Instituto Nacional de Matemática Pura e Aplicada
(IMPA/MCT)

On the Cauchy problem for a nonlocal perturbation of the KdV equation

Borys Yamil Alvarez Samaniego

Tese apresentada para a obtenção do Título de
Doutor em Matemática

Rio de Janeiro

Março, 2002

Resumo

Denotemos por $\mathcal{H}$ a transformada de Hilbert e seja $\eta \geq 0$. Mostraremos que os problemas de valor inicial $u_t + uu_x + u_{xxx} + \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) = 0$, $u(\cdot, 0) = \phi(\cdot)$ e $u_t + \frac{1}{2}(u_x)^2 + u_{xxx} + \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) = 0$, $u(\cdot, 0) = \phi(\cdot)$ são globalmente bem postos em $H^s(\mathbb{R})$, $s \geq 1$, $\eta > 0$. Estudaremos o comportamento limite da solução da primeira equação quando η tende para zero em $H^s(\mathbb{R})$, $s \geq 2$. Além disso, provaremos um teorema de continuação única para a primeira equação em $\mathcal{F}_{3,3}(\mathbb{R}) = H^3(\mathbb{R}) \cap L^2_3(\mathbb{R})$, $\eta > 0$, o que implica perda da propriedade de persistência.

Palavras Chave: Problema de Cauchy, Transformada de Hilbert, Equação KdV.

Abstract

Let $\mathcal{H}$ denote the Hilbert transform and $\eta \geq 0$. We show that the initial value problems $u_t + uu_x + u_{xxx} + \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) = 0$, $u(\cdot, 0) = \phi(\cdot)$ and $u_t + \frac{1}{2}(u_x)^2 + u_{xxx} + \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) = 0$, $u(\cdot, 0) = \phi(\cdot)$ are globally well-posed in $H^s(\mathbb{R})$, $s \geq 1$, $\eta > 0$. We study the limiting behavior of the solutions of the first equation as η tends to zero in $H^s(\mathbb{R})$ and $s \geq 2$. Moreover, we prove a unique continuation theorem for the first equation in $\mathcal{F}_{3,3}(\mathbb{R}) = H^3(\mathbb{R}) \cap L^2_3(\mathbb{R})$, $\eta > 0$, which implies that the persistence property does not hold.

Keywords: Cauchy problem, Hilbert transform, KdV equation.

Agradecimentos

Queria agradecer a Deus e a Nossa Senhora por ter me permitido alcançar esta meta.

Ao meu orientador o Prof. Rafael Iório Jr. por todos os ensinamentos que recebi dele durante os quatro anos de Doutoramento, pela valiosa revisão deste trabalho e pelas sugestões e comentários durante a realização desta Tese.

Aos meus pais Wilson e Maruja e aos meus irmãos Petronio e María de Lourdes por todo o seu amor e apoio incondicional.

A Cristiana Bittencourt pelo seu imenso apoio, por sua compreensão e carinho.

Aos Professores Carlos Isnard, Felipe Linares e Hermano Frid pelos ensinamentos e pela valiosa disposição durante todo este tempo no IMPA, por terem aceito participar da banca e pelos comentários que ajudaram a melhorar a apresentação deste trabalho.

Aos Professores Gustavo Perla (LNCC) e Marcia Scialom (UNICAMP) pelas importantes opiniões e pela sua participação como membros da banca.

Aos Professores André Nachbin e Jorge Zubelli pelos conhecimentos recebidos durante todo este tempo no IMPA.

Aos Professores Nino Endara e Rolando Sáenz da Escuela Politécnica Nacional em Quito por toda a formação básica e ensinamentos que recebi deles.

A todos os meus “hermanos” do IMPA pela amizade e pelas muitas horas de discussão matemática e não matemática em especial para Nuno Luzia, Jaime Orrillo, Guillermo Rodríguez, Feliz Soriano, Xavier Carvajal, Mario da Costa, e Percy Sánchez.

Ao IMPA por ter me concedido o privilégio de estudar nesta Instituição.

Ao CNPq pela bolsa concedida durante o Mestrado e Doutorado no IMPA.

Rio de Janeiro, Brasil
21 de Março de 2002.

Contents

Chapter 1

Introduction

In this Thesis we will consider real valued solutions of the Cauchy problems associated to the following equations

$$u_t + \frac{1}{2}(u_x)^2 + u_{xxx} + \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) = 0,\ u(\cdot, 0) = \phi(\cdot). \tag{1.1}$$

$$w_t + ww_x + w_{xxx} + \eta(\mathcal{H}w_x + \mathcal{H}w_{xxx}) = 0,\ w(\cdot, 0) = \psi(\cdot), \tag{1.2}$$

where $\mathcal{H}$ denotes the Hilbert transform

$$\mathcal{H}f(x) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{f(y)}{y-x}dy, \quad f \in \mathcal{S}(\mathbb{R}), \tag{1.3}$$

$\mathcal{P}$ represents the principal value of the integral and the parameter η is an arbitrary positive number. Some of the properties of the Hilbert transform $\mathcal{H}$ (see [6]) that we will use along this work are: $\mathcal{H} \in B(L^p(\mathbb{R}))$, $p \in (1, \infty)$ and $\widehat{(\mathcal{H}f)}(\xi) = ih(\xi)\hat{f}(\xi)$, for all $f \in H^s(\mathbb{R})$, where

$$h\left(\xi\right) = \begin{cases} -1, & \xi < 0, \\ 1, & \xi > 0. \end{cases} \tag{1.4}$$

Equation (1.2) was derived by Ostrovsky et al. to describe the radiational instability of long waves in a stratified shear flow [16]. It deserves remark that the fourth and fifth terms represent, respectively, amplification and damping. B.F. Feng and T. Kawahara ([5]) studied periodic solutions and solitary waves associated with equation (1.2), from a numerical standpoint. For every $\eta > 0$, a family of solitary wave solutions whose members are distinguished by the number of "humps" is identified numerically. The tails of these waves decay as $O(1/|x|^2)$ when $|x| \to \infty$ independent of the value of η and the number of humps, consequently these waves decay to zero algebraically, which is consistent with Theorem 5.4 that shows that there is an upper limit for the rate of decay of the solution of equation (1.2). Theorem 5.4 implies that solutions do not fall off arbitrarily fast even if the initial condition has this property.

Equation (1.2) can be regarded also as a non local perturbation of a modified version of the generalized Ott-Sudan equation ($w_t + ww_x + w_{xxx} + \mu Bu = 0$, where $\mu > 0$ and $B = -\partial_x^2$, $B = 1$ or $B = -\mathcal{H}\partial_x$) which belongs to a class of nonlinear dissipative equations and it models the unidirectional propagation of small-amplitude nonlinear waves (see [1], [2]). Moreover, it is worth noting that if the Fourier transform of the solution of equation (1.2) has support contained in $\mathbb{R} \setminus [-1, 1]$ for all times, then (1.2) is a nonlinear dissipative equation, as we can see from equation (4.6) below.

The plan of this Thesis is as follows: In Chapter 2 we study the Cauchy problem associated to the linear part of equation (1.1), and we obtain some useful properties of the semigroup associated to the solution. The local theory for equation (1.1) in H^s, $s \geq 1$, and for equation (1.2) in H^s, $s > 1/2$, is developed in Chapter 3. Chapter 4 is devoted to prove global well-posedness of equations (1.1) and (1.2) in H^s, $s \geq 1$, and to study the behavior of the limit of solutions of (1.2) in H^s, $s \geq 2$, when η goes to 0. Moreover, it is worth to say that if u is a solution of problem (1.1) then u_x is a solution of problem (1.2), with $\psi = \phi'$. Therefore global existence of the solution of equation (1.1) in H^s, $s \geq 1$, implies global existence of the solution of equation (1.2) for initial data belonging to the set $\{\psi \in L^2(\mathbb{R})/\exists \phi \in H^1(\mathbb{R}); \psi = \phi'\}$. Finally, in Chapter 5 we obtain a unique continuation theorem for the solution of equation (1.2). We prove that if the solution $w(t)$ is sufficiently smooth ($w(t) \in H^3(\mathbb{R})$) and falls off sufficiently fast as $|x| \to \infty$ ($w(t) \in L^2_3(\mathbb{R})$) for all $t \in [0, T]$, then $w(t) = 0$, for all $t \in [0, T]$.

Notation:

- $\|\cdot\|_s$: the norm in $H^s(\mathbb{R})$, $s \in \mathbb{R}$.
- $\mathcal{H}$: the Hilbert transform.
- $B(X, Y)$: the collection of bounded linear operators from X into Y. If $X = Y$ we write $B(X)$.
- $L^2_r(\mathbb{R}) = (H^r(\mathbb{R}))^\wedge$ and $\mathcal{F}_{s,r}(\mathbb{R}) = H^s(\mathbb{R}) \cap L^2_r(\mathbb{R})$.

Chapter 2

The Linear Equation

In this chapter we consider the Cauchy problem associated to the linear part of equation (1.1), namely

$$v_t + v_{xxx} + \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) = 0, \;\; v(\cdot, 0) = \phi(\cdot), \tag{2.1}$$

where $\phi \in H^s$, for $s \in \mathbf{R}$ and $\eta \geq 0$.

Let $t \geq 0$ and $\xi \in \mathbb{R}$. Taking the Fourier transform in (2.1) and integrating the resulting expression between 0 ant t we get

$$\hat{v}(\xi, t) = \exp[\left(i\xi^3 + \eta(|\xi| - |\xi|^3)\right) t] \; \hat{\phi}(\xi).$$

For $t \geq 0$ and $\xi \in \mathbb{R}$, let

$$F_\eta(t, \xi) = \exp[t \left(i\xi^3 - \eta(|\xi|^3 - |\xi|)\right)]. \tag{2.2}$$

Let $(E_\eta(t))_{t \geq 0}$ be the semigroup on $L^2(\mathbb{R})$ defined by

$$E_\eta(t) f = \mathcal{F}^{-1}(F_\eta(t, \cdot)\hat{f}), \;\; f \in L^2(\mathbb{R}). \tag{2.3}$$

Lemma 2.1. *Let $\eta > 0$. Then, $(E_\eta(t))_{t \geq 0}$ is a C^0 semigroup in $H^s(\mathbb{R})$, $s \in \mathbb{R}$. Moreover,*

$$\|E_\eta(t)\|_{B(H^s;H^s)} \leq e^{\eta t}. \tag{2.4}$$

Proof. It follows easily from the definition of $E_\eta(t)$ that $E_\eta(0) = I$ and $E_\eta(t+t') = E_\eta(t)E_\eta(t')$.
The continuity of the semigroup is a consequence of the inequality

$$|F_\eta(\tau, \xi)| \leq e^{\eta(t+1/2)}, \;\; 0 \leq \tau \leq t + 1/2, \;\; \xi \in \mathbb{R},$$

and the Dominated Convergence Theorem.
Relation (2.4) follows easily from the fact that

$$e^{-2\eta t(|\xi|^3-|\xi|)} \leq e^{2\eta t}, \tag{2.5}$$

for all $\xi \in \mathbb{R}$ and for all $t \geq 0$. ■

Remarks:

1. Let $\phi \in H^s(\mathbb{R})$, then $v(\cdot,t) = E_\eta(t)\phi$ is the unique solution of (2.1), in the class $C(0,\infty; H^s(\mathbb{R})) \cap C^1(0,\infty; H^{s-3}(\mathbb{R}))$.
   The fact that $v(\cdot,t) \in C(0,\infty; H^s(\mathbb{R}))$ follows directly from the continuity of the semigroup $E_\eta(t)$.
   The fact that $v_t \in C(0,\infty; H^{s-3}(\mathbb{R})$ is obtained as follows. Let $t,\tau \geq 0$, $\phi \in H^s(\mathbb{R})$ then

   $$\begin{aligned} \|v_t(t) - v_t(\tau)\|_{s-3}^2 &= \int_{-\infty}^{+\infty} (1+\xi^2)^{s-3} |i\xi^3 - \eta(|\xi|^3 - |\xi|)|^2 \\ &\cdot |e^{t(i\xi^3-\eta(|\xi|^3-|\xi|))} - e^{\tau(i\xi^3-\eta(|\xi|^3-|\xi|))}|^2 |\hat{\phi}(\xi)|^2 d\xi. \end{aligned} \tag{2.6}$$

   We see that $|i\xi^3 - \eta(|\xi|^3 - |\xi|)|^2 \leq c(\eta)(1+\xi^2)^3$ and

   $$|e^{t(i\xi^3-\eta(|\xi|^3-|\xi|))} - e^{\tau(i\xi^3-\eta(|\xi|^3-|\xi|))}| \leq e^{\eta t} + e^{\eta\tau} \leq 2e^{\eta(t+1/2)}, \quad for \ \ \tau \leq t + 1/2.$$

   Then using the Dominated Convergence Theorem we have that $\|v_t(t) - v_t(\tau)\|_{s-3}^2 \to 0$, as $\tau \to t$ and therefore $v_t \in C(0,\infty; H^{s-3}(\mathbb{R}))$.

   Now using the estimates

   $$\left|\frac{e^{h(i\xi^3-\eta(|\xi|^3-|\xi|))} - 1}{h}\right| \leq c(\eta)(1+\xi^2)^{3/2},$$

   $|i\xi^3 - \eta(|\xi|^3 - |\xi|)|^2 \leq c(\eta)(1+\xi^2)^3$ and the Dominated Convergence Theorem, we see that

   $$\lim_{h\to 0} \left\|\frac{v(t+h) - v(t)}{h} + v_{xxx} + \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx})\right\|_{s-3}^2 = 0, \tag{2.7}$$

   Finally, uniqueness follows by taking the Fourier transform of (2.1).

2. When $\eta = 0$, we obtain the unitary group associated with the KdV equation. In this case $\|E_\eta(t)\|_{B(H^s,H^s)} = 1$, for all $t \geq 0$.

Lemma 2.2. *Let $t > 0$, $s \in \mathbb{R}$, $\lambda \geq 0$ and $\eta > 0$ be given. Then, $E_\eta(t) \in B(H^s(\mathbb{R}), H^{s+\lambda}(\mathbb{R}))$. Moreover,*

$$\|E_\eta(t)\phi\|_{s+\lambda} \leq c_\lambda[e^{\eta t} + (1 + \frac{1}{(\eta t)^{\lambda/2}})e^{\frac{\eta t}{8}(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}\lambda}{\eta t}})}]\|\phi\|_s, \tag{2.8}$$

where $\phi \in H^s(\mathbb{R})$ and c_λ is a constant depending only on λ.

Proof. Let $\eta > 0$, $t > 0$, $s \in \mathbb{R}$ and $\lambda \geq 0$. We see that

$$\begin{aligned}
\|E_\eta(t)\phi\|_{s+\lambda}^2 &= \int_{-\infty}^{+\infty} (1+\xi^2)^{s+\lambda}|F_\eta(t,\xi)\hat{\phi}(\xi)|^2 d\xi \\
&= \int_{-\infty}^{+\infty} (1+\xi^2)^{s+\lambda}e^{-2\eta t(|\xi|^3-|\xi|)}|\hat{\phi}(\xi)|^2 d\xi \\
&\leq \sup_{\xi\in\mathbb{R}}[(1+\xi^2)^\lambda e^{-2\eta t(|\xi|^3-|\xi|)}]\|\phi\|_s^2 \\
&\leq c_\lambda \sup_{\xi\in\mathbb{R}}[e^{-2\eta t(|\xi|^3-|\xi|)} + \xi^{2\lambda}e^{-2\eta t(|\xi|^3-|\xi|)}]\|\phi\|_s^2 \\
&\leq c_\lambda[e^{2\eta t} + \sup_{\xi\in\mathbb{R}}\xi^{2\lambda}e^{-2\eta t(|\xi|^3-|\xi|)}]\|\phi\|_s^2.
\end{aligned} \tag{2.9}$$

The function $\xi^{2\lambda}e^{-2\eta t(|\xi|^3-|\xi|)}$ is even, then it is enough to study the supremum of this function for $\xi > 0$. Since $\xi^3 - \xi \geq 1/\sqrt{3}\xi^2 - \xi$, for $\xi \geq 1/\sqrt{3}$, we have that $\xi^{2\lambda}e^{-2\eta t(|\xi|^3-|\xi|)} \leq \xi^{2\lambda}e^{-2\eta t(1/\sqrt{3}\xi^2-\xi)} =: g(\xi)$, for $\xi \geq 1/\sqrt{3}$. If $\xi > 0$, then

$$\begin{aligned}
g'(\xi) = 0 \quad &\Longleftrightarrow \quad \xi^2 - \frac{\sqrt{3}}{2}\xi - \frac{\sqrt{3}}{2\eta t}\lambda = 0 \\
&\Longleftrightarrow \quad \xi = \frac{1}{4}[\sqrt{3} + \sqrt{3 + \frac{8\sqrt{3}\lambda}{\eta t}}].
\end{aligned} \tag{2.10}$$

Let $\xi_0 := \frac{1}{4}\left[\sqrt{3} + \sqrt{3 + \frac{8\sqrt{3}\lambda}{\eta t}}\right] \geq \frac{\sqrt{3}}{2} \geq \frac{1}{\sqrt{3}}$. It follows easily from the definition of ξ_0 that

$$-2\eta t\left(\frac{\xi_0^2}{\sqrt{3}} - \xi_0\right) = \frac{\sqrt{3}}{4}\eta t - \lambda + \frac{\eta t}{4}\sqrt{3 + \frac{8\sqrt{3}\lambda}{\eta t}}. \tag{2.11}$$

Therefore

$$\begin{aligned}
\sup_{\xi\geq 0}\xi^{2\lambda}e^{-2\eta t(\xi^3-\xi)} &\leq c_\lambda e^{2\eta t}+\sup_{\xi\geq 1/\sqrt{3}}\xi^{2\lambda}e^{-2\eta t(\frac{\xi^2}{\sqrt{3}}-\xi)}\\
&\leq c_\lambda e^{2\eta t}+\xi_0^{2\lambda}e^{-2\eta t(\frac{\xi_0^2}{\sqrt{3}}-\xi_0)}\\
&\leq c_\lambda e^{2\eta t}+c_\lambda(3^\lambda+(3+\frac{8\sqrt{3}\lambda}{\eta t})^\lambda)e^{\frac{\eta t}{4}(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}\lambda}{\eta t}})}\\
&\leq c_\lambda e^{2\eta t}+c_\lambda(1+\frac{1}{(\eta t)^\lambda})e^{\frac{\eta t}{4}(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}\lambda}{\eta t}})}. \qquad (2.12)
\end{aligned}$$

Inequalities (2.9) and (2.12) imply

$$\|E_\eta(t)\phi\|_{s+\lambda}^2\leq c_\lambda[e^{2\eta t}+(1+\frac{1}{(\eta t)^\lambda})e^{\frac{\eta t}{4}(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}\lambda}{\eta t}})}]\|\phi\|_s^2. \qquad (2.13)$$

Finally from (2.13) we get (2.8). ■

Lemma 2.3. *Let $t>0$, $s>-1/2$ and $\eta>0$ be given. Then, $E_\eta(t)\in B(L^1(\mathbb{R}),H^s(\mathbb{R}))$. Moreover,*

$$\|E_\eta(t)\psi\|_s\leq c_s(e^{\eta t}+\frac{1}{(\eta t)^{\frac{2s+1}{6}}})\|\psi\|_{L^1}, \qquad (2.14)$$

for all $\psi\in L^1(\mathbb{R})$.

Proof: Let $t>0$, $s>-1/2$, $\eta>0$ and $\psi\in L^1(\mathbb{R})$.

$$\begin{aligned}
\|E_\eta(t)\psi\|_s^2 &= \int_{-\infty}^{+\infty}(1+\xi^2)^s|F_\eta(t,\xi)\hat{\psi}(\xi)|^2d\xi\\
&\leq (2\pi)^{-1}\|\psi\|_{L^1}^2\int_{-\infty}^{+\infty}(1+\xi^2)^s e^{-2\eta t(|\xi|^3-|\xi|)}d\xi. \qquad (2.15)
\end{aligned}$$

Let $I:=\int_{-\infty}^{+\infty}(1+\xi^2)^s e^{-2\eta t(|\xi|^3-|\xi|)}d\xi$. Then,

$$\begin{aligned}
I &= 2\int_0^{+\infty}(1+\xi^2)^s e^{-2\eta t(\xi^3-\xi)}d\xi\\
&\leq 2\Big(\int_0^{\sqrt{2}}3^s e^{2\eta t}d\xi+\int_{\sqrt{2}}^{+\infty}(2\xi^2)^s e^{-2\eta t(\xi^3-\xi)}d\xi\Big)\\
&\leq 2^{3/2}3^s e^{2\eta t}+2^{s+1}\int_{\sqrt{2}}^{+\infty}\xi^{2s}e^{-\eta t\xi^3}d\xi, \qquad (2.16)
\end{aligned}$$

where in inequality (2.16) we used the fact that $\xi^3 - \xi \geq \xi^3/2$, for $\xi \geq \sqrt{2}$. Making the change of variable $x = \eta t \xi^3$ in the integral on the right hand side of (2.16) we obtain

$$\begin{aligned} I &\leq 2^{3/2} 3^s e^{2\eta t} + \frac{2^{s+1}}{3(\eta t)^{\frac{2s+1}{3}}} \int_0^{+\infty} x^{\frac{2s-2}{3}} e^{-x} dx \\ &= 2^{3/2} 3^s e^{2\eta t} + \frac{2^{s+1}}{3(\eta t)^{\frac{2s+1}{3}}} \Gamma\big(\frac{2s+1}{3}\big) \\ &\leq c_s \big(e^{2\eta t} + \frac{1}{(\eta t)^{\frac{2s+1}{3}}}\big). \end{aligned} \tag{2.17}$$

Combining (2.15) and (2.17), and taking the square root of the resulting expression concludes the proof. ■

Chapter 3

Local Theory in $H^s(\mathbb{R}), s \geq 1, \eta > 0$

In this chapter we use Banach's fixed point theorem in a suitable function space, to find a local solution to the integral equation (3.1) in $H^s(\mathbb{R}), s \geq 1$. Proposition 3.1 will give us a local solution for equation (1.1) in $H^s(\mathbb{R}), s \geq 1$. Similar considerations are made for equation (1.2) in $H^s(\mathbb{R}), s > 1/2$.

Theorem 3.1. *Let $\eta > 0$ be fixed and let $\phi \in H^s(\mathbb{R})$, where $s \geq 1$. Then, there exist $T_s(s, \|\phi\|_s, \eta) > 0$ and a unique function $u := u_\eta \in C(0, T_s; H^s(\mathbb{R}))$ satisfying the integral equation*

$$u(\cdot, t) = E_\eta(t)\phi(\cdot) - \frac{1}{2}\int_0^t E_\eta(t-t')(u_x)^2(\cdot, t')dt', \tag{3.1}$$

where $E_\eta(t)$ is defined by (2.3).

Proof: Let $M, T > 0$ be fixed, but arbitrary. T will be conveniently chosen later. Let us start considering the case $s \in [1, 5/2)$. Consider

$$(Af)(t) = E_\eta(t)\phi - \frac{1}{2}\int_0^t E_\eta(t-t')(\partial_x f)^2(t')dt', \tag{3.2}$$

defined on the complete metric space

$$\Xi_s(T) = \big\{f \in C(0, T; H^s(\mathbb{R})); \sup_{t\in[0,T]} \|f(t) - E_\eta(t)\phi\|_s \leq M\big\}. \tag{3.3}$$

i.) First we will prove that if $f \in \Xi_s(T)$ then $Af \in C(0, T : H^s(\mathbb{R}))$. In fact, let

$f \in \Xi_s(T)$, then we have that

$$\begin{aligned}\|(Af)(t)-(Af)(\tau)\|_s &\leq \|(E_\eta(t)-E_\eta(\tau))\phi\|_s + \frac{1}{2}\|\int_0^t E_\eta(t-t')(f_x)^2(t')dt' \\ &\quad -\int_0^\tau E_\eta(\tau-t')(f_x)^2(t')dt'\|_s. \end{aligned} \tag{3.4}$$

The first term on the right hand side of (3.4) goes to zero as $\tau \to t$ because $E_\eta(t)$ is a C^0-semigroup defined on $H^s(\mathbb{R})$. To study the second term on the right hand side of (3.4) let us assume without loss of generality that $\tau > t > 0$. Then

$$\begin{aligned}&\|\int_0^t E_\eta(t-t')(f_x)^2(t')dt' - \int_0^\tau E_\eta(\tau-t')(f_x)^2(t')dt'\|_s \\ &\leq \int_0^t \|(E_\eta(t-t')-E_\eta(\tau-t'))(f_x)^2(t')\|_s dt' + \int_t^\tau \|E_\eta(\tau-t')(f_x)^2(t')\|_s dt'. \end{aligned} \tag{3.5}$$

Denote by $I_1(t,\tau)$ and $I_2(t,\tau)$ the first and the second terms respectively in last expression. By Lemma 2.3 we get

$$\begin{aligned}I_2(t,\tau) &\leq \int_t^\tau c_s\big(e^{\eta(\tau-t')} + \frac{1}{(\eta(\tau-t'))^{\frac{2s+1}{6}}}\big)\|(f_x)^2(t')\|_{L^1}dt' \\ &\leq c_s(M^2+e^{2\eta T}\|\phi\|_s^2)\int_t^\tau \big(e^{\eta(\tau-t')} + \frac{1}{(\eta(\tau-t'))^{\frac{2s+1}{6}}}\big)dt' \\ &= c_s\frac{(M^2+e^{2\eta T}\|\phi\|_s^2)}{\eta}\int_0^{\eta(\tau-t)}\big(e^x+\frac{1}{x^{\frac{2s+1}{6}}}\big)dx, \end{aligned} \tag{3.6}$$

where in the second inequality we have used the fact that $\|(f_x)^2(t')\|_{L^1} \leq \|f(t')\|_1^2$. It follows easily from (3.6) that $I_2(t,\tau)$ tends to zero as $\tau \to t$.
On the other hand, using Lemma 2.3 again we have that

$$\begin{aligned}\|(E_\eta(t-t')-E_\eta(\tau-t'))(f_x)^2(t')\|_s &\leq \|E_\eta(t-t')(f_x)^2(t')\|_s + \|E_\eta(\tau-t')(f_x)^2(t')\|_s \\ &\leq 2c_s\big(e^{\eta(T-t')} + \frac{1}{(\eta(t-t'))^{\frac{2s+1}{6}}}\big)\sup_{t'\in[0,T]}\|f(t')\|_1^2, \end{aligned}$$

where the last term is an integrable function of $t' \in [0,t]$. It follows easily that $\lim_{\tau\to t}\|(E_\eta(t-t')-E_\eta(\tau-t'))(f_x)^2(t')\|_s = 0$, for all $t' \in [0,t]$. Applying the Dominated Convergence Theorem we obtain that $\lim_{\tau\to t} I_1(t,\tau) = 0$. This completes the proof that $Af \in C(0,T;H^s(\mathbb{R}))$.

ii.) Next, we prove that we can choose $T = \tilde{T} > 0$ sufficiently small such that $A(\Xi_s(\tilde{T})) \subset \Xi_s(\tilde{T})$. Let $u \in \Xi_s(T)$. Then,

$$\begin{aligned}
\|(Au)(t) - E_\eta(t)\phi\|_s &\leq \frac{1}{2}\int_0^t \|E_\eta(t-t')(\partial_x u)^2(t')\|_s dt' \\
&\leq c_s \sup_{[0,T]} \|u(t)\|_1^2 \int_0^t \big(e^{\eta(t-t')} + \frac{1}{(\eta(t-t'))^{\frac{2s+1}{6}}}\big)dt' \\
&\leq c_s \sup_{[0,T]} \|u(t)\|_s^2 \frac{1}{\eta}\int_0^{\eta t} \big(e^x + \frac{1}{x^{\frac{2s+1}{6}}}\big)dx \\
&\leq c_s(M^2 + e^{2\eta T}\|\phi\|_s^2)\big(\frac{e^{\eta T}-1}{\eta} + \frac{6\eta^{-\frac{2s+1}{6}}}{-2s+5}T^{\frac{-2s+5}{6}}\big). \qquad (3.7)
\end{aligned}$$

We choose now $T = \tilde{T} > 0$ sufficiently small such that the right hand side of (3.7) is less than M.

iii.) Finally, we will prove that there exists $\hat{T} \in (0, \tilde{T}]$, such that A is a contraction on $\Xi_s(\hat{T})$. Let $t \in [0, T]$, $u, v \in \Xi_s(\tilde{T})$. Defining

$$g(T) := \frac{e^{\eta T}-1}{\eta} + \frac{6}{-2s+5}\eta^{-\frac{2s+1}{6}}T^{\frac{-2s+5}{6}},$$

we get

$$\begin{aligned}
\|Au(t) - Av(t)\|_s &\leq \frac{c_s}{2}\sup_{0\leq t\leq \tilde{T}} \|(\partial_x u)^2(t) - (\partial_x v)^2(t)\|_{L^1} g(\tilde{T}) \\
&\leq \frac{c_s}{2}\sup_{0\leq t\leq \tilde{T}} \big(\|u(t)-v(t)\|_1(\|u(t)\|_1 + \|v(t)\|_1)\big)g(\tilde{T}) \\
&\leq c_s(M + e^{\eta\tilde{T}}\|\phi\|_s)g(\tilde{T}) \sup_{0\leq t\leq \tilde{T}} \|u(t)-v(t)\|_s. \qquad (3.8)
\end{aligned}$$

Taking $\hat{T} \in (0, \tilde{T}]$ such that $c_s(M + e^{\eta\hat{T}}\|\phi\|_s)g(\hat{T}) < 1$, implies **iii.)**.

From **i.)**, **ii.)** and **iii.)** it follows that A has a unique fixed point u in $\Xi_s(\hat{T})$ which satisfies equation (3.1), where $\hat{T} = T_s(s, \eta, \|\phi\|_s) > 0$. The fact that u is the unique solution of equation (3.1) in the class $u \in C(0, T_s : H^s(\mathbb{R}))$ will be a consequence of Proposition 3.2, to be proved below. This concludes the proof of theorem for the case $s \in [1, 5/2)$.

If $s \geq 5/2$, we have that $\phi \in H^s \subset H^2$ and we consider equation (3.1) satisfied by $u \in C(0, T_2; H^2(\mathbb{R}))$, where $T_2 = T_2(s = 2, \eta, \|\phi\|_2) > 0$ is the corresponding time existence with ϕ regarded as an element of $H^2(\mathbb{R})$. We observe that $E_\eta(t)\phi \in H^s(\mathbb{R})$ and $E_\eta(t-t')(u_x)^2(\cdot,t') \in H^s(\mathbb{R})$, for all $t' \in [0,t]$, because of Lemma 2.2. Then, $u(t) \in H^s(\mathbb{R})$, for all $t \in [0,T_2]$. We need to prove that $u \in C(0,T_2;H^s(\mathbb{R}))$. For that purpose we analyse the cases $s \in [5/2,3)$, $s \in [3,7/2)$, and so on.

Let $s \in [5/2,3)$ and $t,\tau \in [0,T_2]$. Assume $\tau > t > 0$, the other cases are treated similarly.

$$
\begin{aligned}
\|u(t)-u(\tau)\|_s &\leq \frac{1}{2}\|\int_0^t E_\eta(t-t')(u_x)^2(\cdot,t')dt' - \int_0^\tau E_\eta(\tau-t')(u_x)^2(\cdot,t')dt'\|_s \\
&\quad + \|(E_\eta(t)-E_\eta(\tau))\phi\|_s \\
&\leq \frac{1}{2}\int_0^t \|(E_\eta(t-t')-E_\eta(\tau-t'))(u_x)^2(\cdot,t')\|_s dt' \\
&\quad + \frac{1}{2}\int_t^\tau \|E_\eta(\tau-t')(u_x)^2(\cdot,t')\|_s dt' + \|(E_\eta(t)-E_\eta(\tau))\phi\|_s. \quad (3.9)
\end{aligned}
$$

The third term on the right hand side of (3.9) goes to zero as $\tau \to t$ because $E_\eta(t)$ is a C^0-semigroup. The first term in (3.9) is treated as the first term of (3.5) using Lemma 2.2, with $\lambda = s-1 \in [3/2,2)$. For the second term in (3.9) we get

$$
\begin{aligned}
&\int_t^\tau \|E_\eta(\tau-t')(u_x)^2(\cdot,t')\|_s dt' \\
&\leq c_\lambda \int_t^\tau \big[e^{\eta(\tau-t')} + \big(1+\frac{1}{(\eta(\tau-t'))^{\lambda/2}}\big)e^{\frac{\eta(\tau-t')}{8}\left(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}\lambda}{\eta(\tau-t')}}\right)}\big]\|(u_x)^2(\cdot,t')\|_1 dt' \\
&\leq \frac{c_\lambda}{\eta}\sup_{t'\in[0,T_2]}\|u(\cdot,t')\|_2^2 \int_0^{\eta(\tau-t)} \big[e^x + \big(1+\frac{1}{x^{\frac{\lambda}{2}}}\big)e^{\frac{x}{8}\left(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}\lambda}{x}}\right)}\big]dx, \qquad (3.10)
\end{aligned}
$$

where $\lambda = s-1 \in [3/2,2)$. Hence the last expression in (3.10) goes to zero as $\tau \to t$.

For $s \in [3,7/2)$, $s \in [7/2,4)$, ... we proceed similarly. So we get $u \in C(0,T_2;H^s)$, for $s \geq 5/2$. When $s \in (2,5/2)$ proceeding in a similar way to the case $s \geq 5/2$ we find that $u \in C(0,T_2;H^s)$. ■

Theorem 3.2. *Let $\eta > 0$ be fixed and let $\phi \in H^s(\mathbb{R})$, where $s > 1/2$. Then, there exist $T_s(s,\|\phi\|_s,\eta) > 0$ and a unique function $u := u_\eta \in C(0,T_s;H^s(\mathbb{R}))$ satisfying the integral equation*

$$
u(\cdot,t) = E_\eta(t)\phi(\cdot) - \frac{1}{2}\int_0^t E_\eta(t-t')(u^2)_x(\cdot,t')dt', \qquad (3.11)
$$

where $E_\eta(t)$ is defined by (2.3).

Proof: Similar to the proof of Theorem 3.1. Use Lemma 2.2 (with $\lambda = 1$) and the fact that $\|(u^2)_x(t)\|_{s-1} \le \|u^2(t)\|_s \le \|u(t)\|_s^2,\ s > 1/2$. ■

Proposition 3.1. *Problem (1.1) (resp. (1.2)) is equivalent to the integral equation (3.1) (resp. (3.11)). More precisely, if $u \in C([0,T]; H^s(\mathbb{R}))$ is a solution of (1.1) then u satisfies (3.1). Conversely, if $u \in C([0,T]; H^s(\mathbb{R}))$ is a solution of (3.1) then $u \in C^1([0,T]; H^{s-3}(\mathbb{R}))$ and satisfies (1.1).*

Proof: The first part was proved in Theorem 3.1 (resp. 3.2). The second part is similar to the proof of Theorem 4.19 (with $\lambda = 1$) in [11]. ■

Proposition 3.2 will prove the continuous dependance on the initial data for problems (1.1) and (1.2). Let us mention a known result we will use.

Lemma 3.1. *Suppose $\beta > 0$, $\gamma > 0$, $\beta + \gamma > 1$, $a \ge 0$, $b \ge 0$, u is nonnegative and $t^{\gamma-1}u(t)$ is locally integrable on $0 \le t < T$. If*

$$u(t) \le a + b \int_0^t (t-s)^{\beta-1} s^{\gamma-1} u(s) ds$$

a.e. in $(0,T)$, then

$$u(t) \le a E_{\beta,\gamma}((b\Gamma(\beta))^{1/\nu} t),$$

where $\nu = \beta + \gamma - 1 > 0$,

$$E_{\beta,\gamma}(s) = \sum_{m=0}^{\infty} c_m s^{m\nu}, \tag{3.12}$$

with $c_0 = 1$ and $c_{m+1}/c_m = \Gamma(m\nu + \gamma)/\Gamma(m\nu + \gamma + \beta)$ for $m \ge 0$. As $s \to \infty$,

$$E_{\beta,\gamma}(s) = O(s^{1/2(\nu/\beta - \gamma)} \cdot \exp(\frac{\beta}{\nu} s^{\nu/\beta})).$$

Proof: See Lemma 7.1.2 in [7]. ■

Proposition 3.2. *Let $\phi, \psi \in H^s(\mathbb{R})$ and let $u, v \in C(0,T; H^s(\mathbb{R}))$ be the corresponding solutions of equation (3.1). Let $M = \sup_{t' \in [0,T]}(\|u(t')\|_s + \|v(t')\|_s)$.*

1.) *If* $s \in [1,2)$, *then*

$$\|u(t)-v(t)\|_s \le e^{\eta T}\|\phi-\psi\|_s E_{\frac{5-2s}{6},1}(\gamma t), \tag{3.13}$$

where

$$\gamma = \Big(\frac{c_s M e^{\eta T}}{2}\frac{(\eta T)^{\frac{2s+1}{6}}+1}{\eta^{\frac{2s+1}{6}}}\Gamma(\frac{5-2s}{6})\Big)^{\frac{6}{5-2s}},$$

and $E_{\frac{5-2s}{6},1}$ *is given by (3.12) (with* $\beta = \frac{5-2s}{6}$ *and* $\gamma = 1$*).*

2.) *If* $s \ge 2$, *then*

$$\|u(t)-v(t)\|_s \le e^{\eta T}\|\phi-\psi\|_s E_{1/2,1}(\gamma t), \tag{3.14}$$

where

$$\gamma = (c_1 M F(T,\eta)\Gamma(1/2))^2$$

and

$$F(T,\eta) = (\frac{2\sqrt{\eta T}+1}{2\sqrt{\eta}})e^{\eta T + \frac{1}{8}\sqrt{3(\eta T)^2 + 8\sqrt{3}\eta T}}.$$

Inequality (3.14) remains true if $\phi,\psi \in H^s(\mathbb{R})$, $s > 1/2$ *and if* $u, v \in C(0,T;H^s(\mathbb{R}))$, $s > 1/2$ *are the corresponding solutions of equation (3.11).*

Proof: Let ϕ,ψ,u,v be as above. Let us start proving **1.)**. Let $s \in [1,5/2)$ and $w(t) = u(t)-v(t)$. Let $M = \sup_{t' \in [0,T]}(\|u(t')\|_s + \|v(t')\|_s)$. From (3.1) we get

$$w(t) = E_\eta(t)(\phi-\psi) - \frac{1}{2}\int_0^t E_\eta(t-t')((u_x)^2(\cdot,t') - (v_x)^2(\cdot,t'))dt'.$$

By (2.14), (2.4) and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\|w(t)\|_s &\le e^{\eta T}\|\phi-\psi\|_s + \frac{c_s}{2}\int_0^t (e^{\eta(t-t')} + \frac{1}{(\eta(t-t'))^{\frac{2s+1}{6}}})\|(v_x)^2(\cdot,t') - (u_x)^2(\cdot,t')\|_{L^1}dt' \\
&\le e^{\eta T}\|\phi-\psi\|_s + \frac{c_s}{2}e^{\eta T}\int_0^t (1 + \frac{1}{(\eta(t-t'))^{\frac{2s+1}{6}}})\|(v_x)^2(\cdot,t') - (u_x)^2(\cdot,t')\|_{L^1}dt' \\
&\le e^{\eta T}\|\phi-\psi\|_s + M\frac{c_s}{2}e^{\eta T}\int_0^t \frac{(\eta T)^{\frac{2s+1}{6}}+1}{\eta^{\frac{2s+1}{6}}}\frac{\|w(\cdot,t')\|_s}{(t-t')^{\frac{2s+1}{6}}}dt'. \qquad (3.15)
\end{aligned}$$

Applying Lemma 3.1 to (3.15), (3.13) follows.

We will now prove **2.)**. Let $s \geq 2$. Let $w(t)$ and M be as before. By (2.8), with $\lambda = 1$, we get

$$\begin{aligned}\|w(t)\|_s &\leq e^{\eta T}\|\phi - \psi\|_s + \frac{c_1}{2}\int_0^t [e^{\eta(t-t')} + \big(1 + \frac{1}{(\eta(t-t'))^{1/2}}\big)e^{\frac{\eta(t-t')}{8}(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}}{\eta(t-t')}})}] \\ &\quad \cdot \|(v_x)^2(\cdot,t') - (u_x)^2(\cdot,t')\|_{s-1}dt'. \qquad (3.16)\end{aligned}$$

Since $s - 1 \geq 1 > 1/2$ we obtain from (3.16)

$$\begin{aligned}\|w(t)\|_s &\leq e^{\eta T}\|\phi - \psi\|_s + \frac{c_1 M}{2}\int_0^t [e^{\eta(t-t')} + \big(1 + \frac{1}{(\eta(t-t'))^{1/2}}\big)e^{\frac{\eta(t-t')}{8}(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}}{\eta(t-t')}})}] \\ &\quad \cdot \|w(\cdot,t')\|_s dt'. \qquad (3.17)\end{aligned}$$

On the other hand,

$$\begin{aligned}&\frac{1}{2}[e^{\eta(t-t')} + \big(1 + \frac{1}{(\eta(t-t'))^{1/2}}\big)e^{\frac{\eta(t-t')\sqrt{3}}{8}+\frac{1}{8}\sqrt{3\eta^2(t-t')^2+8\sqrt{3}\eta(t-t')}}] \\ &\leq \frac{1}{2}[e^{\eta(t-t')} + \big(1 + \frac{1}{(\eta(t-t'))^{1/2}}\big)e^{\eta(t-t')+\frac{1}{8}\sqrt{3\eta^2(t-t')^2+8\sqrt{3}\eta(t-t')}}] \\ &\leq \big(1 + \frac{1}{2\sqrt{\eta(t-t')}}\big)e^{\eta T+\frac{1}{8}\sqrt{3(\eta T)^2+8\sqrt{3}\eta T}}. \qquad (3.18)\end{aligned}$$

Combining (3.17) and (3.18) we get

$$\|w(t)\|_s \leq e^{\eta T}\|\phi - \psi\|_s + c_1 M F(T,\eta)\int_0^t \frac{\|w(\cdot,t')\|_s}{(t-t')^{1/2}}dt'. \qquad (3.19)$$

Finally, applying Lemma 3.1 to (3.19), (3.14) follows.

Now consider equation (3.11). Inequality (3.14) is obtained similarly to item **2.)** for equation (3.1). Use the inequalities

$$\begin{aligned}\|(v^2(\cdot,t') - u^2(\cdot,t'))_x\|_{s-1} &\leq \|v^2(\cdot,t') - u^2(\cdot,t')\|_s \\ &\leq M\|(v-u)(\cdot,t')\|_s\end{aligned}$$

to obtain the same estimate (3.17) for $\|w(t)\|_s$. ■

Chapter 4

Global Theory in $H^s(\mathbb{R}), s \geq 1, \eta > 0$

In this chapter we prove that problems (1.1) and (1.2) are globally well posed in $H^s(\mathbb{R})$, $s \geq 1, \eta > 0$. Lemma 4.1 gives us some a priori estimates to prove the global Theorems 4.1 and 4.2. Lemma 4.2 will be used in Theorem 4.3 to study the limit of solutions of (1.2) when η goes to 0 in H^s, $s \geq 2$.

Lemma 4.1. *Let $\phi \in H^k(\mathbb{R}), k \geq 3$ and let $u \in C([0,T]; H^k)$ be a solution of (1.1), for some $T > 0$. Then, there exists a constant $c > 0$ independent of $\eta > 0$ such that:*

$$\|u\| \leq (\|\phi\| + \sqrt{2T}\|\phi'\|^{5/3}e^{5/3\eta T})e^{(1+\eta)T}, \tag{4.1}$$

$$\|u_x\| \leq \|\phi'\|e^{\eta T}, \tag{4.2}$$

$$\|u_{xx}\| \leq \|\phi''\| \cdot \exp[cT(\eta^{-3}\|\phi'\|^4 e^{4\eta T} + \eta)], \tag{4.3}$$

$$\|u_{xxx}\| \leq \|\phi'''\| \cdot \exp[cT(\eta^{-3}\|\phi'\|^4 e^{4\eta T} + \eta)], \tag{4.4}$$

$$\|u_x\|_j^2 \leq \|\phi'\|_j^2 \cdot \exp[cT(\sup_{[0,T]} \|u_x(t)\|_2 + \eta)], \tag{4.5}$$

where $0 \leq t \leq T$, and $j \geq \max\{3, k-1\}$.

Proof. Let us start proving (4.2). Defining $w := u_x$ and differentiating (1.1) with respect to x, it follows that w satisfies (1.2) with $\psi = \phi'$. Multiplying equation (1.2) by w and integrating over $\mathbb{R}$ we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2 &= -\eta(w,\mathcal{H}w_x)-\eta(w,\mathcal{H}w_{xxx}) \\
&= \eta(\int(|\xi|-|\xi|^3)|\hat{w}(\xi)|^2 d\xi) \\
&= \eta(\int_{|\xi|\leq 1}(|\xi|-|\xi|^3)|\hat{w}(\xi)|^2 d\xi+\int_{|\xi|>1}(|\xi|-|\xi|^3)|\hat{w}(\xi)|^2 d\xi) \qquad (4.6) \\
&\leq \eta(\int_{|\xi|\leq 1}(|\xi|-|\xi|^3)|\hat{w}(\xi)|^2 d\xi) \\
&\leq \eta(\int_{|\xi|\leq 1}|\hat{w}(\xi)|^2 d\xi) \\
&\leq \eta\|w\|^2. \qquad (4.7)
\end{aligned}$$

Integrating the last relation between 0 and t, it gives

$$\|w\|^2 \leq \|\phi'\|^2+2\eta\int_0^t \|w(t')\|^2 dt'. \qquad (4.8)$$

An application of Gronwall's inequality completes the proof of (4.2).

We will now prove (4.1). From equation (1.1) we have that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|^2 &= -\frac{1}{2}((u_x)^2,u)-\eta((\mathcal{H}u_x,u)+(\mathcal{H}u_{xxx},u)) \\
&\leq \frac{1}{2}\|u\|_{L^\infty}\|u_x\|^2+\eta\|u\|^2 \\
&\leq \|u\|^{1/2}\|u_x\|^{5/2}+\eta\|u\|^2 \\
&\leq (\|u\|^2+\|u_x\|^{10/3})+\eta\|u\|^2 \\
&\leq (1+\eta)\|u\|^2+\|\phi'\|^{10/3}e^{10/3\eta T}, \qquad (4.9)
\end{aligned}$$

where the first inequality was obtained in a similar way to (4.7), the second one using the Gagliardo-Nirenberg inequality $(GN1)$ and the third one by Young's inequality (Y) with $p=4$, $p'=4/3$, (see the Appendix). Integrating (4.9) between 0 and t we find that

$$\|u\|^2 \leq \|\phi\|^2+2\|\phi'\|^{10/3}e^{10/3\eta T}T+2(1+\eta)\int_0^t \|u(t')\|^2 dt', \qquad (4.10)$$

so Gronwall's inequality leads to the following expression

$$\|u(t)\|^2 \leq (\|\phi\|^2 + 2\|\phi'\|^{10/3} e^{10/3\eta T} T) \cdot e^{2(1+\eta)T}. \tag{4.11}$$

Taking the square root in (4.11) yields (4.1).

We will now prove (4.3). Let $v := w_x = u_{xx}$, differentiating (1.2) with respect to x, it follows that

$$v_t + (v^2 + wv_x) + v_{xxx} + \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) = 0, \;\; v(\cdot, 0) = \phi''(\cdot), \tag{4.12}$$

multiplying (4.12) by v and integrating over $\mathbb{R}$ we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|^2 &= -(v, wv_x) - (v, v^2) - \eta(v, \mathcal{H}v_x) - \eta(v, \mathcal{H}v_{xxx}) \\ &= (v, wv_x) - \eta(v, \mathcal{H}v_x) - \eta(v, \mathcal{H}v_{xxx}), \end{aligned} \tag{4.13}$$

where we have used the fact that $(v, v^2) = -2(v, wv_x)$.

$$\begin{aligned} -\eta(v, \mathcal{H}v_x) - \eta(v, \mathcal{H}v_{xxx}) &= \eta \int (|\xi| - |\xi|^3)|\hat{v}(\xi)|^2 d\xi \\ &= \eta(\int_{|\xi|\leq 2} (|\xi| - |\xi|^3)|\hat{v}(\xi)|^2 d\xi + \int_{|\xi|>2} (|\xi| - |\xi|^3)|\hat{v}(\xi)|^2 d\xi) \\ &\leq \eta(\int_{|\xi|<1} (|\xi| - |\xi|^3)|\hat{v}(\xi)|^2 d\xi + \int_{|\xi|>2} (|\xi| - |\xi|^3)|\hat{v}(\xi)|^2 d\xi) \\ &\leq \eta(\|v\|^2 + \int_{|\xi|>2} (|\xi| - |\xi|^3)|\hat{v}(\xi)|^2 d\xi) \\ &\leq \eta(\|v\|^2 - \int_{|\xi|>2} \xi^2 |\hat{v}(\xi)|^2 d\xi). \end{aligned} \tag{4.14}$$

In the last inequality we also used the fact that $x - x^3 \leq -x^2$, for $x \geq 2$. Using the estimate (4.14), it follows that the left hand side of (4.13) is bounded by

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq (v, wv_x) + \eta(\|v\|^2 - \int_{|\xi|>2} \xi^2 |\hat{v}(\xi)|^2 d\xi). \tag{4.15}$$

On the other hand,

$$\begin{aligned} (v, wv_x) &\leq \|v\|_{L^\infty} \|w\| \|v_x\| \\ &\leq \|v\|^{1/2} \|v_x\|^{3/2} \|w\| \\ &\leq \|w\|(\epsilon\|v\|^2 + \epsilon^{-1/3}\|v_x\|^2), \end{aligned} \tag{4.16}$$

where the second inequality was obtained using Gagliardo-Nirenberg's inequality $(GN1)$ and the third one by Young's inequality (Y) with $p=4$, $p'=4/3$. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 &\leq \|w\|(\epsilon\|v\|^2+\epsilon^{-1/3}\|v_x\|^2)+\eta(\|v\|^2-\int_{|\xi|>2}\xi^2|\hat{v}(\xi)|^2d\xi)\\ &= (\eta+\epsilon\|w\|)\|v\|^2+\|w\|\epsilon^{-1/3}\|v_x\|^2-\eta\int_{|\xi|>2}\xi^2|\hat{v}(\xi)|^2d\xi\\ &\leq (\eta+\epsilon\|\phi'\|e^{\eta T})\|v\|^2+\|\phi'\|e^{\eta T}\epsilon^{-1/3}\|v_x\|^2-\eta\int_{|\xi|>2}\xi^2|\hat{v}(\xi)|^2d\xi, \quad (4.17)\end{aligned}$$

where in the last inequality we employed (4.2).
Now choosing

$$\epsilon := (\frac{\|\phi'\|e^{\eta T}}{\eta})^3>0, \tag{4.18}$$

we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 &\leq (\eta+\eta^{-3}\|\phi'\|^4e^{4\eta T})\|v\|^2+\eta\|v_x\|^2-\eta\int_{|\xi|>2}\xi^2|\hat{v}(\xi)|^2d\xi\\ &= (\eta+\eta^{-3}\|\phi'\|^4e^{4\eta T})\|v\|^2+\eta\int_{|\xi|\leq 2}\xi^2|\hat{v}(\xi)|^2d\xi\\ &\leq (5\eta+\eta^{-3}\|\phi'\|^4e^{4\eta T})\|v\|^2. \end{aligned}\tag{4.19}$$

Integrating (4.19) between 0 and t gives

$$\|v(t)\|^2\leq\|\phi''\|^2+c(\eta+\eta^{-3}\|\phi'\|^4e^{4\eta T})\int_0^t\|v(t')\|^2dt'. \tag{4.20}$$

Applying Gronwall's inequality and taking the square root we obtain (4.3).

We will now prove (4.4). Let $r:=v_x=w_{xx}=u_{xxx}$. Differentiating (4.12) with respect to x we obtain

$$r_t+3vr+wr_x+r_{xxx}+\eta(\mathcal{H}r_x+\mathcal{H}r_{xxx})=0,\;\; r(\cdot,0)=\phi'''(\cdot), \tag{4.21}$$

multiplying (4.21) by r and integrating over $\mathbb{R}$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|r\|^2 &= -3(vr,r)-(w,rr_x)-\eta((\mathcal{H}r_x,r)+(\mathcal{H}r_{xxx},r))\\ &= 5(w,rr_x)-\eta((\mathcal{H}r_x,r)+(\mathcal{H}r_{xxx},r)),\end{aligned}\tag{4.22}$$

where the last equality was obtained because

$$(w, rr_x) = 1/2(w, (r^2)_x) = -1/2(v, r^2) = -1/2(vr, r).$$

Following the ideas used to prove (4.3) we get

$$\frac{1}{2}\frac{d}{dt}\|r\|^2 \leq 5\|w\|(\epsilon\|r\|^2 + \epsilon^{-1/3}\|r_x\|^2) + \eta(\|r\|^2 - \int_{|\xi|>2} \xi^2|\hat{v}(\xi)|^2 d\xi). \quad (4.23)$$

Using (4.2) and choosing

$$\epsilon := (\frac{5\|\phi'\|e^{\eta T}}{\eta})^3 > 0, \quad (4.24)$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|r\|^2 \leq (\eta^{-3}(5\|\phi'\|e^{\eta T})^4 + 5\eta)\|r\|^2. \quad (4.25)$$

Integrating (4.25) between $[0, t]$ gives

$$\|r(t)\|^2 \leq \|\phi'''\|^2 + c(\eta + \eta^{-3}\|\phi'\|^4 e^{4\eta T}) \int_0^t \|r(\tau)\|^2 d\tau. \quad (4.26)$$

Applying Gronwall's inequality and taking the square root in (4.26) we get (4.4).

Now we will prove (4.5). From equation (1.2) we have

$$\frac{1}{2}\frac{d}{dt}\|w\|_j^2 = -(w, ww_x)_j - \eta((w, \mathcal{H}w_x)_j + (w, \mathcal{H}w_{xxx})_j). \quad (4.27)$$

On the other hand,

$$\begin{aligned} -(w, \mathcal{H}w_x)_j - (w, \mathcal{H}w_{xxx})_j &= \int (|\xi| - |\xi|^3)(1 + \xi^2)^j |\hat{w}(\xi)|^2 d\xi \\ &\leq \|w\|_j^2 - \int_{|\xi|>2} \xi^2((1 + \xi^2)^j |\hat{w}(\xi)|^2 d\xi \quad (4.28) \\ &\leq \|w\|_j^2, \quad (4.29) \end{aligned}$$

where inequality (4.28) was obtained in a similar way to (4.14).

Using inequality (4.29) in (4.27) we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_j^2 &\leq |(w, ww_x)_j| + \eta\|w\|_j^2\\ &\leq c\|w\|_2\|w\|_j^2 + \eta\|w\|_j^2\\ &= (c\|w\|_2 + \eta)\|w\|_j^2, \end{aligned} \tag{4.30}$$

where the second inequality was obtained using Kato's inequality $(K2)$, $j \geq 3$. Integrating (4.30) between 0 and t we get

$$\begin{aligned}\|w(t)\|_j^2 &\leq \|\phi'\|_j^2 + (c\sup_{[0,T]}\|w(t)\|_2 + 2\eta)\cdot\int_0^t \|w(t')\|_j^2 dt'\\ &\leq \|\phi'\|_j^2 + c(\sup_{[0,T]}\|w(t)\|_2 + \eta)\cdot\int_0^t \|w(t')\|_j^2 dt'. \end{aligned} \tag{4.31}$$

Using Gronwall's inequality in (4.31) we see that

$$\|u_x(\cdot,t)\|_j^2 \leq \|\phi'\|_j^2 \cdot \exp[cT(\sup_{[0,T]}\|u_x(\cdot,t)\|_2 + \eta)], \tag{4.32}$$

for $0 \leq t \leq T$. This completes the proof of Lemma 4.1. ■

Remark: Let $\psi \in H^k(\mathbb{R}), k \geq 2$. Let $w \in C([0,T]; H^k)$ be the solution of (1.2), for some $T > 0$. Then, by Lemma 4.1, there exists a constant $c > 0$ independent of $\eta > 0$ such that (4.2) - (4.5) are satisfied with u_x replaced by w, ϕ' replaced by ψ and $j \geq 3$ in (4.5).

Theorem 4.1. *Let $\phi \in H^s(\mathbb{R})$, $s \geq 1$. Then, for each $\eta > 0$, there exists a unique $u_\eta \in C([0,\infty); H^s(\mathbb{R}))$ solution to the problem (1.1), such that $\partial_t u_\eta \in C([0,\infty); H^{s-3}(\mathbb{R}))$.*

Proof. If $s \in \mathbb{Z}^+$, then the result follows from the local theory, Lemma 4.1 and the next remark. By Proposition 3.1, Problem (1.1) is equivalent to the integral equation (3.1), let

$$T^* := \sup\big\{T > 0;\ \exists! u \in C([0,T]; H^s) \text{ satisfying } (3.1)\big\}.$$

We will show that $T^* = \infty$. Suppose that $T^* < \infty$. By Lemma 4.1 we have that

$$\|u(t)\|_s \leq K, \quad \text{for all } t \in [0, T^*), \tag{4.33}$$

where $K = K(\|\phi\|_s, \eta, T^*)$ is a nondecreasing, continuous function of T^*. We claim that there exists $\lim_{t\uparrow T^*} u(t)$ in $H^s(\mathbb{R})$. In fact let $t, \tau \in [0, T^*)$, and suppose $t < \tau$. Using the integral equation (3.1) we find that

$$\begin{aligned} \|u(t) - u(\tau)\|_s &\leq \|\big(E_\eta(t) - E_\eta(\tau)\big)\phi\|_s + \int_0^t \|\big(E_\eta(t-t') - E_\eta(\tau - t')\big)(u_x)^2(t')\|_s dt' \\ &\quad + \int_t^\tau \|E_\eta(\tau - t')(u_x)^2(t')\|_s dt'. \end{aligned} \tag{4.34}$$

Using similar computations to the obtained to estimate (3.4), but using now (4.33), we find that the right hand side of (4.34) tends to zero as $\tau, t \to T^*$. Then, using the Cauchy criterion, we conclude that there exists $\lim_{t\uparrow T^*} u(t)$ in $H^s(\mathbb{R})$. Moreover, the last part shows that the integral representation (3.1) for $u(t)$, is valid for all $t \in [0, T^*]$. Using Theorem 3.1, again, we obtain a contradiction with $T^* < \infty$.

If $s \geq 1$ is not an integer we will use nonlinear interpolation theory. More precisely we will use Theorems 1 and 2 in [4]. A similar proof was given in [3] for the KdV-Kuramoto-Sivashinsky equation. Let $k \geq 2$ be an integer, $k - 1 < s < k$, $B_0^1 = L^2$, $B_0^2 = C(0, T; L^2)$, $B_1^1 = H^k$ and $B_1^2 = C(0, T; H^k)$. Let $\lambda = \frac{k-1}{k} \in (0,1)$ and $\theta = \frac{s}{k} \in (0, 1)$. Then,

$$\begin{aligned} B^1_{\lambda,2} &= [B_0^1, B_1^1]_{\lambda,2} = [L^2, H^k]_{\lambda,2} \approx H^{\lambda k} = H^{k-1}, \\ B^1_{\theta,2} &= [B_0^1, B_1^1]_{\theta,2} = [L^2, H^k]_{\theta,2} \approx H^{\theta k} = H^s. \end{aligned} \tag{4.35}$$

Moreover, since $\theta \geq \lambda$, we get $(\theta, 2) \geq (\lambda, 2)$, as in [4]. Let A be the mapping sending the initial data $\phi \in H^k$ into the unique solution $u \in C(0, T; H^k)$ to problem (1.1) obtained in Theorem 3.1. This mapping satisfies:

(1.) $A : H^k \to C(0, T; H^k)$ is continuous.

(2.) The mapping

$$A : H^{k-1} \to C(0, T; L^2),$$

is Lipschitz in the following sense

$$\|A\phi - A\psi\|_{C(0,T;L^2)} \leq c_0(\|\phi\|_{H^{k-1}} + \|\psi\|_{H^{k-1}})\|\phi - \psi\|_{L^2}, \quad \phi, \psi \in H^{k-1}. \tag{4.36}$$

(3.) And finally,

$$A : H^k \to C(0, T; H^k)$$

is such that

$$\|A\phi\|_{C(0,T;H^k)} \leq c_1(\|\phi\|_{H^{k-1}})\|\phi\|_{H^k}, \;\; \phi \in H^k, \tag{4.37}$$

where $c_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 0, 1$ are continuous non-decreasing functions.

We will prove conditions **(1.)**, **(2.)** and **(3.)** below. Thus, all the hypotheses of Theorems 1 and 2 in [4] are valid in this context. It follows from Theorem 2 in [4] that

$$A : [L^2, H^k]_{\theta,2} \to [C(0,T;L^2), C(0,T;H^k]_{\theta,2}$$

continuously. But $[L^2, H^k]_{\theta,2} \approx H^s$ and by Proposition 3 in [4],

$$[C(0,T;L^2), C(0,T;H^k]_{\theta,2} \subset C(0,T;[L^2,H^k]_{\theta,2}) \approx C(0,T;H^s), \tag{4.38}$$

where the inclusion in (4.38) is continuous. Hence $A : H^s \to C(0,T;H^s)$ continuously.

Since $k \geq 2$, condition **(1.)** follows directly from Proposition 3.2.

Next, we turn to condition **(2.)**. Let ϕ, $\psi \in H^{k-1}(\mathbb{R})$ and $u = A\phi$, $v = A\psi$. Let $w = u - v$.

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2 &= -\frac{1}{2}((u_x + v_x)w_x, w) - \eta(w, \mathcal{H}w_x + \mathcal{H}w_{xxx}) \\ &\leq \frac{1}{2}\|w_x\|(\|u_x\| + \|v_x\|)\|w\|_{L^\infty} \\ &\quad +\eta\big(\|w\|^2 - \int_{|\xi|>2} \xi^2|\hat{w}(\xi)|^2 d\xi\big). \end{aligned} \tag{4.39}$$

On the other hand, we find that

$$\|w_x\|\|w\|_{L^\infty} \leq \|w_x\|\|w_x\|^{1/2}\|w\|^{1/2} = \|w\|^{1/2}\|w_x\|^{3/2} \leq (\epsilon\|w\|^2 + \epsilon^{-1/3}\|w_x\|^2). \tag{4.40}$$

Using estimate (4.40) in (4.39) we have that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2 &\leq \big(\frac{1}{2}(\|\phi'\| + \|\psi'\|)e^{\eta T}\epsilon + \eta\big)\|w\|^2 + \frac{1}{2}\big(\|\phi'\| + \|\psi'\|\big)e^{\eta T}\epsilon^{-1/3}\|w_x\|^2 \\ &\quad -\eta\int_{|\xi|>2} \xi^2|\hat{w}(\xi)|^2 d\xi. \end{aligned} \tag{4.41}$$

Taking

$$\epsilon := \big(\frac{1}{2}\frac{(\|\phi'\| + \|\psi'\|)e^{\eta T}}{\eta}\big)^3 > 0$$

into (4.41) we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 \leq \big[\eta^{-3}\big(\frac{1}{2}(\|\phi'\|+\|\psi'\|)e^{\eta T}\big)^4 + 5\eta\big]\,\|w\|^2. \tag{4.42}$$

Integrating (4.42) between 0 and t and using Gronwall's inequality we get

$$\|w(\cdot,t)\| \leq e^{cT\left(\eta^{-3}(\|\phi'\|^4+\|\psi'\|^4)e^{4\eta T}+\eta\right)}\|\phi-\psi\|. \tag{4.43}$$

This completes the proof of condition **(2.)**.

Finally, we will prove condition **(3.)**. For $k = 1,2,3$, the result follows from Lemma 4.1 , formulas (4.1)-(4.4). For $k \geq 4$, we use (4.5) and we apply induction to $\|u\|_k^2 = \|u\|_{k-1}^2 + \|u_x\|_{k-1}^2$. ■

Theorem 4.2. *Let $\psi \in H^s(\mathbb{R})$, $s \geq 1$. Then, for each $\eta > 0$, there exists a unique $w_\eta \in C([0,\infty); H^s(\mathbb{R}))$ solution to the problem (1.2), such that $\partial_t w_\eta \in C([0,\infty); H^{s-3}(\mathbb{R}))$.*

Proof. The proof is the same as that of Theorem 4.1, the only difference being that now we use

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 &= -\frac{1}{2}((u^2-v^2)_x, w) - \eta(w, \mathcal{H}w_x + \mathcal{H}w_{xxx})\\ &\leq \frac{1}{2}\|w_x\|(\|u\|+\|v\|)\|w\|_{L^\infty}\\ &\quad +\eta(\|w\|^2 - \int_{|\xi|>2}\xi^2|\hat{w}(\xi)|^2 d\xi),\end{aligned}$$

to prove condition **(2.)**. ■

Lemmas 4.2 and 4.4 below, will be used to prove Theorem 4.3 that establishes the convergence of the solutions of the equation (1.2) for $\eta > 0$ to the solution of the KdV equation as η tends to 0.

Lemma 4.2. *Consider the initial value problem (1.1), with $\phi \in H^s$, $s = 2,3$. Let $u \in C([0,T]; H^s)$ be the solution of (1.1) for some $T > 0$. Then, there exists a constant $c > 0$ independent of $\eta > 0$ such that*

$$\|u_{xx}\|^2 \leq \big[(1+\eta T)\cdot e^{c\eta T}P_1(\|\phi\|_1) + P_2(\|\phi\|_2)\big]\cdot e^{c\eta T(1+\|\phi'\|^4 e^{4\eta T})}, \tag{4.44}$$

$$
\begin{aligned}
\|u_{xxx}\|^2 \quad \le \quad & \left[Q_3(\|\phi\|_3) + (1+\eta T)e^{c\eta T}Q_1(\|\phi\|_1)\right] \cdot \exp\left\{\eta T\left[e^{c\eta T}Q_2(\|\phi\|_1)\right.\right. \\
& \left.\left. +(1+(1+\eta T)^{2/3})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T}R_1(\|\phi\|_1)}\right]\right\},
\end{aligned} \tag{4.45}
$$

for $t \in [0,T]$*, and* P_i*,* $i = 1,2$*,* Q_j*,* $j = 1,2,3$*, and* R_k*,* $k = 1,2$*, are nondecreasing functions of their arguments.*

Proof. Let us start proving (4.44). Defining $w = u_x$, it follows that w is a solution of the problem (1.2), with $w(\cdot,0) = \phi'(\cdot)$.
As is well known, the functional

$$
\Phi_2(w) = \frac{1}{2}\int_{-\infty}^{+\infty}[\frac{1}{3}w^3 - (w_x)^2]dx \tag{4.46}
$$

is conserved by the KdV flow and

$$
w_t = -\partial_x(\Phi_2'(w)) - \eta(\mathcal{H}w_x + \mathcal{H}w_{xxx}), \tag{4.47}
$$

where $\Phi_2'(w) = 1/2w^2 + w_{xx}$. Then,

$$
\partial_t\Phi_2(w) = (\Phi_2'(w), w_t) = -\eta(\Phi_2'(w), \mathcal{H}w_x + \mathcal{H}w_{xxx}), \tag{4.48}
$$

so that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty}(\frac{1}{3}w^3 - (w_x)^2)dx \quad = \quad & -\frac{\eta}{2}(w^2, \mathcal{H}w_x) - \frac{\eta}{2}(w^2, \mathcal{H}w_{xxx}) \\
& -\eta(w_{xx}, \mathcal{H}w_x) - \eta(w_{xx}, \mathcal{H}w_{xxx}).
\end{aligned}
$$

Integrating the last equation between 0 and t, we get

$$
\begin{aligned}
& \int_{-\infty}^{+\infty}(\frac{1}{3}w^3 - (w_x)^2)dx - \int_{-\infty}^{+\infty}(\frac{1}{3}(\phi')^3 - (\phi'')^2)dx \\
& = -\eta\int_0^t[(w^2, \mathcal{H}w_x) + (w^2, \mathcal{H}w_{xxx})]d\tau - 2\eta\int_0^t[(w_{xx}, \mathcal{H}w_x) + (w_{xx}, \mathcal{H}w_{xxx})]d\tau.
\end{aligned}
$$

From the last equality we see that

$$\begin{aligned}
\|w_x\|^2 &= \int_{-\infty}^{+\infty}[\frac{1}{3}w^3 - \frac{1}{3}(\phi')^3 + (\phi'')^2]dx + \eta\int_0^t[(w^2, \mathcal{H}w_x) + (w^2, \mathcal{H}w_{xxx})]d\tau \\
&\quad +2\eta\int_0^t[(w_{xx}, \mathcal{H}w_x) + (w_{xx}, \mathcal{H}w_{xxx})]d\tau \\
&= \int_{-\infty}^{+\infty}[\frac{1}{3}w^3 - \frac{1}{3}(\phi')^3 + (\phi'')^2]dx + \eta\int_0^t(w^2, \mathcal{H}w_x)d\tau \\
&\quad +2\eta\int_0^t[-(ww_x, \mathcal{H}w_{xx}) - (w_x, \mathcal{H}w_{xx}) - (w_x, \mathcal{H}w_{xxxx})]d\tau. \qquad (4.49)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|\int_{-\infty}^{+\infty} w^3dx| &\leq \|w\|_{L^\infty}\|w\|^2 \\
&\leq \|w\|^{5/2}\|w_x\|^{1/2} \\
&\leq \epsilon_1\|w_x\|^2 + \epsilon_1^{-1/3}\|w\|^{10/3}, \qquad (4.50)
\end{aligned}$$

where the second inequality was obtained using Gagliardo-Nirenberg's inequality $(GN1)$ and the third one by Young's inequality (Y), with $p = 4$ and $p' = 4/3$. We also find that

$$\int_{-\infty}^{+\infty}(\phi')^3dx \leq \|\phi'\|_{L^\infty}\|\phi\|_2^2 \leq c\|\phi\|_2^3, \qquad (4.51)$$

$$\|\phi''\|^2 \leq \|\phi\|_2^2 \qquad (4.52)$$

and

$$\begin{aligned}
(w^2, \mathcal{H}w_x) &\leq |(w^2, \mathcal{H}w_x)| \leq \|w\|_{L^\infty}\|w\|\|w_x\| \leq \|w\|^{3/2}\|w_x\|^{3/2} \\
&\leq \|w\|^6 + \|w_x\|^2, \qquad (4.53)
\end{aligned}$$

where the last inequality was obtained using Young's inequality with $p = 4/3$ and $p' = 4$. Using the estimates (4.50) - (4.53), it follows that the left hand side of (4.49) is bounded by

$$\begin{aligned}
\|w_x\|^2 &\leq \frac{\epsilon_1}{3}\|w_x\|^2 + \frac{1}{3\epsilon_1^{1/3}}\|w\|^{10/3} + c\|\phi\|_2^3 + \|\phi\|_2^2 + \eta\int_0^t(\|w\|^6 + \|w_x\|^2)d\tau \\
&\quad +2\eta\int_0^t[-(ww_x, \mathcal{H}w_{xx}) - (w_x, \mathcal{H}w_{xx}) - (w_x, \mathcal{H}w_{xxxx})]d\tau. \qquad (4.54)
\end{aligned}$$

Denote by $B(\tau)$ the integrand in the second integral in the last inequality and let $P_2(\|\phi\|_2) := c\|\phi\|_2^3 + \|\phi\|_2^2$. Taking $\epsilon_1 = 1$ in (4.54) and using inequality (4.2) we have that

$$\begin{aligned}
\frac{2}{3}\|w_x\|^2 &\leq ce^{c\eta T}\|\phi'\|^{10/3} + P_2(\|\phi\|_2) + \eta T(\|\phi'\|e^{\eta T})^6 + \eta\int_0^t \big(\|w_x\|^2 + 2B(\tau)\big)d\tau, \\
&\leq (1+\eta T)e^{c\eta T}P_1(\|\phi\|_1) + P_2(\|\phi\|_2) + \eta\int_0^t \big(\|w_x\|^2 + 2B(\tau)\big)d\tau, \qquad (4.55)
\end{aligned}$$

where $P_1(\|\phi\|_1) := c(1 + \|\phi\|_1^4 + \|\phi\|_1^6)$.
Definig $v := w_x$, it follows that

$$\begin{aligned}
B(\tau) &\leq |(wv, \mathcal{H}v_x)| - (v, \mathcal{H}v_x) - (v, \mathcal{H}v_{xxx}) \\
&\leq \|v\|_{L^\infty}\|w\|\|v_x\| + \|v\|^2 - \int_{|\xi|>2} \xi^2|\hat{v}(\xi)|^2 d\xi \\
&\leq \|v\|^{1/2}\|w\|\|v_x\|^{3/2} + \|v\|^2 - \int_{|\xi|>2} \xi^2|\hat{v}(\xi)|^2 d\xi \\
&\leq \|w\|(\epsilon\|v\|^2 + \epsilon^{-1/3}\|v_x\|^2) + \|v\|^2 - \int_{|\xi|>2} \xi^2|\hat{v}(\xi)|^2 d\xi \\
&\leq (5 + \|\phi'\|^4 e^{4\eta T})\|v\|^2, \qquad (4.56)
\end{aligned}$$

where the last inequality was obtained in the same way as (4.19). Using (4.56) in (4.55), we get

$$\|w_x\|^2 \leq (1+\eta T)e^{c\eta T}P_1(\|\phi\|_1) + P_2(\|\phi\|_2) + c\eta(1 + \|\phi'\|^4 e^{4\eta T})\int_0^t \|w_x\|^2 d\tau. \qquad (4.57)$$

Finally, applying Gronwall's inequality to (4.57) we have that

$$\|u_{xx}\|^2 \leq \big[(1+\eta T)\cdot e^{c\eta T}P_1(\|\phi\|_1) + P_2(\|\phi\|_2)\big]\cdot e^{c\eta T(1+\|\phi'\|^4 e^{4\eta T})}. \qquad (4.58)$$

The next step will be to prove (4.45). To this end note that

$$\Phi_4(w) = \int_{-\infty}^{+\infty} (\frac{5}{12}w^4 - 5ww_x^2 + 3w_{xx}^2)dx, \qquad (4.59)$$

is also conserved by the KdV flow. It follows that

$$\Phi_4'(w) = \frac{5}{3}w^3 + 5(w_x)^2 + 10ww_{xx} + 6w_{xxxx}. \tag{4.60}$$

Calculating $\partial_t(\Phi_4(w))$, we obtain

$$\begin{aligned}
\partial_t(\Phi_4(w)) &= (\Phi_4'(w), w_t) \\
&= (\Phi_4'(w), -(\Phi_2'(w))_x) - \eta(\Phi_4'(w), \mathcal{H}w_x + \mathcal{H}w_{xxx}) \\
&= -\eta\big(\frac{5}{3}w^3 + 5(w_x)^2 + 10ww_{xx} + 6w_{xxxx}, \mathcal{H}w_x + \mathcal{H}w_{xxx}\big) \\
&= -\eta\big[\frac{5}{3}(w^3, \mathcal{H}w_x) + \frac{5}{3}(w^3, \mathcal{H}w_{xxx}) + 5((w_x)^2, \mathcal{H}w_x) \\
&\quad +5((w_x)^2, \mathcal{H}w_{xxx}) + 10(ww_{xx}, \mathcal{H}w_x) + 10(ww_{xx}, \mathcal{H}w_{xxx}) \\
&\quad +6(w_{xxxx}, \mathcal{H}w_x) + 6(w_{xxxx}, \mathcal{H}w_{xxx})\big].
\end{aligned} \tag{4.61}$$

On the other hand, we find that:

$$\begin{aligned}
|(w^3, \mathcal{H}w_x)| &\leq \|w\|_{L^\infty}^2\|w\|\|w_x\| \\
&\leq \|w\|^2\|w_x\|^2 \\
&\leq \|w\|^3\|w_{xx}\| \\
&\leq \|w\|^6 + \|w_{xx}\|^2,
\end{aligned} \tag{4.62}$$

where in the second and third inequalities we used Gagliardo-Nirenberg's inequalities $(GN1)$ and $(GN2)$ respectively and in the fourth one we used Young's inequality with $p = p' = 2$.

$$\begin{aligned}
|(w^3, \mathcal{H}w_{xxx})| &= 3|(w^2w_x, \mathcal{H}w_{xx})| \leq 3\|w\|_{L^\infty}^2\|w_x\|\|w_{xx}\| \\
&\leq 3\|w\|\|w_x\|^2\|w_{xx}\| \leq 3\|w\|^2\|w_{xx}\|^2.
\end{aligned} \tag{4.63}$$

$$\begin{aligned}
|((w_x)^2, \mathcal{H}w_x)| &\leq \|w_x\|_{L^\infty}\|w_x\|^2 \leq \|w_x\|^{1/2}\|w_{xx}\|^{1/2}\|w_x\|^2 = \|w_x\|^{5/2}\|w_{xx}\|^{1/2} \\
&\leq \|w\|^{5/4}\|w_{xx}\|^{7/4} \leq \|w\|^{10} + \|w_{xx}\|^2,
\end{aligned} \tag{4.64}$$

where in the last inequality we used (Y) with $p = 8$ and $p' = 8/7$. Next,

$$\begin{aligned}
|((w_x)^2, \mathcal{H}w_{xxx})| &= 2|(w_xw_{xx}, \mathcal{H}w_{xx})| \leq 2\|w_{xx}\|_{L^\infty}\|w_{xx}\|\|w_x\| \leq 2\|w_x\|\|w_{xx}\|^{3/2}\|w_{xxx}\|^{1/2} \\
&\leq 2\|w_x\|(\epsilon_1\|w_{xx}\|^2 + \epsilon_1^{-3}\|w_{xxx}\|^2),
\end{aligned} \tag{4.65}$$

where in the last inequality we used (Y) with $p=4/3$ and $p'=4$. Finally,

$$\begin{aligned}|(ww_{xx},\mathcal{H}w_x)| &\leq \|w\|_{L^\infty}\|w_{xx}\|\|w_x\| \leq \|w\|^{1/2}\|w_x\|^{3/2}\|w_{xx}\| \\ &\leq \|w\|^{1/2}(\|w\|^{1/2}\|w_{xx}\|^{1/2})^{3/2}\|w_{xx}\| = \|w\|^{5/4}\|w_{xx}\|^{7/4} \\ &\leq \|w\|^{10}+\|w_{xx}\|^2, \end{aligned} \tag{4.66}$$

where the last inequality was obtained using Young's inequality with $p=8/7$ and $p'=8$. Using (4.62)-(4.66) we see that the left hand side of (4.61) is bounded by

$$\begin{aligned}\partial_t(\Phi_4(w)) &\leq c\eta\big[\|w\|^6+\|w_{xx}\|^2+\|w\|^2\|w_{xx}\|^2+\|w\|^{10}+\|w_{xx}\|^2 \\ &\quad +\|w_x\|(\epsilon_1\|w_{xx}\|^2+\epsilon_1^{-3}\|w_{xxx}\|^2)\big]-10\eta(ww_{xx},\mathcal{H}w_{xxx}) \\ &\quad -6\eta\big[(w_{xx},\mathcal{H}w_{xxx})+(w_{xx},\mathcal{H}w_{xxxxx})\big]. \end{aligned} \tag{4.67}$$

As a consequence of (4.2) we obtain

$$\begin{aligned}\partial_t(\Phi_4(w)) &\leq \eta e^{c\eta T}Q_1(\|\phi\|_1)+\eta e^{c\eta T}Q_2'(\|\phi\|_1)\|w_{xx}\|^2 \\ &\quad +c\eta\|w_x\|(\epsilon_1\|w_{xx}\|^2+\epsilon_1^{-3}\|w_{xxx}\|^2)-10\eta(ww_{xx},\mathcal{H}w_{xxx}) \\ &\quad -6\eta\big[(w_{xx},\mathcal{H}w_{xxx})+(w_{xx},\mathcal{H}w_{xxxxx})\big], \end{aligned} \tag{4.68}$$

where $Q_1(\|\phi\|_1)=c(\|\phi\|_1^2+\|\phi\|_1^{10})$ and $Q_2'(\|\phi\|_1)=c(1+\|\phi\|_1^2)$.
On the other hand, we have that

$$\begin{aligned}(ww_{xx},\mathcal{H}w_{xxx}) &\leq |(ww_{xx},\mathcal{H}w_{xxx})|=|(wr,\mathcal{H}r_x)|\leq \|r\|_{L^\infty}\|w\|\|r_x\| \\ &\leq \|w\|\|r\|^{1/2}\|r_x\|^{3/2}\leq \|w\|(\epsilon_2\|r\|^2+\epsilon_2^{-1/3}\|r_x\|^2), \end{aligned} \tag{4.69}$$

where $r:=w_{xx}=u_{xxx}$. We also obtain that

$$\begin{aligned}-(w_{xx},\mathcal{H}w_{xxx})-(w_{xx},\mathcal{H}w_{xxxxx}) &= -(r,\mathcal{H}r_x)-(r,\mathcal{H}r_{xxx}) \\ &\leq \|r\|^2-\int_{|\xi|>2}\xi^2|\hat{r}(\xi)|^2d\xi. \end{aligned} \tag{4.70}$$

Hence, using (4.69) and (4.70) in (4.68), it follows that

$$\begin{aligned}\partial_t(\Phi_4(w)) &\leq \eta e^{c\eta T}Q_1(\|\phi\|_1)+\eta e^{c\eta T}Q_2'(\|\phi\|_1)\|r\|^2 \\ &\quad +6\eta\Big[\frac{c\|u_{xx}\|}{6}(\epsilon_1\|r\|^2+\epsilon_1^{-3}\|r_x\|^2)+\frac{c\|u_x\|}{6}(\epsilon_2\|r\|^2+\epsilon_2^{-1/3}\|r_x\|^2) \\ &\quad +\|r\|^2-\int_{|\xi|>2}\xi^2|\hat{r}(\xi)|^2d\xi\Big] \\ &\leq \eta e^{c\eta T}Q_1(\|\phi\|_1)+\eta e^{c\eta T}Q_2''(\|\phi\|_1)\|r\|^2+6\eta\Big[\frac{c}{6}(\|u_{xx}\|\epsilon_1+\|u_x\|\epsilon_2)\|r\|^2 \\ &\quad +\frac{c}{6}(\|u_{xx}\|\epsilon_1^{-3}+\|u_x\|\epsilon_2^{-1/3})\|r_x\|^2-\int_{|\xi|>2}\xi^2|\hat{r}(\xi)|^2d\xi\Big], \end{aligned} \tag{4.71}$$

where $Q_2''(\|\phi\|_1) := Q_2'(\|\phi\|_1) + 6$.
Taking the square root in (4.44) we have

$$\begin{aligned}
\|u_{xx}\| &\leq \big[\sqrt{1+\eta T}e^{c\eta T}\sqrt{P_1(\|\phi\|_1)} + \sqrt{P_2(\|\phi\|_2)}\big]e^{c\eta T e^{c\eta T}(1+\|\phi\|_1^4)} \\
&\leq (1+\sqrt{1+\eta T}e^{c\eta T})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T}(1+\|\phi\|_1^4)} \\
&\leq (1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T}R_1(\|\phi\|_1)}, \qquad (4.72)
\end{aligned}$$

where $R_1(\|\phi\|_1) := 2 + \|\phi\|_1^4$ and $R_2(\|\phi\|_2) := \sqrt{P_1(\|\phi\|_2)} + \sqrt{P_2(\|\phi\|_2)}$ are nondecreasing functions of their arguments.
Combining (4.72), (4.2) and (4.71) we obtain

$$\begin{aligned}
\partial_t(\Phi_4(w)) \leq\ & \eta e^{c\eta T}Q_1(\|\phi\|_1) + \eta e^{c\eta T}Q_2''(\|\phi\|_1)\|r\|^2 \\
&+ c\eta\big[\big((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T}R_1(\|\phi\|_1)}\big)\epsilon_1 + \|\phi'\|e^{\eta T}\epsilon_2\big]\|r\|^2 \\
&+ 6\eta\big\{\frac{c}{6}\big[\big((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T}R_1(\|\phi\|_1)}\big)\epsilon_1^{-3} + \|\phi'\|e^{\eta T}\epsilon_2^{-1/3}\big]\|r_x\|^2 \\
&- \int_{|\xi|>2}\xi^2|\hat{r}(\xi)|^2 d\xi\big\}. \qquad (4.73)
\end{aligned}$$

Choosing

$$\begin{aligned}
\epsilon_1 &= \big(\frac{c}{3}((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T}R_1(\|\phi\|_1)})\big)^{1/3} > 0, \qquad (4.74) \\
\epsilon_2 &= \big(\frac{c}{3}\|\phi'\|e^{\eta T}\big)^3 > 0, \qquad (4.75)
\end{aligned}$$

it follows from (4.73) that

$$\begin{aligned}
\partial_t(\Phi_4(w)) \leq\ & \eta e^{c\eta T}Q_1(\|\phi\|_1) + \eta e^{c\eta T}Q_2''(\|\phi\|_1)\|r\|^2 \\
&+ c\eta\big[\big((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T}R_1(\|\phi\|_1)}\big)^{4/3} + \|\phi'\|^4 e^{4\eta T}\big]\|r\|^2 \\
&+ 6\eta\big(\|r_x\|^2 - \int_{|\xi|>2}\xi^2|\hat{r}(\xi)|^2 d\xi\big). \qquad (4.76)
\end{aligned}$$

Since

$$\|r_x\|^2 - \int_{|\xi|>2}\xi^2|\hat{r}(\xi)|^2 d\xi = \int_{|\xi|\leq 2}\xi^2|\hat{r}(\xi)|^2 d\xi \leq 4\|r\|^2,$$

we obtain

$$\begin{aligned}
\partial_t(\Phi_4(w)) \leq\ & \eta e^{c\eta T}Q_1(\|\phi\|_1) + \eta e^{c\eta T}Q_2(\|\phi\|_1)\|r\|^2 \\
&+ c\eta\big((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T}R_1(\|\phi\|_1)}\big)^{4/3}\|r\|^2, \qquad (4.77)
\end{aligned}$$

where $Q_2(\|\phi\|_1) := Q_2''(\|\phi\|_1) + 24 + c\|\phi\|_1^4$.
Integrating (4.77) in the time interval $[0,t]$ we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty}\big(\frac{5}{12}w^4 - 5ww_x^2 + 3w_{xx}^2\big)dx \;\leq\; & \int_{-\infty}^{+\infty}\big(\frac{5}{12}(\phi')^4 - 5\phi'(\phi'')^2 + 3(\phi''')^2\big)dx \\ & +\eta T e^{c\eta T}Q_1(\|\phi\|_1) + \big\{\eta e^{c\eta T}Q_2(\|\phi\|_1) \\ & +c\eta\big((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T}R_1(\|\phi\|_1)}\big)^{4/3}\big\} \\ & \cdot\int_0^t \|r(\tau)\|^2 d\tau. \qquad (4.78)\end{aligned}$$

On the other hand:

$$\int(\phi')^4dx \leq \|\phi'\|_{L^\infty}^2\|\phi\|_1^2 \leq c\|\phi'\|_1^2\|\phi\|_2^2 \leq c\|\phi\|_2^4. \qquad (4.79)$$

$$\Big|\int\phi'(\phi'')^2dx\Big| \leq \|\phi'\|_{L^\infty}\|\phi\|_2^2 \leq c\|\phi\|_2^3. \qquad (4.80)$$

$$\int(\phi''')^2dx \leq \|\phi\|_3^2. \qquad (4.81)$$

$$\begin{aligned}\int w^4dx \;&\leq\; \|w\|_{L^\infty}^2\|w\|^2 \leq \|w\|^3\|w_x\| \leq c\|w\|^{7/2}\|w_{xx}\|^{1/2} \\ &\leq\; c(\epsilon_1\|w_{xx}\|^2 + \epsilon_1^{-1/3}\|w\|^{14/3}), \qquad (4.82)\end{aligned}$$

where in the last inequality we used Young's inequality, with $p=4$ and $p'=4/3$.

$$\begin{aligned}\int w(w_x)^2dx \;&\leq\; \|w_x\|_{L^\infty}\|w\|\|w_x\| \leq \|w\|\|w_x\|^{3/2}\|w_{xx}\|^{1/2} \\ &\leq\; c\|w\|^{7/4}\|w_{xx}\|^{5/4} \leq c(\epsilon_2\|w_{xx}\|^2 + \epsilon_2^{-5/3}\|w\|^{14/3}), \qquad (4.83)\end{aligned}$$

the last inequality was obtained using Young's inequality with $p=8/5$ and $p'=8/3$.
Using (4.79)-(4.83) in (4.78) we obtain

$$\begin{aligned}3\|r\|^2 \;\leq\; & c\big[\|\phi\|_2^4 + \|\phi\|_2^3 + \|\phi\|_3^2 + \epsilon_1\|r\|^2 + \epsilon_1^{-1/3}(\|\phi'\|e^{\eta T})^{14/3} + \epsilon_2\|r\|^2 \\ & +\epsilon_2^{-5/3}(\|\phi'\|e^{\eta T})^{14/3}\big] + \eta T e^{c\eta T}Q_1(\|\phi\|_1) + c\eta\big\{e^{c\eta T}Q_2(\|\phi\|_1) \\ & +\big((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T}R_1(\|\phi\|_1)}\big)^{4/3}\big\}\int_0^t\|r(\tau)\|^2d\tau. \quad (4.84)\end{aligned}$$

Taking $\epsilon_1 = 1/c = \epsilon_2 > 0$, defining $Q_3(\|\phi\|_3) := c(\|\phi\|_3^2 + \|\phi\|_3^3 + \|\phi\|_3^4)$ and $Q_4(\|\phi\|_1) := c\|\phi\|_1^{14/3}$, we get

$$\begin{aligned}\|r\|^2 \;\le\;& Q_3(\|\phi\|_3) + e^{c\eta T} Q_4(\|\phi\|_1) + \eta T e^{c\eta T} Q_1(\|\phi\|_1) + \eta\big\{e^{c\eta T} Q_2(\|\phi\|_1) \\ &+ \big((1 + (1+\eta T)^{2/3}) R_2(\|\phi\|_2) e^{c\eta T e^{c\eta T} R_1(\|\phi\|_1)}\big)\big\} \int_0^t \|r(\tau)\|^2 d\tau. \end{aligned} \tag{4.85}$$

Gronwall's inequality applied to (4.85) implies that

$$\begin{aligned}\|u_{xxx}\|^2 \;\le\;& \big[Q_3(\|\phi\|_3) + (1+\eta T) e^{c\eta T} Q_1(\|\phi\|_1)\big] \cdot \exp\big\{\eta T \big[e^{c\eta T} Q_2(\|\phi\|_1) \\ &+ (1 + (1+\eta T)^{2/3}) R_2(\|\phi\|_2) e^{c\eta T e^{c\eta T} R_1(\|\phi\|_1)}\big]\big\}, \end{aligned} \tag{4.86}$$

where $Q_1(\|\phi\|_1)$ now stands for $Q_1(\|\phi\|_1) + Q_4(\|\phi\|_1)$. This concludes the proof. ■

Lemma 4.3. *Let $s > 3/2$ be fixed, $\phi \in H^s$, and let $w_\eta \in C(0, T_s : H^s)$ be the solution of (1.2) with $\eta > 0$. Then, there exists a $T_s' > 0$ depending on s and $\|\phi\|_s$, but not on $0 < \eta < 1$, such that w_η can be extended to the interval $[0, T_s']$, and there is a function $\rho \in C([0, T_s']; \mathbb{R})$ such that*

$$\|w_\eta(t)\|_s^2 \le \rho(t), \quad \rho(0) = \|\phi\|_s^2, \quad t \in [0, T_s']. \tag{4.87}$$

Proof. Using $(w, w_{xxx})_s = 0$, $(K1)$ and (4.29), it follows easily from equation (1.2) that

$$\partial_t \|w_\eta(t)\|_s^2 \le 2\eta \|w_\eta(t)\|_s^2 + C_s \|w_\eta(t)\|_s^3. \tag{4.88}$$

Since $\eta \in (0,1)$, we have that

$$\partial_t \|w_\eta(t)\|_s^2 \le C_s(\|w_\eta(t)\|_s^2 + \|w_\eta(t)\|_s^3). \tag{4.89}$$

Then, $\|w_\eta(t)\|_s^2 \le \rho(t)$ in $[0, T_s']$, where $\rho(t)$ satisfies the differential equation

$$\partial_t \rho(t) = C_s(\rho(t) + \rho(t)^{3/2}), \quad \rho(0) = \|\phi\|_s^2. \tag{4.90}$$

Solving the last differential equation we have that

$$\rho(t) = \frac{e^{C_s t}\|\phi\|_s^2}{(1 + \|\phi\|_s - e^{C_s/2 \cdot t}\|\phi\|_s)^2}, \tag{4.91}$$

for $t \in [0, T_s']$, where $T_s' < 2/C_s \cdot ln((1 + \|\phi\|_s)/(\|\phi\|_s))$. ■

Lemma 4.4. *Let $s \geq 2$ be fixed, $\phi \in H^s$, and let $w_\eta \in C(0, T : H^s)$ be the solution of (1.2) with $0 < \eta < 1$. Then, there exists a constant $C = C(s, T, \|\phi\|_2)$, such that*

$$\|w_\eta(t)\|_s^2 \leq \|\phi\|_s^2 \cdot \exp(C(s, T, \|\phi\|_2)T), \tag{4.92}$$

Proof. Using $(w, w_{xxx})_s = 0$, $(K2)$ and (4.29), we obtain easily from equation (1.2)

$$\partial_t \|w_\eta(t)\|_s^2 \leq 2\eta \|w_\eta(t)\|_s^2 + C_s \|w_\eta(t)\|_2 \cdot \|w_\eta(t)\|_s^2. \tag{4.93}$$

From Lemma 4.2 we have that the H^2 norm of $w_\eta(t)$ is bounded by a function of T and $\|\phi\|_2$ but independently of $\eta < 1$. Since $\eta \in (0, 1)$, we obtain from (4.93)

$$\partial_t \|w_\eta(t)\|_s^2 \leq C(s, T, \|\phi\|_2) \cdot \|w_\eta(t)\|_s^2. \tag{4.94}$$

Gronwall's inequality then leads to (4.92). ■

Theorem 4.3. *Let $\eta > 0$ and $\phi \in H^s, s \geq 2$, be given, and let w_η be the solution of the equation (1.2) such that $w_\eta(\cdot, 0) = \phi(\cdot)$. Then, the limit $w_0 = \lim_{\eta \to 0} w_\eta$ exists in $C(0, T; H^s)$ and is the unique solution of (1.2) with $\eta = 0$. Moreover, the map $\phi \in H^s \mapsto w_0 \in C(0, T; H^s)$ is continuous with respect to the corresponding topologies.*

Proof: To prove this theorem we proceed as in [8] or [12].

Let $w^{(j)} = w_{\eta j}, j = 1, 2$, be the solutions of (1.2) with the same initial condition $\phi \in H^s$. Let $w = w^{(1)} - w^{(2)}$, then we obtain

$$w_t + w^{(1)} w_x^{(1)} - w^{(2)} w_x^{(2)} + w_{xxx} + \eta_1(\mathcal{H} w_x + \mathcal{H} w_{xxx}) + (\eta_1 - \eta_2)(\mathcal{H} w_x^{(2)} + \mathcal{H} w_{xxx}^{(2)}) = 0.$$

Multiplying the last equation by w and integrating over $\mathbb{R}$ we get

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|w(t)\|^2 &= -(w, w w_x^{(1)}) - (w, w_x w^{(2)}) - \eta_1(w, \mathcal{H} w_x + \mathcal{H} w_{xxx}) \\
&\quad -(\eta_1 - \eta_2)(w, \mathcal{H} w_x^{(2)} + \mathcal{H} w_{xxx}^{(2)}) \\
&\leq \|w_x^{(1)}\|_{L^\infty} \|w\|^2 + \tfrac{1}{2}\|w_x^{(2)}\|_{L^\infty} \|w\|^2 + \eta_1 \|w\|^2 \\
&\quad + c|\eta_1 - \eta_2|(\|w\| \|w_x^{(2)}\| + \|w_{xx}\| \|w_x^{(2)}\|) \\
&\leq c(\|w^{(1)}\|_2 + \|w^{(2)}\|_2 + \eta_1)\|w\|^2 \\
&\quad + c|\eta_1 - \eta_2|(\|w^{(1)}\|_2 + \|w^{(2)}\|_2)\|w^{(2)}\|_1.
\end{aligned}$$

Applying Gronwall's inequality to the last relation we find

$$\begin{aligned} \|w\|^2 &\leq cT|\eta_1-\eta_2| \sup_{t\in[0,T]} [(\|w^{(1)}\|_2+\|w^{(2)}\|_2)\|w^{(2)}\|_1] \\ &\quad \cdot \exp[cT \sup_{t\in[0,T]} (\|w^{(1)}\|_2+\|w^{(2)}\|_2+\eta_1)]. \end{aligned} \tag{4.95}$$

By Lemma 4.2 the H^2 norms of $w^{(1)}$ and $w^{(2)}$ are bounded by a function of T and $\|\phi\|_2$ but independent of $\eta<1$. From (4.95) and because L^2 is complete, there exists the limit $w_0(t)=\lim_{\eta\to 0} w_\eta(t)$ in L^2 uniformly with respect to $t\in[0,T]$, i.e.

$$\lim_{\eta\to 0}\sup_{t\in[0,T]} \|w_\eta(t)-w_0(t)\|=0, \tag{4.96}$$

and therefore

$$w_0\in C(0,T;L^2). \tag{4.97}$$

Let us prove that $w_0(t)\in H^s$ for all $t\in[0,T]$. In fact, let $t\in[0,T]$ and take a sequence $(\eta_n)_{n\in\mathbb{N}}$ with $\eta_n>0$ and $\eta_n\to 0$ when $n\to\infty$. Since $L^2-lim_{n\to\infty} w_{\eta_n}(t)=w_0(t)$, there exists a subsequence which we again call $(w_{\eta_n}(t))$ such that

$$\lim_{n\to\infty}\widehat{w_{\eta_n}}(t,\xi)=\widehat{w_0}(t,\xi),\quad \xi-a.e. \tag{4.98}$$

Applying Fatou's Lemma, we find that

$$\begin{aligned} \int(1+\xi^2)^s|\widehat{w_0}(t,\xi)|^2d\xi &\leq \liminf_{n\to\infty}\int(1+\xi^2)^s|\widehat{w_{\eta_n}}(t,\xi)|^2d\xi \\ &\leq \|\phi\|_s^2\cdot\exp(C(s,T,\|\phi\|_s)T), \end{aligned} \tag{4.99}$$

where the last inequality is a consequence of inequality (4.92).
We claim that

$$w_\eta(t)\rightharpoonup w_0(t),\ \ in\ \ H^s \tag{4.100}$$

uniformly over $[0,T]$ as $\eta\to 0$. In fact, defining $C^2:=\|\phi\|_s^2\cdot\exp(C(s,T,\|\phi\|_s)T)$, as on the right hand side of (4.92), let $\varphi\in H^{-s}$ and let $\epsilon>0$ be fixed but arbitrary then by density, there exists an element $\varphi_\epsilon\in L^2$ such that $\|\varphi-\varphi_\epsilon\|_{-s}<\epsilon/(3\cdot C)$, then

$$|\varphi(w_\eta(t)-w_0(t))|\leq\|\varphi-\varphi_\epsilon\|_{-s}\|w_\eta(t)-w_0(t)\|_s+|\varphi_\epsilon(w_\eta(t)-\ w_0(t))|. \tag{4.101}$$

Since $\varphi_\epsilon \in L^2$, there exists $\delta_{(\epsilon)} > 0$ such that $0 < \eta < \delta_{(\epsilon)}$ implies $|\varphi_\epsilon(w_\eta(t) - w_0(t))| < \epsilon/3$, for all $t \in [0,T]$.
Using (4.92) and

$$\|w_0(t)\|_s \leq \liminf_{\eta\to 0} \|w_\eta(t)\|_s \leq C \tag{4.102}$$

in inequality (4.101) we have that $\varphi(w_\eta(t) - w_0(t)) \to 0$ as $\eta \to 0$, uniformly in $t \in [0,T]$. Since $w_\eta \in C(0,T;H^s)$, it follows that $w_\eta \in C^\omega(0,T;H^s)$. Then,

$$w_0 \in C^\omega(0,T;H^s). \tag{4.103}$$

Let $\psi \in H^{s-3}$. Then,

$$\begin{aligned}(w_\eta(t),\psi)_{s-3} &= (\phi,\psi)_{s-3} \\ &- \int_0^t (w_\eta(t')\partial_x w_\eta(t') + \partial_x^3 w_\eta(t') + \eta(\mathcal{H}\partial_x w_\eta(t') + \mathcal{H}\partial_x^3 w_\eta(t')),\psi)_{s-3}dt',\end{aligned} \tag{4.104}$$

for all $t \in [0,T]$. From (4.100) we have that

$$\begin{aligned}\partial_x^3 w_\eta &\rightharpoonup \partial_x^3 w_0, \ \ in \ \ H^{s-3}, \\ \mathcal{H}\partial_x w_\eta &\rightharpoonup \mathcal{H}\partial_x w_0, \ \ in \ \ H^{s-1}, \\ \mathcal{H}\partial_x^3 w_\eta &\rightharpoonup \mathcal{H}\partial_x^3 w_0, \ \ in \ \ H^{s-3}\end{aligned} \tag{4.105}$$

uniformly over $[0,T]$ as $\eta \to 0$.
On the other hand, $f_n \rightharpoonup f$ in H^r, $g_n \rightharpoonup g$ in H^r implies $f_n g_n \rightharpoonup fg$ in H^r, for $r > 1/2$. Then $w_\eta(t')\partial_x w_\eta(t') \rightharpoonup w_0(t')\partial_x w_0(t')$ in H^{s-1}, for all $t' \in [0,T]$.
By (4.92) we see that

$$\begin{aligned}\|w_\eta(t')\partial_x w_\eta(t')\|_{s-3} &\leq \|w_\eta(t')\partial_x w_\eta(t')\|_{s-1} \leq \|w_\eta(t')\|_s^2 \\ &\leq \|\phi\|_s^2 \cdot \exp(C(s,T,\|\phi\|_2)T),\end{aligned} \tag{4.106}$$

for all $t' \in [0,T]$. Then, using the Dominated Convergence Theorem, we get

$$\lim_{\eta\to 0} \int_0^t (w_\eta(t')\partial_x w_\eta(t'),\psi)_{s-3}dt' = \int_0^t (w_0(t')\partial_x w_0(t'),\psi)_{s-3}dt'. \tag{4.107}$$

Using (4.105),(4.107) and letting $\eta \to 0$ in (4.104), we obtain

$$(w_0(t),\psi)_{s-3} = (\phi,\psi)_{s-3} - \int_0^t (w_0(t')\partial_x w_0(t') + \partial_x^3 w_0(t'),\psi)_{s-3}dt', \tag{4.108}$$

for all $\psi \in H^{s-3}$ and $t \in [0,T]$. Since $t \in [0,T] \longmapsto w_0(t)\partial_x w_0(t) + \partial_x^3 w_0(t) \in H^{s-3}$ is weakly continuous, it is strongly measurable, thus

$$w_0(t) = \phi - \int_0^t w_0(t')\partial_x w_0(t') + \partial_x^3 w_0(t')dt' \tag{4.109}$$

exists as a Bochner integral. Then

$$w_0 \in AC([0,T]; H^{s-3}). \tag{4.110}$$

As the equation (4.108) is valid for all $\psi \in H^{s-3}$, using the Fundamental Theorem of Calculus we have that

$$\partial_t w_0(t) = -w_0(t)\partial_x w_0(t) - \partial_x^3 w_0(t), \ \ t \in [0,T], \tag{4.111}$$

with the derivative respect to t calculated in the H^{s-3} norm. By Theorem 6.11 in [11], there is a unique solution for the KdV equation in the class $w_0 \in C([0,T] : L^2) \cap C^{\omega}([0,T]; H^s) \cap AC([0,T]; H^{s-3})$ and this unique solution belongs to $w_0 \in C([0,T]; H^s)$, as we asserted. To establish that the solution depends continuously on the initial data, we proceed as in [3]. This completes the proof of the theorem. ■

Chapter 5

Global Theory in $\mathcal{F}_{r,s}(\mathbb{R})$ and Decay Properties of the Solution of (1.2)

Some decay properties of the solution of equation (1.2) for $\eta > 0$ are obtained, similar to those obtained for the Benjamin Ono equation (see [8]). Theorem 5.4 is a unique continuation theorem for equation (1.2). It implies loss of persistence for equation (1.2) in $\mathcal{F}_{3,3}$, while for BO this occurs in $\mathcal{F}_{4,4}$.

Lemma 5.1. *Let $F_\eta(t,\xi)$ be defined by (2.2), where $t \geq 0$, $\xi \in \mathbb{R}$ and $\eta \geq 0$. Then,*

$$\partial_\xi F_\eta(t,\xi) = F_\eta(t,\xi)t\big[3i\xi^2 - \eta h(\xi)(3\xi^2-1)\big], \tag{5.1}$$

$$\partial_\xi^2 F_\eta(t,\xi) = F_\eta(t,\xi)t^2\big[3i\xi^2 - \eta h(\xi)(3\xi^2-1)\big]^2 + 6t\xi(i-\eta h(\xi))F_\eta(t,\xi) + 2\eta t\delta, \tag{5.2}$$

and

$$\begin{aligned}\partial_\xi^3 F_\eta(t,\xi) \;=\;& F_\eta(t,\xi)t^3\big[3i\xi^2 - \eta h(\xi)(3\xi^2-1)\big]^3 + 6t(i-\eta h(\xi))F_\eta(t,\xi)\\ &+F_\eta(t,\xi)t^2\big[36\xi^3(\eta^2-1) - 72i\eta h(\xi)\xi^3 + 12i\eta h(\xi)\xi - 12\eta^2\xi\big]\\ &+6t^2\xi(i-\eta h(\xi))\big[3i\xi^2 - \eta h(\xi)(3\xi^2-1)\big]F_\eta(t,\xi) + 2\eta t\delta',\end{aligned} \tag{5.3}$$

where δ is the Dirac delta distribution and $h(\xi)$ is the sign function.

Proof: This result follows easily using the chain rule. ■

Lemma 5.2. *Let $\eta \geq 0$ be fixed. Then,*

1. *$E_\eta(t):\mathcal{F}_{r,r}\to\mathcal{F}_{r,r}$, for $r=0,1$ is a C^0 semigroup and we have that*

$$\|E_\eta(t)\phi\|_{\mathcal{F}_{r,r}}\le e^{c\eta t}\Theta_{\eta,r}(t)\|\phi\|_{\mathcal{F}_{r,r}}, \tag{5.4}$$

for all $\phi\in\mathcal{F}_{r,r}$, where $\Theta_{\eta,r}(t)$ is a polynomial of degree r with positive coefficients depending only on η and r.

2. *If $r\ge 2$ and $\phi\in\mathcal{F}_{r,r}$, the function $E_\eta(t)\phi\in C([0,\infty);\mathcal{F}_{r,r})$ if and only if*

$$(\partial_\xi^j\hat{\phi})(0)=0,\ \ j=0,1,...,r-2. \tag{5.5}$$

In this case we obtain also an expression like (5.4).

Proof: Similar to the proof of Theorem 2.4 in [8]. ■

Now let us prove a local result for equation (1.2) in $\mathcal{F}_{2,1}(\mathbb{R})$.

Theorem 5.1. *Let $\eta>0$ and $\phi\in\mathcal{F}_{2,1}(\mathbb{R})$. Then, there exist $T(\|\phi\|_{\mathcal{F}_{2,1}},\eta)>0$ and a unique function $u:=u_\eta\in C(0,T;\mathcal{F}_{2,1}(\mathbb{R}))$ satisfying the integral equation (3.11).*

Proof: Let $M,T>0$. Consider the map

$$(Af)(t)=E_\eta(t)\phi-\frac{1}{2}\int_0^t E_\eta(t-t')\partial_x f(t')^2dt', \tag{5.6}$$

defined on the complete metric space

$$\Xi_{2,1}(T)=\big\{f\in C(0,T;\mathcal{F}_{2,1}(\mathbb{R}));\ \sup_{t\in[0,T]}\|f(t)-E_\eta(t)\phi\|_{\mathcal{F}_{2,1}}\le M\big\}. \tag{5.7}$$

Since $\xi^J e^{-\eta t(\xi^3-\xi)}\le \xi^J e^{-\eta t/2\xi^3}$, for $J\in\mathbb{N}$ and $\xi\ge\sqrt{2}$, we have the inequality

$$\sup_{\xi\ge 0}\xi^J e^{-\eta t(\xi^3-\xi)}\le c(J)\big((\frac{1}{\eta t})^{J/3}+e^{\eta t}\big),\ \ J\in\mathbb{N}. \tag{5.8}$$

i.) Let $f\in\Xi_{2,1}(T)$. We will prove that $Af\in C(0,T:\mathcal{F}_{2,1}(\mathbb{R}))$. We remark that $E_\eta(t)$ is a C^0 semigroup on $\mathcal{F}_{2,1}(\mathbb{R})$ and moreover,

$$\|E_\eta(t)\phi\|_{\mathcal{F}_{2,1}}\le e^{c\eta t}\Theta_{\eta,1}(t)\|\phi\|_{\mathcal{F}_{2,1}}, \tag{5.9}$$

for all $\phi\in\mathcal{F}_{2,1}$, where $\Theta_{\eta,1}(t)$ is a polynomial of degree one with positive coefficients depending only on η. The last assertion can be proved using Lemma 2.1 and the

expression (5.1).
By Theorem 3.2 and since $E_\eta(t)$ is a C^0 semigroup in $\mathcal{F}_{2,1}(\mathbb{R})$, it is enough to show that

$$F(t) := \int_0^t E_\eta(t-t')\partial_x f(t')^2 dt' \quad \in \quad C([0,T]; L_1^2(\mathbb{R})).$$

Let $\tau > t > 0$.

$$\begin{aligned}\|x\big(F(t)-F(\tau)\big)\| \;&\le\; \int_0^t \|x\big(E_\eta(\tau-t')-E_\eta(t-t')\big)\partial_x f(t')^2\|dt' \\ &\quad + \int_t^\tau \|xE_\eta(\tau-t')\partial_x f(t')^2\|dt' \qquad (5.10)\end{aligned}$$

Let $g(t') := \partial_x f(t')^2$, for all $t' \in [0,T]$. We remark that $f(t')f_x(t') \in H^1 \cap L_1^2$, for all $t' \in [0,T]$. Let $t' \in [0,T]$. Then,

$$\begin{aligned}\|\partial_x(f(t')f_x(t'))\| \;&\le\; \|\partial_x f(t')\|_{L^\infty}\|\partial_x f(t')\| + \|f(t')\|_{L^\infty}\|\partial_x^2 f(t')\| \\ &\le\; c(\|f(t')\|_2\|f(t')\|_1 + \|f(t')\|_1\|f(t')\|_2) \\ &\le\; c\|f(t')\|^2_{\mathcal{F}_{2,1}} \qquad (5.11)\end{aligned}$$

and

$$\begin{aligned}\|xf(t')f_x(t')\| \;&\le\; \|f_x(t')\|_{L^\infty}\|xf(t')\| \le c\|f(t')\|_2\|f(t')\|_{L_1^2} \\ &\le\; c\|f(t')\|^2_{\mathcal{F}_{2,1}}. \qquad (5.12)\end{aligned}$$

Hence, for every $t' \in [0,T]$, we have that

$$\begin{aligned}\|xE_\eta(\tau-t')g(t')\| \;&=\; \|\partial_\xi F_\eta(\tau-t',\xi)\hat{g}(t',\xi) + F_\eta(\tau-t',\xi)\partial_\xi\hat{g}(t',\xi)\| \\ &\le\; \|F_\eta(\tau-t',\xi)(\tau-t')\big[(3i-3\eta h(\xi))\xi^2+\eta h(\xi)\big]\hat{g}(t',\xi)\| \\ &\quad +\|F_\eta(\tau-t',\xi)\partial_\xi\hat{g}(t',\xi)\| \\ &\le\; c(\eta)(\tau-t')\|F_\eta(\tau-t',\xi)\xi\|_{L^\infty}\|\xi\hat{g}(t',\xi)\| \\ &\quad +\eta(\tau-t')\|F_\eta(\tau-t',\xi)\hat{g}(t',\xi)\| + e^{\eta(\tau-t')}\|\partial_\xi\hat{g}(t',\xi)\| \\ &\le\; c(\eta)\Big(\frac{(\tau-t')^{2/3}}{\eta^{1/3}} + (\tau-t'+1)e^{\eta(\tau-t')}\Big)\|f(t')\|^2_{\mathcal{F}_{2,1}}, \qquad (5.13)\end{aligned}$$

where in the last inequality we used (5.11), (5.12) and (5.8) with $J=1$. Hence the second integral on the right hand side of (5.10) tends to zero as τ tends to t.
On the other hand, we know that $g(t') \in \mathcal{F}_{1,1}(\mathbb{R})$, for all $t' \in [0,T]$. Since $E_\eta(t)$:

$\mathcal{F}_{1,1} \to \mathcal{F}_{1,1}$ is a C^0 semigroup, we conclude that $\|x\big(E_\eta(\tau-t')-E_\eta(t-t')\big)g(t')\|$ tends to zero as $\tau \to t$. Now using the estimate (5.13), the triangular inequality and the Dominated Convergence Theorem, it follows that the first integral on the right hand side of (5.10) tends to zero as $\tau \to t$.

ii.) Now we will prove that $A(\Xi_{2,1}(\tilde{T})) \subset \Xi_{2,1}(\tilde{T})$, for $T=\tilde{T}>0$ small enough. Let $f \in \Xi_{2,1}(T)$. By using the proof of Theorem 3.2, it is enough to show that we can choose $T=\tilde{T}>0$ sufficiently small such that $\|x\big(Af(t)-E_\eta(t)\phi\big)\| \le M/2$, for all $t \in [0,T]$. Let $t \in [0,T]$. Then,

$$
\begin{aligned}
\|x\big(Af(t)-E_\eta(t)\phi\big)\| &\le \frac{1}{2}\int_0^t \|xE_\eta(t-t')\partial_x f(t')^2\|dt' \\
&\le c(\eta)\Big(\frac{T^{2/3}}{\eta^{1/3}}+(T+1)e^{\eta T}\Big)\sup_{t'\in[0,T]}\|f(t')\|^2_{\mathcal{F}_{2,1}}\cdot T \\
&\le c(\eta)\Big(\frac{T^{2/3}}{\eta^{1/3}}+(T+1)e^{\eta T}\Big)\Big(M+e^{c\eta T}\sup_{t'\in[0,T]}\Theta_{\eta,1}(t')\|\phi\|_{\mathcal{F}_{2,1}}\Big)^2\cdot T
\end{aligned}
$$

where the second inequality was obtained using (5.13).

iii.) Next, we will prove that the map A is a contraction defined on $(\Xi_{2,1}(\hat{T}))$, for some $\hat{T}\in(0,\tilde{T}]$. Let $t\in[0,\tilde{T}]$, $u,v\in(\Xi_{2,1}(\tilde{T}))$. Then,

$$
\|x\big(Au(t)-Av(t)\big)\| \le \int_0^t \|xE_\eta(t-t')\partial_x\big(u(t')^2-v(t')^2\big)\|dt'. \tag{5.14}
$$

On the other hand, we have that

$$
\begin{aligned}
\|\partial_x\big(u(t')^2-v(t')^2\big)\| &\le \|u(t')^2-v(t')^2\|_1 \\
&\le \big(\sup_{[0,\tilde{T}]}\|u(t')\|_1+\sup_{[0,\tilde{T}]}\|v(t')\|_1\big)\|u(t')-v(t')\|_1 \\
&\le 2\big(M+e^{c\eta\tilde{T}}\sup_{t'\in[0,\tilde{T}]}\Theta_{\eta,1}(t')\|\phi\|_{\mathcal{F}_{2,1}}\big)\sup_{[0,\tilde{T}]}\|u(t')-v(t')\|_{\mathcal{F}_{2,1}} \\
&\le c(\eta)\big(M+e^{c\eta\tilde{T}}(\tilde{T}+1)\|\phi\|_{\mathcal{F}_{2,1}}\big)\sup_{[0,\tilde{T}]}\|u(t')-v(t')\|_{\mathcal{F}_{2,1}} \qquad (5.15)
\end{aligned}
$$

and

$$
\begin{aligned}
\|x\partial_x\big(u(t')^2-v(t')^2\big)\| &\le 2\big(\|x\partial_x u(t')(u(t')-v(t'))\|+\|xv(t')\partial_x\big(u(t')-v(t')\big)\|\big) \\
&\le 2\big(\|\partial_x u(t')\|_{L^\infty}\|u(t')-v(t')\|_{L^2_1}+\|xv(t')\|\|\partial_x\big(u(t')-v(t')\big)\|_{L^\infty}\big) \\
&\le c\big(\|u(t')\|_2\|u(t')-v(t')\|_{L^2_1}+\|v(t')\|_{L^2_1}\|u(t')-v(t')\|_2\big) \\
&\le c(\eta)\big(M+e^{c\eta\tilde{T}}(\tilde{T}+1)\|\phi\|_{\mathcal{F}_{2,1}}\big)\sup_{[0,\tilde{T}]}\|u(t')-v(t')\|_{\mathcal{F}_{2,1}}, \qquad (5.16)
\end{aligned}
$$

for all $t' \in [0,\tilde{T}]$. Taking the Fourier transform of the expression inside the norm in the integral in (5.14), using (5.1), (2.5), (5.8) with $J=2$, (5.15) and (5.16) it follows from (5.14) that

$$\begin{aligned}\|x\big(Au(t)-Av(t)\big)\| &\leq c(\eta)\big(M+e^{c\eta\tilde{T}}(\tilde{T}+1)\|\phi\|_{\mathcal{F}_{2,1}}\big)\big(\frac{\tilde{T}^{1/3}}{\eta^{2/3}}+(\tilde{T}+1)e^{\eta\tilde{T}}\big)\cdot\tilde{T}\\ &\quad\cdot\sup_{[0,\tilde{T}]}\|u(t')-v(t')\|_{\mathcal{F}_{2,1}}.\end{aligned}$$

From **i.)**, **ii.)** and **iii.)**, it follows that A has a unique fixed point $u \in \Xi_{2,1}(\hat{T})$, $\hat{T}>0$, satisfying the integral equation (3.11). Uniqueness of the solution of (3.11) in $C(0,T;\mathcal{F}_{2,1}(\mathbb{R}))$ is a consequence of Theorem 3.2. ■

Next Lemma is similar to Theorem 5.2.3, obtained in [1], for a regularized version of the Ott-Sudan equation.

Lemma 5.3. *Let $\eta>0$ and $\phi \in \mathcal{F}_{2,1}(\mathbb{R})$. Let $u \in C(0,T;\mathcal{F}_{2,1}(\mathbb{R}))$ be the solution of the integral equation (3.11). Then, $\partial_x^J u \in C((0,T];L_1^2(\mathbb{R}))$, for $J=0,1,2,3$. Moreover, $\mathcal{H}\partial_x u, \mathcal{H}\partial_x^3 u \in C((0,T];L_1^2(\mathbb{R}))$.*

Proof: First, we remark that $\phi \in \mathcal{F}_{2,1}(\mathbb{R})$ implies that $x\partial_x^J E_\eta(t)\phi \in L^2(\mathbb{R})$, for $J=0,1,2,3$ and $t \in (0,T]$. The last assertion can be proved taking the Fourier transform of $x\partial_x^J E_\eta(t)\phi$ and using expressions: (5.1) and (5.8). Moreover, it is not difficult, taking the Fourier transform, to prove that $x\partial_x^J E_\eta(\cdot)\phi \in C((0,T];L^2(\mathbb{R}))$, for $J=0,1,2,3$.
Let $t \in (0,T]$. Let us call $g(t') := u(t')\partial_x u(t')$, for all $t' \in [0,T]$. Then,

$$\|x\partial_x u(t)\| \leq \|x\partial_x E_\eta(t)\phi\| + \int_0^t \|x\partial_x E_\eta(t-t')g(t')\|dt'. \tag{5.17}$$

We have that

$$\|x\partial_x E_\eta(t)\phi\| \leq \|F_\eta(t,\xi)\hat{\phi}(\xi)\| + \|\xi\partial_\xi F_\eta(t,\xi)\hat{\phi}(\xi)\| + \|\xi F_\eta(t,\xi)\partial_\xi\hat{\phi}(\xi)\|, \tag{5.18}$$

where

$$\|F_\eta(t,\xi)\hat{\phi}(\xi)\| \leq e^{\eta t}\|\phi\|, \tag{5.19}$$

$$\begin{aligned}\|\xi\partial_\xi F_\eta(t,\xi)\hat{\phi}(\xi)\| &\leq c(\eta)\|tF_\eta(t,\xi)\xi^3\hat{\phi}(\xi)\| + \eta t\|\xi F_\eta(t,\xi)\hat{\phi}(\xi)\|\\ &\leq c(\eta)\big(\frac{t^{2/3}}{\eta^{1/3}}+te^{\eta t}\big)\|\phi\|_2 + \eta te^{\eta t}\|\phi\|_1\\ &\leq c(\eta)\big(\frac{t^{2/3}}{\eta^{1/3}}+te^{\eta t}\big)\|\phi\|_{\mathcal{F}_{2,1}}\end{aligned} \tag{5.20}$$

and

$$\|\xi F_\eta(t,\xi)\partial_\xi\hat{\phi}(\xi)\| \leq \|\xi F_\eta(t,\xi)\|_{L^\infty}\|x\phi\| \leq c\big(\frac{1}{(\eta t)^{1/3}}+e^{\eta t}\big)\|\phi\|_{\mathcal{F}_{2,1}}. \tag{5.21}$$

On the other hand, we find that

$$\begin{aligned}\|x\partial_x E_\eta(t-t')g(t')\| \;&\leq\; \|F_\eta(t-t',\xi)\hat{g}(t',\xi)\|+\|\xi\partial_\xi F_\eta(t-t',\xi)\hat{g}(t'\xi)\| \\ &\quad +\|\xi F_\eta(t-t',\xi)\partial_\xi\hat{g}(t'\xi)\|. \end{aligned}\tag{5.22}$$

Applying (5.11) and (5.12) to $g(t')=u(t')\partial_x u(t')$, we obtain

$$\|F_\eta(t-t',\xi)\hat{g}(t',\xi)\| \leq \|F_\eta(t-t',\xi)\|_{L^\infty}\|g(t',\xi)\| \leq ce^{\eta(t-t')}\|u(t')\|^2_{\mathcal{F}_{2,1}}, \tag{5.23}$$

$$\begin{aligned}\|\xi\partial_\xi F_\eta(t-t',\xi)\hat{g}(t'\xi)\| \;&\leq\; \big(c(\eta)(t-t')\|\xi^2F_\eta(t-t',\xi)\|_{L^\infty} \\ &\quad +\eta(t-t')\|F_\eta(t-t',\xi)\|_{L^\infty}\big)\|\xi\hat{g}(t',\xi)\| \\ &\leq\; c(\eta)\big(\frac{(t-t')^{1/3}}{\eta^{2/3}}+(t-t')e^{\eta(t-t')}\big)\|u(t')\|^2_{\mathcal{F}_{2,1}} \end{aligned}\tag{5.24}$$

and

$$\begin{aligned}\|\xi F_\eta(t-t',\xi)\partial_\xi\hat{g}(t'\xi)\| \;&\leq\; \|\xi F_\eta(t-t',\xi)\|_{L^\infty}\|\partial_\xi\hat{g}(t'\xi)\| \\ &\leq\; c\big(\frac{1}{(\eta(t-t'))^{1/3}}+e^{\eta(t-t')}\big)\|u(t')\|^2_{\mathcal{F}_{2,1}}. \end{aligned}\tag{5.25}$$

Using (5.18)-(5.25) in (5.17), we have that

$$\begin{aligned}&\|x\partial_x u(t)\| \leq c(\eta)\big((1+t)e^{\eta t}+\frac{t^{2/3}}{\eta^{1/3}}+\frac{1}{(\eta t)^{1/3}}\big)\|\phi\|_{\mathcal{F}_{2,1}} \\ &+c(\eta)\int_0^t \big((1+t-t')e^{\eta(t-t')}+\frac{(t-t')^{1/3}}{\eta^{2/3}}+\frac{1}{(\eta(t-t'))^{1/3}}\big)\|u(t')\|^2_{\mathcal{F}_{2,1}}dt'. \end{aligned}\tag{5.26}$$

Then, $x\partial_x u(t)\in L^2(\mathbb{R})$, for all $t\in(0,T]$. The fact that $x\partial_x u\in C((0,T];L^2(\mathbb{R}))$ is obtained using the expressions (5.18)-(5.25) and the Dominated Convergence Theorem. Now we consider the cases $J=2,3$. Let $t\in(0,T]$. Then,

$$\|x\partial_x^J u(t)\| \leq \|x\partial_x^J E_\eta(t)\phi\| + \int_0^t \|x\partial_x^J E_\eta(t-t')g(t')\|dt'. \tag{5.27}$$

It is easy to see that

$$\begin{aligned}\|x\partial_x^J E_\eta(t-t')g(t')\| &\leq J\|\xi^{J-1}F_\eta(t-t',\xi)\hat{g}(t',\xi)\| + \|\xi^J\partial_\xi F_\eta(t-t',\xi)\hat{g}(t',\xi)\| \\ &\quad +\|\xi^J F_\eta(t-t',\xi)\partial_\xi\hat{g}(t',\xi)\|. \end{aligned} \tag{5.28}$$

Next, we will estimate each term on the right hand side of (5.28):

$$\begin{aligned}\|\xi^{J-1}F_\eta(t-t',\xi)\hat{g}(t',\xi)\| &\leq \|\xi^{J-2}F_\eta(t-t',\xi)\|_{L^\infty}\|\xi\hat{g}(t',\xi)\| \\ &\leq c\|\xi^{J-2}F_\eta(t-t',\xi)\|_{L^\infty}\|u(t')\|^2_{\mathcal{F}_{2,1}} \\ &\leq c(J)\big(\frac{1}{(\eta(t-t'))^{\frac{J-2}{3}}} + e^{\eta(t-t')}\big)\|u(t')\|^2_{\mathcal{F}_{2,1}}. \end{aligned} \tag{5.29}$$

$$\begin{aligned}&\|\xi^J\partial_\xi F_\eta(t-t',\xi)\hat{g}(t',\xi)\| \\ &\leq c(\eta)(t-t')\big(\|\xi^{J+1}F_\eta(t-t',\xi)\|_{L^\infty} + \|\xi^{J-1}F_\eta(t-t',\xi)\|_{L^\infty}\big)\|\xi\hat{g}(t',\xi)\| \\ &\leq c(\eta,J)\big(\frac{1}{\eta^{\frac{J+1}{3}}(t-t')^{\frac{J-2}{3}}} + \frac{(t-t')^{\frac{4-J}{3}}}{\eta^{\frac{J-1}{3}}} + (t-t')e^{\eta(t-t')}\big)\|u(t')\|^2_{\mathcal{F}_{2,1}}. \end{aligned} \tag{5.30}$$

$$\begin{aligned}\|\xi^J F_\eta(t-t',\xi)\partial_\xi\hat{g}(t',\xi)\| &\leq \|\xi^{J-1}F_\eta(t-t',\xi)\|_{L^\infty}\|\xi\partial_\xi\hat{g}(t',\xi)\| \\ &\leq c(J)\big(\frac{1}{(\eta(t-t'))^{\frac{J-1}{3}}} + e^{\eta(t-t')}\big)\|\xi\partial_\xi\hat{g}(t',\xi)\|, \end{aligned} \tag{5.31}$$

where

$$\begin{aligned}&\|\xi\partial_\xi\hat{g}(t',\xi)\| = \|u(t')\partial_x u(t') + x(\partial_x u(t'))^2 + xu(t')\partial_x^2 u(t')\| \\ &\leq c\big(\|u(t')\|_{L^\infty}\|u(t')\|_1 + \|\partial_x u(t')\|_{L^\infty}\|x\partial_x u(t')\| + \|xu(t')\|_{L^\infty}\|u(t')\|_2\big) \\ &\leq c\big(\|u(t')\|^2_{\mathcal{F}_{2,1}} + \|u(t')\|_2\|x\partial_x u(t')\| + \|xu(t')\|^{1/2}\|\partial_x(xu(t'))\|^{1/2}\|u(t')\|_2\big) \\ &\leq c\big(\|u(t')\|^2_{\mathcal{F}_{2,1}} + \|u(t')\|_2\|x\partial_x u(t')\| \\ &\quad +\|u(t')\|^{3/2}_{\mathcal{F}_{2,1}}(\|u(t')\|^{1/2} + \|x\partial_x u(t')\|^{1/2})\big). \end{aligned} \tag{5.32}$$

On the other hand, if $J=2,3$, it is easy to see that

$$\int_0^t \frac{dt'}{(t-t')^{1/3}} = \frac{3}{2}t^{2/3}, \tag{5.33}$$

$$\int_0^t \frac{1}{(t-t')^{\frac{J-1}{3}}}\cdot\frac{1}{t'^{1/3}}dt' < +\infty, \tag{5.34}$$

$$\int_0^t \frac{1}{(t-t')^{\frac{J-1}{3}}}\cdot\frac{1}{t'^{1/6}}dt' < +\infty. \tag{5.35}$$

Using the expressions (5.26), (5.28)-(5.35) in (5.27), and since $x\partial_x^J E_\eta(t)\phi \in L^2(\mathbb{R})$, for $J = 2, 3$, and $t \in (0, T]$, we find that $x\partial_x^J u(t) \in L^2(\mathbb{R})$, for all $t \in (0, T]$, $J = 2, 3$. Finally, $\mathcal{H}\partial_x u, \mathcal{H}\partial_x^3 u \in C((0, T]; L_1^2(\mathbb{R}))$, are consequence of the previous results and the fact that $[x, \mathcal{H}]\partial_x g = 0$, for all $g \in L_s^2(\mathbb{R})$, $s > 1/2$. ■

Corollary 5.1. *Let $\eta > 0$ and let $u \in C(0, T; \mathcal{F}_{2,1}(\mathbb{R}))$ be the solution of the equation (1.2). Then, $\partial_t u \in C((0, T]; L_1^2(\mathbb{R}))$.*

Proof: It follows immediately from Lemma 5.3 and the next remark

$$\begin{aligned}
\|x\big(u(t)\partial_x u(t) - u(t_0)\partial_x u(t_0)\big)\| &\leq \|xu(t)\|\|\partial_x\big(u(t) - u(t_0)\big)\|_{L^\infty} \\
&\quad + \|x\big(u(t) - u(t_0)\big)\|\|\partial_x u(t_0)\|_{L^\infty} \\
&\leq \|u(t)\|_{L_1^2}\|u(t) - u(t_0)\|_2 + \|u(t) - u(t_0)\|_{L_1^2}\|u(t_0)\|_2. \blacksquare
\end{aligned}$$

Next, we prove a global existence theorem for equation (1.2) in $\mathcal{F}_{2,1}(\mathbb{R})$, similar to the proof of Theorem 5.2.10 in [1], where it is proved that the regularized Ott-Sudan equation $w_t + ww_x + w_{xxx} - \mu\mathcal{H}w_x + \epsilon\mathcal{H}w_{xxx} = 0$, with $\epsilon, \mu > 0$ is globally well possed in $\mathcal{F}_{2,1}(\mathbb{R})$. In that case the amplification and damping terms give, because of their signs, $(xw, \mathcal{H}\partial_x(xw)) \leq 0$ and $-(xw, \mathcal{H}\partial_x^3(xw)) \leq 0$ so (see Theorem 5.2 below) $\frac{1}{2}\frac{d}{dt}\|xw(t)\|^2 \leq -(xw, xww_x) - (xw, xw_{xxx}) - \eta(xw, \mathcal{H}w) + 3\eta(xw, \mathcal{H}w_{xx})$, but in our case that can not be done.

Theorem 5.2. *Let $\phi \in \mathcal{F}_{2,1}(\mathbb{R})$. Then, for each $\eta > 0$ there exists a unique solution $u = u_\eta \in C([0, \infty); \mathcal{F}_{2,1}(\mathbb{R}))$ of equation (1.2) such that $\partial_t u \in C((0, \infty); \mathcal{F}_{-1,1}(\mathbb{R}))$.*

Proof: To prove global existence for equation (1.2) in $\mathcal{F}_{2,1}(\mathbb{R})$, it is enough to combine Theorem 4.2 with the next computations.

$$\frac{1}{2}\frac{d}{dt}\|xu(t)\|^2 = -(xu, xuu_x) - (xu, xu_{xxx}) - \eta(xu, x(\mathcal{H}u_x + \mathcal{H}u_{xxx})),$$

$$\begin{aligned}
|(xu, xuu_x)| &\leq \|u_x\|_{L^\infty}\|xu\|^2 \leq c\|u\|_2\|u\|_{L_1^2}^2, \\
-(xu, xu_{xxx}) &= -(xu, [x, \partial_x^3]u) = 3(xu, u_{xx}) \leq 3\|u\|_{L_1^2}\|u\|_2 \leq c\|u\|_{\mathcal{F}_{2,1}}^2,
\end{aligned}$$

where by Lemma 4.1, $\|u(t)\|_2 \leq F(T, \eta, \|\phi\|_2)$, for all $t \in [0, T]$. Moreover,

$$\begin{aligned}
-(xu, x\mathcal{H}u_x) &= -(xu, [x, \mathcal{H}\partial_x]u) - (xu, \mathcal{H}\partial_x(xu)) = (xu, Hu) - (xu, \mathcal{H}\partial_x(xu)), \\
-(xu, x\mathcal{H}u_{xxx}) &= -(xu, [x, \mathcal{H}\partial_x^3]u) - (xu, \mathcal{H}\partial_x^3(xu)) = 3(xu, \mathcal{H}u_{xx}) - (xu, \mathcal{H}\partial_x^3(xu)),
\end{aligned}$$

where we have used the identities

$$\begin{aligned}
[x, \mathcal{H}\partial_x]u &= [x, \mathcal{H}]\partial_x u - \mathcal{H}[\partial_x, x]u, \\
[x, \mathcal{H}]\partial_x u &= 0, \\
[\partial_x, x]u &= u
\end{aligned}$$

and

$$\begin{aligned}
[x, \mathcal{H}\partial_x^3]u &= [x, \mathcal{H}]\partial_x^3 u - \mathcal{H}[\partial_x^3, x]u, \\
[x, \mathcal{H}]\partial_x^3 u &= 0, \\
[\partial_x^3, x]u &= 3\partial_x^2 u.
\end{aligned}$$

Therefore

$$-(xu, x(\mathcal{H}u_x + \mathcal{H}u_{xxx})) \leq \|xu\|\|u\| + \|xu\|\|u\|_2 + \|xu\|^2 \leq 3\|u\|_{\mathcal{F}_{2,1}}^2. \blacksquare$$

Theorem 5.3. *Let $\eta > 0$ be fixed and let $T > 0$. Assume that $u \in C([0,T]; \mathcal{F}_{2,2}(\mathbb{R}))$ is the solution of (1.2). Then, $\hat{u}(t, 0) = 0$, for all $t \in [0,T]$.*

Proof: Multiplying (1.2) by x^2 we obtain

$$\partial_t(x^2 u) = -x^2 u \partial_x u - x^2 \partial_x^3 u - \eta x^2(\mathcal{H}\partial_x u + \mathcal{H}\partial_x^3 u). \tag{5.36}$$

By assumption $x^2 u(t) \in L^2(\mathbb{R})$, for all $t \in [0,T]$. Then, we have that

$$\|x^2 u \partial_x u\| \leq \|\partial_x u\|_{L^\infty} \|x^2 u\| \leq \|u\|_2 \|x^2 u\|, \tag{5.37}$$

and therefore $\gamma(t) := x^2(u\partial_x u)(t) \in L^2(\mathbb{R})$, for all $t \in [0,T]$. It follows easily that $\gamma \in C([0,T]; L^2(\mathbb{R}))$, in fact

$$\begin{aligned}
\|\gamma(t) - \gamma(t_0)\| &\leq \|x^2 u(t)\|\|\partial_x(u(t) - u(t_0))\|_{L^\infty} + \|\partial_x u(t_0)\|_{L^\infty}\|x^2(u(t) - u(t_0))\| \\
&\leq \|u(t)\|_{\mathcal{F}_{2,2}}\|u(t) - u(t_0)\|_2 + \|u(t_0)\|_2\|u(t) - u(t_0)\|_{\mathcal{F}_{2,2}}. \qquad (5.38)
\end{aligned}$$

Applying the Fourier transform in (5.36) we get

$$\partial_t \partial_\xi^2 \hat{u}(t, \xi) = \widehat{\gamma(t)}(\xi) + i\partial_\xi^2(\xi^3 \hat{u}(t, \xi)) - \eta \partial_\xi^2[h(\xi)(-\xi + \xi^3)\hat{u}(t, \xi)]. \tag{5.39}$$

Since $u(t) \in \mathcal{F}_{2,2}$ for all $t \in [0,T]$, it is easy to see that

$$\begin{aligned}\widehat{\beta(t)}(\xi) \;&:=\; \partial_\xi^2(\xi^3\hat{u}(t,\xi)) = 6\xi\hat{u}(t,\xi) + 6\xi^2\partial_\xi\hat{u}(t,\xi) + \xi^3\partial_\xi^2\hat{u}(t,\xi)\\ &\in C([0,T]; L^2_{-3}(\mathbb{R})).\end{aligned} \tag{5.40}$$

And similarly, we have that

$$\partial_\xi^2[h(\xi)(-\xi+\xi^3)\hat{u}(t,\xi)] = -2\delta(\xi)\hat{u}(t,0) + \widehat{\Gamma(t)}(\xi), \tag{5.41}$$

where

$$\begin{aligned}\widehat{\Gamma(t)}(\xi) \;&=\; -2h(\xi)\partial_\xi\hat{u}(t,\xi) - \xi h(\xi)\partial_\xi^2\hat{u}(t,\xi) + 6\xi h(\xi)\hat{u}(t,\xi) + 6\xi^2 h(\xi)\partial_\xi\hat{u}(t,\xi)\\ &+\xi^3 h(\xi)\partial_\xi^2\hat{u}(t,\xi) \in C([0,T]; L^2_{-3}(\mathbb{R})).\end{aligned} \tag{5.42}$$

From (5.39) - (5.41) we obtain

$$\partial_t\partial_\xi^2\hat{u}(t,\xi) = \widehat{\gamma(t)}(\xi) + i\widehat{\beta(t)}(\xi) + 2\eta\delta(\xi)\hat{u}(t,0) - \eta\widehat{\Gamma(t)}(\xi). \tag{5.43}$$

Integrating now (5.43) between 0 and t, we find that

$$2\eta\delta(\xi)\int_0^t \hat{u}(t',0)dt' \in C([0,T]; L^2_{-3}(\mathbb{R})). \tag{5.44}$$

The last expression implies that

$$\int_0^t \hat{u}(t',0)dt' = 0, \quad for \;\; all \;\; t \in [0,T], \tag{5.45}$$

and therefore $\hat{u}(t,0) = 0$, for all $t \in [0,T]$. ■

Theorem 5.4. *Let $\eta > 0$ be fixed and let $T > 0$. Assume that $u \in C([0,T]; \mathcal{F}_{3,3}(\mathbb{R}))$ is the solution of (1.2). Then, $u(t) = 0$, for all $t \in [0,T]$.*

Proof: Multiplying (1.2) by x^3 we obtain

$$\partial_t(x^3u) = -x^3u\partial_x u - x^3\partial_x^3 u - \eta x^3(\mathcal{H}\partial_x u + \mathcal{H}\partial_x^3 u). \tag{5.46}$$

By assumption $x^3u \in L^2(\mathbb{R})$. Then, we have that

$$\|x^3u\partial_x u\| \le \|\partial_x u\|_{L^\infty}\|x^3u\| \le \|u\|_2\|x^3u\|. \tag{5.47}$$

Then, $\gamma(t) = x^3(u\partial_x u)(t) \in L^2(\mathbb{R})$, for all $t \in [0,T]$. Similar to Theorem 5.3 we have that $\gamma \in C([0,T]; L^2(\mathbb{R}))$. Taking the Fourier transform in (5.46) we find that

$$\partial_t \partial_\xi^3 \hat{u}(t,\xi) = -i\widehat{\gamma(t)}(\xi) + i\partial_\xi^3(\xi^3 \hat{u}(t,\xi)) - \eta\partial_\xi^3(h(\xi)(-\xi+\xi^3)\hat{u}(t,\xi)). \tag{5.48}$$

We easily see that

$$\begin{aligned}\partial_\xi^3(\xi^3\hat{u}(t,\xi)) &= 6\hat{u}(t,\xi) + 18\xi\partial_\xi\hat{u}(t,\xi) + 9\xi^2\partial_\xi^2\hat{u}(t,\xi) + \xi^3\partial_\xi^3\hat{u}(t,\xi)\\ &\in C([0,T]; L^2_{-3}(\mathbb{R})),\end{aligned} \tag{5.49}$$

and

$$\begin{aligned}\partial_\xi^3(h(\xi)(-\xi+\xi^3)\hat{u}(t,\xi)) &= \partial_\xi\big(-2\delta(\xi)\hat{u}(t,0) - 2h(\xi)\partial_\xi\hat{u}(t,\xi) - \xi h(\xi)\partial_\xi^2\hat{u}(t,\xi)\\ &\quad +6\xi h(\xi)\hat{u}(t,\xi) + 6\xi^2 h(\xi)\partial_\xi\hat{u}(t,\xi) + \xi^3 h(\xi)\partial_\xi^2\hat{u}(t,\xi)\big)\\ &= -2\delta'(\xi)\hat{u}(t,0) - 4\delta(\xi)\partial_\xi\hat{u}(t,0) + \widehat{\Gamma(t)}(\xi),\end{aligned} \tag{5.50}$$

where

$$\begin{aligned}\widehat{\Gamma(t)}(\xi) &= -h(\xi)\partial_\xi^2\hat{u}(t,\xi) - \xi h(\xi)\partial_\xi^3\hat{u}(t,\xi) + 6h(\xi)\hat{u}(t,\xi)\\ &\quad +18\xi h(\xi)\partial_\xi\hat{u}(t,\xi) + 9\xi^2 h(\xi)\partial_\xi^2\hat{u}(t,\xi)\\ &\quad +\xi^3 h(\xi)\partial_\xi^3\hat{u}(t,\xi) \in C([0,T]; L^2_{-3}(\mathbb{R})).\end{aligned} \tag{5.51}$$

From (5.48) - (5.51) we get

$$\begin{aligned}-i\partial_t\partial_\xi^3\hat{u}(t,\xi) &= -\widehat{\gamma(t)}(\xi) + \partial_\xi^3(\xi^3\hat{u}(t,\xi)) + i\eta\widehat{\Gamma(t)}(\xi)\\ &\quad -2i\eta\delta'(\xi)\hat{u}(t,0) - 4i\eta\delta(\xi)\partial_\xi\hat{u}(t,0).\end{aligned} \tag{5.52}$$

Integrating (5.52) between 0 and t, we have that

$$-2i\eta\delta'(\xi)\int_0^t \hat{u}(t',0)dt' - 4i\eta\delta(\xi)\int_0^t \partial_\xi\hat{u}(t',0)dt' \in C([0,T]; L^2_{-3}(\mathbb{R})). \tag{5.53}$$

Then,

$$\int_0^t \partial_\xi\hat{u}(t',0)dt' = \int_0^t \hat{u}(t',0)dt' = 0,$$

for all $t \in [0,T]$. The last expression implies that

$$\partial_\xi\hat{u}(t,0) = \hat{u}(t,0) = 0, \tag{5.54}$$

for all $t \in [0,T]$.

On the other hand, we have that u satisfies the integral equation

$$u(t,\cdot) = E_\eta(t)\phi(\cdot) - \frac{1}{2}\int_0^t E_\eta(t-t')\partial_x u^2(t',\cdot)dt'. \tag{5.55}$$

Denoting $v := u^2$, $w := \partial_x v$ and taking the Fourier transform in (5.55) we get

$$\hat{u}(t,\xi) = F_\eta(t,\xi)\hat{\phi}(\xi) - \frac{1}{2}\int_0^t F_\eta(t-t',\xi)\hat{w}(t',\xi)dt'. \tag{5.56}$$

Derivating three times equation (5.56), we obtain

$$\begin{aligned}\partial_\xi^3\hat{u}(t,\xi) &= \partial_\xi\big(\partial_\xi^2 F_\eta(t,\xi)\hat{\phi}(\xi) + 2\partial_\xi F_\eta(t,\xi)\partial_\xi\hat{\phi}(\xi) + F_\eta(t,\xi)\partial_\xi^2\hat{\phi}(\xi)\\ &\quad -\frac{1}{2}\int_0^t \partial_\xi^2 F_\eta(t-t',\xi)\hat{w}(t',\xi)dt' - \int_0^t \partial_\xi F_\eta(t-t',\xi)\partial_\xi\hat{w}(t',\xi)dt'\\ &\quad -\frac{1}{2}\int_0^t F_\eta(t-t',\xi)\partial_\xi^2\hat{w}(t',\xi)dt'\big).\end{aligned}$$

Then,

$$\begin{aligned}\partial_\xi^3\hat{u}(t,\xi) &= \partial_\xi^3 F_\eta(t,\xi)\hat{\phi}(\xi) + 3\partial_\xi^2 F_\eta(t,\xi)\partial_\xi\hat{\phi}(\xi) + 3\partial_\xi F_\eta(t,\xi)\partial_\xi^2\hat{\phi}(\xi)\\ &\quad +F_\eta(t,\xi)\partial_\xi^3\hat{\phi}(\xi) - \frac{1}{2}\int_0^t \partial_\xi^3 F_\eta(t-t',\xi)\hat{w}(t',\xi)dt'\\ &\quad -\frac{3}{2}\int_0^t \partial_\xi^2 F_\eta(t-t',\xi)\partial_\xi\hat{w}(t',\xi)dt' - \frac{3}{2}\int_0^t \partial_\xi F_\eta(t-t',\xi)\partial_\xi^2\hat{w}(t',\xi)dt'\\ &\quad -\frac{1}{2}\int_0^t F_\eta(t-t',\xi)\partial_\xi^3\hat{w}(t',\xi)dt'.\end{aligned} \tag{5.57}$$

Since $\eta > 0$, $\phi, u, v \in \mathcal{F}_{3,3}$ and using Lemma 5.1, it follows easily that

$$\partial_\xi F_\eta(t,\xi)\partial_\xi^2\hat{\phi}(\xi) = F_\eta(t,\xi)t\big(3i\xi^2 - \eta h(\xi)(3\xi^2-1)\big)\partial_\xi^2\hat{\phi}(\xi) \quad \in C([0,T];L^2(\mathbb{R})) \tag{5.58}$$

and

$$F_\eta(t,\xi)\partial_\xi^3\hat{\phi}(\xi) \in C([0,T];L^2(\mathbb{R})). \tag{5.59}$$

On the other hand, we can easily see that

$$\partial_\xi^2\hat{w}(t',\xi) = i\big(2\partial_\xi\hat{v}(t'\xi) + \xi\partial_\xi^2\hat{v}(t',\xi)\big),$$

and

$$\partial_\xi^3 \hat{w}(t',\xi) = i\big(3\partial_\xi^2 \hat{v}(t',\xi) + \xi \partial_\xi^3 \hat{v}(t',\xi)\big).$$

Then,

$$-\frac{3}{2}\int_0^t \partial_\xi F_\eta(t-t',\xi)\partial_\xi^2 \hat{w}(t',\xi)dt' \in C([0,T];L^2(\mathbb{R})), \tag{5.60}$$

and

$$-\frac{1}{2}\int_0^t F_\eta(t-t',\xi)\partial_\xi^3 \hat{w}(t',\xi)dt' \in C([0,T];L^2(\mathbb{R})). \tag{5.61}$$

Similarly, we find that

$$\partial_\xi^3 F_\eta(t,\xi)\hat{\phi}(\xi) = f_1(t,\xi) + 2\eta t\delta'(\xi)\hat{\phi}(\xi), \tag{5.62}$$

where $f_1(t,\xi) \in C([0,T];L^2(\mathbb{R}))$.
By (5.58) - (5.62) and making similar considerations to the other terms in (5.57), we obtain from (5.57) that

$$\begin{aligned}\partial_\xi^3 \hat{u}(t,\xi) \;\;&=\;\; f(t,\xi) + 2\eta t\delta'(\xi)\hat{\phi}(\xi) + 6\eta t\delta(\xi)\partial_\xi\hat{\phi}(\xi) - \frac{1}{2}\int_0^t 2\eta(t-t')\delta'(\xi)\hat{w}(t',\xi)dt' \\ &\quad -\frac{3}{2}\int_0^t 2\eta(t-t')\delta(\xi)\partial_\xi \hat{w}(t',\xi)dt',\end{aligned} \tag{5.63}$$

where $f(\cdot,\xi) \in C([0,T];L^2(\mathbb{R}))$. Since

$$\delta(\xi)\partial_\xi\hat{\phi}(\xi) = \delta(\xi)\partial_\xi\hat{\phi}(0) = 0,$$

we have that

$$\begin{aligned}\partial_\xi^3 \hat{u}(t,\xi) \;\;&=\;\; f(t,\xi) + \big(2\eta t\hat{\phi}(\xi) - \eta\int_0^t (t-t')\hat{w}(t',\xi)dt'\big)\delta'(\xi) \\ &\quad -3\eta\int_0^t (t-t')\partial_\xi \hat{w}(t',0)dt'\delta(\xi).\end{aligned} \tag{5.64}$$

Since $\partial_\xi^3 \hat{u}(t,\xi)$ and $f(t,\xi)$ are measurable functions for all $t \in [0,T]$, it follows from equation (5.64) that

$$\int_0^t (t-t')\partial_\xi \hat{w}(t',0)dt' = 0, \text{ for all } t \in [0,T]. \tag{5.65}$$

Let $t \in [0,T]$. Since $u(t) \in \mathcal{F}_{3,3}$ it is easy to see that $xu(t) \in L_2^2(\mathbb{R})$. Then $\widehat{xu(t)} \in H^2(\mathbb{R})$ and therefore $xu(t,\cdot) \in L^1(\mathbb{R})$. On the other hand, we have that

$$\begin{aligned}\int |x\partial_x u(t)^2|dx &= 2\int |xu(t)\partial_x u(t)|dx \leq 2\|\partial_x u(t)\|_{L^\infty}\int |xu(t,x)|dx \\ &\leq 2\|u(t)\|_2\|xu(t,\cdot)\|_{L^1}. \end{aligned} \tag{5.66}$$

Then,

$$\begin{aligned}\partial_\xi \hat{w}(t,0) &= -\frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}} x\partial_x u^2(t,x)dx = \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}} u^2(t,x)dx \\ &= \frac{i}{\sqrt{2\pi}}\|u(t)\|^2. \end{aligned} \tag{5.67}$$

Combining (5.65) and (5.67) we get

$$\int_0^t (t-t')\|u(t')\|^2 dt' = 0, \text{ for all } t \in [0,T]. \tag{5.68}$$

It follows easily from (5.68) that $\|u(t)\| = 0$, for all $t \in [0,T]$ and this concludes the proof of the theorem. ■

Appendix

Kato's Inequalities:

1. Let $s > 3/2$. If u is real valued, then

$$|(u, uDu)_s| \leq C_s \|Du\|_{s-1} \|u\|_s^2. \tag{$K1$}$$

2. If $k \geq 2$ and u is real valued, then

$$|(u, uDu)_k| \leq C_k \|u\|_2 \|u\|_k^2. \tag{$K2$}$$

Gagliardo-Nirenberg's inequalities:

$$\|u\|_{L^\infty} \leq \|u\|^{1/2} \|Du\|^{1/2}, \quad u \in H^s(\mathbb{R}), \quad s \geq 1, \tag{$GN1$}$$

$$\|D^j u\| \leq C \|u\|^{1-j/m} \|D^m u\|^{j/m}, \quad 0 \leq j \leq m, \quad u \in H^m(\mathbb{R}). \tag{$GN2$}$$

Young's inequality:

Let $a, b \geq 0$, $\epsilon > 0$, $1/p + 1/p' = 1, 1 < p < \infty$. Then,

$$ab \leq \epsilon \cdot a^p + \epsilon^{-\frac{1}{p-1}} \cdot b^{p'}. \tag{Y}$$

References

[1] E. A. Alarcón, *The Cauchy problem for the generalized Ott-Sudan equation*, Doctoral Thesis, IMPA, 1990.

[2] E. A. Alarcón, *Existence and finite dimensionality of the global attractor for a class of nonlinear dissipative equations*, Proceedings of the Royal Society of Edinburgh, 123A, (1993), pp. 893-916.

[3] H. A. Biagioni, J. L. Bona, R. J. Iório Jr., M. Scialom, *On the Korteweg-de Vries-Kuramoto-Sivashinsky equation*, Advances in Differential Equations, Vol.1, Number 1, (1996), pp. 1-20.

[4] J. Bona and R. Scott, *Solutions of the Korteweg-de Vries Equation in Fractional Order Sobolev Spaces*, Duke Mathematical Journal, Vol 43, No. 1, (1976), pp. 87-99.

[5] Bao-Feng Feng, T. Kawahara, *Multi-hump stationary waves for a Korteweg-de Vries equation with nonlocal perturbations*, Physica D, 137, (2000), pp. 237-246.

[6] J. Duandikoetxea, *Análisis de Fourier*, Ediciones de la Universidad Autónoma de Madrid, 1991.

[7] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics 840, Springer Verlag, Berlin, 1981.

[8] R. J. Iório, Jr., *On the Cauchy Problem for the Benjamin-Ono Equation,* Comm. PDE, 11, (1986), pp. 1031-1081.

[9] R. J. Iório, Jr., *The Benjamin-Ono Equation in Weighted Sobolev Spaces,* J. Math. Anal. Appl., Vol.157, No. 2, (1991), pp. 577-590.

[10] R. J. Iório, Jr., *KdV, BO and Friends in Weighted Sobolev Spaces*, *Function-Analytic Methods for Partial Differential Equations*, Lecture Notes in Mathematics, Springer-Verlag, vol. 1450 (1990) pp. 105-121.

[11] R. J. Iório, Jr., Valèria de Magalhães Iório, *Fourier Analysis and Partial Differential Equations*, Cambridge studies in advanced mathematics, 70, (2001).

[12] T. Kato, *Nonstationary Flows of Viscous and Ideal Fluids in R^3*, Journal of functional Analysis, 9 (1972), pp. 296-305.

[13] T. Kato,*On the Cauchy Problem for the (Generalized) Korteweg-de Vries Equation*, Studies in Applied Mathematics, Advances in Mathematics Supplementary Studies, Vol. 8,(1983), pp. 93-128.

[14] T. Kato and H. Fujita, *On the non-stationary Navier-Stokes system*, Rend. Sem. Mat. Univ. Padova, 32 (1962), pp. 243-260.

[15] C. E. Kenig, G. Ponce, and L. Vega, *Well- posedness of the initial value problem for the Korteweg-de Vries Equation*, Journal of the American Mathematical Society, Vol. 4, Number 2, (1991), pp. 323-347.

[16] L.A. Ostrovsky, Yu. A. Stepanyants, L. Sh. Tsimring, *Radiation instability in a stratified shear flow*, Int. J. Non-Linear Mech., 19 (1984), pp. 151-161.