincides with the integral rational fiber $\eta^*(1:t)$ of the original fibration η, for each t in $\mathbb{A}^1(k)$, and over $(0:1)$ is a linear combination of smooth rational curves*

$$\begin{aligned}\widetilde{\eta}^*(0:1) = {} & 4\,\widetilde{A}^{(0)} + \widetilde{A}_1^{(1)} + 3\,\widetilde{A}_2^{(1)} + 2\,\widetilde{A}_1^{(2)} + 6\,\widetilde{A}_2^{(2)} + 3\,\widetilde{A}_1^{(3)} + 9\,\widetilde{A}_2^{(3)} + 4\,\widetilde{A}_1^{(4)} + \\ & 12\,\widetilde{A}_2^{(4)} + 5\,\widetilde{A}_1^{(5)} + 11\,\widetilde{A}_2^{(5)} + 6\,\widetilde{A}_1^{(6)} + 10\,\widetilde{A}_2^{(6)} + 7\,\widetilde{A}_1^{(7)} + 9\,\widetilde{A}_2^{(7)} + 8\,\widetilde{A}^{(8)}\end{aligned}$$

whose intersection configurations are obtained from the diagram:



or equivalently from the Coxeter-Dynkin diagram:



However the surface $\widetilde{S}$ still contains an horizontal contractible curve and hence it is not a minimal surface. Indeed it can be seen in the following result.

Theorem 4.5. *The birational transform $\widetilde{H} \subset \widetilde{S}$ of the curve*

$$H := \{(0:1:0)\} \times \mathbb{P}^1(k)$$

is an horizontal smooth rational curve of self-intersection number -1. By blowing down on $\widetilde{S}$ the bunch

$$\{\widetilde{H}, \widetilde{A}_1^{(1)}, \widetilde{A}_1^{(2)}, \widetilde{A}_1^{(3)}, \widetilde{A}_1^{(4)}, \widetilde{A}_1^{(5)}, \widetilde{A}_1^{(6)}, \widetilde{A}_1^{(7)}, \widetilde{A}^{(8)}, \widetilde{A}_2^{(7)}, \widetilde{A}_2^{(6)}, \widetilde{A}_2^{(5)}, \widetilde{A}_2^{(4)}, \widetilde{A}_2^{(3)}, \widetilde{A}_2^{(2)}, \widetilde{A}_2^{(1)}\}$$

we obtain a minimal surface isomorphic to the projective plane.

Before presenting the proof we notice two interesting geometric properties of the horizontal curve H on S.

As we can see in the proof of Proposition 3.33, H is the cross-section that meets each fiber $\eta^*(1:t)$, with $t \in k$, in its Weierstrass point. On the other hand, it is also the indeterminacy locus of the rational double cover $S \dashrightarrow S'$, previously constructed.

Proof. First of all we notice that S is a rational surface as we can see from the assignment

$$(x:y:z) \mapsto ((x:y:z), (z^4 : y^3z - x^4))$$

that defines a birational map between $\mathbb{P}^2(k)$ and S, whose inverse is induced by the natural projection

$$\mathbb{P}^2(k) \times \mathbb{P}^1(k) \longrightarrow \mathbb{P}^2(k).$$

The composed rational map

$$\mathbb{P}^2(k) \dashrightarrow \mathbb{P}^1(k) \text{ defined by } (x:y:z) \mapsto (z^4 : y^3z - x^4)$$

is birational equivalent to η and fails to be regular in the point $(0:1:0)$. So it is necessary to resolve this indeterminacy, by a chain of blowups, in order to obtain an alternate realization of the relative minimal model $\widetilde{\eta} : \widetilde{S} \to \mathbb{P}^1$.

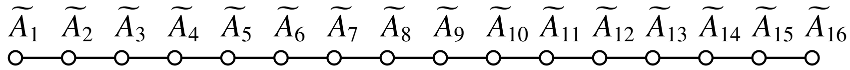
Computation shows that this indeterminacy can be resolved by a chain of sixteen blowups. These blowups provide, in the fiber of the indeterminacy, the smooth rational curves

$$\{\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_8, \widetilde{A}_9, \widetilde{A}_{10}, \widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{A}_{13}, \widetilde{A}_{14}, \widetilde{A}_{15}, \widetilde{A}_{16}\},$$

of self-intersection numbers

$$-2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -1$$

respectively, with intersection configuration according to the Coxeter-Dynkin diagram:



In addition, the morphism obtained by the resolution of the indeterminacy sends $\widetilde{A}_1, \ldots, \widetilde{A}_{15}$ on $(0:1)$ and $\widetilde{A}_{16}$ on $\mathbb{P}^1(k)$. Moreover, by comparing the fibers over $(0:1)$ of this morphism and of $\widetilde{\eta}$ we may relate these smooth rational curves with the curves that arise in Theorem 4.4, as follows.

$$\begin{aligned}
&\widetilde{A}_1 = \widetilde{A}_2^{(1)}, \widetilde{A}_2 = \widetilde{A}_2^{(2)}, \widetilde{A}_3 = \widetilde{A}_2^{(3)}, \widetilde{A}_4 = \widetilde{A}_2^{(4)}, \widetilde{A}_5 = \widetilde{A}_2^{(5)},\\
&\widetilde{A}_6 = \widetilde{A}_2^{(6)}, \widetilde{A}_7 = \widetilde{A}_2^{(7)}, \widetilde{A}_8 = \widetilde{A}^{(8)}, \widetilde{A}_9 = \widetilde{A}_1^{(7)}, \widetilde{A}_{10} = \widetilde{A}_1^{(6)},\\
&\widetilde{A}_{11} = \widetilde{A}_1^{(5)}, \widetilde{A}_{12} = \widetilde{A}_1^{(4)}, \widetilde{A}_{13} = \widetilde{A}_1^{(3)}, \widetilde{A}_{14} = \widetilde{A}_1^{(2)}, \widetilde{A}_{15} = \widetilde{A}_1^{(1)}
\end{aligned}$$

On the other hand, these sixteen blowups also resolve the indeterminacy of the birational map $\mathbb{P}^2(k) \dashrightarrow S$. By looking to the induced isomorphism we conclude that $\widetilde{A}_{16}$ is exactly the birational transform of H. □

Remark 4.6. A curios phenomenon, that never occurs in zero characteristic, may be assigned by the proof of the last theorem. The rational map $\mathbb{P}^2(k) \dashrightarrow \mathbb{P}^1(k)$ defined by the assignment

$$(x : y : z) \mapsto (z^4 : y^3 z - x^4)$$

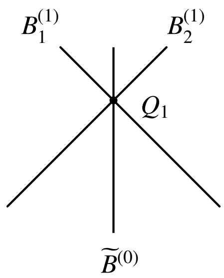
provides a covering of the projective plane, over an algebraically closed field of characteristic three, by a family of quartic curves whose smooth points are always inflection points, that is, a family of non-classical curves. Thus, according to [FS], it make sense to inquire about a possible classification of this phenomenon.

Before presenting the other kind of fibration by nonsmooth curves we will study the regular minimal model of η'.

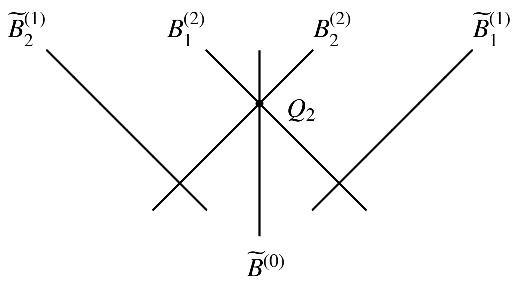
As in the previous fibration, since the fiber of η' over $(1 : t)$ is integral and does not admit singular points of the total space S', for each $t \in k$, it remains to analyze the fiber $\widetilde{\eta'}^*(0 : 1)$.

Let us fix $B^{(0)} := \eta'^*(0 : 1)$. It is a curve on S', containing its unique singularity $Q_0 := ((1 : 0 : 0), (0 : 1))$, given by the polynomial equation "$z = 0$".

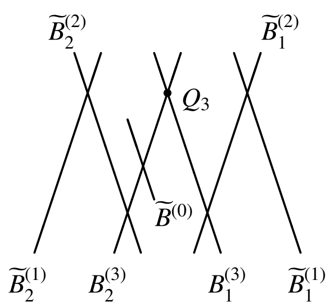
By blowing up Q_0 we obtain the exceptional divisor as union of two rational smooth curves $B_1^{(1)}$ and $B_2^{(1)}$ containing a common point Q_1, which is the unique singularity of the total space and belongs to the strict transform $\widetilde{B}^{(0)}$ of $B^{(0)}$. This configuration can be seen in the following figure, where the intersections are always transversal.



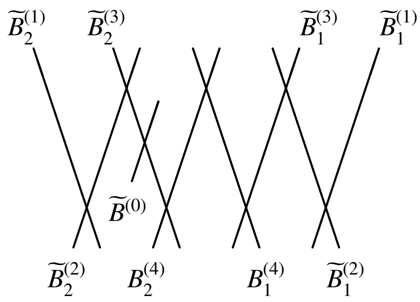
By blowing up Q_1 we obtain the exceptional divisor as union of two rational smooth curves $B_1^{(2)}$ and $B_2^{(2)}$ containing a common point Q_2, which is the unique singularity of the total space and still belongs to $\widetilde{B}^{(0)}$. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{B}_1^{(1)}$ and $\widetilde{B}_2^{(1)}$ are the strict transforms of $B_1^{(1)}$ and $B_2^{(1)}$, respectively.



By blowing up Q_2 we obtain the exceptional divisor as union of two rational smooth curves $B_1^{(3)}$ and $B_2^{(3)}$ containing a common point Q_3, which is the unique singularity of the total space. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{B}_1^{(2)}$ and $\widetilde{B}_2^{(2)}$ are the strict transforms of $B_1^{(2)}$ and $B_2^{(2)}$, respectively.



By blowing up Q_3 we obtain the exceptional divisor as union of two rational smooth curves $B_1^{(4)}$ and $B_2^{(4)}$, which do not admit singularities of the total space. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{B}_1^{(3)}$ and $\widetilde{B}_2^{(3)}$ are the strict transforms of $B_1^{(3)}$ and $B_2^{(3)}$, respectively.



By looking to the charts of each blowup we can compute the fiber of $\widetilde{\eta}$ over $(0:1)$ as follows.

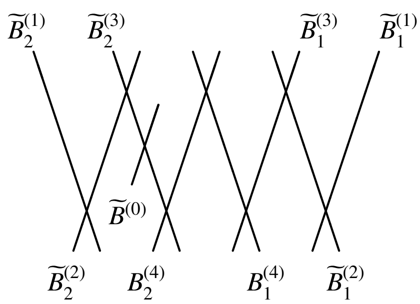
$$\widetilde{\eta}^*(0:1) = 3\,\widetilde{B}^{(0)} + \widetilde{B}_1^{(1)} + 2\,\widetilde{B}_2^{(1)} + 2\,\widetilde{B}_1^{(2)} + 4\,\widetilde{B}_2^{(2)} + 3\,\widetilde{B}_1^{(3)} + 6\,\widetilde{B}_2^{(3)} + 4\,\widetilde{B}_1^{(4)} + 5\,\widetilde{B}_2^{(4)}$$

Since a fiber meets its components with intersection number zero, we can calculate the self-intersection numbers of the components from the intersection numbers of pairs of different components. Indeed we obtain all components with self-intersection two. Thus, there are no curves in the fibers of $\widetilde{\eta}$ with self-intersection -1. We summarize:

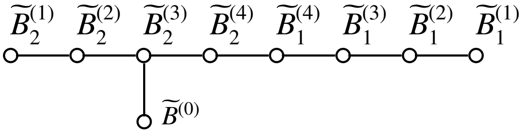
Theorem 4.7. *The fibration $\widetilde{\eta'} : \widetilde{S'} \to \mathbb{P}^1(k)$ is the minimal proper regular model of the fibration by nonsmooth curves $\eta' : S' \to \mathbb{P}^1(k)$. Its fiber over $(1:t)$ coincides with the integral rational fiber $\eta'^*(1:t)$ of the original fibration η', for each $t \in \mathbb{A}^1(k)$, and over $(0:1)$ is a linear combination of smooth rational curves*

$$\widetilde{\eta}^*(0:1) = 3\,\widetilde{B}^{(0)} + \widetilde{B}_1^{(1)} + 2\,\widetilde{B}_2^{(1)} + 2\,\widetilde{B}_1^{(2)} + 4\,\widetilde{B}_2^{(2)} + 3\,\widetilde{B}_1^{(3)} + 6\,\widetilde{B}_2^{(3)} + 4\,\widetilde{B}_1^{(4)} + 5\,\widetilde{B}_2^{(4)}$$

whose intersection configurations are obtained from the diagram:



or from the Coxeter-Dynkin diagram:



However the surface $\widetilde{S'}$ still contains an horizontal contractible curve and hence it is not a minimal surface. Indeed it can be seen in the following result.

Theorem 4.8. *The birational transform $\widetilde{H'} \subset \widetilde{S'}$ of the curve*

$$H' := \{(1:0:0)\} \times \mathbb{P}^1(k)$$

is an horizontal smooth rational curve of self-intersection number -1. By blowing down on $\widetilde{S'}$ the bunch

$$\{\widetilde{H'}, \widetilde{B}_1^{(1)}, \widetilde{B}_1^{(2)}, \widetilde{B}_1^{(3)}, \widetilde{B}_1^{(4)}, \widetilde{B}_2^{(4)}, \widetilde{B}_2^{(3)}, \widetilde{B}_2^{(2)}, \widetilde{B}_2^{(1)}\}$$

we obtain a minimal surface isomorphic to the projective plane.

Proof. First of all we notice that S' is a rational surface as we can seen from the assignment

$$(x:y:z) \mapsto ((x:y:z), (z^3 : y^3 - zx^2))$$

that defines a birational map between $\mathbb{P}^2(k)$ and S', whose inverse is induced by the natural projection

$$\mathbb{P}^2(k) \times \mathbb{P}^1(k) \longrightarrow \mathbb{P}^2(k).$$

The composed rational map

$$\mathbb{P}^2(k) \dashrightarrow \mathbb{P}^1(k) \text{ defined by } (x:y:z) \mapsto (z^3 : y^3 - zx^2)$$

is birational equivalent to η' and fails to be regular in the point $(1:0:0)$. So in order to obtain an alternative realization of the relative minimal model $\widetilde{\eta'}$ of η', it is necessary to resolve this indeterminacy by a chain of blowups.

Computations shows that this indeterminacy can be resolved by a chain of nine blowups. These blowups provide the smooth rational curves in the fiber of the indeterminacy

$$\{\widetilde{B}_1, \widetilde{B}_2, \widetilde{B}_3, \widetilde{B}_4, \widetilde{B}_5, \widetilde{B}_6, \widetilde{B}_7, \widetilde{B}_8, \widetilde{B}_9\}$$

of self-intersection numbers

$$-2, -2, -2, -2, -2, -2, -2, -2, -1$$

respectively, with intersection configuration according to the Coxeter-Dynkin diagram:

$$\overset{\widetilde{B}_1}{\circ} - \overset{\widetilde{B}_2}{\circ} - \overset{\widetilde{B}_3}{\circ} - \overset{\widetilde{B}_4}{\circ} - \overset{\widetilde{B}_5}{\circ} - \overset{\widetilde{B}_6}{\circ} - \overset{\widetilde{B}_7}{\circ} - \overset{\widetilde{B}_8}{\circ} - \overset{\widetilde{B}_9}{\circ}$$

In addition, the morphism obtained by the resolution of the indeterminacy sends $\widetilde{B}_1, \ldots, \widetilde{B}_8$ on $(0 : 1)$ and $\widetilde{B}_9$ on $\mathbb{P}^1(k)$. Moreover, by comparing the fibers over $(0 : 1)$ of this morphism and of $\widetilde{\eta}$ we may relate these smooth rational curves with the curves that arise in Theorem 4.7, as follows.

$$\widetilde{B}_1 = \widetilde{B}_2^{(1)}, \widetilde{B}_2 = \widetilde{B}_2^{(2)}, \widetilde{B}_3 = \widetilde{B}_2^{(3)}, \widetilde{B}_4 = \widetilde{B}_2^{(4)},$$
$$\widetilde{B}_5 = \widetilde{B}_1^{(4)}, \widetilde{B}_6 = \widetilde{B}_1^{(3)}, \widetilde{B}_7 = \widetilde{B}_1^{(2)}, \widetilde{B}_8 = \widetilde{B}_1^{(1)}$$

On the other hand, these nine blowups also resolve the indeterminacy of the birational map $\mathbb{P}^2(k) \dashrightarrow S'$. By looking to the induced isomorphism we conclude that $\widetilde{B}_9$ is exactly the birational transform of H'. □

Now we present a fibration by nonsmooth curves whose general fiber does not admit a rational double cover over a plane projective cuspidal cubic.

Let us consider the surface

$$S \subset \mathbb{P}^2(k) \times \mathbb{P}^1(k)$$

given by the bihomogeneous polynomial

$$sZY^3 - tZ^4 - tZX^3 - sX^4$$

and the morphism

$$\eta : S \longrightarrow \mathbb{P}^1(k)$$

induced by the second projection $((x : y : z), (s : t)) \mapsto (s : t)$.

For each $t \in k$ the Jacobian criterion provides that the fiber over $(1 : t)$, which is identified with the rational and integral plane projective quartic given by the homogeneous polynomial

$$ZY^3 - tZ^4 - tZX^3 - X^4,$$

admits $(0 : t^{1/3} : 1)$ as unique singularity. In addition, it is unibranch of multiplicity and singularity degree three. We still notice that in the affine open subset of $\mathbb{P}^1(k)$, whose points are of the form $(1 : t)$ with $t \in k$, the restriction of the fibration η can be obtained by the restricted morphism

$$\pi^{-1}(C) \to C,$$

where π is the two dimensional fibration by nonsmooth curves constructed in the previous section and C is the affine plane curve given by the polynomial $x - y$ in $k[x, y]$. Hence η does not admit a factorization by a rational double cover followed by a fibration by plane projective cuspidal cubics, as we can see in Theorem 4.1.

On the other hand the fiber over $(0:1)$ is identified with a non-reduced projective plane curve given by the polynomial $Z(X+Z)^3$.

By the Jacobian criterion, though each special fiber is singular the total space S has only two singular points, namely

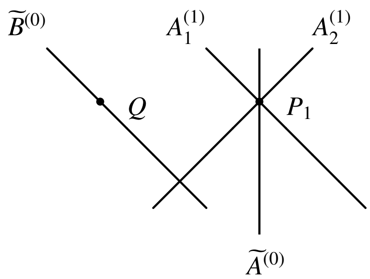
$$P := ((0:1:0),(0:1)) \text{ and } Q := ((2:1:1),(0:1)).$$

As in the previous fibration, we just need to describe the desingularization morphism $\widetilde{S} \to S$, obtained by blowups, and determine the fiber of the morphism

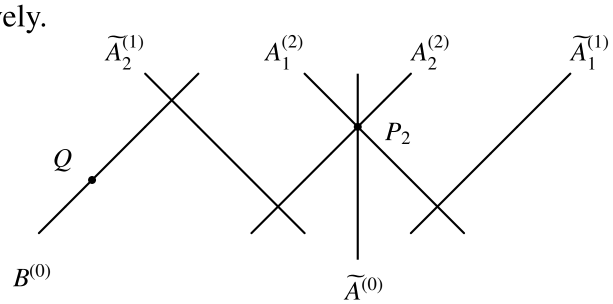
$$\widetilde{\eta} : \widetilde{S} \longrightarrow S \longrightarrow \mathbb{P}^1(k)$$

over $(0:1)$. To do this we denote by $A^{(0)}$ and $B^{(0)}$ the two components "$z=0$" and "$z+x=0$" of the fiber $\eta^*(0:1)$, respectively.

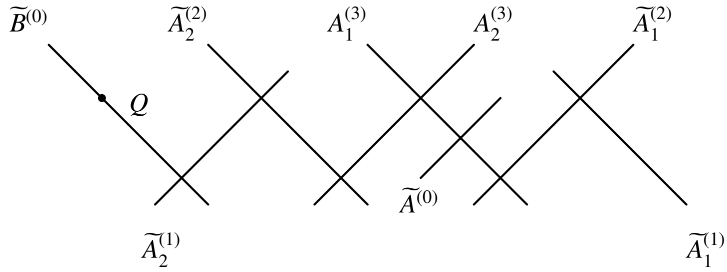
By blowing up P we obtain the exceptional divisor as union of two rational smooth curves $A_1^{(1)}$ and $A_2^{(1)}$ containing a common point P_1, which is the unique singularity of the total space contained in the exceptional divisor, and belongs to the strict transform $\widetilde{A}^{(0)}$ of $A^{(0)}$. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{B}^{(0)}$ is the strict transform of $B^{(0)}$.



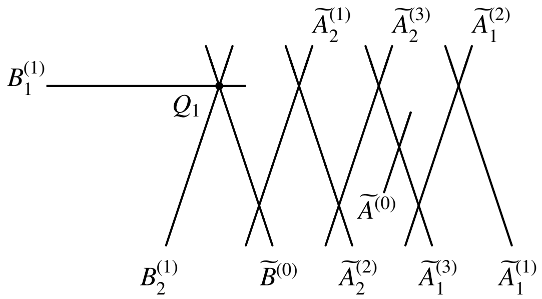
By blowing up P_1 we obtain the exceptional divisor as union of two rational smooth curves $A_1^{(2)}$ and $A_2^{(2)}$ containing a common point P_2, which is the unique singularity of the total space contained in the exceptional divisor, and still belongs to $\widetilde{A}^{(0)}$. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{A}_1^{(1)}$ and $\widetilde{A}_2^{(1)}$ are the strict transforms of $A_1^{(1)}$ and $A_2^{(1)}$, respectively.



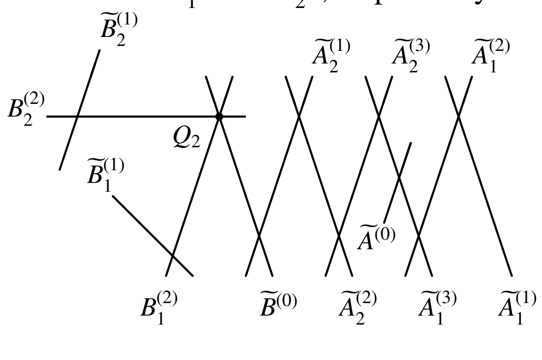
By blowing up P_2 we obtain the exceptional divisor as union of two rational smooth curves $A_1^{(3)}$ and $A_2^{(3)}$ not containing a singularity of the total space. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{A}_1^{(2)}$ and $\widetilde{A}_2^{(2)}$ are the strict transforms of $A_1^{(2)}$ and $A_2^{(2)}$, respectively.



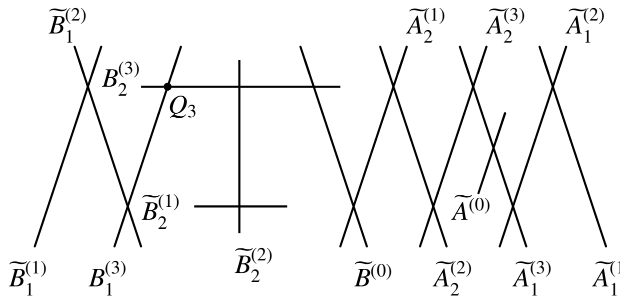
By blowing up Q we obtain the exceptional divisor as union of two rational smooth curves $B_1^{(1)}$ and $B_2^{(1)}$ containing a common point Q_1, which is the unique singularity of the total space and belongs to $\widetilde{B}^{(0)}$. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{A}_1^{(3)}$ and $\widetilde{A}_2^{(3)}$ are the strict transforms of $A_1^{(3)}$ and $A_2^{(3)}$, respectively.



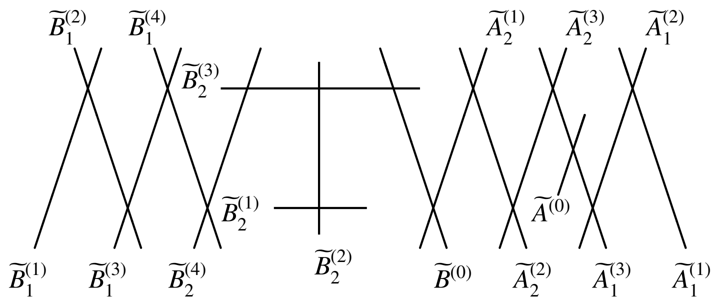
By blowing up Q_1 we obtain the exceptional divisor as union of two rational smooth curves $B_1^{(2)}$ and $B_2^{(2)}$ containing a common point Q_2, which is the unique singularity of the total space and belongs to $\widetilde{B}^{(0)}$. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{B}_1^{(1)}$ and $\widetilde{B}_2^{(1)}$ are the strict transforms of $B_1^{(1)}$ and $B_2^{(1)}$, respectively.



By blowing up Q_2 we obtain the exceptional divisor as union of two rational smooth curves $B_1^{(3)}$ and $B_2^{(3)}$ containing a common point Q_3, which is the unique singularity of the total space. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{B}_1^{(2)}$ and $\widetilde{B}_2^{(2)}$ are the strict transforms of $B_1^{(2)}$ and $B_2^{(2)}$, respectively.



By blowing up Q_3 we obtain the exceptional divisor as union of two rational smooth curves $\widetilde{B}_1^{(4)}$ and $\widetilde{B}_2^{(4)}$ not containing singularities of the total space. This configuration can be seen in the following figure, where the intersections are always transversal and $\widetilde{B}_1^{(3)}$ and $\widetilde{B}_2^{(3)}$ are the strict transforms of $B_1^{(3)}$ and $B_2^{(3)}$, respectively.



By looking to the charts of each blowup we can compute the fiber of $\widetilde{\eta}$ over $(0:1)$ as follows.

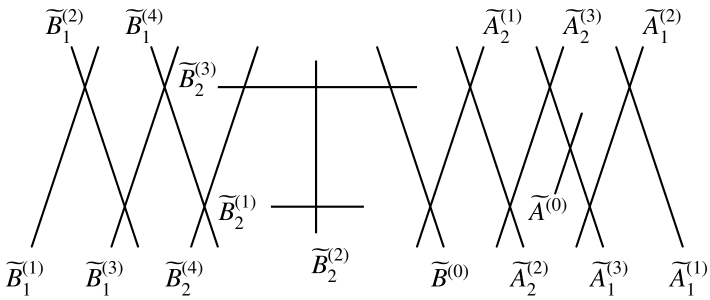
$$\begin{aligned}\widetilde{\eta}^*(0:1) = \widetilde{A}^{(0)} + \widetilde{A}_1^{(1)} + 3\,\widetilde{A}_2^{(1)} + 2\,\widetilde{A}_1^{(2)} + 3\,\widetilde{A}_2^{(2)} + 3\,\widetilde{A}_1^{(3)} + 3\,\widetilde{A}_2^{(3)} + 3\,\widetilde{B}^{(0)} + \\ \widetilde{B}_1^{(1)} + 2\,\widetilde{B}_2^{(1)} + 2\,\widetilde{B}_1^{(2)} + 4\,\widetilde{B}_2^{(2)} + 3\,\widetilde{B}_1^{(3)} + 6\,\widetilde{B}_2^{(3)} + 4\,\widetilde{B}_1^{(4)} + 5\,\widetilde{B}_2^{(4)}.\end{aligned}$$

Since a fiber meets its components with intersection number zero, we can compute the self-intersection numbers of the components from the intersection numbers of pairs of different components. Indeed we obtain the self-intersection of $\widetilde{A}_j^{(i)}$ and $\widetilde{B}_j^{(l)}$ equal to -2, for all $i = 1,2,3$, $l = 1,2,3,4$ and $j = 1,2$, and $\widetilde{A}^{(0)} \cdot \widetilde{A}^{(0)} = -3 = \widetilde{B}^{(0)} \cdot \widetilde{B}^{(0)}$. Thus, there are no curves in the fibers of $\widetilde{\eta}$ with self-intersection -1. We summarize:

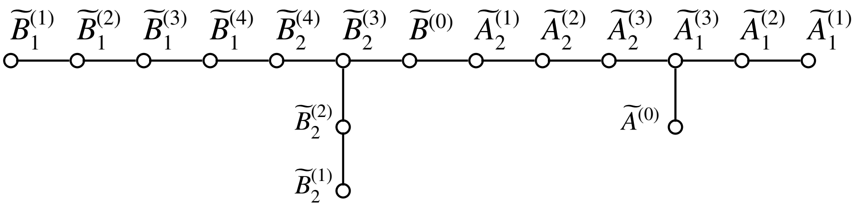
Theorem 4.9. *The fibration $\widetilde{\eta} : \widetilde{S} \to \mathbb{P}^1(k)$ is the minimal proper regular model of the fibration by nonsmooth curves $\eta : S \to \mathbb{P}^1(k)$. Its fibers over $(1 : t)$ coincide with the integral fibers $\eta^*(1 : t)$ of the original fibration η, for each $t \in \mathbb{A}^1(k)$, and over $(0 : 1)$ is a linear combination of smooth rational curves*

$$\begin{aligned}\widetilde{\eta}^*(0:1) = {} & \widetilde{A}^{(0)} + \widetilde{A}^{(1)}_1 + 3\,\widetilde{A}^{(1)}_2 + 2\,\widetilde{A}^{(2)}_1 + 3\,\widetilde{A}^{(2)}_2 + 3\,\widetilde{A}^{(3)}_1 + 3\,\widetilde{A}^{(3)}_2 + 3\,\widetilde{B}^{(0)} + \\ & \widetilde{B}^{(1)}_1 + 2\,\widetilde{B}^{(1)}_2 + 2\,\widetilde{B}^{(2)}_1 + 4\,\widetilde{B}^{(2)}_2 + 3\,\widetilde{B}^{(3)}_1 + 6\,\widetilde{B}^{(3)}_2 + 4\,\widetilde{B}^{(4)}_1 + 5\,\widetilde{B}^{(4)}_2\end{aligned}$$

whose intersection configurations are obtained from the following diagram.



or from the Coxeter-Dynkin diagram:



However the surface $\widetilde{S}$ still contains horizontal contractible curves and hence it is not a minimal surface. Indeed it can be seen in the following result.

Theorem 4.10. *The birational transforms $\widetilde{H}_1, \widetilde{H}_2 \subset \widetilde{S}$ of the curves*

$$H_1 := \{(0:1:0)\} \times \mathbb{P}^1(k)$$
$$H_2 := \{(2:1:1)\} \times \mathbb{P}^1(k)$$

are horizontal smooth rational curves of self-intersection number -1. By blowing down on $\widetilde{S}$ the bunches

$$\{\widetilde{H}_1, \widetilde{A}^{(1)}_1, \widetilde{A}^{(2)}_1, \widetilde{A}^{(3)}_1, \widetilde{A}^{(3)}_2, \widetilde{A}^{(2)}_2, \widetilde{A}^{(1)}_2\}$$
$$\{\widetilde{H}_2, \widetilde{B}^{(1)}_1, \widetilde{B}^{(2)}_1, \widetilde{B}^{(3)}_1, \widetilde{B}^{(4)}_1, \widetilde{B}^{(4)}_2, \widetilde{B}^{(3)}_2, \widetilde{B}^{(2)}_2, \widetilde{B}^{(1)}_2\}$$

we obtain a minimal surface isomorphic to the projective plane.

Proof. First of all we notice that S is a rational surface as we can see from the assignment

$$(x:y:z) \mapsto ((x:y:z),(z^4+x^3z:y^3z-x^4))$$

that defines a birational map between $\mathbb{P}^2(k)$ and S, whose inverse is induced by the natural projection

$$\mathbb{P}^2(k)\times\mathbb{P}^1(k) \longrightarrow \mathbb{P}^2(k).$$

The composed rational map

$$\mathbb{P}^2(k) \dashrightarrow \mathbb{P}^1(k) \text{ defined by } (x:y:z) \mapsto (z^4+x^3z:y^3z-x^4)$$

is birational equivalent to $\eta : S \to \mathbb{P}^1(k)$ and fails to be regular in the points $(0:1:0)$ and $(2:1:1)$. So it is necessary to resolve these indeterminacies, by a chain of blowups, in order to obtain an alternative realization of the relative minimal model $\widetilde{\eta} : \widetilde{S} \to \mathbb{P}^1(k)$.

Computations shows that the indeterminacies $(0:1:0)$ and $(2:1:1)$ can be resolved by a chain of seven and nine blowups respectively.

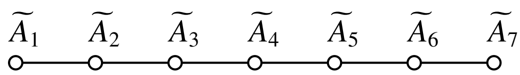
The seven blowups over $(0:1:0)$ provides, in its fiber, the smooth rational curves

$$\{\widetilde{A}_1,\widetilde{A}_2,\widetilde{A}_3,\widetilde{A}_4,\widetilde{A}_5,\widetilde{A}_6,\widetilde{A}_7\},$$

of self-intersection numbers

$$-2,-2,-2,-2,-2,-2,-1$$

respectively, with intersection according to the Coxeter-Dynkin diagram:



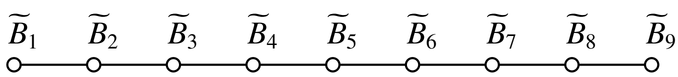
The nine blowups over $(2:1:1)$ provides, in its fiber, the smooth rational curves

$$\{\widetilde{B}_1,\widetilde{B}_2,\widetilde{B}_3,\widetilde{B}_4,\widetilde{B}_5,\widetilde{B}_6,\widetilde{B}_7,\widetilde{B}_8,\widetilde{B}_9\},$$

of self-intersection numbers

$$-2,-2,-2,-2,-2,-2,-2,-2,-1$$

respectively, with intersection according to the Coxeter-Dynkin diagram:



In addition, their images under the morphism obtained by the resolution of the indeterminacies are $(0:1)$ for all $\widetilde{A}_i$ and $\widetilde{B}_j$, with $i = 1, \ldots, 6$ and $j = 1, \ldots, 8$, and $\mathbb{P}^1$ for $\widetilde{A}_7$ and $\widetilde{B}_9$. Moreover, by comparing the fibers over $(0:1)$ of the resolved morphism and of $\widetilde{\eta}$ we may relate these smooth rational curves with the curves that arisen in Theorem 4.9, as follows.

$$\widetilde{A}_1 = \widetilde{A}_2^{(1)}, \widetilde{A}_2 = \widetilde{A}_2^{(2)}, \widetilde{A}_3 = \widetilde{A}_2^{(3)}, \widetilde{A}_4 = \widetilde{A}_1^{(3)}, \widetilde{A}_5 = \widetilde{A}_1^{(2)}, \widetilde{A}_6 = \widetilde{A}_1^{(1)}$$

and

$$\widetilde{B}_1 = \widetilde{B}_2^{(1)}, \widetilde{B}_2 = \widetilde{B}_2^{(2)}, \widetilde{B}_3 = \widetilde{B}_2^{(3)}, \widetilde{B}_4 = \widetilde{B}_2^{(4)},$$
$$\widetilde{B}_5 = \widetilde{B}_1^{(4)}, \widetilde{B}_6 = \widetilde{B}_1^{(3)}, \widetilde{B}_7 = \widetilde{B}_1^{(2)}, \widetilde{B}_8 = \widetilde{B}_1^{(1)}$$

On the other hand, these seven and nine blowups also resolve the indeterminacies of the birational map $\mathbb{P}^2(k) \dashrightarrow S$. By looking to the induced isomorphism we conclude that $\widetilde{A}_7$ and $\widetilde{B}_9$ are exactly the birational transforms of H_1 and H_2 respectively. $\square$

Bibliography

[An] Angermüller, G., *Die Wertehalbgruppe einer ebenen irreduziblen algebroiden Kurve,* Math. Z., 153 (1977), 267–282.

[Ar] Artin, E., *Algebraic numbers and algebraic functions,* New York: Gordon and Breach (1967).

[BS] H. Bedoya and K.-O. Stöhr, *An algorithm to calculate discrete invariants in function fields*, J. Number Theory **27** (1987), 197–232.

[B] E. Bertini, *Sui sistemi lineari*, Instit. Lombardo Accad. Sci. Lett.Rend. A Instituto(II) **15**(1882), 24–28

[BM] E. Bombieri and D. Mumford, *Enriques' classification of surfaces in characteristic p. III*, Invent. Math. **35** (1976), 197–232.

[BN] H. Borges Neto, *Mudança de gênero e classificação de corpos de gênero 2*, Tese de Doutorado, Instituto de Matemática Pura e Aplicada, Rio de Janeiro (1979).

[C] C. Chevalley, *Introduction to the Theory of Algebraic Functions of One Variable*, AMS Math. Surveys, New York, 1951.

[EGA IV] A. Grothendieck, J. Dieudonné, *Éléments de gomtrie algbrique : IV. Étude locale des schémas et des morphismes de schémas,* Publications Mathématiques de l'IHES, 4, 8, 11, 17, 20, 24, 28, 32 (1960-1967).

[FS] E. S. Freitas and K.-O. Stöhr, *Non-classical Gorenstein curves of arithmetic genus three and four*, Math. Z. **218** (1995), 479–502.

[G] D. Gorenstein, *An arithmetic theory of adjoint plane curves*, Trans. Amer. Math. Soc. **72** (1952), 414–436.

[Ha] R. Hartshorne. *Generalized Divisors on Gorenstein Curves and a Theorem of Noether,* J. Math. Kyoto Univ **26**, 375–386 (1986).

[H] H. Hironaka. *On the Arithmetic Genera and Effective Genera of Algebraic Curves,* Mem. Kyoto **30**, 177–195 (1957).

[K] S.L. Kleiman, *Bertini and its two fundamental theorems*, Rend. Circ. Mat. Palermo (2) Suppl. **55** (1998), 9–37.

[Ko] K. Kodaira, *On Compact Analytic Surfaces: II*, Annals of Mathematics, Vol. 77, No. 3 (1963), 563–626

[La] W.E. Lang, *Quasi-elliptic surfaces in characteristic three*, Ann. Sci. École Norm. Sup. (4) **12** (1979), 473–500.

[Li] S. Lichtenbaum, *Curves over discrete valuation rings*, Amer. J. Math. **90** (1968), 380–405.

[Liu] Q. Liu, *Algebraic Geometry and Arithmetic Curves,* Oxford Graduate Texts in Mathematics. Oxford University Press, New-York, (2006). North Holland, Amsterdam, (1993).

[Ma] T. Matsusaca, *The Theorem of Bertini on Linear Systems in Modular Fields,* Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **26** (1951), 51–62.

[M] D. Mumford, *Introduction,* in "Oscar Zariski: Colected Papers I," Hironaka and Mumford (ed.), MIT Press (1972), 3–5.

[N] A. Néron, *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux,* Publications Mathématiques de l'IHES, 21 (1964) 1–128.

[Q] C. S. Queen, *Nonconservative function fields of genus one. I,* Arch. Math., 22 (1971) 612–623.

[R] M. Rosenlicht, *Equivalence relations on algebraic curves*, Ann. of Math. (2) **56** (1952), 169–171.

[Sc] S. Schröer, *Singularities appearing on generic fiber of morphisms between smooth schemes,* Michigan Math. J. 56 (2008), 55-76.

[Se] J.-P. Serre, *Groupes Algébriques et Corps de Classes*, Hermann, Paris, 1959.

[Sh1] I.R. Shafarevich, *"Lectures on minimal models and birational transformations of two-dimensional schemes"*, Tata Institute, Bombay 1966.

[Sh2] I.R. Shafarevich, *"Basic Algebraic Geometry. 1"*, Springer-Verlag, Berlin, 1994.

[St] H. Stichtenoth, *Zur Konservativität algebraischer Funktionenkörper*, J. Reine Angew. Math. **301** (1978), 30–45.

[S1] K.-O. Stöhr, *On singular primes in function fields*, Arch. Math. **50** (1988), 156–163.

[S2] K.-O. Stöhr, *On Bertini's theorem in characteristic p for families of canonical curves in* $\mathbb{P}^{(p-3)/2}$, Proc. London Math. Soc. (3) **89** (2004), 291–316.

[S3] K.-O. Stöhr, *On the poles of regular differentials of singular curves*, Bol. Soc. Bras. Mat. **24** (1993), 105–136.

[S4] K.-O. Stöhr, *On Bertini's theorem for fibrations by plane projective quartic curves in characteristic five*, Journal of Algebra **315** (2007), 502–526.

[SVi] K.-O. Stöhr and M.L.T. Villela, *Nonconservative function fields of genus* $(p+1)/2$, Manuscripta Math. **66** (1989), 61–71.

[SVo] K.-O. Stöhr and J. Voloch, *Weierstrass points and curves over finite fields*, Proc. London Math. Soc. (3) **52** (1986), 1–19.

[T] J. Tate, *Genus change in inseparable extensions of function fields*, Proc. Amer. Math. Soc. **3** (1952), 400–406.

[V] M.L.T. Villela, *Corpos de Funções Algébricas Não Conservativos de Gênero* $(p+1)/2$, Tese de Doutorado, Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro (1984).

[WITO] K.-i. Watanabe, T. Ishikawa, S. Tachibana and K. Otsuka, *On tensor products of Gorenstein rings*, J. Math. Kyoto Univ. **9** (1969), 413–423.

[Z1] O. Zariski, *The theorem of Bertini on the variable singular points of a linear system of varieties*, Trans. Amer. Math. Soc. **56** (1944), 130–140.

[Z2] O. Zariski, *The concept of a simple point of an abstract algebraic variety*, Trans. Amer. Math. Soc. **62** (1947), 1–52.