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C^1 dynamics far from tangencies

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Introduction

In this thesis we study the dynamics of C^1 diffeomorphisms far away from homoclinic tangencies, taht is, such that no diffeomorphism in a neighborhood exhibits a non-transverse intersection between the stable manifold and the unstable manifold of some periodic point. There are two main sets of results, having in common the general theme that *diffeomorphisms far away from tangencies resemble hyperbolic diffeomorphisms.*

In the first part of the work we study the ergodic measures of diffeomorphisms far away from homoclinic tangencies. We show that every ergodic measure has at most one vanishing Lyapunov exponent, and the Osledets splitting corresponding to positive, zero, and negative exponents is dominated.

In fact we prove that Pesin theory (existence of smooth local stable and local unstable manifolds) holds in this C^1 setting: the usual $C^{1+Holder}$ regularity assumption can be replaced by the condition that this system is fay away from homoclinic tangencies. Morever, some shadowing lemma holds, and every hyperbolic ergodic measure is the weak limit of a sequence of atomic invariant measures supported on periodic orbits belonging to the same homoclinic class.

By means of a result announced recently by Díaz and Gorodetski, we deduce that for C^1 generic diffeomorphisms far away from tangencies, every chain recurrent class C is either hyperbolic or has a non-hyperbolic ergodic invariant measure. In particular, if C is an aperiodic class, then every ergodic measure supported in it is non-hyperbolic. That gives a partial answer to a conjecture given by Díaz and Gorodetski in the case of diffeomorphisms far away from tangencies.

In the second part of the work we study the so-called C^1 Newhouse phenomenon: existance of infinitely many periodic sinks or sources for a residual subset of some C^1 open set of diffeomorphisms. We prove that if the C^1 Newhouse phenomenon occurs for diffeomorphisms far away from tangencies, then those periodic sinks/sources must be related to some homoclinic class of codimension 1. In fact, the homoclinic class is the Hausdorff limit of a sequence of periodic sinks/sources. This is in contrast with the only known example of C^1 Newhouse phenomenon, due to Bonatti and Díaz, which correspond to diffeomorphisms close to homoclinic tangencies, and for which the periodic sinks/sources are often related to aperiodic classes.

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ERGODIC MEASURES FAR AWAY FROM TANGENCIES

JIAGANG YANG

Abstract. We show that for C^1 diffeomorphisms far away from homoclinic tangencies, every ergodic invariant measure has at most one zero Lyapunov exponent, and the Oseledets splitting corresponding to positive, zero, and negative exponents is dominated. When the invariant ergodic measure is hyperbolic (all exponents non-zero), then almost every point has a local stable manifold and a local unstable manifold both of which are differentiably embeded disks. Moreover, a version of the classical shadowing lemma holds, so that the hyperbolic measure is the weak limit of a sequence of atomic measures supported on periodic orbits belonging to the same homoclinic class.

Together with a recent result of [6], this allows us to prove that there exists a residual subset R of C^1 diffeomorphisms far away from tangencies such that for any $f \in R$, every chain recurrence class either is hyperbolic, or admits a non-hyperbolic ergodic measure. In particular, if the chain recurrent class is aperiodic, then every ergodic invariant measure supported in it is non-hyperbolic.

1. Introduction

In his famous paper [14], the first time Oseledets gave the definition and existence of Lyapunov exponents for any invariant measure: for an ergodic measure μ of a diffeomorphism f, there exist $k \in \mathbb{N}$, real numbers $\lambda_1 > \cdots > \lambda_k$, and for $\mu-$almost all $x \in M$, there exists a splitting $T_xM = E_x^1 \oplus \cdots \oplus E_x^k$ of the tangent space, such that the splitting is invariant under Df, and

$$\lim_{n\to\pm\infty} \frac{1}{n} log\|Df^n(x)v_i\| = \lambda_i, \quad v_i \in E_x^i \setminus \{0\}.$$

We call λ_i the Lyapunov exponent of μ and the splitting $E^1 \oplus \cdots \oplus E^k$ Oseledets splitting. Usually the splitting is just defined on a full measure subset, not continuous just measurable changed with the points. In fact, for any measurable bundle on M, Oseledets proved the existence of Lyapunov exponents for any invariant measure.

Since then, Lyapunov exponents have played a key role in studying the ergodic behavior of a dynamical system, understanding the Lyapunov exponents also become to one of the classical problems of the theory of differential dynamical systems. Especially when all the Lyapunov exponents are not vanishing, such kind of ergodic measure is called hyperbolic measure and which attracts a lot of attention.

Here we prove that if the diffeomorphism is far away from homoclinic tangencies, the Lyapunov exponents of its ergodic measures can be given a good description. Here a diffeomorphism is far away from homoclinic tangencies means that no diffeomorphism in a neighborhood exhibits a non-transverse intersection between the stable manifold and the unstable manifold of some periodic point.

Theorem 1: *Suppose f is far away from tangencies and μ is an ergodic measure of f, then*

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- *either μ is hyperbolic with index i and the index i Oseledets splitting is a dominated splitting,*
- *or μ has just one zero Lyapunov exponent, and the Oseledets splitting corresponding to negative, zero and positive Lyapunov exponents is a dominated splitting.*

Remark: *The definition of dominated splitting is given in section 1, since dominated splitting is always continuous, so the above special kind of Oseledets splitting is always continuous.*

By the definition, the tangent space of almost every point of a hyperbolic measure is splitted as the sum of two subspaces which are exponentially contracted or expanded by all large enough iterated of the derivative, it means the hyperbolic measure has some 'weak' hyperbolic property on the tangent space. Pesin showed that with some additional regularity assumption on the diffeomorphism (C^2 or $C^{1+Holder}$), the hyperbolic ergodic measure shares many properties with hyperbolic set, for example, there exists a family of local stable manifolds on a positive subset which is continuous and with uniform size, such property is called the stable manifold theorem; Katok gave also a shadowing lemma, with it he proved that the hyperbolic ergodic measure is the weak limit of a sequence of atomic invariant measures supported on periodic orbits belonging to the same homoclinic class, such property is called Katok's closing lemma. Along this direction several deeper results have been proved, such as entropy formula, dimension theory etc, all these results are called Pesin theory.

Now usually we call a hyperbolic measure together with the diffeomorphism a non-uniform hyperbolic system, Pesin theory has been proved a very important and powerful tool to understand the non-uniform hyperbolic system. But there is a restriction because the Pesin theory always needs the diffeomorphism be $C^{1+Holder}$, for C^1 diffeomorphism the arguments fail to work (see [18]).

In [1], they begin to consider C^1 Pesin theory, they proved that with a dominated assumption on the tangent space, the stable manifold theorem is still true, and if the diffeomorphism is 'tame', then there exist a lot of hyperbolic ergodic measures.

In this paper we treat Pesin theory as a theory derives topological information from hyperbolic measure, it means that we just consider the stable manifold theorem and Katok's closing lemma. With such understanding, we show that when the diffeomorphism is C^1 far from tangencies, C^1 Pesin theory is still true. It means that we can replace the regularity assumption about the diffeomorphism in the C^2 Pesin theory by a weak assumption on the diffeomorphism. The precisely statement is following:

Theorem 2: *Suppose $f \in C^1(M) \setminus \overline{HT}$ and μ is a hyperbolic ergodic measure of f, then C^1 Pesin theory is true:*

a) *there exists a compact positive measure subset Λ^s (resp. Λ^u) which has continuous and uniform size of stable (resp. unstable) manifolds, and almost every point has a local stable manifold and a local unstable manifold both of which are differentiably embeded disks.*
b) *μ is the weak limit of a sequence of invariant measures μ_n supported by periodic orbits p_n with index i, and the periodic orbits are homoclinic related with each other.*

Remark:

- *The stable manifolds we get usually is not absolutely continuous, that's because the absolutely continuous property heavily depends on distortion which just holds under C^2 assumption.*
- *In fact, from the proof, it's easy to see the above theorem can be stated in the following classical way of Pesin theory in the C^2 case:*

  Suppose f is far away from tangencies and μ is an ergodic measure of f, then there exists a family of compact set $\Lambda_0 \subset \Lambda_1 \subset \cdots$ with positive measure such that $f^{+(-)}(\Lambda_i) \subset \Lambda_{i+1}$, $\mu(\bigcup_i \Lambda_i) = 1$ and they satisfy the following properties:
  - *for every Λ_i, there exist local continuous stable and unstable manifolds on it with uniform size;*
  - *for every Λ_i, there exist $\varepsilon_i > 0$, $L_i > 0$ and $N_i \in \mathbb{N}$, such that if there exist $x \in \Lambda_i$ and $m > N_i$ satisfying $f^m(x) \in \Lambda_i$ and $d(x, f^m(x)) < \varepsilon_i$, then there exists periodic point p with period m and $d(f^j(x), f^j(p)) < L_i \cdot d(x, f^m(x))$ for $0 \le j < m$. Moreover, some point in the periodic orbit we get has uniform size of local stable and local unstable manifolds and the size just depend on the compact set Λ_i.*

Corollary 1: *Suppose f is far away from tangencies, C is a chain recurrence class of f without periodic point, then any ergodic measure μ with supp$(\mu) \subset C$ is non-hyperbolic.*

In [6], Díaz and Gorodetski started to consider the generic existence of non-hyperbolic ergodic measure and gave the following conjecture:

Conjecture 1: *There exists a generic subset R in $C^1(M)$ such that for any $f \in R$ and C chain recurrent class of f, either C is hyperbolic or there exists a non-hyperbolic ergodic measure μ with support in C.*

[6] shows that for C^1 residual diffeomorphisms, if its some homoclinic class contains periodic points with different indices, then there exists a non-hyperbolic ergodic measure with support in this homoclinic class. With their result, we prove conjecture 1 for diffeomorphisms far from tangencies:

Theorem 3: *There exists a residual subset R in $C^1(M) \setminus \overline{HT}$ such that for any $f \in R$ and any chain recurrence class C of f,*

- *either C is hyperbolic,*
- *or there exists a non-hyperbolic ergodic measure μ with support contained in C.*

The structure of this paper is following: in § 2 we give some definitions and notations, theorem 1 is proved in § 3, in § 4, we give the proof of theorem 2 and corollary 1, in § 5 we give some basic C^1 generic properties theorem 3 is proved in § 6.

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2. Definitions and Notations

Let M be a compact boundlessness Riemannian manifold with $dim(M) = d \geq 2$. Let $Per(f)$ denote the set of periodic points of f and $\Omega(f)$ the non-wondering set of f, for $p \in Per(f)$, $\pi(p)$ means the period of p. If p is a hyperbolic periodic point, the index of p is the dimension of the stable bundle. We denote $Per_i(f)$ the set of the index i periodic points of f, and we call a point x is an index i preperiodic point of f if there exists a family of diffeomorphisms $g_n \xrightarrow{C^1} f$, where g_n has an index i periodic point p_n and $p_n \longrightarrow x$. $P_i^*(f)$ is the set of index i preperiodic point of f, it's easy to know $\overline{P_i(f)} \subset P_i^*(f)$.

Let Λ be an invariant compact set of f, we say Λ is an index i fundamental limit if there exists a family of diffeomorphisms g_n C^1 converging to f, p_n is an index i periodic point of g_n and $Orb(p_n)$ converge to Λ in *Hausdorff* topology. So if $\Lambda(f)$ is an index i fundamental limit, we have $\Lambda(f) \subset P_i^*(f)$.

For two points $x, y \in M$ and some $\delta > 0$, we say there exists a δ-pseudo orbit connects x and y means that there exist points $x = x_0, x_1, \cdots, x_n = y$ such that $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, 1, \cdots, n-1$, we denote it $x \underset{\delta}{\dashv} y$. We say $x \dashv y$ if for any $\delta > 0$ we have $x \underset{\delta}{\dashv} y$ and denote $x \vdash\!\dashv y$ if $x \dashv y$ and $y \dashv x$. A point x is called a chain recurrent point if $x \vdash\!\dashv x$. $CR(f)$ denotes the set of chain recurrent points of f, it's easy to know that $\vdash\!\dashv$ is an closed equivalent relation on $CR(f)$, and every equivalent class of such relation should be compact and is called chain recurrent class. Let K be a compact invariant set of f, if x, y are two points in K, we'll denote $x \underset{K}{\dashv} y$ if for any $\delta > 0$, we have a δ -pseudo orbit in K connects x and y. If for any two points $x, y \in K$ we have $x \underset{K}{\dashv} y$, we call K a chain recurrent set. Let C be a chain recurrent class of f, we call C is an aperiodic class if C does not contain periodic point.

Let Λ be an invariant compact set of f, for $0 < \lambda < 1$ and $1 \leq i < d$, we say Λ has an index $i - (l, \lambda)$ dominated splitting if we have a continuous invariant splitting $T_\Lambda M = E \oplus F$ where $dim(E_x) = i$ for any $x \in \Lambda$ and $\| Df^l|_E(x) \| \cdot \| Df^{-l}|_F(f^l x) \| < \lambda$ for all $x \in \Lambda$. For simplicity, sometimes we just call $\Lambda(f)$ has an index i dominated splitting. A compact invariant set can have many dominated splittings, but for fixed i, the index i dominated splitting is unique.

Remark 2.1. *Suppose μ is an ergodic measure of diffeomorphism f, and $supp(\mu)$ has an index i dominated splitting $E \oplus F$, since the bundles E, F are continuous, they are measurable bundles, consider the Lyapunov exponents for μ on bundle E and F respectively, denote $\lambda_1 \leq \cdots \lambda_i \leq \lambda_{i+1} \leq \cdots \leq \lambda_d$ the exponents of μ, by the definition of dominated splitting, the vectors of F expands faster than the vectors of E, so the exponents for μ on bundle E are smaller than the exponents for μ on bundle F, it implies $\lambda_1, \cdots, \lambda_i$ are the exponents for μ on bundle E and $\lambda_{i+1}, \cdots, \lambda_d$ are the exponents for μ on bundle F.*

We say an ergodic invariant measure μ of diffeomorphism f has type (i, k) if $\#\{$negative Lyapunov exponents of $\mu\} = i$ and $\#\{$vanishing Lyapunov exponents of $\mu\} = k$. In particular, if $k = 0$, we say μ has index i.

We say a diffeomorphism f has C^r tangency if $f \in C^r(M)$, f has hyperbolic periodic point p and there exists a non-transverse intersection between $W^s(p)$ and $W^u(p)$. HT^r is the set of the diffeomorphisms which have C^r tangency, usually we just use HT denote HT^1. We call a diffeomorphism f is far away from tangency if $f \in C^1(M) \setminus \overline{HT}$. The following proposition shows the relation between dominated splitting and far away from tangencies.

Proposition 2.2. *([19]) f is C^1 far away from tangencies if and only if there exists (l, λ) such that $P_i^*(f)$ has index $i - (l, \lambda)$ dominated splitting for $0 < i < d$.*

Usually dominated splitting is not a hyperbolic splitting, Mañé showed that in some special case, one bundle of the dominated splitting is hyperbolic.

Proposition 2.3. *([13]) Suppose $\Lambda(f)$ has an index i dominated splitting $E \oplus F$ $(i \neq 0)$, if $\Lambda(f) \bigcap P_j^*(f) = \phi$ for $0 \leq j < i$, then E is a contracting bundle.*

3. Proof of theorem 1

At first we need the following special statement of ergodic closing lemma which is a little stronger than the original statement given in [13] and whose proof will be given in § 3.1.

Lemma 3.1. *(New statement of Ergodic closing lemma) Suppose μ is a type (i, k) ergodic measure of f, then for any $i \leq j \leq i + k$, $supp(\mu) \subset P_j^*$ and there exists a family of diffeomorphisms g_n, such that:*

1) *: $g_n \xrightarrow{C^1} f$,*
2) *: g_n has periodic point p_n with index j, let μ_n denote the invariant atom measure on $Orb_{g_n}(p_n)$, we have $\mu_n \xrightarrow{*-weak} \mu$.*

Proof of theorem 1: We divide the proof into two cases:

a) μ is hyperbolic with index i;
b) μ has type (i, k) where $k \neq 0$.

In the case a), by lemma 3.1 and proposition 2.2, $supp(\mu) \subset P_i^*$ and $supp(\mu)$ has index i dominated splitting $E_i^{cs} \oplus E_{i+1}^{cu}$. By the definition of dominated splitting and remark 2.1, the Lyapunov exponents for μ on bundle E_i^{cs} are strictly smaller than the Lyapunov exponents for μ on bundle E_{i+1}^{cu}, since the number of negative exponents of μ is i, the Lyapunov exponents for μ on bundle E_i^{cs} are negative and the dominated splitting $E_i^{cs} \oplus E_{i+1}^{cu}$ is the Oseledts splitting corresponding to the positive and negative exponents.

In the case b), at first we'll show $k = 1$.

If $k > 1$, by lemma 3.1, $supp(\mu) \subset P_i^* \bigcap P_{i+1}^* \bigcap \cdots \bigcap P_{i+k}^*$, then by proposition 2.2 and $f \in (\overline{HT})^c$, $supp(\mu)$ has index i dominated splitting $E_i^{cs} \oplus E_{i+1}^{cu}$; index $i+1$ dominated splitting $E_{i+1}^{cs} \oplus E_{i+2}^{cu}$; $\cdots$, and index $i+k$ dominated splitting $E_{i+k}^{cs} \oplus E_{i+k+1}^{cu}$. Denote

$$E_{i+1,1}^c = E_{i+1}^{cs} \bigcap E_i^{cu},\ E_{i+2}^c = E_{i+2}^{cs} \bigcap E_{i+1}^{cu},\ \cdots,\ E_{i+k}^c = E_{i+k}^{cs} \bigcap E_{i+k-1}^{cu},$$

then $supp(\mu)$ has a new dominated splitting

$$T_{supp(\mu)}M = E_i^{cs} \oplus E_{i+1,1}^c \oplus E_{i+2,1}^c \oplus \cdots \oplus E_{i+k,1}^c \oplus E_{i+k+1}^{cu}.$$

We denote $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_d$ the Lyapunov exponents of μ, since μ has type (i,k), $\lambda_{i+1} = \cdots = \lambda_{i+k} = 0$, and we have $\int \ln \|Df(v_x)\|/\|v_x\| d\mu(x) = \lambda_j = 0$ where $v_x \in E_{j,1}^c(x) \setminus \{0\}$ $_{i<j\le i+k}$. In fact for any $n > 0$, $\int \ln \|Df^n(v_x)\|/\|v_x\| d\mu(x) = \sum_{t=0}^{n-1} \int \ln \|Df^{t+1}(v_x)\|/\|Df^t(v_x)\| d\mu(x) = n\lambda_j = 0$ where $v_x \in E_{j,1}^c(x) \setminus \{0\}$ $_{i<j\le i+k}$. Since $E_{i+1,1}^c \oplus E_{i+2,1}^c$ is a dominated splitting, there exists $l \in \mathbb{N}$ such that $\|Df^l(v_x)\|/\|v_x\| < \|D^l f(w_x)\|/\|w_x\|$ for any $x \in supp(\mu)$ and $v_x \in E_{i+1,1}^c(x), w_x \in E_{i+2,1}^c(x)$, so we have $0 = \lambda_{i+1} = \int \ln \|Df^l(v_x)\|/\|v_x\| d\mu(x) < \int \ln \|Df^l(w_x)\|/\|w_x\| d\mu(x) = \lambda_{i+2} = 0$, that's a contradiction.

So μ has type $(i,1)$, by lemma 3.1, $supp(\mu) \subset P_i^* \bigcap P_{i+1}^*$, using proposition 2.2 and above argument, $supp(\mu)$ has the following dominated splitting $T_{supp(\mu)}M = E_i^{cs} \oplus E_{i+1,1}^c \oplus E_{i+2}^{cu}$, using remark 2.1, with the same argument in case (a), the Lyapunov exponents for μ on bundle E_i^{cs} are smaller than the Lyapunov exponent for μ on bundle $E_{i+1,1}^c$, and the Lyapunov exponent for μ on bundle $E_{i+1,1}^c$ are smaller than the Lyapunov exponents for μ on bundle E_{i+2}^{cu}, since μ has type $(i,1)$, we know that the dominated splitting $E_i^{cs} \oplus E_{i+1,1}^c \oplus E_{i+2}^{cu}$ is also the Oseledets splitting corresponding to the negative, zero, positive exponents. □

3.1. A new version of Mañé's ergodic closing lemma.

Proof : Suppose the theorem is wrong, then the measure is not trivial and there exists j with $i \le j \le i+k$ which does not satisfy the theorem.

In the following we'll get the contradiction by showing that $\#\{$negative Lyapunov exponents of $\mu\} > j \ge i$ or $\#\{$positive Lyapunov exponents of $\mu\} > d-j \ge d-(i+k)$, because we know that μ has exactly i number of negative exponents and $d-(i+k)$ number of positive exponents. In order to prove this, we need show that there is a positive measure subset such that for every point in this subset, on its tangent space, the tangent map Df exponentially contracting a subspace with dimension larger than j or the tangent map exponentially expanding a subspace with dimension larger than $d-j$.

Lemma 3.2. *(Ergodic closing lemma) Suppose μ is an ergodic measure of f, then there exists a family of diffeomorphisms g_n, such that:*

1) *: $g_n \xrightarrow{C^1} f$,*
2) *: g_n has periodic point p_n, let μ_n denote the invariant atom measure on $Orb_{g_n}(p_n)$, we have $\mu_n \overset{*-weak}{\longrightarrow} \mu$.*

From Mañé's ergodic closing lemma, there always exists a family of diffeomorphisms $g_n \xrightarrow{C^1} f$ where g_n has an invariant measure μ_n supported on periodic orbit $p_n(g_n)$ and $\mu_n \overset{*-weak}{\longrightarrow} \mu$, suppose the periodic points' indices are all the same and strictly bigger than j.

Denote $j_0 = \min_{t\ge j}\{t$: exists a family of diffeomorphisms $g_n \xrightarrow{C^1} f$ where g_n has an invariant measure μ_n supported on index t periodic orbit $p_n(g_n)$ and $\mu_n \overset{*-weak}{\longrightarrow} \mu$ $\}$, then $j_0 > j$. Choose such a family of diffeomorphisms $\{g_n\}$ which has periodic point $\{p_n(g_n)\}$ with index j_0 and $Orb_{g_n}(p_n)$ supports an invariant measure μ_n for g_n satisfying $\mu_n \overset{*-weak}{\longrightarrow} \mu$, since μ is not trivial, $\lim_{n\to\infty} \pi_{g_n}(p_n(g_n)) \longrightarrow \infty$.

Denote $E^s_{j_0,n}(Orb(p_n))$ the contracting subspace on $Orb(p_n)$ with dimension j_0, then we get a family of periodic linear maps $\{Dg_n|_{E^s_{j_0,n}(Orb(p_n))}\}$.

Definition 3.3. *The above sequence of periodic linear maps $\{Dg_n|_{E^s_{j_0,n}(Orb(p_n))}\}$ over $\mathbb{R}^{j_0}$ is called uniformly periodic contracting if there exists $\varepsilon > 0$ such that for any n large enough and any periodic linear map $\{A_1, \cdots, A_{\pi_{pn}(g_n(p_n))}\}$ over $\mathbb{R}^{j_0}$ satisfying $\|A_j - Dg_n|_{E^s_{j_0,n}(g_n^{j-1}(p_n))}\| < \varepsilon$, we have all the eigenvalues of $\prod_{j=1}^{\pi_{pn}} A_j < 1$.*

Now we'll show that the above sequence of periodic linear maps $\{Dg_n|_{E^s_{j_0,n}(Orb(p_n))}\}$ we've got is uniformly periodic contracting. At first, we need the well known Franks lemma:

Lemma 3.4. *$g_n \xrightarrow{C^1} f$, suppose p_n is a periodic point of g_n, $A|_{Orb(p_n)}$ is an ε-perturbation of $\{Dg_n|_{Orb(p_n)}\}$, then for any neighborhood U of $Orb(p_n)$, there exists g_n' such that $g_n' \equiv g_n$ on $(M \setminus U)\bigcup Orb(p_n)$, $d_{C^1}(g_n, g_n') < \varepsilon$ and $\{Dg_n'|_{orb(p_n)}\} = \{A|_{Orb(p_n)}\}$.*

As a corollary of Franks lemma, we can show that the family of periodic linear maps is uniformly periodic contracting:

Corollary 3.5. *There exists $\varepsilon > 0$ such that for any periodic linear map $\{A_1, \cdots, A_{\pi_{gn}(p_n)}\}$ over $\mathbb{R}^{j_0}$ satisfying $\|A_j - Dg_n|_{E^s_{j_0,n}(g_n^{j-1}(p_n))}\| < \varepsilon$, we have all the eigenvalues of $\prod_{j=1}^{\pi_{pn}} A_j < 1$.*

Proof : If the sequence is not uniformly periodic contracting, there exists $\varepsilon_{n_j} \longrightarrow 0$ and a sequence of periodic linear maps $\{(A_{n_j,1}, \cdots, A_{n_j,\pi_{gn_j}(g_{n_j}(p_{n_j}))})\}_j$ over $\mathbb{R}^{j_0}$ such that $\|A_{n_j,k} - Dg_{n_j}|_{E^s_{j_0,n_j}(g_{n_j}^{k-1}(p_{n_j}))}\| < \varepsilon_{n_j}$ and one eigenvalue of $\prod_{k=1}^{\pi_{gn_j}(p_{n_j})} A_{n_j,k} > 1$.

Now we claim that replace by another sequence of periodic linear maps over $\mathbb{R}^{j_0}$, we can always suppose $\prod_{k=1}^{\pi_{gn_j}(p_{n_j})} A_{n_j,k}$ has index $j_0 - 1$.

Proof of the claim: We can choose a new sequence periodic linear map $\{(B_{n_j,1}, \cdots, B_{n_j,\pi_{gn_j}(g_{n_j}(p_{n_j}))})\}_j$ over $\mathbb{R}^{j_0}$ such that $\|B_{n_j,k} - Dg_{n_j}|_{E^s_{j_0,n_j}(g_{n_j}^{k-1}(p_{n_j}))}\| < \varepsilon_{n_j}$, all the eigenvalues of $\prod_{k=1}^{\pi_{gn_j}(p_{n_j})} B_{n_j,k} \leq 1$ except one real or a couple of complex eigenvalues with norm 1.

If $\prod_{k=1}^{\pi_{gn_j}(p_{n_j})} B_{n_j,k}$ has only one real eigenvalue with norm 1, after small perturbation, we get a new periodic linear map $\{(\widetilde{A}_{n_j,1}, \cdots, \widetilde{A}_{n_j,\pi_{gn_j}(p_{n_j})})\}_j$ over $\mathbb{R}^d$ such that $\prod_{k=1}^{\pi_{gn_j}(p_{n_j})} \widetilde{A}_{n_j,k}$ has index $j_0 - 1$.

If $\prod_{k=1}^{\pi_{gn_j}(p_{n_j})} B_{n_j,k}$ has a couple of complex eigenvalues with norm 1, lemma 3.7 of [4] shows after small perturbation, we can let the two complex eigenvalues to be real with norm 1, then by another perturbation, we get a new periodic linear map $\{(\widetilde{A}_{n_j,1}, \cdots, \widetilde{A}_{n_j,\pi_{gn_j}(p_{n_j})})\}_j$ over $\mathbb{R}^d$ such that $\prod_{k=1}^{\pi_{gn_j}(p_{n_j})} \widetilde{A}_{n_j,k}$ has index $j_0 - 1$.

By above arguments, we can always get a new sequence of periodic linear maps $\{(\widetilde{A}_{n_j,1},\cdots,\widetilde{A}_{n_j,\pi_{g_{n_j}}(p_{n_j})})\}_j$ over $\mathbb{R}^{j_0}$ which satisfying $\|\widetilde{A}_{n_j,k}-Dg_{n_j}(g_{n_j}^{k-1}(p_{n_j}))\|<2\varepsilon_{n_j}$ and $\prod_{k=1}^{\pi_{g_{n_j}}(p_{n_j})}\widetilde{A}_{n_j,k}$ has index j_0-1.

Replace the sequence of periodic linear maps $\{(A_{n_j,1},\cdots,A_{n_j,\pi_{g_{n_j}}(g_{n_j}(p_{n_j}))})\}_j$ over $\mathbb{R}^{j_0}$ by the sequence of periodic linear maps $\{(\widetilde{A}_{n_j,1},\cdots,\widetilde{A}_{n_j,\pi_{g_{n_j}}(p_{n_j})})\}_j$ over $\mathbb{R}^{j_0}$ and finished the proof of the claim. □

Now use Franks lemma, by ε_{n_j} perturbation, we can get a new diffeomorphism g'_{n_j} such that $Orb_{g_{n_j}}(p_{n_j}(g_{n_j}))$ is an index j_0-1 periodic orbit of g'_{n_j}, that's a contradiction with the definition of j_0. □

For such kind of uniformly contracting periodic linear maps, [13] gave the following lemma:

Lemma 3.6. *([13] Lemma II.4): $g_n\xrightarrow{C_1}f$, suppose p_n is index j_0 periodic point of g_n and $\lim_{n\to\infty}\pi_{g_n}(p_n)\longrightarrow\infty$. If the sequence of periodic linear maps $\{Dg_n|_{E^s_{j_0,n}(Orb(p_n))}\}$ is uniformly periodic contracting, then there exist $l>0$, $N_0>0$ and $\lambda<1$ such that for any periodic orbit p_n with period $\pi(p_n)>N_0$, we have*

$$\prod_{i=0}^{[\frac{\pi(pn)}{l}]}\|Dg^l|_{E^s_{j_0,n}(g_n^{il}(p_n))}\|<\lambda^{[\frac{\pi(pn)}{l}]}. \tag{3.1}$$

Remark 3.7. *Under the same assumption with lemma 3.6, and $l>0$, $N_0>0$, $\lambda<1$ given there, for any periodic orbit p_n with period $\pi(p_n)>N_0$ and any $k>0$, we have*

$$\prod_{i=0}^{k[\frac{\pi(pn)}{l}]}\|Dg^l|_{E^s_{j_0,n}(g_n^{il}(p_n))}\|<\lambda^{k[\frac{\pi(pn)}{l}]}. \tag{3.2}$$

That's because we can consider the new sequence of periodic linear maps

$$\{(Dg_n|_{E^s_{j_0,n}(Orb(p_n))});\ (Dg_n|_{E^s_{j_0,n}(Orb(g_n(p_n)))});\ \cdots;\ (Dg_n|_{E^s_{j_0,n}(Orb(g_n^{\pi(pn)-1}(p_n)))})\}.$$

Then (3.1) *is true for $g_n^{k\cdot l\cdot[\frac{\pi(pn)}{l}]}(p_n)$ where $k>0$.*

Now we need the following well known Pliss lemma:

Lemma 3.8. *(Pliss lemma) For $K>0$ and $\lambda<\lambda_1<0$, there exists $\delta>0$ such that for any sequence $\{a_n\}$ satisfying $\|a_n\|<K$ and $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}a_j<\lambda$, there exist $\{N_t\}$ and a subsequence $\{a_{n_i}\}$ such that $\frac{1}{m}\sum_{j=1}^{m}a_{n_i+j}<\lambda_1$ for any $m\in\mathbb{N}$ and $\liminf_{t\to\infty}\frac{\#\{a_{n_i};1<n_i\leq N_t\}}{N_t}>\delta$.*

For the uniformly contracting periodic linear maps $\{Dg_n|_{E^s_{j_0,n}(Orb(p_n))}\}$, remark 3.6 gives parameters $l>0$, $N_0>0$ and $\lambda<1$, choose $\lambda<\lambda_0<1$, by (3.2) and lemma 3.8 (Pliss lemma), with the fact $\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\delta_{g_n^j(p_n)}\longrightarrow\mu_n$ where μ_n is the ergodic measure on $Orb_{g_n}(p_n)$, if denote $\Lambda_n=\{y\in Orb(p_n):\prod_{i=0}^{m}\|Dg_n^l|_{E^s_{n,j_0}(g_n^{il}(y))}\|<\lambda_0^m\}$, then for $\pi(p_n)$ big enough, there exists a uniformly number $\delta>0$, such that $\mu_n(\Lambda_n)>\delta$.

Proposition 3.9. *Suppose X is a compact metric space, denote $CX=\{K:K$ is compact subset of $X\}$, the space CX with Hausdorff topology is still a compact space.*

Since Λ_n is compact, with proposition 3.9, there is a compact set Λ such that $\lim_{n\to\infty} \Lambda_n \longrightarrow \Lambda$. It's easy to know that $\mu(\Lambda) > \delta$ and for every point $y \in \Lambda$, there exists a j_0 dimension space $E_{j_0}(y)$ in it's tangent space such that $\prod_{i=0}^{m} \|Dg_n^l|_{E_{j_0}(f^{il}(y))}\| < \lambda_0^m$ for $m \geq 0$, so every point in Λ has at least j_0 number of negative Lyapunov exponents. Since the measure μ is ergodic and Λ has positive measure, μ has at least j_0 number of negative Lyapunov exponents. That's a contradiction, since μ has just i number of negative Lyapunov exponents and $i \leq j < j_0$. □

4. Proof of theorem 2 and corollary 1

Before we give the proof, we need the following lemma which claims that with a dominated assumption, the C^1 Pesin theory stated in theorem 2 is true. Such idea was given in [1] and the name of C^1 Pesin theory was given there at first. Here we cite one of their result (the stable manifold theorem) and add another new property (similar with Katok's closing lemma in C^2 case), we put them together and call C^1 Pesin theory.

Lemma 4.1. *$f \in C^1(M)$, suppose μ is a hyperbolic ergodic measure of f with index i and there exists an i-dominated splitting on supp(μ), then C^1 Pesin theory is true:*

a) *there exists a compact positive measure subset Λ^s (resp. Λ^u) which has continuous and uniform size of stable (resp. unstable) manifolds,*
b) *μ is the weak limit of a sequence of invariant measures μ_n supported by periodic orbits p_n with index i, and supp(μ) is contained in every homoclinic class $H(p_n, f)$.*

a) is given in [1] at first, we state it here just in order to make the statement more complete. b) generalizes [12]'s result to C^1, in the proof of b) we use a shadowing lemma given in [9] which is similar with the shadowing lemma for C^2 Pesin theory.

Proof of theorem 2: By lemma 3.1, we know that *supp*(μ) $\subset P_i^*(f)$, by proposition 2.2, *supp*(μ) has index i dominated splitting, recall that μ has index i, theorem 1 is a simple corollary of lemma 4.1. □

Proof of corollary 1: If there is a hyperbolic ergodic measure μ with *supp*(μ) $\subset C$, by the fact $f \in (\overline{HT})^c$ and b) of theorem 1, there is a periodic point in C, that's a contradiction. □

The proofs of lemma 4.1 is given in § 3.1.

4.1. C^1 Pesin theory.

In this subsection we'll give the proof of lemma 4.1. a) of theorem 1 was given in [1], but for completeness, we still give a proof here.

Proof of a): We just prove the stable manifold theorem, the proof for unstable manifold theorem is the same. Denote $E^{cs} \oplus E^{cu}$ the index i dominated splitting on *supp*(μ), it's easy to know that they are a Oseledets splitting, and the Lyapunov exponents on bundle E^{cs} are smaller than the Lyapunov exponents on bundle E^{cu}. Since the dimension of E^{cs} is i and μ is a hyperbolic ergodic measure with index i, we know that the Lyapunov exponents on bundle E^{cs} (resp. E^{cu}) are negative (resp. positive). So from the sub-ergodic theorem, there exists $\lambda > 0$ such that:

$$\lim_{n\to\infty}\int \frac{1}{n}\ln\|Df^n|_{E^{cs}(x)}\|d\mu(x) < -\lambda < 0, \tag{4.1}$$

choose N big enough such that $\int \ln\|Df^N|_{E^{cs}(x)}\|d\mu(x) < -\lambda < 0$, from Birkhopf ergodic theorem, there exists a μ full measure subset A^s such that for any $x \in A^s$ we have:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\ln\|Df^N|_{E^{cs}(f^{jN}(x))}\| < -\lambda. \tag{4.2}$$

Choose $0 < \lambda_0 < \lambda$, denote Λ^s the set such that for any $y \in \Lambda^s$ we have $\frac{1}{n}\sum_{j=0}^{n-1}\ln\|Df^N|_{E^{cs}(f^{jN}(y))}\| < -\lambda_0$ for any $n > 0$. It's easy to know that Λ^s is a closed set, lemma 3.8 (Pliss lemma) gives $\delta > 0$, we'll show that $\mu(\Lambda^s) > \delta$. The proof of the following result is very easy and we just omit here.

Lemma 4.2. *There exists a μ full measure subset A_0^s such that for any $x \in A_0^s$, we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)} \overset{weaktoplogy}{\longrightarrow} \mu.$$

We can suppose $A_0^s \subset A^s$ always, by Λ^s is compact and lemma 4.2, for $x \in A_0^s$, there exists $\{N_t\}$ such that $\mu(\Lambda^s) > \lim_{t\to\infty}\frac{1}{N_t}\sum_{j=0}^{N_t-1}\delta_{f^j(x)}(\Lambda^s) > \delta > 0$.

Now we'll show that the positive measure set Λ^s has continuous and uniform size of stable manifold.

Let $I_1^{s,(u)} = (-1,1)^{i,(n-i)}$ and $I_\varepsilon^{s,(u)} = (-\varepsilon,\ \varepsilon)^{i,(n-i)}$, denote by $Emb^1(I^{s(u)}, M)$ the set of C^1-embedding of $I_1^{s(u)}$ on M, recall by [11] that $\widetilde{\Lambda}$ has index i dominated splitting $\widetilde{E}\oplus\widetilde{F}$ implies the following.

Lemma 4.3. *There exist two continuous function $\Phi^{cs} : \widetilde{\Lambda} \longrightarrow Emb^1(I^s, M)$ and $\Phi^{cu} : \widetilde{\Lambda} \longrightarrow Emb^1(I^u, M)$ such that, with $W_\varepsilon^{cs}(x) = \Phi^{cs}(x)I_\varepsilon^s$ and $W_\varepsilon^{cu}(x) = \Phi^{cu}(x)I_\varepsilon^u$, the following properties hold:*

a) *$T_xW_\varepsilon^{cs} = \widetilde{E}(x)$ and $T_xW_\varepsilon^{cu} = \widetilde{F}(x)$,*
b) *For all $0 < \varepsilon_1 < 1$, there exists ε_2 such that $f(W_{\varepsilon_2}^{cs}(x)) \subset W_{\varepsilon_1}^{cs}(f(x))$ and $f^{-1}(W_{\varepsilon_2}^{cu}(x)) \subset W_{\varepsilon_1}^{cu}(f^{-1}(x))$.*
c) *For all $0 < \varepsilon < 1$, there exists $\delta > 0$ such that if $y_1,\ y_2 \in \widetilde{\Lambda}$ and $d(y_1,\ y_2) < \delta$, then $W_\varepsilon^{cs}(y_1) \pitchfork W_\varepsilon^{cu}(y_2) \neq \phi$.*

The following lemma given by [17] shows that there really exists an uniform continuous stable manifold on Λ^s.

Lemma 4.4. *([17]) For any $0 < \lambda < 1$, there exists $\varepsilon > 0$ such that for $x \in \widetilde{\Lambda}$ which satisfies $\prod_{j=0}^{n-1}\|Df^{N_1}|_{\widetilde{E}(f^{jN_1}x)}\| \leq \lambda^n$ for all $n > 0$, then $diam(f^n(W_\varepsilon^{cs})) \longrightarrow 0$, i.e. the central stable manifold of x with size ε is in fact a stable manifold.*

Proof of b): Here we should use a special shadowing lemma given by [9]:

Lemma 4.5. *([9], theorem 1.1): Let $f \in C^1(M)$, assume that Λ is a closed invariant set of f and there is a continuous invariant splitting $T_\Lambda M = E \oplus F$ on Λ, i.e. $Df(E_x) = E_{f(x)}$ and $Df(F_x) = F_{f(x)}$ for $x \in \Lambda$. For any $\lambda_1 < \lambda_2 < 1$ there exist $L > 0$, $\delta_0 > 0$, N_1 such that for any $\delta < \delta_0$ if we have an orbit segment $(x, f(x), \cdots, f^{nN_1}(x))$ satisfies the following properties:*

$$\begin{aligned}\prod_{i=0}^{s-1} \|Df^{N_1}|_{E(f^{jN_1}(x))}\| &\leq (\lambda_1)^s \text{ for } 0 \leq s \leq n-1,\\ \prod_{i=0}^{s-1} \|Df^{-N_1}|_{E(f^{(n-j)N_1}(x))}\| &\leq (\lambda_1)^s \text{ for } 0 \leq s \leq n-1,\\ d(x, f^{nN_1}(x)) &< \delta,\end{aligned}$$

then there exists a periodic point p with period nN_1 and $L\delta$-shadows $(x, f(x), \cdots, f^{nN_1}(x))$.

Now from the proof of a), there also exists a positive measure subset Λ^u such that for any $x \in \Lambda^u$, $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{j=0}^{n-1} \ln \|Df^{-N}|_{E^{cu}(f^{-jN}(x))}\| < -\lambda$. Since μ is ergodic, there exists n_0 such that $\Lambda^{su} = f^{n_0}(\Lambda^s) \bigcap \Lambda^u$ has positive measure, now from the proof of a), for any $x \in \Lambda^{su}$,

$$\frac{1}{n}\sum_{j=0}^{n-1} \ln \|Df^{-N}|_{E^{cu}(f^{-jN}(x))}\| < -\lambda_0; \quad \frac{1}{n}\sum_{j=0}^{n-1} \ln \|Df^{N}|_{E^{cs}(f^{jN-n_0}(x))}\| < -\lambda_0.$$

Choose n_1 big enough and $1 > \lambda_1 > \lambda_0$, for $N_1 = n_1 \cdot N$ and any $x \in \Lambda^{su}$, we have

$$\frac{1}{n}\sum_{j=0}^{n-1} \ln \|Df^{N_1}|_{E^{cs}(f^{jN_1}(x))}\| < -\lambda_1; \quad \frac{1}{n}\sum_{j=0}^{n-1} \ln \|Df^{-N_1}|_{E^{cu}(f^{-jN_1}(x))}\| < -\lambda_1. \tag{4.3}$$

Now we need the following result:

Lemma 4.6. *There exists a subset $\Lambda_0 \subset \Lambda^{su}$, such that $\mu(\Lambda_0) = \mu(\Lambda^{su})$ and for any $x \in \Lambda_0$*

(A) *x is a recurrent point, i.e. there exists $0 < i_1 < i_2 < \cdots i_n < \cdots$ such that $f^{i_nN_1}(x) \in \Lambda^{su}$ and $\lim\limits_{n\to\infty} d(x, f^{i_nN_1}(x)) \longrightarrow 0$.*

(B) $\lim\limits_{n\to\infty} \frac{1}{i_nN_1}\sum\limits_{i=0}^{i_nN_1-1} \delta_{f^i(x)} \longrightarrow \mu$.

Remark 4.7. *Above lemma can be proved by Poincaré recurrence theorem and Birkhoff ergodic theorem, and in fact, we can show that for any $x \in \Lambda_0$, $supp(\mu) \subset Orb^+(x)$.*

Fix a point $x \in \Lambda_0$, by (A) of lemma 4.6, we can choose i_n such that $d(x, f^{i_nN_1}(x)) < d_1$, by lemma 4.5 and (4.3), there a periodic point p_n with period i_nN_1 which $L \cdot d(x, f^{i_nN_1}(x))$-shadows $(x, f(x), \cdots, f^{i_nN_1}(x))$, by (B) of lemma 4.6, $\lim\limits_{n\to\infty} \frac{1}{i_nN_1}\sum\limits_{i=0}^{i_nN_1-1} \delta_{f^i(p_n)} \longrightarrow \mu$.

Now we claim that the above family of periodic points $\{p_n\}$ will have uniform size of stable and unstable manifold.

Proof of the claim: For n big enough, $Orb(p_n)$ is in a small neighborhood of $supp(\mu)$, denote $\widetilde{\Lambda} = supp(\mu) \bigcup(\bigcup\limits_n Orb(p_n))$, then $\widetilde{\Lambda}$ has index i dominated splitting also.

Choose $1 > \lambda_2 > \lambda_1$, then there exists a δ_0 such that for any two points $y_1, y_2 \in \widetilde{\Lambda}$ and $d(y_1, y_2) < \delta_0$, we have $\ln \|Df^{N_1}|_{E^{cs}(y_1)}\| - \ln \|Df^{N_1}|_{E^{cs}(y_2)}\| < \lambda_2 - \lambda_1$. We know that p_n has period $i_n N_1$ and $L \cdot d(x, f^{i_n N_1}(x))$-shadows $(x, f(x), \cdots, f^{i_n N_1}(x))$, with $\lim_{n\to\infty} d(x, f^{i_n N_1}(x)) \longrightarrow 0$ and (4.3), for n big enough, we have $\prod_{j=0}^{m-1} \|Df^{N_1}|_{\widetilde{E}(f^{jN_1}p_n)}\| \le \lambda_2^m$ and $\prod_{j=0}^{m-1} \|Df^{-N_1}|_{\widetilde{E}(f^{-jN_1}p_n)}\| \le \lambda_2^m$ for $0 \le m \le i_n N_1$, since p_n is periodic point with period $i_n N_1$, we know that

$$\prod_{j=0}^{m-1} \|Df^{N_1}|_{\widetilde{E}(f^{jN_1}p_n)}\| \le \lambda_2^m; \prod_{j=0}^{m-1} \|Df^{-N_1}|_{\widetilde{E}(f^{-jN_1}p_n)}\| \le \lambda_2^m \text{for } \ m \ge 0 \tag{4.4}$$

Now by lemma 4.3 and 4.4, p_n has uniform size of stable manifold and unstable manifold. □

Since $p_n \longrightarrow x$, and p_n has uniform size of stable and unstable manifold, by (3) of lemma 4.3, when n, m big enough, $W^s_{loc}(p_n) \pitchfork W^u_{loc}(p_m) \ne \phi$ and $W^s_{loc}(p_n) \pitchfork W^u_{loc}(p_m) \ne \phi$, so p_n and p_m are homoclinic related. Replace by a subsequence, we can suppose $\{p_n\}$ are all homoclinic related with each other, so $\{p_n\}$ and x all belong to the same homoclinic class, by remark 4.7, $supp(\mu) \subset Orb^+(x)$, so we get that $supp(\mu)$ and $\{p_n\}$ all belong to the same homoclinic class. □

5. C^1 Generic Properties

Here at first we'll state some well known C^1 generic properties.

Lemma 5.1. *There exists a C^1 residual subset R such that for any $f \in R$, the following properties are right:*

1) *all the periodic points are hyperbolic and the intersection between stable manifold and unstable manifold of periodic points are always transverse,*
2) *([5]) suppose C is a chain recurrent class of f, if C contains a periodic point p, then $C = H(p, f)$,*
3) *([5]) suppose Λ is an index i fundamental limit of f, then there exists a family of index i periodic points $\{p_n\}$ such that $\lim_{n\to\infty} Orb(p_n) \overset{Hausdorff}{\longrightarrow} \Lambda$.*
4) *([6]) if C is a homoclinic class contains periodic points with different indexes, then there exists a non-trivial non-hyperbolic ergodic measure with support in C.*

The following result is given by Shaobo Gan, a proof can be found in [21].

Lemma 5.2. *$f \in C^1(M)$ and $\{p_n\}$ is a family of index i periodic points of f satisfying $\lim_{n\to\infty} \pi(p_n) \longrightarrow \infty$, if $\{p_n\}$ is index stable, then there exists a subsequence $\{p_{i_n}\}$ such that p_{i_m} and p_{i_n} are homoclinic related for $n \ne m$, so especially, if we have $\lim_{n\to\infty} Orb(p_n) \longrightarrow \Lambda$, then Λ is contained in the homoclinic class of an index i periodic point.*

Corollary 5.3. *Suppose $f \in R$, C is a chain recurrent class of f and $\Lambda \subset C$ is an index i fundamental limit, if C doesn't contain index i periodic point, then Λ is index $i-1$ or $i+1$ fundamental limit.*

Proof : Suppose C doesn't contain index i periodic point, then especially Λ is not an orbit of index i periodic orbit. By above argument and 3) of 5.1, there exists a family of index i periodic point $\{p_n\}$ satisfying $\lim_{n\to\infty} Orb(p_n) = \Lambda$ and $\lim_{n\to\infty} \pi(p_n) = \infty$.

By lemma 5.2, the family of periodic points are not index stable, with [4]'s argument and Franks lemma, there exists a subsequence of periodic orbits $\{Orb(p_{n_j})\}$ and a family of diffeomorphisms $g_{n_j} \xrightarrow{C^1} f$ such that $Orb(p_{n_j})$ is index $i+1$ or $i-1$ periodic points of g_{n_j}. So Λ is also an index $i+1$ or $i-1$ fundamental limit. $\square$

6. Proof of theorem 3:

At first we need the following lemma whose proof is given in § 6.1:

Lemma 6.1. *There exists a generic subset R in $C^1(M) \setminus \overline{HT}$ such that for any $f \in R$ and $C(f)$ is a homoclinic class whose periodic points are all hyperbolic and have an unique index i, then*

- *either C is a hyperbolic set,*
- *or there exists a non-hyperbolic ergodic measure μ with $supp(\mu) \subset C$.*

Proof of theorem 3: Suppose $f \in R$ and C is a chain recurrent class of f, we can always suppose C is not trivial ($\#(C) = \infty$) since if $\#(C)$ is finite, C is a periodic orbit, by 1) of lemma 5.1, C is a hyperbolic periodic orbit, so there is only one invariant measure with support on C and the measure is hyperbolic.

We divide the proof into three cases:

1) C is an aperiodic class;
2) C contains periodic points and all the periodic points in C have the same index;
3) C contains index different periodic point.

In the case 1), Corollary 1 shows any ergodic measure μ with support on C is not hyperbolic and has just 1 zero Lyapunov exponent.

In the case 2), lemma 6.1 shows that either C is hyperbolic or there exists a non-hyperbolic ergodic measure μ with $supp(\mu) \subset C$.

In the case 3), we need the generic property 4) of lemma 5.1 which was proved in [6] shows that there always exists a non-hyperbolic ergodic measure μ with $supp(\mu) \subset C$. $\square$

6.1. Proof of lemma 6.1.

Proof : Here we suppose that C is not hyperbolic and all the ergodic measures with support on C are hyperbolic, we'll show the contradiction.

Suppose C contains index i $(i \neq 0, d)$ periodic point p, then $C \subset H(p,f) \subset P_i^*$, by proposition 2.2, C has an index i dominated splitting $E_i^{cs} \oplus E_{i+1}^{cu}$. Since C is not hyperbolic, the splitting is not hyperbolic splitting, we can suppose the bundle E_i^{cs} is not hyperbolic, by proposition 2.3, there exists $j < i$ such that $C \bigcap P_j^* \neq \phi$, it means there exist $g_n \xrightarrow{C^1} f$ and p_n index j periodic points of g_n such that $p_n \xrightarrow{C^1} x \in C$, from the definition of chain recurrent class, it's easy to know that $\limsup_{n\to\infty} Orb(p_n) \subset C$ and the set is an index j fundamental limit, denote $i_0 = \min\{j : C \text{ contains index } j \text{ fundamental limit}\}$, then we have $C \bigcap P_j^* = \phi$ for $j < i_0$.

Choose $\Lambda_0 \subset C$ an index i_0 fundamental limit, by proposition 2.2, Λ_0 has an index i_0 dominated splitting $E_{i_0}^{cs} \oplus E_{i_0+1}^{cu}$, by proposition 2.3 and the definition of i_0, the bundle $E_{i_0}^{cs}$ is contracting, we denote $E_{i_0}^{cs}$ by $E_{i_0}^{s}$ since now. By generic properties in lemma 5.1, there exists a family of index i_0 periodic points $\{p_n\}$ such that $\lim_{n\to\infty} Orb(p_n) \longrightarrow \Lambda_0$. By lemma 5.3, $\{p_n\}$ cannot be index stable and Λ_0 is an index

i_0+1 fundamental limit also, so $\Lambda \subset P^*_{i_0+1}$. By proposition 2.2 again, Λ_0 has index i_0+1 dominated splitting $E^{cs}_{i_0+1} \oplus E^{cu}_{i_0+2}$, denote $E^c_{i_0+1,1} = E^{cs}_{i_0+1} \bigcap E^{cu}_{i_0+1}$, then Λ_0 has the following dominated splitting $E^s_{i_0} \oplus E^c_{i_0+1,1} \oplus E^{cu}_{i_0+2}$.

Since Λ_0 is an index i_0 fundamental limit, that means the bundle $E^c_{i_0+1,1}$ is not contracting, now we need the following lemma whose proof is easy and we just omit.

Lemma 6.2. *suppose Λ is a compact invariant subset of f with dominated splitting $E \oplus F$ and the bundle $E(\Lambda)$ is not contracting, then there exists a point $x \in \Lambda$ such that $\|Df^n|_{E(x)}\| \geq 1$ for $n \geq 0$.*

By the above lemma there exists $x \in \Lambda_0$ such that $\prod_{i=0}^{n-1} \|Df|_{E^c_{i+1,1}(f^i(x))}\| \geq 1$ for $n \geq 0$ (since $dim(E^c_{i+1,1}) = 1$), choose a converge subsequence from $\{\sum_{j=0}^{n-1} \delta_{f^j(x)}\}_{n=1}^{\infty}$ and suppose $\lim_{j\to\infty} \sum_{j=0}^{n-1} \delta_{f^j(x)} \longrightarrow \nu_0$, then ν_0 is an invariant measure with $supp(\nu_0) \subset \omega(x) \subset \Lambda_0$ such that $\int_{\Lambda_0} \|Df|_{E^c_{i+1,1}}\| d\nu_0 \geq 0$. By ergodic decomposition theorem on Λ_0, we can suppose there exists an ergodic measure ν with support on Λ_0 satisfying $\int_{\Lambda_0} \|Df|_{E^c_{i_0+1,1}}\| d\nu \geq 0$. Denote $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$ the Lyapunov exponents of ν, then $\lambda_{i_0+1} = \int_{\Lambda_0} \|Df|_{E^c_{i_0+1,1}}\| d\nu \geq 0$. Recall that we have supposed ν is hyperbolic, so $\lambda_{i_0+1} > 0$, that means ν has index smaller than i_0+1, by $f \in R \subset C^1(M) \setminus \overline{HT}$ and theorem 1, C contains periodic points with index smaller than i_0+1. Recall that $i_0 < i$, C contains index different periodic points, that's a contradiction. $\square$

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NEWHOUSE PHENOMENON AND HOMOCLINIC CLASSES

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Abstract. We show that for a C^1 residual subset of diffeomorphisms far away from tangency, every non-trivial chain recurrent class that is accumulated by sources ia a homoclinic class contains periodic points with index 1 and it's the Hausdorff limit of a family of sources.

Contents

1. Introduction

In the middle of last century, with many remarkable work, hyperbolic diffeomorphisms have been understood very well, but soon people discovered that the set of hyperbolic diffeomorphisms are not dense among differential dynamics, two kinds of counter examples were described, one associated with heterdimension cycle was given by R.Abraham and Smale [3] and then given by Shub [40] and Mañé [28], another counter example associated with homoclinic tangency was given by Newhouse [31] [32]. In fact, Newhouse got an open set $\mathcal{U} \subset C^2(M)$ where $dim(M) = 2$ such that there exists a C^2 generic subset $R \subset \mathcal{U}$ and for any $f \in R$, f has infinite sinks or sources. Such complicated phenomena (there exist an open set $\mathcal{U}$ in $C^r(M)$ and a generic subset $R \subset \mathcal{U}$, such that any $f \in R$ has infinite sinks or sources) is called C^r Newhouse phenomena today, and we say C^r Newhouse phenomena happens at $\mathcal{U}$.

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In last 90's, some new examples of Newhouse phenomena were found, [33] generalized Newhouse phenomena to high dimensional manifold ($dimM > 2$) but with the same topology $C^r(r > 1)$. [7] used a new tool 'Blender' to show the existence of C^1 Newhouse phenomena on manifold with $dim(M) > 2$. Until now, all the construction of C^r Newhouse phenomena relate closely with homoclinic tangency, more precisely, all the open set $\mathcal{U}$ given by the construction above which happens Newhouse phenomena there will have $\mathcal{U} \subset \overline{HT}$. We hope that it's a necessary condition for C^r Newhouse phenomena happens at $\mathcal{U}$. Palis states it as a conjecture.

Conjecture (Palis): *If C^r Newhouse phenomena happens at $\mathcal{U}$, then $\mathcal{U}$ is contained in $\overline{HT^r}$.*

When $r = 1$ and M is a compact surface, with Mañé's work [29], Pujals' conjecture is equivalent with the famous C^1 Palis strong conjecture.

C^1 **Palis strong conjecture** : *Diffeomorphisms of M exhibiting either a homoclinic tangency or heterodimensional cycle are C^1 dense in the complement of the C^1 closure of hyperbolic systems.*

In the remarkable paper [36] they proved C^1 Palis strong conjecture on $C^1(M)$ when M is a boundless compact surface, so in such case Pujals' conjecture is right. In [37] they gave many relations between C^2 Newhouse phenomena and $\overline{HT^1}$. In this paper we just consider C^1 Newhouse phenomena, and we show that if C^1 Newhouse phenomena happens in an open set $\mathcal{U} \subset C^1(M) \backslash \overline{HT^1}$, it should have some special properties, in fact, in [7] they found an open set $\mathcal{U} \subset \overline{(HT^1)}$ and there exists a generic subset $R \subset \mathcal{U}$ such that any $f \in R$ has infinite sinks or sources stay near a chain recurrent class, and such class does not contain any periodic points, such kind of chain recurrent class is called aperiodic class now. Here we proved that in $\overline{HT}^c$, if there exists Newhouse phenomena, the sinks or sources will just stay near a special kind of homoclinic class.

Theorem 1 *There exists a generic subset $R \subset C^1(M) \backslash \overline{HT^1}$, such that for $f \in R$ and C is any non-trivial chain recurrent class of f, if $C \bigcap P_0^* \neq \phi$, C should be a homoclinic class containing index 1 periodic points and C is an index 0 fundamental limit.*

Theorem 1 means that if we want to disprove the existence of Newhouse phenomena in $C^1(M) \setminus \overline{HT}$, we just need study the homoclinic class containing index 1 periodic point.

In §3 we'll state some generic properties. In §4 we'll introduce a special minimal non-hyperbolic set and theorem 1 will be proved in §5.

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2. Notations and definitions

Let M be a compact boundless Riemannian manifold, since when M is a surface [36] has proved that hyperbolic diffeomorphisms are open and dense in $C^1(M) \setminus \overline{HT}$, we suppose $dim(M) = d > 2$ in this paper. Let $Per(f)$ denote the set of periodic points of f and $\Omega(f)$ the non-wondering set of f, for $p \in Per(f)$, $\pi(p)$ means the period of p. If p is a hyperbolic periodic point, the index of p is the dimension of the stable bundle. We denote $Per_i(f)$ the set of the index i periodic points of f, and we call a point x is an index i preperiodic point of f if there exists a family of diffeomorphisms $g_n \xrightarrow{C^1} f$, where g_n has an index i periodic point p_n and $p_n \longrightarrow x$. $P_i^*(f)$ is the set of index i preperiodic points of f, it's easy to know $\overline{P_i(f)} \subset P_i^*(f)$.

Let Λ be an invariant compact set of f, we say Λ is an index i fundamental limit if there exists a family of diffeomorphisms g_n C^1 converging to f, p_n is an index i periodic point of g_n and $Orb(p_n)$ converge to Λ in *Hausdorff* topology. So if $\Lambda(f)$ is an index i fundamental limit, we have $\Lambda(f) \subset P_i^*(f)$.

For two points $x, y \in M$ and some $\delta > 0$, we say there exists a δ-pseudo orbit connects x and y means that there exist points $x = x_0, x_1, \cdots, x_n = y$ such that $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, 1, \cdots, n-1$, we denote it $x \underset{\delta}{\dashv} y$. We say $x \dashv y$ if for any $\delta > 0$ we have $x \underset{\delta}{\dashv} y$ and denote $x \vdash\!\dashv y$ if $x \dashv y$ and $y \dashv x$. A point x is called a chain recurrent point if $x \vdash\!\dashv x$. $CR(f)$ denotes the set of chain recurrent points of f, it's easy to know that $\vdash\!\dashv$ is an closed equivalent relation on $CR(f)$, and every equivalent class of such relation should be compact and is called chain recurrent class. Let K be a compact invariant set of f, if x, y are two points in K, we'll denote $x \underset{K}{\dashv} y$ if for any $\delta > 0$, we have a δ -pseudo orbit in K connects x and y. If for any two points $x, y \in K$ we have $x \underset{K}{\dashv} y$, we call K a chain recurrent set. Let C be a chain recurrent class of f, we call C is an aperiodic class if C does not contain periodic point.

Let Λ be an invariant compact set of f, for $0 < \lambda < 1$ and $1 \leq i < d$, we say Λ has an index $i - (l, \lambda)$ dominated splitting if we have a continuous invariant splitting $T_\Lambda M = E \oplus F$ where $dim(E_x) = i$ for any $x \in \Lambda$ and $\| Df^l|_E(x) \| \cdot \| Df^{-l}|_F(f^l x) \| < \lambda$ for all $x \in \Lambda$. For simplicity, sometimes we just call $\Lambda(f)$ has an index i dominated splitting. A compact invariant set can have many dominated splittings, but for fixed i, the index i dominated splitting is unique.

We say a diffeomorphism f has C^r tangency if $f \in C^r(M)$, f has hyperbolic periodic point p and there exists a non-transverse intersection between $W^s(p)$ and $W^u(p)$. HT^r is the set of the diffeomorphisms which have C^r tangency, usually we just use HT denote HT^1. We call a diffeomorphism f is far away from tangency if $f \in C^1(M) \setminus \overline{HT}$. The following proposition shows the relation between dominated splitting and far away from tangency.

Proposition 2.1. *([42]) f is C^1 far away from tangency if and only if there exists (l, λ) such that $P_i^*(f)$ has index $i - (l, \lambda)$ dominated splitting for $0 < i < d$.*

Usually dominated splitting is not a hyperbolic splitting, Mañé showed that in some special case, one bundle of the dominated splitting is hyperbolic.

Proposition 2.2. *([29]) Suppose $\Lambda(f)$ has an index i dominated splitting $E \oplus F$ $(i \neq 0)$, if $\Lambda(f) \bigcap P_j^*(f) = \phi$ for $0 \leq j < i$, then E is a contracting bundle.*

3. Generic properties

For a topology space X, we call a set $R \subset X$ is a generic subset of X if R is countable intersection of open and dense subsets of X, and we call a property is a generic property of X if there exists some generic subset R of X holds such property. Especially, when $X = C^1(M)$ and R is a generic subset of $C^1(M)$, we just call R is C^1 generic, and we call any generic property of $C^1(M)$ 'a C^1 generic property' or 'the property is C^1 generic'.

Here we'll state some well known C^1 generic properties.

Proposition 3.1. *There is a C^1 generic subset R_0 such that for any $f \in R_0$, one has*

1) *f is Kupka-Smale (every periodic point p in $Per(f)$ is hyperbolic and the invariant manifolds of periodic points are everywhere transverse).*
2) *$CR(f) = \Omega = \overline{Per(f)}$.*
3) *$P_i^*(f) = \overline{P_i(f)}$*
4) *any chain recurrent set is the Hausdorff limit of periodic orbits.*
5) *any index i fundamental limit is the Hausdorff limit of index i periodic orbits of f.*
6) *any chain recurrent class containing a periodic point p is the homoclinic class $H(p, f)$.*
7) *Suppose C is a homoclinic class of f, and $j_0 = min\{j : C \bigcap Per_j(f) \neq \phi\}$, $j_1 = max\{j : C \bigcap Per_j(f) \neq \phi\}$, then for any $j_0 \leq j \leq j_1$, we have $C \bigcap Per_j(f) \neq \phi$.*

By proposition 3.1, for any f in R_0, every chain recurrent class C of f is either an aperiodic class or a homoclinic class. If $\#C = \infty$, we call C is non-trivial.

Let $R = R_0 \setminus \overline{HT}$, we'll show that the generic subset R of $\overline{HT}^c$ will satisfy theorem 1.

4. A special minimal set

Let $f \in R$, C is a non-trivial chain recurrent class of f, and $j_0 = \min\{j : C \bigcap P_j^* \neq \phi\}$.

Definition 4.1. *: An invariant compact subset Λ of f is called minimal if all the invariant compact subsets of Λ are just Λ and ϕ. An invariant compact subset Λ of f is called minimal index j fundamental limit if Λ is an index j fundamental limit and any invariant compact subset $\Lambda_0 \subsetneq \Lambda$ is not an index j fundamental limit.*

Lemma 4.2. *If $C \bigcap P_j^* \neq \phi$, there always exists a minimal index j fundamental limit in C.*

Proof Let $H = \{\tilde{\Lambda} : \tilde{\Lambda} \subset C$ is an index j fundamental limit$\}$ and we order H by inclusion. Suppose $x \in C \bigcap P_j^*$, then there exist $g_n \xrightarrow{C^1} f$, p_n is index j periodic point of g_n and $p_n \longrightarrow x$. Denote $\Lambda_x = \lim Orb(P_n)$, then Λ_x is an index j fundamental limit. It's easy to know Λ_x is a chain recurrent set and $\Lambda_x \subset C$, so $\Lambda_x \in H$. It means $H \neq \phi$.

Let $H_\Gamma = \{\Lambda_\lambda : \lambda \in \Gamma\}$ be a totally ordered chain of H. Then $\Lambda_\infty = \bigcap_{\lambda \in \Gamma} \Lambda_\lambda$ is a compact invariant set, in fact, there exists $\{\lambda_i\}_{i=1}^\infty$ such that $\Lambda_{\lambda_i} \supset \Lambda_{\lambda_{i+1}}$ and $\Lambda_\infty = \bigcap_{i=1}^\infty \Lambda_{\lambda_i}$.

We claim that Λ_∞ is an index j fundamental limit also.

Proof of the claim From generic property 5) of proposition 3.1 and $f \in R$, for any $\varepsilon > 0$, there exists periodic point p_i such that $p_i \in Per_j(f)$ and $d_H(Orb(p_i), \Lambda_{\lambda_i}) < \frac{\varepsilon}{2}$. When i is big enough, we'll have $d_H(\Lambda_{\lambda_i}, \Lambda_\infty) < \frac{\varepsilon}{2}$, so for any $\varepsilon > 0$, there exists $p_i \in Per_j(f)$such that $d_H(Orb(p_i), \Lambda_\infty) < \varepsilon$. □

Now by Zorn's lemma, there exists a minimal index j fundamental limit in C. □

Suppose Λ is a minimal index j_0 fundamental limit of C, the main aim of this section is the following lemma.

Lemma 4.3. *Suppose $f \in R$, C is a non-trivial chain recurrent class of f, $j_0 = \min\{j : C \bigcap P_j^* \neq \phi\}$. Let Λ be any minimal index j_0 fundamental limit in C, then*

a) *either Λ is a non-trivial minimal set with partial hyperbolic splitting $T|_\Lambda M = E^s_{j_0} \oplus E^c_1 \oplus E^u_{j_0+2}$,*
b) *or C contains a periodic point with index j_0 or $j_0 + 1$ and C is an index j_0 fundamental limit.*

We postpone the proof of lemma 4.3 to §4.4, before that, I'll give or introduce some results at first. In §4.1 I'll give a proof of Shaobo Gan's lemma, in §4.2 I'll introduce Liao's selecting lemma and prove a weakly selecting lemma, in §4.3 I'll introduce a powerful tool 'transition' given by [BDP].

4.1. **Shaobo Gan's lemma.** Let $GL(d)$ be the group of linear isomorphisms of R^d, we call ξ a periodic sequence of linear map if $\xi : Z \longrightarrow Gl(d)$ is a sequence of isomorphisms of R^d and there exists $n_0 \geq 1$ such that $\xi_{j+n_0} = \xi_j$ for all j. We denote $\pi(\xi) = min\{n : \xi_{j+n} = \xi_j \text{ for all } j\}$ the period of ξ, and we call ξ has index i if the map $\prod_{j=0}^{\pi(\xi)-1} \xi_j$ is hyperbolic and has index i, we say ξ is contracting if ξ has index d. We denote $E^{s(u)}$ the stable (unstable) bundle of ξ.

Suppose η is a periodic sequence of linear maps also, we call η is an ε-perturbation of ξ if $\pi(\eta) = \pi(\xi)$ and $\| \eta_j - \xi_j \| \leq \varepsilon$ for any j.

Let $\{\xi^\alpha\}_{\alpha\in\mathcal{A}}$ be a family of periodic sequence of linear maps with index i, we call it is bounded if there exists $K > 0$ such that for any $\alpha \in \mathcal{A}$ and any $j \in \mathbb{Z}$, we have $\| \xi_j^{(\alpha)} \| < K$. For a family of bounded periodic sequences of linear maps $\{\xi^\alpha\}_{\alpha\in\mathcal{A}}$, we say it's index stable if $\xi^{(\alpha)}$ has index i for all $\alpha \in \mathcal{A}$, and there exists $\varepsilon_0 > 0$ such that $\#\{\alpha |$ there exists $\eta^{(\alpha)}$ is ε_0-perturbation of $\xi^{(\alpha)}$ and $\eta^{(\alpha)}$ has index different with $i\} < \infty$. Especially, if it's index d stable, we call $\xi^{(\alpha)}|_{\alpha\in\mathcal{A}}$ is uniformly contracting.

Suppose $f \in C^1(M)$ and $\{p_n(f)\}$ is a family of hyperbolic periodic points of f with index i, we say $p_n(f)$ is index i stable if $\{Df|_{Orb(p_n)}\}_{n=1}^\infty$ is index i stable and $\lim_{n\to\infty} \pi(p_n) = \infty$.

Remark 4.4. *Pliss has proved that if $\{p_n(f)\}$ is index i stable, then $i \neq 0, d$.*

The following lemma was given by Shaobo Gan, and the proof comes from him also.

Lemma 4.5. *([15]) $f \in C^1(M)$, suppose $\{p_n(f)\}$ is index i stable, then there exists a subsequence $\{p_{n_j}\}_{j=1}^\infty$ such that p_{n_j} and $p_{n_{j+1}}$ are homoclinic related.*

Here we just prove the following weaker statement of Gan's lemma.

Lemma 4.6. *(Weaker statement of Gan's lemma) Suppose $f \in R$, Λ is a non-trivial chain recurrent set of f, $\{p_n(f)\}$ is index i stable and $\lim_{n\to\infty} Orb(p_n) = \Lambda$, then there exists a subsequence $\{p_{n_j}(f)\}_{j=1}^\infty$ such that $p_{n_j}(f)$ and $p_{n_{j+1}}(f)$ are homoclinic related.*

Before we prove lemma 4.6, we'll give a few lemmas which will be used in the proof.

Lemma 4.7. *Suppose* $A=\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$ *is a hyperbolic linear map with index* i $(i \neq 0, d)$*, where* $B \in GL(R^i)$ *is a contracting map and* $D \in GL(R^{d-i})$ *is a expanding map. If there exists* $B' \in GL(R^i)$ *an* ε*-perturbation of* B *and* B' *has index different with* i*, then* $A'=\begin{pmatrix} B' & C \\ 0 & D \end{pmatrix}$ *is an* ε*-perturbation of* A *with index different with* i*. In fact, we'll have* $ind(A') = ind(B')$*.*

With lemma 4.7, the following lemma is obvious.

Lemma 4.8. *Suppose* $\{\xi^{(n)}\}_{n=1}^{\infty}$ *is index* i *stable, then* $\{\xi^{(n)}|_{E^s(\xi^{(n)})}\}_{n=1}^{\infty}$ *is stable contracting, and at the same time,* $\{\xi^{(n)}|_{E^u(\xi^{(n)})}\}_{n=1}^{\infty}$ *is stable expanding.*

In [29] Mañé has given a necessary condition for bounded stable contracting sequence.

Lemma 4.9. *(Mañé) If* $\{\xi^{(n)}\}_{n=1}^{\infty}$ *is stable contracting and bounded, then there exist* $N_0, l_0, 0 < \lambda_0 < 1$ *such that if* $\pi(\xi^{(n)}) > N_0$ *we'll have*

$$\prod_{j=0}^{[\frac{\pi(\xi_n)}{l_0}]-1} \| \prod_{t=0}^{l_0-1} \xi^{(n)}_{(jl_0+t)+s} \| \leq \lambda_0^{[\frac{\pi(\xi^{(n)})}{l_0}]}$$

for any $0 \leq s < \pi(\xi^{(n)})$*.*

Proof of lemma 4.6: Since $\Lambda \subset P_i^*$ and f is far away from tangency, by proposition 2.1, Λ has an index $i-(l,\lambda)$ dominated splitting $T|_{\Lambda}M = E \oplus F$. In order to make the proof more simiplier, here we just suppose $l = 1$. Choose a small open neighborhood U of Λ, when U is small enough, $\widetilde{\Lambda} = \bigcap\limits_{j\in\mathbb{Z}} f^j(\overline{U})$ has an index $i-(1,\widetilde{\lambda})$ dominated splitting $T_{\widetilde{\Lambda}}M = \widetilde{E} \oplus \widetilde{F}$ where $\lambda < \widetilde{\lambda} < 1$ and $\widetilde{E}|_{\Lambda} = E$, $\widetilde{F}|_{\Lambda} = F$.

Since $\lim\limits_{n\to\infty} Orb(P_n) = \Lambda$, we can always suppose $Orb(p_n) \subset \overline{U}$, so $Orb(P_n) \subset \widetilde{\Lambda}$ and $E^s|_{Orb(p_n)} = \widetilde{E}|_{Orb(p_n)}$, $F^u|_{Orb(p_n)} = \widetilde{F}|_{Orb(p_n)}$.

By lemma 4.8, we know that $\{Df|_{E^s(Orb(p_n))}\}_{n=1}^{\infty}$ is stable contracting and $\{Df|_{E^u(Orb(p_n))}\}_{n=1}^{\infty}$ is stable expanding. By lemma 4.9, there exist$N_0, l_0, 0 < \lambda_0 < 1$ such that if $\pi(p_n(f)) > N_0$, we have

$$\prod_{j=0}^{[\frac{\pi(p_n)}{l_0}]-1} \|Df^{l_0}|_{E^s(f^{jl_0}p_n)}\| \leq \lambda_0^{[\frac{\pi(p_n)}{l_0}]} \tag{4.1}$$

$$\prod_{j=0}^{[\frac{\pi(p_n)}{l_0}]-1} \|Df^{-l_0}|_{F^u(f^{-jl_0}p_n)}\| \leq \lambda_0^{[\frac{\pi(p_n)}{l_0}]} \tag{4.2}$$

Since $\lim\limits_{n\to\infty} Orb(p_n) = \Lambda$ and Λ is not trivial, we have $\lim\limits_{n\to\infty} \pi(p_n) \longrightarrow \infty$, then we can always suppose all the p_n satisfy (4.1) and (4.2). For simplicity, we suppose $l_0 = 1$ here.

Choose some $\varepsilon > 0$ and $\lambda_1 < 1$ such that $\max\{\widetilde{\lambda},\ \lambda_0\} + \varepsilon < \lambda_1^2 < \lambda_1 < 1$. Now we'll state Pliss lemma in a special context.

Lemma 4.10. *(Pliss[34]) Given* $0 < \lambda_0 + \varepsilon < \lambda_1 < 1$ *and* $Orb(p_n) \subset \widetilde{\Lambda}$ *such that for some* $m \in \mathbb{N}$*, we have* $\prod\limits_{j=0}^{t-1} \|Df|_{E^s(f^j p_n))}\| \leq (\lambda_0 + \varepsilon)^t$ *for all* $s \geq m$*, there exists a sequence* $0 \leq n_1 < n_2 < \cdots$ *such that* $\prod\limits_{j=n_r}^{t-1} \|Df|_{E^s(f^j p_n))}\| \leq \lambda_1^{t-n_r}$ *for all* $t \geq n_r$*,* $r = 1, 2, \cdots$ *.*

Remark 4.11. *The sequence* $\{n_j\}_{j=1}^{\infty}$ *we get above is called the* λ_1*-hyperbolic time for bundle* $E^s|_{Orb(p_n)}$*. By* (4.1),(4.2)*, when* n *is big enough,* $Orb(p_n)$ *will satisfy the assumption of Pliss lemma, so by lemma 4.10, there exists* $q_n^+ \in Orb(p_n)$ *such that* $\prod_{j=0}^{t-1}\|Df|_{E^s(f^j q_n^+)}\| \le \lambda_1^t$ *and* $q_n^- \in Orb(p_n)$ *such that* $\prod_{j=0}^{t-1}\|Df^{-1}|_{F^u(f^{-j} q_n^-)}\| \le \lambda_1^t$ *for all* $t > 0$*.*

Let's denote

$$S_{n,+} = \{m \in \mathbb{Z} : \prod_{j=0}^{s-1}\|Df|_{E^s(f^{m+j}p_n)}\| \le \lambda_1^s \text{ for all } s > 0\},$$

$$S_{n,-} = \{m \in \mathbb{Z} : \prod_{j=0}^{s-1}\|Df^{-1}|_{F^u(f^{m-j}p_n)}\| \le \lambda_1^s \text{ for all } s > 0\}.$$

Then $S_{n,+}$ is the set of λ_1 hyperbolic time for bundle $E^s|_{Orb(p_n)}$ and $S_{n,-}$ is the set of hyperbolic time for bundle $F^u|_{Orb(p_n)}$. From remark 4.11, the set $S_{n,+}$ and $S_{n,-}$ are not empty. We denote $S_n = S_{n,+}\bigcap S_{n,-}$.

Lemma 4.12. $S_n \ne \phi$*.*

Proof: Here for $a, b \in \mathbb{Z}$ and $a < b$, we denote $(a,b)_{\mathbb{Z}} = \{c|\ c \in \mathbb{Z} \textit{ and } a < c < b\}$.

Now suppose the lemma is false, we can choose $\{b_{n,s},\ b_{n,s+1}\} \subset S_{n,-}$ such that we have $b_{n,s+1} > b_{n,s}$, $(b_{n,s}, b_{n,s+1})_{\mathbb{Z}}\bigcap S_{n,-} = \phi$ and $a_{n,t} \in (b_{n,s}, b_{n,s+1})_{\mathbb{Z}}\bigcap S_{n,+}$, then $b_{n,s}, b_{n,s+1} \notin S_{n,+}$.

We claim that for $0 < k \le b_{n,s+1} - b_{n,s} - 1$, we have $\prod_{j=0}^{k-1}\|Df^{-1}|_{F^u(f^{b_{n,s}+j+1}p_n)}\| \ge \lambda_1^k$.

Proof of the claim: We'll use induction to give a proof.

When $k = 1$, since $b_{n,s} + 1 \notin S_{n,-}$, we have $\|Df^{-1}|_{F^u(f^{b_{n,s}+1}p_n)}\| > \lambda_1$.

Now suppose the claim is true for all $1 \le k \le k_0 - 1$ where $1 < k_0 \le b_{n,s+1} - b_{n,s} - 1$, and we suppose the claim is false for k_0, it means that

$$\prod_{j=0}^{k_0-1}\|Df^{-1}|_{F^u(f^{b_{n,s}+j+1}p_n)}\| \le \lambda_1^{k_0}. \tag{4.3}$$

Then by the assumption above that the claim is true for $1 \le k \le k_0 - 1$, we have

$$\prod_{j=0}^{k-1}\|Df^{-1}|_{F^u(f^{b_{n,s}+j+1}p_n)}\| \ge \lambda_1^k \tag{4.4}$$

By (4.3) and (4.4), we get that $\prod_{j=k}^{k_0-1}\|Df^{-1}|_{F^u(f^{b_{n,s}+j+1}p_n)}\| < \lambda_1^{k_0-k}$ for $1 \le k \le k_0 - 1$. It's equivalent to say that

$$\prod_{j=0}^{k-1}\|Df^{-1}|_{F^u(f^{b_{n,s}+k_0-j}p_n)}\| < \lambda_1^k \quad \text{for } 1 \le k \le k_0 - 1 \tag{4.5}$$

By (4.3) and (4.5), we get that

$$\prod_{j=0}^{k-1}\|Df^{-1}|_{F^u(f^{b_{n,s}+k_0-j}p_n)}\| \le \lambda_1^k \quad \text{for } 1 \le k \le k_0 \tag{4.6}$$

When $k > k_0$,by (4.6) and the fact $b_{n,s} \in S_{n,-}$, we have

$$\prod_{j=0}^{k-1} \|Df^{-1}|_{F^u(f^{b_{n,s}+k_0-j})}\| = \prod_{j=0}^{k_0-1} \|Df^{-1}|_{F^u(f^{b_{n,s}+k_0-j})}\| \cdot \prod_{j=0}^{k-k_0-1} \|Df^{-1}|_{F^u(f^{b_{n,s}-j})}\| < \lambda_1^{k_0} \cdot \lambda_1^{k-k_0} = \lambda_1^k,$$

it means $b_{n,s} + k_0 \in S_{n,-}$, it's a contradiction since $b_{n,s} + k_0 \in (b_{n,s}, b_{n,s+1})_{\mathbb{Z}}$, so we finish the induction. $\square$

By the claim above, for $0 < k \leq b_{n,s+1} - b_{n,s} - 1$, we have

$$\prod_{j=0}^{k-1} \|Df^{-1}|_{F^u(f^{b_{n,s}+j+1}p_n)}\| \geq \lambda_1^k. \tag{4.7}$$

Since on $\widetilde{\Lambda}$, $\widetilde{E} \oplus \widetilde{F}$ is an index $i - (1, \widetilde{\lambda})$ dominated splitting, we have

$$\prod_{j=0}^{k-1} (\|Df|_{\widetilde{E}(f^{b_{n,s}+j}p_n)}\| \cdot \|Df^{-1}|_{\widetilde{F}(f^{b_{n,s}+j+1}p_n)}\|) < \widetilde{\lambda}^k.$$

By (4.7) and $\widetilde{E}|_{Orb(p_n)} = E^s|_{Orb(p_n)}$,$\widetilde{F}|_{Orb(p_n)} = F^u|_{Orb(p_n)}$, we'll get

$$\prod_{j=0}^{k-1} \|Df|_{E^s(f^{b_{n,s}+j}p_n)}\| < \frac{\widetilde{\lambda}^k}{\lambda_1^k} \underset{(\widetilde{\lambda}<\lambda_1^2<1)}{<} \lambda_1^k \quad \text{for } 1 < k \leq b_{n,s+1} - b_{n,s} - 1. \tag{4.8}$$

When $k > b_{n,s+1} - b_{n,s} - 1$, let $k = (a_{n,t} - b_{n,s}) + (k - a_{n,t})$, by (4.8) and $a_{n,t} \in S_{n,+}$,

$$\begin{aligned} \prod_{j=0}^{k-1} \|Df|_{E^s(f^{b_{n,s}+j}p_n)}\| &= \prod_{j=0}^{a_{n,t}-b_{n,s}-1} \|Df|_{E^s(f^{b_{n,s}+j}p_n)}\| \cdot \prod_{j=0}^{k-a_{n,t}-1} \|Df|_{E^s(f^{a_{n,t}+j}p_n)}\| \\ &< \lambda_1^{a_{n,t}-b_{n,s}} \cdot \lambda_1^{k-a_{n,t}} = \lambda_1^{k-b_{n,s}} \end{aligned} \tag{4.9}$$

By (4.8) and (4.9), we get $b_{n,s} \in S_{n,+}$, so $S_{n,+} \bigcap S_{n,-} \neq \phi$, it's a contradiction with our assumption, so we finish the proof of lemma 4.12. $\square$

Now let's continue the proof of lemma 4.6, we need the following two lemmas to show that for $a_n \in S_n$, the point $f^{a_n}(p_n)$ will have uniform size of stable manifold and unstable manifold.

Let $I_1 = (-1, 1)^i$ and $I_\varepsilon = (-\varepsilon,\ \varepsilon)^i$, denote by $Emb^1(I, M)$ the set of C^1-embedding of I_1 on M, recall by [21] that $\widetilde{\Lambda}$ has a dominated splitting $\widetilde{E} \oplus \widetilde{F}$ implies the following.

Lemma 4.13. *There exist two continuous function $\Phi^{cs} : \widetilde{\Lambda} \longrightarrow Emb^1(I, M)$ and $\Phi^{cu} : \widetilde{\Lambda} \longrightarrow Emb^1(I, M)$ such that, with $W^{cs}_\varepsilon(x) = \Phi^{cs}(x)I_\varepsilon$ and $W^{cu}_\varepsilon(x) = \Phi^{cu}(x)I_\varepsilon$, the following properties hold:*

a) *$T_x W^{cs}_\varepsilon = \widetilde{E}(x)$ and $T_x W^{cu}_\varepsilon = \widetilde{F}(x)$,*
b) *For all $0 < \varepsilon_1 < 1$, there exists ε_2 such that $f(W^{cs}_{\varepsilon_2}(x)) \subset W^{cs}_{\varepsilon_1}(f(x))$ and $f^{-1}(W^{cu}_{\varepsilon_2}(x)) \subset W^{cu}_{\varepsilon_1}(f^{-1}(x))$.*
c) *For all $0 < \varepsilon < 1$, there exists $\delta > 0$ such that if $y_1,\ y_2 \in \widetilde{\Lambda}$ and $d(y_1,\ y_2) < \delta$, then $W^{cs}_\varepsilon(y_1) \pitchfork W^{cu}_\varepsilon(y_2) \neq \phi$.*

Corollary 4.14. *([36]) For any $0 < \lambda < 1$, there exists $\varepsilon > 0$ such that for $x \in \widetilde{\Lambda}$ which satisfies $\prod_{j=0}^{n-1} \|Df|_{\widetilde{E}(f^j x)}\| \leq \lambda^n$ for all $n > 0$, then $diam(f^n(W^{cs}_\varepsilon)) \longrightarrow 0$, i.e. the central stable manifold of x with size ε is in fact a stable manifold.*

Now for λ_1, using corollary 4.14, we can get an $\varepsilon > 0$. It means that for any $a_n \in S_n$, denote $q_n = f^{a_n}(p_n)$, then $W^{cs}_\varepsilon(q_n)$ is a stable manifold and $W^{cu}_\varepsilon(q_n)$ is an unstable manifold. For this $\varepsilon > 0$, use c) of lemma 4.13, we can fix a δ. Choose a subsequence $\{n_i\}$ such that $d(q_{n_i}, q_{n_{i+1}}) \leq \delta$, then by c) of lemma 4.13, we know $W^{cu}_\varepsilon(q_{n_i}) \pitchfork W^{cs}_\varepsilon(q_{n_{i+1}}) \neq \phi$ and $W^{cu}_\varepsilon(q_{n_{i+1}}) \pitchfork W^{cs}_\varepsilon(q_{n_i}) \neq \phi$. Since the local central stable manifold and local central unstable manifold of q_{n_i} have dynamical meaning, we know that $Orb(q_{n_i})$ and $Orb(q_{n_{i+1}})$ are homoclinic related. □

Remark 4.15. *In the proof of lemma 4.6 we suppose the set Λ has 1-step dominated splitting, that means $l = 1$, and we suppose $l_0 = 1$ there also, they are just in order to make the proof more simplier. In the rest part of the paper, usually we don't use such assumption any more, if we use it we'll point out.*

Now let's consider a sequence of periodic points which are not index stable.

Lemma 4.16. *Suppose $f \in R$, $\lim\limits_{n\to\infty} g_n = f$, $\{p_n(g_n)\}_{n=1}^\infty$ is a family of index i periodic points ($i \neq 0, d$) and $\lim\limits_{n\to\infty} \pi(p_n) \longrightarrow \infty$. If there exist $\lambda_n \longrightarrow 1^-$ and $\lim\limits_{n\to\infty} l_n \longrightarrow \infty$ such that $\lim\limits_{n\to\infty} \frac{\pi(p_n)}{l_n} \longrightarrow \infty$ and $\prod\limits_{j=0}^{[\frac{\pi(p_n)}{l_n}]-1} \|Dg_n^{l_n}|_{E^s(g_n^{jl_n}(p_n))}\| \geq \lambda_n^{[\frac{\pi(p_n)}{l_n}]}$, then for any $\varepsilon > 0$ and $N > 0$, there exists an $n_0 > N$ and g'_{n_0} is an ε-perturbation of g_{n_0} such that $p_{n_0}(g_{n_0})$ is an index $i-1$ periodic point of g'_n.*

Proof: Fix N, consider the periodic sequence of linear maps $\{\xi^n : \ \xi^n = Dg_n|_{E^s(Orb(p_n))}\}_{n\geq N}$, they are all contracting maps. We claim that $\{\xi^n\}$ are not stable contracting.

Proof of the claim: If $\{\xi^n\}$ is stable contracting, by lemma 4.9, there exist $N_0,\ l_0,\ 0 < \lambda_0 < 1$ such that if $\pi(\xi^n) > N_0$, we have

$$\prod_{j=0}^{[\frac{\pi(p_n)}{l_0}]-1} \|Dg_n^{l_0}|_{E^s(g_n^{jl_0}p_n)}\| \leq \lambda_0^{[\frac{\pi(p_n)}{l_0}]} \tag{4.10}$$

Choose some N_1 big enough such that for $n \geq N_1$, we have $\lambda_n \geq\ \lambda^* >\ \lambda_0$ for some $\lambda^* \in\ (\lambda_0, 1)$, then by $\lim\limits_{n\to\infty} \frac{\pi(p_n)}{l_n} \longrightarrow \infty$ and $\lim\limits_{n\to\infty} l_n \longrightarrow \infty$, when n is big enough, we have $\pi(p_n) \gg\ l_n \gg\ \max\{l_0, N_0\}$ and from $\prod\limits_{j=0}^{[\frac{\pi(p_n)}{l_0}]-1} \|Dg_n^{l_n}|_{E^s(g_n^{jl_n}p_n)}\| \geq \lambda_n^{[\frac{\pi(p_n)}{l_n}]} > (\lambda^*)^{[\frac{\pi(p_n)}{l_n}]}$, we'll get $\prod\limits_{j=0}^{[\frac{\pi(p_n)}{l_0}]-1} \|Dg_n^{l_0}|_{E^s(g_n^{jl_0}p_n)}\| \geq \lambda_0^{[\frac{\pi(p_n)}{l_n}]} > \lambda_0^{[\frac{\pi(p_n)}{l_0}]}$, It's a contradiction with (4.10). □

Since $\{\xi^n\}_{n\geq N}$ isn't stable contracting, for $\varepsilon > 0$, there exists a sequence $\{n_i\}$ and $\{\eta^{n_i}\}$ such that η^{n_i} is an ε-perturbation of ξ^{n_i} and η^{n_i} has index smaller than i. Since $\{\xi^{n_i}\}$ is bounded and $\lim\limits_{n\to\infty} \pi(p_n) \longrightarrow \infty$, by [10]'s work, for n_i big enough, we can in fact get η^{n_i} with index $i-1$. By lemma 4.7, there exists $\{A|_{Orb(p_n)}\}_{n\geq 0}$ an ε-perturbation of $\{Dg_n|_{Orb(p_n)}\}$ such that $\{A|_{Orb(p_n)}\}$ has index $i-1$. Now we need the following version of Franks lemma.

Lemma 4.17. *(Franks lemma) Suppose p_n is a periodic point of g_n, $A|_{Orb(p_n)}$ is an ε-perturbation of $\{Dg_n|_{Orb(p_n)}\}$, then for any neighborhood U of $Orb(p_n)$, there exists g'_n such that $g'_n \equiv g_n$ on $(M \setminus U)\bigcup Orb(p_n)$, $d_{C^1}(g_n, g'_n) < \varepsilon$ and $\{Dg'_n|_{orb(p_n)}\} = \{A|_{Orb(p_n)}\}$.*

From Franks lemma, we can change the derivative map along $T_{Orb(p_{n_i})}M$ to be $\{A|_{Orb(p_n)}\}$ and get a new map g'_{n_i} such that $p_{n_i}(g_{n_i})$ is index $i-1$ periodic point of g'_{n_i}. $\square$

4.2. **Weakly selecting lemma.** Liao's selecting lemma is a powerful shadowing lemma for non-uniformly hyperbolic system, with it, we can not only get a lot of periodic points like what the standard shadowing lemma can do, we can even let the periodic points have hyperbolic property as weak as we like. Liao at first used this lemma to study minimal non-hyperbolic set and proved the Ω-stable conjecture for diffeomorphisms in dimension 2 and for flow without singularity in dimension 3. [16] [17] [19] [41] use the same idea proved structure(Ω) stability conjecture for flows without singularity in any dimension. Until now, the most important papers about selecting lemma are [18],[44], [45] and there contain more details about selecting lemma.

In this subsection and the next, we'll show what will happen if all the conditions in weakly selecting lemma are satisfied. The main result in this subsection is lemma 4.21 (The weakly selecting lemma). Now let's state the selecting lemma at first.

Proposition 4.18. *(Liao) Let Λ be a compact invariant set of f with index $i-(l,\lambda)$ dominated splitting $E^{cs}\oplus F^{cu}$. Assume that*

a) *there is a point $b\in\Lambda$ satisfying $\prod_{j=0}^{n-1}\|Df^l|_{E^{cs}(f^{jl}b)}\|\geq 1$ for all $n\geq 1$.*

b) *(The tilda condition) there are λ_1 and λ_2 with $\lambda<\lambda_1<\lambda_2<1$ such that for any $x\in\Lambda$ satisfying $\prod_{j=0}^{n-1}\|Df^l|_{E^{cs}(f^{jl}x)}\|\geq {\lambda_2}^n$ for all $n\geq 1$, $\omega(x)$ contains a point $c\in\Lambda$ satisfying $\prod_{j=0}^{n-1}\|Df^l|_{E^{cs}(f^{jl}c)}\|\leq \lambda_1^n$ for all $n\geq 1$.*

Then for any λ_3 and λ_4 with $\lambda_2<\lambda_3<\lambda_4<1$ and any neighborhood U of Λ, there exists a hyperbolic periodic orbit $Orb(q)$ of f of index i contained entirely in U with a point $q\in Orb(q)$ such that

$$\prod_{j=0}^{m-1}\|Df^l|_{E^{cs}(f^{jl}q)}\|\leq \lambda_4^m, \quad for\ m=1,\cdots,\ \pi_l(q) \tag{4.11}$$

$$\prod_{j=\pi_l(q)-m}^{\pi_l(q)-1}\|Df^l|_{E^{cs}(f^{jl}q)}\|\geq \lambda_3^m \quad for\ m=1,\cdots,\ \pi_l(q) \tag{4.12}$$

where $\pi_l(q)$ is the period of q for the map f^l. The similar assertion for F^{cu} holds respecting f^{-1}.

Remark 4.19. *It's easy to know $\pi(q)\geq\pi_l(q)$. Since $f^{l\cdot\pi_l(q)}(q)=q$, it's obvious that (4.11) and (4.12) are true for all $m\in\mathbb{N}$. In the selecting lemma, when λ_3 and λ_4 are fixed, we can indeed find a sequence of periodic points $\{q_n\}$ satisfying (4.11) and (4.12) and $\overline{\lim_{n\to\infty}}Orb(q_n)\subset\Lambda$. If f is a Kupuka-Smale diffeomorphism, especially when $f\in R$, we can let $\lim_{n\to\infty}\pi_l(q_n)\longrightarrow\infty$, then we'll have $\lim_{n\to\infty}\pi(q_n)\longrightarrow\infty$ at the same time.*

Corollary 4.20. *$f\in R$, let Λ be a compact chain recurrent set of f with index $i-(l_0,\lambda)$ dominated splitting $E^{cs}\oplus F^{cu}$ $(1\leq\ i\leq\ d-1)$. Assume that the splitting satisfies all the conditions of selecting lemma for all $l_n=nl_0$ $(n\in\mathbb{N})$ but with the same parameters $\lambda<\lambda_1<\lambda_2<1$, then for any sequence*

$\{(\lambda_{n,3},\ \lambda_{n,4})\}_{n=1}^{\infty}$ *satisfying* $\lambda_2 < \lambda_{1,3} < \lambda_{1,4} < \lambda_{2,3} < \lambda_{2,3} < \cdots$ *where* $\lambda_{n,3} \longrightarrow 1^-$, *there exists a family of periodic points* $\{q_n(f)\}$ *with index* i *such that*

a) $\lim\limits_{n\to\infty} \pi_{l_n}(q_n(f)) \longrightarrow \infty$.

b)

$$\prod_{j=0}^{m-1} \|Df^{l_n}|_{E^s(f^{jl_n}q_n)}\| \leq \lambda_{n,4}^m \tag{4.13a}$$

$$\prod_{j=\pi_{l_n}(q_n)-m}^{\pi_{l_n}(q_n)-1} \|Df^{l_n}|_{E^s(f^{jl_n}q_n)}\| \geq \lambda_{n,3}^m \quad \textit{for } m \in \mathbb{N} \tag{4.13b}$$

c) $\overline{\lim\limits_{n\to\infty}} Orb(q_n) \subset \Lambda$.

d) $\Lambda \subset H(q_n(f))$ *for all* n.

Proof : At first, let's fix $\lambda_2 < \lambda_{1,3} < \lambda_{1,4} < 1$ and a small neighborhood U of Λ small enough such that the maximal invariant set $\widetilde{\Lambda}$ of $\overline{U}$ has index $i-(l_0,\widetilde{\lambda})$ dominated splitting with $\widetilde{\lambda} < \lambda_2$, we denote the dominated splitting still by $E_i^{cs} \oplus F_{i+1}^{cu}$. (If q is an index i periodic point in $\widetilde{\Lambda}$, then we denote $E_i^{cs} \oplus F_{i+1}^{cu}|_{Orb(q)} = E^s \oplus F^u|_{Orb(q)}$). Now using selecting lemma, with remark 4.19, we can find a family of periodic points $\{q_{1,m}(f)\}_{m=1}^{\infty}$ with index i satisfying b), $\overline{\lim\limits_{n\to\infty}}(q_{1,m}) \subset \Lambda$, $\lim\limits_{m\to\infty} \pi_{l_1}(q_{1,n}) \longrightarrow \infty$ and $Orb(q_{1,m}(f)) \subset \widetilde{\Lambda}$.

Since $\widetilde{\Lambda}$ has an index $i-(l_1,\widetilde{\lambda})$ dominated splitting $E_{\widetilde{\Lambda}}^{cs} \oplus F_{\widetilde{\Lambda}}^{cu}$, from (4.13b) we can know

$$\prod_{j=\pi_{l_1}(q_{1,m})-t+1}^{\pi_{l_1}(q_{1,m})} \|Df^{-l_1}|_{F^{cu}(f^{jl_1}q_{1,m})}\| \leq \widetilde{\lambda}^t / \prod_{j=\pi_{l_1}(q_{1,m})-t}^{\pi_{l_1}(q_{1,m})-1} \|Df^{l_1}|_{E^{cs}(f^{jl_1}q_{1,m})}\| \leq (\frac{\widetilde{\lambda}}{\lambda_{1,3}})^t \text{ for } {\scriptstyle (t\in\mathbb{N})},$$

it equivalent with

$$\prod_{t=0}^{m-1} \|Df^{-l_1}|_{F^{cu}(f^{-jl_1}q_{1,m})}\| \leq (\frac{\widetilde{\lambda}}{\lambda_{1,3}})^t \quad \text{for } t \in \mathbb{N}. \tag{4.14}$$

From (4.13a), (4.13b), by lemma 4.13, Corollary 4.14 and $\frac{\widetilde{\lambda}}{\lambda_{1,3}} < 1$, we can know that for some ε_1, $q_{1,n}$ will have uniformly size of stable manifold $W_{\varepsilon_1}^s(q_{1,n})$ and uniform size of unstable manifold $W_{\varepsilon_1}^u(q_{1,n})$ and there exists a subsequence $\{q_{1,n_j}\}_{j=1}^{\infty}$ such that they are homoclinic related with each other, so $H(q_{1,n_1}) = H(q_{1,n_2}) = \cdots$, with $\overline{\lim\limits_{j\to\infty}} Orb(q_{1,n_j}) \subset \Lambda$, we know $\Lambda \bigcap H(q_{1,n_j}) \neq \phi$. Since $f \in R$, $H(q_{1,n_j})$ should be a chain recurrent class. Because Λ is a chain recurrent set, we have $\Lambda \subset H(q_{1,n_j})$, let $q_1 = q_{1,n_j}$ for some j big enough, then q_1 satisfies $a)$, $d)$.

Now consider $0 < \lambda_2 < \lambda_{2,3} < \lambda_{2,4} < 1$, $E_\Lambda^{cs} \oplus F_\Lambda^{cu}$ is obviously an index $i-(l_2,\lambda)$ dominated splitting of Λ and by the assumption, the splitting satisfy the conditions of selecting lemma for l_2, $\lambda < \lambda_1 < \lambda_2 < 1$, so repeat the above argument, we can get a family of periodic points $\{q_{2,n}(f)\}_{n=1}^{\infty}$ satisfying $b)$, $d)$, $\overline{\lim\limits_{n\to\infty}} Orb(q_{2,n}) \subset \Lambda$, $\Lambda \subset H(q_{2,1},f) = \cdots = H(q_{2,n},f) = \cdots$ and $\lim\limits_{n\to\infty} \pi_{l_2}(q_{2,n}(f)) \longrightarrow \infty$. When n_0 is big enough, we'll have $\pi_{l_2}(q_{2,n_0}) > \pi_{l_1}(q_1)$ and $Orb(q_{2,n_0})$ is near Λ more than $Orb(q_1)$. Let $q_2 = q_{2,n_0}$, continue the above argument for l_n and $\lambda_2 < \lambda_{n,3} < \lambda_{n,4} < 1$, we can get $\{q_n\}_{n=1}^{\infty}$ which we need. $\square$

The following weakly selecting lemma shows when the conditions of the above corollary will be satisfied.

Lemma 4.21. *(Weakly selecting lemma) Let $f \in R$, Λ be a compact invariant set of f with index $i-(l_0,\ \lambda)$ dominated splitting $E^{cs} \oplus F^{cu}$ $(1 \le i \le d-1)$. Assume that*

a) *(Non-hyperbolic condition) the bundle E^{cs} is not contracting,*
b) *(Strong tilda condition) there are $\lambda_2 < 1$ and $l'_0 > 1$ such that for any $x \in \Lambda$, $\omega(x)$ contains a point $c \in \Lambda$ satisfying $\prod_{j=0}^{n-1} \|Df^{l'_0}|_{E^{cs}(f^{jl'_0}c)}\| \le \lambda_2^n$ for all $n \ge 1$.*

Then for any $l_n = n \cdot (l_0 \cdot l'_0)$ and any sequence $\{(\lambda_{n,3}, \lambda_{n,4})\}_{n=1}^{\infty}$ satisfying $\max\{\lambda^{l'_0}, \lambda_2\} < \lambda_{1,3} < \lambda_{1,4} < \cdots < \lambda_{n,3} < \lambda_{n,4} < \cdots$ where $\lambda_{n,3} \longrightarrow 1^-$, there exists a family of periodic points $\{q_n(f)\}$ with index i such that

- $\lim_{n\to\infty} \pi_{l_n}(q_n(f)) \longrightarrow \infty$
- $\prod_{j=0}^{m-1} \|Df^{l_n}|_{E^s(f^{jl_n}q_n)}\| \le \lambda_{n,4}^m$ *and* $\prod_{j=\pi_{l_n}(q_n)-m}^{\pi_{l_n}(q_n)-1} \|Df^{l_n}|_{E^s(f^{jl_n}q_n)}\| \ge \lambda_{n,3}^m$ *for $m \ge 1$*
- $\overline{\lim_{n\to\infty}} Orb(q_n) \subset \Lambda$
- $\Lambda \subset H(q_n(f))$ *for $n \ge 1$.*

Proof Since $E^{cs}_\Lambda \oplus F^{cu}_\Lambda$ is a (l_0, λ) dominated splitting and $l_1 = l_0 \cdot l'_0$, it should be a $(l_1, \lambda^{l'_0})$ dominated splitting also. Choose $\lambda'_2,\ \lambda_1$ such that $\max\{\lambda^{l'_0},\ \lambda_2\} < \lambda_1 < \lambda'_2 < \lambda_{1,3}$, we'll show that the splitting $E^{cs}_\Lambda \oplus F^{cu}_\Lambda$ and the $l_1,\ \lambda^{l'_0} < \lambda_1 < \lambda'_2 < 1$ will satisfy all conditions of corollary 4.20, equivalent, we'll show the splitting $E^{cs}_\Lambda \oplus F^{cu}_\Lambda$, l_n and $\lambda^{l'_0} < \lambda_1 < \lambda'_2 < 1$ will satisfy the condition of selecting lemma for all $n \ge 1$.

0) Since $E^{cs}_\Lambda \oplus F^{cu}_\Lambda$ is a $(l_1,\ \lambda^{l'_0})$ dominated splitting and $l_n = n \cdot l_1$, $E^{cs}_\Lambda \oplus F^{cu}_\Lambda$ is a $(l_n,\ \lambda^{l'_0})$ dominated splitting also.
1) Here we need the following lemma:

   Lemma 4.22. *Let Λ be a compact invariant set of f, E^{cs}_Λ is an continuous invariant bundle on Λ, and $dim(E^{cs}(x)) = i$ for any $x \in \Lambda$ where $i \ne 0$, suppose $l \in \mathbb{N}$, if for any $x \in \Lambda$, there exists an n such that $\prod_{j=0}^{n-1} \|Df^l|_{E^{cs}(f^{jl}x)}\| < 1$, then E^{cs}_Λ is a contracting bundle.*

   Since we know E^{cs}_Λ is continuous but not contracting, so for any l_n, there exists b_n, such that $\prod_{j=0}^{n-1} \|Df^l_n|_{E^{cs}(f^{jl_n}b_n)}\| \ge 1$ for all $m \ge 1$.
2) For any $x \in \Lambda$, $\omega(x)$ contains a point $c_n \in \Lambda$ such that $\prod_{j=0}^{nl_0m-1} \|Df^{l'_0}|_{E^{cs}(f^{jl'_0}c_n)}\| \le \lambda_2^{nl_0m}$ for all $m \ge 1$, since
$$\prod_{j=0}^{nl_0m-1} \|Df^{l'_0}|_{E^{cs}(f^{jl'_0}c_n)}\| \ge \prod_{j=0}^{m-1} \|Df^{nl_0l'_0}|_{E^{cs}(f^{jnl_0l'_0}c_n)}\| = \prod_{j=0}^{m-1} \|Df^{l_n}|_{E^{cs}(f^{jl_n}c_n)}\|,$$
   we have that $\prod_{j=0}^{m-1} \|Df^{l_n}|_{E^{cs}(f^{jl_n}c_n)}\| \le \lambda_2^{mnl_0} \le \lambda_2^m$ for all $m \ge 1$.

Remark 4.23. *In b) of weakly selecting lemma, we don't give any restriction on x, so b) is in fact more stronger than the tilda condition, that's why we call the condition b) in weakly selecting lemma the strong tilda condition.*

By 0), 1), 2) above and corollary 4.20, we proved the lemma. □

4.3. **Transition.** Transition was introduced in [6] at first, there they consider a special linear system with a special property called transition and use it to study homoclinic class. Here I prefer to use a little different way to state it, the notation and definition are basically copy from [6]. The main result in this subsection is corollary 4.26. We begin by giving some definitions.

Given a set $\mathcal{A}$, a word with letters in $\mathcal{A}$ is a finite sequence of $\mathcal{A}$, its length is the number of letters composing it. The set of words admits a natural semi-group structure: the product of the word $[a] = (a_1, \cdots, a_n)$ by $[b] = (b_1, \cdots, b_l)$ is $[a] \cdot [b] = (a_1, \cdots, a_n, b_1, \cdots, b_l)$. We say that a word $[a]$ is not a power if $[a] \neq [b]^k$ for every word $[b]$ and $k > 1$.

Here we'll use some special words. Let's fix $f \in C^1(M)$, for any $x \in Per(f)$, we write $[x] = (f^{\pi(x)-1}(x)), \cdots, x)$ and $\{x\} = (Df(f^{\pi(x)-1}(x)), \cdots, Df(x))$. We call a word $[a] = (a_k, \cdots, a_1)$ with letters in M is a finite ε-pseudo orbit if $d(f(a_i), a_{i+1}) \leq \varepsilon$ for $1 \leq i \leq k-1$, if $\varepsilon = 0$, that means $f(a_i) = a_{i+1}$ for $1 \leq i \leq k-1$, then we call $[a]$ is a finite segment of orbit. We always denote $\{a\} = (Df(a_k), \cdots, Df(a_1))$.

Suppose we have a finite orbit $[a] = (a_n, \cdots, a_1)$ and an ε-pseudo orbit $[b] = (b_l, \cdots, b_1)$, we say $[b]$ is δ-shadowed by $[a]$ if $n = l$ and $d(a_i, b_i) \leq \varepsilon$ for $1 \leq i \leq n$. We say $\{a\}$ is δ-close to $\{b\}$ if $n = l$ and $\|Df(a_i) - Df(b_i)\| \leq \delta$ for $1 \leq i \leq n$.

Suppose $H(p, f)$ is a non-trivial homoclinic class, we say $H(p, f)$ has ε-transition property if : for any finite hyperbolic periodic points $p_1, \cdots, p_n$ in $H(p, f)$ which are homoclinic related with each other, there exist finite orbits $[t^{i,j}] = (t^{i,j}_{k(i,j)}, \cdots, t^{i,j}_1)$ for any $(i, j) \in \{1, \cdots, n\}^2$ where $k(i, j)$ is the length of $[t^{i,j}]$, such that, for every $m \in \mathbb{N}$, $l = (i_1, \cdots, i_m) \in \{1, \cdots, n\}^m$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m$ where the word $((i_1, \alpha_1), \cdots, (i_m, \alpha_m))$ with letters in $\mathbb{N} \times \mathbb{N}$ is not a power, the pseudo orbit $[w(l, \alpha)] = [t^{i_m,i_1}] \cdot [p_{i_m}]^{\alpha_m} \cdot [t^{i_{m-1},i_m}] \cdot [p_{i_{m-1}}]^{\alpha_{m-1}} \cdot \cdots \cdot [t^{i_1,i_2}] \cdot [p_{i_1}]^{\alpha_1}$ is an ε-pseudo orbit and there is a periodic orbit $Orb(q(l, \alpha)) \subset H(p, f)$ such that:

a) the length of $[w(l, \alpha)]$ is $\pi(q(l, \alpha))$ and $[q(l, \alpha)]$ ε-shadow the pseudo orbit $[w(l, \alpha)]$.
b) the word $\{q(l, \alpha)\}$ is ε-close to $\{w(l, \alpha)\}$.
c) there exists a word $\{\widetilde{t}^{t_j,t_{i+1}}\} = (T^{i_j,i_{j+1}}_{k(i_j,i_{j+1})}, \cdots, T^{i_j,i_{j+1}}_1)$ with letters in $GL(R^d)$ ε close to $\{t^{i_j,t_{j+1}}\}$, let $T^{i_j,i_{j+1}} = T^{i_j,i_j+1}_{k(i_j,i_j+1)} \cdot \cdots \cdot T^{i_j,i_j+1}_1$, then
$$T^{i_j,i_{j+1}}(E^s(q_{i_j})) = E^s(q_{i_{j+1}}), \quad , T^{i_j,i_{j+1}}(E^u(q_{i_j})) = E^u(q_{i_{j+1}}).$$

We say $H(p, f)$ has transition property if $H(p, f)$ has ε-transition property for any $\varepsilon > 0$.

Lemma 4.24. *([6]) $f \in C^1(M)$, suppose p is an index i $(i \neq 0, d)$ hyperbolic periodic point of f, then $H(p, f)$ has transition property.*

Lemma 4.25. *$f \in R$, suppose p is an index i $(i \neq 0, d)$ hyperbolic periodic point of f and $H(p, f)$ is not trivial. Suppose there exists a family of periodic point $\{p_n\}_{n=1}^{\infty}$ with index i in $H(p, f)$ homoclinic related with p and $l_n \longrightarrow \infty$, $\lambda_n \longrightarrow 1^-$ such that $\pi_{l_n}(p_n) \longrightarrow \infty$ and $\prod_{j=0}^{\pi_{l_n}(p_n)-1} \|Df^{l_n}|_{E^s(f^{jl_n}(p_n))}\| \geq \lambda_n^{\pi_{l_n}(p_n)}$, then $H(p, f)$ is an index $i-1$ fundamental limit.*

Proof : We claim that we can find $q_n(g_n)$ is periodic point of g_n with index i such that:

1) $\lim_{n\to\infty} g_n = f$.
2) $Orb_{g_n}(q_n)$ is periodic orbit of f also ($f|_{Orb_{g_n}(q_n)} = g_n|_{Orb_{g_n}(q_n)}$), so we just denote it $Orb(q_n)$, then we have $Orb(q_n) \subset H(p,f)$ and $\lim_{n\to\infty} Orb(q_n) = H(p,f)$.
3) $\lim_{n\to\infty} \frac{\pi(q_n)}{l_n} \longrightarrow \infty$
4) $\prod_{j=0}^{[\frac{\pi(q_n)}{l_n}]-1} \|Dg_n^{l_n}|_{E^s_{g_n}(g_n^{jl_n}(q_n))}\| \geq \lambda_n^{[\frac{\pi(q_n)}{l_n}]}$

Proof of the claim: Choose $\varepsilon_n \longrightarrow 0^+$, let's fix n_0 at first and choose an $\varepsilon > 0$ such that $\lambda_{n_0} + 2\varepsilon < 1$. There exists $N_0 \gg n_0$ such that for any $n \geq N_0$, we'll have $l_n \gg l_{n_0}$ and $\lambda_n > \lambda_{n_0} + 2\varepsilon$, then by $\prod_{j=0}^{\pi_{l_n}(p_n)-1} \|Df^{l_n}|_{E^s(f^{jl_n}p_n)}\| \geq \lambda_n^{\pi_{l_n}(p_n)}$, we have $\prod_{j=0}^{ml_{n_0}\pi_{l_n}(p_n)-1} \|Df^{l_n}|_{E^s(f^{jl_n}p_n)}\| \geq \lambda_n^{ml_{n_0}\pi_{l_n}(p_n)}$ for $m \geq 1$, then we get

$$\prod_{j=0}^{ml_n\pi_{l_n}(p_n)-1} \|Df^{l_{n_0}}|_{E^s(f^{jl_{n_0}}p_n)}\| \geq (\lambda_{n_0}+2\varepsilon)^{ml_{n_0}\pi_{l_n}(p_n)} \text{ for } m \geq 1. \tag{4.15}$$

Since $f \in R$, there exists a family of periodic points $\{q_i'\}_{i=1}^N$ with index i, which are ε_{n_0}-dense in $H(p,f)$ and they are homoclinic related with p and $\{p_n\}_{n=1}^\infty$. Now use ε_{n_0}-transition property for $\{q_0'(= p_{N_0}), q_1', \cdots, q_N'\}$, then for $\{i,j\} \in \{0,1,\cdots,N\}^2$, there exists finite orbit $[t^{i,j}] = (t^{i,j}_{k(i,j)}, \cdots, t^{i,j}_1)$ such that for $l = (0,1,\cdots,N)$ and $\alpha_m = (m\cdot l_{n_0}, 1, \cdots, 1)$, the pseudo orbit $[w(l,\alpha_m)] = [t^{N,0}]\cdot[q_N']\cdot\cdots\cdot [t^{0,1}]\cdot[q_0']^{m\cdot l_{N_0}\frac{l_{N_0}\cdot\pi_{l_{N_0}}(p_{N_0})}{\pi(p_{N_0})}}$ is an ε_{n_0}-pseudo orbit and is ε_{n_0}-shadowed by periodic orbit $[q(l,\alpha_m)]$ whose index is i, where $Orb(q(l,\alpha_m))) \subset H(p,f)$ and $\{q(l,\alpha_m)\}$ is ε_{n_0}-near $\{w(l,\alpha_m)\}$.

Consider the word $\{\widetilde{w}(l,\alpha_m)\} = \{\widetilde{t}^{N,0}\}\cdot\{q_N'\}\cdots\cdot\{\widetilde{t}^{N,0}\}\cdot\{q_0'\}^{ml_0}$, it's ε_{n_0} near $\{w(l,\alpha_m)\}$, so $\{\widetilde{w}(l,\alpha_m)\}$ is $2\varepsilon_{n_0}$ near with $\{q(l,\alpha_m)\}$. Now use lemma 4.17 (Franks lemma), we can get a C^1 diffeomorphism $g_{(l,\alpha_m)}$ such that $d(g_{(l,\alpha_m)}, f) < 2\varepsilon_{n_0}$, $Orb_f(q(l,\alpha_m))$ is also orbit of $g_{(l,\alpha_m)}$, and $\{Dg_{(l,\alpha_m)}|_{Orb(q(l,\alpha_m))}\} = \{\widetilde{w}(l,\alpha_m)\}$. By c) of transition property, $E_f^{s(u)}(q_0')$ is invariant bundle of $\{\widetilde{w}(l,\alpha_m)\}$, so they are invariant bundle of g_{l,α_m}, that means $Dg_{(l,\alpha_m)}^{\pi(q(l,\alpha_m))}(E_f^s(q_0')) = E_f^s(q_0')$ and $Dg_{(l,\alpha_m)}^{\pi(q(l,\alpha_m))}(E_f^u(q_0')) = E_f^u(q_0')$. It's easy to know when m is big enough, $E_f^{s(u)}(q_0')$ is stable(unstable) bundle for $g_{(l,\alpha_m)}$, so when m is big enough, $q_{(l,\alpha_m)}$ would be an index i hyperbolic periodic point of $g_{(l,\alpha_m)}$.

Now choose m big enough and let $q_{n_0} = q(l,\alpha_m)$, $g_{n_0} = g_{(l,\alpha_m)}$, it's easy to know q_{n_0}, g_{n_0} satisfy 1), 2). About 3), let's notice that $\pi(q_n) \geq ml_{n_0}$ and m can be chosen arbitrary big. 4) comes from (4.15) and m is big enough. □

Now let's continue the proof of lemma 4.25, by the above claim and lemma 4.16, for any $\varepsilon > 0$ and $N > 0$, there exist an $n_0 > N$ and g_{n_0}' is ε-perturbation of g_{n_0} such that $Orb(q_{n_0})$ is index $i-1$ periodic orbit of g_{n_0}' and $Orb(q_{n_0})$ is ε_{n_0}-dense in $H(p,f)$. Since ε and ε_{n_0} can be arbitrarily small, we get that $\lim_{n\to\infty} g_{n_j}' = f$, $Orb(q_{n_j})$ is index $i-1$ periodic orbit of g_{n_j}' and $\lim_{j\to\infty} Orb(q_{n_j}) = H(p,f)$, so $H(p,f)$ is an index $i-1$ fundamental limit. □

Then main result of this subsection is the following corollary.

Corollary 4.26. *$f \in R$, C is a chain recurrent class of f, Λ is compact invariant set of f with index $i-(l,\lambda)$ dominated splitting $E^{cs} \oplus F^{cu}$ ($1 \leq i \leq d$) and assume they satisfy all the assumption of weakly selecting lemma, then C contains index i periodic point and C is an index $i-1$ funadamental limit.*

Proof : It's just a corollary from Lemma 4.21 (weakly selecting lemma) and lemma 4.25. □

4.4. **Proof of lemma 4.3. Proof** : When Λ is trivial ($\#(\Lambda) < \infty$), Λ is a periodic orbit, since Λ is an index j_0-fundamental limit, it should be an index j_0 hyperbolic periodic orbit, so C contains an index j_0 periodic point and it's an index j_0 fundamental limit.

Now we suppose Λ is not trivial, by generic property 5 of proposition 3.1, there exists a family of index j_0 periodic points $\{p_n(f)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} Orb(p_n(f)) = \Lambda$. Since Λ is not trivial, we have $\pi(p_n(f)) \longrightarrow \infty$.

If Λ isn't an index j_0+1 fundamental limit, we know that $\{p_n(f)\}$ is index j_0 stable, then by lemma 4.6 (Gan's lemma), there exits a subsequence $\{p_{n_i}(f)\}_{i=1}^{\infty}$ such that $p_{n_i}(f)$ and $p_{n_j}(f)$ are homoclinic related, so $H(p_{n_1}, f) = H(p_{n_2}, f) = \cdots$, especially, by $\lim_{n\to\infty} Orb(p_n(f)) = \Lambda$, we know that $\Lambda \subset H(p_{n_1}, f)$, by generic property 6) of proposition 3.1, $C = H(p_{n_1}, f)$, so C contains index j_0 periodic point and it's an index j_0 fundamental limit.

So from now, we suppose Λ is an index $j_0 + 1$ fundamental limit also, then $\Lambda \subset P^*_{j_0} \bigcap P^*_{j_0+1}$, since f is far away from tangency, by proposition 2.1, Λ has an index j_0 dominated splitting $E^{cs}_{j_0}(\Lambda) \oplus E^{cu}_{j_0+1}(\Lambda)$ and an index $j_0 + 1$ dominated splitting $E^{cs}_{j_0+1}(\Lambda) \oplus E^{cu}_{j_0+2}(\Lambda)$. Let $E^c_1(\Lambda) = E^{cu}_{j_0+1}(\Lambda) \bigcap E^{cs}_{j_0+1}(\Lambda)$, then on Λ we have the following dominated splitting: $T|_\Lambda M = E^{cs}_{j_0}(\Lambda) \oplus E^c_1(\Lambda) \oplus E^{cu}_{j_0+2}(\Lambda)$. Since $C \bigcap P^*_j = \phi$ for $j < j_0$, by proposition 2.2, $E^{cs}_{j_0}$ is in fact contracting, so we prefer denoting it $E^s_{j_0}$. Now on Λ we have the dominated splitting $T|_\Lambda M = E^s_{j_0}(\Lambda) \oplus E^c_1(\Lambda) \oplus E^{cu}_{j_0+2}(\Lambda)$.

Remark 4.27. *Since Λ is index j_0 fundamental limit, $E^c_1(\Lambda)$ is not contracting, that means that the bundle $(E^s_{j_0} \oplus E^c_1)|_\Lambda$ is not contracting also.*

When $j_0 + 1 = d$, especially, the dominated splitting on Λ should be $T|_\Lambda M = E^s_{j_0}(\Lambda) \oplus E^c_1(\Lambda)$. In this case, if Λ is not minimal, there exists an $x_0 \in \Lambda$ such that $\omega(x_0) \subsetneq \Lambda$. By the definition of Λ and $j_0 = d - 1$, $\omega(x_0)$ is an index d fundamental limit but not index j fundamental limit for $j < d$. With the generic property (5) of proposition 3.1, $\omega(x_0)$ can be converged by a family of sinks $\{p_n(f)\}$, by remark 4.4, $\pi(p_n(f))$ should be bounded (If it's not bounded, there exist $p_{n_0}(f)$ and $g_{n_0} \overset{C^1}{\sim} f$ such that $g_{n_0}|_{Orb_f(p_{n_0}(f))} = f|_{Orb_f(p_{n_0}(f))}$ and $Orb(p_{n_0}(f))$ is a periodic orbit of g with index smaller than d, that means $\omega(x_0)$ is an fundamental limit with index smaller than d, it's a contradiction). That means $\omega(x_0)$ is trivial, so it's a periodic orbit. Since f is a Kupuka-Smale diffeomorphism and $\omega(x_0)$ is an index d fundamental limit, we can know that $\omega(x_0)$ is an index d hyperbolic periodic orbit, then C contains a sink, it means C itself is just the orbit of sink and $C = \omega(x_0)$, that's a contradiction with C is not trivial, so we proved Λ is minimal when $j_0 + 1 = d$.

Now we just consider $j_0 + 1 < d$, we claim that with all the assumptions above on Λ, then either Λ is minimal, or C contains periodic points with index $j_0 + 1$ and C is an index j_0 fundamental limit.

Proof of claim: Suppose Λ is not minimal, it means that there exists $x_0 \in \Lambda$ such that $\omega(x_0) \neq \Lambda$. Consider the set of compact chain recurrent subset of Λ: $\{\Lambda_\alpha : \ \Lambda_\alpha \subsetneq \Lambda\}_{\alpha\in\mathcal{A}}$, since $\omega(x_0) \in \{\Lambda_\alpha\}_{\alpha\in\mathcal{A}}$, $\mathcal{A} \neq \phi$, by generic property (4) of proposition 3.1, Λ_α is a fundamental limit. By the definition of j_0 and Λ, Λ_α is an index j_α fundamental limit with $j_\alpha \geq j_0 + 1$. Denote $\mathcal{B}$=$\{\beta \in \mathcal{A}$, Λ_β does not contain index j fundamental limit for $j > j_0 + 1\}$.

Remark 4.28. *: For any $\beta \in \mathcal{B}$, Λ_β is an index j_0+1 fundamental limit, on Λ_β we have an index j_0+1 dominated splitting $E^{cs}_{j_0+1}(\Lambda_\beta) \oplus E^{cu}_{j_0+2}(\Lambda_\beta)$. Since we have Λ_β does not contain any index j fundamental limit for all $j \neq j_0+1$, by proposition 2.2, the index j_0+1 dominated splitting is in fact a hyperbolic splitting, that means Λ_β is a hyperbolic set.*

Now we divide the proof of the claim to three subcases: $\#(B)=0$, $\#(B)=N_1<\infty$ and $\#(B)=\infty$.

Case A: $\#(B)=0$.

That means for all $\alpha \in \mathcal{A}$, Λ_α contains an index j_α fundamental limit Λ^*_α for some $j_\alpha > j_0+1$.

Now we need the following two results.

Lemma 4.29. *([45]) Assume $f \in R$, let Λ be an index i fundamental limit of f $(1 \leq i \leq d-1)$, $E^{cs}_i(\Lambda) \oplus E^{cu}_{i+1}(\Lambda)$ is an index $i-(l,\lambda)$ dominated splitting on Λ given by proposition 2.1, then*

1) *either for any $\mu \in (\lambda,1)$, there exists $c \in \Lambda$ such that $\prod_{j=0}^{n-1} \|Df^l|_{E^{cs}_i(f^{jl}c)}\| \leq \mu^n$ for $n \geq 1$,*
2) *or E^{cs}_i splits into a dominated splitting $V^{cs}_{i-1} \oplus V^c_1$ with $dim(V^c_1)=1$ such that for any $\mu \in (\lambda,1)$, there is $c' \in \Lambda$ such that $\prod_{j=0}^{n-1} \|Df^l|_{V^{cs}_{i-1}(f^{jl}c')}\| \leq \mu^n$ for all $n \geq 1$.*

Lemma 4.30. *Let Λ be an invariant compact set of f, with two dominated splitting $E^{cs} \oplus F^{cu}$ and $\widetilde{E}^{cs} \oplus \widetilde{F}^{cu}$, if $dim(E^{cs}) \leq dim(\widetilde{E}^{cs})$, then $E^{cs} \subset \widetilde{E}^{cs}$.*

Choose $\mu_0 \in (\lambda,1)$, since Λ^*_α is an index j_α fundamental limit, proposition 2.1 gives an index $j_\alpha-(l,\lambda)$ dominated splitting $E^{cs}_{j_\alpha} \oplus F^{cu}_{j_\alpha+1}$ on Λ^*_α.

If 1) of lemma 4.29 is true for Λ^*_α, then there exists $c \in \Lambda^*_\alpha$ such that $\prod_{j=0}^{n-1} \|Df^l|_{E^{cs}_{j_\alpha}(f^{jl}c)}\| \leq \mu_0^n$ for $n \geq 1$. On Λ^*_α we have another dominated splitting $(E^s_{j_0} \oplus E^c_1) \oplus E^{j_0+2}_{cu}$ induced from Λ. Since $dim(E^s_{j_0} \oplus E^c_1) = j_0+1 < j_\alpha = dim(E^{cs}_{j_\alpha})$, be lemma 4.30, $E^s_{j_0} \oplus E^c_1 \subset E^{cs}_{j_\alpha}$, so we have $\prod_{j=0}^{n-1} \|Df^l|_{E^s_{j_0} \oplus E^c_1(f^{jl}c)}\| \leq \mu_0^n$ for $n \geq 1$.

If 2) of lemma 4.29 is true for Λ^*_α, then there exists c' such that $\prod_{j=0}^{n-1} \|Df^l|_{V^{cs}_{j_\alpha-1}(f^{jl}c')}\| \leq \mu_0^n$ for $n \geq 1$, recall that $dim(E^s_{j_0} \oplus E^c_1) = j_0+1 \leq j_\alpha-1 = dim(V^{cs}_{j_\alpha-1})$, by lemma 4.30, $E^s_{j_0} \oplus E^c_1 \subset V^{cs}_{j_\alpha-1}(\Lambda^*_\alpha)$, so we have $\prod_{j=0}^{n-1} \|Df^l|_{E^s_{j_0} \oplus E^c_1(f^{jl}c')}\| \leq \mu_0^n$ for $n \geq 1$.

Remark 4.31. *: By the above arguments, we know that for any $\alpha \in \mathcal{A} \setminus B$, and for any $\mu_0 \in (\lambda,1)$, there exists $c \in \Lambda_\alpha$ such that*

$$\prod_{j=0}^{n-1} \|Df^l|_{E^s_{j_0} \oplus E^c_1(f^{jl}c)}\| \leq \mu_0^n \quad \text{for } n \geq 1. \tag{4.16}$$

By remark 4.27and remark 4.31, the index $j_0+1-(l,\lambda)$ dominated splitting $(E^s_{j_0} \oplus E^c_1) \oplus E^{cu}_{j_0+2}$ on Λ satisfies all the conditions of weakly selecting lemma, by corollary 4.26, C contains index j_0+1 periodic point and C is an index j_0 fundamental limit.

Case B: $\#(B) = N_1 < \infty$

Let $\mathcal{B}$=$\{\beta_1, \cdots, \beta_{N_1}\}$, fix $\lambda < \mu_0 < 1$, then by the argument in case A, for any $\beta \in \mathcal{A} \setminus B$, there exists $c \in \Lambda$ satisfies (4.16).

For $\beta_i \in \mathcal{B}$, Λ_{β_i} should be a hyperbolic set where the bundle $E^s_{j_1} \oplus E^c_1|_{\Lambda_{\beta_i}}$ is a contracting bundle, so there exists l' such that for any $x \in \Lambda_{\beta_i}$, $\|Df^{l'}|_{(E^s_{i_0} \oplus E^c_1)(x)}\| < 1/2$.

Let $l_0 = l \cdot l'$ and $1 > \mu_1 > \max\{\mu_0, \frac{1}{2}\}$, then for any Λ_α ($\alpha \in \mathcal{A}$), there exists a point $c \in \Lambda_\alpha$ such that $\prod_{j=0}^{n-1} \|Df^{l_0}|_{E^s_{j_0} \oplus E^c_1(f^{jl_0}c)}\| \le \mu_1^n$. With remark 4.27, the index $j_0 + 1 - (l, \lambda)$ dominated splitting $(E^s_{j_0} \oplus E^c_1) \oplus E^{cu}_{j_0+2}$ on Λ satisfies all the conditions of weakly selecting lemma, by corollary 4.26, C contains index $j_0 + 1$ periodic point and C is an index j_0 fundamental limit.

Case C: $\#(B) = \infty$

In remark 4.28, we have shown that for any $\beta \in \mathcal{B}$, Λ_β is a hyperbolic chain recurrent set with index $j_0 + 1$. Then there exists a family of periodic points $\{p_{\beta,n}\}_{n=1}^\infty$ in C with index $j_0 + 1$ and $\lim_{n\to\infty} Orb(p_{\beta,n}) = \Lambda_\beta$ (by shadowing lemma). If Λ_β is trivial, that means it's an index $j_0 + 1$ periodic orbit, we can let $Orb(p_{\beta,n}) = \Lambda_\beta$ for $n \ge 1$; if Λ_β is not trivial, we can let $\pi(p_{\beta,n}) \longrightarrow \infty$.

We have the following two subcases.

- Subcase $C.1$: There exists $\delta > 0$ such that for any $\Lambda_\beta,\ \beta \in \mathcal{B}$, there exists a family of periodic points $\{p_{\beta,n}\}_{n=1}^\infty$ such that $\lim_{n\to\infty} Orb(p_{\beta,n}) = \Lambda_\beta$ and $\mid Df^{\pi(p_{\beta,n})}|_{E^c_1(p_{\beta,n})} \mid < e^{-\delta\pi(p_{\beta,n})}$.
- Subcase $C.2$: For any $\frac{1}{m} > 0$, there exist $\beta_m \in \mathcal{B}$ and a family of periodic points $\{p_{\beta_m,n}\}_{n=1}^\infty$ satisfying $\lim_{n\to\infty} Orb(p_{\beta_m,n}) = \Lambda_\beta$ and $\mid Df^{\pi(p_{\beta_m,n})}|_{E^c_1(p_{\beta_m,n})} \mid > e^{-\frac{1}{m}\pi(p_{\beta_m,n})}$.

In the subcase $C.1$, let's fix $1 > \mu_1 > \mu_0 > e^{-\delta}$. For $\beta \in \mathcal{B}$, recall that $dim(E^c_1(\Lambda)) = 1$ and $\mid Df^{\pi(p_{\beta,n})}|_{E^c_1(p_{\beta,n})} \mid < e^{-\delta\pi(p_{\beta,n})}$, we'll get $\prod_{i=0}^{\pi(p_{\beta,n})-1} \mid Df|_{E^c_1(p_{\beta,n})} \mid < e^{-\delta\pi(p_{\beta,n})}$, that means for any $s \ge 1$, we have $\prod_{i=0}^{s\pi(p_{\beta,n})-1} \mid Df|_{E^c_1(p_{\beta,n})} \mid < e^{-s\delta\pi(p_{\beta,n})}$ for $s \ge 1$. By lemma 4.10 (Pliss lemma) there exists $x_{\beta,n} \in Orb(p_{\beta,n})$ such that $\mid Df^s|_{E^c_1(x_{\beta,n})} \mid = \prod_{i=0}^{s-1} \mid Df|_{E^c_1(f^i(x_{\beta,n}))} \mid < \mu_0^s$ for $s \ge 1$. Suppose $\lim_{n\to\infty} x_{\beta,n} \longrightarrow c_\beta$ where $c_\beta \in \Lambda_\beta$, then $\prod_{i=0}^{s-1} \mid Df|_{E^c_1(f^i(c_\beta))} \mid < \mu_0^s$ for $s \ge 1$. Notice that $E^s_{j_0}|_\Lambda$ is dominated by $E^c_1|_\Lambda$ and $\mu_1 > \mu_0$, there exists $l' \gg 1$ doesn't depend on β such that $\prod_{i=0}^{t-1} \|Df^{l'}|_{E^c_1 \oplus E^s_{j_0}(f^{il'}(c_\beta))}\| < \mu_1^t$ for $t \ge 1$.

For $\alpha \in \mathcal{A} \setminus B$, by the argument in case A, there exists $c_\alpha \in \mathcal{A}_\alpha$ such that $\prod_{i=0}^{t-1} \|Df^{l_0}|_{E^c_1 \oplus E^s_{j_0}(f^{il_0}(c_\alpha))}\| < \mu_1^t$ for $t \ge 1$.

Let $l_1 = l' \cdot l_0$, then for any $\alpha \in \mathcal{A}$, there exists $c_\alpha \in \mathcal{A}$ such that $\prod_{i=0}^{t-1} \|Df^{l_1}|_{E^c_1 \oplus E^s_{j_0}(f^{il_1}(c_\alpha))}\| < \mu_1^t$ for $t \ge 1$. With remark 4.27, the index $j_0 + 1 - (l, \lambda)$ dominated splitting $(E^s_{j_0} \oplus E^c_1) \oplus E^u_{j_0+2}$ on Λ satisfies all the conditions of weakly selecting lemma. By Corollary 4.26, C contains index j_0+1 periodic point and C is an index j_0 fundamental limit.

In the subcase $C.2$, since Λ_{β_m} is a hyperbolic set, we can always suppose $\{p_{\beta_m,n}\}_{n=1}^{\infty}$ is homoclinic related with each other and $p_{\beta_m,n}\in C$, so C contains index j_0+1 periodic points. Now we'll show C is an index j_0 fundamental limit also.

We claim that there exists a subsequence $\{\beta_{m_t}\}_{t=1}^{\infty}\subset\{\beta_m\}$ and for every β_{m_t} there exists $p_{\beta_{m_t},n_t}\in\{p_{\beta_{m_t},n}\}_{n=1}^{\infty}$ such that $\lim\limits_{t\to\infty}\pi(p_{\beta_{m_t},n_t})\longrightarrow\infty$.

Proof of the claim: Let $\mathcal{B}_0$=$\{\beta_m:\ \Lambda_{\beta_m}$ is given in subcase $C.2$ and Λ_{β_m} is not trivial. $\}$

If $\#(\mathcal{B}_0)=\infty$, then for any $\beta_{m_t}\in\mathcal{B}_0$, by $\Lambda_{\beta_{m_t}}$ is not trivial, we'll have $\lim\limits_{n\to\infty}\pi(p_{\beta_{m_t},n})\longrightarrow\infty$, so when n is big enough, we can let $\pi(p_{\beta_{m_t},n})$ arbitrarily big.

If $\#(\mathcal{B}_0)<\infty$, then for $\beta_m\notin\mathcal{B}_0$, Λ_{β_m} is an index j_0+1 periodic orbit and $Orb(p_{\beta_m,n})\equiv\Lambda_{\beta_m}$ for $n\geq 1$. Since f is a Kupka-Smale diffeomorphism, the number of periodic points with fixed boundary of period should be finite, by the fact $\#(\mathcal{B}\setminus\mathcal{B}_0)=\infty$, there are infinite of m such that Λ_m is index j_0+1 periodic orbits, then we can choose Λ_{β_m} is an index j_0+1 periodic orbit with arbitrarily big period. $\square$

Now for simiplicity, we denote $p_{\beta_{m_t},n_t}$ by p_{β_m,n_m}.

For $\{p_{\beta_m,n_m}\}_{m=1}^{\infty}$, we have $\lim\limits_{m\to\infty}\pi(p_{\beta_m.n_m})\longrightarrow\infty$ and

$$|Df^{\pi(p_{\beta_m,n_m})}|_{E_1^c(p_{\beta_m,n_m})}|>e^{-\frac{1}{m}\pi(p_{\beta_m,n_m})}. \tag{4.17}$$

Choose $\{l_m\}_{m=1}^{\infty}$ carefully, we'll have $\lim\limits_{m\to\infty}l_m\longrightarrow\infty$, $\lim\limits_{m\to\infty}\frac{\pi(p_{\beta_m,n_m})}{l_m}\longrightarrow\infty$ and $\frac{l_m}{m}\longrightarrow 0^+$ (after replacing $\{p_{\beta_m,n_m}\}_{m=1}^{\infty}$ by a subsequence, we can always do this). Since $\pi_{l_m}(p_{\beta_m,n_m})\geq\frac{\pi(p_{\beta_m,n_m})}{l_m}$, we have

$$\lim_{m\to\infty}\pi_{l_m}(p_{\beta_m,n_m})\longrightarrow\infty. \tag{4.18}$$

By (4.17) and the fact $l\cdot\pi_l(p)$ is always a multiple of $\pi(p)$ for any period point p and $l\geq 1$, we have

$$|Df^{l_m\cdot\pi_{l_m}(p_{\beta_m,n_m})}|_{E_1^c(p_{\beta_m,n_m})}|>e^{-\frac{1}{m}l_m\cdot\pi_{l_m}(p_{\beta_m,n_m})},$$

it's equivalent with

$$\prod_{i=0}^{\pi_{l_m}(p_{\beta_m,n_m})-1}\|Df^{l_m}|_{E_1^c(f^{il_m}(p_{\beta_m,n_m}))}\|\geq e^{-\frac{l_m}{m}\cdot\pi_{l_m}(p_{\beta_m,n_m})},$$

then we get

$$\prod_{i=0}^{\pi_{l_m}(p_{\beta_m,n_m})-1}\|Df^{l_m}|_{(E_1^c\oplus E_{j_0}^s)(f^{il_m}(p_{\beta_m,n_m}))}\|\geq e^{-\frac{l_m}{m}\cdot\pi_{l_m}(p_{\beta_m,n_m})},$$

since $\lim\limits_{m\to\infty}\frac{l_m}{m}\longrightarrow 0^+$ and by (4.18), lemma 4.25 tells us C is an index j_0 fundamental limit, this finishes the proof of the claim. $\square$

Now let's continue the proof of lemma 4.3, by the above argument, we can suppose Λ is minimal, not trivial, it's an index j_0 and j_0+1 fundamental limit with dominated splitting $E_{j_0}^s\oplus E_1^c\oplus E_{j_0+2}^{cu}|_{\Lambda}$ where $E_{j_0+2}^{cu}(\Lambda)\neq\phi$.

If $E^{cu}_{j_0+2}(\Lambda)$ is not expanding, by lemma 4.22, we can know that there exists a point $b \in \Lambda$ such that $\prod_{i=0}^{n-1} \|Df^{-l}|_{E^{cu}_{j_0+2}(f^{(i+1)l}b)}\| \geq 1$, since $(E^s_{j_0} \oplus E^c_1) \oplus E^{cu}_{j_0+2}|_\Lambda$ is an index $j_0+1-(l,\lambda)$ fundamental limit, it means that

$$\prod_{i=0}^{n-1} \|Df^l|_{E^s_{j_0}\oplus E^c_1(f^{il}(b))}\| \cdot \prod_{i=0}^{n-1} \|Df^{-l}|_{E^{cu}_{j_0+2}(f^{(i+1)l}(b))}\| \leq \lambda^n, \quad \text{for } n \geq 1,$$

so $\prod_{i=0}^{n-1} \|Df^l|_{E^s_{j_0}\oplus E^c_1(f^{il}(b))}\| \leq \lambda^n$ for all $n \geq 1$. Since Λ is minimal, the index j_0+1 dominated splitting on Λ satisfies strong tilda condition, by remark 4.27, it also satisfies the non-hyperbolic condition, so it satisfies all the conditions of weakly selecting lemma, then by corollary 4.26, C contains index j_0+1 periodic point and it's an index j_0 fundamental limit. $\square$

5. Proof of theorem 1

In order to prove theorem 1, we need the following lemma whose proof has been postponed to the end of this section.

Lemma 5.1. *Let $f \in R$, C is any non-trivial chain recurrent class of f, suppose $\Lambda \subset C$ is a non-trivial minimal set with a codimension-1 partial hyperbolic splitting $T_\Lambda M = E^c_1 \oplus E^u_2$ where $dim(E^c_1|_\Lambda) = 1$ and $E^c_1(\Lambda)$ is not contracting, then C is a homoclinic class containing index 1 periodic point and C is an index 0 fundamental limit.*

Remark 5.2. *in* [9]*, they show that for $f \in R$, if C is a chain recurrent class of f with a codimension-1 dominated splitting $T_C M = E^c_1 \oplus E^u_2$ where $dim(E^c_1|_C) = 1$ and $E^c_1|_C$ is not hyperbolic, then C should be a homoclinic class. We generalize this result to minimal set with Crovisier's work on central curves.*

Proof of theorem 1: Suppose $C \bigcap P^*_0 \neq \phi$, let Λ be an minimal index 0 fundamental limit, then Λ is not trivial (if Λ is trivial, Λ should be an orbit of source, then C itself is source also, that contradicts with C is not trivial)). By lemma 4.3, either C is a homoclinic class containing index 1 periodic point and C is an index 0 fundamental limit or Λ is a non-trivial minimal set with codimension-1 partial hyperbolic splitting $T_\Lambda M = E^c_1 \oplus E^u_2$ where $E^c_1|_\Lambda$ is not trivial. In the first case we've proved theorem 1, in the second case, by lemma 5.1, we also prove theorem 1. $\square$

In §5.1, we'll introduce some properties for codimension-1 partial hyperbolic splitting set, in §5.2 we'll introduce Crovisier's central model for the invariant compact set with partial hyperbolic splitting whose central bundle is 1-dimension and non-hyperbolic. In §5.3 I'll give the proof of lemma 5.1.

5.1. **Some properties for codimension-1 partial hyperbolic splitting.** Let $f \in R$, Λ is a given non-trivial minimal set of f with a codimension-1 partial hyperbolic splitting $T_\Lambda M = E^u \oplus E^c_1$, where $dim(E^c_1(\Lambda)) = 1$ and the bundle $E^c_1|_\Lambda$ is not hyperbolic. In this section we always suppose the dominated splitting is 1-step and the bundle E^u is 1-step expanding, it means that there exists $0 < \lambda < 1$ such that for any $v^u \in E^u(x)$, $v^c \in E^c_1(x)$ where $|v^u| = |v^c| = 1$, $x \in \Lambda$, we have $\frac{|Df(v^c)|}{|Df(v^u)|} < \lambda$, $|Df(v^u)| > \lambda^{-1}$. Fix a small neighborhood U_0 of Λ, then the maximal invariant set $\Lambda_0 = \bigcap_{j=-\infty}^{\infty} f^j(\overline{U_0})$ has also a codimension-1 partial hyperbolic splitting $\widetilde{E^u} \oplus \widetilde{E^c_1}$, the dominated splitting is 1-step and the bundle $\widetilde{E^u}|_{\Lambda_0}$ is also 1-step expanding. We say $E_1^c(\Lambda)$ has an f-orientation if $E_1^c|_\Lambda$ is orientable and Df preserves the orientation. If $E_1^c|_\Lambda$ has an f-orientation, we choose U_0 small enough such that $\widetilde{E_1^c}(\Lambda)$ has an f-orientation also.

Here we should notice the reader that in this section, all the argument will take place just in U_0, and we can suppose U_0 is small enough such that it satisfies all the properties which we need.

When U_0 is small enough, we can extend the bundle $\widetilde{E^u}|_{\Lambda_0}$ and $\widetilde{E_1^c}|_{\Lambda_0}$ to $\overline{U_0}$ such that for any $x \in \overline{U_0}$, $T_xM = \widetilde{E^u}(x) \oplus \widetilde{E_1^c}(x)$, and if $E_1^c|_\Lambda$ is orientable, $\widetilde{E_1^c}|_{\overline{U_0}}$ is orientable also. In fact, no matter $\widetilde{E_1^c}|_{\overline{U_0}}$ is orientable or not, we can always locally define an orientation of $\widetilde{E_1^c}|_{\overline{U_0}}$, it means that there exists $\delta_0 > 0$ such that for any $x \in \overline{U_0}$, we can give an orientation for the bundle $\widetilde{E_1^c}|_{B_{\delta_0}(x)} \bigcap \overline{U}_0$.

For every point $x \in \overline{U_0}$, we define two kinds of cones on its tangent space $C_a^i(x) = \{v | v \in T_xM$, there exists $v' \in \widetilde{E^i}(x)$ such that $d(\frac{v}{|v|}, \frac{v'}{|v'|}) < a\}_{i=c,u}$. When a small enough, $C_a^c \bigcap C_a^u = \phi$, $Df(C_a^u(x)) \subset C_a^u(f(x))$ and $Df^{-1}(C_a^c(x)) \subset C_a^c(f^{-1}(x))$ for $x \in \Lambda_0$.

We say a submanifold D^i $(i = c, u)$ tangents with cone C_a^i if $dim(D^i) = d - 1$ when $i = u$ and $dim(D^i) = 1$ when $i = c$ and for $x \in D^i$, $T_xD^i \subset C_a^i(x)$. For simplicity, sometimes we call it i-disk, especially when $i = c$, we just call D^c a central curve. We say an i-disk D^i has centrer x with size δ if $x \in D^i$, and respecting the Riemannian metric restricting on D^i, the ball centered on x with radius δ is in D^i. We say an i-disk D^i has center x with radius δ if $x \in D^i$, and respecting the Riemannian metric restricting on D^i, the distance between any point $y \in D^i$ and x is smaller than δ.

The following lemma shows some well-known results, it depends on a simple fact: locally the splitting $\widetilde{E_1^c} \oplus \widetilde{E^u}|_{\overline{U}_0}$ looks like linear. [9] 's subsection 4.1 gives many details about such view, from lemma 4.8 in [9], it would be very easy to get the following properties, so here we 'll not give a proof.

Lemma 5.3. *: Let $f \in R$, Λ is a non-trivial minimal set of f with a codimension-1 partial hyperbolic splitting $T_\Lambda M = E_1^c \oplus E^u$ where the bundle $E_1^c|_\Lambda$ is not hyperbolic. $U_0, \delta_0, C_a^u, C_a^c$ are defined by the above argument. Let U be any small neighborhood of Λ satisfying $\overline{U} \subset U_0$, there exist two neighborhoods U_2, U_1 of Λ such that $\Lambda \subset U_2 \subset \overline{U_2} \subset U_1 \subset \overline{U_1} \subset U \subset U_0$ and there exist a_0 small enough and $0 < \delta_{1,3} < \delta_{1,2} < \delta_{1,1} < \delta_0/2$ such that they satisfy the following properties:*

P1 *For any $x \in \overline{U_2}$, we have $B_{2\delta_{1,1}}(x) \subset U_1$, and for any $x \in \overline{U_1}$, we have $B_{2\delta_{1,1}}(x) \subset U$, then for any i-disk D^i $(i = c, u)$ with center $x \in \overline{U_1}$ and radius $2\delta_{1,1}$ we'll have $D^i \subset U$.*

For any $x \in \overline{U_1}$, $\widetilde{E_1^c}|_{B_{2\delta_{1,1}}(x)}$ is orientable, we can choose an orientation and call the direction right, then the orientation of $\overline{E_1^c}|_{B_{2\delta_{1,1}}(x)}$ will give an orientation for central curves in $B_{2\delta_{1,1}}(x)$. We suppose $\delta_{1,1}$ is small enough such that any central curve in $B_{2\delta_{1,1}}(x)$ will not intersect with itself.

For two points $y_1, y_2 \in B_{2\delta_{1,1}}(x)$, we say y_1 is on the x-right of y_2 if there exists a central curve $l \subset B_{2\delta_{1,1}}(x)$ connects y_1 and y_2 and in l, y_1 is on the right of y_2. Then since any central curve in $B_{2\delta_{1,1}}(x)$ is not self-intersection, y_2 is not on x-right of y_1 anymore. Usually, we just simply call y_1 is on the right of y_2.

P2) *Let $\Lambda_1 = \bigcap_{i=-\infty}^{\infty} f^i(\overline{U_1})$, apply lemma 4.13 on Λ_1, we can get the following two kinds of submanifolds families: the local unstable manifolds $W^{uu}_{loc}(x)_{\ x\in\Lambda_1}$ and the local central curves $W^c_{loc}(x)_{\ x\in\Lambda_1}$.*

Choose $\delta_{1,1}$ properly (small enough) we can suppose $W^i_{loc}(x)$ $_{(i=uu,c)}$ has size $\delta_{1,1}$, let $W^i_{\delta_{1,1}}(x)$ be the ball in $W^i_{loc}(x)$ with central x and radius $\delta_{1,1}$, then we have $W^i_{\delta_{1,1}}(x)$ $_{(x\in\Lambda_1, i=c,uu)}$ always tangents with cone $C^i_{a_0}$.

In fact, for $\Lambda_1^+ = \bigcap_{i=0}^{\infty} f^i(\overline{U_1})$, any $x \in \Lambda_1^+$ will have uniform size of unstable manifold $W^{uu}_{\delta_{1,1}}(x)$ which tangents with cone $C^{uu}_{a_0}$.

P3) *By the property of strong unstable manifolds, for $y_1, y_2 \in \Lambda_1^+$, if we have $W^{uu}_{\delta_{1,1}/2}(y_1) \bigcap W^{uu}_{\delta_{1,1}/2}(y_2) \neq \phi$, then $y_1 \in W^{uu}_{\delta_{1,1}}(y_2)$ and $y_2 \in W^{uu}_{\delta_{1,1}}(y_1)$. There exists $0 < \lambda < 1$ such that for any smooth curve $l \subset W^{uu}_{\delta_{1,1}}(x)$ where $x \in \Lambda_1^+$, we'll have $length(f^{-1}(l)) < \lambda \cdot length(l)$.*

P4) *For any central curve D^c and u-disk D^u in U with centers in Λ_1 and radius smaller than $2\delta_{1,1}$, we have $\#\{z|\ z \in D^c \bigcap D^u\} \leq 1$. If $D^c \bigcap D^u \neq \phi$, then they are transverse intersect with each other.*

P5) *For any $x \in \overline{U_1}$, $y \in B_{\delta_{1,3}}(x) \bigcap \Lambda_1$, $D^i_{\delta_{1,2}}$ is an i-disk with center y and radius $\delta_{1,2}$, then $D^i_{\delta_{1,2}} \subset B_{\delta_{1,1}}(x)$.*

For $z \in B_{\delta_{1,3}}(x)$ and $l^{c+}_{\delta_{1,2}}(z)$ is a central curve at the right of z with length $\delta_{1,2}$ and z is one of its extreme points, suppose $l^{c-}_{\delta_{1,2}}(z)$ is a central curve at the left of z with length $\delta_{1,2}$ and z is one of its extreme points, let $l^c_{\delta_{1,2}}(z) = l^{c+}_{\delta_{1,2}}(z) \bigcup l^{c-}_{\delta_{1,2}}(z)$, then $\#\{l^c_{\delta_{1,2}}(z) \bigcap W^{uu}_{\delta_{1,2}}(y)\} = 1$ and they are transverse intersect. Suppose $z \notin W^{uu}_{\delta_{1,2}}(y)$, then if $l^{c+}_{\delta_{1,2}} \bigcap W^{uu}_{\delta_{1,2}}(y) \neq \phi$, we say z is at x-left of y; if $l^{c-}_{\delta_{1,2}} \bigcap W^{uu}_{\delta_{1,2}}(y) \neq \phi$, we say z is at x-right of y. It's easy to show when z is at x-right of y, it's not at x-right of y anymore.

For simplicity, we just call z at the left of $W^{uu}_{loc}(y)$ or the right of $W^{uu}_{loc}(y)$.

P6) *For any $x \in \overline{U_1}$, any $\delta < \delta_{1,2}$, there exists $\delta^* \ll \delta$ such that for $y \in B_{\delta^*}(x) \bigcap \Lambda_1$, if we have $z \in B_{\delta^*}(x) \bigcap \Lambda_1$ also, then $\#\{l^c_\delta(z) \bigcap W^{uu}_{\delta_{1,2}}(y)\} = 1$ and they are transverse intersect ($l^c_\delta(z)$ is defined in P5).*

P7) *For any $0 < \delta^* < 2\delta_{1,1}$, there exists a δ^{**}such that if Γ is a central curve in $\overline{U_1}$ with $length(\Gamma) < 2\delta_{1,1}$, for $x, y \in \Gamma$ and suppose the segment in Γ connecting x and y has length bigger than δ^*, then $d(x, y) > \delta^{**}$.*

P8) *For any $x \in \overline{U_1}$, any central curve l in $B_{\delta_{1,2}}(x)$ will have length smaller than $\delta_{1,1}$.*

For $y \in B_{\delta_{1,2}}(x) \bigcap \Lambda_1^+$, we can let $W^{uu}_{\delta_{1,1}}(y) \bigcap B_{\delta_{1,2}}(x)$ always just have one connected components, and $W^{uu}_{\delta_{1,1}/2}(y)$ divides $B_{\delta_{1,2}}(x)$ into two connected components: the left part and the right part.

If $z_1, z_2 \in B_{\delta_{1,2}}(x)$ are on the different side of $B_{\delta_{1,2}}(x) \bigcap W^{uu}_{\delta_{1,1}/2}(y)$ and there is a central curve $l \subset B_{\delta_{1,2}}(x)$ connecting them, then $\#\{l \bigcap W^{uu}_{\delta_{1,1}/2}(y)\} = 1$.

P9) *Let $x \in \overline{U_1}$, suppose $y_1, y_2 \in B_{\delta_{1,2}}(x) \bigcap \Lambda_1^+$ and there exists a central curve l in $B_{\delta_{1,2}}(x)$ connects them, so by P8) $length(l) < \delta_{1,1}$, now we know $W^{uu}_{\delta_{1,1}/2}(y_1) \bigcap W^{uu}_{\delta_{1,1}/2}(y_2) = \phi$ (otherwise $y_1 \in W^{uu}_{\delta_{1,1}}(y_2)$, then $\#\{l \bigcap W^{uu}_{\delta_{1,1}}(y_1)\} \geq 2$, it contradicts with P4), it means $W^{uu}_{\delta_{1,1}/2}(y_1)$ and $W^{uu}_{\delta_{1,1}/2}(y_2)$ divide $B_{\delta_{1,2}}(x)$ into three connected components. Suppose y_1 is at x-left of y_2, then for any point $z \in \Lambda_1^+$ which are on the left of $W^{uu}_{\delta_{1,1}/2}(y_2) \bigcap B_{\delta_{1,2}}(x)$ and on the right of $W^{uu}_{\delta_{1,1}/2}(y_1) \bigcap B_{\delta_{1,2}}(x)$, we have $W^{uu}_{\delta_{1,1}/2}(z) \bigcap W^{uu}_{\delta_{1,1}/2}(y_i) = \phi$ $_{(i=1,2)}$ and $W^{uu}_{\delta_{11}/2}(z) \bigcap l \neq \phi$.*

P10) *A C^1 curve Γ in $\overline{U_1}$ is called a central segment if $f^i(\Gamma) \subset \overline{U_1}$ for all $i \in \mathbb{Z}$ and it always tangents with $C^c_{a_0}$. Then $\Gamma \subset \Lambda_1$ and it's easy to know that for any $x \in \Gamma$, we have $T_x\Gamma = \widetilde{E^c_1}(x)$. On Γ we have normally hyperbolic splitting $\widetilde{E^c_1} \oplus \widetilde{E^u}|_\Gamma$ since $T_x\Gamma = \widetilde{E^c_1}(x)$, by the property of normally hyperbolic manifold, $\bigcup_{x\in\Gamma} W^{uu}_{\delta_{1,1}/2}(x)$ is a submanifold (dim = d) with boundary, we denote it $W^u_{\delta_{1,1}/2}(\Gamma)$.*

P11) *For any $\varepsilon > 0$, if we have a family of central segment $\{\Gamma_n\}_{n=1}^{\infty}$ with length$(\Gamma_n) > \varepsilon$, there exists $\delta > 0$ such that vol$(W^u_{\delta_{1,1}/2}(\Gamma_n)) > \delta$, so we can find $n_i \neq n_j$ such that $W^u_{\delta_{1,1}/2}(\Gamma_{n_i}) \bigcap W^u_{\delta_{1,1}/2}(\Gamma_{n_j}) \neq \phi$.*

5.2. **Crovisier's central model.** In this subsection, let's fix $U, U_1, U_2, \Lambda_1, \delta_0/2 > \delta_{1,1} > \delta_{1,2} > \delta_{1,3} > 0$, and a_0 given by lemma 5.3, we'll introduce Crovisier's central model. By his work, we can get some dynamical property for the central curve $W^c_{\delta_{1,1}}(x)$ where $x \in \Lambda_1$. The main result in this subsection is lemma 5.11.

Definition 5.4. *A central model is a pair $(\widetilde{K}, \widetilde{f})$ where*

a) *$\widetilde{K}$ is a compact metric space called the base of the central model.*
b) *$\widetilde{f}$ is a continuous map from $\widetilde{K} \times [0,1]$ into $\widetilde{K} \times [0,\infty)$*
c) *$\widetilde{f}(\widetilde{K} \times \{0\}) = \widetilde{K} \times \{0\}$*
d) *$\widetilde{f}$ is a local homeomorphism in a neighborhood of $\widetilde{K} \times \{0\}$: there exists a continuous map $g :\ \widetilde{K} \times [0,1] \longrightarrow \widetilde{K} \times [0,\infty)$ such that $\widetilde{f} \circ \widetilde{g}$ and $\widetilde{g} \circ \widetilde{f}$ are identity maps on $\widetilde{g}^{-1}(\widetilde{K} \times [0,1])$ and $\widetilde{f}^{-1}(\widetilde{K} \times [0,1])$ respectively.*
e) *$\widetilde{f}$ is a skew product: there exits two map $\widetilde{f}_1 :\ \widetilde{K} \longrightarrow \widetilde{K}$ and $\widetilde{f}_2 :\ \widetilde{K} \times [0,1] \longrightarrow [0,\infty)$ respectively such that for any $(x,t) \in \widetilde{K} \times [0,1]$, one has $\widetilde{f}(x,t) = (\widetilde{f}_1(x), \widetilde{f}_2(x,t))$.*

f general doesn't preserve $\widetilde{K} \times [0,1]$, so the dynamics outside $\widetilde{K} \times \{0\}$ is only partially defined.

The central model $(\widetilde{K}, \widetilde{f})$ has a chain recurrent central segment if it contains a segment $I = \{x\} \times [0,a]$ contained in a chain recurrent class of $f|_{\widetilde{K}\times[0,1]}$.

A subset $S \subset \widetilde{K} \times [0,1]$ of a product $\widetilde{K} \times [0,\infty)$ is a strip if for any $x \in \widetilde{K}$, the intersection $S \bigcap \{x\} \times [0,\infty)$ is a non-trivial interval.

In his remarkable paper [13], Crovisier got the following important result.

Lemma 5.5. *([13] Proposition 2.5) Let $(\widetilde{K}, \widetilde{f})$ be a central model with a chain transitive base, then the two following properties are equivalent:*

a) *There is no chain recurrent central segment.*
b) *There exists some strip S in $\widetilde{K} \times [0,1]$ that is arbitrarily small neighborhood of $\widetilde{K} \times \{0\}$ and it's a trapping region for $\widetilde{f}$ or $\widetilde{f}^{-1}$: either $\widetilde{f}(Cl(S)) \subset Int(S)$ or $\widetilde{f}^{-1}(Cl(S)) \subset Int(S)$.*

Remark 5.6. *If the central model $(\widetilde{K}, \widetilde{f})$ has a chain recurrent central segment and $\widetilde{K} \times \{0\}$ is transitive, from Crovisier's proof, we can know for any small neighborhood V of $\widetilde{K} \times \{0\}$, there exists a segment $x \times [0,a]_{a\neq 0}$ contained in the same chain recurrent class of $\widetilde{f}|_V$ with $\widetilde{K} \times \{0\}$.*

An open strip $S \subset \widetilde{f} \times [0,1]$ satisfying $\widetilde{f}(Cl(S)) \subset Int(S)$ or $\widetilde{f}^{-1}(Cl(S)) \subset Int(S)$ will be called a trapping strip.

Definition 5.7. *Let f be a diffeomorphism of a manifold M, $\Lambda, \Lambda_1, U, U_0, U_1, U_2, a_0, \delta_0/2 > \delta_{1,1} > \delta_{1,2} > \delta_{1,3} > 0$ are given in §5.1, where Λ_1 is a partial hyperbolic invariant compact set of f having a 1-dimensional central bundle. A central model $(\widetilde{\Lambda}_1, \widetilde{f})$ is a central model for (Λ_1, f) if there exists a continuous map $\pi : \widetilde{\Lambda}_1 \times [0, \infty) \longrightarrow M$ such that:*

a) *π semi-conjugate $\widetilde{f}$ and f : $f \circ \pi = \pi \circ \widetilde{f}$ on $\widetilde{\Lambda}_1 \times [0, 1]$*
b) *$\pi(\widetilde{\Lambda}_1 \times \{0\}) = \Lambda_1$*
c) *The collection of map $t \longrightarrow \pi(\widetilde{x}, t)$ is a continuous family of C^1 embedding of $[0, \infty)$ into M, parameterized by $\widetilde{x} \in \widetilde{\Lambda_1}$.*
d) *For any $\widetilde{x} \in \widetilde{\Lambda_1}$, the curve $\pi(\widetilde{x}, [0, \infty)) \subset U$ has length bigger than $\delta_{1,2}$ but smaller than $2\delta_{1,1}$, it's tangent at the point $x = \pi(\widetilde{x}, 0) \in \Lambda_1$ to the central bundle and it's a central curve (that means the curve $\pi(\widetilde{x}, [0, \infty))$ tangents with the central cone $C^c_{a_0}$).*

Remark 5.8. *From now, if $(\widetilde{\Lambda}_1, \widetilde{f})$ is a central model for (Λ_1, f) and π is the projection map, we'll denote the central model as $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$. Here I should notice the reader that π in this paper has two different meanings, one denote the period of periodic point and another denote the projection map of central model. If there is any confusion, I'll point out.*

The following lemma shows the relation between central model and a set with codimension-1 partial hyperbolic splitting.

Lemma 5.9. *([Cr2]) $\Lambda, \Lambda_1, U, U_1$ are given in §5.1, then there exists a central model $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$ for (Λ_1, f). Let's denote $\widetilde{\Lambda} \subset \widetilde{\Lambda}_1$ which satisfies $\pi^{-1}(\Lambda) \bigcap (\widetilde{\Lambda}_1 \times \{0\}) = \widetilde{\Lambda} \times \{0\}$, then $(\widetilde{\Lambda}, \widetilde{f}, \pi)$ is a central model for (Λ, f), and $\widetilde{\Lambda} \times \{0\}$ is minimal.*

Remark 5.10. 1) *When the cental bundle $\widetilde{E^c_1}(\Lambda_1)$ has an f-orientation (it means that $\widetilde{E^c_1}(\Lambda_1)$ is orientable and Df preserves such orientation), we call the orientation 'right', then we can get two central models $(\widetilde{\Lambda_1^+}, \widetilde{f}^+, \pi^+)$ and $(\widetilde{\Lambda_1^-}, \widetilde{f}^-, \pi^-)$ for (Λ_1, f), we call them the right model and the left model, where $\pi^i{}_{(i=+,-)}$ is a bijection between $\widetilde{\Lambda}_1^i \times \{0\}$ and Λ_1, and for $\widetilde{x}^i \in \widetilde{\Lambda}_1^i$, $\pi(\widetilde{x}^i \times [0, \infty))$ is a half of central curve at the right $(i = +)$ or left $(i = -)$ of $x = \pi(\widetilde{x}^i \times \{0\})$.*

2) *If f doesn't preserve any orientation of $\widetilde{E^c_1}(\Lambda_1)$, then $\pi : \widetilde{\Lambda}_1 \longrightarrow \Lambda_1$ is two-one: any point $x \in \Lambda_1$ has two preimages $\widetilde{x}^-$ and $\widetilde{x}^+$ in $\widetilde{\Lambda}_1$, the homeomorphism σ of $\widetilde{\Lambda}_1$ which exchanges the preimages $\widetilde{x}^+$ and $\widetilde{x}^-$ of any point $x \in \Lambda_1$ commutes with $\widetilde{f}$.*

In § 5.1, we know any point $x \in \Lambda_1$ has a local orientation, then $\pi(\widetilde{x}^+ \times [0, \infty))$ is a central curve on the right of x, $\pi(\widetilde{x}^- \times [0, \infty))$ is on the left of x, the union of them is a central curve with central at x and radius $\delta_{1,1}$.

The following lemma is the main result in this subsection, it's similar with [Cr]'s proposition 3.6, but a little stronger.

Lemma 5.11. *$f \in R$, Λ is a non-trivial minimal set with a codimension-1 partial hyperbolic splitting $E^c_1 \oplus E^u$ where $dim(E^c_1(\Lambda)) = 1$ and $E^c_1(\Lambda)$ is not hyperbolic. Let U, U_1, Λ_1 be given in §5.1, by lemma 5.9, (Λ_1, f) has a central model $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$, then we can choose U_1 properly such that*

a) *either $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$ has a trapping region,*

b) *or Λ is contained in a homoclinic class C, C contains periodic points with index 1 and it's an index 0 fundamental limit.*

Proof : Let $\widetilde{\Lambda} \subset \widetilde{\Lambda}_1$ satisfy $\widetilde{\Lambda} \times \{0\} = \pi^{-1}(\Lambda) \bigcap \widetilde{\Lambda}_1 \times \{0\}$, then $(\widetilde{\Lambda}, \widetilde{f}, \pi)$ is a central model for (Λ, f). Since now, we just denote $\widetilde{\Lambda} \times \{0\}$ by $\widetilde{\Lambda}$.

At first, let's suppose $(\widetilde{\Lambda}, \widetilde{f}, \pi)$ has no trapping region, then by remark 5.6, for any small neighborhood V of $\widetilde{\Lambda}$ in $\widetilde{\Lambda} \times [0, 1]$, there exists a chain recurrent central segment $x \times I$ in V respecting the map $\widetilde{f}$. By Crovisier's result ([Cr], proposition 3.6), there exits a family of periodic points $\{p_n\}$ such that they all belong to the same chain recurrent class with Λ and $\lim_{n\to\infty} Orb(p_n) = \Lambda$, so $\Lambda \subset H(p_n, f)_{n\geq 1}$. When n is big enough, $Orb(p_n) \subset \Lambda_1$, so $Orb(p_n)$ has a codimension-1 partial hyperbolic splitting $\widetilde{E}^c_1 \oplus \widetilde{E}^u|_{Orb(p_n)}$, that means p_n is an index 1 periodic point.

Now we claim that $H(p_n, f)$ is an index 0 fundamental limit.

Proof of the claim: The argument is exactly the same with the case C in the proof of lemma 4.3, so here we just give a sketch of the proof, we divide the proof to two cases.

A) : there exists $\delta > 0$ such that for any p_n, we have $|Df^{\pi(p_n)}|_{\widetilde{E}^c_1(p_n)}| < e^{-\delta\pi(p_n)}$.

B) : for any $\frac{1}{m} > 0$, there exists p_{n_m} such that $|Df^{\pi(p_{n_m})}|_{\widetilde{E}^c_1(p_{n_m})}| > e^{-\frac{1}{m}\pi(p_{n_m})}$.

In the first case, we use weakly selecting lemma, in case B, we use lemma 4.25. □

Now we suppose $(\widetilde{\Lambda}, \widetilde{f}, \pi)$ has a trapping region S, we can suppose $\widetilde{f}(Cl(s)) \subset Int(S)$ always. Choose $\widetilde{\Lambda}_2$ an open neighborhood of $\widetilde{\Lambda}$ in $\widetilde{\Lambda}_1$ small enough, we can get an open strip S_2 for $\widetilde{\Lambda}_2$ (here open respect $\widetilde{\Lambda}_2 \times [0, 1]$) such that:

a) for any $\widetilde{x} \in \widetilde{\Lambda}$, $\widetilde{x} \times [0, 1] \bigcap S = \widetilde{x} \times [0, 1] \bigcap S_2$,

b) for any $\widetilde{x} \in \widetilde{\Lambda}_2$ and $\widetilde{f}(\widetilde{x}) \in \widetilde{\Lambda}_2$, we have $\widetilde{f}(Cl((\widetilde{x} \times [0, 1]) \bigcap S_2)) \subset (\widetilde{f}(\widetilde{x}) \times [0, 1]) \bigcap S_2$.

Choose U^* neighborhood of Λ small enough, let $\Lambda^* = \bigcap_{-\infty}^{\infty} f^i(\overline{U}^*)$, then $\Lambda^* \subset \Lambda_1$, let $\widetilde{\Lambda}^* \subset \widetilde{\Lambda}_1$ satisfies $\widetilde{\Lambda}^* = \pi^{-1}(\Lambda^*) \bigcap \widetilde{\Lambda}_1$, we'll have $\widetilde{\Lambda}^* \subset \widetilde{\Lambda}_2$. Then consider the central model $(\widetilde{\Lambda}^*, \widetilde{f}, \pi)$ for (Λ^*, f), $S_2 \bigcap (\widetilde{\Lambda}^* \times [0, 1])$ is a trapping region for $(\widetilde{\Lambda}^*, \widetilde{f}, \pi)$.

Now replace U_1 by U^* and Λ_1 by Λ^*, we get a trapping region for $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$. □

5.3. **Proof of lemma 5.1.** Now we suppose Λ is a non-trivial minimal set with a codimension-1 partial hyperbolic splitting $E^c_1 \oplus E^u$ where $dim(E^c_1) = 1$ and $E^c_1(\Lambda)$ is not hyperbolic. We divide the proof of lemma 5.1 into two cases: $E^c_1(\Lambda)$ has an f-orientation or not.

Proof of lemma 5.1 ($E^c_1(\Lambda)$ has an f-orientation)

Let U_0 be the small neighborhood of Λ given in §5.1 such that we can extend the splitting $E^c_1 \oplus E^u|_\Lambda$ to $\overline{U}_0$, we denote the splitting $T_xM = \widetilde{E}^c_1 \oplus \widetilde{E}^u$ $(x \in \widetilde{U}_0)$. Suppose U is any small neighborhood of Λ such that $\overline{U} \subset U_0$, then from lemma 5.3, we can get open sets U_2, U_1 and $\Lambda_1 = \bigcap_{i=-\infty}^{\infty} f^i(\overline{U}_1)$, $a_0 > 0, 0 < \delta_{1,3} < \delta_{1,2} < \delta_{1,1} < \delta_0/2$ such that they satisfy properties P1-P11 of lemma 5.3 there.

Since $E^c_1(\Lambda)$ has an f-orientation, $\widetilde{E}^c_1(\Lambda_1)$ has an f-orientation also, by remark 5.10 we get two central models: the right central model $(\widetilde{\Lambda}^+_1, \widetilde{f}^+, \pi^+)$ and the left central model $(\widetilde{\Lambda}^-_1, \widetilde{f}^-, \pi^-)$, where for any

$\widetilde{x}^+ \in \widetilde{\Lambda}_1^+$, $\pi^+(\widetilde{x}^+ \times [0,\infty))$ is a central curve at the right of $x = \pi^+(\widetilde{x}^+ \times \{0\})$ and $\delta_{1,2} < length(\pi^+(\widetilde{x}^+ \times [0,\infty))) < 2\delta_{1,1}$, so $\pi^+(\widetilde{x}^+ \times [0,\infty)) \subset B_{2\delta_{1,1}}(x) \subset U$. For any $\widetilde{x}^- \in \widetilde{\Lambda}^-$, we have the similar property.

At first, we consider the right central model $(\widetilde{\Lambda}_1^+, \widetilde{f}^+, \pi^+)$, if the right central model doesn't have trapping region, by lemma 5.11, Λ is contained in a homoclinic class $H(p,f)$ which contains an index 1 periodic point and the homoclinic class is an index 0 fundamental limit, then we've proved lemma 5.1, so now we suppose that there exists a trapping region S^+ for the right central model. By the similar argument for the left central model, we can suppose it has a trapping region S^- also.

Claim: Λ is an index 0 fundamental limit.

Proof of the claim: If Λ is not an index 0 fundamental limit, since Λ has a codimension-1 dominated splitting, Λ should be an index 1 fundamental limit. By generic property 5 of proposition 3.1, there exists a family of index 1 periodic points $\{p_n\}$ such that $\lim\limits_{n\to\infty} Orb(p_n) = \Lambda$ and they are index stable, then by Gan's lemma, there exists a subsequence of periodic points $\{p_{n_m}\}_{m=1}^{\infty}$ in C. Now with the same argument of the case C in the proof of lemma 4.3, we can show Λ satisfies weakly selecting lemma, by weakly selecting lemma 4.21, Λ is an index 0 fundamental limit, that's a contradiction. $\square$

Since Λ is an index 0 fundamental limit, by generic property 5) of proposition 3.1, there exists a family of sources $\{p_n\}_{n=1}^{\infty}$ of f satisfying $\lim\limits_{n\to\infty} Orb(p_n) = \Lambda$. We can suppose $Orb(p_n) \subset U_2$ always and let $\widetilde{p}_n^i \in \widetilde{\Lambda}_1^i$ $_{(i=+,-)}$ such that $\pi^{(i)}(\widetilde{p}_n^i \times \{0\}) = p_n$, then $(\widetilde{f}^i)^{\pi(p_n)}(\widetilde{p}_n^i) = \widetilde{p}_n^i$. Denote $\widetilde{p}_n^{+(-)} \times I_n^{+(-)} = (\widetilde{p}_n^{+(-)} \times [0,\infty)) \bigcap S^{+(-)}$ and $\gamma_n^{+(-)} = \pi^{+(-)}(\widetilde{p}^{+(-)} \times I_n^{+(-)})$, let $\gamma_= \gamma_n^+ \bigcup \gamma_n^-$, then γ_n is a central curve with center at p_n. Since $length(\gamma_n^{+(-)}) < 2\delta_{1,1}$, we have $\gamma_n \subset B_{2\delta_{1,1}}(p_n) \subset U_1$.

We've suppose $S^{\pm}$ is a trapping region, then $\widetilde{f}^{+(-)}(\overline{S^{+(-)}}) \subset Int(S^{+(-)})$ or $(\widetilde{f}^{+(-)})^{-1}(\overline{S^{+(-)}}) \subset Int(S^{+(-)})$. In the first case, we say the trapping region is 1-step contracting, in the second case we say it's 1-step expanding. When S^i is 1-step contracting case, we have $(\widetilde{f}^i)^{\pi(p_n)}(\widetilde{p}_n^i \times \overline{I}_n^i) \subset \widetilde{p}_n^i \times I_n^i$, so $f^{\pi(p_n)}(\overline{\gamma_n^i}) \subset \gamma_n^i$ for $i = +,-$ and there exists $\delta > 0$ doesn't depend on n such that $length(\gamma_n^i \setminus f^{\pi(p_n)}(\overline{\gamma_n^i})) > \delta$ for all $n \geq 1$. If S^i is 1-step expanding, we'll still have $length(\gamma_n^i \setminus f^{-\pi(p_n)}(\overline{\gamma_n^i})) > \delta$ for all $n \geq 1$.

Since γ_n^i is either expanding or contracting for $f^{\pi(p_n)}$, let $\Gamma_n^i = \bigcap\limits_{j=-\infty}^{\infty} f^{j\pi(p_n)}(\gamma_n^i)$ $_{(i=+,-)}$, we'll have $f^{\pi(p_n)}(\Gamma_n^i) = \Gamma_n^i$ $_{(i=+,-)}$ where Γ_n^i's extreme points are periodic points. When Γ_n^i is not trivial, we denote q_n^i $_{(i=+,-)}$ the extreme periodic point different with p_n, if Γ_n^i is trivial, we just let $q_n^i = p_n$. We let $\Gamma_n = \Gamma_n^+ \bigcup \Gamma_n^-$ and $h_n^i = \gamma_n^i \setminus \Gamma_i^n$ $_{(i=+,-)}$, then $\Gamma_n \subset \Lambda_1$, $h_n^i \subset U_1$. It's easy to know that h_n^i is in the stable (unstable) manifold of q_n^i if S^i is 1-step contracting (expanding). And since f is a Kupka-Smale diffwomorphism, $f^{\pi(p_n)}|_{\Gamma_n}$ is also a Kupka-Smale diffeomorphism and just has finite sinks and sources (respect $f^{\pi(p_n)}|_{\Gamma_n}$).

Lemma 5.12. *If $\Gamma_n \bigcap \Gamma_m \neq \phi$, then $\Gamma_n \bigcap \Gamma_m$ is a connected central curve, and $\Gamma_n \bigcup \Gamma_m$ is a central segment.*

Proof : We need prove some lemmas at first.

Lemma 5.13. *let $x \in \Gamma_n \bigcap \Gamma_m$ and x is not a periodic point, $x_1 \in \Gamma_n$ is the nearest periodic point at the left of x and $x_2 \in \Gamma_n$ is the nearest periodic point at the right of x. Denote $I_n \subset \Gamma_n$ the segment connecting x_1 and x_2, then $I_n \subset \Gamma_m$.*

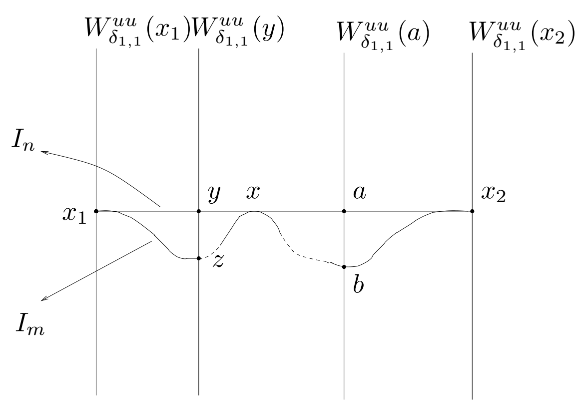
Proof : By the assumption, $f^{\pi(p_n)}$ has no any other fixed point in I_n, so for x_1 and x_2, one of them is sink for $f^{\pi(p_n)}|_{\Gamma_n}$ and another is source for $f^{\pi(p_n)}|_{\Gamma_n}$. We suppose x_1 is the source, then $\lim_{i\to\infty} f^{i\pi(p_n)}(x) \longrightarrow x_2$ and $\lim_{i\to\infty} f^{-i\pi(p_n)}(x) \longrightarrow x_1$. Since Γ_m is a periodic central segment with period $\pi(p_m)$ and $x \in \Gamma_m$, we have $f^{i\pi(p_n)\pi(p_m)}(x) \in \Gamma_m$ for all $i \in \mathbb{Z}$, so $x_2 = \lim_{i\to\infty} f^{i\pi(p_n)\pi(p_m)}(x) \in \Gamma_m$ and $x_1 = \lim_{i\to\infty} f^{-i\pi(p_n)\pi(p_m)}(x) \in \Gamma_m$.

Now denote I_m the central segment in Γ_m connecting x_1 and x_2.

We claim that $I_n = I_m$.

Proof of the claim: If it's not true, there exists $y \in Int(I_n)$, $z \in W^{uu}_{\delta_{1,1}}(y) \bigcap I_m$ and $z \neq y$.

For any $\varepsilon > 0$, consider $a = f^{i\pi(p_n)\pi(p_m)}(y)$ where i is very big, then $a \in I_n$ and it's near x_2 very much. Let $b \in W^{uu}_{\delta_{1,1}}(a) \bigcap I_m$, recall that I_n and I_m are tangent at $\widetilde{E^c_1}(x_2)$, when i is big enough, there exists a curve l in $W^{uu}_{\delta_{1,1}}(a)$ connecting a and b with $length(l) < \varepsilon$.



Now it's easy to know $f^{-i\pi(p_n)\pi(p_m)}(b) \in W^{uu}_{\delta_{1,1}}(y) \bigcap \Gamma_m$. By P4 of lemma 5.3, $\#\{W^{uu}_{\delta_{1,1}}(y) \bigcap \Gamma_m\} = 1$, so $f^{-i\pi(p_n)\pi(p_m)}(b) = z$, then $f^{-i\pi(p_n)\pi(p_m)}(l)$ is a curve connecting y and z, by P3 of lemma 5.3, we'll have $length(f^{-i\pi(p_n)\pi(p_m)}(l)) < \varepsilon \cdot \lambda^{i\pi(p_n)\pi(p_m)}$.

Since ε can be chosen arbitrarily small, we get $y = z$, that's a contradiction. $\square$

By the claim, we finish the proof of lemma 5.13. $\square$

We still need the following result.

Lemma 5.14. *Let $x \in \Gamma_n \bigcap \Gamma_m$ and x be a fixed point of $f^{\pi(p_n)}|_{\Gamma_n}$ and $f^{\pi(p_m)}|_{\Gamma_m}$, suppose Γ_n and Γ_m both have points on the right of x. Let $x_n \in \Gamma_n$ be the nearest fixed point of $f^{\pi(p_n)}|_{\Gamma_n}$ on the right of x and $x_m \in \Gamma_m$ be the nearest fixed point of $f^{\pi(p_m)}|_{\Gamma_m}$ on the right of x. Denote $I_n \subset \Gamma_n$ the central segment in Γ_n connecting x and x_n, $I_m \subset \Gamma_m$ the central segment in Γ_m connecting x and x_m, then $I_n = I_m$.*

Proof : At first, we claim that either $W^{uu}_{\delta_{1,1}}(x_n) \bigcap I_m \neq \phi$ or $W^{uu}_{\delta_{1,1}}(x_m) \bigcap I_n \neq \phi$.

Proof of the claim: Suppose $W^{uu}_{\delta_{1,1}}(x_n)\bigcap I_m \neq \phi$, we know that x_m is on the left of $W^{uu}_{\delta_{1,1}}(x_n)$, recall that x_m is on the right of x, so by P9 of lemma 5.3, $W^{uu}_{\delta_{1,1}}(x_m)\bigcap I_n \neq \phi$. □

Now we suppose $W^{uu}_{\delta_{1,1}}(x_n)\bigcap I_m = y \neq \phi$, then $y \in I_m \setminus \{x\}$, it's easy to know $f^{-i\pi(p_n)\pi(p_m)}(y) \in W^{uu}_{\delta_{1,1}}(x_n)\bigcap I_m$ for $i \geq 1$, so $f^{-i\pi(p_n)\pi(p_m)}(y) = y$. But $\lim\limits_{i\to\infty} f^{-i\pi(p_n)\pi(p_m)}(y) \longrightarrow x_n$, so $x_n = y$. It means that $x_n \in I_m \setminus \{x\}$, so $x_n = x_m$. By the same argument in lemma 5.13, we can prove $I_n = I_m$. □

Now let's continue the proof of lemma 5.12.

Let $\Gamma = \Gamma_n \bigcap \Gamma_m$, $x \in \Gamma$ be the left extreme point of Γ, then by lemma 5.13, x should be a periodic point and on the left of x, there doesn't contain points of at least one of the segment Γ_n or Γ_m. Let $y \in \Gamma$ be the right extreme point of Γ, then on the right of y, there doesn't contain points of at least one of the segments Γ_n or Γ_m.

When $x = y$, Γ_n and Γ_m are on different side of x, $\Gamma_n \bigcup \Gamma_m$ is obviously a central segment.

When $x \neq y$, let I be the maximal central curve in Γ containing x, let z be the right extreme point in I, by lemma 5.13, z should be a periodic point. If $z \neq y$, y is on the right of z and $y \in \Gamma_n \bigcap \Gamma_m$, so by lemma 5.14, I will contain a central segment on the right of z, that's a contradiction with the maximalicity of I, so $z = y$. It means that $I = \Gamma_n \bigcap \Gamma_m$ is an interval, and x, y are its extreme points on the left and right, and Γ_n and Γ_m can not both have points on the left of x, they can not both have points on the right of y also, it's easy to see now that $\Gamma_n \bigcup \Gamma_m$ is a central curve. □

Now we divide the proof of lemma 5.1 to three cases depending on the contracting or expanding properties of the two central models.

Case A: *Two central models have 1-step expanding properties.*

In this case, for any γ_n, we have $f^{-i}(\gamma_n) \in U_1$ for $i \geq 1$, it means $\gamma \subset \Lambda_1^+$, and any $x \in \gamma_n$ will have uniform size of unstable manifold $W^{uu}_{\delta_{1,1}}(x)$. Let $W^u_{\delta_{1,1}/2}(\gamma_n) = \bigcup\limits_{x\in\gamma_n} W^{uu}_{\delta_{1,1}/2}(x)$, by the property of normally hyperbolic submanifold, $W^u_{\delta_{1,1}/2}(\gamma_n)$ is a submanifold ($dim = d$) with boundary, it's easy to know that $W^u_{\delta_{1,1}/2}(\gamma_n)$ has uniform size, that means there exists an $\varepsilon > 0$ such that $B_\varepsilon(p_n) \subset W^u_{\delta_{1,1}/2}(\gamma_n)$ for all $n \geq 1$. Suppose $\lim\limits_{n\to\infty} p_n = p \in \Lambda$, then when n is big enough, $p \in B_\varepsilon(p_n) \subset W^u_{\delta_{1,1}/2}(\gamma_n)$, so $\lim\limits_{i\to\infty} f^{-i\pi(p_n)}(p_n) \longrightarrow$ some periodic point $z \in \Gamma_n$, it means $z \in \Lambda$. But Λ is a non-trivial minimal set of f, that's a contradiction. □

Case B: *Left central model is 1-step contracting and the right central model is 1-step expanding.*

Let's consider γ_n^+, with the same argument in case A, it has uniform size of unstable manifold $W^u_{\delta_{1,1}/2}(\gamma_n^+) = \bigcup\limits_{x\in\gamma_n^+} W^{uu}_{\delta_{1,1}/2}(x)$ (it's because $length(\gamma_n^+) > length(h_n^+) > \delta$), so there exists an $\varepsilon > 0$ such that $Vol(W^u_{\delta_{1,1}/2}(\gamma_n^+)) > \varepsilon$.

Now we claim that for any sequence $\{n_i\}_{i=1}^\infty$, there exists i_0 and a sequence $i_0 < i_1 < i_2 < \cdots$ such that for any $j > 0$, $W^u_{\delta_{1,1}/2}(\gamma^+_{n_{i_j}})\bigcap W^u_{\delta_{1,1}/2}(\gamma^+_{n_{i_0}}) \neq \phi$.

Proof of the claim: Suppose that the claim is not true, then we can find a subsequence $\{n_{i_j}\}_{j=1}^{\infty}$ such that $W^u_{\delta_{1,1}/2}(\gamma^+_{n_{i_{j_0}}})\bigcap W^u_{\delta_{1,1}/2}(\gamma^+_{n_{i_j}})=\phi$ for $j_0\in\mathbb{N}$ and $j>j_0$, it's a contradiction with $Vol(W^u_{\delta_{1,1}/2}(\gamma^+_{n_i}))>\varepsilon$, since we'll have $Vol(M)>\sum\limits_j Vol(W^u_{\delta_{1,1}/2}(\gamma^+_{n_{i_j}}))=\infty$. $\square$

By the above claim, we can find a subsequence $\{n_i\}_{i=1}^{\infty}$ such that for any $i_0\in\mathbb{N}^+$, we can get $W^u_{\delta_{1,1}/2}(\gamma^+_{n_i})\bigcap W^u_{\delta_{1,1}/2}(\gamma^+_{n_{i_0}})\neq\phi$ for $i\geq i_0$. Since f is a Kupka-Smale diffeomorphism, on Γ_{n_i} it just has finite periodic points. So when we fix i_0, we can let i big enough such that $p_{n_i}\notin\gamma_{n_{i_0}}$. It means that we can choose a subsequence $\{(\Gamma_{n_i},\Gamma_{m_i})\}_{i=0}^{\infty}$ such that $p_{m_i}\notin\Gamma_{n_i}$, $W^u_{\delta_{1,1}/2}(\gamma^+_{n_i})\bigcap W^u_{\delta_{1,1}/2}(\gamma^+_{m_i})\neq\phi$ and $\lim\limits_{i\to\infty}(p_{n_i})=\lim\limits_{i\to\infty}(p_{m_i})=x_0$ for some $x_0\in\Lambda$.

Since $W^u_{\delta_{1,1}/2}(\gamma^+_{n_i})\bigcap W^u_{\delta_{1,1}/2}(\gamma^+_{m_i})\neq\phi$, suppose $y_i\in W^u_{\delta_{1,1}/2}(\gamma^+_{n_i})\bigcap W^u_{\delta_{1,1}/2}(\gamma^+_{m_i})$, then

$$\lim_{j\to\infty} f^{-j\pi(p_{n_i})\pi(p_{m_i})}(y_i)\longrightarrow\Gamma^+_{n_i} \text{ and } \lim_{j\to\infty} f^{-j\pi(p_{n_i})\pi(p_{m_i})}(y_i)\longrightarrow\Gamma^+_{m_i},$$

so $\Gamma^+_{n_i}\bigcap\Gamma^+_{m_i}\neq\phi$, by lemma 5.12, $\Gamma_{n_i}\bigcup\Gamma_{m_i}$ is a central segment.

For simplicity, we suppose p_{m_i} is on the right of p_{n_i} for all $i\in\mathbb{N}$, the proof of the other case is similar. Since $p_{m_i}\notin\Gamma_{n_i}$ and $\Gamma_i=\Gamma_{n_i}\bigcup\Gamma_{m_i}$ is a central curve. p_{m_i} is on the right of $q^+_{n_i}$ also. Recall that $q^+_{n_i}$ is a source for $f^{\pi(p_{n_i})}|_{\Gamma_{n_i}}$, and $h^+_{n_i}$ belongs to its basin, so $h^+_{n_i}\bigcap W^{uu}_{\delta_{1,1}/2}(p_{m_i})=\phi$.

Remark 5.15. *: We don't know $h^+_{n_i}\subset\Gamma_{m_i}$ here.*

We know that $h^+_{n_i}$ is a central curve on the right of $q^+_{n_i}$ with length bigger than δ, by property P6 of lemma 5.3, there exists a δ^* such that $d(q^+_{n_i},p_{m_i})>\delta^*$.(Since if $d(q^+_{n_i},p_{m_i})<\delta^*$, we have $l^+_\delta(q^+_{n_i})\bigcap W^{uu}_{\delta_{1,1}/2}(p_n)\neq\phi$ where $l^+_\delta(q^+_{n_i})$ is any central curve at the right of $q^+_{n_i}$ with length δ and $q^+_{n_i}$ is the left extreme point of it, with the fact that p_{m_i} is on the right of $q^+_{n_i}$, we'll have $h^+_n\bigcap W^{uu}_{\delta_{1,1}/2}(p_{m_i})\neq\phi$, that's a contradiction because $h^+_{n_i}\subset W^u(q^+_{n_i})$). So especially, in the central segment Γ_i, the distance between p_{n_i} and p_{m_i} is bigger than δ^*. By property P7 of lemma 5.3, there exists $\delta^{**}>0$ such that $d(p_{n_i},p_{m_i})>\delta^{**}$, it's a contradiction with $\lim\limits_{i\to\infty}(p_{n_i})=\lim\limits_{i\to\infty}(p_{m_i})=x_0\in\Lambda$. $\square$

Case C: *The two central models have 1-step contracting properties.*

In this case, replace by a subsequence, we can suppose for $\{\Gamma_n\}_{n=1}^{\infty}$, we have $p_n\notin\bigcup\limits_{i<n}\Gamma_i$.

Lemma 5.16. *There exists n_0 big enough such that for any $n_1,n_2>n_0$, $n_1\neq n_2$, we always have $W^u_{\delta_{1,1}/2}(\Gamma_{n_1})\bigcap W^u_{\delta_{1,1}/2}(\Gamma_{n_2})=\phi$.*

Proof Suppose the lemma is not true, then we can choose n_1 and n_2 arbitrarily big and satisfying $W^u_{\delta_{1,1}/2}(\Gamma_{n_1})\bigcap W^u_{\delta_{1,1}/2}(\Gamma_{n_2})\neq\phi$, then it's easy to know $\Gamma_{n_1}\bigcap\Gamma_{n_2}\neq\phi$ and $\Gamma_{n_1}\bigcup\Gamma_{n_2}$ is a central curve. We can suppose $n_2>n_1$, then by the assumption of $\{\Gamma_n\}_{n=1}^{\infty}$, we have $p_{n_2}\notin\Gamma_{n_1}$.

We just suppose p_{n_2} is on the right of p_{n_1}, since $\Gamma=\Gamma_{n_1}\bigcup\Gamma_{n_2}$ is a central curve and $p_{n_2}\notin\Gamma_{n_1}$, we can know p_{n_2} is on the right of $q^+_{n_1}$ also, and $q^+_{n_1}\in\Gamma_{n_2}$.

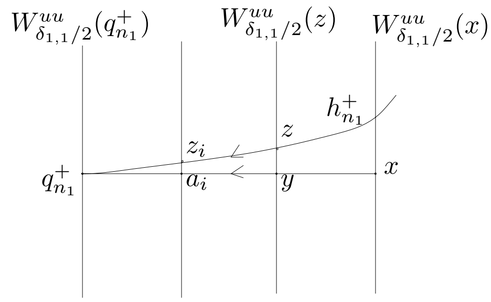
We know that there exists a $\delta>0$ such that $length(h^{+(-)}_n)>\delta$ for all $n\geq 1$. And for such δ, by proposition P6 of lemma 5.3, there exists $0<\delta^*\ll\delta$ such that for any $x,y\in\Lambda_1$, if $d(x,y)<\delta^*$, we have $\#\{W^{uu}_{\delta_{1,1}/2}(x)\bigcap l^c_\delta(y)\}=1$ where $l^c_\delta(y)$ is a central curve with center y and on the two sides of y both have length δ.

Suppose $x \in \Gamma_m$ is the nearest periodic point on the right side of $q^+_{n_1}$, and let $I \subset \Gamma_m$ the central segment in Γ_m connecting $q^+_{n_1}$ and x.

Now we claim that $length(I) > \delta^*$.

Proof of the claim: If $length(I) \leq \delta^*$, then $d(q^+_{n_1}, x) \leq \delta^*$ also. By the facts that x is on the right of $q^+_{n_i}$ and $h^+_{n_1}$ is a central curve with length bigger than δ, we have $h^+_{n_1} \bigcap W^{uu}_{\delta_{1,1}/2}(x) \neq \phi$. Then for any $y \in Int(I)$, $W^{uu}_{\delta_{1,1}/2}(y) \bigcap h^+_{n_1} \neq \phi$.

It's easy to know $I \nsubseteq h^+_{n_1}$ since $h^+_{n_1}$ contains no periodic point, so there exists $z \in h^+_{n_1}$ such that $W^{uu}_{\delta_{1,1}/2}(z) \bigcap Int(I) = y \neq z$.



Because the two central models are 1-step contracting, $q^+_{n_1}$ is a sink for $f^{\pi(p_{n_1})}|_{\Gamma_{n_1}}$, then it's also a sink for $f^{\pi(p_{n_1})\pi(p_{n_2})}|_{\Gamma}$ where $\Gamma = \Gamma_{n_1} \bigcup \Gamma_{n_2}$. We can choose i big enough, such that $z_i = f^{i\pi(p_{n_1})\pi(p_{n_2})}(z)$ near $q^+_{n_1}$ very much, let $a_i = W^{uu}_{\delta_{1,1}/2}(z_i) \bigcap I$. Since $h^+_{n_1}$ and I are tangent at $q^+_{n_1}$ on $\widetilde{E^c_1}(q^+_{n_1})$, for any $\varepsilon > 0$, when i big enough, there exists a curve $l \subset W^{uu}_{\delta_{1,1}/2}(z_i)$ connecting a_i and z_i and $length(l) < \varepsilon$. Since $f^{-i\pi(p_{n_1})\pi(p_{n_2})}(a_i) \in W^{uu}_{\delta_{1,1}/2}(z) \bigcap I$, that means $f^{-i\pi(p_{n_1})\pi(p_{n_2})}(a_i) = y$ and $f^{-i\pi(p_{n_1})\pi(p_{n_2})}(l)$ is a curve connecting z and y. By property P3 of lemma 5.3, $length(f^{-i\pi(p_{n_1})\pi(p_{n_2})}(l)) < \varepsilon\lambda^i$. Since i can be chosen arbitrarily big, we can get $y = z$, that's a contradiction. $\square$

Since $length(I) > \delta^*$, the segment in Γ connecting p_{n_1} and p_{n_2} will have length bigger than δ^* also, by property P7 of lemma 5.3, there exists $\delta^{**} > 0$ such that $d(p_{n_1}, p_{n_2}) > \delta^{**}$. But recall that $\lim_{n\to\infty} p_n \longrightarrow x_0 \in \Lambda$ and n_1, n_2 can be chosen arbitrarily big, we can get $d(p_{n_1}, p_{n_2}) < \delta^{**}$, that's a contradiction. $\square$

With lemma 5.16, we can chosen $\{\Gamma_n\}_{n=1}^{\infty}$ such that if $n \neq m$, $W^u_{\delta_{1,1}/2}(\Gamma_n) \bigcap W^u_{\delta_{1,1}/2}(\Gamma_n) = \phi$. Then by property P11 of lemma 5.3, $\lim_{n\to\infty} length(\Gamma_n) = 0$.

Choose n_0 big enough such that for $m \geq n_0$, $d(p_m, p_{n_0}) < \delta^*/4$ and $length(\Gamma_m) < \delta^*/4$, we can suppose p_m is on the right of p_{n_0}, then by $W^u_{\delta_{1,1}/2}(\Gamma_n) \bigcap W^u_{\delta_{1,1}/2}(\Gamma_n) = \phi$, we know that p_m is on the right of $q^+_{n_0}$ and q^-_m is on the right of $q^+_{n_0}$ also.

Since $d(q^+_{n_0}, q^-_m) \leq d(q^+_{n_0}, p_{n_0}) + d(q^-_m, p_m) + d(p_{n_0}, p_m) < length(\Gamma_{n_0}) + \delta^*/4 + length(\Gamma_m) < \delta^*$, by Property P6 of lemma 5.3 and $length(h^+_{n_0}) > \delta$, $length(h^-_m) > \delta$, we can get $h^+_{n_0} \pitchfork W^{uu}_{\delta_{1,1}/2}(q^-_m) \neq \phi$ and $h^+_m \pitchfork W^{uu}_{\delta_{1,1}/2}(q^+_{n_0}) \neq \phi$. Recall that $h^+_{n_0} \subset W^s(q^+_{n_0})$ and $h^-_m \subset W^s(q^-_m)$, we can know $q^+_{n_0}$ and q^-_m are in the same homoclinic class.

When $m \longrightarrow \infty$, by $length(\Gamma_m) \longrightarrow 0$ and $\lim_{m\to\infty} p_m \longrightarrow x_0 \in \Lambda$, we have $q^-_m \longrightarrow x_0$ also, so $x \in H(q^+_{n_0}, f)$ and then $\Lambda \subset H(q^+_{n_0}, f)$.

Now we'll prove $H(q_{n_0}^+, f)$ is an index 0 fundamental limit.

Recall that $Orb(q_{n_0}^+) \subset U$ and U can be chosen arbitrarily small, so in fact we've proved that there exists a family of periodic points q_n with index 1 such that $\lim_{n\to\infty} Orb(q_n) = \Lambda$ and $\Lambda \subset H(q_1, f) = H(q_2, f) = \cdots$.

By the same argument with case C in the proof of lemma 4.3, we can prove $H(q_1, f)$ is an index 0 fundamental limit. $\square$

Now let's keep on proving the other case of lemma 5.1.

Proof of lemma 5.1($E_1^c(\Lambda)$ has no any f-orientation):

In this case, we just have one central model, but locally we still have orientation for $\widetilde{E_1^c}(\Lambda_1)$, and the two sides have the same dynamical property: they are both 1-step expanding or they are both 1-step contracting. All the other argument is the same with the case where $E_1^c(\Lambda)$ has an f-orientation. $\square$

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