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# The Lyapunov exponents of conservative continuous-time dynamical systems

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À memória do meu Pai
À minha Mãe
À minha Marisa...

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**Abstract**

We prove that for a $C^1$-generic (dense $G_\delta$) subset of all the conservative vector fields on 3-dimensional compact manifolds without singularities, we have for $\mu$-a.e. point $p \in M$ that either the Lyapunov exponents at $p$ are zero or $X$ is an Anosov vector field where $\mu$ is the Lebesgue measure. We also prove a similar version of the previous result in the setting of conservative non-autonomous linear differential systems in the $C^0$ topology. Finally we prove that for a $C^1$-dense subset of all the conservative vector fields on 3-dimensional compact manifolds with singularities, we have for $\mu$-a.e. $p \in M$ that either the Lyapunov exponents at $p$ are zero or $p$ belongs to a compact invariant set with ($m_p$-)dominated splitting for the linear Poincaré flow.

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# Chapter 1

# Introduction and statement of the results

Lyapunov exponents measure the exponential behavior of the tangent map of a dynamical system and if they are non-null together with Hölder regularity and the *Pesin Theory* of non-uniformly hyperbolic systems we get a rich information about geometric properties of the system, namely stable/unstable manifold theory for $\mu$-a.e. point in $M$ and this geometric tools are the base of most of the central results on dynamical systems nowadays. So it is of utmost importance detect when do Lyapunov exponents vanish.

A central result in this direction for discrete systems is the *Mañé-Bochi* theorem, which provides a $C^1$-residual set of area-preserving diffeomorphisms on surfaces where either we have Anosov systems or for $\mu$-a.e. point zero Lyapunov exponents. This theorem was announced in the beginning of the 1980's by Mañé (1948-1995) in [10] but there was only available a sketch of a proof, see [11], the complete proof due to Bochi appeared in [3].

Motivated by these results, Bochi-Viana in [4] extend this result to a large class of discrete systems: volume preserving diffeomorphisms with arbitrary dimension, symplectic maps and also linear cocycles. For a survey of the theory see [5].

Highly inspired by their results we explore here the continuous-time case by following closely the strategy for the proof of discrete case doing the natural adaptations and developing the required techniques for perturbations of vector fields. Our first result is the analogous to the *Mañé-Bochi* theorem for volume-preserving vector fields in 3-dimensional compact manifolds without singularities;

**Theorem 1** *There is a residual* $\mathfrak{R} \subseteq \mathfrak{X}^1_\mu(M)^*$ *such that if* $X \in \mathfrak{R}$ *then we have:*

*(a)* $X$ *is Anosov or*

*(b) Zero Lyapunov exponents for* $\mu$*-a.e.* $p \in M$.

Then we treat the more general setting of *linear differential systems* over continuous flows in compact Hausdorff spaces. For that we consider a dynamics in the base given by a $C^0$ flow $\varphi^t : X \to X$ and a dynamics in the 2-dimensional fiber bundles given by a continuous-time $C^0$ cocycle $\Phi^t : X \to \mathbb{GL}(2,\mathbb{R})$. For the *area-preserving* case, say when $\Phi^t(p) \in \mathbb{SL}(2,\mathbb{R})$, we induce the flow $A(\varphi^t(p)) = \frac{d}{ds}\Phi^s(p)|_{s=t} \circ [\Phi^t(p)]^{-1}$ which is *traceless.* We denote by $\Phi^t_A$ the cocycle associated to $A$. Analogously we also consider the setting of *modified area-preserving* cocycles $\Phi^t(p) \in \mathbb{GL}(2,\mathbb{R})$ satisfying for $p \notin \mathrm{Fix}(\varphi^t)$, $\det\Phi^t(p) = \frac{a(p)}{a(\varphi^t(p))}$ for some non-null sub-exponential continuous function $a : X \to \mathbb{R}$ and for $p \in \mathrm{Fix}(\varphi^t)$ we have $\det\Phi^t(p) = 1$ for all $t \in \mathbb{R}$. This set mimics the volume preserving flows in 3-dimensional manifolds eventually with fixed points. In both settings we have:

**Theorem 2** *There is a residual $\mathfrak{R}$ such that if $A \in \mathfrak{R}$ then for $\mu$-a.e. $p \in X$:*

*(a) $\Phi^t_A(p)$ has a dominated splitting or*

*(b) the Lyapunov exponents are zero.*

Finally using the ideas we developed to prove the previous Theorems, jointly with some simple observations and again in the 3-dimensional setting we are able to prove the following:

**Theorem 3** *There is a dense $\mathfrak{D} \subseteq \mathfrak{X}^1_\mu(M)$ such that if $X \in \mathfrak{D}$, then there exists $X^t$-invariant sets $D$ and $O$ satisfying $\mu(D \cup O) = 1$ and*

*(a) For $p \in O$ we have zero Lyapunov exponents.*

*(b) $D$ is a countable increasing union of compact invariant sets $\Lambda_{m_n}$ admitting a $m_n$-dominated splitting for the linear Poincaré flow.*

# Chapter 2

# Preliminaries

## 2.1 Notation

We denote by $\mathfrak{X}^1_\mu(M)$ the set of all vector fields $X : M \longrightarrow TM$ defined on a 3-dimensional compact, connected, without boundary $C^\infty$ Riemannian manifold $M$. We assume that $\mathfrak{X}^1_\mu(M)$ is endowed with the $C^1$ topology. The measure $\mu$, associated to the volume form $\omega$, will be called Lebesgue measure. We have an associated flow $X^t$ which is the infinitesimal generator of $X$, that is $\frac{dX^t}{dt}|_{t=s}(p) = X(X^s(p))$. This flow has a tangent map $DX^t_p$ which is the solution of the non-autonomous linear differential equation $\dot{u}(t) = DX_{X^t(p)} \cdot u(t)$ called the *linear variational equation*. The subset of $\mathfrak{X}^1_\mu(M)$ formed by the vector fields without singularities will be denoted by $\mathfrak{X}^1_\mu(M)^*$.

## 2.2 Oseledets's Theorem for 3-flows

The Oseledets's Theorem [14] is valid in the setting of discrete-time *cocycles* (for a prove see [12]). It holds in particular for any dynamical cocycle over a diffeomorphism $f : M \to M$ defined by a continuous map $F(p, v) = (f(p), Df_p \cdot v)$ which verifies $\Pi \circ F = f \circ \Pi$ where $\Pi : TM \to M$ is the canonical projection and $F(p, \cdot)$ is linear on the fiber $T_pM$. Oseledets's Theorem asserts that we have for $\mu$-a.e. point $p \in M$ a splitting $T_pM = E^1_p \oplus ... \oplus E^{k(p)}_p$ (Oseledets splitting) and real numbers $\lambda_1(p) > ... > \lambda_{k(p)}(p)$ (Lyapunov exponents) such that $Df_p(E^i_p) = E^i_{f(p)}$ and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n_p \cdot v^i\| = \lambda_i(p)$$

for any $v^i \in E^i_p - \{0\}$ and $i = 1, ..., k(p)$. Consider $X \in \mathfrak{X}^1_\mu(M)$ and the associated flow $X^t : M \to M$. Since Oseledets's Theorem is an asymptotic result and $DX^r_{(\cdot)}$, for fixed $r$, is an uniformly bounded operator we may replace the tangent map $DX^t_p = DX^r_{X^n(p)} \circ DX^n_p$ by the least integer time-$n$ map,

$DX_p^n$, and reformulate Oseledets's Theorem. Oseledets's Theorem allow us to conclude also that:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log|\det(DX_p^t)| = \sum_{i=1}^{k(p)} \lambda_i(p).dim(E_p^i) \tag{2.1}$$

which is related to the sub-exponential decrease of the angle between any subspaces of the Oseledets splitting along $\mu$-a.e. orbits. Since we have $DX_p^t(X(p)) = X(X^t(p))$, we already know one of the Oseledets subspaces, $\mathbb{R}X(p)$, and also that its associated Lyapunov exponent is zero. For the other two, in the conservative setting on 3-manifolds we have $|\det(DX_p^t)| = 1$, so by (2.1) we have $\lambda_1(p) + \lambda_3(p) = 0$, hence either $\lambda_1(p) = -\lambda_3(p) > 0$ or both are zero. The former case gives for $\mu$-a.e. $p \in M$ two directions $E_p^u$ and $E_p^s$ respectively associated to the positive Lyapunov exponent and the negative one with asymptotic exponential behavior. We denote by $\mathfrak{O}(X)$ the Oseledets points, $\mathfrak{O}^+(X) \subseteq \mathfrak{O}(X)$ the points with positive Lyapunov exponent and $\mathfrak{O}^0(X) \subseteq \mathfrak{O}(X)$ the points with all Lyapunov exponents zero. We note that $\mathfrak{O}^+(X) = \mathfrak{O}(X) - \mathfrak{O}^0(X)$. When there is no ambiguity we denote $\mathfrak{O}(X)$ by $\mathfrak{O}$ omitting the vector field.

## 2.3 The linear Poincaré flow

Let $R$ be the set of non-singular points for the vector field $X$. $X$ induces a decomposition of the tangent bundle in a way that each fiber $T_pM$ has a splitting $N_p \oplus \mathbb{R}X(p)$ where $N_p = (\mathbb{R}X(p))^\perp$ is the normal sub-bundle for $p \in R$. Consider the automorphism of vector bundles:

$$\begin{array}{rccc} DX^t: & T_RM & \longrightarrow & T_RM \\ & (p,v) & \longmapsto & (X^t(p), DX^t(p)\cdot v) \end{array}$$

In spite of $R$ being $X^t$-invariant and $\mathbb{R}X(p)$ being $DX^t$-invariant, there is no reason for the sub-bundle $N_R$ to be $DX^t$-invariant. So consider the quotient space $\widetilde{N}_R = T_RM/\mathbb{R}X(R)$ of equivalence classes which is isometrically isomorphic to $N_R$ via $\phi : N_R \longrightarrow \widetilde{N}_R$. The restriction map $DX^t\mid_{\widetilde{N}_R}$is $DX^t$-invariant. There exists an unique map $P_X^t(p) : N_R \longrightarrow N_R$ such that the diagram commutes:

$$\begin{array}{ccc} N_R & \xrightarrow{P_X^t} & N_R \\ \phi\downarrow & & \phi\downarrow \\ \widetilde{N}_R & \xrightarrow{DX^t} & \widetilde{N}_R \end{array}$$

The linear map $P_X^t(p) : N_p \longrightarrow N_{X^t(p)}$ is defined by

$$P_X^t(p)\cdot v = \Pi_{X^t(p)} \circ DX^t(p)\cdot v,$$

where $\Pi_{X^t(p)}$ denotes the canonical projection on $N_{X^t(p)}$. The linear map is also a flow since $P_X^{t+s}(p) = P_X^t(X^s(p)) \circ P_X^s(p)$.

We call $\{P_X^t(p)\}_{t\in\mathbb{R}}$ the *linear Poincaré flow* at $p$ associated to the vector field $X$ and this notion was first introduced by Doering in [7] to prove the hyperbolicity of robustly transitive 3-dimensional flows.

In our setting, if we have an Oseledets point $p$ with $X(p) \neq 0$ and $p \in \mathfrak{O}^+$, the Oseledets splitting on $T_pM$ induces a $P_X^t(p)$-invariant splitting on $N_p$, say $\Pi_p(E_p^\sigma) = N_p^\sigma$ for $\sigma = u, s$. If $p \in \mathfrak{O}^0$, then the $P_X^t$-invariant splitting will be trivial, that is, it is just the normal sub-bundle.

The next lemma shows that the dynamics remains the same.

**Lemma 2.3.1** *The Lyapunov exponents of $P_X^t(p)$ associated to the subspaces $N_p^u$ and $N_p^s$ are respectively $0 \leq \lambda_u(p)$ and $\lambda_s(p) \leq 0$.*

**Proof:** Consider a vector $n^u \in N_p^u$ and denote by $\theta_t = \angle(X(X^t(p)), E^u_{X^t(p)})$. Then

$$\lim_{t\to\pm\infty} \frac{1}{t}\log\|P_X^t(p)\cdot n^u\| = \lim_{t\to\pm\infty}\frac{1}{t}\log\|\Pi_{X^t(p)} \circ DX_p^t \cdot (\alpha.X(p) + v^u)\|$$

for some $\alpha \in \mathbb{R}$ and $v^u \in E_p^u$. But then

$$\begin{aligned} & \lim_{t\to\pm\infty}\frac{1}{t}\log\|\alpha.\Pi_{X^t(p)}\circ DX_p^t\cdot(X(p)) + \Pi_{X^t(p)}\circ DX_p^t\cdot v^u)\| = \\ = & \lim_{t\to\pm\infty}\frac{1}{t}\log\|\alpha.\Pi_{X^t(p)}\circ X(X^t(p)) + \Pi_{X^t(p)}\circ DX_p^t\cdot v^u)\| = \\ = & \lim_{t\to\pm\infty}\frac{1}{t}\log(sin(\theta_t).\|DX_p^t\cdot v^u\|) = \\ = & \lim_{t\to\pm\infty}[\frac{1}{t}\log sin(\theta_t) + \frac{1}{t}\log\|DX_p^t\cdot v^u\|] = \lambda_u(p), \end{aligned}$$

and analogously for $N_p^s$. $\square$

Therefore to decrease the Lyapunov exponents associated to the tangent flow we decrease the Lyapunov exponents associated to the linear Poincaré flow.

In this conservative context we may restate the Oseledets's Theorem for the linear Poincaré flow as;

**Theorem 2.3.2** *Let $X \in \mathfrak{X}_\mu^1(M)$. For $\mu$-a.e. $p \in M$ there exists the upper Lyapunov exponent $\lambda^+(X,p)$ defined by the limit $\lim\limits_{t\to+\infty}\frac{1}{t}log\|P_X^t(p)\|$ that is a non-negative measurable function of $p$. For $\mu$-a.e. point $p \in \mathfrak{O}^+$ there is a splitting of the normal bundle $N_p = N_p^u \oplus N_p^s$ which varies measurably with $p$ such that:*

*If* $\overrightarrow{0} \neq v \in N_p^u$*, then* $\lim\limits_{t\to\pm\infty} \frac{1}{t} log\|P_X^t(p)\cdot v\| = \lambda^+(X,p)$.

*If* $\overrightarrow{0} \neq v \in N_p^s$*, then* $\lim\limits_{t\to\pm\infty} \frac{1}{t} log\|P_X^t(p)\cdot v\| = -\lambda^+(X,p)$.

*If* $\overrightarrow{0} \neq v \notin N_p^u, N_p^s$*, then*

$$\lim_{t\to+\infty} \frac{1}{t} log\|P_X^t(p)\cdot v\| = \lambda^+(X,p) \text{ and } \lim_{t\to-\infty} \frac{1}{t} log\|P_X^t(p)\cdot v\| = -\lambda^+(X,p).$$

Next we recall a Lemma due to Doering [7] that relates the hyperbolicity of the linear Poincaré flow with the hyperbolicity of the tangent flow. Here the compactness of $\Lambda$ plays an important role.

**Lemma 2.3.3** *Let $\Lambda$ be a $X^t$-invariant and compact set. Then $\Lambda$ is hyperbolic for the flow if and only if the linear Poincaré flow is hyperbolic on $\Lambda$.*

## 2.4 Dominated Splitting for the Linear Poincaré flow

Let $\Lambda$ be a $X^t$-invariant subset of $M$. A splitting of the normal bundle $N = N^1 \oplus N^2$ has $m$-dominated structure for the linear Poincaré flow if it is $P_X^t$-invariant and we may find an *uniform* $m \in \mathbb{N}$ such that for any point $p \in \Lambda$ the following inequality holds:

$$\Delta(p,m) = \frac{\|P_X^m(p)|N_p^1\|}{\|P_X^m(p)|N_p^2\|} \leq \frac{1}{2} \tag{2.2}$$

In this case we say that we have a *m-dominated splitting*. A few words about this definition; If we take $p \in \mathfrak{O}^+(X)$ an Oseledets regular point for $X^t$, with a non-trivial splitting (i.e. not all Lyapunov exponents zero) positive Lyapunov exponents only guarantee that we will see expansion for large iterates, say for $P_X^{m(p)}(p)$, but this function $m(\cdot)$ varies from point to point, and possibly is not bounded along the orbit of $p$. So the information given by the Oseledets's Theorem is blind to uniformity.

If $\Lambda$ has a dominated splitting, then it varies continuously from point to point and we may extend the splitting to the closure. Moreover, the decomposition is unique and the angle between the two subspaces is bounded away from zero on $\Lambda$.

Next we define a $X^t$-invariant set $\Lambda_m(X)$ has

$\{p \in \mathfrak{O}^+(X) : p$ has $m$-dominated splitting for the linear Poincaré flow$\}$

Let $\Gamma_m(X) := \mathfrak{O}^+(X) - \Lambda_m(X)$.

The set of points in $\mathfrak{O}^+(X)$ that do not have $m$-dominated splitting, i.e. (2.2) does not hold we denote by $\Delta_m(X)$. Then for some $p \in \Delta_m(X)$ maybe there exists some iterate $X^t(p)$ that has $m$-dominated splitting, so we consider the saturated set $\underset{t\in\mathbb{R}}{\cup} X^t(\Delta_m(X))$ that is equal to $\Gamma_m(X)$.

## 2.5 Ergodicity of sets with dominated splitting for vector fields in $\mathfrak{X}^2_\mu(M)^*$

First we present a result which is a version of a classical theorem of Bowen. Our result says that any hyperbolic set, not necessarily a basic piece, of a non-Anosov conservative $C^2$ 3-flow without fixed points has zero measure. For that we adapt Theorem 14 in [5] to our context.

**Proposition 2.5.1** *Let $X \in \mathfrak{X}^2_\mu(M)^*$ and $\Lambda_m$ be a $X^t$-invariant set with m-dominated splitting for the linear Poincaré flow. Then $\mu(\overline{\Lambda_m}) = 0$ or $X$ is Anosov.*

For the proof we will need the following lemma which says that dominated splitting on the conservative setting without singularities is tantamount to hyperbolicity.

**Lemma 2.5.2** *For $X \in \mathfrak{X}^1_\mu(M)^*$, if $\Lambda_m \neq \emptyset$ then $\overline{\Lambda_m}$ is a hyperbolic set.*

**Proof:** Any $p \in \Lambda_m$ has m-dominated splitting for the linear Poincaré flow. Since we do not have singularities and we have constant dimensions of the sub-bundles this splitting extends to the closure and we get $\Delta(p, m) \leq \frac{1}{2}$ for every $p \in \overline{\Lambda_m}$. Of course for any $i \in \mathbb{N}$ we have $\Delta(p, i.m) \leq \frac{1}{2^i}$. For every $t \in \mathbb{R}$ we may write $t = i.m + r$ and since $\|P^r_X\|$ is bounded, say by M, we take $C = 2^{\frac{r}{m}}.M^2$ and $\sigma = 2^{-\frac{1}{m}}$ to get $\Delta(p, t) \leq C.\sigma^t$ for every $p \in \Lambda_m$ and $t \in \mathbb{R}$. We denote by $\alpha_t$ the angle $\angle(N^u_{X^t(p)}, N^s_{X^t(p)})$.

**Claim 2.5.1** *If $\Lambda_m$ has m-dominated splitting, then for all $p \in \Lambda_m$ we have $\alpha_t \geq \beta > 0$.*

Since $P^m_X(p)$ is a linear isomorphism its co-norm

$$m(P^m_X(p)) := \inf_{\|v\|=1} \|P^m_X(p) \cdot v\|$$

is given by $\|[P^m_X(p)]^{-1}\|^{-1}$. Let $u \in N^u_p$ and $s \in N^s_p$ be unit vectors. Since $\sin(\frac{\alpha_0}{2}) = \frac{\|u-s\|}{2}$, we prove that $\|u - s\|$ is bounded away from zero. By dominated splitting we have $2\|P^m_X(p) \cdot s\| \leq \|P^m_X(p) \cdot u\|$ so

$$2\|P^m_X(p) \cdot s\| \leq \|P^m_X(p) \cdot (u - s + s)\| \leq \|P^m_X(p) \cdot (u - s)\| + \|P^m_X(p) \cdot (s)\|$$

and we get
$$\|P_X^m(p)\cdot s\| \le \|P_X^m(p)\|\|(u-s)\|.$$
Since $\|[P_X^m(p)]^{-1}\|^{-1} \le \|P_X^m(p)\cdot s\|$ we obtain
$$\|[P_X^m(p)]^{-1}\|^{-1} \le \|P_X^m(p)\|\|(u-s)\|$$
and therefore,
$$\|[P_X^m(p)]^{-1}\|^{-1}\|P_X^m(p)\|^{-1} \le \|u-s\|.$$
Now we just note that $P_X^m(p)$ is continuous and $M$ is compact and the claim is proved.
Since we do not have singularities there exists $K>1$ such that for all $p\in M$, $K^{-1}\le \|X(p)\| \le K$. Since the flow is conservative we have:
$$sin(\alpha_0)=\|P_X^t(p)|_{N_p^u}\|.\|P_X^t(p)|_{N_p^s}\|.sin(\alpha_t).\frac{\|X(X^t(p))\|}{\|X(p)\|}.$$
So;
$$\begin{aligned}\|P_X^t(p)|_{N_p^s}\|^2 &= \frac{sin(\alpha_0)}{sin(\alpha_t)}.\frac{\|X(p)\|}{\|X(X^t(p))\|}.\Delta(p,t)\le\\ &\le \Delta(p,i.m+r).sin(\beta)^{-1}.K^2 \le \sigma^t.C.sin(\beta)^{-1}.K^2\end{aligned}$$
Analogously we get
$$\begin{aligned}\|P_X^{-t}(p)|_{N_p^u}\|^2 &= \frac{sin(\alpha_t)}{sin(\alpha_0)}.\frac{\|X(X^t(p))\|}{\|X(p))\|}.\Delta(p,t)\le\\ &\le \Delta(p,i.m+r).sin(\beta)^{-1}.K^2 \le \sigma^t.C.sin(\beta)^{-1}.K^2,\end{aligned}$$
and we have $\overline{\Lambda_m}$ hyperbolic for the linear Poincaré flow. Now by Lemma 2.3.3 we conclude that $\overline{\Lambda_m}$ is hyperbolic for the flow.

□

Now we use standard smooth ergodic and hyperbolic dynamics theory to prove Proposition 2.5.1.

Denote by $\mu_u$ the 1-dimensional Lebesgue measure on the unstable manifold. Suppose that the norm is adapted to get the hyperbolicity constants $C=1$ and $\sigma\in(0,1)$. In what follows we assume that $\mu(\Lambda)>0$.

**Lemma 2.5.3** *There exists a segment of orbit $(x_t)_{t>0}$ on $\Lambda$ such that $\mu_u(W_\epsilon^u(x_t)-\Lambda)\underset{t\to\infty}{\to}0$.*

**Proof:** There exists $x \in \Lambda$ such that $\mu_u(W^u_\epsilon(x) \cap \Lambda) > 0$, otherwise $\forall x \in \Lambda$ $\mu_u(W^u_\epsilon(x) \cap \Lambda) = 0$ and since $X^t$ is twice differentiable we get absolute continuity along unstable manifolds and by a Fubinni disintegration argument we contradict $\mu(\Lambda) > 0$. We take this point $x$ and since $\mu_u(W^u_\epsilon(x) \cap \Lambda) > 0$ there exists $y \in \Lambda$ with density one on $W^u_\epsilon(x) \cap \Lambda$. We define $x_n = X^n(y)$ and we get $\mu_u(X^{-n}(W^u_\epsilon(x_n))) \underset{n\to+\infty}{\longrightarrow} 0$. Therefore,

$$\frac{\mu_u(X^{-n}(W^u_\epsilon(x_n)) - \Lambda)}{\mu_u(W^u_\epsilon(x_n))} \underset{t\to\infty}{\longrightarrow} 0$$

**Claim 2.5.2** *Let $x_1, x_2 \in W^u_\epsilon(x_n)$ such that $d_u(x_1, x_2) < D$. There exists $K > 0$ such that for all $t \geq 0$ we have $\frac{\|DX^{-t}_{x_1}|_{E^u_{x_2}}\|}{\|DX^{-t}_{x_1}|_{E^u_{x_2}}\|} \leq K$.*

To prove the claim we use a standard application of the bounded distortion properties. The sub-bundle $E^u$ is $\nu$-Hölder so we can define a $(C, \nu)$-Hölder function on $\Lambda$ as $\varphi(x) = \log\|DX^{-1}_x|_{E^u_x}\|$. Since $\max_{x\in\Lambda}\|DX^{-r}_x|E^u_x\|$ is bounded and our result is asymptotic we consider time-1 maps,

$$\begin{aligned}
&\log\frac{\|DX^{-n}_{x_1}|_{E^u_{x_1}}\|}{\|DX^{-n}_{x_2}|_{E^u_{x_2}}\|} \leq \\
\leq\ &\log\prod_{i=0}^{n-1}\|DX^{-1}_{X^{-i}(x_1)}|_{E^u_{X^{-i}(x_1)}}\| - \log\prod_{i=0}^{n-1}\|DX^{-1}_{X^{-i}(x_2)}|_{E^u_{X^{-i}(x_2)}}\| = \\
=\ &\sum_{i=0}^{n-1}(\varphi(X^{-i}(x_1)) - \varphi(X^{-i}(x_2))) \leq \sum_{i=0}^{n-1} C.d_u(X^{-i}(x_1), X^{-i}(x_2))^\nu \leq \\
\leq\ &\sum_{i=0}^{n-1} C.\sigma^i d_u(x_1, x_2)^\nu \leq CD^\nu\sum_{i=0}^{n-1}\sigma^{i\nu} \leq CD^\nu\sum_{i=0}^{\infty}\sigma^{i\nu} \leq CD^\nu S,
\end{aligned}$$

where $S$ is sum of the geometric series. We take $K' = CD^\nu S$ and $K = e^{K'}$ and the claim is proved.

Now we have:

$$\frac{\mu_u(W^u_\epsilon(x_t) - \Lambda)}{\mu_u(W^u_\epsilon(x_t))} \leq K\frac{\mu_u(X^{-t}(W^u_\epsilon(x_t) - \Lambda))}{\mu_u(W^u_\epsilon(x_t))} \underset{t\to\infty}{\longrightarrow} 0$$

therefore $\mu_u(W^u_\epsilon(x_t) - \Lambda) \underset{t\to\infty}{\longrightarrow} 0$. $\square$

**Claim 2.5.3** *There exists $x_0 \in \Lambda$ such that $W^u_\epsilon(x_0) \subseteq \Lambda$.*

Let $(x_t)_{t>0}$ be the orbit given by Lemma 2.5.3. Since $\Lambda$ is closed and $x_t \in \Lambda$ we define $x_0 = \lim_{t\to\infty} x_t \in \Lambda$. By continuity of the unstable manifolds we get $W^u_\epsilon(x_t) \to W^u_\epsilon(x_0)$ and by Lemma 2.5.3, $\mu_u(W^u_\epsilon(x_t) - \Lambda) \underset{t\to\infty}{\longrightarrow} 0$ so we conclude that $\mu_u(W^u_\epsilon(x_0) - \Lambda) = 0$. But $\Lambda$ is closed and $W^u_\epsilon(x_0)$ is open on $W^u(x_0)$ so $W^u_\epsilon(x_0) - \Lambda$ is an open set on $W^u(x_0)$ with zero measure, therefore empty and $W^u_\epsilon(x_0) \subseteq \Lambda$. The claim is proved.

**Lemma 2.5.4** *There exists a hyperbolic periodic orbit $X^t(p) \in \Lambda$ such that $W^u(p) \subseteq \Lambda$.*

**Proof:** By continuation of hyperbolic sets we may define the maximal invariant set $\tilde{\Lambda} = \bigcap_{t\in\mathbb{R}} X^t(U_\beta)$ for any neighborhood $U_\beta$ of $\Lambda$ such that $d_H(\Lambda, U_\beta) < \beta$. Consider a point $x_0$ given by Claim 2.5.3 and a small transversal section $\Sigma_0$ to $\{X^t(x_0)\}_{t\in\mathbb{R}}$ at $t = 0$. Since we can always suppose that the measure is supported on $\Lambda$, the induced measure, $\tilde{\mu}$, defined on transversal sections (see section 3.1.2) verifies $\tilde{\mu}(\Sigma_0 \cap \Lambda) > 0$ so by Poincaré recurrence we have for $\tilde{\mu}$-a.e. $x \in \Sigma_0 \cap \Lambda$ a time $s$ such that $X^s(x) \in \Sigma_0$. If all points in $\Sigma_0$ are $\alpha$-close to $x_0$ we get a *$\alpha$-pseudo periodic orbit*. By the shadowing lemma we know that given any $\beta > 0$, there exists $\alpha > 0$ such that any $\alpha$-pseudo orbit in $\Lambda$ is $\beta$-shadowed by an orbit in $M$. Take an adequate $\beta$ so we obtain an orbit, $\bigcup_{t\in\mathbb{R}} X^t(p)$, in $\tilde{\Lambda}$, that shadows the $\alpha$-pseudo periodic orbit. Since, by expansiveness, this orbit is unique and $X^s(p)$ also shadows, we get $X^s(p) = p$ and this orbit is periodic. $p \in \tilde{\Lambda}$ is hyperbolic because it belong to $\tilde{\Lambda}$, therefore have stable/unstable manifolds that are close to the stable/unstable manifolds of $x_0$, we may suppose transversality between $W^u_\epsilon(x_0)$ and $W^s_\epsilon(p)$. Claim 2.5.3 guarantees that $W^u_\epsilon(x_0) \subseteq \Lambda$ and by Palis $\lambda$-lemma it converges to $W^u(p)$. Since $\Lambda$ is closed we get $W^u(p) \subseteq \Lambda$. $\square$

Abbreviate $\Lambda_p = \overline{W^u(p)}$ and define $W^s_\epsilon(\Lambda_p) := \bigcup_{z\in\Lambda_p} W^s_\epsilon(z)$.

**Lemma 2.5.5** *$W^s_\epsilon(\Lambda_p)$ is $X^t$-invariant and it is an open neighborhood of $\Lambda_p$.*

**Proof:** For $t > 0$ take $\delta \in (\sigma^t\epsilon, \epsilon)$, so:

$$X^t(W^s_\delta(\Lambda_p)) \subseteq X^t(\overline{W^s_\delta(\Lambda_p)}) \subseteq X^t(W^s_\epsilon(\Lambda_p)) \subseteq W^s_{\sigma^t\epsilon}(\Lambda_p)) \subseteq W^s_\delta(\Lambda_p)).$$

By the volume preserving property we get:

$$\mu[W^s_\delta(\Lambda_p) - X^t(W^s_\delta(\Lambda_p))] = 0.$$

Since $\mu[W^s_\delta(\Lambda_p) - X^t(W^s_\delta(\Lambda_p))] \geq \mu[W^s_\delta(\Lambda_p) - X^t(\overline{W^s_\delta(\Lambda_p)})]$ we conclude that:

$$\mu[W^s_\delta(\Lambda_p) - X^t(\overline{W^s_\delta(\Lambda_p)})] = 0.$$

Since $W^s_\delta(\Lambda_p)$ is open, $X^t(\overline{W^s_\delta(\Lambda_p)})$ is close and $\mu$ is Lebesgue we get that the open set $W^s_\delta(\Lambda_p) - X^t(\overline{W^s_\delta(\Lambda_p)})$ is an empty set, and since it contains $W^s_\delta(\Lambda_p) - X^t(W^s_\epsilon(\Lambda_p))$ we get $W^s_\delta(\Lambda_p) - X^t(W^s_\epsilon(\Lambda_p)) = \emptyset$.

Since this is true for all $\delta < \epsilon$ we conclude $W^s_\epsilon(\Lambda_p) - X^t(W^s_\epsilon(\Lambda_p)) = \emptyset$ so:

$$W^s_\epsilon(\Lambda_p) = X^t(W^s_\epsilon(\Lambda_p)). \tag{2.3}$$

And we have the $X^t$ invariance.

For the second part of the lemma, we prove that $\Lambda_p \cap W^u(z) = W^u(z)$ for any $z \in \Lambda_p$. We note that $\Lambda_p \cap W^u(z)$ is closed on $W^u(z)$, let us see that it is also open; Take $z \in \Lambda_p$, by definition of $\Lambda_p$ there exists $\{z_n\}_{n\in\mathbb{N}} \in W^u(p)$ such that $z_n \underset{n\to\infty}{\to} z$. $W^u_\epsilon(z_n) \subseteq W^u(p) \subseteq \Lambda_p$ and this local unstable manifolds verifies $W^u_\epsilon(z_n) \underset{n\to\infty}{\to} W^u_\epsilon(z)$. Now, since $W^u_\epsilon(z_n) \subseteq \Lambda_p$ and $\Lambda_p$ is a close set, this imply that $z$ belongs to the interior of $\Lambda_p \cap W^u(z)$. Therefore $\Lambda_p \cap W^u(z) = W^u(z)$ so the union of all unstable manifolds of points of $\Lambda_p$ is $\Lambda_p$ itself. Since the local stable manifolds vary continuously with the point we get that $W^s_\epsilon(\Lambda_p)$ has an open neighborhood of $\Lambda_p$.

□

**Proof:** (Proposition 2.5.1) Consider a set $\Lambda_m$ with $m$-dominated splitting, by Lemma 2.5.2 we get $\overline{\Lambda_m}$ hyperbolic. We take $\Lambda = \overline{\Lambda_m}$ and we follow previous lemmas assuming $\mu(\Lambda) > 0$.

By (2.3) we get $W^s_\epsilon(\Lambda_p) = \underset{t>0}{\cap} X^t(W^s_\epsilon(\Lambda_p))$ but $\underset{t>0}{\cap} X^t(W^s_\epsilon(\Lambda_p)) = \Lambda_p$, therefore $W^s_\epsilon(\Lambda_p) = \Lambda_p$. Again by Lemma 2.5.5, we have $W^s_\epsilon(\Lambda_p)$ open so $\Lambda_p$ is open and is also closed, therefore $\Lambda_p = M$, but $\Lambda_p \subseteq \Lambda$, finally $\Lambda = M$ and $X$ is Anosov. □

The conservative flow $X^t$ is called *aperiodic* if $\mu(Per(X^t)) = 0$.

**Lemma 2.5.6** *There exists $D \subseteq \mathfrak{X}^1_\mu(M)^*$ such that $D$ is $C^1$-dense and if $X \in D$, $X^t$ is aperiodic, $X$ is of class $C^2$ and all its sets with dominated splitting for the linear Poincaré flow have zero or full measure.*

**Proof:** We take the $C^2$-residual given by Robinson version of Kupka-Smale theorem, see [15]. This residual set of vector fields is of class $C^2$ and the associated flows have countable periodic points. Since $\mathfrak{X}^2_\mu(M)$ is a Baire space, it follows that we have a $C^2$-dense set $D$, therefore $C^1$-dense, of vector fields with countable periodic orbits on $\mathfrak{X}^2_\mu(M)$. By Zuppa theorem, see [16], $\mathfrak{X}^2_\mu(M)$ is $C^1$-dense on $\mathfrak{X}^1_\mu(M)$, so $D$ is $C^1$-dense on $\mathfrak{X}^1_\mu(M)$ and all vector fields in $D$ are $C^2$. Since, by Proposition 2.5.1, hyperbolic sets have zero or full measure and $X^t$ is aperiodic the lemma is proved. □

## 2.6 Strategy for the proof of Theorem 1

Given $X \in \mathfrak{X}^1_\mu(M)$, let $\lambda^+(X,p) = \lim\limits_{t\to+\infty} \frac{1}{t}\log\|P^t_X(p)\|$ be the upper Lyapunov exponent which exists $\mu$-a.e. $p \in M$ by Oseledets's Theorem. When there is no ambiguity we denote $\lambda^+(X,p)$ by $\lambda^+(p)$.

We define the "entropy function" by the integration over $M$ of the upper Lyapunov exponent:

$$\begin{array}{rccc} LE: & \mathfrak{X}^1_\mu(M) & \longrightarrow & [0,+\infty) \\ & X & \longmapsto & \int_M \lambda^+(p)d\mu(p) \end{array}$$

**Remark 2.6.1** *Let $f: W \longrightarrow \mathbb{R}$ where $W$ is a topological space. $f$ is upper semicontinuous iff for every $\delta$ the set $\{x : f(x) < \delta\}$ is open. Moreover, the infimum of continuous functions is an upper semicontinuous function.*

**Lemma 2.6.1** *$LE(X) = \inf\limits_{n\geq 1} \frac{1}{n}\int_M log\|P^n_X(p)\|d\mu(p)$, therefore it is upper semicontinuous.*

**Proof:**

$$\begin{aligned} LE(X) &= \int_M \lambda^+(p)d\mu(p) = \lim_{t\to+\infty}\frac{1}{t}\int_M \log\|P^t_X(p)\|d\mu(p) = \\ &= \lim_{n\to+\infty}\frac{1}{n}\int_M \log\|P^n_X(p)\|d\mu(p). \end{aligned}$$

Now the sequence $x_n(X) = \int_M \log\|P^n_X(x)\|d\mu(p)$ is sub-additive, therefore satisfies

$$\lim_{n\to+\infty}\frac{x_n(X)}{n} = \inf_{n\geq 1}\frac{x_n(X)}{n}.$$

Thus $LE(X) = \inf\limits_{n\geq 1}\frac{x_n(X)}{n}$, and since each $x_n(X)$ is a continuous function, by remark 2.6.1, $LE(X)$ is upper semicontinuous. $\square$

Formally we will prove the following;

**Proposition 2.6.2** *Let $X \in \mathfrak{X}^1_\mu(M)^*$ be of class $C^2$, aperiodic and with hyperbolic sets of zero measure. Let $\epsilon, \delta > 0$ be given. Then there exists a conservative vector field $Y$ $\epsilon - C^1$-close to $X$, such that $LE(Y) < \delta$.*

We assume Proposition 2.6.2 and prove Theorem 1:

**Proof:** (of Theorem 1)

By Lemma 2.5.6 we have a dense set such that every $X$ is $C^2$, aperiodic and with hyperbolic sets having full or zero measure.

The set of conservative Anosov vector fields, denoted by $\mathbf{A}$, is open.

For all $k \in \mathbb{N}$ the set $\mathbf{A}_k = \{X \in \mathfrak{X}^1_\mu(M)^* : LE(X) < k^{-1}\}$ is open by Lemma 2.6.1 and Remark 2.6.1. By Proposition 2.6.2 with $\delta = k^{-1}$ we get $\mathbf{A}_k$ dense in $\mathbf{A}^c$, so the set:

$$\mathfrak{R} = \bigcap_{k\in\mathbb{N}} \mathbf{A} \cup \mathbf{A}_k$$

is a $C^1$-residual set. But $\mathfrak{R} = \mathbf{A} \cup \bigcap_{k\in\mathbb{N}} \mathbf{A}_k = \mathbf{A} \cup \{X \in \mathfrak{X}^1_\mu(M)^* : LE(X) = 0\}$, therefore for $X \in \mathfrak{R}$ we have either that $X$ is an Anosov vector field or $LE(X) = \int_M \lambda^+ d\mu(p) = 0$. This last equality implies that $\mu$-a.e. $p \in M$ has zero Lyapunov exponents. $\square$

In the next four Chapters we prove Proposition 2.6.2. The maint steps are as follows. First, we take advantage of the lack of domination behavior to mix the Oseledets directions along one orbit, and this causes a decay on the norm of $P^t_Y(p)$ for $Y$ $C^1$-close to $X$. Being a $C^1$ perturbation allows us to shrink its support, which is crucial to our proof. Then we make this strategy global by almost covering $M$ with self-disjoint flow boxes, which control most of the orbits. We estimate $LE(Y) = \int_M \lambda^+(p) d\mu(p)$ considering a finite time $t$ by Lemma 2.6.1. Then we split $M$ into two sets, the one satisfying $\|P^t_Y(p)\| < e^{t\delta}$ and the other negligible with respect to $\mu$.

# Chapter 3

# Perturbation of vector fields

## 3.1 Auxiliary lemmas

### 3.1.1 A conservative straightening-out lemma

The following theorem, due to Dacorogna and Moser [6], will be used to obtain a conservative local change of coordinates which trivialize a vector field.

**Theorem 3.1.1** *(Dacorogna-Moser) Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with $C^5$ boundary $\partial\Omega$ and $g, f : \overline{\Omega} \to \mathbb{R}$ positive functions of class $C^s$ ($s \geq 2$). Then there exists a diffeomorphism $\varphi : \Omega \to \varphi(\Omega) \subseteq \mathbb{R}^n$ with $\varphi$ of class $C^s$ and satisfying the partial differential equation:*

$$detD\varphi_q g(\varphi(q)) = \lambda f(q), \tag{3.1}$$

*for all $q \in \Omega$ where $\lambda = \int g / \int f$. We also have $\varphi = Id$ at $\partial\Omega$.*

Denote by $X^{[0,t]}(p) = \{X^s(p) : s \in [0,t]\}$. We say that the a segment of an orbit $X^{[0,m]}(p)$ is *straightened-out* if $X^{[0,m]}(p) \subseteq \{(x,0,0) : x \in \mathbb{R}\}$. Denote by $\mathfrak{N}_{X^t(p)}$ the normal plane at $X(X^t(p))$. Denote by $B(X^t(p), r)$ the ball with radius $r$ centered at $X^t(p)$ inside $X(X^t(p))^\perp = \mathfrak{N}_{X^t(p)}$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the constant vector field defined by $T(x,y,z) = (c,0,0)$ for some $c > 0$ and $\mathfrak{F}$, $\mathfrak{C}$ be the flowboxes $\mathfrak{F} := X^{[0,1]}(B(p,r))$ and $\mathfrak{C} := T^{[0,1]}(B(p,r))$.

**Lemma 3.1.2** *(Conservative flowbox theorem) Given a vector field $X \in \mathfrak{X}^s_\mu(M)$ (for $s \geq 2$) and a non-singular point $p \in M$ (eventually periodic with period $\tau > 1$), there exists a conservative $C^s$ diffeomorphism $\Psi : \mathfrak{C} \to \mathfrak{F}$ such that $X = \Psi_* T$.*

**Proof:** Assume that $p = (0,0,0)$ and $X(p) \subseteq \{(x,0,0) : x \in \mathbb{R}\}$. Let $X_1(x,y,z)$ be the projection into the first coordinate of $X(x,y,z)$. We define a function $g : B(p,r) \to \mathbb{R}$ such that $g(y,z) := X_1(0,y,z)$ for $(0,y,z) \in B(p,r)$

(see Figure 3.1). Since $g \in C^s$ we apply Theorem 3.1.1 to $\Omega = B(p,r) \subseteq \mathbb{R}^2$ so there exists a diffeomorphism $\varphi : \Omega \to \varphi(\Omega) \subseteq \mathbb{R}^2$ with $\varphi, \varphi^{-1}$ of class $C^s$ and satisfying the partial differential equation $g(\varphi(\overline{y},\overline{z}))\det D\varphi_{\overline{y},\overline{z}} = \lambda$, for all $(\overline{y},\overline{z}) \in \Omega$ where $\lambda = \int g / \int 1$, and $\varphi|_{\partial\Omega} = Id$. Now we define the $C^s$ change

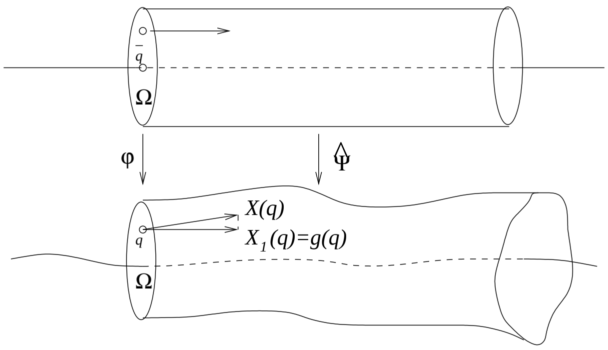


Figure 3.1: Construction of a conservative change of coordinates straightening-out *all* orbits.

of coordinates by:

$$\begin{array}{rccc} \hat{\Psi}: & \mathbb{R}\times\Omega & \longrightarrow & M \\ & (\overline{x},\overline{y},\overline{z}) & \longmapsto & X^{\lambda^{-1}\overline{x}}((0,\varphi(\overline{y},\overline{z})) \end{array}$$

First we claim that:

$$\det D\hat{\Psi}_{(0,\overline{y},\overline{z})} = 1 \text{ for all } (0,\overline{y},\overline{z}) \in \mathbb{R}\times\Omega. \tag{3.2}$$

Let $\Pi_i$ denote the projection into the $i^{th}$-coordinate, for $i = 1,2,3$.
Note that, $\frac{\partial X_1^0}{\partial \overline{y}}(0,\varphi(\overline{y},\overline{z})) = \frac{\partial}{\partial \overline{y}}\Pi_1(0,\varphi(\overline{y},\overline{z})) = 0$ and for $i = 2,3$ we have that, $\frac{\partial X_i^0}{\partial \overline{y}}(0,\varphi(\overline{y},\overline{z})) = \frac{\partial}{\partial \overline{y}}\Pi_i(0,\varphi(\overline{y},\overline{z})) = \frac{\partial}{\partial \overline{y}}(\varphi_i(\overline{y},\overline{z})))$. For $\overline{z}$ we proceed analogously. Now we use these computations to derive,

$$D\hat{\Psi}_{(0,\overline{y},\overline{z})} = \begin{pmatrix} \frac{1}{\lambda}X_1(X^0(0,y,z)) & 0 & 0 \\ \frac{1}{\lambda}X_2(X^0(0,y,z)) & \frac{\partial\varphi_1}{\partial\overline{y}}|_{(\overline{y},\overline{z})} & \frac{\partial\varphi_1}{\partial\overline{z}}|_{(\overline{y},\overline{z})} \\ \frac{1}{\lambda}X_3(X^0(0,y,z)) & \frac{\partial\varphi_2}{\partial\overline{y}}|_{(\overline{y},\overline{z})} & \frac{\partial\varphi_2}{\partial\overline{z}}|_{(\overline{y},\overline{z})} \end{pmatrix}$$

and we get $\det D\hat{\Psi}_{(0,\overline{y},\overline{z})} = \frac{1}{\lambda}X_1((0,y,z))\det D\varphi_{(\overline{y},\overline{z})} = \frac{g(y,z)}{\lambda}\det D\varphi_{(\overline{y},\overline{z})} = 1$ by using (3.1) of Theorem 3.1.1. Therefore (3.2) is proved. Now we will see that:

$$\det D\hat{\Psi}_{(\overline{x}_0,\overline{y}_0,\overline{z}_0)} = 1 \text{ for all } (\overline{x}_0,\overline{y}_0,\overline{z}_0) \in \mathbb{R}\times\Omega.$$

We have:

$$\hat{\Psi}(\overline{x},\overline{y},\overline{z}) = X^{\lambda^{-1}\overline{x}_0}[X^{\lambda^{-1}(\overline{x}-\overline{x}_0)}((0,\varphi(\overline{y},\overline{z})))] = X^{\lambda^{-1}\overline{x}_0}[\hat{\Psi}(\overline{x}-\overline{x}_0,\overline{y},\overline{z})],$$

so

$$D\hat{\Psi}_{(\overline{x},\overline{y},\overline{z})} = DX^{\lambda^{-1}\overline{x}_0}_{\Psi(\overline{x}-\overline{x}_0,\overline{y},\overline{z})} D\hat{\Psi}_{(\overline{x}-\overline{x}_0,\overline{y},\overline{z})}.$$

Evaluated at $\overline{x} = \overline{x}_0$ we get:

$$D\hat{\Psi}_{(\overline{x}_0,\overline{y},\overline{z})} = DX^{\lambda^{-1}\overline{x}_0}_{\Psi(0,\overline{y},\overline{z})} D\hat{\Psi}_{(0,\overline{y},\overline{z})}.$$

We use (3.2) and the fact that the flow $X^t$ is volume preserving to conclude that:

$$\det D\hat{\Psi}_{(\overline{x}_0,\overline{y}_0,\overline{z}_0)} = 1.$$

Finally take $c := \lambda$ and consider the constant vector field $T := (\lambda, 0, 0)$. Let $(x, y, z) = \hat{\Psi}(\overline{x}, \overline{y}, \overline{z})$.
We have:

$$\begin{aligned}
\hat{\Psi}_* T(x,y,z) &= D\hat{\Psi}_{(\overline{x},\overline{y},\overline{z})} T(\overline{x},\overline{y},\overline{z})) = \\
&= \begin{pmatrix} \frac{1}{\lambda} X_1(X^{\lambda^{-1}\overline{x}}(0,y,z)) & \frac{\partial X_1^{\lambda^{-1}\overline{x}}}{\partial \overline{y}}(0,\varphi(\overline{y},\overline{z})) & \frac{\partial X_1^{\lambda^{-1}\overline{x}}}{\partial \overline{z}}(0,\varphi(\overline{y},\overline{z})) \\ \frac{1}{\lambda} X_2(X^{\lambda^{-1}\overline{x}}(0,y,z)) & \frac{\partial X_2^{\lambda^{-1}\overline{x}}}{\partial \overline{y}}(0,\varphi(\overline{y},\overline{z})) & \frac{\partial X_2^{\lambda^{-1}\overline{x}}}{\partial \overline{z}}(0,\varphi(\overline{y},\overline{z})) \\ \frac{1}{\lambda} X_3(X^{\lambda^{-1}\overline{x}}(0,y,z)) & \frac{\partial X_3^{\lambda^{-1}\overline{x}}}{\partial \overline{y}}(0,\varphi(\overline{y},\overline{z})) & \frac{\partial X_3^{\lambda^{-1}\overline{x}}}{\partial \overline{z}}(0,\varphi(\overline{y},\overline{z})) \end{pmatrix} \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix} = \\
&= (X_1(X^{\lambda^{-1}\overline{x}}(0,y,z)), X_2(X^{\lambda^{-1}\overline{x}}(0,y,z)), X_3(X^{\lambda^{-1}\overline{x}}(0,y,z))) = \\
&= X(\hat{\Psi}(\overline{x},\overline{y},\overline{z}))
\end{aligned}$$

therefore $X = \hat{\Psi}_* T$. $\square$

### 3.1.2 More notation, definitions and lemmas

**Coordinates:** For technical reasons, given by previous section, it is useful to take $X$ of class $C^2$, therefore we consider the vector field $X$ in Proposition 2.6.2 of class $C^2$. Given $p \in M$ and a small $r > 0$ let $q \in X^{[0,m]}(B(p,r))$. We will use the conservative flowbox theorem to get a $C^2$ change of coordinates $\Psi := \hat{\Psi}^{-1}$, hence, a $C^1$ vector field which has all orbits straight-out, i.e., $X^{[0,1]}(B(q,r'))$ is sent into $T^{[0,1]}(B(q,r'))$ by $\Psi$. Since for these change of coordinates we fix $t \in [0,1]$, $M$ is compact and $\Psi \in C^2$ we conclude that exists $\Theta_1 := \max\{\|D\Psi_p\|, \|D\Psi_p^{-1}\| : p \in M\}$. We take $\Theta_2 := \max\{\|D^2\Psi_p^{-1}\| : p \in M\}$ and also $\Theta := \max\{\Theta_1, \Theta_2\}$.

**Perturbations and metrics:** All the perturbations in this paper will be developed using the trivial coordinates given by $\Psi$. $\Theta$ will be useful to control the size of the perturbation. Therefore if $\epsilon > 0$, $X := \Psi_* T$ and $Z := T + P$ is a perturbation such that $\|P\|_{C^1} \leq \frac{\epsilon}{\Theta}$, then $\|Y - X\|_{C^1} = \|\Psi_* P\|_{C^1} \leq \epsilon$. According to Moser's Theorem, (see [13] Lemma 2), given a volume form $\omega$ there exists an atlas $\mathfrak{A} = \{\alpha_i : U_i \to \mathbb{R}^3\}$, such that $(\alpha_i)_*\omega = dx \wedge dy \wedge dz$, moreover by compactness of $M$ we can take $\mathfrak{A}$ finite. The Riemannian norm at $T_xM$ will not be used, instead we consider the norm $\|v\|_x := \|(D\alpha_i)_x \cdot v\|$. Given two linear maps $A(t): T_pM \to T_{X^t(p)}M$ and $B(t): T_qM \to T_{X^t(q)}M$ we estimate the distance between them by using the atlas $\mathfrak{A}$ and translating the base points to $(0,0,0) \in \mathbb{R}^3$. Therefore

$$\|A(t) - B(t)\| = \|(D\alpha_{X^t(p)})_{X^t(p)} A(t) (D\alpha_p)_p^{-1} - (D\alpha_{X^t(q)})_{X^t(q)} B(t) (D\alpha_q)_q^{-1}\|.$$

Analogously we estimate the distance between the linear Poincaré flows based at different points.

**Holonomy of linear flows:** Assume that $p = (0,0,0)$ and also that the segment of orbit $X^{[0,T]}(p)$ is straight-out. In this case the tangent flow at $p$ has the following simple form:

$$DX_p^t = \begin{pmatrix} \frac{\partial X_1^t}{\partial x} & \frac{\partial X_1^t}{\partial y} & \frac{\partial X_1^t}{\partial z} \\ 0 & \frac{\partial X_2^t}{\partial y} & \frac{\partial X_2^t}{\partial z} \\ 0 & \frac{\partial X_3^t}{\partial y} & \frac{\partial X_3^t}{\partial z} \end{pmatrix}_p = \begin{pmatrix} x(t) & y(t) & z(t) \\ 0 & a(t) & b(t) \\ 0 & c(t) & d(t) \end{pmatrix}, \tag{3.3}$$

where $x(t) = \|X(X^t(p))\| \|X(p)\|^{-1}$. Hence we have the following action,

$$\begin{array}{cccc} P_X^t: & N_p & \longrightarrow & N_{X^t(p)} \\ & (y,z) & \longmapsto & \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}. \end{array}$$

Let $q = (0,y,z) \in \mathfrak{N}_p$ and $t \in [0,T]$, then:

$$\begin{aligned} X^t(q) &= X^t(p) + DX_p^t \cdot q + O^{\geq 2}(q) = \\ &= X^t(p) + (y(t)y + z(t)z, 0, 0) + (0, P_X^t(p) \begin{pmatrix} y \\ z \end{pmatrix}) + O^{\geq 2}(q), \end{aligned}$$

where $O^{\geq 2}(q)$ is the remainder of the Taylor expansion. So for $|y|, |z|$ small $X^t(0,y,z)$ is approximately $X^t(0,0,0) + (0, P_X^t(0,0,0) \cdot \begin{pmatrix} y \\ z \end{pmatrix})$.

**Measures at transversal sections:** In this context we may consider the time arrival function $\tau(p,t): \mathfrak{N}_p \to \mathfrak{N}_{X^t(p)}$ which is a well defined continuous function, due to the *implicit function theorem*. For $B \subseteq \mathfrak{N}_p$ denote by $X^{\tau(p,t)(B)}(B)$ the set,

$$\{X^{\tau(p,t)(q)}(q) : q \in B\} \subseteq \mathfrak{N}_{X^t(p)}.$$

Given $\delta > 0$, there exists $B$ sufficiently small such that $X^{\tau(p,t)(B)}(B)$ is the intersection of the self-disjoint flowbox $X^{[0,t+\delta]}(B)$ with $\mathfrak{N}_{X^t(p)}$.

Let $X^{\tau(p,t)}: \mathfrak{N}_p \to \mathfrak{N}_{X^t(p)}$ be the Poincaré map between two sections. Given $n_1, n_2 \in \mathfrak{N}_p$ and using the volume form $\omega$ we can define a pair of 2-forms by $\hat{\omega}_p(n_1,n_2) := \omega_p(X(p), n_1, n_2)$ and $\overline{\omega}_p(n_1,n_2) = \omega_p(X(p)\|X(p)\|^{-1}, n_1, n_2)$. The 2-form $\hat{\omega}$ is the interior product of the volume form $\omega$ by the vector field, i.e., $\hat{\omega}_p := (i_X\omega)_p$. Denoting $P_t(\cdot) = X^{\tau(p,t)(\cdot)}(\cdot)$ we have $P_t^*\hat{\omega}_p = \hat{\omega}_p$. The measure $\overline{\mu}$ induced by the 2-form $\overline{\omega}$ is not necessarily $P_t$-invariant, however both measures are equivalent. We call $\overline{\mu}$ the Lebesgue measure at normal sections or *modified area.* In fact we have for $n_1, n_2 \in \mathfrak{N}_p$:

$$\begin{aligned}
P_t^*\overline{\omega}_p(n_1,n_2) &= \overline{\omega}_{X^t(p)}((DP_t)_p \cdot n_1, (DP_t)_p \cdot n_2) = \\
&= \omega_{X^t(p)}(\frac{X(X^t(p))}{\|X(X^t(p))\|}, (DP_t)_p \cdot n_1, (DP_t)_p \cdot n_2) = \\
&= \omega_p(\frac{X(p)}{\|X(p)\|}\frac{\|X(p)\|}{\|X(X^t(p))\|}, n_1, n_2) = \\
&= x(t)^{-1}\omega_p(\frac{X(p)}{\|X(p)\|}, n_1, n_2) = \\
&= x(t)^{-1}\overline{\omega}_p(n_1,n_2).
\end{aligned}$$

By conservativeness of the flow we have $|\det P_X^t(p)|.\|X(X^t(p))\|\|X(p)\|^{-1} = 1$, so $|\det P_X^t(p)| = x(t)^{-1}$. Therefore we can give an explicit expression for the infinitesimal distortion area factor of the linear Poincaré flow, which is expressed by the following lemma.

**Lemma 3.1.3** *Given $\nu > 0$ and $T > 0$, there exists $r > 0$ such that for any measurable set $K \subseteq B(p,r) \subseteq \mathfrak{N}_p$ we have $|\overline{\mu}(K) - x(t).\overline{\mu}(X^{\tau(p,t)(K)}(K))| < \nu$ for all $t \in [0,T]$.*

**Proof:** We choose $r > 0$ such that $|\det P_X^t(p) - \det(DP_t)_q| < \nu x(t)^{-1}$ for every $q \in \mathfrak{N}_p$ with $\|q-p\| < r$ and $t \in [0,T]$. Let $P_t(q) = X^t(p) + P_X^t(p)\cdot q + O^{\geq 2}(q)$.

Since

$$\overline{\mu}(P_t(U)) = \int_{P_t(U)} d\overline{\mu} = \int_U \det(DP_t)_q d\overline{\mu}(q)$$

and

$$\overline{\mu}(P_X^t(p)(U)) = \int_{P_X^t(p)(U)} d\overline{\mu} = \int_U \det P_X^t(p) d\overline{\mu}(q)$$

we get:

$$|\overline{\mu}(P_t(U)) - \overline{\mu}(P_X^t(p)(U))| = |\int_U (\det P_X^t(p) - \det(DP_t)_q) d\overline{\mu}(q)| < \nu x(t)^{-1}.$$

Now we just have to note that

$$\begin{aligned}
& |x(t).\overline{\mu}(P_t(U)) - \overline{\mu}(U)| \leq \\
\leq \ & |x(t)\overline{\mu}(P_t(U)) - x(t)\overline{\mu}(P^t_X(p)(U))| + |x(t)\overline{\mu}(P^t_X(p)(U)) - \overline{\mu}(U)| \leq \\
\leq \ & \nu + |x(t)\mu_{\overline{\omega}}(P^t_X(p)(U)) - \overline{\mu}(U)| = \nu + |x(t)\mu_{P^t_X(p)^*\overline{\omega}}(U) - \overline{\mu}(U)| = \\
= \ & \nu + |x(t)\int_U \det P^t_X(p) d\overline{\mu} - \overline{\mu}(U)| = \nu.
\end{aligned}$$

And the lemma is proved. □

Given an open set $U \subseteq B(p,r)$ we are going to make perturbations supported on the flow box $\underset{t\in[0,n]}{\cup} X^{\tau(p,t)(U)}(U)$. We start with a ball $B_{r'}(p')$ centered in $p'$ with radius $r'$, such that $B_{r'}(p') \subseteq \mathfrak{N}_p$, then we straight-out the orbit of $p'$. However, $B(p',r') \subseteq \mathfrak{N}_{p'}$ is different from $B_{r'}(p')$ unless the vector field is horizontal. We define explicit perturbations with support contained in $X^t(B(p',r'))$ for $t \in [0,n]$. Let $\eta_1, \eta_2 > 0$, we define a flow box:

$$\mathfrak{F} = \underset{t\in[0,n-\eta_2]}{\cup} X^{\tau(p,t)(B(X^{\eta_2}(p'),(1-\eta_1)r'))}(B(X^{\eta_2}(p'),(1-\eta_1)r'))$$

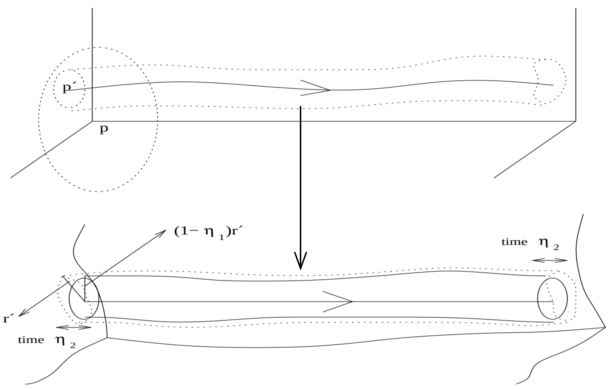


Figure 3.2: Considering a perpendicular section at $X^{\eta_2}(p')$.

The next lemma says that given any sufficient thin flow box $\underset{t\in[0,n]}{\cup} X^{\tau(p,t)}(B(p,r))$ (where $X^{[0,n]}(p)$ is a straightened-out orbit), a point $p' \in B(p,r)$, a small ball $B_{r'}(p') \subseteq \mathfrak{N}_p$ and the flow box $\underset{t\in[0,n]}{\cup} X^{\tau(p,t)(B_{r'}(p'))}((B_{r'}(p')))$, we can always take another flow box with right-angle sections contained in $\underset{t\in[0,n]}{\cup} X^{\tau(p,t)(B_{r'}(p'))}((B_{r'}(p')))$ and very close to it. This can be done because $r > 0$ can be considered close to zero. We leave the proof to the reader (see Figure 3.2).

**Lemma 3.1.4** *Take $n \in \mathbb{N}$. For every $\eta_1, \eta_2 > 0$ there exists $r > 0$ such that for all $r' < r$ we have $B(X^{\eta_2}(p'), (1-\eta_1)r') \subseteq \mathfrak{N}_{X^{\eta_2}(p')}$ and*

$$\mathfrak{F} \subseteq \underset{t\in[0,n]}{\cup} X^{\tau(p,t)(B_{r'}(p'))}(B_{r'}(p')).$$

## 3.2 Realizable linear flows

The next definition adapts the definition of realizable sequence given by Bochi in [3] and will also be central in the proof of our theorem, in broad terms we consider *modified* area-preserving linear flows acting in the normal bundle at $p$, $L^t(p) : N_p \to N_{X^t(p)}$ that do exactly what we want, and ask whether they are ($\gamma$-almost $C^1$) realizable as the linear Poincaré flow of $Y$, $\epsilon$-$C^1$-close to $X$, computed on small transversal neighborhoods of one point.

**Definition 3.2.1** *Given $X \in \mathfrak{X}^1_\mu(M)$, $\epsilon > 0$, $0 < \kappa < 1$ and a non-periodic point $p$, we say that the modified area-preserving sequence of linear flows:*

$$N_p \xrightarrow{L_0} N_{X^1(p)} \xrightarrow{L_1} N_{X^2(p)} \xrightarrow{L_2} \dots \xrightarrow{L_{n-2}} N_{X^{n-1}(p)} \xrightarrow{L_{n-1}} N_{X^n(p)}$$

*is a $(\epsilon, \kappa)$-**realizable linear flow of length** $n$ **at** $p$ if:*

*$\forall \gamma > 0$, $\exists r > 0$ such that for any open set $\emptyset \neq U \subseteq B(p, r) \subseteq \mathfrak{N}_p$, there exists:*

*(a) A measurable set $K \subseteq U$,*

*(b) A zero divergence vector field $Y$, $\epsilon$-$C^1$-close to $X$, such that:*

*(i) $Y^t = X^t$ outside the self-disjoint flowbox $\underset{t\in[0,n]}{\cup} X^{\tau(p,n)(U)}(U)$ and $DX_q = DY_q$ for every $q \in U, X^{\tau(p,n)(U)}(U)$;*

*(ii) $\overline{\mu}(K) > (1-\kappa)\overline{\mu}(U)$;*

*(iii) If $q \in K$, then $\|P^1_Y(Y^j(q)) - L_j\| < \gamma$ for $j \in \{0, 1, ..., n-1\}$.*

**Remark 3.2.1** *In the definition of realizable flow we consider integer iterates, but there is no restriction if we consider any intermediate linear flow, like $L_j : N_{X^{t_j}(p)} \longrightarrow N_{X^{t_{j+1}}(p)}$ with $t_j < t_{j+1}$ and $\sum_{j=0}^{n-1} t_j = n$. The point $p$ may also be periodic, but with period larger than $n$. The realizability is with respect to the $C^1$ topology.*

Now we exhibit how to produce some elementary realizable linear flows, the linear Poincaré flow itself and also juxtaposition of realizable linear flows are realizable linear flows.

**Lemma 3.2.1** *Let $X \in \mathfrak{X}^1_\mu(M)$ and $p \in M$ be a non-periodic point.*

*(1) Given any $t \in \mathbb{R}$, $P_X^t(p)$ (trivial linear flow) is $(\epsilon, \kappa)$-realizable of length $t$ for every $\epsilon$ and $\kappa$.*

*(2) Let $\{L_0, ..., L_{n-1}\}$ be $(\epsilon, \kappa_1)$-realizable of length $n$ at $p$ and $\{L_n, ..., L_{n+m-1}\}$ be $(\epsilon, \kappa_2)$-realizable of length $m$ at $X^n(p)$. For $\kappa$ such that $\kappa < \kappa_1 + \kappa_2 < 1$ the linear flow $\{L_0, ..., L_{n+m-1}\}$ is $(\epsilon, \kappa)$-realizable.*

**Proof:**

(1) Given $\gamma > 0$, by continuity of the linear Poincaré flow, there exists a sufficiently small $r > 0$ such that for all $q \in B(p, r)$ we have the inequality $\|P_X^1(X^j(q)) - P_X^1(X^j(p))\| < \gamma$ for $j = \{0, ..., n-1\}$. Let $r > 0$ be also sufficiently small such that $X^{[0,t]}(B(p, r))$ is a self-disjoint flowbox. For any open set $U \subseteq B(p, r)$, we choose $K \subseteq U$ satisfying (ii) of Definition 3.2.1 and $Y$ equal to $X$. So (i) of Definition 3.2.1 follows by choice of $Y$ and $r$, (ii) follows by choice of $K$ and (iii) is clearly true.

(2) Let $r_1, r_2$ be the radius according to Definition 3.2.1 related to realizable linear flows $\{L_0, ..., L_{n-1}\}$ and $\{L_n, ..., L_{n+m-1}\}$ respectively. We take any non-empty open set $U \subseteq B(p, r_1)$, if we have $X^{\tau(B(p,r_1),n)}(B(p, r_1)) \subseteq B(X^n(p), r_2)$ fine, otherwise we choose a smaller $r < r_1$. Given $\nu > 0$, decrease $r > 0$ if necessary, by using Lemma 3.1.3, to get $|\overline{\mu}(K) - x(t)\overline{\mu}(X^{\tau(p,t)(K)})(K)| < \nu$ for all $t \in [0, n]$ and any measurable set $K \subseteq B(p, r)$. By definition and choice of the radius $r > 0$, we have that the flowbox $\{X^{\tau(p,t)(U)}(U) : t \in [0, n+m]\}$ is self-disjoint, again by definition, given any $U \subseteq B(p, r)$ we get a measurable $K_1 \subseteq U$ and a vector field $Y_1$ satisfying (i), (ii) and (iii) of Definition 3.2.1. Also for any non-empty open subset of $B(X^n(p), r_2)$, in particular for $X^{\tau(p,n)(U)}(U)$, we get a measurable $\hat{K}_2 \subseteq X^{\tau(p,n)(U)}(U) =: \hat{U}$ and a vector field $Y_2$ satisfying (i), (ii) and (iii) of Definition 3.2.1.
Now define the vector field $Y = Y_1$ in the flowbox $\{X^{\tau(p,t)(U)}(U) : t \in [0, n]\}$, $Y = Y_2$ in the flowbox $\{X^{\tau(X^n(p),t+n)(U)}(U) : t \in [0, m]\}$ and $Y = X$ elsewhere. $Y$ is $C^1$ because by definition $(DY_1)_q = DX_q = (DY_2)_q$ for any $q \in X^{\tau(p,n)(U)}(U)$, so $Y$ and $U$ verifies (i). To check (ii) we define $K := K_1 \cap K_2$ where $K_2$ is such that $X^{\tau(p,n)(K_2)}(K_2) = \hat{K}_2$. By Lemma 3.1.3 we get $x(n)\overline{\mu}(\hat{U}) < \overline{\mu}(U) + \nu$ and also $\overline{\mu}(U - K_2) < x(n)\overline{\mu}(\hat{U} - \hat{K}_2) + \nu$. So we get:

$$\begin{aligned}
\overline{\mu}(U - K) &= \overline{\mu}(U - (K_1 \cap K_2)) \leq \overline{\mu}(U - K_1) + \overline{\mu}(U - K_2) < \\
&< \kappa_1\overline{\mu}(U) + x(n)\overline{\mu}(\hat{U} - \hat{K}_2) + \nu < \\
&< \kappa_1\overline{\mu}(U) + x(n)\kappa_2\overline{\mu}(\hat{U}) + \nu < \\
&< \kappa_1\overline{\mu}(U) + \kappa_2\overline{\mu}(U) + \kappa_2\nu + \nu = \\
&= \kappa\overline{\mu}(U) + (1 + \kappa_2)\nu.
\end{aligned}$$

Therefore $\overline{\mu}(K) = \overline{\mu}(U) - \overline{\mu}(U - K) > (1 - \kappa)\overline{\mu}(U) - (1 + \kappa_2)\nu$ and the result follows considering a sufficient small $\nu$. Finally (iii) follows by definition. $\square$

The next lemma says that we only have to prove realizability on balls.

**Lemma 3.2.2** *We only have to prove the realizability of the sequence of linear flows $\{L_0, ..., L_{n-1}\}$ for $U = B(p', r')$ where $B(p', r') \subseteq B(p, r)$.*

**Proof:** We use Vitali's covering lemma and we cover the open set $U$ with a finite number of balls $\{B(p_i, r_i)\}_{i=1,...,m}$ such that $\overline{\mu}(U - \cup_{i=1}^{m} B(p_i, r_i))$ is as small as we want. By hypothesis, for each $B(p_i, r_i)$ there exists a measurable set $K_i \subseteq B(p_i, r_i)$ and a zero divergence vector field $Y_i$, $\epsilon$-$C^1$-close to $X$, such that:

(a) $Y_i^t = X^t$ outside the flowbox $\{X^{\tau(p_i,t)(B(p_i,r_i))}(B(p_i, r_i)) : t \in [0, n]\}$ and $DX_q = (DY_i)_q$ for every $q \in B(p_i, r_i), X^{\tau(p,n)(B(p_i,r_i))}(B(p_i, r_i))$;

(b) $\overline{\mu}(K_i) > (1 - \kappa)\overline{\mu}(B(p_i, r_i))$;

(c) For every $q \in K_i$ we have $\|P^1_{Y_i}(Y_i^j) - L_j\| < \gamma$ for $j \in \{0, 1, ..., n-1\}$.

Then we may define the same objects in $U$ by taking $K = \underset{i=1,..,n}{\cup} K_i$; and $Y = X$ outside the self-disjoint flowbox $\{X^{\tau(p,t)(B(p_i,r_i))}(B(p_i, r_i)) : t \in [0, s]\}$ and $Y = Y_i$ inside it and the lemma is proved. $\square$

### 3.2.1 Small rotations

Next we will construct realizable linear flows of time-1 length at $p$, which rotate by a small angle $\xi$ the action of the linear Poincaré flow, i.e., $L_0 := P^1_X(p) \circ R_\xi$ where $R_\xi$ is a rotation of angle $\xi$. We may expect that increasing the length will allow us to rotate by a larger angle but unfortunately this is not possible, because the size of the perturbation depends on the dynamics, on the angles between the bundles and also on the change of coordinates given by Lemma 3.1.2.

Denote the rotation matrix by:

$$R_\theta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{pmatrix}.$$

Given a 3×3 matrix $A$ denote by $\hat{A}$ the 2×2 matrix obtained after removing the first row and first column from $A$ so:

$$\hat{R}_\theta := \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}.$$

**Lemma 3.2.3** *Given $X \in \mathfrak{X}^2_\mu(M)$, a non-periodic point $p \in M$, $\epsilon > 0$, $0 < \kappa < 1$, $\gamma > 0$ and a fixed time $T = 1$. There exists $r > 0$ (depending on $p$), an angle $\xi$ (not depending on $p$) and a zero divergence vector field $Y$, $\epsilon$-$C^1$-close to $X$ such that:*

*(a) $Y - X$ is supported in the flowbox $\{X^{\tau(p,t)(B(p,r))}(p)(B(p,r)) : t \in [0,1]\}$,*

*(b) For every $q \in B(p, r\sqrt{1-\kappa})$ we have $\|P^1_Y(q) - P^1_X(p) \circ \hat{R}_\xi\| < \gamma$.*

**Proof:** We take $p \in M$ a non-periodic point. Let $C := \max\{\|DX^1_p\| : p \in M\}$ and suppose that this constant is valid for any vector field $\epsilon$-$C^1$-close to $X$. Using Lemma 3.1.2 we obtain a $C^2$ conservative diffeomorphism $\Psi : \mathfrak{F} \to \mathfrak{C}$. Consider the constant $\Theta$ defined at section 3.1.2 and suppose that this constant is also valid for any vector field $\epsilon$-$C^1$- close to $X$.
Next we define two $C^\infty$ *bump-functions* $g$ and $G$. Note that for $q \in B(p,r)$ the orbit $X^{[0,1]}(q)$ in general crosses $\mathfrak{N}_{X^1(p)}$, nevertheless for all $\eta > 0$ exists a small $r > 0$ such that we have $X^{[0,1-\eta/2]}(q) \cap \mathfrak{N}_{X^1(p)} = \emptyset$, for all $q \in B(p,r)$.
Let $\eta > 0$ be sufficiently small to get:

(i) $\|R_{\xi\alpha} - R_\xi\| < \gamma/8C$ for $\alpha \in [1-\eta, 1-\eta/2]$ and also for all $q \in B(p,r)$ we have $X^{[0,\alpha]}(q) \cap \mathfrak{N}_{X^1(p)} = \emptyset$.

Now let $g : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function such that $g(t) = 0$ for $t < 0$, $g(t) = t$ for $t \in [\eta, 1-\eta]$ and $g(t) = \alpha$ for $t \geq 1-\eta/2$ where $\alpha \in ]1-\eta, 1-\eta/2[$ is fixed. We take a sufficiently small $r > 0$ such that for all $q \in B(p,r)$ we have:

(ii) $\|\Pi_{Y^1(q)} - \Pi_{X^1(p)}\| < \frac{\gamma}{2C}$ for any vector field $Y = X$ outside $\mathfrak{F}$ and $\epsilon$-$C^1$-close to $X$;

(iii) $\|D\Psi_p - Id\| < \frac{\gamma}{16\Theta}$;

(iv) For $q \in B(p,r)$ and any vector field $Z = T$ outside $\mathfrak{C}$ and $\epsilon$-$C^1$-close to $X$ we have;

1. $\|D\Psi^{-1}_{Z^1(\Psi(q))} - D\Psi^{-1}_{Z^1(\Psi(p))}\| < \frac{\gamma}{24C\Theta}$;
2. $\|DZ^1_{\Psi(q)} - DZ^1_{\Psi(p)}\| < \frac{\gamma}{24\Theta^2}$ and
3. $\|D\Psi_q - D\Psi_p\| < \frac{\gamma}{24C\Theta}$.

(v) $|y|, |z| < \frac{c\epsilon}{\Theta^2 \ddot{g}(x)\xi}$ for any $\xi$ such that $0 < \xi < 1$;

(vi) $|y|, |z| < \frac{\epsilon}{\Theta}$.

We take the angle $\xi$ such that,

$$\xi < \frac{\epsilon(1 - \sqrt{1-\frac{\kappa}{2}})}{2\Theta^2}.$$

For $r > 0$ satisfying the properties above let $G : \mathbb{R} \to [0,1]$ be a $C^\infty$ function such that $G(\rho) = 1$ for $\rho \leq r\sqrt{1-\frac{\kappa}{2}}$ and $G(\rho) = 0$ for $\rho \geq r$. Note that $\max|\dot{G}| \leq \frac{2}{(1-\sqrt{1-\frac{\kappa}{2}})r}$. Let $\rho = \sqrt{y^2+z^2}$ and consider the rotation flow acting on $\mathfrak{N}_p$ defined by:

$$R_{\xi g(t)G(\rho)}(0,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\xi g(t)G(\rho)) & -sin(\xi g(t)G(\rho)) \\ 0 & sin(\xi g(t)G(\rho)) & cos(\xi g(t)G(\rho)) \end{pmatrix} \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$$

Denote the time derivative by $\dot{R}_{\xi g(t)G(\rho)} = \frac{d}{dt}(R_{\xi g(t)G(\rho)})$. A simple computation shows that,

$$\dot{R}_{\xi g(t)G(\rho)}.R^{-1}_{\xi g(t)G(\rho)} = \xi \dot{g}(t)G(\rho) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.4}$$

**Inducing the adequate vector field:**

We consider the flow $T^t$ associated to the vector field $T$ and we define for $q = (0,y,z) \in B(p,r)$, $\Upsilon(t,q) := T^t(R_{\xi g(t)G(\rho)}(q))$. We denote $R_{\xi g(t)G(\rho)}(q)$ by $R_t(q)$. Let $H(t,q) := (t, R_t(q))$ and $F(t, R_t(q)) := T^t(R_t(q))$ so we obtain $\Upsilon(t,q) = F \circ H(t,q)$. We take time derivatives at $t = s$:

$$\begin{aligned} \frac{d}{dt}\Upsilon(t,q)|_{t=s} &= \frac{d}{dt}T^t(R_t(q))|_{t=s} = DF_{H(s,q)} \cdot DH_s = \\ &= \begin{pmatrix} \partial_1 F & \partial_2 F \end{pmatrix}_{H(s,q)} \begin{pmatrix} \partial_1 H \\ \partial_2 H \end{pmatrix}_s = \\ &= \begin{pmatrix} T(T^s(R_s)) & DT^s_{R_s(q)} \end{pmatrix} \begin{pmatrix} 1 \\ \dot{R}_s(q) \end{pmatrix}, \end{aligned}$$

and we get

$$\frac{d}{dt}\Upsilon(t,q)|_{t=s} = T(T^s \circ R_s(q)) + DT^s_{R_s(q)} \circ \dot{R}_s(q).$$

So the $C^1$ vector field $Z$ is defined in flowbox coordinates by:

$$Z(\cdot) = T(\cdot) + DT^s_{R_s(q)} \circ \dot{R}_s(R_{-s} \circ T^{-s})(\cdot)$$

Let $T^s(R_s(q)) = (cs, y, z)$. Since $T^{-s}(cs,y,z) = (0,y,z)$ and $DT^s_{R_s(q)} = Id$ by (3.4) we obtain that the $C^1$-perturbation is defined by $Z = T + P$ where:

$$P(x,y,z) = \xi \dot{g}(x) G(\sqrt{y^2+z^2})(0,-z,y). \tag{3.5}$$

**Properties of $Z = T + P$:**

$Z^t$ is volume preserving;

$$\begin{aligned} \text{div}Z(x,y,z) &= \text{div}T(x,y,z) - \frac{\partial G}{\partial y}\xi \dot{g}(x) + \frac{\partial G}{\partial z}\xi \dot{g}(x) = \\ &= \xi \dot{g}(x)[-\dot{G}\frac{y}{\sqrt{y^2+z^2}}z + \dot{G}\frac{z}{\sqrt{y^2+z^2}}y] = 0. \end{aligned}$$

We also have that the support of the perturbation $P$ is $B(p,r)\times[0,c\alpha]$.

**Estimation of the $C^1$ norm of the perturbation $P$:**

By (vi) $|y|,|z|<\epsilon/\Theta$ so $\|P\|_{C^0}\leq\epsilon/\Theta$. To compute the $C^1$ norm we take derivatives:

$$DP_{(x,y,z)}=\begin{pmatrix}0 & 0 & 0\\ -\xi\ddot g(x)G(\rho)zc^{-1} & -\xi\dot g(x)\frac{\partial G}{\partial y}z & -\xi\dot g(x)[\frac{\partial G}{\partial z}z+G(\rho)]\\ \xi\ddot g(x)G(\rho)yc^{-1} & \xi\dot g(x)[\frac{\partial G}{\partial y}y+G(\rho)] & \xi\dot g(x)\frac{\partial G}{\partial z}y\end{pmatrix} \quad (3.6)$$

For the first column we use (v), but we must verify also that the other terms are unaffected by the choice of $r>0$ small. We take, for example, $\frac{\partial G}{\partial y}z$ and polar coordinates $(y,z)=(\rho\cos(\beta),\rho\sin(\beta))$ then we have,

$$\begin{aligned}\frac{\partial G}{\partial y}z &= \frac{\partial G(\sqrt{y^2+z^2})}{\partial y}z=\dot G\frac{yz}{\sqrt{y^2+z^2}}\leq\dot G\frac{2\rho^2}{\rho}\leq\\ &\leq \frac{2\rho}{(1-\sqrt{1-\frac{\kappa}{2}})r}\leq\frac{2}{(1-\sqrt{1-\frac{\kappa}{2}})}.\end{aligned}$$

For the other three terms, $\frac{\partial G}{\partial y}y$, $\frac{\partial G}{\partial z}y$ and $\frac{\partial G}{\partial z}z$ we proceed analogously. Since $G$ and $\dot g$ are bounded and $\xi<\frac{\epsilon(1-\sqrt{1-\frac{\kappa}{2}})}{2\Theta^2}$ we get $\|DP\|_{C^0}<\epsilon/\Theta^2$. Note that we are allowed to take $y,z$ close to zero without interfering with the size of the perturbation. This is a key property of the $C^1$ topology. Next we make use of this fact to get properties (a) and (b). The perturbation is defined, in the original coordinates, by $P_1(\cdot)=D\Psi^{-1}_{\Psi(\cdot)}P(\Psi(\cdot))$. We have $\dot g(x)=0$ for $x\geq 1-\eta/2$ and $x\leq 0$. For $t\in[0,1-\eta/2]$ and $q\in B(p,r)$ we guarantee that $X^t(q)$ does not intersect $\mathfrak{N}_{X^1(p)}$, so by (i) above we get that $P_1=Y-X$ is supported inside the flowbox $\underset{t\in[0,1]}{\cup}X^{\tau(p,t)(B(p,r))}(p)(B(p,r))$ and (a) follows. Now we are interested in $\kappa$'s close to zero and since $\Psi|_{\partial B(p,r)}=Id$ we have $\Psi(B(p,r\sqrt{1-\kappa}))\subseteq B((0,0,0),r\sqrt{1-\frac{\kappa}{2}})$. Therefore for $q\in B(p,r\sqrt{1-\kappa})$ we have $Z^1(\Psi(q))=(c,R_{\xi\alpha}(\Psi(q)))$, so:

$$DZ^1_{\Psi(q)}=R_{\xi\alpha}. \quad (3.7)$$

We use (iii) and $X=\Psi_*T$ to get:

$$\begin{aligned}&\|DX^1_pR_{\xi\alpha}-D\Psi^{-1}_{Z^1(\Psi(p))}DZ^1_{\Psi(p)}D\Psi_p\|=\\ =\;&\|D\Psi^{-1}_{T^1(\Psi(p))}DT^1_{\Psi(p)}D\Psi_pR_{\xi\alpha}-D\Psi^{-1}_{T^1(\Psi(p))}DT^1_{\Psi(p)}R_{\xi\alpha}D\Psi_p\|\leq\\ \leq\;&\|D\Psi^{-1}_{T^1(\Psi(p))}\|\|DT^1_{\Psi(p)}\|\|D\Psi_pR_{\xi\alpha}-R_{\xi\alpha}D\Psi_p\|\leq\\ \leq\;&\Theta\|(D\Psi_p-Id)R_{\xi\alpha}+R_{\xi\alpha}(Id-D\Psi_p)\|\leq\gamma/8.\end{aligned}$$

Therefore:

$$\|DX^1_pR_{\xi\alpha}-D\Psi^{-1}_{Z^1(\Psi(p))}DZ^1_{\Psi(p)}D\Psi_p\|\leq\frac{\gamma}{8}. \quad (3.8)$$

Since $Y=\Psi_* Z$ we get $DY^1_q = D\Psi^{-1}_{Z^1(\Psi(q))} DZ^1_{\Psi(q)} D\Psi_q$ so using (iv) we get:

$$\begin{aligned}
& \|DY^1_q - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_p\| = \\
= & \|D\Psi^{-1}_{Z^1(\Psi(q))} DZ^1_{\Psi(q)} D\Psi_q - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_p\| \leq \\
\leq & \|D\Psi^{-1}_{Z^1(\Psi(q))} DZ^1_{\Psi(q)} D\Psi_q - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(q)} D\Psi_q\| + \\
+ & \|D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(q)} D\Psi_q - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_q\| + \\
+ & \|D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_q - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_p\| = \\
= & \|(D\Psi^{-1}_{Z^1(\Psi(q))} - D\Psi^{-1}_{Z^1(\Psi(p))}) DZ^1_{\Psi(q)} D\Psi_q\| + \\
+ & \|D\Psi^{-1}_{Z^1(\Psi(p))} (DZ^1_{\Psi(q)} - DZ^1_{\Psi(p)}) D\Psi_q\| + \\
+ & \|D\Psi^{-1}_{Z^1(\Psi(q))} DZ^1_{\Psi(p)} (D\Psi_q - D\Psi_p)\| \leq \\
\leq & \|D\Psi^{-1}_{Z^1(\Psi(q))} - D\Psi^{-1}_{Z^1(\Psi(p))}\| \|DZ^1_{\Psi(q)}\| \|D\Psi_q\| + \\
+ & \|D\Psi^{-1}_{Z^1(\Psi(p))}\| \|DZ^1_{\Psi(q)} - DZ^1_{\Psi(p)}\| \|D\Psi_q\| + \\
+ & \|D\Psi^{-1}_{Z^1(\Psi(q))}\| \|DZ^1_{\Psi(p)}\| \|D\Psi_q - D\Psi_p\| \leq \\
\leq & \|DZ^1\| \|D\Psi\| \frac{\gamma}{24C\Theta} + \|D\Psi\| \|D\Psi^{-1}\| \frac{\gamma}{24\Theta^2} + \|D\Psi^{-1}\| \|DZ^1\| \frac{\gamma}{24C\Theta} \leq \\
\leq & \frac{\gamma}{8}.
\end{aligned}$$

Therefore:

$$\|DY^1_q - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_p\| \leq \frac{\gamma}{8} \tag{3.9}$$

and (3.8) and (3.9) imply $\|DY^1_q - DX^1_p R_{\xi\alpha}\| \leq \frac{\gamma}{4}$. Jointly with (i) above we get:

$$\begin{aligned}
\|DY^1_q - DX^1_p R_\xi\| &\leq \|DY^1_q - DX^1_p R_{\xi\alpha}\| + \|DX^1_p R_{\xi\alpha} - DX^1_p R_\xi\| \leq \\
&\leq \gamma/4 + \|DX^1_p\| \|R_{\xi\alpha} - R_\xi\| \leq \\
&\leq \gamma/2.
\end{aligned}$$

Finally we use (ii) and we obtain:

$$\begin{aligned}
\|P^1_Y(q) - P^1_X(p) \circ \hat{R}_\xi\| &= \|\Pi_{Y^1(q)} \circ DY^1_q - \Pi_{X^1(p)} \circ DX^1_p R_\xi\| = \\
&= \|\Pi_{Y^1(q)} \circ DY^1_q - \Pi_{Y^1(q)} \circ DX^1_p R_\xi\| + \\
&+ \|\Pi_{Y^1(q)} \circ DX^1_p R_\xi - \Pi_{X^1(p)} \circ DX^1_p R_\xi\| \leq \\
&\leq \|\Pi_{Y^1(q)}\| \|DY^1_q - DX^1_p R_\xi\| + \\
&+ \|\Pi_{Y^1(q)} - \Pi_{X^1(p)}\| \|DX^1_p R_\xi\| \leq \\
&\leq \gamma.
\end{aligned}$$

**Estimation of the $C^1$ norm of $P_1$:**

Above in (vi) we consider $|y|, |z| < \epsilon/\Theta$ and we choose $\xi < \frac{\epsilon(1-\sqrt{1-\frac{\kappa}{2}})}{2\Theta^2}$ and we obtain $\|DP\|_{C^0} < \epsilon/\Theta^2$. Now since $P_1(q) = D\Psi^{-1}_{\Psi(q)} P(\Psi(q))$ and we have

$(DP_1)_q = D^2\Psi^{-1}_{\Psi(q)} DP_{\Psi(q)} D\Psi_q$ we obtain that $\|P_1\|_{C^0} \leq \Theta\|P\|_{C^0} \leq \epsilon$ and also that $\|DP_1\|_{C^0} \leq \|D^2\Psi^{-1}\|\|DP\|\|D\Psi\| \leq \Theta^2\|DP\| \leq \epsilon$. We conclude that,

$$\|Y - X\|_{C^1} = \|P_1\|_{C^1} \leq \epsilon,$$

and the lemma is proved. $\square$

**Lemma 3.2.4** *Given $X \in \mathfrak{X}^2_\mu(M)$, a non-periodic point $p \in M$, $\epsilon > 0$, $0 < \kappa < 1$, $\gamma > 0$ and a fixed time $T = 1$. There exists $r > 0$ (depending on $p$), an angle $\xi$ (not depending on $p$) and a zero divergence vector field $Y$, $\epsilon$-$C^1$-close to $X$ such that:*

*(a) $Y - X$ is supported in the flowbox $\underset{t\in[0,1]}{\cup} X^{\tau(p,-t)(B(p,r))}(B(p,r))$,*

*(b) For every $q \in X^{\tau(p,-1)(B(p,\sqrt{1-\kappa}r))}(B(p,(1-\kappa)r))$ we have the following inequality $\|P^1_Y(q) - \hat{R}_\xi \circ P^1_X(X^{-1}(p))\| < \gamma$.*

**Proof:** We proceed like in Lemma 3.2.3, this time for $X^{-t}$ finding a change of coordinates $\hat{\Psi}(x,y,z) = X^{-\lambda^{-1}x}(0,\varphi(y,z))$. Then we consider $R^{-1}_{\xi g(t)G(\rho)}$, $t > 0$ and we find $Z$. We define $Y = \Psi_* Z$ and we get:

$$\begin{aligned} P^1_Y(q) &= [P^{-1}_Y(Y^1(q))]^{-1} \approx [P^{-1}_X(p) \circ \hat{R}^{-1}_\xi]^{-1} = \\ &= \hat{R}_\xi \circ [P^{-1}_X(p)]^{-1} = \hat{R}_\xi \circ P^1_X(X^1(p)), \end{aligned}$$

and the arguments are equal to the ones used in the proof of Lemma 3.2.3. $\square$

Now we use the two previous lemmas to build some realizable linear flows.

**Lemma 3.2.5** *Given $X \in \mathfrak{X}^2_\mu(M)$, $\epsilon > 0$, $0 < \kappa < 1$, a non-periodic point $p$ and a fixed time $T = 1$. Then there exists an angle $\xi$ (not depending on $p$) such that $L_0 = P^1_X(p) \circ \hat{R}_\xi$ and $L_0 = \hat{R}_\xi \circ P^1_X(p)$ are $(\epsilon,\kappa)$-realizable linear flows of length 1 at $p$.*

**Proof:** We prove that $L_0 = P^1_X(p) \circ \hat{R}_\xi$ is $(\epsilon,\kappa)$-realizable. Let $\gamma > 0$. By Lemma 3.2.2 we may choose the open set $U$ to be a ball, say $B(p',r') \subseteq B(p,r)$. Now we apply Lemma 3.2.3 and we get a zero divergence vector field $Y$, $\epsilon$-$C^1$-close to $X$ such that $Y - X$ is supported inside the flowbox defined by $\{X^{\tau(p',t)(B(p',r'))}(B(p',r')) : t \in [0,1]\}$ and for every $q \in B(p', r'\sqrt{1-\kappa})$ we have $\|P^1_Y(q) - P^1_X(p') \circ \hat{R}_\xi\| < \gamma$. Note that since $r > 0$ is small the arrival time for points at $B(p,r)$ is almost 1.
The support of the perturbation is contained in the flowbox. For the perturbation $P$ defined in Lemma 3.2.3 we have $DP_{\Psi(q)} = [0]$ for $\Psi(q) \in B(0,r')$, $T^t(B(0,r'))$ and for $t \geq 1 - \eta/2$ so $DX_q = DY_q$ for any $q$ belonging to $B(p',r')$ and also to $X^{\tau(p,1)(B(p',r'))}(B(p',r'))$. Therefore (i) on Definition 3.2.1 is true.

We take $K \subseteq U$ equal to $\overline{B}(p', r'\sqrt{1-\kappa})$ and we get $\frac{\overline{\mu}(K)}{\overline{\mu}(U)} = \frac{\pi(1-\kappa)r'^2}{\pi.r'^2} = 1-\kappa$ and (ii) follows. Finally (iii) follows from (b) of Lemma 3.2.3 and the continuity of the linear Poincaré flow.
For $L_0 = \hat{R}_\xi \circ P_X^1(p)$ we proceed analogously now using Lemma 3.2.4. Given any open set $\hat{U} \subseteq \mathfrak{N}_{X^{-1}(p)}$ we take $U = X^{\tau(X^{-1}(p),1)(\hat{U})}(\hat{U}) \subseteq \mathfrak{N}_p$, then we measure theoretically fill up this open set $U$ by taking a finite union of balls $\{B_i\}_{i=1,\ldots,m}$. We denoted this covering by $\mathfrak{C}$. Let $\hat{\mathfrak{C}} \subseteq \hat{U}$ be such that $X^{\tau(X^{-1}(p),1)(\hat{\mathfrak{C}})}(\hat{\mathfrak{C}}) = \mathfrak{C}$. Of course $\overline{\mu}(\hat{U} - \hat{\mathfrak{C}})$ can be made as small as we want, and the realizability follows. □

We continue to produce realizable linear flows.

**Lemma 3.2.6** *Given $X \in \mathfrak{X}^2_\mu(M)$, $\epsilon > 0$, $0 < \kappa < 1$ and a non-periodic point $p$, there exists an angle $\xi$ such that for $|\xi_i| < \xi$, $i = 1, 2$;*

$$N_p \xrightarrow{P_X^1(p)\circ \hat{R}_{\xi_1}} N_{X^1(p)} \xrightarrow{P_X^r(p)} N_{X^{1+r}(p)} \xrightarrow{\hat{R}_{\xi_2}\circ P_X^1(X^{r+2}(p))} N_{X^{r+2}(p)}$$

*is a $(\epsilon, \kappa)$-realizable linear flow of length $r+2$ at $p$.*

**Proof:** Take $\gamma > 0$. By Lemma 3.2.5 for $\kappa_1 < \kappa$ we get $\xi$ such that $P_X^1(p) \circ \hat{R}_{\xi_1}$ and $\hat{R}_{\xi_2} \circ P_X^1(X^{r+2}(p))$ are $(\epsilon, \kappa_1)$-realizable. By Lemma 3.2.1 (1) the trivial flow $P_X^r$ is $(\epsilon, \kappa_1)$-realizable. Now if $\kappa_1 < \kappa/3$ then we use Lemma 3.2.1 (2) and obtain the $(\epsilon, \kappa)$-realizability. □

### 3.2.2 Large rotations

Now we find conditions under which we can rotate by large angles. In the previous section we were able to rotate by time-1, so what happens if we increase time? We want to rotate by an angle $2\pi$, thus we take a time $m$ such that $\xi m = 2\pi$. But $\xi$ is in general very small, so $m$ must be very large. Note that the choice of $m$ may affect the norm of the perturbation because $\|\Psi\|$, for $\Psi$ given by Lemma 3.1.2, depends on $m$ and in general increases with $m$. Furthermore the dynamics along the orbit may also obstruct the construction of a small norm perturbation. Let us consider a situation for which this last problem is minimized, say when we have simultaneously:

(a) No domination conditions for each bundle, i.e. $P_X^t(p)$ is "almost conformal" for all $t \in [0, m]$.

(b) The angle between $N^u_{X^t}(p)$ and $N^s_{X^t}(p)$ is larger than a fixed $\xi$ for all $t \in [0, m]$.

Even if we have properties (a) and (b) our perturbations may not have a small $C^1$-norm, because as we already said, the entries $y(t)$ and $z(t)$ of $DX^t_p$

(see (3.3)) may obstruct the construction of a vector field $Y - X$ with small norm. So we will concatenate several small rotations, however this concatenation worsens $\kappa$. In [3] this problem is bypassed by using a nested rotation lemma (Lemma 3.7 in [3]) and here we will adapt this procedure. Note that if we had $y(t), z(t)$ bounded, then under conditions (a) and (b) we could perform large rotations with just one single perturbation. In fact this is what we did when we carry out the development of perturbations for linear differential systems, see section 7.6.2.

Since we will juxtapose several rotations beginning with a ball it turns out that after the first time-1 iteration, by the linear Poincaré flow, it will became an ellipse. We consider vector fields which induce elliptical rotations over normal sections, so take the elliptical rotation flow defined by:

$$E_{\xi g(t)G(\rho)} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & a\cos(\xi g(t)G(\rho)) & -a\sin(\xi g(t)G(\rho)) \\ 0 & d\sin(\xi g(t)G(\rho)) & d\cos(\xi g(t)G(\rho)) \end{pmatrix},$$

where $d \geq a$ are the axis of the ellipse. Let $E = \sqrt{\frac{d}{a}}$. As in [3] we call $E$ the eccentricity of the ellipse, so eccentricity close to one is equivalent to be almost conformal. A simple computation shows that:

$$\dot{E}_{\xi g(t)G(\rho)}.E^{-1}_{\xi g(t)G(\rho)} = \xi \dot{g}(t)G(\rho) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -E^{-2} \\ 0 & E^2 & 0 \end{pmatrix}. \tag{3.10}$$

The number $E$ measures how large is the norm of a modified area-preserving linear map that sends a ball into the ellipse. The ball $B(p,1)$ is mapped into an ellipse denoted by $\mathfrak{B}(p) \subseteq \mathfrak{N}_p$. Let $0 < \zeta < 1$ and we denote by $\mathfrak{B}(p,\zeta)$ the ellipse $\mathfrak{B}(p)$ after a shrinking by a factor of $\zeta$.

**Lemma 3.2.7** *Given $X \in \mathfrak{X}^2_\mu(M)$, a non-periodic point $p \in M$, $\epsilon > 0$, $0 < \kappa < 1$, $\gamma > 0$ a fixed time $T = 1$ and $E \geq 1$, then there exists $r > 0$ and $\hat{\epsilon} > 0$ such that if $\mathfrak{B}(p,r)$ is an ellipse with eccentricity less than E, $diam(\mathfrak{B}(p,r)) < \epsilon$ and $\|P^1_X(p)\hat{E}_\xi - P^1_X(p)\| < \hat{\epsilon}$ for $\xi > 0$, we may find a $C^1$ zero divergence vector field $Y$, $\epsilon$-$C^1$-close to $X$ such that:*

*(a) $Y - X$ is supported in the flowbox $\underset{t\in[0,1]}{\cup} X^{\tau(p,t)(\mathfrak{B}(p,r))}(\mathfrak{B}(p,r))$,*

*(b) For every $q \in \mathfrak{B}(p, r\sqrt{1-\kappa})$ we have $\|P^1_Y(q) - P^1_X(p) \circ \hat{E}_\xi\| < \gamma$.*

**Proof:** The proof is the same of Lemma 3.2.3, but the angle $\xi$ depends also on $E$, because rotations of ellipses with large $E$ imply large perturbations. By (3.10) we get $\|\dot{E}_{\xi g(t)G(\rho)}.E^{-1}_{\xi g(t)G(\rho)}\| \leq E^2\xi$. Let $\hat{\epsilon} > 0$ be sufficiently small.

So we have $\|P_X^1(p)\hat{E}_\xi - P_X^1(p)\| < \hat{\epsilon}$, therefore $a, d \approx 1$. Consequently $E^2 \approx 1$ and we rotate approximately $\xi$. $\square$

The next lemma says that if we fix a small ellipse in a small ball $B(p,r) \subseteq \mathfrak{N}_p$ and consider its arrival into $\mathfrak{N}_{X^1(p)}$, then this set is almost the image by the linear Poincaré flow at $p$ of the same ellipse modulo translations. A similar statement is proved in Lemma 3.6 of [3] (see Figure 3.3).

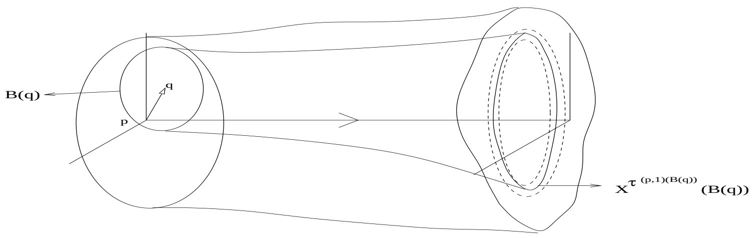


Figure 3.3: For small $r > 0$, $X^{\tau(p,1)(\mathfrak{B}(q))}(\mathfrak{B}(q))$ is almost an ellipse.

**Lemma 3.2.8** *Let $X^t : M \longrightarrow M$ be a $C^1$-flow, $\zeta \in ]0,1[$ (near 1), $E \geq 1$. There exists $r > 0$ such that, for all ellipses $\mathfrak{B}(q) \subseteq B(p,r) \subseteq \mathfrak{N}_p$ with eccentricity less or equal than $E$ we have:*

*(A)* $P_X^1(p)(\mathfrak{B}(q,\zeta) - q) + X^{\tau(p,1)(q)}(q) \subseteq X^{\tau(p,1)(\mathfrak{B}(q))}(\mathfrak{B}(q))$

*(B)* $X^{\tau(p,1)(\mathfrak{B}(q))}(\mathfrak{B}(q)) \subseteq P_X^1(p)(\mathfrak{B}(q, 2-\zeta) - q) + X^{\tau(p,1)(q)}(q)$.

Since $P_X^t(p)$ is modified area-preserving, we measure the non-conformality using its norm $\|P_X^t(p)\|$ in the following way. Suppose that $P_X^t(p)$ has a matrix representation $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $d \geq a$. Then the eccentricity of $P_X^t(p)(B(p,1))$ is $E = \sqrt{\frac{d}{a}}$. Since $\|P_X^t(p)\| = d$, and by volume-preserving we have $a^{-1} = d.x(t)$ we conclude that $E = \sqrt{\frac{d}{a}} = \sqrt{d^2 x(t)} = d\sqrt{x(t)} = \|P_X^t(p)\|\sqrt{x(t)}$.

In next lemma we adapt Lemma 3.3 of [3].

**Lemma 3.2.9** *Given $X \in \mathfrak{X}_\mu^2(M)$, a non-periodic point $p \in M$, $\epsilon > 0$, $0 < \kappa < 1$, $\gamma > 0$ and $E \geq 1$, suppose that for a fixed $n \in \mathbb{N}$ we have $\|P_X^j(p)\| \leq E\sqrt{x^{-1}(j)}$ for $j = 0, 1, ..., n-1$. Then there exists $\hat{\epsilon} > 0$ such that if:*

$$N_p \xrightarrow{L_0} N_{X^1(p)} \xrightarrow{L_1} \dots \xrightarrow{L_{n-1}} N_{X^n(p)}$$

*is a sequence of linear flows satisfying:*

*(a)* $L_{j-1} \circ ... \circ L_0(B(p,1)) = P_X^j(p)(B(p,1))$ *for* $j = 0, 1, ..., n-1$*;*

*(b)* $\|P_X^1(X^j(p)) - L_j\| < \hat{\epsilon}$ *for* $j = 0, 1, ..., n-1$,

*then* $\{L_0, L_1, ..., L_{n-1}\}$ *is a* $(\epsilon, \kappa)$*-realizable linear flow.*

**Proof:** Let $\gamma > 0$ be given. Take $\hat{\epsilon}$ given by Lemma 3.2.7 and depending on $X$, $\epsilon$, $\kappa$, $E$ and $\gamma/3$.

**Choice of r:**

We choose $r_0 > 0$ such that:

(1) $\|P_X^j(q)\| < 2E\sqrt{x^{-1}(j)}$ for all $q \in B(p, r_0\sqrt{1-\kappa})$ and for $j = 0, 1, ..., n-1$;

(2) For all $q \in X^{\tau(p,j)(B(p,r_0))}(B(p,r_0))$ we have $\|P_X^1(q) - P_X^1(X^j(p))\| < \frac{\gamma}{3E}$;

(3) Any vector field $Y$ $\epsilon$-$C^1$-close to $X$ and also such that $X = Y$ outside $X^{[0,n]}(B(p,r_0))$ satisfy: $\|P_Y^1(Y^j(q)) - P_Y^1(x_j)\| < \frac{\gamma}{3}$, for any $q \in B(p,r_0)$ and $x_j \in \mathfrak{N}_{X^j(p)}$.

By hypothesis $\|P_X^j(p)\| \leq E\sqrt{x^{-1}(j)}$ and also we have by *(a)* $L_{j-1} \circ ... \circ L_0(B(p,1)) = P_X^j(p)(B(p,1))$ for $j = 0, 1, ..., n-1$, then we take $E_j$ the elliptical rotation with eccentricity less or equal than $E$ and define,

$$L_j := P_X^1(X^j(p)) \circ \hat{E}_j.$$

Now we choose $\kappa_0 < \kappa$ by taking $\lambda \in ]0,1[$ near 1 satisfying $\lambda^{4n}(1-\kappa_0) > 1-\kappa$. We take $\zeta \in ]0,1[$, such that $\zeta \in ]\lambda, 1[$ and $2-\zeta \in ]1, \lambda^{-1}[$. By Lemma 3.2.8 there exists $r_1 > 0$ such that for all ellipses $\mathfrak{B}(q) \subseteq B(p, r_1) \subseteq \mathfrak{N}_p$ with eccentricity less or equal than $E$ we have:

(A) $P_X^1(p)(\mathfrak{B}(q,\zeta) - q) + X^{\tau(p,1)(q)}(q) \subseteq X^{\tau(p,1)(\mathfrak{B}(q))}(\mathfrak{B}(q))$

(B) $X^{\tau(p,1)(\mathfrak{B}(q))}(\mathfrak{B}(q)) \subseteq P_X^1(p)(\mathfrak{B}(q, 2-\zeta) - q) + X^{\tau(p,1)(q)}(q)$.

Again by Lemma 3.2.8 we have for all $j \in \{1, ..., n-1\}$ that there exists $r_{j+1} > 0$ such that for any ellipse $\mathfrak{B}(q) \subseteq B(X^j(p), r_{j+1}) \subseteq \mathfrak{N}_{X^j(p)}$ with eccentricity less or equal than $E$ we have:

$$X^{\tau(X^j(p),1)(\mathfrak{B}(q))}(\mathfrak{B}(q)) \supseteq P_X^1(X^j(p))(\mathfrak{B}(q,\zeta) - q) + X^{\tau(X^j(p),1)(q)}(q).$$

After applying Lemma 3.2.8 $n$ times, we choose $r_{n+1} > 0$ such that:

$$X^{\tau(p,j)(B(p,r_{n+1}))(B(p,r_{n+1}))} \subseteq B(X^j(p), r_{j+1}) \text{ for } j \in \{1, ..., n-1\}.$$

We define the value of $r > 0$ in Definition 3.2.1 by:

$$r := \frac{\min\{r_i\sqrt{x(j)}\}_{i=0}^{n+1}}{3E}.$$

**Defining Y and K:**

By Lemma 3.2.2 we consider $U = B(p', r') \subseteq B(p, r)$. We define a sequence of ellipses $\mathfrak{B}_s^j \subseteq \mathfrak{N}_{X^j(p)}$ for $j \in \{0, 1, ..., n-1\}$ all of eccentricity $\leq E$ by:

$\mathfrak{B}^0_s = B(p', sr')$ for $s \in ]0,1]$;

$\mathfrak{B}^j_s = P^j_X(p')(B(p', sr') - p') + X^{\tau(p,j)(p')}(p')$ for $j \in \{1, ..., n-1\}$.

Denote $X^{\tau(p,j)(p')}(p') = p'_j$. It follows from $P^j_X(B(p,r)) \subseteq B(X^j(p), E\sqrt{x^{-1}(j)}r)$, $X^{\tau(p,j)(B(p,r))}(B(p,r)) \subseteq B(X^j(p), 2E\sqrt{x^{-1}(j)}r)$ and from the choice of $r$ that,

$$\mathfrak{B}^j_s \subseteq B(X^j(p), 3E\sqrt{x^{-1}(j)}r) \subseteq B(X^j(p), r_{j+1}) \text{ for all } j = \{0, ..., n-1\}.$$

These ellipses are in the conditions of Lemma 3.2.8 so for all $j \in \{0, 1, ..., n-1\}$ we have:

$$\mathfrak{B}^{j+1}_{s\lambda^{-1}} \supseteq \mathfrak{B}^{j+1}_{s(2-\zeta)} \supseteq X^{\tau(X^j(p),1)(\mathfrak{B}^j_s)}(\mathfrak{B}^j_s) \supseteq \mathfrak{B}^{j+1}_{s\zeta} \supseteq \mathfrak{B}^{j+1}_{s\lambda}.$$

We apply Lemma 3.2.7 to $p'_j$, $\kappa_0$, $\mathfrak{B}^j_s$ and $E_j$, with $s = \lambda^n$. So there exists an angle $\xi(\hat{\epsilon})$ and a vector field $Y_j \in \mathfrak{U}$ such that:

(a) $Y_j - X$ is supported in the flowbox $\underset{t\in[0,1]}{\cup} X^{\tau(p'_j,t)(\mathfrak{B}^j_{\lambda^n})}(\mathfrak{B}^j_{\lambda^n})$;

(b) For every $q_j \in \mathfrak{B}^j_{\lambda^n\sqrt{1-\kappa_0}}$ we have $\|P^1_Y(q_j) - P^1_X(p'_j) \circ \hat{E}_\xi\| < \gamma/3$.

We get $Y_j$ for $j = 0, ..., n-1$ with disjoint supports and define $Y := \sum_{j=0}^{n-1} Y_j$. Defining $K := \overline{\mathfrak{B}}^0_{\lambda^{2n}\sqrt{1-\kappa_0}} = \overline{B}(p', \lambda^{2n}\sqrt{1-\kappa_0})$ we get,

$$\frac{\overline{\mu}(K)}{\overline{\mu}(U)} = \frac{\pi(\lambda^{2n}\sqrt{1-\kappa_0}r')^2}{\pi r'^2} = \lambda^{4n}(1-\kappa_0) > 1-\kappa.$$

Let us see that when we iterate we have a nested sequence, i.e., for all $q \in K$, we have $Y^j(q) \in \mathfrak{B}^j_{\lambda^n\sqrt{1-\kappa_0}}$ for all $j \in \{0, 1, ..., n-1\}$.
We have $Y^{\tau(p',1)(\overline{\mathfrak{B}}^0_s)}(\overline{\mathfrak{B}}^0_s) \subseteq \overline{\mathfrak{B}}^1_{s(2-\zeta)} \subseteq \overline{\mathfrak{B}}^1_{s\lambda^{-1}}$, so for every $j = \{1, 2, ..., n-1\}$ we obtain $Y^{\tau(p',j)(\overline{\mathfrak{B}}^0_s)}(\overline{\mathfrak{B}}^0_s) \subseteq \overline{\mathfrak{B}}^j_{s\lambda^{-j}} \subseteq \overline{\mathfrak{B}}^j_{s\lambda^{-n}}$. Hence for $s = \lambda^{2n}\sqrt{1-\kappa_0}$ we get $Y^{\tau(p',1)(K)}(K) \subseteq \overline{\mathfrak{B}}^j_{\lambda^n\sqrt{1-\kappa_0}}$, and the orbit of $q$ will be always inside the domain of the rotations.
Finally we prove that for all $q \in K$ we have $\|P^1_Y(Y^j(q)) - L_j\| < \gamma$ by using (3), (b) and (2),

$$\begin{aligned}
\|P^1_Y(Y^j(q)) - L_j\| &= \|P^1_Y(Y^j(q)) - P^1_X(X^j(p)) \circ \hat{E}_j\| \leq \\
&\leq \|P^1_Y(Y^j(q)) - P^1_Y(q_j)\| + \|P^1_Y(q_j) - P^1_X(p'_j) \circ \hat{E}_j\| + \\
&+ \|P^1_X(p'_j) \circ \hat{E}_j - P^1_X(X^j(p)) \circ \hat{E}_j\| \leq \\
&\leq \gamma/3 + \gamma/3 + \|P^1_X(p'_j) - P^1_X(X^j(p))\|\|\hat{E}_j\| \leq \gamma
\end{aligned}$$

and the lemma is proved. $\square$

# Chapter 4

# Exchange of the Oseledets directions along an orbit segment

When we have an orbit without dominated splitting, along an orbit segment $X^{[0,m]}(p)$ of this orbit it may occur an "exchange on the dominance" during a period of time $r$, i.e. for $c > 1$,

$$\Delta(X^t(p), r) = \frac{\|P_X^r(X^t(p))|N^s_{X^t(p)}\|}{\|P_X^r(X^t(p))|N^u_{X^t(p)}\|} \geq c.$$

Therefore the dynamics sends vectors near $N^u_{X^r(p)}$ into vectors near $N^s_{X^{t+r}(p)}$ during that period. The next simple lemma, whose prove may be found in [3], explicit this behavior. Denote by $n_t^\sigma \in N^\sigma_{X^t(p)}$ two unit vectors, for $\sigma = u, s$.

**Lemma 4.0.10** *Given an angle $\xi$, there exists $c > 1$, such that if we have $\Delta(X^t(p), r) > c$ then there exists a non-zero vector $v \in N_{X^t(p)}$ such that $\angle(v, n_t^u) < \xi$ and $\angle(P_X^r(X^t(p)).v, n^s_{t+r}) < \xi$.*

The next lemma gives us sufficient conditions under which we may apply Lemma 3.2.9.

**Lemma 4.0.11** *Let $\xi > 0$ and $d > 1$ be given, there exists $E > 1$ such that: If for all $t \in [0, m]$: $\angle(N^u_{X^t(p)}, N^s_{X^t(p)}) > \xi$ and $d^{-1} \leq \frac{\|P_X^t(p)|_{N_p^s}\|}{\|P_X^t(p)|_{N_p^u}\|} \leq d$ then $\|P_X^t(p)\| \leq E\sqrt{x(t)^{-1}}$ for all $t \in [0, m]$.*

**Proof:** We define $\angle(N^u_{X^t(p)}, N^s_{X^t(p)}) =: \xi_t > \xi$ for all $t \in [0, m]$. By volume preserving we get

$$\|P_X^t(p)|_{N_p^s}\| = x(t)^{-1}\|P_X^t(p)|_{N_p^u}\|^{-1}\frac{\sin\xi_0}{\sin\xi_t},$$

therefore

$$\|P_X^t(p)|_{N_p^s}\|^2 = x(t)^{-1}\frac{\|P_X^t(p)|_{N_p^s}\|}{\|P_X^t(p)|_{N_p^u}\|}\sin^{-1}\xi_t \le x(t)^{-1}d.\sin^{-1}\xi$$

and we obtain $\|P_X^t(p)|_{N_p^s}\| \le \sqrt{x(t)^{-1}d.\sin^{-1}\xi}$.
Analogously:

$$\|P_X^t(p)|_{N_p^u}\|^2 \le x(t)^{-1}\frac{\|P_X^t(p)|_{N_p^u}\|}{\|P_X^t(p)|_{N_p^s}\|}\sin^{-1}\xi_t \le x(t)^{-1}d.\sin^{-1}\xi$$

and we also obtain $\|P_X^t(p)|_{N_p^u}\| \le \sqrt{x(t)^{-1}d.\sin^{-1}\xi}$.
We conclude that $\|P_X^t(p)\| \le \sqrt{2x(t)^{-1}d.\sin^{-1}\xi}$, for all $t \in [0,m]$, so the statement holds taking $E = \sqrt{2d.\sin^{-1}\xi}$. $\square$

Now we are able to mix the Oseledets subspaces by small perturbations along orbits with lack of hyperbolicity.

**Lemma 4.0.12** *Let $X \in \mathfrak{X}^2_\mu(M)$, $\epsilon > 0$ and $\kappa \in (0,1)$. There exists $m \in \mathbb{N}$, such that for every $p \in \Delta_m(X)$ there exists a $(\epsilon,\kappa)$-realizable linear flow such that:*

$$L^m(N_p^u) = N^s_{X^m(p)}.$$

**Proof:** First we set up the constants. Take $\xi > 0$ the minimum of the angles satisfying simultaneously Lemma 3.2.5 and Lemma 3.2.6 and depending on $X$, $\epsilon$ and $\kappa/2$. Take $C := \max\{\|DX^{\pm 1}\| : p \in M\}$ and $c$ given by Lemma 4.0.10 depending on the angle $\xi$. It also will be useful to take $c > C^2$. Lemma 4.0.11 gives us $E > 1$ depending on $\xi$ and $d = 2c^2$. Let $\hat{\epsilon} > 0$ depending on $X$, $\epsilon$, $\kappa$ and $E$ given by Lemma 3.2.9. Let $\beta > 0$ be such that for $\xi_0 < \beta$, $\|R_{\xi_0} - Id\| \le C^{-1}E^{-2}\hat{\epsilon}$. Finally we take a very large $m \in \mathbb{N}$ satisfying $m \ge \frac{2\pi}{\beta}$.
**I - Angle between Oseledets subspaces small:**
If along the orbit segment there is a time $r$ such that the angle between $N^u_{X^r(p)}$ and $N^s_{X^r(p)}$ is less than $\xi$ say:

$$\text{For some } r \in [0,T] \text{ we have } \angle(N^u_{X^r(p)}, N^s_{X^r(p)}) < \xi. \tag{4.1}$$

We take advantage of this fact and define a realizable linear flow of length 1 in the following way; If $r < m-1$ the linear flow is based at $X^r(p)$ and defined by $L_0 := P_X^1(X^r(p)) \circ R_\xi$ and if $r > m-1$ the linear flow is based at $X^{r-1}(p)$ and defined by $L_0 := R_\xi \circ P_X^1(X^{r-1}(p))$. Now we use Lemma 3.2.5 and concatenate from right and left, if necessary, with trivial realizable linear flows by using (1) of Lemma 3.2.1. We obtain $L^m(N_p^u) = N^s_{X^m(p)}$.
**II - Locally $N^s$ dominates $N^u$:**
Now we suppose that:

$$\text{For some } 0 \le r+t \le m \text{ we have } \Delta(X^t, r) > c. \tag{4.2}$$

We use Lemma 4.0.10 and there exists a vector $v \in N_{X^t(p)}$ such that $\angle(v, n_t^u) < \xi$ and $\angle(P_X^r(X^t(p)) \cdot v, n_{t+r}^s) < \xi$. Since $\xi$ is small we apply Lemma 3.2.5 to both extremes at $X^t(p)$ and at $X^{t+r}(p)$. By choice of $c$ we get $r > 2$ and so we have disjoint perturbations. Therefore our first rotation allow us to send $N_{X^t(p)}^u$ onto $v.\mathbb{R}$, the dynamics help us and maps this direction into $P_X^r(X^t(p)) \cdot v$ in time $r$, finally and another rotation sends $P_X^r(X^t(p)) \cdot v.\mathbb{R}$ onto $N_{X^{t+r}(p)}^s$. Now we use Lemma 3.2.1 and concatenate the three realizable linear flows, say rotation-trivial-rotation, by using Lemma 3.2.6 and we get $L^m(N_p^u) = N_{X^m(p)}^s$.
**III - Lack of dominance behavior:**
We suppose that we do not have both (4.1) and (4.2). We set up the conditions of Lemma 4.0.11. Since $\Delta(p, m) \geq \frac{1}{2}$ and (4.2) is false we conclude that,

$$\Delta(X^r(p), t) = \Delta(X^{t+r}(p), m - t - r)^{-1} \Delta(p, m) \Delta(p, r)^{-1} \geq (2c^2)^{-1},$$

therefore since $d = 2c^2$ and we get:

$$d^{-1} \leq \frac{\|P_X^t(p)|_{N_{X^r(p)}^s}\|}{\|P_X^t(p)|_{N_{X^r(p)}^u}\|} \leq d \text{ for all } r, t \text{ such that } 0 \leq r + t \leq m.$$

In particular for $r = 0$ we have $\forall t \in [0, m]$: $\angle(N_{X^t(p)}^u, N_{X^t(p)}^s) > \xi$ we use Lemma 4.0.11 and conclude that $\|P_X^t(p)\| \leq E\sqrt{x(t)^{-1}}$ for all $t \in [0, m]$.

Take $\xi_1, \xi_2, ..., \xi_{m-1}$ such that each $\xi_j$ is less than $\beta$ and also take $\sum_{j=0}^{m-1} \xi_j = \angle(N_p^u, N_p^s)$. We define:

$$\begin{array}{rccc} L_j: & N_{X^j(p)} & \longrightarrow & N_{X^{j+1}(p)} \\ & v & \longmapsto & P_X^{j+1}(p) \circ R_{\xi_j} \circ [P_X^j(p)]^{-1} \cdot v \end{array}$$

Let us see that we are in the hypotheses of Lemma 3.2.9:
Since by definition $L_{j-1} \circ ... \circ L_0 = P_X^j(p) \circ R_{\sum_{j=0}^{m-1} \xi_j}$ we have

$$L_{j-1} \circ ... \circ L_0(B(p, 1)) = P_X^j(p)(B(p, 1))$$

and it verifies (a). Now we have:

$$\begin{aligned} \|P_X^1(X^j(p)) - L_j\| &\leq \|P_X^1(X^j(p)) - P_X^{j+1}(p) \circ R_{\xi_j} \circ [P_X^j(p)]^{-1}\| = \\ &= \|P_X^1(X^j(p))[Id - P_X^j(p) \circ R_{\xi_j} \circ [P_X^j(p)]^{-1}]\| \leq \\ &\leq \|P_X^1(X^j(p))\| \|P_X^j(p)[Id - R_{\xi_j}][P_X^j(p)]^{-1}]\| \leq \\ &\leq \|P_X^1(X^j(p))\| \|P_X^j(p)\| \|[P_X^j(p)]^{-1}\| \|Id - R_{\xi_j}\| \leq \\ &\leq CE\sqrt{x^{-1}(j)} E\sqrt{x(j)} \|Id - R_{\xi_j}\|. \end{aligned}$$

In last inequality we use $\|P_X^{-t}\| \leq E\sqrt{x(t)}$. Therefore we have:

$$\|P_X^1(X^j(p)) - L_j\| \leq CE^2 \|Id - R_{\xi_j}\| \leq \hat{\epsilon}$$

and (b) is true, so by Lemma 3.2.9 we have the realizability of our linear flow therefore:

$$L^m(N_p^u) = L_{m-1} \circ ... \circ L_0(N_p^u) = P_X^m(p) \circ R_{\sum_{j=0}^{m-1} \xi_j}(N_p^u) = P_X^m(p).N_p^s = N_{X^m(p)}^s,$$

which proves the lemma. $\square$

# Chapter 5

# Lowering the norm - Local procedure

Now we consider two lemmas, the first one we adapt Lemma 3.12 from [3], to the continuous-time case. This lemma gives us information about *when* we have a recurrence to a positive measure set. The second lemma is an elementary result and it relates the original norm with a new norm which is better when we do the computations.

**Lemma 5.0.13** *Let $X^t : M \to M$ be a measurable $\mu$-invariant flow, $\Delta \subseteq M$ a positive measure set, its saturate $\Gamma = \bigcup_{t \in \mathbb{R}} X^t(\Delta)$ and $\gamma > 0$. There exists a measurable function $T : \Gamma \to \mathbb{R}$ such that for $\mu$-a.e. $p \in \Gamma$, all $t \geq T(p)$ and every $\tau \in [0,1]$ there exists some $s \in [0,t]$ satisfying $|\frac{s}{t} - \tau| < \gamma$ and $X^s(p) \in \Delta$.*

**Proof:** We denote the characteristic function on the set $\Delta$ by $\chi_\Delta$. Let $B(t) = \int_0^t \chi_\Delta(X^r(p))dr$. We claim that for $\mu$-a.e. $p \in \Gamma$ we have $\lim_{t\to\infty} \frac{1}{t} B(t) = a > 0$. The existence of the limit follows from Birkhoff's ergodic theorem, the positivity follows from the diffeomorphism version [3], where we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_\Delta(f^j(p)) = a > 0.$$

Note that $\varphi(p) = \int_0^1 \chi_\Delta(X^t(p))dt$ is integrable, then:

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1}{t}\int_0^t \chi_\Delta(X^r(p))dr &= \lim_{n\to\infty}\frac{1}{n}\int_0^n \chi_\Delta(X^r(p))dr = \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int_j^{j+1} \chi_\Delta(X^r(p))dr = \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int_0^{1} \chi_\Delta(X^{\tilde r+j}(p))d\tilde r = \\
&= \int_0^1 [\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} \chi_\Delta((X^1)^j(X^{\tilde r}(p)))]d\tilde r = \\
&= \int_0^1 a d\tilde r = a > 0.
\end{aligned}
$$

Let $p \in \Gamma$ and $\epsilon \in (0, a)$ satisfying $(a+\epsilon)/(1+\gamma) < a - \epsilon$. Since we have $\lim\limits_{t\to\infty} \frac{1}{t}B(t) = a > 0$ we choose $t_0$ such that $|B(t)/t - a| < \epsilon$ for all $t \geq t_0$ and take $T(p)$ such that $\frac{t_0}{\gamma(a-\epsilon)} < T(p)$. The proof follows by contradiction, suppose that:

$$
\exists t \geq T(p), \exists \tau \in [0,1] : \forall s \in [t(\tau-\gamma), t(\tau+\gamma)], X^s(p) \notin \Delta.
$$

Take $[\alpha, \beta] = [t(\tau-\gamma), t(\tau+\gamma)] \cap [0,t]$, $\beta > t\gamma + \alpha$.

If $\alpha > t_0$, then:

$$
a - \epsilon < \frac{B(\beta)}{\beta} = \frac{B(\alpha)}{\beta} \leq \frac{B(\alpha)}{\alpha + t\gamma} = \frac{B(\alpha)}{\alpha(1+t\gamma/\alpha)} < \frac{a+\epsilon}{1+\gamma\frac{t}{\alpha}} < a - \epsilon,
$$

which is a contradiction, so $\alpha \leq t_0$ and in this case;

$$
a - \epsilon < \frac{B(\beta)}{\beta} < \frac{\alpha}{\alpha + t\gamma} < \frac{t_0}{\gamma t} < a - \epsilon
$$

we obtain again a contradiction. $\square$

Consider $p, q := X^t(p) \in \Gamma$ and the map $P : N_p \to N_q$ with matrix written relative to the *Oseledets basis* (given by $\{n_p^u, n_p^s\}$ and $\{n_q^u, n_q^s\}$) as:

$$
P = \begin{pmatrix} a^{uu} & a^{us} \\ a^{su} & a^{ss} \end{pmatrix}.
$$

Let $\|P\|_{\max} = \max\{|a^{uu}|, |a^{us}|, |a^{su}|, |a^{ss}|\}$.

**Lemma 5.0.14** *(a)* $\|P\| \leq 4\frac{1}{sin\angle(N_p^u, N_p^s)}\|P\|_{max}$, *(b)* $\|P\|_{max} \leq \frac{1}{sin\angle(N_q^u, N_q^s)}\|P\|$

**Proof:** See [4], Lemma 4.5. $\square$

**Lemma 5.0.15** *Let $X \in \mathfrak{X}^3_\mu(M)$, with $X^t$ aperiodic and the measure of hyperbolic sets zero. Let $\epsilon, \delta > 0$, $0 < \kappa < 1$. There exists a measurable function $T : M \to \mathbb{R}$ such that for $\mu$-a.e. $p \in M$ and every $t \geq T(p)$, there exists a $(\epsilon, \kappa)$-realizable linear flow at $p$ with length $t$ such that $\|L^t(p)\| \leq e^{t\delta}$.*

**Proof:** First we take $m \in \mathbb{R}$ large enough given by Lemma 4.0.12 depending on $X, \epsilon, \kappa/2$, abbreviate $\Gamma = \Gamma^+_m(X)$ and $\Delta = \Delta_m(X)$. The union of $\Gamma$ with the Oseledets points with zero Lyapunov exponents is a full measure set, otherwise we could get a positive measure set with $m$-dominated splitting and by Proposition 2.5.1 $X$ is Anosov and this contradicts the hypothesis of hyperbolic sets have zero measure. We suppose $\mu(\Gamma) > 0$ because if $\mu(\Gamma) = 0$ then for $\mu$-a.e. point $p \in M$ we have $\lambda^+(p) = 0$ and the proof is over because a trivial linear flow does the work. Remember that $\Gamma = \underset{t\in\mathbb{R}}{\cup} X^t(\Delta)$. For $\mu$-a.e. $p \in \Gamma$ the Oseledets's Theorem gives us $Q(p)$ such that $\forall t \geq Q(p)$ we have:

(1) $\frac{1}{t}\log\|P^t_X(p) \cdot n^u\| < \lambda^+(p) + \delta$ for all $n^u \in N^u_p - \{0\}$;

(2) $\frac{1}{t}\log\|P^t_X(p) \cdot n^s\| < -\lambda^+(p) + \delta$ for all $n^u \in N^s_p - \{0\}$;

(3) $\log\frac{1}{sin\angle(N^u_{X^t(p)}, N^s_{X^t(p)})} < t\delta$.

By using Lemma 5.0.13 with $\tau = 1/2$ we get recurrence to $\Delta$ approximately in the middle of the orbit segment, but to get good estimates to the norm of the linear flow $L^t$, points in the orbit after this time, must also satisfy (1) and (2), that is why we consider the following sets; We define $B_n := \{p \in \Gamma : Q(p) \leq n\}$ for $n \in \mathbb{N}$, of course that $B_n \subseteq B_{n+1}$ and $\mu(\Gamma - B_n) \underset{n\to\infty}{\to} 0$. Consider a family of sets defined by:

$$C_0 := \emptyset, C_1 := \underset{t\in\mathbb{R}}{\cup} X^t(\Delta \cap X^{-m}(B_1)), ..., C_n := \underset{t\in\mathbb{R}}{\cup} X^t(\Delta \cap X^{-m}(B_n)), ...$$

Clearly $C_n \underset{n\to\infty}{\to} \Gamma$, so the measurable function $T : \Gamma \to \mathbb{R}$ will be $\mu$-a.e. defined on each $C_n - C_{n-1}$ for $n \in \mathbb{N}$. Taking $c > 2\max\{\|DX^1_p\| : p \in M\}$ yields the Lyapunov exponents of any $p \in \mathfrak{O}$ less than $c$. For $p$ we have non-null Lyapunov exponents so we have the Oseledets 1-dimensional subspaces $N^u_p$ and $N^s_p$. Let $\gamma = \min\{1/6, \delta/c\}$. Now we use Lemma 5.0.13 substituting $\Delta$ by $\Delta \cap X^{-m}(B_n)$ and $\Gamma$ by $\underset{t\in\mathbb{R}}{\cup} X^t(\Delta \cap X^{-m}(B_n))$, so by this lemma for each $n$ there exists a measurable function $T_n : C_n \to \mathbb{R}$ such that for $\mu$-a.e $p \in C_n$, and for all $t \geq T_n(p)$, there exists some $s \in [0, t]$ satisfying $X^s(p) \in \Delta \cap X^{-m}(B_n)$ and $|\frac{s}{t} - \frac{1}{2}| < \gamma$. Now we define a sufficiently large $T(p)$ for $p \in C_n - C_{n-1}$;

$$T(p) \geq \max\{T_n(p), \frac{m}{\gamma}, 6Q(p), \frac{1}{\delta}\log\frac{4}{sin\angle(N^u_p, N^s_p)}\} \tag{5.1}$$

Let $p \in C_n - C_{n-1}$ and $t \geq T(p)$, since $t \geq T(p) \geq T_n(p)$ by (a) above $X^s(p) \in \Delta$, so by Lemma 4.0.12 we may define a $(\epsilon, \kappa/2)$-realizable linear flow $L_1 : N_{X^s(p)} \longrightarrow N_{X^{s+m}(p)}$, sending $N^u_{X^s(p)}$ into $N^s_{X^{s+m}(p)}$. Now we concatenate from right and left with trivial linear flows and by Lemma 3.2.1 we obtain a $(\epsilon, \kappa)$-realizable linear flow defined by:

$$N_p \xrightarrow{L_0} N_{X^s(p)} \xrightarrow{L_1} N_{X^{s+m}(p)} \xrightarrow{L_2} N_{X^t(p)}$$

with $L_0 = P^s_X(p)$ and $L_2 = P^{t-m-s}_X(X^{s+m}(p))$. Now we estimate $\|L^t(p)\|$, and for that we consider the linear maps relatively to a suitable unitary basis $\{n^u_{X^r(p)}, n^s_{X^r(p)}\}$ for $r \in [0, t]$ that is invariant for the linear Poincaré flow, so they have the form:

$$L_2 = \begin{pmatrix} c^{uu} & 0 \\ 0 & c^{ss} \end{pmatrix}, L_1 = \begin{pmatrix} b^{uu} & b^{us} \\ b^{su} & b^{ss} \end{pmatrix}, L_0 = \begin{pmatrix} a^{uu} & 0 \\ 0 & a^{ss} \end{pmatrix}.$$

The key observation is that $b^{uu} = 0$, and this is the reason why we send $N^u_{X^s(p)}$ into $N^s_{X^{s+m}(p)}$. Hence we will be able to get all entries of the product matrix small, whereas if $b^{uu} \neq 0$ this could not be done. So consider the product matrix:

$$L^t(p) = \begin{pmatrix} 0 & a^{uu}b^{us}c^{ss} \\ a^{ss}b^{su}c^{uu} & a^{ss}b^{ss}c^{ss} \end{pmatrix}.$$

**Claim 5.0.1** *For $p \in C_n - C_{n-1}$ and $t \geq T(p)$ we have:*

*(a) $log|a^{uu}| < \frac{1}{2}t(\lambda^+(p) + 4\delta)$;*

*(b) $log|a^{ss}| < \frac{1}{2}t(-\lambda^+(p) + 4\delta)$;*

*(c) $log|c^{uu}| < \frac{1}{2}t(\lambda^+(p) + 4\delta)$;*

*(d) $log|c^{ss}| < \frac{1}{2}t(-\lambda^+(p) + 4\delta)$.*

**Proof:** (of the claim) For (a) we have $s > t(1/2 - \gamma) > t/3 > T(p)/3 \geq Q(p)$ so by Oseledets's Theorem we have $log|a^{uu}| = log|P^s_X(p) \cdot n^u_p| < s(\lambda^+(p) + \delta)$ and also $log|a^{ss}| = log|P^s_X(p) \cdot n^s_p| < s(-\lambda^+(p) + \delta)$. Since $\gamma\lambda^+(p) < \gamma c < \delta$ and $\gamma < 1/2$ we obtain;

$$s(\lambda^+(p) + \delta) < t(1/2 + \gamma)(\lambda^+(p) + \delta) < t(\lambda^+(p)/2 + \delta/2 + \lambda^+(p)\gamma + \gamma\delta) <$$

$$< t(\lambda^+(p)/2 + \delta/2 + \delta + \delta/2) < \frac{1}{2}t(\lambda^+(p) + 4\delta)$$

and (a) follows. We note that (b) is analogous to (a) by taking $-\lambda^+(p)$ instead. For (c) we make use of the fact that $X^s(p) \in X^{-m}(B_n)$, therefore $X^{s+m}(p) \in B_n$ and by definition of $B_n$, $Q(X^{s+m}(p)) \leq n$, so we will have the approximation rate given by Oseledets's Theorem if $t - m - s > n$. By (5.1) for

$t \geq T(p)$ we have $-m/t \geq -\gamma$, $-s/t > -\frac{1}{2} - \gamma$ and we know that $-\gamma \geq -1/6$ so:

$$t - m - s = t(1 - \frac{m}{t} - \frac{s}{t}) > t(\frac{1}{2} - 2\gamma) > \frac{t}{6} > Q(p) \geq n.$$

Now:

$$\begin{aligned} log|c^{uu}| &= log|P_X^{t-m-s}(X^{s+m}(p)) \cdot n^u_{X^{s+m}(p)}| < (t-m-s)(\lambda^+(p)+\delta) < \\ &< t(1 - m/t - s/t)(\lambda^+(p)+\delta) < t(\gamma + 1/2)(\lambda^+(p)+\delta) = \\ &= t(\gamma\lambda^+(p) + \gamma\delta + \lambda^+(p)/2 + \delta/2) < \\ &< t(\delta + \delta/2 + \lambda^+(p)/2 + \delta/2) = \frac{1}{2}t(\lambda^+(p) + 4\delta). \end{aligned}$$

Again (d) is analogous to (c) by taking $-\lambda^+(p)$ instead and the claim is proved. □ Now we estimate $\|L_1\|_{\max}$. First note that;

$$s + m > t(1/2 - \gamma + m/t) > t(1/2 - \gamma) > t/6 > Q(p) \geq n,$$

so again by Oseledets's Theorem (3) we have an estimate for the angle, i.e.,

$$sin^{-1}\angle(N^u_{X^{s+m}(p)}, N^s_{X^{s+m}(p)}) < e^{(s+m)\delta} < e^{t\delta}.$$

Since $L_1$ is $(\epsilon, \kappa)$-realizable we conclude that $\|L_1 - P_X^m(X^s(p))\|$ is small, therefore since $t > T(p) \geq m/\gamma$ and $\gamma c < \delta$ we have $\|L_1\| < e^{mc} < e^{t\gamma c} < e^{t\delta}$. By Lemma 5.0.14 (b) we get:

$$\|L_1\|_{\max} \leq sin^{-1}\angle(N^u_{X^{s+m}(p)}, N^s_{X^{s+m}(p)})\|L_1\| \leq e^{2t\delta}.$$

Now we give estimates for each of the entries of the product matrix:

$$\begin{aligned} |a^{uu}b^{us}c^{ss}| &\leq e^{\frac{1}{2}t(\lambda^+(p)+4\delta)+2t\delta+\frac{1}{2}t(-\lambda^+(p)+4\delta)} = e^{6t\delta}. \\ |a^{ss}b^{su}c^{uu}| &\leq e^{\frac{1}{2}t(-\lambda^+(p)+4\delta)+2t\delta+\frac{1}{2}t(\lambda^+(p)+4\delta)} = e^{6t\delta}. \\ |a^{ss}b^{ss}c^{ss}| &\leq e^{\frac{1}{2}t(-\lambda^+(p)+4\delta)+2t\delta+\frac{1}{2}t(-\lambda^+(p)+4\delta)} \leq e^{-t\lambda^+(p)+6t\delta} \leq e^{6t\delta}. \end{aligned}$$

This implies the inequality $\|L^t(p)\|_{\max} < e^{6t\delta}$. Again by Lemma 5.0.14 (a) we have:

$$\|L^t(p)\| \leq 4\frac{1}{sin\angle(N^u_p, N^s_p)}\|L^t(p)\|_{\max}.$$

But $t \geq T(p) \geq \frac{1}{\delta}\log\frac{4}{sin\angle(N^u_p,N^s_p)}$ so $\frac{4}{sin\angle(N^u_p,N^s_p)} \leq e^{t\delta}$ and we get $\|L^t(p)\| \leq e^{7t\delta}$. Replacing $\delta$ by $\delta/7$ we conclude that $\|L^t(p)\| \leq e^{t\delta}$ and the lemma is proved. □

## 5.1 Realizing vector fields

Let $X \in \mathfrak{X}^2_\mu(M)^*$, with $X^t$ aperiodic and also with all hyperbolic sets with zero Lebesgue measure. Given $\epsilon, \delta > 0$ and $0 < \kappa < 1$, we suppose that

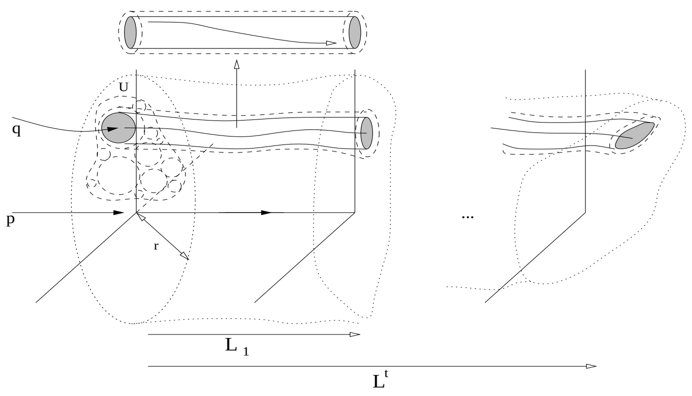


Figure 5.1: Realizing vector fields given a linear flow $L^t$.

$m$ is large enough to satisfy Lemma 4.0.12. By Lemma 5.0.15 there exists a measurable function $T : M \to \mathbb{R}$ such that for $\mu$-a.e. $p \in M$, for every $t \geq T(p)$, there exists a $(\epsilon, \kappa)$-realizable linear flow at $p$ with length $t$ such that $\|L^t(p)\| \leq e^{t\delta}$. By Definition 3.2.1 $\forall \gamma > 0$, $\exists r(p,t) > 0$ such that for all open set $\emptyset \neq U \subseteq B(p,r)$, there exist:

(a) A measurable set $K \subseteq U$,

(b) A zero divergence $C^1$ vector field $Y$, $\epsilon$-$C^1$-close to $X$, such that:

(i) $Y = X$ outside the self-disjoint flowbox $\{X^{\tau(p,s)(U)}(U) : s \in [0,t]\}$ and for every $q \in U, X^{\tau(p,t)(U)}(U)$ we have $DX_q = DY_q$;

(ii) $\overline{\mu}(K) > (1-\kappa)\overline{\mu}(U)$;

(iii) If $q \in K$, then $\|P_Y^t(q) - L^t\| < \gamma$.

By (iii) and $\|L^t(p)\| \leq e^{t\delta}$ we conclude that $\|P_Y^t(q)\| \leq e^{\delta t} + \gamma$ for all $q \in K$, and we note that $\gamma$ is very small. So the vector field $Y$ is the one who *realizes* the property of having small norm for the orbit of $p$, and this property is shared by large percentage of points inside any open set inside $\mathfrak{N}_p$ near $p$ (see Figure 5.1). This property is crucial because after we perturb $X$ the point $p$ may no longer be in $\mathfrak{O}(Y)$, however most points (relatively to Lebesgue measure) near $p$ have norm close to the norm of $p$, therefore small norm.

# Chapter 6

# Lowering the norm - Global procedure

## 6.1 Sections of flows and special flows

Now we use the local construction of realizable linear flows with small norm to get a conservative vector field $Y$ near $X$ with $LE(Y)$ small.

First we define a special flow built under a ceiling function $h$. Consider:

(a) A measure space $\Sigma$;

(b) A map $R : \Sigma \to \Sigma$;

(c) A measure $\tilde{\mu}$ defined in $\Sigma$;

(d) An integrable function $h : \Sigma \to \mathbb{R}^+$, with $h(x) \geq \alpha > 0$ for all $x \in \Sigma$ and $\int_\Sigma h(x) d\tilde{\mu}(x) = 1$.

Consider the flow on the product space $M_h \subseteq \Sigma \times \mathbb{R}$ where $M_h$ is the set below the graph of $h(x)$, on which the dynamics is defined by:

$$\begin{array}{rccc} S^s : & \Sigma \times \mathbb{R} & \longrightarrow & \Sigma \times \mathbb{R} \\ & (x, r) & \longmapsto & (R^n(x), r + s - \sum_{i=0}^{n-1} h(R^i(x))) \end{array}$$

and $n \in \mathbb{Z}$ is uniquely defined by:

$$\sum_{i=0}^{n-1} h(R^i(x)) \leq r + s < \sum_{i=0}^{n} h(R^i(x)).$$

Informally speaking this flow $S^t$ moves any point $(y, r) \in M_h$ to $(y, r + s)$ at time-one speed until hits the graph of $h$ after that the point returns to the base $\Sigma$ and proceed its journey, see Figure 6.1.

The following lemma, see [1], gives a representation of an aperiodic flow by a flow build from a ceiling function $h$.

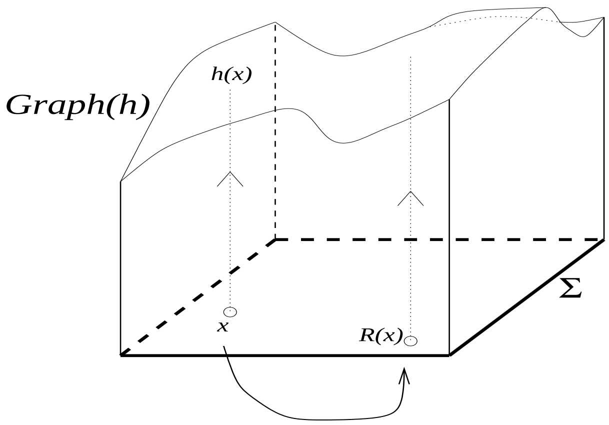


Figure 6.1: A Kakutani castle or a special representation of a flow.

**Lemma 6.1.1** *(Ambrose-Kakutani) Any aperiodic flow $X^t : M \to M$ is isomorphic to some special flow.*

The isomorphism is given by a measure preserving transformation $W : M \to M_h$. The measure $\mu^* = W^*\mu$ is decomposed into the product of Lebesgue measure in $\mathbb{R}$ and an $R$-invariant measure $\tilde{\mu}$ in $\Sigma$, i.e.:

$$\int_{M_h} f(x,s)d\mu^* = \int_{\Sigma}\int_0^{h(x)} f(x,s)DDS\tilde{\mu}(x). \tag{6.1}$$

So we have a simplified representation of our flow $X^s(p)$. In what follows we consider that our flow have this representation. Given a special flow over a section $\Sigma$ the set $Q = \bigcup\limits_{t\in\mathbb{R}} X^t(\Sigma)$ is called *Kakutani castle* and the *tower of height i*, denoted by $T_i$, is the set below the graph of $h(B_i)$ where $B_i = \{x \in \Sigma : h(x) = i\}$ so $T_i = \underset{t\in[0,i]}{\cup} X^t(B_i)$.

Next we consider lemma which is a time-continuous version of Lemma 4.1 of [3]:

**Lemma 6.1.2** *Let $X^t : M \to M$ be $\mu$-preserving aperiodic flow. For every positive measure set $U \subseteq M$ and every $h \in \mathbb{R}$, there exists a $\tilde{\mu}$-positive measure section $B \subseteq U$ such that:*

*(a) $X^{[0,h]}(B)$ is a self-disjoint flow box;*

*(b) $B$ is maximal (i.e. no set containing $B$ and with larger measure has the same properties as $B$).*

**Proof:** Suppose that for all $B_1 \subseteq U$ with $\tilde{\mu}$ positive measure, we have $\tilde{\mu}(X^{[0,n[}(B_1)\Delta X^{]0,n]}(B_1)) = 0$, therefore $\tilde{\mu}$-a.e. $x \in B_1$ is fixed for $X^t$ or periodic with period less then $n$ but $X^t$ is aperiodic, so there exists $B_1 \subseteq U$ with $\tilde{\mu}(B_1) > 0$, such that $X^{[0,n]}(B_1)$ is a self-disjoint flow box.

If $\mu(U - X^{[-n,n]}(B_1)) = 0$ there is no chance of getting a set $B_1$ with larger measure. If $\mu(U - X^{[-n,n]}(B_1)) > 0$, then we extract $B_2 \subseteq \{U - X^{[-n,n]}(B_1)\}$ such that $X^{[0,n]}(B_2)$ is a self-disjoint flow box and repeat by method of exhaustion. $\square$

For $\mu$-generic point $p$, Lemma 5.0.15 gives us $T(p)$, which, in general is very large. Hence Lemma 6.1.2 will be very useful to avoid overlapping of perturbations.

## 6.2 The construction of an adequate section

Now we prove Proposition 2.6.2 and for that purpose the next step is the construction of a special flow over a section. Consider $X \in \mathfrak{X}^1_\mu(M)$, of class $C^2$, aperiodic, with hyperbolic sets measuring zero, $\epsilon$ and $\delta$ given by Proposition 2.6.2.

For all $Y$ $\epsilon - C^1$ close to $X$ we define $C := \max\limits_{p \in M} \|P^1_Y(p)\|$.

We take $\kappa = \delta^2$. Using the measurable function given by Lemma 5.0.15 we define:

$$Z_h = \{p \in M : T(p) \leq h\}. \tag{6.2}$$

Of course that $\mu(M - Z_h) \underset{h \to \infty}{\longrightarrow} 0$ so taking $h$ sufficiently large guarantees

$$\mu(M - Z_h) < \delta^2. \tag{6.3}$$

We intend to build a special flow with ceiling function with height not less than $h$ and section inside $Z_h$. Since $Z_h$ has almost full measure by Lemma 6.1.2 we get a $\tilde{\mu}$-positive measure set $B \subseteq Z_h$. The function $h(x)$ satisfy $h(x) \geq h$ and since $x \in B \subseteq Z_h$ we have $h \geq T(x)$ so we are in the conditions of Lemma 5.0.15. Let $\hat{Q}$ be the castle with base $B$, i.e. $\hat{Q} = \bigcup\limits_{t \in \mathbb{R}} X^t(B)$.

We have $\hat{Q} \supset Z_h$ in measure theoretical sense, this follows because if by contradiction there exists $U_1 \subseteq Z_h$ with $\mu(U_1) > 0$ and $U_1 \cap \hat{Q} = \emptyset$, then by Lemma 6.1.2 we could extract a section $B_1 \subseteq U_1$ with $\tilde{\mu}(B_1) > 0$ and $X^{[0,h]}(B_1)$ would be a self-disjoint flow box. But since $\hat{Q} = \bigcup\limits_{t \in \mathbb{R}} X^t(B)$ and $\bigcup\limits_{t \in \mathbb{R}} X^t(B) \cap U_1 = \emptyset$ we contradict the maximality of $B$. So by (6.3) we get the inequality:

$$\mu(\hat{Q}^c) \leq \delta^2. \tag{6.4}$$

Define the subcastle $Q \subseteq \hat{Q}$ by excluding the towers of $\hat{Q}$ with height bigger than $3h$ and we (like in [3] Lemma 4.2) obtain:

**Lemma 6.2.1** $\mu(\hat{Q} - Q) < 3\delta^2$.

**Proof:** Let $B_i = \{x \in B : h(x) = i\}$, $T_i = \underset{t\in[0,i]}{\cup} X^t(B_i)$ and $\hat{Q} = \underset{i\geq h}{\cup} T_i$. Take $i, j \in \mathbb{R}$ with $i \geq 2h$ and $j \in [h, i-h]$. We have, by definition of tower, that $\underset{j\in[h,i-h]}{\cup} X^j(B_i)$ is self-disjoint, furthermore is disjoint from $\underset{t\in[0,h[}{\cup} X^t(B)$, by choice of $i$ and $j$.

$\underset{j\in[h,i-h]}{\cup} X^j(B_i) \subseteq Z_h^c$ in measure theoretical sense, otherwise since the set $\underset{j\in[h,i-h]}{\cup} X^j(B_i)$ is disjoint from $\underset{t\in[0,h]}{\cup} X^t(B)$ we extend $B$ with more elements and obtain $\mu(\underset{j\in[h,i-h]}{\cup} X^j(B_i) \cap Z_h) \neq 0$.

Each $T_i$ for $i \geq 2h$ decomposes into three floors $T_i^1$, $T_i^2$ and $T_i^3$ where:

$T_i^1 \subseteq \underset{t\in[0,h[}{\cup} X^t(B)$ with length $h$;

$T_i^2 \subseteq \underset{t\in[h,i-h[}{\cup} X^t(B)$ with length $\geq i - 2h$;

$T_i^3 \subseteq \underset{t\in[i-h,i]}{\cup} X^t(B)$ with length $h$.

So if $i \geq 3h$, then the length of $T_i^2$ is bigger or equal than $h$, hence by (6.3) we have:

$$\mu(\underset{i\geq 3h}{\cup} T_i) \leq 3\mu(\underset{i\geq 3h}{\cup} T_i^2) \leq 3\mu(Z_h^c) \leq 3\delta^2.$$

□

## 6.3 The zero divergence vector field $Y$ $\epsilon$-$C^1$-close to $X$

Now we make use of the realizability of vector fields and the properties of special flows to construct a conservative vector field $Y$ inside the subcastle $Q$ by gluing a finite number of local perturbations supported on self-disjoint flow boxes.
Next we follow Lemma 4.14 of [4].

**Lemma 6.3.1** *Given $\gamma > 0$, there exists $Y$, $\epsilon$-$C^1$-close to $X$, a castle $U$ for $Y^t$ and a subcastle $K$ for $Y^t$ such that:*

*(a) The castle $U$ is open;*

*(b) $\mu(U - Q) < \gamma$ and $\mu(Q - U) < 2\gamma$;*

*(c) $\mu(U - K) < \kappa(1 + \gamma)$;*

*(d) $Y^t(U) = X^t(U)$ and $Y^t = X^t$ outside the castle $U$;*

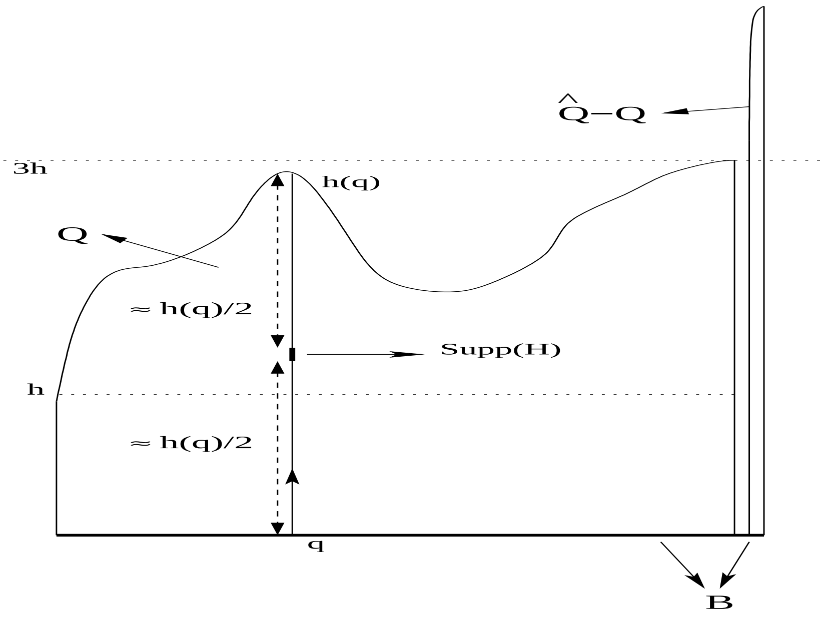


Figure 6.2: The subcastle Q support all perturbations.

*(e) If $q$ is in the base of $K$ and $h(q)$ is the height of the tower of $K$ that contains $q$, then*

$$\|P_Y^{h(q)}(q)\| \le e^{\delta h(q)} + \gamma.$$

**Proof:** The castle $Q$ is a measurable set and since $\mu$ is Borel regular, there exists a compact $J \subseteq Q$ with:

$$\mu(Q - J) < \gamma\mu(\hat{Q}). \tag{6.5}$$

We choose this compact such that it is a $X^t$-castle with the same structure as $Q$ (i.e. preserving the same dynamics of bases and towers as the castle $Q$ do). Now we choose an open castle $V$ such that $J \subseteq V$ with:

$$\mu(V - J) < \gamma\mu(\hat{Q}), \tag{6.6}$$

and also with the same structure of $Q$ and $J$.

For every point $p_1$ in $J \cap B$, we have $p_1 \in B$ and so it follows that $h(p_1) \ge h$. Since $(J \cap B) \subseteq Z_h$ we have $T(p_1) \le h$, therefore $T(p_1) \le h \le h(p_1)$. So we are able to construct a conservative vector field which realizes a linear flow who has the property of having small norm, i.e., for all $t_1 \ge T(p_1)$, and for $\gamma$ fixed, there exists a radius $r_1(p_1, t_1)$ (take a smaller one if we leave the open castle $V$) such that for almost (related with $\kappa = \delta^2$) all point in $U_1 = B(p_1, r_1) \subseteq \mathfrak{N}_{p_1}$, more precisely for all point $q \in K_1 \subseteq U_1$, we have a vector field $Y_1$ supported in a small tubular neighborhood of the orbit segment $X^{[0,t_1]}(p_1)$ such that:

$$\|P^{t_1}_{Y_1}(q)\| \leq e^{\delta t_1} + \gamma.$$

Now we continue by choosing $p_i's$ and by Vitali's arguments we fill up the base of $J$, denoted by $B_J$, by finite pairwise self-disjoint balls $U_i's$ satisfying $\tilde{\mu}(B_J - U_i) \leq \gamma\tilde{\mu}(B_J)$ so,

$$\mu(J - U) \leq \gamma\mu(J). \tag{6.7}$$

$U$ is a $X^t$-castle with section the union of the $U_i's$. So we get a vector field $Y_i$ supported in a small tubular neighborhood of the orbit segment $X^{[0,t_i]}(p_i)$, $\epsilon$-$C^1$-close to $X$ and such that for all $q \in K_i \subseteq U_i$:

$$\|P^{t_i}_{Y_i}(q)\| \leq e^{\delta t_i} + \gamma.$$

Now we define $Y = Y_i$ inside the flow box $\underset{t\in[0,t_i]}{\cup} X^t(U_i)$ and $Y = X$ outside. Since these flow boxes are pairwise disjoint, the vector field is well defined and it is $\epsilon$-$C^1$-close to $X$. Note that $V$ is also a castle for $Y^t$, $U$ is also a $Y^t$-subcastle of the $Y^t$-castle $V$ and have for base the union of all $U_i's$. We take $K$ the $Y^t$-subcastle with section (base of the castle) the union of $K_i's$. By construction of $U$ we get (a), (d) and (e).

Now we prove (b). By (6.7) and since by (ii) of Definition 3.2.1 $\overline{\mu}(K_i) > (1-\kappa)\overline{\mu}(U_i)$, we obtain $\mu(U-K) < \kappa\mu(U)$. By (6.6) and recalling that $V \supset U$ and $J \subseteq K$ we get:

$$\mu(U - Q) < \mu(V - J) < \gamma\mu(\hat{Q}) \leq \gamma.$$

To prove that $\mu(Q - U) < 2\gamma$ we use (6.5) and (6.7) so:

$$\mu(Q - U) \leq \mu(Q - J) + \mu(J - U) < 2\gamma\mu(\hat{Q}) < 2\gamma.$$

Finally for (c) since (b) implies:

$$\mu(U) < \mu(Q) + \mu(U - Q) < 1 + \gamma,$$

we use the inequality $\mu(U - K) < \kappa\mu(U)$ and get:

$$\mu(U - K) < \kappa(1 + \gamma).$$

□

## 6.4 Computing $LE(Y)$

By Lemma 2.6.1 we have:

$$LE(Y) = \inf_{n\geq 1} \int_M \frac{1}{n}\log\|P^n_Y(p)\|d\mu(p).$$

The next inequality is valid for any positive integer in particular for $t = h\delta^{-1}$ (we may assume that this is an integer),

$$LE(Y) \leq \int_M \frac{1}{t}\log\|P_Y^t(p)\|d\mu(p).$$

By what we did above for orbit segments inside the castle $K$ and starting in the base, we guarantee small upper Lyapunov exponent, so we define the set of points whose orbit stay for a long time in $K$ by,

$$G = \{p \in M : Y^s(p) \in K \ \forall s \in [0,t]\}.$$

Its complementary set is:

$$G^c = \{p \in M : \exists s \in [0,t] : p \in Y^{-s}(K^c)\}.$$

**Lemma 6.4.1** *For $p \in G$ we have $\|P_Y^t(p)\| < e^{t(1+6logC)\delta}$.*

**Proof:** Let $p \in G$. We split the orbit segment $X^{[0,t]}(p)$ by return-times at $B_K$ (the section of the castle $K$), say $t = b + r_n + ... + r_2 + r_1 + a$ where all $X^a(p), X^{r_1+a}(p), X^{r_2+r_1+a}(p), ..., X^{\sum_{i=1}^n r_i+a}(p)$ are in the base $B_K$. By restriction of height $a, b, r_i \in ]0, 3h]$ except when $p \in B_K$, where $a = 0$, and $X^t(p) \in B_K$, where $b = 0$ (see Figure 6.3).

Note that,

$$\begin{aligned}
\|P_Y^t(p)\| &= \|P_Y^{b+\sum_{i=1}^n r_i+a}(p)\| \leq \\
&\leq \|P_Y^b(X^{\sum_{i=1}^n r_i+a}(p)))\| \times \|P_Y^{r_n}(X^{\sum_{i=1}^{n-1} r_i+a}(p)))\| \times ... \\
&... \times\|P_Y^{r_1}(X^a(p))\| \times \|P_Y^a(p)\|.
\end{aligned}$$

But these maps are based at points in $B_K$ so we recall Lemma 5.0.15 and get:

$$\begin{aligned}
\|P_Y^t(p)\| &\leq C^{3h}e^{\sum_{i=1}^n r_i\delta}C^{3h} \leq e^{(b+\sum_{i=1}^n r_i+a)\delta}C^{6h} \leq \\
&\leq e^{t\delta}C^{6h} \leq e^{t\delta}C^{6\delta t} \leq e^{t(1+6\log C)\delta}.
\end{aligned}$$

□

We conclude that points in $G$ are controllable, i.e. the norm $\|P_Y^t(p)\|$ is small, but we still do not know what happens outside $G$, however next lemma says that $G^c$ has small measure.

**Lemma 6.4.2** *Let $\gamma < \delta^2 h^{-1}$ then $\mu(G^c) < 9\delta$.*

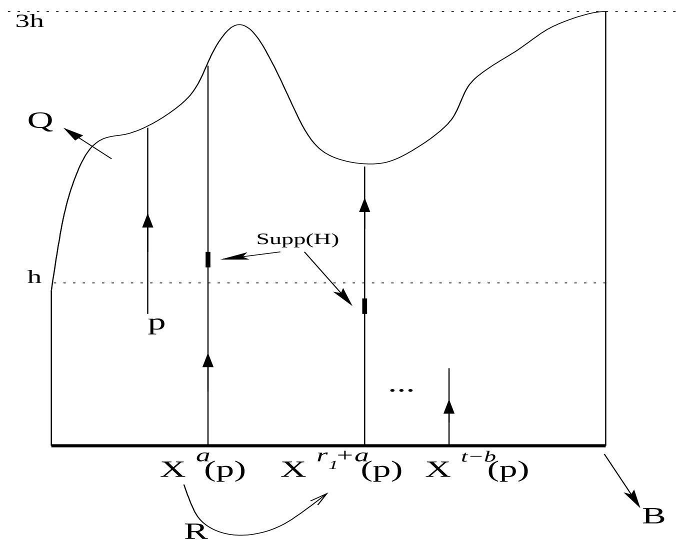


Figure 6.3: The global procedure at G.

**Proof:** Assuming the following inequality we prove the lemma:

$$\mu(\hat{Q} - G) \leq 8\delta. \tag{6.8}$$

We have:

$$\mu(G^c) \leq \mu(\hat{Q}^c) + \mu(\hat{Q} - G).$$

By definition of $\hat{Q}$, see (6.4), we have $\mu(\hat{Q}^c) < \delta^2$ and together with (6.8) we get:

$$\mu(G^c) \leq \delta^2 + 8\delta \leq 9\delta.$$

So let us prove inequality (6.8):

We have:

$$\hat{Q} - G = \hat{Q} \cap \{p \in M : \exists s \in [0,t] : p \in Y^{-s}(K^c)\}.$$

Since:

$$K^c \subseteq (U - K) \cup (Q - U) \cup (\hat{Q} - Q) \cup (M - \hat{Q})$$

we have $\hat{Q} - G$ contained in the union of the following sets:

A=$\{p \in M : \exists s \in [0,t] : p \in Y^{-s}(U - K)\}$;

B=$\{p \in M : \exists s \in [0,t] : p \in Y^{-s}(Q - U)\}$;

C=$\{p \in M : \exists s \in [0,t] : p \in Y^{-s}(\hat{Q} - Q)\}$;

D=$\{p \in M : \exists s \in [0,t] : p \in Y^{-s}(\hat{Q}^c)\}$.

Now we prove that each one of the four sets above have small measure:

**A:**

We have $\mu(A) \leq t\tilde{\mu}(B_{U-K})$ and since $U-K$ is a $Y^t$-castle with towers larger than $h$:

$$\mu(U-K) = \int_{B_{U-K}} \int_0^{h(x)} ds d\tilde{\mu}(x) \geq \int_{B_{U-K}} h d\tilde{\mu}(x) = \tilde{\mu}(B_{U-K})h.$$

We recall that $\kappa = \delta^2$, $\gamma = \delta^2 h^{-1}$ and $t = h\delta^{-1}$ and by Lemma 6.3.1 (c) we obtain:

$$\mu(A) \leq \frac{t}{h}\mu(U-K) \leq \frac{t}{h}\kappa(1+\gamma) = \frac{h\delta^{-1}}{h}\delta^2(1+\frac{\delta^2}{h}) \leq 2\delta.$$

**B:**

By Lemma 6.3.1 (b) we get:

$$\mu(Q-U) \leq 2\gamma \leq 2\delta^2 h^{-1} = 2\delta^2 t^{-1}\delta^{-1} = 2\delta t^{-1},$$

so we have $t\mu(Q-U) < 2\delta$, it follows that:

$$\mu(B) \leq t\mu(Q-U) < 2\delta.$$

**C:**

We have $\mu(C) \leq t\tilde{\mu}(B_{\hat{Q}-Q})$.

For the $X^t$-castle $\hat{Q}-Q$ the towers are large by definition, so the area of its bases are small, we denote the bases of this castle by $B^X_{\hat{Q}-Q}$, so:

$$\tilde{\mu}(B^X_{\hat{Q}-Q}) \leq \frac{1}{3h}\mu(\hat{Q}-Q). \tag{6.9}$$

By Lemma 6.3.1 (b), $\mu(U-Q) < \gamma$ so we use this inequality, Lemma 6.2.1, (6.9) and estimate:

$$\tilde{\mu}(B_{\hat{Q}-Q}) \leq \frac{\mu(\hat{Q}-Q)}{3h} + \gamma \leq \frac{\delta^2}{h} + \gamma.$$

Therefore:

$$\mu(C) \leq t(\frac{\delta^2}{h} + \gamma) = h\delta^{-1}(\frac{\delta^2}{h} + \frac{\delta^2}{h}) = 2\delta.$$

**D:**

We have $Y^s(\hat{Q}^c) - \hat{Q}^c \subseteq Y^s(U-Q)$. By definition of $\hat{Q}$ we get $\mu(\hat{Q}^c) < \delta^2$ and by Lemma 6.3.1 (b) we have $\mu(U-Q) < \gamma$ so,

$$\begin{aligned} \mu(D) &\leq \mu(\hat{Q}^c) + (t-s)\mu(Y^s(\hat{Q}^c) - \hat{Q}^c) \leq \mu(\hat{Q}^c) + (t-s)\mu(Y^s(U-Q)) = \\ &= \mu(\hat{Q}^c) + (t-s)\mu(U-Q) < \delta^2 + t\gamma = \delta^2 + \delta < 2\delta. \end{aligned}$$

Finally we estimate $\mu(\hat{Q}-G)$:

$$\mu(\hat{Q}-G) \leq \mu(A)+\mu(B)+\mu(C)+\mu(D) \leq 2\delta+2\delta+2\delta+2\delta = 8\delta$$

and (6.8) is proved. $\square$

**Proof of Theorem 1:**

Now we finish the proof of Proposition 2.6.2, therefore Theorem 1:

$$\begin{aligned} LE(Y) &= \inf_{n\geq 1}\int_M \frac{1}{n}\log\|P_Y^n(p)\|d\mu(p) \leq \int_M \frac{1}{t}\log\|P_Y^t(p)\|d\mu(p) \leq \\ &\leq \int_G \frac{1}{t}\log\|P_Y^t(p)\|d\mu(p) + \int_{G^c} \frac{1}{t}\log\|P_Y^t(p)\|d\mu(p) \leq \\ &\leq (1+6\log C)\delta\mu(G) + \log C\mu(G^c). \end{aligned}$$

Now we use Lemma 6.4.2 and conclude that,

$$LE(Y) \leq (1+6\log C)\delta + 9\delta\log C = (1+15\log C)\delta.$$

And the Theorem 1 is proved by substitution of $\delta$ by $\frac{\delta}{(1+15\log C)}$ along the proof.

# Chapter 7

# Dichotomy for generic conservative linear differential systems in $C^0$ topology

## 7.1 Basic definitions

Let $X$ be a compact Hausdorff space, $\mu$ a Borel regular measure and $\varphi^t : X \to X$ a one-parameter family of continuous maps for which $\mu$ is $\varphi^t$-invariant.

A cocycle based on $\varphi^t$ is defined by a flow $\Phi^t(p)$ differentiable on the time-parameter $t \in \mathbb{R}$ and continuous on space-parameter $p \in X$, acting on $GL(2, \mathbb{R})$. Together they form the linear skew-product flow:

$$\begin{array}{rccc} \Psi^t : & X \times \mathbb{R}^2 & \longrightarrow & X \times \mathbb{R}^2 \\ & (p, v) & \longmapsto & (\varphi^t(p), \Phi^t(p) \cdot v) \end{array}$$

The flow $\Phi^t$ verifies the *cocycle identity*:

$$\Phi^{t+s}(p) = \Phi^s(\varphi^t(p)) \circ \Phi^t(p),$$

for all $t, s \in \mathbb{R}$ and $p \in X$.

If we define a map $A : X \to GL(2, \mathbb{R})$ in a point $p \in X$ by:

$$A(p) = \frac{d}{ds}\Phi^s(p)|_{s=0}$$

and along the orbit $\varphi^t(p)$ by:

$$A(\varphi^t(p)) = \frac{d}{ds}\Phi^s(p)|_{s=t} \circ [\Phi^t(p)]^{-1}, \tag{7.1}$$

then $\Phi^t(p)$ will be the solution of the linear variational equation:

$$\frac{d}{ds}u(s)_{|s=t} = A(\varphi^t(p)) \cdot u(t), \tag{7.2}$$

and $\Phi^t(p)$ is also called the fundamental matrix. Given a cocycle $\Phi^t$ we can induce the associated $A$ by using (7.1) and given $A$ we can recover the cocycle by solving the linear variational equation (7.2), from which we get $\Phi^t_A$. We are interested in two kind of systems, the ones with $\det\Phi^t = 1$ which we call *area-preserving* or *traceless*, denoted by $GL(2, \mathbb{R}, \mathrm{Tr} = 0)$, and the *modified area-preserving*, denoted by $GL(2, \mathbb{R}, \varphi^t)$, by establishing a link to the flow $\varphi^t$. To define this setting we need to consider a continuous non-negative sub-exponential function $a : X \to \mathbb{R}$ which is positive outside $Fix(\varphi^t)$ and we say that $A$ is modified area-preserving if:

$$\det\Phi^t_A(p) = \frac{a(p)}{a(\varphi^t(p))} \text{ for all } p \notin Fix(\varphi^t) \text{ and } t \in \mathbb{R},$$

$$\det\Phi^t_A(p) = 1 \text{ for all } p \in Fix(\varphi^t).$$

By Liouville formula we get: $e^{\int_0^t \mathrm{Tr}A(\varphi^s(p))ds} = \det\Phi^t(p)$, so,

$$e^{\int_0^t \mathrm{Tr}A(\varphi^s(p))ds} = \frac{a(p)}{a(\varphi^t(p))}.$$

## 7.2 Topology and conservative perturbations

Consider the set of linear differential systems $A$ which are continuous and denote it by $C^0(X, GL(2, \mathbb{R}))$. We endow $C^0(X, GL(2, \mathbb{R}))$ with the uniform convergence topology defined by $\|A - B\|_0 = \max_{p\in X}\|A(p) - B(p)\|$.
We also define a $L^\infty$-topology, this time on the set of measurable and $\mu$-a.e. bounded maps $L^\infty(X, GL(2, \mathbb{R}))$, such that $\|A - B\|_\infty = \mathrm{esssup}\|A(p) - B(p)\|$. Therefore we may speak about conservative $C^0$(or $L^\infty$)-perturbations of systems $A \in C^0(X, GL(2, \mathbb{R}))$ (or $A \in L^\infty(X, GL(2, \mathbb{R}))$) along the orbit $\varphi^t(p)$ as $A+H$ where $H \in C^0(X, GL(2, \mathbb{R}))$ (or $H \in L^\infty(X, GL(2, \mathbb{R}))$) and $\mathrm{Tr}H(\varphi^t(p)) = 0$. This follows by direct application of Liouville formula, because,

$$e^{\int_0^t \mathrm{Tr}A(\varphi^s(p))+\mathrm{Tr}H(\varphi^s(p))ds} = e^{\int_0^t \mathrm{Tr}A(\varphi^s(p))ds} = \det\Phi^t(p).$$

Given a conservative perturbation of $A$, say $A + H$, we denote by $\Phi^t_{A+H}(p)$ the solution of the related linear variational equation (7.2), i.e., of

$$\dot{v}(t) = [A(t) + H(t)] \cdot v(t).$$

## 7.3 Oseledets's Theorem and the entropy function

The Oseledets's Theorem, see [14], has also an analogous version for linear differential systems, (see [8] for a simple proof). Moreover, for our particular

2-dimensional conservative linear differential systems we have the following version;

**Theorem 7.3.1** *Let $\Phi^t$ be as above. For $\mu$-a.e. $p \in X$ there exists the upper Lyapunov exponent $\lambda^+(p)$ defined by the limit $\lim_{t\to+\infty} \frac{1}{t} log\|\Phi^t(p)\|$ that is a non-negative measurable function of $p$. For $\mu$-a.e. point $p \in \mathfrak{O}^+$ there is a splitting of $\mathbb{R}^2 = N_p^u \oplus N_p^s$ which varies measurably with $p$ such that:*

*If $\vec{0} \neq v \in N_p^u$, then $\lim_{t\to\pm\infty} \frac{1}{t} log\|\Phi^t(p)\cdot v\| = \lambda^+(p)$;*

*If $\vec{0} \neq v \in N_p^s$, then $\lim_{t\to\pm\infty} \frac{1}{t} log\|\Phi^t(p)\cdot v\| = -\lambda^+(p)$;*

*If $\vec{0} \neq v \notin N_p^u, N_p^s$, then*
*$\lim_{t\to+\infty} \frac{1}{t} log\|\Phi^t(p)\cdot v\| = \lambda^+(p)$ and $\lim_{t\to-\infty} \frac{1}{t} log\|\Phi^t(p)\cdot v\| = -\lambda^+(p)$.*

Let $\mathfrak{O}^+ := \mathfrak{O}^+(A)$ denote the set of points with non-zero Lyapunov exponents and let $\mathfrak{O}^0(A)$ denote the set of points with both Lyapunov exponents zero.

Note that the symmetry of the Lyapunov exponents follows from (2.1). Consequently, for the area-preserving case we have $\lambda^+(p) = -\lambda^-(p)$. For the modified area-preserving case we have the equality, $\det\Phi^t(p) = \frac{a(p)}{a(\varphi^t(p))}$ and since $a(\cdot)$ is sub-exponential and positive along non-fixed orbits we get $\lambda^+(p) = -\lambda^-(p)$. For fixed points the former equality follows directly from $\det\Phi_A^t = 1$.

We also define the *entropy function* of the system $A$, this time over any measurable, $\varphi^t$-invariant set $\Gamma \subseteq X$ by:

$$\begin{array}{rccc} LE(\cdot,\Gamma): & GL(2,\mathbb{R}) & \longrightarrow & [0,+\infty) \\ & A & \longmapsto & \int_\Gamma \lambda^+(p)d\mu(p) \end{array}$$

using the subadditivity of the norm we obtain:

$$LE(A,\Gamma) = \inf_{n\geq 1}\frac{1}{n}\int_\Gamma \log\|\Phi^n(p)\|d\mu(p).$$

Since $LE(\cdot,\Gamma)$ is the infimum of continuous functions it is upper semicontinuous.

## 7.4 Hyperbolic structures

Let $A$ be a linear differential system over a flow $\varphi^t$, the set $\Lambda \subseteq X$ is said to be *uniformly hyperbolic set* if there exists uniform constants $C > 0$ and

$\sigma \in (0,1)$ such that for $\mu$-a.e. $p \in \Lambda$ there is a $\Phi_A^t(p)$-invariant decomposition $\mathbb{R}^2 = N_p^u \oplus N_p^s$ varying measurably with $p$ and satisfying for $t > 0$ the following equalities: $\|\Phi_A^{-t}(p)|_{N_p^u}\| \leq C\sigma^t$ and $\|\Phi_A^t(p)|_{N_p^s}\| \leq C\sigma^t$. If $\Lambda = X$, then we say that $A$ is *uniformly hyperbolic*.

A $\varphi^t$-invariant set $\Lambda_m \subseteq X$ has *m-dominated splitting* for $A$ if for $\mu$-a.e. $p \in \Lambda_m$, there is a $\Phi_A^t(p)$-invariant decomposition $\mathbb{R}^2 = N_p^u \oplus N_p^s$ varying measurably with $p$ and satisfying, $\frac{\|\Phi_A^m(q)|_{N_q^s}\|}{\|\Phi_A^m(q)|_{N_q^u}\|} \leq \frac{1}{2}$ for any $q = X^t(p)$.
Another definition equivalent to this one is considering contants $C > 0$ and $\sigma \in (0,1)$ such that $\frac{\|\Phi_A^t(q)|_{N_q^s}\|}{\|\Phi_A^t(q)|_{N_q^u}\|} \leq C\sigma^t$. In this case we say that the $\varphi^t$-invariant set has $(C,\sigma)$-dominated splitting.
Let $\Delta(p,m) := \frac{\|\Phi_A^m(p)|_{N_p^s}\|}{\|\Phi_A^m(p)|_{N_p^u}\|}$. We define the following sets:

$\Lambda_m(A) = \{p \in X : \text{the orbit } \varphi^t(p) \text{ has } m\text{-dominated splitting for } \Phi^t\}$;
$\Gamma_m(A) = X - \Lambda_m(A)$;
$\Gamma_m^+(A) = \Gamma_m(A) \cap \mathfrak{O}^+(A)$;
$\Gamma_m^*(A) = \{p \in \Gamma_m^+(A) : p \notin Per(\varphi^t)\}$;
$\Delta_m(A) = \{p \in X : \Delta(p,m) \geq \frac{1}{2}\}$.

Before moving on to the proof of Theorem 2 we would like to recall some facts. First of all note that if $q \in \Gamma_m(A)$, then for some point in the orbit of $q$, say $\varphi^t(q) = p$, we have $\Delta(p,m) \geq 1/2$, and therefore $p \in \Delta_m(A)$. Moreover $\Gamma_m = \bigcup_{t \in \mathbb{R}} \varphi^t(\Delta_m)$. The set $\Delta_m$ is again of utmost importance because that is where we will apply a perturbation to the original system, like we did for vector fields.

## 7.5 Disregarding periodic points in $\mathfrak{O}^+(A)$

Our main objective will be decay $LE(B, \Gamma_m(A))$ for a system $B$ close to the original system $A$. We say that a flow is *aperiodic* if the measure of periodic points is zero, clearly $\varphi^t : \Gamma_m^*(A) \to \Gamma_m^*(A)$ is aperiodic. We will use Ambrose-Kakutani theorem, see [1], which gives us a special representation of $\varphi^t|_{\Gamma_m^*(A)}$. Next we perturb inside $\Gamma_m^*(A)$ to decrease $LE(B, \Gamma_m^*(A))$. However we have no information about $\Gamma_m(A) - \Gamma_m^*(A)$. For our purposes, points in $\Gamma_m(A)$ with zero Lyapunov exponents will not be a problem, whereas the set $\Gamma_m^+(A) - \Gamma_m^*(A)$ may cause some trouble. Lemma 7.5.1 says that $\mu(\Gamma_m^+(A) - \Gamma_m^*(A))$ is small (depending on $m$), therefore we will obtain $LE(B, \Gamma_m^+(A) - \Gamma_m^*(A))$ also small. We note that for a fixed $m \in \mathbb{N}$ we have $p \in \Gamma_m^+(A) - \Gamma_m^*(A)$ if $p$ is periodic, has positive Lyapunov exponent and belongs to $\Gamma_m(A)$.

**Lemma 7.5.1** *For any $\delta > 0$, there exists $m \in \mathbb{N}$ such that we have $\mu(\Gamma_m^+(A) - \Gamma_m^*(A)) < \delta$.*

**Proof:** Let $P$ be the measure of all periodic points in $\mathfrak{O}^+(A)$. If $P = 0$ then there is nothing to prove, so consider $P > 0$. Define

$$\mathrm{Per}(n, \lambda) = \{p \in \mathfrak{O}^+ : \varphi^t(p) = p \text{ for some } t \leq n \text{ and } \lambda^+(p) > \lambda\}.$$

So $\mu(\underset{n\in\mathbb{N}}{\cup}\mathrm{Per}(n, 0)) = P$ therefore for all $\delta > 0$, there exists $n_0, \lambda_0$ such that $\mu(\mathrm{Per}(n_0, \lambda_0)) > P - \delta$.

If $p \in \mathrm{Per}(n_0, \lambda_0)$, then $p$ has $m'$ dominated splitting for some $m'$, therefore there exists a large $m$ such that $\mathrm{Per}(n_0, \lambda_0) \subseteq \cup_{i=1}^m \Lambda_i$ and we get:
$\Gamma_m^+(A) - \Gamma_m^*(A) \subseteq \underset{n\in\mathbb{N}}{\cup}\mathrm{Per}(n, 0) - \cup_{i=1}^m \Lambda_i$ so:

$$\begin{aligned}
\mu(\Gamma_m^+(A) - \Gamma_m^*(A)) &< \mu(\underset{n\in\mathbb{N}}{\cup}\mathrm{Per}(n, 0) - \cup_{i=1}^m \Lambda_i) < \\
&< \mu(\underset{n\in\mathbb{N}}{\cup}\mathrm{Per}(n, 0) - \mathrm{Per}(n_0, \lambda_0)) = \\
&= \mu(\underset{n\in\mathbb{N}}{\cup}\mathrm{Per}(n, 0)) - \mu(\mathrm{Per}(n_0, \lambda_0)) < \delta.
\end{aligned}$$

□

## 7.6 Perturbations of linear differential systems

We begin by lowering $\|\Phi^t_{A+H}(q)\|$ along a segment of the orbit, this is valid in both settings $GL(2, \mathbb{R}, \mathrm{Tr} = 0)$ and $GL(2, \mathbb{R}, \varphi^t)$. In order to achieve this goal we carry out some perturbations which we explains in the next section.

### 7.6.1 Small rotations by time-1 perturbation

**Lemma 7.6.1** *Given a conservative system $A$ and $\epsilon > 0$, there exists an angle $\xi$, such that for all $p \in X$ (non-periodic or with period larger than 1), there exists a system $B$ such that:*

*(a) $\|A - B\| < \epsilon$;*

*(b) $B$ is supported in $\varphi^t(p)$ for $t \in [0, 1]$;*

*(c) $B$ is conservative and*

*(d) $\Phi^1_B(p) = \Phi^1_A(p) \circ R_\xi$, where $R_\xi$ is a rotation of angle $\xi$.*

**Proof:**

Let $0 < \eta < 1$ and $g : \mathbb{R} \to \mathbb{R}$ the *bump-function* defined by $g(t) = 0$ for $t < 0$, $g(t) = t$ for $t \in [\eta, 1 - \eta]$ and $g(t) = 1$ for $t \geq 1$. Define:

$$\Phi^t(p) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \text{ and } R_{\xi g(t)} = \begin{pmatrix} cos(\xi g(t)) & -sin(\xi g(t)) \\ sin(\xi g(t)) & cos(\xi g(t)) \end{pmatrix}.$$

We know that $u(t) = \Phi^t(p)$ is a solution of the linear variational equation (7.2). Take $\Phi^t(p) \cdot R_{\xi g(t)}$ and compute the time derivative:

$$\begin{aligned}
(\Phi^t(p) \cdot R_{\xi g(t)})^{\cdot} &= (\Phi^t(p))^{\cdot} R_{\xi g(t)} + \Phi^t(p) \dot{R}_{\xi g(t)} = \\
&= A(\varphi^t(p))\Phi^t(p) R_{\xi g(t)} + \Phi^t(p)\dot{R}_{\xi g(t)} = \\
&= A(\varphi^t(p))\Phi^t(p) R_{\xi g(t)} + \\
&+ \Phi^t(p)\dot{R}_{\xi g(t)} R_{-\xi g(t)} [\Phi^t(p)]^{-1} \Phi^t(p) R_{\xi g(t)} = \\
&= [A(\varphi^t(p)) + \Phi^t(p) R_{\xi g(t)}^{\cdot} R_{-\xi g(t)} [\Phi^t(p)]^{-1}] \cdot (\Phi^t(p) R_{\xi g(t)}).
\end{aligned}$$

Define $B(\varphi^t(p)) = A(\varphi^t(p)) + H(\varphi^t(p))$ where:

$$H(\varphi^t(p)) = H(\xi, t) = \Phi^t(p) R_{\xi g(t)}^{\cdot} R_{-\xi g(t)} [\Phi^t(p)]^{-1}.$$

We conclude that $v(t) = \Phi^t(p) R_{\xi g(t)}$ is solution of the linear variational equation:

$$\frac{d}{ds} v(s)_{|s=t} = [A(\varphi^t(p)) + H(\xi, t)] \cdot v(t) \tag{7.3}$$

Since,

$$R_{\xi g(t)}^{\cdot} \cdot R_{-\xi g(t)} = \xi \dot{g}(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we easily derive

$$H(\xi, t) = \frac{\xi \dot{g}(t)}{\det \Phi^t(p)} \begin{pmatrix} b(t)d(t) + a(t)c(t) & -b(t)^2 - a(t)^2 \\ d(t)^2 + c(t)^2 & -b(t)d(t) - a(t)c(t) \end{pmatrix}.$$

Hence $\mathrm{Tr} H(\xi, t) = 0$ and the perturbation is conservative according to our definition, and so (c) follows. Moreover since $\dot{g}(t) = 0$ for $t \notin ]0, 1[$, its support is $\varphi^t(p)$ for $t \in [0, 1]$ and (b) is proved. Since $t \in [0, 1]$ and all the terms in the definition of $H(\xi, t)$ are uniformly bounded for all $p \in X$, given any size of perturbation allowed by $\epsilon > 0$ we take $\xi$ sufficiently small to guarantee that $\|H\| < \epsilon$ and obtain (a). Finally, for (d), we note that $v(t) = \Phi^t_A(p) \cdot R_{\xi g(t)}$ is solution of (7.3). So for $t = 1$ we obtain $\Phi^1_B(p) = \Phi^1_A(p) \cdot R_\xi$ and the lemma is proved.

□

**Lemma 7.6.2** *Given a conservative system $A$ and $\epsilon > 0$, there exists an angle $\xi$, such that for all $p \in X$ (non-periodic or with period larger than 1), there exists a system $B$ such that:*

*(a)* $\|A - B\| < \epsilon$*;*

*(b)* $B$ *is supported in* $\varphi^t(p)$ *for* $t \in [0,1]$*;*

*(c)* $B$ *is conservative and*

*(d)* $\Phi^1_B(p) = \tilde{R}_\xi \circ \Phi^1_A(p)$*, where* $\tilde{R}_\xi$ *is an elliptical rotation of angle* $\xi$*.*

**Proof:** We use the same notation of Lemma 7.6.1. Define the one parameter elliptical rotation by:

$$\tilde{R}_{\xi g(t)} = \Phi^t(p) \cdot R_{\xi g(t)} \cdot \Phi^{-t}(\varphi^t(p)) \tag{7.4}$$

Now we consider $\tilde{R}_{\xi g(t)} \cdot \Phi^t(p)$ and take time derivatives:

$$\begin{aligned}
(\tilde{R}_{\xi g(t)} \cdot \Phi^t(p))^{\cdot} &= (\Phi^t(p) \cdot R_{\xi g(t)} \cdot \Phi^{-t}(\varphi^t(p)) \cdot \Phi^t(p))^{\cdot} = (\Phi^t(p) \cdot R_{\xi g(t)})^{\cdot} = \\
&= [A(\varphi^t(p)) + H(\varphi^t(p))] \cdot (\Phi^t(p) R_{\xi g(t)}) = \\
&= [A(\varphi^t(p)) + H(\varphi^t(p))] \cdot (\tilde{R}_{\xi g(t)} \cdot \Phi^t(p)),
\end{aligned}$$

and we reduce to the proof of Lemma 7.6.1. $\square$

**Remark 7.6.1** *We will need Lemma 7.6.2 to perform some small rotations, and we point out that this lemma gives us an elliptical rotation. So after the change of coordinates the angle may decrease depending on how large the norm of this change of coordinates is. However we can always find* $\xi_0 < \xi$ *depending on* $\|\Phi^t_A(p)\|$ *(for* $t \in [0,1]$*) and conclude that the perturbation realizes* $\Phi^1_{A+H}(p) = R_{\xi_0} \cdot \Phi^1_A(p)$*.*

## 7.6.2 Large rotations by time-$m$ perturbation

We recall Lemma 4.0.11 which give us a control of the norm of $\Phi^t(p)$. In the case of $GL(2, \mathbb{R}, \mathrm{Tr}{=}0)$ we take $\sqrt{\frac{a(p)}{a(\varphi^t(p))}} = 1$. Lemma 4.0.11 says that given $\xi > 0$, $p \in X$ and $d > 1$, there exists $E > 1$ such that if for all $t \in [0, m]$, $\angle(N^u_{\varphi^t(p)}, N^s_{\varphi^t(p)}) > \xi$ and $d^{-1} \leq \frac{\|\Phi^t(p)|_{N^s_p}\|}{\|\Phi^t(p)|_{N^u_p}\|} \leq d$, then $\|\Phi^t(p)\| \leq E\sqrt{\frac{a(p)}{a(\varphi^t(p))}}$ for all $t \in [0, m]$.

In order to perform rotations of large angles we could try, under some particular conditions and like we did for vector fields, to concatenate smoothly several time-1 small rotations until obtain the desired angle. Otherwise, which is easier, we could induce a time-$m$ perturbation to generate the rotations of a given large angle, however this cannot be done in general, because again some hyperbolicity in the dynamics obstruct the whole construction.
Under the conditions of Lemma 4.0.11 it is possible to rotate large angles by

time-$m$ keeping the norm of $H(\xi, t)$ for $t \in [0, m]$ small. Since the explicit perturbation is given by,

$$B(\varphi^s(p)) = \frac{d}{dt}[\Phi_A^{t+s}(p) \cdot R_{\xi g(s+t)} \cdot \Phi_A^{-s}(\varphi^s(p))]_{|t=0} \Phi_A^s(p) \cdot R_{\xi g(s)} \cdot \Phi_A^{-s}(\varphi^s(p)),$$

we expect that some control of $\|\Phi_A^t\|$ is needed, so Lemma 4.0.11 will play again an important role.

**Lemma 7.6.3** *Given a conservative system $A$ and $\epsilon, d, \xi > 0$, there exists $m \in \mathbb{N}$, such that if the following conditions are satisfied for $p \in X$ non-periodic or with period larger than $m$ namely,*

*(1) $\angle(N^u_{\varphi^t(p)}, N^s_{\varphi^t(p)}) > \xi$ for all $t \in [0, m]$*

*(2) $d^{-1} \le \frac{\|\Phi^t(p)|_{N_p^s}\|}{\|\Phi^t(p)|_{N_p^u}\|} \le d$,*

*then there exists a system $B$ such that for all $\alpha \in [0, 2\pi]$, we have:*

*(a) $\|A - B\| < \epsilon$;*

*(b) $B$ is supported in $\varphi^t(p)$ for $t \in [0, m]$;*

*(c) $B$ is conservative and*

*(d) $\Phi_B^1(p) = \Phi_A^1(p) \circ R_\alpha$.*

**Proof:**

For any $m \in \mathbb{N}$ we consider $\eta > 0$ close to zero and $g : \mathbb{R} \to \mathbb{R}$ the *bump-function* such that $g(t) = 0$ for $t < 0$, $g(t) = t$ for $t \in [\eta, m-\eta]$ and $g(t) = 1$ for $t \ge m$. We use then the same procedure of Lemma 7.6.1 by defining $R_{\theta g(t)}$. Let $\alpha \in [0, 2\pi]$. Take $\theta < \frac{\epsilon \sin\xi}{4d}$ and $m = \frac{\alpha}{\theta}$. There is no restriction while considering $m \in \mathbb{N}$, by taking a smaller $\theta$. Now fix the function $g$ depending on this $m$. Clearly we will obtain $H$ such that (b), (c) and (d) are verified. We claim that (a) is also true. By hypothesis we have (1) and (2), so by Lemma 4.0.11 we conclude that $\|\Phi^t(p)\| < E\sqrt{\frac{a(p)}{a(\varphi^t(p))}}$ and since $E = \sqrt{2d.\sin^{-1}\xi}$ we get $\theta < \frac{\epsilon}{2E^2}$. Using the same notation of Lemma 7.6.1, the perturbation is defined, for $t \in [0, m]$, by,

$$H(\theta, t) = \frac{\theta \dot{g}(t)}{\det \Phi^t(p)} \begin{pmatrix} b(t)d(t) + a(t)c(t) & -b(t)^2 - a(t)^2 \\ d(t)^2 + c(t)^2 & -b(t)d(t) - a(t)c(t) \end{pmatrix}.$$

Let us see that $\|H\| < \epsilon$ (consider the norm of the maximum):

$$\|H(\theta, t)\| \le$$

$$\leq \frac{\theta\dot{g}(t)}{|\det\Phi^t(p)|}\max_{t\in[0,m]}\{\pm[b(t)d(t)+a(t)c(t)], -b(t)^2-a(t)^2, d(t)^2+c(t)^2\} \leq$$
$$\leq 2\frac{\theta\dot{g}(t)}{|\det\Phi^t(p)|}\|\Phi^t(p)\|^2.$$

Now, by Lemma 4.0.11 we obtain:

$$\|H(\theta,t)\| \leq 2\frac{\theta\dot{g}(t)}{|\det\Phi^t(p)|}2d.\sin^{-1}\xi\frac{a(p)}{a(\varphi^t(p))} \leq 2\frac{\theta\dot{g}(t)}{|\det\Phi^t(p)|}E^2\frac{a(p)}{a(\varphi^t(p))} \leq \epsilon.$$

Which concludes the proof. □

## 7.7 Lowering the norm - Local procedure

The next lemma is analogous to Lemma 4.0.12.

**Lemma 7.7.1** *Let $A$ be a continuous conservative system and $\epsilon > 0$. There exists $m \in \mathbb{N}$ such that, given any $p \in \Gamma^*_m(A)$, there exists $H$ satisfying $\|H\| < \epsilon$ and $\Phi^m_{A+H}(N^u_p) = N^s_{\varphi^m(p)}$.*

**Proof:** Let $\xi > 0$ be given by Lemmas 7.6.1 and 7.6.2 in order to guarantee time-1 $\epsilon$-perturbations. Let $c > 0$ be given by Claim 4.0.10. Let $E > 1$ be given by Lemma 4.0.11 depending on $\xi$ and $d = 2c^2$. Let $m \in \mathbb{N}$ be given by Lemma 7.6.3 and depending on $E$, hence depending on $d$ and $\xi$. Consider $p \in \Gamma^*_m(A)$ and $\Phi^t(p)$ for $t \in [0,m]$.

**Small angle**: If for some $t \in [0,m]$ we have $\angle(N^u_{\varphi^t(p)}, N^s_{\varphi^t(p)}) < \xi$ we use a small rotation by a time-1 perturbation and, if $t+1 \leq m$ we get $H$ such that $\Phi^1_{A+H}(N^u_{\varphi^t(p)}) = N^s_{\varphi^{t+1}(p)}$. If $t+1 > m$ we get $H$ such that $\Phi^{-1}_{A+H}(N^s_{\varphi^t(p)}) = N^u_{\varphi^{t-1}(p)}$ and in both cases $\|H\| < \epsilon$. Consequently we obtain $\Phi^m_{A+H}(N^u_p) = N^s_{\varphi^m(p)}$.
Now we consider the case when there exists $r, t \in \mathbb{R}$ with $0 \leq r+t \leq m$ such that $\Delta(\varphi^t(p), r) > c$. We use Claim 4.0.10 in order to obtain a vector $v \in N_{\varphi^t(p)}$ such that $\angle(v, N^u_{\varphi^t(p)}) < \xi$ and $\angle(\Phi^r(\varphi^t(p))\cdot v, N^s_{\varphi^{t+r}(p)}) < \xi$. Now, since $\xi$ is small we make two small rotations at both extremes $\varphi^t(p)$ and $\varphi^{t+r}(p)$. The choice of $c$ sufficient large guarantees disjoint perturbations. Therefore, our first rotation $\Phi^1_{A+H_1}(\varphi^t(p)) = \Phi^1_A(\varphi^t(p))R_\xi$, induced by the perturbation $H_1$, allows us to send $N^u_{\varphi^t(p)}$ into $v\cdot\mathbb{R}$, the dynamics of $\Phi^r_A$ help us and send this direction into $\Phi^r(\varphi^t(p))\cdot v$ in time $r$ (see Figure 1) and another rotation, $\Phi^1_{A+H_2}(\varphi^{t+r-1}(p)) = R_\xi\cdot\Phi^1_A(\varphi^{t+r-1}(p))$, induced by the perturbation $H_2$, maps $\Phi^r(\varphi^t(p))\cdot(v\cdot\mathbb{R})$ into $N^s_{\varphi^{t+r}(p)}$. Now we concatenate smoothly the five matrix transitions, say,

$$\Phi^{m-(t+r)}_A(\varphi^{t+r}(p))\Phi^{r-1}_{A+H_2}(\varphi^{t+r-1}(p))\Phi^{r-2}_A(\varphi^{t+1}(p))\Phi^1_{A+H_1}(\varphi^t(p))\Phi^t_A(p),$$

and we get $\Phi^m_{A+H}(N^u_p) = N^s_{\varphi^m(p)}$. Note that $H(p) = H_1(p) + H_2(p)$ and $\|H_i\| < \epsilon$, for $i = 1, 2$.

**Large angle**: Finally, we have for all $t \in [0, m]$, $\angle(N^u_{\varphi^t(p)}, N^s_{\varphi^t(p)}) > \xi$ and since for all $r, t \in \mathbb{R}$ with $0 \leq r + t \leq m$ we have $\Delta(\varphi^t(p), r) < c$. Since $p \in \Delta_m$ we also have $\Delta(p, m) \geq 1/2$. Therefore, we conclude that,

$$\Delta(\varphi^t(p), r) = \Delta(\varphi^{t+r}(p), m - t - r)\Delta(p, m)\Delta(p, t)^{-1} \geq \frac{1}{2c^2}$$

So for $t = 0$ and $r \in [0, m]$ we have $(2c^2)^{-1} \leq \Delta(p, r) \leq c$ and since $d = 2c^2$ we obtain,

$$d^{-1} \leq \frac{\|\Phi^r(p)|_{N^s_p}\|}{\|\Phi^r(p)|_{N^u_p}\|} \leq d,$$

for all $r \in [0, m]$. The conditions of Lemma 4.0.11 are now satisfied and by applying Lemma 7.6.3 we are able to use rotations by large angles and therefore $\Phi^m_{A+H}(N^u_p) = N^s_{\varphi^m(p)}$ which proves the lemma. $\square$

In the next lemma we only give an outline of the proof and skip technical arguments already explained in Lemma 5.0.15.

**Lemma 7.7.2** *Let $A$ be a continuous conservative system, $\epsilon > 0$, and $\delta > 0$. There exists $m \in \mathbb{N}$ and a measurable function $T : \Gamma^*_m(A) \to \mathbb{R}$ such that for $\mu$-a.e. $q \in \Gamma^*_m$ and every $t > T(q)$ there exist a traceless $\{H(\varphi^s(q))\}_{s\in\mathbb{R}}$, varying smoothly with $s$ and supported on the segment $\varphi^{[0,t]}(p)$ such that:*

*(a) $\|H\| < \epsilon$,*

*(b) $\frac{1}{t} log\|\Phi^t_{A+H}(q)\| < \delta$.*

**Proof:** First, using Lemma 7.7.1, we choose a sufficiently large $m$ in order to send $N^u_p$ into $N^s_{\varphi^m(p)}$ under $\epsilon$-small $C^0$-perturbation, for Osededets regular points $p \in \Delta_m$. So, for our perturbation $A + H$ we obtain $\Phi^m_{A+H}(p)(N^u_p) = N^s_{\varphi^m(p)}$.

Given $q$ in the saturated set $\Gamma^*_m(A)$ and using a qualitative recurrence result (see [3], lemma 3.12) for all $t > T(q)$ we have to fall into $\Delta_m$ approximately in the middle of the journey, say $\varphi^\tau(q) = p$, for $\tau \approx t/2$. We take $t >> m$. Now we perturb and we get $\Phi^t_{A+H}(q)(N^u_q) = N^s_{\varphi^t(q)}$. The contribution of the exponential growth along the direction $N^u_q$ in the first half, will be annihilated on the other half by an exponential decreasing bundle $N^s_{\varphi^{s+m}(q)}$ implying $\|\Phi^t_{A+H}(q)\| < e^{t\delta}$. That is the reason why we mix the two directions. $\square$

## 7.8 Lowering the norm - Global procedure

For the global case we construct a special flow by using Ambrose-Kakutani theorem over the aperiodic flow $\varphi^t : \Gamma_m^* \to \Gamma_m^*$, but first we use Lemma 7.5.1 to increase $m \in \mathbb{N}$ if necessary and obtain:

$$\mu(\Gamma_m^+(A) - \Gamma_m^*(A)) < \delta. \tag{7.5}$$

Using the measurable function given by Lemma 7.7.2 we define

$$Z_h = \{p \in \Gamma_m^*(A) : T(p) \leq h\}.$$

Of course that $\mu(\Gamma_m^*(A) - Z_h) \underset{h\to\infty}{\longrightarrow} 0$ so holds we take $h$ sufficiently large such that:

$$\mu(\Gamma_m^*(A) - Z_h) < \delta^2 \mu(\Gamma_m^*(A)). \tag{7.6}$$

Let us now increase $h$ and use Oseledets's Theorem, which is an asymptotic result, to get for points $p \in \mathfrak{O}^0(A)$ the inequality:

$$\|\Phi_A^t\| < e^{t\delta} \text{ for all } t \geq h. \tag{7.7}$$

Suppose that we have a ceiling function over a section $B \subseteq Z_h$ satisfying $h(x) \geq h$. We denote by $\hat{Q}$ the Kakutani castle with base $B$. Excluding all towers with height above $3h$ we define a subcastle which we denote by $Q$.
We claim that $\mu(\hat{Q} - Q) < 3\delta^2\mu(\Gamma_m^*(A))$, as in [3] Lemma 4.2. Now we will decay the entropy function $LE(\cdot, \Gamma_m(A))$ at $A$, by a small perturbation $B = A + H$ of the system. We start with a $L^\infty$-perturbation and the idea for the continuous ones comes from noting that $H(\cdot)$ is measurable and therefore, by Lusin's theorem, we have that measurable functions are almost continuous and since we are only interested on almost all points in the base the same result will follow.
The next lemma give us a $L^\infty$ perturbation.

**Lemma 7.8.1** *Let $A$ be a conservative system and $\epsilon, \delta > 0$. Then, there exists $m \in \mathbb{N}$ and a traceless system $H \in L^\infty(X, GL(2,\mathbb{R}))$ such that:*

*(a)* $\|H\|_\infty < \epsilon$*;*

*(b)* $\|H(p)\| = 0$ *for any* $p \notin \Gamma_m(A)$*;*

*(c)* $LE(A + H, \Gamma_m(A)) < \delta$*.*

**Proof:** Suppose that $\mu(\Gamma_m(A)) > 0$, otherwise, there is nothing to prove.
The equality:

$$LE(A + H, \Gamma_m(A)) = \inf_{n\in\mathbb{N}} \frac{1}{n} \int_{\Gamma_m(A)} \log\|\Phi_{A+H}^n(p)\| d\mu(p)$$

will allow us to prove that $LE(A+H,\Gamma_m(A))$ is small by proving that

$$\frac{1}{t}\int_{\Gamma_m(A)}\log\|\Phi^t_{A+H}(p)\|d\mu(p)$$

is small for a large fixed $t=h\delta^{-1}$.

Note that those points that stay for a long time in $Q$ will necessarily have low contribution for $LE(A+H,\Gamma_m(A))$. So we define

$$G=\{p\in\Gamma^*_m(A):\varphi^s(p)\in Q,\forall s\in[0,t[\}$$

and we claim that:

$$\mu(\Gamma^*_m(A)-G)<15\delta \tag{7.8}$$

which is a consequence of Lemma 6.4.2. Note that since $t$ is large and the castle $Q$ has height bounded towers and large measure, the orbit leaves $Q$ often, but by (7.8), it is highly likely to enter $Q$ again. So we split the orbit segment $\varphi^{[0,t]}(p)$ for $p\in G$ by return-times to $B$, say $t=b+r_n+...+r_2+r_1+a$ where all $\varphi^a(p),\varphi^{r_1+a}(p),\varphi^{r_2+r_1+a}(p),...,\varphi^{\sum_{i=1}^n r_i+a}(p)$ are in the base $B$, and $a,b,r_i\in[0,3h[$. Given $q\in B$ the height of its tower $h(q)$ verifies $h(q)\geq h$, but $B\subseteq Z_h$ so $h(q)\geq h\geq T(q)$, therefore Lemma 7.7.2 says that for every $t>T(q)$ there exists a traceless $\{H(\varphi^s(q))\}_{s\in\mathbb{R}}$, varying smoothly with support on the segment $\varphi^{[0,m]}(p)$ such that:

(a) $\|H\|<\epsilon$ and

(b) $\frac{1}{t}\log\|\Phi^t_{A+H}(q)\|<\delta$.

Note that

$$\begin{aligned}\|\Phi^t_{A+H}(p)\| &= \|\Phi^{b+\sum_{i=1}^n r_i+a}_{A+H}(p)\|\leq\\ &\leq \|\Phi^b_{A+H}(\varphi^{\sum_{i=1}^n r_i+a}(p)))\|\cdot(...)\cdot\|\Phi^{r_1}_{A+H}(\varphi^a(p))\|\cdot\|\Phi^a_{A+H}(p)\|.\end{aligned}$$

Take $C=\sup_{p\in M}\|\Phi^1_{A+H}(p)\|$. By (b) and the fact that the towers are smaller than $3h$ we conclude that:

$$\|\Phi^t_{A+H}(p)\|\leq C^{3h}.e^{\sum_{i=1}^n r_i\delta}.C^{3h}\leq e^{(b+\sum_{i=1}^n r_i\delta+a)\delta}C^{6h}\leq e^{t\delta}.C^{6h}$$

and we get $\frac{1}{t}\log\|\Phi^t_{A+H}(p)\|\leq\delta(1+6\log C)$. Therefore, since

$$\Gamma_m(A)\supset\Gamma^+_m(A)\supset\Gamma^*_m(A)\supset G,$$

we obtain:

$$\begin{aligned} LE(A+H,\Gamma_m(A)) &= \int_{\Gamma_m(A)-\Gamma_m^+(A)} \frac{1}{t}\log\|\Phi^t_{A+H}(p)\|d\mu(p) + \\ &+ \int_{\Gamma_m^+(A)-\Gamma_m^*(A)} \frac{1}{t}\log\|\Phi^t_{A+H}(p)\|d\mu(p) + \\ &+ \int_{\Gamma_m^*(A)-G} \frac{1}{t}\log\|\Phi^t_{A+H}(p)\|d\mu(p) + \\ &+ \int_{G} \frac{1}{t}\log\|\Phi^t_{A+H}(p)\|d\mu(p). \end{aligned}$$

Now we use (7.7), (7.5), (7.8) and the fact that $\frac{1}{t}\log\|\Phi^t_{A+H}(p)\| \leq \delta(1+6\log C)$ in order to get:

$$LE(A+H,\Gamma_m(A)) \leq \delta + \delta\log C + 15\delta\log C + (1+6\log C)\delta.$$

Substituting $\delta$ by $\frac{\delta}{(2+22\log C)}$ along the proof we cause a decay on $LE(\cdot,\Gamma_m(A))$ by a $\epsilon$-small perturbation of the original system.

□

In the next lemma we will construct a $C^0$ perturbation.

**Lemma 7.8.2** *Given a continuous conservative system $A$ and $\epsilon,\delta>0$ there exists $m\in\mathbb{N}$ and a continuous traceless system $H_0$ such that $B=A+H_0$ verifies:*

*(a) $\|A-B\|_\infty<\epsilon$;*

*(b) $A(p)=B(p)$ for any $p\notin\Gamma_m(A)$;*

*(c) $LE(B,\Gamma_m(A))<\delta$.*

**Proof:** By Lemma 7.8.1 we obtain $m\in\mathbb{N}$ and a traceless $H\in L^\infty(X,GL(2,\mathbb{R}))$ such that for $t=h\delta^{-1}$ we have:

$$LE(A+H,\Gamma_m(A)) \leq \int_{\Gamma_m(A)} \frac{1}{t}\log\|\Phi^t_{A+H}(p)\|d\mu(p) < \delta.$$

We now use Lusin's theorem which states that for any measurable function, for instance $H$, there is $H_1\in C^0(X,GL(2,\mathbb{R}))$, such that:

(a) $H_1(p)=H(p)$ for any $p\notin\Gamma_m(A)$;

(b) $\|H_1\|_\infty<\epsilon$;

(c) $\mu(E)=\mu(\{p\in M : H_1(p)\neq H(p)\})<\delta t^{-1}$.

Since for points $p \in E$ we do not necessarily have $\text{Tr}H_1(p) = 0$ we change, say the entry $1-1$ of the matrix, obtaining a new matrix $H_0$ this time with $\text{Tr}H_0(p) = 0$. We define the $C^0$ perturbation $B = A + H_0$ which verifies $\text{Tr}A = \text{Tr}B$.

Now we define the sets

$$L = \{p \in X : H_0(p) = H(p)\} \text{ and}$$

$$G_L = \{p \in X : \varphi^s(p) \in \Gamma_m(A) \cap L, \forall s \in [0,t]\}.$$

Clearly $G_L \subseteq \Gamma_m(A)$ and we have:

$$\mu(\Gamma_m(A) - G_L) \leq t\mu(E) \leq \delta. \tag{7.9}$$

Therefore we conclude that,

$$\begin{aligned} LE(B, \Gamma_m(A)) &= \inf_{n\in\mathbb{N}} \int_{\Gamma_m(A)} \frac{1}{n}\log\|\Phi_B^t(p)\|d\mu(p) \leq \\ &\leq \int_{\Gamma_m(A)} \frac{1}{t}\log\|\Phi_B^t(p)\|d\mu(p) = \\ &= \int_{\Gamma_m(A)-G_L} \frac{1}{t}\log\|\Phi_B^t(p)\|d\mu(p) + \int_{G_L} \frac{1}{t}\log\|\Phi_B^t(p)\|d\mu(p) = \\ &= \int_{\Gamma_m(A)-G_L} \frac{1}{t}\log\|\Phi_B^t(p)\|d\mu(p) + \int_{G_L} \frac{1}{t}\log\|\Phi_{A+H}^t(p)\|d\mu(p). \end{aligned}$$

Let $C = \max_{p\in X}\|\Phi_B^1(p)\|$ and since $G_L \subseteq \Gamma_m(A)$ we get:

$$LE(B, \Gamma_m(A)) \leq \mu(\Gamma_m(A) - G_L)\log C + \int_{\Gamma_m(A)} \frac{1}{t}\log\|\Phi_{A+H}^t(p)\|d\mu(p).$$

Now by (7.9) and Lemma 7.8.1 we obtain:

$$LE(B, \Gamma_m(A)) \leq \delta\log C + \delta.$$

We then reconstruct the proof replacing $\delta$ by $\frac{\delta}{(1+\log C)}$.

□

## 7.9 End of the proof of Theorem 2

Denote by $\Gamma_\infty(A)$ the set $\bigcap_{m\in\mathbb{N}} \Gamma_m(A)$. The following lemma will be useful to prove Theorem 2.

**Lemma 7.9.1** *Given a continuous conservative system $A$ and $\epsilon, \delta > 0$, there exists $B$, $\epsilon$-close to $A$ such that:*

$$LE(B, X) < LE(A, X) - \int_{\Gamma_\infty(A)} \lambda^+(A, p) d\mu(p) + \delta.$$

**Proof:** By Lemma 7.8.2 there exists $m \in \mathbb{N}$ and a continuous conservative system $B$ such that:

(a) $\|A - B\|_\infty < \epsilon$;

(b) $A(p) = B(p)$ for any $p \notin \Gamma_m(A)$;

(c) $LE(B, \Gamma_m(A)) < \delta$.

So, we have:

$$\begin{aligned} LE(B, X) &= LE(B, \Gamma_m(A)) + LE(B, X - \Gamma_m(A)) = \\ &= LE(B, \Gamma_m(A)) + LE(A, X - \Gamma_m(A)) \leq \\ &\leq \delta + LE(A, X - \Gamma_\infty(A)) = \delta + LE(A, X) - LE(A, \Gamma_\infty(A)). \end{aligned}$$

□

**Theorem 7.9.2** *Given $A$ in the set of continuous conservative systems, we have that if $A$ is a continuity point of the entropy function $LE(\cdot)$, then for $\mu$-a.e. $p \in X$ the following dichotomy holds:*

*(a) Either the Oseledets splitting is dominated*

*(b) or Lyapunov exponents are zero.*

**Proof:** Take $A$ a continuous linear differential system (area-preserving or modified area-preserving). Suppose that $A$ is a continuity point for $LE(\cdot)$. Suppose $\mu(X - \bigcup_{m\in\mathbb{N}} \Lambda_m(A)) > 0$, otherwise the statement is proved.

So $\mu(X \cap \bigcap_{m\in\mathbb{N}} (\Gamma_m(A))) = \mu(\bigcap_{m\in\mathbb{N}} \Gamma_m(A)) > 0$, and therefore, $\mu(\Gamma_\infty(A)) > 0$.

Consequently we must have that $LE(A, \Gamma_\infty(A)) = 0$, otherwise by Lemma 7.9.1 we break the continuity and get a contradiction. So, for any $p \in \mathfrak{O}(A)$ we have zero Lyapunov exponents or if it has positive ones, then $p \notin \Gamma_\infty(A)$ and therefore it has $m$-dominated splitting for some $m \in \mathbb{N}$.

□

The conclusions of Theorem 7.9.2 are sufficient to guarantee that $A$ is a continuity point of the entropy function. By hypothesis, $X = D \cup O (\text{mod } 0)$, where $D$ are points with dominated splitting and $O$ are points with null exponents. Since $LE(A, O) = 0$ and $LE$ is upper semicontinuous we conclude that $LE(B,O)$ is close to $LE(A,O)$, for $B$ close to $A$. Moreover, $D$ has also dominated splitting for $B$ with rates of dominated splitting close to the ones belonging to $A$.

**Proof of Theorem 2:**

Theorem 2 now follows by using the fact that the set of points of continuity of upper semicontinuous functions is a residual set, see [9].

## 7.10 Some consequences of Theorem 2 - Ergodic flows

**Corollary 7.10.1** *If $\mu$ is ergodic, then there is a residual subset $\mathfrak{R}$ of area-preserving systems such that for every $A \in \mathfrak{R}$ we have $\Phi_A^t$ uniformly hyperbolic or Lyapunov exponents are zero for $\mu$-a.e. point $p \in X$.*

**Proof:** Take the residual $\mathfrak{R}$ given by Theorem 2 and $A \in \mathfrak{R}$. If $\mu(\Lambda_m) = 0$ for all $m$, then the theorem follows, otherwise if $\mu(\Lambda_m) > 0$ for some $m$, we have a full measure set $\Lambda_m$ with $m$-dominated splitting, because $\Lambda_m$ is $\varphi^t$-invariant and $\mu$ is ergodic.

Conservativeness yields $\det\Phi^t(p) = 1$ for all $t \in \mathbb{R}$ and $p \in M$. Given a $\mu$-generic point $p \in M$ we have:

$$sin(\angle(N_p^u, N_p^s)) = \|\Phi^t(p)|_{N_p^u}\|.\|\Phi^t(p)|_{N_p^s}\| sin(\angle(N_{\varphi^t(p)}^u, N_{\varphi^t(p)}^s)). \tag{7.10}$$

Claim 2.5.1 and (7.10) implies that:

$$\|\Phi^t(p)|_{N_p^u}\|.\|\Phi^t(p)|_{N_p^s}\| \geq \sin^{-1}\alpha.$$

Again by dominated splitting, there exists constants $C > 0$ and $\sigma \in (0,1)$ such that:

$$C\sigma^t \geq \frac{\|\Phi^t(p)|_{N_p^s}\|}{\|\Phi^t(p)|_{N_p^u}\|} \geq \sin^{-1}\alpha \|\Phi^{-t}(p)|_{N_p^u}\|^2,$$

and consequently

$$\|\Phi^{-t}(p)|_{N_p^u}\| \leq \sqrt{C\sin^{-1}\alpha}(\sigma^{t/2}).$$

Now it suffices to take the constants of uniform hyperbolicity $C' = \sqrt{C\sin^{-1}\alpha}$ and $\sigma' = \sqrt{\sigma}$ and to proceed analogously for $N^s$ in order to prove the corollary. □

**Corollary 7.10.2** *If $\mu$ is ergodic and $Fix(\varphi^t) = \emptyset$, then there is a residual subset $\mathfrak{R}$ of modified area-preserving systems such that for every $A \in \mathfrak{R}$ either $\Phi_A^t$ is uniformly hyperbolic or $LE(A) = 0$.*

**Proof:** Since $a(\cdot)$ is a non-null continuous function on a compact set $X$, the quotient $\frac{a(\cdot)}{a(\varphi^t(\cdot))}$ has an upperbound $K$ and a lowerbound $K^{-1}$. Conservativeness, dominated splitting and the nonexistence of fixed points for the flow $\varphi^t$ guarantees that:

$$C\sigma^t \geq \frac{\|\Phi^t(p)|_{N_p^s}\|}{\|\Phi^t(p)|_{N_p^u}\|} \geq \sin^{-1}\alpha K^2 \|\Phi^{-t}(p)|_{N_p^u}\|^2,$$

and we proceed as in Corollary 7.10.1. $\square$

# Chapter 8

# Dichotomy for vector fields with singularities

## 8.1 Global dichotomy under additional hypothesis

Consider the following hypothesis again in the 3-dimensional context:

**Hypothesis 8.1.1** *Let $\mathfrak{X}^2_\mu(M)$ and $\Lambda_m$ be a $X^t$-invariant set with $m$-dominated splitting for the linear Poincaré flow. Then $\mu(\overline{\Lambda_m}) = 0$ or $X$ is Anosov.*

Under this hypothesis we prove an analogous to Proposition 2.6.2. The measure of singularities may be neglected by using Lemma 7.5.1, so we do not need to make any perturbations on singularities. Moreover, when we estimate the $C^1$ norm of the perturbation $P$, defined in (3.5), the first column of the matrix was given by,

$$(0, -\xi\ddot{g}(x)G(\sqrt{y^2+z^2})c^{-1}z, \xi\ddot{g}(x)G(\sqrt{y^2+z^2})c^{-1}y).$$

Note that near singularities $c^{-1}$ is very large, however since the radius depends on the point $p$ we can decrease the radius $r(p)$ and control the $C^1$ norm. So the perturbations we developed work equally on this setting.

**Theorem 8.1.2** *Under Hypothesis 8.1.1 there exists a residual $\mathfrak{R} \subseteq \mathfrak{X}^1_\mu(M)$ such that if $X \in \mathfrak{R}$ then we have:*

*(a) $X$ is Anosov or*

*(b) Zero Lyapunov exponents for $\mu$-a.e. $p \in M$.*

## 8.2 Proof of Theorem 3

### 8.2.1 Adapting the proof of Theorem 1

If in Theorem 3.1.1 we take $\Omega$ with $C^\infty$ boundary and $g, f$ also $C^\infty$, the diffeomorphism $\varphi$, provided by Dacorogna-Moser, is also $C^\infty$. So our conservative flowbox theorem guarantee a conservative change of coordinates $\Psi \in C^\infty$. Note that the perturbation $P$, defined in (3.5), is also $C^\infty$, moreover we know by [16] (for other proof see Theorem 3.1 of [2]) that $\mathfrak{X}^\infty_\mu(M)$ is $C^1$-dense in $\mathfrak{X}^1_\mu(M)$. The following Proposition is similar to Proposition 2.6.2. The main difference is *where* the computation of the entropy function is done.

**Proposition 8.2.1** *Let $X \in \mathfrak{X}^\infty_\mu(M)$ and $\epsilon, \delta > 0$. There exists $m \in \mathbb{N}$ and a zero divergence $C^\infty$ vector field $Y$, $\epsilon$-$C^1$-close to $X$ that equals $X$ outside the open set $\Gamma_m(X)$ and such that $LE(Y, \Gamma_m(X)) < \delta$.*

**Proof:** We note that for a fixed $m \in \mathbb{N}$ we have $p \in \Gamma^+_m(X) - \Gamma^*_m(X)$ if $p$ is periodic, has positive Lyapunov exponent and belongs to $\Gamma_m(X)$. We consider again the following simple claim proved in Lemma 7.5.1.

**Claim 8.2.1** *For any $\delta > 0$, there exists $m \in \mathbb{N}$ such that we have $\mu(\Gamma^+_m(X) - \Gamma^*_m(X)) < \delta$.*

To find $m \in \mathbb{N}$ we first proceed like in Lemma 4.0.12, then we take $m \in \mathbb{N}$ sufficiently large to satisfy also Claim 8.2.1. Now we consider the measurable function $T : \Gamma^*_m(X) \to \mathbb{R}$ similar to the function of Lemma 5.0.15. We define $Z_h = \{p \in \Gamma^*_m(X) : T(p) \leq h\}$. Of course that $\mu(\Gamma^*_m(X) - Z_h) \underset{h\to\infty}{\longrightarrow} 0$ so we take $h$ sufficiently large to satisfy $\mu(\Gamma^*_m(X) - Z_h) < \delta^2 \mu(\Gamma^*_m(X))$. Now we increase $h$, if necessary, and use Oseledets's Theorem, which is an asymptotic result, to get for $p \in \mathfrak{O}^0(X)$ the inequality:

$$\|P^t_X(p)\| < e^{t\delta} \text{ for all } t \geq h. \tag{8.1}$$

Clearly $X^t : \Gamma^*_m(X) \to \Gamma^*_m(X)$ is an aperiodic flow. Now we follow the construction of section 6.3 and finally we compute $LE(Y, \Gamma^*_m(X))$. We define again the set of "good" points, $G := \{p \in \Gamma^*_m(X) : Y^s(p) \in K, \forall s \in [0, t]\}$. By Lemma 6.4.2 $\mu(U \cup \Gamma^*_m(X) - G) < 12\delta$. Define $A = A(p, t, Y) := \frac{1}{t}\log\|P^t_Y(p)\|$.

$$\begin{aligned} LE(Y, \Gamma_m(X)) &\leq \int_{\Gamma_m(X)} A d\mu(p) \leq \\ &\leq \int_{\Gamma_m(X) - (U \cup \Gamma^+_m(X))} A d\mu(p) + \int_{U \cup \Gamma^+_m(X) - G} A d\mu(p) + \int_G A d\mu(p). \end{aligned}$$

By (8.1) and since $Y = X$ outside $U$ we obtain,

$$\int_{\Gamma_m(X) - (U \cup \Gamma^+_m(X))} A(p, t, Y) d\mu(p) \leq \int_{\Gamma_m(X) - \Gamma^+_m(X)} A(p, t, X) d\mu(p) \leq \delta.$$

Since $C := \max\{\|P_X^1(p)\| : p \in M\}$ we use Claim 8.2.1 and Lemma 6.4.2 to conclude that, $\int_{U \cup \Gamma_m^+(X) - G} A d\mu(p) \leq 13\delta$. Finally at $G$ our construction allow us to obtain $\int_G A d\mu(p) \leq \delta$ and the proposition is proved. $\square$

### 8.2.2 End of the proof of Theorem 3

Let $\tilde{X} \in \mathfrak{X}_\mu^1(M)$ and $\tilde{\epsilon} > 0$ be given. We will prove that exists $Y \in \mathfrak{X}_\mu^1(M)$, $\tilde{\epsilon}$-$C^1$-close to $X$ satisfying the conclusions of Theorem 3. For $\epsilon = \tilde{\epsilon}/2$, there exists $X \in \mathfrak{X}_\mu^\infty(M)$ $\epsilon$-$C^1$-close to $\tilde{X}$. It suffices to prove Theorem 3 for the vector field $X$ and $\epsilon > 0$.

**Proof:** (of Theorem 3)
Let $X \in \mathfrak{X}_\mu^\infty(M)$ and $\epsilon > 0$. We will find $Y$ $\epsilon$-$C^1$-close to $X$ and a partition $M = D \cup O$ into $Y^t$-invariant sets such that:
(a) For $p \in O$ we have zero Lyapunov exponents.
(b) $D$ is a countable increasing union of compact invariant sets $\Lambda_{m_n}$ admitting a $m_n$-dominated splitting for the Linear Poincaré flow. We define the sequence $\{X_n\}_{n\geq 0} \in \mathfrak{X}_\mu^\infty(M)$, $m_n \in \mathbb{N}$ and eventually $\epsilon_n > 0$ for $n \geq 0$.
Take $X_0 = X$, $\theta > 1$ (near 1) and $\delta_n \underset{n\to 0}{\rightarrow} 0$.
If $\int_{\Gamma_m(X)} \lambda^+(X) d\mu = 0$ for some $m \in \mathbb{N}$, then we are finished by taking $Y = X$, $D = \Lambda_m(X)$ and $O$ a full measure subset of $\Gamma_m(X)$. Otherwise for some $m = m_0$ we have $\int_{\Gamma_{m_0}(X)} \lambda^+(X) d\mu > 0$. Let $\epsilon_0 \in (0, \epsilon/2)$ be sufficiently small such that:

$$\int_{\Gamma_{m_0}(X_0)} \lambda^+(Z) d\mu \leq \theta \int_{\Gamma_{m_0}(X_0)} \lambda^+(X_0) d\mu,$$

for all $Z$ $2\epsilon_0$-$C^1$-close of $X_0$ and $Z = X_0$ outside $\Gamma_{m_0}(X_0)$. $\epsilon_0$ always exists because $LE(\cdot, \Gamma_{m_0}(X_0))$ is upper semicontinuous and $\Gamma_{m_0}(X_0)$ is simultaneously invariant for $X_0^t$ and $Z^t$.
Knowing $X_0$, $m_0$ and $\epsilon_0$ we are going to define $X_1 \in \mathfrak{X}_\mu^\infty(M)$, $m_1 \in \mathbb{N}$ and eventually $\epsilon_1 > 0$.
By Proposition 8.2.1, there exists $m_1 \in \mathbb{N}$ and $X_1 \in \mathfrak{X}_\mu^\infty(M)$ a perturbation of $X_0$ $\epsilon_0$-$C^1$-close, with $X_1 = X_0$ outside $\Gamma_{m_1}(X_0)$ and such that:

$$\int_{\Gamma_{m_1}(X_0)} \lambda^+(X_1) < \delta_1.$$

Suppose that $m_1 \geq m_0$. Note that $\Gamma_{m_1}(X_1) \subseteq \Gamma_{m_0}(X_1) \subseteq \Gamma_{m_0}(X_0)$. If $\int_{\Gamma_{m_1}(X_1)} \lambda^+(X_1) = 0$, then we are finished by taking $Y = X_1$, $D = \Lambda_{m_1}(X_1)$ and $O$ a full measure subset of $\Gamma_{m_1}(X_1)$. Otherwise if $\int_{\Gamma_{m_1}(X_1)} \lambda^+(X_1) > 0$ we choose $\epsilon_1 \in (0, \epsilon_0/2)$ such that $B(X_1, 2\epsilon_1) \subseteq B(X_0, \epsilon_0)$ and also

$$\int_{\Gamma_{m_1}(X_1)} \lambda^+(Z) d\mu \leq \theta \int_{\Gamma_{m_1}(X_1)} \lambda^+(X_1) d\mu,$$

for all $Z$ $2\epsilon_1$-$C^1$-close of $X_1$ and $Z = X_1$ outside $\Gamma_{m_1}(X_1)$.
Recursively knowing $X_{n-1}$, $m_{n-1}$ and $\epsilon_{n-1} \in (0, \epsilon 2^{-n})$ we are going to define $X_n \in \mathfrak{X}^\infty_\mu(M)$, $m_n \in \mathbb{N}$ and eventually $\epsilon_n > 0$.
Again by Proposition 8.2.1, there exists $m_n \in \mathbb{N}$ and $X_n \in \mathfrak{X}^\infty_\mu(M)$ a perturbation of $X_{n-1}$ $\epsilon_{n-1}$-$C^1$-close, with $X_n = X_{n-1}$ outside $\Gamma_{m_{n-1}}(X_{n-1})$ and such that:

$$\int_{\Gamma_{m_n}(X_{n-1})} \lambda^+(X_n) < \delta_n.$$

Suppose that $m_n \geq m_{n-1}$. Now $\Gamma_{m_n}(X_n) \subseteq \Gamma_{m_{n-1}}(X_n) \subseteq \Gamma_{m_{n-1}}(X_{n-1})$. If $\int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n) = 0$, then we are finished by taking $Y = X_n$, $D = \Lambda_{m_n}(Y)$ and $O$ a full measure subset of $\Gamma_{m_n}(Y)$. Otherwise if $\int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n) > 0$ we choose $\epsilon_n \in (0, \epsilon_{n-1}/2)$ so that $B(X_n, 2\epsilon_n) \subseteq B(X_{n-1}, \epsilon_{n-1})$ and also

$$\int_{\Gamma_{m_n}(X_n)} \lambda^+(Z) d\mu \leq \theta \int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n) d\mu,$$

for all $Z$ $2\epsilon_n$-$C^1$-close of $X_n$ and $Z = X_n$ outside $\Gamma_{m_n}(X_n)$.
We continue this procedure and if for some $n \in \mathbb{N}$ we obtain $\int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n) = 0$ we are over, otherwise the sequence $\{X_n\}_{n\geq 0}$ converges $C^1$ to some $Y \in \mathfrak{X}^1_\mu(M)$, moreover since $\epsilon_n < \epsilon/2^n$ we have $Y$ $\epsilon$-$C^1$-close to $X$.
Let $D = \underset{n\in\mathbb{N}}{\cup} \Lambda_{m_n}(X_n)$. Since $\Lambda_{m_n}(X_n) \supseteq \Lambda_{m_{n-1}}(X_{n-1})$ and $Y = X_n$ at $\Lambda_{m_n}(X_n)$, $Y^t$ has $m_n$-dominated splitting at $\Lambda_{m_n}(X_n)$.
Let $\Gamma := [\underset{n\in\mathbb{N}}{\cup} \Lambda_{m_n}(X_n)]^c = \underset{n\in\mathbb{N}}{\cap} \Gamma_{m_n}(X_n)$, clearly $\Gamma \subseteq \Gamma_{m_n}(X_n)$. To finish the proof of Theorem 3 we must see if $\int_\Gamma \lambda^+(Y) d\mu = 0$.
Note that $Y \in B(X_n, 2\epsilon_n)$ for all $n \in \mathbb{N}$. So we have

$$\int_\Gamma \lambda^+(Y) d\mu < \int_{\Gamma_{m_n}(X_n)} \lambda^+(Y) d\mu \leq \theta \int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n) d\mu = \theta\delta_n \underset{n\to\infty}{\rightarrow} 0.$$

We conclude that we have zero Lyapunov exponents in a full measure subset $O$ of $\Gamma$ and Theorem 3 is proved. $\square$

Finally we consider the reason why Theorem 3 is stated for dense subset instead of a residual subset? In [4], we find a strategy developed to obtain a residual subset, unfortunately this are not applied to our case, let us see why. They start with a $C^1$ system which is a continuity point $X$ of the function $LE(\cdot, X)$. Then they define the "jump" of the function at $X$ by $LE(X, \Gamma_\infty(X))$ where $\Gamma_\infty(X) := \underset{m\in\mathbb{N}}{\cap} \Gamma_m(X)$. Of course that being a continuity point implies that the "jump" is zero. So $\mu(\Gamma_\infty(X)) = 0$ or $\lambda^+(p) = 0$ for $\mu$-a.e. point $p \in \Gamma_\infty(X)$ and the statements of Theorem 3 are verified. Note that to estimate a lower bound for the "jump" we perturb the original vector field $X$ like we did to prove Theorem 3. But our conservative flowbox theorem may not be applied to $X$, unless $X$ is of class $C^2$, so this argument only works for $X \in \mathfrak{X}^2_\mu(M)$.

However this set equipped with $C^1$ topology is not a Baire space, so in general residual sets are meaningless.

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