Instituto de Matemática Pura e Aplicada

Essays on Risk Regulation and on Affine Term Structure Models

A thesis submitted in partial fulfillment of the requirements for obtaining the degree of Doctor of Philosophy

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Abstract

This work is composed by two distinct parts. In Chapter 1 we analyze the problem of introduction of capital requirements to cover market and credit risks by a general equilibrium model. We start setting necessary conditions to the existency of equilibrium. Next, we study the social welfare problem in this economy and we determine conditions to the Pareto Efficiency. Finally, we assess the consequences about assets prices, forecast variances of the assets returns, bankrutcy probability of the financial institution and contagion due to such practice.

In the second part (Chapter 2) we investigate the incompleteness of the fixed income market in Brazil. Firstly, we study two classical no-arbitrage multi-factor models, namely the Gaussian model and the Cox-Ingerssoll-Ross model. The results indicate evidences that the bond market is unable to span the fixed income market as a whole. Next, we propose a model (a variation of Unspanned Stochastic Volatility model proposed by Collin-Dufresne & Goldstein, 2001) in which there are souces of uncertainties driving the short term rate volatility not captured by the bond market. This model is a promising candidate to reconcile theory with empirical findings.

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Por fim, sei que é desnecessário dizer mas gostaria de frisar que qualquer erro que por ventura permaneça nesse trabalho é de minha inteira responsabilidade. As pessoas acima listadas só contribuíram com os acertos.

Nos meus tempos de menino
Porém menino sonha demais
Menino sonha com coisas
Que a gente cresce e não vê jamais
Nelson Rufino

Lutei pelo justo, pelo bom e pelo melhor do mundo.
Olga Benário

Como campo de conhecimento a economia ainda carrega a sua deformação ancestral. A saber, a difícil imbricação entre o que é válido na realidade e o que realmente serve a interesses influentes e articulados.
John Kenneth Galbraith

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Chapter 1

Risk Regulation in a Financial Market

1.1 Introduction

Banking crises have continually occurred throughout time. Usually, due to the connection of banks to different economic branches, the bankruptcy of a financial institution is more detrimental to society than that of a nonfinancial one.

In an attempt to reduce the frequency and intensity of such crises, some regulations have been proposed for this sector[1]. The major regulation is certainly the Basel Agreement, which resulted from a process under the heading of the Basel Committee on Banking Supervision.

The Basel Committee was set up in 1974 under the auspices of the Bank for International Settlements (BIS) by the central banks of the G10 members[2]. The main aim of the Committee is to strengthen cooperation between financial supervisors. It should be highlighted that the Committee itself does not have any superior authority over governments, and therefore, its recommendations do not have legal force.

[1]There is no consensus agreement in the academic literature on the reasons for banking regulation. The two major reasons are: the risk of systemic crises and the inability of depositors to monitor banks. See Santos (2000) for a review of the literature on theoretical explanations for banking regulation.

[2]The current members of the Basel Committee are: Belgium, Canada, France, Germany, Italy, Japan, Luxembourg, Netherlands, Spain, Sweden, Switzerland, England, and the United States.

In December 1987, the Committee submitted a document to be considered by the member countries, establishing minimum capital requirements for credit risk. In July 1988, after approval by the G10 member countries, this document (known as the Basel Capital Accord or as the 1988 Basel Accord) was released to banks. Since then, the recommendations of this agreement have been gradually introduced not only into the member countries but also into all countries whose banks are internationally active.

Basically, the 1988 Basel Accord imposes a capital requirement of at least 8%[3] of the risk-adjusted asset, defined as the sum of asset positions multiplied by asset-specific risk weights.

The second step was to define criteria for capital requirements to account for market risk. So, in January 1996, the Amendment to the Capital Accord to Incorporate Market Risks (Basel Committee on Banking Supervision, 1996a)[4] set the minimum capital requirement for a financial institution as the sum of a capital charge to cover credit risk (at least 8% of the risk-adjusted asset) and another charge to cover market risk.

In order to meet the requests of the financial industry, the Basel Accord Amendment of 1996 allowed the use of internal or standard models to gauge market risk. In a standard model, the regulatory authority defines the criteria for minimum capital requirement that should be met by financial institutions. The internal model, however, gives banks the option to use their own risk measurement models to determine their capital charge. Nevertheless, in order to use this model, banks must fulfill a series of requirements. From the quantitative point of view, the 99% confidence interval Value-at-Risk (VaR)[5] over a 10-day horizon is used as the basis for calculating market risk. The capital requirements to cover market risk should be equal to the maximum between: (i) the average VaR on the previous 60 business days multiplied by a factor (known as multiplier), and (ii) the previous day's VaR. However,

[3] In Brazil, the capital requirement is 11% instead of 8%.

[4] For an overview on the Amendment to the Capital Accord to Incorporate Market Risk, see the Basel Committee on Banking Supervision (1996b).

[5] VaR is a risk metric proposed by the J.P. Morgan Bank in 1994 and represents the maximum loss to which a portfolio is subject for a given confidence interval and time horizon. For instance, a one-day 99% VaR of R$ 10 million means that there is only 1 in 100 chance of the portfolio loss to exceed R$ 10 million at the end of the next business day. Undeniably, the widespread use of VaR-based risk management models results from the fact that this risk metric is easy to interpret. For an overview of VaR, see, for instance, Duffie & Pan (1997).

as the factor is always larger than three[6], the value specified in item (i) is almost always larger than the value stipulated in item (ii).

Although Brazil is not a member of the Basel Committee, its banking system follows the principles established by the Basel Accord. In 1994, the National Monetary Council (CMN), through Resolution 2,099[7], took the first step towards adapting the Brazilian financial system to the international standards outlined by the Committee. This rule established that all financial institutions have to hold a minimum total capital equal to 11% of their risk-adjusted assets.

The rules for calculating the required net worth have been changing over time in order to increase their efficiency and to include several types of risks. Up to the end of 2005, financial institutions were required to allocate capital to cover the credit risks of their assets (with a different treatment for credit risk of swap agreements), to cover currency risks and gold investment risks and operations in Reais with fixed interest rates. However, there is yet no capital requirement for covering market risks related to stocks and commodities[8].

The capital charge necessary to cover credit risk follows a model that closely resembles the one proposed by the 1988 Basel Accord. On the other hand, the charge necessary to cover market risk includes two risk factors: (i) currency and gold, and (ii) fixed interest rates.

Coverage of currency risk follows the standard model. In short, capital requirement to cover currency risk corresponds to 50% of the net worth of operations involving gold and assets and liabilities denominated in foreign currency[9].

For the coverage of market risk of fixed interest rates, the Central Bank of Brazil (Bacen) used an intermediate approach between the standard and internal models[10]. The capital charge necessary to cover this type of risk is calculated according to the Committee's guidelines, i.e., the maximum

[6]In Brazil, the multiplication factor ranges between 1 and 3.

[7]Banking regulation rules in Brazil can be obtained from the Central Bank of Brazil (Bacen) website: <http://www.bacen.gov.br>.

[8]Bacen's Communiqué 12,746, dated December 9, 2004 sets the end of 2005 as the deadline for the regulating agency to lay down guidelines for capital requirement to cover market risk that are not provided by the current regulations.

[9]See Bacen's Circular 3,229 dated March 26, 2004

[10]Arcoverde (2000) gives an in-depth description of the method used by Bacen for the establishment of regulations for market risk of fixed interest rates.

value between the VaR over the previous 60 business days multiplied by a factor and the previous day's VaR. Nevertheless, the rule established by Bacen includes two different aspects: (i) the parameters for VaR calculation (covariance matrix of the assets) are stipulated by Bacen on a daily basis, and (ii) the VaR multiplier is a decreasing function of market volatility, i.e., the larger the market turbulence, the lower the multiplier[11].

Regardless of legal requirements, several financial institutions have recently adopted internal VaR-based models for market risk management. Most of this self-discipline process was a demand from stockholders and investors who were concerned with the increase of volatility in a globalized world and who wanted transparency in the management of their resources. Nowadays, even in emergent countries like Brazil, all banks with some market activity calculate their VaR on a daily basis.

The aim of the present study is to assess the economic implications of risk regulation by means of a general equilibrium model. The model used is similar to the one proposed by Daníelsson et al. (2004).

Firstly we analyze the welfare effects of the introduction of capital requirements to cover market risk. Surprisingly, we show that some institutions can be better in a regulated economy (i.e., an economy where all financial institutions must satisfy the risk constraint) than an unregulated economy (i.e., an economy where there are no risk limits). Next, in opposition to the academic consensus that VaR is not an appropriate risk measure, we will see that when it is used with regulatory purposes, it can reduces the financial fragility of the market (defined as the sum of the bankruptcy probabilities of all financial agents) and the contagion.

In Section 1.5 we analyze the effects of peculiar rules established by Bacen. To do this we implemented two changes in the above-mentioned model, namely: (i) the covariance matrix of the assets is defined by the regulating agency, and may therefore not match the market expectation and (ii) the multiplier associated with the VaR constraint is a decreasing function of market volatility. The purpose of the variable factor is to prevent high capital requirements after economic crises (see Arcoverde, 2000).

The consequences of the first change are often ambiguous and depend on how Bacen estimates the covariance matrix. Predicting the behavior of the economy is only possible under some special circumstances. For instance, if Bacen overestimates the volatility of all risky assets, then the negative effects

[11]See Bacen's Circular 2,972 dated March 23, 2000 as well as its Technical Note.

of VaR-based capital requirement shown in Daníelsson et al. (2004) (namely: decrease in equilibrium price and increase in volatility) are enhanced. This suggests that the internal model can be more accurate in estimating market risk than the standard model[12].

The advantage of the second change is that it guarantees equilibrium in critical situations, which does not occur in the original model proposed by Basel.

In the last Section of this chapter we address by a numerical example the economic implications of credit risk regulation[13]. Basically, we show that if the regulating agency improperly choose the risk weights then the financial fraglity can be higher in a regulated economy than an unregulated one.

1.2 Review of the Literature

After the implementation of VaR as the standard procedure for market risk management in the second half of the last decade, a wide range of academic studies (either empirical or theoretical) have been carried out in order to assess the economic consequences of such practice.

Basak & Shapiro (2001) investigated the implications of the investment decision problem when the trader is subject to an exogenous VaR limit. They showed that agents who suffer such restriction divide adverse states of nature into two classes: bad states and intermediate states. Since these agents are only concerned with the probability of loss and not with its magnitude, they opt to protect themselves against intemediate states of nature and become completely vulnerable to bad states. As a result, the expected loss, considering a loss occurred, is larger for the agents that manage risk by means of a VaR model than for those who do not manage the market risk at all. To control the magnitude of losses, the authors suggest the Expected Shortfall[14] as an alternative risk metric, i.e., the expected loss, considering there was a loss. With this metric, the undesirable effects of a VaR-based risk management are eliminated.

[12] Bacen's Communiqué 12,746 sets the end of 2007 as the deadline for the regulating agency to establish the eligibility criteria for the adoption of internal models for market risk and for the validation of these models.

[13] Jackson et al. (1999) review the literature on the effects of minimum capital requirements for credit risk, as established by the 1988 Basel Accord.

[14] For further information about Expected Shortfall, see Acerbi & Tasche (2002).

Daníelsson & Zigrand (2003) used a two-period economy with a continuum of financial institutions characterized by a constant absolute risk aversion coefficient and subject to a VaR constraint They showed that:

- Optimal risk sharing is impaired.
- If all financial institutions are regulated (i.e., if they must satisfy the risk constraint), an equilibrium might not exist.
- Volatility of positive-beta assets in a regulated economy is greater than in an unregulated economy (i.e., an economy where there are no risk limits).
- Prices of positive-beta assets in a regulated economy are lower than prices of assets in an unregulated economy.
- Liquidity in a regulated economy is smaller than in an unregulated economy.

Daníelsson et al. (2004) used a numerical simulation to extend the model proposed by Daníelsson & Zigrand (2003) to a multiperiod environment, and assessed the intensity of adverse impacts of VaR-based risk constraint.

Leippold et al. (2003) considered the implications of VaR-based risk management for a continuous-time economy with intermediate consumption, stochastic opportunity set and heterogeneous attitude towards risk. By using asymptotic approximation methods, they showed that VaR-based risk management can lead banks to increase their exposure to risk in highly volatile states of nature. However, the effects on volatility and expected asset return are ambiguous, depending on the dynamics of the model. On the other hand, the interest rate will always be lower and the Sharpe ratio will always be greater in a regulated economy.

Cuoco & Liu (2004) analyzed the dynamics of the investment and VaR reporting problems faced by financial institutions that are subjected to a VaR-based risk constraint, following the internal modeling approach, considering the effects of adverse selection and moral hazard. They showed that when institutions which regularly underreport its true VaR (the accuracy of the risk measurement model is checked by backtesting) are punished, internal models can be very effective not only in curbing portfolio risk, but also in inducing truthful revelation of this risk.

While many papers have discussed the financial investment problem with credit risk constraint, we are not aware of any that have actually modelled the equilibrium effects of this kind of risk regulation.

Kim & Santomero (1988) investigated the credit risk regulation problem utilizing a simple mean-variance model and concluded that the use of simple capital ratios regulation is an ineffective means to bound the insolvency of banks. As solution to this inefficiency, they derived explicitly the theoretically correct risk weights that minimizes the insolvency of banks and showed that the correct weigths are independent of the individual bank's preferences.

Rochet (1992) studied the consequences of capital regulations on the portfolio choices of banks and obtained results very similar to ones obtained by Kim & Santomero (1988). He showed that the optimal risk weigth must be proportional to the systemics risk of the assets (their betas).

Blum(1999) pointed out that, in a dynamic framework, a capital intertemporal effect can arise which leads to an increase in bank's risk. The key insight is that under binding capital requirements an additional unit of equity tomorrow is more valuable to a bank. If raising equity is excessively costly, the only possibility to increase equity tomorrow is to increase risk today.

1.3 The Basic Model

Consider a two period economy ($t = 0, 1$) according to proposed by Daníelsson & Zigrand (2003). At $t = 0$ agents (financial institutions) invest in $N + 1$ assets that mature at $t = 1$. The asset 0 is risk-free and yields payoff d_0. The risky assets are nonredundant and promise at $t = 1$ a payoff

$$\boldsymbol{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix},$$

that follow a Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

The price of asset i is denoted by q_i. The return on asset i is defined by

$$R_i \equiv \frac{d_i}{q_i}.$$

We follow common modelling practice by endowing financial institutions with their own utility functions (such as in Basak and Shapiro, 2001, for instance). There is a continuum of small agents characterized by a constant coefficient of absolute risk aversion (CARA) h. The population of agents is such that h is uniformly distributed on the interval $[\ell, 1]$. To guarantee that all agents are risk-averse, let us suppose that $\ell > 0$.

Let x^h and y_i^h be the number of units of the risk-free asset and of the risky asset i, respectively, held by financial institution h at $t = 0$. Then the wealth of agent h at time $t = 1$ is

$$W_1^h = d_0 x^h + \sum_i d_i y_i^h.$$

The agents choose the portfolio that maximizes the expected value of their wealth utility $u^h\left(W_1^h\right)$ subject to budget and risk constraints.

The time-zero wealth of an agent of type h comprises initial endowments in the risk-free asset, θ_0^h, as well in the risky assets, $\boldsymbol{\theta}^h = \left(\theta_1^h, \dots, \theta_N^h\right)'$.

The budget constraint of institution h at $t = 0$ is

$$q_0 x^h + \sum_i q_i y_i^h \leq W_0^h,$$

where $W_0^h = q_0\theta_0^h + \sum_i q_i\theta_i^h$ is the initial wealth of agent h.

The role of the regulating agency consists to limiting the set of investments opportunities in the risky assets. That is, the regulating agency introduces a new constraint (hereafter, denominated risk constraint) that can be written as

$$\boldsymbol{y}^h \in \Upsilon, \quad \forall h \in [\ell, 1], \tag{1.1}$$

for some $\Upsilon \subseteq \mathbb{R}^N$. Of course, the regulating agency's aim is to choose Υ so as to minimize the financial fragility of the market, damaging as little as possible the economy.

The investment problem of financial institution h is[15]

$$\begin{array}{cc} \underset{\left(x^h, \boldsymbol{y}^h\right)}{\text{Max}} & \mathcal{E}\left(u^h\left(W_1^h\right)\right) \\ \text{s.a.} & q_0 x^h + \sum_{i=1}^N q_i y_i^h \leq q_0\theta_0^h + \sum_{i=1}^N q_i\theta_i^h, \\ & \boldsymbol{y}^h \in \Upsilon \end{array}$$

[15]Hereafter, when there isn't any doubt about which we want to tell, for $x \in \mathbb{R}$ and $\boldsymbol{y} \in \mathbb{R}^N$ we write simply $(x, \boldsymbol{y})$ on the contrary $(x, \boldsymbol{y}')$.

As the budget constraint is homogeneous of degree zero in prices, we can normalize, without loss of generality, the price of risk-free asset to $q_0 = 1$. Moreover, since u^h is strictly increasing, the budget constraint must be bind. The next lemma is a direct consequence of the properties of a continuous function defined on compact set.

Lemma 1.1 *If Υ is compact and convex then the problem of financial institution has only one solution.*

A competitive equilibrium for the economy in question is an asset price vector $(q_0, \boldsymbol{q}) = (q_0, q_1, \ldots, q_N)$ and a mapping $h \in [\ell, 1] \mapsto \left(x^h, \boldsymbol{y}^h\right)$, such that

1. $\left(x^h, \boldsymbol{y}^h\right)$ solves the problem of financial institution h when assets prices are equal to $(q_0, \boldsymbol{q}')$.
2. Market clearing, that is, $\int_\ell^1 \boldsymbol{y}^h \mathrm{d}h = \boldsymbol{\theta}$ and $\int_\ell^1 x^h \mathrm{d}h = \theta_0$, where $\boldsymbol{\theta} = \int_\ell^1 \boldsymbol{\theta}^h \mathrm{d}h$ is the aggregate amount of risky assets and $\theta_0 = \int_\ell^1 \theta_0^h \mathrm{d}h$ is the aggregate amount of risk-free asset.

1.4 Market Risk - Homogeneous Beliefs

First we suppose that the regulating agency and the financial institutions agree with the parameters (mean and covariace matrix) of the risky assets distribution. Hereafter, we will call this assumption homogeneous belief[16]. Note that homogeneous beliefs is similar to adoption of internal model by the regulating agency.

To limit market risk of financial institutions, the regulating agency follows the standard methodology known as Value-at-Risk (VaR). VaR is defined by

$$VaR^\alpha \equiv -\inf\left\{x \in \mathbb{R}; \mathcal{P}\left[W_1^h - \mathcal{E}\left(W_1^h\right) \leq x\right] > \alpha\right\}, \tag{1.2}$$

where $\mathcal{P}$ is the probability measure corresponding to risky assets payoff distribution, $\mathcal{E}$ is the expected value relative to this measure and α is the significance level adopted (the probability of losses exceeding the VaR)[17]. The

[16]Moreover, we suppose (although it isn't necessary to the results of this Section) that the regulating agency and financial institutions not only agree with the risky assets distribution but also they know this distribution perfectly.

[17]VaR when defined by Equation 1.2 is known as relative VaR, while the absolute VaR is defined as $VaR^\alpha = -\inf\left\{x \in \mathbb{R}; \mathcal{P}\left[W_1^h \leq x\right] > \alpha\right\}$ (see Jorion, 2001).

risk constraint is fixed as a uniform upper bound to VaR, that is,

$$VaR^{\alpha} \leq \overline{VaR}, \tag{1.3}$$

where $\overline{VaR}$ is a VaR exogenous bound set by the regulating agency that depends on a market volatility index. By using normal distribution properties, the risk constraint can be rewritten as an exogenous upper limit for the portfolio variance

$$\Upsilon = \left\{ \boldsymbol{y} \in \mathbb{R}^N ; \boldsymbol{y}' \boldsymbol{\Sigma} \boldsymbol{y} \leq \nu \right\}, \tag{1.4}$$

where the paramater ν, called nonseverity of the risk constrain, depends on α and $\overline{VaR}$.

The next proposition characaterizes the solution of the problem of financial institutions. The demonstration of this proposition can be found in Daníelsson & Zigrand (2003).

Proposition 1.1 *Let* $\left(x^h, \boldsymbol{y}^h\right)$ *be the solution of the problem of financial institution* h *when the price vector of risky assets is* $\boldsymbol{q}$. *We have:*

1. *If* $h \geq \sqrt{\frac{\rho}{\nu}}$ *then*

$$\boldsymbol{y}^h = \frac{1}{h} \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\mu} - r_0 \boldsymbol{q} \right), \tag{1.5}$$

where $\rho = \left(\boldsymbol{\mu} - r_0 \boldsymbol{q} \right)' \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\mu} - r_0 \boldsymbol{q} \right)$ *and* r_0 *is the risk-free rate.*

2. *If* $h < \sqrt{\frac{\rho}{\nu}}$ *then*

$$\boldsymbol{y}^h = \sqrt{\frac{\nu}{\rho}} \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\mu} - r_0 \boldsymbol{q} \right). \tag{1.6}$$

In any case $x^h = \theta_0^h + \sum_i q_i \theta_i^h - \sum_i q_i y_i^h$.

Note that the introduction of the risk constraint prevents optimal risk sharing since all institutions with CARA less than or equal to $\sqrt{\frac{\rho}{\nu}}$ choose the same portfolio.

After solving the problem of the financial institutions, the market clearing condition automatically provides the equilibrium prices, as presented in the following proposition (again, the demonstration is in Daníelsson & Zigrand, 2003):

Proposition 1.2 *Suppose that $R_i > r_0$ for all $i = 1, \ldots, N$. Then, the equilibrium price vector of risky assets is*

$$\boldsymbol{q} = \frac{1}{r_0} \left(\boldsymbol{\mu} - \Psi \boldsymbol{\Sigma} \boldsymbol{\theta} \right), \tag{1.7}$$

where Ψ is the market price of risk scalar (see Daníelsson & Zigrand, 2003). Denoting by $F(\cdot)$ the non-principal branch of the Lambert correspondence[18]*, we have*

$$\Psi = \begin{cases} \frac{1}{\ln \ell^{-1}} & \text{if} \quad 0 \leq \kappa \leq \ell \ln \ell^{-1} \\ -\frac{\kappa+\ell}{\kappa F(-(\kappa+\ell)e^{-1})} & \text{if} \quad \ell \ln \ell^{-1} < \kappa < 1-\ell \\ \text{any number} \geq \frac{1}{1-\ell} & \text{if} \quad \kappa = 1-\ell, \end{cases} \tag{1.8}$$

where

$$\kappa = \sqrt{\frac{\boldsymbol{\theta}' \boldsymbol{\Sigma} \boldsymbol{\theta}}{\nu}}.$$

An equilibrium fails to exist if $\kappa > 1 - \ell$.

Figure 1.1 illustrates Ψ as a function of κ. When $\kappa = 1-\ell$ the equilibrium is undetermined. If exists equilibrium and at least one institution hits the risk constraint then $\ell \ln \ell^{-1} < \kappa < 1 - \ell$, hence Ψ is a strictly increasing function of κ and so a strictly decreasing function of ν. This implies that tighter is the regulation (that is, smaller is ν) lesser will be the risky assets equilibrium prices.

1.4.1 Pareto Efficiency and Welfare of Financial Institutions

A classic and fundamental question in the economic theory is to determine if the equilibrium allocations are or not Pareto efficient. That is, if the economy is in equilibrium, is it possible, using only the initial endowments, to reorganize the distribution of assets such as makes some agent better withuot making some another agent worse? If the answer is positive, then

[18] The non-principal branch of the Lambert correspondence is the inverse of the function $f : (-\infty, -1] \mapsto \left[-e^{-1}, 0\right)$ defined by $f(x) = xe^x$. For more details and properties of the Lambert correspondence see Corless et al (1996).

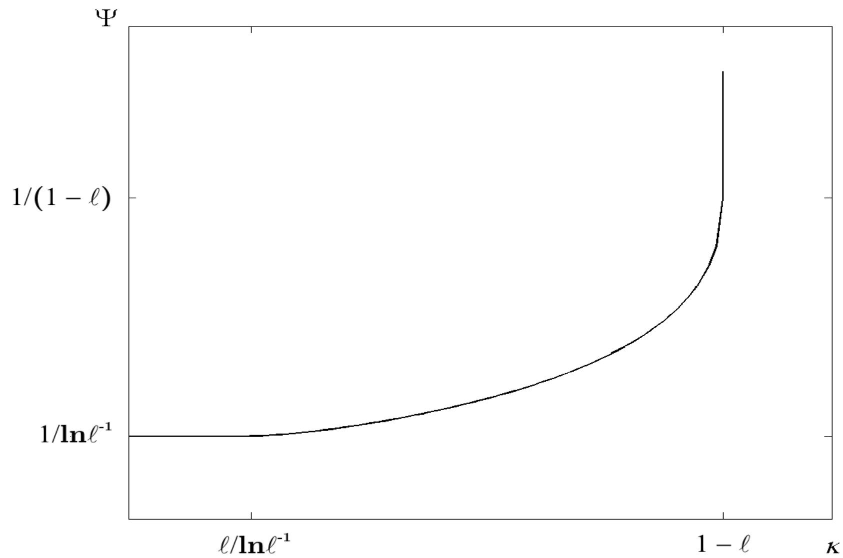

Figure 1.1: Illustration of Ψ.

the equilibrium is not efficient. In this Section we propose a definition of Pareto efficiency accordingly to the intuition explained above and show that for the economy analysed in Section 1.3, the equilibrium allocation complies this criterion.

Definition 1.1 *An allocation* $\left\{\left(x^h, \boldsymbol{y}^h\right)\right\}_{h\in[\ell,1]}$ *is feasible if* $\int_\ell^1 x^h\, dh \leq \theta_0$ *and* $\int_\ell^1 \boldsymbol{y}^h\, dh \leq \boldsymbol{\theta}$[19].

Definition 1.2 *A feasible allocation* $\left\{\left(x^h, \boldsymbol{y}^h\right)\right\}_{h\in[\ell,1]}$ *is Pareto efficiency if there is no other feasible allocation* $\left\{\left(\hat{x}^h, \hat{\boldsymbol{y}}^h\right)\right\}_{h\in[\ell,1]}$ *such as* $\mathcal{E}\left[u^h\left(\hat{x}^h, \hat{\boldsymbol{y}}^h\right)\right] \geq \mathcal{E}\left[u^h\left(x^h, \boldsymbol{y}^h\right)\right]$ *for all* h*, and strict inequality holds for* $h \in H \subseteq [\ell, 1]$ *with* $\mathcal{L}(H) > 0$*, where* $\mathcal{L}$ *is the Lebesgue measure on* $[\ell, 1]$*.*

The next proposition asserts that, in spite of the risk constraint to introduce a fricction on the market, the equilibrium, when it exist, is still efficient.

[19]If $x, y \in \mathbb{R}^N$ then $x \leq y$ means that $x_i \leq y_i$ for all i.

Proposition 1.3 *Suppose that exists equilibrium for the economy with VaR constraint and that* $\boldsymbol{y}^h \geq 0, \quad \forall h$*. Then the equilibrium allocation is Pareto efficient.*

To measure the financial institutions welfare we suppose that we have a linear-in-utilitily welfare function, also called Bergson welfare function (see Varian, 1992), which the weigth of each agent is equal to the inverse of your CARA. That is, we suppose that the regulating agency consider more important the financial institutions less risk averse. Of course, other schemes can be consider such as put the same weights for all institutions or else put the riskier averse institutions with higher weight.

Definition 1.3 *Let* $\left\{\left(x^h, \boldsymbol{y}^h\right)\right\}_{h\in[\ell,1]}$ *be an feasible allocation for the economy under analysis. We define the financial institutions welfare function by:*

$$\Lambda_f\left(\nu\right) = -\int_{\ell}^{1} \frac{\ln\left\{-\mathcal{E}\left[u^h\left(W_1^h\right)\right]\right\}}{h} \mathrm{d}h.$$

Proposition 1.4 *Suppose that for the economy considered here exists equilibrium and at least one financial institution hits the risk constraint, then the financial institutions welfare function is given by:*

$$\Lambda_f\left(\nu\right) = r_0\theta_0 + \mu\theta + \frac{\theta'\Sigma\theta}{4\kappa^2}\left[\left(\kappa\Psi\right)^2 - \left(\ell+\kappa\right)\kappa\Psi + \ell^2\right]$$

Proposition 1.5 *If equilibrium exists and at least one financial institution hits the risk constraint, the financial institutions welfare function is increasing in* ν*.*

Proposition 1.5 tells us that tigher is the risk regulation lower is the welfare of financial institutions as a whole. But, what happens at individual level? Would be possible for an agent to increase its utility in a regulated economy? Proposition 1.6 (below) states that, under certain conditions, the answer to the last question is positive. The intuition is immediate: At a regulated economy, agents less risk averse decrease their positions in riskier assets, then prices of these assets fall, tha makes interesting for other agents to buy them, thus increasing the agents' utility. Therefore, each financial institution maximizes its utility for a certain value of the nonseverity parameter that doesn't correspond necessarily to the situation of an unregulated economy ($\nu = \infty$). Before presenting Proposition 1.6 we are going to estabilish some preliminary calculations and notations.

Denote by $\overline{\nu}$ the maximum value of ν such as at least one institution hits the risk constraint and by $\underline{\nu}$ the lower value of ν for which exists equilibrium, in other words,

$$\overline{\nu} = \frac{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}{(\ell \ln \ell^{-1})^2} \quad \text{and} \quad \underline{\nu} = \frac{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}{(1-\ell)^2}.$$

Consider the following functions:

1. $g_1(\nu) : [\underline{\nu}, \overline{\nu}] \mapsto [\ell, 1]$, defined by $g_1(\nu) = \kappa\Psi + \kappa^3\Psi'(\kappa)\left(\frac{1}{1-\ell} - \frac{1}{\kappa}\right)$,
2. $g_2(\nu) : [\underline{\nu}, \overline{\nu}] \mapsto [\ell, 1]$, defined by $g_2(\nu) = \kappa\Psi$ and
3. $g_3(\nu) : [\underline{\nu}, \overline{\nu}] \mapsto \left[\frac{1-\ell}{\ln \ell^{-1}}, 1\right]$, defined by $g_3(\nu) = \Psi(1-\ell)$;

where $\Psi'(\kappa)$ is the derivate of Ψ, that is

$$\Psi'(\kappa) = \frac{1}{\kappa F\left(-(\kappa+\ell)\,\mathrm{e}^{-1}\right)}\left[\frac{\ell}{\kappa} + \frac{1}{F\left(-(\kappa+\ell)\,\mathrm{e}^{-1}\right)+1}\right].$$

It is easy to see that $g_1(\underline{\nu}) = g_2(\underline{\nu}) = g_3(\underline{\nu}) = 1$. Since κ, Ψ and Ψ' are strictly decreasing functions of ν[20] we have that g_1, g_2 and g_3 are strictly decreasing function of ν too. Figure 1.2 shows graphs of these three functions.

If we fix the market parameters ($\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$) then the welfare of financial institution h is given by its expected utility at $t = 1$:

$$\mathcal{E}\left(u^h\left(W_1^h\right)\right) = r_0\left(\theta_0^h + \boldsymbol{q}\boldsymbol{\theta}^h - \boldsymbol{q}\boldsymbol{y}^h\right) + \boldsymbol{\mu}\boldsymbol{y}^h - h\frac{\boldsymbol{y}^{h\prime}\Sigma\boldsymbol{y}^h}{2}.$$

Therefore, in equilibrium, the welfare of institution h depends on the nonseverity parameter ν. If the aggregate endowment of the risky assets is uniformly distributed between all agents (that is, $\boldsymbol{\theta}^h = \frac{\boldsymbol{\theta}}{1-\ell}$) then, after some algebraic manipulations, it is possible to show that to analyze the welfare of institution h as function of ν is equivalent to study the function $f^h(\nu) : [\underline{\nu}, \overline{\nu}] \mapsto \mathbb{R}$ defined by:

[20] Ψ' is a decreasing function of ν because $\Psi(\kappa)$ is a convexity function thus $\Psi''(\kappa) > 0$. Hence $\Psi'(\kappa)$ is increasing in κ and therefore decreasing in ν.

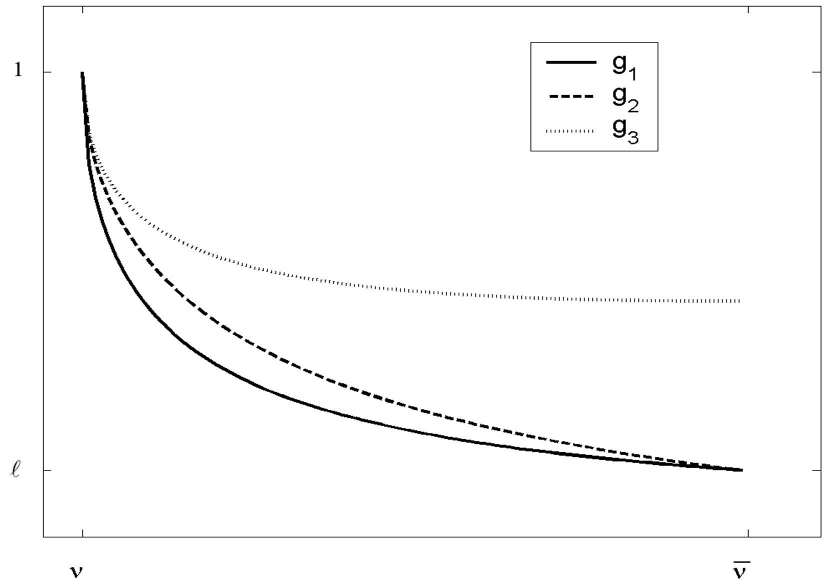

Figure 1.2: Graphs of functions g_1, g_2 and g_3.

$$f^h(\nu) = \begin{cases} \frac{\Psi^2}{2h} - \frac{\Psi}{1-\ell} & \text{if } \nu \geq g_2^{-1}(h) \\ \frac{\Psi}{\kappa} - \frac{h}{2\kappa^2} - \frac{\Psi}{1-\ell} & \text{if } \nu < g_2^{-1}(h) \end{cases} \tag{1.9}$$

Greater is $f^h(\nu)$ greater is the welfare of the institution h.

Now we are apt to present the main result of this Section.

Proposition 1.6 *Let $f^h(\nu)$ defined by Equation 1.9 then*

1. *For $\frac{1-\ell}{\ln \ell^{-1}} < h \leq 1$ we have*
   - *If $g_3^{-1}(h) < \nu \leq \overline{\nu}$ then $f^h(\nu)$ is strictly increasing.*
   - *If $g_2^{-1}(h) < \nu \leq g_3^{-1}(h)$ then $f^h(\nu)$ is strictly decreasing.*
   - *If $g_1^{-1}(h) < \nu \leq g_2^{-1}(h)$ then $f^h(\nu)$ is strictly decreasing.*
   - *If $\underline{\nu} < \nu \leq g_1^{-1}(h)$ then $f^h(\nu)$ is strictly increasing.*
2. *For $\ell \leq h \leq \frac{1-\ell}{\ln \ell^{-1}}$ we have*

- *If $g_2^{-1}(h) < \nu \le \overline{\nu}$ then $f^h(\nu)$ is strictly decreasing.*
- *If $g_1^{-1}(h) < \nu \le g_2^{-1}(h)$ then $f^h(\nu)$ is strictly decreasing.*
- *If $\underline{\nu} < \nu \le g_1^{-1}(h)$ then $f^h(\nu)$ is strictly increasing.*

In any case $f^h(\overline{\nu}) = \frac{1}{\ln \ell^{-1}} \left(\frac{1}{2h \ln \ell^{-1}} - \frac{1}{1-\ell} \right)$ and $f^h(\underline{\nu}) = -\frac{h}{2(1-\ell)^2}$

The next proposition shows that between the tighest level ($\nu = \underline{\nu}$) and the softest level ($\nu = \overline{\nu}$) of regulation, all the financial institutions prefer the last one.

Proposition 1.7 *For all h we have $f^h(\overline{\nu}) \ge f^h(\underline{\nu})$.*

By Proposition 1.7 we have that if $\ell \le h \le \frac{1-\ell}{\ln \ell^{-1}}$ then the maximum of $f^h(\nu)$ occurs when $\nu = g_1^{-1}(h)$. However, if $\frac{1-\ell}{\ln \ell^{-1}} < h \le 1$ there are two possible candidates for the maximum of $f^h(\nu)$: the same $g_1^{-1}(h)$ or $\overline{\nu}$. The next proposition gives conditions that allow us to decide in which of these points the function $f^h(\nu)$ assumes its maximum.

Proposition 1.8 *Let $t(h) : \left[\frac{1-\ell}{\ln \ell^{-1}}, 1\right] \mapsto \mathbb{R}$ defined by*

$$t(h) = \frac{\Psi}{\kappa} - \frac{h}{2\kappa^2} - \frac{\Psi}{1-\ell} - \frac{1}{\ln \ell^{-1}} \left(\frac{1}{2h \ln \ell^{-1}} - \frac{1}{1-\ell} \right),$$

where κ and Ψ are calculated at $\nu = g_1^{-1}(h)$. The function $t(h)$ is strictly decreasing and has only one root. Denoting by h^* this root we have

1. *If $\frac{1-\ell}{\ln \ell^{-1}} \le h \le h^*$ then the maximum of $f^h(\nu)$ occurs when $\nu = g_1^{-1}(h)$.*
2. *If $h^* \le h \le 1$ then the maximum of $f^h(\nu)$ occurs when $\nu = \overline{\nu}$.*

Figures 1.3, 1.4 and 1.5 illustrate the graphs of $f^h(\nu)$ for $h \in \left[\ell, \frac{1-\ell}{\ln \ell^{-1}}\right]$, $h \in \left[\frac{1-\ell}{\ln \ell^{-1}}, h^*\right]$ and $h \in [h^*, 1]$, respectively.

Observe that if the institution is sufficiently little risk averse (that is, $h > h^*$) then the best to this institution is that does not have capital requirement for market risk covering ($\nu \ge \overline{\nu}$)

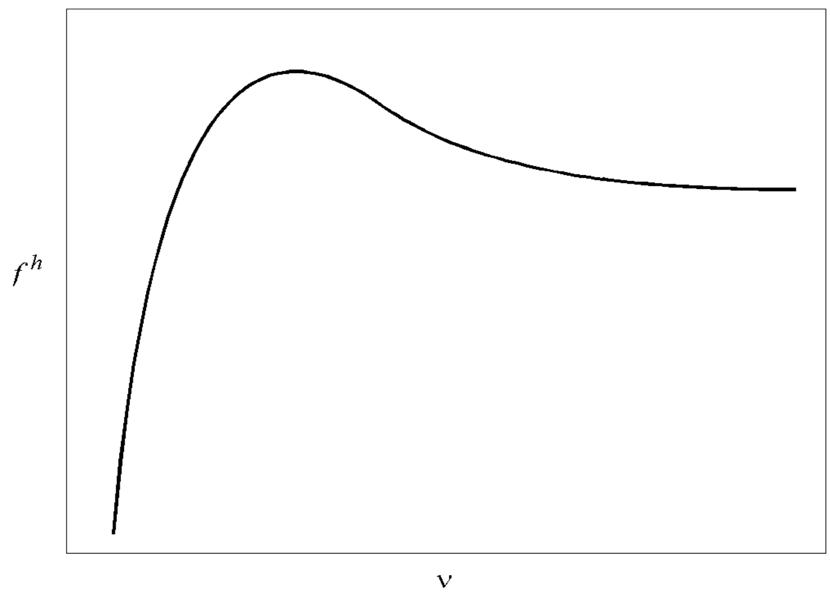

Figure 1.3: Function f^h for $h \in \left[\ell, \frac{1-\ell}{\ln \ell^{-1}}\right]$.

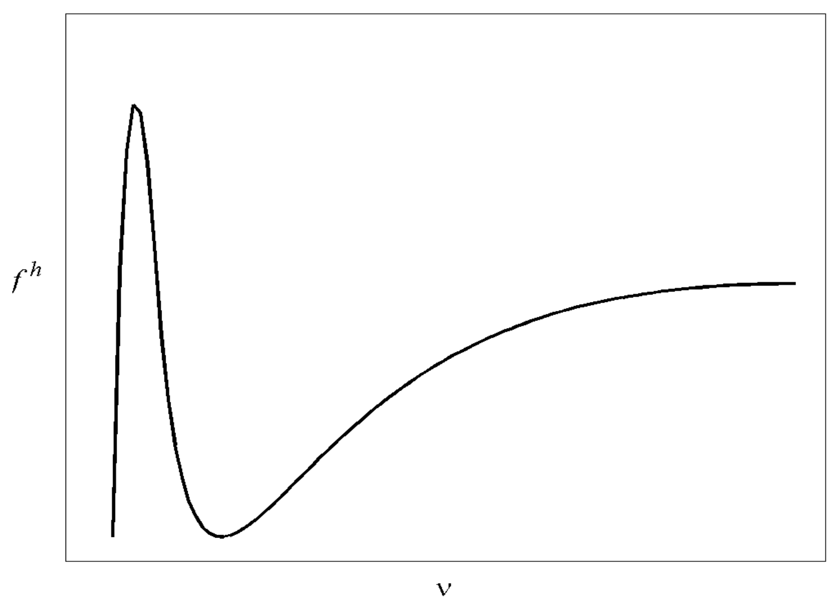

Figure 1.4: Function f^h for $h \in \left[\frac{1-\ell}{\ln \ell^{-1}}, h^*\right]$.

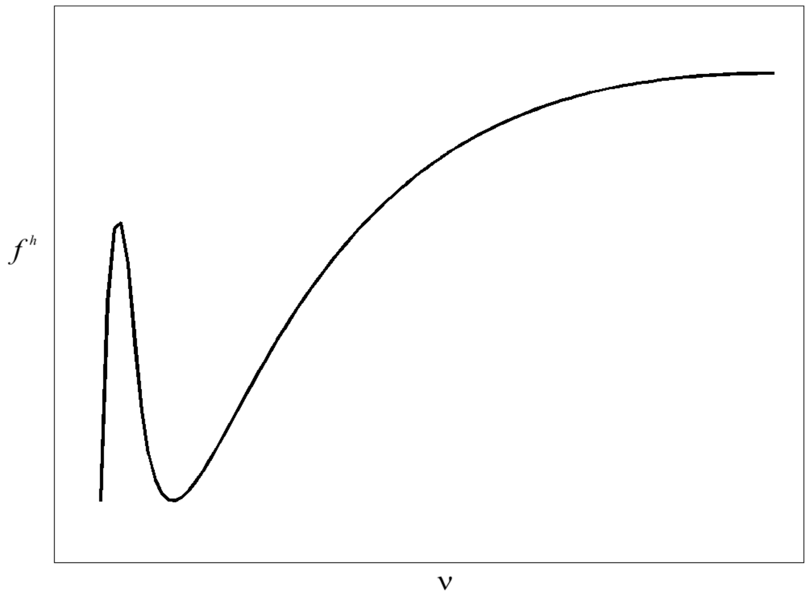

Figure 1.5: Function f^h for $h \in [h^*, 1]$.

1.4.2 Bankruptcy Probability

The financial institution h goes to bankrupty if its wealth at $t = 1$ is less or equal zero. If equilibrium exists and at least one institution reaches the risk constraint the probability of this to occur is

$$pb^h \equiv \mathcal{P}\left[W_1^h < 0\right] = \Phi\left(-\frac{m^h}{s^h}\right),$$

where $m^h = r_0 W_0^h + \Psi \boldsymbol{\theta}' \boldsymbol{\Sigma} y^h$ and $s^h = \sqrt{y^{h\prime} \boldsymbol{\Sigma} y^h}$ are, respectively, the mean and the standard deviation of W_1^h, and Φ represents the cumulative standard normal distribution function. Since Φ is strictly increasing, to analyze the behavior of pb^h as a function of the nonseverity parameter ν, it is enough to study how $\frac{m^h}{s^h}$ varies when the regulating agency modifies ν. Greater is this quotient, lower is the default probability of institution h. Using Propositions 1.1 and 1.2 it is easy to see that in equilibrium we have

1. If $h < g_2(\nu)$ then

$$\frac{m^h}{s^h} = \frac{\kappa r_0 W_0^h}{\sqrt{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}} + \Psi\sqrt{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}.$$

2. If $h \geq g_2(\nu)$ then

$$\frac{m^h}{s^h} = \frac{r_0 W_0^h h}{\Psi\sqrt{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}} + \Psi\sqrt{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}.$$

For the purpose of comparison, the value of this quotient in an unregulated economy is

$$\frac{m^h}{s^h} = \frac{r_0 W_0^h h \ln \ell^{-1}}{\sqrt{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}} + \frac{\sqrt{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}}{\ln \ell^{-1}} \quad \forall h.$$

Proposition 1.9 *Assume that equilibrium exists and at least one institution hits the risk constraint. Let* $\tilde{\nu}$ *be the nonseverity parameter value such as* $\Psi = \tilde{\Psi}$*, where* $\tilde{\Psi} \equiv \sqrt{\frac{h r_0 W_0^h}{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}}$*. That is, considering* Ψ *as function of* ν *we have* $\tilde{\nu} = \Psi^{-1}\left(\tilde{\Psi}\right)$ *(if* $\tilde{\Psi} \leq \frac{1}{\ln \ell^{-1}}$ *set* $\tilde{\nu} = \overline{\nu}$ *and if* $\tilde{\Psi} \geq \frac{1}{1-\ell}$ *set* $\tilde{\nu} = \underline{\nu}$*).*

1. *If* $\tilde{\nu} \leq g_2^{-1}(h)$ *then* $\frac{m^h}{s^h}$ *is a decreasing function of* ν *on the interval* $\left[\underline{\nu}, g_2^{-1}(h)\right]$ *and increasing on the interval* $\left[g_2^{-1}(h), \overline{\nu}\right]$*.*

2. *If* $g_2^{-1}(h) < \tilde{\nu} \leq \overline{\nu}$ *then* $\frac{m^h}{s^h}$ *is a decreasing function of* ν *on the interval* $[\underline{\nu}, \tilde{\nu}]$ *and increasing on the interval* $[\tilde{\nu}, \overline{\nu}]$*.*

3. *If* $\tilde{\nu} > \overline{\nu}$ *then* $\frac{m^h}{s^h}$ *is a decreasing function of* ν*.*

Proposition 1.9 gives interesting conclusions on the effectiveness of the risk regulation (effectiveness understood here as the reduction of the bankruptcy probability):

1. Greater is W_0^h less is $\tilde{\nu}$, then if the institution is highly capitalized, the regulation can increase its bankruptcy probability. On the other hand, if the net worth of an institution is small, then, from the regulating agency point of view, the regulation is always beneficial, since more severe it is, less is the default probability of the institution.

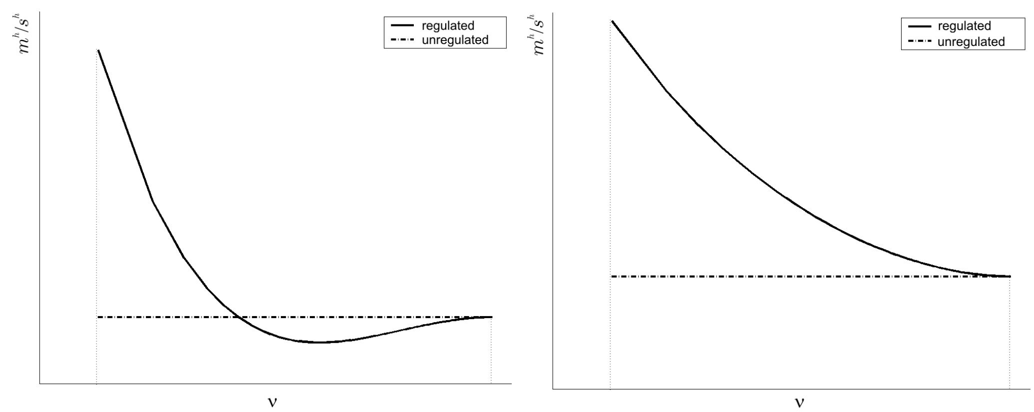

Figure 1.6: Graphs of the function $\frac{m^h}{s^h}$. At (a) $\tilde{\nu} \leq g_2^{-1}(h)$ and at (b) $\tilde{\nu} > \overline{\nu}$.

2. More nervous will be the market more effective will be the regulation.
3. The regulation is more effective for the institutions less risk averse (smaller h). If the institution will be super conservative then the regulation can increase its bankruptcy probability.

Figure 1.6 presents the graphs of $\frac{m^h}{s^h}$ (solid line) for cases 1 and 3 of Proposition 1.9. The horizontal dash-dot line represents the same relation in an unregulated economy.

Evidently, the regulating agency must consider the system as a whole and not an institution in particular. Therefore, it is interesting to analyze the total bankruptcy probability, defined as the sum (integral) of the default probability of all institutions,

$$pgb \equiv \int_{\ell}^{1} pb^h \mathrm{d}h. \tag{1.10}$$

Directly related (and more treatable from the algebraic point of view) with the metric defined by Equation 1.10 we have the integral in h of the quotient $\frac{m^h}{s^h}$,

$$\Lambda_s(\nu) \equiv \int_{\ell}^{1} \frac{m^h}{s^h} \mathrm{d}h. \tag{1.11}$$

If the initial endowment of the assets is uniformly distributed between the agents, then $W_0^h = W_0$ for all h. In this case

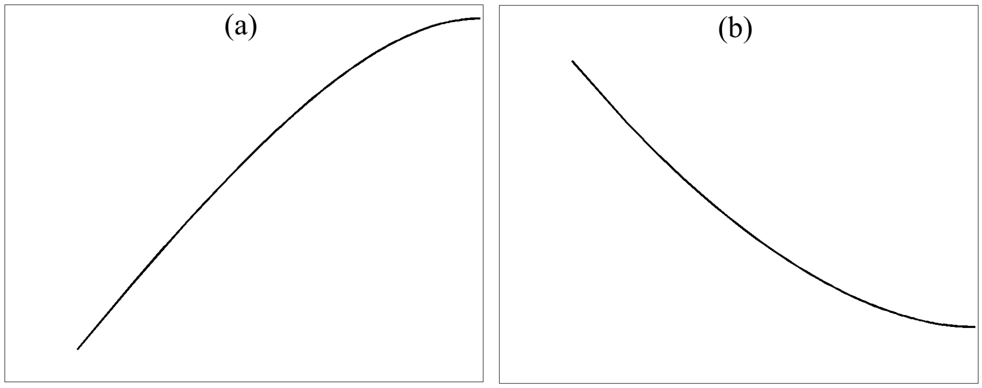

Figure 1.7: Graphs of the function Λ_s. In the letter (a) the level of capitalizaton of the financial institutions is high and in the letter (b) the opposite occurs.

$$\Lambda_s(\nu) = \frac{r_0 W_0}{\sqrt{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}}\left(\frac{\kappa^2\Psi}{2} + \frac{1}{2\Psi} - \kappa\ell\right) + \Psi(1-\ell)\sqrt{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}. \tag{1.12}$$

The first and the second terms of the left side of Equation 1.12 are, respectively, increasing and decreasing functions of ν. Then the phenomenon already observed individually happens again in global level: If the level of capitalization of the financial institutions is high or the degree of market nervousness is low, then the regulation can have contrary effect to the planned one (that is, to increase the financial fragility of the institutions). On the other hand, if the institutions have a small initial wealth or the market is nervous then the risk regulation presents the benefit to diminish the number of bankruptcies. Figure 1.7 shows these two situations.

1.4.3 Financial Market Contagion

In this section we analyze the problem of financial market contagion. Contagion is the transmission of shocks to other financial institutions, beyond any fundamental link among the institutions and beyond common shocks. Contagion can take place both during "good times" and "bad times". Then, contagion does not need to be related to crises. However, contagion has been emphasized during crisis times. Examples of recent contagious episodes are the Tequila crisis of 1994-95, the East Asian crisis of 1997 and the Russian crisis of 1998 (for details about these episodes see Kaminsky & Reinhart, 1998).

Some recent studies have addressed the problem of financial market contagion. Bae et al. (2000) propose an approach to measure contagion based on the co-incidence of extreme returns shocks across countries within a region. Bordo (2000) uses principal component analysis to assess the extent of comovement across all markets. Morris and Shin (2000) suggest an argument stemming from co-ordination failure and switching strategies that offers an explanation for some of the recent currency crises. Tsomocos (2003) characterises contagion and financial fragility as an equilibrium phenomenon.

Based on the model presented in Section 1.3 we develop a new approach to evaluate the contagion in an economic in which financial institutions are subject to market risk constraint. To introduce the possibility of contagion we increase the portfolios space of each financial institution allowing investments among them. To avoid an infinite dimensional optimization problem, instead of a continuum of financial institutions considered in the basic model we suppose that there is a finite number of them. Let's describe in more details a simple version of the contagion model where there is only three financial institutions. Generalizations of this particular case are immediate.

Consider a two period economy with three financial institutions A, B and C. There are two risky assets with payoff $\boldsymbol{d}$ normaly distributed. To become interesting investments of one financial institution in another one we have to introduce a friction on the market. There is many ways that this can be done. Here we opt to prevent that some financial institutions have access to all assets. More specifically, financial institution C can invest in both risky assets and in the risk-free assets. Its portfolio is (x_c, c_1, c_2), where x_c is number of units of the risk-free asset held by C, $\boldsymbol{c} = (c_1, c_2)'$ is the risky asset portfolio of C. The initial endowment of C is $(\theta_0^C, \theta_1^C, \theta_2^C)$. Financial institution B can invest in the risky asset 1, in the risk-free asset and in financial institution C. Its portfolio is (x_B, b_1, z_{BC}) where z_{BC} is the sharing of B in C and its initial endowment is (θ_0^B, θ_1^B). Finally, financial institution A can invest only in B and C and in the risk-free asset. Its portfolio is (x_A, z_{AB}, z_{AC}) where z_{AB} is the sharing of A in B and z_{AC} is the sharing of A in C and its initial endowment is θ_0^B. The CARA of these financial institutions are h_C, h_B and h_A, respectively. To avoid situations where a financial institution fully buy another institution we suppose $h_A \geq h_B \geq h_C$.

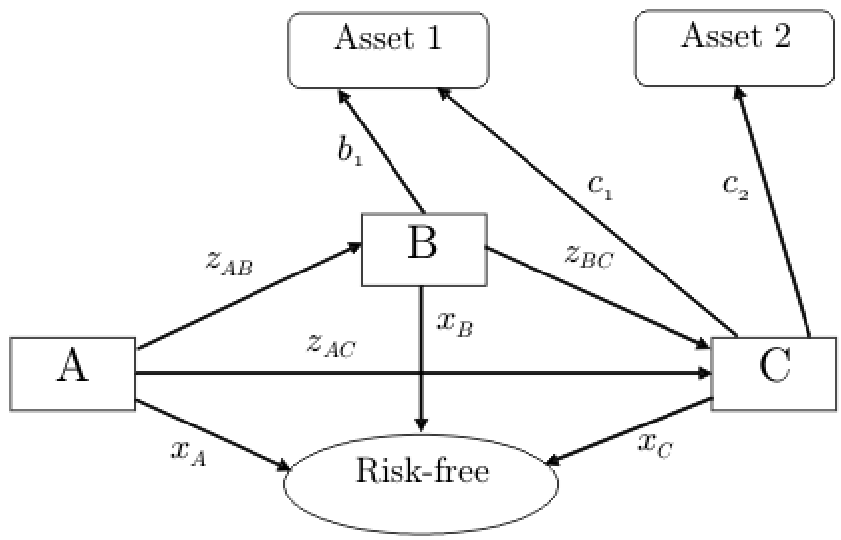

Figure 1.8: Financial market contagion model.

Figure 1.8 illustrate the model.

The wealths of institutions at $t = 1$ are

$$W_C = r_0 x_C + \boldsymbol{d} \cdot \boldsymbol{c},$$

$$\begin{aligned} W_B &= r_0 x_B + d_1 b_1 + z_{BC} W_C \\ &= r_0 x_B + z_{BC} r_0 x_C + \boldsymbol{d} \cdot \boldsymbol{\beta} \quad \text{and} \end{aligned}$$

$$\begin{aligned} W_A &= r_0 x_A + z_{AB} W_B + z_{AC} W_C \\ &= r_0 x_A + z_{AB} r_0 x_B + z_{AB} z_{BC} r_0 x_C + z_{AC} r_0 x_C + \boldsymbol{d} \cdot \boldsymbol{\alpha}, \end{aligned}$$

where

$$\boldsymbol{\beta} = \begin{pmatrix} b_1 + z_{BC} c_1 \\ z_{BC} c_2 \end{pmatrix}$$

and

$$\boldsymbol{\alpha} = \begin{pmatrix} z_{AB} \beta_1 + z_{AC} c_1 \\ z_{AB} \beta_2 + z_{AC} c_2 \end{pmatrix}$$

The budget constraints for A, B and C are respectively:

$$x_A + z_{AC}\tfrac{K_P^C}{1-z_{AB}-z_{BC}} + z_{AB}\tfrac{K_P^C}{1-z_{AB}} = \theta_0^A,$$

$$x_B + q_1 b_1 + z_{BC}\tfrac{K_P^C}{1-z_{AC}-z_{BC}} = \tfrac{K_P^B}{1-z_{AB}} \quad \text{and}$$

$$x_C + \boldsymbol{q}\cdot\boldsymbol{c} = \tfrac{K_P^C}{1-z_{AC}-z_{BC}},$$

where $K_P^C = \theta_0^C + \boldsymbol{q}\cdot\boldsymbol{\theta}$ and $K_P^B = \theta_0^B + q_1\theta_1^B$ are the equity capital of C and B, respectively.

The risk constraints for A, B and C are respectively:

$$\boldsymbol{c}'\boldsymbol{\Sigma}\boldsymbol{c} \le \nu,$$

$$\boldsymbol{\beta}'\boldsymbol{\Sigma}\boldsymbol{\beta} \le \nu \quad \text{and}$$

$$\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha} \le \nu.$$

Each institution maximize the expected value of its wealth utility subject to the budget and risk constraints. To solve the problem of financial institution we proceed in the same way that was done in Section 1.3. If the risky asset price is $\boldsymbol{q}$, then the optimum portfolios are:

1. If $h_C \ge \sqrt{\frac{\rho_C}{\nu}}$ then

$$\boldsymbol{c} = \frac{1}{h_C}\boldsymbol{\Sigma}^{-1}\boldsymbol{e}_C, \tag{1.13}$$

   where $\rho_C = \boldsymbol{e}_C'\boldsymbol{\Sigma}^{-1}\boldsymbol{e}_C$ and $\boldsymbol{e}_C = (\boldsymbol{\mu} - r_0\boldsymbol{q})$.

2. If $h_C < \sqrt{\frac{\rho_C}{\nu}}$ then

$$\boldsymbol{c} = \sqrt{\frac{\nu}{\rho_C}}\boldsymbol{\Sigma}^{-1}\boldsymbol{e}_C, \tag{1.14}$$

for financial institution C.

1. If $h_B \ge \sqrt{\frac{\rho_B}{\nu}}$ then

$$\boldsymbol{\beta} = \frac{1}{h_B}\boldsymbol{\Sigma}_B^{-1}\boldsymbol{e}_B, \tag{1.15}$$

where $\rho_B = \boldsymbol{e}'_B \left(\boldsymbol{B}\boldsymbol{\Sigma}\boldsymbol{B}'\right)^{-1} \boldsymbol{e}_B$, $\boldsymbol{e}_B = (\mu_1 - r_0 q_1, (\boldsymbol{\mu} - r_0 \boldsymbol{q}) \cdot \boldsymbol{c})'$, $\boldsymbol{\Sigma}_B = \boldsymbol{B}\boldsymbol{\Sigma}$ and

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ \\ c_1 & c_2 \end{bmatrix}.$$

2. If $h_B < \sqrt{\frac{\rho_B}{\nu}}$ then

$$\boldsymbol{\beta} = \sqrt{\frac{\nu}{\rho_B}} \boldsymbol{\Sigma}_B^{-1} \boldsymbol{e}_B, \tag{1.16}$$

for financial institution B.

1. If $h_A \geq \sqrt{\frac{\rho_A}{\nu}}$ then

$$\boldsymbol{\beta} = \frac{1}{h_A} \boldsymbol{\Sigma}_A^{-1} \boldsymbol{e}_A, \tag{1.17}$$

where $\rho_A = \boldsymbol{e}'_A \left(\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}'\right)^{-1} \boldsymbol{e}_A$,
$\boldsymbol{e}_A = ((\mu_1 - r_0 q_1) b_1 + z_{BC}(\boldsymbol{\mu} - r_0 \boldsymbol{q}) \cdot \boldsymbol{c}, (\boldsymbol{\mu} - r_0 \boldsymbol{q}) \cdot \boldsymbol{c})'$, $\boldsymbol{\Sigma}_A = \boldsymbol{A}\boldsymbol{\Sigma}$ and

$$\boldsymbol{A} = \begin{bmatrix} \beta_1 & \beta_2 \\ \\ c_1 & c_2 \end{bmatrix}.$$

2. If $h_A < \sqrt{\frac{\rho_A}{\nu}}$ then

$$\boldsymbol{\beta} = \sqrt{\frac{\nu}{\rho_A}} \boldsymbol{\Sigma}_A^{-1} \boldsymbol{e}_A, \tag{1.18}$$

for financial institution A.

To find the equilibrium prices we have to use the market clearing condition:

$$\begin{aligned} c_1 + b_1 &= \theta_1 = \theta_1^B + \theta_1^C \\ c_2 &= \theta_2 = \theta_2^C \end{aligned}$$

To solve this system we have to use numerical methods since that the system is non-linear and there isn't a close form solution.

Now we are ready to define metrics of contagion that allow us to evaluate the impact of risk regulation on the level of financial institutions contagion. Since W_A, W_B and W_C are normal with mean and variance known it is easy to compute the following probabilities:

$$p_A \equiv \mathcal{P}\left[W_A \leq 0\right],$$

$$p_B \equiv \mathcal{P}\left[W_B \leq 0\right] \quad \text{and}$$

$$p_C \equiv \mathcal{P}\left[W_C \leq 0\right].$$

Besides the probabilities above, to measure the contagion we need to compute conditional probabilities like $\mathcal{P}\left[W_i \leq 0 \cap W_j \leq 0\right]$ where $i, j = A, B, C$. Let D_{BC} be the region of the payoff plane such as

$$\boldsymbol{c} \cdot \boldsymbol{d} \leq -r_0 x_C$$

$$\boldsymbol{\beta} \cdot \boldsymbol{d} \leq -r_0 x_B - r_0 z_{BC} x_C$$

then

$$p_{BC} \equiv \mathcal{P}\left[W_B \leq 0 \cap W_C \leq 0\right] = \int_{D_{BC}} N\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right),$$

where $N\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$ is the density probability function of a bidimensional normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

Let D_{AC} be the region of the payoff plane such as

$$\boldsymbol{c} \cdot \boldsymbol{d} \leq -r_0 x_C$$

$$\boldsymbol{\alpha} \cdot \boldsymbol{d} \leq -r_0 x_A - r_0 z_{AB} x_B - r_0 (z_{AC} + z_{AB} z_{BC})$$

then

$$p_{AC} \equiv \mathcal{P}\left[W_A \leq 0 \cap W_C \leq 0\right] = \int_{D_{AC}} N\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right).$$

Finally, let D_{AB} be the region of the payoff plane such as

$$\boldsymbol{\beta} \cdot \boldsymbol{d} \leq -r_0 x_B - r_0 z_{BC} x_C$$

$$\boldsymbol{\alpha} \cdot \boldsymbol{d} \leq -r_0 x_A - r_0 z_{AB} x_B - r_0 (z_{AC} + z_{AB} z_{BC})$$

then

$$p_{AB} \equiv \mathcal{P}\left[W_A \leq 0 \cap W_B \leq 0\right] = \int_{D_{AB}} N\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right).$$

We define the contagion metric of institution i on institution j ($i > j$ in a lexicographic order $i, j = A, B, C$) by the bankruptcy probability of j conditional on the bankruptcy probability of i, that is

$$\mathcal{C}_{CB} = \mathcal{P}\left[W_B \leq 0 | \, W_C \leq 0\right] = \frac{p_{BC}}{p_C} \tag{1.19}$$

$$\mathcal{C}_{CA} = \mathcal{P}\left[W_A \leq 0 | \, W_C \leq 0\right] = \frac{p_{AC}}{p_C} \tag{1.20}$$

$$\mathcal{C}_{BA} = \mathcal{P}\left[W_A \leq 0 | \, W_B \leq 0\right] = \frac{p_{AB}}{p_B} \tag{1.21}$$

It is interesting to compare the probabilities above (that represent the probability of contagion) with the bankruptcy probability of an single institution. For example, if we compare $\mathcal{C}_{CB}$ with p_B we have an idea of the bankruptcy probability of B due to its own operations, that is, we have a notion of the bankruptcy probability of B not due to the contagion of C on B.

The contagion metrics $\mathcal{C}_{CB}$, $\mathcal{C}_{AC}$ and $\mathcal{C}_{AB}$ are increasing functions of the nonseverity parameter ν, i.e, tighter is the regulation smaller is the contagion. Figure 1.9 illutrates $\mathcal{C}_{CB}$ for $\ell = 0.0011$, $\boldsymbol{\theta} = (1.5, 0.9)'$, $\boldsymbol{\mu} = (1.5, 1.2)'$, $r_0 = 1.00013$, $h_A = 0.5$, $h_B = 0.4$, $h_C = 0.1$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.6 & 0.25 \\ 0.25 & 0.4 \end{bmatrix}$$

In Subsection 1.4.1 we show that for $h < h^*$ the financial institution h in an economy without contagion prefer that the regulation is fixed in a specific level $\nu < \overline{\nu}$. If $h > h^*$ then financial institutions h prefer no regulation (that is, $\nu \geq \overline{\nu}$). The intuition is very simple: to get benefit with the regulation these financial institutions would like a level of regulation tigher than $\underline{\nu}$, but in this case there isn't equilibrium. Since it is impossible, they have no gain with regulation hence they prefer $\nu = \overline{\nu}$. Figure 1.10 shows the optimum ν as a function of h.

The same question about the optimum level of regulation for each financial institution can be done to an economy with possibility of contagion.

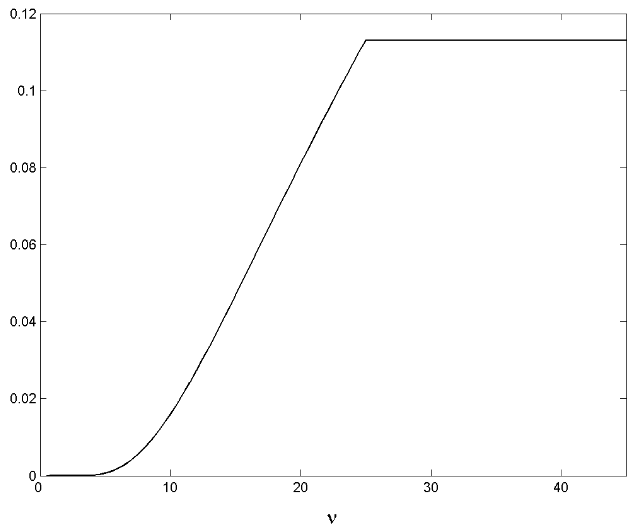

Figure 1.9: Contagion probability of C on B.

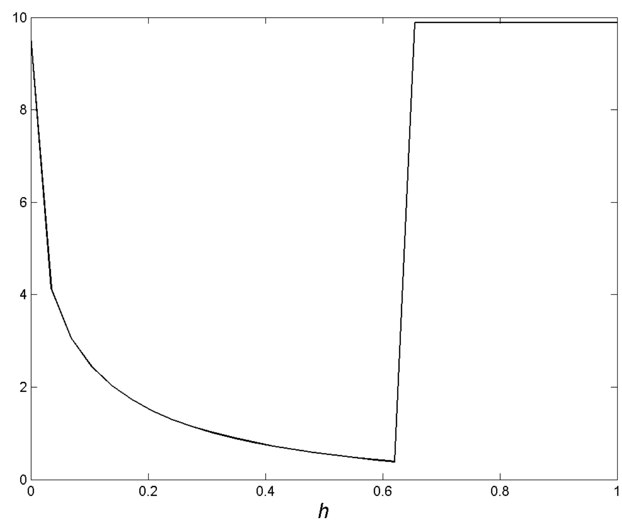

Figure 1.10: Optimum level of regulation (ν) as a function of h in an economy without contagion.

h_B	h_C	σ_1^2	σ_2^2	ν_B
0.4	0.1	0.8	0.2	1.12
0.4	0.1	0.6	0.4	1.88
0.4	0.1	0.2	0.6	2.43
0.3	0.26	0.2	0.25	4.57
0.27	0.26	0.01	0.25	7.93

Table 1.1: Diferrents values of ν_B

Let ν_h the value of the parameter of nonseverity that maximizes the utility function of financial institution h. That is,

$$\nu_h \in \text{Arg Máx } \mathcal{E}\left[u^h(W_h)\right], \tag{1.22}$$

where W_h is the wealth of financial institution h in an equilibrium allocation.

Of course, since financial institution C is the less risk averse one, it has no benefit with regulation, that is, $\nu_{h_C} = \infty$. For institutions B and C ν_h depends on factors like market conditions and the difference among the coefficients of risk aversion. Let's analyze in more details ν_{h_B} when these factors varies. The analyze of ν_{h_A} is very similar and we don't make it here.

On the one hand financial institution B prefers a tighter level of regulation. For example, since B invests on C, it would like that this institution doesn't take excessive risk to prevent that C go to bankrupt. Also, if asset 1 volatility is greater than asset 2 volatility then regulation is benefit to B since the only way that B has to invest on asset 2 is investing in C and the regulation can become the portfolio of C more concentrated on asset 2. On the other hand, it is possible, for example, that institution B wishes to invest a big amount of resources on asset 2 and the preferences of B and C are very similar (that is, $h_B \approx h_C$). But asset 2 is accessible only by institution C which has an upper limit on its investments in asset 2 and institution B has an upper limit on its investments in institution C. Then, in equilibrium, the number of units of asset 2 effectively hold by institution B can be smaller than in unregulated economy. In this case institution B prefer a softer level of regulation.

Table 1.1 shows values of ν_B as a function of h_B, h_C, asset 1 variance and asset 2 variance. The other model parameters are fixed and equals to the same values used in the exercise described in Figure 1.9.

The tighter level of regulation that B prefers occurs when the difference between h_B and h_C is big and asset 1 is much more volatile than asset 2. In this case financial institution B loves regulation because it becomes the portfolio of C concentrated on asset 2. But if the preferences of B and C are very similar and the asset 1 variance is very small then regulation is undesirable for financial institution B since it prejudices C and consequently B too.

Observe that in economy with contagion and incomplete asset structure, institution B wants that institution C has preferences very similar to its. But we showed in Section 1.3 that regulation can affect the effective degree of risk aversion. Then what B desires is that the C effective degree of risk aversion coincides with its.

1.4.4 The Regulating Agency Problem

The regulating agency is responsible for choosing of the regulation level (represented by the nonseverity parameter). The regulating agency would like to work with a regulation level that maximizes Λ_s. However, it cannot disdain the welfare of financial institutions. As we saw in Subsection 1.4.1 the welfare of financial institutions is represented by Λ_f. Thus, the problem of the regulating agency is possible to be described as choosing $\nu > \underline{\nu}$ that maximizes the following function:

$$\Lambda(\nu) \equiv \Lambda_f(\nu) + \lambda \Lambda_s(\nu), \tag{1.23}$$

where λ is a positive constant.

However, the solution of this type of problem has little value since we would have to arbitrate a λ and moreover we are adding utilities with probabilities. The most important is to note that the regulating agency must assume a commitment between:

1. To fix ν sufficently small in order to control the bankruptcy probability. Special attention must be given to the situation in which the level of capitalization of the financial institutions is low.

2. To fix ν sufficently large in order to not impact the financial institutions welfare.

A more realistic approach for the problem of risk regulation consists of extending the model considered here taking in account the existence of lobbies of financial institutions. In other words, we admit that the financial institutions try to persuade the regulating agency that their preferred positions would also serve the regulating agency's interests and perhaps those of the general public. So ν is endogenous and becomes only to be determined by equilibrium conditions. Let us see in general lines as such model can be specified[21].

A lobby can appear when the interests of a group are not perfectly lined up with the ones of the public power. The regulation (whatever may be) is a fertile land for the occurrence of lobbies. In the case of risk regulation the financial institutions have interest in maximizing its utility and the regulating agency desires to guarantee the soudness of the financial system. Not always these interests coincide perfectly.

In Subsection 1.4.2 we show that the regulating agency must consider information about the net worth of financial institutions (represented by W_0[22]) and the level of market nervousness (represented by σ). These variables constitute facts about the world and financial institutions have better information about them than the regulating agency. We assume that only one of these two variables is not perfectly known by the regulating agency and let us denote it by γ. The welfare of the regulating agency is represented by $G(\nu,\gamma)$. A reasonable choice for $G(\nu,\gamma)$ is $\lambda_s(\nu,\gamma)$.

We still assume the existence of only lobby group represented by the financial institutions h such as $h \in H$ where $H \subseteq [\ell, 1]$. The welfare of the lobbyist will be denoted by $U(\nu,\gamma)$ and represents the aggregation of the preferences of the members of the lobby group. Based on Definition 1.3 we can make

$$U(\nu,\gamma) = -\int_H \frac{\ln\left\{-\mathcal{E}\left[u^h\left(W_1^h\right)\right]\right\}}{h}\mathrm{d}h.$$

The lobbyist knows γ. In the first period he sends a message of a set of possibilities B to the regulating agency. The strategy of the lobbyist is described by a function $A(\gamma)$ such as when the lobbyist observes γ he communicates the message $b = A(\gamma)$ for some $b \in B$.

Initially, the regulating agency believes that γ is a random variable distributed in $[\gamma_{min}, \gamma_{max}]$ accordingly to the density $f_\gamma(\gamma)$. Let $\hat{a}(\gamma)$ be the strategy that the regulating agency suspects that is used by the lobbyist. Let $\hat{f}_\gamma\left(\cdot|b,\hat{A}\right)$ be the probability density of the distribution of γ after the regulating agency to interpret the lobbyist's message and to update his beliefs.

[21]For details on lobby theory see, for example, Grossman & Helpman (2002).

[22]Once more we are admitting that $W_0^h = W_0, \quad \forall h$.

Then the regulating agency chooses the regulation level such as

$$\nu^G\left(b|\hat{A}\right) = \underset{\nu}{\arg\max} \int G(\nu,\gamma)\,\hat{f}_\gamma\left(\cdot|b,\hat{A}\right)\mathrm{d}\gamma.$$

We assume that the lobbyist knows how the regulating agency will interpret his messages and also knows the regulating agency's preference. Then the optimun strategy of the lobbyist is constructed by finding, for each value of γ, the message $b \in B$ that maximizes $U\left[\nu^G\left(b|\hat{A}\right),\gamma\right]$. Moreover, in the equilibrium we must have $\hat{a}(\gamma) = a(\gamma)$.

It is easy to see that a full revelation equilibrium occurs if and only if

$$\underset{\nu}{\arg\max}\ U(\nu,\gamma) = \underset{\nu}{\arg\max}\ G(\nu,\gamma) \qquad \forall\gamma \in [\gamma_{min},\gamma_{max}].$$

Also it is easy to see that a babbling equilibrium always exists. A intermediate case and closer to a real situation consists of representing the equilibrium through partitions. In partitions equilibrium, the lobbyist chooses one of a finite number n of messages, say $b \in \{b_1,\ldots,b_n\}$. The regulating agency interprets the message b_i to mean that $\gamma_{i-1} \le \gamma \le \gamma_i$ for some set of numbers $\{\gamma_0,\gamma_1,\ldots,\gamma_n\}$ such as $\gamma_0 = \gamma_{min}$ and $\gamma_n = \gamma_{max}$. When the regulating agency hears the message b_i he updates his beliefs to exclude the possibility that γ lies outside the interval $\left[\gamma_{i-1},\gamma_i\right]$. After that, he chooses the regulation level $\nu(b_i)$ such as

$$\nu(b_i) = \underset{\nu}{\arg\max} \int_{\gamma_{i-1}}^{\gamma_i} G(\nu,\gamma)\,\hat{f}_\gamma(\gamma|b_i)\,\mathrm{d}\gamma.$$

In the problem of regulation just presented the welfare functions of the regulating agency and the lobby are very complex. Then to analyze the behavior of the economy in equilibrium, numerical procedures are necessary. For example, assume that there is only one risky asset with volatility σ. Also assume that the capital of the financial institutions is of public knowledge and take $\gamma = \sigma$. The lobbyist chooses between two messages: calm market and nervous market. In the first case, the regulating agency considers σ uniformly distributed on $[0.02, 0.025]$ and in the second on $[0.025, 0.03]$. The lobby is formed by the financial institutions such as $h \in [0.9, 1]$. In other words, the riskier averse institutions are the ones that are able to influence regulador[23]. If $W_0 = 2$ then, solving numerically the equilibrium problem, we find that the regulating agency chooses $\overline{\nu}$ as regulation level. This means that the regulation do not have some effects on the financial institutions.

This example portraies a common situation: the regulation often is fixed in a little severe level. In practical terms, few financial institutions worry about the risk regulated limit in their investment decisions.

Finally, it is necessary to emphasize that this regulated analysis with lobby is only one initial study, or better saying, a proposal for future studies. Undoubtly more general situations must be considered. For example, including others lobbies and also the cost of the lobbyist activity.

1.5 Market Risk - Heterogeneous Beliefs

In addition to the internal model, the 1996's Amendment of the Basel Accord gives the option of adopting a standard model. In this case, the capital allocation is done through a plan of weight factors of assets as a degree of risk established for the regulating agency. Brazil adopted a hybrid system concerning capital requirement for covering market risk of interests rates. The VaR is used as metric of risk, however, the distribution parameters of the assets returns are fixed by the regulating agency. This situation will be called heterogeneous beliefs.

On our model, heterogeneous beliefs cause an modification of the set Υ. The regulating agency considers that the risky assets payoff at $t = 1$ obeys a normal distribution with mean $\boldsymbol{m}$ and covariance matrix $\boldsymbol{S}$, that implies the following form for Υ:

$$\Upsilon = \{\boldsymbol{y}; \boldsymbol{y}'\boldsymbol{S}\boldsymbol{y} \leq \nu\}.$$

Since $\boldsymbol{S}$ is positive definite by Lemma 1.1, there is only one solution of the financial institution problem.

The hypothesis of heterogeneous beliefs introduces an additional difficulty in the solution of the financial institutions investment problem and consequently also in the problem to find equilibrium prices. We should therefore begin with a particular case. More specifically, we will first analyze an economy in which there exists only one risky asset. In this situation, the matrix

[23] This hypothesis are very common since the institutions riskier averse are in general the biggest one.

algebra consists of operations with real numbers, which substantially facilitate the calculations.

Proposition 1.10 *Consider an economy with only one risky asset, i.e., $N = 1$. Let (μ, σ) be the agents' beliefs about the mean and variance of the risky asset payoff, and θ the net supply of this asset. Let s_t be the expectation of the regulating agency about the variance of the risky asset. Suppose that $R > r_0$, where R is the risk asset return. The solution to the problem of the financial institution h, $\left(x^h, y^h\right)$, when the risky asset price is q, is given by:*

$$y^h = \begin{cases} \frac{1}{h}\sigma^{-1}\left(\mu - r_0 q\right) & \text{if } h \geq \sqrt{\frac{\rho}{\nu}} \\ \sqrt{\frac{\nu}{s}} & \text{if } h < \sqrt{\frac{\rho}{\nu}}, \end{cases} \tag{1.24}$$

where $\rho = \left(\mu - r_0 q\right)^2 \sigma^{-2} s$.

In any case, $x^h = W_0^h - q y^h$.

To finding equilibrium prices, it is necessary, one more time, to use the market clearing condition, as presented in the following proposition.

Proposition 1.11 *Under the same conditions of Proposition 1.10, the equilibrium price of the risky asset is*

$$q = \frac{1}{r_0}\left(\mu - \Psi\sigma\theta\right),$$

where Ψ is given by Equation 1.8 with $\kappa = \theta\sqrt{\frac{s}{\nu}}$. Once again, an equilibrium fails to exist if $\kappa > 1 - \ell$.

After analyzing this particular case, we will now turn our attention to a more general situation in which there are N risky assets. The propositions in the sequel characterize the solutions to the problems of the financial institutions and of finding equilibrium prices.

Proposition 1.12 *Let $\left(x^h, \boldsymbol{y}^h\right)$ be the solution of the problem of financial institution h when the price vector of risky assets is $\boldsymbol{q}$. Let $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ be agents' beliefs about the mean and covariance matrix of risky assets payoffs. We have:*

1. *If $h \geq \sqrt{\frac{\rho}{\nu}}$ then*

$$\boldsymbol{y}^h = \frac{1}{h}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_0\boldsymbol{q}\right), \tag{1.25}$$

where $\rho = \left(\boldsymbol{\mu} - r_0\boldsymbol{q}\right)' \boldsymbol{\Sigma}^{-1}\boldsymbol{S}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_0\boldsymbol{q}\right)$.

2. If $h < \sqrt{\frac{\rho}{\nu}}$ then

$$\boldsymbol{y}^h = \sqrt{\frac{\nu}{\rho}} \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r_0 \boldsymbol{q}) . \qquad (1.26)$$

In any case $x_t^h = W_0^h - \sum_i q_i y_i^h$.

Proposition 1.13 *Assume that $R_i > r_0$ for all i. Then, for the economy specified in this Section, the equilibrium price of the risky assets is*

$$\boldsymbol{q} = \frac{1}{r_0} (\boldsymbol{\mu} - \Psi \boldsymbol{\Sigma} \boldsymbol{\theta}) , \qquad (1.27)$$

where Ψ is given by Equation 1.8 with

$$\kappa = \sqrt{\frac{\boldsymbol{\theta}' \boldsymbol{S} \boldsymbol{\theta}}{\nu}}.$$

An equilibrium fails to exist if $\kappa > 1 - \ell$.

Proposition 1.13 is a generalization of the result obtained by Daníelsson & Zigrand (2003), who considered an economy with risk constraint in which the beliefs of the agents and of the regulating agency are the same, that is, $\boldsymbol{\Sigma} = \boldsymbol{S}$ and $\boldsymbol{\mu} = \boldsymbol{m}$ (see Proposition 1.1). Comparing this economy to the one presented herein, we can infer that the first one represents the regulation via internal models and that the second one is an intermediate approach between the standard model and the internal model adopted in Brazil. The following proposition compares the equilibrium prices in these two situations.

Proposition 1.14 *Consider again an economy with only one risky asset. Suppose also that $\theta > 0$. Then, if $s > \sigma$ and at least one financial institution hits the risk constraint, the risky asset price is higher when internal models, instead of the intermediate approach, are used.*

1.6 Economics Effects of the Market Risk Regulation - Analysis by Simulation

A simple and efficient way to observe the economics effects of a risk regulation consists of carrying through simulation financial market behavior. In this Section we will study an economy of infinite horizon that is a multi-period extension of the two period economy analyzed in Section 1.5. The next Subsection presents the model of infinite periods. Basically, we will set notation and adapt the previous results to this new situation.

1.6.1 The Infinite Horizon Model

Consider an infinite-period economy constructed by the sequence of two-period economies as proposed in Section 1.5. Time is discrete and indexed by $t \in T = \{0, 1, 2, \ldots\}$. At each period t the agents (financial institutions) invest in $N+1$ assets with maturity at $t+1$. Asset 0 is risk-free and yields payoff $d_{0,t+1}$ at $t+1$. The risky assets are non-redundant and promise to yield a payoff at $t+1$

$$\boldsymbol{d}_t = \begin{pmatrix} d_{1,t+1} \\ \vdots \\ d_{N,t+1} \end{pmatrix},$$

which conditioned on the information available up to t follows a Gaussian distribution.

Let q_{it} be the price of asset i at t. The return on asset i between periods t and $t+1$ is defined as

$$R_{i,t+1} \equiv \frac{d_{i,t+1}}{q_{it}}.$$

Let x_t^h and y_{it}^h be the number of units of the risk-free asset and of the risky asset i, respectively, held by financial institution h between periods t and $t+1$. Then the wealth of agent h at time $t+1$ is

$$W_{t+1}^h = d_{0,t+1} x_t^h + \sum_i d_{i,t+1} y_{it}^h.$$

Agents have a very short time horizon, so they choose the portfolio that maximizes the expected value of the wealth utility in the next period subject to budget and risk constraints.

Admit that there is a fixed, deterministic time-invariant net supply of θ_i units of the i^{th} risky asset. Let $\boldsymbol{\theta}$ be the vector that represents the aggregate endowments of the risky assets, i.e.,

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix}.$$

For reasons that will be clearer further ahead, the net supply of the risk-free asset depends on t and will be denoted by θ_{0t}.

Daníelsson & Zigrand (2003) showed that in this economy equilibrium prices depend only on aggregate endowment, no matter how this wealth is distributed between the agents. Therefore, we may suppose the new supply of risky assets at each period belongs to other individuals rather than to the financial institutions[24].

The budget constraint between periods t and $t+1$ is

$$q_{0t}x_t^h + \sum_i q_{it}y_{it}^h \leq d_{0t}x_{t-1}^h + \sum_i d_{it}y_{i,t-1}^h.$$

The regulating agency considers that the payoffs of the risky assets at $t+1$ conditioned on the information available at t follow a normal distribution with mean $\boldsymbol{m}_t$ and covariance matrix $\boldsymbol{S}_t$

$$\Upsilon_t = \left\{ \boldsymbol{y} \in \mathbb{R}^N ; \boldsymbol{y}' \boldsymbol{S}_t \boldsymbol{y} \leq \nu_t \right\}.$$

Observe that we are allowing that the risk constrain varies in the course of time. This situation reflects what it occurs in the Brazilian market, since the Central Bank of Brazil has the power to modify the multiplier.

The investment problem of the financial institution h at time t can be written as

$$\begin{array}{cc} \underset{\left(x_t^h, \boldsymbol{y}_t^h\right)}{\text{Max}} & \mathcal{E}_t\left(u^h\left(W_{t+1}^h\right)\right) \\ \text{s.a.} & q_{0t}x_t^h + \sum_{i=1}^N q_{it}y_{it}^h \leq d_{0t}x_{t-1}^h + \sum_{i=1}^N d_{it}y_{i,t-1}^h \\ & \boldsymbol{y}_t^{h\prime} \boldsymbol{S}_t \boldsymbol{y}_t^h \leq \nu_t, \end{array}$$

where $\mathcal{E}_t$ is the expected value with respect to the agents beliefs at t[25].

In each period t, an equilibrium for the economy in question is an asset price vector $(q_{0t}, q_{1t}, \ldots, q_{Nt}) = (q_{0t}, \boldsymbol{q}_t)$ and a mapping $h \in [\ell, 1] \mapsto \left(x_t^h, \boldsymbol{y}_t^h\right)$, such as

1. $\left(x_t^h, \boldsymbol{y}_t^h\right)$ solves the problem of financial institution h at time t when asset prices are equal to $(q_{0t}, \boldsymbol{q}_t)$.

2. Market clearing, that is, $\int_\ell^1 \boldsymbol{y}_t^h \mathrm{d}h = \boldsymbol{\theta}$ and $\int_\ell^1 x_t^h \mathrm{d}h = \theta_{0t}$.

[24] The same assumption applies to the supply θ_{0t} of the risk-free asset.

[25] All financial institutions expected the same behaviour of risky assets.

Propositions in the sequel characterize the solution of the financial institution problem and of finding the equilibrium price for the economy of infinite periods considered here. The demonstrations of them are similar to the demonstrations of the Propositions 1.12 and 1.13 presented in Section 1.5 and will not be done here.

Proposition 1.15 *Let $\left(x_t^h, \boldsymbol{y}_t^h\right)$ be the solution to the problem of financial institution h at time t when the price vector of risky assets is equal to $\boldsymbol{q}_t$. Let $(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$ be the agents' beliefs about the mean and covariance matrix of risky asset payoffs between t and $t+1$. We have:*

1. *If $h \geq \sqrt{\frac{\rho_t}{\nu_t}}$ then*

$$\boldsymbol{y}_t^h = \frac{1}{h} \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{\mu}_t - r_{0,t+1} \boldsymbol{q}_t\right), \tag{1.28}$$

where $\rho_t = \left(\boldsymbol{\mu}_t - r_{0,t+1} \boldsymbol{q}_t\right)' \boldsymbol{\Sigma}_t^{-1} \boldsymbol{S}_t \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{\mu}_t - r_{0,t+1} \boldsymbol{q}_t\right)$.

2. *If $h < \sqrt{\frac{\rho_t}{\nu_t}}$ then*

$$\boldsymbol{y}_t^h = \sqrt{\frac{\nu_t}{\rho_t}} \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{\mu}_t - r_{0,t+1} \boldsymbol{q}_t\right). \tag{1.29}$$

In any case $x_t^h = \frac{1}{q_{0t}} \left(d_{0t} x_{t-1}^h + \sum_i d_{it} y_{i,t-1}^h - \sum_i q_{it} y_{it}^h\right)$.

Proposition 1.16 *Suppose that $R_{i,t+1} > r_{0,t+1}$ for all i. Then, for the economy specified in this Section, the equilibrium price of risky assets at date t is*

$$\boldsymbol{q}_t = \frac{1}{r_{0,t+1}} \left(\boldsymbol{\mu}_t - \Psi_t \boldsymbol{\Sigma}_t \boldsymbol{\theta}\right), \tag{1.30}$$

where Ψ_t is given by Equation 1.8 with

$$\kappa_t = \sqrt{\frac{\boldsymbol{\theta}' \boldsymbol{S}_t \boldsymbol{\theta}}{\nu_t}}.$$

An equilibrium fails to exist if $\kappa_t > 1 - \ell$, for all t.

Intuitively, Proposition 1.16 shows that the regulating agency acts by changing the average effective risk aversion across all agents (Ψ). Observe that the variable non severity of the risk constraint guarantees the equilibrium in more general situations than in the case where $\overline{\nu}_t = \overline{\nu}$. For instance, in moments of crisis, $\boldsymbol{\theta}' \boldsymbol{S}_t \boldsymbol{\theta}$ tends to increase, however, since $\overline{\nu}_t$ is an increasing function of the market volatility, κ_t is kept under control.

For each t, the total bankruptcy probability is defined naturally as

$$pgb_t = \int_{\ell}^{1} pb_t^h \mathrm{d}h,$$

where pb_t^h is the default probability of financial institution h at $t+1$ conditioned to the information available at t. Hence,

$$pb_t^h = \mathcal{P}_t^d \left[W_{t+1}^h \leq 0 \right],$$

where $\mathcal{P}_t^d$ is the probability measure conditioned on the information available at t corresponding to the payoffs of risky assets distribution.

To compute pb_t^h it is necessary to know x_t^h. According to Proposition 1.15 x_t^h depends on q_{0t} which in equilibrium, can be obtained by Walras' Law:

$$q_{0t} = d_{0t} + \frac{(\boldsymbol{d}_t - \boldsymbol{q}_t)' \boldsymbol{\theta}}{\theta_{0t}}. \tag{1.31}$$

Analytically, it is possible to set θ_{0t} constant over time, as it is done to the net supply of risky assets. However, a computationally simpler procedure is to:

1. Keep the price of the risk-free asset constant and equal to 1 in the unregulated economy.

2. Calculate θ_{0t} using Equation 1.31.

3. Run the same simulation for the regulated economy with the θ_{0t} obtained in the previous item.

Note that this procedure is equivalent to the existence of a monetary authority that controls the supply of risk-free assets to keep the interest rate constant for the unregulated economy.

1.6.2 Dynamics of the Simulation

In order to our results become close to the reality, we admit that neither the financial institutions and the regulating agency know the distribution of asset payoff[26]. Soon they must forecast the mean and the covariance matrix of this distribution in each time t. Therefore, the key issue to the simulation of the economy specified in the previous Section lies in the modeling of three major processes:

1. The data generating process (DGP) for asset payoffs.

2. The belief revision process of the agents;

3. The belief revision process of the regulator.

In choosing the DGP, our primary objective was to mirror important stylized facts regarding financial returns, in particular volatility clustering, unconditional non-normality, and the relative size difference between returns and volatility for equities. Bearing this in mind, we considered a multivariate GARCH(1,1) process to be a DGP. Since multivariate GARCH models are difficult to use due to the large number of parameters and their nonlinear relationships, simplifications have been proposed. Here we use an approach known as BEKK GARCH (see Santos, 2002 for further details on multivariate GARCH models). Data generation was done in $MatLab^{TM}$ using the function full_bekk_simulate.m of the toolbox UCSD GARCH (Sheppard, 1999).

Financial institutions do not know the DGP, then they can only infer it from historical data. Let us admit that the agents update their beliefs about the asset returns according to the Exponentially Weighted Moving Average (EWMA) method, as recommended by $RiskMetrics^{TM}$. Therefore, financial institutions believe that *asset* returns at $t+1$ conditioned on the information available at t *are* normal with mean

$$\boldsymbol{\mu}_t = \begin{pmatrix} \mu_{1t} \\ \vdots \\ \mu_{Nt} \end{pmatrix}$$

[26] That is, in addition to heterogeneous beliefs we are admitting imperfect information.

and covariance matrix

$$\boldsymbol{\Sigma}_t = \begin{pmatrix} \sigma_t^{11} & \cdots & \sigma_t^{1N} \\ \vdots & \ddots & \vdots \\ \sigma_t^{1N} & \cdots & \sigma_t^{NN} \end{pmatrix},$$

where μ_{it} is the expected value of $R_{i,t+1}$ and σ_t^{ij} is the covariance between $R_{i,t+1}$ and $R_{j,t+1}$.

The updating rule for agents' beliefs can be expressed as follows:

$$((\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t), \quad \boldsymbol{R}_{t+1}) \mapsto (\boldsymbol{\mu}_{t+1}, \boldsymbol{\Sigma}_{t+1}),$$

where

$$\sigma_{t+1}^{ij} = \rho\sigma_t^{ij} + (1-\rho)(R_{i,t+1} - \mu_{it})(R_{j,t+1} - \mu_{jt})$$

and

$$\boldsymbol{\mu}_{t+1} = \rho\boldsymbol{\mu}_t + (1-\rho) \ \boldsymbol{R}_{t+1}.$$

The decay factor ρ is set at 0.97 as recommended by $RiskMetrics^{TM}$.

The regulating agency revises its expectations every two months (44 business days). This window is consistent with the average periodicity of parameter updates stipulated by Bacen's Circular 2,972[27]. The regulating agency's belief is still conditional normal, however the covariance matrix and the mean payoff are estimated, respectively, by the sample covariance matrix and by the sample mean of the payoffs of the last 43 observations.

In addition to the update of beliefs, it is necessary to establish a rule for the variable multiplier. This can be achieved by making the nonseverity of the risk constraint, $\overline{\nu}_t$, an increasing function of a market turbulence index. Theoretical sophistications do not add much in this case; so, we take $\overline{\nu}_t$ as a linear function of the variance of a portfolio formed by the supply of risky assets, that is,

$$\overline{\nu}_t = M\boldsymbol{\theta}'\boldsymbol{S}_t\boldsymbol{\theta}, \tag{1.32}$$

where M is a positive constant.

The dynamics of our model are generated in the following fashion. The economy begins with an initial arbitrary set of beliefs $\{(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0); \boldsymbol{S}_0\}$. Based

[27]Bacen revises the parameters of Circular 2,972 within no longer than one month. However, under normal market conditions, the parameters have remained unchanged, on average, for two months.

on these beliefs, agents make their portfolio choices. Given the portfolio choices, the aggregate demand functions can be defined. Together with the aggregate endowments, by market clearing, we can calculate the assets prices. Then, the realizations of payoffs $\{\boldsymbol{d}_1\}$ determine the returns $\boldsymbol{R}_1$ for the risky assets, and the agents and financial institutions update their beliefs. This process is repeated until the simulation ends at $t = T$.

1.6.3 Results

The following table shows the parameters used in the simulation. As the aim here is to show how the assets behave in a regulated economy, we consider the existence of only two risky assets. If we consider $N > 2$ would complicate numerical calculations without adding any important information.

Endowments	$\boldsymbol{\theta} = \begin{pmatrix} 1.9 \\ 0.5 \end{pmatrix}$
Risk-free asset payoff	$r_{0,t+1} = 1.00013, \forall t$
Lowest risk aversion	$\ell = 0.0011$
Linear function coefficient $\overline{\nu}_t$	$M = 40$
Unconditional covariance matrix of risky assets	$\begin{pmatrix} 0.6 & 0.25 \\ 0.25 & 0.4 \end{pmatrix}$
Unconditional mean of risky assets payoff	$\begin{pmatrix} 1.5 \\ 1.2 \end{pmatrix}$

Table 1.2: Parameters used in the simulation.

To make sure the results are not affected by initial conditions, the economy adjusts for 500 periods and only after that we start to record the data. Thus, date 1 corresponds to the $501^{\underline{st}}$ period of the simulation.

The main numerical conclusions of the simulation are shown in a series of figures. First, we analyze the evolution of the total bankruptcy probability of an economy à la Daníelsson et al. (2004), i.e., the agents' beliefs are identical with those of the regulating agency and the risk constraint multiplier is constant or equivalently $\overline{\nu}_t = \overline{\nu}$ for all t. Then, by maintaining the agents' expectations identical with those of the regulating agency, we examine the characteristics of a variable risk constraint multiplier according to the rule defined in Equation 1.32. Finally, we assess the consequences of different beliefs between the agents and the regulating agency. Then to not bias the results with another piece of information, we set, once again, $\overline{\nu}_t = \overline{\nu}$ for all t in the last simulation.

Regulated Economy versus Not Regulated

Initially, let us analyze the behavior of the total bankruptcy probability in a regulated economy and in an unregulated economy in which the agents and the regulating agency share the same beliefs and where $\overline{\nu}_t = \overline{\nu} = 100$. The sampling period corresponds to 250 business days (approximately one year). In order to prevent simulation uncertainties from influencing the results, 1000 independent simulations were made. Thus, the reported data correspond to the mean of these simulations.

Figure 1.11 shows the evolution of the mean of the asset 1 equilibrium price throughout the sampled period. The asset price in an unregulated economy is approximately 12% higher than the price in a regulated economy. Figure 1.12 shows agents' forecast variance for the return of risky asset 1. Clearly, the imposition of a VaR-based risk constraint increases the expected forecast variance. These results mirror those obtained by Daníelsson et al. (2004). Intuitively, the reduction in prices and the increase in volatility occur because risk constraint causes a transfer of risky assets from the least risk-averse agents to the most risk-averse ones. But this only happens if the asset price is reduced, which implies increase in volatility, since payoffs are generated exogenously. Figure 1.13 shows the estimates for the correlation coefficient between the risky assets in both situations analyzed. Risk regulation causes a small decrease in the correlation between assets. This suggests that VaR-based capital requirements can reduce the probability of systemic crises.

Figure 1.14 plots the total bankruptcy probability. The graph shows that the total bankruptcy probability in an unregulated economy is approximately 18% higher than in a regulated one. Basically, this occurs because risk constraint forces the agents to choose portfolios that are more focused on less risky assets. This is probably one of the reasons that justify the use of the VaR as a regulation instrument for risk control.

Multiplier of the Risk Constraint Variable

Let us now analyze the economic implications when risk limit $\overline{\nu}_t$ depends on a market turbulence index. We ran 1000 independent simulations. The

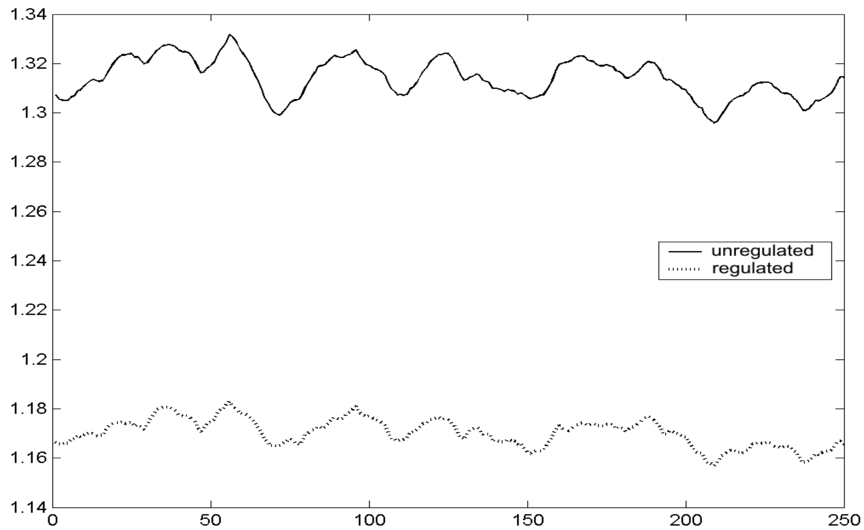

Figure 1.11: Average price of asset 1 in 1000 simulations.

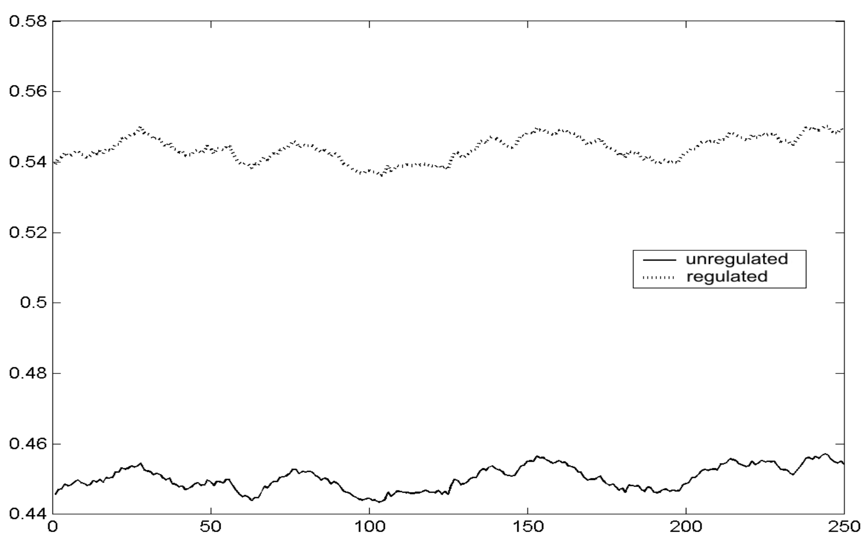

Figure 1.12: Average variance of asset 1 in 1000 simulations.

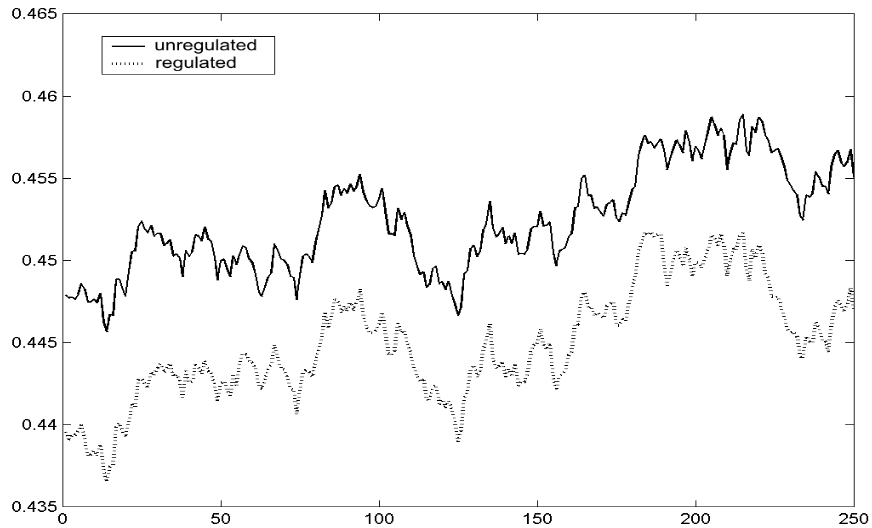

Figure 1.13: Average correlation coefficient in 1000 simulations.

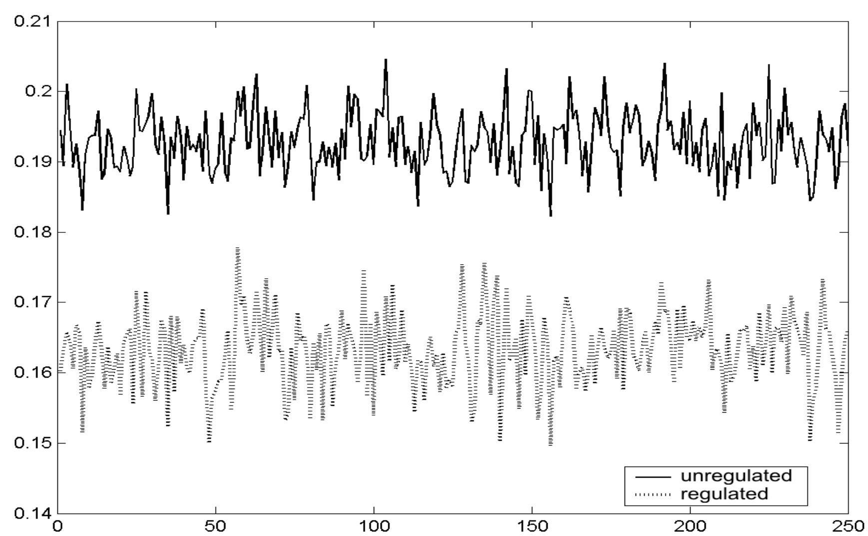

Figure 1.14: Total bankruptcy probability.

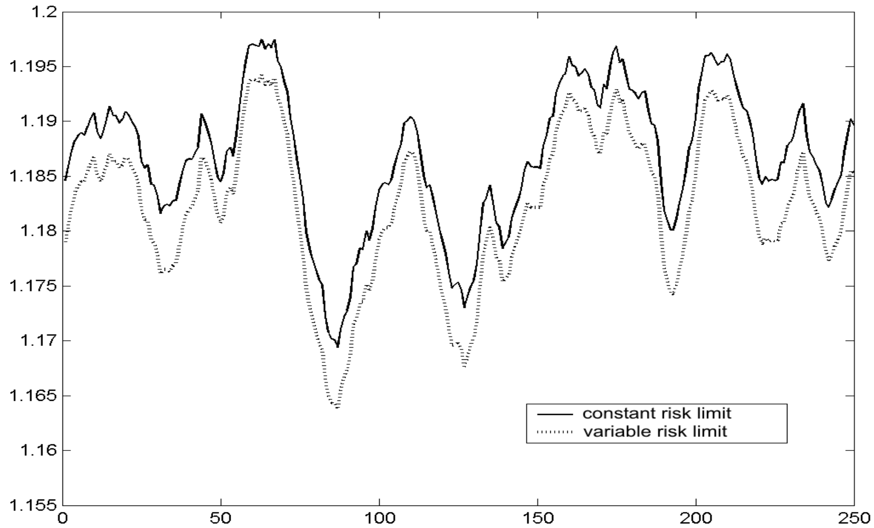

Figure 1.15: Prices with constant and variable risk limits.

reported results correspond to the mean of these 1000 simulations. In each simulation, we first calculated prices and variances using $\overline{\nu}_t$ as specified by Equation 1.32. Afterwards, we performed a new run with the same payoffs generated in the previous one, but making the risk limit constant and equal to the mean risk limits of the 250 observations, i.e., $\overline{\nu} = \sum_{t=1}^{250} \overline{\nu}_t/250$. The aim of this procedure is to compare the effects of a variable risk limit with a constant risk limit, but keeping similar capital requirement in both situations. To simplify, let us consider that the beliefs of the regulating agency and of the agents are the same.

Figure 1.15 shows the equilibrium price series of asset 1. As can be observed, the variable risk limit causes a small decrease in the equilibrium price (around 0.33%). Figure 1.16 shows the variance estimate in both runs. The variable risk limit introduces the undersirable effect of increasing the variance. However, this is a small shortcoming (less than 4%). The explanation for these results is that the regulating agency must be very strict under normal market conditions when the risk limit is variable in order to maintain the same average capital requirement in both situations.

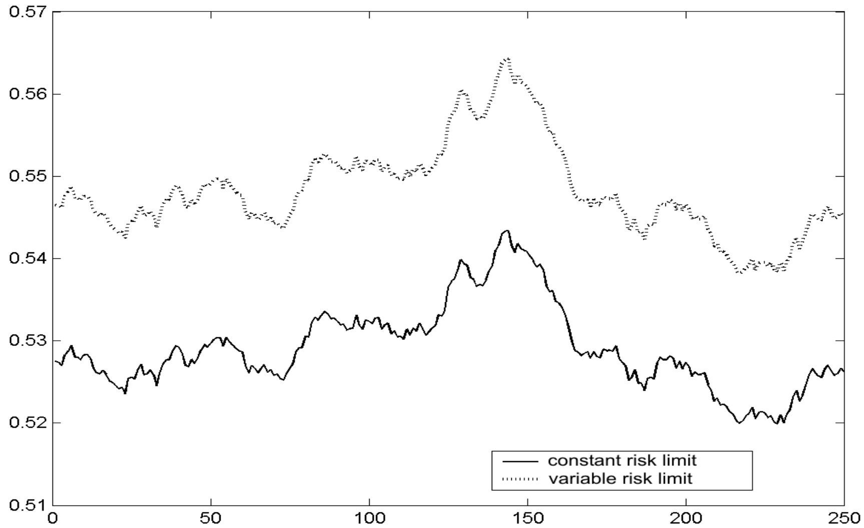

Figure 1.16: Variance with with constant and variable risk limits.

Differences between Beliefs of Regulator and Agents

Finally, let us look at the effects on equilibrium prices and variance when the agents' beliefs do not match those of the regulating agency. To eliminate possible noises with regard to the use of a variable nonseverity risk constraint, consider that $\overline{\nu}_t = 100$ for all t. Again, the reported results correspond to the mean of 1000 simulations.

Let us first analyze two extreme cases:

1. $\boldsymbol{S}_t = \begin{pmatrix} 0.2 & 0.25 \\ 0.25 & 0.1 \end{pmatrix}, \quad \forall t;$

2. $\boldsymbol{S}_t = \begin{pmatrix} 0.9 & 0.25 \\ 0.25 & 0.7 \end{pmatrix}, \quad \forall t.$

In situation 1, the regulating agency underestimates the volatilities, whereas the opposite is true for situation 2[28]. The simulation results in these two cases are shown in Figures 1.17 and 1.18. In situation 1, the equilibrium

[28]If we generate a large number of paths for the payoff of risky assets according to

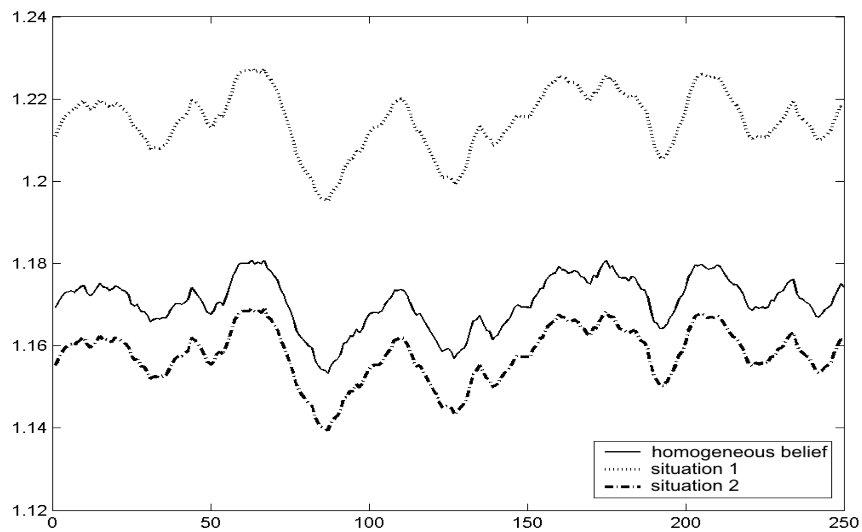

Figure 1.17: Prices with homogeneous beliefs and in two extremes cases.

prices and variances of asset 1 are respectively higher and lower than in the case in which the regulating agency and the agent share the same beliefs. The opposite occurs in situation 2.

As a theoretical exercise, let us analyze an intermediate situation (and more realistic) in which the updating of beliefs of the regulating agency is made according to the rule described in Section 1.6.2. Figures 1.19 and 1.20 respectively show the equilibrium prices and the forecast variance of asset 1. When the beliefs of agents and of the regulating agency differ, the equilibrium price and the forecast variance are higher than in a regulated economy with homogeneous expectations. This shows that the updating method used by the regulating agency underestimates the variance of asset 1, which, as a result, causes an increase in the equilibrium price. Nevertheless, the effect on variance is detrimental. This is mainly due to imperfections in the covariance estimate between assets 1 and 2 or in variance of asset 2.

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the rule defined in Subsection 1.6.2 and using the parameters shown in Table 1.2, then, the variance of risky assets estimated by financial agents (via EWMA) is larger than in situation 1 and smaller than in situation 2.

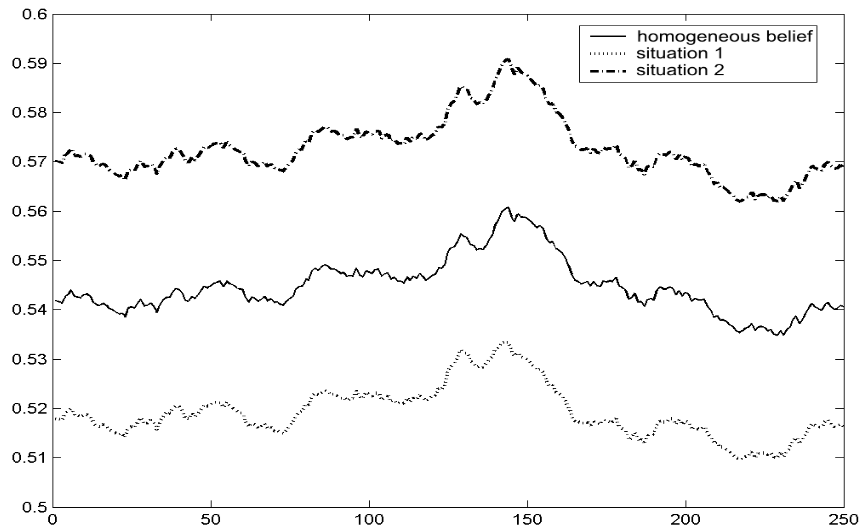

Figure 1.18: Variances with homogeneous beliefs and in two extremes cases.

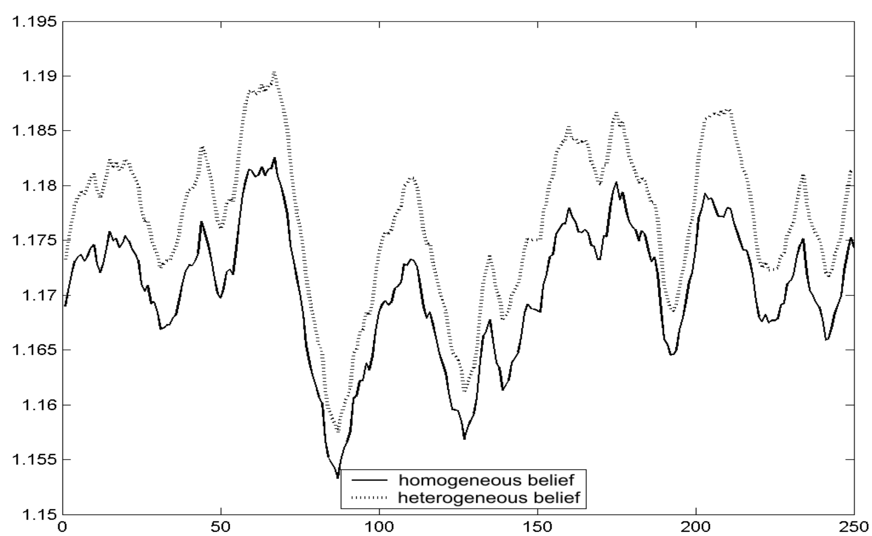

Figure 1.19: Prices with homegenous and heterogenous beliefs.

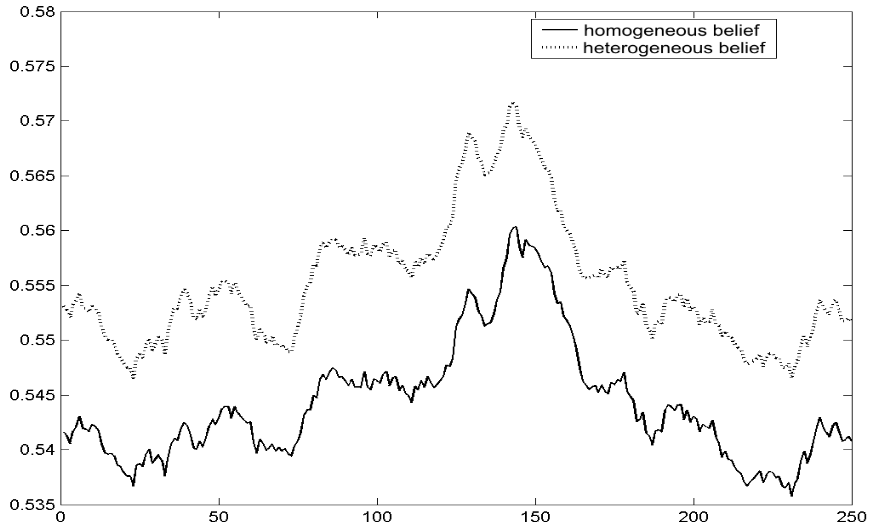

Figure 1.20: Variances with homegenous and heterogenous beliefs.

1.7 Credit Risk

In this Section we will study the economics effects of the capital requirement for covering credit risk. The model is the same presented in Section 1.3. This model, although its simplicity, is sufficiently flexible to cover a series of interest situations. In contrast to the analysis of market risk, that was deep, in the study of credit risk we will work in a more informal way. The main conclusions will be extracted from simple numerical examples.

The Basel proposal to covering credit risk consists in using what is known by risk-adjusted assets. Basically, the idea is to separate the assets of the financial institutions in I groups and in each group to apply an asset-specific risk weight. The positions bought and sold in different assets must be added in absolute value. The result of this account is the Risk-Adjuted Asset (RAA). The RAA must be less or equal to a fraction of the institution net worth.

To facilitate the analysis, we are going to introduce a small alteration in the basic model (see Section 1.3). The wealth of the institution h at $t = 0$ will not come from an initial endowment of assets. It will be an amount of equity capital generated in a previous period being independent of the assets prices[29]. In order to detach this alteration let us denote by K^h, instead of W_0^h, the wealth of the institution h at $t = 0$. Finally, we are also going to assume that the asset supply belongs to other agents and not to the financial institutions.

Made these comments, the risk constraint assumes the following form:

$$\Upsilon = \left\{ \boldsymbol{y} \in \mathbb{R}^N; \beta_1 |q_1 y_1| + \ldots + \beta_N |q_N y_N| \leq K^h \right\},$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N) \in \mathbb{R}^N_{++}$ are weight factors. Of course, if $N > I$ then at least two β's are equal, that is, if there is more assets than groups then at least two assets have the same weigh factor.

When we have only two risky assets and the prices of these assets are positive then the risk constraint are a lozenge as illustrated in the Figure 1.21. The institution h problem can be written as

$$\begin{array}{ll} \underset{\boldsymbol{y}^h}{\text{Min}} & (r_0 \boldsymbol{q} - \boldsymbol{\mu})' \, \boldsymbol{y}^h + h \frac{\boldsymbol{y}^{h\prime} \boldsymbol{\Sigma} \boldsymbol{y}^h}{2} \\ \text{s.a.} & \beta_1 q_1 y_1^h + \beta_2 q_2 y_2^h \leq K^h \\ & \beta_1 q_1 y_1^h - \beta_2 q_2 y_2^h \leq K^h \\ & -\beta_1 q_1 y_1^h + \beta_2 q_2 y_2^h \leq K^h \\ & -\beta_1 q_1 y_1^h - \beta_2 q_2 y_2^h \leq K^h \end{array}$$

Hence to solve the previous problem we have to consider nine different cases (depending on which restrictions are active in the optimum). For example, $\boldsymbol{y}^h = \frac{1}{h} \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r_0 \boldsymbol{q})$ is an interior solution of this problem.

In order to avoid a tedious sequence of calculations in the same way that it was done for market risk, we are going to restrict our analysis to a particular example. Suppose $N = 2$, $K^h = 2$ for all h, $r_0 = 1.00013$, $\boldsymbol{\mu} = (1.5, 1.2)'$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0.6 & 0.25 \\ 0.25 & 0.4 \end{pmatrix}.$$

[29] In a way, it already had occurred in the model of infinite periods presented in Section 1.6, when the wealth of the institutions in any period is resulted of asset payoff of the previous period.

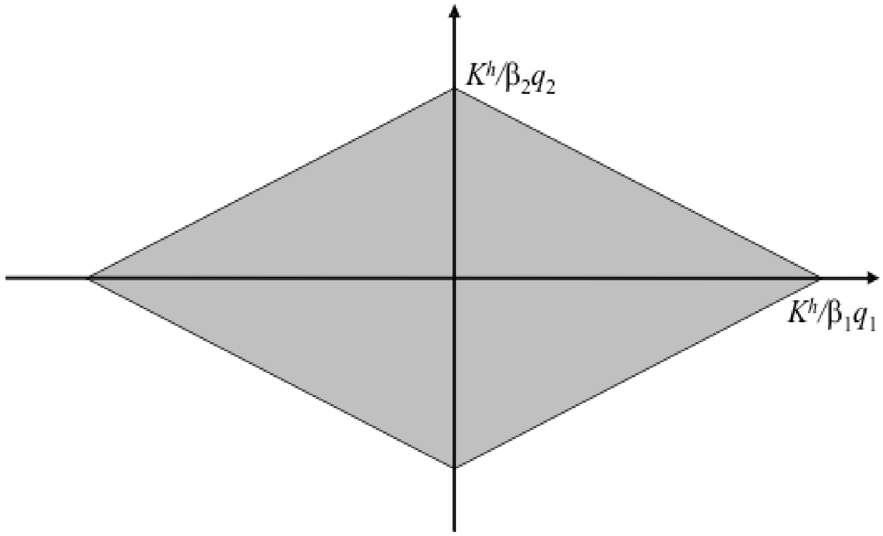

Figure 1.21: Credit risk constraint ($N = 2$).

Figure 1.22 presents the equilibrium prices of assets 1 and 2 as function of β_1 (β_2 fixed and equal to 0.3). Note that, except for small imperfections, these prices are decreasing functions of the weight factor of asset 1. Figure 1.23 illustrates the default probability of financial institutions as function of its CAAR. In Figure 1.23a the weight factors are well adjusted ($\beta_1 = 0.5$ and $\beta_2 = 0.3$) and the regulating agency apparently obtains its intention, namely, decreasing the bankrupt probability of the less risk averse institutions. However, in Figure 1.23b the regulating agency was not very sucessful in the choice of the weight factors ($\beta_1 = 0.5$ and $\beta_2 = 0.05$), since it attributed to asset 2 a weight not compatible with its volatility. As a consequence there was an increase of bankrupt probability of institutions less risk averse in a regulated economy. Finally, Figure 1.24 presents the total bankruptcy probability (the sum of the bankrupt probabilities of all institutions) as function of β_1. Note that exist a region (β_1 between 0.1 and 0.7) in which the regulating agency does not have to act.

1.8 Conclusion

The primary aim of this chapter was to analyze the economic impacts of risk regulation of financial institutions by means of a general equilibrium model.

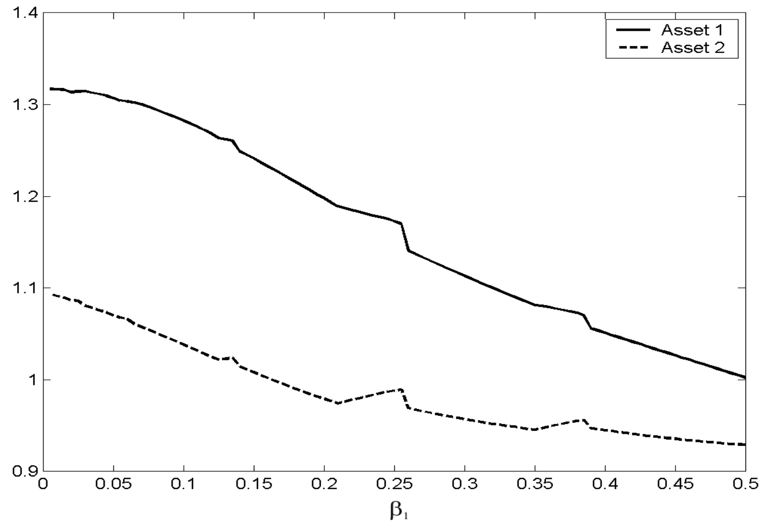

Figure 1.22: Prices of assets 1 e 2 with credit risk constraint.

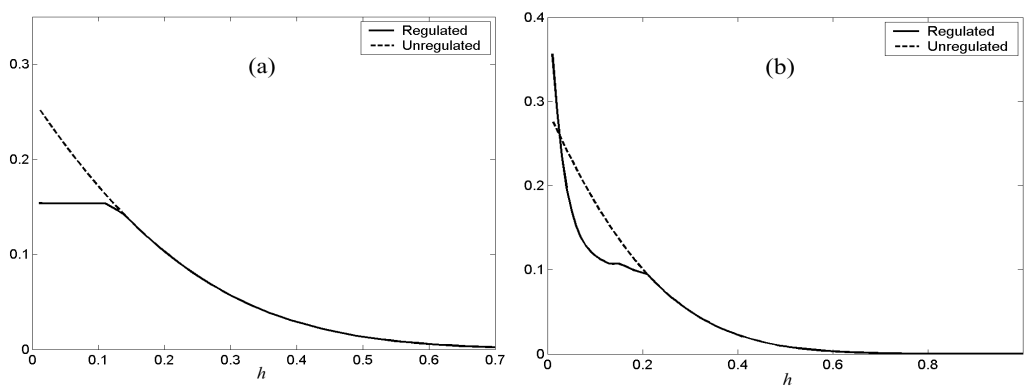

Figure 1.23: Default probability of institution h. In (a) $\beta_1 = 0.5$ and $\beta_2 = 0.3$. In (b) $\beta_1 = 0.5$ and $\beta_2 = 0.05$.

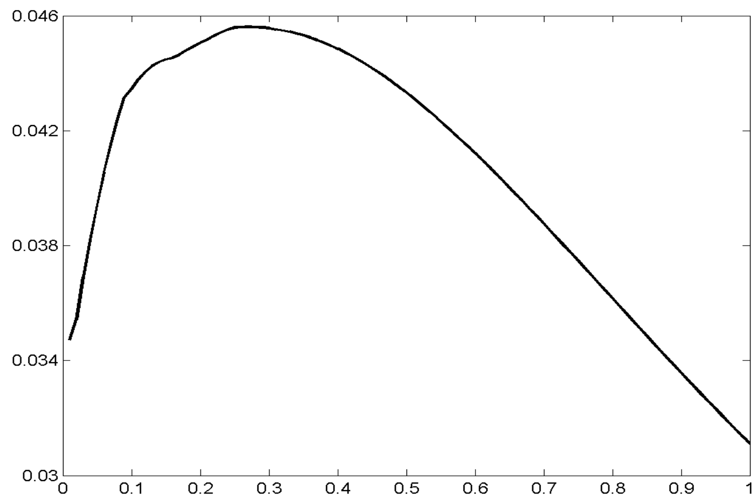

Figure 1.24: Total bankruptcy probability as a function of β_1.

In terms of economic welfare properties in an economy with market risk constraint we showed that:

- Under usual conditions, the equilibrium allocations are Pareto efficient.
- For each institution there is a level of regulation that maximizes its utility. We point out that this level is not necessarily equivalent to the absence of regulation.
- If the net worth of a financial institution is high or the market volatility is small then the VaR-based risk regulation can increase its bankruptcy probability.
- The VaR-based risk regulation decreases the bankruptcy probability only if there is a market problem, for example, high volatility.
- The VaR-based risk regulation decreases the probability of contagion.

In terms of economic impacts of Brazilian peculiarities regarding market risk regulation of financial institutions (i.e. the risk constraint multiplier depends on market volatility and heterogeneous beliefs between the regulating agency and agents), the major conclusions are:

- When nonseverity is variable, prices are lower and variance is higher than in the case in which nonseverity is constant. However, these undesirable effects are negligible. On the other hand, variable nonseverity guarantees the existence of equilibrium in periods of market turbulence. This equilibrium would not be achieved if nonseverity were constant.

- When the regulating agency imposes its beliefs on financial institutions for calculating the risk limit, or when the regulating agency imposes an intermediate model between the standard and internal ones in order to allocate capital to cover market risk, the effects on volatilities and equilibrium prices depend on the method used by the regulating agency to estimate the covariance matrix. This suggests that the adoption of internal models produces a more efficient equilibrium to the economy. In this regard, Bacen, by means of Communiqué 12,746, establishes deadlines for the gradual shift of Brazilian rules to an internal modeling approach. However, one should recall that this method requires much more banking supervision.

- When an intermediate model is adopted for risk regulation (which is the case of Brazil), special attention should be paid to the updating of beliefs, since, if the regulating agency overestimates the volatility of assets, there may be negative effects.

Finally, we demonstrated that when credit risk constraint is present it is very important that the regulating agency set appropriately the risk weights. For some set of risk weigth values the financial fragility is higher in a regulated economy than in an unregulated one.

Appendix - Proofs of Propositons

Proof of Proposition 1.3

The demonstration follows the usual procedure. Assume that the equilibrium allocation $\left\{\left(x^h, \boldsymbol{y}^h\right)\right\}_{h\in[\ell,1]}$ with prices $\boldsymbol{q}$ is not Pareto eficient. Hence, there is another feasible allocation $\left\{\left(\hat{x}^h, \hat{\boldsymbol{y}}^h\right)_{h\in[\ell,1]}\right\}$ that dominates $\left\{\left(x^h, \boldsymbol{y}^h\right)_{h\in[\ell,1]}\right\}$ in Pareto sense. That is, $\mathcal{E}\left[u^h\left(\hat{x}^h, \hat{\boldsymbol{y}}^h\right)\right] \geq \mathcal{E}\left[u^h\left(x^h, \boldsymbol{y}^h\right)\right]$ for all h and exist $H \subseteq [\ell, 1]$ with $\mathcal{L}(H) > 0$ such as if $h \in H$ then the strict inequality holds.

Note that for all h we should have $\hat{x}^h + \boldsymbol{q}\hat{\boldsymbol{y}}^h \geq W_0^h$, where $W_0^h = \theta_0^h + \boldsymbol{q}\theta^h$ is the initial wealth of agent h, since on the contrary, for $\epsilon > 0$ sufficiently small, $\left(\hat{x}^h + \epsilon, \hat{\boldsymbol{y}}^h\right)$ belongs to the restriction set of institution h problem with prices $\boldsymbol{q}$. Since u^h is strictly increasing it would result that $\left(\hat{x}^h + \epsilon, \hat{\boldsymbol{y}}^h\right)$ is preferable to $\left(x^h, \boldsymbol{y}^h\right)$ that would be in contradiction with $\left(x^h, \boldsymbol{y}^h\right)$ to be the optimum of institution h problem with prices $\boldsymbol{q}$. Moreover we must have $\hat{x}^h + \boldsymbol{q}\hat{\boldsymbol{y}}^h > W_0^h$ for all $h \in H$.

Since $\left(\hat{x}^h, \hat{\boldsymbol{y}}^h\right)$ is feasible, we have:

$$\theta_0 + \boldsymbol{q}\theta \geq \int_\ell^1 \hat{x}^h \mathrm{d}h + \boldsymbol{q}\int_\ell^1 \hat{\boldsymbol{y}}^h \mathrm{d}h =$$

$$\int_H \hat{x}^h \mathrm{d}h + \int_{[\ell,1]-H} \hat{x}^h \mathrm{d}h + \boldsymbol{q}\left(\int_H \hat{\boldsymbol{y}}^h \mathrm{d}h + \int_{[\ell,1]-H} \hat{\boldsymbol{y}}^h \mathrm{d}h\right) >$$

$$\int_\ell^1 W_0^h \mathrm{d}h = \theta_0 + \boldsymbol{q}\theta.$$

The contradiction demonstrates the desired result. □

Proof of Proposition 1.4

If equilibrium exists and all institutions reach the risk constraint then

$$\frac{\theta' \boldsymbol{\Sigma} \theta}{(1-\ell)^2} < \nu < \frac{\theta' \boldsymbol{\Sigma} \theta}{\left(\ell \ln \ell^{-1}\right)^2},$$

or, equivalently $\ell \geq \kappa\Psi = \sqrt{\frac{\varrho}{\nu}} < 1$. In these conditions, for one given equilibrium allocation $\left\{\left(x^h, \boldsymbol{y}^h\right)\right\}_{H\in[\ell,1]}$ with prices $\boldsymbol{q}$ we have

$$\Lambda(\nu) = -\int_\ell^1 \frac{\ln\left\{-\mathcal{E}\left[u^h\left(W_1^h\right)\right]\right\}}{h}\mathrm{d}h =$$

$$-\int_\ell^1 \left(\left(\theta_0^h + \boldsymbol{q}\theta^h - \boldsymbol{q}\boldsymbol{y}^h\right) r_0 + \mu\boldsymbol{y}^h - h\frac{\boldsymbol{y}^{h\prime}\boldsymbol{\Sigma}\boldsymbol{y}^h}{2}\right)\mathrm{d}h =$$

$$r_0\theta + \mu\theta - \tfrac{1}{2}\int_\ell^1 h\boldsymbol{y}^{h\prime}\boldsymbol{\Sigma}\boldsymbol{y}^h\mathrm{d}h =$$

$$r_0\theta + \mu\theta - \tfrac{1}{2}\left(\int_\ell^{\Psi\kappa} h\frac{\theta'\boldsymbol{\Sigma}\theta}{\kappa^2}\mathrm{d}h + \int_{\Psi\kappa}^1 \frac{\Psi^2}{h}\theta'\Sigma\theta\mathrm{d}h\right).$$

Calculating the integrals and using the identity $\ln\kappa\psi = \left(\frac{\kappa\psi-\ell}{\kappa} - 1\right)\frac{1}{\kappa}$ we have

$$\Lambda(\nu) = r_0\theta_0 + \mu\theta + \frac{\theta'\boldsymbol{\Sigma}\theta}{4\kappa^2}\left[(\kappa\Psi)^2 - (\ell+\kappa)\,\kappa\Psi + \ell^2\right].$$

□

Proof of Proposition 1.5

Since κ is a decreasing function of ν and Ψ is an increasing function of κ, to show that Λ is an increasing function of ν is sufficient to show that

$$f(\kappa) = (\kappa\Psi)^2 - 2\,(\ell+\kappa)\,\kappa\Psi + \ell^2$$

is a decreasing function of κ. Consider the quadratic polynomial $p(x) = x^2 - 2(\ell+\kappa)x + \ell^2$. This polynomial have two positive real roots:

$$x_1 = \ell + \kappa - \sqrt{\kappa^2 + 2\kappa\ell} \quad \text{and} \quad x_2 = \ell + \kappa + \sqrt{\kappa^2 + 2\kappa\ell}.$$

When κ increases x_1 decreases and x_2 increases (see Figure 1.25). Since $\kappa\Psi$ is an increasing function of κ and $\kappa\Psi < \kappa + \ell$ follows that when κ increases, $(\kappa\Psi)^2 - (\ell+\kappa)\,\kappa\Psi + \ell^2$ decreases.

□

Proof of Proposition 1.6

Since κ is a strictly decreasing function of ν, to verify the intervals where $f^h(\nu)$ is increasing or decreasing it is enough to analyze f^h as a function of κ.

If $\underline{\nu} \leq \nu \leq g_2^{-1}(h)$ then $h \leq g_2(\nu) = \kappa\Psi$, hence $f^h(\nu) = \frac{\Psi}{\kappa} - \frac{h}{2\kappa^2} - \frac{\Psi}{1-\ell}$ and

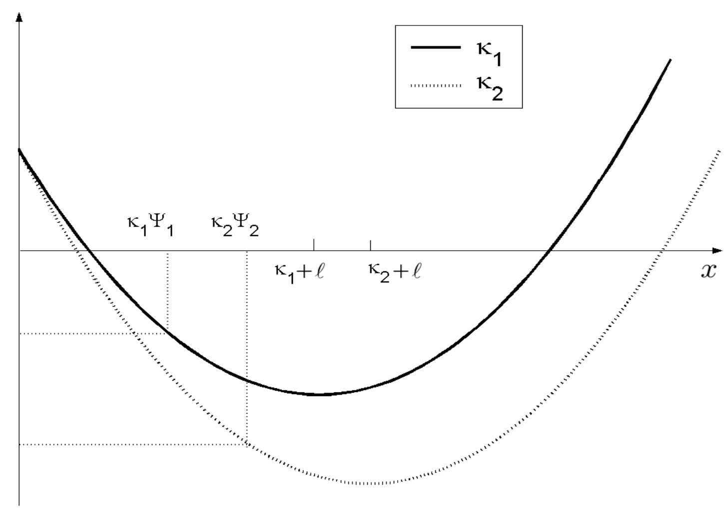

Figure 1.25: Polynomial $p(x) = x^2 - 2(\ell + \kappa)x + \ell^2$ $(\kappa_1 < \kappa_2)$.

$$\frac{\partial f^h}{\partial \kappa} = \Psi'(\kappa)\left(\frac{1}{\kappa} - \frac{1}{1-\ell}\right) + \frac{h}{\kappa^3} - \frac{\Psi}{\kappa^2}.$$

Thus, if $\nu \leq g_1^{-1}(h)$ then $\frac{\partial f^h}{\partial \kappa} < 0$ hence f^h is a strictly decreasing function of κ and therefore a strictly increasing function of ν. Case $g_1^{-1}(h) \leq \nu \leq g_2^{-1}(h)$, a similar argument shows that f^h is a strictly decreasing function of ν.

If $g_2^{-1}(h) < \nu \leq \overline{\nu}$ then

$$\frac{\partial f^h}{\partial \kappa} = \Psi'(\kappa)\left(\frac{\Psi}{h} - \frac{1}{1-\ell}\right).$$

We have to consider two cases:

1. If $\ell \leq h \leq \frac{1-\ell}{\ln \ell^{-1}}$ then $g_3(\nu) > h$. Therefore $\frac{\partial f^h}{\partial \kappa} > 0$ then f^h is a strictly increasing function of κ and a strictly decreasing of ν.

2. If $\frac{1-\ell}{\ln \ell^{-1}} \leq h \leq 1$ then the equation $g_3(\nu) = h$ has only one solution. Therefore, if $g_3^{-1}(h) < \nu \leq \overline{\nu}$ then f^h is strictly increasing function of

ν. On the other hand, if $g_2^{-1}(h) < \nu \leq g_3^{-1}(h)$ then f^h is a strictly decreasing function of ν.

□

Proof of Proposition 1.7

It is sufficient to show that

$$\frac{1}{2}\left(\frac{1}{h\left(\ln \ell^{-1}\right)^2} + \frac{h}{\left(1-\ell\right)^2}\right) \geq \frac{1}{\left(1-\ell\right)\left(\ln \ell^{-1}\right)}.$$

But the minimum of the left side of the previous equation occurs at $h = \frac{1-\ell}{\ln \ell^{-1}}$ and is equal to $\frac{1}{(1-\ell)(\ln \ell^{-1})}$. □

Proof of Proposition 1.8

The function $t(h)$ is continuous, moreover using the elementary differential calculus it is possible after tedious manipulation to prove that:

1. $t\left(\frac{1-\ell}{\ln \ell^{-1}}\right) > 0$ and
2. $t(1) < 0$.

By Bolzano's theorem the function $t(h)$ has at least one real root on the interval $\left[\frac{1-\ell}{\ln \ell^{-1}}, 1\right]$. To show that it is the only root we have to prove that $t(h)$ is strictly decreasing. We can write $t(h)$ as the difference between two functions: $t(h) = t_2(h) - t_1(h)$ where

$$t_1(h) = \frac{1}{\ln \ell^{-1}}\left(\frac{1}{2h \ln \ell^{-1}} - \frac{1}{1-\ell}\right) \quad \text{and}$$

$$t_2(h) = \frac{\Psi}{\kappa} - \frac{h}{2\kappa^2} - \frac{\Psi}{1-\ell} \text{ with } \kappa \text{ and } \Psi \text{ computed at } \nu = g_1^{-1}(h).$$

Therefore,

$$\frac{\partial t_1}{\partial h} = -\frac{1}{2\left(h \ln \ell^{-1}\right)^2} \quad \text{and}$$

$$\frac{\partial t_2}{\partial h} = -\frac{1}{2\kappa^2},$$

where to compute the last derivate we use the fact that at $\nu = g_1^{-1}(h)$, $\frac{\partial t_2}{\partial \kappa} = 0$. Hence we must demonstrate that $\frac{\partial t_2}{\partial h} \leq \frac{\partial t_1}{\partial h}$. But this occurs because

$$\underset{h}{\text{Max}} \ \frac{\partial t_2}{\partial h} = \underset{h}{\text{Min}} \ \frac{\partial t_1}{\partial h} = -\frac{1}{2(1-\ell)^2}.$$

The other affirmations of the proposition are immediate consequences of the behavior of $t(h)$. □

Proof of Proposition 1.9

If $\nu < g_2^{-1}(h)$ then

$$\frac{\partial \frac{m^h}{s^h}}{\partial \nu} = \frac{r_0 W_0^h}{\sqrt{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}}\frac{\partial \kappa}{\partial \nu} + \sqrt{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}\frac{\partial \Psi}{\partial \nu},$$

since $\frac{\partial \kappa}{\partial \nu} < 0$ and $\frac{\partial \Psi}{\partial \nu} < 0$ we have $\frac{\partial \frac{m^h}{s^h}}{\partial \nu} < 0$.
If $\nu \geq g_2^{-1}(h)$ then

$$\frac{\partial \frac{m^h}{s^h}}{\partial \nu} = \frac{\partial \Psi}{\partial \nu}\left(-\frac{r_0 W_0^h h}{\Psi^2\sqrt{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}} + \sqrt{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}}\right).$$

Hence, when $\nu \leq \tilde{\nu}$ we have $\frac{\partial \frac{m^h}{s^h}}{\partial \nu} < 0$ and when $\nu \geq \tilde{\nu}$ we have $\frac{\partial \frac{m^h}{s^h}}{\partial \nu} > 0$. □

Proof of Proposition 1.10

As the utility of agent h is strictly increasing, the budget constraint should be bind. Therefore, the wealth of the institution h at $t = 1$ is given by:

$$W_1^h = \left(W_0^h - q_t y^h\right) r_0 + d y^h.$$

Since agents have a constant absolute risk aversion coefficient, without loss of generality, we can suppose that the utility of institution h has the form

$$u^h(x) = -\mathrm{e}^{-hx}.$$

Thus, the investment problem of institution h is

$$\begin{array}{ll} \underset{y^h}{\text{Max}} & \mathcal{E}\left(-\mathrm{e}^{-hW_1^h}\right) \\ \text{s.a.} & s\left(y^h\right)^2 \leq \nu. \end{array}$$

By the normality hypothesis of the payoff one has that $-hw_1^h$ is also normal with mean $-h\left[\left(W_0^h - qy^h\right) r_0 + \mu y^h\right]$ and variance $h^2\left(y^h\right)^2\sigma$. Therefore, $-\mathrm{e}^{-hW_1^h}$ is lognormal with mean

$$-\mathrm{e}^{-h\left[\left(W_0^h - qy^h\right)r_0 + \mu y^h\right] + \frac{h^2\left(y^h\right)^2\sigma}{2}}.$$

Taking logarithms, eliminating the terms that do not depend on y^h and dividing by h, the problem of financial institution h becomes

$$\begin{array}{ll} \underset{y^h}{\text{Min}} & \left(qr_0 - \mu\right) y^h + \frac{h\left(y^h\right)^2\sigma}{2} \\ \text{s.a.} & s\left(y^h\right)^2 \leq \nu, \end{array}$$

whose Lagrangean is

$$L\left(y^h, \lambda^h\right) = \left(qr_0 - \mu\right) y^h + \frac{h\left(y^h\right)^2\sigma}{2} + \lambda^h\left[s\left(y^h\right)^2 - \nu\right].$$

The 1$^{\underline{\text{st}}}$ order condition is

$$y^h = \left(h\sigma + 2s\lambda^h\right)^{-1}\left(\mu - qr_0\right).$$

If $h \geq \sqrt{\frac{\rho}{\nu}}$ then is easy to show that $y^h = (h\sigma)^{-1}\left(\mu - qr_0\right)$ satisfies the 1$^{\underline{\text{st}}}$ order condition with the risk constraint not bind ($\lambda^h = 0$).

If $h < \sqrt{\frac{\rho}{\nu}}$ then the risk constraint should be active, hence $\lambda^h = \frac{\mu - qr_0}{2\sqrt{s\nu}} - \frac{h\sigma}{2s}$ implying that $y^h = \sqrt{\frac{\nu}{s}}$.

By the convexity, the first-order condition is also sufficient. According to Lemma 1.1 the only solution to the problem of financial institution h is given by Equation 1.24.

□

Proof of Proposition 1.11

Let $\Psi = \frac{(\mu - qr_0)}{\theta\sigma}$ and define the following function:

$$f\left(\Psi\right) = \begin{cases} \Psi \ln \ell^{-}1 & \text{if } 0 < \Psi < \frac{\ell}{\kappa} \\ \frac{\kappa\Psi - \ell}{\kappa} - \Psi \ln \kappa\Psi & \text{if } \frac{\ell}{\kappa} \leq \Psi < \frac{1}{\kappa} \\ \frac{1-\ell}{\kappa} & \text{if } \Psi \geq \frac{1}{\kappa}. \end{cases}$$

The market clearing condition is equivalent to finding Ψ such as $f\left(\Psi\right) = 1$, that is, the equilibrium prices are given by $q = \frac{\mu - \Psi\sigma\theta}{r_0}$, where Ψ is the solution of $f\left(\Psi\right) = 1$. Solving this equation, we have:

1. If $0 \leq \kappa < \ell \ln \ell^{-1}$ then $\Psi = \frac{1}{\ln \ell^{-1}}$.

2. If $\ell \ln \ell^{-1} \leq \kappa < 1 - \ell$ then

$$f\left(\frac{\ell}{\kappa}\right) \leq 1 < f\left(\frac{1}{\kappa}\right),$$

in other words, the solution is in the central branch of f. Therefore, we have to solve the following equation:

$$\frac{\kappa\Psi - \ell}{\kappa} - \Psi \ln \kappa\Psi = 1,$$

which is equivalent to

$$z \ln z = -\left(\kappa + \ell\right), \tag{1.33}$$

where $z = \kappa\Psi \mathrm{e}^{-1}$.

Equation 1.33 can be written as $F\left(-\left(\kappa + \ell\right)\mathrm{e}^{-1}\right) = z \ln z$. Substituting this equality in 1.33 one has

$$\Psi = -\frac{\kappa + \ell}{\kappa F\left(-\left(\kappa + \ell\right)\mathrm{e}^{-1}\right)}.$$

3. If $\kappa = 1 - \ell$ then Ψ is any real number greater than $1/\kappa$.

4. If $\kappa > 1 - \ell$ then there is no Ψ that satisfies the market clearing condition, i.e., an equilibrium fails to exist. □

Proof of Propostition 1.12

Following the same procedure used in the demonstration of the Proposition 1.10 one can easily see that the Lagrangean of the problem of financial institution h is

$$L\left(\boldsymbol{y}^h, \lambda^h\right) = \left(r_0 \boldsymbol{q} - \boldsymbol{\mu}\right)\boldsymbol{y}^h + \frac{h \boldsymbol{y}^{h\prime} \boldsymbol{\Sigma} \boldsymbol{y}^h}{2} + \lambda^h \left(\boldsymbol{y}^{h\prime} \boldsymbol{S} \boldsymbol{y}^h - \nu\right).$$

The 1st order condition is

$$\boldsymbol{y}^h = \left(h\boldsymbol{\Sigma} + 2\lambda^h \boldsymbol{S}\right)^{-1}\left(\boldsymbol{\mu} - r_0 \boldsymbol{q}\right). \tag{1.34}$$

If $h \geq \sqrt{\frac{\rho}{\nu}}$ then $\boldsymbol{y}^h = \frac{1}{h}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\boldsymbol{q})$ is the solution of agent h problem with risk constraint not bind ($\lambda^h = 0$).

If $h < \sqrt{\frac{\rho}{\nu}}$ then the risk constraint should be active, i.e, $\boldsymbol{y}^{h\prime}\boldsymbol{S}\boldsymbol{y}^h = \nu$. To solve the optimization problem in this case we could substitute Equation 1.34 at the risk constraint to find λ^h and then to find the optimal portfolio. This process involves serious technical difficulties since we must determine the roots of a polynomial of order $2N$. A simpler method is to use the results obtained for the one-dimensional case and infer a possible optimizing solution y^h. Then, one should check whether this candidate satisfies the first-order condition.

When $N = 1$ we show that all the institutions with the active risk constraint behaved as if their CAAR were equal to $\sqrt{\frac{\rho}{\nu}}$,i.e., optimal risk sharing was impaired. Based on this observation, a natural choice for y^h is given by Equation 1.26. It is easy to see that this choice of y^h satisfies the $1^{\underline{\text{st}}}$ order condition with the risk constraint active. Again, by convexity, the $1^{\underline{\text{st}}}$ order condition is also sufficient. By Lemma 1.1 results that the (only) solution of the financial institution h problem is given by Equations 1.25 and 1.26. □

Proof of Proposition 1.13

We want to show that $\boldsymbol{p} \equiv \boldsymbol{\Sigma}^{-1}(\mu - r_0\boldsymbol{q}) = \Psi\boldsymbol{\theta}$ with Ψ defined by Equation 1.8. When $0 \leq \kappa \leq \ell \ln \ell^{-1}$ (which corresponds to the case in which no institutions reach the risk constraint) or when $\kappa \geq 1 - \ell$ (which corresponds to the case in which risk constraints of all institutions are active, with no equilibrium if strict inequality holds), the demonstration is trivial. Let us focus now on the intermediate situation, i.e, $\ell \ln \ell^{-1} < \kappa < 1 - \ell$, in other words, some institutions have an active risk constraint and others don't.

The market clearing condition gives

$$\int_{\ell}^{\sqrt{\frac{\rho}{\nu}}} \sqrt{\frac{\nu}{\rho}}\boldsymbol{p}\mathrm{d}h + \int_{\sqrt{\frac{\rho}{\nu}}}^{1} \frac{\boldsymbol{p}}{h}\mathrm{d}h = \boldsymbol{\theta},$$

or

$$\left[1 - \ell\sqrt{\frac{\nu}{\rho}} - \ln\left(\sqrt{\frac{\rho}{\nu}}\right)\right]\boldsymbol{p} = \boldsymbol{\theta}.$$

Hence, vectors $\boldsymbol{p}$ and $\boldsymbol{\theta}$ are collinear. Let us make $\boldsymbol{p} = \Psi\boldsymbol{\theta}$, then Ψ should satisfy the following equation:

$$\frac{\kappa\Psi - \ell}{\kappa} - \Psi \ln \kappa\Psi = 1,$$

which is identical to equation for Ψ obtained in the one-dimensional case (see the proof of Proposition 1.11) which concludes the proof. □

Proof of Proposition 1.14

The demonstration is an immediate consequence of the fact that Ψ (defined by Equation 1.8) is an increasing function of κ on the interval $[0, 1-\ell]$. □

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Chapter 2

Applications of Affine Models on the Brazilian Fixed Income Market

2.1 Introduction

Fixed income options are intensively traded in numerous markets around the world. Their popularity comes from their usefulness on the management and control of interest rates risk. They are usually priced with the use of sophisticated arbitrage-free term structure models (see Heath et al., 1992). A prerequisite for those models to work is their adjustment to capture the cross-sectional and dynamics of the bonds yields. However, it is not clear that efficiently capturing information from bonds yields necessarily guarantees efficiency on the pricing of options written on this market. In other words, can in general underlying market dynamics completely characterize its corresponding option market dynamics?

We try to answer this question by analyzing the underlying and option markets of the most popular Brazilian fixed income instrument, the ID-Future.

The issue of reconciling underlying and derivatives market information has been addressed before on some recent studies. Chernov and Ghysels (2000), Ait Sahalia et al. (2001) and Pan (2002) have worked with dynamic asset pricing models estimated based on a joint time series of S&P500 spot and options data. Jagannathan et al. (2003) studies relative pricing of caps and swaptions for dynamic term structure models estimated based only on US swaps. Heidari and Wu (2003) show, through principal component analysis, that while three factors are enough to capture the dynamics of US interest rate swaps, three additional factors are necessary to capture the joint dynamics of swaps and swaptions. Bikbov and Chernov (2005) compare Gaussian and Stochastic Volatility affine models using a joint data set of eurodollar futures and option prices. Li and Zhao (2005) test the ability of quadratic term structure models on pricing and hedging caps when they are estimated based only on US swaps data. Almeida et al. (2005) show that affine term structure models estimated based on joint US swaps/caps data are more capable to explain excess bond returns than when estimated based only on swaps data.

On the context of the Brazilian fixed income market, we observe that the IDI option market contains sources of information independent from ID-Futures market. We start presenting regression results supporting the existence of dynamic factors driving option prices which are not spanned by the bonds yields. After, we analysed two classical interest rate models: the Gaussian and the Cox-Ingersoll-Ross (CIR) models. The former model presents a very simple volatility structure that is unable to capture the information contained in the option market. The last model is the simplest interest rate model that presents stochastic volatility. When we calibrated this model using options on the estimation procedure we note a decrease on call price error compared with the error when we use only ID-Futures on the estimation procedure. In other words, there is information on IDI options that are not on bonds yields.

With this information in hands, we propose a theoretical model which takes into account the existence of independent dynamic sources driving the option market but not the underlying market. Moreover, as we shall see, this model attributes to the additional independent factors, the ability of driving the volatility of the underlying fixed income instruments. Models like that have been first proposed by Collin-Dufresne & Goldstein (2001), and are denominated Unspanned Stochastic Volatility (USV) models.

Besides the analysis of the incompleteness of the Brazilian fixed income market, this work make important contributions to pricing IDI options. IDI options have a peculiar characteristic which is not shared by usual fixed income international options: Their payoff depends on the integral of the short-term rate along the path between the trading date t and the option maturity date T. Usually the payoff of bond vanilla options depends on the short-term rate (or more generally the state vector) evaluated only at the maturity date T. In an usual terminology IDI option is a kind of asian options.

IDI options have been priced before with the use of single factor term structure models. Silva (1997) adopted the Black et al. (1990) model, Vieira Neto & Valls (1999) adopted the Vasicek (1977) model, Fajardo & Ornelas (2003) adopted the CIR model (Cox et al., 1985), and Gluckstern et al. (2002) and Almeida et al. (2002) adopted the Hull & White (1990) model. Note that our approach generalizes theirs on at least two points. First, while they adopt single factor models, we adopt a multi-factor model. It is well-known that in order to capture term structure dynamics we need at least two factors, and in general three factors[1]. Second, while the other studies were interested in pricing the cross sectional of options on a single day, we are interested in analyzing bonds yields and options prices dynamics using model estimated parameters based on ID-Future and IDI options. This approach demands much more from the model than the simple calculation of options prices on a single day, and posterior comparison with market prices.

Another important innovation of this work is the pricing of IDI options in the CIR model based on a numerical Laplace transform inversion. Fajardo & Ornelas (2003) priced IDI options with the use of a single factor CIR model. They obtained a close formula for the call price, but they relied on a very strong assumption, namely the short term interest rate at any two differents times are independent random variables. Not surprisingly, the results that they got were very poor.

Finally, we obtained a theoretical pricing formulas for IDI options under a multi-factor dynamic term structure models that exhibit USV.

2.2 Data and Market Description

In the following two subsections we explain how ID-Futures and IDI options work. For more details about these contracts we recommend the Brazilian Mercantile & Future Exchange (BM&F) webpage[2].

[1]See Litterman and Scheinkman (1991) for a seminal factor analysis on term structure data.

[2]http://www.bmf.com.br/indexenglish.asp

2.2.1 ID-Futures

The One-Day Interbank Deposit Future Contract (ID-Future) with maturity T is a future contract whose underlying asset is the accumulated daily ID rates[3] capitalized between the trading time t ($t \leq T$) and T. The contract size corresponds to R$ 100,000.00 (one hundred thousand Brazilian Real) discounted by the accumulated rate negociated between the buyer and the seller of the contract. Then, if one buys an ID-Future at a price $\overline{ID}$[4] at time t and holds it until the maturity T, his gain/loss is

$$100000 \cdot \left(\frac{\prod_{i=1}^{\zeta(t,T)} (1 + ID_i)^{(1/252)}}{\left(1 + \overline{ID}\right)^{\zeta(t,T)/252}} - 1 \right),$$

where ID_i denotes the ID rate $i-1$ days after the trading time t. The function $\zeta(t,T)$ represents the numbers of days between times t and T[5].

This contract is very similar to a zero coupon bond. The only difference is that it presents a sequence of cash flows paid every day. Each daily cash flow is the difference between the settlement price[6] on the current day and the settlement price on the day before corrected by the ID rate of the day before.

BM&F is the entity that offers the ID-Future. The number of authorized contract-maturity months is fixed by BM&F (on average, there are about twenty authorized contract-maturity months for each day but only around ten are liquid). Contract-maturity months are the first four months subsequent to the month in which a trade has been made and, after that, the months that initiate each following quarter. Expiration date is the first business day of the contract-maturity month.

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[3] The ID rate is the average one-day interbank borrowing/lending rate, calculated by CETIP (Central of Custody and Financial Settlement of Securities) every workday. The ID rate is expressed in effective rate per annum, based on 252 business-days.

[4] The ID-Future is quoted in interest rate per annum based on 252 bussiness-days.

[5] Without any loss of generality, in this paper, we associate the continuously-compounded ID rate to the short term rate r_t. Then the gain/loss can be written as $100000 \cdot \left(e^{\int_t^T (r_u - \overline{r}) du} - 1 \right)$, where $\overline{r} = \ln(1 + \overline{ID})$.

[6] The settlement price at time t of a ID-Future with maturity T is equal to R$ 100,000.00 discounted by its closing price quotation.

2.2.2 IDI and its Option Market

The IDI index is actually defined as the accumulated ID rate. Using the association between the short term rate r_t and the continuously-compounded ID rate we can write the IDI index value as the exponential of the accumulated short term interest rate

$$IDI_t = IDI_0 \cdot e^{\int_0^t r_u \mathrm{d}u}. \tag{2.1}$$

This index has been fixed to the value of 100000 points in January 2, 1997. It has actually been resettled to its initial value sometimes, most recently in January 2, 2003. This index is computed every workday by BM&F.

An IDI option with time of maturity T is an European option where the underlying asset is the IDI and whose payoff depends on IDI_T. When the strike is K, the payoff of an IDI option is $L_c(T) = (IDI_T - K)^+$ for a call and $L_p(T) = (K - IDI_T, 0)^+$ for a put.

BM&F is also the entity that offers the IDI option. Strike prices (expressed in index points) and the number of authorized contract-maturity months are established by BM&F. Contract-maturity months can happen to be any month and expiration date is the first business day of the maturity month. A series is just a set of characteristics of the option contract, which determine its expiration date and strike price. The series is identified by a specific code established by BM&F. On average, there are about 30 authorized series within each day for call options and ten for put options, but not more than ten call options series and only two or three put options series are liquid. The liquidity of put options is low, being common finding a sequence of days where no put option trade took place.

2.2.3 Data

Data consists of time series of yields of ID-Futures for all different liquid maturities, and values of IDI options for different strikes and maturities. The data covers the period from January 02, 2003 to December 30, 2005.

BM&F maintains a daily historical database with the price and number of trades of every ID-Future and IDI option that have been traded in some day. With the ID-Future database and a time series of ID interest rates, it is direct by cubic interpolation to estimate the interest rates for fixed maturities for all trading days. For each fixed time to maturity, a reference bond is a zero coupon bond with that time to maturity. We adopt the fixed times to maturity of 1, 21, 63, 126, 189, 252 and 378 days. For the options, we select two diferents data base. The first is formed by the more liquid IDI call in each day with price greater or equal to 150. The second is constituted by a synthetic at-the-money IDI call[7] with time to maturity equals to 95 days[8]. The former option data base is used to test the ability of the models to pricing IDI calls. The latter data base is used to calibrate the models.

After excluding weekends, holidays, and workdays where no deal took place, we have a total of 748 yields for the reference bonds for each fixed maturity and 748 prices of IDI call options in our sample.

2.3 Evidence that Bonds do not Span the Fixed Income Market

For motivate the model proposed in Section 2.6 we present an empirical evidence of extra factors driving option prices not existent on the bonds market. Specifically we run regressions where the dependent variable is the price of the synthetic at-the-money IDI call, while the independent variables are the yields of the reference bonds for all the previously fixed maturities, 1, 21, 63, 126, 189, 252, and 378 days. Setting up the notation, let cs_t represent the time t price of the synthetic at-the-money IDI call. Let also $rb_t(\tau)$ represent the time t yield of the reference bond with time to maturity τ, expressed in years. We basically run two types of regressions. The first, a standard multiple linear regression:

$$
\begin{aligned}
& cs_t = \beta_0 + \beta_1 rb_t(\tfrac{1}{252}) + \beta_2 rb_t(\tfrac{21}{252}) + \beta_3 rb_t(\tfrac{63}{252}) \\
& +\beta_4 rb_t(\tfrac{126}{252}) + \beta_5 rb_t(\tfrac{189}{252}) + \beta_6 rb_t(1) + \beta_7 rb_t(1.5) + \epsilon_t^1 = \qquad (2.2) \\
& \beta_0 + \beta \cdot rb_t + \epsilon_t^1,
\end{aligned}
$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_7)$ and $rb_t = \big(rb_t(\tfrac{1}{252}), rb_t(\tfrac{21}{252}), \ldots, rb_t(1.5)\big)$.

[7] An at-the-money option is an option with moneyness equal to one. The moneyness is defined by the quocient between the present value of the strike and the IDI at the current time.

[8] In each day, the synthetic at-the-money IDI call is obtained by an interpolation scheme on Black volatility for the time to maturity equals to 95 days. Again, we elimated options with price less than 150.

The second, a non-linear regression:

$$cs_t = a \cdot rb_t + b \cdot rb_t^2 + c \cdot rb_t^3 + d \cdot \mathrm{e}^{rb_t} + \epsilon_t^2, \tag{2.3}$$

where a, b, c and d are seven-dimensional vectors, and powers and exponentials of rb's are calculated using matrix algebra, meaning that operations are performed on each element of the vector at the same time.

The R^2s (or explained variance) are respectively 14.23% and 45.48% for linear and non-linear regressions[9]. Results of these regressions suggest the existence of factors driving options dynamics with sources of uncertainty independent of the underlying market.

2.4 The Gaussian Model

2.4.1 Model Specification

The uncertainty in the economy is characterized by a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions. We assume the existence of a pricing measure $\mathbb{Q}$ under which discounted bond prices are martingales. The Gaussian model is specified through by defining the short term rate r_t as a sum of N normal random variables:

$$r_t = \phi_0 + \sum_{i=1}^{N} X_t^i, \tag{2.4}$$

where the dynamics of process X is given by

$$\mathrm{d}X_t = -\kappa X_t \mathrm{d}t + \rho \mathrm{d}W_t^{\mathbb{Q}}, \tag{2.5}$$

with $W^{\mathbb{Q}}$ being an N-dimensional brownian motion under $\mathbb{Q}$, κ a diagonal matrix with κ_i in the i_{th} diagonal position, and ρ is a matrix responsible for correlation among the X factors. The connection between martingale probability measure $\mathbb{Q}$ and physical probability measure $\mathbb{P}$ is given by Girsanov's Theorem with an essentially affine (Duffee, 2002)[10] market price of risk

$$\mathrm{d}W_t^{\mathbb{P}} = \mathrm{d}W_t^{\mathbb{Q}} - \lambda_X X_t \mathrm{d}t, \tag{2.6}$$

[9]The non-linear regression, is non-linear on the ID yields but can be solved by transformations of variables which turn the regression into a linear one. In this sense, the R^2 of this non-linear regression is that of a corresponding transformed linear multiple regression.

[10]Constrained for admissibility purposes (see Dai and Singleton, 2000).

where λ_X is an $N \times N$ matrix and $W^{\mathbb{P}}$ is a brownian motion under $\mathbb{P}$.

An IDI option is just an asian option whose payoff is a function of the instantaneous rate (the underlying) over the whole of its life-time. Then to price it, we have to know the distribution of the integral of instantaneous rate. The next Lemma characterizes this distribution under the Gaussian model.

Lemma 2.1 *Let $Y(t,T) = \int_t^T r_u du$. Then, under the measure $\mathbb{Q}$ and conditional on the sigma field $\mathcal{F}_t$, Y is normally distributed with mean $M(t,T)$ and variance $V(t,T)$, where*

$$M(t,T) = \phi_0\tau + \sum_{i=1}^{N} \frac{1-\mathrm{e}^{-\kappa_i\tau}}{\kappa_i} X_t^i \tag{2.7}$$

and

$$\begin{aligned} V(t,T) = {} & \textstyle\sum_{i=1}^N \frac{1}{\kappa_i^2}\left(\tau + \frac{2}{\kappa_i}\mathrm{e}^{-\kappa_i\tau} - \frac{1}{2\kappa_i}\mathrm{e}^{-2\kappa_i\tau} - \frac{3}{2\kappa_i}\right)\sum_{j=1}^N \rho_{ij}^2 + \\ & +2\textstyle\sum_{i=1}^N \sum_{k>i} \frac{1}{\kappa_i\kappa_k}\left(\tau + \frac{\mathrm{e}^{-\kappa_i\tau}-1}{\kappa_i} + \frac{\mathrm{e}^{-\kappa_k\tau}-1}{\kappa_k} - \frac{\mathrm{e}^{-(\kappa_i+\kappa_k)\tau}-1}{\kappa_i+\kappa_k}\right)\sum_{j=1}^N \rho_{ij}\rho_{kj}, \end{aligned} \tag{2.8}$$

where $\tau = T - t$.

2.4.2 Pricing Zero Coupon Bonds

Let $P(t,T)$ denote the time t price of a zero coupon bond maturing at time T, paying one monetary unit. It is well known that Multi-factor Gaussian models offer closed-form formulas for zero coupon bond prices. The next lemma presents a simple proof of this fact for the particular model in hand.

Lemma 2.2 *The price at time t of a zero coupon bond maturing at time T is*

$$P(t,T) = \mathrm{e}^{A(t,T)+B(t,T)'X_t}, \tag{2.9}$$

where $A(t,T) = -\phi_0\tau + \frac{1}{2}V(t,T)$ and $B(t,T)$ is a column vector with $-\frac{1-\mathrm{e}^{-\kappa_n\tau}}{\kappa_n}$ in the n_{th} element.

Using Equation 2.9 and Itô's lemma we can obtain the dynamics of a bond price under the martingale measure $\mathbb{Q}$

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r_t\mathrm{d}t + B(t,T)'\rho\mathrm{d}W_t^{\mathbb{Q}}. \tag{2.10}$$

To hold this bond, the investors will ask for an instantaneous expected excess return ($z^i(t,T)$). Then, under the physical measure, the bond price dynamics is

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = (r_t + z^i(t,T))\mathrm{d}t + B(t,T)'\rho \mathrm{d}W_t^{\mathbb{P}}. \tag{2.11}$$

Applying Girsanov's Theorem to change measures we have

$$z^i(t,T) = B(t,T)'\rho\lambda_X X_t. \tag{2.12}$$

2.4.3 Pricing IDI Options

IDI options have been priced before with the use of single factor term structure models[11]. We directly generalize those models adopting multiple factors supporting multiple movements driving the term structure as suggested to be true by empirical factor analysis (see Litterman and Scheinkman, 1991). Option pricing is provided in what follows.

Denote by $c(t,T)$ the time t price of a call option on the IDI, with time of maturity T and strike price K, then

$$\begin{aligned} c(t,T) &= E_t^{\mathbb{Q}}\left[\mathrm{e}^{-Y(t,T)}L_c(T)\right] \\ &= E_t^{\mathbb{Q}}\left[\left(IDI_t - K\mathrm{e}^{-Y(t,T)}\right)^+\right] \\ &= IDI_t E_t^{\mathbb{Q}}\left[\left(1-\tfrac{K}{IDI_T}\right)^+\right] \\ &= \int_{y\geq \log\frac{K}{IDI_t}} \left(IDI_t - K\mathrm{e}^{-y}\right) f_{Y|\mathcal{F}_t}(y)\mathrm{d}y. \end{aligned} \tag{2.13}$$

where $f_{Y|\mathcal{F}_t}(y)$ is the probability density of $Y(t,T)|\mathcal{F}_t$

By Lemma 2.1 we know that $Y|\mathcal{F}_t$ is normally distributed. Using a simple property of normal distribuition it is easy to compute $c(t,T)$, how demonstrated in the following Lemma.

Lemma 2.3 *The price at time t of the above mentioned option is*

$$c(t,T) = IDI_t\Phi(d) - KP(t,T)\Phi(d - \sqrt{V(t,T)}), \tag{2.14}$$

[11] Silva (1997) adopted the Black et al. (1990) model, Vieira Neto & Valls (1999) adopted the Vasicek (1977) model, and Fajardo & Ornelas (2003) adopted the CIR (1985) model.

where Φ *denotes the cumulative normal distribution function, and* d *is given by*

$$d = \frac{\log \frac{IDI_t}{K} - \log P(t,T) + V(t,T)/2}{\sqrt{V(t,T)}}. \tag{2.15}$$

If $p(t,T)$ is the price at time t of the IDI put with strike K and maturity T then, by the put-call parity, we have

$$p(t,T) = KP(t,T)\Phi(\sqrt{V(t,T)} - d) - IDI_t\Phi(-d).$$

2.4.4 Parameters Estimation

In this Subsection we estimate a three factor Gaussian model using Brazilian ID-Futures data and IDI option prices. The model parameters are estimated using a maximum likelihood procedure (see Appendix A for details) with two different strategies:

- First, we adopt only ID-Futures data. The reference market instrument observed without error are the reference ID bonds maturiting at 1, 126 and 252 days. Then to find the states vector at each time t we have to solve the following linear system:

$$\begin{aligned} rb_t(0.00397) &= -\tfrac{A(0.00397,\phi)}{0.00397} - \tfrac{B(0.00397,\phi)'}{0.00397}X_t \\ rb_t(0.5) &= -\tfrac{A(0.5,\phi)}{0.5} - \tfrac{B(0.5,\phi)'}{0.5}X_t \\ rb_t(1) &= -\tfrac{A(1,\phi)}{1} - \tfrac{B(1,\phi)'}{1}X_t. \end{aligned} \tag{2.16}$$

  For the reference ID bond with maturities 21, 63, 189 and 378 days, we assume observations with gaussian errors u_t uncorrelated along time:

$$\begin{aligned} rb_t(0.0833) &= -\tfrac{A(0.0833,\phi)}{0.0833} - \tfrac{B(0.0833,\phi)'}{0.0833}X_t + u_t(0.0833) \\ rb_t(0.25) &= -\tfrac{A(0.25,\phi)}{0.25} - \tfrac{B(0.25,\phi)'}{0.25}X_t + u_t(0.25) \\ rb_t(0.75) &= -\tfrac{A(0.75,\phi)}{0.75} - \tfrac{B(0.75,\phi)'}{0.75}X_t + u_t(0.75) \\ rb_t(1.5) &= -\tfrac{A(1.5,\phi)}{1.5} - \tfrac{B(1.5,\phi)'}{1.5}X_t + u_t(1.5). \end{aligned} \tag{2.17}$$

The Jacobian matrix is

$$Jac_t = \begin{bmatrix} -\frac{B(0.00397,\phi)'}{0.00397} \\ \\ -\frac{B(0.5,\phi)'}{0.5} \\ \\ -\frac{B(1,\phi)'}{1} \end{bmatrix}; \tag{2.18}$$

- Second, we introduce options information on the estimation procedure. Again, we assume that the yields of the reference ID bond with maturities 1, 126 and 252 days are observed without error. Then it is possible to obtain the values of the state vector at each time t solving the linear system 2.16. The difference between the two strategies appears only on the errors definitions. In the second strategy we include options price errors[12], that is, for each t the vector u_t has length five and its components are

$$\begin{aligned} rb_t(0.0833) &= -\tfrac{A(0.0833,\phi)}{0.0833} - \tfrac{B(0.0833,\phi)'}{0.0833}X_t + u_t(0.0833) \\ rb_t(0.25) &= -\tfrac{A(0.25,\phi)}{0.25} - \tfrac{B(0.25,\phi)'}{0.25}X_t + u_t(0.25) \\ rb_t(0.75) &= -\tfrac{A(0.75,\phi)}{0.75} - \tfrac{B(0.75,\phi)'}{0.75}X_t + u_t(0.75) \\ rb_t(1.5) &= -\tfrac{A(1.5,\phi)}{1.5} - \tfrac{B(1.5,\phi)'}{1.5}X_t + u_t(1.5) \\ cs_t &= c(t,T) + u_t(\text{call}). \end{aligned} \tag{2.19}$$

In both cases the transition probability from X_{t-1} to X_t in the real world is

$$p(X_t|X_{t-1};\phi) = \Psi\left(X_t, m_{\Delta t}, V_{\Delta t}\right),$$

with:

- $\Psi\left(x,m,V\right) = \left((2\pi)^{3/2}|V|\right)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}(x-m)'V^{-1}(x-m)}$,
- $m_{\Delta t} = \mathrm{e}^{(\rho B-\kappa)\Delta t}X_{t-1}$ and

[12] Alternatively, we can use implied volatility instead of prices to construct the options errors.

- $V_{\Delta t} = \Xi(\Delta t)\left[\int_0^{\Delta t} \Xi^{-1}(u)\rho\left(\Xi^{-1}(u)\rho\right)' \mathrm{d}u\right]\Xi(\Delta t)'$, where $\Xi(t)$ is the exponential of the matrix $(\rho B - \kappa)t$.

2.4.5 Empirical Results

Tables 2.1 and 2.2 present respectively the values of the parameters estimated for the model which does not adopt options and for the model that adopts options on its estimation procedure[13]. Standard deviations are obtained by the BHHH method [14]. Note that the majority of the parameters is significant at a 95% confidence interval (parameters whose ratio column presents bold values), except for the risk premia parameters, which are usually the hardest ones to pin down (see for instance Duffee, 2002 or Dai and Singleton, 2002 and 2000 for comparisons of results). Note also that the introduction of options information has a little effect on parameters estimation.

The mean absolute interest errors for the reference ID bond with maturities 21, 63, 126 and 378 days are respectively 0.18%, 0.07%, 0.02%, 0.12% and 0.28% both when we don't use options on the estimation procedure and when we use them. The mean absolute price error of the liquid IDI calls are 17.53% (when we use only the ID reference bonds) and 17.43% (when options are taken into account). Just for comparison purposes, using US LIBOR and swaps data to price the cap market, Jaganathan et al. (2003) presented a three-factor model which obtains an average relative pricing error of 30% on the first year of their sample period (February, 1995 to June, 1999), achieving errors higher than 50% on the remaining portion of their sample. Figure 2.1 shows the observed and the theoretical option price (estimated use only ID reference bonds) for the liquid calls[15]. Observe that Gaussian model sub-estimates the prices of liquid IDI calls.

The results obtained suggest that the use of options to calibrate Gaussian model has no effects on the parameters estimation and consequently on pricing bonds and options. Therefore, the Gaussian model is not appropriated to capture the information about the fixed income market contained on op-

[13]Parameters don't show on the tables are fixed equals to zero.

[14]See Davidson & MacKinnon (1993).

[15]Due the little difference between the two sets of parameters estimated, the theoretical IDI call price when synthetic calls are taken into account in the esimation procedure are very close to the theoretical option price when we use only ID reference bonds in the parameters estimation.

Parameter	Value	Standard Error	ratio $\frac{\text{abs(Value)}}{\text{Std Error}}$
κ_{11}	6.3518	0.0039	**1609.29**
κ_{22}	1.6206	0.0021	**775.07**
κ_{33}	0.0003	0.0000	**54991.03**
ρ_{11}	0.0916	0.0012	**78.93**
ρ_{21}	-0.0215	0.0011	**19.41**
ρ_{22}	0.0401	0.0004	**98.52**
ρ_{31}	-0.0008	0.0003	**2.54**
ρ_{32}	-0.0193	0.0001	**149.27**
ρ_{33}	0.0109	0.0001	**144.66**
$\lambda_X(11)$	-3.1285	124.0011	0.03
$\lambda_X(21)$	0.4265	69.3241	0.01
$\lambda_X(22)$	0.0052	11.1908	0.00
$\lambda_X(31)$	-1.9877	38.4728	0.05
$\lambda_X(32)$	2.5641	10.1220	0.25
$\lambda_X(33)$	-0.7475	7.8156	0.10
ϕ_0	0.18	-	-

Table 2.1: Parameters and standard errors for the Gauss model estimated using only ID-Future reference bonds.

Parameter	Value	Standard Error	ratio $\frac{\text{abs(Value)}}{\text{Std Error}}$
κ_{11}	6.3521	0.0019	**3371.18**
κ_{22}	1.6232	0.0022	**745.86**
κ_{33}	0.0003	0.0000	**7623834**
ρ_{11}	0.0923	0.0011	**84.29**
ρ_{21}	-0.0215	0.0009	**23.70**
ρ_{22}	0.0404	0.0003	**130.23**
ρ_{31}	-0.0008	0.0002	**4.20**
ρ_{32}	-0.0194	0.0001	**174.36**
ρ_{33}	0.0109	0.0001	**152.14**
$\lambda_X(11)$	-3.1471	117.7430	0.03
$\lambda_X(21)$	0.4286	66.7649	0.01
$\lambda_X(22)$	0.0052	11.4283	0.00
$\lambda_X(31)$	-1.9990	37.6906	0.05
$\lambda_X(32)$	2.5783	10.1888	0.25
$\lambda_X(33)$	-0.7518	7.8895	0.10
ϕ_0	0.18	-	-

Table 2.2: Parameters and standard errors for the Gauss model estimated using ID-Future reference bonds and synthetic IDI call option.

tions (a well-known empirical fact)[16]. The problem with Gaussian model is that the volatility structure of the interest rate is very simple which becomes impossible to the maximum likelihood estimator to extract information from options.

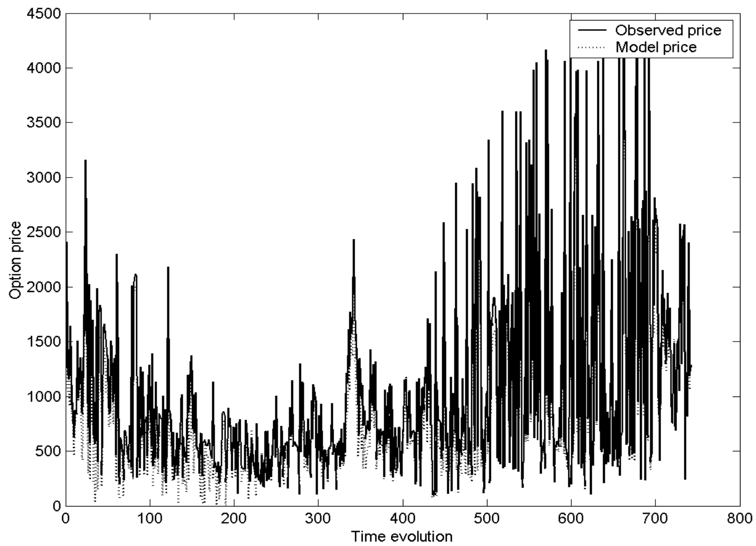

Figure 2.1: Observed IDI call price and model IDI call price (parameters estimated using only ID reference bonds).

2.5 The CIR Model

2.5.1 Model Specification

In the CIR model, the short term rate process is specified as a sum of N Feller process:

[16]We tried to introduce options information on the estimation procedure by assuming that the reference market instrument observed without error are the reference ID bonds maturiting at 1 and 252 days and the synthetic at-the-money IDI call option. But this strategy worsens the mean absolute value of the options prices errors. Probably, it happened because the errors for the other reference bonds are higher since we inverted only two ID reference bonds. For future work we will intend to calibrate a four-factor Gaussian model (three factors to invert ID reference bonds and one factor to invert options).

$$r_t = \phi_0 + \sum_{n=1}^{N} X_n(t), \tag{2.20}$$

where ϕ_0 is a constant and the dynamics of process X_n is given by

$$\mathrm{d}X_n(t) = \kappa_n \left(\theta_n - X_n(t)\right) \mathrm{d}t + \sigma_n \sqrt{X_n(t)} \mathrm{d}W_n^{\mathbb{Q}}(t), \quad n = 1, \ldots, N, \tag{2.21}$$

with $W^{\mathbb{Q}} = \left(W_1^{\mathbb{Q}}, \ldots, W_N^{\mathbb{Q}}\right)$ being an N-dimensional Brownian motion under $\mathbb{Q}$; κ_n, θ_n and σ_n are positive constants satisfying the Feller's condition $2\kappa_n\theta_n > \sigma_n^2$ for all n.

Further, we assume that the market price of risk process $\lambda^X(t) = \left(\lambda_1^X(t), \ldots, \lambda_N^X(t)\right)$ has the particular functional form

$$\lambda_n^X(t) = \frac{\lambda_n}{\sigma_n} \sqrt{X_n(t)}, \quad n = 1, \ldots, N. \tag{2.22}$$

Then, the connection between martingale probability measure $\mathbb{Q}$ and physical probability measure $\mathbb{P}$ is given by Girsanov's Theorem

$$\mathrm{d}W_n^{\mathbb{P}}(t) = \mathrm{d}W_n^{\mathbb{Q}}(t) - \lambda_n^X(t)\mathrm{d}t. \tag{2.23}$$

Substituing (2.23) in (2.21) we obtain the dynamics of X_n in the real world

$$\mathrm{d}X_n(t) = \overline{\kappa}_n \left(\overline{\theta}_n - X_n(t)\right) \mathrm{d}t + \sigma_n \sqrt{X_n(t)} \mathrm{d}W_n^{\mathbb{P}}(t), \quad n = 1, \ldots, N, \tag{2.24}$$

where $\overline{\kappa}_n = \kappa_n - \lambda_n$ and $\overline{\theta}_n = \frac{\kappa_n\theta_n}{\kappa_n - \lambda_n}$.

The probability density of X_n at time T under $\mathbb{Q}$, conditional on its value at the current time t, is given by (see Cox et al., 1985)

$$f_{X_n(T)|X_n(t)}(x) = c_n \mathrm{e}^{-u_n - v_n} \left(\frac{v_n}{u_n}\right)^{\frac{q_n}{2}} I_{q_n}\left(2\left(u_n v_n\right)^{\frac{1}{2}}\right), \tag{2.25}$$

where

$$c_n = \frac{2\kappa_n}{\sigma_n^2 \left(1 - \mathrm{e}^{-\kappa_n(T-t)}\right)}, \tag{2.26}$$

$$u_n = c_n X_n(t) \mathrm{e}^{-\kappa_n(T-t)}, \tag{2.27}$$

$$v_n = c_n x, \tag{2.28}$$

$$q_n = \frac{2\kappa_n\theta_n}{\sigma_n^2} - 1, \tag{2.29}$$

and $I_{q_n}(\cdot)$ is the modified Bessel function of the first kind of order q_n.

2.5.2 Pricing Zero Coupon Bonds

From Brigo & Mercurio (2001) we know that the time t price of a zero coupon bond maturing at time T is

$$P(t,T) = \mathrm{e}^{A(t,T)+B(t,T)'X_t}, \tag{2.30}$$

where

$$A(t,T) = -\phi_0\tau + \sum_{n=1}^{N} \frac{2\kappa_n\theta_n}{\sigma_n^2} \log\left(\frac{2\gamma_n \mathrm{e}^{\frac{(\kappa_n+\gamma_n)\tau}{2}}}{2\gamma_n + (\kappa_n+\gamma_n)\left(\mathrm{e}^{\tau\gamma_n}-1\right)}\right), \tag{2.31}$$

with $\gamma_n = \sqrt{\kappa_n^2 + 2\sigma_n^2}$ and $B(t,T)$ is a column vector with

$$B_n(t,T) = \frac{2\left(\mathrm{e}^{\tau\gamma_n}-1\right)}{2\gamma_n + (\kappa_n+\gamma_n)\left(\mathrm{e}^{\tau\gamma_n}-1\right)}, \tag{2.32}$$

in the n_{th} element.

Using Equation 2.30 and Itô's lemma we can obtain the dynamics of a bond price under the martingale measure $\mathbb{Q}$

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r_t\mathrm{d}t - B(t,T)'\mathrm{diag}\left(\left[\sigma_1\sqrt{X_1(t)},\ldots,\sigma_N\sqrt{X_N(t)}\right]\right)\mathrm{d}W^{\mathbb{Q}}, \tag{2.33}$$

where for $x \in \mathbb{R}^N$, $\mathrm{diag}(x)$ stands for a diagonal matrix with x_n in the n_{th} diagonal position.

To hold this bond, the investors will ask for an instantaneous expected excess return. Then, under the physical measure, the bond price dynamics is

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = \left(r_t + z^i(t,T)\right)\mathrm{d}t - B(t,T)'\mathrm{diag}\left(\left[\sigma_1\sqrt{X_1(t)},\ldots,\sigma_N\sqrt{X_N(t)}\right]\right)\mathrm{d}W^{\mathbb{P}}. \tag{2.34}$$

Applying Girsanov's Theorem to change measures we have

$$z^i(t,T) = -B(t,T)'\mathrm{diag}\left([\lambda_1,\ldots,\lambda_N]\right)X_t. \tag{2.35}$$

2.5.3 Pricing IDI Options

Accordingly to the explanation in Subsection 2.4.3 to pricing IDI options we have to know the distribuiton of $Y(t,T)$ under $\mathbb{Q}$ conditioned on the information available at time t. If r_t is a Gaussian process then $Y|\mathcal{F}_t$ is normally distributed. But in this case there isn't a close form for $f_{Y|\mathcal{F}_t}$[17]. Nevertheless, the Laplace transform of $Y(t,T)|\mathcal{F}_t$ can be calculated. Following Leblanc & Scaillet (1998) we have

$$\mathcal{L}\left(Y|\mathcal{F}_t\right)(s) = \mathrm{e}^{\zeta\phi_0\tau}\prod_{n=1}^{N}\mathcal{L}\left(Y_n|\mathcal{F}_t\right)(s), \tag{2.36}$$

where $\mathcal{L}\left(Y_n|\mathcal{F}_t\right)(s) = E_t^{\mathbb{Q}}\left[\mathrm{e}^{-sY_n}\right]$ is the Laplace Transform of $Y_n(t,T) = \int_t^T X_n(u)\mathrm{d}u$ that is given by[18]

$$\mathcal{L}\left(Y_n|\mathcal{F}_t\right)(s) = \left[\frac{2\tilde{\gamma}_n\mathrm{e}^{\frac{(\kappa_n+\tilde{\gamma}_n)\tau}{2}}}{2\tilde{\gamma}_n + (\kappa_n+\tilde{\gamma}_n)\left(\mathrm{e}^{\tau\tilde{\gamma}_n}-1\right)}\right]^{\frac{2\kappa_n\theta_n}{\sigma_n^2}} \cdot \mathrm{e}^{\frac{2\left(\mathrm{e}^{\tau\tilde{\gamma}_n}-1\right)}{2\tilde{\gamma}_n+(\kappa_n+\tilde{\gamma}_n)\left(\mathrm{e}^{\tau\tilde{\gamma}_n}-1\right)}},$$

with $\tilde{\gamma}_n = \tilde{\gamma}_n(s) = \sqrt{\kappa_n^2 + 2\sigma_n^2 s}$.

At this point, the difficulty arises because to implement this procedure we need a tool of numerical Laplace transform inversion.

Nagaradjasarma (2003) reviews the main approaches proposed in literature for pricing asian options on interest rate in the CIR model. As point out above, the problem consists in inverting a Laplace transform. Each method presents its advantages and drawbacks. We implemented the follow methods: Laguerre series expansion (Dufresne, 2000), Widder (1946) and Abate & Whitt (1995). The last method is the most satisfactory both on the processing time and on precision point of view.

Observe that after to invert the Laplace transform (Equation 2.36) we would need a new numerical procedure, namely, the integration of the function $\left(IDI_t - K\mathrm{e}^{-y}\right)f_{Y|\mathcal{F}_t}(y)$ on the interval $\left[\log\frac{K}{IDI_t}, +\infty\right)$ (see Equation

[17] See Dufresne (2001) for a proof that $Y(t,T)|\mathcal{F}_t$ has a continuos version.

[18] The Laplace Transform of $Y_n(t,T)$ is just the time t bond price maturing at T when short term rate follows a one-dimensional CIR process with parameters $(\kappa_n, s\theta_n, \sqrt{s}\sigma_n)$ starting from sr_t.

2.13). Nevertheless, Dassios & Nagaradjasarma (2003) showed that is possible to compute the call price in only one step as described in the next lemma.

Lemma 2.4 *For $\mu \geq 0$, denote by $\mathcal{G}(s,\mu)$ the inverse Laplace transform of*

$$\frac{E_t^{\mathbb{Q}}\left(\mathrm{e}^{-(\mu+s)}\right)}{s} = \frac{\mathcal{L}\left(Y|\mathcal{F}_t\right)(\mu+s)}{s}$$

with respect to $s > 0$. Hence

$$c(t,T) = K\mathcal{G}\left(\log\frac{K}{IDI_t}, 1\right) - IDI_t\mathcal{G}\left(\log\frac{K}{IDI_t}, 0\right) - KP(t,T) + IDI_t. \tag{2.37}$$

2.5.4 Parameters Estimation

In this Subsection we estimate two versions of a three factor CIR model using a maximum likelihood procedure (see Appendix A for details) in a similar way that was done to Gaussian model. The first one employs only ID-Futures data and the second one employs both ID-Futures data and IDI option prices. Let us describe in more detalis these two strategies.

For both versions we assume that the ID reference bonds maturiting at 1, 126 and 252 days are observed without errors. Then to find the states vector at each time t we have to solve the linear system 2.16 with $A(t,T)$ and $B(t,T)$ given by Equations 2.31 and 2.32, respectively. Moreover, the Jacobian matrix is specified by Equation 2.18.

The difference between the two versions appears only on the errors definitions. Whereas to the first version we use errors definided by Equations 2.17, to the second version the errors are defined by Equations 2.19.

Since the dynamics of X in the real world is a Feller process and the X_n's are independent, the transition probability from $X(t-1)$ to $X(t)$ in the real world is

$$p(X(t)|X(t-1);\phi) = \prod_{n=1}^{3} f_{X_n(t)|X_n(t-1)}(x)\Bigg|_{x=X_n(t)},$$

with $f_{X_n(t)|X_n(t-1)}(\cdot)$ given by Equation 2.25 changing κ_n by $\overline{\kappa}_n$ and θ_n by $\overline{\theta}_n$.

2.5.5 Empirical Results

Tables 2.3 and 2.4 present the values of the parameters (estimated respectively without and with options information) as well the asymptotic standard deviations to test their significance. Except for the risk premia parameter λ_3, all parameters is significant at a 95% confidence interval. The mean absolute error of the yields of the reference bonds maturiting at 21, 63, 189 and 378 days are respectively 0.21%, 0.10%, 0.03%, and 0.14% when we don't use options and 0.20%, 0.10%, 0.03% and 0.14% when we use them.

Parameter	Value	Standard Error	ratio $\frac{\text{abs(Value)}}{\text{Std Err.}}$
κ_1	78.5620	0.2646	**296.88**
κ_2	0.0001	0.0000	**14.47**
κ_3	1.7183	0.0117	**146.36**
θ_1	0.0113	0.0001	**121.55**
θ_2	35.3915	0.4814	**73.52**
θ_3	0.0825	0.0003	**242.76**
σ_1	0.3958	0.0113	**34.91**
σ_2	0.0619	0.0003	**239.84**
σ_3	0.3172	0.0048	**66.62**
λ_1	33.4554	4.4172	**7.57**
λ_2	1.8281	0.1524	**12.00**
λ_3	-0.0013	3.7908	0.00
ϕ_0	0	-	-

Table 2.3: Parameters and standard errors for the CIR model estimated using only ID-Future reference bonds.

The mean absolute error of the liquid IDI calls are 67.70% (without options in the estimation procedure) and 32.44% (with options). These results point out an improvement on pricing calls when IDI options are taken into account in the parameters estimation. Figure 2.2 shows the theoretical options prices and the real prices. Observe that without options the model super-estimate options prices. This happens because the estimator has no information about the options (which has information about volatility) and consequently it understands the jumps in the interest rates as a high volatility. Figure 2.3 illustrates the instantaneous volatility of increments on the

Parameter	Value	Standard Error	ratio $\frac{\text{abs(Value)}}{\text{Std Err.}}$
κ_1	58.1921	0.0120	**4829.88**
κ_2	0.0000	0.0000	**19.00**
κ_3	1.6668	0.0057	**294.83**
θ_1	0.0109	0.0000	**359.85**
θ_2	38.1596	0.0048	**7878.28**
θ_3	0.0843	0.0001	**655.39**
σ_1	0.3869	0.0029	**133.52**
σ_2	0.0507	0.0001	**413.38**
σ_3	0.1975	0.0011	**187.93**
λ_1	37.0680	3.1359	**11.82**
λ_2	-0.6328	0.0894	**7.08**
λ_3	0.0002	1.2450	0.00
ϕ_0	0	-	-

Table 2.4: Parameters and standard errors for the CIR model estimated using ID-Future reference bonds and options data.

short term rate for the two sets of parameters estimated. Note that without options the estimator super-estimate the volatility.

The empirical results of the CIR model calibration permit us to conclude that exist extra factos driving the volatility of the interest rate not present on the bonds market.

2.6 The SVCEG Model

2.6.1 Model Specification and Bonds Price

Building on Collin-Dufresne and Goldstein (2001) and Casassus et al. (2005), we provide in this Section the complete theoretical characterization of a dynamic term structure model exhibiting USV to be implemented using simultaneous data from the ID-Futures and IDI-options markets. We denominate this model the Stochastic Volatility Conditional Extended Gaussian Model (SVCEG) which is described below.

The model is within the class of affine models analysed by Duffie and Kan (1996). It presents two stochastic factors, X_t and Z_t that compose

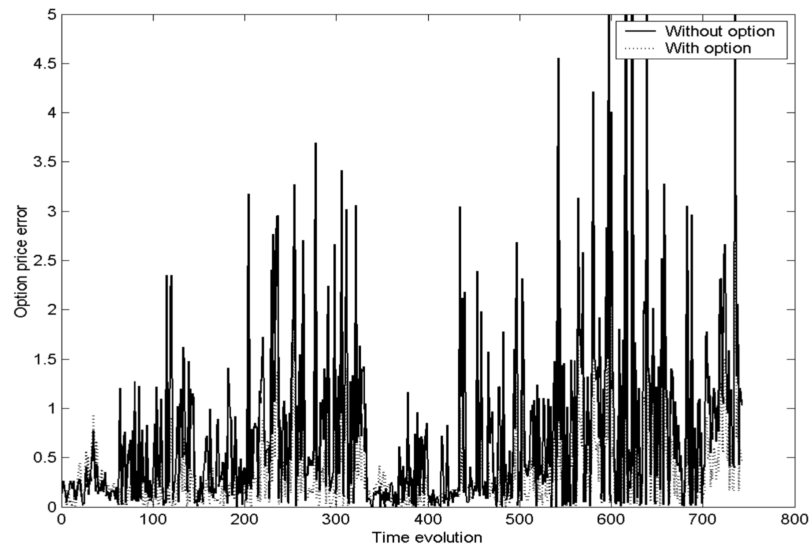

Figure 2.2: Options prices erros on the CIR model.

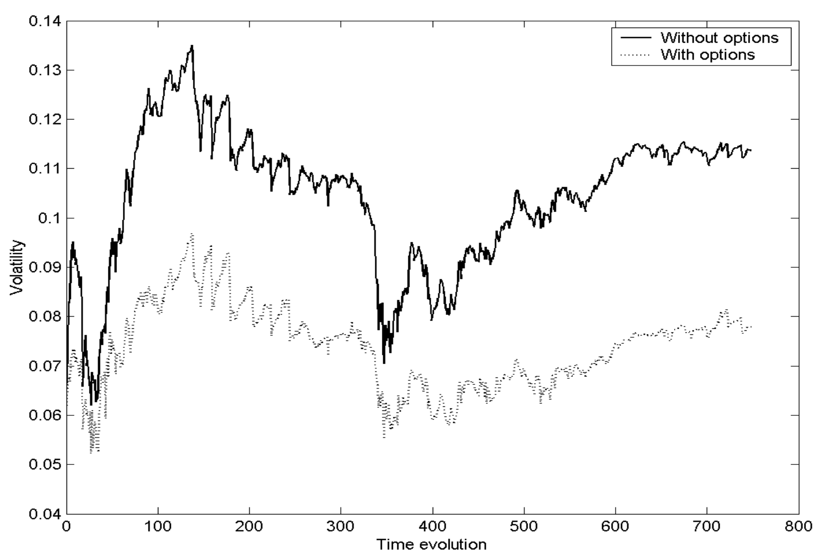

Figure 2.3: Instantaneous volatility of the increments on the short term rate on the CIR model.

the short term rate r_t, one stochastic factor v_t which represents the instantaneous volatility of factor Z_t, and a conditionally deterministic factor θ_t which represents the time varying long term mean of factor Z_t:

$$\begin{aligned}
r_t &= \phi_0 + X_t + Z_t, \\
\mathrm{d}X_t &= \eta(\mu - X_t)dt + \sigma \mathrm{d}W_X^{\mathbb{Q}}(t), \\
\mathrm{d}Z_t &= \kappa(\theta_t - Z_t)dt + \sqrt{v_t}\mathrm{d}W_Z^{\mathbb{Q}}(t), \\
\mathrm{d}\theta_t &= (\gamma_t - 2\kappa\theta_t + \tfrac{v_t}{\kappa})\mathrm{d}t \quad \text{and} \\
\mathrm{d}v_t &= (\alpha - \beta v_t)\mathrm{d}t + \delta\sqrt{v_t}\mathrm{d}W_v^{\mathbb{Q}}(t);
\end{aligned} \tag{2.38}$$

where $W_X^{\mathbb{Q}}$, $W_Z^{\mathbb{Q}}$ and $W_v^{\mathbb{Q}}$ are independent brownian motions. Note that the volatility v_t follows a CIR process and by the independence assumption if we condition on the path of volatility we have θ_t a deterministic function and the short rate would follow a two factor extended Gaussian process with time-varying long term mean θ_t. It directly follows from Casassus et al. (2005) that the time t price of a zero coupon bond maturing at time T is given by:

$$P(t,T) = \mathrm{e}^{A(t,T) + B_X(\tau)X_t + B_Z(\tau)Z_t + B_\theta(\tau)\theta_t}, \tag{2.39}$$

where

$$\begin{aligned}
B_X(\tau) &= -\tfrac{1-\mathrm{e}^{-\eta\tau}}{\eta}, \\
B_Z(\tau) &= -\tfrac{1-\mathrm{e}^{-\kappa\tau}}{\kappa}, \\
B_\theta(\tau) &= -\tfrac{(1-\mathrm{e}^{-\kappa\tau})^2}{2\kappa}, \\
A(t,T) &= -\phi_0\tau + A_X(t,T) + A_Z(t,T), \\
A_X(t,T) &= -\left(\mu - \tfrac{\sigma^2}{2\eta^2}\right)(B_X(\tau) + \tau) - \tfrac{\sigma^2}{4\eta}B_X(\tau)^2, \\
A_Z(t,T) &= \textstyle\int_t^T \gamma_s B_\theta(T-s)\mathrm{d}s \quad \text{and} \\
\tau &= T - t.
\end{aligned} \tag{2.40}$$

Note that the price of the bond does not depend directly on the volatility variable creating an incomplete market where options are actually needed to hedge against the uncertainty of the volatility, not covered by the cross section of bond prices.

In order to relate the brownian motions under the risk neutral measure to the brownian motions under the physical measure, we have to define a parametric form for the risk premia charged by investors, which under a dynamic term structure model is represented by the market price of risk. In the SVCEG model we work with an extended affine market price of risk (see Cheridito et al. 2003):

$$\mathrm{d}W_X^{\mathbb{Q}}(t) = \mathrm{d}W_X^{\mathbb{P}}(t) + \frac{1}{\sigma}\left(\lambda_0^X + \lambda_1^X X_t\right)\mathrm{d}t, \tag{2.41}$$

$$\mathrm{d}W_Z^{\mathbb{Q}}(t) = \mathrm{d}W_Z^{\mathbb{P}}(t) + \frac{1}{\sqrt{v_t}}\left(\lambda_0^Z + \lambda_1^Z Z_t\right)\mathrm{d}t \tag{2.42}$$

and

$$\mathrm{d}W_v^{\mathbb{Q}}(t) = \mathrm{d}W_v^{\mathbb{P}}(t) + \frac{1}{\delta\sqrt{v_t}}\left(\lambda_0^v + \lambda_1^v v_t\right)\mathrm{d}t. \tag{2.43}$$

Then, under the physical probability measure the dynamics of (X_t, Z_t, v_t) is

$$\begin{aligned}
\mathrm{d}X_t &= \tilde{\eta}\left(\tilde{\mu} - X_t\right)\mathrm{d}t + \sigma\mathrm{d}W_X^{\mathbb{P}}(t)\\
\mathrm{d}Z_t &= \tilde{\kappa}\left(\tilde{\theta}_t - Z_t\right)\mathrm{d}t + \sqrt{v_t}\mathrm{d}W_Z^{\mathbb{P}}(t) \quad \text{and}\\
\mathrm{d}v_t &= (\tilde{\alpha} - \tilde{\beta}v_t)\mathrm{d}t + \delta\sqrt{v_t}\mathrm{d}Z_t^{\mathbb{P}},
\end{aligned}$$

where $\tilde{\eta} = \eta - \lambda_1^X$, $\tilde{\mu} = \frac{\eta\mu + \lambda_0^X}{\eta - \lambda_1^X}$, $\tilde{\kappa} = \kappa - \lambda_1^Z$, $\tilde{\theta}_t = \frac{\kappa\theta_t + \lambda_0^Z}{\tilde{\kappa}}$, $\tilde{\alpha} = \alpha + \lambda_0^v$ and $\tilde{\beta} = \beta - \lambda_1^v$.

The risk neutral bond price dynamics is

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r_t\mathrm{d}t + \sigma B_X(\tau)\mathrm{d}W_X^{\mathbb{Q}} + B_Z(\tau)\sqrt{v_t}\mathrm{d}W_Z^{\mathbb{Q}},$$

once more we can see that bond prices are insensitive to volatility-risk and hence cannot be used to hedge it. Under the physical measure, the bond price dynamics is

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = (r_t + z^i(t,T))\mathrm{d}t + \sigma B_X(\tau)\mathrm{d}W_X^{\mathbb{P}} + B_Z(\tau)\sqrt{v_t}\mathrm{d}W_Z^{\mathbb{P}},$$

where the instantaneous expected excess return is given by

$$z^i(t,T) = B_X(\tau)(\lambda_0^X + \lambda_1^X X_t) + B_Z(\tau)(\lambda_0^Z + \lambda_1^Z Z_t).$$

2.6.2 Pricing IDI Options

At time t, an IDI call with time of maturity T, and strike K can be priced by the same technique applied by Hull & White (1987) in one of the seminal papers on stochastic volatility models: By the independence of the brownian motions $W_Z^{\mathbb{Q}}$ and ${W_v}^{\mathbb{Q}}$, conditioning on the volatility path, it does not affect the distribution of $W_Z^{\mathbb{Q}}$ and we can use the law of iterated expectations to obtain the price as a double expectation. The inner is going to present a Black & Scholes type of analytical formula (see Section 2.4), while the external expectation integrates the volatility distribution, essentially a non-central χ^2 distribution. The model can be extended to deal with correlation between the brownian motions $W_Z^{\mathbb{Q}}$ and ${W_v}^{\mathbb{Q}}$ (see Casassus et al., 2005).

In the sequel, we present a series of lemmas which culminates in obtaining the price of an IDI option as a function of the state variables under the SVCEG model. These results will be useful when implementing a dynamic version of the model.

Let $\mathcal{F}_{t,T}^{v}$ be the σ-field that represents the information on the volatility process between times t and T, i.e $\mathcal{F}_{t,T}^{v} = \sigma\left\{v_u : u \in [t,T]\right\}$. Denote by $\mathcal{G}_{t,T}$ the σ-field generated by the union of the σ-fields $\mathcal{F}_{t,T}^{v}$ and $\mathcal{F}_t$, i.e. $\mathcal{G}_{t,T} = \sigma\left\{\mathcal{F}_t \cup \mathcal{F}_{t,T}^{v}\right\}$. The following lemma is a generalization of Lemma 2.1.

Lemma 2.5 *Let $Y(t,T) = \int_t^T r_u \mathrm{d}u$, where r_t presents the dynamics described by Equation 2.38. Then conditional on $\mathcal{G}_{t,T}$, $Y(t,T)$ is normally distributed with mean $M(t,T)$ and variance $V(t,T)$ given by:*

$$M(t,T) = \phi_0\tau + M_X(t,T) + M_Z(t,T)$$

and

$$V(t,T) = V_X(t,T) + V_Z(t,T),$$

with

$$M_X(t,T) = \tau\mu + \frac{1}{\eta}(X_t - \mu)\left(1 - \mathrm{e}^{-\eta\tau}\right), \tag{2.44}$$

$$V_X(t,T) = \frac{\sigma^2}{2\eta^3}\left(4\mathrm{e}^{-\eta\tau} - \mathrm{e}^{-2\eta\tau} + 2\eta\tau - 3\right), \tag{2.45}$$

$$M_Z(t,T) = \frac{1 - \mathrm{e}^{-\kappa(T-t)}}{\kappa} Z_t + \int_t^T \left(1 - \mathrm{e}^{-\kappa(T-u)}\right)\theta_u \mathrm{d}u, \quad \textit{and} \tag{2.46}$$

$$V_Z(t,T) = \frac{1}{\kappa^2}\int_t^T \left(1 - \mathrm{e}^{-\kappa(T-u)}\right)^2 v_u \mathrm{d}u, \tag{2.47}$$

where $v_u : u \in [t,T]$ *is the path of the volatility conditional on* $\mathcal{G}_{t,T}$ *and*

$$\theta_u = \mathrm{e}^{-2\kappa(u-t)}\left(\theta_t + \int_t^u \mathrm{e}^{-2\kappa(t-s)}\left(\gamma_s + \frac{v_s}{\kappa}\right)\mathrm{d}s\right), \quad t \le u \le T. \tag{2.48}$$

Lemma 2.6 *The time* t *price of a zero coupon bond maturing a time* T *can be written as*

$$P(t,T) = \mathrm{e}^{-\phi_0\tau - M(t,T) + \frac{V(t,T)}{2}},$$

that is,

$$M_Z(t,T) = \frac{V_Z(t,T)}{2} - A_Z(t,T) + B_z(\tau)Z_t + B_\theta(\tau)\theta_t \tag{2.49}$$

and

$$M_X(t,T) = \frac{V_X(t,T)}{2} - A_X(t,T) - B_X(\tau)X_t. \tag{2.50}$$

Lemma 2.7 *The time* t *price of a call option on the IDI with time to maturity* T *and strike price* K *is*

$$c(t,T) = \mathrm{E}^{\mathbb{Q}}\left[f\left(IDI_t, K, t, T, V(t,T)\right)|\mathcal{F}_t\right], \tag{2.51}$$

where

$$f\left(IDI_t, K, t, T, V(t,T)\right) =$$

$$IDI_t\Phi\left(d\right) - KP(t,T)\Phi\left(d - \sqrt{V(t,T)}\right),$$

with

$$d = \frac{\log\frac{IDI_t}{K} - \log P(t,T) + V(t,T)/2}{\sqrt{V(t,T)}}. \tag{2.52}$$

If the call option is at-the-money (i.e., $IDI_t = KP(t,T)$) Equation 2.51 simplify to

$$c(t,T) = \mathrm{E}^{\mathbb{Q}}\left[IDI_t\left(2\Phi\left(\frac{\sqrt{V(t,T)}}{2}\right) - 1\right)\middle|\mathcal{F}_t\right]. \tag{2.53}$$

Finally, using the well known fact that an at-the-money option is almost a linear function on Black's volatility, we obtain

$$c(t,T) = IDI_t\left[2\Phi\left(\frac{\sqrt{\mathrm{E}^{\mathbb{Q}}\left(V(t,T)|\mathcal{F}_t\right)}}{2}\right) - 1\right]. \tag{2.54}$$

Lemma 2.8

$$\mathrm{E}^{\mathbb{Q}}\left(V(t,T)|\mathcal{F}_t\right) = V_X(t,T) + \frac{v_t}{\kappa^2}c_1(t,T) + \frac{\alpha}{\beta\kappa^2}c_2(t,T),$$

where:

- $c_1(t,T) = \frac{1-\mathrm{e}^{-\beta\tau}}{\beta} - 2\frac{\mathrm{e}^{-\beta\tau}-\mathrm{e}^{-\kappa\tau}}{\kappa-\beta} + \frac{\mathrm{e}^{-\beta\tau}-\mathrm{e}^{-2\kappa\tau}}{2\kappa-\beta}$ *and*
- $c_2(t,T) = \frac{1}{\kappa}\left(-\frac{3}{2} + 2\mathrm{e}^{-\kappa\tau} - \frac{\mathrm{e}^{-2\kappa\tau}}{2}\right) + \tau - c_1(t,T)$.

Note that Lemma 2.7 completely characterizes the price of an IDI option as a function of the state variables $(X_t, Z_t, \theta_t, v_t)$ while Lemma 2.8 combined with Equation 2.54 gives an approximation to the option price which depends only on the stochastic volatility variable v_t, as long as the option is at-the-money. For an empirical application, the approximation should be nice because stochastic volatility can be explicitly extracted from option prices, and its risk premia might be studied under the SVCEG model.

2.6.3 Parameters Estimation

Once more, we use a maximum likelihood estimation to obtain the parameters of the model specified in Subsection 2.6.1. Denote by $IDIspot \in \mathbb{R}^H$ the vector of spot IDI. To simplify the estimation procedure, we consider $\gamma_t = \gamma$ for all t and taking $\Delta t = 1/252$ years. Assume that the vector parameter is $\phi = (\phi_0, \eta, \mu, \sigma, \kappa, \gamma, \alpha, \beta, \delta, \lambda_0^X, \lambda_1^X, \lambda_0^Z, \lambda_1^Z, \lambda_0^v, \lambda_1^v)$.

By Equation 2.39 we know that

$$R(t,\tau,\phi) = -\frac{A(\tau,\phi)}{\tau} - \frac{B_X(\tau,\phi)}{\tau}X_t - \frac{B_Z(\tau,\phi)}{\tau}Z_t - \frac{B_\theta(\tau,\phi)}{\tau}\theta_t, \quad (2.55)$$

where $B_X(\tau,\phi)$, $B_Z(\tau,\phi)$ and $B_\theta(\tau,\phi)$ are given by Equation 2.40 and $A(\tau,\phi) = \phi_0\tau + A_X(\tau,\phi) - \frac{\gamma}{2\kappa}\left[\tau + \frac{1}{\kappa}\left(-\frac{3}{2} + 2\mathrm{e}^{-\kappa\tau} - \frac{\mathrm{e}^{-2\kappa\tau}}{2}\right)\right]$.

Since $\Delta t = 1/252$ is small, we discretize the dynamics of the SVCEG model in the following way:

$$X_t = X_{t-1} + \tilde{\eta}\left(\tilde{\mu} - X_{t-1}\right)\Delta t + \sigma\epsilon_X, \quad (2.56)$$

$$Z_t = Z_{t-1} + \tilde{\kappa}\left(\tilde{\theta}_{t-1} - Z_{t-1}\right)\Delta t + \sqrt{v_{t-1}}\epsilon_Z, \quad (2.57)$$

$$\theta_t = \theta_{t-1} + \left(\gamma - 2\kappa\theta_{t-1} + \frac{v_{t-1}}{\kappa}\right)\Delta t \quad \text{and} \tag{2.58}$$

$$v_t = v_{t-1} + \left(\tilde{\alpha} - \tilde{\beta} v_{t-1}\right)\Delta t + \delta\sqrt{v_{t-1}}\epsilon_v, \tag{2.59}$$

where ϵ_X, ϵ_Z and ϵ_v are independent normal distributions with mean zero and variance Δt.

In order to obtain the values of the state vector at each time t we use a recursive procedure. First, we need to choose three instruments to be priced without error in order to invert the state vector. Note that under the SVCEG model we are forced to choose at least one of the instruments priced without error to be an option, because by construction we are not able to extract the volatility state variable from the cross sectional of bond prices alone. Let us assume that the yield of the reference ID bonds with fixed maturity 21 and 252 days and the price of the synthetic IDI call option are observed without error. If we know θ_t then (X_t, Z_t, v_t) can be calculated by Equations 2.54 and 2.55, and Lemma 2.8. Then, θ_{t+1} might be calculated making use of Equation 2.58.

Reference ID bond with maturities 63, 126, 189, and 378 are assumed to be priced with gaussian errors u_t uncorrelated throughout time.

The Jacobian matrix is

$$Jac_t = \begin{bmatrix} -\frac{B_X(0.083,\phi)}{0.083} & -\frac{B_Z(0.083,\phi)}{0.083} & 0 \\ -B_X(1,\phi) & -B_X(1,\phi) & 0 \\ 0 & 0 & \frac{c_1(95/252,\phi) IDIspot_t}{2\kappa^2\sqrt{\upsilon}}\Psi\left(\frac{\sqrt{\upsilon}}{2}, 0, 1\right) \end{bmatrix},$$

with

- $\upsilon = V_X(t,T) + \frac{v_t c_1(95/252,\phi)}{\kappa^2} + \frac{\alpha c_2(95/252,\phi)}{\beta\kappa^2}$ and
- c_1 and c_2 are given by Lemma 2.8.

The transition probability from $(X_{t-1}, Z_{t-1}, v_{t-1})$ to (X_t, Z_t, v_t), under the physical probability measure is

$$p\left((x_t, z_t, v_t)|(x_{t-1}, z_{t-1}, v_{t-1});\phi\right) = \Psi\left((X_t, Z_t, v_t), m_{\Delta t}, V_{\Delta t}\right),$$

where

$$m_{\Delta t} =$$

$$\left(X_{t-1} + \tilde{\eta}(\tilde{\mu} - X_{t-1})\Delta t, Z_{t-1} + \tilde{\kappa}(\tilde{\theta}_{t-1} - Z_{t-1})\Delta t, v_{t-1} + \left(\tilde{\alpha} - \tilde{\beta} v_{t-1}\right)\Delta t\right)',$$

$$V_{\Delta t} = \Delta t \left(\text{diag}\left[\sigma^2, v_{t-1}, \delta^2 v_{t-1}\right]\right)$$

and $\Psi(\cdot)$ is the three-dimensional normal probability density function (see Subsection 2.4.4).

2.6.4 Empirical Results

Table 2.5 presents the values of the parameters and their asymptotic standard deviations. Except for γ and λ_1^Z, all parameters is significant at a 95% confidence interval.

Parameter	Value	Standard Error	ratio $\frac{\text{abs(Value)}}{\text{Std Err.}}$
η	4.3508	0.1195	**36.41**
σ	0.0155	0.0001	**123.54**
κ	0.0000	0.0000	**348.52**
γ	-0.4377	0.8577	0.51
α	0.0014	0.0000	**42.72**
β	0.2856	0.0889	**3.21**
δ	0.0518	0.0002	**241.87**
λ_X^1	-0.8926	0.1644	**5.43**
λ_Z^1	-0.0065	0.7926	0.0081
λ_v^1	-5.3205	0.0904	**58.85**
ϕ_0	0.15	-	-

Table 2.5: Parameters and standard errors for the SVCEG model.

The mean absolute errors of the yields are 0.13%, 0.16%, 0.10% and 0.25%. These errors are greater than those obtained both in the Gaussian and CIR models. Of course this is a consequence of the fact that we only use two factors to compose the instantaneous interest rate in the SVCEG model.

To pricing the liquid IDI calls we use Monte Carlo simulation[19]. In order to obtain the simulated paths we discretized the stochastic differential equations of X_t and Z_t in the same way that was done in Equations 2.56 and 2.57 whereas stochastic differential equation of v_t was discretized via the Milstein Scheme (Kloden & Platen, 1992):

[19]We can not pricing the liquid IDI calls by means of Equation 2.54 since they are not necessarily at-the-money.

$$v_t = v_{t-1} + (\alpha - \beta v_{t-1})\Delta t + \delta\sqrt{v_{t-1}}\epsilon_v + \frac{\delta^2}{4}(\epsilon_v^2 - \Delta t).$$

To reduce the variance of the simulation error, we use antithetic variable technique (Glasserman, 2003).

Figures 2.4 and 2.5 show the observed and the model price of the liquid IDI call. The mean absolute error of the liquid IDI calls is 22%. Observe that this error is lower than CIR model's error and greater than Gaussian model's error. Probably the Gaussian error is lower than SVCEG error because in the first model the yields errors are small. For future works we will intend to extent the SVCEG model to incorporate three factors describing the short term interest rate. Hence we will expect to observe yields errors smaller and consequently an improvement on option pricing.

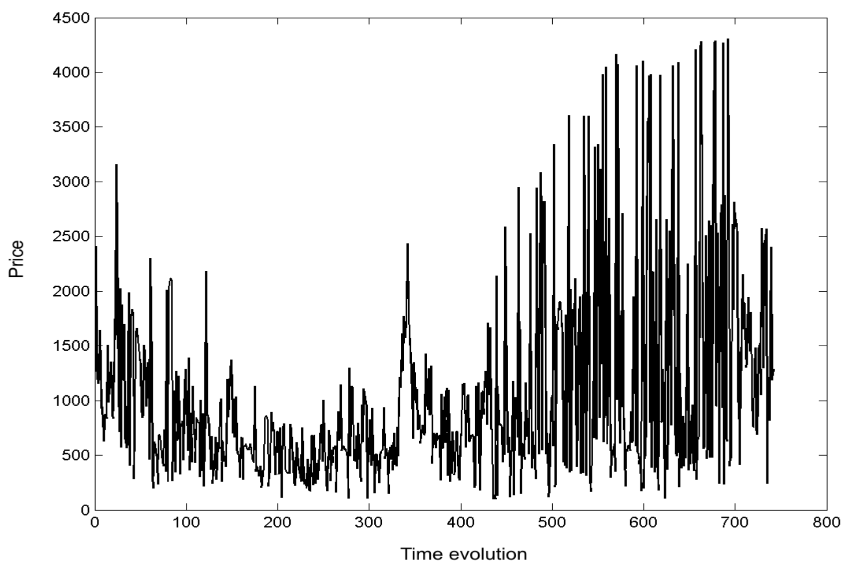

Figure 2.4: Observed price of the liquid IDI call.

Figure 2.6 illustrates the instantaneous volatility of the one-year bond return extracted from the CIR model (without and with options) and the USV model[20]. Note that on average the instantaneous volatility is smaller

[20]For the Gaussian model, the instantaneous volatility of the one-year bond is constant and equal to 7.22%

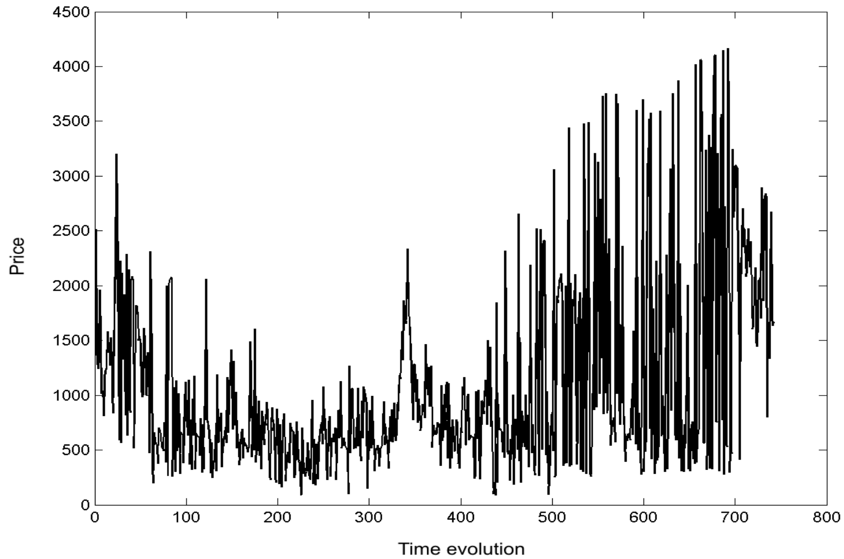

Figure 2.5: USV model price of the liquid IDI call.

and closer to the GARCH volatility one-year bond (Figure 2.7) in the USV model than in the CIR model.

2.7 Conclusion

Recent work on empirical asset pricing theory make use of combined information from both underlying and derivatives markets. Following this trend, we combine information from the Brazilian ID-Future market and its corresponding IDI option market. We obtain from a regression analysis, evidence on the existence of dynamic sources of uncertainty that would be driving the option market independently from the underlying market. Results obtained with the regressions were confirmed through the implementation of a dynamic multi-factor CIR model. The goal was to test the model ability to price options, when its parameters were estimated by two distinct strategies. In the first we took into account only the ID-Future market data whereas in the second we used both ID-Future and IDI options. We observed that an expressive improvement on pricing IDI options when the second estimation strategy was applied.

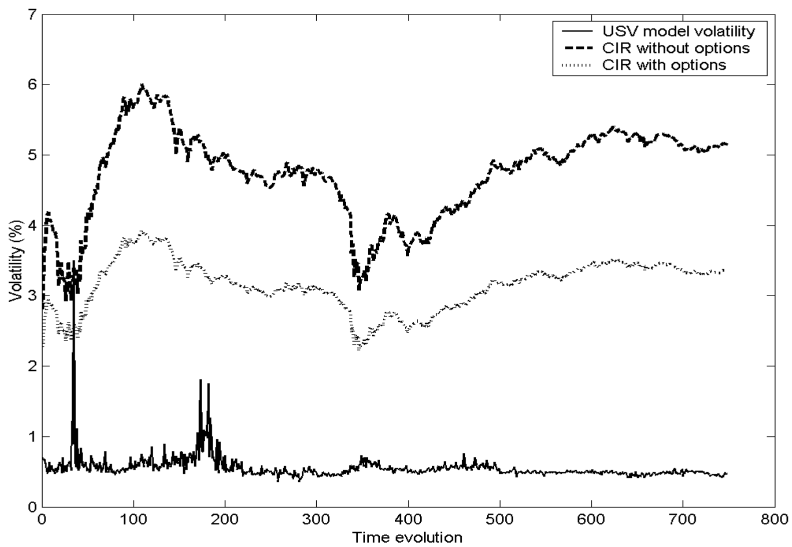

Figure 2.6: Instantaneous volatility of the one-year bond return.

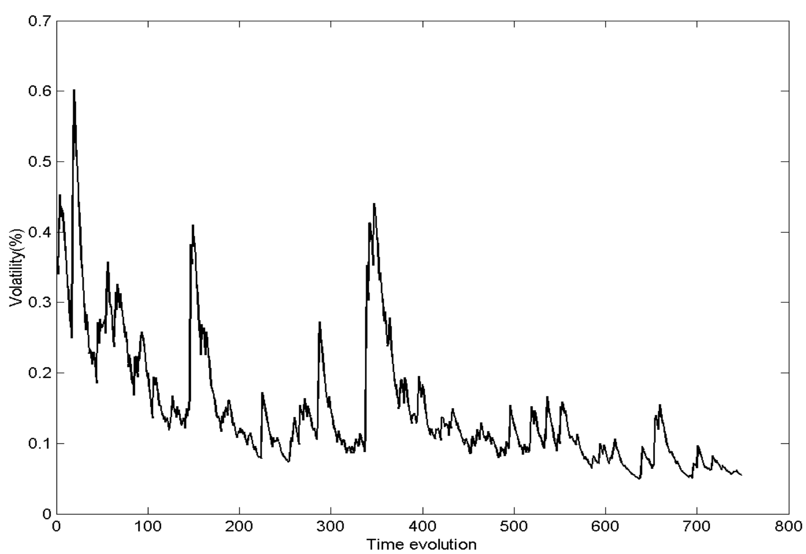

Figure 2.7: Instantaneous volatility of the one-year bond return obtained by GARCH method.

At the same time we implemented a multi-factor Gaussian model. Due to the very simple volatility strucuture of this model we didn't obtain any significant progress on price IDI calls when options were took into account in the estimation procedure. However, aside its widely known imperfections (e.g. short term rate with positive probability and constant volatility through time), the Gaussian model presents the best results in terms of call price and yields errors.

Based on these evidences, we propose a theoretical dynamic term structure model consistent with Unspanned Stochastic Volatility and present detailed results on the pricing of bonds and options under this model, as well as on how to implement it based on a mixed ID-Future market/IDI option database. For future works we will intend to implement a four-factor USV model which three factors will be related with the yields bonds and the other will be related with IDI options. Certainly this model will present yields error smaller than the USV model studied here and probably we will observe an improvement on pricing IDI calls.

Appendix A
Maximum Likelihood Estimation

On this work, we adopt the maximum likelihood estimation procedure described in Chen and Scott (1993). We observe the following reference ID bonds yields along H points in time: $rb_t(1/252)$, $rb_t(21/252)$, $rb_t(63/252)$, $rb_t(126/252)$, $rb_t(189/252)$, $rb_t(1)$ and $rb_t(1.5)$[21]. Let rb represents the $H \times 7$ matrix containing these ID bonds yields for all H points. In addition we observe the price cs_t for an at-the-money call with time to maturity of 95/252 years at the same H points. Let cs be the vector of length H that represents these call prices. The reference ID bonds and the at-the-money call are called reference market instruments. Denote by $rmi = [rb, cs]$ the $H \times 8$ matrix containing the yields and the price of these reference market instruments. Assume that the model parameters are represented by the vector ϕ and that the difference between times $t-1$ and t is Δt. Finally, let $g_i(X_t; t, \phi)$ be the function that represents the relation between the reference market instrument i and the state variables at time t for $i = 1, \ldots, 8$.

In particular, we assume that the reference market instruments i_1, i_2 and i_3 are observed without error. To obtain the values of the state vector at each time t we have to solve a system:

$$\begin{aligned} g_{i_1}(X_t; t, \phi) &= rmi(t, i_1) \\ g_{i_2}(X_t; t, \phi) &= rmi(t, i_2) \\ g_{i_3}(X_t; t, \phi) &= rmi(t, i_3). \end{aligned} \tag{2.60}$$

For the reference market instruments i_4, i_5, i_6, i_7 and i_8, we assume observation with gaussian errors u_t uncorrelated along time:

$$\begin{aligned} & rmi(t, \begin{bmatrix} i_4 & i_5 & i_6 & i_7 & i_8 \end{bmatrix}) - u_t = \\ & \begin{bmatrix} g_{i_4}(X_t; t, \phi) & g_{i_5}(X_t; t, \phi) & g_{i_6}(X_t; t, \phi) & g_{i_7}(X_t; t, \phi) & g_{i_8}(X_t; t, \phi) \end{bmatrix} \end{aligned} \tag{2.61}$$

Now, we can write the log-likelihood function as

$$\begin{aligned} & L(\phi, rb) = \sum_{t=2}^{H} \log p(X_t | X_{t-1}; \phi) - \\ & - \sum_{t=2}^{H} \log |Jac_t| - \frac{H-1}{2} \log |\Omega| - \frac{1}{2} \sum_{t=2}^{H} u_t' \Omega^{-1} u_t, \end{aligned} \tag{2.62}$$

[21] $rb_t(\tau)$ stands for the time t reference ID bonds yields with time to maturity of τ.

where:

1. $Jac_t = \begin{bmatrix} \frac{\partial g_{i_1}(X_t;t,\phi)}{\partial X_t} \\ \frac{\partial g_{i_2}(X_t;t,\phi)}{\partial X_t} \\ \frac{\partial g_{i_3}(X_t;t,\phi)}{\partial X_t} \end{bmatrix}$ is the Jacobian matrix of the transformation defined by Equation 2.60;

2. Ω represents the covariance matrix for u_t, estimated using the sample covariance matrix of the u_t's implied by the extracted state vector along time;

3. $p(X_t|X_{t-1};\phi)$ is the transition probability from X_{t-1} to X_t in the real world (that is, under the measure $\mathbb{P}$).

Our final objective is to estimate the vector of parameters ϕ which maximizes function $L(\phi, rb)$. In order to try to avoid possible local minima we use several different starting values and search for the optimal point by making use of the Nelder-Mead Simplex algorithm for non-linear functions optimization (implemented in the $MatLab^{TM}$ *fminsearch* function) and the gradient-based optimization method (implemented in the $MatLab^{TM}$ *fminunc* function).

Appendix B
Proofs

Proof. of Lemma 2.1
It is not hard to verify by Ito's rule that for each $t < T$ the unique (strong) solution of (2.5) is[22]

$$X_T^i = X_t^i \mathrm{e}^{-\kappa_i(T-t)} + \sum_{j=1}^{N} \rho_{ij} \int_t^T \mathrm{e}^{-\kappa_i(T-s)} \mathrm{d}W_s^j, \quad i = 1, \ldots, N.$$

Then

$$r_T = \phi_0 + \sum_{i=1}^{N} \left(X_t^i \mathrm{e}^{-\kappa_i(T-t)} + \sum_{j=1}^{N} \rho_{ij} \int_t^T \mathrm{e}^{-\kappa_i(T-s)} \mathrm{d}W_s^j \right).$$

Stochastic integration by parts implies that

$$\int_t^T X_u^i \mathrm{d}u = \int_t^T (T-u)\, \mathrm{d}X_u^i + (T-t)\, X_t^i. \tag{2.63}$$

By definition of X, the integral in the right-hand side can be written as

$$\int_t^T (T-u)\, \mathrm{d}X_u^i = -\kappa_i \int_t^T (T-u)\, X_u^i \mathrm{d}u + \sum_{j=1}^{N} \rho_{ij} \int_t^T (T-u)\, \mathrm{d}W_u^j.$$

But

$$\int_t^T (T-u)\, X_u^i \mathrm{d}u =$$

$$= X_t^i \int_t^T (T-u)\, \mathrm{e}^{-\kappa_i(u-t)} \mathrm{d}u + \sum_{j=1}^{N} \rho_{ij} \int_t^T (T-u) \int_t^u \mathrm{e}^{-\kappa_i(u-s)} \mathrm{d}W_s^j \mathrm{d}u.$$

Calculating separately the last two integrals, we have

$$\int_t^T (T-u)\, \mathrm{e}^{-\kappa_i(u-t)} \mathrm{d}u = \left(\frac{T-t}{\kappa_i} + \frac{\mathrm{e}^{-\kappa_i(u-t)} - 1}{\kappa_i^2} \right)$$

[22] In this Appendix we drop the superscript $\mathbb{Q}$ and denote the N-dimensional brownian motion $W^{\mathbb{Q}}$ simply by W.

and, again by integration by parts,

$$\int_t^T (T-u) \int_t^u \mathrm{e}^{-\kappa_i (u-s)} \mathrm{d}W_s^j \mathrm{d}u =$$

$$= \int_t^T \left(\int_t^u \mathrm{e}^{\kappa_i s} \mathrm{d}W_s^j \right) \mathrm{d}_u \left(\int_t^u (T-v)\, \mathrm{e}^{-\kappa_i v} \mathrm{d}v \right) =$$

$$= \left(\int_t^T \mathrm{e}^{\kappa_i u} \mathrm{d}W_u^j \right) \left(\int_t^T (T-v)\, \mathrm{e}^{-\kappa_i v} \mathrm{d}v \right) -$$

$$- \int_t^T \left(\int_t^u (T-v)\, \mathrm{e}^{-\kappa_i v} \mathrm{d}v \right) \mathrm{e}^{\kappa_i u} \mathrm{d}W_u^j =$$

$$= \int_t^T \left(\int_u^T (T-v)\, \mathrm{e}^{-\kappa_i v} \mathrm{d}v \right) \mathrm{e}^{\kappa_i u} \mathrm{d}W_u^j =$$

$$\tfrac{1}{\kappa_i} \int_t^T \left(T - u + \tfrac{\mathrm{e}^{-\kappa_i (T-u)} - 1}{\kappa_i} \right) \mathrm{d}W_u^j.$$

Substituting the previous terms in Equation 2.63, we obtain

$$\int_t^T X_u^i \mathrm{d}u = (T-t)\, X_t^i -$$

$$-\kappa_i \left[X_t^i \left(\tfrac{T-t}{\kappa_i} + \tfrac{\mathrm{e}^{-\kappa_i (T-t)} - 1}{\kappa_i^2} \right) + \textstyle\sum_{j=1}^N \tfrac{\rho_{ij}}{\kappa_i} \int_t^T \left(T - u + \tfrac{\mathrm{e}^{-\kappa_i (T-u)} - 1}{\kappa_i} \right) \mathrm{d}W_u^j \right] +$$

$$+ \textstyle\sum_{j=1}^N \rho_{ij} \int_t^T (T-u)\, \mathrm{d}W_u^j =$$

$$= -\tfrac{\mathrm{e}^{-\kappa_i (T-u)} - 1}{\kappa_i} X_t^i + \textstyle\sum_{j=1}^N \rho_{ij} \int_t^T -\tfrac{\mathrm{e}^{-\kappa_i (T-u)} - 1}{\kappa_i} \mathrm{d}W_u^j =$$

$$= \tfrac{1 - \mathrm{e}^{-\kappa_i (T-u)}}{\kappa_i} X_t^i + \tfrac{1}{\kappa_i} \textstyle\sum_{j=1}^N \rho_{ij} \int_t^T \left(1 - \mathrm{e}^{-\kappa_i (T-u)} \right) \mathrm{d}W_u^j,$$

that is,

$$\int_t^T X_u^i \mathrm{d}u = \frac{1 - \mathrm{e}^{-\kappa_i (T-u)}}{\kappa_i} X_t^i + \frac{1}{\kappa_i} \sum_{j=1}^N \rho_{ij} \int_t^T \left(1 - \mathrm{e}^{-\kappa_i (T-u)} \right) \mathrm{d}W_u^j. \quad (2.64)$$

Then $Y(t,T) = \phi_0 (T-t) + \sum_{i=1}^N \int_t^T X_u^i \mathrm{d}u$ conditional on $\mathcal{F}_t$ is normally distributed (see Duffie, 2001) with mean

$$M(t,T) = \phi_0 (T-t) + \sum_{i=1}^N \frac{1 - \mathrm{e}^{-\kappa_i (T-t)}}{\kappa_i} X_t^i, \quad (2.65)$$

where we only used the fact that the stochastic integral in 2.64 is a martingale. The variance of $Y(t,T)|\mathcal{F}_t$ is

$$V(t,T) = \mathrm{var}^{\mathbb{Q}}\left[\sum_{i=1}^{N}\frac{Y_i}{\kappa_i}|\mathcal{F}_t\right], \tag{2.66}$$

where $Y_i = \sum_{j=1}^{N}\rho_{ij}\int_t^T\left(1-\mathrm{e}^{-\kappa_i(T-u)}\right)\mathrm{d}W_u^j$. Then

$$V(t,T) = \sum_{i=1}^{N}\frac{\mathrm{var}^{\mathbb{Q}}(Y_i|\mathcal{F}_t)}{\kappa_i^2} + 2\sum_{i=1}^{N}\sum_{k>i}\frac{\mathrm{cov}^{\mathbb{Q}}(Y_i, Y_k|\mathcal{F}_t)}{\kappa_i\kappa_k}.$$

Using Ito's isometry we have

$$V(t,T) = \sum_{i=1}^{N}\tfrac{1}{\kappa_i^2}\sum_{j=1}^{N}\rho_{ij}^2\int_t^T\left(1-\mathrm{e}^{-\kappa_i(T-u)}\right)^2\mathrm{d}u +$$
$$+2\sum_{i=1}^{N}\sum_{k>i}\tfrac{1}{\kappa_i\kappa_k}\sum_{j=1}^{N}\rho_{ij}\rho_{kj}\int_t^T\left(1-\mathrm{e}^{-\kappa_i(T-u)}\right)\left(1-\mathrm{e}^{-\kappa_k(T-u)}\right)\mathrm{d}u. \tag{2.67}$$

At this point, simple integration produces

$$V(t,T) = \sum_{i=1}^{N}\tfrac{1}{\kappa_i^2}\left(\tau + \tfrac{2}{\kappa_i}\mathrm{e}^{-\kappa_i\tau} - \tfrac{1}{2\kappa_i}\mathrm{e}^{-2\kappa_i\tau} - \tfrac{3}{2\kappa_i}\right)\sum_{j=1}^{N}\rho_{ij}^2 +$$
$$+2\sum_{i=1}^{N}\sum_{k>i}\tfrac{1}{\kappa_i\kappa_k}\left(\tau + \tfrac{\mathrm{e}^{-\kappa_i\tau}-1}{\kappa_i} + \tfrac{\mathrm{e}^{-\kappa_k\tau}-1}{\kappa_k} - \tfrac{\mathrm{e}^{-(\kappa_i+\kappa_k)\tau}-1}{\kappa_i+\kappa_k}\right)\sum_{j=1}^{N}\rho_{ij}\rho_{kj}, \tag{2.68}$$

where $\tau = T - t$. □

Proof of Lemma 2.2
The martingale condition for bond prices (Duffie, 2001) gives

$$P(t,T) = \mathrm{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_u \mathrm{d}u}|\mathcal{F}_t\right] = \mathrm{E}^{\mathbb{Q}}\left[\mathrm{e}^{-y(t,T)}|\mathcal{F}_t\right]. \tag{2.69}$$

Now the normality of variable $Y(t,T)|\mathcal{F}_t$ (Lemma 2.1), and a simple property of the mean of log-normal distributions complete the proof. □

Proof of Lemma 2.3
By Equation 2.13 the proof consists of a simple calculation of the ordinary

integral $\mathrm{E}^{\mathbb{Q}}\left[\max\left(IDI_t - K\mathrm{e}^{-y}, 0\right)|\mathcal{F}_t\right]$.

$$c\left(t,T\right) = \mathrm{E}^{\mathbb{Q}}\left[\max\left(IDI_t - K\mathrm{e}^{-y}, 0\right)|\mathcal{F}_t\right] =$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V(t,T)}} \max\left(IDI_t - K\mathrm{e}^{-y}, 0\right) \mathrm{e}^{-\frac{(y-M(t,T))^2}{2V(t,T)}} \mathrm{d}y = \tag{2.70}$$

$$= \int_{\log(K/IDI_t)}^{\infty} \frac{1}{\sqrt{2\pi V(t,T)}} \left(IDI_t - K\mathrm{e}^{-y}\right) \mathrm{e}^{-\frac{(y-M(t,T))^2}{2V(t,T)}} \mathrm{d}y.$$

Making the substitution $z = \frac{y-M(t,T)}{\sqrt{V(t,T)}}$ we have:

$$c\left(t,T\right) = \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}} \left(IDI_t - K\mathrm{e}^{-z\sqrt{V(t,T)}-M(t,T)}\right) \mathrm{e}^{-\frac{1}{2}z^2}\mathrm{d}z =$$

$$= IDT_t \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}z^2}\mathrm{d}z - K\int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z\sqrt{V(t,T)}-M(t,T)-\frac{1}{2}z^2}\mathrm{d}z = \tag{2.71}$$

$$= IDI_t\Phi\left(d\right) - K\mathrm{e}^{-M(t,T)+\frac{V(t,T)}{2}} \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\left(z+\sqrt{V(t,T)}\right)^2}\mathrm{d}z.$$

where d is given by Equation 2.15. Making a new substitution $v = z + \sqrt{V\left(t,T\right)}$ and using Lemma 2.2 results in Equation 2.14. □

Proof of Lemma 2.5

By definition of r_t we have

$$Y(t,T) = \phi_0\tau + \int_t^T X_u\mathrm{d}u + \int_t^T Z_u\mathrm{d}u.$$

From Lemma 2.1 we know that $\int_t^T X_u\mathrm{d}u$ conditioned on $\mathcal{F}_t$ is normal with mean and variance given by Equations 2.7 and 2.8 respectively.

By an argument similar to the one presented on the proof of Lemma 2.1, it is simple to show that

$$\int_t^T Z_u\mathrm{d}u = \kappa\int_t^T(T-u)\theta_u\mathrm{d}u - \kappa^2 I(t,T) + \frac{1-\mathrm{e}^{-\kappa(T-t)}}{\kappa}Z_t +$$

$$+ \int_t^T \left(T-u-\kappa\int_t^T(T-s)\mathrm{e}^{-\kappa(s-u)}\mathrm{d}s\right)\sqrt{v_u}\mathrm{d}W_Z^{\mathbb{Q}}(u),$$

where $I(t,T) = \int_t^T(T-u)\int_t^u \mathrm{e}^{-\kappa(u-s)}\theta_s\mathrm{d}s\mathrm{d}u$. Then

$$M_Z(t,T) = \kappa\int_t^T(T-u)\theta_u\mathrm{d}u - \kappa^2 I(t,T) + \frac{1-\mathrm{e}^{-\kappa(T-t)}}{\kappa}Z_t, \tag{2.72}$$

Using Ito's isometry we have

$$V_Z(t,T) = \frac{1}{\kappa^2}\int_t^T \left(1-\mathrm{e}^{-\kappa(T-u)}\right)^2 v_u \mathrm{d}u.$$

Now, changing the order of integration in $I(t,T)$ we obtain

$$I(t,T) = \int_t^T \frac{T-u}{\kappa}\theta_u \mathrm{d}u + \int_t^T \frac{\mathrm{e}^{-\kappa(T-u)}-1}{\kappa^2}\theta_u \mathrm{d}u.$$

Substituting this expression in Equation 2.72 we conclude the proof. □

Proof of Lemma 2.6

Equation 2.50 is a direct consequence of simple algebric manipulations of Equations that define $M_X(t,T)$, $V_X(t,T)$, $B_X(\tau)$ and $A_X(t,T)$.

By Lemma 2.5 we know that

$$M_Z(t,T) = B_Z(\tau)Z_t + \int_t^T \left(1-\mathrm{e}^{-\kappa(T-u)}\right)\theta_u \mathrm{d}u,$$

then is sufficient to show that

$$\int_t^T \left(1-\mathrm{e}^{-\kappa(T-u)}\right)\theta_u \mathrm{d}u = \frac{V_Z(t,T)}{2} - A_Z(t,T) + B_\theta(\tau)\theta_t.$$

By Equation 2.48 we have

$$\begin{aligned}
&\textstyle\int_t^T \left(1-\mathrm{e}^{-\kappa(T-u)}\right)\theta_u \mathrm{d}u = \\
&\textstyle\int_t^T \mathrm{e}^{-2\kappa(u-t)}\left(\theta_t + \int_t^u \mathrm{e}^{-2\kappa(t-s)}\left(\gamma_s + \frac{v_s}{\kappa}\right)\mathrm{d}s\right)\left(1-\mathrm{e}^{-\kappa(T-u)}\right)\mathrm{d}u = \\
&\textstyle\theta_t \mathrm{e}^{2\kappa t}\int_t^T \mathrm{e}^{-2\kappa u}\left(1-\mathrm{e}^{-\kappa(T-u)}\right)\mathrm{d}u + \\
&\textstyle\int_t^T\int_s^T \mathrm{e}^{-2\kappa u}\left(1-\mathrm{e}^{-\kappa(T-u)}\right)\mathrm{e}^{2\kappa s}\left(\gamma_s+\frac{v_s}{\kappa}\right)\mathrm{d}u\mathrm{d}s = \\
&\textstyle\frac{\theta_t}{2\kappa}\left(1-\mathrm{e}^{-\kappa(T-t)}\right)^2 + \frac{1}{2\kappa}\int_t^T\left(\gamma_u+\frac{v_u}{\kappa}\right)\left(1-\mathrm{e}^{-\kappa(T-u)}\right)^2\mathrm{d}u = \\
&\textstyle B_\theta(\tau)\theta_t - A_Z(t,T) + \frac{V_Z(t,T)}{2},
\end{aligned}$$

which concludes the proof. □

Proof of Lemma 2.7
Using the law of iterated expectations we have

$$c(t,T) = \mathrm{E}^{\mathbb{Q}}\left[\max\left(IDI_t - K\mathrm{e}^{-Y(t,T)}, 0\right)|\mathcal{F}_t\right] =$$

$$\mathrm{E}^{\mathbb{Q}}\left[\mathrm{E}^{\mathbb{Q}}\left[\max\left(IDI_t - K\mathrm{e}^{-Y(t,T)}, 0\right)|\mathcal{G}_{t,T}\right]|\mathcal{F}_t\right],$$

by the same argument using in the proof of Lemma 2.3 and now applying Lemmas 2.5 and 2.2 results. □

Proof of Lemma 2.8

$$\mathrm{E}^{\mathbb{Q}}\left(V(t,T)|\mathcal{F}_t\right) =$$

$$V_X(t,T) + \tfrac{1}{\kappa^2}\int_t^T \left(1 - \mathrm{e}^{-\kappa(T-u)}\right)^2 \mathrm{E}^{\mathbb{Q}}\left(v_u|\mathcal{F}_t\right)\mathrm{d}u =$$

$$V_X(t,T) + \tfrac{1}{\kappa^2}\int_t^T \left(1 - \mathrm{e}^{-\kappa(T-u)}\right)^2 \left(v_t\mathrm{e}^{-\beta(u-t)} + \tfrac{\alpha}{\beta}\left(1 - \mathrm{e}^{-\beta(u-t)}\right)\right)\mathrm{d}u,$$

where in last step we have used the property of the mean of a CIR process (see Brigo and Mercurio, 2001). Expanding the terms in the right side and calculating the ordinary integrals give the desired result. □

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