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**ON CONVEX REPRESENTATIONS OF MAXIMAL  
MONOTONE OPERATORS IN NON-REFLEXIVE  
BANACH SPACES**

REPRESENTAÇÕES CONVEXAS DE OPERADORES MONÓTONOS  
MAXIMAIS EM ESPAÇOS DE BANACH NÃO REFLEXIVOS

Author: Maicon Marques Alves

Adviser: Prof. Dr. Benar Fux Svaiter

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Ministério  
da Ciência  
e Tecnologia





*To my wife Daiane*



## Abstract

The main focus of this thesis is to study some generalizations of a theorem of Burachik and Svaiter to non-reflexive Banach spaces, and its theoretical implications in the theory of maximal monotone operators, with special emphasis for maximal monotone operators of type (NI).

**Keywords:** Maximal monotone operators, convex functions, convex representations, non-reflexive Banach spaces, Fitzpatrick functions, Fitzpatrick family, operators of type (NI), extension to the bidual, duality mapping, Gossez generalized duality mapping, surjectivity.



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# Introduction

Maximal monotone operators appear in several branches of applied mathematics such as optimization, partial differential equations, and variational analysis. These operators were object of intense research between 1960 and 1980, when Brezis, Browder, Minty and Rockafellar established the fundamental results about them. The theory of convex representations of maximal monotone operators emerged at the end of the 1980s with the work [20] of S. Fitzpatrick<sup>1</sup>. It took some years until the Fitzpatrick's results were rediscovered by Martínez-Legaz and Théra [35] and Burachik and Svaiter [17]. Since then, this subject has called the attention of several researchers and has been object of intense research in the field of convex analysis, monotone operator theory and optimization [2, 3, 4, 11, 7, 9, 8, 10, 18, 33, 34, 31, 30, 29, 32, 36, 43]. We refer the reader to [17] for exploiting the relationship between convex representations and enlargements of maximal monotone operators.

The main point on Fitzpatrick's result is that it allows the use of convex analysis in the study of maximal monotone operators. The proof of many basic results can be simplified and also new results obtained, mostly in non-reflexive Banach spaces, with this new tool.

In this thesis we will present some of the new results on maximal monotone operators, as well as a simple proof of Rockafellar theorem on the maximality of the subdifferential. This thesis is build around the generalization of a Burachik-Svaiter theorem about maximal monotonicity on reflexive Banach spaces [18]. We generalized this result to non-reflexive Banach spaces. Together with some techniques used for this generalization it allowed us to obtain some other results, which are also presented here. A classical theorem of Rockafellar states that the subdifferential is maximal monotone. Unfortunately, most of the techniques Rockafellar used for proving this fact could not be adapted to our aims. For these reasons, a new and simpler proof of the maximal monotonicity of the subdifferential was the first result we obtained.

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<sup>1</sup><http://www.maths.uwa.edu.au/Members/tributes/fitzpatr>

## Presentation of chapters

**Chapter 1:** In this chapter we present the results of Fitzpatrick and Burachik-Svaiter on convex representations of maximal monotone operators. Fitzpatrick's results are summarized in Theorem 1.1.1. In Section 1.2, we called "The starting point", Burachik-Svaiter's results in this direction of research are presented.

**Chapter 2:** This chapter is the first step toward the study of maximal monotonicity in non-reflexive Banach spaces. We start, in Section 2.1, by presenting a new proof of maximality of the subdifferential of a convex function. This proof is simpler than Rockafellar's classical proof, and makes use of classical results from subdifferential calculus as Brøndsted-Rockafellar's Theorem and Fenchel-Rockafellar duality Theorem. We also observe that the proof can be simplified in reflexive spaces, and that it can be seen as a particular case of a more general maximality result presented in Theorem 2.2.5.

Section 2.2 is devoted to the study of monotone operators representable by convex functions satisfying condition (1.13). Theorem 2.2.5, states that condition (1.13) is a sufficient condition for maximal monotonicity in non-reflexive spaces, generalizing Theorem 1.2.3 for this non-reflexive setting. In Theorem 2.2.7 and Theorem 2.2.8, we shall prove that such operators satisfies a Brøndsted-Rockafellar type property. In this context, we further prove, in Theorem 2.2.10, that maximal monotone operators of type (NI) also satisfies this Brøndsted-Rockafellar type property. In Theorem 2.3.1 and Lemma 2.3.2, of Section 2.3, we prove some additional properties of operators of type (NI) and a sum theorem for this class of operators, respectively.

The results of this chapter were published in [28, 29, 31].

**Chapter 3:** This chapter is about surjectivity of perturbation of maximal monotone operators in non-reflexive Banach spaces. The results presented are from [30]. In a *reflexive* Banach space surjectivity of a monotone operator plus the duality mapping is equivalent to maximal monotonicity. This is a classical result of Rockafellar [38].

In [21] Gossez introduced an "enlarged" version of the duality mapping and proved similar Rockafellar's surjectivity results for the class of maximal monotone operators of type (D), introduced by himself.

The class of maximal monotone operators of type (NI), introduced by S. Simons in [39], encompasses the Gossez type (D) operators.

We shall make use of the analytical tools developed in Chapter 2 in order to obtain surjectivity results of perturbations of operators of type (NI) by using the Gossez “enlarged” duality mapping. The main results of the Chapter are present in Theorem 3.2.3.

**Chapter 4:** Any maximal monotone operator  $T : X \rightrightarrows X^*$  is also a monotone operator  $T : X^{**} \rightrightarrows X^*$  and admits one (or more) maximal monotone extension in  $X^{**} \times X^*$  that (in general) will not be unique.

This chapter approaches the problem of under which conditions a maximal monotone operator  $T : X \rightrightarrows X^*$  has a unique extension to the bidual. The Gossez type (D) maximal monotone operators have a unique maximal monotone extension to the bidual [21, 22, 23, 24].

We will prove that maximal monotone operators of type (NI) admit a unique extension to the bidual and that, for non-linear operators, the condition (NI) is equivalent to the unicity of maximal monotone extension to the bidual. For proving this equivalence we will show that if  $T \subset X \times X^*$  is maximal monotone and convex then  $T$  is an affine subspace. The results of this chapter are from the paper [31].

## Notations

$B_X[0, M]$	closed ball of $X$ with radius $M$
$\text{cl } f$	largest l.s.c. function majorized by $f$
$\text{conv } f$	largest convex function majorized by $f$
$\text{cl conv } f$	largest l.s.c. convex function majorized by $f$
$\text{dom}(f)$	effective domain of $f$
$\delta_A$	indicator function of $A$
$\varphi_T$	Fitzpatrick function of $T$
$\mathcal{F}_T$	Fitzpatrick family of $T$
$J$	duality mapping $J = \partial_{\frac{1}{2}} \ \cdot\ ^2$
$J_\varepsilon$	enlarged duality mapping $J_\varepsilon = \partial_{\varepsilon \frac{1}{2}} \ \cdot\ ^2$
$\Lambda h$	$\Lambda h(x, x^*) = h^*(x^*, x)$
$\pi$	duality product in $X \times X^*$
$\pi_*$	duality product in $X^* \times X^{**}$
$\overline{\mathbb{R}}$	extended real system $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$
$\overline{\mathbb{R}}^X$	set of functions of $X$ into $\overline{\mathbb{R}}$
$R(T)$	range of $T$
$R$	$R : X^{**} \times X^* \rightarrow X^* \times X^{**}, \quad R(x^{**}, x^*) = (x^*, x^{**})$
$\mathcal{S}_T$	$\mathcal{S}$ -function of $T$
$T^{-1}$	inverse of $T$

# Chapter 1

## Basic results and notation

Let  $X$  be a real Banach space with topological dual  $X^*$ . For  $x \in X$  and  $x^* \in X^*$  we will use the notation  $\langle x, x^* \rangle = x^*(x)$ . A point to set operator  $T : X \rightrightarrows X^*$  is a relation on  $X \times X^*$ :

$$T \subset X \times X^*$$

and  $x^* \in T(x)$  means  $(x, x^*) \in T$ . An operator  $T : X \rightrightarrows X^*$  is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \forall (x, x^*), (y, y^*) \in T$$

and it is *maximal monotone* if it is monotone and maximal (with respect to the inclusion) in the family of monotone operators of  $X$  into  $X^*$ .

### 1.1 Fitzpatrick functions

Brezis and Haraux defined in [12], for a maximal monotone operator  $T : X \rightrightarrows X^*$ , the function

$$\beta_T \in \bar{\mathbb{R}}^{X \times X^*}, \beta_T(x, x^*) = \sup_{(y, y^*) \in T} \langle x - y, y^* - x^* \rangle. \quad (1.1)$$

Note that if  $(x, x^*) \in T$  then the above inner product is always nonpositive, being equal to zero for  $(y, y^*) = (x, x^*)$ . So,  $\beta_T = 0$  in  $T$ . If  $(x, x^*) \notin T$ , since  $T$  is maximal monotone we conclude that  $\langle x - y, x^* - y^* \rangle < 0$  for some  $(y, y^*) \in T$  and so  $\beta_T(x, x^*) > 0$ . Therefore, for any  $(x, x^*) \in X \times X^*$

$$\beta_T(x, x^*) \geq 0, \quad \beta_T(x, x^*) = 0 \iff (x, x^*) \in T.$$

The function  $\beta_T$  provides a representation of the maximal monotone operator, but it lacks properties to be explored.

Fitzpatrick defined [20], for a maximal monotone operator  $T$ , the function

$$\varphi_T(x, x^*) = \sup_{(y, y^*) \in T} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle \quad (1.2)$$

$$= \sup_{(y, y^*) \in T} \langle y, x^* \rangle + \langle x, y^* \rangle - \langle y, y^* \rangle. \quad (1.3)$$

Using (1.2) and the previous observations we conclude that

$$\varphi_T(x, x^*) \geq \langle x, x^* \rangle, \quad \varphi_T(x, x^*) = \langle x, x^* \rangle \iff (x, x^*) \in T. \quad (1.4)$$

Note also from (1.3) that  $\varphi_T$ , being a sup of *linear* functions on  $(x, x^*)$ , is convex and lowersemicontinuous. The above equation generalizes, in some sense, Fenchel-Young inequality (2.1). Fitzpatrick also defined a family of convex functions associated with each maximal monotone operator  $T$ ,

$$\mathcal{F}_T = \left\{ h \in \bar{\mathbb{R}}^{X \times X^*} \left| \begin{array}{l} h \text{ is convex and lower semicontinuous} \\ h(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^* \\ (x, x^*) \in T \Rightarrow h(x, x^*) = \langle x, x^* \rangle \end{array} \right. \right\}. \quad (1.5)$$

and proved the next result:

**Theorem 1.1.1** ([20, Theorem 3.10]). *Let  $T : X \rightrightarrows X^*$  be maximal monotone. Then for any  $h \in \mathcal{F}_T$*

$$(x, x^*) \in T \iff h(x, x^*) = \langle x, x^* \rangle \quad (1.6)$$

and  $\varphi_T$  is the smallest element of the family  $\mathcal{F}_T$ .

*Proof.* Inclusion of  $\varphi_T$  in  $\mathcal{F}_T$  has already been proved in (1.4), and in the preceding discussion. To prove that  $\varphi_T$  is minimal in  $\mathcal{F}_T$ , take  $h \in \mathcal{F}_T$  and  $(y, y^*) \in T$ . For any  $(x, x^*)$  and  $p, q \geq 0$ ,  $p + q = 1$ , we have

$$\begin{aligned} \langle px + qy, px^* + qy^* \rangle &\leq h(px + qy, px^* + qy^*) \\ &\leq ph(x, x^*) + qh(y, y^*) \end{aligned} \quad (1.7)$$

where the first inequality follows for the fact that  $h$  majorizes the duality product and the second one from the convexity of  $h$ . Since  $(y, y^*) \in T$ ,  $h(y, y^*) = \langle y, y^* \rangle$ , which combined with (1.7) yields

$$p\langle x, x^* \rangle + q[\langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle] \leq h(x, x^*).$$

Now, taking limit as  $p \rightarrow 0$  ( $q \rightarrow 1$ ) in the above inequality and supremum over  $(y, y^*) \in T$  in the resulting inequality we conclude that  $\varphi_T \leq h$ . Therefore,  $\varphi_T$  is minimal in  $\mathcal{F}_T$  and (1.6) follows from this minimality. □

It is worth to note that by (1.6), each Fitzpatrick function fully characterizes the operator  $T$  which defines the family  $\mathcal{F}_T$ . This is exactly what we mean by a convex representation of a maximal monotone operator  $T$ . For instance, if  $f$  is a proper, lower semicontinuous convex function on  $X$ , then  $h_s \in \mathcal{F}_{\partial f}$ , where  $h_s(x, x^*) = f(x) + f^*(x^*)$ .

## 1.2 The starting point

In this section we will discuss previous results of Burachik and Svaiter, which are the starting point of this thesis. Fitzpatrick function combines duality product and conjugation. Before showing that, we will establish some notation and conventions.

Recall that the *conjugate* of  $f \in \bar{\mathbb{R}}^X$  is

$$f^* \in \bar{\mathbb{R}}^{X^*}, \quad f^*(x^*) = \sup_{x \in X} \langle x, x^* \rangle - f(x). \quad (1.8)$$

From now on we will denote the duality product by  $\pi$ ,

$$\pi \in \mathbb{R}^{X \times X^*}, \quad \pi(x, x^*) = \langle x, x^* \rangle. \quad (1.9)$$

We will also identify  $X$  with its image under the canonical injection into  $X^{**}$ . With these conventions, we have

$$\varphi_T(x, x^*) = (\pi + \delta_T)^*(x^*, x)$$

where  $\delta_T \in \bar{\mathbb{R}}^{X \times X^*}$  is the indicator function of  $T$ :

$$\begin{cases} 0, & \text{on } T, \\ \infty, & \text{otherwise} \end{cases}$$

In [17] Burachik and Svaiter observed that the family  $\mathcal{F}_T$  is closed under the supremum operation and defined its largest element, the so called  $\mathcal{S}$ -function:

$$\mathcal{S}_T \in \bar{\mathbb{R}}^{X \times X^*}, \quad \mathcal{S}_T = \sup_{h \in \mathcal{F}_T} h, \quad (1.10)$$

Hence,  $\mathcal{S}_T \in \mathcal{F}_T$ .

The next theorem gives an “explicit” expression of  $\mathcal{S}_T$  and use this function (together with  $\varphi_T$ ) to provide an alternative characterization of  $\mathcal{F}_T$  (to be used in Theorem 4.3.1).

**Theorem 1.2.1** ([17, Corollary 4.1, Remark 5.4]). *Let  $\varphi_T$  and  $\mathcal{S}_T$  be the Fitzpatrick and  $\mathcal{S}$ -function associated to  $T$ , respectively, as defined in (1.2) and (1.10). Then,*

1.  $\mathfrak{S}_T = \text{cl conv}(\pi + \delta_T)$ ,
2. for any convex lower semicontinuous function  $h \in \bar{\mathbb{R}}^{X \times X^*}$ ,
$$h \in \mathcal{F}_T \iff \varphi_T \leq h \leq \mathfrak{S}_T,$$
3. for any  $(x, x^*) \in X \times X^*$ ,  $\varphi_T(x, x^*) = (\mathfrak{S}_T)^*(x^*, x)$ ,
4. if  $X$  is reflexive, then for any  $(x, x^*) \in X \times X^*$   $\mathfrak{S}_T(x, x^*) = (\varphi_T)^*(x^*, x)$ .

If we define,

$$\Lambda : \bar{\mathbb{R}}^{X \times X^*} \rightarrow \bar{\mathbb{R}}^{X \times X^*}, \quad \Lambda h(x, x^*) := h^*(x^*, x), \quad (1.11)$$

according to the above theorem,  $\Lambda \mathfrak{S}_T = \varphi_T \in \mathcal{F}_T$ . So, it is natural to ask whether  $\Lambda$  maps  $\mathcal{F}_T$  into itself. Burachik and Svaiter also proved that this happens in fact:

**Theorem 1.2.2** ([17, Theorem 5.3]). *Suppose that  $T$  is maximal monotone. Then*

$$\Lambda h \in \mathcal{F}_T, \quad \forall h \in \mathcal{F}_T,$$

that is, if  $h \in \mathcal{F}_T$ , and

$$g : X \times X^* \rightarrow \bar{\mathbb{R}}, \quad g(x, x^*) = h^*(x^*, x),$$

then  $g \in \mathcal{F}_T$ . In a reflexive Banach space  $\Lambda \varphi_T = \mathfrak{S}_T$ .

It is interesting to note that  $\Lambda$  is an order-reversing mapping of  $\mathcal{F}_T$  into itself. This fact suggests that this mapping may have fixed points in  $\mathcal{F}_T$ . Svaiter proved [43] that if  $T$  is maximal monotone, then  $\Lambda$  always has a fixed point in  $\mathcal{F}_T$ .

Note that, by Theorem 1.2.2, if  $h \in \mathcal{F}_T$  then:

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x) &\geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*. \end{aligned} \quad (1.12)$$

In [18] Burachik and Svaiter proved that the converse of this implication holds in a reflexive Banach space:

**Theorem 1.2.3** ([18, Theorem 3.1]). *Let  $X$  be a reflexive Banach space. If  $h \in \bar{\mathbb{R}}^{X \times X^*}$  is proper, convex, l.s.c. and*

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^* \\ h^*(x^*, x) &\geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^* \end{aligned}$$

then

$$T := \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}$$

is maximal monotone and  $h, \Lambda h \in \mathcal{F}_T$ .

The proof of Theorem 1.2.3 was based on Rockafellar's surjectivity theorem (see Theorem 3.0.3). Unfortunately, this technique cannot be used in a non-reflexive setting. Motivated by this problem, in [29] we propose to replace (1.12) by the novel condition:

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle, & \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, & \forall (x^*, x^{**}) \in X^* \times X^{**}. \end{aligned} \tag{1.13}$$

The above condition generalizes (1.12) and allows one to prove Theorem 1.2.3 in non-reflexive spaces, by replacing condition (1.12) by (1.13) and by using the Fenchel-Rockafellar duality Theorem A.0.4.

It still remains as an open question to prove Theorem 1.2.3 in non-reflexive Banach spaces with the original Burachik-Svaiter's condition (1.12). In the last years, a great effort has been spent in trying to answer affirmatively such open question. Theorem 1.2.3 would give a affirmative answer to the celebrate Rockafellar's conjecture on the sum of maximal monotone operators in non-reflexive Banach spaces (see Lemma 2.3.2).

The main focus of this thesis is to study some generalizations of Theorem 1.2.3 to non-reflexive Banach spaces, and its theoretical implications in the theory of maximal monotone operators, with special emphasis for maximal monotone operators of type (NI).

The results presented in this thesis are contained in the works [28, 29, 30, 31, 32], all of them in collaboration with B. F. Svaiter.

## Chapter 2

# Maximality in non-reflexive Banach spaces

This chapter is concerned with the study of maximal monotonicity in non-reflexive Banach spaces. In Section 2.1, we give a new proof of the maximality of the subdifferential of a convex function. Section 2.2 is devoted to the study of monotone operators representable by convex functions satisfying condition (1.13). In Section 2.3 we prove some additional properties of operators of type (NI) and a sum theorem for this class of operators.

The results of this chapter were published in [28, 29, 31].

### 2.1 Maximality of subdifferentials

Recall that the *subdifferential* of  $f$  is the operator  $\partial f : X \rightrightarrows X^*$ ,

$$\partial f(x) = \{x^* \in X^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle, \forall y \in X\}.$$

Using the above definition, it is easy to check that if  $f$  is proper, convex and lower semicontinuous (l.s.c. for short), then  $\partial f$  is monotone and

$$\begin{aligned} f(x) + f^*(x^*) &\geq \langle x, x^* \rangle, \\ f(x) + f^*(x^*) &= \langle x, x^* \rangle \iff x^* \in \partial f(x). \end{aligned} \quad (2.1)$$

First proved by Rockafellar in [37], the maximal monotonicity of the subdifferential of a convex function is still object of study and several authors attempt to give simpler proofs to this fact (see [28] and references therein). Rockafellar's original proof is based on very important tools introduced by himself in [37]. In particular, he has proved a result of weak density for the graph of  $\partial f$  in the graph of  $\partial f^*$  that has been widely used in different situations in convex

analysis (see Theorem 6.1 of [17] for an application) and later on was introduced by Gossez in the context of maximal monotone operators of type (D) [23, 24, 21, 22].

In this section we present a short proof for the maximality of sub-differentials which makes use of classical results from subdifferential calculus like Brøndsted-Rockafellar's Theorem (Theorem A.0.3) and Fenchel-Rockafellar duality Theorem (Theorem A.0.4). We also observe that our proof can be still simplified in reflexive spaces, in particular in finite dimensional spaces, and it can be seen as a particular case of a more general maximality result presented in Theorem 2.2.5.

**Theorem 2.1.1.** *Let  $X$  be a Banach space. If  $f \in \bar{\mathbb{R}}^X$  is proper, convex and l.s.c., then  $\partial f : X \rightrightarrows X^*$  is maximal monotone.*

*Proof.* (Marques Alves-Svaiter) Monotonicity of  $\partial f$  is easy to check. Suppose that  $(x_0, x_0^*) \in X \times X^*$  is such that

$$\langle x - x_0, x^* - x_0^* \rangle \geq 0, \quad \forall x^* \in \partial f(x).$$

Define

$$f_0 \in \bar{\mathbb{R}}^X, \quad f_0(x) := f(x + x_0) - \langle x, x_0^* \rangle. \quad (2.2)$$

Applying Theorem A.0.4 to  $f_0$  and  $g(x) := \frac{1}{2}\|x\|^2$  we conclude that there exists  $x^* \in X^*$  such that

$$\inf_{x \in X} f_0(x) + \frac{1}{2}\|x\|^2 + f_0^*(x^*) + \frac{1}{2}\|x^*\|^2 = 0. \quad (2.3)$$

In particular, there exists a (minimizing) sequence  $\{x_n\}$  such that

$$\begin{aligned} \frac{1}{n^2} &\geq f_0(x_n) + \frac{1}{2}\|x_n\|^2 + f_0^*(x^*) + \frac{1}{2}\|x^*\|^2 \\ &\geq \langle x_n, x^* \rangle + \frac{1}{2}\|x_n\|^2 + \frac{1}{2}\|x^*\|^2 \\ &\geq \frac{1}{2}(\|x_n\| - \|x^*\|)^2 \geq 0, \end{aligned} \quad (2.4)$$

where the second inequality follows from Fenchel-Young inequality. Using the above equation we obtain

$$f_0(x_n) + f_0^*(x^*) - \langle x_n, x^* \rangle \leq 1/n^2.$$

Hence,  $x^* \in \partial_{1/n^2} f_0(x_n)$  and by Theorem A.0.3 it follows that there exist sequences  $\{z_n\}$  in  $X$  and  $\{z_n^*\}$  in  $X^*$  such that

$$z_n^* \in \partial f_0(z_n), \quad \|z_n^* - x^*\| \leq 1/n \quad \text{and} \quad \|z_n - x_n\| \leq 1/n. \quad (2.5)$$

Using the initial assumption, we also obtain

$$\langle z_n, z_n^* \rangle \geq 0. \quad (2.6)$$

Using (2.4) we get

$$\|x_n\| \rightarrow \|x^*\|, \quad \langle x_n, x^* \rangle \rightarrow -\|x^*\|^2, \quad \text{as } n \rightarrow \infty, \quad (2.7)$$

which, combined with (2.5) and (2.6) yields  $x^* = 0$ . Therefore,  $x_n \rightarrow 0$ . As  $f_0$  is l.s.c.,  $x = 0$  minimizes  $f_0(x) + \frac{1}{2}\|x\|^2$  and, using (2.3) we have

$$f_0(0) + f_0^*(0) = 0.$$

Therefore  $0 \in \partial f_0(0)$ , which is equivalent to  $x_0^* \in \partial f(x_0)$ .  $\square$

Notice that in a reflexive Banach space  $X$  (in particular, in finite dimensional vector spaces) the proof of Theorem 2.1.1 can be further simplified by taking a minimum on (2.3). This leads to the existence of  $z \in X$  such that

$$f_0(z) + f_0^*(x^*) = \langle z, x^* \rangle, \quad \frac{1}{2}\|z\|^2 + \frac{1}{2}\|x^*\|^2 + \langle z, x^* \rangle = 0$$

and so  $0 \in \partial f_0(0)$ , which finishes the proof.

It should also be noted that a similar (and simple) proof for Theorem 2.1.1 can be obtained directly from Theorem 2.2.5 by using  $h_s(x, x^*) := f(x) + f^*(x^*)$  as a convex representation for the monotone operator  $\partial f$ .

## 2.2 Maximality and Brøndsted-Rockafellar property

In this section we are interested in the study of maximality of monotone operators representable by convex functions satisfying condition (1.13). A remarkable result is Theorem 2.2.5, in which we prove that condition (1.13) is a sufficient condition for maximal monotonicity in non-reflexive spaces, generalizing Theorem 1.2.3 for this non-reflexive setting.

After proving Theorem 2.2.5 we will study *Brøndsted-Rockafellar property* for maximal monotone operators in non-reflexive Banach spaces. Burachik, Iusem and Svaiter [15] defined the  $\varepsilon$ -enlargement of  $T$  for  $\varepsilon \geq 0$ , as  $T^\varepsilon : X \rightrightarrows X^*$

$$T^\varepsilon(x) = \{x^* \in X^* \mid \langle x - y, x^* - y^* \rangle \geq -\varepsilon \quad \forall (y, y^*) \in T\}. \quad (2.8)$$

It is trivial to verify that  $T \subset T^\varepsilon$ . The  $\varepsilon$ -enlargement is a generalization of the  $\varepsilon$ -subdifferential of a convex function and has both theoretical and practical uses [41, 42, 19, 25, 26, 27]. An important question concerning the study of  $\varepsilon$ -enlargements of a maximal

monotone operator  $T$  is whether an element in the graph of  $T^\varepsilon$  can be approximated by an element in the graph of  $T$ .

A maximal monotone operator  $T : X \rightrightarrows X^*$  has the *Brøndsted-Rockafellar property* if, for any  $\varepsilon > 0$ ,

$$x^* \in T^\varepsilon(x) \Rightarrow \forall \lambda > 0, \exists (\bar{x}, \bar{x}^*) \in T, \quad \|x - \bar{x}\| \leq \lambda, \quad \|\bar{x}^* - x^*\| \leq \varepsilon/\lambda.$$

It does make sense to ask if every maximal monotone operator has Brøndsted-Rockafellar property. This question has been successfully solved for the extension  $\partial_\varepsilon f$ , of  $\partial f$ , by Brønsted and Rockafellar, as is showed in Theorem A.0.3. In the case of a general maximal monotone operator, the answer is affirmative in reflexive Banach spaces [44, 16] but is negative in the non-reflexive case [40].

The operator  $T$  satisfies the *strict Brøndsted-Rockafellar property* [29] if

$$\begin{aligned} x^* \in T^\varepsilon(x), \quad \eta > \varepsilon \Rightarrow \forall \lambda > 0, \exists (\bar{x}, \bar{x}^*) \in T, \\ \|x - \bar{x}\| < \lambda, \quad \|\bar{x}^* - x^*\| < \eta/\lambda. \end{aligned} \quad (2.9)$$

In Theorem 2.2.7 and Theorem 2.2.10 we will prove that maximal monotone operators representable by convex functions satisfying (1.13) and maximal monotone operators of type (NI) satisfies the strict Brøndsted-Rockafellar property.

The results of this section were published in [29, 31].

### 2.2.1 Preliminary results

The main result of this subsection is Theorem 2.2.4. We start by proving some technical results.

**Theorem 2.2.1.** (Marques Alves-Svaiter [29]) *Let  $h \in \bar{\mathbb{R}}^{X \times X^*}$  be a convex and l.s.c. function. If*

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}, \end{aligned}$$

*then for any  $\varepsilon > 0$  there exists  $(\tilde{x}, \tilde{x}^*) \in X \times X^*$  such that*

$$h(\tilde{x}, \tilde{x}^*) + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 < \varepsilon, \quad \|\tilde{x}\|^2 \leq h(0, 0), \quad \|\tilde{x}^*\|^2 \leq h(0, 0),$$

*where the two last inequalities are strict in the case  $h(0, 0) > 0$ .*

*Proof.* If  $h(0, 0) < \varepsilon$  then  $(\tilde{x}, \tilde{x}^*) = (0, 0)$  has the desired properties. The non-trivial case is

$$\varepsilon \leq h(0, 0), \quad (2.10)$$

which we consider now. Using the assumptions on  $h$ , we conclude that for any  $(x, x^*) \in X \times X^*$ ,

$$\begin{aligned} h(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 &\geq \langle x, x^* \rangle + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \\ &\geq -\|x\| \|x^*\| + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \\ &= \frac{1}{2} (\|x\| - \|x^*\|)^2 \geq 0. \end{aligned} \quad (2.11)$$

Analogously, for all  $(z^*, z^{**}) \in X^* \times X^{**}$ ,

$$\begin{aligned} h(z^*, z^{**}) + \frac{1}{2}\|z^*\|^2 + \frac{1}{2}\|z^{**}\|^2 &\geq \langle z^*, z^{**} \rangle + \frac{1}{2}\|z^*\|^2 + \frac{1}{2}\|z^{**}\|^2 \\ &\geq -\|z^*\| \|z^{**}\| + \frac{1}{2}\|z^*\|^2 + \frac{1}{2}\|z^{**}\|^2 \\ &= \frac{1}{2} (\|z^*\| - \|z^{**}\|)^2 \geq 0. \end{aligned} \quad (2.12)$$

Now using Theorem A.0.4 for the Banach space  $X \times X^*$  and  $f, g \in \overline{\mathbb{R}}^{X \times X^*}$ ,

$$f(x, x^*) := h(x, x^*), \quad g(x, x^*) := \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2$$

we conclude that there exists  $(\hat{z}^*, \hat{z}^{**}) \in X^* \times X^{**}$  such that

$$\inf_{(x, x^*)} h(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = -h^*(\hat{z}^*, \hat{z}^{**}) - \frac{1}{2}\|\hat{z}^*\|^2 - \frac{1}{2}\|\hat{z}^{**}\|^2.$$

As the right hand side of the above equation is non positive and the left hand side is non negative, these two terms are zero. Therefore,

$$\inf_{(x, x^*) \in X \times X^*} h(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad (2.13)$$

and

$$h^*(\hat{z}^*, \hat{z}^{**}) + \frac{1}{2}\|\hat{z}^*\|^2 + \frac{1}{2}\|\hat{z}^{**}\|^2 = 0. \quad (2.14)$$

For  $(z^*, z^{**}) = (\hat{z}^*, \hat{z}^{**})$ , all inequalities on (2.12) must hold as equalities. Therefore,

$$\|\hat{z}^*\|^2 = \|\hat{z}^{**}\|^2 = -h(\hat{z}^*, \hat{z}^{**}) \leq h(0, 0), \quad (2.15)$$

where the last inequality follows from the definition of conjugate.

Using (2.13) we conclude that for any  $\eta > 0$ , there exists  $(x_\eta, x_\eta^*) \in X \times X^*$  such that

$$h(x_\eta, x_\eta^*) + \frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta^*\|^2 < \eta. \quad (2.16)$$

If  $h(0, 0) = \infty$ , then, taking  $\eta = \varepsilon$  and  $(\tilde{x}, \tilde{x}^*) = (x_\eta, x_\eta^*)$  we conclude that the theorem holds. Now, we discuss the case  $h(0, 0) < \infty$ . In this case, using (2.15) we have

$$\|\hat{z}^*\| = \|\hat{z}^{**}\| \leq \sqrt{h(0, 0)}. \quad (2.17)$$

Note that from (2.10) we are considering

$$\varepsilon \leq h(0, 0) < \infty. \quad (2.18)$$

Combining (2.14) with (2.16) and using Fenchel-Young inequality (A.2) we obtain

$$\begin{aligned} \eta &> h(x_\eta, x_\eta^*) + \frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta^*\|^2 + h^*(\hat{z}^*, \hat{z}^{**}) + \frac{1}{2}\|\hat{z}^*\|^2 + \frac{1}{2}\|\hat{z}^{**}\|^2 \\ &\geq \langle x_\eta, \hat{z}^* \rangle + \langle x_\eta^*, \hat{z}^{**} \rangle + \frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta^*\|^2 + \frac{1}{2}\|\hat{z}^*\|^2 + \frac{1}{2}\|\hat{z}^{**}\|^2 \\ &\geq \frac{1}{2}\|x_\eta\|^2 - \|x_\eta\|\|\hat{z}^*\| + \frac{1}{2}\|\hat{z}^{**}\|^2 + \frac{1}{2}\|x_\eta^*\|^2 - \|x_\eta^*\|\|\hat{z}^{**}\| + \frac{1}{2}\|\hat{z}^{**}\|^2 \\ &= \frac{1}{2}(\|x_\eta\| - \|\hat{z}^*\|)^2 + \frac{1}{2}(\|x_\eta^*\| - \|\hat{z}^{**}\|)^2. \end{aligned}$$

As the two terms in the last inequality are non negative,

$$\|x_\eta\| < \|\hat{z}^*\| + \sqrt{2\eta}, \quad \|x_\eta^*\| < \|\hat{z}^{**}\| + \sqrt{2\eta}.$$

Therefore, using (2.17) we obtain

$$\|x_\eta\| < \sqrt{h(0, 0)} + \sqrt{2\eta}, \quad \|x_\eta^*\| < \sqrt{h(0, 0)} + \sqrt{2\eta}.$$

For finishing the proof, take in (2.16)

$$0 < \eta < \frac{\varepsilon^2}{2h(0, 0)} \quad (2.19)$$

and let

$$\tau = \frac{\sqrt{h(0, 0)}}{\sqrt{h(0, 0)} + \sqrt{2\eta}} \quad \tilde{x} = \tau x_\eta, \quad \tilde{x}^* = \tau x_\eta^*. \quad (2.20)$$

Then,

$$\|\tilde{x}\| < \sqrt{h(0, 0)}, \quad \|\tilde{x}^*\| < \sqrt{h(0, 0)}.$$

Now, using the convexity of  $h$  and of the square of the norms and (2.16), we have

$$\begin{aligned}
h(\tilde{x}, \tilde{x}^*) + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 &\leq (1 - \tau) h(0, 0) \\
&+ \tau \left( h(x_\eta, x_\eta^*) + \frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta^*\|^2 \right) \\
&< (1 - \tau) h(0, 0) + \tau \eta \\
&= h(0, 0) - \tau(h(0, 0) - \eta).
\end{aligned}$$

Therefore, using also (2.19)

$$\begin{aligned}
\varepsilon - \left( h(\tilde{x}, \tilde{x}^*) + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 \right) &\geq \varepsilon - h(0, 0) + \tau(h(0, 0) - \eta) \\
&> \varepsilon - h(0, 0) + \tau(h(0, 0) - 2\eta) \\
&= \varepsilon - h(0, 0) + \sqrt{h(0, 0)} \left( \sqrt{h(0, 0)} - \sqrt{2\eta} \right) \\
&= \varepsilon - \sqrt{2h(0, 0)\eta} > 0.
\end{aligned}$$

which completes the proof.  $\square$

In Theorem 2.2.1 the origin has a special role. In order to use this theorem with an arbitrary point, define [33], for  $h \in \bar{\mathbb{R}}^{X \times X^*}$  and  $(z, z^*) \in X \times X^*$ ,  $h_{(z, z^*)} \in \bar{\mathbb{R}}^{X \times X^*}$ ,

$$h_{(z, z^*)}(x, x^*) := h(x + z, x^* + z^*) - [\langle x, z^* \rangle + \langle z, x^* \rangle + \langle z, z^* \rangle]. \quad (2.21)$$

Notice that

$$h_{(z, z^*)}(x, x^*) - \langle x, x^* \rangle = h(x + z, x^* + z^*) - \langle x + z, x^* + z^* \rangle. \quad (2.22)$$

The operation  $h \mapsto h_{(z, z^*)}$  preserves many properties of  $h$ , as convexity, lower semicontinuity and can be seen as the action of the group  $(X \times X^*, +)$  on  $\bar{\mathbb{R}}^{X \times X^*}$ , because

$$(h_{(z_0, z_0^*)})_{(z_1, z_1^*)} = h_{(z_0 + z_1, z_0^* + z_1^*)}.$$

The proof of Theorem 2.2.5 will be heavily based on these nice properties of the map  $h \mapsto h_{(z, z^*)}$ . In the next proposition we prove that the class of l.s.c. convex functions  $h \in \bar{\mathbb{R}}^{X \times X^*}$  such that  $h \geq \pi$  and  $h^* \geq \pi_*$  is invariant under the map  $h \mapsto h_{(z, z^*)}$ :

**Proposition 2.2.2.** (Marques Alves-Svaiter [29]) *For any  $h \in \bar{\mathbb{R}}^{X \times X^*}$  it holds that:*

1.  $h$  is proper, convex and l.s.c.  $\iff h_{(z, z^*)}$  is proper, convex and l.s.c.,  $\forall (z, z^*) \in X \times X^*$ ;

2.  $(h_{(z,z^*)})^* = (h^*)_{(z^*,z)}$ , where the rightmost  $z$  is identified with its image under the canonical injection of  $X$  into  $X^{**}$ ;
3.  $h \geq \pi, h^* \geq \pi_* \iff h_{(z,z^*)} \geq \pi, (h_{(z,z^*)})^* \geq \pi_*, \forall (z, z^*) \in X \times X^*$ .

*Proof.* Item 1 is trivial to check. For proving item 2, take  $(x^*, x^{**}) \in X^* \times X^{**}$ . Then, using (2.21) we obtain

$$\begin{aligned}
(h_{(z,z^*)})^*(x^*, x^{**}) &= \sup_{(y, y^*)} \langle y, x^* \rangle + \langle y^*, x^{**} \rangle - h_{(z,z^*)}(y, y^*) \\
&= h^*(x^* + z^*, x^{**} + z) - [\langle x^*, z \rangle + \langle z^*, x^{**} \rangle + \langle z^*, z \rangle] \\
&= (h^*)_{(z^*,z)}(x^*, x^{**}).
\end{aligned} \tag{2.23}$$

It remains to prove item 3. For proving the “if”, note that  $h_{(0,0)} = h$ . For proving the “only if”, use (2.22) to conclude that  $h_{(z,z^*)} \geq \pi$  whenever  $h \geq \pi$ . By the same reasoning,  $(h^*)_{(z^*,z)} \geq \pi_*$  whenever  $h^* \geq \pi_*$  and so that using item 2 we end the proof of item 3.  $\square$

**Corollary 2.2.3.** (Marques Alves-Svaiter [29]) *Let  $h \in \bar{\mathbb{R}}^{X \times X^*}$  be a convex and l.s.c. function. If*

$$\begin{aligned}
h(x, x^*) &\geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*, \\
h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**},
\end{aligned}$$

*then for any  $(z, z^*) \in X \times X^*$  and  $\varepsilon > 0$  there exist  $(\tilde{x}, \tilde{x}^*) \in X \times X^*$  such that*

$$\begin{aligned}
h(\tilde{x}, \tilde{x}^*) &< \langle \tilde{x}, \tilde{x}^* \rangle + \varepsilon, \\
\|\tilde{x} - z\|^2 &\leq h(z, z^*) - \langle z, z^* \rangle, \\
\|\tilde{x}^* - z^*\|^2 &\leq h(z, z^*) - \langle z, z^* \rangle,
\end{aligned}$$

*where the two last inequalities are strict in the case  $\langle z, z^* \rangle < h(z, z^*)$ .*

*Proof.* If  $h(z, z^*) = \langle z, z^* \rangle$  then  $(\tilde{x}, \tilde{x}^*) = (z, z^*)$  satisfy the desired conditions. Assume that

$$0 < h(z, z^*) - \langle z, z^* \rangle. \tag{2.24}$$

Using Proposition 2.2.2 and Theorem 2.2.1 we conclude that there exists  $(\tilde{y}, \tilde{y}^*) \in X \times X^*$  such that

$$\begin{aligned}
h_{(z,z^*)}(\tilde{y}, \tilde{y}^*) + \frac{1}{2}\|\tilde{y}\|^2 + \frac{1}{2}\|\tilde{y}^*\|^2 &< \varepsilon, \\
\|\tilde{y}\|^2 &< h_{(z,z^*)}(0, 0), \\
\|\tilde{y}^*\|^2 &< h_{(z,z^*)}(0, 0).
\end{aligned} \tag{2.25}$$

Using (2.22), we obtain  $h_{(z,z^*)}(0,0) = h(z,z^*) - \langle z, z^* \rangle$ . Let

$$\tilde{x} := \tilde{y} + z, \quad \tilde{x}^* := \tilde{y}^* + z^*.$$

Therefore, using (2.25) and (2.24), we have

$$\|\tilde{x} - z\|^2 < h(z, z^*) - \langle z, z^* \rangle, \quad \|\tilde{x}^* - z^*\|^2 < h(z, z^*) - \langle z, z^* \rangle.$$

For finishing the proof of the corollary, use (2.22) and (2.25) to obtain

$$\begin{aligned} h(\tilde{x}, \tilde{x}^*) - \langle \tilde{x}, \tilde{x}^* \rangle &= h_{(z,z^*)}(\tilde{y}, \tilde{y}^*) - \langle \tilde{y}, \tilde{y}^* \rangle \\ &\leq h_{(z,z^*)}(\tilde{y}, \tilde{y}^*) + \frac{1}{2}\|\tilde{y}\|^2 + \frac{1}{2}\|\tilde{y}^*\|^2 < \varepsilon. \end{aligned}$$

□

Now we come with the main result of this subsection. It is important by itself and will be used in the next sections, specially for proving Theorem 2.2.5.

**Theorem 2.2.4.** (Marques Alves-Svaiter [29]) *Let  $h \in \bar{\mathbb{R}}^{X \times X^*}$  be convex, l.s.c. and*

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}. \end{aligned}$$

If  $(x, x^*) \in X \times X^*$ ,  $\varepsilon > 0$  and

$$h(x, x^*) < \langle x, x^* \rangle + \varepsilon,$$

then, for any  $\lambda > 0$  there exists  $(\bar{x}_\lambda, \bar{x}_\lambda^*) \in X \times X^*$  such that

$$h(\bar{x}_\lambda, \bar{x}_\lambda^*) = \langle \bar{x}_\lambda, \bar{x}_\lambda^* \rangle, \quad \|\bar{x}_\lambda - x\| < \lambda, \quad \|\bar{x}_\lambda^* - x^*\| < \frac{\varepsilon}{\lambda}.$$

*Proof.* First, suppose that  $\lambda = \sqrt{\varepsilon}$ . If  $h(x, x^*) - \langle x, x^* \rangle = 0$ , then  $(x, x^*)$  has the desired properties. So, suppose also that  $h(x, x^*) - \langle x, x^* \rangle > 0$ . Let  $\varepsilon_0 > 0$  and  $\theta \in (0, 1)$  be such that

$$0 < h(x, x^*) - \langle x, x^* \rangle < \varepsilon_0 < \varepsilon, \quad \frac{\sqrt{\varepsilon_0}}{1 + \sqrt{\theta}} < \sqrt{\varepsilon}. \quad (2.26)$$

Define inductively a sequence  $\{(x_k, x_k^*)\}$  as follows: For  $k = 0$ , let

$$(x_0, x_0^*) = (x, x^*). \quad (2.27)$$

Given  $k$  and  $(x_k, x_k^*)$ , use Corollary 2.2.3 to conclude that there exists some  $(x_{k+1}, x_{k+1}^*)$  such that

$$h(x_{k+1}, x_{k+1}^*) - \langle x_{k+1}, x_{k+1}^* \rangle < \theta^{k+1} \varepsilon_0 \quad (2.28)$$

and

$$\begin{aligned}\|x_{k+1} - x_k\| &\leq \sqrt{h(x_k, x_k^*) - \langle x_k, x_k^* \rangle}, \\ \|x_{k+1}^* - x_k^*\| &\leq \sqrt{h(x_k, x_k^*) - \langle x_k, x_k^* \rangle}.\end{aligned}\quad (2.29)$$

Using (2.26) and (2.28) we conclude that for all  $k$ ,

$$0 \leq h(x_k, x_k^*) - \langle x_k, x_k^* \rangle < \theta^k \varepsilon_0, \quad (2.30)$$

which, combined with (2.29) yields

$$\sum_{k=0}^{\infty} \|x_{k+1} - x_k\| < \sqrt{\varepsilon_0} \sum_{k=0}^{\infty} \sqrt{\theta^k}, \quad \sum_{k=0}^{\infty} \|x_{k+1}^* - x_k^*\| < \sqrt{\varepsilon_0} \sum_{k=0}^{\infty} \sqrt{\theta^k}.$$

The second part of (2.26) gives

$$\sum_{k=0}^{\infty} \|x_{k+1} - x_k\| < \sqrt{\varepsilon}, \quad \sum_{k=0}^{\infty} \|x_{k+1}^* - x_k^*\| < \sqrt{\varepsilon}. \quad (2.31)$$

In particular, the sequences  $\{x_k\}$  and  $\{x_k^*\}$  are convergent. Let

$$\bar{x} := \lim_{k \rightarrow \infty} x_k, \quad \bar{x}^* := \lim_{k \rightarrow \infty} x_k^*.$$

Then, using (2.31) we have

$$\|\bar{x} - x\| < \sqrt{\varepsilon}, \quad \|\bar{x}^* - x^*\| < \sqrt{\varepsilon}.$$

Using (2.30) we have

$$\lim_{k \rightarrow \infty} h(x_k, x_k^*) - \langle x_k, x_k^* \rangle = 0.$$

As  $h$  is l.s.c. and the duality product is continuous (in the strong topology of  $X \times X^*$ ),

$$h(\bar{x}, \bar{x}^*) - \langle \bar{x}, \bar{x}^* \rangle \leq 0.$$

Therefore,  $h(\bar{x}, \bar{x}^*) - \langle \bar{x}, \bar{x}^* \rangle = 0$ , which ends the proof of the theorem for  $\lambda = \sqrt{\varepsilon}$ . To prove the general case, use in  $X$  the norm

$$\| \|x\| \| := \frac{\sqrt{\varepsilon}}{\lambda} \|x\|,$$

and apply the previous case in this re-normed space.  $\square$

## 2.2.2 Maximality of representable monotone operators

Next we present one of the main results of this thesis. It generalizes Theorem 1.2.3 for non-reflexive Banach spaces by replacing condition (1.12) by condition (1.13).

**Theorem 2.2.5.** (Marques Alves-Svaiter [29]) *Let  $h \in \bar{\mathbb{R}}^{X \times X^*}$  be a convex and l.s.c. function. If*

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle, & \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, & \forall (x^*, x^{**}) \in X^* \times X^{**}, \end{aligned}$$

then

$$T := \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}$$

is maximal monotone and  $h, \Lambda h \in \mathcal{F}_T$ .

*Proof.* The duality product  $\pi : X \times X^* \rightarrow \mathbb{R}$  is everywhere differentiable and

$$\pi'(x, x^*) = (x^*, x).$$

Suppose that  $h(x, x^*) = \langle x, x^* \rangle = \pi(x, x^*)$ . Then, by Lemma A.0.5  $(x^*, x) \in \partial h(x, x^*)$ , that is,

$$h(x, x^*) + h^*(x^*, x) = \langle (x, x^*), (x^*, x) \rangle. \quad (2.32)$$

Take  $(x, x^*), (y, y^*) \in T$ . Then, as was explained above,

$$(x^*, x) \in \partial h(x, x^*), \quad (y^*, y) \in \partial h(y, y^*).$$

Since  $\partial h$  is monotone,

$$\langle (x, x^*) - (y, y^*), (x^*, x) - (y^*, y) \rangle \geq 0,$$

which gives  $\langle x - y, x^* - y^* \rangle \geq 0$ . Hence,  $T$  is monotone.

Now, if  $h(x, x^*) = \langle x, x^* \rangle$ , then using (2.32) we have  $h^*(x^*, x) = \langle x, x^* \rangle$ . Conversely, if  $h^*(x^*, x) = \langle x, x^* \rangle$ , then by the same reasoning  $h^{**}(x, x^*) = \langle x, x^* \rangle$ . As  $h$  is proper, convex and l.s.c.,  $h(x, x^*) = h^{**}(x, x^*)$ . Thus,

$$T = \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}. \quad (2.33)$$

For proving maximal monotonicity of  $T$ , take  $(z, z^*) \in X \times X^*$  and assume that

$$\langle x - z, x^* - z^* \rangle \geq 0, \quad \forall (x, x^*) \in T. \quad (2.34)$$

Using Theorem 2.2.1 and Proposition 2.2.2 we know that

$$\inf_{(x, x^*) \in X \times X^*} h_{(z, z^*)}(x, x^*) + \frac{1}{2} \|x\|^2 + \frac{1}{2} \|x^*\|^2 = 0.$$

Therefore, there exists a minimizing sequence  $\{(x_k, x_k^*)\}$  such that

$$h_{(z, z^*)}(x_k, x_k^*) + \frac{1}{2} \|x_k\|^2 + \frac{1}{2} \|x_k^*\|^2 < \frac{1}{k^2}, \quad k = 1, 2, \dots \quad (2.35)$$

Note that the sequence  $\{(x_k, x_k^*)\}$  is bounded and

$$\begin{aligned} h_{(z, z^*)}(x_k, x_k^*) - \langle x_k, x_k^* \rangle &\leq h_{(z, z^*)}(x_k, x_k^*) + \|x_k\| \|x_k^*\| \\ &\leq h_{(z, z^*)}(x_k, x_k^*) + \frac{1}{2} \|x_k\|^2 + \frac{1}{2} \|x_k^*\|^2. \end{aligned}$$

Combining the two above inequalities we obtain

$$h_{(z, z^*)}(x_k, x_k^*) < \langle x_k, x_k^* \rangle + \frac{1}{k^2}.$$

Now applying Theorem 2.2.4, we conclude that there for each  $k$  there exists some  $(\bar{x}_k, \bar{x}_k^*)$  such that

$$h_{(z, z^*)}(\bar{x}_k, \bar{x}_k^*) = \langle \bar{x}_k, \bar{x}_k^* \rangle, \quad \|\bar{x}_k - x_k\| < 1/k, \quad \|\bar{x}_k^* - x_k^*\| < 1/k.$$

Then,

$$(\bar{y}_k, \bar{y}_k^*) := (\bar{x}_k + z, \bar{x}_k^* + z^*) \in T,$$

and from (2.34)

$$\langle \bar{x}_k, \bar{x}_k^* \rangle = \langle \bar{y}_k - z, \bar{y}_k^* - z^* \rangle \geq 0.$$

The duality product is uniformly continuous on bounded sets. Since  $\{(x_k, x_k^*)\}$  is bounded and  $\lim_{k \rightarrow \infty} \|x_k - \bar{x}_k\| = \lim_{k \rightarrow \infty} \|x_k^* - \bar{x}_k^*\| = 0$  we conclude that

$$\liminf_{k \rightarrow \infty} \langle x_k, x_k^* \rangle \geq 0.$$

Using (2.35) and the fact that  $h$  majorizes the duality product, we have

$$0 \leq \langle x_k, x_k^* \rangle + \frac{1}{2} \|x_k\|^2 + \frac{1}{2} \|x_k^*\|^2 \leq h_{(z, z^*)}(x_k, x_k^*) + \frac{1}{2} \|x_k\|^2 + \frac{1}{2} \|x_k^*\|^2 < \frac{1}{k^2}.$$

Hence,  $\langle x_k, x_k^* \rangle < 1/k^2$  and  $\limsup_{k \rightarrow \infty} \langle x_k, x_k^* \rangle \leq 0$ , which implies  $\lim_{k \rightarrow \infty} \langle x_k, x_k^* \rangle = 0$ . Combining this result with the above inequalities we conclude that

$$\lim_{k \rightarrow \infty} (x_k, x_k^*) = 0.$$

Therefore,  $\lim_{k \rightarrow \infty} (\bar{x}_k, \bar{x}_k^*) = 0$  and  $\{(\bar{y}_k, \bar{y}_k^*)\}$  converges to  $(z, z^*)$ . As  $h(\bar{y}_k, \bar{y}_k^*) = \langle \bar{y}_k, \bar{y}_k^* \rangle$  and  $h$  is lower semicontinuous,

$$h(z, z^*) \leq \langle z, z^* \rangle.$$

which readily implies  $h(z, z^*) = \langle z, z^* \rangle$ . Therefore  $(z, z^*) \in T$  and so that  $T$  is maximal monotone and  $h \in \mathcal{F}_T$ . Finally, using (2.33) we conclude that  $\Lambda h \in \mathcal{F}_T$ .  $\square$

The duality product is continuous in  $X \times X^*$ . Therefore, if a convex function majorizes the duality product then the convex closure of this function also majorizes it and has the same conjugate. This fact can be used to remove the assumption of lower semicontinuity of  $h$  in Theorem 2.2.5:

**Theorem 2.2.6.** (Marques Alves-Svaiter [29]) *Let  $h \in \bar{\mathbb{R}}^{X \times X^*}$  be a convex function. If*

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle, & \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, & \forall (x^*, x^{**}) \in X^* \times X^{**}, \end{aligned} \quad (2.36)$$

then

$$S := \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}$$

is maximal monotone and  $\text{cl } h, \Lambda h \in \mathcal{F}_S$ .

*Proof.* Define  $\bar{h} := \text{cl } h$ . Then,  $\bar{h}$  is proper, convex, l.s.c. and  $(\bar{h})^* = h^*$ . Since the duality product is continuous,  $\bar{h}$  also satisfies (2.36). Thus, applying Theorem 2.2.5 to  $\bar{h}$ , we have that  $S$  is maximal monotone and  $\text{cl } h, \Lambda h \in \mathcal{F}_S$ .  $\square$

### 2.2.3 Brøndsted-Rockafellar type theorems for representable and of type (NI) operators

The class of maximal monotone operators of type (NI) was introduced by S. Simons for generalizing some results in reflexive Banach spaces to non-reflexive spaces:

**Definition 2.2.1.** ([39]) *A maximal monotone operator  $T : X \rightrightarrows X^*$  is of type (NI) if*

$$\inf_{(y, y^*) \in T} \langle y^* - x^*, y - x^{**} \rangle \leq 0, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

As pointed out before, in this subsection we aim to prove that maximal monotone operators representable by convex function satisfying (1.13) and operators of type (NI) satisfy the strict Brøndsted-Rockafellar property.

**Theorem 2.2.7.** (Marques Alves-Svaiter [29]) *Let  $h \in \bar{\mathbb{R}}^{X \times X^*}$  be a convex and l.s.c. function. If*

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle, & \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, & \forall (x^*, x^{**}) \in X^* \times X^{**}, \end{aligned}$$

then the maximal monotone operator (see Theorem 2.2.5)

$$T := \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}$$

satisfies the strict Brøndsted-Rockafellar property: If  $\eta > \varepsilon$  and  $x^* \in T^\varepsilon(x)$ , that is,

$$\langle x - y, x^* - y^* \rangle \geq -\varepsilon, \quad \forall (y, y^*) \in T,$$

then, for any  $\lambda > 0$  there exists  $(\bar{x}_\lambda, \bar{x}_\lambda^*) \in X \times X^*$  such that

$$\bar{x}_\lambda^* \in T(\bar{x}_\lambda), \quad \|x - \bar{x}_\lambda\| < \lambda, \quad \|x^* - \bar{x}_\lambda^*\| < \frac{\eta}{\lambda}.$$

*Proof.* Assume that  $\eta > \varepsilon > 0$  and

$$\langle x - y, x^* - y^* \rangle \geq -\varepsilon, \quad \forall (y, y^*) \in T.$$

The Fitzpatrick function of  $T$  is

$$\begin{aligned} \varphi_T(x, x^*) &= \sup_{(y, y^*) \in T} \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle \\ &= \sup_{(y, y^*) \in T} -\langle x - y, x^* - y^* \rangle + \langle x, x^* \rangle. \end{aligned}$$

Therefore

$$\varphi_T(x, x^*) \leq \langle x, x^* \rangle + \varepsilon < \langle x, x^* \rangle + \eta.$$

Now recall that, as  $T$  is maximal monotone,  $\varphi_T$  is the smallest element of the family  $\mathcal{F}_T$ . In particular  $h \geq \varphi_T$ . Hence,

$$\varphi_T^* \geq h^*,$$

which implies that  $\varphi_T$  satisfies the hypothesis of Theorem 2.2.4. Thus, there exists  $(\bar{x}_\lambda, \bar{x}_\lambda^*)$  such that

$$\varphi_T(\bar{x}_\lambda, \bar{x}_\lambda^*) = \langle \bar{x}_\lambda, \bar{x}_\lambda^* \rangle, \quad \|x - \bar{x}_\lambda\| < \lambda, \quad \|x^* - \bar{x}_\lambda^*\| < \frac{\eta}{\lambda}.$$

The first equality above says that  $(\bar{x}_\lambda, \bar{x}_\lambda^*) \in T$ , which ends the proof of the theorem.  $\square$

**Theorem 2.2.8.** (Marques Alves-Svaiter [29]) *Let  $h \in \bar{\mathbb{R}}^{X \times X^*}$  be a convex function. If*

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}, \end{aligned}$$

then the maximal monotone operator (see Theorem 2.2.6)

$$S := \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}$$

satisfies the strict Brøndsted-Rockafellar property: If  $\eta > \varepsilon$  and  $x^* \in S^\varepsilon(x)$ , that is,

$$\langle x - y, x^* - y^* \rangle \geq -\varepsilon, \quad \forall (y, y^*) \in S,$$

then, for any  $\lambda > 0$  there exists  $(\bar{x}_\lambda, \bar{x}_\lambda^*) \in X \times X^*$  such that

$$\bar{x}_\lambda^* \in S(\bar{x}_\lambda), \quad \|x - \bar{x}_\lambda\| < \lambda, \quad \|x^* - \bar{x}_\lambda^*\| < \frac{\eta}{\lambda}.$$

In Theorem 2.2.10 we shall prove that the class of maximal monotone operators of type (NI) satisfies the strict Brøndsted-Rockafellar property. The starting point of the proof is a characterization of the class of operators of type (NI) given by the  $\mathfrak{S}$ -function:

**Proposition 2.2.9.** (Marques Alves-Svaiter [31]) *A maximal monotone operator  $T : X \rightrightarrows X^*$  is of type (NI) if, and only if,*

$$(\mathfrak{S}_T)^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

*Proof.* Recall that  $\mathfrak{S}_T = \text{clconv}(\pi + \delta_T)$ . The proof follows directly from the identity below:

$$\begin{aligned} (\mathfrak{S}_T)^*(x^*, x^{**}) &= (\pi + \delta_T)^*(x^*, x^{**}) \\ &= \sup_{(y, y^*) \in T} \langle y, x^* \rangle + \langle y^*, x^{**} \rangle - \langle y, y^* \rangle \\ &= - \inf_{(y, y^*) \in T} \langle x^* - y^*, x^{**} - y \rangle + \langle x^*, x^{**} \rangle \end{aligned}$$

□

**Theorem 2.2.10.** (Marques Alves-Svaiter [31]) *Let  $T : X \rightrightarrows X^*$  be a maximal monotone operator of type (NI). Then,  $T$  satisfies the strict Brøndsted-Rockafellar property: If  $\eta > \varepsilon$  and  $x^* \in T^\varepsilon(x)$ , that is,*

$$\langle x - y, x^* - y^* \rangle \geq -\varepsilon, \quad \forall (y, y^*) \in T,$$

then, for any  $\lambda > 0$  there exists  $(\bar{x}_\lambda, \bar{x}_\lambda^*) \in X \times X^*$  such that

$$\bar{x}_\lambda^* \in T(\bar{x}_\lambda), \quad \|x - \bar{x}_\lambda\| < \lambda, \quad \|x^* - \bar{x}_\lambda^*\| < \frac{\eta}{\lambda}.$$

*Proof.* Recall that  $\mathfrak{S}_T \in \mathcal{F}_T$  and so

$$T = \{(x, x^*) \in X \times X^* \mid \mathfrak{S}_T(x, x^*) = \langle x, x^* \rangle\}.$$

Using the fact that  $\mathfrak{S}_T \in \mathcal{F}_T$  and Proposition 2.2.9, we have that

$$\begin{aligned} \mathfrak{S}_T(x, x^*) &\geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*, \\ (\mathfrak{S}_T)^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}, \end{aligned}$$

Thus, the result follows from Theorem 2.2.7. □

## 2.3 A new characterization and a sum theorem for operators of type (NI)

Proposition 2.2.9 says that if a maximal monotone operator  $T : X \rightrightarrows X^*$  is of type (NI) then there exists  $h \in \mathcal{F}_T$ , namely  $h = \mathcal{S}_T$ , such that

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle, & \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, & \forall (x^*, x^{**}) \in X^* \times X^{**}. \end{aligned} \quad (2.37)$$

In Theorem 2.2.5 we used the fact that if  $h$  satisfies condition (2.37), then it also satisfies the following *variational condition*:

$$\inf_{(x, x^*)} h_{(x_0, x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x_0^*). \quad (2.38)$$

In Theorem 2.3.1 we will show that conditions (2.37) and (2.38) are equivalent and that if *some*  $h \in \mathcal{F}_T$  satisfies condition (2.37), then *all* function in the Fitzpatrick family of  $T$  satisfies condition (2.37). Remember that if  $h \in \mathcal{F}_T$ , then  $h \geq \pi$  holds by definition of  $\mathcal{F}_T$ . In particular, (2.38) provides a sort of *variational characterization* of the class of maximal monotone operators of type (NI).

**Theorem 2.3.1.** (Marques Alves-Svaiter [32]) *Let  $T : X \rightrightarrows X^*$  be maximal monotone. The following conditions are equivalent:*

1.  $T$  is of type (NI),
2. there exists  $h \in \mathcal{F}_T$  such that

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle, & \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, & \forall (x^*, x^{**}) \in X^* \times X^{**}. \end{aligned}$$

3. for all  $h \in \mathcal{F}_T$ ,

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle, & \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, & \forall (x^*, x^{**}) \in X^* \times X^{**}. \end{aligned}$$

4. there exists  $h \in \mathcal{F}_T$  such that

$$\inf_{(x, x^*)} h_{(x_0, x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x_0^*) \in X \times X^*,$$

5. for all  $h \in \mathcal{F}_T$ ,

$$\inf_{(x, x^*)} h_{(x_0, x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x_0^*) \in X \times X^*.$$

*Proof.* First let us prove that item 2 and item 4 are equivalent. Using Theorem 2.2.1 we conclude that item 2 implies item 4. For proving that item 4 implies item 2, first note that, for any  $(z, z^*) \in X \times X^*$ ,

$$h_{(z, z^*)}(0, 0) \geq \inf_{(x, x^*)} h_{(z, z^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2.$$

Therefore, using item 4 we obtain

$$h(z, z^*) - \langle z, z^* \rangle = h_{(z, z^*)}(0, 0) \geq 0.$$

Since  $(z, z^*)$  is an arbitrary element of  $X \times X^*$  we conclude that  $h \geq \pi$ .

For proving that  $h^* \geq \pi_*$ , take some  $(y^*, y^{**}) \in X^* \times X^{**}$ . First, use Fenchel-Young inequality to conclude that for any  $(x, x^*), (z, z^*) \in X \times X^*$ ,

$$h_{(z, z^*)}(x, x^*) \geq \langle x, y^* - z^* \rangle + \langle x^*, y^{**} - z \rangle - (h_{(z, z^*)})^*(y^* - z^*, y^{**} - z).$$

$$\text{As } (h_{(z, z^*)})^* = (h^*)_{(z^*, z)},$$

$$(h_{(z, z^*)})^*(y^* - z^*, y^{**} - z) = h^*(y^*, y^{**}) - \langle y^*, y^{**} \rangle + \langle y^* - z^*, y^{**} - z \rangle.$$

Combining the two above equations we obtain

$$\begin{aligned} h_{(z, z^*)}(x, x^*) &\geq \langle x, y^* - z^* \rangle + \langle x^*, y^{**} - z \rangle \\ &\quad - \langle y^* - z^*, y^{**} - z \rangle + \langle y^*, y^{**} \rangle - h^*(y^*, y^{**}). \end{aligned}$$

Adding  $(1/2)\|x\|^2 + (1/2)\|x^*\|^2$  in both sides of the above inequality we have

$$\begin{aligned} h_{(z, z^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 &\geq \langle x, y^* - z^* \rangle + \langle x^*, y^{**} - z \rangle + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \\ &\quad - \langle y^* - z^*, y^{**} - z \rangle + \langle y^*, y^{**} \rangle - h^*(y^*, y^{**}). \end{aligned}$$

Note that

$$\langle x, y^* - z^* \rangle + \frac{1}{2}\|x\|^2 \geq -\frac{1}{2}\|y^* - z^*\|^2, \quad \langle x^*, y^{**} - z \rangle + \frac{1}{2}\|x^*\|^2 \geq -\frac{1}{2}\|y^{**} - z\|^2.$$

Therefore, for any  $(x, x^*), (z, z^*) \in X \times X^*$ ,

$$\begin{aligned} h_{(z, z^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 &\geq -\frac{1}{2}\|y^* - z^*\|^2 - \frac{1}{2}\|y^{**} - z\|^2 \\ &\quad - \langle y^* - z^*, y^{**} - z \rangle + \langle y^*, y^{**} \rangle - h^*(y^*, y^{**}). \end{aligned}$$

Using now the assumption we conclude that the infimum, for  $(x, x^*) \in X \times X^*$ , at the left hand side of the above inequality is 0. Therefore,

taking the infimum on  $(x, x^*) \in X \times X^*$  at the left hand side of the above inequality and rearranging the resulting inequality we have

$$h^*(y^*, y^{**}) - \langle y^*, y^{**} \rangle \geq -\frac{1}{2}\|y^* - z^*\|^2 - \frac{1}{2}\|y^{**} - z\|^2 - \langle y^* - z^*, y^{**} - z \rangle.$$

Note that

$$\sup_{z^* \in X^*} -\langle y^* - z^*, y^{**} - z \rangle - \frac{1}{2}\|y^* - z^*\|^2 = \frac{1}{2}\|y^{**} - z\|^2.$$

Hence, taking the sup in  $z^* \in X^*$  at the right hand side of the previous inequality, we obtain

$$h^*(y^*, y^{**}) - \langle y^*, y^{**} \rangle \geq 0$$

and item 4 holds. Now, using that item 2 and item 4 are equivalent it is trivial to verify that item 3 and item 5 are equivalent.

The second step is to prove that item 4 and item 5 are equivalent. So, assume that item 4 holds, that is, for some  $h \in \mathcal{F}_T$ ,

$$\inf_{(x, x^*) \in X \times X^*} h_{(x_0, x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x_0^*) \in X \times X^*.$$

Take  $g \in \mathcal{F}_T$ , and  $(x_0, x_0^*) \in X \times X^*$ . First observe that, for any  $(x, x^*) \in X \times X^*$ ,  $g_{(x_0, x_0^*)}(x, x^*) \geq \langle x, x^* \rangle$  and

$$g_{(x_0, x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \geq \langle x, x^* \rangle + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \geq 0.$$

Therefore,

$$\inf_{(x, x^*) \in X \times X^*} g_{(x_0, x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \geq 0. \quad (2.39)$$

As the square of the norm is coercive, there exist  $M > 0$  such that

$$\left\{ (x, x^*) \in X \times X^* \mid h_{(x_0, x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 < 1 \right\} \subset B_{X \times X^*}(0, M),$$

where

$$B_{X \times X^*}(0, M) = \left\{ (x, x^*) \in X \times X^* \mid \sqrt{\|x\|^2 + \|x^*\|^2} < M \right\}.$$

For any  $\varepsilon > 0$ , there exists  $(\tilde{x}, \tilde{x}^*)$  such that

$$\min \{1, \varepsilon^2\} > h_{(x_0, x_0^*)}(\tilde{x}, \tilde{x}^*) + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2.$$

Therefore

$$\begin{aligned} \varepsilon^2 &> h_{(x_0, x_0^*)}(\tilde{x}, \tilde{x}^*) + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 \geq h_{(x_0, x_0^*)}(\tilde{x}, \tilde{x}^*) - \langle \tilde{x}, \tilde{x}^* \rangle \geq 0, \\ M^2 &\geq \|\tilde{x}\|^2 + \|\tilde{x}^*\|^2. \end{aligned} \quad (2.40)$$

In particular,

$$\varepsilon^2 > h_{(x_0, x_0^*)}(\tilde{x}, \tilde{x}^*) - \langle \tilde{x}, \tilde{x}^* \rangle.$$

Now using Theorem 2.2.4 we conclude that there exists  $(\bar{x}, \bar{x}^*)$  such that

$$h_{(x_0, x_0^*)}(\bar{x}, \bar{x}^*) = \langle \bar{x}, \bar{x}^* \rangle, \quad \|\tilde{x} - \bar{x}\| < \varepsilon, \quad \|\tilde{x}^* - \bar{x}^*\| < \varepsilon. \quad (2.41)$$

Therefore,

$$\begin{aligned} h(\bar{x} + x_0, \bar{x}^* + x_0^*) - \langle \bar{x} + x_0, \bar{x}^* + x_0^* \rangle &= h_{(x_0, x_0^*)}(\bar{x}, \bar{x}^*) - \langle \bar{x}, \bar{x}^* \rangle = 0, \\ \text{and } (\bar{x} + x_0, \bar{x}^* + x_0^*) &\in T. \text{ As } g \in \mathcal{F}_T, \end{aligned}$$

$$g(\bar{x} + x_0, \bar{x}^* + x_0^*) = \langle \bar{x} + x_0, \bar{x}^* + x_0^* \rangle,$$

and

$$g_{(x_0, x_0^*)}(\bar{x}, \bar{x}^*) = \langle \bar{x}, \bar{x}^* \rangle. \quad (2.42)$$

Using the first line of (2.40) we have

$$\varepsilon^2 > h_{(x_0, x_0^*)}(\tilde{x}, \tilde{x}^*) + \left[ \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 + \langle \tilde{x}, \tilde{x}^* \rangle \right] - \langle \tilde{x}, \tilde{x}^* \rangle \geq \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 + \langle \tilde{x}, \tilde{x}^* \rangle.$$

Therefore,

$$\varepsilon^2 > \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 + \langle \tilde{x}, \tilde{x}^* \rangle. \quad (2.43)$$

Direct use of (2.41) gives

$$\begin{aligned} \langle \bar{x}, \bar{x}^* \rangle &= \langle \tilde{x}, \tilde{x}^* \rangle + \langle \bar{x} - \tilde{x}, \tilde{x}^* \rangle + \langle \tilde{x}, \bar{x}^* - \tilde{x}^* \rangle + \langle \bar{x} - \tilde{x}, \bar{x}^* - \tilde{x}^* \rangle \\ &\leq \langle \tilde{x}, \tilde{x}^* \rangle + \|\bar{x} - \tilde{x}\| \|\tilde{x}^*\| + \|\tilde{x}\| \|\bar{x}^* - \tilde{x}^*\| + \|\bar{x} - \tilde{x}\| \|\bar{x}^* - \tilde{x}^*\| \\ &\leq \langle \tilde{x}, \tilde{x}^* \rangle + \varepsilon[\|\tilde{x}^*\| + \|\tilde{x}\|] + \varepsilon^2 \end{aligned}$$

and

$$\begin{aligned} \|\bar{x}\|^2 + \|\bar{x}^*\|^2 &\leq (\|\tilde{x}\| + \|\bar{x} - \tilde{x}\|)^2 + (\|\tilde{x}^*\| + \|\bar{x}^* - \tilde{x}^*\|)^2 \\ &\leq \|\tilde{x}\|^2 + \|\tilde{x}^*\|^2 + 2\varepsilon[\|\tilde{x}\| + \|\tilde{x}^*\|] + 2\varepsilon^2 \end{aligned}$$

Combining the two above equations with (2.42) we obtain

$$g_{(x_0, x_0^*)}(\bar{x}, \bar{x}^*) + \frac{1}{2}\|\bar{x}\|^2 + \frac{1}{2}\|\bar{x}^*\|^2 \leq \langle \tilde{x}, \tilde{x}^* \rangle + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 + 2\varepsilon[\|\tilde{x}\| + \|\tilde{x}^*\|] + 2\varepsilon^2$$

Using now (2.43) and the second line of (2.40) we conclude that

$$g_{(x_0, x_0^*)}(\bar{x}, \bar{x}^*) + \frac{1}{2}\|\bar{x}\|^2 + \frac{1}{2}\|\bar{x}^*\|^2 \leq 2\varepsilon M\sqrt{2} + 3\varepsilon^2.$$

As  $\varepsilon$  is an arbitrary strictly positive number, using also (2.39) we conclude that

$$\inf_{(x, x^*) \in X \times X^*} g_{(x_0, x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0.$$

Altogether, we conclude that if item 4 holds then item 5 holds. The converse (item 5 implies item 4) is trivial to verify. Hence item 4 and item 5 are equivalent. As item 2 is equivalent to item 4 and item 3 is equivalent to 5, we conclude that items 2,3,4 and 5 are equivalent.

Now we will prove that item 1 is equivalent to item 3 and conclude the proof of the theorem. First suppose that item 3 holds. Since  $\mathfrak{S}_T \in \mathcal{F}_T$

$$(\mathfrak{S}_T)^* \geq \pi_*.$$

As has already been observed, for any proper function  $h$  it holds that  $(\text{cl conv } h)^* = h^*$ . Therefore

$$(\mathfrak{S}_T)^* = (\pi + \delta_T)^* \geq \pi_*,$$

that is,

$$\sup_{(y, y^*) \in T} \langle y, x^* \rangle + \langle y^*, x^{**} \rangle - \langle y, y^* \rangle \geq \langle x^*, x^{**} \rangle, \forall (x^*, x^{**}) \in X^* \times X^{**} \quad (2.44)$$

After some algebraic manipulations we conclude that (2.44) is equivalent to

$$\inf_{(y, y^*) \in T} \langle x^{**} - y, x^* - y^* \rangle \leq 0, \quad \forall (x^*, x^{**}) \in X^* \times X^{**},$$

that is,  $T$  is type (NI) and so item 1 holds. If item 1 holds, by the same reasoning we conclude that (2.44) holds and therefore  $(\mathfrak{S}_T)^* \geq \pi_*$ . As  $\mathfrak{S}_T \in \mathcal{F}_T$ , we conclude that item 2 holds. As has been proved previously item 2  $\Rightarrow$  item 3.  $\square$

In the next lemma we give a sufficient condition for proving that the sum of maximal monotone operators of type (NI) is of type (NI).

**Lemma 2.3.2.** (Marques Alves-Svaiter [30]) *Let  $T_1, T_2 : X \rightrightarrows X^*$  be maximal monotone and of type (NI). Take*

$$h_1 \in \mathcal{F}_{T_1}, \quad h_2 \in \mathcal{F}_{T_2}$$

and define

$$h \in \bar{\mathbb{R}}^{X \times X^*},$$

$$h(x, x^*) := (h_1(x, \cdot) \square h_2(x, \cdot))(x^*) = \inf_{y^* \in X^*} h_1(x, y^*) + h_2(x, x^* - y^*),$$

$$\Pr_X \text{ dom}(h_i) := \{x \in X \mid \exists x^*, h_i(x, x^*) < \infty\}, \quad i = 1, 2.$$

If

$$\bigcup_{\lambda > 0} \lambda [\Pr_X \text{ dom}(h_1) - \Pr_X \text{ dom}(h_2)] \quad (2.45)$$

is a closed subspace then

$$h \geq \pi, h^* \geq \pi_*, \quad \Lambda h \geq \pi, (\Lambda h)^* \geq \pi_*,$$

$$\begin{aligned} T_1 + T_2 &= \{(x, x^*) \mid \Lambda h(x, x^*) = \langle x, x^* \rangle\} \\ &= \{(x, x^*) \mid h(x, x^*) = \langle x, x^* \rangle\} \end{aligned}$$

and  $T_1 + T_2$  is maximal monotone of type (NI) and

$$\Lambda h, \text{cl } h \in \mathcal{F}_{T_1 + T_2}.$$

*Proof.* Since  $h_1 \in \mathcal{F}_{T_1}$  and  $h_2 \in \mathcal{F}_{T_2}$ ,  $h_1 \geq \pi$  and  $h_2 \geq \pi$ . So

$$h_1(x, y^*) + h_2(x, x^* - y^*) \geq \langle x, y^* \rangle + \langle x, x^* - y^* \rangle = \langle x, x^* \rangle.$$

Taking the inf in  $y^*$  at the left-hand side of the above inequality we conclude that  $h \geq \pi$ .

Let  $(x^*, x^{**}) \in X^* \times X^{**}$ . Using the definition of  $h$  we have

$$h^*(x^*, x^{**}) = \sup_{(z, z^*) \in X \times X^*} \langle z, x^* \rangle + \langle z^*, x^{**} \rangle - h(z, z^*) \quad (2.46)$$

$$\begin{aligned} &= \sup_{(z, z^*, y^*) \in X \times X^* \times X^*} \langle z, x^* \rangle + \langle z^*, x^{**} \rangle - h_1(z, y^*) \\ &\quad - h_2(z, z^* - y^*) \end{aligned} \quad (2.47)$$

$$\begin{aligned} &= \sup_{(z, y^*, w^*) \in X \times X^* \times X^*} \langle z, x^* \rangle + \langle y^*, x^{**} \rangle + \langle w^*, x^{**} \rangle - h_1(z, y^*) \\ &\quad - h_2(z, w^*) \end{aligned} \quad (2.48)$$

where we used the substitution  $z^* = w^* + y^*$  in the last term. So, defining  $H_1, H_2 : X \times X^* \times X^* \rightarrow \bar{\mathbb{R}}$

$$H_1(x, y^*, z^*) = h_1(x, y^*), \quad H_2(x, y^*, z^*) = h_2(x, z^*). \quad (2.49)$$

we have

$$h^*(x^*, x^{**}) = (H_1 + H_2)^*(x^*, x^{**}, x^{**}).$$

Using (2.45), Theorem A.0.4 and (2.49) we conclude that the conjugate of the sum at the right hand side of the above equation is the *exact* inf-convolution of the conjugates. Therefore,

$$h^*(x^*, x^{**}) = \min_{(u^*, y^{**}, z^{**})} H_1^*(u^*, y^{**}, z^{**}) + H_2^*(x^* - u^*, x^{**} - y^{**}, x^{**} - z^{**}).$$

Direct use of definition (2.49) yields

$$H_1^*(u^*, y^{**}, z^{**}) = h_1^*(u^*, y^{**}) + \delta_0(z^{**}), \quad \forall (u^*, y^{**}, z^{**}) \in X^* \times X^{**} \times X^{**}, \quad (2.50)$$

$$H_2^*(u^*, y^{**}, z^{**}) = h_2^*(u^*, z^{**}) + \delta_0(y^{**}), \quad \forall (u^*, y^{**}, z^{**}) \in X^* \times X^{**} \times X^{**}. \quad (2.51)$$

Hence,

$$h^*(x^*, x^{**}) = \min_{u^* \in X^*} h_1^*(u^*, x^{**}) + h_2^*(x^* - u^*, x^{**}). \quad (2.52)$$

Therefore, using that  $h_1^* \geq \pi_*$ ,  $h_2^* \geq \pi_*$ , (2.52) and the same reasoning used to show that  $h \geq \pi$  we have

$$h^* \geq \pi^*.$$

Up to now, we proved that  $h \geq \pi$  and  $h^* \geq \pi_*$  (and  $\Lambda h \geq \pi$ ). So, using Theorem 2.2.6 we conclude that  $S : X \rightrightarrows X^*$ , defined as

$$S = \{(x, x^*) \in X \times X^* \mid \Lambda h(x, x^*) = \langle x, x^* \rangle\},$$

is maximal monotone. Since  $\Lambda h$  is convex and lower semicontinuous,  $\Lambda h \in \mathcal{F}_S$ .

We will prove that  $T_1 + T_2 = S$ . Take  $(x, x^*) \in S$ , that is,  $\Lambda h(x, x^*) = \langle x, x^* \rangle$ . Using (2.52) we conclude that there exists  $u^* \in X^*$  such that

$$h_1^*(u^*, x) + h_2^*(x^* - u^*, x) = \langle x, x^* \rangle.$$

We know that

$$h_1^*(u^*, x) \geq \langle x, u^* \rangle, \quad h_2^*(x^* - u^*, x) \geq \langle x, x^* - u^* \rangle.$$

Combining these inequalities with the previous equation we conclude that these inequalities hold as equalities, and so

$$\begin{aligned} u^* &\in T_1(x), & x^* - u^* &\in T_2(x), & x^* &\in (T_1 + T_2)(x). \\ h_1(x, u^*) &= \langle x, u^* \rangle, & h_2(x, x^* - u^*) &= \langle x, x^* - u^* \rangle, & h(x, x^*) &\leq \langle x, x^* \rangle. \end{aligned}$$

We proved that  $S \subset T_1 + T_2$ . Since  $T_1 + T_2$  is monotone and  $S$  is maximal monotone, we have  $T_1 + T_2 = S$  (and  $\Lambda h \in \mathcal{F}_{T_1 + T_2}$ ). Note

also that  $h(x, x^*) \leq \langle x, x^* \rangle$  for any  $(x, x^*) \in T_1 + T_2 = S$ . As  $h \geq \pi$ , we have equality in  $T_1 + T_2$ . Therefore,

$$T_1 + T_2 \subset \{(x, x^*) \mid h(x, x^*) = \langle x, x^* \rangle\} \subset \{(x, x^*) \mid \text{cl } h(x, x^*) \leq \langle x, x^* \rangle\}.$$

Since  $h \geq \pi$  and the duality product  $\pi$  is *continuous* in  $X \times X^*$ , we also have  $\text{cl } h \geq \pi$ . Hence, using the above inclusion we conclude that  $\text{cl } h$  coincides with  $\pi$  in  $T_1 + T_2$ . Therefore,  $\text{cl } h \in \mathcal{F}_{T_1 + T_2}$  and the rightmost set in the above inclusions is  $T_1 + T_2$ . Hence

$$T_1 + T_2 = \{(x, x^*) \mid h(x, x^*) = \langle x, x^* \rangle\}.$$

Conjugation is invariant under the (lower semicontinuous) closure operation. Therefore,

$$(\text{cl } h)^* = h^* \geq \pi_*$$

and so  $T_1 + T_2$  is of type (NI). We proved already that  $\Lambda h \in \mathcal{F}_{T_1 + T_2}$ . Using item 3 of Theorem 2.3.1 we conclude that  $(\Lambda h)^* \geq \pi_*$ .  $\square$

## Chapter 3

# On the relation between surjectivity of perturbations and operators of type (NI)

In this chapter we are concerned with surjectivity of perturbation of maximal monotone operators in non-reflexive Banach spaces. The results presented here are collected from [30]. In a *reflexive* Banach space the following result due to Rockafellar gives a necessary and sufficient condition for maximal monotonicity in terms of the surjectivity of perturbations by the duality mapping  $J$ :

**Theorem 3.0.3** ([38, Proposition 1]). *Let  $X$  be a reflexive Banach space and  $T : X \rightrightarrows X^*$  be a monotone operator. Then  $T$  is maximal monotone if and only if*

$$R(T(\cdot + z_0) + J) = X^*, \quad \forall z_0 \in X.$$

Here, by  $(x, x^*) \in T(\cdot + z_0)$  we means  $(x + z_0, x^*) \in T$ . Recall that the *duality mapping* is the point to set operator  $J : X \rightrightarrows X^*$  defined by

$$J(x) = \partial \frac{1}{2} \|x\|^2.$$

The point is that  $J$  is surjectivity if and only if  $X$  is reflexive. In order to overcome this difficult, J.-P. Gossez introduced [21] an “enlarged” version of the duality mapping,  $J_\varepsilon : X \rightrightarrows X^*$  defined by

$$J_\varepsilon(x) = \partial_\varepsilon \frac{1}{2} \|x\|^2,$$

and obtained similar results of Theorem 3.0.3 for a special class of maximal monotone operators he introduced in non-reflexive Banach

spaces, the operators of type (D). Notice that for any  $\varepsilon > 0$ ,  $J_\varepsilon$  is always surjectivity.

Recall that a maximal monotone operator  $T : X \rightrightarrows X^*$  is of type (NI) if

$$\inf_{(y, y^*) \in T} \langle y^* - x^*, y - x^{**} \rangle \leq 0, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

The class of maximal monotone operators of type (NI) encompasses the Gossez type (D) operators and was introduced by S. Simons [39] to generalize some results concerning maximal monotonicity in reflexive Banach spaces for non-reflexive Banach spaces.

The general framework of convex representations of maximal monotone operators developed in the previous chapters allows us to characterize the operators of type (NI) by the existence of a Fitzpatrick function in the Fitzpatrick family such that the conjugate majorizes the duality product (see Theorem 2.3.1). In the next sections, we will use this results to obtain surjectivity results for perturbations of maximal monotone operators of type (NI).

### 3.1 Preliminary results

We begin with two elementary technical results which will be useful.

**Proposition 3.1.1.** (Marques Alves-Svaiter [30]) *The following statements holds:*

1. For any  $\varepsilon \geq 0$ , if  $y^* \in J_\varepsilon(x)$ , then  $|\|x\| - \|y^*\|| \leq \sqrt{2\varepsilon}$ .
2. Let  $T : X \rightrightarrows X^*$  be a monotone operator and  $\varepsilon, M > 0$ . Then,

$$(T + J_\varepsilon)^{-1} (B_{X^*}[0, M])$$

*is bounded.*

*Proof.* For proving item 1, let  $\varepsilon \geq 0$  and  $y^* \in J_\varepsilon(x)$ . The desired result follows from the following inequalities:

$$\frac{1}{2}(\|x\| - \|y^*\|)^2 \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y^*\|^2 - \langle x, y^* \rangle \leq \varepsilon.$$

For proving item 2, take  $(z, z^*) \in T$ . If  $x \in (T + J_\varepsilon)^{-1} (B[0, M])$  then there exists  $x^*, y^*$  such that

$$x^* \in T(x), \quad y^* \in J_\varepsilon(x), \quad \|x^* + y^*\| \leq M.$$

Therefore, using Fenchel Young inequality (A.2), the monotonicity of  $T$  and the definition of  $J_\varepsilon$  we obtain

$$\begin{aligned} \frac{1}{2}\|x - z\|^2 + \frac{1}{2}\|x^* + y^* - z^*\|^2 &\geq \langle x - z, x^* + y^* - z^* \rangle \\ &\geq \langle x - z, y^* \rangle \\ &\geq \left[ \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y^*\|^2 - \varepsilon \right] - \|z\|\|y^*\|. \end{aligned}$$

Note also that

$$\|x - z\|^2 \leq \|x\|^2 + 2\|x\|\|z\| + \|z\|^2, \quad \|x^* + y^* - z^*\|^2 \leq (M + \|z^*\|)^2.$$

Combining the above equations we obtain

$$\frac{1}{2}\|z\|^2 + \frac{1}{2}(M + \|z^*\|)^2 \geq \frac{1}{2}\|y^*\|^2 - \|x\|\|z\| - \|z\|\|y^*\| - \varepsilon.$$

As  $y^* \in J_\varepsilon(x)$ , by item 1, we have  $\|x\| \leq \|y^*\| + \sqrt{2\varepsilon}$ . Therefore

$$\frac{1}{2}\|z\|^2 + \frac{1}{2}(M + \|z^*\|)^2 \geq \frac{1}{2}\|y^*\|^2 - 2\|y^*\|\|z\| - \|z\|\sqrt{2\varepsilon} - \varepsilon.$$

Hence,  $y^*$  is bounded. In fact,

$$\|y^*\| \leq 2\|z\| + \sqrt{4\|z\|^2 + 2\left[\|z\|\sqrt{2\varepsilon} + \varepsilon\right] + \|z\|^2 + (M + \|z^*\|)^2}.$$

As we already observed,  $\|x\| \leq \|y^*\| + \sqrt{2\varepsilon}$  and so,  $\|x\|$  is also bounded.  $\square$

Now we will prove that under monotonicity, dense range of some perturbation of a monotone operator is equivalent to surjectivity of that perturbation.

**Lemma 3.1.2.** (Marques Alves-Svaiter [30]) *Let  $T : X \rightrightarrows X^*$  be monotone and  $\mu > 0$ . Then the conditions below are equivalent*

1.  $\overline{R(T(\cdot + z_0) + \mu J_\varepsilon)} = X^*$ , for any  $\varepsilon > 0$  and  $z_0 \in X$ ,
2.  $R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*$ , for any  $\varepsilon > 0$  and  $z_0 \in X$ .

*Proof.* It suffices to prove the lemma for  $\mu = 1$  and then, for the general case, consider  $T' = \mu^{-1}T$ . Now note that for any  $z_0 \in X$  and  $z_0^* \in X^*$ ,  $T - \{(z_0, z_0^*)\}$  is also monotone. Therefore, it suffices to prove that  $0 \in \overline{R(T + J_\varepsilon)}$ , for any  $\varepsilon > 0$  if and only if  $0 \in R(T + J_\varepsilon)$ , for any  $\varepsilon > 0$ . The "if" is easy to check. To prove the "only if", suppose that

$$0 \in \overline{R(T + J_\varepsilon)}, \quad \forall \varepsilon > 0.$$

First use item 2 of Proposition 3.1.1 with  $M = 1/2$  to conclude that there exists  $\rho > 0$  such that

$$(T + J_{1/2})^{-1}(B_{X^*}[0, 1/2]) \subset B_X[0, \rho].$$

By assumption, for any  $0 < \eta < \frac{1}{2}$  there exists  $x_\eta \in X$ ,  $x_\eta^*, y_\eta^* \in X^*$  such that

$$x_\eta^* \in T(x_\eta), \quad y_\eta^* \in J_\eta(x_\eta) \quad \text{and} \quad \|x_\eta^* + y_\eta^*\| < \eta < \frac{1}{2}. \quad (3.1)$$

As  $J_\eta(x_\eta) \subset J_{1/2}(x_\eta)$ ,  $x_\eta \in (T + J_{1/2})^{-1}(x_\eta^* + y_\eta^*)$  and so,

$$\|x_\eta\| \leq \rho, \quad \|y_\eta^*\| \leq \rho + 1.$$

where the second inequality follows from the first one and item 1 of Proposition 3.1.1. Therefore

$$\begin{aligned} \frac{1}{2}\|x_\eta^*\|^2 &\leq \frac{1}{2}(\|x_\eta^* + y_\eta^*\| + \|y_\eta^*\|)^2 \leq \frac{1}{2}\eta^2 + \eta(\rho + 1) + \frac{1}{2}\|y_\eta^*\|^2, \\ \langle x_\eta, x_\eta^* \rangle &= \langle x_\eta, x_\eta^* + y_\eta^* \rangle - \langle x_\eta, y_\eta^* \rangle \leq \rho\eta - \langle x_\eta, y_\eta^* \rangle. \end{aligned}$$

Combining the above inequalities we obtain

$$\frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta^*\|^2 + \langle x_\eta, x_\eta^* \rangle \leq \frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|y_\eta^*\|^2 - \langle x_\eta, y_\eta^* \rangle + \eta(2\rho + 1) + \frac{1}{2}\eta^2.$$

The inclusion  $y_\eta^* \in J_\eta(x_\eta)$ , means that,

$$\frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|y_\eta^*\|^2 - \langle x_\eta, y_\eta^* \rangle \leq \eta. \quad (3.2)$$

Hence, using the two above inequalities we conclude that

$$\frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta^*\|^2 + \langle x_\eta, x_\eta^* \rangle \leq 2\eta(\rho + 1) + \frac{1}{2}\eta^2.$$

For finishing the prove, take an arbitrary  $\varepsilon > 0$ . Choosing  $0 < \eta < 1/2$  such that,

$$2\eta(\rho + 1) + \frac{1}{2}\eta^2 < \varepsilon,$$

we have

$$\frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta^*\|^2 + \langle x_\eta, x_\eta^* \rangle < \varepsilon, \quad x_\eta^* \in T(x_\eta).$$

According to the above inequality,  $-x_\eta^* \in J_\varepsilon(x_\eta)$ . Hence  $0 \in (T + J_\varepsilon)(x_\eta)$ .  $\square$

## 3.2 Main results

As we pointed out in the introduction of the present chapter, in a reflexive Banach space, surjectivity of a monotone operator plus the duality mapping is equivalent to maximal monotonicity. This is a classical result of Rockafellar [38]. For obtaining a partial extension of this result for non-reflexive Banach spaces, we must consider the “enlarged” duality mapping [21].

**Lemma 3.2.1.** (Marques Alves-Svaiter [30]) *Let  $T : X \rightrightarrows X^*$  be monotone and  $\mu > 0$ . If*

$$\overline{R(T(\cdot + z_0) + \mu J_\varepsilon)} = X^*, \quad \forall \varepsilon > 0, z_0 \in X$$

*then  $\bar{T}$ , the closure of  $T$  in the norm-topology of  $X \times X^*$ , is maximal monotone and of type (NI).*

*Proof.* Note that  $T + \mu J_\varepsilon = \mu(\mu^{-1}T + J_\varepsilon)$ . Therefore, it suffices to prove the lemma for  $\mu = 1$  and then, for the general case, consider  $T' = \mu^{-1}T$ . The monotonicity of  $\bar{T}$  follows from the continuity of the duality product.

Using the assumptions on  $T$  and Lemma 3.1.2 we conclude that  $T(\cdot + z_0) + J_\varepsilon$  is onto, for any  $\varepsilon > 0$  and  $z_0 \in X$ . Therefore, for any  $(z_0, z_0^*) \in X \times X^*$  and  $\varepsilon > 0$ , there exists  $x_\varepsilon, x_\varepsilon^*$  such that

$$x_\varepsilon^* + z_0^* \in T(x_\varepsilon + z_0) \quad \text{and} \quad -x_\varepsilon^* \in J_\varepsilon(x_\varepsilon). \quad (3.3)$$

Note that the second inclusion in the above equation is equivalent to

$$\frac{1}{2}\|x_\varepsilon\|^2 + \frac{1}{2}\|x_\varepsilon^*\|^2 \leq \langle x_\varepsilon, -x_\varepsilon^* \rangle + \varepsilon. \quad (3.4)$$

For proving maximal monotonicity of  $\bar{T}$ , suppose that  $(z_0, z_0^*) \in X \times X^*$  is monotonically related to  $\bar{T}$ . As  $T \subset \bar{T}$

$$\langle z - z_0, z^* - z_0^* \rangle \geq 0, \quad \forall (z, z^*) \in T.$$

So, taking  $\varepsilon > 0$  and  $x_\varepsilon \in X, x_\varepsilon^* \in X^*$  as in (3.3) we conclude that

$$\langle x_\varepsilon, x_\varepsilon^* \rangle = \langle x_\varepsilon + z_0 - z_0, x_\varepsilon^* + z_0^* - z_0^* \rangle \geq 0,$$

which, combined with (3.4) yields

$$\frac{1}{2}\|x_\varepsilon\|^2 + \frac{1}{2}\|x_\varepsilon^*\|^2 \leq \varepsilon.$$

Since  $(x_\varepsilon + z_0, x_\varepsilon^* + z_0^*) \in T$ , and  $\varepsilon$  is an arbitrary strictly positive number, we conclude that  $(z_0, z_0^*) \in \bar{T}$ , and  $\bar{T}$  is maximal monotone.

It remains to be proved that  $\bar{T}$  is of type (NI). Consider an arbitrary  $(z_0, z_0^*) \in X \times X^*$  and  $h \in \mathcal{F}_{\bar{T}}$ . Then, using (3.3), (3.4) we conclude that for any  $\varepsilon > 0$ , there exists  $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$  such that

$$h(x_\varepsilon + z_0, x_\varepsilon^* + z_0^*) = \langle x_\varepsilon + z_0, x_\varepsilon^* + z_0^* \rangle, \quad \frac{1}{2}\|x_\varepsilon\|^2 + \frac{1}{2}\|x_\varepsilon^*\|^2 \leq \langle x_\varepsilon, -x_\varepsilon^* \rangle + \varepsilon.$$

The first equality above is equivalent to  $h_{(z_0, z_0^*)}(x_\varepsilon, x_\varepsilon^*) = \langle x_\varepsilon, x_\varepsilon^* \rangle$ . Therefore,

$$h_{(z_0, z_0^*)}(x_\varepsilon, x_\varepsilon^*) + \frac{1}{2}\|x_\varepsilon\|^2 + \frac{1}{2}\|x_\varepsilon^*\|^2 < \varepsilon,$$

that is,

$$\inf h_{(z_0, z_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0.$$

Now, use item 5 of Theorem 2.3.1 to conclude that  $\bar{T}$  is of type (NI).  $\square$

Direct application of Lemma 3.2.1 gives the next corollary.

**Corollary 3.2.2.** (Marques Alves-Svaiter [30]) *If  $T : X \rightrightarrows X^*$  is monotone, closed,  $\mu > 0$  and*

$$\overline{R(T(\cdot + z_0) + \mu J_\varepsilon)} = X^*, \quad \forall \varepsilon > 0, z_0 \in X$$

*then  $T$  is maximal monotone and of type (NI).*

*Proof.* Use Lemma 3.2.1 and the assumption  $T = \bar{T}$ .  $\square$

The next result gives a complete characterization (in non-reflexive Banach spaces) of maximal monotone operators of type (NI) in terms of the surjectivity of perturbations, by  $J$  and  $J_\varepsilon$ .

**Theorem 3.2.3.** (Marques Alves-Svaiter [30]) *If  $T : X \rightrightarrows X^*$  is a closed monotone operator then the conditions bellow are equivalent*

1.  $\overline{R(T(\cdot + z_0) + J)} = X^*$ , for all  $z_0 \in X$ ,
2.  $\overline{R(T(\cdot + z_0) + J_\varepsilon)} = X^*$ , for all  $\varepsilon > 0$ ,  $z_0 \in X$ ,
3.  $R(T(\cdot + z_0) + J_\varepsilon) = X^*$ , for all  $\varepsilon > 0$ ,  $z_0 \in X$ ,
4.  $T$  is maximal monotone and of type (NI).

*Proof.* Item 1 trivially implies item 2. Using Lemma 3.1.2 we conclude that, in particular, item 2 implies item 3. Now use Corollary 3.2.2 to conclude that item 3 implies item 4. Up to now we have  $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ .

For completing the proof we will show that item 4 implies item 1. So, assume that item 4 holds, that is,  $T$  is of type (NI). Take  $z_0^* \in X^*$  and  $z_0 \in X$ . Define  $T_0 = T - \{(z_0, z_0^*)\}$ . Trivially

$$z_0^* \in \overline{R(T(\cdot + z_0) + J)} \iff 0 \in \overline{R(T_0 + J)}.$$

As the class of operators of type (NI) is invariant under translations, in order to prove item 1, it is sufficient to prove that if  $T$  is of type (NI), then  $0 \in \overline{R(T + J)}$ . Let  $h \in \mathcal{F}_T$  and  $\varepsilon > 0$ . Define  $p : X \times X^* \rightarrow \mathbb{R}$ ,

$$p(x, x^*) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2. \quad (3.5)$$

Item 5 of Theorem 2.3.1 ensures us that there exists  $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$  such that

$$h(x_\varepsilon, x_\varepsilon^*) + p(x_\varepsilon, -x_\varepsilon^*) < \varepsilon^2. \quad (3.6)$$

Direct calculations yields  $p \geq \pi$  and  $p^* \geq \pi_*$ . We also know that  $p \in \mathcal{F}_J$  and so  $J$  is of type (NI). Define  $H : X \times X^* \rightarrow \overline{\mathbb{R}}$ ,

$$H(x, x^*) = \inf_{y^* \in X^*} h(x, y^*) + p(x, x^* - y^*).$$

As  $\text{dom}(p) = X \times X^*$ , we may apply Lemma 2.3.2 to conclude that  $T + J$  is of type (NI) and  $\text{cl } H \in \mathcal{F}_{T+J}$ . Using (3.6) we have

$$H(x_\varepsilon, 0) \leq h(x_\varepsilon, x_\varepsilon^*) + p(x_\varepsilon, -x_\varepsilon^*) < \varepsilon^2.$$

So,  $\text{cl } H(x_\varepsilon, 0) \leq H(x_\varepsilon, 0) < \langle x_\varepsilon, 0 \rangle + \varepsilon^2$ . Now use Theorem 2.2.4 to conclude that there exist  $\bar{x}, \bar{x}^*$  such that

$$(\bar{x}, \bar{x}^*) \in T + J, \quad \|\bar{x} - x_\varepsilon\| < \varepsilon, \quad \|\bar{x}^* - 0\| < \varepsilon.$$

So,  $\bar{x}^* \in R(T + J)$  and  $\|\bar{x}^*\| < \varepsilon$ . As  $\varepsilon > 0$  is arbitrary,  $0$  is in the closure of  $R(T + J)$ .  $\square$

The next corollary is an immediate consequence of Theorem 3.2.3.

**Corollary 3.2.4.** (Marques Alves-Svaiter [30]) *If  $T : X \rightrightarrows X^*$  is a closed monotone operator then the conditions below are equivalent:*

- (a)  $\overline{R(T(\cdot + z_0) + \mu J)} = X^*$  for all  $z_0 \in X$  and some  $\mu > 0$ ,
- (b)  $\overline{R(T(\cdot + z_0) + \mu J)} = X^*$  for all  $z_0 \in X$ ,  $\mu > 0$ ,
- (c)  $\overline{R(T(\cdot + z_0) + \mu J_\varepsilon)} = X^*$  for all  $\varepsilon > 0$ ,  $z_0 \in X$  and some  $\mu > 0$ ,
- (d)  $\overline{R(T(\cdot + z_0) + \mu J_\varepsilon)} = X^*$  for all  $\varepsilon > 0$ ,  $z_0 \in X$ ,  $\mu > 0$ ,

(e)  $R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*$  for all  $\varepsilon > 0$ ,  $z_0 \in X$ , and some  $\mu > 0$ ,

(f)  $R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*$  for all  $\varepsilon > 0$ ,  $z_0 \in X$ ,  $\mu > 0$ ,

(g)  $T$  is maximal monotone and of type (NI).

*Proof.* Suppose that item (a) holds. Define  $T' = \mu^{-1}T$  and use Theorem 3.2.3 to conclude that  $T'$  is maximal monotone and of type (NI). Therefore,  $T = \mu T'$  is maximal monotone and of type (NI), which means that (g) holds.

Now assume that item (g) holds, that is,  $T$  is maximal monotone and of type (NI). Then, for all  $\mu > 0$ ,  $\mu^{-1}T$  is maximal monotone and of type (NI), which implies item (b).

Since the implication (b) $\Rightarrow$ (a) is trivial, we conclude that items (a), (b), (g) are equivalent.

The same reasoning shows that items (c), (d), (g) are equivalent and so on.  $\square$

## Chapter 4

# On the uniqueness of the extension to the bidual

Let us start by introducing maximal monotonicity in  $X^{**} \times X^*$ . An operator  $T : X^{**} \rightrightarrows X^*$ , is *monotone* if

$$\langle x^{**} - y^{**}, x^* - y^* \rangle \geq 0, \forall (x^{**}, x^*), (y^{**}, y^*) \in T.$$

An operator  $T : X^{**} \rightrightarrows X^*$  is maximal monotone (in  $X^{**} \times X^*$ ) if it is monotone and maximal (with respect to the inclusion) in the family of monotone operators of  $X^{**}$  into  $X^*$ . The canonical injection of  $X$  into  $X^{**}$  allows one to identify  $X$  with a subset of  $X^{**}$ . Therefore, any maximal monotone operator  $T : X \rightrightarrows X^*$  is also a monotone operator  $T : X^{**} \rightrightarrows X^*$  and admits one (or more) maximal monotone extension in  $X^{**} \times X^*$ . In general this maximal monotone extension will not be unique. In this chapter, we are concerned with the problem:

Under which conditions a maximal monotone operator  $T : X \rightrightarrows X^*$  has a unique extension to the bidual,  $X^{**} \rightrightarrows X^*$ ?

The problem of unicity of maximal extension of a generic monotone operator was studied in details by Martínez-Legaz and Svaiter in [33]. That paper will be an important reference for the present chapter.

The specific problem above mentioned, of uniqueness of extension of a *maximal monotone* operator to the bidual, has been previously addressed by Gossez [21, 22, 23, 24]. He found a condition under which uniqueness of the extension is guaranteed [24]. Latter the condition (NI), was studied by S. Simons in [39]. This condition guarantees the uniqueness of the extension to the bidual and encompasses Gossez type (D) condition.

We will prove that maximal monotone operators of type (NI) admit a unique extension to the bidual and that, for non-linear operators, the condition (NI) is equivalent to the unicity of maximal monotone extension to the bidual. For proving this equivalence we will show that if  $T \subset X \times X^*$  is maximal monotone and convex then  $T$  is an affine manifold.

The results of this chapter are from the paper [31].

## 4.1 Convexity and maximal monotonicity

Recall that a linear (affine) manifold of a real linear space  $Z$  is a set  $A \subset Z$  such that there exists  $V$ , subspace of  $Z$ , and a point  $z_0$  such that

$$A = V + \{z_0\} = \{z + z_0 \mid z \in V\}.$$

The next lemma states that a convex maximal monotone operator is “essentially” linear:

**Lemma 4.1.1.** (Marques Alves-Svaiter [31]) *If  $T : X \rightrightarrows X^*$  is maximal monotone and convex, then  $T$  is affine.*

*Proof.* Take an arbitrary  $(x_0, x_0^*) \in T$  and define

$$T_0 = T - \{(x_0, x_0^*)\}.$$

Note that  $T_0$  is maximal monotone and convex. So, it suffices to prove that  $T_0$  is a linear subspace of  $X \times X^*$ . Take an arbitrary  $(x, x^*) \in T_0$ . First we claim that

$$t(x, x^*) \in T_0, \quad \forall t \geq 0. \quad (4.1)$$

For  $0 \leq t \leq 1$  the above inclusion holds because  $(0, 0) \in T_0$  and  $T_0$  is convex. For the case  $t \geq 1$  let  $(y, y^*) \in T$ . Then,  $t^{-1}(y, y^*) \in T_0$  and so

$$\langle x - t^{-1}y, x^* - t^{-1}y^* \rangle \geq 0.$$

Multiplying this inequality by  $t$  we conclude that  $\langle tx - y, tx^* - y^* \rangle \geq 0$ . As  $(y, y^*)$  is a generic element of  $T_0$ , which is maximal monotone, we conclude that  $t(x, x^*) \in T_0$  and the claim (4.1) holds.

We have just proved that  $T_0$  is a convex cone. Now take an arbitrary pair

$$(x, x^*), (y, y^*) \in T_0.$$

Then

$$(x + y, x^* + y^*) = 2 \left[ \frac{1}{2}(x, x^*) + \frac{1}{2}(y, y^*) \right] \in T_0. \quad (4.2)$$

Since  $(0, 0) \in T_0$ , we have

$$\langle y - (-x), y^* - (-x^*) \rangle = \langle (y + x) - 0, (y^* + x^*) - 0 \rangle \geq 0.$$

Since  $T_0$  is maximal monotone, we conclude that  $-(x, x^*) \in T_0$ . Therefore, using again (4.1) we conclude that  $T_0$  is closed under scalar multiplication. In order to end the proof, combine this result with (4.2) and conclude that  $T_0$  is a linear manifold.  $\square$

This lemma generalizes a result of Burachik and Iusem [14, Lemma 2.14], which states that if a point to point maximal monotone operator is convex and its domain has a non-empty interior, then the operator is affine. Burachik and Iusem also proved that under these assumptions, the operator is defined in the whole space. After the submission of [31], Bauschke, Wang and Yao published, in the arXiv.org preprint server, a preprint [5] with the same result of Lemma 4.1.1.

## 4.2 Basic results and some notation

In this section we use the notation  $\pi_{X \times X^*}$ ,  $\langle \cdot, \cdot \rangle_{X \times X^*}$  for the duality product

$$\pi_{X \times X^*}(x, x^*) = \langle x, x^* \rangle_{X \times X^*} = x^*(x).$$

Whenever the underlying domain of the duality product is clear, we will use the notations  $\pi$  and  $\langle \cdot, \cdot \rangle$ . The indicator function of  $A \subset X$  is  $\delta_{A, X} \in \bar{\mathbb{R}}^X$ ,

$$\delta_{A, X}(x) = \begin{cases} 0, & x \in A \\ \infty, & \text{otherwise.} \end{cases}$$

Whenever the set  $X$  is implicitly defined, we use the notation  $\delta_A$ .

The  $\mathcal{S}$ -function and Fitzpatrick function (as well as the Fitzpatrick family) are still well defined for arbitrary sets (or operators)  $T \subset X \times X^*$ :

$$\mathcal{S}_T \in \bar{\mathbb{R}}^{X \times X^*}, \quad \mathcal{S}_T = \text{cl conv}(\pi + \delta_T), \quad (4.3)$$

$$\varphi_T \in \bar{\mathbb{R}}^{X \times X^*}, \quad \varphi_T(x, x^*) = \sup_{(y, y^*) \in T} \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y^*, y \rangle. \quad (4.4)$$

Martínez-Legaz and Svaiter studied in [33] generic properties of  $\mathcal{S}_T$  and  $\varphi_T$  for arbitrary sets and its relation with monotonicity and maximal monotonicity. They observed that for a generic  $T \subset X \times X^*$

$$\varphi_T(x, x^*) = (\pi + \delta_T)^*(x^*, x) = (\mathcal{S}_T)^*(x^*, x), \quad \forall (x, x^*) \in X \times X^*. \quad (4.5)$$

Therefore, also for an arbitrary  $T$ , one has  $\Lambda\mathcal{S}_T = \varphi_T$ .

It will be useful to define a relation  $\mu$  which characterizes monotonicity, and study monotonicity in the framework of this relation and the classical notion of polarity [6].

Recall that a relation in a set  $V$  is a subset  $\mu$  of  $V \times V$ .

**Definition 4.2.1** ([33]). *The monotone relation  $\mu$  in  $X \times X^*$  is defined as*

$$\mu = \{((x, x^*), (y, y^*)) \in (X \times X^*)^2 \mid \langle x - y, x^* - y^* \rangle \geq 0\}.$$

*Two points  $(x, x^*), (y, y^*) \in X \times X^*$  are monotonically related or in monotone relation if  $(x, x^*) \mu (y, y^*)$ , that is,*

$$\langle x - y, x^* - y^* \rangle \geq 0.$$

*Given  $A \subset X \times X^*$ , the monotone polar (in  $X \times X^*$ ) of  $A$  is the set  $A^\mu$ ,*

$$\begin{aligned} A^\mu &= \{(x, x^*) \in X \times X^* \mid (x, x^*) \mu (y, y^*), \quad \forall (y, y^*) \in A\}, \\ &= \{(x, x^*) \in X \times X^* \mid \langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (y, y^*) \in A\}. \end{aligned} \tag{4.6}$$

We shall need some results of Martínez-Legaz and Svaiter which are scattered along [33], and which we present in the next two theorems:

**Theorem 4.2.1** ([33, Eq. (22), Prop. 2, Prop. 21]). *Let  $A \subset X \times X^*$ . Then*

$$A^\mu = \{(x, x^*) \in X \times X^* \mid \varphi_T(x, x^*) \leq \langle x, x^* \rangle\}, \tag{4.7}$$

*and the following conditions are equivalent*

1.  *$A$  is monotone,*
2.  *$\varphi_A \leq (\pi + \delta_A)$ .*
3.  *$A \subset A^\mu$ .*

*Moreover,  $A$  is maximal monotone if and only if  $A = A^\mu$ .*

Note in the above theorem and in the definition of the Fitzpatrick family, the convenience of defining as in [33, Eq. (12) and below], for  $h \in \bar{\mathbb{R}}^{X \times X^*}$ :

$$\begin{aligned} b(h) &:= \{(x, x^*) \in X \times X^* \mid h(x, x^*) \leq \langle x, x^* \rangle\}, \\ L(h) &:= \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}. \end{aligned} \tag{4.8}$$

**Theorem 4.2.2** ([33, Prop. 36, Lemma 38]). *Suppose that  $A \subset X \times X^*$  is monotone. Then the following conditions are equivalent*

1. *A has a unique maximal monotone extension (in  $X \times X^*$ ),*
2.  *$A^\mu$  is monotone*
3.  *$A^\mu$  is maximal monotone,*

*and if any of these conditions holds, then  $A^\mu$  is the unique maximal monotone extension of A.*

*Moreover, still assuming only A monotone,*

$$\varphi_A \geq \pi \iff b(\varphi_A) = L(\varphi_A) \quad (4.9)$$

*and if these conditions hold, then A has a unique maximal monotone extension,  $A^\mu$ .*

### 4.3 Extension theorems

Let  $T : X \rightrightarrows X^*$  be maximal monotone. The inverse of  $T$  is  $T^{-1} : X^* \rightrightarrows X$ ,

$$T^{-1} = \{(x^*, x) \in X^* \times X \mid (x, x^*) \in T\}. \quad (4.10)$$

Note that  $T^{-1} \subset X^* \times X \subset X^* \times X^{**}$ . The Fitzpatrick function of  $T^{-1}$ , regarded as a subset of  $X^* \times X^{**}$  is, according to (4.4),

$$\begin{aligned} \varphi_{T^{-1}, X^* \times X^{**}}(x^*, x^{**}) &= \sup_{(y^*, y^{**}) \in T^{-1}} \langle x^*, y^{**} \rangle + \langle y^*, x^{**} \rangle - \langle y^*, y^{**} \rangle \\ &= \sup_{(y^*, y) \in T^{-1}} \langle x^*, y \rangle + \langle y^*, x^{**} \rangle - \langle y^*, y \rangle \\ &= (\pi + \delta_T)^*(x^*, x^{**}). \end{aligned}$$

where the last  $x^*$  is identified with its image under the canonical injection of  $X^*$  into  $X^{***}$ . Using the above equations, (4.3) and the fact that conjugation is invariant under the convex-closure operation, we obtain

$$\varphi_{T^{-1}, X^* \times X^{**}} = (\pi + \delta_T)^* = (\mathfrak{S}_T)^*, \quad (4.11)$$

where  $\pi = \pi_{X \times X^*}$  and  $\delta_T = \delta_{T, X \times X^*}$ .

We will use the notation  $(T^{-1})^{\mu, X^* \times X^{**}}$  for denoting the monotone polar of  $T^{-1}$  in  $X^* \times X^{**}$ . Combing the above equation with Theorem 4.2.1 we obtain a simple expression for this monotone polar:

$$(T^{-1})^{\mu, X^* \times X^{**}} = \{(x^*, x^{**}) \in X^* \times X^{**} \mid (\mathfrak{S}_T)^*(x^*, x^{**}) \leq \langle x^*, x^{**} \rangle\}. \quad (4.12)$$

Next we establish our first extension theorem. In order to simplify the notation, define

$$R : X^{**} \times X^* \rightarrow X^* \times X^{**}, \quad R(x^{**}, x^*) = (x^*, x^{**}).$$

Note that  $R(X \times X^*) = X^* \times X$ .

**Theorem 4.3.1.** (Marques Alves-Svaiter [31]) *Let  $T : X \rightrightarrows X^*$  be a maximal monotone operator of type (NI). Then*

1.  $T$  admits a unique maximal monotone extension  $\tilde{T} : X^{**} \rightrightarrows X^*$ ;
2.  $(\mathcal{S}_T)^* = \varphi_{R\tilde{T}}$ ;
3. for all  $h \in \mathcal{F}_T$ ,  $h^* \in \mathcal{F}_{R\tilde{T}}$ , that is,  $h^* \geq \pi_*$  and  $(x^*, x^{**}) \in R\tilde{T} \Rightarrow h^*(x^*, x^{**}) = \langle x^*, x^{**} \rangle$ .

*Proof.* Using Proposition 2.2.9 and (4.11) we have

$$\varphi_{T^{-1}, X^* \times X^{**}}(x^*, x^{**}) = (\mathcal{S}_T)^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

Therefore, using Theorem 4.2.2 and Theorem 4.2.1 for  $A = T^{-1} \subset X^* \times X^{**}$ , we conclude that  $(T^{-1})^{\mu, X^* \times X^{**}}$ , the monotone polar of  $T^{-1}$  in  $X^* \times X^{**}$ , is the *unique* maximal monotone extension of  $T^{-1}$  to  $X^* \times X^{**}$  and

$$(T^{-1})^{\mu, X^* \times X^{**}} = \{(x^*, x^{**}) \in X^* \times X^{**} \mid (\mathcal{S}_T)^*(x^*, x^{**}) = \langle x^*, x^{**} \rangle\}. \quad (4.13)$$

Using the above result and again Proposition 2.2.9, we conclude that

$$(\mathcal{S}_T)^* \in \mathcal{F}_{(T^{-1})^{\mu, X^* \times X^{**}}}.$$

Now, define

$$\tilde{T} = \{(x^{**}, x^*) \in X^{**} \times X^* \mid (x^*, x^{**}) \in (T^{-1})^{\mu, X^* \times X^{**}}\}. \quad (4.14)$$

Note that  $RT = T^{-1}$  and  $R\tilde{T} = (T^{-1})^{\mu, X^* \times X^{**}}$ . Therefore

$$(\mathcal{S}_T)^* \in \mathcal{F}_{R\tilde{T}}. \quad (4.15)$$

Moreover, since  $R$  is a bijection which preserves the duality product, we conclude that  $\tilde{T}$  is the unique maximal monotone extension of  $T$  in  $X^{**} \times X^*$ . This proves Item 1.

Since  $T \subset \tilde{T}$ ,

$$\begin{aligned} \varphi_{R\tilde{T}}(x^*, x^{**}) &= \sup_{(y^*, y^{**}) \in R\tilde{T}} \langle x^*, y^{**} \rangle + \langle y^*, x^{**} \rangle - \langle y^*, y^{**} \rangle \\ &= \sup_{(y^{**}, y^*) \in \tilde{T}} \langle x^*, y^{**} \rangle + \langle y^*, x^{**} \rangle - \langle y^*, y^{**} \rangle \\ &\geq \sup_{(y, y^*) \in T} \langle y, x^* \rangle + \langle y^*, x^{**} \rangle - \langle y, y^* \rangle = (\pi + \delta_T)^*(x^*, x^{**}). \end{aligned}$$

Combining the above equation with the second equality in (4.11) we conclude that  $\varphi_{R\tilde{T}} \geq (\mathfrak{S}_T)^*$ . Using also the fact that  $\varphi_{R\tilde{T}}$  is minimal in  $\mathcal{F}_{R\tilde{T}}$  and (4.15), we obtain  $\varphi_{R\tilde{T}} = (\mathfrak{S}_T)^*$ . This proves Item 2.

By Theorem 1.2.3,  $\varphi_T(x, x^*) = (\mathfrak{S}_T)^*(x^*, x)$ . Therefore,

$$\begin{aligned} (\varphi_T)^*(x^*, x^{**}) &= \sup_{(y, y^*) \in X \times X^*} \langle y, x^* \rangle + \langle y^*, x^{**} \rangle - \varphi_T(y, y^*) \\ &= \sup_{(y, y^*) \in X \times X^*} \langle y, x^* \rangle + \langle y^*, x^{**} \rangle - (\mathfrak{S}_T)^*(y^*, y) \\ &\leq \sup_{(y^{**}, y^*) \in X^{**} \times X^*} \langle y^{**}, x^* \rangle + \langle y^*, x^{**} \rangle - (\mathfrak{S}_T)^*(y^*, y^{**}) \\ &= (\mathfrak{S}_T)^{**}(x^{**}, x^*). \end{aligned}$$

Take  $h \in \mathcal{F}_T$ . By Theorem 1.2.3 one has  $\varphi_T \leq h \leq s_T$ . Using also the fact that conjugation reverts the order, the above equation and Proposition 2.2.9, we conclude that, for any  $(x^*, x^{**})$ ,

$$\langle x^*, x^{**} \rangle \leq (\mathfrak{S}_T)^*(x^*, x^{**}) \leq h^*(x^*, x^{**}) \leq (\varphi_T)^*(x^*, x^{**}) \leq (\mathfrak{S}_T)^{**}(x^{**}, x^*). \quad (4.16)$$

Define

$$g \in \overline{\mathbb{R}}^{X^* \times X^{**}}, \quad g(x^*, x^{**}) := ((\mathfrak{S}_T)^*)^*(x^{**}, x^*).$$

Using (4.15) and Theorem 1.2.2 we conclude that  $g \in \mathcal{F}_{R\tilde{T}}$ . Therefore, using again the maximal monotonicity of  $R\tilde{T}$  in  $X^* \times X^{**}$ , we have

$$g(x^*, x^{**}) = (\mathfrak{S}_T)^{**}(x^{**}, x^*) = \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in R\tilde{T}.$$

Combining the above equations with (4.16) we conclude that  $h^*$  majorizes the duality product in  $X^* \times X^{**}$  and coincides with it in  $R\tilde{T}$ . Since  $h^*$  is also convex and closed, we have  $h^* \in \mathcal{F}_{R\tilde{T}}$ . This proves Item 3.  $\square$

Item 1 in the above theorem was firstly proved in [39], while item 2 and item 3 are taken from [31]. The following partial converse of item 1 of Theorem 4.3.1 is the third main result of this chapter.

**Theorem 4.3.2.** (Marques Alves-Svaiter [31]) *Suppose that  $T : X \rightrightarrows X^*$  is maximal monotone and has a unique extension  $\tilde{T} : X^{**} \rightrightarrows X^*$ . Then either*

$$(\mathfrak{S}_T)^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}, \quad (4.17)$$

*that is,  $T$  is of type (NI), or  $T$  is affine linear and  $T = \text{dom}(\varphi_T)$ .*

*Proof.* Suppose there exists only one  $\tilde{T} \subset X^{**} \times X^*$  maximal monotone extension of  $T$  to  $X^{**} \times X^*$ . If  $T$  is not of type (NI), there exists  $(x_0^*, x_0^{**}) \in X^* \times X^{**}$  such that

$$(\mathfrak{S}_T)^*(x_0^*, x_0^{**}) < \langle x_0^*, x_0^{**} \rangle. \quad (4.18)$$

Since  $R$  is a bijection that preserves the duality product and  $RT = T^{-1}$ , we conclude that  $R\tilde{T}$  is the unique maximal monotone extension of  $T^{-1}$  to  $X^* \times X^{**}$ . Using now Theorem 4.2.2, Theorem 4.2.1 and (4.11) we obtain

$$\begin{aligned} R\tilde{T} &= (T^{-1})^{\mu, X^* \times X^{**}} \\ &= \{(x^*, x^{**}) \in X^* \times X^{**} \mid \varphi_{T^{-1}, X^* \times X^{**}}(x^*, x^{**}) \leq \langle x^*, x^{**} \rangle\} \\ &= \{(x^*, x^{**}) \in X^* \times X^{**} \mid (\mathfrak{S}_T)^*(x^*, x^{**}) \leq \langle x^*, x^{**} \rangle\}. \end{aligned} \quad (4.19)$$

Suppose that

$$(\mathfrak{S}_T)^*(x^*, x^{**}) < \infty. \quad (4.20)$$

Define, for  $t \in \mathbb{R}$ ,

$$p(t) := (x_0^*, x_0^{**}) + t(x^* - x_0^*, x^{**} - x_0^{**}) = (1-t)(x_0^*, x_0^{**}) + t(x^*, x^{**}).$$

Since  $(\mathfrak{S}_T)^*$  is convex, we have the inequality

$$\begin{aligned} (\mathfrak{S}_T)^*(p(t)) - \pi_{X^* \times X^{**}}(p(t)) &\leq (1-t)(\mathfrak{S}_T)^*(x_0^*, x_0^{**}) + t(\mathfrak{S}_T)^*(x^*, x^{**}) \\ &\quad - \pi_{X^* \times X^{**}}(p(t)), \quad \forall t \in [0, 1]. \end{aligned}$$

Since the duality product is continuous, the limit of the right hand side of this inequality, for  $t \rightarrow 0+$  is  $(\mathfrak{S}_T)^*(x_0^*, x_0^{**}) - \langle x_0^*, x_0^{**} \rangle < 0$ . Combining this fact with (4.19) we conclude that for  $t \geq 0$  and small enough,

$$(x_0^*, x_0^{**}) + t(x^* - x_0^*, x^{**} - x_0^{**}) \in R\tilde{T}.$$

Altogether, we proved that

$$\begin{aligned} (\mathfrak{S}_T)^*(x^*, x^{**}) < \infty &\Rightarrow \exists \bar{t} > 0, \forall t \in [0, \bar{t}] \\ (x_0^*, x_0^{**}) + t(x^* - x_0^*, x^{**} - x_0^{**}) &\in R\tilde{T}. \end{aligned} \quad (4.21)$$

Now, suppose that

$$(\mathfrak{S}_T)^*(x_1^*, x_1^{**}) < \infty, \quad \mathfrak{S}_T^*(x_2^*, x_2^{**}) < \infty.$$

Then, using (4.21), we conclude that there exists  $t > 0$  such that

$$(x_0^*, x_0^{**}) + t(x_1^* - x_0^*, x_1^{**} - x_0^{**}) \in R\tilde{T}, \quad (x_0^*, x_0^{**}) + t(x_2^* - x_0^*, x_2^{**} - x_0^{**}) \in R\tilde{T}.$$

Since  $R\tilde{T}$  is (maximal) monotone, the above points are monotonically related (in the sense of Definition 4.2.1) and

$$t^2 \langle x_1^* - x_2^*, x_1^{**} - x_2^{**} \rangle \geq 0.$$

Hence,  $\langle x_1^* - x_2^*, x_1^{**} - x_2^{**} \rangle \geq 0$ . Therefore the set

$$W := \{(x^*, x^{**}) \in X^* \times X^{**} \mid (\mathcal{S}_T)^*(x^*, x^{**}) < \infty\},$$

is monotone. By (4.19),  $R\tilde{T} \subset W$ . Hence  $W = R\tilde{T}$  and

$$\tilde{T} = \{(x^{**}, x^*) \in X^{**} \times X^* \mid (\mathcal{S}_T)^*(x^*, x^{**}) < \infty\}.$$

Since  $(\mathcal{S}_T)^*$  is convex,  $W$  is also convex. Therefore,  $R\tilde{T}$  is convex and maximal monotone. Now, using Lemma 4.1.1 we conclude that  $R\tilde{T}$  is affine. This also implies that  $\tilde{T}$  is affine linear. Since

$$T = \tilde{T} \cap X \times X^*,$$

we conclude that  $T$  is affine and

$$\begin{aligned} T &= \{(x, x^*) \mid (\mathcal{S}_T)^*(x^*, x) < \infty\} \\ &= \{(x, x^*) \mid \varphi_T(x, x^*) < \infty\} \end{aligned}$$

where the last equality follows from Theorem 1.2.3.  $\square$

According to the above theorems, for non-linear maximal monotone operators, condition (4.17) is equivalent to unicity of maximal monotone extension to the bidual.

# Appendix A

## Basics of convex analysis

Let  $X$  be a real Banach space with dual  $X^*$ . We use the notation  $\pi$  and  $\pi_*$  for the duality product in  $X \times X^*$  and in  $X^* \times X^{**}$ , respectively:

$$\begin{aligned} \pi : X \times X^* &\rightarrow \mathbb{R}, & \pi_* : X^* \times X^{**} &\rightarrow \mathbb{R} \\ \pi(x, x^*) &= \langle x, x^* \rangle, & \pi_*(x^*, x^{**}) &= \langle x^*, x^{**} \rangle. \end{aligned} \quad (\text{A.1})$$

The norms on  $X$ ,  $X^*$  and  $X^{**}$  will be denoted by  $\|\cdot\|$ . Whenever necessary, we will identify  $X$  with its image under the canonical injection of  $X$  into  $X^{**}$ .

We denote by  $\bar{\mathbb{R}}$  the extended-real system and by  $\bar{\mathbb{R}}^X$  the set of extended-real valued functions defined on  $X$ :

$$\bar{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}, \quad \bar{\mathbb{R}}^X = \{f : X \rightarrow \bar{\mathbb{R}}\}.$$

A function  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$  is *convex* if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$$

whenever  $x, y \in X$  and  $t \in (0, 1)$ . This is equivalent to say that the epigraph of  $f$ , defined by  $\text{ep}f = \{(x, t) \in X \times \mathbb{R} \mid f(x) \leq t\}$  is a convex subset of  $X \times \mathbb{R}$ . Moreover,  $f$  is *lower semicontinuous* (l.s.c. for short) whenever  $\text{ep}f$  is a closed subset of  $X \times \mathbb{R}$ .

An extended-real valued function is said to be *proper* if  $f > -\infty$  it is not identically  $\infty$ . The *effective domain* of a proper function  $f \in \bar{\mathbb{R}}^X$  is

$$\text{dom}(f) = \{x \in X \mid f(x) < \infty\}.$$

For  $f \in \bar{\mathbb{R}}^X$ ,  $\text{conv } f \in \bar{\mathbb{R}}^X$  is the largest convex function majorized by  $f$ , and  $\text{cl } f \in \bar{\mathbb{R}}^X$  is the largest l.s.c. function majorized by  $f$ . It is trivial to verify that

$$\text{cl } f(x) = \liminf_{y \rightarrow x} f(y), \quad f^* = (\text{conv } f)^* = (\text{cl } \text{conv } f)^*.$$

The functions  $\text{cl } f$  and  $\text{cl conv } f$  are usually called the l.s.c. closure of  $f$  and the convex l.s.c. closure of  $f$ , respectively.

The concept of  $\varepsilon$ -subdifferential of a convex function  $f \in \bar{\mathbb{R}}^X$  was introduced by Brøndsted and Rockafellar [13]. It is a point to set operator  $\partial_\varepsilon f : X \rightrightarrows X^*$  defined at  $x \in X$  by

$$\partial_\varepsilon f(x) = \{x^* \in X^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle - \varepsilon, \forall y \in X\},$$

where  $\varepsilon \geq 0$ .

An special interest is given for the case  $\varepsilon = 0$ . The point to set operator  $\partial f : X \rightrightarrows X^*$  defined by  $\partial f = \partial_0 f$  is called the *subdifferential* of  $f$ . For each  $x \in X$  the elements  $x^* \in \partial f(x)$  are called the *subgradients* of  $f$  at  $x$ . In particular,  $\partial f(x) \subset \partial_\varepsilon f(x)$ , for all  $x \in X$  and  $\varepsilon \geq 0$ .

The next theorem estimates how well the  $\varepsilon$ -subdifferential approximates the subdifferential of a convex function. It is known as *Brøndsted-Rockafellar Theorem*.

**Theorem A.0.3** ([13, Lemma]). *Let  $f \in \bar{\mathbb{R}}^X$  be a proper, convex and l.s.c. function. Given  $x^* \in \partial_\varepsilon f(x)$ , for any  $\lambda > 0$  there exist  $\bar{x} \in X$  and  $\bar{x}^* \in \partial f(\bar{x})$  such that*

$$\|x - \bar{x}\| \leq \lambda, \quad \|x^* - \bar{x}^*\| \leq \frac{\varepsilon}{\lambda}.$$

For a proper convex function  $f \in \bar{\mathbb{R}}^X$ , the *Fenchel-Legendre conjugate* of  $f$  is the function  $f^* \in \bar{\mathbb{R}}^{X^*}$  defined by

$$f^*(x^*) = \sup_{x \in X} \langle x, x^* \rangle - f(x).$$

If  $f$  is proper, convex and l.s.c., then  $f^*$  is proper and  $f$  satisfies the *Fenchel-Young inequality*:

$$f(x) + f^*(x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*. \quad (\text{A.2})$$

Moreover, in this case,  $\partial_\varepsilon f$  and  $\partial f$  can be characterized using  $f^*$ :

$$\begin{aligned} \partial f(x) &= \{x^* \in X^* \mid f(x) + f^*(x^*) = \langle x, x^* \rangle\}, \\ \partial_\varepsilon f(x) &= \{x^* \in X^* \mid f(x) + f^*(x^*) \leq \langle x, x^* \rangle + \varepsilon\}. \end{aligned} \quad (\text{A.3})$$

The subdifferential and the  $\varepsilon$ -subdifferential of the function  $\frac{1}{2}\|\cdot\|^2$  will be of special interest, and will be denoted by  $J : X \rightrightarrows X^*$  and  $J_\varepsilon : X \rightrightarrows X^*$  respectively

$$J(x) = \partial \frac{1}{2}\|x\|^2, \quad J_\varepsilon(x) = \partial_\varepsilon \frac{1}{2}\|x\|^2.$$

Using  $f(x) = (1/2)\|x\|^2$  in (A.3), it is trivial to verify that

$$\begin{aligned} J(x) &= \{x^* \in X^* \mid \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = \langle x, x^* \rangle\} \\ &= \{x^* \in X^* \mid \|x\|^2 = \|x^*\|^2 = \langle x, x^* \rangle\} \end{aligned}$$

and

$$J_\varepsilon(x) = \{Z^* \in X^* \mid \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \leq \langle x, x^* \rangle + \varepsilon\}.$$

The operator  $J$  is widely used in convex analysis in Banach spaces and it is called the *duality mapping* of  $X$ . The operator  $J_\varepsilon$  was introduced by Gossez [21] to generalize some results concerning maximal monotonicity in reflexive Banach spaces to non-reflexive Banach spaces.

In what follows we present the Attouch-Brezis's version of the Fenchel-Rockafellar duality theorem:

**Theorem A.0.4** ([1, Theorem 1.1]). *Let  $X$  be a Banach space and  $f, g \in \bar{\mathbb{R}}^X$  be two proper, convex and l.s.c. functions. If*

$$\bigcup_{\lambda > 0} \lambda [\text{dom}(f) - \text{dom}(g)]$$

*is a closed subspace of  $X$ , then*

$$\inf_{x \in X} f(x) + g(x) = \max_{x^* \in X^*} -f^*(x^*) - g^*(-x^*). \quad (\text{A.4})$$

We finish this appendix with a well known result of the theory of convex functions:

**Lemma A.0.5.** *Let  $E$  be a real topological linear space and  $f : E \rightarrow \bar{\mathbb{R}}$  be a convex function. If  $g : E \rightarrow \mathbb{R}$  is Gateaux differentiable at  $x_0$ ,  $f(x_0) = g(x_0)$  and  $f \geq g$  in a neighborhood of  $x_0$ , then  $g'(x_0) \in \partial f(x_0)$ .*

# Bibliography

- [1] H. Attouch and H. Brezis. Duality for the sum of convex functions in general Banach spaces. In *Aspects of mathematics and its applications*, volume 34 of *North-Holland Math. Library*, pages 125–133. North-Holland, Amsterdam, 1986.
- [2] S. Bartz, H. H. Bauschke, J. M. Borwein, S. Reich, and X. Wang. Fitzpatrick functions, cyclic monotonicity and Rockafellar’s antiderivative. *Nonlinear Anal.*, 66(5):1198–1223, 2007.
- [3] H. H. Bauschke, J. M. Borwein, and X. Wang. Fitzpatrick functions and continuous linear monotone operators. *SIAM J. Optim.*, 18(3):789–809, 2007.
- [4] H. H. Bauschke and X. Wang. A convex-analytical approach to extension results for  $n$ -cyclically monotone operators. *Set-Valued Anal.*, 15(3):297–306, 2007.
- [5] H.H. Bauschke, X. Wang, and L. Yao. Monotone linear relations: maximality and Fitzpatrick functions. *ArXiv:0805.4256v1*, May, 2008.
- [6] G. Birkhoff. *Lattice theory*. Third edition. American Mathematical Society Colloquium Publications, Vol. XXV. American Mathematical Society, Providence, R.I., 1967.
- [7] J. M. Borwein. Maximal monotonicity via convex analysis. *J. Convex Anal.*, 13(3-4):561–586, 2006.
- [8] J. M. Borwein. Maximality of sums of two maximal monotone operators. *Proc. Amer. Math. Soc.*, 134(10):2951–2955, 2006.
- [9] J. M. Borwein. Simon Fitzpatrick memorial volume. *J. Convex Anal.*, 13(3-4):463–476, 2006.
- [10] J. M. Borwein. Maximality of sums of two maximal monotone operators in general Banach space. *Proc. Amer. Math. Soc.*, 135(12):3917–3924 (electronic), 2007.

- [11] R. I. Boţ, S.-M. Grad, and G. Wanka. Maximal monotonicity for the precomposition with a linear operator. *SIAM J. Optim.*, 17(4):1239–1252 (electronic), 2006.
- [12] H. Brezis and A. Haraux. Image d’une somme d’opérateurs monotones et applications. *Israel J. Math.*, 23(2):165–186, 1976.
- [13] A. Brøndsted and R. T. Rockafellar. On the subdifferentiability of convex functions. *Proc. Amer. Math. Soc.*, 16:605–611, 1965.
- [14] R. S. Burachik and A. N. Iusem. On non-enlargeable and fully enlargeable monotone operators. *J. Convex Anal.*, 13(3-4):603–622, 2006.
- [15] R. S. Burachik, A. N. Iusem, and B. F. Svaiter. Enlargement of monotone operators with applications to variational inequalities. *Set-Valued Anal.*, 5(2):159–180, 1997.
- [16] R. S. Burachik and B. F. Svaiter.  $\epsilon$ -enlargements of maximal monotone operators in Banach spaces. *Set-Valued Anal.*, 7(2):117–132, 1999.
- [17] R. S. Burachik and B. F. Svaiter. Maximal monotone operators, convex functions and a special family of enlargements. *Set-Valued Anal.*, 10(4):297–316, 2002.
- [18] R. S. Burachik and B. F. Svaiter. Maximal monotonicity, conjugation and the duality product. *Proc. Amer. Math. Soc.*, 131(8):2379–2383 (electronic), 2003.
- [19] R. Correa and P. Gajardo. Eigenvalues of set-valued operators in Banach spaces. *Set-Valued Anal.*, 13(1):1–19, 2005.
- [20] S. Fitzpatrick. Representing monotone operators by convex functions. In *Workshop/Miniconference on Functional Analysis and Optimization (Canberra, 1988)*, volume 20 of *Proc. Centre Math. Anal. Austral. Nat. Univ.*, pages 59–65. Austral. Nat. Univ., Canberra, 1988.
- [21] J.-P. Gossez. Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs. *J. Math. Anal. Appl.*, 34:371–395, 1971.
- [22] J.-P. Gossez. On the range of a coercive maximal monotone operator in a nonreflexive Banach space. *Proc. Amer. Math. Soc.*, 35:88–92, 1972.

- [23] J.-P. Gossez. On a convexity property of the range of a maximal monotone operator. *Proc. Amer. Math. Soc.*, 55(2):359–360, 1976.
- [24] J.-P. Gossez. On the extensions to the bidual of a maximal monotone operator. *Proc. Amer. Math. Soc.*, 62(1):67–71 (1977), 1976.
- [25] T. T. Hue and J. J. Strodiot. Convergence analysis of a relaxed extragradient-proximal point algorithm application to variational inequalities. *Optimization*, 54(2):191–213, 2005.
- [26] A. Kaplan and R. Tichatschke. Interior proximal method for variational inequalities on non-polyhedral sets. *Discuss. Math. Differ. Incl. Control Optim.*, 27(1):71–93, 2007.
- [27] P.-E. Maingé and A. Moudafi. A proximal method for maximal monotone operators via discretization of a first order dissipative dynamical system. *J. Convex Anal.*, 14(4):869–878, 2007.
- [28] M. Marques Alves and B. F. Svaiter. A new proof for maximal monotonicity of subdifferential operators. *J. Convex Anal.*, 15(2):345–348, 2008.
- [29] M. Marques Alves and B.F. Svaiter. Brøndsted-Rockafellar property and maximality of monotone operators representable by convex functions in non-reflexive Banach spaces. *J. Convex Anal.*, 2(15), 2008.
- [30] M. Marques Alves and B.F. Svaiter. On the surjectivity properties of perturbations of maximal monotone operators in non-reflexive Banach spaces. *Submitted*, 2008.
- [31] M. Marques Alves and B.F. Svaiter. Maximal monotone operators with a unique extension to the bidual. *J. Convex Anal.*, 16(2), 2009.
- [32] M. Marques Alves and B.F. Svaiter. A new old class of maximal monotone operators. *J. Convex Anal.*, 16(4), 2009.
- [33] J.-E. Martínez-Legaz and B. F. Svaiter. Monotone operators representable by l.s.c. convex functions. *Set-Valued Anal.*, 13(1):21–46, 2005.
- [34] J.-E. Martínez-Legaz and B. F. Svaiter. Minimal convex functions bounded below by the duality product. *Proc. Amer. Math. Soc.*, 136(3):873–878 (electronic), 2008.

- [35] J.-E. Martinez-Legaz and M. Théra. A convex representation of maximal monotone operators. *J. Nonlinear Convex Anal.*, 2(2):243–247, 2001. Special issue for Professor Ky Fan.
- [36] S. Reich and S. Simons. Fenchel duality, Fitzpatrick functions and the Kirszbraun-Valentine extension theorem. *Proc. Amer. Math. Soc.*, 133(9):2657–2660 (electronic), 2005.
- [37] R. T. Rockafellar. On the maximal monotonicity of subdifferential mappings. *Pacific J. Math.*, 33:209–216, 1970.
- [38] R. T. Rockafellar. On the maximality of sums of nonlinear monotone operators. *Trans. Amer. Math. Soc.*, 149:75–88, 1970.
- [39] S. Simons. The range of a monotone operator. *J. Math. Anal. Appl.*, 199(1):176–201, 1996.
- [40] S. Simons. Maximal monotone multifunctions of Brøndsted-Rockafellar type. *Set-Valued Anal.*, 7(3):255–294, 1999.
- [41] M. V. Solodov and B. F. Svaiter. A hybrid approximate extragradient-proximal point algorithm using the enlargement of a maximal monotone operator. *Set-Valued Anal.*, 7(4):323–345, 1999.
- [42] M. V. Solodov and B. F. Svaiter. A unified framework for some inexact proximal point algorithms. *Numer. Funct. Anal. Optim.*, 22(7-8):1013–1035, 2001.
- [43] B. F. Svaiter. Fixed points in the family of convex representations of a maximal monotone operator. *Proc. Amer. Math. Soc.*, 131(12):3851–3859 (electronic), 2003.
- [44] D. Torralba. *Convergence épigraphique et changements d'échelle en analyse variationnelle et optimisation*. PhD Thesis. Université de Montpellier II, 1996.