Riemann solutions of balance systems with phase change for thermal flow in porous media.

Adviser: Dan Marchesin
Co-adviser: Johannes Bruining
Student: Wanderson José Lambert

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Contents

Às duas mulheres da minha vida na ordem em que apareceram: Ivete e Milene.

Abstract

This work encompasses two types of results. First we present a new general theory which deals with Riemann solution for a large class of balance equations. This class of equations has interest because it can be applied to model thermal flow with mass interchange between phases in porous media, with important applications in oil recovery.

As applications of this theory, we present three models of steam injection in a horizontal porous media. The systems of equations are based on mass balance, energy conservation and Darcy law of force. We neglect compressibility, heat conductivity and capillarity effects.

In first example we consider steam/water/oil flow in porous medium. We only prepare the formalism for this class of equations.

In second example we consider steam/water/nitrogen flow into a porous medium. We develop the general theory for a 4×4 system of balance equations. We solve the Riemann problem with application to recovery of geothermal energy. A rarefaction evaporation wave is observed.

In third example we consider the steam/water injection into a porous medium. We completely solve the Riemann problem associated with this model. We obtain a rich class of bifurcations in the solutions. A new type of shock, the evaporation shock, is identified in the Riemann solution.

Keywords: Porous medium, steamdrive, Riemann solution, balance equations, multiphase flow, distillation

Resumo

Este trabalho abrange dois tipos de resultados. Primeiramente nós apresentamos uma nova teoria geral que trata da solução de Riemann para uma grande classe de equações de balanço. Esta classe de equações tem interesse porque pode ser aplicada ao modelamento de fluxos térmicos em meios porosos com transferência de massa entre fases, com importantes aplicações na recuperação de óleo.

Como aplicações desta teoria, nós apresentamos trŝ modelos de injeção de vapor em meios porosos horizontais. Os sistemas de equações são baseados no balanço de massa, na conservação de energia e na lei de Darcy da força. Nós negligenciamos a compressibilidade, a condutividade do calor e efeitos de capilaridade.

No primeiro modelo, nós consideramos fluxo de vapor/água/óleo no meio poroso. Nós somente preparamos o formalismo para esta classe de equações para resolvermos o problema de Riemann como um trabalho futuro.

No segundo modelo, nós consideramos fluxo de vapor/água/nitrogênio num meio poroso. Nós desenvolvemos uma teoria geral para um sistema 4×4 de equações de balanço. Resolvemos o problema de Riemann com aplicação à recuperação de energia geotérmica. Uma onda de rarefação de evaporação é observada.

No terceiro exemplo, consideramos a injeção de vapor/água em um meio poroso. Nós resolvemos completamente o problema de Riemann associado com este modelo. Obtemos uma rica classe de bifurcações para as soluções. Um novo tipo de choque, o choque da evaporação, identificado na solução de Riemann.

Palavras-chaves: meios porosos, injeção de vapor, solução de Riemann, equações de balanço, fluxo multifásico, destilação.

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CHAPTER 1

Introduction

A particular class of balance equations are the *conservation laws*. The physical interpretation of these laws is that for any limited spatial domain $\mathbb{D} \subset \mathbb{R}^d$ the change the total amount of any quantity (for example, mass, momentum, energy) is due exclusively to the flow of this quantity through the boundary of the domain.

In the Mathematics literature, a system of conservation laws in one spatial dimension is usually defined as:

$$U_t + F_x = 0, \tag{1.1}$$

where $U \in \Omega \subset \mathbb{R}^m$ represents a state vector field of dependent variables. The function $F := F(U) := (F_1(U), F_2(U), \cdots, F_n(U))^T : \Omega \longrightarrow \mathbb{R}^m$ is the flux of such quantities, generically, it is a $\mathcal{C}^2$ function. The derivatives $U_t = \partial U/\partial t$ and $F_x(U) = \partial F/\partial x$ are interpreted in the distribution sense, so that (1.1) is regarded to be in the weak form; this is important because of the presence of shocks.

An important problem associated to Eq. (1.1) appeared in the work of Riemann in the XIX century. He was interested in the evolution of two gases with different pressures separated by a thin membrane within a tube. Suddenly, the membrane is broken. He analyzed the evolution of the gas flow. In this model he observed rarefaction waves and discontinuous solutions, which are called shocks nowadays. This work originated the study of an important class of Cauchy problems, the so called Riemann problem, governed by equations of type (1.1) with initial data of the form:

$$U(x, t = 0) := \begin{cases} U_0, & \text{if} \quad x < 0 \\ U_1, & \text{if} \quad x > 0 \end{cases}. \tag{1.2}$$

Riemann problems have the important property that solutions are invariant under the scaling $x \to ax$, $t \to at$, for $a > 0$. For non-linear problems (1.1), they play the role of fundamental solutions, in a way analogous to sines and cosines for linear problems.

Conservation laws have been used to model a variety of physical phenomena (such as fluid dynamics, flow in porous medium, road traffic, electromagnetism, etc...) and therefore the theory for this class of equations is well developed, see for example [60, 20, 59, 45] and for numerical methods [46, 36, 34].

In most applications it is useful to utilize a more general form of conservation laws:

$$G_t + F_x = 0, \tag{1.3}$$

where $G := G(U) := (G_1(U), G_2(U), \cdots, G_n(U))^T : \Omega \longrightarrow \mathbb{R}^m$ also are $\mathcal{C}^2$. G is the accumulation term; the equations are interpreted in the distribution sense.

In classical models of flow in porous medium, the system of equations can be written in the form, see [12]:

$$G_t + (uF)_x = 0, \tag{1.4}$$

where u is the total or Darcy speed. However, incompressibility is often assumed in these models, which implies that u is constant spatially. As a result, in such flows (1.4) reduces to the form (1.3).

In many problems, such as transport of hot fluids and gases undergoing mass gain, loss or transfer, flows involving chemical reactions such as combustion balance laws generalizing (1.3) are required to describe the flow. The classical form of balance laws is:

$$G_t + F_x = Q. \tag{1.5}$$

Here $Q := Q(U) := (Q_1(U), Q_2(U), \cdots, Q_m(U))^T : \Omega \longrightarrow \mathbb{R}^m$ are $\mathcal{C}^2$ functions that represent sources, sinks or transference of the physical quantities. Although there are many applications for this class of equations, its theory is not well consolidated.

In our work, we are interested in models for flow in porous media, for such flows Q represents mass transfer of chemical components between phases with no net gain or loss of component mass, as well as conservation of total energy. The variable u representing the Darcy speed also appears, but only in particular way within the flux term:

$$\mathcal{G}_t + (u\mathcal{F})_x = Q. \tag{1.6}$$

The variables in (1.6) are $\mathcal{V} \in \Omega \subset \mathbb{R}^m$ and $u \in \mathcal{R}$. In this model, the variable u is not constant, generically.

System of type (1.6) model thermal compositional flows in porous media. The variable u is called speed because this is its interpretation in many applications; the pair $(\mathcal{V}, u)$ in $\mathbb{R}^{n+1}$ is called *state variable*. $\mathcal{G}$ and $\mathcal{F}$ are the vector-valued functions $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{m+1})^T$: $\Omega \longrightarrow \mathbb{R}^{m+1}$ and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_{m+1})^T$: $\Omega \longrightarrow \mathbb{R}^{m+1}$, where $u\mathcal{F}_i$ is the flux for the conserved quantity $\mathcal{G}_i$ and $\partial\mathcal{G}_i/\partial t$ is the corresponding accumulation term, for $i = 1, 2, \cdots, m+1$. On the right hand side in $Q = (Q_1, Q_2, \cdots, Q_{m+1})^T$: $\Omega \longrightarrow \mathbb{R}^{m+1}$, the first m terms represent mass transfer and Q_{m+1} represents energy conservation. Generically for a thermally isolated system, the first m equations represent mass balance for different chemical components in different phases and the $(m+1)$-*th* equation represents the conservation of total energy, which usually can be represented by setting $Q_{m+1} = 0$. The functions $\mathcal{G}$, $\mathcal{F}$ and Q are continuous in the whole domain Ω, but later we will see that they are only piecewise $\mathcal{C}^2$. The equations are supplemented by relationships expressing local thermodynamic equilibrium in each of the $\mathcal{C}^2$ parts.

The solution $\mathcal{V}(x,t)$ and $u(x,t)$ for $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$ needs to be determined. The terms Q often generate fast variations in the solution due thermodynamic processes; this is another reason to regard the solution as distributions and (1.6) must be taken in the weak sense. Despite the fact that the variable u does not appear in the accumulation term, but only in the flux term and despite the presence of the transfer term Q formally breaking scale invariance, we are able to solve the complete Riemann problem associated to Eq. (1.6). In fact, a general theory to deal this class of equations is proposed in this work, extending recent works of Bruining et. al. [5, 6, 8, 9, 10, 11]. This theory encompasses and generalizes the bifurcation phenomena for systems that change type discovered over the last two decades, see [24, 25, 26].

In Chapter 2, we define the physical situations, in which one or several phases or chemical components are missing, determining the expression of the prevailing local thermodynamic equilibrium. In each of these physical situations, by combining the original equations the balance system (1.6) reduces to simpler systems of conservation laws with fewer equations of the form:

$$\frac{\partial}{\partial t}G(V) + \frac{\partial}{\partial x}uF(V) = 0, \tag{1.7}$$

where the transfer terms Q has cancelled out. In a physical situation governed by (1.7), the corresponding set of variables V is a subset of the set of variables $\mathcal{V}$, while F and G are obtained from $\mathcal{F}$ and $\mathcal{G}$; they have $n+1$ components, $n \leq m$. The physical domain in each physical situation is denoted by Ω^j for $j = I, II, III, \cdots$ and it is a subset of Ω. There are three groups of variables: the basic variables V, called "primary variables"; the variable u, called "secondary variable" because it is obtained from the primary variables; and the "trivial variables" are constant or can be recovered from other variables in a simple way. Notice that the numbers of primary, secondary and trivial variables always add up to $m+1$.

In [15], Colombo et. al studied a problem with phase transitions in 2×2 systems of type (1.1). They considered that the physical domain is formed by two disjoint sub-domains $\mathcal{X}_1$ and $\mathcal{X}_2$. The domains are physical situations, which Colombo called phases. The phase transition is the jump in the solution with left and right states belonging to different phases. Our model is physically more adequate because it includes also infinitesimally small phase transitions, so that $\mathcal{X}_1$ adjoins $\mathcal{X}_2$, as they often are do in actual physics.

The class of equations studied in this work was motivated by practical problems of steam injection in porous medium. Mathematical models for steam injection can be found in the works of Coats [13], Marx et. al. [50], Lake et. al. [23] and Pope et. al. [54]. The rigorous analysis we are pursuing started in the works of Bruining et. al. [5, 6, 8, 9, 10, 11]. Following these initial ideas, we focus our interest in Riemann problems for variants of steam injection. Techniques based on steam injection have much interest because they are used in oil industry. For half a century, direct of cyclic steam injection have been applied as a method for enhancing recovery of oil, mainly for heavier oil. Heavy oil at low temperatures is highly viscous and so recovery is very difficult. Steam injection is used to increase the oil temperature, improving recovery both because the oil viscosity is lowered and because the volatility of light oil is increased, concentrating it in certain regions and allowing it to act as a liquid solvent that displaces the heavy oil, see [4] and [11].

Steam flood is part of the so called *thermal methods*, see [55]. Thermal methods are divided into two large groups. One group encompasses the techniques in which heat is injected into the porous medium, a category encompassing steam injection. In other group heat is generated locally in the porous medium, usually by means of combustion. In oil recovery, combustion utilizes a small part of the oil in the reservoir. Air is injected and the oxygen participates in the chemical reactions. As steam injection, combustion phenomena are modelled by balance equations, see [1], and the works of Coats [14], Crookston et. al. [16]. Modern theory of conservation laws was used for the study of combustion by Mota et. al. [27, 28, 29, 30, 31, 32], focusing on the study of the internal structures of combustion front which is a classical problem in engineering, see Souza et. al. [62].

In Chapter 3, we present the mathematical and physical model for flow of water, steam, volatile and dead oil. Dead oil is a heavy oil with negligible vapor pressure while volatile oil is a light oil with non-negligible vapor pressure. A partial Riemann solution of this problem was found by Bruining and Marchesin [11].

In the last decades, steam injection has been adapted for clean-up of organic contaminants from the subsurface, the so called groundwater remediation. The organic pollutants are referred to as non-aqueous phase liquids; they are divided into two large groups: DNAPL and LNAPL. The DNAPL's are denser than water; examples of DNAPL are chlorinated solvents. It is very difficult to decontaminate soils containing these substances because they can enter deeply underground. The LNAPL's are the substances less dense than water, such as gasoline and oil.

Traditional remediation of contaminated sites consists of groundwater extraction and clean up of the drained liquid; this technique is called pump-and-treat, see [21], [49] and references therein. The material is treated and discharged in another place. This method is very expensive and time-consuming, for example, in the Visalia test site, in California, polluted with creosote (a NAPL fluid), the estimated costs for pumping and treating are $US\$26,000$ per gallon of creosote and the expected time for this procedure is 3250 years. It is inefficient to clean up sites contaminated with DNAPL's (dense non aqueous phase liquids). So removal of contaminants with steam is considered as an alternative; for the same site in Visalia, the estimated cost per gallon of creosote removed with steam is $US\$130$ and the time to clean up is approximately 3 years, see [17].

In [18, 19], Davis described the mechanism of steam injection. For clean up, steam is injected in the soil by using wells. Initially the steam heats the formation around the wells. The steam condenses as the latent heat of vaporization of water is transferred from the steam to the region around the wells. As more steam is injected, the hot water moves into the formation, pushing the water initially present in the rock, which is at the ambient temperature. When the porous medium at the injection region has absorbed enough heat to reach the temperature of the injected steam, steam itself actually enters the medium, pushing the cold water and the bank of condensed steam in front of it. When these hot liquids reach the region that contains the volatile contaminant, the contaminant is displaced. Since the temperature of the volatile contaminant increases part of this contaminant evaporates originating a gaseous flow with organic components.

Notice that in this model there appear several physical situations and that there is mass transfer between the phases. The total energy is conserved, a fact that is modelled by using the conservation law for energy. The conservation of mass in the flow together with Darcy's law for porous media have motivated the introduction of equations of type (1.6). In chapter 2, we propose a general formalism for solving this class of equations. We consider the different physical situations as simple connected subsets state of $\mathbb{R}^m$. Physical situations are adjacent in state space. There exists a linear map E that takes the system of balance laws (1.6) into a system of conservation laws (1.7) associated to each physical situation. After this elimination, further simplifications using thermodynamical constraints may occur in each physical situation is described by a set of variables V that is a subset of the set of variables $\mathcal{V}$. Of course, each of the systems of type (1.7) has fewer equations than the complete system (1.6).

We are interested in the Riemann-Goursat problem where $x = 0$ is the injection point and the data in (1.2) for $x < 0$ is replaced by the injection boundary data at $x = 0$. It is well known that the Eqs. (1.1), (1.3) with initial data of (1.2) exhibits solutions that are constant along lines x/t, i.e., self-similar solutions. By physical considerations, we assume that in our model rarefaction waves occur only within each physical situation, where Eq. (1.6) reduces to equations of type (1.7). Since there is fast transfer of mass in the thin space between the physical situations, we propose shocks linking them. In Sections [2.2] and 2.3, we prove that the solution states are constant along the lines x/t. We also show that we can obtain the rarefaction and shock waves first in the space of primary variables, and that the secondary variable u can be recovered in terms of the primary variables.

In Proposition 2.3.1 we summarize the theory showing that the Riemann problem can be completely obtained in the space of primary variables. Moreover, if a sequence of waves and states solves the Riemann problem in the primary variables, for a given $u_L > 0$ (or $u_R > 0$), then it is also a solution for any other $u_L > 0$ (or $u_R > 0$), i. e., if the Darcy speed u_L in the initial data is modified while $\mathcal{V}_L$ and $\mathcal{V}_R$ are kept fixed, the Darcy speeds along the Riemann solutions are rescaled in the (x, t) plane, while the wave sequences and the Riemann solution in $\mathcal{V}(x, t)$ are kept unaltered. As tools, we prove three generalization of the Triple Shock Rule [25]. These rules are important to compare the speed of shocks of different families and obtain the Riemann solution.

In Section 2.3.3 we generalize the Bethe-Wendroff theorem for shocks between physical situations for equations of type (1.6). As in the classical case, extrema of shock speeds are related to shock/rarefaction resonance. This theorem is very important for the identification of bifurcation loci in the solution. These structures are important because the Riemann problem changes when the left (or right) data is prescribed in different regions in the physical domain relatively to these structures.

In Section 2.4, we define the notion of *regular Rankine-Hugoniot (RH) locus*, which are curves that parametrize shocks. For the class of equations (1.6), we define the notion of strictly hyperbolic physical situation and show a generalization of the Lax theorem for shocks within a physical situation, Proposition 2.4.1. In Section 2.4.3, we prove that the speed u does not change sign in connected branches of the RH locus.

In Appendix C, we show some applications of the general theory. In Appendix C.1, we prove that the formalism for the class of equations (1.6) can be applied to the simpler classical equation (1.3). In Appendix C.2, we obtain an efficient quasi-implicity numerical scheme for equations (1.7) without knowing the variable u. In Appendix C.3, we show that the viscous profile for equations (1.6) can be studied only in the space of primary variables V, without knowing the speed u.

In Chapter 3, we present a physical model for steam and gaseous volatile oil injection into a horizontal, linear porous medium filled with oil and water. We neglect compressibility, heat conductivity and capillarity effects and present a physical model for oil/water/steam injection is based on the mass balance and energy conservation equations as well as on Darcy's. We present the main physical definitions and summarize the equations of the model. This system of equations is a representative example of flow in porous media with mass interchange between phases (in the absence of gravitational effects) Eq. (1.6). The Riemann solution was partially solved in [11]. Our purpose is to show how this model fits within the formalism.

In Chapter 4, using the model proposed by Bruining et. al. [6], we formulate a system that describes the flow of nitrogen, steam and water in a porous medium. The physical model for nitrogen/steam/water injection is based on the mass balance and energy conservation equations. We describe the formalism for solving Riemann problem associated to the a 4×4 system of balance equations. As example of Riemann solution, we show an application for the recovery of geothermal energy. Injecting water and nitrogen into a hot porous rock, the water in the mixture evaporates and rock thermal heat can be recovered even if the rock is not very hot. A new fact is the presence of a rarefaction wave associated to evaporation in the two-phase situation.

In Chapter 5, we obtain the solutions for the basic one-dimensional profiles that appear in the clean up problem or in recovery of geothermal energy. We consider the injection of a mixture of steam and water in several proportions into a porous rock filled with a different mixture of water and steam. We describe completely all possible solutions of the Riemann problem. We find several types of shock between regions and a rich structure of bifurcations. A new type of shock, the evaporation shock, is identified in the Riemann solution. This model generalizes the work of Bruining et. al. [6], where the condensation shock appeared.

Each chapter can be read separately in this thesis. We put in each chapter a brief introduction explaining the content of each chapter, so that they can be read separately in this thesis. More precisely, Chapter 3 requires Chapter 2. Chapters 4 and 5 can be read on their own.

Much remains to be studied for models of type (1.6) as discussed in the conclusion, Chapter 7.

CHAPTER 2

General Formulation

Mass interchange between different phases occurs typically in multiphase flows in porous media. Such flows are not represented by conservation laws or hyperbolic equations both because they have source terms and because they are not evolutionary in all the variables. In the absence of gravitational effects, such flows can be modelled by systems of $m+1$ equations:

$$\frac{\partial}{\partial t}\mathcal{G}(\mathcal{V}) + \frac{\partial}{\partial x}u\mathcal{F}(\mathcal{V}) = \mathcal{Q}(\mathcal{V}), \tag{2.1}$$

where $\mathcal{V} \in \Omega \subset \mathbb{R}^m$ and $u \in \mathcal{R}$. The variable u is called speed because this is its interpretation in many applications; the pair $(\mathcal{V}, u)$ in $\mathbb{R}^{n+1}$ is called *state variable*. $\mathcal{G}$ and $\mathcal{F}$ represent the vector-valued functions $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{m+1})^T$: $\Omega \longrightarrow \mathbb{R}^{m+1}$ and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_{m+1})^T$: $\Omega \longrightarrow \mathbb{R}^{m+1}$, where $u\mathcal{F}_i$ is the flux for the conserved quantity $\mathcal{G}_i$ and $\partial\mathcal{G}_i/\partial t$ is the corresponding accumulation term, for $i = 1, 2, \cdots, m+1$. On the right hand side $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \cdots, \mathcal{Q}_{m+1})^T$: $\Omega \longrightarrow \mathbb{R}^{m+1}$, the first m components represent mass transfer and $\mathcal{Q}_{m+1}$ represents energy transfer due to phase change. Generically for a thermally isolated system, the m first equations represent mass balance for different chemical components in different phases and the $(m+1)$-*th* equation represents the conservation of total energy, which usually can be represented by setting $\mathcal{Q}_{m+1} = 0$. The functions $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{Q}$ are continuous in the whole domain Ω, but later we will see that they are only piecewise smooth.

The solution $\mathcal{V}(x,t)$ and $u(x,t)$ for $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$ needs to be determined. The terms $\mathcal{Q}$ often generate fast variations in the solution, so that in such cases the solution may be regarded as distributions and the equation (2.1) must be taken in the weak sense. This equation has an important feature: the variable u does not appear in the accumulation term, but only in the flux term.

Generically, the equations (2.1) model flows where there are phase changes giving rise to mass exchange between different phases. Such flows encompass particular physical situations, where one or several phases or chemical components are missing. The balance system (2.1) reduces in each particular physical situation to simpler systems of conservation laws with fewer equations of the form:

$$\frac{\partial}{\partial t}G(V) + \frac{\partial}{\partial x}uF(V) = 0, \tag{2.2}$$

where the source term Q is absent. In each physical situation, the corresponding set of variables V is a subset of the set of variables $\mathcal{V}$; F and G are obtained from $\mathcal{F}$ and $\mathcal{G}$; they have $n+1$ components, $n \leq m$.

Typically, the laws of thermodynamics play a central role in models (2.1). Each physical situation where the system of balance equations (2.1) reduces to a system of type (2.2) is actually a *physical situation under local thermodynamical equilibrium or quasi-equilibrium* and the system (2.2) is a *system under local thermodynamical equilibrium or quasi-equilibrium*: this equilibrium is enforced by thermodynamic relationships between the quantities in V. There are three groups of variables in each physical situation. Generically, when physical changes occur in waves under thermodynamical equilibrium, the solution of the system (2.1) and (2.2) are the same; however, when there is only local thermodynamical equilibrium, the solution of (2.1) is an approximation of the (2.2). In a future work, we intend to understand how deviations in thermodynamical equilibrium interfere in the approximation of (2.1). The basic variables V, called "primary variables"; the variable u, called "secondary variable" because it is obtained from the primary variables; and the "trivial variables" are constant or they can be recovered from other variables in a simple way. Notice that the number of primary, secondary and trivial variables always add up to $m+1$.

2.1 Physical situations

We define the *physical situation* Ω^j as a subset of Ω that describes the *j-th* physical situation. The Roman number index $I, II, III, \cdots$ indexes the physical situation; we represent the set of indices by $\mathcal{PS} = \{I, II, III, \cdots\}$. We denote the points of Ω^j by $\mathcal{V}^j$, see Examples 2.1.1 and 2.1.2 below.

The sets Ω^j satisfy:

$$\Omega^j \subset \Omega \qquad \text{and} \qquad \Omega = \bigcup_j \Omega^j. \tag{2.3}$$

The sets $\mathcal{P}^j$ *of primary state variables* are obtained from Ω^j, one subset for each physical situation. The primary variables correspond to the unknowns V^j of a system of conservation laws of the form (2.2). The number of primary variables in $\mathcal{P}^j$ is indicated by $n(j)$. The primary variables in each $\mathcal{P}^j$ are denoted by V^j following the notation in (2.2) to indicate that they depend on the physical situation j. For each physical situation, there are accumulation and flux functions G and F satisfying an equation of the form (2.2); G and F are obtained from $\mathcal{G}$ and $\mathcal{F}$. To indicate that

(G, F) depends also on j, we utilize the index j, so the functions are written as $G^j = (G_1^j, G_2^j, \cdots, G_{n(j)+1}^j)^T$ and $F^j = (F_1^j, F_2^j, \cdots, F_{n(j)+1}^j)^T$.

We denote the *trivial variables* by $\mathbb{V}^j$ and the space of trivial variables by $\mathcal{P}_j$.

Notice that in each physical situation Ω^j, the trivial variables are uniquely determined by the primary variables or solely by the physical situation, so there is a one-to-one correspondence between $\mathcal{P}^j$ and Ω^j, which is denoted by i_j; its inverse i_j^- is defined uniquely, which is the restriction of Ω^j to $\mathcal{P}^j$.

Consider a point $\mathcal{V}^j \in \Omega$ with primary coordinates $V^j \in \mathcal{P}^j$, the restriction i_j satisfies:

$$i_j : \Omega^j \longrightarrow \mathcal{P}^j, \qquad i_j(\mathcal{V}^j) = V^j, . \tag{2.4}$$

Notice that this restriction is a projection of $\mathcal{V}^j$ into $\mathcal{P}^j$.

The reverse is a map from $\mathcal{P}^j$ to $\mathcal{V}^j$, which is:

$$i_j^- : \mathcal{P}^j \longrightarrow \Omega^j, \qquad i_j^-(V^j) = \mathcal{V}^j. \tag{2.5}$$

We define the restriction of $\mathcal{G}$ and $\mathcal{F}$ to Ω^j by $\mathcal{R}_G^j : \Omega^j \longrightarrow \mathbb{R}^{m+1}$ and $\mathcal{R}_F^j : \Omega^j \longrightarrow \mathbb{R}^{m+1}$, respectively. For $\mathcal{V}^j \in \Omega^j$, they satisfy:

$$\mathcal{R}_G^j(\mathcal{V}^j) = \mathcal{G}(\mathcal{V}^j) \qquad \text{and} \qquad \mathcal{R}_F^j(\mathcal{V}^j) = \mathcal{F}(\mathcal{V}^j). \tag{2.6}$$

The original $\mathcal{G}$ and $\mathcal{F}$ are continuous in Ω, but they are not differentiable. However we assume that the restrictions are smooth, specifically, they belong to $\mathcal{C}^2(\Omega^j, \mathbb{R}^{m+1})$. We will see below that the functions (G^j, F^j) for the j-th conservation system can be obtained as a linear combination of $\mathcal{R}_G^j$ and $\mathcal{R}_F^j$, so the pairs (G^j, F^j) belong to $\mathcal{C}^2(\Omega^j, \mathbb{R}^{m+1})$.

Because we are solving problems in one spatial dimension, certain pairs of physical situations may occur contiguously. Let Ω^j and Ω^k with $\mathcal{P}^j$ and $\mathcal{P}^k$ represent the primary variables in one of these pairs, with Ω^j on the left of Ω^k in (x, t) space; we use $(-)$ for j and $(+)$ for k. Whenever there is a non-empty intersection between a pair Ω^j and Ω^k, we denote it by Ω^{jk}, i. e., $\Omega^{jk} = \Omega^j \bigcap \Omega^k$. The elements of Ω^{jk} are denoted by $\mathcal{V}^{jk}$.

In the space of primary variables, this intersection is denoted by Π^{jk} and is given by:

$$\Pi^{jk} = i_j(\Omega^{jk}) = i_k(\Omega^{jk}).$$

Definition 2.1.1. *Let $\mathcal{V} \in \bigcup_k \Omega^k$. We say that $\mathcal{V}$ is a* ***private point*** *of some Ω^j for $j \in \mathcal{PS}$, if there exists no other $l \in \mathcal{PS}$, such that $\mathcal{V} \in \Omega^l$.*

Definition 2.1.2. *Let $\mathcal{V} \in \bigcup_k \Omega^k$. We say that $\mathcal{V}$ is a* ***frontier point*** *of some Ω^j for $j \in \mathcal{PS}$, if there exists at least another $l \in \mathcal{PS}$, such that $\mathcal{V} \in \Omega^l$.*

Remark 2.1.1. *Notice that if $\mathcal{V}$ is a private point, then we can perform the projection in $\mathcal{P}^j$ for some $j \in \mathcal{PS}$, i.e., there exists a $V \in \mathcal{P}^j$, such that $V = i_j(\mathcal{V})$; we call such V too private point. If $\mathcal{V}$ is a frontier point, then in the applications it can belong to different physical situations depending on the necessity, but there always exists at least a $j \in \mathcal{PS}$ and a $V \in \mathcal{P}^j$ such that $V = i_j(\mathcal{V}^j)$; we call such V too frontier point.*

From the continuity of the accumulation and flux functions $\mathcal{F}$ and $\mathcal{G}$ in Ω we obtain:

Lemma 2.1.1. *For all $\mathcal{V}^{jk} \in \Omega^{jk}$, $\mathcal{R}^j_G(\mathcal{V}^{jk}) = \mathcal{R}^k_G(\mathcal{V}^{jk})$ and $\mathcal{R}^j_F(\mathcal{V}^{jk}) = \mathcal{R}^k_F(\mathcal{V}^{jk})$.*

It is important to define the notion of open sets in the topology induced by Ω^j and $\mathcal{P}^j$ relatively to $\mathbb{R}^{n(j)}$. Generically we are interested in solving the system of equations to obtain the primary variables in each physical situation, so it is useful to define the induced topology of Ω^j from the induced topology of $\mathcal{P}^j$ and the inverse function i_j^-, given by (2.5). For each $P \in \mathbb{R}^{n(j)}$ and $\epsilon > 0$, the open balls are the sets of $x \in \mathbb{R}^{n(j)}$ that satisfy $||x - P||_{n(j)} < \epsilon$, denoted by $\mathcal{B}_\epsilon(P)$. The norm $||\cdot||$ is the euclidian norm of $\mathbb{R}^m$, with $m > n(j)$ and $||\cdot||_{n(j)}$ is the induced norm in $R^{n(j)}$.

Definition 2.1.3. *The* ***euclidian norm in the induced topology*** *Ω^j or $\mathcal{P}^j$ is the restriction of $||\cdot||$ in each space, which are denoted by $||\cdot||_{\Omega^j}$ and $||\cdot||_{\mathcal{P}^j}$.*

Definition 2.1.4. *Let $V \in \mathcal{P}^j$, the* ***open balls in the induced topology*** *$\mathcal{P}^j$ are $B^j_\epsilon = \mathcal{P}^j \bigcap \mathcal{B}_\epsilon(V)$, and the* ***open balls in the induced topology*** *Ω^j are $i_j^-(B^j_\epsilon)$ where i_j and i_j^- are defined in (2.4) and (2.5).*

Definition 2.1.5. *Let $D \subset \mathcal{P}^j$. We say that D is an* ***open subset*** *in the space of primary variables, if all $V \in D$ there exists an $\epsilon > 0$ such that $B^j_\epsilon(V) \subset D$. The subset $i_j^-(D)$ is an open subset of Ω^j.*

Definition 2.1.6. *Let $V \in \mathcal{P}^j$. We say that V is an* ***internal point*** *of $\mathcal{P}^j$, if there exists an $\epsilon > 0$ such that $B^j_\epsilon(V) \subset \mathcal{P}^j$. The other points are called* ***boundary points*** *of $\mathcal{P}^j$. If V is an internal (boundary) point for a $\mathcal{P}^j$, then $i_j^-(V)$ is an internal (boundary) point for Ω^j.*

Remark 2.1.2. *Notice that the frontier points are boundary points, however, there are boundary points that are private points. The points that are boundary and frontier points are called* ***exterior boundary****.*

Since the primary variable spaces are projections of the Ω^j for $j \in \mathcal{PS}$, it is not possible to consider the union of different primary variable spaces $\mathcal{P}^j$. However, we can consider the union of some Ω^j. Since we are interested in physical phenomena we consider a connected union of the variable spaces Ω^k, which is denoted by $\bigcup_k \Omega^k$.

Remark 2.1.3. *Since each physical situation contains its frontiers, it is closed in Ω.*

Example 2.1.1.

In Chapter 5, we study the problem of steam injection into a porous medium, proposed originally by Bruining *et. al.* and partially solved in [6]. In this model diffusive, capillarity and heat conductivity effects are ignored. The mass balance equations for liquid water, steam and energy conservation in terms of enthalpies are written as:

$$\frac{\partial}{\partial t}\varphi\rho_W s_w + \frac{\partial}{\partial x}u\rho_W f_w = q_{g\longrightarrow a,w}, \tag{2.7}$$

$$\frac{\partial}{\partial t}\varphi\rho_g s_g + \frac{\partial}{\partial x}u\rho_g f_g = -q_{g\longrightarrow a,w}, \tag{2.8}$$

$$\frac{\partial}{\partial t}\left(\rho_r h_r + \varphi(\rho_W h_w s_w + \rho_g h_g s_g)\right) + \frac{\partial}{\partial x}u(\rho_W h_w f_w + \rho_g h_g f_g) = 0. \tag{2.9}$$

Here φ is the rock porosity assumed to be constant; s_w and s_g are, respectively, the water saturation and steam saturation; u is the Darcy velocity; f_w and f_g are the fractional flow functions for water and steam; ρ_r and ρ_W are the constant rock and water densities. The steam density ρ_g is a function of the temperature T; h_r, h_w and h_g are the rock, water and steam enthalpies per unit mass; these properties depend on temperature (see Appendix B); and $q_{g\longrightarrow a,w}$ is the water mass transfer term or steam condensation term. All variables are defined in Appendix B.

19 There are three physical situations: steam situation, labelled by I where there is only steam above the fixed water boiling temperature; the boiling situation, labelled by II, where water and steam coexist at boiling temperature; the water situation, labelled by III, where there is only water below boiling temperature, see Figure 2.1. In the steam situation the space of primary and trivial variables are $\mathcal{P}^I = \{T\}$ and $\mathcal{P}_I = \{s_w = 0\}$ with $\Omega^I = \{s_w = 0, T\}$; the Darcy speed u is an unknown (secondary variable). In the boiling situation, $\mathcal{P}^{II} = \{s_w\}$, $\mathcal{P}_{II} = \{T^b\}$ and $\Omega^{II} = \{S_w, T^b\}$, where T^b is the boiling temperature at the prevailing temperature; the Darcy speed u is constant (secondary variable) and the flow is governed by the Buckley-Leverett equation. In the water situation, $\mathcal{P}^{III} = \{T\}$, $\mathcal{P}_{III} = \{s_w = 1\}$ and $\Omega^{III} = \{s_w = 1, T\}$; the Darcy speed u is constant (secondary variable).

Example 2.1.2.

In Chapter 4, we study flow of nitrogen and steam in a porous medium with water, proposed originally by Bruining et. al and partially solved in [10], with same simplifications as in the Example 2.1.1. The water, steam, nitrogen mass balance equation and the energy balance equations are:

$$\frac{\partial}{\partial t}\varphi\rho_W s_w + \frac{\partial}{\partial x}u\rho_W f_w = q_{g\longrightarrow a,w}, \tag{2.10}$$
$$\frac{\partial}{\partial t}\varphi\rho_{gw} s_g + \frac{\partial}{\partial x}u\rho_{gw} f_g = -q_{g\longrightarrow a,w}, \tag{2.11}$$
$$\frac{\partial}{\partial t}\varphi\rho_{gn} s_g + \frac{\partial}{\partial x}u\rho_{gn} f_g = 0, \tag{2.12}$$
$$\frac{\partial}{\partial t}\left(\rho_r h_r + \varphi(\rho_W h_w s_w + (\rho_{gw}h_{gw} + \rho_{gn}h_{gn})s_g)\right) + \frac{\partial}{\partial x}u\left(\rho_W h_w f_w + (\rho_{gw}h_{gw} + \rho_{gn}h_{gn})f_g\right) = 0. \tag{2.13}$$

The quantities φ, ρ_W, ρ_r, s_w, s_g, f_w and f_g were defined in the previous example. In the gaseous phase, the concentration of steam is ρ_{gw} (steam mass per unit gas volume) and the concentration of nitrogen is ρ_{gn}; in the presence of liquid water thermodynamical considerations specify the dependence of these concentrations solely on temperature, see Appendix A.

We consider an example illustrating how the accumulation and flux functions G and F depend on the physical situation. We describe the gaseous physical situation, where $s_g = 1$ and $s_w = 0$. We assume that nitrogen and steam behave as ideal gases with densities denoted by ρ_{gN} and ρ_{gW}, see Appendix A. We also assume that there are no volume effects due to mixing, namely the volumes of the components are additive, i.e., $\rho_{gw}/\rho_{gW}(T) + \rho_{gn}/\rho_{gN}(T) = 1$. This fact allows to define the steam and nitrogen gas composition, respectively, as:

$$\psi_{gw} = \rho_{gw}/\rho_{gW}(T) \qquad \text{and} \qquad \psi_{gn} = \rho_{gn}/\rho_{gN}(T), \tag{2.14}$$

so the compositions are additive, i.e., $\psi_{gw} + \psi_{gn} = 1$. We use ψ_{gw} as the independent variable. Therefore, there are three unknowns to be determined: temperature T, gas composition ψ_{gw} and speed u. The saturation is trivial, i.e., $s_g = 1$ and $s_w = 0$.

We rewrite equations (2.10)-(2.13) using Eq. (2.14). Since $s_w = 0$, $f_w = 0$, from Eq. (2.10) we obtain that $q_{g\longrightarrow a,w}$ vanishes, so the system is written as:

$$\frac{\partial}{\partial t}\varphi\frac{M_W p_{at}}{R}\psi_{gw}T^{-1} + \frac{\partial}{\partial x}u\frac{M_W p_{at}}{R}\psi_{gw}T^{-1} = 0,$$
$$\frac{\partial}{\partial t}\varphi\frac{M_N p_{at}}{R}(1-\psi_{gw})T^{-1} + \frac{\partial}{\partial x}u\frac{M_N p_{at}}{R}(1-\psi_{gw})T^{-1} = 0,$$
$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + \psi_{gw}\frac{M_W p_{at}}{RT}h_{gW} + \psi_{gn}\frac{M_N p_{at}}{RT}h_{gN}\right) + \frac{\partial}{\partial x}u\left(\psi_{gw}M_W h_{gW} + \psi_{gn}M_N h_{gN}\right)\frac{p_{at}}{RT} = 0,$$

where $\hat{H}_r = \hat{H}_r(T)$, $h_{gW} = h_{gW}(T)$, $h_{gN} = h_{gN}(T)$. The constants are described in Appendix A. Notice that $(M_W p_{at}/R$ and $M_N p_{at}/R$ are constants that cancel out of the first two equations

In the single-phase gaseous situation, labelled by I, the spaces of primary and trivial variables are $\mathcal{P}^I = \{\psi_{gw}, T\}$ and $\mathcal{P}_I = \{s_w = 0\}$ with $\Omega^I = \{s_w = 0, \psi_{gw}, T\}$; the Darcy speed u is an unknown (secondary variable). In the two-phase situation, labelled by II, the spaces of primary and trivial variables are $\mathcal{P}^{II} = \{s_w, T\}$ and $\mathcal{P}_{II} = \{\psi_{gw}(T)\}$ with $\Omega^{II} = \{s_w, \psi_{gw}(T), T\}$; the Darcy speed u is an unknown (secondary variable). In this case the gas composition is given in terms of temperature by Clausius-Clapeyron and Raoult's laws, see [44, 68].

In the single-phase liquid situation, labelled by III, the spaces of primary and trivial variables are $\mathcal{P}^{III} = \{T\}$ and $\mathcal{P}_{III} = \{s_w = 1, \psi_{gw}(T)\}$ with $\Omega^{III} = \{s_w = 1, \psi_{gw}(T), T\}$; the (secondary variable) Darcy speed is a constant. Even though there is no gas in this physical situation, it is useful to define the gas composition by continuity from Eq (2.14).

In each physical situation, it is necessary to calculate the solution for the primary variables. In Figure 2.1, we show the physical situations in the primary state space variable for I, II and III. In Ω, Eqs. (2.10)-(2.13) represent a system with four equations and four variables: the temperature, the gas saturation, the gas composition and the speed.

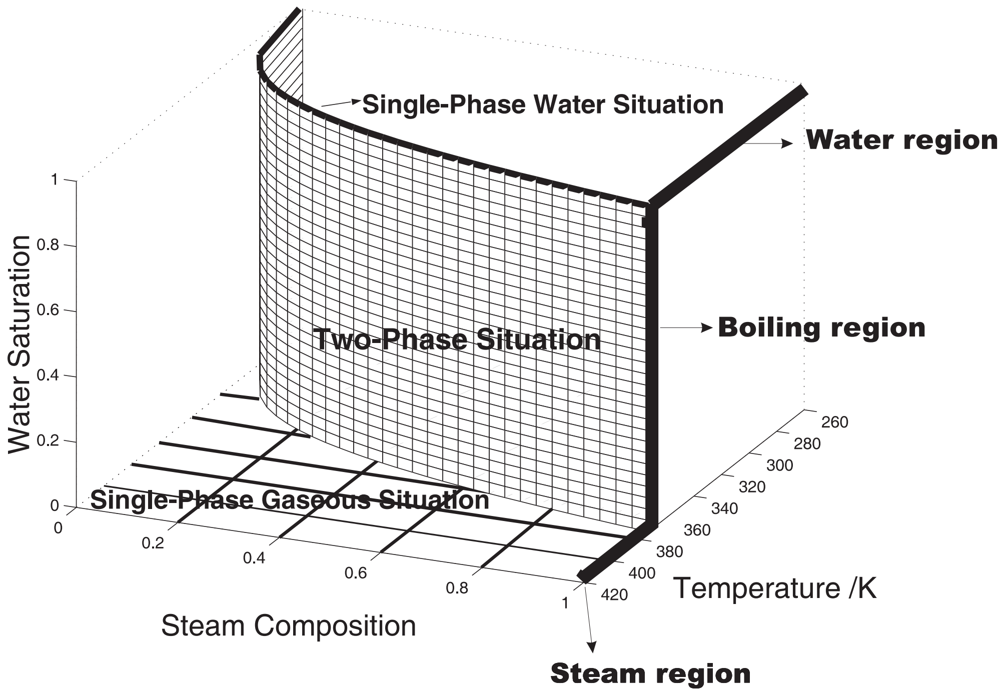


Figure 2.1: State space, physical situations and their domains in Example 2.1.2. The bold straight lines represent the nitrogen-free situations, that are: the water situation for $s_w = 1$; the boiling situation for s_w non-constant; the steam situation for $s_w = 0$. This Riemann problem is solved in Chapter 5.

2.1.1 Accumulations and fluxes for a physical situation

Generically, there is no unique choice of (G^j, F^j). As a simple example, consider $G^j = (G_1^j, G_2^j, \cdots, G_{n(j)+1}^j)$ and $F^j = (F_1^j, F_2^j, \cdots, F_{n(j)+1}^j)$ satisfying (2.2) for $\mathcal{V}^j \in \Omega^j$. A permutation in the components of (G^j, F^j) yields the same solution of (2.2). However, it is necessary to select a choice for the (G^j, F^j) for each $j \in \mathcal{PS}$.

Property 2.1.1. *We are interested in solving problems where there exists a linear map* $E : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{n+1}$ *that satisfies:*

$$E(Q(\mathcal{V})) = 0, \quad \forall\, \mathcal{V} \in \Omega, \tag{2.15}$$

where $Q(\mathcal{V})$ *is a* $(m+1) \times 1$ *vector and* 0 *is a* $(n+1) \times 1$ *vector. So the vector* $Q(\mathcal{V}) \in \mathcal{K}(E)$*, where* $\mathcal{K}(E)$ *is the kernel of* E*.*

For each chemical component, this map acts on (2.1) by taking the equations for all phases and adding them. The resulting equation (2.2) represents the conservation of mass of this component in all phases, so that the source terms add to zero.

Definition 2.1.7. *We say that a linear map* $E : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{n+1}$ *has maximal rank associated to the system* (2.1)*, or* ***maximal rank****, if it satisfies* (2.15) *and:*

$$m + 1 = rank(E) + dim(span\{Q(\mathcal{V})\}), \tag{2.16}$$

where $span\{Q\}$ *is the vector space generated by* Q*. The rank of* E *is denoted by* $n+1$*. We call the linear map* E ***internal transfer matrix****.*

There exists a class of linear maps $E : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{n+1}$ *that satisfy* (2.15) *and* (2.16) *and denote by* $\mathcal{E}$*:*

$$\mathcal{E} = \{E : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{n+1} \quad \textit{that satisfy } (2.15) \textit{ and } (2.16)\}. \tag{2.17}$$

From (2.16), for $E \in \mathcal{E}$, notice that $\mathcal{K}(E) = span\{Q(\mathcal{V})\}$.

Definition of (G^j, F^j)

Since this pair of functions is defined only in the physical situation $j \in \mathcal{PS}$ and in this physical situation we are interested only in the primary variables, it is necessary to utilize $\mathcal{R}_G^j$ and $\mathcal{R}_F^j$ defined in (2.6) and the relationship between the dependent variable $\mathcal{V}^j$ and the corresponding primary variable V^j. Let $E \in \mathcal{E}$ be a linear map, then for $\mathcal{V}^j$ and the corresponding primary variable V^j, the pair (G_E^j, F_E^j) is:

$$G_E^j(V^j) = E\mathcal{R}_G^j(\mathcal{V}^j) \qquad \text{and} \qquad F_E^j(V^j) = E\mathcal{R}_F^j(\mathcal{V}^j). \tag{2.18}$$

The number $n(j)$ represents the dimension of (G_E^j, F_E^j), i.e., the number of primary variables to be determined.

Remark 2.1.4. *Notice that the* $n(j)$ *is the same for any* $j \in \mathcal{PS}$*, so in what follows we only write* n *to indicate that it does not depend on the physical situation.*

From Lemma 2.1.1 and from the continuity of the linear map E we obtain:

Proposition 2.1.1. *Consider two pairs* (G^j_E, F^j_E) *and* (G^k_E, F^k_E) *obtained from the same linear map* E *from* (2.18)*, such that* $\Pi^{jk} \neq \emptyset$*, then:*

$$G^j_E(V) = G^k_E(V) \quad and \quad F^j_E(V) = F^k_E(V) \qquad \forall\, V \,\in\, \Pi^{jk}. \tag{2.19}$$

Thus we can see that the cumulative and flux terms can be understood only as functions of V for V belonging to some $\mathcal{P}^j$.

Consider the balance system (2.1) for $\mathcal{V}^j \in \Omega^j$, in a specified physical situation. Applying the linear map $E \in \mathcal{E}$ in this system and using $\mathcal{R}^j_G$ and $\mathcal{R}^j_F$ defined in (2.6) and (2.18), we obtain the following system of conservation laws (2.2) for which the unknowns are the primary variables V^j associated to $\mathcal{V}^j$ that is:

$$\frac{\partial}{\partial t} G^j_E + \frac{\partial}{\partial x} u F^j_E = 0. \tag{2.20}$$

We would like the solution of the resulting conservation laws not to depend on $E \in \mathcal{E}$, i.e., the solution of (2.2) should be the same for some $E \in \mathcal{E}$ (at this stage, we are not interested yet in the solution associated to initial and boundary conditions. We are interested only in the form of the differential equation). To guarantee this independence, we utilize the following Lemma:

Lemma 2.1.2. *Let* A *and* B *be two vector spaces;* $E_1 : A \longrightarrow B$ *and* $E_2 : A \longrightarrow B$ *two linear maps. If* $\mathcal{K}(E_1) = \mathcal{K}(E_2)$*, there exists an isomorphism* $\mathbb{I} : B \longrightarrow B$ *such that* $E_1 = \mathbb{I} E_2$*.*

Using Lemma 2.1.2, we obtain the following Proposition for $E \in \mathcal{E}$:

Proposition 2.1.2. *Let* $E_1, E_2 \in \mathcal{E}$ *be two different matrices. Then for* $(\mathcal{V}^j, u) \in \Omega^j \times \mathbb{R}$ *for each* $j \in \mathcal{PS}$*, the corresponding* $(V^j, u) \in \mathcal{P}^j \times \mathbb{R}$ *is the solution of*

$$\frac{\partial}{\partial t} G^j_{E_1} + \frac{\partial}{\partial x} u F^j_{E_1} = 0. \tag{2.21}$$

if, only if, is solution of

$$\frac{\partial}{\partial t} G^j_{E_2} + \frac{\partial}{\partial x} u F^j_{E_2} = 0, \tag{2.22}$$

where V^j *is the primary variable associated to* $\mathcal{V}^j$*.*

Proof: Since $E_1, E_2 \in \mathcal{E}$, then, by definition of $\mathcal{E}$, we know that $\mathcal{K}(E_1) = \mathcal{K}(E_2)$. From Lemma 2.1.2 there exists an isomorphism $\mathbb{I} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ such that $E_1 = \mathbb{I} E_2$. Assume that (V^j, u) is a solution of (2.21). So using Def. 2.18 and $E_1 = \mathbb{I}^{-1}\mathbb{I} E_1 = \mathbb{I}^{-1} E_2$, Eq. (2.21) can be written as:

$$\mathbb{I}^{-1}\left(E_2\left(\frac{\partial}{\partial t}\mathcal{R}^j_G + \frac{\partial}{\partial x} u \mathcal{R}^j_F\right)\right) = 0.$$

Since $\mathbb{I}^{-1}$ is also an isomorphism, $\mathcal{K}(\mathbb{I}^{-1}) = 0$, so we obtain that:

$$E_2 \left(\frac{\partial}{\partial t} \mathcal{R}^j_G + \frac{\partial}{\partial x} u \mathcal{R}^j_F \right) = 0. \tag{2.23}$$

That is (2.22). Since it is valid for E_1 and E_2 in $\mathcal{E}$, the assertion follows. □

Corollary 2.1.1. *The solution of* (2.2) *does not depend on the linear map $E \in \mathcal{E}$.*

Thus we can choose any $E \in \mathcal{E}$ and since the solution of conservation law does not depend on E we rewrite (G^j_E, F^j_E) as (G^j, F^j).

Example 2.1.3. *For a conservation law in the form* (2.2)*, the transfer matrix E is any non-singular matrix.*

For the balance system (2.7)-(2.9) *and* (2.10)-(2.13) *the transfer matrices are :*

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textit{and} \qquad E = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.1.2 Riemann problem

We use $W^j = (V^j, u)$, for $V^j \in \mathcal{P}^j$ for some $j \in \mathcal{PS}$ to represent the state variables, where u is the secondary variable and V^j are the primary state variables; similarly, $\mathcal{W} = (\mathcal{V}, u)$ for $\mathcal{V} \in \Omega$ represents the state variables, u is the secondary variable and $\mathcal{V}$ are the primary together with the trivial state variables. We are interested in the Riemann problem associated to (2.1), that is the solution of these equations with initial data

$$\begin{cases} \mathcal{W}_L = (\mathcal{V}_L, u_L) & \text{if } x > 0, \\ \mathcal{W}_R = (\mathcal{V}_R, \cdot) & \text{if } x < 0. \end{cases} \tag{2.24}$$

Since the system (2.1) has an infinite characteristic speed associated to u, only one of the speeds u_L or u_R is given (in this case we have chosen u_L). Later, it will be clear that the other speed on the right hand side can be determined from the other equations in the system.

The general solution of the Riemann problem associated to Eq. (2.2) consists of a sequence of elementary waves: rarefactions and shocks. These waves may be separated by constant states. The solutions are studied in the sections that follow.

2.1.3 Wave Sequences and Riemann Solution

A Riemann solution is a sequence of elementary waves w_k (shocks and rarefactions) for $k = 1, 2, \cdots, m$ and constant states W_k for $k = 1, 2, \cdots, m$.

$$\mathcal{W}_L \equiv \mathcal{W}_0 \xrightarrow{w_1} \mathcal{W}_1 \xrightarrow{w_2} \cdots \xrightarrow{w_m} \mathcal{W}_m \equiv \mathcal{W}_R. \tag{2.25}$$

We represent any state by $\mathcal{W}$. The wave w_k has left and right states $\mathcal{W}_{k-1}$ and $\mathcal{W}_k$ and speeds $\xi_k^- < \xi_k^+$ in case of rarefaction waves and $v = \xi_k^- = \xi_k^+$ in case of shock waves. The left state of the first wave w_1 is $(\mathcal{V}_L, u_L)$ and the right state of w_m is $(\mathcal{V}_R, u_R)$, where u_R needs to be found. In the Riemann solution it is necessary that $\xi_k^+ \leq \xi_{k+1}^-$; this inequality is called *geometrical compatibility*. When $\xi_k^+ < \xi_{k+1}^-$ there is a separating constant state W_{k+1} between w_k and w_{k+1}; in this sequence the wave w_k is indicated by $\mapsto$. If $\xi_k^+ = \xi_{k+1}^-$ there is no actual constant state in physical space, so the wave w_k is a composite with w_{k+1}; it is indicated by $\twoheadrightarrow$.

Different physical situations are separated by shocks respecting the geometrical compatibility. When it is useful to emphasize the waves in the sequence (2.25) rather than the states, we use the notation:

$$w_1 \rightarrowtail w_2 \rightarrowtail \cdots \rightarrowtail w_m, \tag{2.26}$$

where for each w_k, $\rightarrowtail$ represents $\mapsto$ for the case $\xi_k^+ < \xi_{k+1}^-$ and $\twoheadrightarrow$ for the case $\xi_k^+ = \xi_{k+1}^-$.

2.2 Characteristic speeds

Since the cumulative and flux terms are not smooth between physical situations, we calculate the characteristic speeds only within the physical situation. The system of conservation reduces to the conservation form (2.2), thus it is possible to find the characteristic speeds, and then the rarefaction waves. These waves are important to solve the Riemann problem; the characteristic speeds are also important to propose adequate numerical methods to solve Eq. (2.1) and condition for numerical stability, see [40].

For a smooth wave we differentiate all equations in (2.2) with respect to their variables, obtaining a system:

$$B\frac{\partial}{\partial t}\begin{pmatrix} V \\ u \end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix} V \\ u \end{pmatrix} = 0; \tag{2.27}$$

the matrices B and A are the derivatives of $G(V)$ and $uF(V)$ with respect to $W = (V, u)$, where $V = V^j \in \mathcal{P}^j$ are the primary variables for some $j \in \mathcal{PS}$ and u is the speed. Since $G(V)$ does not depend on u, the matrix $\partial G/\partial W$ has a zero column, ie.

$$\frac{\partial G}{\partial W} = \begin{pmatrix} \frac{\partial G_1}{\partial V_1} & \frac{\partial G_1}{\partial V_2} & \cdots & \frac{\partial G_1}{\partial V_n} & 0 \\ \frac{\partial G_2}{\partial V_1} & \frac{\partial G_2}{\partial V_2} & \cdots & \frac{\partial G_2}{\partial V_n} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial G_n}{\partial V_1} & \frac{\partial G_n}{\partial V_2} & \cdots & \frac{\partial G_n}{\partial V_n} & 0 \\ \frac{\partial G_{n+1}}{\partial V_1} & \frac{\partial G_{n+1}}{\partial V_2} & \cdots & \frac{\partial G_{n+1}}{\partial V_n} & 0 \end{pmatrix} = \left(\frac{\partial G}{\partial V} \,\Big|\, 0\right) := B, \tag{2.28}$$

and

$$\frac{\partial uF}{\partial W} = \begin{pmatrix} u\frac{\partial F_1}{\partial V_1} & u\frac{\partial F_1}{\partial V_2} & \cdots & u\frac{\partial F_1}{\partial V_n} & F_1 \\ u\frac{\partial F_2}{\partial V_1} & u\frac{\partial F_2}{\partial V_2} & \cdots & u\frac{\partial F_2}{\partial V_n} & F_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u\frac{\partial F_n}{\partial V_1} & u\frac{\partial F_n}{\partial V_2} & \cdots & u\frac{\partial F_n}{\partial V_n} & F_n \\ u\frac{\partial F_{n+1}}{\partial V_1} & u\frac{\partial F_{n+1}}{\partial V_2} & \cdots & u\frac{\partial F_{n+1}}{\partial V_n} & F_{n+1} \end{pmatrix} = \left(u\frac{\partial F}{\partial V} \;\middle|\; F \right) := A, \tag{2.29}$$

where $\partial G/\partial V$ and $u\partial F/\partial V$ represent the n first columns of B and A, 0 and F represents the $(n+1)$-*th* column of B and A.

Remark 2.2.1. *Let $V^* \in \mathcal{P}^j$. If V^* is private point, the derivatives $\partial G/\partial V$ and $\partial F/\partial V$ are well defined at V^*, because if the point is an internal point the cumulative terms G and flux terms F are $\mathcal{C}^2$. On the other hand, if V^* is a boundary point the derivatives are defined laterally.*

However, if V^* is a frontier point, i.e., there exists another $k \in \mathcal{PS}$, with $k \neq j$, such that $V^* \in \mathcal{P}^k$, at V^* the derivatives are not well defined, because the derivatives can be different in neighboring physical situations. Since by Rem. 2.1.3 *each physical situation is closed, and we are interested in the rarefaction waves in a determined physical situation, we define the derivatives laterally in each $\mathcal{P}^j$.*

The eigenvalues λ and right eigenvectors $\vec{r} = (r_1, r_2, \cdots, r_{n+1})^T$ with $n+1$ components of the following system are the rarefaction wave speeds and the characteristic directions, respectively:

$$A\vec{r} = \lambda B\vec{r}. \tag{2.30}$$

Hereafter, the word “eigenvectors” means “right eigenvectors”. Similarly the left eigenvector $\vec{\ell} = (\ell_1, \ell_2, \cdots, \ell_{n+1})$ satisfies:

$$\vec{\ell}A = \lambda\vec{\ell}B \qquad \text{or} \qquad A^T\vec{\ell}^T = \lambda B\vec{\ell}^T.$$

To find λ, we need to solve the characteristic equation

$$det(A - \lambda B) = 0. \tag{2.31}$$

Whenever necessary, we write the right and left eigenvectors without the upper arrow, i.e., $r = \vec{r}$ and $\ell = \vec{\ell}$.

Remark 2.2.2. *We utilize an index p to indicate a particular characteristic speed family. The eigenvalue, right eigenvector and left eigenvector of the $p-$**family** are denoted by λ_p, $r^p = (r_1^p, r_2^p, \cdots, r_n^p, r_{n+1}^p)$ and $\ell^p = (\ell_1^p, \ell_2^p, \cdots, \ell_n^p, \ell_{n+1}^p)$.*

We define the set of indices n as $\mathcal{C}$:

$$\mathcal{C} = \{1, 2, \cdots, n+1\}. \tag{2.32}$$

Definition 2.2.1. *The* ***integral curve*** *of the p-family is the solution of :*

$$\left(\frac{dV}{d\xi},\frac{du}{d\xi}\right)=r^p, \quad i.e, \quad \frac{dV_i}{d\xi}=r_i^p \quad \text{for all } i\in\mathcal{C} \quad \text{and} \quad \frac{du}{d\xi}=r_{n+1}^p. \tag{2.33}$$

When ξ increases satisfying:

$$\xi=\lambda_p(V(\xi),u(\xi)), \tag{2.34}$$

the integral curve defines a rarefaction curve. If in addition to (2.34)*, the variable ξ satisfies $x=\xi t$ the integral curve defines rarefaction waves in the (x,t) plane.*

Lemma 2.2.1. *Assume that $u\neq 0$ and that $F_k(V)\neq 0$ for some fixed $k\in\mathcal{C}$ for all V. The eigenvalue, right and left eigenvectors for the system* (2.27)*, with B and A given by Eqs.* (2.28) *and* (2.29) *have the form:*

$$\lambda=u\vartheta(V), \tag{2.35}$$

$$r=(g_1(V),g_2(V)\cdots,g_n(V),ug_{n+1}(V)) \quad \text{and} \quad \ell=(\ell_1(V),\ell_2(V),\cdots,\ell_{n+1}(V)), \tag{2.36}$$

where ϑ, g_i and ℓ_i (for all $i\in\mathcal{C}$) are functions that depend only on V. Moreover, there are at most n eigenvalues and eigenvectors associated to this system of $n+1$ equations.

Proof: The eigenvalues λ of (2.27) are the roots of $det(A-\lambda B)=0$, for B and A given by Eq. (2.28) and (2.29). This characteristic equation is:

$$det(A-\lambda B)=det\left(u\frac{\partial F}{\partial V}(V)-\lambda\frac{\partial G}{\partial V}(V)\;\Big|\;F(V)\right)=0. \tag{2.37}$$

Since $u\neq 0$, we divide the first n columns by u and define $\vartheta=\lambda/u$, then after some calculations Eq. (2.37) reduces to:

$$det\left(\frac{\partial F}{\partial V}(V)-\vartheta\frac{\partial G}{\partial V}(V)\;\Big|\;F(V)\right)=0. \tag{2.38}$$

Since Eq (2.38) depends on V only, the eigenvalues ϑ are functions of V, i.e., $\vartheta=\vartheta(V)$, they do not depend on u. Using that $\lambda=u\vartheta$, the eigenvalues have the form (2.35).

The eigenvectors are solutions of $(A-\lambda B)r=0$. Writing $r=(r_1,\cdots,r_{n+1})^T$, we obtain:

$$\left(u\left(\frac{\partial F}{\partial V}(V)-\vartheta(V)\frac{\partial G}{\partial V}(V)\right)\;\Big|\;F(V)\right)r=0. \tag{2.39}$$

Since there is an index k such that $F_k(V)\neq 0$, then we can write r_{n+1} as:

$$r_{n+1}=u\frac{1}{F_k(V)}\sum_{l=1}^{n}\left(\frac{\partial F_k}{\partial V_l}(V)-\vartheta(V)\frac{\partial G_k}{\partial V_l}(V)\right)r_l, \tag{2.40}$$

Substituting r_{n+1} given by Eq. (2.40) into Eq. (2.39), we obtain a linear system in the unknowns r_j for $j=1,2,\cdots,n$, where we can cancel u, showing that the r_l for $l=1,2,\cdots,n$ depend on V only. So (2.36.a) follows from (2.40).

For $\ell = (\ell_1, \ell_2, \cdots, \ell_{n+1})$, we solve:

$$\ell \cdot \left(u \left(\frac{\partial F}{\partial V}(V) - \vartheta(V) \frac{\partial G}{\partial V}(V) \right) \;\Big|\; F(V) \right) = 0,$$

then ℓ is solution of the following system:

$$u \left(\frac{\partial F_1}{\partial V_l} - \vartheta \frac{\partial G_1}{\partial V_l} \right) \ell_1 + \cdots + u \left(\frac{\partial F_{n+1}}{\partial V_l} - \vartheta \frac{\partial G_{n+1}}{\partial V_l} \right) \ell_{n+1} = 0 \quad \text{for } l = 1, \cdots, n \tag{2.41}$$

$$F_1 \ell_1 + F_2 \ell_2 + \cdots + F_{n+1} \ell_{n+1} = 0. \tag{2.42}$$

Since $u \neq 0$, we divide Eqs. (2.41) by u, then we obtain a system with coefficients that depend only on the variables V, leading to (2.36.b). □

Remark 2.2.3. *The hypothesis that there exists an index k such that $F_k(V) \neq 0$ for all V is sufficient, but it is not necessary. However in the most of problems this hypothesis is satisfied.*

Proposition 2.2.1. *Assume that locally the eigenvector r associated to a certain family forms a local vector field. Then we calculate the primary variables V on the rarefaction waves in the (x, t) plane independently of u, i.e., first we obtain the primary variables in the classical way, i.e., solving (2.33.b), and then we calculate the secondary variable u in terms of the primary variables from:*

$$u = u^- exp(\gamma(\xi)), \quad \gamma(\xi) = \int_{\xi^-}^{\xi} g_{n+1}(V(\eta)) d\eta, \tag{2.43}$$

where $\xi = x/t$, $\xi^- = \lambda(W^-)$ and u^- is the "first" value of u on the rarefaction wave, i.e., $u = u^-$ for $\xi = \xi^-$.

Proof: Using (2.33) and (2.34) and since $g_i(V) = r_i$ for $i = 1, 2, \cdots, n$ we obtain V independently of u by solving the system of differential equations:

$$\frac{dV}{d\xi} = \left(\frac{dV_1}{d\xi}, \frac{dV_2}{d\xi}, \cdots, \frac{dV_n}{d\xi} \right) = (g_1, g_2, \cdots, g_n), \quad \text{with } \xi^- = \lambda(W^-).$$

After obtaining $V(\xi)$, we use the expression for the last component of r in (2.36.a) to solve $du/d\xi = u g_{n+1}$, yielding (2.43). From Lemma 2.2.1, we know that λ has the form $u\vartheta(V)$, so using Eqs. (2.34) and (2.43), we obtain ξ implicitly as:

$$\xi = u^- \vartheta(V(\xi)) exp(\gamma(\xi)). \tag{2.44}$$

Since ξ depends only on u^- and V on the integral curves, the results follows. □

Definition 2.2.2. *Since only the first n coordinates of eigenvectors are important to obtain the integral curves in the space V (i.e. independently of u), it is useful to define for any p-family the structures, see Rem.* 2.2.2*:*

$$\hat{r}^p = (r_1^p, r_2^p, \cdots, r_n^p) \quad \textit{and} \quad \hat{\ell}^p = (\ell_1^p, \ell_2^p, \cdots, \ell_n^p), \tag{2.45}$$

and

$$\hat{\lambda}^{p,-}(V) := \lambda_p(W)/u^- \quad \textit{and} \quad \hat{\lambda}^{p,+}(V) := \lambda_p(W)/u^+. \tag{2.46}$$

Corollary 2.2.1. *Assume that $u^- \neq 0$, so we perform the change of variables:*

$$\xi = \frac{x}{t} \longrightarrow \hat{\xi} = \frac{x}{u^- t}, \tag{2.47}$$

The system (2.2) *can be written in the space of variables* $(x, T) = (x, u^- t)$.

Proof: In V space, one can prove that the eigenvalues and eigenvectors have the form $\lambda^-(V)$, with r given in Eq. (2.36.a). From Eqs. (2.34) and (2.44), it follows that $\hat{\xi}^-$ and $\hat{\xi}$ satisfy:

$$\hat{\xi}^- = \vartheta(V^-) \quad \text{and} \quad \hat{\xi} = \vartheta(V(\hat{\xi}))exp(\gamma(V(\xi))). \tag{2.48}$$

Thus in the (x, T) plane, the variables V and $\hat{\xi}$ do not depend on u, so the rarefaction curve does not depend on u. The speed keeps the form (2.43). $\square$

Corollary 2.2.2. *If $r_{n+1} \equiv 0$ in a region, then from Prop.* 2.2.1 *u is constant along the corresponding rarefaction curve.*

2.3 Shock waves

Typically, discontinuities or shocks appear in solutions of Riemann (or Cauchy) problems of non-linear hyperbolic or hyperbolic-elliptic equations (see [60]). The Rankine-Hugoniot (*RH*) condition for shocks needs to be satisfied as it expresses conservation of mass. For a given state W^-, the set of states W^+ that satisfies the *RH* condition is called *Rankine-Hugoniot locus (RH locus) of* W^-, which is denoted by $\mathcal{RH}(W^-)$.

2.3.1 Shocks and the Rankine-Hugoniot condition

The main feature of the class of equations (2.1) is the existence of separate regions where the system of balance equations (2.1) reduces to different systems of conservation equations (2.2). To obtain the complete Riemann problem solution it is necessary to link these regions. Because often the term Q gives rise to fast changes between regions, we consider shocks linking these different regions.

We apply a linear map E given in Def. 2.1.7 on the system (2.1). Since $E[Q] = 0$, we obtain a system in conservative form:

$$\frac{\partial}{\partial t} E\mathcal{G}(\mathcal{V}) + \frac{\partial}{\partial x}\left(uE\mathcal{F}(\mathcal{V})\right) = 0, \tag{2.49}$$

where $\mathcal{V} = \mathcal{V}^j \in \Omega^j$ for some $j \in \mathcal{PS}$.

Remark 2.3.1. *We should use a single E linking neighboring physical situations. Notice that it is possible to use a single E for the complete system yielding the simplest global formulation. However, we have proved that the weak solution does not depend on E, so we can use different linear maps E in the interior of each physical situation, see Proposition* 2.1.2.

The RH condition applied to Eq. (2.49) is written as:

$$v^s \left(E\mathcal{G}(\mathcal{V}^+) - E\mathcal{G}(\mathcal{V}^-) \right) = u^+ E\mathcal{F}(\mathcal{V}^+) - u^- E\mathcal{F}(\mathcal{V}^-), \tag{2.50}$$

where $(\mathcal{V}^-, u^-)$ is the state on the left of the shock and $(\mathcal{V}^+, u^+)$ is the state on the right of the shock; v^s is the shock speed. We specify the left conditions for the variables $\mathcal{V}$ and u, but as we will see, the right conditions are specified only for one component of $\mathcal{V}^+$. The speed u^+ is always obtained from the RH condition (2.50).

The define the *cumulative mass transfer in the shocks* can be defined as:

$$[Q(\mathcal{V})] := u^+ \mathcal{F}(\mathcal{V}^+) - u^- \mathcal{F}(\mathcal{V}^-) - v^s \left(\mathcal{G}(\mathcal{V}^+) - \mathcal{G}(\mathcal{V}^-) \right). \tag{2.51}$$

Once the Riemann solution is computed, (2.51) calculates the rate of transfer of each chemical component in a shock.

We consider shocks of two different types: shocks within regions in a certain physical situation and shocks between different physical situations. In both cases V^- and V^+ are the primary variables associated to $\mathcal{V}^-$ and $\mathcal{V}^+$, respectively.

Shock within a physical situation

Since $\mathcal{V}^+, \mathcal{V}^- \in \Omega^j$ for some $j \in \mathcal{PS}$, using the definition of $\mathcal{R}_G^j$ and $\mathcal{R}_F^j$ given in Eq. (2.6) and the relationship (2.18), the RH condition (2.50) can be written as:

$$v^s \left(G^j(V^+) - G^j(V^-) \right) = u^+ F^j(V^+) - u^- F^j(V^-). \tag{2.52}$$

Shock between distinct physical situations

Since $\mathcal{V}^+ \in \Omega^j$ and $\mathcal{V}^- \in \Omega^k$ for some pair $j, k \in \mathcal{PS}$ with $k \neq j$, using the definition of $\mathcal{R}_G^j$, $\mathcal{R}_G^k$ $\mathcal{R}_F^j$ and $\mathcal{R}_F^k$ given in Eq. (2.6) and the relationship (2.18), the RH condition (2.50) can be written as:

$$v^s \left(G^j(V^+) - G^k(V^-) \right) = u^+ F^j(V^+) - u^- F^k(V^-). \tag{2.53}$$

In the following sections, we will always indicate functions of the shock left state by the superscript $(-)$ and functions of the right state of the shock by $(+)$. This notation encompasses both (2.52) and (2.53).

2.3.2 The Rankine-Hugoniot locus and shock waves

Notice that in the RH condition (2.52) or (2.53) for each fixed state $(-)$, there is a degree of freedom in the variables V, that is a 1-dimensional structure, generically. This structure is called the *RH locus*.

Writing Eq. (2.53) as a linear homogeneous system, we obtain:

$$\begin{pmatrix} [G_1] & -F_1^+ & F_1^- \\ [G_2] & -F_2^+ & F_2^- \\ \vdots & \vdots & \vdots \\ [G_n] & -F_n^+ & F_n^- \\ [G_{n+1}] & -F_{n+1}^+ & F_{n+1}^- \end{pmatrix} \begin{pmatrix} v^s \\ u^+ \\ u^- \end{pmatrix} = 0, \qquad \text{or} \qquad \mathcal{M} \begin{pmatrix} v^s \\ u^+ \\ u^- \end{pmatrix} = 0, \tag{2.54}$$

where $[G_i]$ for $i \in \mathcal{C}$ is given by:

$$[G_i] = G_i^j(V^+) - G_i^k(V^-), \; F_i^+ = F_i^j(V^+), \;\; F_i^- = F_i^k(V^-), \tag{2.55}$$

The 3×3 minors of the matrix $\mathcal{M}$ are denoted by $\mathcal{M}_{pqs}$:

$$\mathcal{M}_{pqs} = \begin{pmatrix} [G_p] & -F_p^+ & F_p^- \\ [G_q] & -F_q^+ & F_q^- \\ [G_s] & -F_s^+ & F_s^- \end{pmatrix} \qquad \text{for all distinct} \quad p,\, q \text{ and } s \text{ in } \mathcal{C}. \tag{2.56}$$

The homogeneous linear system (2.54) has a non-trivial solution, if only if

$$\mathcal{H}_{pqs} = det\left(\mathcal{M}_{pqs}\right) = 0 \qquad \text{for all distinct} \quad p,\, q \text{ and } s \text{ in } \mathcal{C}. \tag{2.57}$$

Definition 2.3.1. *The RH locus is given by implicit expressions in the primary variables. For (V^-, u^-) fixed in some physical situation, the RH locus in the projected space V consists of the V^+ that satisfy $det(\mathcal{M}_{pqs}) = 0$ for all distinct p, q and s in $\mathcal{C}$. Notice that V^+ can lie in the same or in a distinct physical situation relatively to V^-. We denote the RH locus of a state V^- as $\mathcal{RH}(V^-)$.*

Notice that there are $\mathbb{C}^3_{n+1}$ equations satisfying (2.57); however, under certain hypotheses, the following Lemma guarantees that we only need to solve a set of $(n-1)$ equations of type (2.57).

Since we want to characterize the solution of (2.57), it is useful to define for each $i \in \mathcal{C}$, the vector

$$\mathcal{D}_i \equiv \mathcal{D}_i(V^-; V^+) \equiv \left([G]_i, -F_i^+, F_i^-\right). \tag{2.58}$$

Using the notation (2.58) we obtain the simple lemma:

Lemma 2.3.1. *Let V^- be fixed. Assume that there exists two linearly independent vectors $\mathcal{D}_l$ and $\mathcal{D}_m$ for $l, m \in \mathcal{C}$ in a neighborhood $\mathcal{B}^j_\delta(V^+)$. If there are $(n-1)$ distinct minors $\mathcal{M}_{lms}$ for $s \in \mathcal{C} - \{l, m\}$ such that $det(\mathcal{M}_{lms}) = 0$, then it follows that $det(\mathcal{M}_{pqs}) = 0$ for all distinct p, q and s in $\mathcal{C}$.*

Proof: There are $(n-1)$ indices $s \in \mathcal{C} - \{l, m\}$ such that $det(\mathcal{M}_{lms}) = 0$.

Since $\mathcal{D}_l$ and $\mathcal{D}_m$ are L.I., it follows that for all $s \in \mathcal{C}$ the vector $\mathcal{D}_s$ is a linear combination of $\mathcal{D}_l$ and $\mathcal{D}_m$. Since any minor $\mathcal{M}_{pqs}$ with distinct p, q and s in $\mathcal{C}$ has the form:

$$\mathcal{M}_{pqs} = \begin{pmatrix} \mathcal{D}_p \\ \mathcal{D}_q \\ \mathcal{D}_s \end{pmatrix},$$

it follows that $det(\mathcal{M}_{pqs}) = 0$ and the lemma is proved. $\square$

Definition 2.3.2. *Consider $V^- \in \mathcal{P}^j$ and $V^+ \in \mathcal{P}^k$. Assume that $V \in \mathcal{P}^k$ belongs to a 1-dimensional locus where the vectors $\mathcal{D}_p$ and $\mathcal{D}_q$ are linearly dependent (LD) for some $p \neq q \in \mathcal{C}$. We call this locus in Ω a **linear degeneracy** and denote it by $\mathfrak{X}_{pq}(V^-;V^+)$.*

Example 2.3.1. *In Section 4.5, we show that in the single-phase gaseous situation (see Example 2.1.2), the accumulations and fluxes for steam and nitrogen are:*

$$G_1 = \frac{\varphi\theta_W\psi_{gw}}{T}, \quad F_1 = \frac{\varphi\theta_W\psi_{gw}}{T}, \quad G_2 = \frac{\varphi\theta_N\psi_{gn}}{T} \quad and \quad F_2 = \frac{\varphi\theta_N\psi_{gn}}{T},$$

where $\psi_{gn} = 1 - \psi_{gw}$. The constans θ_W, θ_N are given by by Eq. (4.24). The terms $\mathcal{D}_1$ and $\mathcal{D}_2$ are:

$$\begin{aligned}
\mathcal{D}_1 &= \left[\varphi\theta_W\left(\frac{\psi_{gw}^+}{T^+} - \frac{\psi_{gw}^-}{T^-}\right) \qquad -\frac{\varphi\theta_W\psi_{gw}^+}{T^+} \qquad \frac{\varphi\theta_W\psi_{gw}^-}{T^-}\right]\\
\mathcal{D}_2 &= \left[\varphi\theta_N\left(\frac{\psi_{gn}^+}{T^+} - \frac{\psi_{gn}^-}{T^-}\right) \qquad -\frac{\varphi\theta_N\psi_{gn}^+}{T^+} \qquad \frac{\varphi\theta_N\psi_{gn}^-}{T^-}\right].
\end{aligned}$$

Let (T^-, ψ_{gw}^-) be fixed. It is easy to see that in the plane (T, ψ_{gw}) the straight line $(T, \psi_{gw} = \psi_{gw}^-)$ is the linear degeneracy $\mathfrak{X}_{12}(V^-)$. Of course, this straight line belongs to the RH locus for $(-)$.

Remark 2.3.2. *If a linear degeneracy occurs a more detailed analysis is required.*

Corollary 2.3.1. *Given $V^- \in \mathcal{P}^j$ for some $j \in \mathcal{PS}$, assume that for $V^+ \in \mathcal{RH}(V^-)$ and in a neighborhood $\mathcal{B}_\delta^j(V^+)$ there exists at least a pair of linearly independent (LI) vectors $\mathcal{D}_p$ and $\mathcal{D}_q$ with $p, q \in \mathcal{C}$ of the form (2.58). Assume also that for all $l \neq m \in \mathcal{C}$ and $V^+ \in \mathcal{B}_\delta^j(V^+) - \mathfrak{X}_{lm}(V^-;V^+)$ the vectors $\mathcal{D}_l$ and $\mathcal{D}_m$ are linearly independent. Then the RH locus is a 1-dimensional structure and it is obtained as the solution of $(n-1)$ equations:*

$$det\begin{pmatrix}\mathcal{D}_p\\ \mathcal{D}_q\\ \mathcal{D}_r\end{pmatrix} = 0, \qquad \forall\, r \in \mathcal{C} - \{p, q\}. \tag{2.59}$$

Moreover, $\mathfrak{X}_{lm}(V^-;V^+)$ for $l \neq m \in \mathcal{C}$ belongs to the RH locus.

Definition 2.3.3. *The minimum set formed by the triplet of indices that define implicitly the RH locus are called **RH indices** and they are denoted by $\mathcal{C}_{RH}$. There are $n-1$ triplets, numbered from $\overrightarrow{1}$ to $\overrightarrow{n-1}$. There is no a unique choice for $\mathcal{C}_{RH}$.*

So using Def. (2.57), the *RH* locus is:

$$\mathcal{H} = 0 \qquad \text{for all} \quad l \in \{\overrightarrow{1}, \overrightarrow{2}, \cdots, \overrightarrow{n-1}\} \in \mathcal{C}_{RH}, \tag{2.60}$$

where each number l corresponds to a triplet in $\mathcal{C}_{RH}$.

Let us assume that $\mathcal{RH}(V^-)$ was found. Now we want to determine u^+. Let $p, q \in \mathcal{C}$. Given $V^- \in \Omega^k$ for some $k \in \mathcal{PS}$, if $\mathcal{D}_p$ and $\mathcal{D}_q$ are LI in a neighborhood $\mathcal{B}^j_\delta(V^+)$, then:

$$det \begin{pmatrix} [G_p] & -F_p^+ \\ [G_q] & -F_q^+ \end{pmatrix} \neq 0. \tag{2.61}$$

The shock and speeds are obtained by solving the following system on the RH locus:

$$\begin{pmatrix} [G_p] & -F_p^+ \\ [G_q] & -F_q^+ \end{pmatrix} \begin{pmatrix} v^s \\ u^+ \end{pmatrix} = \begin{pmatrix} -u^- F_p^- \\ -u^- F_q^- \end{pmatrix}. \tag{2.62}$$

In this case v^s and u^+ can be written as function of u^- and $\mathcal{V}^-$ as:

$$v^s = u^- \frac{F_p^+ F_q^- - F_q^+ F_p^-}{F_q^+[G_p] - F_p^+[G_q]} \quad \text{and} \quad u^+ = u^- \frac{F_q^-[G_p] - F_p^-[G_q]}{F_q^+[G_p] - F_p^+[G_q]}. \tag{2.63}$$

Of course, we could have chosen to solve v^s and u^- in terms of u^+.

Remark 2.3.3. *In the definitions that follow, all wave structures can be obtained solely in the space of primary variables V. Using $u^+ := u^+(V^-, u^-; V^+)$ and $v^s := v^s(V^-, u^-; V^+)$, we define:*

$$\frac{u^+}{u^-} = \chi(V^-; V^+), \quad \hat{v}^-(V^-; V^+) := \frac{v^s}{u^-} \quad \textit{and} \quad \hat{v}^+(V^-; V^+) := \frac{v^s}{u^- \chi(V^-; V^+)}. \tag{2.64}$$

Let us index the rarefaction curves in the Riemann solution by l for $l = 1, 2, \cdots, \rho_1$; the shocks are indexed by m for $m = 1, 2, \cdots \rho_2$. From the Sections 2.2 and 2.3.2, we have proved the following:

Proposition 2.3.1. *Let u_L be positive. The primary variables V in the shock and rarefaction curves do not depend on the left speed $u_L > 0$. If a sequence of waves and states solve the Riemann problem in the primary variables, for a given $u_L > 0$ (or $u_R > 0$), then it is also a solution for any other $u_L > 0$ (or $u_R > 0$):*

$$\begin{cases} (\mathcal{V}_L, u_L) & \textit{if } x < 0 \\ (\mathcal{V}_R, \cdot) & \textit{if } x > 0. \end{cases} \tag{2.65}$$

Moreover, assume that for each m there are $p, q \in \mathcal{C}$ such that the following inequality is satisfied: $F_{q,m}^+[G_{p,m}] - F_{p,m}^+[G_{q,m}] \neq 0$. Then u_R is given by:

$$u_R = u_L \prod_{l=1}^{\rho_1} exp\left(\int_{\xi^{-,l}}^{\xi^{+,l}} g_{n+1}^l(V(\eta)) d\eta \right) \prod_{m=1}^{\rho_2} \frac{F_{q,m}^-[G_{p,m}] - F_{p,m}^-[G_{q,m}]}{F_{q,m}^+[G_{p,m}] - F_{p,m}^+[G_{q,m}]}. \tag{2.66}$$

where g_{n+1}^l is the $(n+1)$-th component of eigenvector r, $\xi^{-,l}$ and $\xi^{+,l}$ are the first and the last values of ξ^l associated to the l-th rarefaction wave.

In the Proposition above, $G_{j,m}$ and $F_{j,m}$ represent the *j-th* components $(j \in \mathcal{C})$ of G and F on the *m-th* shock wave. Similarly $exp\left(\int_{\xi_{0,l}}^{\xi^l} g_{n+1}(V(\eta))d\eta\right)$ is computed along the l-rarefaction curve, see (2.43).

In Proposition 2.3.1, instead of u_L we could have prescribed the right speed u_R and obtain u_L and the other speeds as function of V and u_R.

Proposition 2.3.2. *Assume that $u_L \neq 0$. If the speed u_L in the initial data is modified while $\mathcal{V}_L$ and $\mathcal{V}_R$ are kept fixed, the speed u_R as well as the Riemann solutions are rescaled in the (x,t) plane, while the wave sequences and the values of $\mathcal{V}$ are kept unaltered.*

Proof: The rarefaction case was proved in Corollary 2.2.1. In the shock case, performing the change of variable (2.47) with $u^- = u_L$ and using (2.52):

$$\frac{v^s}{u_L}(G^+ - G^-) = \frac{u^+}{u_L}F^+ - \frac{u^-}{u_L}F^-. \tag{2.67}$$

Defining $U := u/u_L$ and $\hat{v}^L = v^s/u_L$, we have:

(1) If there exists a rarefaction before u^-, from Corollary 2.2.1, we see that $u^- = u_L\mathcal{T}(V)$, where $\mathcal{T}(V) := exp(\gamma(V(\xi)))$ depends only on V.

(2) If there is no rarefaction with $u^- = u_L$, we conclude that (2.67) can be rewritten as:

$$\hat{v}^L(G^+ - G^-) = U^+(V)F^+ - U^-(V)F^-, \tag{2.68}$$

where $U^-(V) = \mathcal{T}(V)$ if (1) is satisfied or $U^-(V) = 1$ if (2) is satisfied. Since $U^+(V)$ depends only on V, using Corollary 2.2.1 we obtain the result. $\square$

This Proposition leads to an important result:

Corollary 2.3.2. *Assume the same hypotheses of Proposition 2.3.2. Then the Riemann solution can be obtained in each physical situation first in the primary variables V. The solution for the speed can be obtained in terms of $\mathcal{V}$ and u_L by Eq. (2.66). The same is true for rarefaction and shock speeds.*

We can now prove a generalization of the Triple Shock Rule [25]:

Proposition 2.3.3. *(Triple Shock Rule.) Consider three states in different physical situations with two speeds namely (V^M, u^+), (V^+, u^+) and (V^-, u^-). Assume that $V^- \in \mathcal{RH}(V^+)$, $V^M \in \mathcal{RH}(V^-)$ and $V^+ \in \mathcal{RH}(V^M)$, with shock speeds $v^{+,-}$, $v^{-,M}$ and $v^{M,+}$. Then, either:*

(1) $v^{+,-} = v^{M,-} = v^{+,M}$; or

(2) $G(V^+) - G(V^-)$ and $G(V^+) - G(V^M)$ are LD.

Proof: The *RH* conditions can be written as:

$$v^{+,-}(G(V^-) - G(V^+)) = u^-F(V^-) - u^+F(V^+), \tag{2.69}$$

$$v^{-,M}(G(V^M) - G(V^-)) = u^+F(V^M) - u^-F(V^-), \tag{2.70}$$

$$v^{M,+}(G(V^+) - G(V^M)) = u^+F(V^+) - u^+F(V^M). \tag{2.71}$$

Add (2.71), (2.70) and (2.69). We obtain:

$$(v^{M,+} - v^{-,M})\left(G(V^+) - G(V^M)\right) + (v^{-,M} - v^{+,-})\left(G(V^+) - G(V^-)\right) = 0. \tag{2.72}$$

Thus (1) or (2) are satisfied. $\square$

For future use we define $\beth := \beth(V^-, u^-; V^M, u^M; V^*, u^*; V^+, u^+)$ as:

$$\beth := (v^{M,*} - v^{-,+})\left(G(V^+) - G(V^-)\right) + (v^{-,M} - v^{M,*})\left(G(V^M) - G(V^-)\right); \tag{2.73}$$

The limitation of the triple shock rule is that, in several applications we do not know the intermediate speed. In such cases we need the following result:

Proposition 2.3.4. *(Quadruple Shock Rule I.) Consider two physical situations, with three primary points and three speeds, determining four states, namely (V^-, u^-) in Ω^k, (V^+, u^+) and (V^+, u^*) in Ω^j and (V^M, u^M) either in Ω^j or Ω^k. Assume that there are shocks between the following pairs of states:*

(i) (V^-, u^-) and (V^+, u^+) with speed $v^{-,+}$,

(ii) (V^-, u^-) and (V^M, u^M) with speed $v^{-,M}$,

(iii) (V^M, u^M) and (V^+, u^*) with speed $v^{M,*}$.

We conclude that:

R1) If $\beth \neq 0$ and at least two shock speeds coincide, i.e., one of the following shock speed equalities is satisfied:

$$\textit{either} \quad v^{-,+} = v^{-,M} \quad \textit{or} \quad v^{-,+} = v^{M,*} \quad \textit{or} \quad v^{-,M} = v^{M,*}. \tag{2.74}$$

Then:

(1) $u^* = u^+$;

(2) all three shock speeds are equal:

$$v^{-,+} = v^{-,M} = v^{M,*}. \tag{2.75}$$

R2) On the other hand, if $\beth = 0$, then $u^* = u^+$.

Proof: Assume first $G(V^+) - G(V^-)$ and $G(V^M) - G(V^-)$ are LI. Disregarding the indices that indicate the physical situations in the cumulative and flux terms, the *RH* condition for (V^-, u^-)-(V^+, u^+), (V^-, u^-)-(V^M, u^M) and (V^M, u^M)-(V^+, u^*) can be written, respectively as:

$$v^{-,+}(G(V^+) - G(V^-)) = u^+F(V^+) - u^-F(V^-), \tag{2.76}$$

$$v^{-,M}(G(V^M) - G(V^-)) = u^MF(V^M) - u^-F(V^-), \tag{2.77}$$

$$v^{M,*}(G(V^+) - G(V^M)) = u^*F(V^+) - u^MF(V^M). \tag{2.78}$$

Assume now that Eq. (2.74.a) is satisfied. Substituting $v^{-,+} = v^{-,M} = v$ in Eqs. (2.76) and (2.77) and subtracting the resulting equations, we obtain:

$$v(G(V^+) - G(V^M)) = u^+F(V^+) - u^MF(V^M), \tag{2.79}$$

From Eq. (2.63), notice that the shock speeds depend solely of the primary variables and speed on the left state and on the primary variable of the right state. From Eqs. (2.78) and (2.79), we can see that the left and right states as well as left speed are the same, so $u^* = u^+$. Eq. (2.75) is satisfied also because the shock speed depends just on the initial and final states.

To prove $R2$, we subtract (2.76) from the sum of (2.77) with (2.78), obtaining:

$$\daleth = (u^+ - u^*)F(V^+).$$

If $\daleth = 0$, then $u^* = u^+$ and the result follows.

The other cases are proved similarly. $\square$

Remark 2.3.4. *The equality $\daleth = 0$ is a particular case of the Triple Shock Rule.*

Generically, we do not know if $V^+ \in \mathcal{RH}(V^-) \bigcap \mathcal{RH}(V^M)$. Therefore, we rewrite Prop. 2.3.4 in the more practical form:

Proposition 2.3.5. *(Quadruple Shock Rule II.) Consider two physical situations Ω^j and Ω^k for $k, j \in \mathcal{PS}$, with four primary points and four speeds, determining four states, namely (V^-, u^-) in Ω^k, (V^+, u^+) in Ω^j, (V^M, u^M) and (V^*, u^*) either in Ω^j or Ω^k. Assume that there are shocks between the following pairs of states:*

(i) (V^-, u^-) and (V^+, u^+) with speed $v^{-,+}$,

(ii) (V^-, u^-) and (V^M, u^M) with speed $v^{-,M}$,

(iii) (V^M, u^M) and (V^*, u^*) with speed $v^{M,*}$,

Assume also that the following conditions are satisfied:

(I) $G(V^+) - G(V^-)$ and $G(V^*) - G(V^M)$ are LI;

(II) two shock speeds coincide, i.e., at least one of the following shock speed equalities is satisfied:

$$\textit{either} \quad v^{-,+} = v^{-,M} \quad \textit{or} \quad v^{-,+} = v^{M,*} \quad \textit{or} \quad v^{-,M} = v^{M,*}. \tag{2.80}$$

(III) V^+ and V^* have a coordinate V_i with coinciding values;

(IV) The curve $\mathcal{RH}(V^M)$ does not possess two different points with the same coordinate V_i having coinciding values.

Then

(1) V^*=V^+;

(2) $u^* = u^+$;

(3) all three shock speeds are equal:

$$v^{-,+} = v^{-,M} = v^{M,*}. \tag{2.81}$$

Proof: Disregarding the indices that indicate the physical situations in the accumulation and flux terms, the RH conditions for (V^-,u^-)-(V^+,u^+), (V^-,u^-)-(V^M,u^M) and (V^M,u^M)-(V^*,u^*) are written, respectively as:

$$v^{-,+}(G(V^+) - G(V^-)) = u^+F(V^+) - u^-F(V^-), \tag{2.82}$$

$$v^{-,M}(G(V^M) - G(V^-)) = u^MF(V^M) - u^-F(V^-), \tag{2.83}$$

$$v^{M,*}(G(V^*) - G(V^M)) = u^*F(V^*) - u^MF(V^M). \tag{2.84}$$

Assume that now Eq. (2.80.a) is satisfied. Substituting $v^{-,+} = v^{-,M} = v$ in Eqs. (2.82) and (2.83) and subtracting the resulting equations, we obtain:

$$v(G(V^+) - G(V^M)) = u^+F(V^+) - u^MF(V^M). \tag{2.85}$$

Notice that Eqs. (2.85) and (2.84) define implicitly the RH locus from (V^M,u^M). Since the RH locus in the variable V depends solely on V^M and the accumulation and flux functions, the RH locus defined by Eqs. (2.85) and (2.84) coincide. Because (III) and (IV) are satisfied, it follows that $V^* = V^+$.

Now from Eqs. (2.64), notice that the and shock speeds depend solely on the primary variables and speed of the left state and on the primary variable of the right state. From Eqs. (2.84) and (2.85), we can see that the left and the right states are the same for each expression and define the same RH locus, so $u^* = u^+$ and Eq. (2.81) is satisfied.

The other cases are proved similarly. □

2.3.3 Extension of the Bethe-Wendroff theorem

There are loci where the solutions change topology such as: secondary bifurcations, coincidences, double contacts, inflections, hystereses and interior boundary contacts. So the Riemann problem cannot be solved by the classical construction [43]. The Bethe-Wendroff theorem is an important tool to characterize such loci.

We prove an extension of the important Bethe-Wendroff theorem for shocks between regions in different situations. We recall that the left eigenvector ℓ does not depend on u. We recall the notation $[G] = (G^+ - G^-)$, where $G^+ = G(V^+)$ and $G^- = G(V^-)$.

Proposition 2.3.6. *Let $v^s(W^+;W^-)$ be the shock speed between different physical situations. Assume that $\ell^p(V^+) \cdot [G] \neq 0$. Then v^s has a critical point at W^+ and so $\hat{v}^+(V^-,V^+)$ has a critical point at V^+, if and only if:*

$$\hat{v}^+(V^-;V^+) = \lambda^{p,+}(V^+) \qquad \textit{for the family } p, \tag{2.86}$$

where $\lambda^{p,+}(V)$ is given by Eq. (2.46) and $\hat{v}^+(V^-;V^+)$ is given by (2.64.c).

Proof: Assuming that the RH curve can be parametrized by ζ in a neighborhood of (V^+,u^+), we can write the RH condition as:

$$v^s(G(V(\zeta)) - G^-) = u(\zeta)F(V(\zeta)) - u^-F^-, \tag{2.87}$$

where $v^s := v^s(\zeta)$. Differentiating (2.87) with respect to ζ we obtain:

$$\frac{dv^s}{d\zeta}(G(V(\zeta)) - G^-) + v^s \frac{\partial G(V(\zeta))}{\partial W}\frac{dW}{d\zeta} = \frac{\partial\,(u(\zeta)F(V(\zeta)))}{\partial W}\frac{dW}{d\zeta}, \tag{2.88}$$

Setting ζ^+ such that $(V(\zeta^+), u(\zeta^+)) = W(\zeta^+) = W^+ = (V^+, u^+)$, Eq. (2.88) yields:

$$[G]\frac{dv^s}{d\zeta} + v^s \frac{\partial G}{\partial W}\frac{dW}{d\zeta} = \frac{\partial\,(uF)}{\partial W}\frac{dW}{d\zeta}, \tag{2.89}$$

where $W = (V, u)$. Assume first that (2.86) is satisfied. Notice that if $\hat{v}^+(V^-, V^+) = \hat{\lambda}^+$, then for $(V^+, u = u^+)$, we have $\lambda = u^+\hat{\lambda}^+$ and $v^s(V^-, u^-; V^+) = \lambda(V^+, u^+)$ (we dropped the family index p). Substituting them in (2.89) we obtain at $(+) = (V^+, u^+)$ the expression (2.89) with $v^s := u\lambda/u$. Let ℓ the left eigenvector associated to $\hat{\lambda}^+$; taking the inner product of (2.89) at (V^+, u^+) by ℓ, we obtain:

$$\ell \cdot [G]\frac{dv^s}{d\zeta} + \ell \cdot \left(u\frac{\lambda}{u}\frac{\partial G}{\partial W} - \frac{\partial\,(uF)}{\partial W}\right)\frac{dW}{d\zeta} = 0. \tag{2.90}$$

Since ℓ is an eigenvector associated to λ, the second term of (2.90) is zero and it follows that:

$$\ell \cdot [G]\frac{dv^s}{d\zeta} = 0.$$

Since by hypothesis $\ell \cdot [G] \neq 0$, we obtain that $dv^s/d\zeta = 0$ and the shock speed is critical.

On other hand, assume that v^s has a critical point, so $dv^s/d\zeta = 0$ and Eq. (2.89) reduces to:

$$v^s \frac{\partial G}{\partial W}\frac{dW}{d\zeta} = \frac{\partial\,(uF)}{\partial W}\frac{dW}{d\zeta}, \quad \text{or} \quad \left(\frac{\partial\,(uF)}{\partial W} - v^s\frac{\partial G}{\partial W}\right)\frac{dW}{d\zeta} = 0. \tag{2.91}$$

Eq. (2.91) has a solution if, only if,

$$v^s(V^-, u^-; V^+) = \lambda(V^+, u^+) \qquad \text{so} \qquad \hat{v}^+(V^-; V^+) = \hat{\lambda}^+(V^+).$$

□

Following Oleinik [53] and Liu [47, 48], we can obtain a useful result for equations in the form (2.2):

Proposition 2.3.7. *Let $\hat{v}^+(V^+; V^-)$ be the shock speed. Assume that $\ell^+ \cdot (G^+ - G^-) \neq 0$ for ℓ, r and λ belong to the same family p. Assume that $\hat{v}^+$ has a critical point at V^+. Then the following relations are true:*

i) If $\hat{v}^+$ has a minimum:

$$\frac{(\ell^+ \cdot \partial_W G^+ r^+)\,(\nabla\lambda \cdot r^+)}{\ell^+ \cdot [G]} > 0. \tag{2.92}$$

ii) If $\hat{v}^+$ has a maximum:

$$\frac{(\ell^+ \cdot \partial_W G^+ r^+)\,(\nabla\lambda \cdot r^+)}{\ell^+ \cdot [G]} < 0, \tag{2.93}$$

where $\partial_W G^+ := (\partial G/\partial W)\,(V^+)$, $\ell^+ := \ell(V^+)$ and $r := r(V^+, u^+)$. Moreover the inequalities (2.92) and (2.93) do not depend on the speed u.

Proof: Assume that the shock and rarefaction curves can be parametrized by ζ in a neighborhood of (V^+, u^+) (if the curves exist, these parametrizations exists in a small neighborhood of (V^+, u^+)).

Differentiating Eq. (2.88) again we obtain:

$$\begin{aligned} &\frac{d^2 v^s}{d^2\zeta}(G(V(\zeta)) - G^-) + 2\frac{dv^s}{d\zeta}\frac{\partial G(V(\zeta))}{\partial W}\frac{dW}{d\zeta} + v^s\frac{\partial^2 G(V(\zeta))}{\partial^2 W}\left(\frac{dW}{d\zeta}, \frac{dW}{d\zeta}\right) + \\ &+ v^s\frac{\partial G(V(\zeta))}{\partial W}\frac{d^2 W}{d^2\zeta} = \frac{\partial^2\left(u(\zeta)F(V(\zeta))\right)}{\partial^2 W}\left(\frac{dW}{d\zeta}, \frac{dW}{d\zeta}\right) + \frac{\partial^2\left(u(\zeta)F(V(\zeta))\right)}{\partial^2 W}\frac{d^2 W}{d^2\zeta}, \end{aligned} \tag{2.94}$$

Set $\zeta = \zeta^+$ and denote $W(\zeta^+) = W^+$. Notice that since $\ell^+ \cdot [G] \neq 0$ and $\hat{v}^+$ has a critical point at V^+, by Proposition 2.3.6 the equality $dv^s/d\zeta = 0$ is satisfied at V^+ and $dW/d\zeta = r := r^+$. So (2.94) reduces to:

$$[G]\frac{d^2 v^s}{d^2\zeta} + \lambda\left(\frac{\partial^2 G}{\partial^2 W}(r,r) + \frac{\partial G}{\partial W}\frac{d^2 W}{d^2\zeta}\right) = \frac{\partial^2 (uF)}{\partial^2 W}(r,r) + \frac{\partial^2 (uF)}{\partial^2 W}\frac{d^2 W}{d^2\zeta}, \tag{2.95}$$

We recall that Eq. (2.30) can be written as:

$$\lambda\frac{\partial G}{\partial W}r = \frac{\partial (uF)}{\partial W}r. \tag{2.96}$$

Differentiating (2.96) with respect to ζ along a rarefaction curve starting at (V^+, u^+), we obtain:

$$\frac{d\lambda}{d\zeta}\frac{\partial G}{\partial W}r + \lambda\frac{\partial^2 G}{\partial^2 W}(r,r) + \lambda\frac{\partial G}{\partial W}\frac{dr}{d\zeta} = \frac{\partial^2 (uF)}{\partial^2 W}(r,r) + \frac{\partial (uF)}{\partial W}\frac{dr}{d\zeta}, \tag{2.97}$$

where $d\lambda/d\zeta = \nabla\lambda \cdot r$ and we set $\zeta = \zeta^+$ in (2.97). Manipulations of Eqs. (2.95) and (2.97) yield:

$$[G]\frac{d^2 v^s}{d^2\zeta} - \frac{\partial G}{\partial W}r + \left(u\lambda\frac{\partial G}{\partial W}\nabla\lambda \cdot r - \frac{\partial (uF)}{\partial W}\right)\left(\frac{d^2 W}{d^2\zeta} - \frac{dr}{d\zeta}\right) = 0 \tag{2.98}$$

We multiply (2.98) by the left eigenvector ℓ. Notice that the last term in (2.98) vanishes, i.e.:

$$\ell \cdot \left(u\lambda\frac{\partial G}{\partial W} - \frac{\partial (uF)}{\partial W}\right)\left(\frac{d^2 W}{d^2\zeta} - \frac{dr}{d\zeta}\right) = 0, \tag{2.99}$$

because from Lemma 2.2.1, ℓ is eigenvector associated to $\hat{\lambda}^{+}$. The system (2.98) reduces to:

$$\ell \cdot [G]\frac{d^2 v^s}{d^2\zeta} - \left(\ell \cdot \frac{\partial G}{\partial W} r\right) \nabla\lambda \cdot r = 0, \tag{2.100}$$

after some manipulations, we see that the inequalities (2.92) and (2.93) are satisfied.

Notice that the inequalities (2.92) and (2.93) do not depend on the speed u because:

i)- The matrix $\partial_W G^{+}$ has the form (2.28) and by Lemma 2.2.1, the right eigenvectors r have the form (2.36.a), so the expression $\partial_W G^{+} r$ is independent of u.

ii)- From Lemma 2.2.1, we know that ℓ does not depend on the speed u, so $\ell \cdot \partial_W G^{+} r$ does not depend on u.

Since G is solely a function of V, the claim follows. $\square$

Lemma 2.3.2. *Assume that the shock and rarefaction curves can be parametrized by ζ in a neighborhood $\mathcal{B}_{\delta}^{j}(V^{+}, u^{+})$, with $\zeta = \zeta^{+}$ at (V^{+}, u^{+}). Assume also that G is smooth, V^{-} is sufficiently close to V^{+} and $\hat{v}^{+}(V^{-}; V^{+}) = \hat{\lambda}^{+,p}(V^{+})$ for some family p for λ, ℓ and r, then we approximate $\ell^{+} \cdot [G]$ by:*

$$\ell^{+} \cdot [G] \simeq \mathcal{A}(\zeta^{-}; \zeta^{+}) = \ell^{+} \partial_W G^{+} r^{+} (\zeta^{+} - \zeta^{-}), \tag{2.101}$$

where $V^{-} = V(\zeta^{-})$, $V^{+} = V(\zeta^{+})$ and $|\zeta^{+} - \zeta^{-}| < \epsilon$.

Proof: Since G is smooth, we can approximate $[G]$ by:

$$[G] = \partial_W G^{+} \frac{dW}{d\zeta}(\zeta^{+} - \zeta^{-}) + \mathbb{O}\left((\zeta^{+} - \zeta^{-})^2\right).$$

Disregarding terms of order 2, since $\hat{v}^{+}(V^{-}; V^{+}) = \hat{\lambda}^{+}(V^{+})$, by Proposition 2.3.6 it follows that $dW(\zeta^{+})/d\zeta = r(V^{+}, u^{+})$, so Eq. (2.101) is satisfied and the Lemma follows. $\square$

Notice that it is not necessary to know u^{-} and u^{+}, because (2.101) does not depend on the speed u.

Proposition 2.3.8. *Assume the hypotheses of Proposition 2.3.7. Then the following relationships are valid:*

i) If $\mathcal{A}(\xi^{-}, \xi^{+}) > 0$, then $\hat{v}^{+}$ has a minimum.

ii) If $\mathcal{A}(\xi^{-}, \xi^{+}) < 0$, then $\hat{v}^{+}$ has a maximum.

Proof: First, let V^{+} and V^{-} satisfy hypotheses of Lemma 2.3.2. From Eqs. (2.100) and (2.101), we know that $d^2 v^s/d^2\zeta$, satisfies:

$$\frac{d^2 v^s}{d^2\zeta} \simeq \frac{\ell^{+} B^{+} r^{+}}{\ell^{+} B^{+} r^{+} (\zeta^{+} - \zeta^{-}) \nabla\lambda \cdot r^{+}} = \frac{\nabla\lambda \cdot r^{+}}{\zeta^{+} - \zeta^{-}}, \tag{2.102}$$

Observe that if $\nabla\lambda \cdot r^{+} > 0$, then by the *geometrical compatibility* it is necessary that $\zeta^{+} > \zeta^{-}$. On the other hand, if $\nabla\lambda \cdot r^{+} < 0$, then $\zeta^{+} < \zeta^{-}$, by the same argument. Using Eq. (2.102), we obtain that $d^2 v^s/d^2\zeta > 0$, i.e., v^s (and $\hat{v}^{+}$) is minimum at V^{+}.

Now let V^- and V^+ be such that $\mathcal{A}(\xi^-,\xi^+) > 0$. Since $\ell^+ B^+ r^+ \nabla\lambda \cdot r^+$ depends solely on V^+, it is necessary to analyze only $\ell^+ \cdot [G]$. Since $\mathcal{A}(\xi^-,\xi^+) > 0$, $\ell^+ \cdot [G]$ has the same sign as $\ell^+ B^+ r^+ (\zeta^+ - \zeta^-)$, then $d^2 v^s / d^2 s > 0$ and v^s (and so $\hat{v}^+$) is minimum at V^+.

Now let V^- and V^+ be such that $\mathcal{A}(\xi^-,\xi^+) < 0$, then $\ell^+ \cdot [G]$ and $\ell^+ B^+ r^+ (\zeta^+ - \zeta^-)$ have opposite signs, so $d^2 v^s / d^2 s < 0$ and v^s (and $\hat{v}^+$) is maximum at V^+. $\square$

Secondary bifurcation manifold

The RH locus is obtained implicitly by $\mathcal{H}_l(V^-, V^+) = 0$ for $l \in \mathcal{C}_{RH}$, where $\mathcal{H}_l : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$, and V^- and V^+ are the primary variables. Notice that $\mathcal{RH}$ is 1-dimensional locus, defined by expressions (2.59). However, sometimes these implicit expressions fail to define a curve for V^+ in the different physical situations, typically the curve self-intersects. We call these points V^+ **secondary bifurcation** and, from the implicit function theorem, they are the $(+)$ states for which there exists a $(-)$ state such that the following equalities are satisfied:

$$\mathcal{H}_l = 0 \qquad \text{for } l \in \{\overrightarrow{1}, \overrightarrow{2}, \cdots, \overrightarrow{n-1}\} \tag{2.103}$$

and

$$J_{\mathcal{H}} = \frac{\partial \mathcal{H}_l}{\partial V^+} \quad \text{does not have maximal rank} \quad \text{for all } l \in \{\overrightarrow{1}, \overrightarrow{2}, \cdots, \overrightarrow{n-1}\}, \tag{2.104}$$

where $\mathcal{H}_l$ is defined implicitly by (2.57) and $J_{\mathcal{H}}$ is the Jacobian matrix of the vectors $\mathcal{H}_l$ for $l \in \{\overrightarrow{1}, \overrightarrow{2}, \cdots, \overrightarrow{n-1}\}$ corresponds to a triplet of $\mathcal{C}_{RH}$. The Jacobian is an $n \times (n-1)$ matrix in the following form:

$$\begin{pmatrix} \dfrac{\partial \mathcal{H}_{\overrightarrow{1}}}{\partial V_1} & \dfrac{\partial \mathcal{H}_{\overrightarrow{1}}}{\partial V_2} & \cdots & \dfrac{\partial \mathcal{H}_{\overrightarrow{1}}}{\partial V_n} \\ \dfrac{\partial \mathcal{H}_{\overrightarrow{2}}}{\partial V_1} & \dfrac{\partial \mathcal{H}_{\overrightarrow{2}}}{\partial V_2} & \cdots & \dfrac{\partial \mathcal{H}_{\overrightarrow{2}}}{\partial V_n} \\ \vdots & \vdots & \vdots & s \\ \dfrac{\partial \mathcal{H}_{\overrightarrow{n-1}}}{\partial V_1} & \dfrac{\partial \mathcal{H}_{\overrightarrow{n-1}}}{\partial V_2} & \cdots & \dfrac{\partial \mathcal{H}_{\overrightarrow{n-1}}}{\partial V_n} \end{pmatrix} \tag{2.105}$$

The Jacobian does not have maximal rank if any two minors of size $(n-1) \times (n-1)$ have determinant zero. Following the same ideas to obtain RH locus, we can prove that:

Corollary 2.3.3. *The secondary bifurcation is an $(n-1)$ dimensional structure in the space of n variables V^+, i.e., it is generically codimension-1 in the space V^+.*

2.4 Regular *RH* locus

Definition 2.4.1. *Given a state $V^- \in \mathcal{P}^j$ for $j \in \mathcal{PS}$, we say that the RH locus for V^- is* ***regular*** *if for all $k \in \mathcal{PS}$ (including $k = j$) and for each $V^+ \in \mathcal{P}^k$ with $V^+ \neq V^-$ the shock speed v^s and the speed u are finite and there are at least two distinct indices $p, q \in \mathcal{C}$ depending on k and j such that:*

1.

$$F_q[G_p] - F_p[G_q] \neq 0; \tag{2.106}$$

2. $[G_q] \neq 0$ *and the following limit exist:*

$$\lim_{V^+ \longrightarrow V^-} \frac{[G_p]}{[G_q]}. \tag{2.107}$$

3. *The following inequality is satisfied:*

$$F_p^- \neq F_q^- \lim_{V^+ \longrightarrow V^-} \frac{[G_p]}{[G_q]}. \tag{2.108}$$

Remark 2.4.1. *When V^- is a frontier point, then the limits in* (2.107) *and* (2.108) *may not be well defined. However, since we are interested in obtaining the solution in the physical situation $\mathcal{P}^j$ we regard the limits as lateral limits.*

2.4.1 Small shocks

A shock between two states W^- and W^+ is said to be small if there exists a sufficiently small $\epsilon > 0$ such that $|W^+ - W^-| < \epsilon$. For this type of shock we can prove important results that can be extended to more general shocks.

Speed for regular small shocks

Lemma 2.4.1. *Assume that F and G are $\mathcal{C}^2$ for $V \in \mathcal{P}^j$. In a regular system, when the primary variable V^+ converges to V^- the speed u^+ converges to u^-.*

Proof:
Let V^- be fixed. From eq. (2.63.b), we define for all $V^+ \in \mathcal{P}^j$ the expression:

$$u^+ = u^- \frac{F_q^-[G_p] - F_p^-[G_q]}{F_q^+[G_p] - F_p^+[G_q]}. \tag{2.109}$$

We can rewrite (2.109) as:

$$(u^+ - u^-)\left(F_q^-[G_p] - F_p^-[G_q]\right) = u^+\left([F_q][G_p] - [F_p][G_q]\right). \tag{2.110}$$

Assume that $[G_q] \neq 0$. Dividing (2.110) by $[G_q]$, we obtain:

$$(u^+ - u^-)\left(F_q^- \frac{[G_p]}{[G_q]} - F_p^-\right) = u^+ \left([F_q]\frac{[G_p]}{[G_q]} - [F_p]\right). \tag{2.111}$$

Let $V^+ \in \mathcal{RH}(V^-)$ tend to V^- and remember that the limit (2.107) exists and (2.108) is satisfied. From Eqs. (2.106)-(2.108), u^+ is bounded. Notice that $[F_q]$ and $[F_p]$ tend to zero when V^+ tends to V^-, so $u = u^+ = u^-$. $\square$

Extension of Lax theory

Often shocks between regions are not weak and the cumulative and flux terms are not smooth; the Lax theory cannot be extended to this type of shocks. However, within a physical situation and under certain hypotheses we can obtain a theory for the Riemann solution similar to Lax's. We consider a physical situation $j \in \mathcal{PS}$.

We say that the system (2.2) represents a strictly hyperbolic physical situation if there are n distinct eigenvalues $\lambda_p = u\vartheta_p(V)$ that satisfy:

$$\vartheta_1(V) < \vartheta_2(V) < \cdots < \vartheta_n(V). \tag{2.112}$$

We know that if (2.112) is satisfied, there exist n right r^p (and n left ℓ^p) eigenvectors that are LI.

From (2.36) of Lemma 2.2.1, the right and left eigenvectors of the system (2.2) have the form $r^p = (g_1^p(V), \cdots, g_n^p(V), u g_{n+1}^p(V))$ and $\ell^p = (\ell_1^p(V), \cdots, \ell_n^p(V), \ell_{n+1}^p(V))$. Notice that neither the right nor the left vectors form bases for the space (V, u).

Nevertheless, the left and right eigenvectors satisfy the following lemma:

Lemma 2.4.2. *Consider $\lambda_i \neq \lambda_j$, and the associate right and left eigenvectors r^i and ℓ^j, then:*

$$\ell^j \partial_W G r^i = 0. \tag{2.113}$$

We can prove now that if the system (2.2) is satisfied in the regular case there exists a local basis of shock curves in the space of primary variables in each physical situation.

Proposition 2.4.1. *Let $W^- = (V^-, u^-)$ be any point in phase space with V^- in the interior of $\mathcal{P}^j$. Then there are n $\mathcal{C}^2$ one-parameter curves; of states $V_k = V_k(\theta)$ for $k = 1, 2, \cdots, n$ and θ sufficiently small, where $V_k(0) = V^-$. The points $V_k(\theta)$ satisfy the RH condition* (2.52)*. Moreover the following relationships are satisfied:*

$$V_k(\theta) = V^- + \theta \hat{r}^k(V^-) + \frac{\theta^2}{2}\partial_V \hat{r}^k(V^-)\hat{r}^k(V^-) + \mathbb{O}(\theta^3) \tag{2.114}$$

and

$$v_k^s = \lambda_k(W^-) + \frac{\theta^2}{2}\partial_W \lambda_l(W^-) \cdot r^k(W^-) + \mathbb{O}(\theta^2). \tag{2.115}$$

We can also obtain the speed $u_k = u_k(V(\theta))$ from $V_k(\theta)$ and Eq. (2.64.b)*, for $k = 1, 2, \cdots, n$.*

Proof: We follow the same idea of the original theorem of Lax [43]. We can write:

$$
\begin{aligned}
v^s(G(V)-G(V^-)) &= \int_0^1 \frac{d}{d\sigma}(G(V^-+\sigma(V-V^-)))d\sigma \\
&= \int_0^1 \partial_V G(V^-+\sigma(V-V^-))(V-V^-)d\sigma
\end{aligned}
\tag{2.116}
$$

Similarly, from Eq. (2.64.a), we can rewrite $uF(V)-u^-F(V^-)$. Using $u=u^-\chi$, with $\chi:=\chi(V^-,V)$ we define $\hat{W}-\hat{W}^-=(V-V^-,\chi-1)$, and $\mathbb{F}$ as follows:

$$
\begin{aligned}
uF(V)-u^-F(V^-) &= u^-\int_0^1 \frac{d}{d\sigma}\Big((1+\sigma(\chi-1))F(V^-+\sigma(V-V^-))\Big)d\sigma \\
&= \mathbb{F}(\hat{W}-\hat{W}^-),
\end{aligned}
\tag{2.117}
$$

where $\mathbb{F}:=\mathbb{F}(W^-,W^+)$ is an $(n+1)\times(n+1)$ matrix given by:

$$
\mathbb{F}:=u^-\int_0^1 \left((1+\sigma(\chi-1))\partial_V F(V^-+\sigma(V-V^-)), F(V^-+\sigma(V-V^-))\right)d\sigma.
$$

and $\hat{W}-\hat{W}^-=(V-V^-,\chi-1)$.

From Eq. (2.116) we can see that $v^s(G(V)-G(V^-))$ does not depend on u. However, it is necessary rewrite it in the variable $\hat{W}$, so we define the $(n+1)\times(n+1)$ matrix $\mathbb{G}(V)$ as:

$$
\mathbb{G}(V):=\int_0^1 \left(\partial_V G(V^-+\sigma(V-V^-),0)\right)d\sigma.
$$

Thus, we can rewrite $v^s(G(V)-G(V^-))$ as:

$$
v^s(G(V)-G(V^-)) = v^s\mathbb{G}(V)(\hat{W}-W^-).
\tag{2.118}
$$

Using Eqs. (2.117) and (2.118), the Rankine-Hugoniot condition (2.50) becomes:

$$
\left(\mathbb{F}(V)-v^s\mathbb{G}(V)\right)(V-V^-,\chi-1)=0.
\tag{2.119}
$$

Using (2.119), we know that a state $V\in\mathcal{N}$ belongs to the RHL locus of V^-, if, only if, there exists a $k\in\{1,2,\cdots,n\}$ such that:

$$
v_k^s(V^-,u^-;V)=\mu_k(u^-,V).
\tag{2.120}
$$

and that $(V-V^-,\chi-1)$ is a right eigenvector of $\mathbb{F}(V)-\mu\mathbb{G}(V)$ associated to $\mu_k(u^-,V)$.

Notice that when V converges to V^- by the Lemma 2.4.1 the speed u converges to u^-, so the matrices $\mathbb{F}(V)\longrightarrow A$ and $\mathbb{G}(V)\longrightarrow B$, with B and A given by Eq. (2.28) and (2.29). Since $A-\lambda B$ has n distinct eigenvalues that satisfy (2.112), and the functions $\mathbb{F}(V)$ and $\mathbb{G}(V)$ are continuous, there exists a neighborhood of $\mathcal{N}$ of V^- in the topology of $\mathcal{P}^j$ and n distinct functions $\mu_k(V,V^-,u^-)$ that are the eigenvalues of $\mathbb{F}(V)-\mu\mathbb{G}(V)$, that satisfy $\mu_k(V^-,V^-,u^-)=\lambda(V^-,u^-)$.

We denote the left eigenvectors of the system $\mathbb{F}(V) - \mu\mathbb{G}(V)$ by $(L^k(V))^T$. Since $(L^k(V))^T$ is not a basis in (V, u) space, for $(V - V^-, \chi - 1)$ to be a right eigenvector, the condition:

$$L^j(V)\mathbb{G}(V - V^-, \chi - 1) = 0 \qquad j \neq k \tag{2.121}$$

is only necessary. Notice that the first n coordinates of the eigenvectors are uniquely defined by Eq. (2.121). Moreover, if $V \in \mathcal{N}$ belongs to the RHL of V^-, the speed u is given by Eq. (2.64.a), thus Eq. (2.121) defines uniquely the eigenvectors of $\mathbb{F}(V) - \mu\mathbb{G}(V)$.

Since $\mathbb{G}$ has a zero column, Eq. (2.121) does not depend on $\chi - 1$, and Eq. (2.121) is a system with $(n-1)$ equations in n unknowns V. This system which can be written as:

$$\Phi_k(V) = \mathbb{M}_k(V)(V - V^-) = 0, \tag{2.122}$$

where

$$\mathbb{M}_k(V) = \begin{pmatrix} \left(L^1(V)\right)^T \\ \vdots \\ \left(L^{k-1}(V)\right)^T \\ \left(L^{k+1}(V)\right)^T \\ \vdots \\ \left(L^n(V)\right)^T \end{pmatrix} \hat{\mathbb{G}}, \tag{2.123}$$

with $\hat{\mathbb{G}} = \int_0^1 \partial_V G(V^- + \sigma(V - V^-))d\sigma$.

Notice that at V^- we obtain:

$$\Phi_k(V^-) = 0 \qquad \text{and} \qquad \partial_V \Phi_k(V^-) = M_k(V^-). \tag{2.124}$$

Since $L(V^-) = \ell(V^-)$ and $\hat{\mathbb{G}}$ have rank n, the matrix $M_k(V)$ also rank $n-1$, yielding that the matrix $\partial_V \Phi_k(V^-)$ has rank $n-1$. By the implicit function theorem, there exists a one-parameter family of functions $V_k(\epsilon)$, for $1 \leq k \leq n$ and $|\epsilon| \leq \epsilon_1$ small enough, with

$$V_k(0) = V^-, \tag{2.125}$$

and the n different waves with shock speed satisfying:

$$\lim_{V \longrightarrow V^-} v^s(V^-, u^-; V) = \lambda_k(V^-, u^-) \qquad \text{for } 1 \leq k \leq n. \tag{2.126}$$

Letting ϵ tend to zero in Eqs. (2.121), we obtain that $dV(0)/d\epsilon$ is parallel to $\hat{r}^k(V^-)$. After a change in parametrization, we obtain

$$dV(0)/d\epsilon = \hat{r}^k(V^-). \tag{2.127}$$

After obtaining V, Eq. (2.87) determines u. Differentiating (2.89), using that v^s satisfies Eq. (2.87), from (2.127) it follows that:

$$\left(\frac{dV}{d\epsilon}, \frac{du}{d\epsilon}\right)(0) = r^k(V^-) \qquad \text{for } k = 1, 2, \cdots, n. \tag{2.128}$$

We drop the index k in the expressions. Now using Eqs. (2.94) and (2.97) we can prove the expansion (2.115). To do so, first we subtract (2.94) from (2.97) and set $\epsilon = 0$ we obtain:

$$2\frac{dv^s}{d\epsilon}\left(\ell\frac{\partial G}{\partial W}r\right) - \frac{\partial G}{\partial W}r\nabla\lambda\cdot r + \left(\lambda\frac{\partial G}{\partial W} - \frac{\partial\,(uF)}{\partial W}\right)\left(\frac{d^2W}{d^2\epsilon} - \frac{dr}{d\epsilon}\right) = 0, \tag{2.129}$$

with $\lambda = \lambda(V^-, u^-)$ and $r = r^k(V^-, u^-)$ for some $k = 1, 2, \cdots, n$. Multiplying Eq. (2.129) by ℓ and using (2.99), we obtain:

$$2\frac{dv^s}{d\epsilon}\left(\ell\frac{\partial G}{\partial W}r\right) = \left(\ell\frac{\partial G}{\partial W}r\right)\nabla\lambda\cdot r, \tag{2.130}$$

Since $\ell\partial_W Gr \neq 0$, we get:

$$\frac{dv^s}{d\epsilon} = \frac{1}{2}\nabla\lambda\cdot r. \tag{2.131}$$

Substituting (2.131) in Eq. (2.129) and noticing that $dr/d\epsilon = \partial_W r\cdot r$, we obtain:

$$\left(\lambda\frac{\partial G}{\partial W} - \frac{\partial\,(uF)}{\partial W}\right)\left(\frac{d^2W}{d^2\epsilon} - \frac{dr}{d\epsilon}\right). \tag{2.132}$$

Therefore, there exists a real number α such that

$$\left(\frac{d^2V}{d^2\epsilon}, \frac{d^2u}{d^2\epsilon}\right) = \frac{d^2W}{d^2\epsilon} = \left(\frac{d\hat{r}}{d\epsilon} + \alpha\hat{r}, u\frac{dr_{n+1}}{d\epsilon} + \alpha u r_{n+1}\right). \tag{2.133}$$

Changing the parametrization by setting:

$$\epsilon = \theta - \frac{1}{2}\alpha\theta^2. \tag{2.134}$$

Then using Taylor expansion, Eqs. (2.126) and (2.131) yield Eq. (2.114):

$$\begin{aligned} v^s &= \lambda + \frac{\epsilon}{2}\nabla\lambda\cdot r + \mathbb{O}(\epsilon^2), \\ &= \lambda + \frac{\theta}{2}\nabla\lambda\cdot r + \mathbb{O}(\epsilon^2). \end{aligned}$$

Eq. (2.115) is obtained by using Eq. (2.133):

$$\begin{aligned} V(\epsilon) &= V^- + \epsilon\hat{r} + \frac{\epsilon^2}{2}\left(\partial_V\hat{r}\cdot\hat{r} + \alpha\hat{r}\right) + \mathbb{O}(\epsilon^3) \\ &= V^- + \theta\hat{r} + \frac{\theta^2}{2}\partial_V\hat{r}\cdot\hat{r} + \mathbb{O}(\epsilon^3). \qquad \square \end{aligned}$$

Under certain hypotheses, we can prove the Theorem of Lax to construct the local Riemann solution within each physical situation.

2.4.2 Other degeneracies of RH locus

In addition to the linear degeneracy 2.3.2, we study another degeneracy that seldom occurs. To do so, we utilize the notation:

$$\mathcal{D}_{pq}(V^-, V^+) := F_p^+[G_q] - F_q^+[G_p].$$

Proposition 2.3.1 is valid if non zero denominators can be found for Eq. (2.66). So it is necessary to study the behavior of the solution when $\mathcal{D}_{pq}(V^-, V^+) = 0$ for all $p, q \in \mathcal{C}$. For a fixed pair $p, q \in \mathcal{C}$, we can follow the same steps of Proposition 4.4.3 and prove:

Proposition 2.4.2. *If $\mathcal{D}_{pq}(V^-, V^+) = 0$ for all $p, q \in \mathcal{C}$, it follows that:*

$$[G_k] = \rho_1 F_k^+ \qquad and \qquad F_k^- = \rho_2 F_k^+, \quad \forall\, k \in \mathcal{C}.$$

where ρ_1 and ρ_2 are numbers depending on $[G]$, F^- and F^+. Moreover the shock speed v^s satisfies:

$$v^s = \frac{\chi(V^-; V^+) - \rho_2}{\rho_1} u^-,$$

where $\chi(V^-; V^+)$ is given by Eq. (2.64.a).

2.4.3 The sign of u^+ on regular Rankine-Hugoniot loci

In the previous Propositions, it is necessary that u is never zero. Moreover, it is useful that u and the injection speed u^- have the same sign. From Eq. (2.43), these properties are clearly true on the integral curves (and so on rarefaction waves), but they are not obvious on shock waves. We now prove a sign invariance for the *connected branches* of the RHL containing (V^-, u^-), that we denote by $CRHL$.

To do so, for a fixed V^- in some $\mathcal{P}^j$, we define for each $V \in \mathcal{P}^k$ (including $k = j$) and for all index pairs $p, q \in \mathcal{C}$, the following continuous functions $\Xi_{pq} := \Xi_{pq}(V, V^-)$ from the denominator and numerator of the expression for u given in (2.64.b):

$$\Xi_{pq} = \left\{F_p^-(G_q - G_q^-) - F_q^-(G_p - G_p^-)\right\}\left\{F_p(G_q - G_q^-) - F_q(G_p - G_p^-)\right\}, \qquad (2.135)$$

where the cumulative and flux terms depend on j and k and $G = G(V)$, $G^- = G(V^-)$, $F = F(V)$ and $F^- = F(V^-)$. The shock curves can occur between different physical situations, therefore it is useful to represent the states on RHL as belonging to Ω^j. Using (2.18), we can write $\Xi_{pq} := \Xi_{pq}(\mathcal{V}^-, \mathcal{V}^+)$.

Remark 2.4.2. *Notice that for $\mathcal{V}^-$ and $\mathcal{V}^+ \in \mathcal{RH}$ there exists at least a pair $p, q \in \mathcal{C}$ such that $\hat{X}_{pq}(\mathcal{V}^-, \mathcal{V}^+) \neq 0$ in the regular case. By the hypothesis of regular case, the speed u is finite, then using the Eq. (2.64.b) we can see that the denominator of expression (2.64.b) is zero, if only if, its numerator is zero. We conclude that if Ξ_{pq} is zero for all p, q, then the numerator of (2.64.b) is zero for any p, q, that is $\mathcal{D}_{pq} = 0$ for all p, q. It is impossible because it violates the hypothesis 1 of Definition* 2.4.1.

It follows from Prop. 2.1.1 that:

Lemma 2.4.3. *Ξ_{pq} are continuous functions for all $p, q \in \mathcal{C}$.*

Remark 2.4.3. *Notice that for u^+ given by (2.63), the inequality $u^+u^- < 0$ is satisfied if, only if, $\Xi_{pq}(V^-, V^+) < 0$.*

Remark 2.4.4. *For two fixed states $\mathcal{V}^-$ and $\mathcal{V}^+ \in \mathcal{RH}(V^-)$ for all pairs $p, q \in \mathcal{C}$ either $\Xi_{pq}(\mathcal{V}^-, \mathcal{V}^+) \leq 0$ or $\Xi_{pq}(\mathcal{V}^-, \mathcal{V}^+) \geq 0$. This fact occurs because for points in RH locus the speed is uniquely defined.*

Lemma 2.4.4. *For $\mathcal{V}^- \in \Omega^j$, the expression Ξ_{pq} given by Eq. (2.135) satisfies:*

1. *$\Xi_{pq} = 0$ for $\mathcal{V}^+ = \mathcal{V}^-$ and for all $p, q \in \mathcal{C}$;*

2. *There exists an $\epsilon > 0$ and and indices $p, q \in \mathcal{C}$ such that for $\mathcal{V} \in CRHL \bigcap B^j_\epsilon(\mathcal{V}^-)$, the inequality $\Xi_{pq}(\mathcal{V}^-, \mathcal{V}^+) > 0$ is satisfied.*

Proof: The assertion (1) is immediate. Because $u^+ = u^-$ when $\mathcal{V}^+$ converges to $\mathcal{V}^-$, there exists an $\epsilon > 0$ such that for each $\mathcal{V} \in CRHL \bigcap B_\epsilon(\mathcal{V}^-)$, it follows that $\Xi_{pq}(\mathcal{V}^-, \mathcal{V}) \geq 0$. Moreover, since we assume that the system is regular in the sense of Definition 2.4.1, for each $V \in B_\epsilon(\mathcal{V}^-) \bigcap CRHL \backslash \mathcal{V}^-$ there exists at least a pair $p, q \in \mathcal{C}$ such that $\Xi_{pq}(\mathcal{V}^-, \mathcal{V}) > 0$, and (2) is valid. □

Definition 2.4.2. *For fixed $\mathcal{V}^- \in \Omega^j$, we utilize Ξ_{pq} given in (2.135) to define the two disjoint sets:*

$$\mathcal{J}^{-,0} := \left\{ \begin{array}{l} \mathcal{V}^k \in CRHL \bigcap \Omega^k, \text{ for all } k = \mathcal{PS}, \text{ such that for all } p, q \in \mathcal{C} \\ \Xi_{pq}(\mathcal{V}^-, \mathcal{V}^l) \leq 0, \text{ but there is at least a pair } l, m \in \mathcal{C}, \text{ such that } \Xi_{lm}(\mathcal{V}^-, \mathcal{V}^k) < 0 \end{array} \right\} \tag{2.136}$$

and

$$\mathcal{J}^{+,0} := \left\{ \begin{array}{l} \mathcal{V}^k \in CRHL \bigcap \Omega^k, \text{ for all } k \in \mathcal{PS}, \text{ such that for all } p, q \in \mathcal{C} \\ \Xi_{pq}(\mathcal{V}^-, \mathcal{V}^k) \geq 0, \text{ but there is at least a pair } l, m \in \mathcal{C}, \text{ such that } \Xi_{lm}(\mathcal{V}^-, \mathcal{V}^k) > 0 \end{array} \right\} \tag{2.137}$$

These two sets form a disjoint decomposition of $CRHL$ in Ω as summarized in the Lemma below:

Lemma 2.4.5. *$\mathcal{J}^{-,0} \bigcap \mathcal{J}^{+,0} = \emptyset$ and $\mathcal{J}^{-,0} \bigcup \mathcal{J}^{+,0} = CRHL(\mathcal{V}^-) \backslash \mathcal{V}^-$.*

Proof: Let a $\mathcal{V} \in \mathcal{J}^{-,0} \bigcap \mathcal{J}^{+,0}$. If $\mathcal{V} \in \mathcal{J}^{-,0}$ then $\Xi_{pq}(\mathcal{V}^-, \mathcal{V}) \leq 0$ for all $p, q \in \mathcal{C}$ and there exists at least a index pair l, m such that $\Xi_{lm}(\mathcal{V}^-, \mathcal{V}) < 0$; similarly, if $\mathcal{V} \in \mathcal{J}^{+,0}$, then there exists a index pair $i, j \in \mathcal{C}$ such that $\Xi_{ij}(\mathcal{V}^-, \mathcal{V}) > 0$. From Remark 2.4.4, we know that it does not occurs, so $\mathcal{J}^{-,0} \bigcap \mathcal{J}^{+,0} = \emptyset$.

Consider now $\mathcal{V} \in CRHL(\mathcal{V}^-)$. It is immediate that $\mathcal{V}$ belongs to the $\mathcal{J}^{-,0} \bigcap \mathcal{J}^{+,0}$. Notice that $\mathcal{V}^-$ there is no in $\mathcal{J}^{-,0} \bigcap \mathcal{J}^{+,0}$ because of Lemma 2.4.4. □

From the continuity of $\Xi_{pq}(\mathcal{V}^-, \mathcal{V})$ we obtain the following lemma.

Lemma 2.4.6. *The sets $\mathcal{J}^{-,0}$ and $\mathcal{J}^{+,0}$ are connected and open in CRHL.*

Since CRHL is connected and from Lemma 2.4.4 we know that there is a pair $p, q \in \mathcal{C}$ such that $\Xi_{pq}(\mathcal{V}^-, \mathcal{V})$ for $\mathcal{V}$ belongs to the $CRHL(\mathcal{V}^-)$ in a neighborhood, the Lemmas 2.4.5 and 2.4.6 we yield:

Proposition 2.4.3. *Let any $\mathcal{V}^- \in \Omega^j$. The speed u has the same sign of the speed u^- for the state $\mathcal{V}^-$ on the CRHL.*

We show some applications of the theory developed in Appendix C.

CHAPTER 3

Mathematical modelling of thermal oil recovery with distillation

We present a physical model for steam and gaseous volatile oil injection into a horizontal, linear porous medium filled with oil and water. The oil in the porous medium consists of dead oil and dissolved volatile oil. Dead oil is an oil with negligible vapor pressure while volatile oil is an oil with substantial vapor pressure. The model is based on mass balance and energy conservation equations as well as on Darcy's law. We present the main physical definitions and summarize the equations of the model. This system of equations is a representative example of flow in porous media with mass interchange between phases (in the absence of gravitational effects). Such flows are not represented by conservation laws or hyperbolic equations both because they have source terms and because they are not evolutionary in all the variables. The model contains a system of 6 equations: 5 equations for mass balance and one equation for energy conservation, which is generically represented by:

$$\frac{\partial}{\partial t}\mathcal{G}(\mathcal{V}) + \frac{\partial}{\partial x}u\mathcal{F}(\mathcal{V}) = \mathcal{Q}(\mathcal{V}), \tag{3.1}$$

where $\mathcal{V} = (s_w, s_g, T, v_{ov}, y_{gv}) \subset \Omega \subset \mathbb{R}^5$ represents the 5 dependent variables to be determined for all x and $t \geq 0$. The cumulative and flux terms are $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_6\}$ and $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_6\}$. Here s_w and s_g are the water and gas saturations, T is the temperature, v_{ov} and y_{gv} are the volatile oil volume fraction in the oleic gaseous phases, respectively. The real-valued dependent variable u is the Darcy speed. In (3.1), $u\mathcal{F}_i$ is the flux for the conserved quantity $\mathcal{G}_i$ and $\partial\mathcal{G}_i/\partial t$ is the corresponding accumulation term, in each of the 6 equations. The source term $Q = \{Q_1, Q_2, \cdots, Q_6\}$ represents the water and volatile oil mass transfer between the phases; the component Q_6 vanishes because it represents the conservation of energy total. The functions $\mathcal{G}_i$ and $\mathcal{F}_i$ are continuous in the whole domain Ω, but later we will see that they are only piecewise smooth. In the (x,t) plane the terms Q often generate fast variations in the variables $\mathcal{V}(x,t)$, so that $Q(\mathcal{V}(x,t))$ may be regarded as distribution. Eq. (3.1) has another important feature: the variable u does not appear in the accumulation term, but only in the flux term.

Thermodynamics plays a very important role in the physical phenomena modelled by Eq. (3.1). The thermodynamical behavior of interest can be divided into systems in equilibrium or in quasi-equilibrium. For such types of behavior we can use the Gibb's phase rule, which determines the number of thermodynamical degrees of freedom:

$$f = c - p + 2, \tag{3.2}$$

where f is the number of thermodynamical degrees of freedom, c is the number of components and p is the number of phases. Such flows encompass particular physical situations, where one or several phases or chemical components are missing. We study the possible physical component and phase mixture situations in thermodynamical equilibrium or quasi-equilibrium that appear in the Riemann solution. For equilibrium situations, $Q = 0$. In the situations in quasi-equilibrium Q is not zero but very small, so that the thermodynamical degrees of freedom still satisfy Eq. (3.2) in all physical situations. Finally, Q has fast variation in the regions between the physical situations. In the model considered in this work, there is a map E that can be applied to the balance system (3.1). This map reduces (3.1) to simpler systems of conservation laws, that is, E eliminates the terms Q; F and G are obtained from $\mathcal{F}$ and $\mathcal{G}$ through E and reduces to equation of form (2.2) After this elimination, further simplifications may occur in each physical situation, so that each situation is described by a set of variables V that is a subset of the set of variables $\mathcal{V}$. Each of the systems of type (2.2) has fewer equations than the complete system (3.1). It is useful to define $W = (V, u)$. Although Eq. (2.2) is a conservation law, it is not a hyperbolic system, because the variable u appears solely in the flux term. Indeed, Eq. (2.2) has an infinite speed mode associated to u. Under certain hypotheses we show that if $u > 0$ the Riemann solution in the variables $\mathcal{V}$ does not depend on u. Thus the unknown variables in the system of equation for each physical situation are called *primary variables*, to be determined from the corresponding version of (2.2). Because the variable u can be obtained from the primary variables, we call it a *secondary variable.* The variables that are constant or that are obtained in a simple way from the primary variables are called *trivial variables*.

3.1 Physical and Mathematical Models

We consider the injection of steam and volatile oil into a linear horizontal porous rock cylinder with constant porosity and absolute permeability, containing water and oil initially. The oil consists of dead oil with or without volatile oil. There are three chemical components: water, volatile oil and dead oil, (w, v, d) as well as three immiscible phases: aqueous (liquid water), oleic (liquid oil) and gaseous, (w, o, g). We use the following convention in subscripts for physical properties: the first subscript (w, o, g) refers to the phase and the second subscript (w, v, d) refers to the component.

We use simple mixing rules. We disregard any heat of mixing both in the gaseous phase between steam and volatile oil and in the oleic phase between volatile oil and dead oil. Moreover we disregard any volume contraction effects resulting from mixing. The concentration $\left[kg/m^3\right]$ of (dead) volatile oil in the oleic phase is denoted as (ρ_{od}) ρ_{ov}. In the same way we define the concentration of the volatile oil (water vapor) in the gaseous phase as ρ_{gv} $\left(\rho_{gw}\right)$. For ideal fluids we obtain

$$\frac{\rho_{ov}}{\rho_V} + \frac{\rho_{od}}{\rho_D} = 1, \qquad \frac{\rho_{gw}}{\rho_{gW}} + \frac{\rho_{gv}}{\rho_{gV}} = 1, \tag{3.3}$$

where the capital subscripts indicates that the component is in a pure phase. The pure liquid densities of volatile and dead oil in the oleic phase, denoted by ρ_V, ρ_D $[kg/m^3]$, are considered to be independent of temperature; the pure vapor densities of steam and volatile oil in the gaseous phase, denoted by ρ_{gW} and ρ_{gV}, are considered to obey the ideal gas law, i.e.,

$$\rho_{gW} = \frac{M_W P}{RT}, \qquad \rho_{gV} = \frac{M_V P}{RT}, \tag{3.4}$$

where M_W, M_V denote the molar weights of water and volatile oil respectively. T is the variable temperature and the gas constant is $R = 8.31[J/mol/K]$. The pressure P is not a variable in this problem, but the fixed prevailing pressure value.

We define x_{ov} as the mole fraction of volatile oil in the oleic phase and y_{gv} the mole fraction of volatile oil in the vapor phase. In some physical situations, in addition to temperature and pressure, it is necessary to specify another thermodynamical variable: either x_{ov} or y_{gv}, which are explained in the following text.

Using Clausius-Clapeyron law, we can write the the pure water vapor pressure P_w as:

$$P_w\left(T\right) = P_o \exp\left(\frac{-M_W \Lambda_W\left(T_b^w\right)}{R}\left(\frac{1}{T} - \frac{1}{T_b^w}\right)\right), \tag{3.5}$$

where $\Lambda_W\left(T_b^w\right)$ $[J/kg]$ is its evaporation heat at its normal boiling temperature T_b^w $[K]$, at P_o, the atmospheric pressure. Eq. (3.5) is valid for any temperature in situations where there exists liquid water.

We also use Clausius-Clapeyron law for the vapor pressure of pure volatile oil:

$$P_V\left(T\right) = P_o \exp\left(\frac{-M_V \Lambda_V\left(T_b^v\right)}{R}\left(\frac{1}{T} - \frac{1}{T_b^v}\right)\right), \tag{3.6}$$

where $\Lambda_V\left(T_b^v\right)$ is its evaporation heat at its normal boiling temperature T_b^v and atmospheric pressure. Eq. (3.6) is valid for any temperature below T_b^v. In addition, we use Raoult's law, which states that the vapor pressure of volatile oil is equal to its pure vapor pressure given by Eq. (3.6) times the mole fraction x_{ov} of volatile oil dissolved in the oil phase. Therefore we obtain

$$P_v\left(T\right) = x_{ov} P_o \exp\left(\frac{-M_V \Lambda_V\left(T_b^v\right)}{R}\left(\frac{1}{T} - \frac{1}{T_b^v}\right)\right). \tag{3.7}$$

We assume that the prevailing pressure P is the sum of the two vapor pressures: P_w and P_v.

We can use Eq. (3.3.a) and define the volatile oil volume fraction and the dead oil volume fraction in the oleic phase respectively, as:

$$v_{ov} = \rho_{ov}/\rho_V, \quad 1 - v_{ov} = \rho_{od}/\rho_D, \tag{3.8}$$

Similarly, we can use Eq. (3.3.b) and define the volatile oil volume fraction and the steam water in the gaseous phase respectively, as:

$$y_{gv} = \rho_{gv}/\rho_{gV}, \quad 1 - y_{gv} = \rho_{gw}/\rho_{gW}, \tag{3.9}$$

We can derive an equation that relates the oleic phase density to the mole fraction x_{ov}. From the definition of the mole fraction (moles volatile oil / total moles) we immediately observe that

$$x_{ov} = \frac{\rho_{ov}/M_V}{\rho_{ov}/M_V + \rho_{od}/M_D}. \tag{3.10}$$

The concentration of steam and volatile oil in the gaseous phase is given by:

$$\rho_{gw} = \frac{M_W P_w}{RT}, \quad \rho_{gv} = \frac{M_V P_v}{RT}. \tag{3.11}$$

In the physical situations in which there exists volatile oil in the oleic phase, we can use Eqs. (3.7), (3.9.a) and (3.11.b), we derive the following expression for the mole fraction of volatile oil in the vapor phase y_{gv} as a function of x_{ov} and T:

$$y_{gv}\left(T, x_{ov}\right) = \frac{P_v\left(T\right)}{P} = \frac{P_o}{P} x_{ov} \exp\left(\frac{-M_V \Lambda_V\left(T_b^v\right)}{R}\left(\frac{1}{T} - \frac{1}{T_b^v}\right)\right). \tag{3.12}$$

We assume that the rock is filled with a mixture of water, oil and gas, i.e.:

$$s_w + s_g + s_o = 1, \tag{3.13}$$

where s_w, s_g and s_o are the oil, gas and water saturations, i.e., the pore volume fraction is filled with the liquid water, gaseous and oleic phases respectively. We usually write s_o in terms of s_g and s_w.

We assume that the fluids are incompressible and that the pressure changes are so small that they do not affect the physical properties of the fluids. Darcy's law for multiphase flow relates pressure gradient with its Darcy speed:

$$u_w = -\frac{kk_{rw}}{\mu_w}\frac{\partial p}{\partial x}, \qquad u_g = -\frac{kk_{rg}}{\mu_g}\frac{\partial p}{\partial x} \qquad \text{and} \qquad u_o = -\frac{kk_{ro}}{\mu_o}\frac{\partial p}{\partial x}. \tag{3.14}$$

The relative permeability functions k_{rw}, k_{rg} and k_{ro} are considered to be functions of the saturations (see Appendix A); μ_w, μ_g and μ_o are the viscosities of the aqueous, gaseous and oleic phases. The pressure in the aqueous, gaseous and oleic phases is taken to be identical, p, because the capillary pressure between the different phase is neglected. The fractional flow functions for water phase, gaseous phase and oleic phase are defined by:

$$f_w = \frac{k_{rw}/\mu_w}{d}, \qquad f_g = \frac{k_{rg}/\mu_g}{d} \qquad \text{and} \qquad f_o = \frac{k_{ro}/\mu_o}{d}, \tag{3.15}$$

where $d = k_{rw}/\mu_w + k_{rg}/\mu_g + k_{ro}/\mu_o$.

Using Darcy's law (3.14) and Eq. (3.15):

$$u_w = uf_w, \qquad u_g = uf_g \qquad \text{and} \qquad u_o = uf_o, \quad \text{where } u = u_w + u_g + u_o \tag{3.16}$$

is the *total* or *Darcy speed*.

3.1.1 Balance Equations.

We can write the mass balance equations for liquid water, steam in the gaseous phase, volatile oil in the gaseous phase, volatile oil in the oleic phase and dead oil conservation respectively as:

$$\varphi\frac{\partial}{\partial t}\rho_W s_w + \frac{\partial}{\partial x}\rho_W u_w = q_{g\longrightarrow a,w}, \tag{3.17}$$

$$\varphi\frac{\partial}{\partial t}\rho_{gw} s_g + \frac{\partial}{\partial x}\rho_{gw} u_g = -q_{g\longrightarrow a,w}, \tag{3.18}$$

$$\varphi\frac{\partial}{\partial t}\rho_{gv} s_g + \frac{\partial}{\partial x}\rho_{gv} uf_g = q_{g\longrightarrow o,v}, \tag{3.19}$$

$$\varphi\frac{\partial}{\partial t}\rho_{ov} s_o + \frac{\partial}{\partial x}\rho_{ov} u_o = -q_{g\longrightarrow o,v}, \tag{3.20}$$

$$\varphi\frac{\partial}{\partial t}\rho_{od} s_o + \frac{\partial}{\partial x}\rho_{od} u_o = 0, \tag{3.21}$$

where $q_{g\longrightarrow a,w}$ and $q_{g\longrightarrow o,v}$ are the water and the volatile oil condensation rates. The steam condenses into liquid water and the volatile oil dissolves into the dead oil.

The conservation of energy in terms of enthalpies per unit mass is given as

$$\begin{aligned}&\frac{\partial}{\partial t}\big(H_r + \varphi s_w \rho_W h_W + \varphi s_o\left(\rho_{ov} h_{oV} + \rho_{od} h_{oD}\right) + \varphi s_g\left(\rho_{gw} h_{gW} + \rho_{gv} h_{gV}\right)\big)\\ &+ \frac{\partial}{\partial x}\big(u_w \rho_W h_W + u_o\left(\rho_{ov} h_{oV} + \rho_{od} h_{oD}\right) + u_g\left(\rho_{gw} h_{gW} + \rho_{gv} h_{gV}\right)\big) = 0.\end{aligned} \tag{3.22}$$

Using u_w, u_g and u_o given by Eq. (3.16) in Eqs. (3.17)-(3.22), we obtain:

$$\varphi\frac{\partial}{\partial t}\rho_W s_w + \frac{\partial}{\partial x}u\rho_W f_w = q_{g\longrightarrow a,w}, \tag{3.23}$$
$$\varphi\frac{\partial}{\partial t}\rho_{gw} s_g + \frac{\partial}{\partial x}u\rho_{gw} f_g = -q_{g\longrightarrow a,w}, \tag{3.24}$$
$$\varphi\frac{\partial}{\partial t}\rho_{gv} s_g + \frac{\partial}{\partial x}u\rho_{gv} f_g = q_{g\longrightarrow o,v}, \tag{3.25}$$
$$\varphi\frac{\partial}{\partial t}\rho_{ov} s_o + \frac{\partial}{\partial x}u\rho_{ov} f_o = -q_{g\longrightarrow o,v}, \tag{3.26}$$
$$\varphi\frac{\partial}{\partial t}\rho_{od} s_o + \frac{\partial}{\partial x}u\rho_{od} f_o = 0, \tag{3.27}$$
$$\begin{aligned}&\frac{\partial}{\partial t}\big(H_r + \varphi s_w\rho_W h_W + \varphi s_o\left(\rho_{ov}h_{oV} + \rho_{od}h_{oD}\right) + \varphi s_g\left(\rho_{gw}h_{gW} + \rho_{gv}h_{gV}\right)\big)\\ &\qquad + \frac{\partial}{\partial x}u\big(f_w\rho_W h_W + f_o\left(\rho_{ov}h_{oV} + \rho_{od}h_{oD}\right) + f_g\left(\rho_{gw}h_{gW} + \rho_{gv}h_{gV}\right)\big) = 0.\end{aligned} \tag{3.28}$$

3.1.2 Reduced system of equations

The system of balance laws (3.23)-(3.28) is written in the conservative form by adding (3.23) to (3.24) and (3.25) to (3.26):

$$\varphi\frac{\partial}{\partial t}\left(\rho_W s_w + \rho_{gw}s_g\right) + \frac{\partial}{\partial x}u\left(\rho_W f_w + f_g\rho_{gw}\right) = 0, \tag{3.29}$$
$$\varphi\frac{\partial}{\partial t}\left(\rho_{gv} s_g + \rho_{ov}s_o\right) + \frac{\partial}{\partial x}u\left(\rho_{gv} f_g + \rho_{ov}f_o\right) = 0, \tag{3.30}$$
$$\varphi\frac{\partial}{\partial t}\rho_{od} s_o + \frac{\partial}{\partial x}\rho_{od}u f_o = 0,$$
$$\begin{aligned}&\frac{\partial}{\partial t}\big(H_r + \varphi s_w\rho_W h_W + \varphi s_o\left(\rho_{ov}h_{oV} + \rho_{od}h_{oD}\right) + \varphi s_g\left(\rho_{gw}h_{gW} + \rho_{gv}h_{gV}\right)\big) +\\ &\qquad + \frac{\partial}{\partial x}u\big(f_w\rho_W h_W + f_o\left(\rho_{ov}h_{oV} + \rho_{od}h_{oD}\right) + f_g\left(\rho_{gw}h_{gW} + \rho_{gv}h_{gV}\right)\big) = 0.\end{aligned}$$

Eq. (3.29) is the conservation of component water in the steam and gaseous phases; Eq. (3.30) is the conservation of volatile oil in the gaseous and oleic phases. Notice that it is possible to substitute Eq. (3.27) by the the conservation of total oil, which is obtained by adding Eqs. (3.25)-(3.27):

$$\varphi\frac{\partial}{\partial t}\left(\rho_{gv}s_g + (\rho_{od} + \rho_{ov})s_o\right) + \frac{\partial}{\partial x}u\left(\rho_{gv}f_g + (\rho_{ov} + \rho_{od})f_o\right) = 0 \tag{3.31}$$

The map E is applied to the cumulative and flux vectors generically given by $\mathcal{G}$ and $\mathcal{F}$ in Eq. (3.1). There is one such map E from (3.23)-(3.28) to (3.29), (3.30) (3.27), (3.28) and another to (3.29), (3.30), (3.31), (3.28). They are:

$$E_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.32}$$

The isomorphism (basis change) taking E_1 to E_2 and its inverse taking E_2 to E_1 are:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this work we will use E_1.

3.2 Physical situations

This model exhibits a rich structure of physical situations depending on the injection and initial conditions. Although certain physical situations seldom occur, for completeness we mention all of them. In the formalism proposed in Chapter [2], we can identify 12 physical situations Ω^j, 12 space of primary variables $\mathcal{P}^j$ and the 12 space of trivial variables $\mathcal{P}_j$ for $j = I, II, \cdots, XII$.

3.2.1 Three phases.

In this case we have three phases and three components. Using Gibb's phase rule (3.2), we see that there are two thermodynamical degrees of freedom. Since the pressure is fixed, the only degree of freedom is the temperature. Thus x_{ov} and y_{gv} are uniquely defined by the temperature T, as explained now.

From Eqs. (3.5), (3.7) and $P = P_w(T) + P_v(T)$, for the mole fraction in the liquid phase x_{ov} we find an expression depending on the temperature:

$$x_{ov}(T) = \frac{\frac{\rho_{ov}}{M_V}}{\frac{\rho_{ov}}{M_V} + \frac{\rho_{od}}{M_D}} = \frac{P - P_o \exp\left(\frac{-M_W \Lambda_W (T_b^w)}{R}\left(\frac{1}{T} - \frac{1}{T_b^w}\right)\right)}{P_o \exp\left(\frac{-M_V \Lambda_V (T_b^v)}{R}\left(\frac{1}{T} - \frac{1}{T_b^v}\right)\right)}. \tag{3.33}$$

Since x_{ov} is function of temperature, we use Eq. (3.8.a) to obtain v_{ov} as a function of temperature only:

$$v_{ov} = \frac{x_{ov}\rho_D M_V}{x_{ov}\rho_D M_V + (1 - x_{ov})\rho_V M_D}, \tag{3.34}$$

where x_{ov} is obtained from expression (3.33). The system of equations reduces to (3.29), (3.30) (3.27), (3.28) (alternatively to the system (3.29), (3.30), (3.31), (3.28)).

The mole fraction of volatile oil in the vapor phase y_{gv} is obtained using Eq. (3.12).

The primary variables to be determined are s_w, s_g, T, so $\mathcal{P}^I = \{s_w, s_g, T\}$; the speed u is not constant, it is the secondary variable; and the trivial variables are v_{ov} and y_{gv}, so $\mathcal{P}_I = \{v_{ov}, y_{gv}\}$. The physical situation is characterized by $\Omega^I = \{s_w, s_g, T, v_{ov}, y_{gv}\}$.

Notice for this physical situation $i_I(s_w, s_g, T, v_{ov}, y_{gv}) := (s_w, s_g, T)$; the terms $\mathcal{R}_G^T$ and $\mathcal{R}_F^T$ are functions with 6 components given by:

$$\mathcal{R}_G^T = \big(\varphi\rho_W s_w, \varphi\rho_{gw}s_g, \varphi\rho_{gv}s_g, \varphi\rho_{ov}s_o, \varphi\rho_{od}s_o,\\ H_r + \varphi s_w\rho_W h_W + \varphi s_o\left(\rho_{ov}h_{oV} + \rho_{od}h_{oD}\right) + \varphi s_g\left(\rho_{gw}h_{gW} + \rho_{gv}h_{gV}\right)\big).$$

and

$$\mathcal{R}_F^T = \big(\rho_W f_w, \rho_{gw}f_g, \rho_{gv}f_g, \rho_{ov}f_o, \rho_{od}f_o,\\ f_w\rho_W h_W + f_o\left(\rho_{ov}h_{oV} + \rho_{od}h_{oD}\right) + f_g\left(\rho_{gw}h_{gW} + \rho_{gv}h_{gV}\right)\big).$$

3.2.2 Two phases

In this section, the number of phases is two, i.e., $p = 2$ in Gibb's phase rule (3.2).

Steam phase and liquid water phase.

There is one component. Using Gibb's phase rule (3.2), we can see that $T = T_b^w$ is constant. There is no oil, so v_{ov} is not defined a priori and $y_{gv} = 0$. If it is necessary to obtain the complete solution, we define the variables using limits in the variable v_{ov} from the contiguous situations.

The primary variable to be determined is s_w, so $\mathcal{P}^{II} = \{s_w\}$; the speed u is constant; the trivial variables are $s_g = 0$, $T = T_b^w$, $y_{gv} = 0$ and v_{ov}, so $\mathcal{P}_{II} = \{s_g = 0, T = T_b^w, y_{gv} = 0, v_{ov}\}$. The physical situation is characterized by $\Omega^{II} = \{sw, s_g = 0, T = T_b^w, v_{ov}, y_{gv} = 0\}$.

The system of equations reduces to a single conservation law, Buckley-Leverett's equation:

$$\varphi\frac{\partial}{\partial t}s_w + u\frac{\partial}{\partial x}f_w = 0. \tag{3.35}$$

Notice for this physical situation $i_I(s_w, s_g, T, v_{ov}, y_{gv}) := (s_w)$; the terms $\mathcal{R}_G^T$ and $\mathcal{R}_F^T$ are functions with 6 components given by:

$$\mathcal{R}_G^T = \big(\varphi\rho_W s_w, \varphi\rho_{gw}s_g, 0, 0, 0, H_r + \varphi s_w\rho_W h_W + \varphi s_g\rho_{gw}h_{gW}\big).$$

and

$$\mathcal{R}_F^T = \big(\rho_W f_w, \rho_{gw}f_g, 0, 0, 0, f_w\rho_W h_W + f_g\rho_{gw}h_{gW}\big).$$

Gaseous and oleic phases.

There are three components, water, volatile and dead oil. Using Gibb's phase rule (3.2), in addition to the temperature T, we need to find another thermodynamical variable, v_{ov}. However, notice that we could utilize as a primary variable the quantity $v_{od} = 1 - v_{ov}$. Generically, there is no unique choice between primary and trivial variables.

Inverting Eq. (3.34), we obtain x_{ov} as function of v_{ov}:

$$x_{ov} = \frac{v_{ov}\rho_v M_D}{v_{ov}\rho_v M_D + (1 - v_{ov})\rho_D M_V}. \tag{3.36}$$

In this physical situation, y_{gv} is given by Eq. (3.12) as a function of T and x_{ov}. The variable y_{gv} can be obtained using Eqs. (3.36), (3.8.a) and (3.8.b); notice that x_{ov} can be written as function of v_{ov}. Thus from Eq. (3.12) y_{gv} depends on v_{ov} and T.

The primary variables to be determined are s_g, T, v_{ov}. so $\mathcal{P}^{III} = \{s_g, T, v_{ov}\}$; the speed u is not constant; finally the trivial variables are $s_w = 0$ and y_{gv}, so $\mathcal{P}_{III} = \{s_w = 0, y_{gv}(T, v_{ov})\}$. The physical situation is characterized by $\Omega^{III} = \{s_w = 0, s_g, T, v_{ov}, y_{gv}(T, v_{ov})\}$.

Using Eqs. (3.9.a) and (3.9.b), we can rewrite the system (3.29), (3.30), (3.27), (3.28) in the variables s_g, T, v_{ov} and u, and obtain the following system of conservation for steam, volatile oil, dead oil and energy:

$$\varphi\frac{\partial}{\partial t}\rho_{gw}s_g + \frac{\partial}{\partial x}uf_g\rho_{gw} = 0, \tag{3.37}$$

$$\varphi\frac{\partial}{\partial t}\left(\rho_{gv}s_g + \rho_V v_{ov}s_o\right) + \frac{\partial}{\partial x}u\left(\rho_{gv}f_g + v_{ov}\rho_V f_o\right) = 0, \tag{3.38}$$

$$\varphi\frac{\partial}{\partial t}\rho_D(1 - v_{ov})s_o + \frac{\partial}{\partial x}\rho_D(1 - v_{ov})uf_o = 0, \tag{3.39}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(H_r + \varphi s_o\left(v_{ov}\rho_V h_{oV} + \rho_D(1 - v_{ov})h_{oD}\right) + \varphi s_g\left(\rho_{gw}h_{gW} + \rho_{gv}h_{gV}\right)\Big) + \\ &\qquad + \frac{\partial}{\partial x}u\Big(f_o\left(v_{ov}\rho_V h_{oV} + \rho_D(1 - v_{ov})h_{oD}\right) + f_g\left(\rho_{gw}h_{gW} + \rho_{gv}h_{gV}\right)\Big) = 0,\end{aligned} \tag{3.40}$$

where $\rho_{gv} = y_{gv}\rho_{gV}$, $\rho_{gw} = (1 - y_{gv})\rho_{gW}$ and y_{gv} depend on T and v_{ov}.

A particular case occurs in the absence of dead oil. Then $v_{ov} = 1$ trivially and Eq. (3.39) disappears.

Notice for this physical situation $i_I(s_w, s_g, T, v_{ov}, y_{gv}) := (s_g, T, v_{ov})$; the terms $\mathcal{R}_G^T$ and $\mathcal{R}_F^T$ are functions with 6 components given by:

$$\begin{aligned}\mathcal{R}_G^T = \Big(&0, \varphi\rho_{gw}s_g, \varphi\rho_{gv}s_g, \varphi\rho_V v_{ov}s_o, \varphi\rho_D(1 - v_{ov})s_o, \\ &H_r + \varphi s_o\left(\rho_V v_{ov}h_{oV} + \rho_D(1 - v_{ov})h_{oD}\right) + \varphi s_g\left(\rho_{gw}h_{gW} + \rho_{gv}h_{gV}\right)\Big).\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_F^T = \Big(&0, \rho_{gw}f_g, \rho_{gv}f_g, \rho_V v_{ov}f_o, \rho_D(1 - v_{ov})f_o, \\ &f_o\left(\rho_V v_{ov}h_{oV} + \rho_D(1 - v_{ov})h_{oD}\right) + f_g\left(\rho_{gw}h_{gW} + \rho_{gv}h_{gV}\right)\Big).\end{aligned}$$

Liquid oleic and aqueous phases.

There are three components. Using the Gibb's phase rule (3.2), in addition to the temperature T, we have another thermodynamical degree of freedom; we choose v_{ov} to be this variable. In this physical situation y_{gv} is meaningless, but if it is necessary, it can be obtained for continuity in phase space as function of T and v_{ov} using Eq. (3.12).

The primary variables to be determined are s_w, T, v_{ov}, so $\mathcal{P}^{IV} = \{s_w, T, v_{ov}\}$; the speed u is not constant; and the trivial variables are $s_g = 0$ and y_{gv}, so $\mathcal{P}_{IV} = \{s_g = 0, y_{gv}\}$. The physical situation is characterized by $\Omega^{IV} = \{s_w, s_g = 0, T, v_{ov}, y_{gv}\}$.

The system (3.29), (3.30) (3.27), (3.28) reduces to the following system of conservation for water, volatile oil, dead oil and energy:

$$\varphi\frac{\partial}{\partial t}\rho_W s_w + \frac{\partial}{\partial x}u\rho_W f_w = 0, \tag{3.41}$$

$$\varphi\frac{\partial}{\partial t}v_{ov}\rho_V s_o + \frac{\partial}{\partial x}uv_{ov}\rho_V f_o = 0, \tag{3.42}$$

$$\varphi\frac{\partial}{\partial t}\rho_D(1 - v_{ov})s_o + \frac{\partial}{\partial x}\rho_D(1 - v_{ov})uf_o = 0, \tag{3.43}$$

$$\begin{aligned}\frac{\partial}{\partial t}\big(H_r + \varphi s_w\rho_W h_W + \varphi s_o\left(v_{ov}\rho_V h_{oV} + \rho_D(1 - v_{ov})h_{oD}\right)\big) +\\ + \frac{\partial}{\partial x}u\big(f_w\rho_W h_W + f_o\left(v_{ov}\rho_V h_{oV} + \rho_D(1 - v_{ov})h_{oD}\right)\big) = 0.\end{aligned} \tag{3.44}$$

Two particular cases arise: one of them corresponds to absence of dead oil, the other to the absence of volatile oil.

Liquid water phase and gaseous steam / volatile oil phase.

There are two components, in the absence of dead oil. Using Gibb's phase rule (3.2), we see that there are two thermodynamical degrees of freedom. Since the pressure is fixed, the only degree of freedom is the temperature. Since there is no liquid volatile oil v_{ov} is not defined. On the other hand, since $P_v(T) = P - P_w(T)$, y_{gv} is uniquely defined by the temperature T, see Sec. 3.2.1.

The primary variables to be determined are s_w, s_g, T, so $\mathcal{P}^V = \{s_w, s_g, T\}$; the speed u is not constant; and the trivial variables are $v_{ov} = 0$ and y_{gv}, so $\mathcal{P}_V = \{v_{ov}, y_{gv}\}$. The physical situation is characterized by $\Omega^V = \{s_w, s_g, T, v_{ov} = 0, y_{gv}\}$.

The system (3.29), (3.30) (3.27), (3.28) reduces to the following system of conservation laws for water, volatile oil and energy:

$$\varphi\frac{\partial}{\partial t}\rho_W s_w + \frac{\partial}{\partial x}u\left(\rho_W f_w + f_g\rho_{gw}\right) = 0, \tag{3.45}$$

$$\varphi\frac{\partial}{\partial t}\rho_{gv}s_g + \frac{\partial}{\partial x}u\rho_{gv}f_g = 0, \tag{3.46}$$

$$\frac{\partial}{\partial t}\big(H_r + \varphi s_w\rho_W h_W + \varphi s_g\left(\rho_{gw}h_{gW} + \rho_{gv}h_{gV}\right)\big) + \frac{\partial}{\partial x}u\big(f_w\rho_W h_W + f_g\left(\rho_{gw}h_{gW} + \rho_{gv}h_{gV}\right)\big) = 0. \tag{3.47}$$

The steam fraction in the gaseous phase is always positive. One particular case arises, corresponding to the absence of volatile oil.

3.2.3 One phase

In the following subsections, the physical situations consist of a single phase, so $p = 1$ in the Gibb's phase rule (3.2).

Liquid oleic phase.

There are two components. Using the Gibb's phase rule (3.2) we can see that in addition to the temperature T there is another thermodynamical degree of freedom. The suitable thermodynamical variable is v_{ov}. As in the previous physical situation, y_{gv} is meaningless, but it can be obtained from continuity in phase space as a function of T and v_{ov} using Eq. (3.12).

The primary variables to be determined are T, v_{ov}, so $\mathcal{P}^{VI} = \{T, v_{ov}\}$; the speed u is not constant; the trivial variables are $s_w = 0$, $s_g = 0$ and y_{gv}, so $\mathcal{P}_{VI} = \{s_w = 0, s_g = 0, y_{gv}\}$. The physical situation is characterized by $\Omega^{VI} = \{s_w = 0, s_g = 0, T, v_{ov}, y_{gv}\}$.

The system of equations (3.29), (3.30) (3.27), (3.28) reduces to the conservation laws for volatile oil, dead oil and energy:

$$\varphi\frac{\partial}{\partial t}v_{ov}\rho_V + \frac{\partial}{\partial x}uv_{ov}\rho_V = 0, \tag{3.48}$$

$$\varphi\frac{\partial}{\partial t}(1 - v_{ov})\rho_D + \frac{\partial}{\partial x}u(1 - v_{ov})\rho_D = 0, \tag{3.49}$$

$$\frac{\partial}{\partial t}\big(H_r + \varphi\big(v_{ov}\rho_V h_{oV} + (1 - v_{ov})\rho_D h_{oD}\big) + \frac{\partial}{\partial x}u\big(v_{ov}\rho_V h_{oV} + (1 - v_{ov})\rho_D h_{oD}\big) = 0. \tag{3.50}$$

Liquid aqueous phase.

There is one component. Using the Gibb's phase rule (3.2), the only thermodynamical variable is the temperature T. Since there is no oil in this situation, v_{ov} and y_{gv} are not defined in principle, but we obtain them as limits of Eqs. (3.33) and (3.12).

Since ρ_W does not depend on T, one can show that u is constant spatially.

The primary variable to be determined by the system of equations is T, so $\mathcal{P}^{VII} = \{T\}$; the speed u is constant; and the trivial variables are $s_w = 1$, $s_g = 0$ and v_{ov} and y_{gv} so $\mathcal{P}_{VII} = \{s_w = 1, s_g = 0, v_{ov}, y_{gv}\}$. The physical situation is characterized by $\Omega^{VII} = \{s_w = 1, s_g = 0, T, v_{ov}, y_{gv}\}$.

The system of equations (3.29), (3.30), (3.27), (3.28) reduces to the single scalar equation for conservation laws of energy:

$$\frac{\partial}{\partial t}(H_r + \varphi\rho_W h_W) + u\frac{\partial}{\partial x}\rho_W h_W = 0. \tag{3.51}$$

Gaseous steam/volatile oil.

There are two components. Since there is no liquid oil v_{ov} is not defined, but we obtain it as a limit of Eq. (3.33). Using the Gibb's phase rule (3.2), in addition to the temperature T, we have another thermodynamical degree of freedom; we choose y_{gv} as this variable.

The primary variables to be determined are T, y_{gv}, so $\mathcal{P}^{VIII} = \{T, y_{gv}\}$; the speed u is not constant; the trivial variables are $s_w = 0$, $s_g = 1$ and v_{ov}, $\mathcal{P}_{VIII} = \{s_w = 0, s_g = 1, v_{ov}\}$. The physical situation is characterized by $\Omega^{VIII} = \{s_w = 0, s_g = 1, T, v_{ov}, y_{gv}\}$.

The system of equations governing this flow reduces to the conservation laws of steam, of volatile oil in the gaseous phase and of energy:

$$\varphi\frac{\partial}{\partial t}(1 - y_{gv})\rho_{gW} + \frac{\partial}{\partial x}u(1 - y_{gv})\rho_{gW} = 0, \tag{3.52}$$

$$\varphi\frac{\partial}{\partial t}y_{gv}\rho_{gV} + \frac{\partial}{\partial x}u y_{gv}\rho_{gV} = 0, \tag{3.53}$$

$$\frac{\partial}{\partial t}\left(H_r + \varphi\left((1 - y_{gv})\rho_{gW}h_{gW} + y_{gv}\rho_{gV}h_{gV}\right)\right) + \frac{\partial}{\partial x}u\left((1 - y_{gv})\rho_{gW}h_{gW} + y_{gv}\rho_{gV}h_{gV}\right) = 0. \tag{3.54}$$

Pure steam phase.

In this case there are one phase and one component. Using the Gibb's phase rule (3.2), the only degree of freedom is the temperature. Since there is no oil, v_{ov} and y_{gv} are not defined, but they can be obtained as limits of Eqs. (3.33) and (3.12).

The primary variables to be determined is T, so $\mathcal{P}^{IX} = \{T\}$; the speed u is not constant; and the trivial variables are $s_w = 0$, $s_g = 1$, v_{ov} and y_{gv}, so $\mathcal{P}_{IX} = \{s_w = 0, s_g = 1, v_{ov}, y_{gv}\}$. The physical situation is characterized by $\Omega^{VIII} = \{s_w = 0, s_g = 1, T, v_{ov}, y_{gv}\}$.

The system reduces to the conservation laws of steam and of energy:

$$\varphi\frac{\partial}{\partial t}\rho_{gw} + \frac{\partial}{\partial x}u\rho_{gw} = 0, \tag{3.55}$$

$$\frac{\partial}{\partial t}(H_r + \varphi\rho_{gw}h_{gW}) + \frac{\partial}{\partial x}u\rho_{gw}h_{gW} = 0. \tag{3.56}$$

The next cases consists of single phase and single components, so in the Gibb's phase rule $p = 1$ and $c = 1$, so there is only two thermodynamical degrees of freedom. Since the pressure is fixed, the temperature T is chosen. This will be the primary variable.

Pure liquid volatile oil.

Since there is only volatile oil in the oleic phase, $v_{ov} = 1$. Since there is no gaseous phase y_{gv} is not defined, but they are obtained by continuity as a limit process.

The primary variable is T, so $\mathcal{P}^X = \{T\}$. The speed u is constant; the trivial variables are $s_w = 0$, $s_g = 0$, $v_{ov} = 1$ and y_{gv}, so $\mathcal{P}_X = \{s_w = 0, s_g = 0, v_{ov} = 1, y_{gv}\}$. The physical situation is characterized by $\Omega^X = \{s_w = 0, s_g = 0, T, v_{ov} = 1, y_{gv}\}$.

The system of equations reduces to the conservation laws of energy:

$$\frac{\partial}{\partial t}(H_r + \varphi\rho_V h_{oV}) + u\frac{\partial}{\partial x}\rho_V h_{oV} = 0. \tag{3.57}$$

Pure gaseous volatile oil.

Since there is only volatile oil in the gaseous phase, thus $y_{gv} = 1$; Since there is no liquid oil v_{ov} is not defined, but can be obtained by continuity as a limit process.

The primary variable is the T, so $\mathcal{P}^{XI} = \{T\}$. The speed u is not constant; the trivial variables are $s_g = 0$, $s_w = 0$, v_{ov} and $y_{gv} = 1$, so $\mathcal{P}_{XI} = \{s_w = 0, s_g = 0, v_{ov}, y_{gv} = 1\}$. The physical situation is characterized by $\Omega^{XI} = \{s_w = 0, s_g = 0, T, v_{ov}, y_{gv} = 1\}$.

The system of equations reduces to the conservation laws of gaseous volatile oil and of energy:

$$\varphi\frac{\partial}{\partial t}\rho_{gV} + \frac{\partial}{\partial x}u\rho_{gV} = 0, \tag{3.58}$$

$$\frac{\partial}{\partial t}(H_r + \varphi\rho_{gV}h_{gV}) + \frac{\partial}{\partial x}u\rho_{gV}h_{gV} = 0. \tag{3.59}$$

Pure liquid dead oil.

Since the the vapor pressure is negligible in the dead oil, it occurs just in the oleic phase and y_{gv} is not defined, but it is obtained by continuity as a limit process.

The primary variable is T, so $\mathcal{P}^{XII} = \{T\}$. The speed u is constant; the trivial variables are $s_w = 0$, $s_g = 0$, $v_{ov} = 0$ and y_{gv}, so $\mathcal{P}_{XI} = \{s_w = 0, s_g = 0, v_{ov} = 0, y_{gv}\}$. The physical situation is characterized by $\Omega^{XI} = \{s_w = 0, s_g = 0, T, v_{ov} = 0, y_{gv}\}$.

The system of equations reduces to to the conservation laws of energy:

$$\frac{\partial}{\partial t}(H_r + \varphi\rho_D h_{oD}) + u\frac{\partial}{\partial x}\rho_D h_{oD} = 0. \tag{3.60}$$

CHAPTER 4

The Riemann solution for the injection of steam and nitrogen

The frequent and widespread occurrence of contamination due to spills and leaks of organic materials, such as petroleum products, that occur during their transport, storage and disposal constitute a menace to our high-quality groundwater resources. In spite of increased awareness of the environmental impacts of oil spills, it appears to be impossible to avoid these accidents, so it is necessary to develop techniques for groundwater remediation.

Steam injection is widely studied in Petroleum Engineering, see [10]. In Chapter 5, we study the steam and water injection in several proportions into a porous medium saturated with a different mixture of steam and water. A disadvantage of pure steam injection is the ecological impact of high temperatures. This can be alleviated if we co-inject nitrogen leading to a lower temperatures. Within of such a mixture, the boiling temperature of water depends on nitrogen concentration and it is lower than water regular boiling temperature.

Another application is the recovery of geothermal energy. Injecting water and nitrogen into a hot porous rock, the water in the mixture evaporates and rock thermal heat can be recovered even if the rock is not very hot.

We present a physical model for steam, nitrogen and water flow based on mass balance and energy conservation equations. We present the main physical definitions and equations of the model. We study the three possible physical phase mixture situations. For each physical situation, we reduce the original four balance equations system to be presented in Section 4.2 to systems of conservation laws of type (2.2):

$$\frac{\partial}{\partial t}G(V) + \frac{\partial}{\partial x}uF(V) = 0,$$

supplemented by appropriate thermodynamic constraints between variables, such as Raoult's law and others described in Appendix A. Here $V = (V_1, V_2) : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{D} \subset$

$\mathbb{R}^3$ is a subset of the variables: gas saturation s_g, steam composition ψ_{gw} and temperature T and it represents the unknowns in each physical situation; $G = (G_1, G_2, G_3) : \mathbb{D} \longrightarrow \mathbb{R}^3$ and $F = (F_1, F_2, F_3) : \mathbb{D} \longrightarrow \mathbb{R}^3$ are the accumulation vector and the flux vector, respectively; u is a total velocity.

The state of the general system is represented by (s_g, ψ_{gw}, T, u). Systems of conservation equations of the type (2.2) have an important feature: the variable u does not appear in the accumulation terms, it appears isolated in the flux terms, thus such systems have an infinite speed mode associated to u; despite this fact and the presence of source terms in the original balance equations we are able to solve the associated Riemann problem, which is a mathematical novelty. Another surprise in this model is the presence of a rarefaction wave associated to evaporation in the two-phase situation.

In Section 4.1, we present the physical model that describes the injection of steam and nitrogen in a one-dimensional horizontal porous rock initially filled with water liquid. In Section 4.2, we present the model equations for balance of water, steam, nitrogen and energy. The full governing equations are not a system of conservation laws, since there is a water mass transfer term, which is the condensation or evaporation of water between the liquid and gaseous phases. In Section 4.3, we describe three physical situations in which the balance system is rewritten as conservation systems of type (2.2). In Section 4.4, we study the type of waves that appear in the solution and some general results for 3×3 systems; a more complete theory is established in [2]. In Secs. 4.5, 4.6 and 4.7, we obtain the waves in each physical situation, it is remarkable that there is an evaporation rarefaction wave in the two-phase situation. In Section 4.8, we study the shocks between different regions. In Section 4.9, we show an example of the Riemann solution for a mixture of steam and nitrogen in the *spg* injected into a rock initially filled with water.

4.1 Physical model

We consider the injection of steam and nitrogen in a one-dimensional horizontal porous rock core initially filled with water at temperature T^0 . The core consists of the rock with constant porosity φ and absolute permeability k (see Appendix A). We assume that the fluids are incompressible and that the pressure variations along the core are so small that they do not affect the physical properties of the fluids.

Darcy's law for multiphase flow relates the pressure gradient in each fluid phase with its seepage speed:

$$u_w = -\frac{kk_{rw}}{\mu_w}\frac{\partial p_w}{\partial x}, \qquad u_g = -\frac{kk_{rg}}{\mu_g}\frac{\partial p_g}{\partial x}. \tag{4.1}$$

The water and gas relative permeability functions k_{rw} and k_{rg} are considered to be functions of their respective saturations (see Appendix A); μ_w and μ_g are the viscosities of liquid and gaseous phases; p_w and p_g are the pressures in the liquid and gaseous phases. The capillary pressure P_c and the capillary diffusion coefficient Ω

are:

$$P_c = P_c(s_w) = p_g - p_w \qquad \text{and} \qquad \Omega = -f_w \frac{kk_{rg}}{\mu_g}\frac{dP_c}{ds_w} \geq 0. \tag{4.2}$$

The fractional flow functions for water and steam are defined by:

$$f_w = \frac{k_{rw}/\mu_w}{k_{rw}/\mu_w + k_{rg}/\mu_g}, \qquad f_g = \frac{k_{rg}/\mu_g}{k_{rw}/\mu_w + k_{rg}/\mu_g}. \tag{4.3}$$

Using Darcy's law (3.14) and the definition of P_c in Eq. (4.2.a), Eqs. (4.2) and (3.15) yield:

$$u_w = uf_w - \Omega\frac{\partial s_w}{\partial x}, \qquad u_g = uf_g - \Omega\frac{\partial s_g}{\partial x}, \quad \text{where } u = u_w + u_g \tag{4.4}$$

is the *total* or *Darcy velocity*. We will see that Ω acts as a capillary diffusion coefficient; s_w is the water saturation, i.e., the pore fraction filled with water; similarly, s_g is the gas saturation.

4.2 The model equations

Neglecting molecular diffusion effects, we write the equations of mass balance for water, steam and nitrogen as:

$$\frac{\partial}{\partial t}\varphi\rho_W s_w + \frac{\partial}{\partial x}u_w\rho_W = +q_{g\longrightarrow a,w}, \tag{4.5}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gw} s_g + \frac{\partial}{\partial x}u_{gw}\rho_{gw} = -q_{g\longrightarrow a,w}, \tag{4.6}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gn} s_g + \frac{\partial}{\partial x}u_{gn}\rho_{gn} = 0, \tag{4.7}$$

where $q_{g\longrightarrow a,w}$ is the water mass source term (i.e., the condensation rate between the steam and water phases); here ρ_W is the water density, which is assumed to be constant, ρ_{gw} (ρ_{gn}) denote the concentration of steam (nitrogen) in the gaseous phase (mass per unit gas volume); the concentration dependence on temperature is specified by Raoult's laws and other thermodynamical relationships given in Appendix A. The saturations s_w and s_g add to 1. By (4.3), the same is true for f_w and f_g.

Since there are 4 unknowns (T, s_g, ψ_{gw} and u) and 3 equations, it is necessary to specify another equation, representing energy conservation, based on an enthalpy formulation, see [2, 3]. We include longitudinal heat conduction, but neglect heat losses to the surrounding rock. We ignore adiabatic compression and decompression effects. Thus the energy conservation is given by:

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + \rho_W h_W s_w + (\rho_{gw}h_{gW} + \rho_{gn}h_{gN})s_g\right) + \\ \frac{\partial}{\partial x}\left(u_w\rho_W h_W + u_g(\rho_{gw}h_{gW} + \rho_{gn}h_{gN})\right) = \frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right), \tag{4.8}$$

here H_r is the rock enthalpy per unit volume and $\hat{H}_r = H_r/\varphi$; h_W, h_{gW} and h_{gN} are the enthalpies per unit mass of water in the liquid aqueous phase, water and nitrogen in the gaseous phase; these enthalpies depend on temperature as described by Eqs. $(A.3)$ and $(A.4)$. The composite conductivity of the rock/fluid system $\kappa > \kappa_r > 0$ is given by:

$$\kappa = \kappa_r + \varphi(s_w \kappa_w + s_g \kappa_g). \tag{4.9}$$

We define H_W and H_g, the effective water and gas enthalpies per unit volume as:

$$H_W = \rho_W h_W \qquad \text{and} \qquad H_g = \rho_{gw} h_{gW} + \rho_{gn} h_{gN}, \tag{4.10}$$

so from (4.4), (4.10), we can rewrite (4.5)-(4.8) as:

$$\frac{\partial}{\partial t}\varphi\rho_W s_w + \frac{\partial}{\partial x}u\rho_W f_w = \frac{\partial}{\partial x}\left(\rho_W \Omega \frac{\partial s_w}{\partial x}\right) + q_{g\longrightarrow a,w}, \tag{4.11}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gw} s_g + \frac{\partial}{\partial x}u\rho_{gw} f_g = \frac{\partial}{\partial x}\left(\rho_{gw} \Omega \frac{\partial s_g}{\partial x}\right) - q_{g\longrightarrow a,w}, \tag{4.12}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gn} s_g + \frac{\partial}{\partial x}u\rho_{gn} f_g = \frac{\partial}{\partial x}\left(\rho_{gn} \Omega \frac{\partial s_g}{\partial x}\right), \tag{4.13}$$

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + H_w s_w + H_g s_g\right) + \frac{\partial}{\partial x}u\Big(H_w f_w + H_g f_g\Big) =$$
$$= \frac{\partial}{\partial x}\left(\left(H_g - H_w\right)\Omega\frac{\partial s_g}{\partial x}\right) + \frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right). \tag{4.14}$$

We are interested in scales dictated by field reservoirs. The effect of spatial second derivative terms (capillary pressure, heat conductivity, etc) is to widen the heat condensation front as well as other shocks, while the convergence of the characteristics tries to sharpen them. The balance of these effects yields the width of these fronts. In the field this width is typically a few tenth of centimeters; on the other hand, the distance between injection and production wells is of the order of 1000 meters. Thus this width is negligible, so we can set it to zero and simplify our analysis with no error of practical importance. From now on we disregard the diffusive terms in (4.11)-(4.14), resulting in the following balance system:

$$\frac{\partial}{\partial t}\varphi\rho_W s_w + \frac{\partial}{\partial x}u\rho_W f_w = +q_{g\longrightarrow a,w}, \tag{4.15}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gw} s_g + \frac{\partial}{\partial x}u\rho_{gw} f_g = -q_{g\longrightarrow a,w}, \tag{4.16}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gn} s_g + \frac{\partial}{\partial x}u\rho_{gn} f_g = 0, \tag{4.17}$$

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + H_w s_w + H_g s_g\right) + \frac{\partial}{\partial x}u\Big(H_w f_w + H_g f_g\Big) = 0. \tag{4.18}$$

4.3 Physical situations

There are three regions in different physical situations. Initially, there is only water in the core; we call this situation the single-phase liquid situation, *spl*. We can inject a superheated gas to form a region containing only gaseous steam and nitrogen; we call this the single-phase gaseous situation, *spg*. There is also the two-phase situation, *tp*, where there is a mixture of liquid water, gaseous nitrogen and steam at boiling temperature. The latter depends on the concentration of nitrogen in the gas. We assume that each situation is in local thermodynamical equilibrium, so we can use Gibbs' phase rule (3.2).

4.3.1 Single-phase gaseous situation - *spg*

In this situation there are two components and one gaseous phase, i.e., $c = 2$ and $p = 1$, so from Gibbs' phase rule (3.2) there are three thermodynamical degrees of freedom. Since in our thermodynamical model the pressure is fixed, there are two unknown thermodynamical variables: temperature and gas composition.

We assume that nitrogen and steam in the gaseous phase behave as ideal gases with densities depending on T denoted by ρ_{gN} and ρ_{gW}, see $(A.9)$ in Appendix A. We use Raoult's law (see [68]) and assume that there are no volume effects due to mixing, so that the volumes of the components are additive:

$$\rho_{gw}/\rho_{gW}(T) + \rho_{gn}/\rho_{gN}(T) = 1. \tag{4.19}$$

This fact allows to define the steam and nitrogen gas composition, respectively, as:

$$\psi_{gw} = \rho_{gw}/\rho_{gW}(T) \qquad \text{and} \qquad \psi_{gn} = \rho_{gn}/\rho_{gN}(T), \quad \text{so} \quad \psi_{gw} + \psi_{gn} = 1, \tag{4.20}$$

thus when the steam or nitrogen composition is known, the other composition is obtained trivially from (4.20.c). We use ψ_{gw} as the basic state variable. The compositions ρ_{gw} and ρ_{gn} are functions of temperature and composition; they are obtained from (4.20.a) and (4.20.b).

Therefore, there are three unknowns to be determined: temperature T, gas composition ψ_{gw} and speed u. The saturations are trivial, $s_g = 1$ and $s_w = 0$.

We rewrite equations (4.15)-(4.18) using Eqs. (4.20.a) and (4.20.b). Since $s_w = 0$, $f_w = 0$, so Eq. (4.15) disappears and we obtain that $q_{g\longrightarrow a,w}$ vanishes; thus the remaining system becomes:

$$\frac{\partial}{\partial t}\varphi\theta_W\psi_{gw}T^{-1} + \frac{\partial}{\partial x}u\theta_W\psi_{gw}T^{-1} = 0, \tag{4.21}$$

$$\frac{\partial}{\partial t}\varphi\theta_N\psi_{gn}T^{-1} + \frac{\partial}{\partial x}u\theta_N\psi_{gn}T^{-1} = 0, \tag{4.22}$$

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + \psi_{gw}H_{gW} + \psi_{gn}H_{gN}\right) + \frac{\partial}{\partial x}u\left(\psi_{gw}H_{gW} + \psi_{gn}H_{gN}\right) = 0; \tag{4.23}$$

we have substituted ρ_{gW} and ρ_{gN} from Eq. $(A.9)$, and defined θ_W, θ_N, $H_{gW}(T)$ and $H_{gN}(T)$:

$$\theta_W = \frac{M_W p_{at}}{R}, \quad \theta_N = \frac{M_N p_{at}}{R}, \quad H_{gW}(T) = \frac{\theta_W h_{gW}}{T} \quad \text{and} \quad H_{gN}(T) = \frac{\theta_N h_{gN}}{T}, \tag{4.24}$$

where M_N, M_W are the molar masses of nitrogen and water; R is the universal gas constant; p_{at} is the atmospheric pressure; h_{gW} and h_{gN} are functions of T given in Appendix A.

Remark 4.3.1. *The spg does not exist for any pair (T, ψ_{gw}), because of thermodynamic constraints. The tp situation occurs for states where the composition is given in terms of the temperature T as:*

$$\psi_{gw}(T) = \rho_{gw}(T)/\rho_{gW}(T), \tag{4.25}$$

where ρ_{gw} and ρ_{gW} given by Eqs. $(A.8.a)$, $(A.9.a)$. The composition $\psi_{gw}(T)$ is the projection of the tp on the plane $\{T, \psi_{gw}\}$. As we assume thermodynamical equilibrium, the solution of Eq. (4.25) defines the boundary of the physical region in the plane $\{T, \psi_{gw}\}$, see Fig. (4.1.a).

We define the spg physical region as:

$$\Gamma = \left\{(T, \psi_{gw}) \in \ spg \ that \ satisfy \ \psi_{gw} \leq \rho_{gw}(T)/\rho_{gW}(T)\right\}. \tag{4.26}$$

*For each ψ_{gw}, the temperature that satisfies (4.25) is called **saturation temperature**. The curve $\partial\Gamma$ is formed by the pairs $(T, \psi_{gw}(T))$ that satisfy (4.25).*

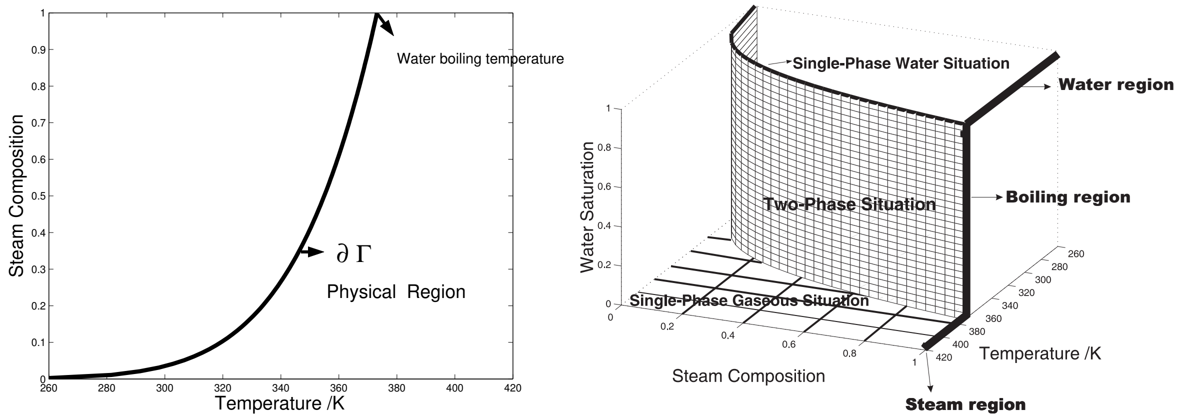


Figure 4.1: *a) Left:* The physical region lies to the right of the curve defined by (4.25). The continuous graph represents the saturation temperature for the mixture of water and nitrogen. *b) Right:* Phase space for $V = (s_W, \psi_{gw}, T)$ and physical situations. The bold lines represent the physical regions for water and steam without nitrogen, see Chapter 5.

4.3.2 Two-phase situation - *tp*

In the *tp* there are two components, $c = 2$, and two-phases, $p = 2$; using Gibbs' phase rule (3.2), there are two thermodynamical degrees of freedom, temperature and pressure (which is fixed). The compositions in Eqs. (4.27) and (4.28) depend on temperature, and vice-versa inverting Eq. (4.25). As the pressure is given, the boiling temperature of water is specified by these compositions. When pure steam is injected the condensation temperature is constant $T^b = 373.15$, see Appendix A. We have three variables to be determined: temperature, saturation and total Darcy velocity.

There is a linearly dependent equation in the system (4.15)-(4.18). We add (4.15) and (4.16) and replace (4.15)-(4.18) by:

$$\frac{\partial}{\partial t}\varphi\left(\rho_W s_w + \rho_{gw} s_g\right) + \frac{\partial}{\partial x} u\left(\rho_W f_w + \rho_{gw} f_g\right) = 0, \tag{4.27}$$

$$\frac{\partial}{\partial t}\varphi\left(\rho_{gn} s_g\right) + \frac{\partial}{\partial x} u(\rho_{gn} f_g) = 0, \tag{4.28}$$

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + H_W s_w + H_g s_g\right) + \frac{\partial}{\partial x} u(H_W f_w + H_g f_g) = 0, \tag{4.29}$$

The compositions ρ_{gw} and ρ_{gn} are functions of temperature only that can be obtained from Eqs. $(A.8.a)$ and $(A.8.b)$ in Appendix A.

4.3.3 Single-phase liquid situation - *spl*

In the *spl* there is one component and one phase, so from Gibbs' phase rule (3.2), there are two thermodynamical degrees of freedom, temperature and pressure (which is fixed). Since $s_w = 1$ and $s_g = 0$, we have $f_w = 1$ and $f_g = 0$. From (4.12), $q_{g\longrightarrow a,w}$ vanishes. One can prove that the total Darcy velocity u is independent of position, so (4.15)-(4.18) become:

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + H_w\right) + u_W \frac{\partial H_w}{\partial x} = 0, \tag{4.30}$$

where we use u_W to indicate that the velocity is spatially constant in the *spl*; V is just the temperature T. We assume that rock and water enthalpy depend linearly on temperature, so we can rewrite (4.30) as:

$$\frac{\partial}{\partial t} T + \lambda_T^W \frac{\partial T}{\partial x} = 0, \quad \text{where} \quad \lambda_T^W = \frac{u_W}{\varphi} \frac{C_W}{C_W + \hat{C}_r}, \tag{4.31}$$

and C_W is the water rock heat capacity and $\hat{C}_r$ is the rock heat capacity C_r divided by φ. All quantities are given in Appendix A.

4.3.4 Primary, secondary and trivial variables

Fig. 4.1.b shows the 3 physical situations in the variables $V = (T, \psi_{gw}, s_w)$. As basic variable, we choose s_w instead of s_g for convenience.

For each physical situation there are three groups of variables. One group encompasses the variables V in the system (2.2). We call the variables in V "primary variables" or basic variables. The variable u is called "secondary variable" because we will see that it is obtained from the primary variables. The trivial variables are the variables that depend on primary variables in a simple way (for example, in the *tp* the steam composition depends on temperature) or are obtained trivially (for example, in the *spg* the gas saturation is trivial and its value is 1). In Table 4.3.4 we summarize these variables.

Physical situation \ Variable types	Primary	Secondary	Trivial
Single-phase gaseous situation	T, ψ_{gw}	u	$s_g = 1$
Two-phase situation	s_g, T	u	$\psi_{gw}(T)$
Single-phase liquid situation	T	u	$s_g = 0$, ψ_{gw}

Table 1: Classification according to situation.

4.4 General theory of Riemann Solutions

We are interested in the Riemann problem associated to (4.15)-(4.18), that is the solution of these equations with initial data

$$\begin{cases} (s_g, \psi_{gw}, T, u)_L & \text{if } x > 0 \\ (s_g, \psi_{gw}, T, \cdot)_R & \text{if } x < 0, \end{cases} \tag{4.32}$$

The speed $u_L > 0$ is specified at the injection point. In the next sections we show that u_R can be obtained as function of u_L and the primary variables.

The general solution of the Riemann problem associated to Eq. (2.2) consists of a sequence of elementary waves, rarefactions and shocks; they are studied in the sections that follow, [60]. We assume that the cumulative and flux functions have continuous second derivative ($\mathcal{C}^2$) within each physical situation and that are only continuous between the physical situations.

4.4.1 Characteristic speeds in each physical situation

System of conservation laws in different forms must be used to find the characteristic speeds. If we assume that the solution is sufficiently smooth, we differentiate all equations in (2.2) with respect to their variables, obtaining a system of the form 2.27)

$$B\frac{\partial}{\partial t}\begin{pmatrix} V \\ u \end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix} V \\ u \end{pmatrix} = 0, \tag{4.33}$$

where the matrices B and A (which depend on V) are the derivatives of $G(V)$ and $uF(V)$ with respect to the variables V and u. Since $G(V)$ does not depend on u, the last column in the matrix B is zero. The characteristic values $\lambda_i := \lambda_i(V, u)$ and vectors $\vec{r}^i := \vec{r}^i(V, u)$, (where i is the label of each eigenvector) for the following system are the rarefaction wave speeds and directions:

$$A\vec{r}^i = \lambda_i B\vec{r}^i \quad \text{where } \lambda_i \text{ is obtained by solving} \quad det(A - \lambda_i B) = 0. \tag{4.34}$$

Similarly the left eigenvectors $\vec{\ell}^i = (\ell_1^i, \ell_2^i, \ell_3^i)$ satisfy:

$$\vec{\ell}^i A = \lambda_i \vec{\ell}^i B, \quad \text{for the same } \lambda_i. \tag{4.35}$$

Remark 4.4.1. *We consider here a general 3×3 system because it is the simplest non-trivial system of type* (2.2). *Generically, the right and left eigenvectors have three components: two primary variables and one secondary variable. However, in the spl situation we do not utilize this formalism because the system* (2.2) *reduces to a single equation.*

Hereafter , the word “eigenvectors” represents “right eigenvectors”. For brevity, we will write the right and left eigenvectors without the upper arrow, i.e., we replace $\vec{r}$ by r and $\vec{\ell}$ by ℓ.

Remark 4.4.2. *In Lemma* [2.2.1], *we prove that the left eigenvectors ℓ^i do not depend on the Darcy speed u.*

Definition 4.4.1. *For each i, the **integral curves** in the (V, u) plane are solutions of*

$$\left(\frac{dV}{d\xi}, \frac{du}{d\xi}\right) = r, \quad i.e, \quad \frac{dV_1}{d\xi} = r_1, \quad \frac{dV_2}{d\xi} = r_2 \quad and \quad \frac{du}{d\xi} = r_3. \tag{4.36}$$

When ξ satisfies:

$$\xi = \lambda(V(\xi), u(\xi)), \tag{4.37}$$

the integral curves define rarefaction curves. If in addition to (4.37), *the variable ξ increases and satisfies $x = \xi t$ the integral curves define rarefaction waves in the (x, t) plane.*

Remark 4.4.3. *In Secs.* 4.5 *and* 4.7, *we calculate the eigenvalues and eigenvectors for the nitrogen-steam-water model and show that they have the following form:*

$$\lambda = u(\xi)\hat{\lambda}(V(\xi)) \qquad and \qquad r = (\eta_1(V(\xi)), \eta_2(V(\xi)), u(\xi)\eta_3(V(\xi))), \tag{4.38}$$

where $\hat{\lambda}$ and η_j for $j = 1, 2, 3$ do not depend on u. It is useful to define:

$$\hat{\lambda}^{i,-}(V) := \lambda_i/u^-, \quad \hat{\lambda}^{i,+}(V) := \lambda_i/u^+ \quad and \quad \hat{r}^i = (r_1^i, r_2^i) \quad for\ i = 1, 2, \tag{4.39}$$

where u^- is the Darcy speed for V^- and u^+ is the Darcy speed for V^+. The primary variables V^- and V^+ are the first and last value of V in the rarefaction wave. Then we can see from Eq. (4.36) *that (V, u) along the rarefaction wave satisfy:*

$$\left(\frac{dV_1}{d\xi}, \frac{dV_2}{d\xi}\right) = \Big(\eta_1(V(\xi)), \eta_2(V(\xi))\Big) \qquad and \qquad \frac{du}{d\xi} = u\eta_3(V(\xi)). \tag{4.40}$$

Since $V = (V_1, V_2)$ does not depend on u we can solve first (4.40.a) *obtaining $V(\xi)$ and then substitute it in* (4.40.b) *to obtain $u(\xi)$:*

$$u(\xi) = u^- exp\left(\int_{\xi^-}^{\xi} \eta_3(V(\sigma))d\sigma\right). \tag{4.41}$$

Here u^- is the first value for u on this integral curve, and ξ^- is chosen such that $\xi^- = \lambda(V^-, u^-)$. Notice that ξ is given implicitly by (4.37)*; using* (4.41) *this implicit relation is:*

$$\xi = u^-\hat{\lambda}(V(\xi))exp\left(\int_{\xi^-}^{\xi} \eta_3(V(\sigma))d\sigma\right). \tag{4.42}$$

ξ depends on u^- and V, but it does not depend on the Darcy speed in the rarefaction curve. When $V = V^+$, i.e., the last V in the rarefaction wave, we denote $\xi = \xi^+$.

Remark 4.4.4. *Assume that $u^- > 0$, so we perform the change of variable:*

$$\xi = \frac{x}{t} \longrightarrow \hat{\xi} = \frac{x}{u^-t}, \tag{4.43}$$

The system (4.33) *can be written in terms of the independent variables $(x, T) = (x, u^-t)$. In this space, one can prove that the eigenvalues and eigenvectors have the form $\lambda^{i,-}(V)$; r_1^i and r_2^i are the first two components of the vector r^i in Eq.* (4.38.*b*)*. From Eqs.* (4.37) *and* (4.42)*, it follows that $\hat{\xi}^-$ and $\hat{\xi}$ satisfy:*

$$\hat{\xi}^- = \hat{\lambda}(V^-), \quad and \quad \hat{\xi} = \hat{\lambda}(V(\hat{\xi}))exp\left(\int_{\hat{\xi}^-}^{\hat{\xi}} \eta_3(V(\sigma))d\sigma\right). \tag{4.44}$$

It follows that in (x, T) the variables V and $\hat{\xi}$ do not depend on u, so the rarefaction curves too do not depend on u. The Darcy speed keeps the form (4.41)*.*

4.4.2 Shock waves

Frequently, discontinuities or shocks appear in solution of the Riemann (or Cauchy) problems of non-linear hyperbolic or hyperbolic-elliptic equations (see [60]). The Rankine-Hugoniot condition for shocks (RH) needs to be satisfied.

The main feature of this class of equations is the existence of separated regions where the system of balance equations (4.15)-(4.18) reduces to different systems of conservation equations (2.2). In the nitrogen/steam/water flow problem considered in this paper, there are three different situations: the *spg*, the *tp* and the *spl*. To obtain the complete Riemann problem solution it is necessary to link these regions. As the water mass source term acts in the infinitesimal space between physical situations, we propose the existence of shocks linking these different regions. Thus we divide the study of the shocks into different groups: shocks within a physical situation and shocks between different physical situations.

For a fixed $W^- = (V^-, u^-)$ the **Rankine-Hugoniot Locus**, RH locus, parametrizes the discontinuous solutions of Eq. (4.15)-(4.18), ie., it consists of the $W^+ = (V^+, u^+)$ that satisfy the following RH condition and it is denoted by $\mathcal{RH}(W^-)$, see [2.3.1]:

$$v^s\left(G^+(V^+) - G^-(V^-)\right) = u^+F^+(V^+) - u^-F^-(V^-), \tag{4.45}$$

where (V^+, u^+) is the state at the right of the shock and (V^-, u^-) is the state at the left of the shock; v^s is the shock speed; G^+ (G^-) and F^+ (F^-) are the accumulation and flux terms at the right (left) of the shock, which in general have different expressions. We specify the state (V^-, u^-) on the left hand side, but at the right u^+ is not specified, it is obtained from the RH condition (4.45). From the continuity of F and G at the interface between different physical situations is clear that $W^+ = W^-$ is always a solution of (4.45), i.e., the $(-)$ state always belongs to a branch of RH locus.

We define the function $\mathcal{H} = \mathcal{H}(W^-; W^+)$ as:

$$\mathcal{H} := v^s \left(G^+(V^+) - G^-(V^-)\right) - u^+F^+(V^+) + u^-F^-(V^-), \tag{4.46}$$

notice that the RH locus of W^- are the states W that satisfy $\mathcal{H}(W^-; W^+) := 0$.

Since in this work we are interested in connected branches (i.e., that contain the $(-)$ state), we use the following criterion for admissibility of shocks instead of the viscosity profile criterion:

Definition 4.4.2. *(T.P. Liu.* [47,48]*) We call* ***shock curve*** *the parts V of Rankine-Hugoniot curve where the shock speed decreases when V moves away from V^-. When we consider the waves in (x,t) also, each point of the shock curve represents* ***shock wave****. The shock curve parametrizes the $(+)$ states of admissible shocks waves $(-)$, $(+)$.*

Rankine-Hugoniot locus

For a fixed $W^- = (V^-, u^-)$, we obtain the RH locus, or $\mathcal{RH}(W^-)$, from Eq. (4.45), which can be written as:

$$v^s[G_1] = u^+F_1^+ - u^-F_1^-, \tag{4.47}$$
$$v^s[G_2] = u^+F_2^+ - u^-F_2^-, \tag{4.48}$$
$$v^s[G_3] = u^+F_3^+ - u^-F_3^-, \tag{4.49}$$

where $[G_i] = G_i^+ - G_i^-$, $G_i^\pm = G_i^\pm(V^\pm)$ and $F_i^\pm = F_i^\pm(V^\pm)$ for $i = 1,2,3$. We rewrite the system (4.47)-(4.49) as:

$$\begin{pmatrix} [G_1] & -F_1^+ & F_1^- \\ [G_2] & -F_2^+ & F_2^- \\ [G_3] & -F_3^+ & F_3^- \end{pmatrix} \begin{pmatrix} v^s \\ u^+ \\ u^- \end{pmatrix} = 0, \quad \text{which requires} \quad det \begin{pmatrix} [G_1] & -F_1^+ & F_1^- \\ [G_2] & -F_2^+ & F_2^- \\ [G_3] & -F_3^+ & F_3^- \end{pmatrix} = 0 \tag{4.50}$$

to have a non-trivial solution. Eq. $(4.50.b)$ yields the implicit expression $\mathcal{H}_e = 0$ with:

$$\mathcal{H}_e := [G_1](F_3^+F_2^- - F_3^-F_2^+) + [G_2](F_1^+F_3^- - F_1^-F_3^+) + [G_3](F_2^+F_1^- - F_2^-F_1^+). \tag{4.51}$$

Since we can obtain the RH locus in V, it more useful denote it by $\mathcal{RH}(V^-)$ instead of $\mathcal{RH}(W^-)$. Generically, the expression $\mathcal{H}_e(V^-; V^+) = 0$ defines a 1-dimensional structure, see Section 4.4.2. Notice that there are only 2 primary variables in V^+ for the *spg* and the *tp* (see the following Section for definition). In the *spl* there is only one scalar equation with the temperature as the primary variable, so $\mathcal{RH}(T^-)$ is the whole temperature physical range.

Solving the system (4.50), we obtain v^s and u^+ as functions of V^-, V^+ and u^-:

$$v^s = u^- \frac{F_2^+ F_1^- - F_1^+ F_2^-}{F_1^+[G_2] - F_2^+[G_1]} = u^- \frac{F_3^+ F_2^- - F_2^+ F_3^-}{F_2^+[G_3] - F_3^+[G_2^+]} = u^- \frac{F_1^+ F_3^- - F_3^+ F_1^-}{F_3^+[G_1] - F_1^+[G_3]}, \tag{4.52}$$

$$u^+ = u^- \frac{F_1^-[G_2] - F_2^-[G_1]}{F_1^+[G_2] - F_2^+[G_1]} = u^- \frac{F_2^-[G_3] - F_3^-[G_2]}{F_2^+[G_3] - F_3^+[G_2]} = u^- \frac{F_3^-[G_1] - F_1^-[G_3]}{F_3^+[G_1] - F_1^+[G_3]}. \tag{4.53}$$

Of course, Eqs. (4.52) and (4.53) are valid if each denominator is non-zero.

Remark 4.4.5. *In the definitions that follow, all wave structures can be obtained in the space of primary variables V. Using $u^+ = u^+(V^-, u^-; V^+)$ and $v^s := v^s(V^-, u^-; V^+)$, we define:*

$$\frac{u^+}{u^-} = \chi(V^-; V^+), \quad \hat{v}^-(V^-; V^+) := \frac{v^s}{u^-} \quad \textit{and} \quad \hat{v}^+(V^-; V^+) := \frac{v^s}{u^-\chi(V^-; V+)}. \tag{4.54}$$

We define the set of unordered index pairs for Eqs. (4.52) and (4.53) as:

$$\mathbb{P} = \{\{1,2\}, \{3,1\}, \{2,3\}\}. \tag{4.55}$$

From this Section 4.4.2 and Rems. 4.4.3 and 4.4.4, we have proved the following:

Proposition 4.4.1. *Let u_L be positive. The primary variables V in the shock and rarefaction curves do not depend on the left Darcy speed $u_L > 0$. If a sequence of waves and states solve the Riemann problem in the primary variables, for a given $u_L > 0$ (or $u_R > 0$), then it is also a solution for any other $u_L > 0$ (or $u_R > 0$):*

$$\begin{cases} (V_L, \cdot) & \textit{if } x < 0 \\ (V_R, \cdot) & \textit{if } x > 0. \end{cases} \tag{4.56}$$

Moreover, assume that for each m there is $\{p, q\} \in \mathbb{P}$ such that the following inequality is satisfied: $F^+_{q,m}[G_{p,m}] - F^+_{p,m}[G_{q,m}] \neq 0$. Then u_R is given by:

$$u_R = u_L \prod_{l=1}^{\rho_1} exp\left(\int_{\xi^{-,l}}^{\xi^{+,l}} \eta_3^l(V(\sigma))d\sigma\right) \prod_{m=1}^{\rho_2} \frac{F^-_{q,m}[G_{p,m}] - F^-_{p,m}[G_{q,m}]}{F^+_{q,m}[G_{p,m}] - F^+_{p,m}[G_{q,m}]}, \tag{4.57}$$

where η_3^l is the third component of eigenvector r, $\xi^{-,l}$ and $\xi^{+,l}$ are the first and the last values of ξ^l associated to the l-th rarefaction wave.

In the Proposition above, $G_{j,m}$ and $F_{j,m}$ represent the j-th components ($j = \{1,2,3\}$) of G and F on the m-th shock wave. Similarly $exp\left(\int_{\xi^{-,l}}^{\xi^{+,l}} \eta_3^l(V(\sigma))d\sigma\right)$ is computed along the l-rarefaction curve.

A very important result follows from the Proposition 2.3.2 is:

Corollary 4.4.1. *Assume the same hypothesis of Proposition 2.3.2, then the Riemann solution can be obtained in each physical situation first in the primary variables V. Then the Darcy speed can be obtained at any point of the space (V, u) in terms of V and u_L by an equation analogous to Eq. (4.57).*

Remark 4.4.6. *Corollary* 4.4.1 *states that the secondary variable u can be obtained from the primary variables, so the former usually will not appear in figures.*

Proposition 4.4.2. *(Quadruple Shock Rule.) In the tp and the spg situations, consider four primary points and three Darcy speeds, determining four states:* (V^-, u^-) *in the tp,* (V^+, u^+) *in the spg and* (V^M, u^M) *and* (V^*, u^*) *both in the tp or the spg. Assume that there are shocks between the following pairs of states:*
(i) (V^-, u^-) *and* (V^+, u^+) *with speed* $v^{-,+}$*,*
(ii) (V^-, u^-) *and* (V^M, u^M) *with speed* $v^{-,M}$*,*
(iii) (V^M, u^M) *and* (V^*, u^*) *with speed* $v^{M,*}$*,*
such that two speeds coincide, i.e., at least one of the following equalities is satisfied:

$$\text{either} \quad v^{-,+} = v^{-,M} \quad \text{or} \quad v^{-,+} = v^{M,*} \quad \text{or} \quad v^{-,M} = v^{-,+}. \tag{4.58}$$

If the following conditions (I) *to* (III) *are satisfied:*
(I) $G(V^+) - G(V^-)$ *and* $G(V^*) - G(V^M)$ *are LI;*
(II) V^+ *and* V^* *have one coordinate* V_i *with coinciding values;*
(III) $\partial\mathcal{H}_e/\partial V_j \neq 0$ *for* $j \neq i$ *for all* $V \in \mathcal{RH}(V^-)$*,*
then
(1) V^*=V^+*;*
(2) $u^* = u^+$*;*
(3) all the three speeds are equal:

$$v^{-,+} = v^{-,M} = v^{M,*}. \tag{4.59}$$

Proof: Disregarding the index in accumulation and flux terms, the RH conditions for (V^-, u^-)-(V^+, u^+), (V^-, u^-)-(V^M, u^M) and (V^M, u^M)-(V^*, u^*) are respectively:

$$v^{-,+}(G(V^+) - G(V^-)) = u^+F(V^+) - u^-F(V^-), \tag{4.60}$$

$$v^{-,M}(G(V^M) - G(V^-)) = u^MF(V^M) - u^-F(V^-), \tag{4.61}$$

$$v^{M,*}(G(V^*) - G(V^M)) = u^*F(V^*) - u^MF(V^M). \tag{4.62}$$

Assume that now Eq. (4.58.a) is satisfied. Substituting $v^{-,+} = v^{-,M} = v$ in Eqs. (4.60) and (4.61) and subtracting Eq. (4.60) from (4.61), we obtain:

$$v(G(V^+) - G(V^M)) = u^+F(V^+) - u^MF(V^M). \tag{4.63}$$

Notice that Eqs. (4.63) and (4.62) define implicitly the RH locus by $\mathcal{H}_e(V^M; V^+) = 0$ in the variables V^M and V^+. Since the RH locus depends solely on V^M and the accumulation and flux functions, we obtain that both RH locus defined by Eqs. (4.63) and (4.62) coincide. Since (I) to (III) are satisfied, then $V^* = V^+$.

Now from Eq. (4.53), we notice for a fixed u^- that the Darcy and shock speeds depend solely on V^- and V^+. From Eqs. (4.62) and (4.63), we can see that the $(-)$ and $(+)$ states are the same for each expression and that they define the same RH locus, so $u^* = u^+$ and Eq. (4.59) is satisfied.

The other cases are proved similarly. □

Degeneracies of RH Locus

We utilize the notation:

$$\mathcal{D}_{ij}(V^-,V^+) = F_i^+[G_j] - F_j^+[G_i]. \tag{4.64}$$

Proposition 4.4.1 is valid if the denominator of Eq. (4.57) is non zero for some $\{i,j\} \in \mathbb{P}$. So it is necessary to study the behavior of the solution when $\mathcal{D}_{ij}(V^-,V^+) = 0$ for all $\{i,j\} \in \mathbb{P}$. For a fixed pair $\{i,j\} \in \mathbb{P}$, it easy to prove that:

Lemma 4.4.1. *Let $\{V^-,V^+\}$ satisfy $\mathcal{H}_e(V^-;V^+) = 0$, where $\mathcal{H}_e$ is given by Eq. (4.51). If $\mathcal{D}_{ij}(V^-,V^+) = 0$, then one of following conditions is satisfied:*

$$(i) \quad F_i^+F_j^- - F_i^+F_j^- = 0, \tag{4.65}$$

or

$$(ii) \quad \mathcal{D}_{12}(V^-,V^+) = \mathcal{D}_{31}(V^-,V^+) = \mathcal{D}_{23}(V^-,V^+) = 0. \tag{4.66}$$

From this Lemma it follows immediately that:

Corollary 4.4.2. *Let $\{V^-,V^+\}$ satisfy $\mathcal{H}_e(V^-;V^+) = 0$. If $\mathcal{D}_{ij}(V^-,V^+)$ vanishes for two index pairs $\{i,j\} \in \mathbb{P}$, then it vanishes for all pairs.*

Proposition 4.4.3. *If $\mathcal{D}_{ij}(V^-,V^+) = 0$ for all $\{i,j\} \in \mathbb{P}$ and $(F_1^+,F_2^+,F_3^+) \neq 0$, we obtain:*

$$[G_k] = \rho_1 F_k^+ \qquad \text{and} \qquad F_k^- = \rho_2 F_k^+, \quad \text{for } k = 1,2,3. \tag{4.67}$$

where ρ_1 and ρ_2 are constants depending on $[G]$, F^- and F^+. Moreover, for χ defined in (4.54.a), the shock speed v^s satisfies:

$$v^s = u^- \frac{\chi(V^-;V^+) - \rho_2}{\rho_1}. \tag{4.68}$$

Proof: Let $u^- > 0$. Since $\mathcal{D}_{ij}(V^-,V^+) = 0 \; \forall \; \{i,j\} \in \mathbb{P}$, it follows that:

$$(F_2^+[G_3] - F_3^+[G_2])e_1 + (F_3^+[G_1] - F_1^+[G_3])e_2 + (F_1^+[G_2] - F_2^+[G_1])e_3 = 0, \tag{4.69}$$

where e_i for $i = 1,2,3$ is the canonical basis for $\mathbb{R}^3$. Eq. (4.69) can be written as:

$$(F_1^+,F_2^+,F_3^+) \times ([G_1],[G_2],[G_3]) = 0, \tag{4.70}$$

where $\times$ represents the curl. Since Eq. (4.70) is satisfied, it follows that $[G]$ is parallel to F^+, so there is a constant ρ_1 so Eq. (4.67.a) is satisfied. Substituting $[G] = \rho_1 F^+$ into the RH condition (4.45), we obtain:

$$v^s \rho_1 F_1^+ = u^+F_1^+ - u^-F_1^-, \tag{4.71}$$
$$v^s \rho_1 F_2^+ = u^+F_2^+ - u^-F_2^-, \tag{4.72}$$
$$v^s \rho_1 F_3^+ = u^+F_3^+ - u^-F_3^-, \tag{4.73}$$

If $F_i^- = 0$ for some $i = 1,2,3$, it follows that:

$$v^s = u^+/\rho_1. \tag{4.74}$$

If $F_i^- \neq 0$ for all $i = 1,2,3$, multiplying Eq. (4.71) by F_2^- and (4.72) by $-F_1^-$ and adding the results, we obtain:

$$v^s \varphi \rho_1 (F_1^+ F_2^- - F_2^+ F_1^-) = u^+ (F_1^+ F_2^- - F_2^+ F_1^-). \tag{4.75}$$

Let us assume temporarily that $F_1^+ F_2^- - F_2^+ F_1^- \neq 0$, so Eq. (4.74) is satisfied. Substituting v^s given by Eq. (4.74) into Eq. (4.71) we obtain $u^- F_1^- = 0$. Since $F_1^- \neq 0$, generically, it follows that $u^- = 0$, which is false. So $F_1^+ F_2^- - F_2^+ F_1^- = 0$.

Similar calculations for Eqs. (4.72), (4.73) and (4.71), (4.73) show that:

$$F_1^+ F_2^- - F_2^+ F_1^- = F_2^+ F_3^- - F_3^+ F_2^- = F_1^+ F_3^- - F_3^+ F_1^- = 0. \tag{4.76}$$

Since Eq. (4.76) is satisfied, there exists a constant ρ_2 such that $F^- = \rho_2 F^+$. Eq. (4.68) can be obtained by substituting $[G] = \rho_1 F^+$ and $F^- = \rho_2 F^+$ in the RH condition (4.45). □

Remark 4.4.7. *If* $(F_1^+, F_2^+, F_3^+) = 0$ *and* $([G_1], [G_2], [G_3]) \neq 0$, *it is easy to prove that:*

$$F_k^- = \rho_3 [G_k] \quad \text{and} \quad v^s = u^- \rho_3, \tag{4.77}$$

where ρ_3 *is a constant that depends on* $[G]$ *and* F^-.

Corollary 4.4.3. *The states* $\{V^-, V^+\}$ *satisfying the RH condition* (4.45), *for which* $\mathcal{D}_{ij}(V^-, V^+) = 0$ *for all* $\{i,j\} \in \mathbb{P}$, *satisfy also:*

$$F_i^+ F_j^- - F_j^+ F_i^- = 0, \quad \forall \{i,j\} \in \mathbb{P}. \tag{4.78}$$

We notice that the system (4.78) has always the trivial solution $V^+ = V^-$.

4.4.3 Bifurcation loci

Assume that there are no degeneracies any RH locus. For standard conservation laws, there are loci where the solutions change topology such as: secondary bifurcation, coincidence, double contact, inflection, hysteresis and interior boundary contact. The proof of this change depends on the Bethe-Wendroff Theorem 2.3.6, that in this model can be written as:

Proposition 4.4.4. *Assume that* F *and* G *are* $\mathcal{C}^2$. *Let* $v^s(W^+; W^-)$ *be the shock speed between different physical situations. Assume that* $\ell^i(V^+) \cdot (G^+(V^+) - G^-(V^-)) \neq 0$. *Then* v^s *has a critical point at* W^+ *(and* $\hat{v}^+(V^-; V^+)$ *has a critical point at* V^+*), if and only if:*

$$\hat{v}^+(V^-; V^+) = \lambda^{i,+}(V^+) \qquad \text{for} \quad i = 1 \text{ or } 2, \tag{4.79}$$

where $\lambda^+(V)$ *is given by Eq.* (4.39) *and* $v^+(V^-; V^+)$ *is given by* (4.54).

Secondary bifurcation manifold

The RH locus for a fixed V^- is obtained implicitly by $\mathcal{H}_e(V^-;V^+) = 0$, where $\mathcal{H}_e : \mathbb{R}^4 \longrightarrow \mathbb{R}$, and V^+ are the primary variables. At some points, these implicit expression fail to define a curve for V^+ in the space of primary variables. We call the set of these points the **secondary bifurcation locus**. From the implicit function theorem, they are the " $+$ " states for which there exists a " $-$ " state such that the following equalities are satisfied:

$$\mathcal{H}_e(V^-;V^+) = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{H}_e}{\partial V_j^+} = 0, \quad \text{for} \quad j = 1 \text{ and } 2, \tag{4.80}$$

where $\mathcal{H}_e$ is defined by Eq. (4.51) and $\partial \mathcal{H}_e/\partial V_j^+$ for $j = 1,2$ are:

$$\frac{\partial \mathcal{H}_e}{\partial V_j^+} = \left(\frac{\partial F_3^+}{\partial V_j^+} F_2^- - F_3^- \frac{\partial F_2^+}{\partial V_j^+}, \frac{\partial F_1^+}{\partial V_j^+} F_3^- - F_1^- \frac{\partial F_3^+}{\partial V_j^+}, \frac{\partial F_2^+}{\partial V_j^+} F_1^- - F_2^- \frac{\partial F_1^+}{\partial V_j^+} \right) [G]^T +$$
$$+ \left(F_3^+ F_2^- - F_3^- F_2^+, F_1^+ F_3^- - F_1^- F_3^+, F_2^+ F_1^- - F_2^- F_1^+ \right) \left(\frac{\partial G^+}{\partial V_j^+} \right)^T, \tag{4.81}$$

where $[G] = ([G_1], [G_2], [G_3])$ and $\partial G/\partial V = (\partial G_1/\partial V, \partial G_2/\partial V, \partial G_3/\partial V)$.

Equation (4.80) yields an equivalent expression for the secondary bifurcation.

Proposition 4.4.5. *A state V^- belongs to the bifurcation manifold for the family i when there exists a state V^+ such that:*

$$V^+ \in \mathcal{RH}(V^-) \quad \textit{with} \quad \hat{v}^+(V^-;V^+) = \hat{\lambda}^{i,+}(V^+) \quad \textit{and} \quad \ell^i(V^+) \cdot [G] = 0. \tag{4.82}$$

Proof: We drop the family index i. Assume that (4.80) is satisfied. Rearranging the terms, we can rewrite $\partial \mathcal{H}_e/\partial V_j^+$ for $j = 1,2$:

$$\frac{\partial \mathcal{H}_e}{\partial V_j^+} = \left(F_3^- [G_2] - F_2^- [G_3] \right) \frac{\partial F_1^+}{\partial V_j^+} - \left(F_3^- F_2^+ - F_3^+ F_2^- \right) \frac{\partial G_1^+}{\partial V_j^+} + \left(F_1^- [G_3] - F_3^- [G_1] \right) \frac{\partial F_2^+}{\partial V_j^+}$$
$$- \left(F_3^+ F_1^- - F_3^- F_1^+ \right) \frac{\partial G_2^+}{\partial V_j^+} + \left(F_2^- [G_1] - F_1^- [G_2] \right) \frac{\partial F_3^+}{\partial V_j^+} - \left(F_1^+ F_2^- - F_1^- F_2^+ \right) \frac{\partial G_3^+}{\partial V_j^+}. \tag{4.83}$$

Notice that $\lambda^+ = u^+ \hat{\lambda}^+$, so the matrix $\partial_W(uF) - \lambda \partial_W G$, is written at (V^+, u^+) as:

$$\begin{pmatrix} u^+ \left(\frac{\partial F_1^+}{\partial V_1^+} - \hat{\lambda}^+ \frac{\partial G_1^+}{\partial V_1^+} \right) & u^+ \left(\frac{\partial F_1^+}{\partial V_2^+} - \hat{\lambda}^+ \frac{\partial G_1^+}{\partial V_2^+} \right) & F_1^+ \\ u^+ \left(\frac{\partial F_2^+}{\partial V_1^+} - \hat{\lambda}^+ \frac{\partial G_2^+}{\partial V_1^+} \right) & u^+ \left(\frac{\partial F_2^+}{\partial V_2^+} - \hat{\lambda}^+ \frac{\partial G_2^+}{\partial V_2^+} \right) & F_2^+ \\ u^+ \left(\frac{\partial F_3^+}{\partial V_1^+} - \hat{\lambda}^+ \frac{\partial G_3^+}{\partial V_1^+} \right) & u^+ \left(\frac{\partial F_3^+}{\partial V_2^+} - \hat{\lambda}^+ \frac{\partial G_3^+}{\partial V_2^+} \right) & F_3^+ \end{pmatrix} \tag{4.84}$$

Setting $\hat{v}^+(V^-;V^+) = \hat{\lambda}^+(V^+)$ in (4.84) and assuming that $\mathcal{D}_{ij}(V^-,V^+) = F_i^-[G_j] - F_j^-[G_i] \neq 0$ for all $\{i,j\} \in \mathbb{P}$, we use Eqs. (4.52) and (4.54) to write an equivalent but convenient expression for $\hat{v}^+$ in each matrix element, obtaining:

$$\begin{pmatrix} u^+\left(\frac{\partial F_1^+}{\partial V_1^+} - \frac{F_3^-F_2^+ - F_3^+F_2^-}{F_3^-[G_2]-F_2^-[G_3]}\frac{\partial G_1^+}{\partial V_1^+}\right) & u^+\left(\frac{\partial F_1^+}{\partial V_2^+} - \frac{F_3^-F_2^+ - F_3^+F_2^-}{F_3^-[G_2]-F_2^-[G_3]}\frac{\partial G_1^+}{\partial V_2^+}\right) & F_1^+ \\ u^+\left(\frac{\partial F_2^+}{\partial V_1^+} - \frac{F_3^+F_1^- - F_3^-F_1^+}{F_1^-[G_3]-F_3^-[G_1]}\frac{\partial G_2^+}{\partial V_1^+}\right) & u^+\left(\frac{\partial F_2^+}{\partial V_2^+} - \frac{F_3^+F_1^- - F_3^-F_1^+}{F_1^-[G_3]-F_3^-[G_1]}\frac{\partial G_2^+}{\partial V_2^+}\right) & F_2^+ \\ u^+\left(\frac{\partial F_3^+}{\partial V_1^+} - \frac{F_1^+F_2^- - F_1^-F_2^+}{F_2^-[G_1]-F_1^-[G_2]}\frac{\partial G_3^+}{\partial V_1^+}\right) & u^+\left(\frac{\partial F_3^+}{\partial V_2^+} - \frac{F_1^+F_2^- - F_1^-F_2^+}{F_2^-[G_1]-F_1^-[G_2]}\frac{\partial G_3^+}{\partial V_2^+}\right) & F_3^+ \end{pmatrix} \tag{4.85}$$

Since $\partial\mathcal{H}_e/\partial V_j^+ = 0$ for $j = 1, 2$, from Eq (4.83), it follows that for ℓ given by

$$\ell = \left(F_3^-[G_2] - F_2^-[G_3],\ \ F_1^-[G_3] - F_3^-[G_1],\ \ F_2^-[G_1] - F_1^-[G_2]\right), \tag{4.86}$$

the inner products of columns 1 and 2 by ℓ are zero. Since at (V^-,V^+) the expression $\mathcal{H}_e(V^-)$ vanishes, after some calculations, we obtain:

$$\ell \cdot (F_1^+, F_2^+, F_3^+) = 0.$$

First let us consider the case $\mathcal{D}_{ij}(V^-,V^+) \neq 0$ for all $\{i,j\} \in \mathbb{P}$, so it is clear that ℓ is a left eigenvector of the system. One can prove that all eigenvectors for $\lambda(V^+) = v^s(V^-,V^+)$ are parallel to ℓ. Notice that $\ell \cdot ([G_1],[G_2],[G_3] = 0$, because this equality satisfies the RH locus $\mathcal{H}_e = 0$, with $\mathcal{H}_e$ given by (4.51).

If $\mathcal{D}_{ij}(V^-,V^+) = 0$ for a pair $\{i,j\} \in \mathbb{P}$, using Lemma 4.4.1, the relationships $F_i^+F_j^- - F_j^+F_i^- = 0$ or $\mathcal{D}_{ij}(V^-,V^+) = 0$ for all $\{i,j\} \in \mathbb{P}$ are satisfied.

Assume that $\mathcal{D}_{ij}(V^-,V^+) \neq 0$ for some $\{i,j\} \in \mathbb{P}$. For concreteness we set $i = 3$ and $j = 2$; the other cases can be proved similarly. Since $F_1^-[G_3] - F_3^-[G_1] \neq 0$ and the matrix (4.85) has the form (a_{ij}) for $i,j = 1,2,3$, and we can write a_{11}, a_{12} and a_{13}, respectively as:

$$u^+\left(\frac{\partial F_1^+}{\partial V_1^+} - \frac{F_3^+F_1^- - F_3^-F_1^+}{F_1^-[G_3]-F_3^-[G_1]}\frac{\partial G_1^+}{\partial V_1^+}\right), \quad u^+\left(\frac{\partial F_1^+}{\partial V_2^+} - \frac{F_3^+F_1^- - F_3^-F_1^+}{F_1^-[G_3]-F_3^-[G_1]}\frac{\partial G_1^+}{\partial V_2^+}\right) \quad \text{and} \quad F_1^+.$$

Substituting ℓ in (4.86) by the the following vector:

$$\ell = \left(0,\ \ F_1^-[G_3] - F_3^-[G_1],\ \ F_2^-[G_1] - F_1^-[G_2]\right), \tag{4.87}$$

it is easy to prove that this vector ℓ is in the kernel of the transpose of the matrix (4.85).

Finally assume that $\mathcal{D}_{ij} = 0$ for all $\{i,j\} \in \mathbb{P}$. Since ℓ is a left eigenvector of the matrix (4.85) it follows that $\ell \cdot (F_1^+, F_2^+, F_3^+) = 0$. From Eq. (4.67.$a$) in the Proposition 4.4.3, we know that $([G_1],[G_2],[G_3]) = (\rho_1 F_1^+, \rho_1 F_2^+, \rho_1 F_3^+)$ for any constant $\rho_1 \in \mathbb{R}$, so:

$$\ell \cdot [G] = \ell \cdot (\rho_1 F^+) = \rho_1 \ell \cdot F^+ = 0.$$

The converse can be proved similarly by reversing the calculations. $\square$

Coincidence loci

There are two important types of speed coincidence: coincidence between eigenvalues and coincidence between eigenvalues and shock speeds. These structures are important because the Riemann solution changes when the L (or R) data is prescribed in different regions relatively to these curves.

Coincidence between eigenvalues in each physical situation. Notice that it is possible to analyze the system in the variables V without taking into account the Darcy speed u. At the coincidence locus between eigenvalues the Darcy speeds also coincide, so the coincidence in the V space consists of the states in a certain physical situation such that

$$\hat{\lambda}^1(V) = \hat{\lambda}^2(V).$$

Double contact curves

It consists of the states V^- for which there is a state V^+ such that the shock joining V^- and V^+ has speed coinciding with the characteristic speed of family i at $(-)$ and of family j at $(+)$:

$$V^+ \in \mathcal{RH}(V^-) \quad \text{with} \quad \hat{\lambda}^{i,-}(V^-) = \hat{v}^-(V^-, V^+) \quad \text{and} \quad \hat{v}^+(V^-, V^+) = \hat{\lambda}^{j,+}(V^+).$$

Notice that V^- and V^+ can be in the same or in different physical situations.

Inflection curves

The rarefaction curves are useful to construct rarefaction waves where the eigenvalue varies monotonically; the inflection curve is the curve where the monotonicity fails, thus rarefaction curves stop at this curve. Since u varies along the rarefaction wave, it is impossible to disregard the effect of u. Any state $W = (V, u)$ on the inflection curve satisfies:

$$\nabla \lambda_i(W) \cdot r^i(W) = 0. \tag{4.88}$$

However, the Darcy speed can be isolated in Eq. (4.88), so it is easy to prove that in the space of primary variables, the inflection curve of family i, for $i = 1, 2$, consists of the states V satisfying the equation:

$$\nabla_V \hat{\lambda}^i \cdot \hat{r} = -\hat{\lambda}^i \eta_3, \tag{4.89}$$

where $\eta_i(V)$ for $i = 1, 2, 3$ are the coordinates for the eigenvector r given in Eq. (4.38.b).

Interior boundary contact (extension of the boundary)

Because of the presence of physical region boundaries it is important to obtain the states V joined to points V' on the physical boundary by shock waves that are characteristic at V for the family i. These states satisfy:

$$V \in \mathcal{RH}(V') \quad \text{with } V' \text{ on the boundary and } \hat{\lambda}^i(V) = \hat{v}^-(V, V').$$

4.4.4 Admissibility of shocks

Frequently, for a fixed state V^- there are points in the RH curve of V^- that separate potentially admissible shocks from non-admissible shocks. At these points the shock speed coincides with the characteristic speed and the shock speed has an extremum, see Proposition 4.4.4, so the Riemann solution usually changes. There are two type of such shocks: left characteristic and right characteristic shocks. In this structure, the shock has speed equal to the characteristic speed at the left or at the right state.

Left characteristic shock.

For a fixed state V^-, these points are the V^+ such that:

$$V^+ \in \mathcal{RH}(V^-) \quad \text{with} \quad \hat{v}^+(V^-;V^+) = \hat{\lambda}^{j,+}(V^+) \quad \text{for family } j = 1 \text{ or } 2.$$

Notice that V^- and V^+ can be in the same or different physical situations.

Right characteristic shock.

For a fixed V^+, these points are the V^- such that:

$$V^- \in \mathcal{RH}(V^+) \quad \text{with} \quad \hat{v}^-(V^-,V^+) = \hat{\lambda}^{j,+}(V^+) \quad \text{for family } j = 1 \text{ or } 2.$$

Notice that V^- and V^+ can be in the same or in different physical situations.

4.5 Elementary waves in the single-phase gaseous situation

4.5.1 Characteristic speed analysis

We simplify the system (4.21)-(4.23) by substituting (4.21) by (4.21) divided by θ_W and (4.21) by a certain linear combination of (4.21) with (4.22):

$$\frac{\partial}{\partial t}\varphi\psi_{gw}T^{-1} + \frac{\partial}{\partial x}u\psi_{gw}T^{-1} = 0, \tag{4.90}$$

$$\frac{\partial}{\partial t}\varphi T^{-1} + \frac{\partial}{\partial x}uT^{-1} = 0, \tag{4.91}$$

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + \psi_{gw}H_{gW} + \psi_{gn}H_{gN}\right) + \frac{\partial}{\partial x}u(\psi_{gw}H_{gW} + \psi_{gn}H_{gN}) = 0, \tag{4.92}$$

where we have used the equality $\psi_{gn} = 1 - \psi_{gw}$.

We differentiate Eqs. (4.90)-(4.92) and we rewrite it in the form (4.33) with $V = (\psi_{gw}, T)$ as:

$$B\frac{\partial}{\partial t}\begin{pmatrix}\psi_{gw}\\ T\\ u\end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix}\psi_{gw}\\ T\\ u\end{pmatrix} = 0, \tag{4.93}$$

where (indicating the differentiation relative to temperature by $'$):

$$B = \varphi \begin{pmatrix} T^{-1} & -\psi_{gw}T^{-2} & 0 \\ 0 & -T^{-2} & 0 \\ (H_{gW} - H_{gN}) & \hat{C}_r + \psi_{gw}H'_{gW} + \psi_{gn}H'_{gN} & 0 \end{pmatrix}, \tag{4.94}$$

$$A = \begin{pmatrix} uT^{-1} & -u\psi_{gw}T^{-2} & \psi_{gw}T^{-1} \\ 0 & -uT^{-2} & T^{-1} \\ u(H_{gW} - H_{gN}) & u(\psi_{gw}H'_{gW} + \psi_{gn}H'_{gN}) & \psi_{gw}H_{gW} + \psi_{gn}H_{gN} \end{pmatrix}; \tag{4.95}$$

in (4.94) the constant $\hat{C}_r$ is the rock heat capacity divided by φ, see Appendix A.

Using (4.94) and (4.95), we notice that $(u - \varphi\lambda)$ is repeated in the first column of $A - \lambda B$. Factoring this term, the determinant of $A - \lambda B$ is $(u - \varphi\lambda)$ times the determinant of

$$\begin{pmatrix} T^{-1} & -\psi_{gw}T^{-2}(u - \varphi\lambda) & \psi_{gw}T^{-1} \\ 0 & -(u - \varphi\lambda)T^{-2} & T^{-1} \\ (H_{gW} - H_{gN}) & (u - \varphi\lambda)(\psi_{gw}H'_{gW} + \psi_{gn}H'_{gN}) - \varphi\lambda\hat{C}_r & \psi_{gw}H_{gW} + \psi_{gn}H_{gN} \end{pmatrix}, \tag{4.96}$$

so an eigenvalue and eigenvector for the system (4.93) are

$$\lambda_c = u/\varphi, \qquad r_c = (1, 0, 0)^T, \tag{4.97}$$

which correspond to fluid transport and the wave for this eigenvalue is a contact discontinuity. The composition ψ_{gw} changes but the speed and the temperature are constant. There is a single vector for λ_c, because the matrix (4.96) with $\lambda = \lambda_c$ has rank 3, generically. We denote this eigenvalue by λ_c to indicate that in the associated wave only the composition changes.

We find the other characteristic speed λ in (4.96). If $\psi_{gw} \neq 0$, we add the first line to the second line multiplied by $-\psi_{gw}$ obtaining:

$$\begin{pmatrix} T^{-1} & 0 & 0 \\ T^{-1} & -(u - \varphi\lambda)T^{-2} & T^{-1} \\ (H_{gW} - H_{gN}) & (u - \varphi\lambda)(\psi_{gw}H'_{gW} + \psi_{gn}H'_{gN}) - \varphi\lambda\hat{C}_r & \psi_{gw}H_{gW} + \psi_{gn}H_{gN} \end{pmatrix}. \tag{4.98}$$

The determinant of (4.98) is $-T^{-2}$ times

$$\Big[(u - \varphi\lambda)(\psi_{gw}H'_{gW} + \psi_{gn}H'_{gN}) - \varphi\lambda\hat{C}_r + (u - \varphi\lambda)T^{-1}(\psi_{gw}H_{gW} + \psi_{gn})H_{gN}\Big]. \tag{4.99}$$

Setting (4.99) to zero we obtain:

$$\lambda_T = \frac{u}{\varphi}\left(1 - \frac{\hat{C}_r T}{\mathbb{F}}\right), \tag{4.100}$$

where using (4.24), $\mathbb{F} := \mathbb{F}(T)$ is given by:

$$\mathbb{F}(T) = \left(\psi_{gw}\rho_{gW}(T)h'_{gW}(T) + \psi_{gn}\rho_{gN}(T)h'_{gN}(T) + \hat{C}_r\right)T. \tag{4.101}$$

As ρ_{gW}, ρ_{gN}, h'_{gW}, h'_{gN} and $\hat{C}_r$ are positive and ψ_{gw} and ψ_{gn} are non-negative, $\mathbb{F}(T)$ is always positive. The eigenvector associated to λ_T is

$$r_T = (0, \mathbb{F}, u\hat{C}_r). \tag{4.102}$$

This eigenvector is unique, because substituting λ_T in (4.98) we see that the resulting matrix has rank 2. The rarefaction wave has constant composition ψ_{gw}. Notice that $\lambda_T < \lambda_c$ in the physical range, so we can order these waves: the slowest wave is a thermal wave, with speed λ_T; the fastest is a compositional wave, with speed λ_c.

Behavior of the gaseous thermal inflection curve

To obtain the rarefaction curves in the *spg*, we need to study the sign of $\nabla\lambda_T \cdot r_T$. After a lengthly calculation, we obtain from (4.100) and (4.102)

$$\nabla\lambda_T \cdot r_T = \frac{uC_rT^2}{\varphi}\Xi(\psi_{gw}, T), \quad \text{where} \quad \Xi = \psi_{gw}\rho_{gW}h''_{gw} + (1-\psi_{gw})\rho_{gN}h''_{gn}. \tag{4.103}$$

As u, $\hat{C}_r$, T and φ are positive, we need to study the sign of Ξ.

The gaseous thermal inflection curve is denoted by $\mathcal{I}_T$; for Γ given by (4.26) it is defined by:

$$\mathcal{I}_T = \left\{(T, \psi_{gw}) \in \Gamma \text{ that satisfy } \nabla\lambda_T \cdot r_T = 0\right\}. \tag{4.104}$$

We plot the physical region and $\mathcal{I}_T$ in Figure (4.2.*a*), showing the signs of Ξ. In Figure 4.2.*b*, we plot the horizontal rarefaction lines associated to λ_c and the vertical rarefaction lines associated to λ_T, see (4.102). The waves associated to λ_c are contact discontinuities.

4.5.2 Shocks and contact discontinuities

Using (4.45) in Eqs. (4.90)-(4.92), the RH condition is:

$$v^s\varphi\left(\psi^+_{gw}/T^+ - \psi^-_{gw}/T^-\right) = u^+\psi^+_{gw}/T^+ - u^-\psi^-_{gw}/T^-, \tag{4.105}$$

$$v^s\varphi\left(1/T^+ - 1/T^-\right) = u^+/T^+ - u^-/T^-, \tag{4.106}$$

$$\begin{aligned} v^s\varphi\Big(H^+_r + \psi^+_{gw}H^+_{gW} + (1-\psi^+_{gw})H^+_{gN} - (H^-_r + \psi^-_{gw}H^-_{gW} + (1-\psi^-_{gw})H^-_{gN})\Big) = \\ u^+(\psi^+_{gw}H^+_{gW} + (1-\psi^+_{gw})H^+_{gN}) - u^-(\psi^-_{gw}H^-_{gW} + (1-\psi^-_{gw})H^-_{gN}), \end{aligned} \tag{4.107}$$

where $H^\pm_{gW} = \theta_W h_{gW}(T^\pm)/T^\pm$ and $H^\pm_{gN} = \theta_N h_{gN}(T^\pm)/T^\pm$.

In (4.105)-(4.107), there is a contact discontinuity with speed (4.97); the Darcy speed and the temperature are constant on this wave, i.e., $u = u^-$ and $T = T^-$.

The other shock occurs with ψ_{gw} constant; it is a thermal shock with speed v^s and Darcy speed u^+ on the right side given by:

$$v^s = \frac{u^-}{T^-}\left\{\Upsilon(T^-;T^+)(T^+ - T^-) + T^+\right\} \qquad \text{and} \qquad u^+ = \frac{u^-}{\varphi}\Upsilon(T^-;T^+),$$

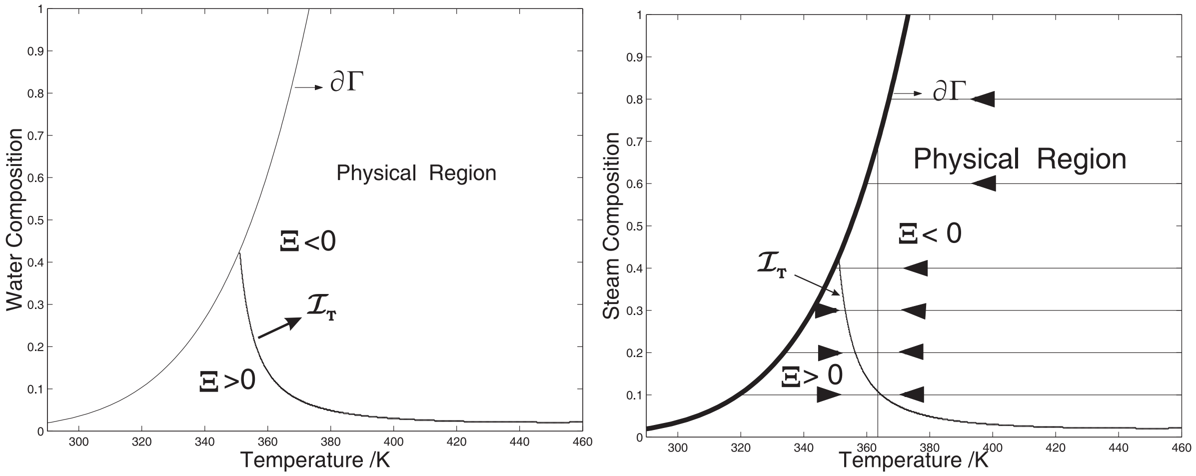


Figure 4.2: *a)-left:* The single-phase gaseous physical region Γ, and inflection locus. *b)-right:* Rarefaction curves. The horizontal rarefaction curves are associated to λ_T; we indicate with an arrow the direction of increasing speed. The vertical lines are contact discontinuity curves associated to λ_c, in which ψ_{gw} changes, T and u are constant.

where Υ is defined as follows, with θ_W, θ_N, H_{gW} and H_{gN} given by Eq. (4.24):

$$\Upsilon = \frac{\psi_{gw}\theta_W\left(h^+_{gW} - h^-_{gW}\right) + (1-\psi_{gw})\theta_N\left(h^+_{gN} - h^-_{gN}\right)}{T^-\left(\hat{H}^+_r - \hat{H}^-_r\right) + \psi_{gw}\theta_W\left(h^+_{gW} - h^-_{gW}\right) + (1-\psi_{gw})\theta_N\left(h^+_{gN} - h^-_{gN}\right)}. \tag{4.108}$$

The RH and rarefaction curves are contained in the horizontal lines in Figure (4.2.*b*).

4.6 Contact discontinuities in the single-phase liquid situation

Eq. (4.31) is linear in the *spl*, so a unique wave is associated to λ^W_T, given by (4.31.*b*). This wave is a contact discontinuity and there is no genuine rarefaction wave.

Remark 4.6.1. *In the spl there exists a cooling discontinuity between a state with temperature T^- and another state with temperature T^+. For the Riemann data: T^- if $x<0$ and T^+ if $x>0$, the solution is T^- if $x/t<\lambda^W_T$ and T^+ if $x/t>\lambda^W_T$.*

4.7 Elementary waves in the two-phase situation

4.7.1 Characteristic speed analysis

By means of a procedure similar to that in Section 4.4.1, we write the system (4.27)-(4.29) in the form (4.33), with $V = (s_g, T)$, as:

$$B\frac{\partial}{\partial t}\begin{pmatrix} s_g \\ T \\ u \end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix} s_g \\ T \\ u \end{pmatrix} = 0, \tag{4.109}$$

with

$$B = \begin{pmatrix} \varphi(\rho_{gw} - \rho_W) & \varphi\rho'_{gw}s_g & 0 \\ \varphi\rho_{gn} & \varphi\rho'_{gn}s_g & 0 \\ \varphi(H_g - H_W) & \varphi(\hat{C}_r + C_W + s_g(C_g - C_W)) & 0 \end{pmatrix}, \tag{4.110}$$

$$A = \begin{pmatrix} u(\rho_{gw} - \rho_W)\frac{\partial f_g}{\partial s_g} & u\left((\rho_{gw} - \rho_W)\frac{\partial f_g}{\partial T} + \rho'_{gw}f_g\right) & \rho_W + f_g(\rho_{gw} - \rho_W) \\ u\rho_{gn}\frac{\partial f_g}{\partial s_g} & u\left(\rho_{gn}\frac{\partial f_g}{\partial T} + \rho'_{gn}f_g\right) & f_g\rho_{gn} \\ u(H_g - H_W)\frac{\partial f_g}{\partial s_g} & u\left(C_W + f_g(C_g - C_W) + (H_g - H_W)\frac{\partial f_g}{\partial T}\right) & f_g(H_g - H_W) + H_w \end{pmatrix}, \tag{4.111}$$

where $C_g = \partial H_g/\partial T$ and the constant C_W are the gas and water heat capacities given in Appendix A.

In order to use Eq. (4.34.b) for system (4.109), we calculate $A - \lambda B$ as:

$$\begin{pmatrix} \gamma_2\gamma_1 & u\gamma_2\frac{\partial f_g}{\partial T} + \rho'_{gw}\gamma_4 & \rho_W + f_g\gamma_2 \\ \rho_{gn}\gamma_1 & u\rho_{gn}\frac{\partial f_g}{\partial T} + \rho'_{gn}\gamma_4 & f_g\rho_{gn} \\ \gamma_3\gamma_1 & C_W(u - \lambda\varphi) + (C_g - C_W)\gamma_4 - \varphi\lambda\hat{C}_r + u\gamma_3\frac{\partial f_g}{\partial T} & f_g\gamma_3 + H_w \end{pmatrix}, \tag{4.112}$$

where we have defined

$$\gamma_1 = \gamma_1(s, T, u) = u\frac{\partial f_g}{\partial s_g} - \lambda\varphi \qquad \text{and} \qquad \gamma_4 := \gamma_4(s, T, u) = uf_g - \lambda\varphi s_g, \tag{4.113}$$

$$\gamma_2 := \gamma_2(T) = \rho_{gw} - \rho_W, \qquad \text{and} \qquad \gamma_3 := \gamma_3(T) = H_g - H_W. \tag{4.114}$$

Setting $\gamma_1 = 0$ in (4.112), its determinant vanishes, and we find the Buckley-Leverett characteristic speed and characteristic vector:

$$\lambda_s = \frac{u}{\varphi}\frac{\partial f_g}{\partial s_g}, \qquad r_s = (1, 0, 0)^T, \tag{4.115}$$

On this rarefaction curve T and u are constant, only s_g changes, hence the subscript s in λ_s and r_s is of a saturation wave. The eigenvector associated to the eigenvalue (4.115) is unique: substituting λ_s in (4.112) leads to a matrix with rank 2, generically.

We factor γ_1 in first column, for $\gamma_1 \neq 0$. The other eigenvalue of (4.112) is the root of

$$det \begin{pmatrix} \gamma_2 & u\frac{\partial f_g}{\partial T}\gamma_2 + \rho'_{gw}\gamma_4 & \rho_W + f_g\gamma_2 \\ \rho_{gn} & u\rho_{gn}\frac{\partial f_g}{\partial T} + \rho'_{gn}\gamma_4 & f_g\rho_{gn} \\ \gamma_3 & C_W(u-\lambda\varphi) + \gamma'_3\gamma_4 - \varphi\lambda\hat{C}_r + u\frac{\partial f_g}{\partial T}\gamma_3 & f_g\gamma_3 + H_W \end{pmatrix} = 0. \tag{4.116}$$

We consider first the case $\rho_{gn} \neq 0$, the case $\rho_{gn} = 0$ is considered later. We divide the second row by ρ_{gn} and we obtain the eigenvalue:

$$\lambda_e = \frac{u}{\varphi}\frac{f_g\left[(\gamma'_3 - \left(\rho'_{gn}/\rho_{gn}\right)\gamma_3)\rho_W - H_W(\rho'_{gw} - \left(\rho'_{gn}/\rho_{gn}\right)\gamma_2)\right] + C_W\rho_W}{\rho_W(C_W + \hat{C}_r) + s_g\left[(\gamma'_3 - \left(\rho'_{gn}/\rho_{gn}\right)\gamma_3)\rho_W - H_W(\rho'_{gw} - \left(\rho'_{gn}/\rho_{gn}\right)\gamma_2)\right]}, \tag{4.117}$$

with eigenvector

$$r_e = (-\varpi_1, \bar{\gamma}_1, -u\varpi_3)^T, \tag{4.118}$$

where

$$\bar{\gamma}_1 = \frac{\partial f_g}{\partial s_g} - \frac{f_g\left[(\gamma'_3\rho_{gn} - \rho'_{gn}\gamma_3)\rho_W - H_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2)\right] + C_W\rho_W\rho_{gn}}{\rho_W\rho_{gn}(C_W + \hat{C}_r) + s_g\left[(\gamma'_3\rho_{gn} - \rho'_{gn}\gamma_3)\rho_W - H_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2)\right]}, \tag{4.119}$$

$$\bar{\gamma}_4 = \frac{C_W(f_g - s_g)\rho_W\rho_{gn} + \hat{C}_r f_g\rho_W\rho_{gn}}{\rho_W\rho_{gn}(C_W + \hat{C}_r) + s_g\left[(\gamma'_3\rho_{gn} - \rho'_{gn}\gamma_3)\rho_W - H_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2)\right]}, \tag{4.120}$$

$$\varpi_1 = \bar{\gamma}_4\left(\frac{\left(\rho'_{gn}\right)\rho_W - f_g(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2)}{\rho_{gn}\rho_W}\right) + \frac{\partial f_g}{\partial T}, \tag{4.121}$$

$$\varpi_3 = \frac{\bar{\gamma}_1\bar{\gamma}_4\left(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2\right)}{\rho_W\rho_{gn}}, \tag{4.122}$$

and γ_2 is given in (4.114) .

In Figs. (4.5.a) and (4.5.b) we can see that in the region where $\lambda_s > \lambda_e$, along the rarefaction waves the temperature, gaseous water saturation and u increase; and in the region where $\lambda_s < \lambda_e$ along the rarefaction waves the temperature and u increase and the gaseous water saturation decreases. We utilize the subscript e in λ_e, r_e to indicate that this eigenpair is associated to an evaporation wave.

Coincidence curve

The coincidence curve between λ_e and λ_s is:

$$\mathbb{C}_{s,e} = \left\{(T, s_g) \in tp \text{ such that } \lambda_s(T, s_g) = \lambda_e(T, s_g)\right\}. \tag{4.123}$$

Differentiating λ_e in (4.117) with respect to s_g, equating it to zero and isolating $\partial f_g/\partial s_g$, we obtain:

Lemma 4.7.1. *On the coincidence curve $\mathbb{C}_{s,e}$, the derivative $\partial\lambda_e/\partial s_g$ vanishes.*

Thus coincidence between eigenvalues occurs where λ_e is a stationary. As in Prop. 5.6.4, we can prove the following Proposition:

Proposition 4.7.1. *Lines with fixed T intercept the coincidence curve $\mathbb{C}_{s,e}$ twice. At the lowest value of s_g, λ_e has a minimum and at the higher value λ_e has a maximum.*

Thus, in the tp, when the temperature T varies two branches of the coincidence curve are generated as sketched in Figure (4.4.a).

Proof: We verify this proposition geometrically. First, we define

$$\mathbb{A} = -\left(\left(\left(H_W - H_g\right)'\rho_{gn} - \rho'_{gn}(H_W - H_g)\right)\rho_W - H_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}(\rho_{gw} - \rho_W))\right). \tag{4.124}$$

One can verify that $\mathbb{A} > 0$ for all T in the tp physical range. We multiply and divide λ_e from Eq. (4.117) by $\mathbb{A}$ obtaining:

$$\lambda_e = \frac{u}{\varphi}\frac{f_g - f_e}{s_g - s_e}, \quad \text{where} \quad f_e \equiv \frac{\rho_W\rho_{gn}C_W}{\mathbb{A}} \quad \text{and} \quad s_e \equiv \frac{\rho_W\rho_{gn}\left(C_W + \hat{C}_r\right)}{\mathbb{A}}. \tag{4.125}$$

For the coincidence points with fixed u, it follows from (4.115):

$$\frac{\partial f_g}{\partial s_g} = \frac{f_g - f_e}{s_g - s_e}. \tag{4.126}$$

Eq. (4.126) defines the tangency points of the secant from the point (s_e, f_e) to the graph of f_g. One can prove that $s_e > 1$ and $f_e > 1$. In Figure 4.3, we plot a possible point (s_e, f_e). At the first point where λ_e and λ_T coincide the speed is a minimum and at the second point the speed is maximum. □

Behavior of the two-phase evaporation inflection curve

To obtain the evaporation rarefaction curves we need to study the sign of $\nabla\lambda_e \cdot r_e$. After a lengthly calculation, we get

$$\nabla\lambda_e \cdot r_e = \frac{u}{\varphi}\frac{\bar{\gamma}_1\bar{\gamma}_4}{s_g - s_e}\left(\mathbb{A}' - \mathbb{A}\left(\frac{\rho'_{gn}}{\rho_{gn}}\right) - C_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2)\right), \tag{4.127}$$

where $\bar{\gamma}_1$, $\bar{\gamma}_4$ are defined in Eqs. (4.119) and (4.120). The expression for $\mathbb{A}$ is given by (4.124) and $\mathbb{A}'$ is its derivative with respect to temperature.

One can verify that $\mathbb{A}' - \mathbb{A}\rho'_{gn}/\rho_{gn} - C_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2) > 0$, so the points where $\nabla\lambda_e \cdot r_e$ vanishes satisfy:

$$\lambda_e = \lambda_s \qquad \text{or} \qquad \frac{f_g}{s_g} = \frac{C_W}{C_W + \hat{C}_r}. \tag{4.128}$$

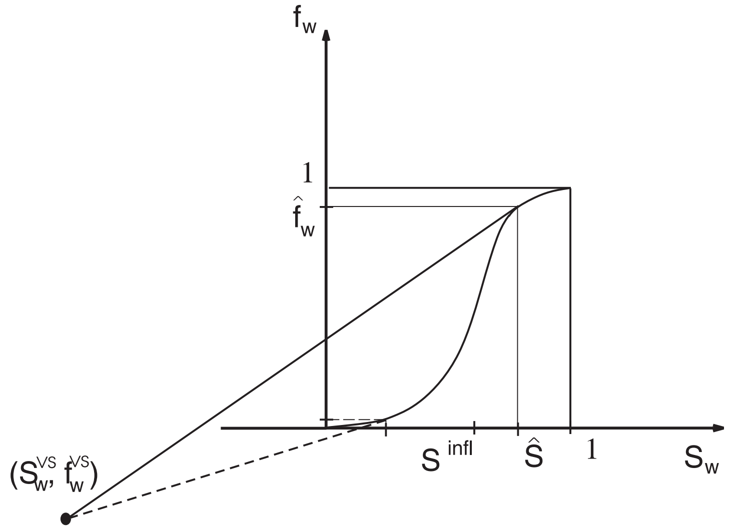


Figure 4.3: The coincidence between λ_e and λ_s occurs at the tangency point on the graph of f_g from (s_e, f_e). Since $\mathbb{A} > 0$, s_e and f_e are positive, the first coincidence point is a maximum while the second point is a minimum for λ_e.

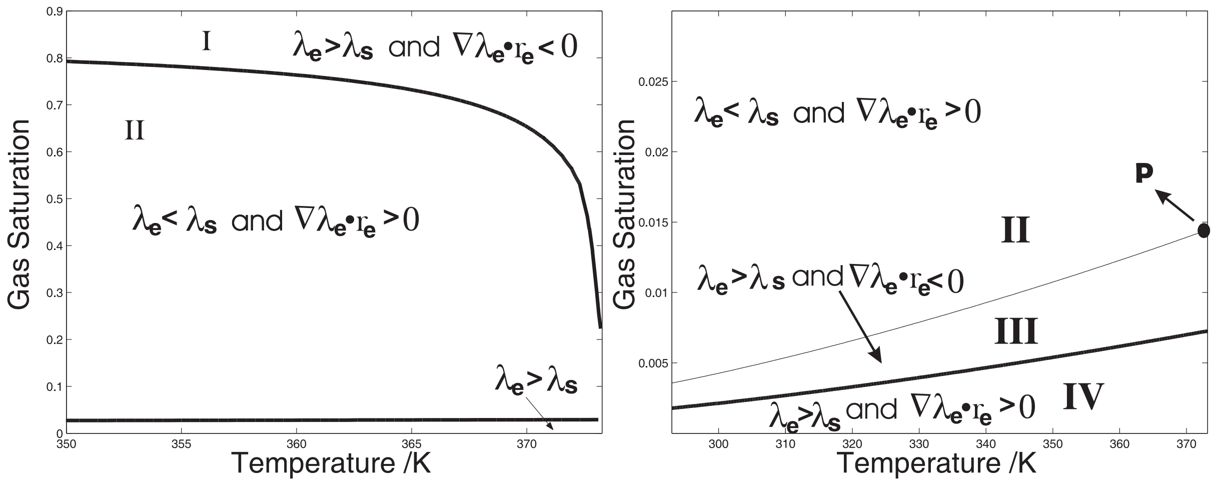


Figure 4.4: *a)-left:* Coincidence curve $\mathbb{C}_{s,e}$. Relative sizes of λ_s and λ_e in a piece of the *tp* state space. The almost horizontal coincidence curve $\lambda_e = \lambda_s$ is not in scale, because it is very close to the axis $S_g = 0$. *b)-right:* A zoom of the region below the coincidence curve. In both figures, all curves form the inflection curve, subdividing *tp* situation in four parts.

We define the curve $\mathbb{I}_e$ as:

$$\mathbb{I}_e := \{(T, s_g) \in tp \text{ satisfying Eq. } (4.128.b)\}.$$

We denote the inflection curve in the tp of λ_e by $\mathcal{I}_e$; it is defined by:

$$\mathcal{I}_e = \left\{(T, s_g) \text{ satisfying } (4.128.a) \text{ and } (4.128.b)\right\} \tag{4.129}$$

The following Lemma follows from Eqs. (4.128) and (4.123).

Lemma 4.7.2. *The coincidence curve $\mathbb{C}_{s,e}$ is contained in $\mathcal{I}_e$. Moreover $\mathcal{I}_e = \mathbb{C}_{s,e} \bigcup \mathbb{I}_e$.*

The solid curves in Fig. 4.4 are the $\mathcal{I}_e$, where we also plot the sign of $\nabla\lambda_e \cdot r_e$. In Figure 4.5, we draw rarefaction curves in the tp.

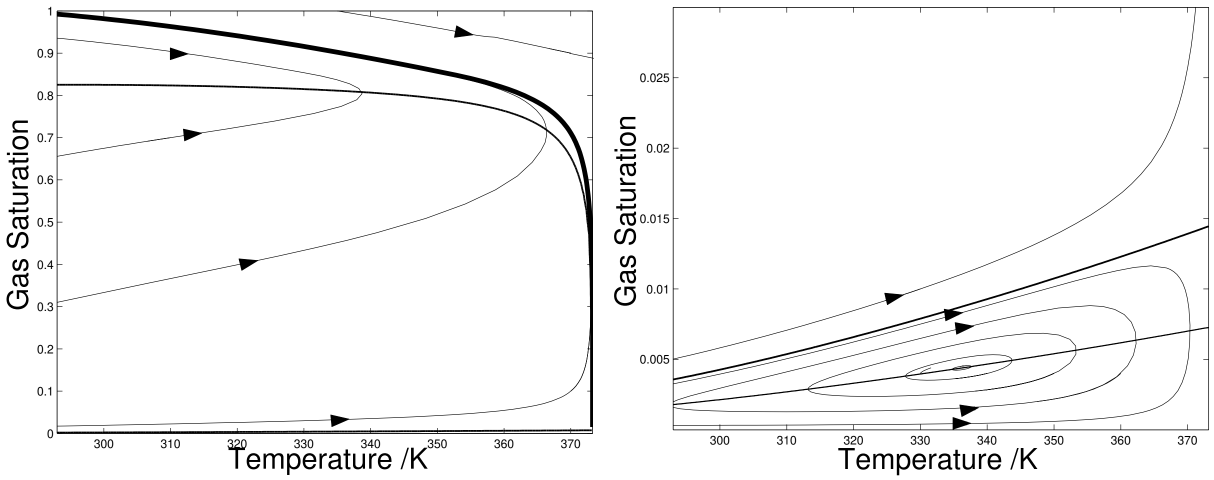


Figure 4.5: *a)-left:* The rarefaction curves projected in the plane T, s_g. The thin curves without arrows are $\mathbb{C}_{s,e}$. The bold curve indicates an invariant curve for the rarefaction field, which a rarefaction curve that reaches the point P in Fig. (4.4.b). *b)-right:* The rarefaction curves in the regions III and IV shown in the Figure (4.4.b). The arrows indicate the direction of increasing speed.

For the case $\rho_{gn} = 0$ but $\rho_{gn}^{'} \neq 0$, the matrix (4.116) reduces to

$$\begin{pmatrix} \gamma_2 & u\frac{\partial f_g}{\partial T}\gamma_2 + \rho_{gw}^{'}\gamma_4 & \rho_W + f_g\gamma_2 \\ 0 & \rho_{gn}^{'}\gamma_4 & 0 \\ \gamma_3 & C_W(u - \lambda\varphi) + \gamma_3^{'}\gamma_4 + \varphi\lambda\hat{C}_r + u\frac{\partial f_g}{\partial T}\gamma_3 & f_g\gamma_3 + H_W \end{pmatrix}. \tag{4.130}$$

The eigenvalue and eigenvector are

$$\lambda_e\Big|_{\rho_{gn}=0} = \frac{u\, f_g}{\varphi\, s_g}, \qquad r_e\Big|_{\rho_{gn}=0} = \left(-\bar{\gamma}_4\rho_{gn}^{'}\left(f_g\gamma_2 + 1\right), 1, \bar{\gamma}_4\rho_{gn}^{'}\gamma_2\right). \tag{4.131}$$

The case $\rho_{gn} = 0$ and $\rho_{gn}^{'} = 0$, with no nitrogen, is studied in [6 and Chapter 5.

4.7.2 Shocks in the two-phase situation

As in the spg, we use Eqs. (4.45) and (4.27)-(4.29) to obtain the RH condition:

$$v^s\varphi\left((s_W^+ - s_W^-)\rho_W + s_g^+\rho_{gw}^+ - s_g^-\rho_{gw}^-\right) = \left(u^+ f_w^+ - f_w^- u^-\right)\rho_W + u^+ f_g^+\rho_{gw}^+ - u^- f_g^-\rho_{gw}^-, \tag{4.132}$$

$$v^s\varphi\left(s_g^+\rho_{gn}^+ - s_g^-\rho_{gn}^-\right) = u^+ f_g^+\rho_{gn}^+ - u^- f_g^-\rho_{gn}^-, \tag{4.133}$$

$$v^s\varphi\left(\hat{H}_r^+ - \hat{H}_r^- + H_W^+ - H_W^- + s_g^+(H_g^+ - H_W^+) - s_g^-(H_g^- - H_W^-)\right) = u^+(f_w^+ H_W^+ + f_g^+ H_g^+) - u^-(f_w^- H_W^- + f_g^- H_g^-). \tag{4.134}$$

The isothermal branch of the RH curve is obtained for constant temperature T, i.e., $T^+ = T^- = T$. In this case the shock speed is

$$v^s = \frac{u^-}{\varphi}\frac{f_w(s_w^+, T) - f_w(s_w^-, T)}{s_w^+ - s_w^-}.$$

On this wave, the Darcy speed is constant. This is the Buckley-Leverett shock; the RH and rarefaction curves associated to λ_s lie on the same straight line.

The non-isothermal wave is obtained after a lengthly calculation. We fix the left state and the temperature of the right state, with $T^+ \neq T^-$. Following [10], we can obtain the shock speed v^s and Darcy speed u^+ at the right independently of s_g^+. We rewrite (4.132)-(4.134) as:

$$\left(v^s\varphi s_g^+ - u^+ f_g^+\right)\left(\rho_{gw}^+ - \rho_W\right) = v^s\varphi s_g^-\left(\rho_{gw}^- - \rho_W\right) - u^-\left(f_w^-\rho_W + f_g^-\rho_{gw}^-\right) + u^+\rho_W, \tag{4.135}$$

$$\left(v^s\varphi s_g^+ - u^+ f_g^+\right)\rho_{gn}^+ = v^s\varphi s_g^-\rho_{gn}^- - u^- f_g^-\rho_{gn}^-, \tag{4.136}$$

$$\left(v^s\varphi s_g^+ - u^+ f_g^+\right)\left(H_g^+ - H_W^+\right) = v^s\varphi\left(\hat{H}_r^- - \hat{H}_r^+ - H_W^+ + s_W^- H_W^- + s_g^- H_g^-\right) + u^+ H_W^+ - u^-(f_w^- H_W^- + f_g^- H_g^-), \tag{4.137}$$

where we have used $s_w = 1 - s_g$ and $f_w = 1 - f_g$.

As $\rho_W > \rho_{gw}$ and $H_W > H_g$ in the temperature physical range and we assume that $\rho_{gn} \neq 0$ (the term ρ_{gn} is zero only at the water boiling temperature), we add (4.135) divided by $\rho_W^+ - \rho_{gw}^+$ to (4.136) divided by ρ_{gn}^+. Similarly, we add (4.136) divided by ρ_{gn}^+ to (4.137) divided by $H_W^+ - H_g^+$ and we obtain:

$$\frac{v^s\varphi\left(s_g^-(\rho_{gw}^- - \rho_W)\right) - u^-\left(f_w^-\rho_W + f_g^-\rho_{gw}^-\right) + u^+\rho_W}{\rho_{gw} - \rho_W^+} = \frac{v^s\varphi s_g^-\rho_{gn}^- - u^- f_g^-\rho_{gn}^-}{\rho_{gn}^+}, \tag{4.138}$$

$$\frac{v^s\varphi\left(\hat{H}_r^- - \hat{H}_r^+ - H_W^+ + s_W^- H_W^- + s_g^- H_g^-\right) + u^+ H_W^+ - u^-(f_w^- H_W^- + f_g^- H_g^-)}{H_g^+ - H_W^+} = \\ = \frac{v^s\varphi s_g^-\rho_{gn}^- - u^- f_g^-\rho_{gn}^-}{\rho_{gn}^+}. \tag{4.139}$$

From (4.138) and (4.139), we obtain v^s and u^+ in terms of s_g^-, T^-, u^- and T^+ as:

$$v^s = \frac{u^-}{\varphi}\frac{\rho_{gn}^+(\Pi_2 H_W^+ - \Pi_1\rho_W) - f_g^-\rho_{gn}^-(\rho_{gw}^+ H_W^+ - H_g^+\rho_W)}{\rho_{gn}^+(\Pi_4 H_W^+ - \Pi_3\rho_W) - s_g^-\rho_{gn}^-(\rho_{gw}^+ H_W^+ - H_g^+\rho_W)} \tag{4.140}$$

and

$$u^+ = u^-\left(\frac{v^s\varphi}{u^-}\left(s_g^-\frac{\rho_{gn}^-}{\rho_{gn}^+}(\rho_{gw}^+ - \rho_W) - \Pi_4\right) + \Pi_2 - f_g^-\frac{\rho_{gn}^-}{\rho_{gn}^+}(\rho_{gw}^+ - \rho_W)\right)/\rho_W, \tag{4.141}$$

where $\Pi_1(s_g^-, T^-)$, $\Pi_2(s_g^-, T^-)$, $\Pi_3(s_g^-, T^-; T^+)$ and $\Pi_4(s_g^-, T^-)$ are given by:

$$\Pi_1 = f_w^- H_W^- + f_g^- H_g^-, \qquad \Pi_2 = f_w^-\rho_W + f_g^-\rho_{gw}^-, \tag{4.142}$$

$$\Pi_3 = \hat{H}_r^- - \hat{H}_r^+ - H_W^+ + s_W^- H_W^- + s_g^- H_g^- \quad \text{and} \quad \Pi_4 = s_g^-(\rho_{gw}^- - \rho_W). \tag{4.143}$$

After finding v^s and u^+, we substitute them in (4.133) and divide the resulting equation by $u^+\rho_{gn}^+$, rearrange the terms, obtaining an implicit equation for s_g^+:

$$\frac{f_g(s_g^+, T^+) - f^*}{s_g^+ - s^*} = \hat{v}^s\varphi, \tag{4.144}$$

where

$$f^* = \hat{u}^-\left(\rho_{gn}^-/\rho_{gn}^+\right)f_g^- \quad \text{and} \quad s^* = \left(\rho_{gn}^-/\rho_{gn}^+\right)s_g^-,$$

with $\hat{v}^- = v^s/u^-$ and $\hat{u}^- = u^-/u^+$.

As v^s and u^+ do not depend on s_g^+, the solution of Eq. (4.144) is the intersection of the graph of $f_g(s_g^+)$ with a linear function of s_g^+. From the shape of the graph of f_g, we see that (4.144) can have one, two or three solutions. Each intersection point belongs to the Rankine-Hugoniot curve $\mathcal{RH}(V^-)$. When T^+ varies each intersection generates a branch of the Rankine-Hugoniot curve.

This is called the non-isothermal Rankine-Hugoniot curve; some examples are shown in Figure 4.6.

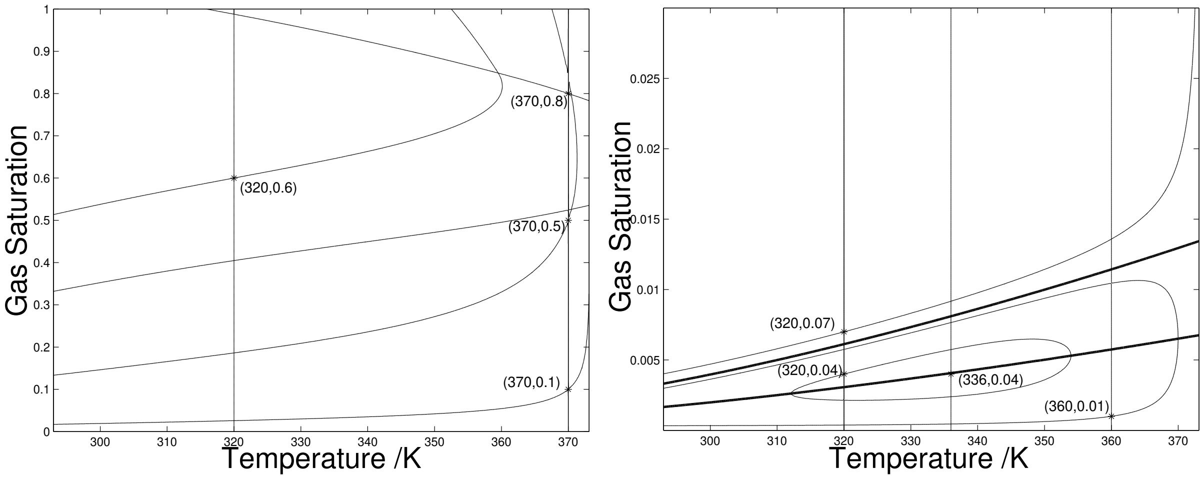


Figure 4.6: *a)-left:* The RH curves projected in the plane T, s_g for $(-)$ points marked. *b)- right:* The region shown in Figure $(4.4.b)$. Each RH locus is formed by a non-isothermal curve and a vertical line which is the isothermal Buckley-Leverett RH locus. For the point $(336, 0.04)$ the Rankine-Hugoniot locus reduces to the isothermal curve only.

4.8 Shocks between regions

We consider now shocks that separate physical situations. In Section 4.4, we gave the RH condition (4.45) for shocks between regions. The main feature is that the accumulation and flux terms have different expressions at each shock side.

4.8.1 Shock between the gaseous and the two-phase situations

For the left state we need to specify steam composition ψ_{gw}^-, the temperature T^- and the Darcy speed u^-. For the right state, we specify the temperature T^+; the steam composition is a function of temperature (see (4.20)) and the Darcy speed is obtained from the other variables. The RH condition (4.45) is written as:

$$v^s\varphi\left(\rho_W S_w^+ + \rho_{gw}^+ s_g^+ - \psi_{gw}^-\rho_{gW}^-\right) = u^+\left(\rho_W f_w^+ + \rho_{gw}^+ f_g^+\right) - u^-\psi_{gw}^-\rho_{gW}^-, \qquad (4.145)$$

$$v^s\varphi\left(\rho_{gn}^+ s_g^+ - \psi_{gn}^-\rho_{gN}^-\right) = u^+\rho_{gn}^+ f_g^+ - u^-\psi_{gn}^-\rho_{gN}^-, \qquad (4.146)$$

$$v^s\varphi\left(\hat{H}_r^+ - \hat{H}_r^- + H_w^+ s_w^+ + H_g^+ s_g^+ - \hat{H}_g^-\right) = u^+\left(H_w^+ f_w^+ + H_g^+ f_g^+\right) - u^-\hat{H}_g^-, \qquad (4.147)$$

where $\hat{H}_g^- = \psi_{gw}^- H_{gW}^- + \psi_{gn}^- H_{gN}^-$ with H_{gW} and H_{gN} given by Eq. (4.24).

The system (4.145)-(4.147) is similar to (4.132)-(4.134). As in that case, we eliminate the saturation and we obtain two linear equations in two unknowns v^s and u^+:

$$\frac{v^s\varphi\left(\psi_{gw}^-\rho_{gW}^- - \rho_W\right) - u^-\psi_{gw}^-\rho_{gW}^- + u^+\rho_W}{\rho_{gw}^+ - \rho_W} = \frac{(v^s\varphi - u^-)\,\psi_{gn}^-\rho_{gN}^-}{\rho_{gn}^+}, \tag{4.148}$$

$$\frac{v^s\varphi\left(\hat{H}_r^- - \hat{H}_r^+ - H_W^+ + \hat{H}_g^-\right) + u^+H_W^+ - u^-\hat{H}_g^-}{H_g^+ - H_W^+} = \frac{(v^s\varphi - u^-)\,\psi_{gn}^-\rho_{gN}^-}{\rho_{gn}^+}, \tag{4.149}$$

We obtain expressions for v^s and u^+ as functions of the left state (ψ_{gw}^-, T^-, u^-) and T^+:

$$v^s = \frac{u^-}{\varphi}\frac{\psi_{gn}^-\rho_{gN}^-\left(\rho_{gw}^+H_W^+ - H_g^+\rho_W\right) - \rho_{gn}^+\left(\psi_{gw}^-\rho_{gW}^-H_W^+ - H_g^+\rho_W\right)}{\Xi_1\rho_W - \Xi_2H_W^+}, \tag{4.150}$$

$$u^+ = \frac{u^-\left(\psi_{gw}^-\rho_{gW}^-\rho_{gn}^+ - \psi_{gn}^-\rho_{gN}^-(\rho_{gw}^+ - \rho_W)\right) - v^s\varphi\Xi_2}{\rho_W\rho_{gn}^+}, \tag{4.151}$$

where Ξ_1 and Ξ_2 are given by:

$$\Xi_1 = \left(\hat{H}_r^- - \hat{H}_r^+ - H_W^+ + \hat{H}_g^-\right)\rho_{gn}^+ - \psi_{gn}^-\rho_{gN}^-\gamma_2^+, \quad \Xi_2 = \left(\psi_{gw}^-\rho_{gW}^- - \rho_W\right)\rho_{gn}^+ - \psi_{gn}^-\rho_{gN}^-\gamma_2^+.$$

After finding v^s and u^+, we use Eq. (4.146) to obtain an equation similar to (4.144) for s_g^+:

$$v^s\varphi = \frac{f_g(s_g^+, T^+) - f^\S}{s_g^+ - s^\S}, \tag{4.152}$$

where

$$s^\S = \psi_{gn}^-\rho_{gN}^-/\rho_{gn}^+ \qquad \text{and} \qquad f^\S = \hat{u}^-\psi_{gn}^-\rho_{gN}^-/\rho_{gn}^+.$$

Geometrically, Eq. (4.152) represents tangency points of the secant from the point $(s^\S, f^\S)$ to the graph f_g for fixed T^+ with slope $v^s\varphi$.

The Secondary Bifurcation.

The secondary bifurcation consists of the pairs of states for which the Prop. 4.4.4 fails, see Sec. 4.4.3. The definition of secondary bifurcation is given by Eqs. (4.80)-(4.81). From Eq. (4.152), we define for T^-, ψ_{gw}^- in the *spg* and T^+, s_g^+ in the *tp*:

$$F(T^-, \psi_{gw}^-, u^-; T^+, s_g^+, u^+) = u^+\rho_{gn}^+f_g(s_g^+, T^+) - v^s\varphi\left(s_g^+\rho_{gn}^+ - \psi_{gn}^-\rho_{gN}^-\right) + u^-\psi_{gn}^-\rho_{gN}^-. \tag{4.153}$$

As v^s and u^+ do not depend on s_g^+ we obtain a simple expression for $\partial F/\partial s_g^+$; for brevity, we rewrite T^+ as T, s_g^+ as s_g and u^+ as u:

$$\frac{\partial F}{\partial s_g} = \left(v^s\varphi - u\frac{\partial f_g}{\partial s_g}\right)\rho_{gn},$$

so $\partial F/\partial s_g = 0$ yields:

$$v^s = \frac{u}{\varphi}\frac{\partial f_g}{\partial s_g}. \tag{4.154}$$

Since both v^s and u depend on T, the expression obtained for $\partial F/\partial T$ is complicated; a lengthly calculation yields:

$$\frac{\partial F}{\partial T} = \frac{\partial v^s}{\partial T}\varphi\left(s_g\rho_{gn} - \psi_{gn}^-\rho_{gN}^-\right) + v^s\varphi s_g\rho'_{gn} - u\frac{\partial f_g}{\partial T}\rho_{gn} - \frac{\partial u}{\partial T}f_g\rho_{gn} - uf_g\rho'_{gn}. \tag{4.155}$$

Denoting the numerator and denominator of $v^s\varphi/u^-$ in Eq. (4.150) by n^{vs} and d^{vs}, respectively, we obtain:

$$\frac{\partial v^s}{\partial T} = \frac{u^-}{\varphi}\frac{\left(\partial n^{vs}/\partial T\right)d^{vs} - n^{vs}\left(\partial d^{vs}/\partial T\right)}{(d^{vs})^2}, \tag{4.156}$$

where $\partial d^{vs}/\partial T =$:

$$\left(\hat{H}_r^- - \hat{H}_r - H_W + \hat{H}_g^-\right)\rho'_{gn} - (C_r + C_W)\rho_{gn} + \psi_{gn}^-\rho_{gN}^-\rho'_{gw}(1 - H_w) - \rho'_{gn}\psi_{gw}^-\rho_{gw}^-\rho_W - \Xi_2 C_W,$$

$$\begin{aligned}\frac{\partial n^{vs}}{\partial T} = \psi_{gn}^-\rho_{gN}^-\left(\rho'_{gw}H_W + \rho_{gw}C_W - C_g\rho_W\right) - \rho'_{gn}\left(\psi_{gw}^-\rho_{gw}^-H_W - H_g\rho_w\right)\\ -\rho_{gn}\left(\psi_{gw}^-\rho_{gW}^-C_W - C_g\rho_W\right).\end{aligned}$$

Denoting the numerator of u in Eq. (4.151) by N_u, we obtain that:

$$\frac{\partial u}{\partial T} = \frac{\left(\partial n^u/\partial T\right)\rho_{gn}\rho_W - N_u\rho'_{gn}\rho_W}{(\rho_{gn}\rho_W)^2}, \tag{4.157}$$

where

$$n^u = u^-\left(\psi_{gw}^-\rho_{gW}^-\rho'_{gn} - \psi_{gN}^-\rho'_{gw}\right) - \frac{\partial v^s}{\partial T}\varphi\Xi_2 - v^s\varphi\frac{\partial\Xi_2}{\partial T}.$$

4.8.2 Shock between the two-phase and the liquid situations

The *spl* can be considered a limit case of the *tp* situation, with $s_w = 1$ and $s_g = 0$. The left state is (s_g^-, T^-, u^-); in the right state, we specify only the temperature T^+; the Darcy speed is obtained from the other variables and the steam composition is a function of temperature (see (4.20)). The RH condition (4.45) is written as:

$$\varphi v^s\left(\rho_W - \rho_W s_W^- - \rho_{gW}^- s_g^-\right) = u^+\rho_W - u^-\rho_W f_w^- - u^-\rho_{gw}^- f_g^-, \tag{4.158}$$

$$-\varphi v^s\rho_{gn}^- s_g^- = -u^-\rho_{gn}^- f_g^-, \tag{4.159}$$

$$\varphi v^s\left(\hat{H}_r^+ - \hat{H}_r^- + H_w^+ - H_g^- s_g^- - s_w^- H_w^-\right) = u^+H_w^+ - u^-\left(H_w^- f_w^- + H_g^- f_g^-\right). \tag{4.160}$$

From (4.159), we obtain that

$$v^s = \frac{u^-}{\varphi}\frac{f_g^-}{s_g^-}, \tag{4.161}$$

Substituting v^s given by (4.161) in Eq. (4.158) we obtain :

$$u^+ = u^-. \tag{4.162}$$

There is no mass transfer between the *tp* situation and the *spl* situation, so we can consider the *spl* as a particular case of the *tp*.

Lemma 4.8.1. *If there is a shock between the tp " $-$ " and the spl " $+$ " with $T^+ \neq T^-$ (i.e., a non-isothermal shock), then*

$$\frac{f_g^-}{s_g^-} = \frac{C_W}{\hat{C}_r + C_W}. \tag{4.163}$$

Proof: We rewrite (4.160) using (4.161) and (4.162) as:

$$\left(H_r^+ + H_W^+\right)\frac{u^- f_g^-}{s_g^-} - u^- H_W^+ = \frac{u^- f_g^-}{s_g^-}\left(H_r^- + s_W^- H_W^- + s_g^- H_g^-\right) - u^-(f_w^- H_W^- + f_g^- H_g^-),$$

manipulating this equation we obtain

$$\left(H_r^+ + H_w^+\right)\frac{u^- f_g^-}{s_g^-} - u^- H_w^+ = \frac{u^- f_g^-}{s_g^-}\left(H_r^- + (1 - s_g^-)H_w^- + s_g^- H_g^-\right) - u^-(f_w^- H_w^- + f_g^- H_g^-),$$

$$\left(H_r^+ - H_r^- + H_w^+ - H_w^-\right)\frac{f_g^-}{s_g^-} = \left(H_w^+ - H_w^-\right),$$

$$\frac{f_g^-}{s_g^-} = \frac{(H_w^+ - H_w^-)}{(H_r^+ - H_r^- + H_w^+ - H_w^-)} = \frac{C_W}{\hat{C}_r + C_W},$$

where we have used that $H_w = C_W(T - T^0)$ and $\hat{H}_r = \hat{C}_r(T - T^0)$. $\square$

Remark 4.8.1. *We call the $(-)$ states that satisfy* (4.163) *the condensation curve; on this curve there is non-isothermal mass transfer between the gaseous and liquid phases. From Lemma* 4.8.1 *and Eq. (4.128), we obtain the following:*

Corollary 4.8.1. *In the tp, the inflection curve $\mathcal{I}_e$ is the union of coincidence locus $\mathbb{C}_{s,e}$ with the "condensation curve".*

Lemma 4.8.2. *In the set of $(-)$ states that satisfy Eq.* (4.163)*, the evaporation characteristic speed in the tp equals the contact discontinuity speed in liquid situation, i.e.,*

$$\lambda_e = \frac{u^b}{\varphi}\frac{C_W}{C_W + \hat{C}_R}.$$

Proof: Since f_g/s_g satisfies Eq. (4.163), we substitute $f_g = s_g C_W/(C_W + \hat{C}_r)$ in Eq. (4.117). After some calculations we obtain the result.

Proposition 4.8.1. *The Riemann solution for any* $(-)$ *state,* $(-) = (s^-, T^-, \cdot)$ *that satisfies Eq.* (4.163) *and* $(+)$ *states,* $(+) = (0, T^+, \cdot)$ *in the spl consists of a shock between* $(-)$ *and* $(+)$.

Proof: From Lemma 4.8.1, if the $(-)$ belongs to the condensation curve, the shock speed between regions is

$$v^s = \frac{u^-}{\varphi} \frac{C_W}{\hat{C}_r + C_W},$$

which is the same as the the contact discontinuity speed in the *spl*. □

The following corollary follows immediately:

Corollary 4.8.2. *The Riemann solution for left states* $L = (s_L, T_L, \cdot)$ *in regions III or IV (see Fig.* 4.4.*b) and right states* $R = (0, T_R, \cdot)$ *in the spl consists of a Buckley-Leverett shock from L to* $(0, T_L, \cdot)$ *followed by a thermal contact discontinuity in the spl with speed* λ_T^W *given by* (4.31) *and* $u_L = u_R$.

4.9 The Riemann solution for geothermal energy recovery

We show an example of Riemann solution. We consider the injection of a two-phase mixture of water, steam and nitrogen into a rock containing superheated steam ($\psi_R = 1$) at a temperature $T_R > T^b$:

$$\begin{cases} \left(s_g, \psi(T), T, u\right)_L & \text{if } x = 0 \text{ (the injection point)}, \qquad \text{with} \quad u_L > 0 \\ \left(s_g = 1, 1, T, \cdot\right)_R & \text{if } x > 0 \end{cases} \tag{4.164}$$

From Prop. 4.4.1, the projection of the solution into the primary variables does not depend on u_L. We can subdivide the *tp* region in 4 subregions, relatively to the left state L. For L in regions I and II of Figure (4.4.*a*) there are 3 subregions with the following property: for any L in a given subregion the Riemann problem with data R of form (4.164) has the same sequence of waves, see Figure 4.7. In 4.9.3, we utilize results of Section 5.6.2 to obtain the Riemann solution.

4.9.1 Subdivision of *tp*

For the data (4.164) with T_R fixed, the Riemann solutions have the same wave sequence and structure in for L each subregion labelled $\mathcal{L}_1$ to $\mathcal{L}_4$ of Figure 4.7; subregions III and IV will be divide in Chapter 6. A very important task is to obtain the curves that bound subregions: the curve E_2, see Fig. (4.7.*a*) and Sec. 4.9.1; the curve E_1, see Figs. (4.7.*a*) and (4.7.*b*) and Sec. 4.9.1; the curve E_3, see Fig. (4.7.*a*) and Sec. 4.9.1.

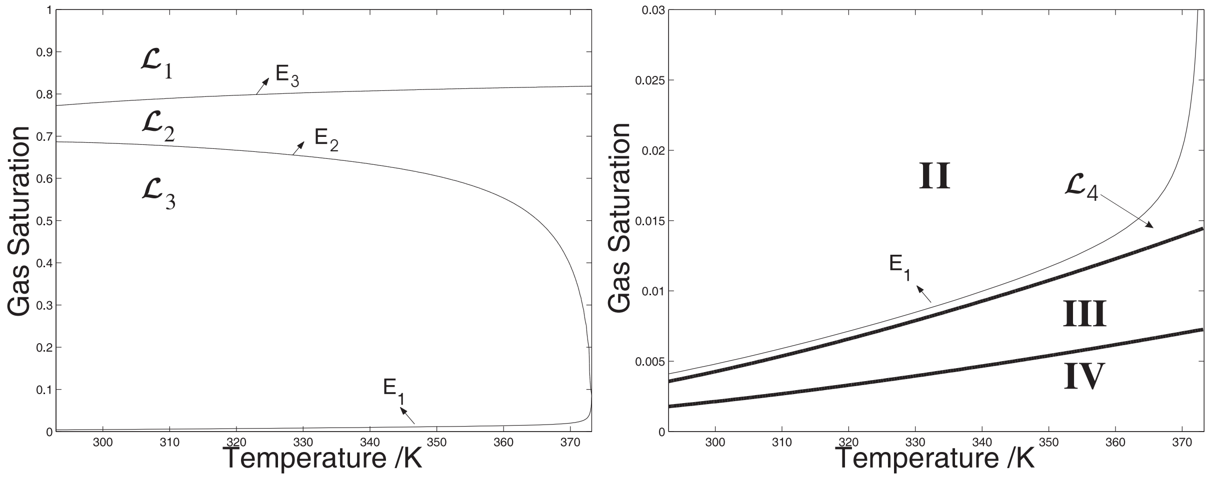


Figure 4.7: *a)-left* The 3 subregions in I and II for a Riemann data of form (4.164). The curves are defined in Secs. 4.9.1, 4.9.1 and 4.9.1. *b)-right* Zoom of regions II to IV. Each curve is explained in Section 4.9.1

Remark 4.9.1. *The evaporation wave speed λ_e for states in $\mathcal{L}_2$ and $\mathcal{L}_3$ in tp is larger than the thermal wave speed in the spg. From geometrical compatibility, at the right of the evaporation shock there is no intermediate thermal wave (rarefaction or shock) in the spg, i.e., the shock between regions should reach a state of the form $(1, \psi_{gw}, T_R)$.*

Curve E_2

This curve is defined as the states V^- in the *tp* such that the evaporation rarefaction speed $\lambda_e(V^-)$ equals the speed $v^{BG}(V^-; V^+)$ of the shock to $V^+ = (1, T_R, \psi_{gw})$ in the *spg*. Notice that this curve is an extension of the boundary, see Section 4.4.3.

We obtain ψ_{gw} numerically. For each V^- state of the form $(S, \psi_{gw}(T), T)$ in the *tp*, we search the state in the *RH* locus starting at V^- with right temperature T_R. These restrictions yield a state in the *spg* with steam composition ψ_{gw} as unknown.

Curve E_1

We are interested in the Riemann solution for left states in I or II. From geometrical compatibility, to solve the Riemann problem we choose first the slowest wave whenever possible. In II, the slowest wave is the evaporation wave. Since we are interested in connecting a left state at lower temperature to a state at higher temperature, we use an evaporation rarefaction curve instead of a shock, see Section 4.7. We can do this until the speed of the evaporation rarefaction curve equals the speed of the shock between regions; this occurs when the rarefaction curve crosses E_2.

The integral curves starting at certain left states do not cross E_2, rather they reach directly the steam region at boiling temperature. We define the curve E_1 as the evaporation rarefaction curve that crosses E_2 at water boiling temperature. Notice that for left state in II above E_1, the evaporation rarefaction curve always crosses E_2.

Curve E_3

For states V^- above the curve E_2, the evaporation rarefaction speed is larger than $v^{BG}(V^-;V^+)$, for $V^+ = (1, T_R, \psi_{gw})$; in Figures (4.4.*a*) and (4.8.*a*) we show the relatives sizes of v^{BG}, λ_e and λ_s. When s_g tends to 1 the Buckley-Leverett shock speed is zero, so there are transition curves, $v^{BL} = v^{BG}$, $v^{BL} = v^T$ and $v^{BG} = v^T$. From the Prop. 4.4.2, we obtain a bifurcation curve E_3 where $v^T = v^{BL} = v^{BG}$. For states above this curve, the Buckley-Leverett shock is slower than v^{BG} and v^T, so there is no direct shock between the *tp* and the *spg* situations.

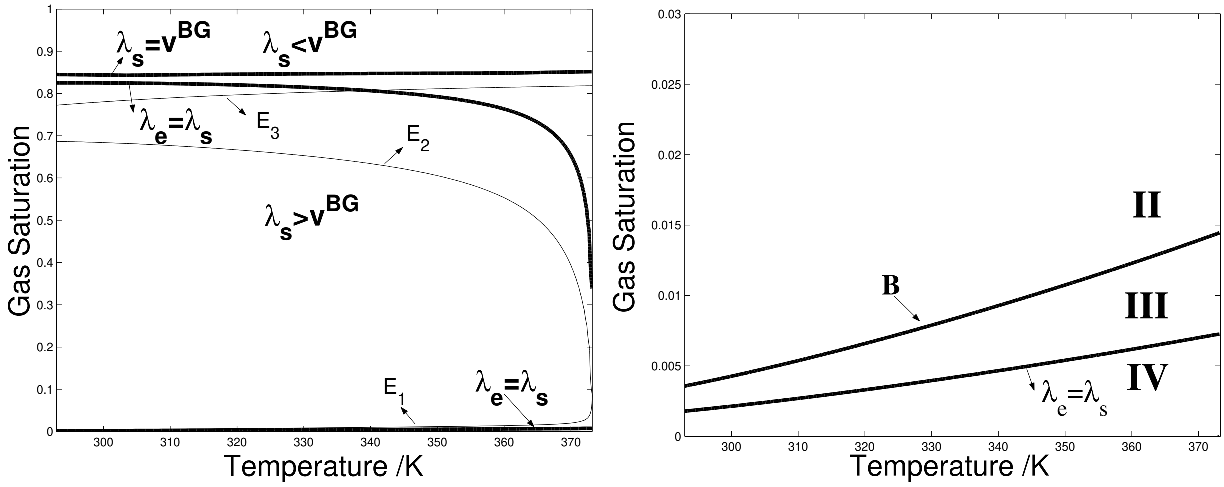


Figure 4.8: *a)-left)* The relevant curves for the Riemann solution for the left states in the *tp* situation. *b)-right)* The curve B is a bifurcation curve; for left states V out of this curve, the shock and rarefactions starting in V do not cross B.

4.9.2 The Riemann solution

We use the notation $V_L = \left(s_g, \psi_{gw}(T), T\right)_L$ and $V_R = \left(s_g = 1, 1, T\right)_R$. All the variables for the intermediate states are written out. The states are labelled by 1, 2, etc. We use the following nomenclature: evaporation rarefaction is R_E, the Buckley-Leverett shock and rarefaction are S_{BL} and R_{BL}, the shock between the *tp* and the *spg* is S_{BG} and the contact compositional discontinuity is S_C. We use the notation for wave sequences established in Section 2.1.3.

V_L in $\mathcal{L}_4$.

There is an evaporation rarefaction curve from V_L up to $V_1 = (s_L, 1, T^b)$, where T^b is the boiling temperature. For this state there is no nitrogen and so $\psi_{gw} = 1$. The solution after the intermediate state $(s_M, 1, T^b)$ was found in Chapter 5: an evaporation shock between the boiling situation to the steam situation appears, denoted by S_{VS}.

The solution consists of the waves $R_e \mapsto R_{BL} \twoheadrightarrow S_{VS}$.

$$V_L \xrightarrow{R_e} V_1 \xrightarrow{R_{BL}} V_2 \xrightarrow{S_{VS}} V_R. \tag{4.165}$$

V_L in $\mathcal{L}_3$.

There is a rarefaction from V_L up to $V_1 = (s^*, \psi_{gw}(T^*), T^*)$ in tp; V_1 is a state where $\lambda_e(V_1) = v^{BL}(V_1; V_2)$, with $V_2 = (1, \psi_{gw}, T_R)$ in spg. We have described how to obtain ψ_{gw} in Section 4.9.1. Finally, there is a compositional contact discontinuity at temperature T_R with speed v^C from V_2 to V_R.

The solution consists of the waves $R_E \twoheadrightarrow S_{BG} \mapsto S_C$ with sequence:

$$V_L \xrightarrow{R_e} V_1 \xrightarrow{S_{BG}} V_2 \xrightarrow{S_C} V_R. \tag{4.166}$$

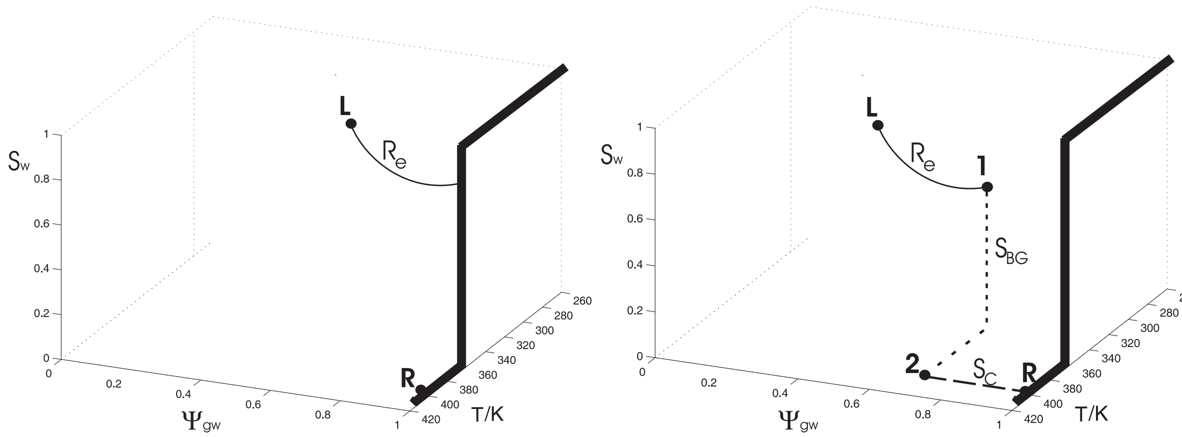


Figure 4.9: Riemann solutions in phase space, omitting the surface shown in Fig. 4.1. *a) Left:* Solution (4.165) for $V_L \in \mathcal{L}_4$, Sec. 4.9.2. *b) Right:* Solution (4.166) for $V_L \in \mathcal{L}_3$, Sec. 4.9.2. The numbers 1 and 2 indicate the intermediate states V in wave sequence.

V_L in $\mathcal{L}_2$.

Let $V_1 = (1, \psi_{gw}, T_R)$ be a state in the spg situation. Since $\lambda_e(V_L) > v^{BL}(V_L; V_1)$, there is a shock between V_L and V_1 with speed v^{BG}. From the state V_1 there is compositional contact discontinuity with speed v^C to V_R.

The solution consists of the wave $S_{BG} \mapsto S_C$ with sequence:

$$V_L \xrightarrow{S_{BG}} V_1 \xrightarrow{S_C} V_R. \tag{4.167}$$

V_L **in** $\mathcal{L}_3$**.**

Since v^{BG} and v^{BL} are smaller than v^T, there is a Buckley-Leverett saturation shock between V_L and $V_1 = (1, \psi_{gw}(T_L), T_L)$. From V_1 there is a thermal shock with speed v^T to $V_2 = (1, \psi_{gw}(T_L), T_R)$ and finally there is a compositional contact discontinuity with speed v^C to V_R.

The solution consists of the waves $S_{BL} \twoheadrightarrow S_T \mapsto S_C$ with sequence:

$$V_L \xrightarrow{S_{BL}} V_1 \xrightarrow{S_T} V_2 \xrightarrow{S_C} V_R. \tag{4.168}$$

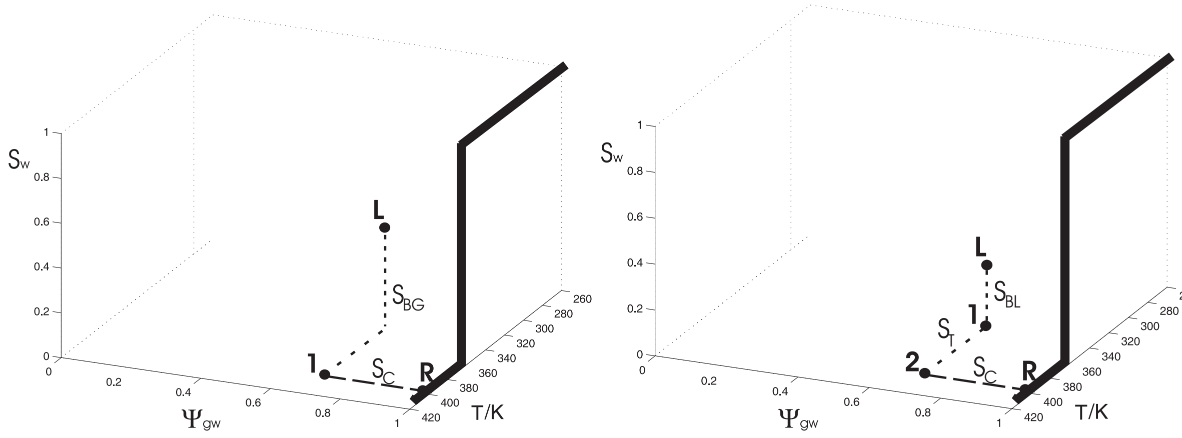


Figure 4.10: Riemann solutions in phase space, removing the surface shown in Fig. 4.1. *a) Left:* Solution (4.167) for $V_L \in \mathcal{L}_2$, Sec. 4.9.2. *b) Right:* Solution (4.168) for $V_L \in \mathcal{L}_1$, Sec. 4.9.2. The numbers 1 and 2 indicate the intermediate states V in wave sequence.

4.9.3 V_L in III and IV

The integral curves or RH locus starting at any point in III or IV do not reach the water boiling temperature. Moreover, the RH locus between the tp and the spg situations for points in III or IV do not reach temperatures above the temperature T_L. So the only possibility to get to the right state is by crossing the curve B. However, the evaporation wave does not cross the curve B; to cross it we need a Buckley-Leverett shock. The Buckley-Leverett shock for an initial state below B and a final state above B is faster than the evaporation wave λ_e. Moreover, it is faster than the shock between regions. So with a Buckley-Leverett shock of this type, it is only possible to reach states of form $(1, \psi_{gw}(T), T)$ with $T < T^b$. Since the thermal wave is slower than the Buckley-Leverett shock, so it is impossible to construct the Riemann solution satisfying the geometrical compatibility principle for waves only in the tp situation.

To obtain the Riemann solution, we utilize the results of Section 5.6.2, so it is postponed to Chapter 6

CHAPTER 5

Riemann solution for steam and water flow

In this section, we use part of the theory developed for conservation laws to solve a system of balance equations for steam and water flow in a porous medium, see [20], [45], [59] and [60] for hyperbolic systems. The solution exhibits an intriguing yet systematic structure. It is desirable to obtain a theory for balance equations as complete as that for conservation laws; combustion phenomena are also modelled by balance equations, see [1] and references theirein.

This class of balance equations has appeared in mathematical models for clean-up, see [6]. Soil and groundwater contamination due to spills of non-aqueous phase liquids (NAPL's) have received a great deal of attention from society, because, in general, these components can cause damage to the ecosystem and environmental impact to a large area around the spills. Removal of contaminants with steam is considered to be an attractive groundwater remediation technique. We consider here a model for steam injection presented in [6]. Steam injection is widely studied in Petroleum Engineering, see [6] and references theirein.

In the recovery of geothermal energy, a steam producing well crosses porous rock containing water at high pressure and temperature. The water usually boils in the rock, forming a steam vaporization front.Because of the high pressures and pressure gradients, the nature of the flow equations changes: for single phase steam or water flow the asymptotic solution depends on $x/\sqrt{t}$, see [37]. In many situations, a two-phase region develops, precisely as in the case analyzed in our work for small pressure gradients. The precise asymptotic behavior for for flow of water and steam containing a two-phase region seems to be an open problem flow under high pressures. Tsypkin et. al., [66]. Studies in steam and water injection is a related application in several works: Bruining [6], Kondrashov [35], Kulikoskii [37], [38] and Tsypkin et. al. [63]-[67].

We consider the constant rate injection of a mixture of steam/water in a specified proportion into a porous medium filled with another homogeneous steam/water mixture. We study all possible proportions of steam and water as boundary and initial conditions for the problem. We present a physical model for steam injection based on mass balance and energy conservation equations. We present the main physical definitions and equations; we refer to [6] for more details. We study the three possible physical phase mixture situations: single-phase gaseous situation, which represents a region with superheated steam, called steam region, *sr*; a two-phase situation, which represents a region where the water and steam coexist called boiling region, *br*; and a single-phase liquid situation which represents a region with water only, called water region, *wr*. We reduce the three balance equations system presented in Sec. 5.1 to a system of conservation laws of form (2.2):

$$\frac{\partial}{\partial t}G(V) + \frac{\partial}{\partial x}uF(V) = 0,$$

where $V = (V_1) : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \Omega \subset \mathbb{R}$ represents the variables to be determined; $G = (G_1, G_2) : \Omega \longrightarrow \mathbb{R}^2$ and $F = (F_1, F_2) : \Omega \longrightarrow \mathbb{R}^2$ are the accumulation vector and the flux vector, respectively; $u : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$, $u = u(x,t)$ is the total velocity. It is useful to define $U = (V, u)$. The vector V represents the water saturation s_w and the temperature T. The state of the system is represented by (s_w, T, u). Eq. (2.2) has an important feature, the variable u does not appear in the accumulation term, it appears isolated in the flux therm, therefore this equation has an infinite speed mode associated to u. Nevertheless we are able to solve the complete Riemann problem associated to Eq. (2.2). Moreover, under certain hypotheses it is possible to solve numerically the problem, in [40] Lambert et. al. consider a model in the balance form for nitrogen and steam injection.

In [6], Bruining et. al. considered as initial condition for the Riemann problem, a porous rock filled with water at a temperature T^0, in which a mixture of water and steam at saturation temperature (boiling temperature) in given proportions is injected. The main feature was the existence of a Steam Condensation Front (SCF), which is a shock between the *br* and the *wr*. The analysis of the shock between each pair of regions is important because bifurcations occur and frequently non-classical structures appear in the solution.

In this work, we completely solve the Riemann problem. We study the three possible physical phase mixture situations: single-phase superheated steam gaseous situation, in the *sr*; a two-phase situation where the water and steam coexist in equilibrium in the *br*; and a single-phase *wr*.

The Riemann problem A is the injection of a mixture of water and steam at boiling temperature in a porous rock filled with steam at temperature above the boiling temperature (superheated steam). In this case, a new wave, a vaporization shock (VS), appears between the *br* and the *sr*. In the Riemann problem B, we inject liquid water in a porous rock containing water and steam at boiling temperature. These initial and boundary conditions are the reverse of those considered in [6]. There is a water evaporation shock (WES) between the *sr* and the *br*. In the Riemann problem C, we inject superheated steam in a porous rock containing water and steam at boiling temperature. There appears a condensation shock (CS) between the *sr* and the *br*. This

Riemann solution is the most interesting solution; it has a rich bifurcation structure. We obtain two bifurcation curves. The first bifurcation is the TCS locus, where the left thermal characteristic speed in the sr coincides with the condensation shock speed v^{CS}. The second bifurcation is the CSS locus, where the condensation shock speed v^{CS} coincides with the right saturation characteristic speed λ_s^b in the br. These two bifurcation curves intersect at a point, the double bifurcation SHB (see Fig. 5.8). This state is very important because it is an organizing point between several different phase mixtures. In the Riemann problem D, we inject water at a temperature below its boiling temperature in a porous rock containing superheated steam. There is a br between the wr and the sr. So the Riemann solution consists of a combination of the waves in the Riemann problem B and C. In the Riemann problem E, we inject superheated steam in a porous rock containing water below its boiling temperature. As in the Riemann problem D, there is a br between the wr and the sr. For this Riemann problem, the solution is obtained combining the Riemann solution B and the solution F, see [6] and [39].

In Sec. 5.1, we present the mathematical and physical formulations of the injection problem in terms of balance equations. In Sec. 5.2, we consider separately each region in different physical situations under thermodynamical equilibrium and we rewrite the corresponding balance equations in conservative form. In Sec. 5.4, we study the shock and rarefaction waves that occur in each physical situation separately. In Sec. 5.5, we study the shocks in the transitions between regions. In Sec. 5.6, we present the solution of the Riemann problem for the five types of injection. In Appendix A, we describe notation and physical quantities appearing in the physical model.

5.1 Mathematical and Physical model

5.1.1 The model equations

Ignoring diffusive effects, the mass balance equation for liquid water and steam read:

$$\frac{\partial}{\partial t}\varphi\rho_w s_w + \frac{\partial}{\partial x}\rho_w u_w = +q, \tag{5.1}$$

$$\frac{\partial}{\partial t}\varphi\rho_g s_g + \frac{\partial}{\partial x}\rho_g u_g = -q, \tag{5.2}$$

where φ is the rock porosity assumed to be constant; s_w and s_g are the water and steam saturations; ρ_w is the water density, which is assumed to be constant for simplicity; the steam density ρ_g is a function of the temperature T (i.e, we neglect the effects of gas compressibility) and decreases with temperature; the term q is the mass transfer between the gaseous and liquid water; u_w and u_g are the water and steam phase velocity.

Disregarding heat conductivity, the energy balance equation can be written as:

$$\frac{\partial}{\partial t}\varphi(\hat{H}_r + \rho_w h_w s_w + \rho_g h_g s_g) + \frac{\partial}{\partial x}(u_w\rho_w h_w + u_g\rho_g h_g) = 0, \tag{5.3}$$

where H_r is the rock enthalpy per unit volume and h_w and h_g are the water and gas enthalpies per unit mass, respectively, and $\hat{H}_r = H_r/\varphi$. The enthalpies and ρ_g are functions only of temperature and their expressions are found in Appendix A. From these expressions, one can see that the enthalpies are increasing functions and that h_g is a convex function.

Remark 5.1.1. *In [6], Bruining et. al have defined the water and gas enthalpies per unit volume as:*

$$H_w = \rho_w h_w \qquad \text{and} \qquad H_g = \rho_g \rho_g. \tag{5.4}$$

From the expressions for ρ_w, ρ_g, h_w and h_g in Appendix A, we can prove that there is a temperature $\widetilde{T} < T^b$ where H_w and H_g satisfy:

(i) $H_w(\widetilde{T}) = H_g(\widetilde{T})$;
(ii) $H_w(T) < H_g(T)$ *if* $T < \widetilde{T}$;
(iii) $H_w(T) > H_g(T)$ *if* $T > \widetilde{T}$.

The gas and water enthalpies per unit volume increase when the temperature increases. Moreover, $h_g > h_w$ for all T in the physical range.

5.1.2 Physical Model

To determine the fluid flow rate, we use Darcy's law for multiphase flow without gravity and capillary pressure effects:

$$u_w = -\frac{kk_{rw}}{\mu_w}\frac{\partial p}{\partial x}, \qquad u_g = -\frac{kk_{rg}}{\mu_g}\frac{\partial p}{\partial x}, \tag{5.5}$$

where k is the absolute permeability for rock (see Appendix A); the relative permeability functions k_{rw} and k_{rg} are considered to be power functions of their respective effective saturations (see Appendix A); μ_w and μ_g are the viscosity of liquid water and the viscosity of steam and they are functions of temperature; p is the common pressure of the liquid and gaseous phase. We define the fractional flow functions for water and steam depending on the saturation and temperature as follows, see Figure 5.1:

$$f_w = \frac{k_{rw}/\mu_w}{d}, \qquad f_g = \frac{k_{rg}/\mu_g}{d} \qquad \text{where} \qquad d = \frac{k_{rw}}{\mu_w} + \frac{k_{rg}}{\mu_g}. \tag{5.6}$$

Using (5.6) in Darcy's law (5.5)

$$u_w = uf_w, \qquad u_g = uf_g, \qquad \text{where} \qquad u = u_w + u_g, \tag{5.7}$$

and u is the total or Darcy velocity. The saturations s_w and s_g add to 1.

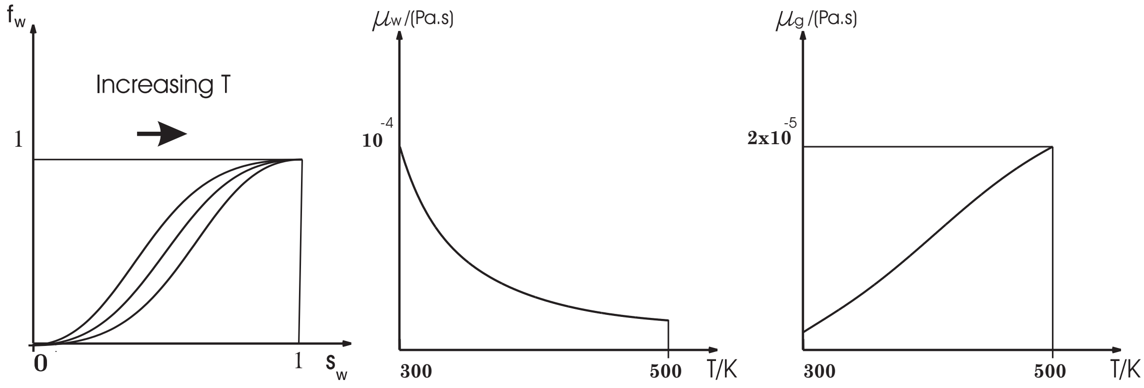


Figure 5.1: *a) Left:* The shape of water fractional flux f_w, originating from typical $k_{rw}(s_w)$ and $k_{rg}(s_g)$, given in $(B.6)$ for different values of the temperature T. The separation between the curves is tiny in reality. *b) Center:* Water viscosity $\mu_w(T)$. *c) Right:* Gas viscosity $\mu_g(T)$.

5.2 Regions under thermodynamical equilibrium

As we will see later, the five zones can be organized in three regions in different physical situations where the fluids are in thermodynamic equilibrium, which is often specified by an equation of state (EOS). Each physical situation determines the structure of the governing system of equations. One region consists of steam only, with temperature at least T^b (the condensation temperature of pure water, which is around $373.15K$ at atmospheric temperature), where we must determine two variables: temperature and Darcy velocity u. The steam saturation is $s_g = 1$ ($s_w = 0$). There is a second region consisting of steam and water, with liquid water saturation s_w and gas saturation s_g both less than 1. We must determine two variables: the velocity and saturation (either s_w or s_g, because they add to 1); the temperature here is known and its value is $T = T^b$. Finally there is a region of liquid water, where we must also determine two variables: temperature and velocity. The saturation is known: $s_g = 0$ and $s_w = 1$.

We summarize these regions as follows:

$s_w \backslash T$	$T > T^b$	$T = T^b$	$T < T^b$
$s_w = 0$	superheated steam zone	hot steam zone	3
$s_w < 1$	1	hot steam-water zone	4
$s_w = 1$	2	hot water zone	cold water zone

Table 1: Classification according to saturation and temperature.

We call "steam region" (represented by "*sr*") the superheated steam zone. We call "boiling region" (represented by *br*) the hot steam zone together with the hot steam-water zone and the hot water zone. We call "water region" (represented by "*wr*") the hot water zone together with the cold water zone. In [6], there is no region with steam above the boiling temperature; the *sr* and *wr* are called "hot region" and "liquid water region" respectively.

Notice that the hot steam zone at boiling temperature belongs to both *sr* and *br*; also, the hot water zone belongs to both *br* and *wr*.

Remark 5.2.1. *Because of thermodynamical equilibrium, steam cannot exist at temperatures lower than T^b; similarly, there is no liquid water at a temperature above T^b. Thus the regions with numbers* 1-4 *in Table 1 do not exist because of our requirement of thermodynamic equilibrium. Regions* 1-4 *would represent the following unstable mixtures: (1) superheated steam with water, (2) superheated water, (3) steam below T^b and (4) steam-water below T^b.*

5.3 Equations in conservative form

From the previous discussion, we notice that in each region under thermodynamic equilibrium there are two variables to be determined in the system (5.1)-(5.3); the other variables are trivial. For example, in the *br* the temperature and Darcy speed are determined by the system of equations, but the saturation is trivial, its value is $s_w = 0$. Thus we can rewrite the system (5.1)-(5.3) as a system of two conservation laws and two variables as follows. We add Eq. (5.1) to (5.2) and use (5.7):

$$\frac{\partial}{\partial t}\varphi\left(\rho_w s_w + \rho_g s_g\right) + \frac{\partial}{\partial x}u\left(\rho_w f_w + \rho_g f_g\right) = 0. \tag{5.8}$$

Using (5.7) in the energy conservation equation (5.3), it becomes:

$$\frac{\partial}{\partial t}\varphi(\hat{H}_r + \rho_w h_w s_w + \rho_g h_g s_g) + \frac{\partial}{\partial x}u(\rho_w h_w f_w + \rho_g h_g f_g) = 0. \tag{5.9}$$

We will use (5.8)-(5.9) from now on. Not only this system models the flow in each region under thermodynamic equilibrium, but it also determines the shocks between regions (see Sec. 5), when supplemented by appropriate thermodynamic equations of state.

As initial conditions, we assume that the porous rock is full of a mixture of water and steam (saturation $s_w(x,t=0) = s_R$) with constant temperature $T(x,t=0) = T_R$. As boundary conditions at the injection point at the left of the porous rock, the total injection rate u_L is specified as a constant. The constant water-steam injection ratio needs to be given too, which is (s_L, T_L). It is specified in terms of the water fractional flow $f_w(s_L, T_L)$ at the injection point.

5.4 Elementary waves under thermodynamical equilibrium

We consider rarefaction and shocks waves for (5.8)-(5.9) in each region.

5.4.1 Steam region -*sr*

The temperature is high, $T > T^b$, so there is only steam, $s_w = 0$, (notice that from Eq. (5.1), $q \equiv 0$) and the state (s_w, T, u) can be represented by $(0, T, u)$. The system (5.8)-(5.9) reduces to

$$\frac{\partial}{\partial t}\varphi\rho_g + \frac{\partial}{\partial x}u\rho_g = 0, \tag{5.10}$$

$$\frac{\partial}{\partial t}\varphi(\hat{H}_r + \rho_g h_g) + \frac{\partial}{\partial x}u\rho_g h_g = 0. \tag{5.11}$$

Rarefaction wave

Assuming that all dependent variables are smooth, we can differentiate (5.10) and (5.11) with respect to their variables:

$$\varphi\rho_g'\frac{\partial T}{\partial t} + u\rho_g'\frac{\partial T}{\partial x} + \rho_g\frac{\partial}{\partial x}u = 0, \tag{5.12}$$

$$\varphi\left(\hat{H}_r' + C_g\right)\frac{\partial T}{\partial t} + uC_g\frac{\partial T}{\partial x} + \rho_g h_g\frac{\partial}{\partial x}u = 0, \tag{5.13}$$

where prime denotes derivative relative to T.

We use the notation $\hat{C}_r = \hat{H}_r' = d\hat{H}_r/dT$ for the effective rock heat capacity divided by φ and $C_g = \partial(\rho_g h_g)/\partial T$ for the steam heat capacity per unit volume; we assume that the effective rock heat capacity $\hat{C}_r$ is constant (see Appendix A). We rewrite (5.12)-(5.13) as:

$$B\frac{\partial}{\partial t}\begin{pmatrix} T \\ u \end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix} T \\ u \end{pmatrix} = 0, \tag{5.14}$$

where

$$B = \begin{pmatrix} \varphi\rho_g' & 0 \\ \varphi(\hat{C}_r + C_g) & 0 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} u\rho_g' & \rho_g \\ uC_g & \rho_g h_g \end{pmatrix}. \tag{5.15}$$

The characteristic speed λ and the eigenvector $\vec{r} = (r_1, r_2)^T = (dT, du)^T$ in the following system are the speed of rarefaction waves and the characteristic direction, respectively:

$$det(A - \lambda B) = 0, \qquad A\vec{r} = \lambda B\vec{r}. \tag{5.16}$$

We find only one characteristic speed and vector:

$$\lambda_T^g(T,u) = \frac{u}{\varphi}\frac{\rho_g C_g - \rho_g'\rho_g h_g}{\rho_g(\hat{C}_r + C_g) - \rho_g'\rho_g h_g} = \frac{u}{\varphi}\frac{\rho_g c_g}{\hat{C}_r + \rho_g c_g}, \quad \text{and} \quad \vec{r}_T = \left(1, \frac{u\hat{C}_r}{T(\hat{C}_r + \rho_g c_g)}\right)^T, \tag{5.17}$$

where $\rho_g = \rho_g(T)$, $h_g = h_g(T)$, $C_g = C_g(T)$, and the derivatives relative to temperature are $\rho_g' = \rho_g'(T)$, $c_g = h_g'(T)$ and $\hat{C}_r$ is constant; we used the equality $\rho_g' = -\rho_g/T$ which follows from Eq. $(B.4)$. The notation for this wave has subscript T because it is a thermal wave; the saturation ($s_w = 0$) stays constant, but the temperature T and the speed u change. We obtain the thermal rarefaction curve in (T, u) space from $\vec{r}_T$ in (5.17):

$$\frac{du}{dT} = u\frac{\hat{C}_r}{T(\hat{C}_r + \rho_g(T)c_g(T))} \qquad \text{or} \qquad \frac{du}{u} = \frac{dT}{T\left(1 + \rho_g(T)c_g(T)/\hat{C}_r\right)}. \tag{5.18}$$

Remark 5.4.1. *Normally $\rho_g c_g \ll \hat{C}_r$, so we can approximate λ_T^g by:*

$$\lambda_T^g \simeq \frac{u}{\varphi}\frac{\rho_g c_g}{\hat{C}_r}. \tag{5.19}$$

In Appendix A, we obtain a better approximation for the characteristic speed λ_T^g.

The rarefaction wave in the $\{x, t\}$ plane is the solution of the following equations:

$$\left(\frac{dT}{d\xi}, \frac{du}{d\xi}\right)^T = \vec{r}_T, \qquad \text{with} \qquad \xi = \frac{x}{t} = \lambda_T^g(u(\xi), T(\xi)). \tag{5.20}$$

From Eqs. $(5.17.c)$ and $(5.20.a)$, $dT/d\xi$ satisfies:

$$\frac{dT}{d\xi} = 1,$$

so we write T as:

$$T = \xi - \xi^- + T^-, \tag{5.21}$$

where, from Eqs $(5.20.b)$, ξ^- satisfies:

$$\xi^- = \lambda_T^g(u^-, T^-),$$

Remark 5.4.2. *If we make the same approximation used in Eq.* (5.19) *from Remark* 5.4.1 *we obtain for Eq.* (5.18)*:*

$$\frac{du}{u} = \frac{dT}{T} \qquad or \qquad \frac{u}{T} = \frac{u^-}{T^-}, \tag{5.22}$$

where u^- and T^- are any points in the br, so we obtain that u/T is constant on rarefaction wave, i.e., Eq. (5.22) *defines a Riemann invariant. So from* (5.21)*, the speed u in the plane $\{x, t\}$ is given by:*

$$u = \frac{u^-}{T^-}\left(\xi - \xi^- + T^-\right).$$

In Appendix A, we obtain a better approximation for the rarefaction wave (5.18).

Remark 5.4.3. *In the sr, the temperature decreases from left to right along the thermal rarefaction wave. In the Section* 5.4.1 *we consider a thermal steam shocks; analogously, on the right of such a shock the temperature is higher than on the left.*

Thermal steam shock

We assume now that $T^+ > T^- \geq T^b$. Let us consider the thermal discontinuity with speed v_T^g between the $(-)$ state $(0,T^-,u^-)$ and the $(+)$ state $(0,T^+,u^+)$. For such a *thermal steam shock*, Eqs. (5.10)-(5.11) yield the following Rankine-Hugoniot (RH) condition:

$$v_T^g = \frac{u^+\rho_g^+ - u^-\rho_g^-}{\varphi(\rho_g^+ - \rho_g^-)} = \frac{u^+\rho_g^+h_g^+ - u^-\rho_g^-h_g^-}{\varphi\left((\hat{H}_r^+ + \rho_g^+h_g^+) - (\hat{H}_r^- + \rho_g^-h_g^-)\right)}, \tag{5.23}$$

where $h_g^\pm = h_g(T^\pm)$, $\hat{H}_r^\pm = \hat{H}_r(T^\pm)$ and $\rho_g^\pm = \rho_g(T^\pm)$. From the second equality in Eq. (5.23), we obtain u^+ as a function of u^-:

$$u^+ = u^-\frac{(\hat{H}_r^+ - \hat{H}_r^-)/\rho_g^+ + h_g^+ - h_g^-}{(\hat{H}_r^+ - \hat{H}_r^-)/\rho_g^- + h_g^+ - h_g^-}; \tag{5.24}$$

it is easy to see that the denominator of (5.24) is positive. Moreover $u^+ > u^-$.

We substitute (5.24) in Eq. (5.23); since u^+ is function of u^-, we obtain $v_T^g = v_T^g(T^-,u^-;T^+)$ or $v_T^g = v_T^g(T^-;T^+,u^+)$:

$$v_T^g = \frac{u^-}{\varphi}\frac{h_g^+ - h_g^-}{(\hat{H}_r^+ - \hat{H}_r^-)/\rho_g^- + h_g^+ - h_g^-} = \frac{u^+}{\varphi}\frac{h_g^+ - h_g^-}{(\hat{H}_r^+ - \hat{H}_r^-)/\rho_g^+ + h_g^+ - h_g^-}. \tag{5.25}$$

Remark 5.4.4. *Notice that we can rewrite* (5.25.a) *as:*

$$v_T^g = v_T^g(T^-,u^-;T^+) = \frac{u^-}{\varphi}\frac{\left(h_g^+ - h_g^-\right)/(T^+ - T^-)}{\hat{C}_r/\rho_g^- + \left(h_g^+ - h_g^-\right)/(T^+ - T^-)}.$$

Defining $\lambda_g^\pm = \lambda_T^g(T^\pm,u^\pm)$, from convexity of $h_g(T)$ it follows that $v_T^g < \lambda_g^-$ for $T^+ > T^-$. Using (5.25.b) *we see that $\lambda_g^+ < v_T^g < \lambda_g^-$ if $T^+ > T^-$, so the steam shock satisfies the Lax condition. The equality $v_T^g = \lambda_g^-$ holds if, only if, $T^+ = T^-$.*

5.4.2 Boiling region - *br*

Because the temperature is constant and equal to the boiling temperature T^b it can be shown (see [6]), that there is no mass exchange between phases and that the system (5.8)-(5.9) reduces to a single scalar equation with fixed $u = u^b$:

$$\varphi\frac{\partial}{\partial t}s_w + u^b\frac{\partial}{\partial x}f_w = 0. \tag{5.26}$$

Eq. (5.26) supports classical Buckley-Leverett rarefaction and shock waves.

Saturation shocks

Consider a $(-)$ state (s_w^-, T^b, u^b) and a $(+)$ state (s_w^+, T^b, u^b); we obtain the following RH condition, where u^b is the common Darcy velocity, $T = T^b$ and we use the nomenclature $f_w^b(s_w) = f_w(s_w, T^b)$:

$$v_s^b(s_w^-, u^b; s_w^+) = v_s(s_w^-, T^b, u^b; s_w^+) = \frac{u^b}{\varphi}\frac{f_w^b(s_w^+) - f_w^b(s_w^-)}{s_w^+ - s_w^-} = \frac{u^b}{\varphi}\frac{f_g^b(s_g^+) - f_g^b(s_g^-)}{s_g^+ - s_g^-}. \quad (5.27)$$

A particular shock for (5.26) separates a mixture of steam and water on the left from pure water on the right, both at boiling temperature. Following [6] we call it the *Hot Isothermal Steam-Water shock* (or *HISW*) between the $(-)$ state (s_w^-, T^b, u^b) and the $(+)$ state $(s_w^+ = 1, T^b, u^b)$. It has speed $v_{g,w}^b$ given by:

$$v_{g,w}^b(s_w^-, u^b) = v_{g,w}(s_w^-, T^b, u^b) = \frac{u^b}{\varphi}\frac{1 - f_w^b(s_w^-)}{1 - s_w^-} = \frac{u^b}{\varphi}\frac{f_g^b(s_g^-)}{s_g^-}. \quad (5.28)$$

Another particular shock for (5.26) separates pure water on the left from a mixture of steam and water on the right, both at boiling temperature. We call it the *Hot Isothermal Water-Steam shock* (or *HIWS*) between the $(-)$ state (s_w^-, T^b, u^b) and the $(+)$ state $(s_w^+ = 0, T^b, u^b)$. It has speed $v_{g,s}^b$ given by

$$v_{g,s}^b(s_w^-, u^b) = v_{g,s}(s_w^-, T^b, u^b) = \frac{u^b}{\varphi}\frac{f_w^b(s_w^-)}{s_w^-}. \quad (5.29)$$

Notice that $v_{g,s}(s_w = 0, T^b, u^b) = v_{g,w}(s_w = 1, T^b, u^b) = 0$.

Saturation rarefaction waves

We will denote by λ_s^b the speed of propagation of saturation waves in the *br*. It is obtained from Eq. (5.26) as:

$$\lambda_s^b = \lambda_s(s_w, T^b, u^b) = \frac{u^b}{\varphi}\frac{\partial f_w^b}{\partial s_w}(s_w). \quad (5.30)$$

5.4.3 Water region - *wr*

The system (5.8)-(5.9) reduces to a scalar equation, with constant u^w and $s_w = 1$:

$$\varphi\frac{\partial}{\partial t}\left(\hat{H}_r(T) + \rho_w h_w(T)\right) + u^w\frac{\partial}{\partial x}\rho_w h_w(T) = 0. \quad (5.31)$$

Between a $(-)$ state $(1, T^0, u^w)$ and a $(+)$ state $(1, T, u^w)$, the following RH condition for the thermal discontinuity is valid:

$$v_T^w = \frac{u^w}{\varphi}\frac{\rho_w\left(h_w - h_w^0\right)}{\hat{H}_r + \rho_w h_w - (\hat{H}_r^0 + \rho_w h_w^0)} = \frac{u^w}{\varphi}\frac{C_w}{\hat{C}_r + C_w}, \quad (5.32)$$

where u^w is Darcy speed in the wr and $C_w = \rho_w \partial h_w/\partial T$; the second equality is obtained taking into account $(B.5)$. If $T^0 = T^b$ or $T = T^b$, then $u^b = u^w$. From (5.32), the discontinuity is a contact wave and there is no other characteristic speed in this region.

5.5 Shocks between regions

Within shocks separating regions there is no thermodynamic equilibrium, so q is not zero; however we can still use the system (5.8)-(5.9), because in each region the number of variable to be determined in the system (5.1)-(5.3) is at most 2. This system contains another variable, namely the mass transfer term q. However this variable is not essential to obtain the Riemann solution. It is useful to define the cumulative mass transfer function:

$$\mathcal{Q}(x,t) = \int^x q(\xi,t)d\xi, \tag{5.33}$$

where this integral should be understood in the distribution sense. From Eq. (5.33), we can write $q = \partial\mathcal{Q}(x,t)/\partial x$.

We also define $\mathcal{Q}^-(t) = \mathcal{Q}(x^-,t)$ and $\mathcal{Q}^+(t) = \mathcal{Q}(x^+,t)$, where x^- and x^+ are the points immediately on the left and right of the transition between regions. We define the accumulative balance as the difference between $\mathcal{Q}^+(t)$ and $\mathcal{Q}^-(t)$ and denote it by $[\mathcal{Q}]$.

We can rewrite the system (5.1)-(5.3) (in distribution sense) as:

$$\frac{\partial}{\partial t}G(s_w,T) + \frac{\partial}{\partial x}\left(uF(s_w,T) - Q(s_w,T)\right) = 0, \tag{5.34}$$

where $Q = (\mathcal{Q}(V), -\mathcal{Q}(V), 0)^T$; $G = (G_1, G_2, G_3)^T$ and $F = (F_1, F_2, F_3)^T$. The components of F and G are readily obtained from Eqs. (5.1)-(5.3) using (5.7).

The shock waves are discontinuous solutions of Eq. (5.34) and satisfy the RH condition:

$$s(G(s_w^+,T^+) - G(s_w^-,T^-)) = u^+F(s_w^+,T^+) - u^-F(s_w^-,T^-) - [\mathcal{Q}], \tag{5.35}$$

This term should be understood in the sense of distributions.

5.5.1 Water Evaporation Shock

WES - This is the discontinuity between a $(-)$ state $(1, T^-, u^-)$ in the wr and a $(+)$ state (s_w^+, T^b, u^+) in the br. It satisfies the following RH conditions for the speed v^{WES}, obtained from Eqs. (5.8)-(5.9):

$$u^+(\rho_w f_w^+ + \rho_g^b f_g^+) - \varphi v^{WES}(\rho_w s_w^+ + \rho_g^b s_g^+) = u^-\rho_w - \varphi v^{WES}\rho_w, \tag{5.36}$$

$$u^+\left(\rho_w h_w^b f_w^+ + \rho_g^b h_g^b f_g^+\right) - u^-\rho_w h_w^- = v^{WES}\varphi\left(\hat{H}_r^b + \rho_w h_w^b s_w^+ + \rho_g^b h_g^b s_g^+ - \hat{H}_r^- - \rho_w h_w^-\right). \tag{5.37}$$

where $f_w(s_w = 1, \cdot) = 1$, $f_w^+ = f_w(s_w^+, T^b)$ and $f_g = 1 - f_w$.

The Darcy speed u^+ is found from u^- using (5.36) and (5.37) as:

$$u^+ = u^- \frac{(\hat{H}_r^b - \hat{H}_r^-) + s_w^+ \rho_w \left(h_w^b - h_w^-\right) + s_g^+ \rho_g^b \left(h_g^b - h_w^-\right)}{(\hat{H}_r^b - \hat{H}_r^-)\left(f_g^+ \left(\rho_g^b/\rho_w\right) + f_w^+\right) + \left(\rho_g^b (s_w^+ - f_w^+) + \rho_w f_w^+\right)\left(h_w^b - h_w^-\right) + s_g^+ \rho_g^b \left(h_g^b - h_w^-\right)}. \tag{5.38}$$

Eq (5.38) is always valid because $T^- < T^b$, so each term in the denominator is positive. The terms $\hat{H}_r^b - \hat{H}_r^-$ and $h_w^b - h_w^-$ are positive because the enthalpies increase with temperature. The term $h_g^b - h_w^-$ is positive because $h_g^b > h_w^b$ and since $h_w^b > h_w^-$ the positivity follows.

Since we can write u^+ as function of u^w, we obtain $v^{WES} = v^{WES}(T^-, u^-; s_w^+)$:

$$v^{WES} = \frac{u^+}{\varphi} \frac{f_g^+ \rho_g^b \left(h_g^b - h_w^-\right) + f_w^+ \rho_w \left(h_w^b - h_w^-\right)}{\hat{H}_r^b - \hat{H}_r^- + s_g^+ \rho_g^b \left(h_g^b - h_w^-\right) + s_w^+ \rho_w \left(h_w^b - h_w^-\right)}. \tag{5.39}$$

The denominator of v^{WES} in (5.39) is never zero because each term in the sum is positive.

Lemma 5.5.1. *Define $\mathcal{L}(T)$ as:*

$$\mathcal{L}(T) = \rho_g^b (h_g^b - h_w^T) - \rho_w \left(h_w^b - h_w^T\right), \tag{5.40}$$

where $h_w^T = h_w(T)$. There is a unique temperature $T^\dagger < T^b$ such that $\mathcal{L}(T^\dagger) = 0$. Moreover, $\mathcal{L}(T) < 0$ for $T < T^\dagger$ and $\mathcal{L}(T) > 0$ for $T > T^\dagger$.

Proof: We define:

$$\mathcal{L}^b = \rho_w h_w^b - \rho_g^b h_g^b = H_w^b - H_g^b,$$

where H_w and H_g are defined in 5.4. From (i)-(iii) of Remark 5.4, we can see that $\mathcal{L}^b > 0$. Since $\mathcal{L}^b$ does not depend on T, if T is small, we obtain that

$$\rho_w h_w^b - \rho_g^b h_g^b = \mathcal{L}^b > h_w^T(\rho_w - \rho_g^T) \Longrightarrow \mathcal{L}(T) = h_w^T(\rho_w - \rho_g^T) - \mathcal{L}^b < 0,$$

On the other hand, if $T = T^b$ we obtain from Eq. (5.40) that

$$\mathcal{L}(T^b) = \mathcal{L}^b > 0,$$

so there is almost a temperature $T^\dagger$ such that $\mathcal{L}(T^\dagger) = 0$. Moreover, the temperature $T^\dagger$ is unique, because if we differentiate $\mathcal{L}(T)$:

$$\frac{d\mathcal{L}(T)}{dT} = C_w(\rho_w - \rho_g^b)/\rho_w > 0,$$

i.e, $\mathcal{L}(T)$ is a monotonic function, so there is only a temperature that satisfies $\mathcal{L}(T^\dagger) = 0$. $\square$

Substituting $s_g^+ = 1 - s_w^+$, $f_g^+ = 1 - f_w^+$, using Lemma 5.5.1 for $T \neq T^\dagger$, we rewrite Eq. (5.39)

$$v^{WES} = \frac{u^+}{\varphi}\frac{f_w^+ - f_w^{WES}}{s_w^+ - s_w^{WES}}, \; f_w^{WES} \equiv \frac{\rho_g^b(h_g^b - h_w^-)}{\mathcal{L}(T^-)}, \; s_w^{WES} \equiv \frac{\hat{H}_r^b - \hat{H}_r^- + \rho_g^b(h_g^b - h_w^-)}{\mathcal{L}(T^-)}, \tag{5.41}$$

Eqs. (5.41) are the basis for a graphical construction of the WES, see Fig. 5.2.*b*.

For $T^- < T^\dagger$, f_w^{WES} and s_w^{WES} are negative, while for $T^- > T^\dagger$, $f_w^{WES} > 1$ and $s_w^{WES} > 1$. When $T^- = T^\dagger$, we obtain that $\rho_g^b(h_g^b - h_w^-) = \rho_w\left(h_w^b - h_w^-\right)$, so Eq. (5.39) reduces to:

$$v^{WES}(1, T^- = T^\dagger, u^w; s_w^+) = \frac{u^+}{\varphi}\frac{\rho_w\left(h_w^b - h_w^\dagger\right)\left(f_w^b + f_g^b\right)}{\hat{H}_r^b - H_r^\dagger + \rho_w(h_w^b - h_w^\dagger)\left(s_w + s_g\right)} = \frac{u^-}{\varphi}\frac{C_w}{\hat{C}_r + C_w}. \tag{5.42}$$

Notice that if $u^- = u^+$, $v^{WES} = v_T^w$ in the *wr*, see Eq (5.32).

5.5.2 Vaporization Shock

VS - It is a discontinuity between a $(-)$ state (s_w^-, T^b, u^-) in the *br* and a $(+)$ state $(0, T^+ > T^b, u^+)$ in the *sr*. The Vaporization Shock satisfies the following RH conditions with speed v^{VS} obtained from Eqs. (5.8)-(5.9):

$$v^{VS}\varphi(\rho_g^+ - \rho_g^b s_g^- - \rho_w s_w^-) = u^+\rho_g^+ - u^-(\rho_w f_w^- + \rho_g^b f_g^-), \tag{5.43}$$

$$v^{VS}\varphi(H_r^+ - H_r^b + \rho_g^+ h_g^+ - s_g^-\rho_g^b h_g^b - s_w^-\rho_w h_w^b) = u^+\rho_g^+ h_g^+ - u^-(\rho_g^b h_g^b f_g^- + \rho_w h_w^b f_w^-), \tag{5.44}$$

where $h_g^+ = h_g(T^+)$, $h_w^+ = h_w(T^+)$, $H_r^+ = H_r(T^+)$, $\rho_g^+ = \rho_g(T^+)$ and $f_w^- = f_w(s_w^-, T^b)$.

Since $T > T^b$, we obtain v^{VS} as the following fraction, which has positive denominator:

$$v^{VS} = v^{VS}(s_w^-, u^-; T^+) = \frac{u^-}{\varphi}\frac{f_g^-\rho_g^b\left(h_g^+ - h_g^b\right) + f_w^-\rho_w\left(h_g^+ - h_w^b\right)}{\hat{H}_r^+ - \hat{H}_r^b + s_g^-\rho_g^b\left(h_g^+ - h_g^b\right) + s_w^-\rho_w\left(h_g^+ - h_w^b\right)}. \tag{5.45}$$

We rewrite Eq. (5.45) in a shorter form. Substituting $s_g = 1 - s_w$ and $f_g = 1 - f_g$, we define $\mathbb{A}(T^+) = \rho_g^b\left(h_g^+ - h_g^b\right) - \rho_w\left(h_g^+ - h_w^b\right)$.

Lemma 5.5.2. *If $T^+ > T^b$, the water and rock enthalpies satisfy the following inequality: $\rho_g^b\left(h_g^+ - h_g^b\right) < \rho_w\left(h_g^+ - h_w^b\right)$.*

Proof: From Remark 5.1.1, we know that $h_w^b < h_g^b$. Using the expression for $\mathbb{A}$ we obtain:

$$\rho_w\left(h_g^+ - h_w^b\right) - \rho_g^b\left(h_g^+ - h_g^b\right) > \rho_w\left(h_g^+ - h_g^b\right) - \rho_g^b\left(h_g^+ - h_g^b\right) = (\rho_w - \rho_g^b)\left(h_g^+ - h_g^-\right),$$

but $(\rho_w - \rho_g^b) > 0$ because $\rho_w >> \rho_g$ and $(h_g^+ - h_g^b) > 0$ since h_g increases with temperature. $\square$

From above Lemma 5.5.2, we obtain that $\mathbb{A} < 0$. We multiply and divide (5.45) by $\mathbb{A}$ and we obtain:

$$v^{VS} = \frac{u^-}{\varphi}\frac{f_w^- - f_w^{VS}}{s_w^- - s_w^{VS}}, \quad \text{where} \quad f_w^{VS} \equiv \frac{\rho_w\left(h_g^+ - h_w^b\right)}{\mathbb{A}(T^+)}, \; s_w^{VS} \equiv \frac{\hat{H}_r^+ - \hat{H}_r^b + \rho_g^b\left(h_g^+ - h_g^b\right)}{\mathbb{A}(T^+)}; \tag{5.46}$$

notice that f_w^{VS} and s_w^{VS} are negative. Eqs. (5.46) are the basis for a graphical construction of VS, see Fig. 5.2.*a*.

The Darcy speed u^+ in the *sr* is the fraction with positive denominator:

$$u^+ = u^-\frac{A^+(\rho_w(s_g^- - f_g^-) + \rho_g^+ f_g^-) + B^+(\rho_g^b(s_w^- - f_w^-) + f_w^-\rho_g^+) + C^+\left(f_g^-\rho_g^b + f_w^-\rho_w\right)}{\left(A^+ s_g^- + B^+ s_w^- + C^+\right)\rho_g^+}, \tag{5.47}$$

where:

$$A^+ = \rho_g^b\left(h_g^+ - h_g^b\right), \quad B^+ = \rho_w\left(h_g^+ - h_w^b\right), \quad C^+ = \hat{H}_r^+ - \hat{H}_r^b. \tag{5.48}$$

5.5.3 Condensation Shock

CS - This is the discontinuity between a $(-)$ state $(0, T^- > T^b, u^-)$ in the *sr* and a $(+)$ state (s_w^+, T^b, u^+) in the *br*. It is the reverse of the shock VS. From (5.47):

$$u^+ = u^-\frac{\left(A^- s_g^+ + B^- s_w^+ + C^-\right)\rho_g^-}{A^-(\rho_w(s_g^+ - f_g^+) + \rho_g^- f_g^+) + B^-(\rho_g^b(s_w^+ - f_w^+) + f_w^+\rho_g^-) + C^-\left(f_g^+\rho_g^b + f_w^+\rho_w\right)}, \tag{5.49}$$

A^-, B^- and C^- are obtained from A^+, B^+ and C^+ in (5.48) by substituting T^+ by T^-.

Lemma 5.5.3. *The denominator of* (5.49) *is always positive.*

Proof: We calculate:

$$\begin{aligned} A^-\rho_w f_g + B^-\rho_g^b f_w &= \left(h_g^- - h_g^b\right)\rho_g^b\rho_w f_g + \left(h_g^- - h_w^b\right)\rho_w\rho_g^b f_w, \\ &= h_g^-\rho_g^b\rho_w f_g - h_g^b\rho_w f_g + h_g^-\rho_w\rho_g^b f_w - h_w^b\rho_g^b f_w, \\ &= h_g^-\rho_g^b\rho_w(f_g + f_w) - (h_g^b\rho_w f_g + h_w^b\rho_g^b f_w), \\ &= h_g^-\rho_g^b\rho_w - (h_g^b\rho_w f_g + h_w^b\rho_g^b f_w), \end{aligned} \tag{5.50}$$

where A^- and B^- are given in (5.48) and $f_w = f_w^+$ and $f_g = f_g^+$.

Similarly, we calculate:

$$A^-\rho_w s_g + B^-\rho_g^b s_w = h_g^-\rho_g^b\rho_w - (h_g^b\rho_w s_g + h_w^b\rho_g^b s_w), \tag{5.51}$$

where $s_w = s_w^+$ and $s_g = s_g^-$. So, from (5.51) and (5.50):

$$
\begin{aligned}
A^-\rho_w(s_g - f_g) + B^-\rho_g^b(s_w - f_w) &= h_g^b\rho_w(f_g - s_g) + h_w^b\rho_g^b(f_w - s_w), \\
&= \rho_w\rho_g^b\left(h_g^b(f_g - s_g) + h_w^b(f_w - s_w)\right), \\
&= \rho_w\rho_g^b\left(h_g^b - h_w^b\right)(s_w - f_w), \qquad (5.52)
\end{aligned}
$$

where $\rho_w\rho_g^b\left(h_g^b - h_w^b\right) > 0$.

We rewrite the denominator of (5.49) as:

$$
C^-\left(f_g^b\rho_g^b + f_w^b\rho_w\right) + \rho_w\rho_g^b\left(h_g^b - h_w^b\right)(s_w - f_w) + A^-\rho_g^- f_g + B^-\rho_g^- f_w; \qquad (5.53)
$$

notice that

$$
B^-\rho_g^- - \rho_w\rho_g^b\left(h_g^b - h_w^b\right) = \rho_w H_g^- - \rho_g^- H_w^b - \left(\rho_w H_g^b - \rho_g^b H_w^b\right), \qquad (5.54)
$$

where H_g and H_w are defined in 5.1.1. From the same Remark, we know that $H_g^- > H_g^b$ because $T^- > T^b$, (see (5.54)) and $\rho_g^b > \rho_g^-$ because ρ_g is decreasing, we have:

$$
B^-\rho_g^- - \rho_w\rho_g^b\left(h_g^b - h_w^b\right) > \left(\rho_w H_g^b - \rho_g^b H_w^b\right) - \left(\rho_w H_g^b - \rho_g^b H_w^b\right) = 0, \qquad (5.55)
$$

so from (5.53) and (5.55) we obtain:

$$
\begin{aligned}
&C^-\left(f_g^b\rho_g^b + f_w^b\rho_w\right) + \rho_w\rho_g^b\left(h_g^b - h_w^b\right)(s_w - f_w) + A^-\rho_g^- f_g + B^-\rho_g^- f_w, \\
&\quad = C^-\left(f_g^b\rho_g^b + f_w^b\rho_w\right) + \rho_w\rho_g^b\left(h_g^b - h_w^b\right)s_w + A^-\rho_g^- f_g + f_w(B^-\rho_g^- - \rho_w\rho_g^b\left(h_g^b - h_w^b\right)), \\
&\quad > C^-\left(f_g^b\rho_g^b + f_w^b\rho_w\right) + \rho_w\rho_g^b\left(h_g^b - h_w^b\right)s_w + A\rho_g^- f_g > 0. \qquad (5.56)
\end{aligned}
$$

The last equality is because $h_g^b > h_w^b$ and each term is positive. So the denominator of (5.49) is always positive. $\square$

Since the CS is reverse of the VS and u^+ is a function of u^-, we obtain $v^{CS} = v^{CS}(T^-, u^-; s_w^+)$:

$$
v^{CS} = \frac{u^+}{\varphi}\frac{f_g^+\rho_g^b\left(h_g^- - h_g^b\right) + f_w^+\rho_w\left(h_g^- - h_w^b\right)}{\hat{H}_r^- - \hat{H}_r^b + s_g^+\rho_g^b\left(h_g^- - h_g^b\right) + s_w^+\rho_w\left(h_g^- - h_w^b\right)} \quad \text{or} \quad v^{CS} = \frac{u^+}{\varphi}\frac{f_w^+ - f_w^{CS}}{s_w^+ - s_w^{CS}}. \qquad (5.57)
$$

Since the CS shock is reverse of the VS, f_w^{CS} and s_w^{CS} are obtained from f_w^{VS} and s_w^{VS} in (5.46) by substituting T^+ by T^-; notice that the denominator in (5.57.a) is never zero.

5.5.4 Steam condensation front

SCF - This is the discontinuity between a $(-)$ state $(s_w^- < 1, T^b, u^-)$ in the *br* and a $(+)$ state $(1, T^+, u^+)$ in the *wr*. It is the reverse of the WES, so $v^{SCF} = v^{SCF}(s_w^-, u^-; T^+)$ is given by:

$$v^{SCF} = \frac{u^-}{\varphi} \frac{f_g^- \rho_g^b \left(h_g^b - h_w^+\right) + f_w^- \rho_w \left(h_w^b - h_w^+\right)}{\hat{H}_r^b - \hat{H}_r^+ + s_g^- \rho_g^b \left(h_g^b - h_w^+\right) + s_w^- \rho_w \left(h_w^b - h_w^+\right)} = \frac{u^-}{\varphi} \frac{f_w^- - f_w^{SCF}}{s_w^- - s_w^{SCF}}. \tag{5.58}$$

Here (5.58) is derived as Eq. (5.41), with f_w^{SCF} and s_w^{SCF} obtained from f_w^{WES} and s_w^{WES} in (5.41) by substituting T^+ by T^-.

From Eqs. (5.36)-(5.37), we can find u^+ as the following fraction with positive denominator:

$$u^+ = u^- \frac{(\hat{H}_r^b - \hat{H}_r^+)\left(f_g^- \left(\rho_g^b/\rho_w\right) + f_w^-\right) + \left(\rho_g^b(s_w^- - f_w^-) + \rho_w f_w^-\right)\left(h_w^b - h_w^+\right) + s_g^- \rho_g^b \left(h_g^b - h_w^+\right)}{\hat{H}_r^b - \hat{H}_r^+ + s_w^- \rho_w \left(h_w^b - h_w^+\right) + s_g^- \rho_g^b \left(h_g^b - h_w^+\right)}. \tag{5.59}$$

5.6 The Riemann Solution

The Riemann problem is the solution of (5.8)-(5.9) with initial data

$$\begin{cases} L = (s_L, T_L, u_L) & \text{if } x > 0 \\ R = (s_R, T_R, \cdot) & \text{if } x < 0, \end{cases} \tag{5.60}$$

where $s := s_w$ is the water saturation. We will see that the speed u cannot be prescribed on both sides. Given a speed on one side the other one is obtained by solving the system (5.1)-(5.3); in this case we have chosen to prescribe u_L.

We consider in this paper the Riemann problem for all initial data; we divide the data as:

Riemann Problem	L state	R state
Data A	Steam and Water, $T_L = T^b$	Steam, $T_R > T^b$
Data B	Water, $T_L < T^b$	Steam and Water, $T_R = T^b$
Data C	Steam, $T_L > T^b$	Steam and Water, $T_R = T^b$
Data D	Water, $T_L < T^b$	Steam, $T_R > T^b$
Data E	Steam, $T_L > T^b$	Water, $T_R < T^b$
Data F	Steam and water, $T_L = T^b$	Water, $T_R < T^b$

The Riemann problem with Data F for $T_R = T^0$ was solved in [6]; in that paper, T^0 is an arbitrary temperature where the water and rock enthalpies were made to vanish; we do not repeat this Riemann problem here.

The rarefaction waves are denoted by R_T for thermal rarefactions in *sr* and R_s for (Buckley-Leverett) saturation rarefactions in *br*. The shocks in regions under thermodynamic equilibrium are denoted with a single subscript, S_T for thermal shocks in

sr, S_s for (Buckley-Leverett) saturation shocks , S_G for HIWS in br and S_W for thermal discontinuities in wr. We recall that the shocks between regions are WES, VS, CS and SCF.

5.6.1 Riemann Problem A

Water injection.

First, we inject water with temperature T^b, i.e., $L = (1, T^b, u_L)$ in a porous rock filled with superheated steam, i.e., $R = (0, T_R > T^b, u_R)$, which is a sr. In the br, generated by the L state, the flow is governed by a Buckley-Leverett equation. It is well known that the Buckley-Leverett rarefaction has speed λ_s^b given by (5.30) from $s_w = 1$ to $s_w = s^*$, which is defined by:

$$\frac{\partial f_w}{\partial s_w}(s^*, T^b) = \frac{f_w(s^*, T^b)}{s^*}. \tag{5.61}$$

There is a saturation shock and the solution is continued by a shock in the br.

Since the temperature increases in the rarefaction joining the $(-)$ state $(0, T^b, u^-)$ to the $(+)$ state $(0, T^+ > T^b, u^+)$, see Rem. 5.4.3, there is a shock with speed v_T^g given by (5.25).

Lemma 5.6.1. *The speed $v_{g,s}^b$ of the HIWS shock between (s^*, T^b, u^-) and $(0, T^b, u = u^-)$ given by (5.29) is larger than the speed v_T^g of the shock between $(0, T^b, u^-)$ and $(0, T > T^b, u^+)$.*

Proof: From Eq. (5.25) we have $u^- = u^b$ and we can see easily that $v_T^g < u^b/\varphi$. From (5.29) and (5.61), we have that $v_{g,s}^b$ for s^* satisfies $v_{g,s}^b > u^b/\varphi$. So we conclude that $v_T^g < v_{g,s}^b$. $\square$

Thus we conclude that there is a shock with speed v^{VS} between the br and the sr. Let f_w^{VS} and s_w^{VS} be given by (5.46); from the Sec. 2.1.3, we find a saturation $\hat{s} > s^*$ defined by the following equality, (Fig. 5.2.a):

$$v^{VS}(\hat{s}) = \frac{u^b}{\varphi} \frac{f_w(\hat{s}, T^b) - f_w^{VS}}{\hat{s} - s_w^{VS}} = \frac{u^b}{\varphi} \frac{\partial f_w}{\partial s_w}(\hat{s}, T^b). \tag{5.62}$$

Lemma 5.6.2. *The saturations $\hat{s}$ and s^* satisfy $s^* < \hat{s}$*

Proof: The result follows from shape of f_w, i.e., a function convex between $s_w = 0$ and s^{inf}; and concave between s^{inf} and $s_w = 1$ with $f_w(1, \cdot) = 1$.

Using Eq. (5.62) we can prove the following Lemma:

Lemma 5.6.3. *The solutions of (5.62) are the points where the derivative of v^{VS} with respect to S_w vanishes.*

Proof: The points that satisfy (5.62) are tangency points of the secant from the point (S_w^{VS}, f_w^{VS}) to the graph f_w, so the derivative vanishes. (An analytic proof for this Lemma is similar to proof of the Lemma 5.6.6). $\square$

Moreover, we can verify the following Proposition:

Proposition 5.6.1. *There are two saturation values that satisfy* (5.62) *for each* $T^+ > T^b$. *The largest value satisfies* (5.62) *and maximizes* v^{VS} *while the other minimizes* v^{VS}*; because of geometrical compatibility, we choose the largest value and we denote it by* $\hat{s}$. *It is called "Hot-Bifurcation saturation I" or HBI.*

Proof: We verify geometrically this proposition; equation (5.62) represents tangency points of the secant from the point (S_w^{VS}, f_w^{VS}) to the graph f_w, for each fixed T^-. In figure 5.2, we plot a possible point (S_w^{VS}, f_w^{VS}).

The largest water saturation value satisfying (5.62) maximizes v^{VS}. This value is denoted $\hat{S}$, and the other saturation minimizes the shock speed v^{VS}.

The analytical proof for this proposition can be obtained following the steps of Proposition 5.6.4, however this proof is longer than the geometrical verification. Notice that in Proposition 5.6.4, the chosen water saturation minimizes v^{CS}. $\square$

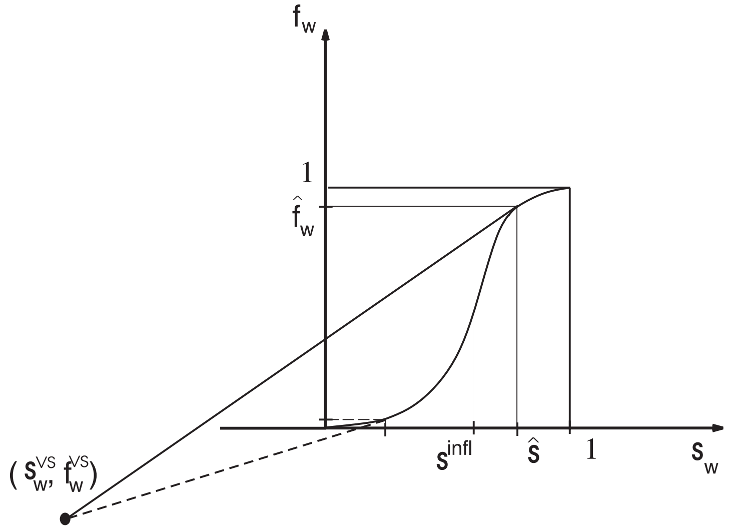


Figure 5.2: The coincidence between v^{VS} and λ_S^b. As $\mathbb{A} < 0$, so $S_w^{VS} < 0$ and $f_w^{VS} < 0$ are negative, the largest water saturation point that satisfies (5.62) maximizes and the other point minimizes v^{VS}.

It is necessary that $v^{VS}(\hat{s}) > v_T^g$; otherwise, the geometrical compatibility in Sec. 2.1.3 says that the vaporization shock VS does not exists. This is summarized as follows:

Lemma 5.6.4. *The vaporization shock between* $(-)$ *state* (s_w^-, T^b, u^-) *and* $(+)$ *state* $(1, T^+, u^+)$ *with speed* v^{VS} *given by* (5.45) *is larger than* v_T^g *for* $s^* \leq s_w^- \leq 1$ *and* $T^+ > T^b$. *(We recall that* s^* *is defined in Eq.* (5.61)*).*

Proof: From (5.45), v^{VS} is written as:

$$v^{VS}(S_w) = \frac{u^b}{\varphi} \frac{f_g^- \rho_g^b \left(h_g^+ - h_g^b\right) + f_w^- \rho_w \left(h_g^+ - h_w^b\right)}{\hat{H}_r^+ - \hat{H}_r^- + s_g \rho_g^b \left(h_g^+ - h_g^b\right) + s_w \rho_w \left(h_g^+ - h_w^b\right)}, \tag{5.63}$$

where $s_w = s_w^-$ and $s_g = s_g^-$. As to $s^* \leq s_w \leq 1$, we have $f_w(s_w) > s_w$, so from Lemma 5.5.2 and (5.63) we obtain:

$$v^{VS}(S_w) > \frac{u^b}{\varphi} \frac{f_g^- \rho_g^b \left(h_g^+ - h_g^b\right) + f_w^- \rho_g^b \left(h_g^+ - h_g^b\right)}{\hat{H}_r^+ - \hat{H}_r^- + s_g \rho_g^b \left(h_g^+ - h_g^b\right) + s_w \rho_g^b \left(h_g^+ - h_g^b\right)}; \tag{5.64}$$

using $f_w + f_g = 1$ and $S_w + S_g = 1$ in (5.64) we write finally:

$$v^{VS}(S_w) > \frac{u^b}{\varphi} \frac{\rho_g^b \left(h_g^+ - h_g^b\right)}{\hat{H}_r^+ - \hat{H}_r^- + \rho_g^b \left(h_g^+ - h_g^b\right)} = v_T^g. \qquad \square \tag{5.65}$$

From (5.29), we conclude that $v_{g,s}^b$ and s_w decrease together. From (5.25), v_T^g is constant. So we expect that there is a saturation s^{**} where $v_T^g = v_{g,s}^b(s^{**}, s_w = 0)$, which is called "Hot-Bifurcation II saturation", or HBII.

Remark 5.6.1. *Let $a, b, c, d \in \mathbb{R}^*$, if*

$$\frac{a}{b} = \frac{c}{d}, \qquad \longrightarrow \qquad \frac{a}{b} = \frac{c - \alpha a}{d - \alpha b} = \frac{a - \alpha c}{b - \alpha d} = \frac{c}{d}, \qquad \textit{for } \alpha \neq k.$$

One can verify the following:

Proposition 5.6.2. *For each fixed T^+, there is a unique saturation s^{**} for which*

$$v_T^g(T^b, u^-; T^+) = v_{g,s}^b(u^-; s^{**}) = v^{VS}(s^{**}, u^-; T^+). \tag{5.66}$$

Furthermore s^{**} is defined in terms of T^+ by any of the following equivalent equalities:

1. $v_T^g(T^b, u^-; T^+) = v_{g,s}^b(u^-; s^{**})$;
2. $v_{g,s}^b(u^-; s^{**}) = v^{VS}(s^{**}, u^-; T^+)$;
3. $v_T^g(T^b, u^-; T^+) = v^{VS}(s^{**}, u^-; T^+)$.

Proof:
(1) First we assume that $v_T^g = v_{g,s}^b$:
We set $T^b = T^-$ and $T = T^+$, so we can write

$$v_T^g = \frac{u^-}{\varphi} \frac{\rho_g^- (h_g^+ - h_g^-)}{\hat{H}_r^+ - \hat{H}_r^- + \rho_g^- (h_g^+ - h_g^-)} = \frac{u^-}{\varphi} \frac{f_w}{S^{**}} = v_{g,s}^b, \tag{5.67}$$

where $f_w = f_w(s^{**}, T^b)$.

We multiply and divide the third term of (5.67) by $\rho_w \left(h_g^+ - h_w^-\right)$, which is positive:

$$v_T^g = \frac{u^-}{\varphi} \frac{\rho_g^- (h_g^+ - h_g^-)}{\hat{H}_r^+ - \hat{H}_r^- + \rho_g^- (h_g^+ - h_g^-)} = \frac{u^-}{\varphi} \frac{f_w \rho_w \left(h_g^+ - h_w^-\right)}{s^{**} \rho_w \left(h_g^+ - h_w^-\right)} = v_{g,s}^b. \tag{5.68}$$

From the Remark 5.6.1, we obtain:

$$v_T^g = \frac{u^-}{\varphi} \frac{\rho_g^- (h_g^+ - h_g^-) + f_w \rho_w \left(h_g^+ - h_w^-\right)}{\hat{H}_r^+ - \hat{H}_r^- + \rho_g^- (h_g^+ - h_g^-) + s^{**} \rho_w \left(h_g^+ - h_w^-\right)} = \frac{u^-}{\varphi} \frac{f_w}{s^{**}} = v_{g,s}^b. \tag{5.69}$$

We multiply and divide the third equation of (5.69) by $\rho_w \left(h_g^+ - h_w^-\right)$, so v_T^g equals:

$$\frac{u^-}{\varphi} \frac{\rho_g^- (h_g^+ - h_g^-) + f_w \rho_w \left(h_g^+ - h_w^-\right)}{\hat{H}_r^+ - \hat{H}_r^- + \rho_g^- (h_g^+ - h_g^-) + s^{**} \rho_w \left(h_g^+ - h_w^-\right)} = \frac{u^-}{\varphi} \frac{f_w \rho_g^- (h_g^+ - h_g^-)}{s^{**} \rho_g^- (h_g^+ - h_g^-)} = v_{g,s}^b. \tag{5.70}$$

From the Remark 5.6.1, and using that $1 - f_w(s^{**}) = f_g$ and $1 - s^{**} = s_g^{**}$ we obtain:

$$v_T^g = \frac{u^-}{\varphi} \frac{f_g \rho_g^- (h_g^+ - h_g^-) + f_w \rho_w \left(h_g^+ - h_w^-\right)}{\hat{H}_r^+ - \hat{H}_r^- + s_g^{**} \rho_g^- (h_g^+ - h_g^-) + s^{**} \rho_w \left(h_g^+ - h_w^-\right)} = v^{VS} = v_{g,s}^b. \tag{5.71}$$

so $v_T^g = v_{g,s}^b$ implies that $v_T^g = v_{g,s}^b = v^{VS}$.

(2) We assume that $v_{g,s}^b = v^{VS}$ so:

$$v^{VS} = \frac{u^-}{\varphi} \frac{f_g \rho_g^- (h_g^+ - h_g^-) + f_w \rho_w \left(h_g^+ - h_w^-\right)}{\hat{H}_r^+ - \hat{H}_r^- + s_g^{**} \rho_g^- (h_g^+ - h_g^-) + s^{**} \rho_w \left(h_g^+ - h_w^-\right)} = \frac{u^-}{\varphi} \frac{f_w}{s^{**}} = v_{g,s}^b \tag{5.72}$$

Performing the above calculation in reverse order, we see that $v^{VS} = v_{g,s}^b$ implies that $v^{VS} = v_{g,s}^b = v_T^g$.

(3) We assume that $v_T^g = v^{VS}$, so:

$$\frac{\rho_g^- (h_g^+ - h_g^-)}{\hat{H}_r^+ - \hat{H}_r^- + \rho_g^- (h_g^+ - h_g^-)} = \frac{u^-}{\varphi} \frac{f_g \rho_g^- (h_g^+ - h_g^-) + f_w \rho_w \left(h_g^+ - h_w^-\right)}{\hat{H}_r^+ - \hat{H}_r^- + s_g^{**} \rho_g^- (h_g^+ - h_g^-) + s^{**} \rho_w \left(h_g^+ - h_w^-\right)}. \tag{5.73}$$

From the Remark 5.6.1, we obtain:

$$\frac{\rho_g^- (h_g^+ - h_g^-)}{\hat{H}_r^+ - \hat{H}_r^- + \rho_g^- (h_g^+ - h_g^-)} = \frac{f_w \left(\rho_w \left(h_g^+ - h_w^-\right) - \rho_g^- (h_g^+ - h_g^-)\right)}{s^{**} \left(\rho_w \left(h_g^+ - h_w^-\right) - \rho_g^- (h_g^+ - h_g^-)\right)}. \tag{5.74}$$

Since $\rho_w\left(h_g^+ - h_w^-\right) - \rho_g^-(h_g^+ - h_g^-) \neq 0$ (see Lemma 5.5.2), then:

$$\frac{\rho_g^-(h_g^+ - h_g^-)}{\hat{H}_r^+ - \hat{H}_r^- + \rho_g^-(h_g^+ - h_g^-)} = \frac{f_w}{s^{**}}. \tag{5.75}$$

so $v_T^g = v^{VS}$ implies that $v_T^g = v_{g,s}^b = v^{VS}$. $\square$

Lemma 5.6.5. *For any $T^+ > T^b$ in the physical range, the saturation s^{**} given by Eq.* (5.66) *satisfies $s^{**} < \hat{s}$.*

Proof: The saturation s^{**} satisfies Eq. (1) of Proposition 5.6.2. Geometrically, this Equation represent a line from a point (s_w^{VS}, f_w^{VS}) passing on the point $(0,0)$ and intersecting the graph f_w for each fixed T^+, where s_w^{VS} and f_w^{VS} are given by Eq. (5.46.c) and (5.46.b) respectively. Notice that this intersection occurs always because $s_w^{VS} < f_w^{VS} < 0$. Moreover, from Figure 5.3 we can notice that $s^{**} \leq \hat{s}$ and $s^{**} = \hat{s}$ if, only if, $s^* = s^{**} = \hat{s}$, and from Lemma 5.6.2 follows that $s^* < \hat{s}$, so $s^{**} < \hat{s}$. Since T^+ is arbitrary the Lemma is proved. $\square$

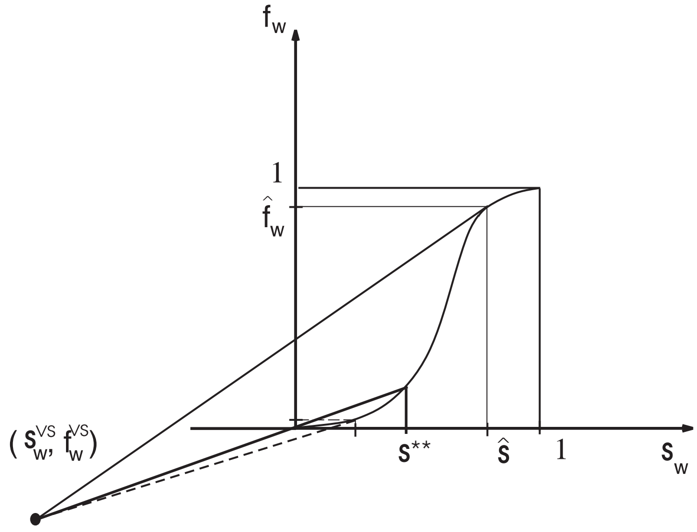


Figure 5.3: The saturation s^{**} obtained from a line from a point (s_w^{VS}, f_w^{VS}) passing on the point $(0,0)$ and intersecting the graph f_w.

Solution. Now we can describe the possible solutions for Riemann Data A:
For $\hat{s} \leq s_L \leq 1$. The waves $R_s \twoheadrightarrow VS$, with $\hat{s}$ given by Eq. (5.62) and the sequence:

$$L = (s_L, T^b, u_L) \xrightarrow{R_s} (\hat{s}, T^b, u_L) \xrightarrow{VS} (0, T_R, u_R) = R. \tag{5.76}$$

For $s^{**} \leq s_L < \hat{s}$. The wave VS with s^{**} given by Eq. (5.66) and the sequence:

$$L = (s_L, T^b, u_L) \xrightarrow{VS} (0, T_R, u_R) = R. \tag{5.77}$$

For $s_L < s^{**}$**.** The waves $S_G \mapsto S_T$, with sequence:

$$(s_L, T^b, u_L) \xrightarrow{S_G} (0, T^b, u_L) \xrightarrow{S_T} (0, T_R, u_R). \tag{5.78}$$

Wave behavior on bifurcations

Notice that the Buckley-Leverett rarefaction wave R_S disappears on the vertical dotted line from $\hat{S}$ that separates region I from II; we call it the "$R_S - S_{VS}$ boundary bifurcation". On the vertical dotted line from S^{**} separating region II from III, the condensation shock S_{VS} between regions disappears and a saturation S_G and a thermal shock S_T appear in the *br*; we call this line the "$S_T - S_{VS}$ bifurcation".

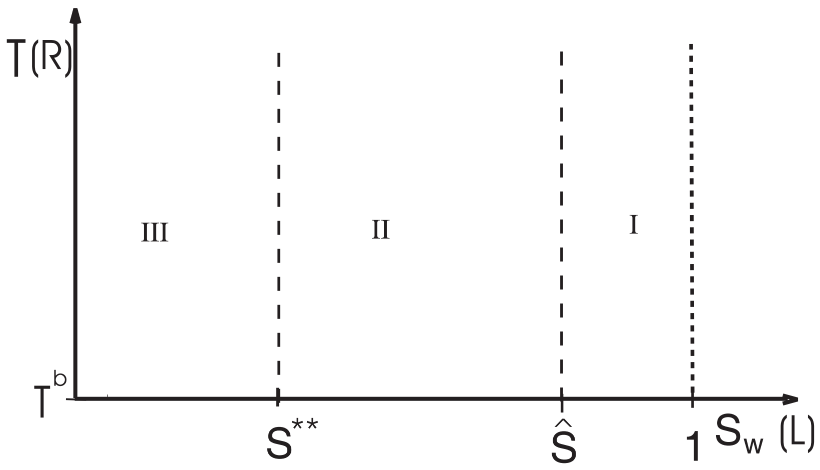


Figure 5.4: The dotted line is the boundary of physical range. The dashed lines are bifurcation loci. The temperature is higher than the water boiling temperature (superheated steam) or is equal to boiling temperature. The saturations S^{**} and $\hat{S}$ are given in Eqs. (5.66) and (5.62); the vertical lines from $\hat{S}$ and S^{**} are the "$R_S - S_{VS}$ bifurcation" and the "$S_T - S_{VS}$ bifurcation".

5.6.2 Riemann Problem B

Since the flow in the *wr* is governed by the linear Eq. (5.31) with constant characteristic speed v_T^w given by (5.32), this wave is a contact discontinuity.

For the Riemann problem $L = (1, T_L \leq T^b, u_L)$ and $R = (s_R, T^b, u_R)$, we have the following:

Proposition 5.6.3. *For each* $(-)$ *state* $(1, T^- < T^b, u^-)$ *and* $(+)$ *state* (s_w^+, T^+, u^+)*, there is a water saturation* $s^\flat = s^\flat(T^-)$*, such that for* s_w^+ *satisfying* $s^\flat \leq s_w^+ \leq 1$*, the shock speeds* v^{WES} *and* v_T^w *satisfy:*

$$v_T^w \geq v^{WES}, \qquad s^\flat \leq s_w^+ \leq 1; \tag{5.79}$$

the equality occurs only if $s_w^+ = 1$ *or* $s_w^+ = s^\flat$. *We call* $s^\flat$ *a "Cold Bifurcation saturation I" or CBI. Moreover, there is a water saturation* $s_{\dagger\dagger} = s_{\dagger\dagger}(T^-)$ *satisfying* $s^\flat < s_{\dagger\dagger}$ *such that:*

$$\lambda_s^b(s_{\dagger\dagger}) = v^{WES}(T^-, u^-; s_{\dagger\dagger}). \tag{5.80}$$

Proof: We define the functions $\Theta(T^-; s_w^+)$ and $\Psi(T^-; s_w^+)$ as:

$$\Theta(T^-; s_w^+) := v_T^w - v^{WES}(T^-; s_w^+), \tag{5.81}$$

$$\Psi(T^-; s_w^+) := \lambda_s^b(T^-; s_w^+) - v^{WES}(T^-; s_w^+). \tag{5.82}$$

The proposition is true, because from Figure 5.5 the curve Ψ is above the curve Θ. $\square$

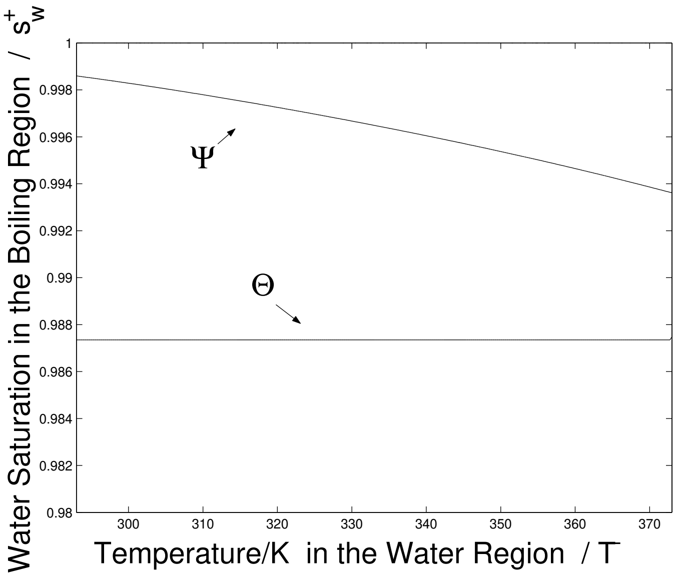


Figure 5.5: The functions $\Theta(T^-; s_w^+)$ and $\Psi(T^-; s_w^+)$ in the $\{T^-, s_w^+\}$.

In the nomenclature of [26], the state $s_w = 1$ is the left-extension of $s_{\dagger\dagger}$ with speed λ_s^b. Also, $s_{\dagger\dagger}$ here coincides with $s_{\dagger\dagger}$ obtained in [6]. This saturation maximizes v^{WES} (and consequently v^{SCF}); we call $s_{\dagger\dagger}$ the "Cold Bifurcation saturation II" or CBII.

Remark 5.6.2. *Notice that from Sec.* 5.5.1, f_w^{WES} *and* s_w^{WES} *are negative if* $T^- < T^\dagger$ *and positive if* $T^- > T^\dagger$ *(see Fig.* 5.2.*b). The solution behavior is the same in both cases.*

Remark 5.6.3. *Notice that from Section* 5.5.1, f_w^{WES} *and* S_w^{WES} *are negative if* $T < T^\dagger$ *and positive if* $T > T^\dagger$ *(see Figure* 5.6*). Nevertheless the solution behavior is the same, i.e., the point* $T^\dagger$ *is not a bifurcation.*

Prop. 5.6.3 yields the following Corollaries used to obtain the solution in the *br*:

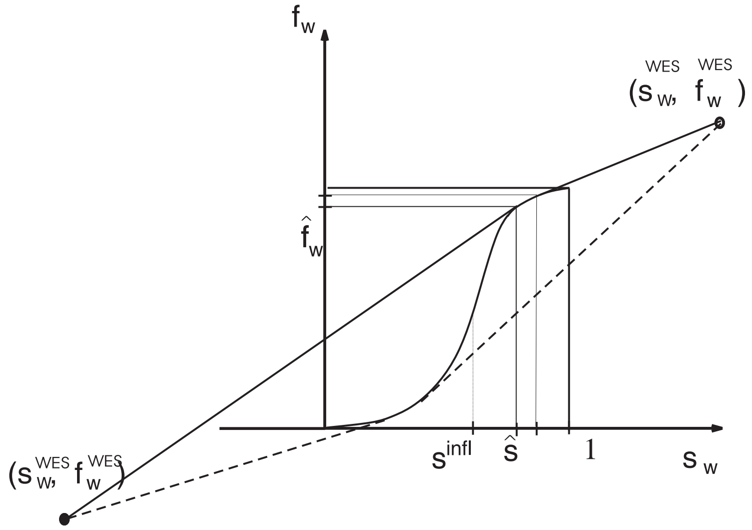


Figure 5.6: We represent two possible points $s_w^{WES}, f_w^{WES})$. If $T < T^\dagger$, both s_w^{WES} and f_w^{WES} are negative; otherwise both are larger than 1.

Corollary 5.6.1. *If $s^{infl}(T^b) < s_w^+ < s_{\dagger\dagger}$, the solution continues in the br as a rarefaction to s_w^+. If $s_w^+ < s^{infl}$ the rarefaction continues to $s^\S$, where $s^\S$ is defined by the second equality in:*

$$v^{\S,+} = \frac{\partial f_w}{\partial s_w}(s^\S, T^b) = \frac{f_w(s_w^+, T^b) - f_w(s^\S, T^b)}{s_w^+ - s^\S}, \tag{5.83}$$

where $s^{infl} = s^{infl}(T^b)$ is the inflection saturation defined by:

$$\left.\frac{\partial^2}{\partial s_w^2} f_w(s_w, T^b)\right|_{s_w = s^{infl}} = 0. \tag{5.84}$$

Proof: From Section 5.4.3 and [6], we know that v_T^w satisfies:

$$\frac{v_T^w \varphi}{u^w} = \frac{C_w}{\hat{C}_r + C_w} < 1,$$

the last inequality holds because $\hat{C}_r$ and C_w are positive, and from Eq. (5.79) in Proposition 5.6.3 we obtain that $v^{WES} < 1$.

If $s_w > s^*$ where s^* satisfies (5.61), the rarefaction wave is always a solution but at s^* we know that:

$$\frac{\partial f_w\,(s^*)}{\partial s_w} > 1.$$

As v^{WES} satisfy (5.80) we obtain that:

$$\frac{\partial f_w(s_{\dagger\dagger}, T^b)}{\partial s_w} < \frac{\partial f_w(s^*, T^b)}{\partial s_w},$$

so if $s^{infl} < s_w^+$ there is a rarefaction from $s_{\dagger\dagger}$ to s_w^+ and if $s_w^+ < s^{infl}$ there is a rarefaction from $s_{\dagger\dagger}$ to $s^{\S}$ that satisfies (5.83) . $\square$

Corollary 5.6.2. *As the left state temperature T^- tends to the water boiling temperature, the water saturation $s_{\dagger\dagger}$ tends to 1, i.e., the limit of $s_{\dagger\dagger}$ lies in the wr.*

Proof: We take the limit $T^- \longrightarrow T^b$ in Eq. (5.38), and obtain:

$$u^b = u^w \frac{s_g \rho_g^b (h_g^b - h_w^g)}{s_g \rho_g^b (h_g^b - h_w^g)} = u^w,$$

because $h_g^b > h_w^g$. Substituting $u^b = u^w$ in Eq. (5.39) and taking the limit $T^- \longrightarrow T^b$:

$$v^{WES} = \frac{u^b}{\varphi} \frac{f_g^b}{S_g}.$$

From Eq. (5.80), after simplifications, we obtain that at $(s_{\dagger\dagger}, T)$:

$$\frac{\partial f_w}{\partial s_w} = \frac{1 - f_w}{1 - s_{\dagger\dagger}}, \tag{5.85}$$

Eq. (5.85) has two solutions, one of which is $s_{\dagger\dagger} = 1$. The other solution $s_{\dagger\dagger,2}$ satisfies $s_{\dagger\dagger,2} < s^{infl}$, so only $s_{\dagger\dagger} = 1$ is important for the solution. $\square$

Solution. Now we can describe the possible solutions for Riemann Data B:

For $s_R > s_{\dagger\dagger}$. As $s_{\dagger\dagger}$ satisfies (5.80), $(u^b/\varphi)\partial f_w(s_R, T^b)/\partial s_w < v^{WES}$, i.e, the shock v^{WES} is faster than the characteristic speed in the *br*, the wave sequence is:

$$L = (1, T_L, u_L) \xrightarrow{WES} (s_R, T^b, u_R) = R. \tag{5.86}$$

For $s^{infl}(T^b) < s_R < s_{\dagger\dagger}$. The waves $WES \twoheadrightarrow R_s$, with sequence:

$$L = (1, T_L, u_L) \xrightarrow{WES} (s_{\dagger\dagger}, T^b, u_R) \xrightarrow{R_s} (s_R, T^b, u_R) = R. \tag{5.87}$$

For $s_R < s^{infl}(T^b)$. The waves $WES \twoheadrightarrow R_s \twoheadrightarrow S_s$ with sequence:

$$L = (1, T_L, u_L) \xrightarrow{WES} (s_{\dagger\dagger}, T^b, u_R) \xrightarrow{R_s} (s^{\S}, T^b, u_R) \xrightarrow{S_s} (s_R, T^b, u_R) = R, \tag{5.88}$$

where $s^{\S}$ is given by Eq. (5.83).

Wave behavior on bifurcations

The Buckley-Leverett rarefaction wave R_S appears on the horizontal dotted line from $s_{\dagger\dagger}$ that separates region I from II; we call this line the "$S_{WES} - R_S$ bifurcation". On the horizontal dotted line from S^{infl} separating the region II from III, a Buckley-Leverett shock S_S appears; we call this line the "$R_S - S_S$ bifurcation".

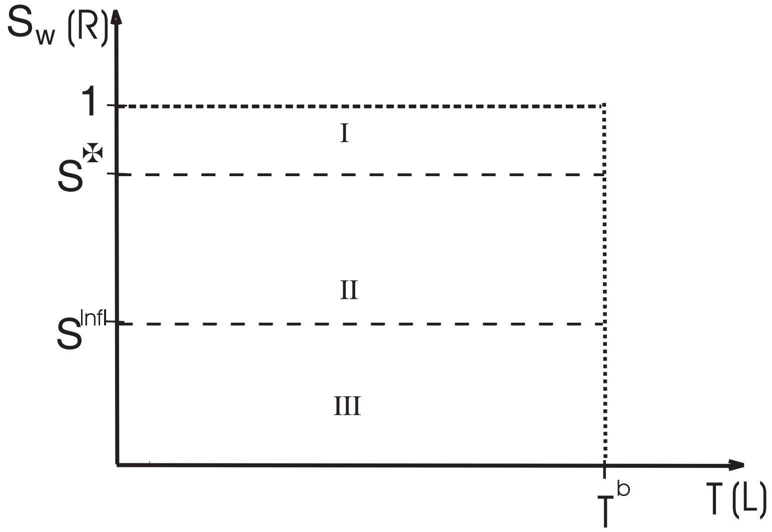


Figure 5.7: The dotted line is the boundary of physical range. The dashed lines divides are the bifurcation loci. The temperature is lower than the water boiling temperature (liquid water) or is equal to water boiling temperature. The saturations $s_{\dagger\dagger}$ and s^{infl} are given in Eqs. (5.66) and (5.62); the horizontal lines from $s_{\dagger\dagger}$ and s^{infl} are the "$S_{WES} - R_S$ bifurcation" and the "$R_S - S_S$ bifurcation".

5.6.3 Riemann Problem C

Let $L = (0, T_L > T^b, u_L)$ and $R = (s_R, T^b, u_R)$. In this Riemann Problem, there are two relevant bifurcation curves. The first bifurcation occurs at the points where the thermal rarefaction speed λ_T^g (Eq. (5.17.a)) equals the shock speed v^{CS} (Eq. (5.57)). The other bifurcation appears at the points where the speed v^{CS} coincides with the Buckley-Leverett rarefaction speed λ_s^b (Eq. (5.30)) in the *br*.

Moreover, there is a point where all previous speeds coincide. It is a double bifurcation point in the $\{T^-; s_w^+\}$ plane of left state temperatures T^- in *sr* and right saturation s_w^+ in *br* (see Sec. 5.6.3). This point is denoted by $(\hat{T}; s^\dagger)$ and it should be understood as the projection of $(s_w = 0, \hat{T}; s^\dagger, T^b)$ onto the $\{T^-; s_w^+\}$ plane.

Definition of $\hat{T}$ and $s^\dagger$

Let $\Upsilon = \Upsilon(T^-; s_w^+)$ be defined as:

$$\Upsilon = \frac{\rho_g^- c_g^-}{\hat{C}_r + \rho_g^- c_g^-} - \left(\frac{u^+}{u^-}\right) \frac{f_w^+ - f_w^{CS}}{s_w^+ - s_w^{CS}}. \tag{5.89}$$

The fraction u^+/u^- is obtained from Eq. (5.47). This fraction does not depend on u^- or u^+, showing that also Υ does not depend on u^- (or u^+). Moreover, at the points $(T^-; s_w^+)$ where $\Upsilon = 0$, the equality $\lambda_T^g = v^{CS}$ holds.

Now we can define the thermal coincidence as the curve where the left thermal wave speed λ_T^g coincides with the condensation shock v^{CS}; it is denoted by TCS locus:

$$TCS = \{(T,s) \mid \Upsilon(T,s) = 0, \text{ for } T \in sr \text{ and } s \in br\}. \tag{5.90}$$

Analogously, we define $\Lambda = \Lambda(T^-; s_w^+)$ as:

$$\Lambda = \frac{f_w^+ - f_w^{CS}}{s_w^+ - s_w^{CS}} - \frac{\partial f_w^b}{\partial s_w}, \tag{5.91}$$

where f_w^{CS} and s_w^{CS} depend on T^-, see (5.46). Now we can define the CSS locus as the curve where $v^{CS} = \lambda_s^b$ and for each T^- fixed, $v^{CS}(T^-; s_w^+)$ is understood as a function of s_w^+ that is minimized (see Prop. 5.6.4) :

$$CSS = \{(T^-; s_w^+) \mid \Lambda = 0 \mid v^{CS} \text{ is minimum; } T^- \in sr, s_w^+ \in br\}. \tag{5.92}$$

In Fig. 5.8, the TCS and CSS loci are shown as curves in the plane $\{T, s\}$. The horizontal axis represents the states in the *sr* and the vertical axis represents the states in the *br*. The two loci intersect transversally at $(\hat{T}; s^\dagger)$, the double bifurcation point "SHB". It can be obtained numerically using root finders. The temperature $\hat{T}$ satisfies $T^b < \hat{T}$, and the saturation $s^\dagger$ satisfies $0 < s^\dagger < 1$.

For the Riemann solution, we need to study the relationships between T_L and $\hat{T}$ at $(s_w = 0, T^-)$, and between s_R and $s^\dagger$ at (s_w^+, T^b) .

Let us define

$$\Xi(T^-; s_w^+) := \frac{f_w^+ - f_w^{CS}}{s_w^+ - s_w^{CS}} := \frac{f_g^+ \rho_g^b \left(h_g^- - h_g^b\right) + f_w^+ \rho_w \left(h_g^- - h_w^b\right)}{\hat{H}_r^- - \hat{H}_r^b + s_g^+ \rho_g^b \left(h_g^- - h_g^b\right) + s_w^+ \rho_w \left(h_g^- - h_w^b\right)} \tag{5.93}$$

the second equality originates from the definition f_w^{CS} and s_w^{CS} given in Eq. (5.41).

Lemma 5.6.6. *The solutions of* (5.92) *are the points where* $\partial\Xi/\partial s_w$ *is zero:*

Proof: We differentiate (5.93) with respect to s_w, equate it to zero and isolate $\partial f_w/\partial s_w$ to obtain the result. We notice that $\partial f_g/\partial s_w = -\partial f_w/\partial s_w$ and $\partial s_g/\partial s_w = -1$. □

Lemma 5.6.7. *For all* $0 \leq s_w \leq 1$ *the following inequality is valid*

$$\frac{f_w}{s_w} \leq \frac{\partial f_w(s^{infl}, \cdot)}{\partial s_w} \tag{5.94}$$

Proof: The extremum for f_w/s_w is obtained for $\partial/\partial s_w(f_w/s_w) = 0$; performing this differentiation, we obtain that $f_w/s_w = \partial f_w/\partial s_w$, which yields

$$\frac{f_w}{s_w} \leq \max_{s_w} \frac{\partial f_w}{\partial s_w} = \frac{\partial f_w(s^{infl}, \cdot)}{\partial s_w}. \qquad \square$$

The *CSS* locus is obtained using the following proposition:

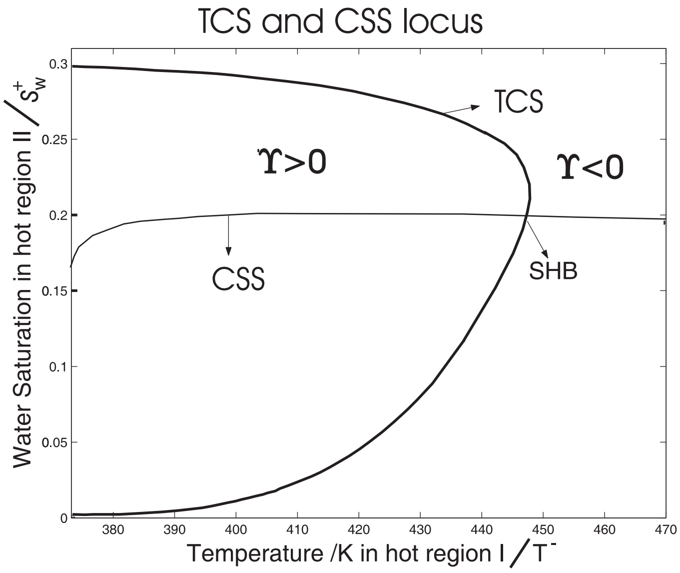


Figure 5.8: Schematic phase space. The intersection of the TCS and CSS loci is the SHB at $(\hat{T}; s^{\dagger})$. The horizontal axis represents the *sr* and the vertical axis represents the *br*, so *SHB* represents two points: $\hat{T}$ is the projection of $(s_w = 0; \hat{T})$ on the *sr*; $s^{\dagger}$ is the projection of $(s_w = s^{\dagger}; T^b)$ on the *br*; between the TCS locus and the water saturation axis $\Upsilon > 0$, so $\lambda_T^g > v^{CS}$; in the complementary region $\lambda_T^g < v^{CS}$.

Proposition 5.6.4. *There are always two water saturation values satisfying* $\Lambda(T, s) = 0$ *for each fixed* $T > T^b$. *The smallest value minimizes* Ξ *(and consequently* v^{CS}*), while the other value maximizes* Ξ *(and* v^{CS}*). We define* $s^{\bullet}$ *as the smaller water saturation value.*

The point $(T, s^{\bullet})$ belongs to CSS locus.

Proof: The function $\partial f_w / \partial s_w$ is continuous and satisfies the following relationship, where $\partial f_w / \partial s_w$ is evaluated at $T = T^b$

$$\frac{\partial f_w (s_w = 0)}{\partial s_w} = \frac{\partial f_w (s_w = 1)}{\partial s_w} = 0. \tag{5.95}$$

From equation (5.93) with $s_w = 0$ it follows that:

$$0 < \frac{\rho_g^b \left(h_g^- - h_g^b \right)}{\hat{H}_r^- - \hat{H}_r^b + \rho_g^b \left(h_g^- - h_g^b \right)} < 1. \tag{5.96}$$

From equation (5.93) with $s_w = 1$ we have:

$$0 < \frac{\rho_w \left(h_g^- - h_w^b \right)}{\hat{H}_r^- - \hat{H}_r^b + \rho_w \left(h_g^- - h_w^b \right)} < 1. \tag{5.97}$$

From Lemma 5.5.2 we know that $\rho_w(h_g^+ - h_w^b > \rho_g^b(h_g^+ - h_g^b)$, from the fact that $f_w > s_w$ we obtain that $f_g < s_g$ and from Lemma 5.6.7 it follows that:

$$\frac{f_g^+ \rho_g^b \left(h_g^- - h_g^b\right) + f_w^+ \rho_w \left(h_g^- - h_w^b\right)}{\hat{H}_r^- - \hat{H}_r^b + s_g^+ \rho_g^b \left(h_g^- - h_g^b\right) + s_w^+ \rho_w \left(h_g^- - h_w^b\right)} <$$
$$< \frac{\partial f_w(s^{infl})}{\partial s_w} \frac{\rho_w(h_g^+ - h_w^b)}{\hat{H}_r^T - \hat{H}_r^b + \rho_w(h_g^+ - h_w^b)} < \frac{\partial f_w(s^{infl})}{\partial s_w}, \tag{5.98}$$

so from the mean value theorem, for each fixed T, there is at least one point between $s_w = 0$ and $s_w = s^{infl}$ and at least another point between $s_w = s^{infl}$ and $s_w = 1$ where $\Xi(T;s)$ vanishes.

We need to show that there exist exactly 2 saturations that satisfy $\Xi(T;s) = 0$. To do so we note that:

$$\frac{\partial^2 f_w(s_w)}{\partial s_w^2} = 0 \quad \Longleftrightarrow \quad s_w = s^{infl}. \tag{5.99}$$

Moreover $\partial^2 f_w / \partial s_w^2$ satisfies:

$$\begin{cases} \frac{\partial^2 f_w(s_w)}{\partial s_w^2} > 0, & \text{if } s_w < s^{infl} \\ \frac{\partial^2 f_w(s_w)}{\partial s_w^2} < 0, & \text{if } s_w > s^{infl}, \end{cases} \tag{5.100}$$

but from Lemma 5.6.6 we know that if s_w satisfies $\Xi(T,s) = 0$, then s_w is a extremum for (5.93), i.e,

$$\Xi = \frac{\partial f_w}{\partial s_w} \quad \Longleftrightarrow \quad \frac{\partial \Xi}{\partial s_w} = 0, \tag{5.101}$$

where Ξ is the function defined in (5.93).

Claim 1: The smaller water saturation for which $\Xi(T,s)$ vanishes is a minimum for $\Xi(T,s)$.

Proof: We differentiate Ξ with respect to s_w and obtain after manipulations that:

$$\frac{\partial \Xi}{\partial s_w} = \frac{B - A}{D^2} \left\{ \frac{\partial f_w}{\partial s_w} - \Xi \right\}, \tag{5.102}$$

where A and B are given in (5.48) and D is the denominator of Ξ. The term D satisfies $D \neq 0$, and from Lemma 5.5.2 we know that $B - A > 0$, so the sign of (5.102) depends only on the sign of:

$$\Delta = \frac{\partial f_w}{\partial s_w} - \Xi, \tag{5.103}$$

where Δ is a continuous function.

From (5.95) and (5.96), we know that $\partial f_w / \partial s_w - \Xi < 0$ for $s_w = 0$ or 1. If $\Xi(T,s) = 0$ is satisfied we obtain that $\partial \Xi / \partial s_w = 0$ and then $\Delta = 0$; on the other hand if $\Delta = 0$ this point satisfies $\Xi(T,s) = 0$ and $\partial \Xi / \partial s_w = 0$.

We know that there are at least 2 water saturations where (5.92) is satisfied. We denote the smaller one by $s^{\bullet}$, so we obtain that $\Delta < 0$ for $0 \leq s_w < s^{\bullet}$.

At the $s^{\bullet}$ where (5.92) is satisfied we obtain that:

$$\frac{\partial \Xi}{\partial s_w} = 0 \qquad \text{and} \qquad \frac{\partial^2 f_w}{\partial s_w^2} > 0, \tag{5.104}$$

so there is a neighborhood $V^{s^{\bullet}}$ that satisfies:

$$\begin{cases} \Xi > \frac{\partial f_w}{\partial s_w}, & \text{if } s_w \in V^{s^{\bullet}} \ / \ s_w < s^{\bullet} \\ \Xi < \frac{\partial f_w}{\partial s_w}, & \text{if } s_w \in V^{s^{\bullet}} \ / \ s_w > s^{\bullet}. \end{cases} \tag{5.105}$$

So for $s \in V^{s^{\bullet}}$, $s > s^{\bullet}$ the graph of v^{CS} lies below the graph of $\partial f_w / \partial s_w$. The point $s^{\bullet}$ minimizes Ξ.

Claim 2: There is another water saturation point $s^{\bullet\bullet}$ that satisfies (5.92); $s^{\bullet\bullet}$ maximizes Ξ and satisfies $s^{\bullet\bullet} > s^{infl}$.

Proof: From Eq. (5.102) and (5.105) we know that Ξ increases for $s_w > s^{\bullet}$. From Eq. (5.105), $\Xi < \partial fw / \partial s_w$ for $s_w \in V^{s^{\bullet}}$ and $s_w > s^{\bullet}$, as Ξ is continuous we obtain that $s^{\bullet\bullet}$ satisfies:

$$s^{\bullet\bullet} > s^{infl}. \tag{5.106}$$

If (5.106) is not satisfied, $s^{\bullet\bullet}$ is a extremum point for Ξ and satisfies:

$$\frac{\partial \Xi \left(s^{\bullet\bullet}\right)}{\partial s_w} = 0 \qquad \text{and} \qquad \frac{\partial^2 f_w \left(s^{\bullet\bullet}\right)}{\partial s_w^2} > 0. \tag{5.107}$$

As $\Xi < \partial f_w / \partial s_w$ for $s_w > s^{\bullet}$, and $\Xi(s^{\bullet\bullet}) = \partial f_w / \partial s_w$, so there is a neighborhood $\hat{V}^{\bullet\bullet}$ of $s^{\bullet\bullet}$ that satisfies:

$$\frac{\partial \Xi}{\partial s_w} > \frac{\partial^2 f_w}{\partial s_w^2} \geq 0 \qquad \text{for } s_w \in \hat{V}^{\bullet\bullet} \ / \ s_w < s^{\bullet\bullet}. \tag{5.108}$$

Taking the limit $s_w \longrightarrow s^{\bullet\bullet}$ in Eq. (5.108) we obtain that:

$$0 = \frac{\partial \Xi}{\partial s_w} > \frac{\partial^2 f_w}{\partial s_w^2} \geq 0. \tag{5.109}$$

For $s^{\bullet\bullet}$ to satisfy Eq. (5.109), it is necessary $s^{\bullet\bullet} = s^{infl}$, but from Eq. (5.98) we obtain that $\Xi(s^{infl}) < \partial f_w(s^{infl}) / \partial s_w$, so $s^{\bullet\bullet} > s^{infl}$.

The point $s^{\bullet\bullet}$ maximizes Ξ, because Ξ increases for $s_w < s^{\bullet\bullet}$ and as $\partial \Xi(s^{\bullet\bullet}) / \partial s_w = 0$ and $\partial^2 f_w(s^{\bullet\bullet}) / \partial s_w^2 < 0$, so there is a neighborhood $V^{s^{\bullet\bullet}}$ where Ξ satisfies:

$$\begin{cases} \Xi < \frac{\partial f_w}{\partial s_w}, & \text{if } s_w \in V^{s^{\bullet\bullet}} \ / \ s_w < s^{\bullet\bullet} \\ \Xi > \frac{\partial f_w}{\partial s_w}, & \text{if } s_w \in V^{s^{\bullet\bullet}} \ / \ s_w > s^{\bullet\bullet}. \end{cases} \tag{5.110}$$

For $s \in V^{s^{\bullet\bullet}}$, $s > s^{\bullet\bullet}$ the graph of Ξ lies above the graph of $\partial f_w/\partial s_w$, so from (5.102) we obtain that:

$$\begin{cases} \frac{\partial \Xi}{\partial s_w} > 0, & \text{if } s_w \in V^{s^{\bullet\bullet}} \ / \ s_w < s^{\bullet\bullet} \\ \frac{\partial \Xi}{\partial s_w} > 0 < 0, & \text{if } s_w \in V^{s^{\bullet\bullet}} \ / \ s_w > s^{\bullet\bullet}. \end{cases} \tag{5.111}$$

The point $s^{\bullet\bullet}$ maximizes Ξ.

Claim 3: There is no other point that satisfies (5.92).

Proof: If this point exists, we denote it by $s^{\bullet\bullet\bullet}$,; it should minimize locally Ξ because $\Xi > \partial f_w/\partial s_w$ for $s_w > s^{\bullet\bullet}$, then from (5.102) we obtain that Ξ decreases. Moreover, Ξ would satisfy:

$$\Xi > \frac{\partial f_w}{\partial s_w}, \ \text{for} \ s^{\bullet\bullet} < s_w < s^{\bullet\bullet\bullet}, \tag{5.112}$$

and as $\partial^2 f_w/\partial s_w^2 < 0$ for all point $s_w > s^{infl}$ (including $s^{\bullet\bullet\bullet}$) and as $\partial\Xi(s^{\bullet\bullet\bullet})/\partial s_w = 0$ there should exist a neighborhood $V^{s^{\bullet\bullet\bullet}}$ where

$$\Xi > \frac{\partial f_w}{\partial s_w} \qquad \text{for} \ s_w \in V^{s^{\bullet\bullet\bullet}} \backslash s^{\bullet\bullet\bullet}. \tag{5.113}$$

So from (5.113), we obtain that Ξ is tangent in $s^{\bullet\bullet\bullet}$, moreover from (5.102)

$$\frac{\partial \Xi}{\partial s_w} < 0, \qquad \text{for} \ s_w \in V^{s^{\bullet\bullet\bullet}}. \tag{5.114}$$

Since $\Xi > \partial f_w/s_w$ for $s_w < s^{\bullet\bullet\bullet}$ and as $\Xi(s^{\bullet\bullet\bullet}) = \partial f_w(s^{\bullet\bullet\bullet})/\partial s_w$ is is necessary that:

$$\frac{\partial \Xi}{\partial s_w} < \frac{\partial^2 f_w}{\partial s_w^2}, \qquad \text{for} \ \hat{V} \subseteqq V^{S^{\bullet\bullet\bullet}} \tag{5.115}$$

where $\hat{V}$ is a neighborhood where (5.115) is satisfied.

Taking the limit $s_w \longrightarrow s^{\bullet\bullet\bullet}$ in (5.115) we obtain

$$0 = \frac{\partial \Xi(s^{\bullet\bullet\bullet})}{\partial s_w} \le \frac{\partial^2 f_w}{\partial s_w^2} < 0. \tag{5.116}$$

so we conclude that there is no other point that satisfies (5.92).

There are only two points that satisfy (5.92) for each fixed T^-. $\square$

From Prop. 5.6.4, one can prove:

Corollary 5.6.3. *For each fixed T in the br, the solution after $s^\bullet$ continues as a rarefaction in the br.*

Proof: From Proposition 5.6.4, $s^\bullet$ minimizes Ξ. From eq. (5.96), we know that

$$\Xi(s_w = 0) < 1, \tag{5.117}$$

So for $s_w = s^\bullet$ we obtain:

$$\frac{\partial f_w}{\partial s_w} = \Xi < 1 < \frac{1 - f_w(s_w^{\ltimes})}{1 - s_w^{\ltimes}} = \frac{\partial f_w(s_w^{\ltimes})}{\partial s_w}, \tag{5.118}$$

$s^{\ltimes}$ is the smallest value where the rarefaction can be sketched in the *br*. $\square$

Prop. 5.6.4 implies that for each fixed temperature T^- there are two saturations that satisfy $\Lambda = 0$, i.e. $\lambda_s^b = v^{CS}$, but we choose the one that belongs to the CSS and denote it by $s^\star$.

In Fig. 5.9.*a*, we obtain the water saturation $s^\star$, for each T^-. We represent this map of T^- into $s^\star$, which is used in Item (2) of the following proposition, as:

$$s^\star := s^\star(T^-). \tag{5.119}$$

Proposition 5.6.5. *If $T^- < \hat{T}$, then $\lambda_T^g > v^{CS}$. Thus there is a shock between $(s_w = 0, T^-, u^-)$ to $(s^\star, T^b, u^+)$. Furthermore, the solution continues in the br as a rarefaction.*

Proof: The proof consist of two steps:

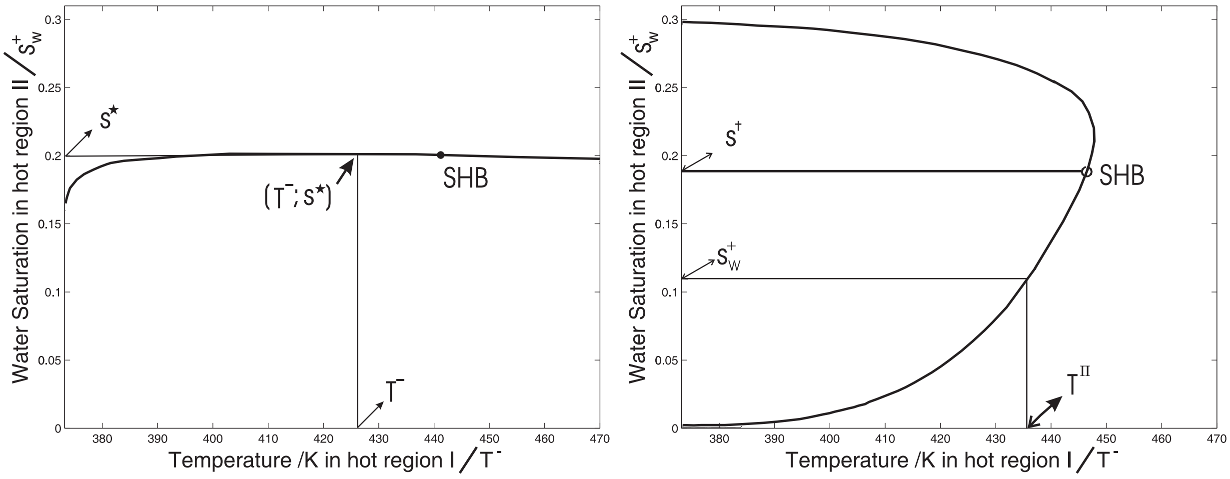


Figure 5.9: *a) Left:* The graph of $\Lambda = 0$. For each T^-, the vertical line crosses the graph at a saturation $s^\star = s^\star(T^-)$. *b)-Right:* The value $T^{\Pi}(s_R)$ is obtained from the TCS locus. We plot a horizontal line from s_w^+, the point where this line intersects the graph Υ is the point $(T^{\Pi}(s_w^+), s_w^+)$.

(1) *The characteristic speed λ_T^g is larger than the shock speed v^{CS} at $(T_L, s^\star)$.* From Figs. 5.9.*a* and 5.8, we can see that $\Upsilon > 0$ at $(T^-, s^\star)$, so $\lambda_T^g > v^{CS}$. Thus there is a shock between $(0, T^-, u^-)$ and $(s^\star, T^b, u^+)$.

(2) *The solution continues as a rarefaction in the br.* As $s^\star$ satisfies $\Lambda = 0$, this fact follows from Cor. 5.6.3. $\square$

Prop. 5.6.5 yields:

Corollary 5.6.4. *When T^- tends to T^b, the saturation $s^\star$ tends to s_{wc}, the connate water saturation (see Appendix A); thus the solution is continuous when T tends to T^b between the sr and the br.*

Proof: Using $\Lambda = 0$, where Λ is given by Eq. (5.91), taking the limit of $T_L \longrightarrow T^b$, we obtain:

$$\frac{f_w(s_w, T^b)}{s_w} = \frac{\partial f_w(s_w, T^b)}{\partial s_w}. \tag{5.120}$$

The smallest value where (5.120) is satisfied is $s_w = s_{wc}$, so $s^\star = s_{wc}$ and the solution is continuous between the *sr* and the *br*. $\square$

Remark 5.6.4. *When $T^- = T^b$, the left state lies in the br. This solution was obtained in [6]. In that case, there was a region with connate water saturation in the br. From Cor.* 5.6.4, *our solution agrees with that in [6]. Notice that the connate water in this region is immobile, but this water evaporates when the vaporization shock advances.*

Fix $s_w^+ < s^\dagger$. Using the TCS locus, we obtain $T^\sqcap(s_w^+)$ as a function of s_w^+. For each s_w^+ we draw a horizontal line. We project the intersection of this horizontal line and the TCS onto the horizontal axis to obtain $T^\sqcap$; we denote this mapping by:

$$T^\sqcap := T^\sqcap(s_w^+). \tag{5.121}$$

It is important that $T^\sqcap(s_w^+)$ is monotone for $s_w^+ < s^\dagger$.

Proposition 5.6.6. *For $s_w^+ < s^\dagger$ and $T^- > \hat{T}$, there is a rarefaction from $(0, T^-, u^-)$ to $(0, T^\sqcap, u^\sqcap)$. At $(0, T^\sqcap, u^\sqcap)$ the following speeds coincide:*

$$\lambda_T^g(T^\sqcap, u^\sqcap) = v^{CS}(T^\sqcap, u^\sqcap; s_w^+), \tag{5.122}$$

so there is a left characteristic shock between $(0, T^\sqcap, u^\sqcap)$ and (s_w^+, T^b, u^+) with speed v^{CS}.

Proof: In Fig. 5.9.*b*, we plot an example of s_w^+ and its respective $T^\sqcap(s_w^+)$. Since the temperature decreases from left to right along the thermal rarefaction wave, from Rem. (5.4.3), this wave is a rarefaction. $\square$

Corollary 5.6.5. *When s_w^+ tends to* 0 *in br, the temperature $T^\sqcap$ converges to T^b; thus the solution is continuous between the br and the sr.*

Proof: Taking the limit of $s_w^+ \longrightarrow 0$ in Eq. (5.89) we obtain:

$$\Upsilon(s_w = 0) = \frac{\rho_g c_g}{\hat{C}_r + \rho_g c_g} - \frac{\rho_g^b\left(h_g^- - h_g^b\right)}{\rho_g^b\left(h_g^- - h_g^b\right) + \hat{C}_r}. \tag{5.123}$$

We multiply the second term on the right hand side by $(T^- - T^b)/(T^- - T^b)$ and we obtain:

$$\Upsilon(s_w = 0) = \frac{\rho_g c_g}{\hat{C}_r + \rho_g c_g} - \frac{\rho_g^b\left(h_g^- - h_g^b\right)/\left(T^- - T^b\right)}{\rho_g^b\left(h_g^- - h_g^b\right)/\left(T^- - T^b\right) + \hat{C}_r}. \tag{5.124}$$

Taking the limit $T^- \longrightarrow T^b$ in (5.124), we obtain:

$$\lim_{T^- \longrightarrow T^b} \Upsilon(S_w = 0) = \frac{\rho_g^b c_g^b}{\hat{C}_r + \rho_g^b c_g^b} - \lim_{T^- \longrightarrow T^b} \frac{\rho_g^b \left(h_g^- - h_g^b\right) / \left(T^- - T^b\right)}{\rho_g^b \left(h_g^- - h_g^b\right) / \left(T^- - T^b\right) + \hat{C}_r} =$$
$$= \frac{\rho_g^b c_g^b}{\hat{C}_r + \rho_g^b c_g^b} - \frac{\rho_g^b c_g^b}{\hat{C}_r + \rho_g^b c_g^b} = 0. \tag{5.125}$$

Thus $T^- = T^b$ belongs to the TCS locus. Moreover, Eq. (5.123) defines a curve with $S_R = 0$ and T^- in the *sr*; we can prove that this curve is monotonic (see Figure 5.10), therefore this is the only solution. $\square$

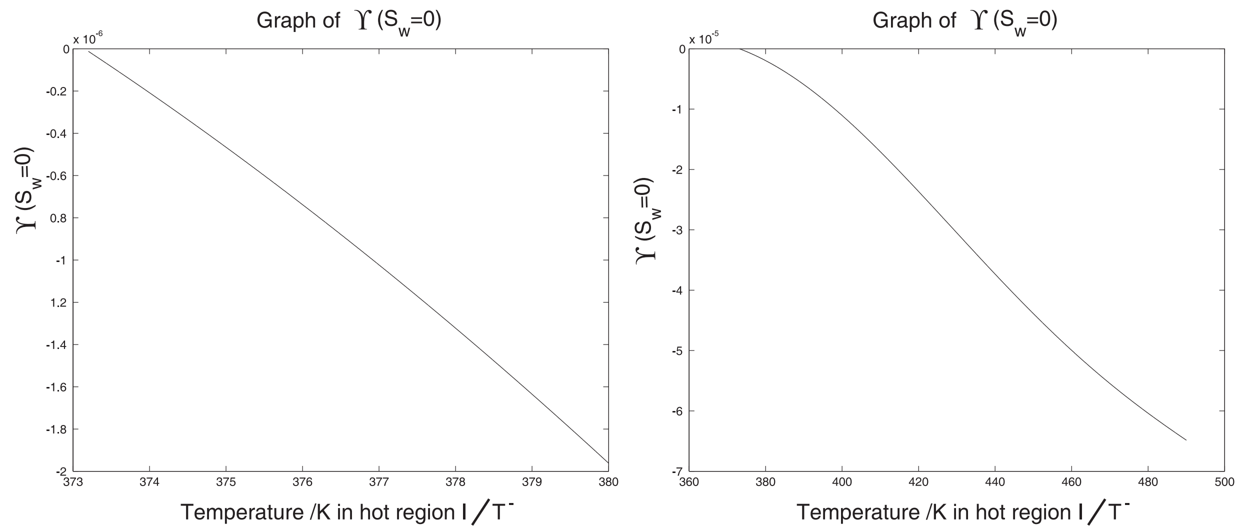


Figure 5.10: Left: plot of Eq. (5.123) for $T \in [373.15K, 380K]$. Right: the same plot for $T \in [373.15K, 490K]$.

In Eq. (5.121), we find $s^\star = s^\star(T^-)$; also $T^\Pi = T^\Pi(s_w^+)$ from Eq. (5.119), see Fig. 5.9.*b*. Using $s^\star$ and T^Π, four possible solution candidates arise:

(i) $T^- < T^\Pi$, $s^\star < s_w^-$. A shock from $(0, T^-, u^-)$ to $(s^\star, T^b, u^+)$, continuing to (s_w^+, T^b, u^+) through a Buckley-Leverett rarefaction.

(ii) $T^- < T^\Pi$, $s^\star > s_w^+$. A shock from $(0, T^-, u^-)$ to (s_w^+, T^b, u^+) with speed v^{CS}.

(iii) $T^- > T^\Pi$, $s^\star < s_w^+$. A rarefaction from $(0, T^-, u^-)$ to $(0, T^\Pi, u^\Pi)$ followed by a shock to $(s^\star, T^b, u^+)$ continuing to (s_w^+, T^b, u^+) through a Buckley-Leverett rarefaction.

(iv) $T^- > T^\Pi$, $s^\star > s_w^+$. A rarefaction from $(0, T^-, u^-)$ to $(0, T^\Pi, u^\Pi)$, followed by a shock to $(s^\star, T^b, u^+)$ with speed v^{CS}.

Proposition 5.6.7. *For left temperature T^- and fixed right saturation s_w^+ satisfying $T^- < \hat{T}$ and $s_w^+ < s^\dagger$, (i) and (iii) do not occur.*

Proof: From Fig. 5.11.*a*, we separate the water saturation in the *br* (the right side states of the Riemann problem) in two intervals, s_{RegI} and s_{RegII}. The first one is $s_w^+ \in [s_{wc}, s^\dagger,]$; the second one is $s_w^+ \in [s_w = 0, s_{wc}]$. Recall Eq. (5.121), which defines T^Π as function of s_w^+. So we define $T_{RegI} = T^\Pi(s_{RegI})$; similarly, we define $T_{RegII} = T^\Pi(s_{RegII})$. One can verify from Fig. 5.11.*a* that $T^\Pi(s_w^+)$ is monotone increasing.

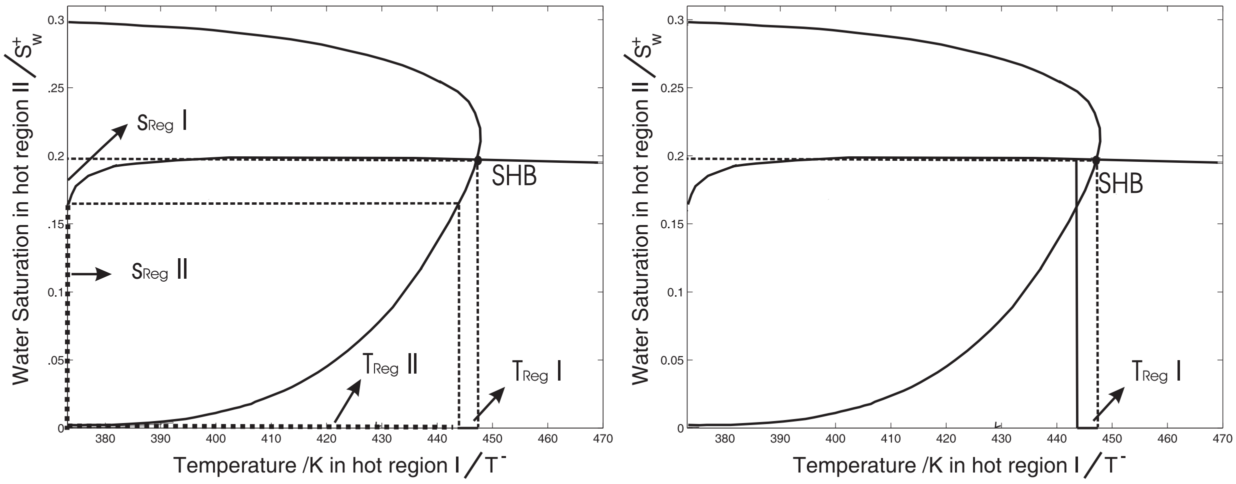


Figure 5.11: *a)-Left:* The map T^Π defines T_{RegI} and T_{RegII}. The saturation lies in the *br*, which is subdivided in $s_{RegI} = [s_{wc}, s^\dagger]$ and in $s_{RegII} = [0, s_{wc}]$. The corresponding intervals in the *sr* are T_{RegI} and T_{RegII}. *b)-Right:* Mapping from T_{RegI} to *br*. Each $T^- \in T_{RegI}$ defines a $s^\star$ in the *br* (see Prop. 5.6.6); notice that $s^\star$ satisfies $s^\star \geq s^\dagger$, which is the saturation at SHB.

Fig. 5.11.*b* represents the mapping from T_{RegI} to the *br*. Each value of $T^- \in T_{RegI}$ defines a value for $s^\star$ in the *br* through the function $s^\star(T^-)$ in Eq. (5.119). The function $s^\star(T^-)$ is not monotone and all values of $s^\star = s^\star(T^-)$ for $T^- \in T_{RegI}$ are larger or equal to $s^\dagger$, the saturation at the SHB point. Notice also that the mapping $s^\star(T^-)$ for $T^- \in T_{RegI} \bigcup T_{RegII}$ is onto s_{RegI}. So we obtain that $s_w^+ < s^\star(T^-)$ for $T^- \in T_{RegI}$, so (*i*) and (*iii*) do not occur. $\square$

Solution: (summarized in Fig 5.13).

(I) For $T_L > \hat{T}$ and $s_R > s^{infl} > s^\dagger$. The waves $R_T \twoheadrightarrow CS \twoheadrightarrow R_s \twoheadrightarrow S_s$ with sequence:

$$L = (0, T_L, u_L) \xrightarrow{R_T} (0, \hat{T}, \hat{u}) \xrightarrow{CS} (s^\dagger, T^b, u_R) \xrightarrow{R_s} (s^\S, T^b, u_R) \xrightarrow{S_s} (s_R, T^b, u_R) = R, \quad (5.126)$$

where $(\hat{T}, s^\dagger)$ is the SHB point and $s^\S$ is given by Eq. (5.83).

(II) For $T_L > \hat{T}$ and $s^{infl} > s_R > s^\dagger$. The waves $R_T \twoheadrightarrow CS \twoheadrightarrow R_s$ with sequence:

$$L = (0, T_L, u_L) \xrightarrow{R_T} (0, \hat{T}, \hat{u}) \xrightarrow{CS} (s^\dagger, T^b, u_R) \xrightarrow{R_s} (s_R, T^b, u_R) = R. \quad (5.127)$$

(III) For $T^\Pi < T_L$ and $s_R < s^\star(T_L)$. The solution is sketched in Fig. 5.12.*a*, based on Props. 5.6.7 and 5.6.6; the waves are $R_T \twoheadrightarrow CS$ with sequence:

$$L = (0, T_L, u_L) \xrightarrow{R_T} (0, T^\Pi < \hat{T}, u^\Pi) \xrightarrow{CS} (s_R, T^b, u_R) = R. \quad \square \quad (5.128)$$

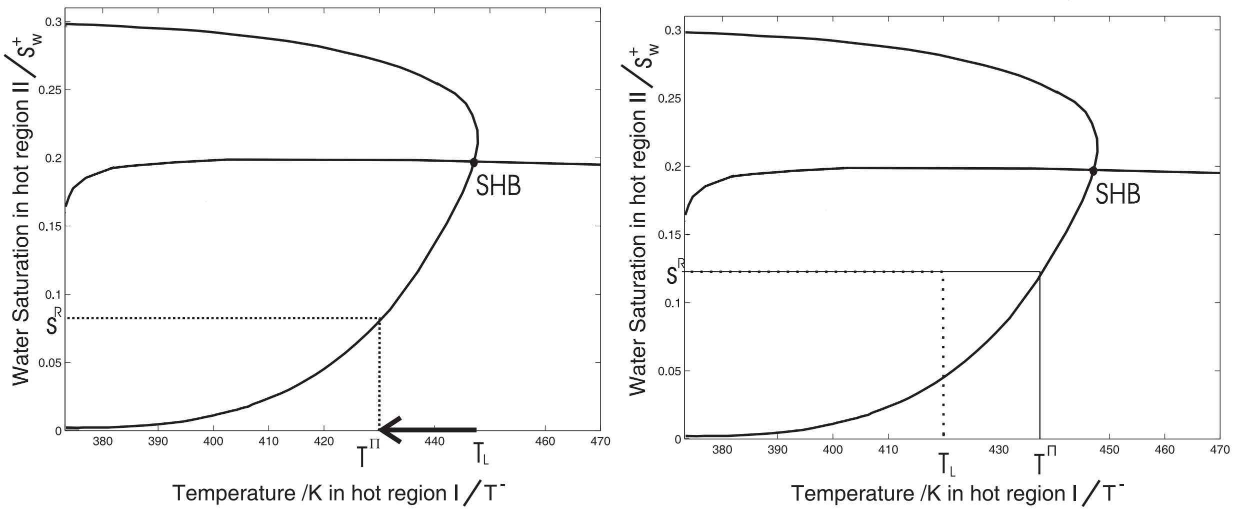


Figure 5.12: *a)-Left: Riemann Solution*. the point T_L in the horizontal axis represents $(s_L = 0, T_L)$. The rarefaction from $(s_L = 0, T_L)$ to $(s_L = 0, T^{\Pi})$ is represented by a line and an arrow to indicate the direction of increasing speed; the rarefaction is followed by a shock from $(s_w = 0, T^{\Pi})$ to (s_R, T^b) with speed v^{CS} with construction shown by dotted lines. *b)-Right:* the dotted line represents the shock from $(s_L = 0, T_L, u_L)$ to (s_R, T^b, u_R) with speed v^{CS}. We draw the solution for a left state $(s_L = 0, T_L)$, which we represent by T_L.

(IV) For $T^{\Pi} > T_L$ and $s_R < s^\star(T_L)$, the solution is sketched in Fig. 5.12.*b*. It is the wave CS with sequence:

$$L = (0, T_L, u_L) \xrightarrow{CS} (s_R, T^b, u_R) = R. \tag{5.129}$$

(V) For $T_L < \hat{T}$ and $s^{infl} > s_R > s^\star(T_L)$, where the mapping $s^\star = s^\star(T_L)$ is defined in Eq. (5.119). We obtain the waves $CS \twoheadrightarrow R_s$ with sequence:

$$L = (0, T_L, u_L) \xrightarrow{CS} (s^\star, T^b, u_R) \xrightarrow{R_s} (s_R, T^b, u_R) = R. \tag{5.130}$$

(VI) For $T_L < \hat{T}$ and $s_R > s^{infl} > s^\dagger$. See (5.130). The waves $CS \twoheadrightarrow R_s \twoheadrightarrow S_s$ with sequence:

$$L = (0, T_L, u_L) \xrightarrow{CS} (s^\star, T^b, u_R) \xrightarrow{R_s} (s^\S, T^b, u_R) \xrightarrow{S_s} (s_R, T^b, u_R) = R, \tag{5.131}$$

where $s^\S$ is given by Eq. (5.83) with $s_w^+ = s_R$.

Remark 5.6.5. *We remark that the CS is a double sonic transitional wave, see* [58].

Wave behavior on bifurcations

Notice that the Buckley-Leverett shock S_S disappears on the line that separates I from II; we call this line "$R_S - S_S$ bifurcation". On the line that separates I from VI, the thermal rarefaction R_T in *sr* disappears; we call this line the "$R_T - S_{CS}$ bifurcation". On the line that separates II and V, the thermal rarefaction R_T in the *br* also disappears, so this line is the "$R_T - S_{CS}$ bifurcation". On the line that separates V from VI the Buckley-Leverett shock S_S disappears, so this line is the "$R_S - S_S$ bifurcation". The "$R_T - S_{CS}$ bifurcation" is the vertical line from SHB separating I, II from V, VI and the "$R_T - S_{CS}$ bifurcation" is the horizontal line from S^{infl} separating I, VI from II, V.

On the line that separates II from III the Buckley-Leverett rarefaction R_S disappears; we call this line the "$R_S - S_{CS}$ bifurcation". On the curve that separates III from IV, the thermal rarefaction R_T disappears; we call it the"$R_T - S_{SC}$ bifurcation". Finally, on the curve that separates IV from V the Buckley-Leverett rarefaction R_S disappears; we call it the "$S_{SCS} - R_S$ bifurcation".

5.6.4 Riemann Problem D

We inject pure water at temperature $T_L < T^b$, i.e, the left state is $(0, T_L, u_L)$; on the right we have pure steam at temperature $T_R > T^b$.

Before describing our proposed Riemann solution, it is necessary to prove that there is no possible Riemann solution with a direct shock between *sr* and the *wr*. Using the RH condition (5.36)-(5.37) with $s_w^+ = 0$ and T^b replaced by $T^+ > T^b$, we obtain this hypothetical "complete water evaporation shock", labelled $CWES$, with speed v^{CWES}. The superscript $+$ $(-)$ in the following equations represents the temperature

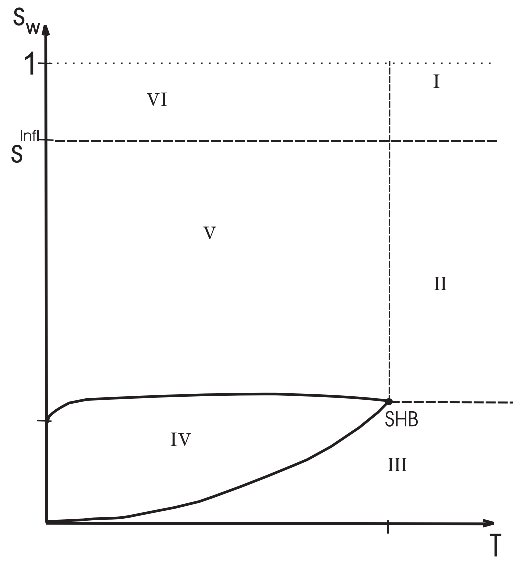


Figure 5.13: Phase diagram for Riemann Problem C. The dotted line delimits the physical range, the dashed lines are bifurcation loci. The continuous curves are parts of the TCS and the CSS bifurcations loci. The horizontal axis represents the left states $(0, T_L, u_L)$ in *sr*; the vertical axis represents the right states (s_R, T^b, u_R) in *br*. The solutions are given in *I-VI*, Sec. 5.6.3.

T^+ (T^-). The speed of such shock between $(s_w = 1, T^-, u^w)$ to $(s_w = 0, T^+ > T^b, u)$ would be:

$$v^{CWES}(T^-, u^-; s_w^+ = 0, T^+) = \frac{u^w}{\varphi} \frac{\rho_w(h_g^+ - h_w^-)}{H_r^+ - H_r^- + \rho_w(h_g^+ - h_w^-)}. \tag{5.132}$$

Proposition 5.6.8. *Complete Evaporation. For any $T^- < T^b < T^+$, if there exists a complete water evaporation shock from $(1, T^-, u^-)$ to $(0, T^+ > T^b, u^+)$ with speed $v^{CWES}(T^-, u^-; s_w^+ = 0, T^+)$ given by (5.132), then this shock satisfies:*

$$v^{CWES} > v_T^w, \tag{5.133}$$

where v_T^w is the speed of thermal discontinuity given by Eq. (5.32).

Proof: The speed in the liquid water region v_T^w is written as:

$$v_T^w = \frac{u^w}{\varphi} \frac{C_w}{\hat{C}_r + C_w} = \frac{u^w}{\varphi} \frac{C_w}{\hat{C}_r + C_w} \frac{(T^+ - T^-)}{(T^+ - T^-)} = \frac{u^w}{\varphi} \frac{\rho_w(h_w^+ - h_w^-)}{H_r^+ - H_r^- + \rho_w(h_w^+ - h_w^-)}.$$

Using v^{CWES} given by (5.132) and the relationship $\rho_w(h_g^+ - h_w^-) > \rho_w(h_w^+ - h_w^-)$, it follows that $v^{CWES} > v_T^w$. $\square$

If this shock exists, it would not satisfy the Oleinik condition for entropy [53]. Therefore we conclude that instead of a shock there is a *br* between the $(-)$ and $(+)$ state. The solution is constructed using results from Sections 5.6.1 and 5.6.2. Since $s_w^+ = 0$, from Sec. 5.6.2 we obtain:

Proposition 5.6.9. *The saturation $s_{\dagger\dagger}$ in* (5.80) *is larger than $\hat{s}$ (defined in* (5.62)*), thus there is a rarefaction from* $(s_{\dagger\dagger}, T^b, u^+)$ *to* $(\hat{s}, T^b, u^+)$.

Proof: For $s_w = 1$, we obtain from Eq. (5.39) and (5.58) that v^{WES} and v^{VS} satisfy:

$$v^{WES} = \frac{u^b}{\varphi}\frac{\rho_w(h_w^b - h_w^-)}{\hat{H}_r^b - \hat{H}_r^- + \rho_w(h_w^b - h_w^-)} = \frac{u^b}{\varphi}\frac{C_w}{\hat{C}_r + C_w},$$

$$v^{VS} = \frac{u^b}{\varphi}\frac{\rho_w(h_g^+ - h_w^b)}{\hat{H}_r^+ - \hat{H}_r^b + \rho_w(h_g^+ - h_w^b)};$$

we can show as in Proposition 5.6.8 for $s_w = 1$ that

$$v^{VS} > v^{WES}. \tag{5.134}$$

The inequality Eq. (5.134) is satisfied for s_w, since that $f_w \geq s_w$.

We know that $s_{\dagger\dagger}$ is given by (5.80) and for this saturation $f_w(s_{\dagger\dagger}) > s_{\dagger\dagger}$, so Eq. (5.134) is satisfied from $s_w = 1$ to at least $s_w = s_{\dagger\dagger}$. As $\partial f_w/\partial s_w$ increases for s_w decreasing from $s_w = 1$ to s^{infl}, the water saturation $s_{\dagger\dagger}$ that satisfies (5.80) is larger than $\hat{s}$, where $\hat{s}$ satisfies (5.62), because $f_w \geq s_w$ and (5.134) is satisfied. □

Corollary 5.6.6. *The solution in the br consists of a rarefaction from* $(s_{\dagger\dagger}, T^b, u^+)$ *to* $(\hat{s}, T^b, u^+)$.

Proof: As $s_{\dagger\dagger} > \hat{s}$ and from Section 5.6.1, we know that the solution consists of a rarefaction from $s_w = 1$ to $s_w = \hat{s}$ in the *br*, we conclude that exists a rarefaction from $s_{\dagger\dagger}$ to $\hat{s}$ in this region. □

Solution. The solution consists of the waves $WES \twoheadrightarrow R_s \twoheadrightarrow CS$ with sequence:

$$L = (1, T_L, u_L) \xrightarrow{WES} (s_{\dagger\dagger}, T^b, u_R) \xrightarrow{R_s} (\hat{s}, T^b, u_R) \xrightarrow{SC} (0, T_R, u_R) = R. \tag{5.135}$$

5.6.5 Riemann Problem E

We inject pure steam at temperature $T_L > T^b$, i.e, $L = (0, T_L > T^b, u_L)$; on right we have pure water at $T_R < T^b$, i.e., the right state is $R = (1, T_R < T^b, u_R)$. We use results from Section 5.6.4 to obtain the solution.

We remark that the vaporization shock, VS, is the reverse of the condensation shock, CS, so there could exist a hypothetical "complete condensation shock", labelled CCS, with speed v^{CCS}. This shock would be obtained using the RH condition (5.43)-(5.44) with the $(-)$ state replaced by $(sw = 1, T^+ < T^b, u^+)$, and the $(+)$ state replaced by $(s_w^- = 0, T^- > T^b, u^-)$. The speed of such shock would be:

$$v^{CCS}(T^-, u^-; s_w^+ = 1, T^+) = \frac{u^-}{\varphi}\frac{\rho_g^-(h_g^+ - h_w^-)}{H_r^+ - H_r^- + \rho_g^+(h_g^+ - h_w^-)}. \tag{5.136}$$

The following fact indicates that instead of this shock, there is always a *br* between the *sr* and the *wr*.

Proposition 5.6.10. *Complete Condensation. For any $T^- \geq T^b$ the complete condensation shock from $(s_w = 0, T^-, u^-)$ to $(s_w = 1, T^+, u^+)$ with speed $v^{CCS}(T, u^-; s_w^+ = 1)$ satisfies:*

$$v^{CCS} > \lambda_T^g(T^-, u^-). \tag{5.137}$$

Proof: We divide the numerator and denominator of (5.136) by $(T^+ - T^-)$:

$$\begin{aligned} v^{CCS}(s_w = 1) &= \frac{u}{\varphi} \frac{\left(\rho_g^+(h_g^+ - h_w^-)\right) / (T^+ - T^-)}{\hat{C}_r + \left(\rho_g^+(h_g^+ - h_w^-)\right) / (T^+ - T^-)} \\ &> \frac{u}{\varphi} \frac{\left(\rho_g^+(h_g^+ - h_g^-)\right) / (T^+ - T^-)}{\hat{C}_r + \left(\rho_g^+(h_g^+ - h_g^-)\right) (T^+ - T^-)} > \frac{u}{\varphi} \frac{\rho_g^+ c_g^+}{\hat{C}_r + \rho_g^+ \rho_g^+} = \lambda_T^g, \end{aligned} \tag{5.138}$$

the inequality above are valid because $h_g^0 > h_w^0$ and $c_g' = \partial^2 h_g / \partial T^2 < 0$ for $T^- > T^b$. □

Corollary 5.6.7. *There is a br between the states $(s_w = 0, T^-, u^-)$ and $(s_w = 1, T^0, u^w)$.*

Proof: From Proposition 5.6.10, there exists a *br* between $(s_w = 0, T^-, u^-)$ and $(s_w = 1, T^0, u^w)$, because if this region did not exist, as Eq. (5.137) is satisfied it would be possible to sketch a rarefaction from $(s_w = 0, T^-, u^-)$ to $(s_w = 0, T^b, u^b)$ in the *br*. □

In [6], Bruining et al. obtained the solution for injection of water and steam at boiling temperature in a porous rock with water at temperature T^0. In that work, the water and rock enthalpies were made to vanish at temperature T^0, so the formulae for shock speeds are slightly different from formulae in present paper; however both are equivalent since the enthalpy is defined in up to a constant. In that paper, two saturations denoted by $s_\dagger$ and $s_{\dagger\dagger}$ were found both satisfying Eq. (5.80); the choice for $s_{\dagger\dagger}$ was also made. In [6], cases I and II are correct, but there are some mistakes that influence the solution in case III. The first relevant mistake is the statement that the saturation $S_\dagger$ maximizes v^{SCF} (Remark 11). The correct statement is:

Proposition 5.6.11. *There are two saturation values that satisfy Eq. (3.7) in [6] for each fixed $T < T^b$. The smallest $S_\dagger$ minimizes v^{SCF}. The other $S_{\dagger\dagger}$ maximizes v^{SCF}, but it is irrelevant.*

Moreover the solution is stable if we vary the left state, it follows that:

Corollary 5.6.8. *In the limit as S_L tends to zero, the wave speed $v_{g,w}^b$ in the br given by Eq. (2.8) in [1] converges to zero, so the solution for the Riemann problem reduces a cooling discontinuity in the liquid water region.*

From Proposition 5.6.11 and Corollary 5.6.8, we see that the Figures 3.1 and 3.4 in [6] contain an error; we correct them in Figure 5.14.

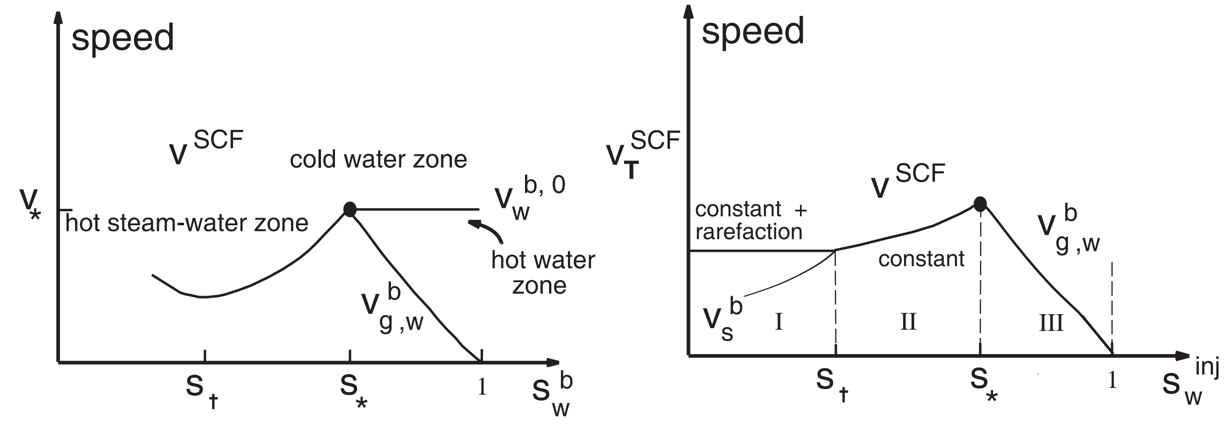
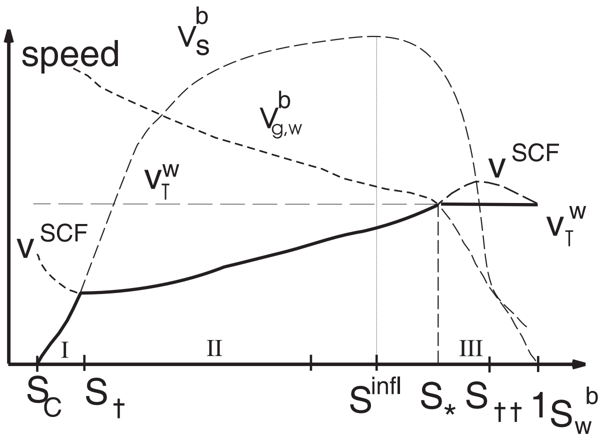


Figure 5.14: *a)-Left:* Corrected schematic bifurcation near S^* (for fixed u^0, T^0 versus S_w^b. The shock speed SCF is minimum at $S_\dagger$; the shock speed $v_{g,w}^b$ tends to zero when S_w^b tends to 1. *b)-Right:* The corrected structure of steam-water zone below solid curve marked by v_s^b, v_T^{SCF}, v^{SCF} given in Eqs (2.7), (2.24) with $S_w = S_\dagger$, Eq. (2.24) with $S_\dagger < S^{inj} < S_*$, and Eq. (2.8) respectively. The figures are not drawn to scale.

The wave speed diagram in Figure 4.5 of [6] contains an error also. The correct diagrams are given in Figure 5.15. There are two diagrams because the saturation shock speed and the thermal shock are different for injected saturations larger than S_*. These diagrams are drawn out of scale for illustrative purposes, because the characteristic speed v_s^g can be much larger than the other wave speeds. Figs. 3.1 and 3.4 in [6] contain an error also; those figures were summarized in Figure 4.5 in [6], which is corrected here in Figure 5.15. The Riemann solution for the cases I and II are correct, however the Riemann solution for case III contains an error. The mistake is the statement that the saturation shock speed λ_s^b is faster than v^{SCF} in cases I and II. The correct statement is that $v_{g,w}^b$ speed converges to zero if the water saturation at the left state tends to zero, as summarized in Corollary 5.6.8 above. The thermal wave speed is drawn in Figure 5.15.a) and in 5.15.b) we draw the saturation wave speed.



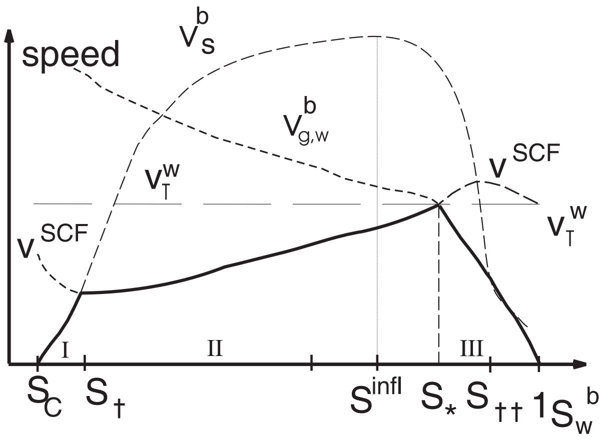


Figure 5.15: *a)-Left:* The diagram represents the thermal wave speed; *b)-Right:* The diagram represents the saturation wave speed. They correct the diagram presented in Figure 4.5 of [6]. In both, the solid curve represents the wave speeds used in the Riemann solution. The characteristic speed v_s^b, Eq. (2.7), is the Buckley-Leverett characteristic speed in the *br*; the shock speed $v_{g,w}^b$ is the HISW speed, Eq. (2.8), and represents the Buckley-Leverett shock for a water saturation s_w to $s_w = 0$; v^{SCF} is the condensation shock speed between the *br* and the *wr*, Eq. (2.24); $v_w^{b,0}$ is the cooling contact discontinuity speed in the liquid water region, Eq. (2.15). Notice that the temperature shock speed tends to v_T^W when water saturation tends to $S_w = 1$ (liquid water region) and the saturation shock speed tends to zero, thus the solution converges to the solution in the liquid water region.

The solution diagram for case III given in Figure 4.4 of [6] must be modified. The strength of saturation shock tends to zero when the injection saturation tends to 1, while the speed of cooling discontinuity does not change. In Figure 5.16 we show the schematic solution for case III for three different injection saturations.

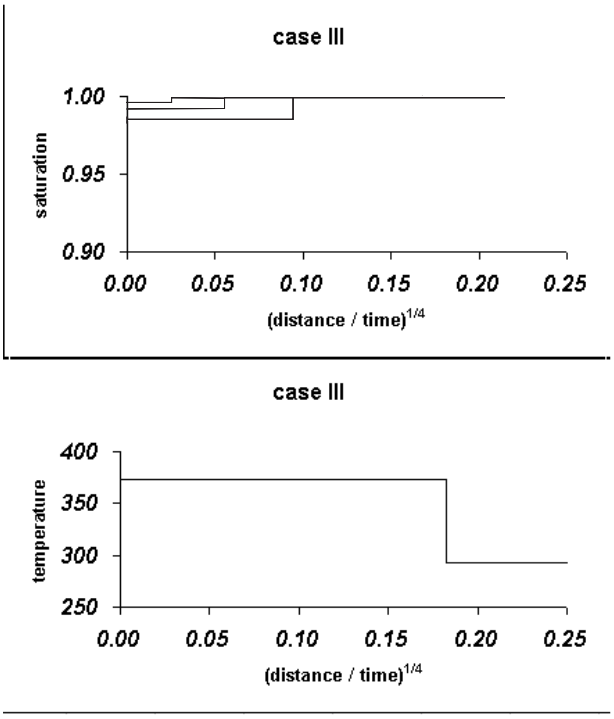


Figure 5.16: Steam injection for three high water saturation values. We get a constant state upstream, the Buckley-Leverett saturation shock and the cooling discontinuity. These waves have distinct speeds; notice that $v^b_{g,w}$ tends to zero when the injection saturation tends to 1, while the speed of cooling discontinuity does not change.

To complete the Riemann solution, we obtain a relationship between $s^\star$ and $s^\dagger$:

Proposition 5.6.12. *The water saturation $s^\star$, defined in* (5.119)*, obtained in the br by a shock from $(0, T^-, u^-)$ to $(s^\star, T^b, u^+)$ satisfies the following inequality:*

$$s^\star < s_\dagger,$$

so from $(s^\star, T^b, u^+)$ the solution continues as a rarefaction to $(s_\dagger, T^b, u^+)$.

Proof: We analyze the relationship between the shock speeds v^{CS} and v^{SCF}. It is possible show that v^{SCF} is faster than v^{CS}. Since $(T^-, s^\star)$ belongs to CSS so v^{CS} satisfies:

$$v^{CS} = \frac{u^b}{\varphi} \frac{\partial f_w(s^\star, T^b)}{\partial s_w},$$

and v^{SCF} satisfies Eq. (5.80), we obtain that:

$$\frac{\partial f_w(s^\star, T^b)}{\partial s_w} < \frac{\partial f_w(s_\dagger, T^b)}{\partial s_w},$$

as $\partial f_w / \partial s_w$ is monotone increasing for $s_{wc} < s_w < s_w^{infl}$, we obtain the result. □

Solution. We obtain the Riemann solution using the results in [6], [39] and (5.131), (5.126).

For $T_L < \hat{T}$**.** As in (5.131), there is a shock from $L = (0, T^-, u^-)$ to $(s^\star, T^b, u^+)$ where $s^\star = s^\star(T_L)$ is given by (5.119) and $s_\dagger$ satisfies Eq. (5.80). The waves are $CS \twoheadrightarrow R_s \twoheadrightarrow SCF$ with sequence:

$$L = (0, T_L, u_L) \xrightarrow{CS} (s^\star, T^b, u_R) \xrightarrow{R_s} (s_\dagger, T^b, u_R) \xrightarrow{SCF} (s_R, T^b, u_R) = R. \tag{5.139}$$

For $T_L > \hat{T}$**.** The waves $R_T \twoheadrightarrow CS \twoheadrightarrow R_s \twoheadrightarrow SCF$ with sequence:

$$L = (0, T_L, u) \xrightarrow{R_T} (0, \hat{T}, u) \xrightarrow{CS} (s^\dagger, T^b, u^b) \xrightarrow{R_s} (s_\dagger, T^b, u^b) \xrightarrow{SCF} (s, T < T^b, u_R) = R, \tag{5.140}$$

where $(\hat{T}, s^\dagger)$ is the SHB point.

CHAPTER 6

Riemann solution for problem of Chapter 4 for V_L in III and IV.

Here we utilize results of Chapter 4 and Chapter 5. First we can identify subregions $\mathcal{L}_5$ and $\mathcal{L}_6$ in $III \bigcup IV$. The only way to reach *spl* region from III and IV is by a Buckley-Leverret shock, see Section 4.9.3. Notice that the boundary between III and IV is the coincidence curve $\lambda_s = \lambda_e$, in which λ_s and λ_e are given by Eqs. (4.115) and (4.117). Moreover, there exists a coincidence curve between the evaporation wave and the hot isothermal steam water, HISW, with speed $v^b_{g,w}$ and given by Eq. (5.28); we denote this coincidence curve by $\mathcal{C}_{BLE}$, see Fig. 6.1.

In next Riemann solutions, s represents the water saturation.

6.1 V_L in $\mathcal{L}_6$

In this region, HISW is slower than the evaporation wave, so we can reach the *spl* directly by a HISW. It is easy to prove that:

Lemma 6.1.1. *The following inequalities are valid for* (s_w, T) *in* $\mathcal{L}_6$*:*

$$v^b_{g,w}(s_w, T) < v^{WES}(s_w = 1, T; s_{\dagger\dagger}) < v^w_T, \tag{6.1}$$

where $s_{\dagger\dagger}$ *is obtained from Eq.* (5.80).

The Riemann solution consists of the waves $S^{BL} \mapsto WES \twoheadrightarrow R_s \twoheadrightarrow CS$ with sequence:

$$L = (s_L, T_L, u_L) \xrightarrow{BL} (1, T_L, u_L) \xrightarrow{WES} (s_{\dagger\dagger}, T^b, u_R) \xrightarrow{R_s} (\hat{s}, T^b, u_R) \xrightarrow{SC} (0, T_R, u_R) = R. \tag{6.2}$$

6.2 V_L in $\mathcal{L}_5$

In this region, HISW is faster than the evaporation wave. From a state in $\mathcal{L}_5$, we reach $\mathcal{C}_{BLE}$ using a condensation rarefaction. From this state in the coincidence curve, we can reach the *spl* directly by a HISW.

The Riemann solution consists of the waves $R^e \twoheadrightarrow S^{BL} \mapsto WES \twoheadrightarrow R_s \twoheadrightarrow CS$ with sequence:

$$L = (s_L, T_L, u_L) \xrightarrow{R_s} (s, T, u) \xrightarrow{BL} (1, T, u) \xrightarrow{WES} (s_{\dagger\dagger}, T^b, u_R) \xrightarrow{R_s} (\hat{s}, T^b, u_R) \xrightarrow{SC} (0, T_R, u_R) = R, \tag{6.3}$$

where (s, T) are the states in the coincidence curve $\mathcal{C}_{BLE}$ obtained from the evaporation rarefaction wave from (s_L, T_L) and u is the respective speed.

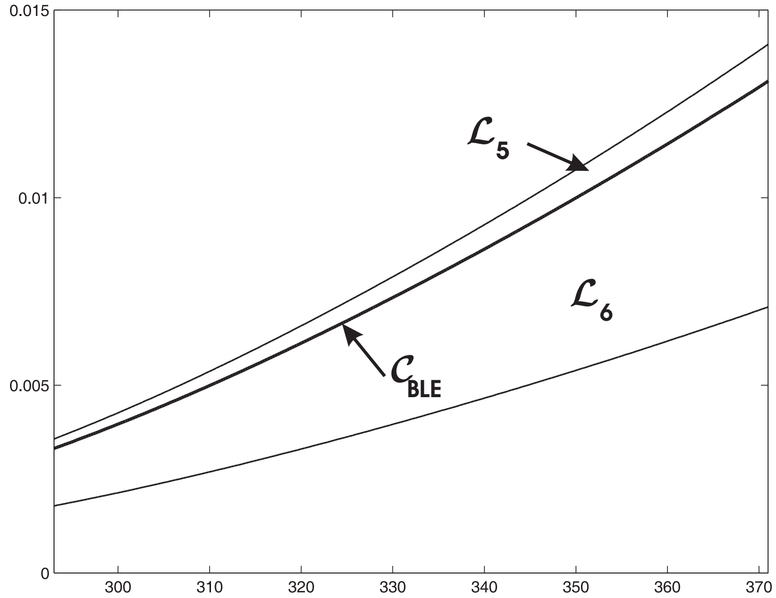


Figure 6.1: Subregions $\mathcal{L}_5$ and $\mathcal{L}_6$. $\mathcal{C}_{BLE}$ is the coincidence curve $v^b_{g,w} = \lambda_e$. For states in $\mathcal{L}_5$, we have that $v^b_{g,w} < \lambda_e$; for states in $\mathcal{L}_6$, we have that $v^b_{g,w} > \lambda_e$.

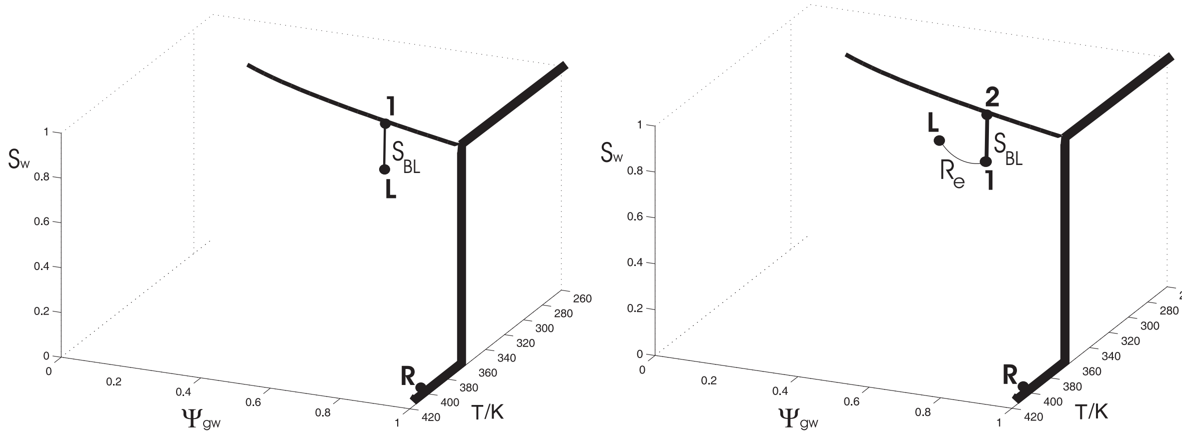


Figure 6.2: Riemann solutions in phase space, omitting the surface shown in Fig. 4.1. *a) Left:* Solution (6.2) for $V_L \in \mathcal{L}_6$, Sec. 6.1. *b) Right:* Solution (6.3) for $V_L \in \mathcal{L}_5$, Sec. 6.2. The numbers 1 and 2 indicate intermediate states V in wave sequence. The waves after *spl* situation are not represented in this figure, but can be obtained in the previous Chapter.

CHAPTER 7

Summary and Conclusions

We have described a global formalism for balance laws of form (2.1). We show a systematic theory for the Riemann solution for general Riemann problems for a wide class of balance equations with phase changes.

We have also obtained the solution of the Riemann problem for the injection of a mixture nitrogen/steam/water into a porous rock filled with steam above boiling temperature. We show a systematic theory for the Riemann solution for a 3×3 balance system. The set of solutions depends L^1 continuously on the Riemann data.

We have also described completely all possible solutions of the Riemann problem for the injection of a mixture of steam and water in several proportions and temperature into a core filled with a different mixture of steam and water in all proportions, (of course, the temperature must be lower than the thermodynamical critical temperature of water). The set of solutions depends L^1 continuously on the Riemann data. We found several types of shock between regions and systematized a scheme to find the solution from these shocks. A new type of shock, the evaporation shock, was identified.

As a future work we would like obtain the Riemann solution for the problem described in Chapter 3. We are interested also in considering the gravitational and capillarity pressure effects and the effects of pressure changes in terms of physical properties.

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APPENDIX A

Physical quantities; symbols and values for the Nitrogen Problem

Table 2, Summary of physical input parameters and variables			
Physical quantity	*Symbol*	*Value*	*Unit*
Water, steam fractional functions	f_w, f_g	Eq. (3.15) .	[m^3 /m^3]
Porous rock permeability	k	1.0×10^{-12}.	[m^3]
Water, steam relative permeabilities	k_{rw}, k_{rg}	Eq. (A.10).	[m^3 /m^3]
Water, steam end point permeabilities	k'_{rw}, k'_{rg}	Eq. $(A.10.a)$, $(A.10.b)$.	[m^3]
Pressure	p_{at}	1.0135×10^5.	[Pa]
Water Saturation Pressure	p^{sat}	Eq. $(A.7)$.	[Pa]
Water, steam phase velocity	u_w, u_g	Eq. (3.14) .	[m^3 /(m^2 s)]
Total Darcy velocity	u	$u_w + u_g$, Eq. (3.16) .	[m^3 /(m^2 s)]
Effective rock heat capacity	C_r	2.029×10^6.	[J /(m^3 K)]
Steam and nitrogen enthalpies	h_{gW}, h_{gN}	Eqs. $(A.3)$, $(A.4)$.	[J /m^3]
Water enthalpy	h_W	Eq. $(A.2)$.	[J /m^3]
Rock enthalpy	H_r	Eq. $(A.1)$.	[J /m^3]
Water, steam saturations	s_w, s_g	Dependent variables.	[m^3 /m^3]
Connate water saturation	s_{wc}	0.15.	[m^3 /m^3]
Temperature	T	Dependent variable.	[K]
Water, steam thermal conductivity	κ_w , κ_g	0.652 , 0.0208.	[W/(mK)]
Rock, composite thermal conductivity	κ_r , κ	1.83 , Eq. (4.9) .	[W/(mK)]
Water, steam viscosity	μ_w , μ_g	Eq. (A.5) , Eq. (A.6) .	[Pa s]
Steam and nitrogen densities	ρ_{gw}, ρ_{gn}	Eq. $(A.8.a)$, $(A.8.b)$.	[kg/m^3]
Constant water density	ρ_W	998.2.	[kg/m^3]
Steam and nitrogen gas composition	ψ_{gw}, ψ_{gn}	Dependent variables.	$[-]$
Universal gas constant	R	8.31	$[J/mol/K]$
Nitrogen and water molar masses	M_N, M_W	0.28, 0.18	$[kg/mol]$
Rock porosity	φ	0.38.	[m^3 /m^3]

A.1 Temperature dependent properties of steam and water

The rock enthalpy H_r and $\hat{H}_r$ are given by:

$$H_r = (1-\varphi)C_r(T-\bar{T}) \qquad \text{and} \qquad \hat{H}_r = H_r/\varphi. \tag{A.1}$$

The water enthalpy per mass unit h_W is:

$$h_W = C_W T/\rho_w. \tag{A.2}$$

The steam enthalpy $h_{gW}[J/kg]$ as a function of temperature is approximated by

$$\begin{aligned} h_{gW}(T) = &-2.20269\times 10^7 + 3.65317\times 10^5 T - 2.25837\times 10^3 T^2 + 7.3742T^3 \\ &- 1.33437\times 10^{-2}T^4 + 1.26913\times 10^{-5}T^5 - 4.9688\times 10^{-9}T^6 - \overline{h}_w. \end{aligned} \tag{A.3}$$

The nitrogen enthalpy $h_{gN}[J/kg]$ as a function of temperature is approximated by

$$h_{gN}(T) = 975.0T + 0.0935T^2 - 0.476\times 10^{-7}T^3 - \overline{h}_{gN}. \tag{A.4}$$

These constants enthalpies $\overline{h}_w$ and $\overline{h}_{gN}$ are chosen such that $h_w(T)$, $h_{gN}(T)$ vanish at a reference temperature $\overline{T} = 293K$.

The temperature dependent liquid water viscosity μ_w $[Pas]$ is approximated by

$$\mu_w = -0.0123274 + \frac{27.1038}{T} - \frac{23527.5}{T^2} + \frac{1.01425\times 10^7}{T^3} - \frac{2.17342\times 10^9}{T^4} + \frac{1.86935\times 10^{11}}{T^5}. \tag{A.5}$$

We assume that that the viscosity of the gas is independent of the composition.

$$\mu_g = 1.8264\times 10^{-5}\left(\frac{T}{T^b}\right)^{0.6}. \tag{A.6}$$

The water saturation pressure as a function of temperature is given as

$$\begin{aligned} p^{sat} = 10^3(&-175.776 + 2.29272T - 0.0113953T^2 + 0.000026278T^3 \\ &-0.0000000273726T^4 + 1.13816\times 10^{-11}T^5)^2 \end{aligned} \tag{A.7}$$

The corresponding concentrations ρ_{gw}, ρ_{gn} are calculated with the ideal gas law:

$$\rho_{gw} = \frac{M_W\, p^{sat}}{RT},\ \rho_{gn} = \frac{M_N\ \left(p_{at} - p^{sat}\right)}{RT}, \tag{A.8}$$

The pure phase densities are:

$$\rho_{gW}(T) = \frac{M_W p_{at}}{RT}, \qquad \rho_{gN}(T) = \frac{M_N p_{at}}{RT}, \tag{A.9}$$

where the gas constant $R = 8.31[J/mol/K]$, M_W and M_N are the the nitrogen and water molar mass.

For simplicity the liquid water density is assumed to be constant at $998.2.kg/m^3$.

A.2 Constitutive relations

The relative permeability functions k_{rw} and k_{rg} are considered to be power functions of their respective saturations, i.e.

$$k_{rw} = \begin{cases} k'_{rw}\left(\frac{s_w - s_{wc}}{1 - s_{wc} - s_{gr}}\right)^{n_w} \\ 0 \end{cases}, k_{rg} \begin{cases} k'_{rg} = \left(\frac{s_g - s_{gr}}{1 - s_{wc} - s_{gr}}\right)^{n_g} & \text{for} \quad s_{wc} \leq s_w \leq 1, \\ 1 & \text{for} \quad s_w < s_{wc}. \end{cases} \tag{A.10}$$

For the computations we take $n_w = 2$ and $n_g = 2$. The end point permeabilities k'_{rw}, k'_{rg} are 0.5 and 0.95 respectively. The connate water saturation s_{wc} is given in the table.

APPENDIX B

Physical quantities; symbols and values for the steam injection

B.1 Temperature dependent properties of steam and water

We use reference [1] to obtain all the temperature dependent properties below. The water and steam densities used to obtain the enthalpies are defined at the bottom.

The steam enthalpy h_g [J/kg] as a function of temperature is approximated by

$$\begin{aligned} h_g = &-2.20269 \times 10^7 + 3.65317 \times 10^5 T - 2.25837 \times 10^3 T^2 + 7.3742 T^3 \\ &-1.33437 \times 10^{-2} T^4 + 1.26913 \times 10^{-5} T^5 - 4.9688 \times 10^{-9} T^6. \end{aligned} \tag{B.1}$$

We also use the temperature dependent steam viscosity

$$\begin{aligned} \mu_g = &-5.46807 \times 10^{-4} + 6.89490 \times 10^{-6} T - 3.39999 \times 10^{-8} T^2 + 8.29842 \times 10^{-11} T^3 \\ &-9.97060 \times 10^{-14} T^4 + 4.71914 \times 10^{-17} T^5. \end{aligned} \tag{B.2}$$

The temperature dependent water viscosity μ_w is approximated by

$$\mu_w = -0.0123274 + \frac{27.1038}{T} - \frac{23527.5}{T^2} + \frac{1.01425 \times 10^7}{T^3} - \frac{2.17342 \times 10^9}{T^4} + \frac{1.86935 \times 10^{11}}{T^5}. \tag{B.3}$$

We assume that the steam is a ideal gas, so the steam density is a function of temperature:

$$\rho_g(T) = p \frac{M_{H_2O}}{R} \frac{1}{T}, \tag{B.4}$$

where M_{H_2O} is the water molecular mass $[Kg/m^3]$, p is the pressure atmospheric [Pa] and R=8.31 [J/mol K]. The quantity pM_{H_2O}/R is a constant.

The liquid water density is constant, and the value is $998.2Kg/m^3$.
We define $\hat{H}_r$ and the water enthalpy per mass unit h_w respectively as:

$$\hat{H}_r(T) = (1-\varphi)/\varphi C_r T \qquad \text{and} \qquad h_w = C_w T/\rho_w. \tag{B.5}$$

B.2 Constitutive relations

The relative permeability functions k_{rg} and k_{rw} are considered to be power functions of their respective saturations *i.e.*

$$k_{rg} = \left(\frac{s_g}{1-s_{wc}}\right)^{n_g} \quad \text{and} \quad k_{rw} \quad = \begin{cases} \left(\frac{s_w - s_{wc}}{1-s_{wc}}\right)^{n_w} & \text{for} \quad s_w \geq s_{wc}\,, \\ 0 & \text{for} \quad 0 \leq s_w \leq s_{wc}, \end{cases} . \tag{B.6}$$

For the computations we take $n_w = 4, = n_g = 2$. The connate water saturation s_{wc} is given in Table 2 below.

Table 2, Summary of physical input parameters and variables

Physical quantity	*Symbol*	*Value*	*Unit*
Water, steam fractional functions	f_w, f_g	Eq. (5.6) .	[m^3/m^3]
Porous rock permeability	k	1.0×10^{-12}.	[m^3]
Water, steam relative permeabilities	k_{rw}, k_{rg}	Eq. (B.6) .	[m^3/m^3]
Pressure	p	1.0135×10^5.	[Pa]
Mass condensation rate	q	Eqs (5.1)-(5.2).	[kg /(m^{3}s)]
Water, steam phase velocity	u_w, u_g	Eq. (5.5) .	[m^3/(m^{2}s)]
Total Darcy velocity	u	$u_w + u_g$, Eq (5.7).	[m^3/(m^{2}s)]
Water and rock heat capacity	C_w, C_r	4.22×10^6, 2.029×10^6.	[J/(m^{3}K)]
Steam and water enthalpies	h_g, h_w	Eqs. $(B.1)$, $(B.5.b)$.	[J/m^3]
Rock enthalpy	H_r	$(1-\varphi)C_r T$.	[J/m^3]
Water, steam saturations	s_w, s_g	Dependent variables.	[m^3/m^3]
Connate water saturation	s_{wc}	0.15.	[m^3/m^3]
Temperature	T	Dependent variable.	[K]
Boiling point of water–steam	T^b	$\simeq 373.15$.	[K]
Water, steam thermal conductivity	κ_w, κ_g	0.652, 0.0208.	[W/(mK)]
Rock, composite thermal conductivity	κ_r, κ	1.83 .	[W/(mK)]
Water, steam viscosity	μ_w, μ_g	Eqs. (B.3) , (B.2).	[Pa s]
Water, steam densities	ρ_w, ρ_g	998.2, Eq. (B.4) .	[kg/m^3]
Rock porosity (constant)	φ	0.38.	[m^3/m^3]

B.3 Approximation for $T + M_{\rho_g} c_g(T)/C_r$ in the sr

As the temperature in the sr changes only from $T^b = 373,15K$ to around $T = 400K$, from the expression for c_g we can approximate $T + (M_{\rho_g} h_g)/\hat{C}_r$ by a linear expression

$aT+b$ (or, more generally, the by $f(T)$), where a e b are given in $(B.8)$.

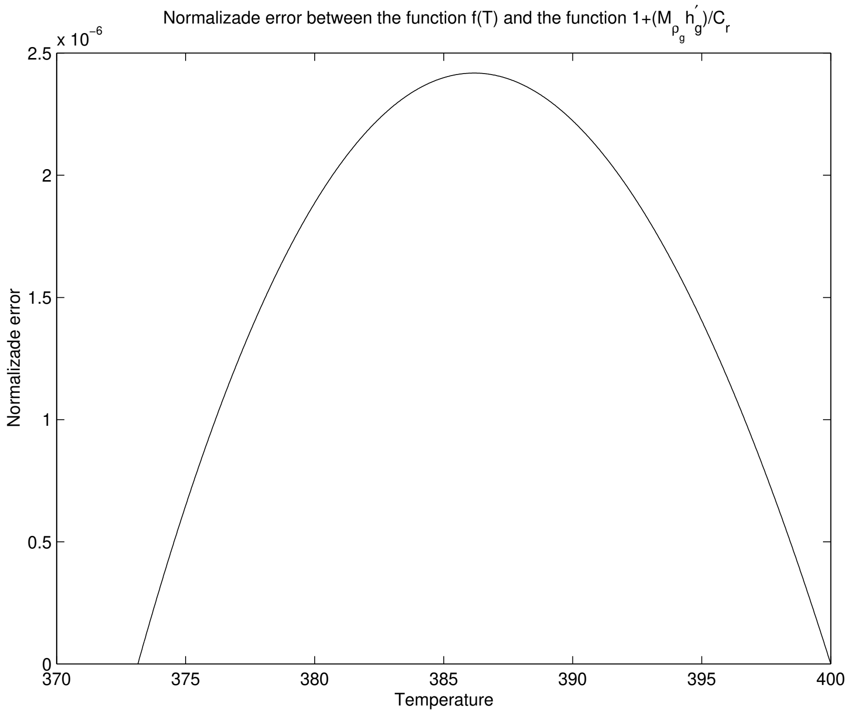
$$f(T) = aT + b. \tag{B.7}$$

where

$$a = \frac{C_{400} - C_{T^b}}{400 - T^b} \qquad \text{and} \qquad b = \frac{400C_{T^b} - T^bC_{400}}{400 - T^b}, \tag{B.8}$$

and

$$C_{400} = 400 + \frac{M_{\rho_g} c_g(400)}{C_r} \qquad C_{T^b} = T^b + \frac{M_{\rho_g} c_g(T^b)}{C_r}. \tag{B.9}$$



Solving $(B.9)$ and $(B.8)$ to find a and b we have

$$a \cong 0.99952824228873 \qquad \text{and} \qquad b \cong 0.28502504243858. \tag{B.10}$$

Solving (5.18) with linear denominator $aT+b$ (see Appendix B) given by $(B.7)$, we have

$$u = u^0 \sqrt[a]{aT+b}, \tag{B.11}$$

where u^0 is the speed at the beginning of the rarefaction wave.

We use here $(B.7)$ in λ_T^g and $\vec{r}_T$. After manipulations, we have

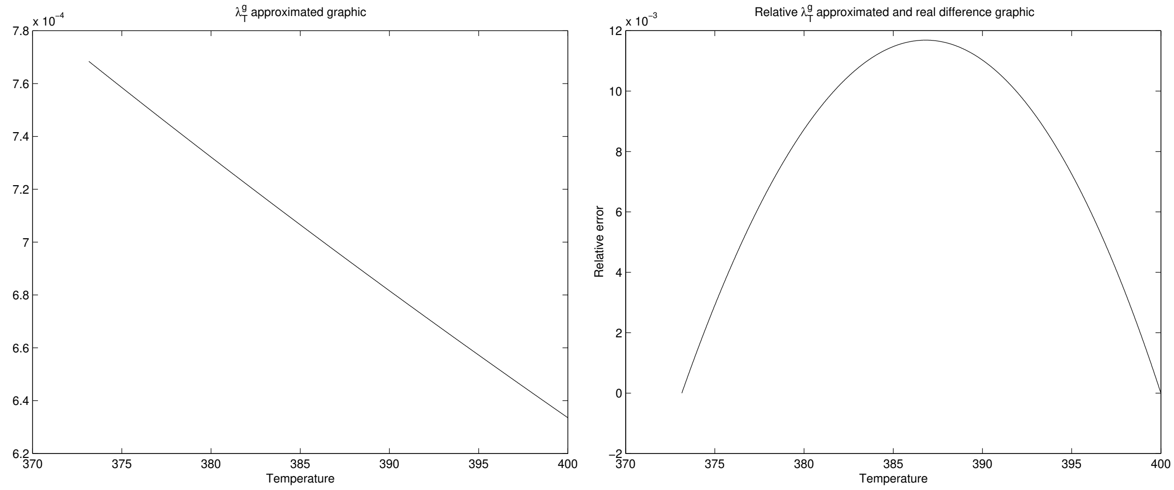
$$\lambda_T^g \cong \frac{u}{\varphi} \frac{M_\rho c_g(T)}{\hat{C}_r(aT+b)}, \quad \vec{r}_T \cong (1, \frac{u}{aT+b}) \quad \text{and} \quad c_g(T) = \frac{((a-1)T+b)\hat{C}_r}{M_{\rho_g}}, \tag{B.12}$$

where a e b are given in Eq. $(B.7)$.

Finally, from λ_T and $c_g(T)$ given in $(B.12)$ we have

$$\lambda_T^g \cong \frac{u}{\varphi}\frac{(a-1)T+b}{(aT+b)}. \tag{B.13}$$

So from $(B.12)$ and $(B.13)$, $(B.16)$ can be written as



$$\nabla\lambda_T^g \cdot \vec{r}_T = \frac{u}{\varphi}\frac{(a-1)T}{(aT+b)^2}, \tag{B.14}$$

Thus $\nabla\lambda_T^g \cdot \vec{r}_T$ is negative for temperatures from $373.15K$ to $400K$, so along rarefaction curves defined by (5.17.a) there are only decreasing temperatures; physically this means that we are injecting steam at low temperature into the steam at higher temperatures.

Thus the inflection locus in the plane (T,u) consists of two parts

$$u = 0 \qquad \text{and} \qquad T = 0, \tag{B.15}$$

which are physically irrelevant.

B.3.1 Rarefaction wave behavior

Lemma B.3.1. *If T decreases, so u decreases on rarefaction curve in direction where the rarefaction exists.*

Proof: As du/dT satisfies $(5.18.a)$, so

$$\frac{du}{dT} > 0,$$

because all terms on the right hand side in $(5.18.a)$ are positive.

So, if T increases then u increases; similarly, if T decreases then u decreases. From Appendix B, equation $(5.4.3)$ T decreases along the rarefaction curve, so u decreases on the rarefaction curve. $\square$

We study the rarefaction wave behavior. In this study, we find the region when λ_T increases and when it decreases along integral curves, and the inflection locus where λ_T is stationary. To do so, we need to calculate

$$\nabla \lambda_T^g \cdot \vec{r}_T. \tag{B.16}$$

We use λ_T^g and $\vec{r}_T$ given by $(5.17.a)$ and $(5.17.b)$, respectively, in Eq. $(B.16)$:

$$\nabla \lambda_T^g \cdot \vec{r}_T = \frac{u}{\varphi} \frac{M_{\rho_g} \hat{C}_r T c_g'(T)}{(\hat{C}_r T + M_{\rho_g} c_g(T))^2}, \tag{B.17}$$

so in this case we also have that λ_T^g decreases when the temperature increases.

APPENDIX C

Applications

In Section $C.1$ we show that the formalism developed to class of equations (2.2) can be applied to a hyperbolic system of form:

$$\frac{\partial \hat{G}(V)}{\partial t} + u\frac{\partial \hat{F}(V)}{\partial x} = 0, \qquad u = \text{const.} \tag{C.1}$$

as a particular case. In Section $C.2$, we show an application to numerical method; in Section $C.3$ we find the travelling waves independently on the variable u. We will drop hats.

C.1 Hyperbolic waves in a physical situation

A very nice property of the formalism developed for shocks and rarefaction structures of Eq. (2.2) is that it can be applied to the hyperbolic system in the form $(C.1)$, illustrating that it is a particular system of the more general class (2.2).

To do so, we consider the system $(C.1)$ with n state variables, where the cumulative terms are $G = (G_1, G_2, \cdots, G_n)$ and the flux terms are $F = (F_1, F_2, \cdots, F_n)$.

Assume that there is no degeneracies. The RH condition for Eq. $(C.1)$ is:

$$v^s[G_i] = u[F_i] \qquad \text{for} \quad i = 1, 2, \cdots, n, \tag{C.2}$$

where $[G_i] = G_i(V^+) - G_i(V^-)$ and $[G_i] = F_i(V^+) - F_i(V^-)$. The RH locus satisfies:

$$[G_i][F_j] = [G_j][F_i] \qquad \text{for all} \quad i, j = 1, 2, \cdots, n. \tag{C.3}$$

If $[G_1] \neq 0$ and $[F_1] \neq 0$, then the system of equations in $(C.3)$ reduces to:

$$[G_1][F_j] = [G_j][F_1] \qquad \text{for} \quad j = 2, 3, \cdots, n. \tag{C.4}$$

The eigenvalues λ and right eigenvectors r of Eq. $(C.1)$ are obtained by solving (2.30) with A and B given by:

$$B = \begin{pmatrix} \frac{\partial G_1}{\partial V_1} & \frac{\partial G_1}{\partial V_2} & \cdots & \frac{\partial G_1}{\partial V_n} \\ \frac{\partial G_2}{\partial V_1} & \frac{\partial G_2}{\partial V_2} & \cdots & \frac{\partial G_2}{\partial V_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial G_n}{\partial V_1} & \frac{\partial G_n}{\partial V_2} & \cdots & \frac{\partial G_n}{\partial V_n} \end{pmatrix} \text{ and } A = \begin{pmatrix} u\frac{\partial F_1}{\partial V_1} & u\frac{\partial F_1}{\partial V_2} & \cdots & u\frac{\partial F_1}{\partial V_n} \\ u\frac{\partial F_2}{\partial V_1} & u\frac{\partial F_2}{\partial V_2} & \cdots & u\frac{\partial F_2}{\partial V_n} \\ \vdots & \vdots & \vdots & \vdots \\ u\frac{\partial F_n}{\partial V_1} & u\frac{\partial F_n}{\partial V_2} & \cdots & u\frac{\partial F_n}{\partial V_n} \end{pmatrix}.$$

We can rewrite $(C.1)$ in the form (2.2). To do so, we assume that u is the secondary variable and V are the n primary variables. The system can be written as:

$$\frac{\partial G(V)}{\partial t} + \frac{\partial u F(V)}{\partial x} = 0, \tag{C.5}$$

$$\frac{\partial u}{\partial x} = 0, \tag{C.6}$$

where Eq. $(C.1)$ represents the n first equations and $(C.6)$ yields that u is constant in the space. (Notice that in the classical hyperbolic equations $u \equiv 1$).

The matrix $\mathcal{M}$ defined in (2.54) for the system $(C.5)$-$(C.6)$ is:

$$\begin{pmatrix} [G_1] & -F_1^+ & F_1^- \\ [G_2] & -F_2^+ & F_2^- \\ \vdots & \vdots & \vdots \\ [G_n] & -F_n^+ & F_n^- \\ 0 & -1 & 1 \end{pmatrix}. \tag{C.7}$$

Assuming the same hypotheses of Corollary 2.3.1 and without loss of generality assuming that $p = 1$ and $q = 2$, i.e., $\mathcal{D}_1$ and $\mathcal{D}_2$ are L.I., the RH locus are the solutions of

$$det \begin{pmatrix} [G_1] & -F_1^+ & F_1^- \\ [G_2] & -F_2^+ & F_2^- \\ [G_i] & -F_i^+ & F_i^- \end{pmatrix} = 0, \tag{C.8}$$

for $i = 3, 4, \cdots, n+1$, where $G_{n+1} = 0$ and $F_{n+1} = 1$.

Notice that these are $n-2$ restrictions. For $i = n+1$ in Eq. $(C.8)$, we need to solve:

$$det \begin{pmatrix} [G_1] & -F_1^+ & F_1^- \\ [G_2] & -F_2^+ & F_2^- \\ 0 & -1 & 1 \end{pmatrix} = 0,$$

that yields:

$$[G_2][F_1] = [G_1][F_2]. \tag{C.9}$$

For $i = 3, 4, \cdots, n$, we can write $(C.8)$ as:

$$F_1^+([G_2]F_i^- - [G_i]F_2^-) + F_2^+([G_i]F_1^- - [G_1]F_i^-) + F_i^+([G_1]F_2^- - [G_2]F_1^-) = 0 \tag{C.10}$$

After some manipulations we obtain:

$$F_1^+([G_2][F_i]-[G_i][F_2])+F_2^+([G_i][F_1]-[G_1][F_i])+F_i^+([G_1][F_2]-[G_2][F_1])=0. \quad (C.11)$$

Applying $(C.9)$ in Eq. $(C.11)$, we obtain:

$$F_1^+([G_2][F_i]-[G_i][F_2])+F_2^+([G_i][F_1]-[G_1][F_i])=0. \quad (C.12)$$

(i) First we assume that $[F_1]=0$. From Eq. $(C.12)$, it follows that

$$F_1^+([G_2][F_i]-[G_i][F_2])-F_2^+[G_1][F_i]=0. \quad (C.13)$$

If $[F_2]\neq 0$ then $[G_1]=0$. Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are assumed to be L.I., then $F_1^+\neq 0$ and for all $i=3,4,\cdots,n$:

$$[G_2][F_i]-[G_i][F_2]=0. \quad (C.14)$$

So $(C.9)$ together $(C.14)$ is the RH locus for $(C.5)$, $(C.6)$ and coincides with the RH locus of (C.1). If $[F_2]=0$, then it follows that for $i=3,4,\cdots,n$:

$$\left(F_1^+[G_2]-F_2^+[G_1]\right)[F_i]=0. \quad (C.15)$$

(ii) Let us assume that $[F_i]=0$ for all $i=3,4,\cdots,n$, then $(C.4)$ is satisfied. If $F_1^+[G_2]-F_2^+[G_1]=0$, $(C.9)$ we know that $F_1^-[G_2]-F_2^-[G_1]=0$, implying that $\mathcal{D}_1$ and $\mathcal{D}_2$ are linearly dependent and contradicting the hypotheses.

Finally assume that $[F_1]\neq 0$; substituting $[G_2]$ in $(C.12)$ by $[G_1][F_2]/[F_1]$ it follows that:

$$\begin{aligned}
&F_1^+\left([G_1]\frac{[F_2]}{[F_1]}[F_i]-[G_i][F_2]\right)+F_2^+\left([G_i][F_1]-[G_1][F_i]\right)=0,\\
&\frac{F_1^+[F_2]}{[F_1]}\left([G_1][F_i]-[G_i][F_1]\right)+F_2^+\left([G_i][F_1]-[G_1][F_i]\right)=0,\\
&\left([G_1][F_i]-[G_i][F_1]\right)\left(\frac{[F_2][F_1^+]-F_2^+[F_1]}{[F_1]}\right)=0.
\end{aligned} \quad (C.16)$$

If $[F_2][F_1^+]-F_2^+[F_1]\neq 0$, then for all $i=3,4,\cdots,n$ we obtain:

$$[F_i][G_1]=[G_i][F_1],$$

that is the RH locus. On the other hand, if $[F_2][F_1^+]-F_2^+[F_1]=0$, then:

$$F_2^+F_1^- - F_2^-F_1^+ = 0 \quad (C.17)$$

Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are assumed to be LI, then:

$$det\begin{pmatrix}[G_1] & -F_1^+\\ [G_2] & -F_2^+\end{pmatrix}\neq 0 \quad \text{or} \quad det\begin{pmatrix}[G_1] & F_1^-\\ [G_2] & F_2^-\end{pmatrix}\neq 0 \quad \text{or} \quad det\begin{pmatrix}F_1^- & -F_1^+\\ F_2^- & -F_2^+\end{pmatrix}\neq 0. \quad (C.18)$$

From $(C.17)$ and $(C.9)$, we see that $(C.18)$ cannot be satisfied. Thus we conclude that the RH locus of $(C.1)$ coincides with the RH locus of the system $(C.5)$, $(C.6)$, defined by Eq. $(C.8)$.

We have just proved the following Proposition:

Proposition C.1.1. *Under the same hypotheses of Corollary* 2.3.1*, the RH locus of* (C.1) *defined by Eq.* (C.4) *coincides with the RH locus of the system* (C.5), (C.6).

Proposition C.1.2. *The eigenvalues, the right and the left eigenvectors of the system* (C.5), (C.6) *are* $\lambda = \hat{\lambda}$*,* $r = (\hat{r}, 0)$ *and* $\ell = (\ell, \ell_{n+1})$*, where* $\hat{\lambda}$ *is the eigenvalue and* $\hat{r}$ *and* $\hat{\ell}$ *are the right and left eigenvectors of the system* (C.1). *The coordinate* ℓ_{n+1} *is determined by the system* (C.5), (C.6).

Proof: The eigenvalues of the system (C.5), (C.6) are the solutions of:

$$det \begin{pmatrix} u\frac{\partial F_1}{\partial V_1} - \lambda\frac{\partial G_1}{\partial V_1} & u\frac{\partial F_1}{\partial V_2} - \lambda\frac{\partial G_1}{\partial V_2} & \cdots & u\frac{\partial F_1}{\partial V_n} - \lambda\frac{\partial G_1}{\partial V_n} & F_1 \\ u\frac{\partial F_2}{\partial V_1} - \lambda\frac{\partial G_2}{\partial V_1} & u\frac{\partial F_2}{\partial V_2} - \lambda\frac{\partial G_2}{\partial V_2} & \cdots & u\frac{\partial F_2}{\partial V_n} - \lambda\frac{\partial G_2}{\partial V_n} & F_2 \\ \vdots & \vdots & \vdots & \vdots & \\ u\frac{\partial F_n}{\partial V_1} - \lambda\frac{\partial G_n}{\partial V_1} & u\frac{\partial F_n}{\partial V_2} - \lambda\frac{\partial G_n}{\partial V_2} & \cdots & u\frac{\partial F_n}{\partial V_n} - \lambda\frac{\partial G_n}{\partial V_n} & F_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = 0, \qquad (C.19)$$

which reduces to:

$$det \begin{pmatrix} u\frac{\partial F_1}{\partial V_1} - \lambda\frac{\partial G_1}{\partial V_1} & u\frac{\partial F_1}{\partial V_2} - \lambda\frac{\partial G_1}{\partial V_2} & \cdots & u\frac{\partial F_1}{\partial V_n} - \lambda\frac{\partial G_1}{\partial V_n} \\ u\frac{\partial F_2}{\partial V_1} - \lambda\frac{\partial G_2}{\partial V_1} & u\frac{\partial F_2}{\partial V_2} - \lambda\frac{\partial G_2}{\partial V_2} & \cdots & u\frac{\partial F_2}{\partial V_n} - \lambda\frac{\partial G_2}{\partial V_n} \\ \vdots & \vdots & \vdots & \vdots \\ u\frac{\partial F_n}{\partial V_1} - \lambda\frac{\partial G_n}{\partial V_1} & u\frac{\partial F_n}{\partial V_2} - \lambda\frac{\partial G_n}{\partial V_2} & \cdots & u\frac{\partial F_n}{\partial V_n} - \lambda\frac{\partial G_n}{\partial V_n} \end{pmatrix} = 0, \qquad (C.20)$$

which yields the eigenvalues of the system (C.1). The right and left eigenvectors satisfy (C.19), with $r = (\hat{r}, 0)$ and $\ell = (\hat{\ell}, \ell_{n+1})$, where $\hat{\lambda}$ is the eigenvalue and $\hat{r}$ and $\hat{\ell}$ are the right and left eigenvectors of the system (C.1) and

$$\ell_{n+1} = \hat{\ell}_1\hat{F}_1 + \hat{\ell}_1 F_2 + \cdots + \hat{\ell}_n F_n,$$

where $\hat{\ell}_i$ for $i = 1, 2, \cdots, n$ are the coordinates of left eigenvector $\hat{\ell}$. □

From Propositions C.1.1 and C.1.2 we obtain:

Corollary C.1.1. *The wave structures for* (C.1) *coincide with the wave structures for* (C.5), (C.6) .

C.2 Numerical Method

We propose a numerical method to solve Eq. (2.2) for all contiguous physical situations. Because the speed u does not appear in $G(V)$, there is an infinite speed mode associated to u. Therefore the usual explicit finite difference numerical methods for solving hyperbolic equations cannot be applied to Eq. (2.2). Bruining et. al. have applied In [40], we have applied a similar method to the Riemann problem of nitrogen and steam injection.

C.2.1 General Formulation

We represent the spatial position of any quantity by k and the time by symbol n. We denote the time interval by Δt and the spatial interval by Δx.

For the numerical approximation, we use

$$V^{k,n} = V(k\Delta x, n\Delta t) \qquad \text{and} \qquad u^{k,n} = u(k\Delta x, n\Delta t), \qquad k \in \mathbb{Z},\ n \in \mathbb{N},$$

here $k\Delta x$ and $n\Delta t$ are, respectively, the position and the time where V and u are evaluated.

We denote the variables to be found at time $n+1$ and position k by

$$V = V(k\Delta x, (n+1)\Delta t) \qquad \text{and} \qquad u = u(k\Delta x, (n+1)\Delta t). \tag{C.21}$$

For simplicity, we abbreviate the known quantities

$$G(V^{k,n}) = G^{k,n}, \qquad F(V^{k,n}) = F^{k,n}, \qquad G = G(V) \qquad \text{and} \qquad F = F(V). \tag{C.22}$$

C.2.2 Numerical Method for the System

Using the notation of Section C.2.1, we propose an implicit method to approximate Eq. (2.2) given by the following difference scheme:

$$\frac{G - G^{k,n}}{\Delta t} = -\frac{uF + u^{k-1,n+1}F^{k-1,n+1}}{\Delta x}. \tag{C.23}$$

where u and V are the unknowns defined in (C.21). These equations are utilized to find the unknown V and u at time $n+1$ and position k.

Proposition C.2.1. *If there exists a j such that $F_j(V) \neq 0$ for each V in the domain, the variables V can be obtained from the implicit scheme (C.23) without calculating u.*

Proof: We introduce the new variable $\hat{u}$ given by:

$$\hat{u} = \frac{u\Delta t}{\Delta x}. \tag{C.24}$$

So the system (C.23) can be rewritten as:

$$G(V) - G^{k,n} = -\hat{u}F + \hat{u}^{k-1,n+1}F^{k-1,n+1}. \tag{C.25}$$

Since by hypothesis there exists a j such that $F_j(V) \neq 0$, we can obtain u as function of V as:

$$\hat{u} = \frac{G_j^{k,n} - G_j(V) + \hat{u}^{k-1,n+1}F_j^{k-1,n+1}}{F_j}. \tag{C.26}$$

Notice that the equation C.26 is valid for some time n. Substituting (C.26) in the remaining equations of system (C.25) we obtain for all $i \neq j$:

$$\left(G_i(V) - G_i^{k,n}\right) F_j = \left(G_j^{k,n} - G_j(V)\right) F_i + \hat{u}^{k-1,n+1}F_i^{k-1,n+1}F_j. \tag{C.27}$$

Notice that we know $u^{0,n}$ for some time n, because we know the speed of injection point. From Eq. $(C.26)$ we can obtain $u^{k-1,n+1}$ as function of $u^{0,n+1}$ and V. After some calculations we obtain:

$$\hat{u}^{k-1,n+1} = \frac{\sum_{m=1}^{k-1} G_j^{m,n} - G_j^{m,n+1}(V) + \hat{u}^{0,n+1} F_j^{0,n+1}}{F_j^{k-1,n+1}}. \tag{C.28}$$

Substituting $(C.28)$ in $(C.27)$, we obtain for each $i \neq j$:

$$\begin{aligned}
&\left(G_i(V) - G_i^{k,n}\right) F_j F_j^{k-1,n+1} - \left(G_j^{k,n} - G_j(V)\right) F_i F_j^{k-1,n+1} \\
&- \left(\sum_{m=1}^{k-1} G_j^{m,n} - G_j^{m,n+1}(V)\right) F_i^{k-1,n+1} F_j - \hat{u}^{0,n+1} F_j^{0,n+1} F_i^{k-1,n+1} F_j = 0.
\end{aligned} \tag{C.29}$$

□

Remark C.2.1. *Since the matrix $\partial G/\partial W$ is singular, it is not possible to use Newton method to obtain (V,u) in Eq. $(C.23)$. An advantage in utilizing the scheme $(C.29)$ is that u does not appear in this scheme, so we can linearize the reduced system or solve it by a non-linear method, such as the Newton method.*

C.3 Travelling waves and viscous profiles.

The system of equations (2.1) generically is obtained by simplifications from more complex systems with diffusive terms of form:

$$\frac{\partial}{\partial t}\mathcal{G}(\mathcal{V}) + \frac{\partial}{\partial x} u\mathcal{F}(\mathcal{V}) = Q(\mathcal{V}) + \frac{\partial}{\partial x}\mathbb{B}\frac{\partial \mathcal{V}}{\partial x}, \tag{C.30}$$

The $m \times m+1$ matrix $\mathbb{B} := \mathbb{B}(\mathcal{V})$ is called *viscosity matrix*; it can represent molecular diffusion, capillarity effects, heat conduction and other physical phenomena.

The viscosity profile criterion is widely used for admissibility of physical shocks. As a future work we will utilize this method to solve more complicated models where elliptic regions appear, see [24, 25, 26] and [61].

The travelling waves are solutions of $(C.30)$ that depend only on the parameter $\xi = x - v^S t$, where v^S is the shock speed to be found. We rewrite this similarity equation as:

$$\mathcal{W} = \mathcal{W}(x - v^S t), \tag{C.31}$$

where we recall that $\mathcal{W} = (\mathcal{V}, u)$.

For the analysis of travelling waves and viscosity profile, we define the cumulative condensation distribution $\mathcal{Q}(x,t)$ and cumulative condensation $\mathcal{Q}^+(t)$ by

$$\mathcal{Q}(x,t) = \int_{-\infty}^{x} Q(x',t)dx', \qquad \mathcal{Q}^+(t) = \int_{-\infty}^{+\infty} Q(x',t)dx'. \tag{C.32}$$

From Eq. $(C.32)$, we can write $Q = \partial\mathcal{Q}(x,t)/\partial x$. We also define $\mathcal{Q}^-(t) = \mathcal{Q}(x^-,t)$ and $\mathcal{Q}^+(t) = \mathcal{Q}(x^+,t)$, where x^- and x^+ are the points immediately on the left and right of the transition between regions.

We can rewrite Eq. $(C.30)$ using the parameter ξ and we obtain:

$$-v^s\frac{d\mathcal{G}}{d\xi} + \frac{d(u\mathcal{F})}{d\xi} = \frac{d\mathcal{Q}}{d\xi} + \frac{d\mathbb{B}}{d\xi}\frac{d\mathcal{V}}{d\xi} \tag{C.33}$$

Integrating $(C.33)$ em ξ and setting that the limits satisfy:

$$\lim_{\xi\longrightarrow-\infty}(\mathcal{V}(\xi), u(\xi), \mathcal{Q}(\mathcal{V}(\xi))) = (\mathcal{V}^L, u^L, \mathcal{Q}^L), \qquad \lim_{\xi\longrightarrow-\infty}\frac{d\mathcal{V}}{d\xi} = 0, \tag{C.34}$$

$$\lim_{\xi\longrightarrow+\infty}(\mathcal{V}(\xi), u(\xi), \mathcal{Q}(V(\xi))) = (\mathcal{V}^R, u^R, \mathcal{Q}^R), \qquad \lim_{\xi\longrightarrow+\infty}\frac{d\mathcal{V}}{d\xi} = 0. \tag{C.35}$$

We obtain the shock speed with $[\mathcal{Q}] = \mathcal{Q}^R - \mathcal{Q}^L$:

$$-v^s(\mathcal{G}(\mathcal{V}^R) - \mathcal{G}(\mathcal{V}^L)) + u^R\mathcal{F}(\mathcal{V}^R) - u^L\mathcal{F}(\mathcal{V}^L) = [\mathcal{Q}], \tag{C.36}$$

and the ordinary system of equations that determines the orbit of travelling wave with $\mathcal{G}^L = \mathcal{G}(\mathcal{V}^L)$ and $\mathcal{F}^L = \mathcal{F}(\mathcal{V}^L)$:

$$\mathbb{B}\frac{d(\mathcal{V}(\xi))}{d\xi} = v^s(\mathcal{G}^L - \mathcal{G}(\mathcal{V}(\xi))) + u(\xi)\mathcal{F}(\mathcal{V}(\xi)) - u^L\mathcal{F}^L - (\mathcal{Q}(\mathcal{V}(\xi)) - \mathcal{Q}^L), \tag{C.37}$$

$$\frac{d\mathcal{Q}}{d\xi} = Q. \tag{C.38}$$

Lemma C.3.1. *Applying $E \in \mathcal{E}$ defined in* (2.17), $(C.36)$ *reduces to the RH condition* (2.50).

Lemma C.3.2. *If there exists an i such that $\mathcal{F}_i(\mathcal{V}) \neq 0$ for each $\mathcal{V}$ (see Remark* 2.2.3*) along the orbit of the system* $(C.37)$*, then in the variables V this orbit does not depend explicitly on u.*

Proof: The *i-th* line of the system $(C.37)$ is:

$$\mathbb{B}_i\frac{d(\mathcal{V}(\xi))}{d\xi} = v^s(\mathcal{G}_i^L - \mathcal{G}_i(\mathcal{V}(\xi))) + u(\xi)\mathcal{F}_i(\mathcal{V}(\xi)) - u^L\mathcal{F}_i^L - (\mathcal{Q}_i(\mathcal{V}(\xi)) - \mathcal{Q}_i^L), \tag{C.39}$$

where $\mathbb{B}_i$ represents the *i-th* line of the matrix $\mathbb{B}$. Assume that $\mathcal{F}_i \neq 0$ for some i, we can obtain the Darcy speed as:

$$u(\xi) = \frac{u^L\mathcal{F}_i^L - \mathbb{B}_i d(\mathcal{V}(\xi))/d\xi - v^s(\mathcal{G}_i^L - \mathcal{G}_i(\mathcal{V}(\xi))) - (\mathcal{Q}_i(\mathcal{V}(\xi)) - \mathcal{Q}_i^L)}{\mathcal{F}_i(\mathcal{V}(\xi))}, \tag{C.40}$$

After same calculations, for fixed i and $k = 1, 2, \cdots, (i-1), (i+1), \cdots, m$, we can write (C.37) as:

$$\frac{d\mathbb{B}_i}{d\xi}\mathcal{F}_k(\mathcal{V}(\xi)) - \frac{d\mathbb{B}_{k,}}{d\xi}\mathcal{F}_i(\mathcal{V}(\xi)) = u^L\left(\mathcal{F}_i^L\mathcal{F}_k(\mathcal{V}(\xi)) - \mathcal{F}_i^L\mathcal{F}_k(\mathcal{V}(\xi))\right) + \\ + v^s\left\{\left(\mathcal{G}_i^L - \mathcal{G}_i(\mathcal{V}(\xi))\mathcal{F}_k(\mathcal{V}(\xi))\right) - \left(\mathcal{G}_k^L - \mathcal{G}_k(\mathcal{V}(\xi))\mathcal{F}_i(\mathcal{V}(\xi))\right)\right\} + \\ + (\mathcal{Q}_k(\mathcal{V}(\xi)) - \mathcal{Q}_k^L)\mathcal{F}_i(\mathcal{V}(\xi)) - (\mathcal{Q}_i(\mathcal{V}(\xi)) - \mathcal{Q}_i^L)\mathcal{F}_k(\mathcal{V}(\xi)). \tag{C.41}$$

Since the number of equations and variables is the same, the proposition is proved. □

Corollary C.3.1. *Within each physical situation it is possible to apply $E \in \mathcal{E}$. In this case, the orbits of the system* (C.41) *can be calculated only in the space of primary variables.*