Numerical boundary corrector methods and analysis for a second order elliptic PDE with highly oscillatory periodic coefficients with applications to porous media

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Aos meus pais, Carlos e Beth.

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Resumo

Nesta tese são propostos novos métodos numéricos de multi-escalas para a aproximação de equações diferenciais parciais elíticas lineares de segunda ordem, com coeficientes periódicos rapidamente oscilatórios em uma micro escala. O objetivo principal destes métodos é capturar as oscilações da solução que ocorrem na micro escala, mas sem resolver o problema diretamente em uma malha fina comparada a micro escala. Os métodos aqui propostos são baseados em expansões assintóticas e em teoria de homogeneização. Considera-se uma expansão assintótica de primeira ordem da solução do problema original incluindo o corretor de fronteira, e posteriormente é feita a aproximação numérica de cada termo fazendo uso da teoria de elementos finitos.

A originalidade do trabalho se encontra no desenvolvimento de aproximações numéricas para os corretores de fronteira. Apresentamos uma análise de convergência rigorosa para os métodos propostos, que pode ser dividida em duas partes. Primeiramente estimamos o erro entre a expansão assintótica e a solução exata. Aqui podemos destacar a obtenção de estimativas de erro com hipóteses mais fracas que as encontradas na literatura. Posteriormente é estimado o erro entre a expansão assintótica e sua aproximação numérica. Esta análise é feita através da teoria de elementos finitos e inclui novas estimativas de erro ao se fazer uso de uma formulação mista para aproximar a derivada normal de uma dada equação.

Abstract

We develop a numerical discretization for linear elliptic equations with rapidly oscillating coefficients. The major goal is to develop a numerical scheme on a mesh size $h > \epsilon$ (or $h >> \epsilon$), capturing the solution oscillations occurring in a scale ϵ. The proposed method is based on asymptotic expansion and a novel treatment on the boundary corrector term. We obtain discretization errors of $O(h^2 + \epsilon^{3/2} + \epsilon h)$ and $O(h + \epsilon)$ for the L^2 norm and the broken semi-norm H^1, respectively. Numerical results are presented.

Contents

Chapter 1

Preliminaries

1.1 Introduction

Under suitable hypothesis, the governing equations for the miscible displacement of one incompressible fluid by another in a porous medium are given by

$$
\begin{aligned}
&\nabla \cdot v = \nabla \cdot k \nabla p = q \\
&\phi \partial_t S - \nabla \cdot D \nabla S + v \cdot \nabla f_w(s_w) = (\tilde{S} - S) q
\end{aligned}
\tag{1.1}
$$

where v and p are the total fluid velocity and pressure, S is the saturation of the invading fluid, $\tilde{S}$ is the specified injection or the resident production saturation of the injected fluid, q is the total volumetric flow rate at the well, k is the permeability, ϕ is the porosity, and D is the diffusion-dispersion tensor; see Equations (3.32) and (3.33) in [27]. The permeability k oscillates in a very fine scale ϵ compared to the reservoir size, and the use of a fine mesh compared to ϵ to solve Equation (1.1) is very expensive and often impossible, see Section 2 in [28]. The standard solution to circumvent this problem is to up-scale the permeability; see [14, 19]. However, in this process the information on $v = k\nabla p$ is lost; see [35]. In this thesis we propose a numerical algorithm to approximate the solution of Equation (1.1) that provide a good approximation for $k\nabla p$, but requires much less computational effort than approximating the equation using a fine mesh compared to ϵ. More specifically, this thesis is concerned with the development of numerical methods to approximate u_ϵ, the weak solution of the problem:

$$
L_\epsilon u_\epsilon = -\frac{\partial}{\partial x_i}(a_{ij}(x/\epsilon)\frac{\partial}{\partial x_j}u_\epsilon) = f \text{ in } \Omega, \quad u_\epsilon = 0 \text{ on } \partial\Omega, \tag{1.2}
$$

where $a(y) = (a_{ij}(y))$ is a positive symmetric definite matrix, $x, y \in \mathbb{R}^2$ and $\epsilon \in (0,1)$ is the periodicity parameter. We assume $a_{ij} \in L^\infty_{\text{per}}(Y)$ i.e. $a_{ij} \in L^\infty(\mathbb{R}^2)$ and Y-periodic, $Y = (0,1)^2$, and that there exists a positive constant γ_a such that $a_{ij}(y)\xi_i\xi_j \geq \gamma_a\|\xi\|^2$ for all $\xi \in \mathbb{R}^2$ and $y \in Y$. Here we have changed the notation in order to follow the standard notation used in literature for multiscale methods.

We note that when $h > \epsilon$ standard finite element methods do not yield good numerical approximations; see [32]. Recently new numerical methods have been proposed for solving Problem (1.2) such as the multi-scale finite element methods [24, 31, 4, 13, 26], the residual-free bubble function methods [11, 5, 6, 45, 12], and the generalized FEM for homogenization problems [46]. There are also related methods for the case the homogenized equation is not known; see HMM [20, 21, 2] and [23, 25]. The methods proposed here are designed to work with a mesh size $h > \epsilon$ (or $h >> \epsilon$). In addiction, as opposed to the methods [5, 31, 45, 4, 11], they are strongly based on asymptotic expansions of u_ϵ.

One of the first mathematical tools used to handle this problem was homogenization theory [8, 9]. Based on this theory, a first order expansion of u_ϵ plus a boundary corrector term are considered, i.e.

$$
u_\epsilon(x) \approx u_0(x) + \epsilon u_1(x, x/\epsilon) + \epsilon\theta_\epsilon(x),
$$

and then each term is numerically approximated. The core of this thesis is the design and analysis of numerical boundary correctors for approximating θ_ϵ, defined as the weak solution of

$$
\frac{\partial}{\partial x_i}(a_{ij}(x/\epsilon)\frac{\partial}{\partial x_j}\theta_\epsilon) = 0 \text{ in } \Omega, \quad \theta_\epsilon = u_1 \text{ on } \partial\Omega.
$$

We note that the coefficients $a_{ij}(x/\epsilon)$ and the boundary values $-u_1(x, \frac{x}{\epsilon})$ in the above equation are highly oscillatory, hence obtainning a good discrete approximation for θ_ϵ is not a trivial problem. We propose an analytical approximation for θ_ϵ, denoted by ϕ_ϵ, which satisfies the oscillating boundary condition and is suitable for numerical approximation. In order to perform the convergence analysis we consider a slightly modification of the theoretical approximation for the boundary corrector proposed in [3, 41]; see Remark 2.1.2.

In this thesis a rigorous convergence analysis for the numerical methods is considered and performed in two parts. First we prove error estimates between u_ϵ and $u_0+\epsilon u_1+\epsilon\phi_\epsilon$ in L^2 and H^1 norms. We note that Propositions 2.1.1 and 3.2.1 generalize respectively, Propositions 2.1 and 2.3 from [41] to the case $a_{ij} \in L^\infty_{per}(Y)$ and $\Omega \subset \mathbb{R}^3$. The second part of the convergence analysis is based on finite elements theory. The main difficulty here lies in the fact that we use the trace on $\partial\Omega$ of finite elements approximations of PDEs in Ω as boundary condition in subsequent problems. We then develop error estimates for the discrete Lagrange multipliers in the case of $W^{1,p}$ spaces; see Proposition 5.1.3.

This thesis is organized as follows. In the rest of Chapter 1 we recall some background from Sobolev spaces, regularity theory for second order elliptic equations, and finite elements methods. Chapter 2 introduces the asymptotic expansion of u_ϵ, describes a theoretical approximation for the boundary corrector term, and presents the main theorems for estimating the errors due to the asymptotic expansion approximation. Chapter 3 presents the proof of the main theorems from Chapter 2. Chapter 4 describes the two numerical methods developed for approximating u_ϵ. The first method is designed for the case when Ω is a rectangular domain and when bilinear finite elements are used to approximate u_0. The second method introduces the Lagrange multiplier space used to approximate $\partial_\eta u_0$ on $\partial\Omega$, allowing the generalization of the method to the case where the domain Ω is a convex polygon with rational boundary normals, see the Appendix. We note also that we are able to prove the L^2 convergence for the second method under weaker assumption on u_0, than in the first method. Chapter 5 develops the error analysis due to the finite element approximation, and Chapter 6 presents numerical results and conclusions. Finally, in the Appendix we present a generalized version of the second method from Chapter 4 for the case when Ω is a convex polygon with rational boundary normals.

1.2 Sobolev Spaces

Let $D \subset \mathbb{R}^n$ be a open set, m be a non-negative integer, and $\alpha \in \mathbb{R}^n$ such that α_i is a non-negative integer. Define the norm

$$\|u\|_{m,p,D} = \sum_{|\alpha| \leq m} \|\mathcal{D}^\alpha u\|_{L^p(D)} \tag{1.3}$$

where $\mathcal{D}^\alpha = \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n}$, and $|\alpha| = \sum_i \alpha_i$. For $s \in [0, \infty)$, $s = m + \sigma$ and $\sigma \in (0,1)$ we set

$$\|u\|_{s,p,D} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(D)} + \sum_{|\alpha|=m} \int_D \int_D \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} dx dy \right)^{1/p}.$$

Definition 1.2.1 *$W^{s,p}(D)$ is the closure of $C^\infty(D)$ with respect to the norm $\| \cdot \|_{s,p,D}$.*

Definition 1.2.2 *$W_0^{s,p}(D)$ is the closure of $C_0^\infty(D)$ with respect to the norm $\| \cdot \|_{s,p,D}$.*

Definition 1.2.3 *$W_0^{-s,p'}(D)$ is the dual space of $W_0^{s,p}(D)$, with 1/p+1/p'=1.*

We also introduce the following norms and semi-norms

$$\begin{aligned} \|v\|_{m,\infty,D} &= \max_{|\alpha| \leq m} \{\text{ess.} \sup_{x \in D} |\partial^\alpha v(x)|\}, \\ |v|_{m,\infty,D} &= \max_{|\alpha| = m} \{\text{ess.} \sup_{x \in D} |\partial^\alpha v(x)|\}, \end{aligned}$$

and for $1 \leq q < \infty$

$$|v|_{m,q,D} = \left(\int_B \sum_{|\alpha|=m} |\mathcal{D}^\alpha v|^q dx \right)^{1/q}.$$

We now present important results concerning Sobolev spaces used in this thesis.

Theorem 1.2.1 *(The Sobolev embedding theorem) Let D be a domain in $\mathbb{R}^n$, j and m be nonnegative integers and p satisfy $1 \le p < \infty$.*

Part I *If D has the cone property, see [1], then there exists the following embeddings:*

Case A. Suppose $mp < n$

$$W^{j+m,p}(D) \hookrightarrow W^{m,q}(D), \qquad p \le q \le np/(n-mp),$$

Case B. Suppose $mp = n$

$$W^{j+m,p}(D) \hookrightarrow W^{m,q}(D), \qquad p \le q < \infty,$$

Moreover, if $p = 1$ the above embedding exists with $q = \infty$ as well; in fact,

$$W^{j+n,1}(D) \hookrightarrow C^j(D), \qquad p \le q < \infty,$$

Case C. Suppose $mp > n$

$$W^{j+m,p}(D) \hookrightarrow C^j(D), \qquad p \le q < \infty,$$

Part II *The results above are valid provided the spaces W are replaced by their respectively spaces W_0.*

Proof: See Theorem 5.4 in [1]. □

Theorem 1.2.2 *Let D be a bounded open subset of $\mathbb{R}^2$ whose boundary $\partial D = \cup_i^N \Gamma_i$ is a polygon and $\gamma_i : (0,1) \to \Gamma_i$ is a parametrization of Γ_i. Then the mapping $u \to u|_{\Gamma_i}$ is a linear continuous mapping from $W^{1,p}(D)$ onto the subspace $\prod_i^N W^{1-1/p,p}(\Gamma_i)$ defined by:*

(a) *no extra condition when $1 < p < 2$,*

(b) $u|_{\Gamma_i}(s_i) = u|_{\Gamma_{i+1}}(s_j)$, *$1 \le i \le N$ when $2 < p < \infty$*

(c) $\int_0^{\delta_i} \frac{|u(\gamma_i(s)) - u(\gamma_{i+1}(-s)|^2}{s} ds < \infty, \quad 1 \le i \le N$ *when $p = 2$.*

Proof: See Theorem 1.5.2.3 in [29]. □

Given two Banach spaces $B_1 \hookrightarrow B_0$, for any $u \in B_0$ and $t > 0$, let

$$K(t,u) = \inf_{v \in B_1} (\|u - v\|_{B_0} + t\|v\|_{B_1})$$

For $0 < \sigma < 1$ and $1 \leq p < \infty$ define the norm

$$\|u\|_{[B_0,B_1]_{\sigma,p}} = \left(\int_0^\infty t^{-\sigma p - 1} K(t,u)^p dt \right)^{1/p}. \tag{1.4}$$

The set

$$[B_0, B_1]_{\sigma,p} = \{u \in B_0;\ \|u\|_{[B_0,B_1]_{\sigma,p}} < \infty\}$$

is a Banach space with norm (1.4).

Theorem 1.2.3 *Let $0 < s < 1$. If D has a Lipschitz boundary, then*

$$W^{m+s,p}(D) = [W^{m,p}(D), W^{m+1,p}(D)]_{s,p}$$

and the norms are equivalent.

Proof: See Theorem 12.2.3 in [10]. □

We also have:

Theorem 1.2.4 *Suppose $A_i, B_i,\ i = 0,1$ are two pairs of Banach spaces such that $A_1 \hookrightarrow A_0$ and $B_1 \hookrightarrow B_0$. Let T be a linear operator that maps A_i to B_i, $i = 0,1$. Then T maps $[A_0, A_1]_{\sigma,p}$ to $[B_0, B_1]_{\sigma,p}$. Moreover,*

$$\|T\|_{\mathcal{L}([A_0,A_1]_{\sigma,p},[B_0,B_1]_{\sigma,p})} \leq \|T\|^{1-\sigma}_{\mathcal{L}(A_0,B_0)} \|T\|^{\sigma}_{\mathcal{L}(A_1,B_1)}.$$

Proof: See Proposition 12.1.5 in [10] □

Let $a < b \in \mathbb{R}$, we define the space

$$H^{1/2}_{00}((a,b)) = [L^2((a,b)), H^1_0((a,b))]_{1/2,2}.$$

1.2.1 Periodic Sobolev Spaces

Let $Y = (0,1)^n$, $G = \{[0,\infty) \times (0,1)\}$, and $C^\infty_{per}(Y)$ be the subset of $C^\infty(\mathbb{R}^n)$ of the Y-periodic functions. We introduce the following spaces

Definition 1.2.4 *$W^{s,p}_{per}(Y)$ is the closure of $C^\infty_{per}(Y)$, with respect to the norm $\|\cdot\|_{s,p,Y}$, in the case $p = 2$ we also use the notation $H^s_{per}(Y)$.*

Definition 1.2.5 *$H^1_{per}(G)$ is the closure of $\{\varphi \in C^\infty(G) \cap H^1(G),\ \varphi(y_1,\cdot)$ is $(0,1)^{n-1}$-periodic $\forall\ y_1 \in [0,\infty)\}$, with respect to the norm $\|\cdot\|_{1,G}$.*

In the case $Y = (0,1)^2$

Definition 1.2.6 *$H^{1/2}_{per}((0,1)) = \{g \in H^{1/2}((0,1));\ \exists\ \varphi \in H^1_{per}(Y)$ and $g = \varphi|_{0\times(0,1)}\}$.*

Interesting properties concerning the space $H^1_{per}(Y)$, are given by the following propositions.

Proposition 1.2.1 *Let $u \in H^1_{per}(Y)$. Then, u has the same trace on the opposite faces of Y.*

Proof: See Proposition 3.49 in [16]. □

Given $g \in L^p(Y)$, let $g^\#$ denotes the Y-periodic extension of g to $\mathbb{R}^n$.

Proposition 1.2.2 *Let $u \in H^1_{per}(Y)$. Then $u^\# \in H^1_{loc}(\mathbb{R}^n)$.*

Proof: See Proposition 3.50 in [16]. □

1.3 Regularity Results

In this section we present some results from the regularity theory for second order linear elliptic equations. More specifically, we are interested in the following equations

$$\nabla \cdot A(x)\nabla u = f, \text{ in } D, \text{ and } u = 0 \text{ on } \partial D, \tag{1.5}$$

and

$$\nabla \cdot A(x)\nabla u = 0, \text{ in } D, \text{ and } u = g \text{ on } \partial D, \tag{1.6}$$

where $D \subset \mathbb{R}^n$ is a bounded open convex domain, $g \in H^{1/2}(\partial D)$, $f \in H^{-1}(D)$, and $A(x) = (a_{ij}(x))$, $a_{ij} \in L^\infty(D)$ and $a_{ij}(x)\xi_i\xi_j \geq \gamma_a\|\xi\|^2$, a.e. in D.

Theorem 1.3.1 *Let $u \in H_0^1(D)$ be defined as the weak solution of equation (1.5). Assume $a_{ij} \in C^{0,1}(D)$ and $f \in L^2(D)$. Then $u \in H^2(D)$ and there exists a constant $c > 0$ such that*

$$\|u\|_{2,2,D} \leq c\|f\|_{0,2,D}.$$

Proof: See Theorems 3.1.3.1 and 3.2.1.2 in [29]. □

Theorem 1.3.2 *Let $u \in H_0^1(D)$ be defined as the weak solution of equation (1.5). Assume D is bounded open convex polygon , $a_{ij}(x) = c_{ij}$, $c_{ij} \in \mathbb{R}$, $f \in L^p(D)$ and $u \in W^{2,p}(D)$, for $p > 2$. Then there exists a constant $c > 0$ such that*

$$\|u\|_{2,p,D} \leq c\|f\|_{0,p,D}.$$

Proof: From Theorem 4.3.2.4 and Remark 4.3.4.5 in [29] we obtain

$$\|u\|_{2,p,D} \leq c(\|f\|_{0,p,D} + \|u\|_{0,p,D}),$$

and from the Sobolev embedding theorem $\|u\|_{0,p,D} \leq c\|u\|_1$. Hence the theorem follows from Theorem 1.3.1. □

Theorem 1.3.3 *Let $u \in H^1(D)$ be defined as the weak solution of equation (1.6). Assume $g \in H^{1/2}(\partial D) \cap C^0(\partial D)$. Then $u \in C^0(D)$ and*

$$\min_{x \in \partial D} u(x) \leq \min_{x \in D} u(x) \leq \max_{x \in D} u(x) \leq \max_{x \in \partial D} u(x)$$

Proof: See Theorem 2.5 and (3.5) in [40]. □

Theorem 1.3.4 *Let D be a bounded domain of class $W^{2,q}$, see pg 7 [36], and $u \in H_0^1(D)$ be the weak solution of equation (1.5). Assume $a_{ij} \in W^{1,q}(D)$, for $q > n$, then*

$$u \in W^{2,q}(D) \cap C^{1,\alpha}(\overline{D}), \quad \alpha = 1 - n/q.$$

Proof: See Theorem 15.1 in [36]. □

Theorem 1.3.5 *Let $D \subset \mathbb{R}^2$ be a bounded convex polygonal domain, $\partial D = \cup_i^N \Gamma_i$, where Γ_i are line segments, and define $u \in H_0^1(D)$ as the weak solution of the equation (1.5). Assume that $u \in H^2(D)$, then $\frac{\partial u}{\partial \eta} \in H_{00}^{1/2}(\Gamma_i)$ and there exists a constant c such that*

$$\sum_{i=1}^{N} \left\| \frac{\partial u}{\partial \eta} \right\|_{H_{00}^{1/2}(\Gamma_i)} \leq c\|u\|_2,$$

where $\frac{\partial}{\partial \eta}$ denotes the normal derivative.

Proof: See Theorem A.2 in [44]. $\square$

The following theorem is a generalization of Theorem 6.1 [44]

Theorem 1.3.6 *Let $D \subset \mathbb{R}^2$ be a polygonal convex domain, $\partial D = \cup_i^N \Gamma_i$, where Γ_i are line segments, and $g \in H^{1/2}(\partial D)$. Define define $u \in H^1(D)$ as the weak solution of the equation (1.6). Assume $a_{ij} = c_{ij}$, $c_{ij} \in \mathbb{R}$, then there exists a constant c such that*

$$\|u\|_{0,2,D} \leq c \sum_{i=1}^{N} \|g\|_{H^{-1/2}(\Gamma_i)}$$

Proof: By the Green formula we obtain for all $v \in H_0^1(D) \cap H^2(D)$

$$\begin{aligned} \int_D (-\nabla \cdot A\nabla v)u &= \int_D A_{ij}\partial_{x_i} u \partial_{x_j} v dx - \int_{\partial D} \partial_{\eta_A} v g ds \\ &= -\int_{\partial D} \partial_{\eta_A} v g ds, \quad \text{by (1.6)}, \end{aligned} \tag{1.7}$$

where $\partial_{\eta_A} v = (A\nabla v) \cdot \eta$. Let $v \in H_0^1(D)$ be the weak solution of the problem

$$-\nabla \cdot A\nabla v = u, \quad \text{in} \quad D, \quad \text{and} \quad v = 0 \quad \text{on} \quad \partial D. \tag{1.8}$$

Since D is convex, regularity theory ensures that $v \in H_0^1(D) \cap H^2(D)$ and that

$$\|v\|_{2,2,D} \leq c\|u\|_{0,2,D}.$$

From the trace theorem we have that $\partial_{\eta_A} v \in H_{00}^{1/2}(\Gamma_i)$ and that

$$\begin{aligned} \|\partial_{\eta_A} v\|_{H_{00}^{1/2}(\Gamma_i)} &\leq c\|v\|_{2,2,D} \\ &\leq c\|u\|_{0,2,D}. \end{aligned} \tag{1.9}$$

Applying the definition of v, (1.8), we have

$$\begin{aligned} \|u\|_{0,2,D}^2 &= \int_D (-\nabla \cdot A\nabla v) u dx \\ &\leq \sum_{i=1}^N \|\partial_{\eta_A} v\|_{H_{00}^{1/2}(\Gamma_i)} \|g\|_{H^{-1/2}(\Gamma_i)}, \quad \text{by (1.7)} \\ &\leq c \sum_{i=1}^N \|u\|_{0,2,D} \|g\|_{H^{-1/2}(\Gamma_i)}. \end{aligned} \tag{1.10}$$

□

1.3.1 Equations with Periodic Boundary Conditions

Let $Y = (0,1)^n$, $A(y) = a_{ij}(y)$ and $a_{ij} \in L^\infty_{per}(Y)$. We define the u, the weak solution of the equation

$$-\nabla \cdot A\nabla u = f, \quad \text{in } Y, \quad u \text{ is } Y\text{-periodic}, \tag{1.11}$$

as the solution $u \in W_{per}(Y) = \{v \in H^1_{per}(Y); \int_Y v dy = 0\}$ of the following variational problem

$$\int_Y a_{ij} \partial_{y_i} u \partial_{y_j} v = \langle f, v \rangle \tag{1.12}$$

Theorem 1.3.7 *where $f \in (W_{per}(Y))'$. Then problem (1.12) has a unique solution and*

$$\|u\|_{W_{per}(Y)} \leq \alpha^{-1} \|f\|_{(W_{per}(Y))'}.$$

Proof: See Theorem 4.26 in [16] □

Theorem 1.3.8 *Let $A(y) = a_{ij}(y)$, $a_{ij} \in L^\infty_{per}(Y)$, $h \in (L^2_{per}(Y))^n$. Let $u \in H^1_{per}(Y)/\mathbb{R}$ be the solution of Equation (1.12), for $f = \nabla \cdot h$, and recall the notation $u^\#$ from Section 1.2.1. Then $u^\#$ is the unique solution of*

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^n} a^\#_{ij}(y) \partial_{y_i} u^\# \partial_{y_j} v dy = \int_{\mathbb{R}^n} h \cdot \nabla v dy, \quad \forall\ v \in C_0^\infty(\mathbb{R}^n) \\ &u^\# \quad Y - \textit{periodic}, \quad \textit{and} \quad \textstyle\int_Y u^\# = 0. \end{aligned}$$

Proof: See Theorem 5.4 in [16]. □

Theorem 1.3.9 *The following equation*

$$L_1\phi = F \ in \ Y, \quad \phi \in H^1_{per}(Y).$$

admits a unique solution in $H^1_{per}(Y)/\mathbb{R}$ iff

$$\int_Y F dy = 0.$$

Proof: Note that $\langle F, \cdot \rangle : H^1_{per}(Y)/\mathbb{R} \to \mathbb{R}$ define by

$$\langle F, \varphi \rangle = \int_Y F\varphi dy,$$

belongs to $(H^1_{per}(Y)/\mathbb{R})'$ if and only if $\int_Y F dy = 0$. Hence the theorem follows from Theorem 1.3.7 □

Let $G = \{[0,\infty) \times [0,1]\}$ and define $v \in H^1_{per}(G)$ as the solution of the problem

$$\begin{aligned} -\nabla_y \cdot a(y_1, y_2)\nabla_y v &= 0 \text{ in } G, \\ v(0, y_2) &= g(y_2), \end{aligned} \tag{1.13}$$

where $g \in H^{1/2}_{per}((0,1))$, $a(y) = (a_{ij}(y))$, $a_{ij} \in L^\infty_{per}((0,1)^2)$ and $a_{ij}(y)\xi_i\xi_j \leq c\|\xi\|^2$. The above problems have been studied by several authors, see [43, 38, 35, 41]. Here we give a theorem that guarantees the existence of a unique solution for the above equation.

Theorem 1.3.10 *There exists a unique solution $v \in H^1_{loc}(G)$ for Equation (1.13). Furthermore, there exists constants $\gamma, c > 0$ and a unique $\chi \in \mathbb{R}$ such that*

$$exp(\gamma y_1)\partial_{y_i} v \in L^2(G) \quad i = 1, 2,$$

and

$$|v(y) - \chi| \leq c exp(\gamma y_1) \ as \ \ y_1 \to \infty$$

Proof: See Theorem 10.1 section 10.4 in [38] and Theorem 3 in [43]. □

Lemma 1.3.1 *A function* $\mathbf{v} \in L^2_{per}(Y)^2$, $(\mathbf{v} \in L^2_{per}(Y)^3)$ *satisfies*

$$\nabla \cdot \mathbf{v} = \mathbf{0}, \tag{1.14}$$

and $\int_Y v_i dy = 0$ *iff there exists a function* $\phi \in H^1_{per}(Y)$ $(\phi \in H^1_{per}(Y)^3)$ *such that:*

$$\mathbf{v} = curl\phi. \tag{1.15}$$

Proof: ($\Rightarrow$) Consider the discrete Fourier transform associated to v_i (see [33]), writing Equations (1.14), (1.15) in terms of Fourier coefficients we obtain

$$k_1 \hat{v}_1(k) = k_2 \hat{v}_2(k), \tag{1.16}$$

$$\hat{v}_1(k) = ick_2 \hat{\phi}(k) \quad \text{and} \quad \hat{v}_2(k) = ick_1 \hat{\phi}(k). \tag{1.17}$$

Take

$$\hat{\phi}(k) = \begin{cases} \hat{v}_1(k)/ick_2 & \text{if} \quad k_1, k_2 \neq 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

Since $\int_Y v_i dy = 0$, we obtain $\hat{v}_i(0,0) = 0$ and by Equation (1.16) we immediately get that relation (1.17) is satisfied $\forall\ k \in \mathbb{N}^2$. In addition,

$$\sum_{k \in Z^2} (1 + |k|^2) \hat{\phi}(k)^2 \leq c \sum_{k \in Z^2} \hat{v}_2(k)^2 + \sum_{k \in Z^2} \hat{v}_1(x)^2 \leq c \|\mathbf{v}\|^{\mathbf{2}}_{\mathbf{0,2,D}}$$

hence $\phi \in H^1_{per}(Y)$ (see Proposition 3.194 in [33]).

($\Leftarrow$) In this case we note that v_1, v_2 defined by equation (1.17) satisfies (1.16). And since $\phi \in H^1_{per}(Y)$,

$$\|v_i\|^2_{0,2,D} \leq c \sum_{k \in Z^2} \hat{v}_i(k)^2 \leq \sum_{k \in Z^2} (1 + |k|^2) \hat{\phi}(k)^2 \leq c \|\phi\|^2_1.$$

The case $(\mathbf{v} \in L^2_{per}(Y)^3)$ follows from the proof of Theorem 3.4 in [30], replacing continuous Fourier transforms by discrete Fourier transforms. □

1.4 Finite Elements Theory

Let $D \subset \mathbb{R}^2$ polygonal domain, and denote by $\mathcal{T}^h(D)$ a regular partition of the domain D by triangular or rectangular elements K_i. Let $\mathcal{P}_t(K_i)$ be the space of polynomials of degree less or equal to t, then we define the finite element space $V^h(D) = \{\phi \in H^1(D), \phi|_{K_i} \in \mathcal{P}_t(K_i)\}$.

We also define the non-conforming norms related to a partition $\mathcal{T}_h(D) = K_1, K_2, ..., K_N$ of D by

$$\|v\|_{m,p,h} = \sqrt{\sum_{K_j \in \mathcal{T}_h(D)} \|v\|^2{}_{W^{m,p}(K_j)}}.$$

We now recall some important results from finite element theory.

The standard finite element interpolation operator $\mathcal{I}^h : C^0(D) \to V^h(D)$ is defined by

$$\mathcal{I}^h v(z) = v(z),$$

for all $z \in D$, z being a vertex of an element $K_i \in \mathcal{T}^h(D)$.

Theorem 1.4.1 *Assume that $v \in W^{m,p}(D)$, with $m - 2/p > 0$. Then for $K \in \mathcal{T}^h(D)$ there exists a constant c such that*

$$\|v - \mathcal{I}^h v\|_{l,p,K} \leq ch^{m-l} \|v\|_{m,p,K},$$

for $0 \leq l \leq m$.

Proof: See Theorem 4.4.4 in [10]. □

In the case of non-continuous functions the interpolation operator $\mathcal{I}^h$ is not well defined. For this reason, we introduce the interpolation operator $\tilde{\mathcal{I}}_h : L^2(D) \to V^h(D)$ defined as follows. Given z a nodal point of $\mathcal{T}_h(D)$, let $w_z = \cup_{\overline{K}_i \cap z \neq \emptyset} K_i$, and $P_{0,z} : w_z \to V^h(D)|_{w_z}$ be the L^2 projector to $V^h(D)|_{w_z}$. Then set

$$\tilde{\mathcal{I}}^h_c v(z) = P_{0,z} v(z).$$

Theorem 1.4.2 *Assume that $v \in W^{m,p}(D)$. Then there exists a constant c such that*

$$\|v - \tilde{\mathcal{I}}_h v\|_{l,p,h} \leq ch^{m-l} \|v\|_{m,p,D},$$

for $0 \leq l \leq m$.

Proof: See Theorem 4.8.7 in [10]. □

We also introduce the interpolation operator $\Pi : L^2(D) \to V^h(D)$, defined as follows. If z is an interior nodal point of $\mathcal{T}_h(D)$ then $\Pi(z) = \tilde{\mathcal{I}}_h(z)$, if z is a nodal point and $z \in \partial D$ then let $w_z = \cup_{\overline{K}_i \cap z \neq \emptyset} \overline{K}_i \cap \partial D$, and $P_{0,z} : w_z \to V^h(D)|_{w_z}$ be the L^2 projector to $V^h(D)|_{w_z}$. Then set

$$\Pi v(z) = P_{0,z} v(z).$$

(see (2.3) in [47]). Π has the following property:

$$\text{if } v \in L^2(D) \text{ and } v|_{\partial D} \in V^h(D)|_{\partial D} \ \text{ then } \Pi v|_{\partial D} = v|_{\partial D}.$$

Lemma 1.4.1 *Let $\Pi : L^2(D) \to V^h(D) \subset H^1(D)$ be defined as above, then there exists a constant c depending only on D such for $v \in L^2(D)$*

$$\|v - \Pi v\|_{0,2,D} \leq c\|v\|_{0,2,D},$$

and for $v \in H^1(D)$

$$\|v - \Pi v\|_{1,2,D} \leq c\|v\|_{1,2,D}.$$

Proof: See Theorem 2 in [47]. □

Theorem 1.4.3 *(Inverse estimate) Assume that $\phi \in V^h(D)$, with $m - 2/p > 0$. Then for $K \in \mathcal{T}^h(D)$, there exists a constant c such that*

$$\|\phi^h\|_{l,p,K} \leq c h^{m-l+min(0,\frac{2}{p}-\frac{2}{q})} \|\phi^h\|_{m,q,K},$$

for $0 \leq m \leq l$.

Proof: See Theorem 4.5.11 in [10]. □

Consider the following problem

$$-\nabla \cdot A\nabla\psi = f \text{ in } D \quad \psi = 0 \text{ on } \partial D \tag{1.18}$$

where the matrix $(A = (a_{ij}(x)))$ is symmetric positive definite, and assume that there exist constants γ_1 and γ_2, such that $\gamma_1\|\xi\|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \gamma_2\|\xi\|^2$ for all $\xi \in \mathbb{R}^2$ and $x \in D$.

We define the weak solution of the above equation as the solution of the following variational problem. Find $\psi \in H_0^1(D)$ such that

$$\int_D a_{ij}\partial_i\psi\partial_j\phi dx = \int_D f\phi dx, \quad \forall\ \phi \in H_0^1(D), \tag{1.19}$$

and define the finite element approximation $\psi^h \in V^h(D) \cap H_0^1(D)$ as the solution of

$$\int_D a_{ij}\partial_i\psi^h\partial_j\phi^h dx = \int_D f\phi^h dx, \quad \forall\ \phi^h \in V^h(D) \cap H_0^1(D). \tag{1.20}$$

The error estimates between ψ and ψ^h are given by:

Theorem 1.4.4 *Let ψ and ψ^h be defined by equations (1.19) and (1.20), respectively. Assume that $a_{ij} \in W^{1,q}(D)$ for $q \geq 2$. Assume also that Problem (1.18) has the following regularity, there exists $\mu > 2$, such that $\psi \in W^{2,q}$ if $f \in L^q$ for $1 < q < \mu$. Then there exists a constant c such that*

$$\|\psi - \psi^h\|_{1,p,D} \leq ch\|\psi\|_{2,p,D},$$

for $l = 1$ and $2 \leq p \leq \infty$,

$$\|\psi - \psi^h\|_{0,p,D} \leq ch^2\|\psi\|_{2,p,D},$$

for $l = 1$ and $2 \leq p < \infty$, and

$$\|\psi - \psi^h\|_{0,\infty,D} \leq ch^2\ ln(h)\|\psi\|_{2,\infty,D}.$$

Proof: See Chapter 7 in [10]. □

Theorem 1.4.5 *Let ψ and ψ^h be defined by equations (1.19) and (1.20), respectively. Assume that $a_{ij} \in L^\infty(D)$. Then there exists a constant c such that*

$$\|\psi - \psi^h\|_{m,2,D} \leq ch^{1-m}\|\psi\|_{1,2,D},$$

for $m = 1$ or 2.

Proof: See Theorems 5.4.4 and 5.4.8 in [10]. □

1.5 Notation

Throughout this thesis when we use the norm $\| \cdot \|_{s,q,D}$ we do not make reference to the domain D, or to the coefficient q when $D = \Omega$, or $q = 2$, respectively We always use the Einstein summation convention, i.e. repeated indices indicate summation, except for the index k. In what follows c denotes a generic constant independent of ϵ and mesh parameters.

Chapter 2

Theoretical Approximation

2.1 Asymptotic Expansions Using Multiple Scales

Let u_ϵ be the solution of Problem 1.2. We shall consider the following anzats

$$u_\epsilon(x) = u_0(x, x/\epsilon) + \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x, x/\epsilon) + \cdots, \tag{2.1}$$

where the functions $u_j(x, y)$ are Y-periodic in y. Applied to the functions $u_j(x, x/\epsilon)$ the operator ∂_{x_i} becomes $\partial_{x_i} + 1/\epsilon\partial_{y_i}$, therefore we can write the operator L_ϵ from (1.2) as

$$L_\epsilon = \epsilon^{-2}L_1 + \epsilon^{-1}L_2 + \epsilon^0 L_3, \tag{2.2}$$

where

$$\begin{aligned} L_1 &= -\partial_{y_i}(a_{ij}(y)\partial_{y_j}), \\ L_2 &= -\partial_{y_i}(a_{ij}(y)\partial_{x_j}) - \partial_{x_i}(a_{ij}(y)\partial_{y_j}) \end{aligned}$$

and

$$L_3 = -\partial_{x_i}(a_{ij}(y)\partial_{x_j}).$$

Applying (2.1) and (2.2) in Equation (1.2), and matching the terms of the same order in ϵ up order one we have

$$L_1 u_0 = 0, \tag{2.3}$$

$$L_1 u_1 + L_2 u_0 = 0, \tag{2.4}$$

$$L_1 u_2 + L_2 u_1 + L_3 u_0 = f. \tag{2.5}$$

Theorem 1.3.9 implies that the only Y-periodic solution of (2.3) is

$$u_0(x, y) = u_0(x), \tag{2.6}$$

and hence (2.4) reduces to

$$L_1 u_1 = (\partial_{y_i} a_{ij}(y))\, \partial_{x_j} u_0(x). \tag{2.7}$$

We obtain u_1 by exploring the separation of variables on the right hand side of (2.7). By Theorem 1.3.9 there exists a unique weak solution $\chi^j \in H^1_{per}(Y)$ of the following problem

$$\begin{array}{c} L_1 \chi^j = -\frac{\partial}{\partial y_i} a_{ij}(y) \;\text{ in }\; Y, \;\; \chi^j \;\text{ is } Y\text{-periodic} \\ \int_Y \chi^j dy = 0, \end{array} \tag{2.8}$$

therefore the general solution of (2.7) is given by

$$u_1(x, \frac{x}{\epsilon}) = -\chi^j \left(\frac{x}{\epsilon}\right) \frac{\partial u_0}{\partial x_j}(x) + \tilde{u}_1(x). \tag{2.9}$$

In this thesis we choose $\tilde{u}_1 = 0$ in (2.9), leading to

$$u_1(x, \frac{x}{\epsilon}) = -\chi^j \left(\frac{x}{\epsilon}\right) \frac{\partial u_0}{\partial x_j}(x). \tag{2.10}$$

Consider u_2 in equation (2.5) as an unknown and x as a parameter. Then by Theorem 1.3.9 there exists a solution u_2 for (2.5) iff

$$\int_Y (L_2 u_1 + L_3 u_0 - f) dy = 0. \tag{2.11}$$

We observe that

$$\begin{aligned} \int_Y L_2 u_1 dy &= -\partial_{x_i} \int_Y a_{ik}(y) \partial_{y_k} u_1 dy \\ &= -\partial_{x_i} \int_Y a_{ik}(y) \partial_{y_k} \left(\chi^j \left(\frac{x}{\epsilon}\right) \frac{\partial u_0}{\partial x_j}(x)\right) dy \end{aligned} \tag{2.12}$$

where we have used (2.9) to obtain the last equation. Finally from (2.11) and (2.12) we obtain

$$-\frac{1}{|Y|}\int_Y \left(a_{ij} - a_{ik}\partial_{y_k}\chi^j\right)\frac{\partial^2 u_0}{\partial x_i \partial x_j}dy = f, \tag{2.13}$$

where $|Y| = \int_Y 1dy$. The above equation is know as the homogenized equation. Define the matrix

$$\begin{aligned} A_{ij} &= \frac{1}{|Y|}\int_Y \left(a_{ij}(y) - a_{ik}(y)\partial_{y_k}\chi^j(y)\right)dy \\ &= \frac{1}{|Y|}\int_Y a_{lm}(y)\frac{\partial}{\partial y_l}(y_i - \chi^i)\frac{\partial}{\partial y_m}(y_j - \chi^j)dy, \quad \text{by (2.8).} \end{aligned} \tag{2.14}$$

It is easy to check that the matrix A is symmetric positive definite, and from (2.13) we define $u_0 \in H_0^1(\Omega)$ as the weak solution of

$$-\nabla.A\nabla u_0 = f \text{ in } \Omega, \quad u_0 = 0 \text{ on } \partial\Omega. \tag{2.15}$$

Note that we are looking toward an approximation for u_ϵ and that $u_0+\epsilon u_1$ does not satisfy the zero Dirichlet boundary condition on $\partial\Omega$, imposed for u_ϵ. In order to overcome this, the boundary corrector term, $\theta_\epsilon \in H^1(\Omega)$, is introduced as the solution of

$$-\nabla \cdot a(x/\epsilon)\nabla\theta_\epsilon = 0 \text{ in } \Omega, \quad \theta_\epsilon = -u_1(x, \frac{x}{\epsilon}) \text{ on } \partial\Omega, \tag{2.16}$$

and we obtain $u_0 + \epsilon u_1 + \epsilon\theta_\epsilon \in H_0^1(\Omega)$. See Propositions 3.1.1 and 3.3.1 for error estimates between u_ϵ and $u_0 + \epsilon u_1 + \epsilon\theta_\epsilon$ in norms $\|\cdot\|_1$ and $\|\cdot\|_0$, respectively.

We also define the term u_2, although it is not used in the numerical method, it appears in the proof of Proposition 3.2.1. Set

$$b_{ij} = -a_{ij} + a_{ik}\frac{\partial\chi^j}{\partial y_k} + \frac{\partial}{\partial y_k}(a_{ki}\chi^j),$$

observe that $\overline{b}_{ij} = A_{ij}$, where $\overline{b}_{ij} = \int_Y b_{ij}dy$. Define $\chi^{ij} \in H^1_{per}(Y)$ as the weak solution with zero average over Y of

$$\nabla_y \cdot a\nabla_y\chi^{ij} = b_{ij} - \overline{b}_{ij}, \tag{2.17}$$

and let

$$u_2(x, \frac{x}{\epsilon}) = -\chi^{ij}\left(\frac{x}{\epsilon}\right)\frac{\partial^2 u_0}{\partial x_i \partial x_j}(x). \tag{2.18}$$

2.1.1 Boundary Corrector Approximation

The coefficients $a_{ij}(x/\epsilon)$ and the boundary values $-u_1(x, \frac{x}{\epsilon})$ in the Equation (2.16) are highly oscillatory, hence to obtain a good discrete approximation for θ_ϵ is not a trivial problem. We propose an analytical approximation for θ_ϵ, denoted by ϕ_ϵ, which satisfies the oscillating boundary condition and is suitable for numerical approximation. The approximation proposed here is similar to the ones used in [3, 41].

Note that u_0 vanishes on $\partial\Omega$, therefore $\nabla u_0|_{\partial\Omega} = \eta\partial_\eta u_0$, where η denotes the unity outward normal vector to $\partial\Omega$ and $\partial_\eta u_0$ denotes the unity outward derivative of u_0 on $\partial\Omega$. Hence in order to obtain the approximation ϕ_ϵ for θ_ϵ, we introduce the following decomposition $\theta_\epsilon = \tilde{\theta}_\epsilon + \bar{\theta}_\epsilon$ where

$$(2.19)\quad -\nabla \cdot a(x/\epsilon)\nabla\tilde{\theta}_\epsilon = 0 \text{ in } \Omega, \quad \tilde{\theta}_\epsilon = (\chi^j(\frac{x}{\epsilon})\eta_j - \chi^*)\partial_\eta u_0 \text{ on } \partial\Omega$$

and

$$(2.20)\qquad -\nabla \cdot a(x/\epsilon)\nabla\bar{\theta}_\epsilon = 0 \text{ in } \Omega, \quad \bar{\theta}_\epsilon = \chi^*\partial_\eta u_0 \text{ on } \partial\Omega,$$

where $\chi^*|_{\Gamma_k} = \chi^*_k$, $k \in \{e, w, n, s\}$ are properly chosen constants, and $\Gamma_e = \{1\} \times [0,1]$, $\Gamma_w = \{0\} \times [0,1]$, $\Gamma_n = [0,1] \times \{1\}$, and $\Gamma_s = [0,1] \times \{0\}$. In Remark 2.1.1 we show that $\chi^*\partial_\eta u_0$ and $\chi^j(\frac{x}{\epsilon})\eta_j\partial_\eta u_0 \in H^{1/2}(\partial\Omega)$, therefore the Problems (2.19) and (2.20) are well posed. Later in this section we define the functions $\tilde{\phi}_\epsilon$ and $\bar{\phi}_\epsilon$, which are the approximations for $\tilde{\theta}_\epsilon$ and $\bar{\theta}_\epsilon$ respectively, and define $\phi_\epsilon = \tilde{\phi}_\epsilon + \bar{\phi}_\epsilon$.

Remark 2.1.1 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygon and assume $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$. We have by Theorem 1.3.5 that $\partial_\eta u_0|_{\Gamma_k} \in H^{1/2}_{00}(\Gamma_k)$ and $\|\partial_\eta u_0\|_{H^{1/2}_{00}(\Gamma_k)} \leq \|u_0\|_2$, therefore*

$$\|\chi^*\partial_\eta u_0\|_{H^{1/2}(\partial\Omega)} \;\leq\; c(\chi^*)\|u_0\|_2.$$

Note also that $u_1(x, \frac{x}{\epsilon}) = -\chi^j\left(\frac{x}{\epsilon}\right)\frac{\partial u_0}{\partial x_j}(x)$ and $\frac{\partial u_1}{\partial x_l} = -\left(\frac{\partial \chi^j}{\partial x_l}\right)\frac{\partial u_0}{\partial x_j} - \chi^j\left(\frac{\partial^2 u_0}{\partial x_l \partial x_j}\right)$. If we assume $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(Y)$, for $p \geq 2$ and $q > 2$ or $p > 2$ and $q \geq 2$, by a direct application of Sobolev embedding Theorem (5.4 [1]) we obtain $u_1 \in H^1(\Omega)$. In addition, from regularity theory of elliptic equations we obtain $\chi^j \in L^\infty(Y) \cap H^1(Y)$ (see Theorems 1.3.4 and 1.3.8), hence we also have $u_1|_{\Gamma_k} \in H^{1/2}_{00}(\Gamma_k)$.

Calculating the Constants χ_k^*

We define the constants χ_k^* such that the function $\tilde{\phi}_\epsilon$ decays exponentially to zero away from the boundary and satisfies the Dirichlet boundary condition $\tilde{\phi}_\epsilon(x) = -u_1(x, \frac{x}{\epsilon}) - \chi^* \partial_\eta u_0(x)$ for $x \in \partial\Omega$.

Associated to each side of Ω define the functions v_k, $k \in \{e, w, n, s\}$ as

1. Let $G_e = \{(-\infty, 0] \times [0,1]\}$ and v_e the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y v_e = 0 \text{ in } G_e, \\
&v_e(0, y_2) = \chi^1(1/\epsilon, y_2) \quad \text{for} \quad 0 < y_2 < 1, \\
&v_e(y_1, \cdot) \quad [0,1]\text{-periodic for } \quad -\infty < y_1 < 0, \\
&\text{and} \quad \partial_{y_i} v_e \exp(-\gamma y_1) \in L^2(G_e) \quad i = 1, 2.
\end{aligned}$$

2. Let $G_w = \{[0, \infty) \times [0,1]\}$ and v_w the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y v_w = 0 \quad \text{in} \quad G_w, \\
&v_w(0, y_2) = -\chi^1(0, y_2) \quad \text{for} \quad 0 < y_2 < 1, \\
&v_w(y_1, \cdot) \quad [0,1]\text{-periodic for} \quad 0 < y_1 < \infty, \\
&\text{and} \quad \partial_{y_i} v_w \exp(\gamma y_1) \in L^2(G_w) \quad i = 1, 2.
\end{aligned}$$

3. Let $G_n = \{[0,1] \times (-\infty, 0]\}$ and v_n the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y v_n = 0 \quad \text{in} \quad G_n, \\
&v_n(y_1, 0) = \chi^2(y_1, 1/\epsilon) \quad \text{for} \quad 0 < y_1 < 1, \\
&v_n(\cdot, y_2) \quad [0,1]\text{-periodic for } \quad -\infty < y_2 < 0, \\
&\text{and} \quad \partial_{y_i} v_n \exp(-\gamma y_2) \in L^2(G_n) \quad i = 1, 2.
\end{aligned}$$

4. Let $G_s = \{[0,1] \times [0, \infty)\}$ and v_s the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y v_s = 0 \quad \text{in} \quad G_s, \\
&v_s(y_1, 0) = -\chi^2(y_1, 0) \quad \text{for} \quad 0 < y_1 < 1, \\
&v_s(\cdot, y_2) \quad [0,1]\text{-periodic for} \quad 0 < y_2 < \infty, \\
&\text{and} \quad \partial_{y_i} v_n \exp(\gamma y_2) \in L^2(G_s) \quad i = 1, 2.
\end{aligned}$$

By Theorem 1.3.10 there exists constants χ_k^*, such that

$$|v_k(y) - \chi_k^*| \le c\exp(\gamma y \cdot \eta_k) \text{ as } \; y \cdot \eta_k \to -\infty,$$

where η_k denotes the unity outward normal on Γ_k.

Approximating $\tilde{\theta}_\epsilon$

We note by Remark 2.1.1 that $(u_1(x, \frac{x}{\epsilon}) - \chi^* \partial_\eta u_0)|_{\Gamma_k} \in H_{00}^{1/2}(\Gamma_k)$. Thus, we can split $\tilde{\theta}_\epsilon = \sum_{k \in \{e,w,n,s\}} \tilde{\theta}_\epsilon^k$ where

(2.21) $$L_\epsilon \tilde{\theta}_\epsilon^k = 0 \text{ in } \Omega, \quad \text{and} \quad \tilde{\theta}_\epsilon^k = \begin{cases} -u_1(x, \frac{x}{\epsilon}) - \chi^* \partial_\eta u_0 & \text{on } \Gamma_k \\ 0 & \text{on } \partial\Omega \setminus \Gamma_k. \end{cases}$$

We approximate $\tilde{\theta}_\epsilon^k$ by $\tilde{\phi}_\epsilon^k$ given by

(2.22) $$\begin{aligned} \tilde{\phi}_\epsilon^e(x_1, x_2) &= \varphi_e(x_1) \left(v_e(\frac{x_1 - 1}{\epsilon}, \frac{x_2}{\epsilon}) - \chi_e^* \right) \frac{\partial u_0}{\partial x_1}(x_1, x_2), \\ \tilde{\phi}_\epsilon^w(x_1, x_2) &= -\varphi_w(x_1) \left(v_w(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) - \chi_w^* \right) \frac{\partial u_0}{\partial x_1}(x_1, x_2), \\ \tilde{\phi}_\epsilon^n(x_1, x_2) &= \varphi_n(x_2) \left(v_n(\frac{x_1}{\epsilon}, \frac{x_2 - 1}{\epsilon}) - \chi_n^* \right) \frac{\partial u_0}{\partial x_2}(x_1, x_2), \\ \tilde{\phi}_\epsilon^s(x_1, x_2) &= -\varphi_s(x_2) \left(v_s(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) - \chi_s^* \right) \frac{\partial u_0}{\partial x_2}(x_1, x_2), \end{aligned}$$

where φ_k are nonnegative smooth functions satisfying

$$\varphi_e(s) = \varphi_n(s) = \begin{cases} 1 & \text{if } s \in [2/3, 1] \\ 0 & \text{if } s \in [0, 1/3], \end{cases} \qquad \varphi_w(s) = \varphi_s(s) = \begin{cases} 0 & \text{if } s \in [2/3, 1] \\ 1 & \text{if } s \in [0, 1/3]. \end{cases}$$

Hence

(2.23) $$\tilde{\phi}_\epsilon = \sum_{k \in \{e,w,n,s\}} \tilde{\phi}_\epsilon^k$$

approximates $\tilde{\theta}_\epsilon$, and $\tilde{\phi}_\epsilon = \tilde{\theta}_\epsilon$ on the boundary of Ω.

Remark 2.1.2 *We note that the approximation considered here is slight different from the one considered in [41]. There it is defined*

$$\tilde{\phi}_\epsilon^e(x_1, x_2) = \varphi_e(x_1) \left(v_e(\frac{x_1 - 1}{\epsilon}, \frac{x_2}{\epsilon}) - \chi_e^* \right) V(x_2)$$

where $V \in C_0^\infty((0,1))$ and $V(x_2)$ approximates $\partial_\eta u_0|_{\Gamma_e}$. Similar for the other functions $\tilde{\phi}_\epsilon^k$.

Approximating $\bar{\theta}_\epsilon$

The boundary condition imposed on Equation (2.20) does not depend on ϵ. An effective approximation for $\bar{\theta}_\epsilon$ is given by $\bar{\phi} \in H^1(\Omega)$ the weak solution of

$$-\nabla \cdot A\nabla\bar{\phi} = 0 \text{ in } \Omega, \quad \bar{\phi} = \chi^* \partial_\eta u_0 \text{ on } \partial\Omega. \tag{2.24}$$

By Propositions 3.1.3 and 3.2.2, we have that $\bar{\phi}$ is a good approximation for $\bar{\theta}_\epsilon$ only on the L^2 norm, since $\|\bar{\phi} - \bar{\theta}_\epsilon\|_0$ is $O(\epsilon)$ and $\|\bar{\phi} - \bar{\theta}_\epsilon\|_1$ is $O(1)$. We note, however, that the asymptotic expansion considered here to approximate u_ϵ is given by $u_0 + \epsilon u_1 + \epsilon\bar{\theta}_\epsilon + \epsilon\tilde{\theta}_\epsilon$, and by a triangular inequality we obtain $\|u_\epsilon - u_0 - \epsilon u_1 - \epsilon\bar{\phi} - \epsilon\tilde{\theta}_\epsilon\|_1 \leq c\epsilon + \|u_\epsilon - u_0 - \epsilon u_1 - \epsilon\theta_\epsilon\|_1$. Hence, when estimating the error on the H^1 norm between u_ϵ and its theoretical approximation, the contribution due to the approximation of $\bar{\theta}_\epsilon$ by $\bar{\phi}$ is $O(\epsilon)$.

Approximating u_ϵ

We finally define the theoretical approximation for u_ϵ as $u_0 + \epsilon u_1 + \epsilon\phi_\epsilon$, where

$$\phi_\epsilon = \tilde{\phi}_\epsilon + \bar{\phi}. \tag{2.25}$$

Note that $\phi_\epsilon|_{\partial\Omega} = \theta_\epsilon|_{\partial\Omega}$, therefore $u_0 + \epsilon u_1 + \epsilon\phi_\epsilon = 0$ on $\partial\Omega$. The following theorems provide error estimates between u_ϵ and $u_0 - \epsilon u_1 - \epsilon\phi_\epsilon$ on the H^1 and L^2 norms. Theorem 2.1.1 estimates the error on the H^1 norm, while Theorems 2.1.2 and 2.1.3 estimate the error on the L^2 norm. Theorem 2.1.2 assumes more regularity on u_0 and less regularity on a that is assumed in Theorem 2.1.3.

Theorem 2.1.1 *Let u_ϵ be the solution of the Problem (1.2), u_0, u_1 and ϕ_ϵ defined by Equations (2.15), (2.10) and (2.25), respectively. Assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{2,p}(\Omega)$, $\chi^j \in W^{1,q}_{per}(Y)$, v_e and $\nabla(v_e - \chi^*_e) exp(-\gamma y_1) \in L^s(G_e)$, for $1/s + 3/p \leq 1$, $s \geq 2$ and $1/p + 1/q \leq 1/2$. We also assume similar hypothesis for the other functions v_k. Then there exists a constant c independent of ϵ such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\phi_\epsilon(\cdot)\|_1 \leq c\epsilon\|u_0\|_{2,p}.$$

Proof: See Section 3.1 □

Theorem 2.1.2 *Let u_ϵ be the solution of Problem (1.2), u_0, u_1, ϕ_ϵ, $\bar{\phi}$ and χ^{ij} defined by Equations (2.15), (2.10), (2.25), (2.24) and (2.17) , respectively. Assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{3,p}(\Omega)$, and $\bar{\phi} \in W^{2,p}(\Omega)$ and $\chi^{ij} \in W^{1,q}_{per}(Y)$, for $p > 2$ and $1/p + 1/q \leq 1/2$. Assume also $\chi^j \in W^{1,\infty}(Y)$, v_e and $\nabla(v_e - \chi^*_e) exp(-\gamma y_1) \in L^\infty(G_e)$. We also assume similar hypothesis for the other functions v_k. Then there exists a constant c independent of ϵ such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\phi_\epsilon(\cdot)\|_0 \leq c\epsilon^{3/2}\|u_0\|_{3,p}.$$

Proof: See Section 3.2 □

Theorem 2.1.3 *Let u_ϵ be the solution of Problem (1.2), u_0, u_1 and ϕ_ϵ be defined by Equations (2.15), (2.10) and (2.25), respectively. Assume $a_{ij} \in C^{1,\beta}_{per}(Y)$, $\beta > 0$, $u_0 \in H^3(\Omega)$. Then there exists a constant c independent of ϵ such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\phi_\epsilon(\cdot)\|_0 \leq c\epsilon^{3/2}\|u_0\|_3.$$

Proof: See Section 3.3 □

Remark 2.1.3 *Due to the Proposition 3.1.2, which under the hypothesis of Theorems 2.1.2 and 2.1.3 gives that $\|\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon\|_0$ is $O(\epsilon^{1/2})$, we obtain a factor $\epsilon^{3/2}$ in these theorems, rather than ϵ^2 as in Propositions 3.2.1 and 3.3.1.*

Chapter 3

Theoretical error estimates

Here we prove Theorems 2.1.1, 2.1.3 and 2.1.2.

3.1 Proof of Theorem 2.1.1

By the triangular inequality we have

$$\begin{aligned} |u_\epsilon - u_0 - u_1 - \phi_\epsilon|_{1,h} &\leq |u_\epsilon - u_0 - u_1 - \theta_\epsilon|_1 \\ &\quad + \epsilon|\bar{\theta}_\epsilon - \bar{\phi}|_1 + \epsilon|\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon|_1, \end{aligned}$$

and the theorem follows from Propositions 3.1.1, 3.1.2 and 3.1.3. □

We now prove the propositions used in the proof of Theorem 2.1.1. The following proposition gives the same error estimate of Theorem 2.2 in [3], however here we assume $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(\Omega)$ for $1/p + 1/q \leq 1/2$ while in Theorem 2.2 in [3] it is assumed $u_0 \in W^{2,\infty}(\Omega)$ and $\chi^j \in H^1_{per}(\Omega)$. It also generalizes Proposition 2.1 from [41] where it is assumed $a_{ij} \in C^{1,\beta}_{per}(Y)$, $u_0 \in H^2(\Omega)$ and $\Omega \subset \mathbb{R}^2$. We note here that Theorem 1.1 from [37] gives conditions concerning the discontinuities of the functions a_{ij} such that $\chi^j \in W^{1,\infty}_{per}(Y)$. Finally, we observe that in the case $a_{ij} \in C^{1,\beta}_{per}(Y)$ a error estimate similar to Proposition 3.1.1 can be obtained in the case a zero Neumann boundary condition is used to define u_ϵ; see [42].

Proposition 3.1.1 *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a convex domain, u_ϵ be the solution of Problem (1.2) and u_0, u_1, and θ_ϵ be defined by Equations (2.15), (2.10), and (2.16), respectively. Assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{2,p}(\Omega)$ and*

$\chi^j \in W^{1,q}_{per}(Y)$ for $1/p+1/q \leq 1/2$. Then there exists a constant c independent of u_0 and ϵ, such that

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\theta_\epsilon(\cdot)\|_1 \leq c\epsilon\|u_0\|_{2,p}.$$

Proof: Define

$$\begin{aligned} v_0(x,y) &= a(y)\nabla_x u_0(x) + a(y)\nabla_y u_1(x,y) \\ &= a(y)(\nabla_y y_j - \nabla_y \chi^j(y))\frac{\partial u_0}{\partial x_j}(x). \end{aligned} \tag{3.1}$$

From the definition of χ^j we have

$$\int_Y \left(a(y)(e_j - \nabla_y\chi^j(y)) - Ae_j\right)\nabla_y\phi(y)dy = 0, \quad \forall\ \phi \in H^1_{per}(Y).$$

Since the vector $a(y)(e_j - \nabla_y\chi^j(y)) - Ae_j$ is Y periodic and has zero average entries over Y, by Lemma 1.3.1 there exists $\phi_j(y) \in H^1_{per}(Y)$ with zero average over Y such that

$$a(y)(\nabla_y y_j - \nabla_y\chi^j(y)) - Ae_j = -curl_y\phi_j(y). \tag{3.2}$$

Let

$$\phi(x,y) = \phi_j(y)\frac{\partial u_0}{\partial x_j}(x) \tag{3.3}$$

and define

$$\begin{aligned} v_1(x,y) &= -curl_x\phi(x,y) \\ &= \begin{pmatrix} -\phi_j(y)\frac{\partial^2 u_0}{\partial x_2\partial x_j}(x) \\ \phi_j(y)\frac{\partial^2 u_0}{\partial x_1\partial x_j}(x) \end{pmatrix}. \end{aligned}$$

Observe that $a \in L^\infty(Y)^{2\times 2}$ and if $d = 2$ $|curl_y\phi_j|_{0,q} = |\phi_j|_{1,q}$. Since $\chi^j \in W^{1,q}_{per}(Y)$ and ϕ_j has zero average over Y, we apply a Poincare inequality to obtain

$$\|\phi_j\|_{1,q,Y} \leq c|curl_y\phi_j|_{0,q,Y} \leq c(\|\chi^1\|_{1,q,Y} + \|\chi^2\|_{1,q,Y}).$$

In the case $d = 3$ by Remark 3.11 in [30] $\phi_j \in W^{1,q}_{per}(Y)^3$. From hypothesis $u_0 \in W^{2,p}(\Omega)$ for $1/p + 1/q \leq 1/2$, hence $v_1(x, x/\epsilon) \in L^2(\Omega)$ and $\|v_1\|_0 \leq c(\|\chi^1\|_{1,q,Y} + \|\chi^2\|_{1,q,Y})\|u_0\|_{2,p}$. Moreover, by Lemma 1.3.1,

$$\nabla_x \cdot v_1(x,y) = 0, \tag{3.4}$$

and simple calculations give

$$
\begin{aligned}
\nabla_y \cdot v_1(x,y) &= \nabla_y \cdot curl_x \left(\phi_j(y)\partial_{x_j} u_0(x)\right) \\
&= -\nabla_x \cdot curl_y \left(\phi_j(y)\partial_{x_j} u_0(x)\right) \\
&= -\nabla_x \cdot v_0(x,y) - f. \qquad (3.5)
\end{aligned}
$$

Let

$$z_\epsilon(x) = u_\epsilon(x) - u_0(x) - \epsilon u_1(x, x/\epsilon)$$

and

$$\eta_\epsilon(x) = a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x, x/\epsilon) - \epsilon v_1(x, x/\epsilon).$$

Then

$$
\begin{aligned}
& a(x/\epsilon)\nabla z_\epsilon(x) - \eta_\epsilon(x) \\
& = a(x/\epsilon)\nabla u_\epsilon(x) - a(x/\epsilon)\nabla_x u_0(x) - \epsilon a(x/\epsilon)\nabla_x u_1(x, x/\epsilon) \\
& \quad - a(x/\epsilon)\nabla_y u_1(x, x/\epsilon) - a(x/\epsilon)\nabla u_\epsilon(x) + v_0(x, x/\epsilon) + \epsilon v_1(x, x/\epsilon) \\
& = \epsilon(v_1(x, x/\epsilon) - a(x/\epsilon)\nabla_x u_1(x, x/\epsilon)),
\end{aligned}
$$

and so

$$\|a(\cdot/\epsilon)\nabla z_\epsilon - \eta_\epsilon\|_0 \le \epsilon \|v_1(\cdot, \cdot/\epsilon) - a(\cdot/\epsilon)\nabla_x u_1(\cdot, \cdot/\epsilon)\|_0. \qquad (3.6)$$

Given $g \in L^2(\Omega)$, let $w_\epsilon \in H_0^1(\Omega)$ be the solution of

$$\int_\Omega a(x/\epsilon)\nabla w_\epsilon(x)\nabla\psi(x)dx = \int_\Omega g(x)\psi(x)dx, \quad \forall \psi \in H_0^1(\Omega), \qquad (3.7)$$

hence

$$
\begin{aligned}
\int_\Omega g(z_\epsilon - \epsilon\theta_\epsilon)dx &= \int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot \nabla(z_\epsilon - \epsilon\theta_\epsilon)dx \\
&= \int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot \nabla z_\epsilon dx - \epsilon \int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot \nabla\theta_\epsilon dx \\
&= \int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot \nabla z_\epsilon dx. \qquad (3.8)
\end{aligned}
$$

Now observe that

$$\int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot \nabla z_\epsilon dx = \int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot (\nabla z_\epsilon - \eta_\epsilon)dx + \int_\Omega \eta_\epsilon \cdot \nabla w_\epsilon dx. \qquad (3.9)$$

In order to estimate the second term on the right hand side of (3.9) we apply the definition of η_ϵ to obtain

$$\begin{aligned}\int_\Omega \eta_\epsilon \cdot \nabla w_\epsilon dx &= \int_\Omega (a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x,x/\epsilon) - \epsilon v_1(x,x/\epsilon)) \cdot \nabla w_\epsilon(x)dx \\ &= \int_\Omega f w_\epsilon dx - \int_\Omega (v_0(x,x/\epsilon) - \epsilon v_1(x,x/\epsilon)) \cdot \nabla w_\epsilon(x)dx. \qquad (3.10)\end{aligned}$$

We note that

$$\begin{aligned}\int_\Omega v_1(x,x/\epsilon) \cdot \nabla w_\epsilon(x)dx &= \int_\Omega \nabla \cdot v_1(x,x/\epsilon) w_\epsilon(x)dx \\ &= \int_\Omega (\nabla_x + 1/\epsilon \nabla_y) \cdot v_1(x,y)|_{(y=x/\epsilon)} w_\epsilon(x)dx \\ &= \frac{-1}{\epsilon}\int_\Omega (\nabla_x \cdot v_0 + f) w_\epsilon dx, \qquad (3.11)\end{aligned}$$

where we have used (3.4) and (3.5) to obtain (3.11). Using the definition of v_0 we have

$$\int_\Omega v_0(x,x/\epsilon) \cdot \nabla w_\epsilon(x)dx = \int_\Omega a(x/\epsilon)(e_j - \nabla_y \chi^j(x/\epsilon))\frac{\partial u_0}{\partial x_j}(x) \cdot \nabla w_\epsilon(x)dx,$$

and by the chain rule we obtain

$$\begin{aligned}\int_\Omega v_0(x,x/\epsilon) \cdot \nabla w_\epsilon dx &= \int_\Omega a(x/\epsilon)(e_j - \nabla_y \chi^j(x/\epsilon)) \cdot \nabla \left(\frac{\partial u_0}{\partial x_j} w_\epsilon(x)\right) dx \\ &\quad - \int_\Omega a(x/\epsilon)(e_j - \nabla_y \chi^j(x/\epsilon)) \cdot \left(w_\epsilon \nabla \frac{\partial u_0}{\partial x_j}(x)\right) dx. \qquad (3.12)\end{aligned}$$

In this paragraph we evaluate the first term on the right hand side of (3.12). Let $(\frac{\epsilon}{3}Y_i)_{i=1,\ldots,i_m}$ be a finite set of translated cells of $\frac{\epsilon}{3}Y$, recovering $\overline{\Omega}$, and consider a partition of unity ρ_i, such that supp$\rho_i \subset \frac{2\epsilon}{3}Y_i$, where $\frac{2\epsilon}{3}Y_i$ denotes the cell $\frac{2\epsilon}{3}Y$ centered in $\frac{\epsilon}{3}Y_i$. We note that

$$\text{supp}(\rho_i w_\epsilon) \subset \frac{2\epsilon}{3}Y_i \subset \epsilon Y_i \qquad (3.13)$$

hence

$$\int_\Omega a(x/\epsilon)(e_j - \nabla_y \chi^j(x/\epsilon)) \cdot \nabla(\frac{\partial u_0}{\partial x_j} w_\epsilon(x))dx$$

$$= \sum_{i=1:i_m} \int_{\epsilon Y_i} a(x/\epsilon)(e_j - \nabla_y \chi^j(x/\epsilon)) \cdot \nabla(\rho_i \frac{\partial u_0}{\partial x_j} w_\epsilon(x))dx$$

$$= 0. \tag{3.14}$$

Here to obtain (3.14) we first note that $\chi^j \in W^{1,q}_{per}(Y)$, $H^1_{per}(Y) \hookrightarrow W^{1,q'}_{per}(Y)$ for $1/q' = 1 - 1/q$, and (2.8) implies

$$\int_Y a_{ij}(y)\partial_{y_l}(\chi^j - y_j)\partial_{y_m}\psi = 0, \quad \forall\ \psi \in W^{1,q'}_{per}(Y).$$

Since $1/p + 1/q \leq 1/2$ and (3.13) holds we obtain $\rho_i \partial_{x_j} u_0 w_\epsilon \in W^{1,q'}_{per}(\epsilon Y_i)$ and (3.14) follows.

For the second term on the right hand side of equation (3.12), we use the definition of v_0 and it follows that

$$-\int_\Omega a(x/\epsilon)(e_j - \nabla_y \chi^j(x/\epsilon)) \cdot \left(w_\epsilon \nabla \frac{\partial u_0}{\partial x_j}(x) \right) dx = -\int_\Omega \nabla_x \cdot v_0(x, x/\epsilon) w_\epsilon(x) dx.$$

Hence

$$\int_\Omega v_0(x, x/\epsilon) \cdot \nabla w_\epsilon(x) dx = -\int_\Omega \nabla_x \cdot v_0(x, x/\epsilon) w_\epsilon(x) dx. \tag{3.15}$$

From equations (3.10), (3.11) and (3.15) we obtain

$$\int_\Omega \eta_\epsilon \cdot \nabla w_\epsilon dx = 0,$$

and from (3.9)

$$\int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot \nabla z_\epsilon dx = \int_\Omega a(\cdot/\epsilon)(\nabla z_\epsilon - \eta_\epsilon) \cdot \nabla w_\epsilon) dx. \tag{3.16}$$

From equations (3.8) and (3.16) we have

$$\begin{aligned} \left| \int_\Omega g(z_\epsilon - \epsilon\theta_\epsilon) dx \right| &\leq c \|a(\cdot/\epsilon)\nabla z_\epsilon - \eta_\epsilon\|_0 \|w_\epsilon\|_1 \\ &\leq \epsilon \|v_1(\cdot, \cdot/\epsilon) - a(\cdot/\epsilon)\nabla_x u_1(\cdot, \cdot/\epsilon)\|_0 \|g\|_{-1}, \quad \text{by (3.6)}. \end{aligned}$$

Dividing by $\|g\|_{-1}$ and taking the supremum for $g \neq 0$ we get

$$\begin{aligned} \|z_\epsilon(x) - \epsilon\theta_\epsilon\|_1 &\leq c\epsilon \|v_1(\cdot, \cdot/\epsilon) - a(\cdot/\epsilon)\nabla_x u_1(\cdot, \cdot/\epsilon)\|_0 \\ &\leq c\epsilon(\|\chi^1\|_{1,q,Y} + \|\chi^2\|_{1,q,Y})\|u_0\|_{2,p}. \end{aligned}$$

□

The following remark is used in the proof of Proposition 3.2.2.

Remark 3.1.1 *Let $f \in H^{-1}(\Omega)$, $g \in H^{1/2}(\partial\Omega)$ and define $u_\epsilon \in H^1(\Omega)$ as the weak solution of the following problem*

$$L_\epsilon u_\epsilon = f \quad in \quad \Omega, \quad u_\epsilon = g \quad on \quad \partial\Omega.$$

It is easy to see that Proposition 3.1.1 extends immediately to this case if u_0, defined as the solution of

$$-\nabla . A \nabla u_0 = f \quad in \quad \Omega, \qquad u_0 = g \quad on \quad \partial\Omega,$$

belongs to $W^{2,p}(\Omega)$.

The following corollary follows from Proposition 3.1.1 and is used in the proof of Proposition 3.2.2.

Corollary 3.1.1 *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a convex domain, u_ϵ and u_0 be defined by Equations (1.2) and (2.15), respectively. Assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{m,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(Y)$ for $(m-1)p > 2$ and $1/p + 1/q \leq 1/2$. Then there exists a constant c independent of u_0 and ϵ such that*

$$\|u_\epsilon - u_0\|_0 \leq c\epsilon \|u_0\|_{m,p}.$$

Proof: The hypothesis $u_0 \in W^{m,p}(\Omega)$, $(m-1)p > d$ implies $\partial_{x_i} u_0 \in C(\Omega)$, and $\chi^j \in C(Y)$ see Remark 2.1.1, therefore $\|u_1\|_0 \leq c\|u_0\|_{m,p}$. From the maximum principle $\|\theta_\epsilon\|_{0,\infty} \leq \|\partial_{x_i} u_0\|_{0,\infty,\partial\Omega} \|\chi^i\|_{0,\infty,\partial\Omega}$, and hence the corollary follows from Proposition 3.1.1. □

The following proposition estimates the H^1 norm of $\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon$, and is used in the proof of Theorem 3.1.

Proposition 3.1.2 *Let u_0, $\tilde{\theta}_\epsilon$ and $\tilde{\phi}_\epsilon$ be defined by Equations (2.15), (2.19) and (2.23), respectively, and the functions v_k be defined as in Subsection 2.1.1. Assume $u_0 \in W^{2,p}(\Omega)$, and v_e and $\nabla(v_e - \chi^*_e) exp(-\gamma y_1) \in L^s(G_e)$ for $s \geq 2$ and $1/s + 3/p \leq 1$. We also assume similar hypothesis for the other functions v_k. Then there exists positive constants $0 < \delta(p, s) \leq 1/2$, and $c(\delta, \gamma)$ independent of ϵ such that*

$$\begin{aligned} \|\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon\|_1 \quad &\leq \quad c(\delta, \gamma)\epsilon^\delta \|a\|_{0,\infty} \|u_0\|_{2,p} \max_k \left(\|\nabla(v_k - \chi^*_k) exp(-\gamma y \cdot \eta^k)\|_{0,s,G_k} \right. \\ &\qquad \left. + \|v_k - \chi^*_k\|_{1,s,G_k} \right). \end{aligned}$$

In addition, when $p, s \to \infty$ then $\delta \to 1/2$ with $c(\delta, \gamma)$ bounded independent of δ.

Proof: By definition

$$\|\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon\|_1 \leq \sum_{k\in\{e,w,n,s\}} \|\tilde{\theta}^k_\epsilon - \tilde{\phi}^k_\epsilon\|_1.$$

Consider the case $k = e$, the other cases are treated in a similar way, and the notation $v^\epsilon_e(x) = v_e(\frac{x_1-1}{\epsilon}, \frac{x_2}{\epsilon})$ and $a^\epsilon(x) = a(x/\epsilon)$. Let $g \in H^1_0(\Omega)$, then we apply the definition of $\tilde{\phi}^e_\epsilon$ to obtain

$$\begin{aligned}
\int_\Omega a^\epsilon \nabla(\tilde{\theta}^e_\epsilon - \tilde{\phi}^e_\epsilon)\nabla g dx &= \int_\Omega -a^\epsilon \nabla \left((v^\epsilon_e - \chi^*_e)\varphi_e \frac{\partial u_0}{\partial x_1}\right) \nabla g dx \\
&= -\int_\Omega \left(\varphi_e \frac{\partial u_0}{\partial x_1} a^\epsilon \nabla (v^\epsilon_e - \chi^*_e)\right) \nabla g dx \\
&\quad -\int_\Omega \left((v^\epsilon_e - \chi^*_e) a^\epsilon \nabla \left(\varphi_e \frac{\partial u_0}{\partial x_1}\right)\right) \nabla g dx.
\end{aligned} \tag{3.17}$$

We note that due to the Sobolev embedding theorem, the integrals above are well defined. For the first term on the right hand side of Equation (3.17) we have

$$\begin{aligned}
\int_\Omega \left(\varphi_e \frac{\partial u_0}{\partial x_1} a^\epsilon \nabla (v^\epsilon_e - \chi^*_e)\right) \nabla g dx &= \int_\Omega a^\epsilon \nabla (v^\epsilon_e - \chi^*_e) \nabla \left(\varphi_e \frac{\partial u_0}{\partial x_1} g\right) dx \\
&\quad - \int_\Omega a^\epsilon \nabla (v^\epsilon_e - \chi^*_e) \cdot g \nabla \left(\varphi_e \frac{\partial u_0}{\partial x_1}\right) dx.
\end{aligned} \tag{3.18}$$

In this paragraph, we estimate the first term of the right hand side of (3.18). Let $I_i = \{(i-1)\epsilon/3 - \epsilon/10 < x_2 < i\epsilon/3 + \epsilon/10, \}$, $i_m = 1 + \sup_{i\in\mathbb{N}}(i3/\epsilon < 1)$, and consider a partition of unity ρ_i of Ω, subject to $(0,1) \times I_i$. Let I^ϵ_i be the interval centered in I_i with $|I^\epsilon_i| = \epsilon$. Since $\operatorname{supp}(\rho_i g) \subset [0,1] \times I^\epsilon_i$ we have

$$\begin{aligned}
&\int_\Omega a^\epsilon \nabla (v^\epsilon_e - \chi^*_e) \nabla \left(\varphi_e \frac{\partial u_0}{\partial x_1} g\right) dx \\
&\qquad = \sum_{i\leq i_m} \int_0^1 \int_{I^\epsilon_i} a^\epsilon \nabla (v^\epsilon_e - \chi^*_e) \nabla \left(\rho_i \varphi_e \frac{\partial u_0}{\partial x_1} g\right) dx_2 dx_1 \\
&\qquad = 0,
\end{aligned} \tag{3.19}$$

where to arrive in (3.19) we have used the definition of v_e and arguments similar to the ones used to obtain (3.14).

For the second term on the right hand side of equation (3.18), applying a Cauchy inequality we have

$$\left|\int_\Omega a^\epsilon \nabla(v_e^\epsilon - \chi_e^*) \cdot \nabla\left(\varphi_e \frac{\partial u_0}{\partial x_1}\right) g dx\right|$$

$$\leq \|a\|_\infty |\varphi_e \nabla u_0|_{1,p} \left\|\nabla v_e^\epsilon \exp(-\gamma \frac{x_1 - 1}{\epsilon})\right\|_{0,s}$$

$$\left(\frac{\epsilon}{\gamma}\right)^{1/l} \left\|(\gamma/\epsilon)^{1/l} \exp(\gamma \frac{x_1 - 1}{\epsilon}) g\right\|_{0,l}, \tag{3.20}$$

where $1/l = 1 - 1/p - 1/s$. Taking $y_1 = (x_1 - 1)/\epsilon$ and $y_2 = x_2/\epsilon$, and exploring the $[0,1]$-periodicity of $v_e(y_1, \cdot)$ we have

$$\left\|\nabla(v_e^\epsilon - \chi_e^*) \exp(-\gamma \frac{x_1 - 1}{\epsilon})\right\|_{0,s}^s$$

$$\leq (\frac{1}{\epsilon} + 1) \int_{-1/\epsilon}^0 \int_0^1 |\nabla_y v_e \exp(-\gamma y_1)|^s \epsilon^{2-s} dy_2 dy_1$$

$$\leq c\epsilon^{(1-s)} \|\nabla_y v_e \exp(-\gamma y_1)\|_{0,s,G_e}^s. \tag{3.21}$$

Let $g_n \in C_0^\infty(\Omega)$, $g_n \to g$ in H^1 and $I_n = (0,1) \cap |g_n| > 0$, then integrating by parts in x_1

$$\left\|(\gamma/\epsilon)^{1/l} \exp(\gamma \frac{x_1 - 1}{\epsilon}) g_n\right\|_{0,l}$$

$$= \left(\int_0^1 \int_{I_n} \frac{\gamma}{\epsilon} \exp(l\gamma \frac{x_1 - 1}{\epsilon}) |g_n|^l dx_1 dx_2\right)^{1/l}$$

$$= \left(-\int_0^1 \int_{I_n} \frac{1}{l} \exp\left(l\gamma \frac{x_1 - 1}{\epsilon}\right) \frac{\partial |g_n|^l}{\partial x_1} dx_1 dx_2\right)^{1/l} \tag{3.22}$$

$$\leq c\left(\left\|\exp\left(l\gamma \frac{x_1 - 1}{\epsilon}\right)\right\|_{0,r'} \|g_n\|_{0,s'(l-1)}^{l-1} \left\|\frac{\partial g_n}{\partial x_1}\right\|_0\right)^{1/l} \tag{3.23}$$

$$\leq c(\Omega)(s'(l-1))^{(l-1)/l} \left(\frac{\epsilon}{r' l \gamma}\right)^{1/(r'l)} |g_n|_1^2. \tag{3.24}$$

From (3.22) to (3.23) we have used a Cauchy inequality with $1/r'+1/s'=1/2$. In order to obtain (3.24), we note that the last inequality in the proof of Lemma 5.10 in [1] states

$$\begin{aligned}
\|g_n\|_{0,s'(l-1)} &\leq 2^{(t-1)/t}\left(\frac{2t-t}{2-t}\right)\|g_n\|_{1,t}, \quad \text{for } 2t/(2-t)=s'(l-1),\ 1\leq t<2 \\
&\leq 2^{(t-1)/t}\left(\frac{2t-t}{2-t}\right)\mathrm{vol}(\Omega)^{(1/t-1/2)}\|g_n\|_1, \text{ by Theorem 2.8 in [1]} \\
&\leq c(\Omega)\left(\frac{2t-t}{2-t}\right)|g_n|_1, \text{ by the Poincare inequality.}
\end{aligned}$$

Hence (3.24) follows from (3.23). Taking the limit $n\to\infty$ we obtain inequality (3.24) for g.

Since $1/s+3/p<1$, there exists $r'>2$ such that $1/lr'+1/l+1/s-1>0$, and hence from (3.18), (3.19), (3.20), (3.21) and (3.24) it follows

$$\begin{aligned}
\int_\Omega \varphi_e\frac{\partial u_0}{\partial x_1}a^\epsilon\nabla(v_e^\epsilon-\chi_e^*)\nabla g dx &\leq c(\Omega,\gamma)(s'(l-1))^{(l-1)/l}\epsilon^{\delta'}\|a\|_\infty|\varphi_e\nabla u_0|_{1,p} \\
&\quad \|\nabla(v_e-\chi_e^*)\exp(-\gamma y_1)\|_{0,s,G_e}|g|_1,
\end{aligned} \tag{3.25}$$

where $\delta'=1/lr'+1/l+1/s-1$.

For estimating the second term on the right hand side of (3.17), we apply a Cauchy inequality with $1/r+1/p=1/2$ to obtain

$$\begin{aligned}
&\left|\int_\Omega(v_e^\epsilon-\chi_e^*)a^\epsilon\nabla\left(\varphi_e\frac{\partial u_0}{\partial x_1}\right)\cdot\nabla g dx\right| \\
&\qquad\leq \|a\|_{0,\infty}\left|\varphi_e\frac{\partial u_0}{\partial x_1}\right|_{1,p}\left(\epsilon\int_{G_e}(v_e-\chi_e^*)^r dy\right)^{1/r}|g|_1 \\
&\qquad\leq c(r)\epsilon^{1/r}\|a\|_{0,\infty}\left|\varphi_e\frac{\partial u_0}{\partial x_1}\right|_{1,p}\|v_e^\epsilon-\chi_e^*\|_{1,G_e}|g|_1,
\end{aligned} \tag{3.26}$$

where we have used the Sobolev embedding Theorem to obtain the last inequality.

Taking $g=\tilde\theta_\epsilon^e-\tilde\phi_\epsilon^e$ and using the ellipticity of a

$$\begin{aligned}
|\tilde\theta_\epsilon^e-\tilde\phi_\epsilon^e|^2_{H_0^1(\Omega)} &\leq \gamma_a^{-1}\int_\Omega(a^\epsilon\nabla(\tilde\theta_\epsilon^e-\tilde\phi_\epsilon^e))\cdot\nabla(\tilde\theta_\epsilon^e-\tilde\phi_\epsilon^e)dx \\
&\leq \frac{c(r)}{\gamma_a}\epsilon^\delta\|a\|_{0,\infty}|\varphi_e\nabla u_0|_{1,p}\,(\|\nabla(v_e-\chi_e^*)\exp(-\gamma y_1)\|_{0,s,G_e} \\
&\quad +\|\nabla(v_e-\chi_e^*)\|_{1,G_e})\,|\tilde\theta_\epsilon^e-\tilde\phi_\epsilon^e|_{H_0^1(\Omega)},
\end{aligned}$$

where $\delta = \min\{\delta', 1/r\}$.

Observe that $s, p \to \infty$, implies $l \to 1$. Choosing $s' = 1/(l-1)$ in Inequality (3.24) we have that $(s'(l-1))^{(l-1)/l}\left(\epsilon/(r'l\gamma)\right)^{1/(r'l)} \to \epsilon^{1/2}/(2\gamma)$. In inequality (3.26) $p \to \infty$ implies $1/r \to 1/2$ and $c(r)\epsilon^{1/r} \to c\epsilon^{1/2}$. □

Finally, we prove the last proposition used in the proof of Theorem 3.1. Proposition 3.1.3 estimates the H^1 norm of $\bar{\phi} - \bar{\theta}_\epsilon$.

Proposition 3.1.3 *Let Ω be a convex polygon, and the functions u_0, $\bar{\theta}_\epsilon$ and $\bar{\phi}$ be defined by Equations (2.15), (2.20) and (2.24), respectively. Assume that $u_0 \in H^2(\Omega)$, then there exists a positive constant c independent of ϵ and u_0 such that*

$$|\bar{\phi} - \bar{\theta}_\epsilon|_1 \leq c\frac{\|a\|_{0,\infty,Y}}{\gamma_a}\|u_0\|_2.$$

Proof: Consider the notation $a^\epsilon(x) = a(x/\epsilon)$, the same will be used for a_{ij}. Since $(\bar{\phi} - \bar{\theta}_\epsilon) = 0$ on $\partial\Omega$ we have

$$\int_\Omega a^\epsilon_{ij}\frac{\partial(\bar{\phi} - \bar{\theta}_\epsilon)}{\partial x_i}\frac{\partial(\bar{\phi} - \bar{\theta}_\epsilon)}{\partial x_j}dx = \int_\Omega a^\epsilon_{ij}\frac{\partial\bar{\phi}}{\partial x_i}\frac{\partial(\bar{\phi} - \bar{\theta}_\epsilon)}{\partial x_j}dx$$

$$\leq \|a\|_{0,\infty,Y}\left(\int_\Omega |\nabla\bar{\phi}|^2dx\right)^{1/2}\left(\int_\Omega |\nabla(\bar{\phi} - \bar{\theta}_\epsilon)|^2dx\right)^{1/2},$$

and from the ellipticity of a we obtain

$$|\bar{\phi} - \bar{\theta}_\epsilon|_1 \leq \frac{\|a\|_{0,\infty,Y}}{\gamma_a}|\bar{\phi}|_1.$$

The regularity theory gives that $|\bar{\phi}|_1 \leq c\|\chi^*\partial_\eta u_0\|_{H^{1/2}(\partial\Omega)}$, and since Ω is a convex polygon by Remark 2.1.1

$$|\bar{\phi} - \bar{\theta}_\epsilon|_1 \leq c\|u_0\|_2.$$

□

3.2 Proof of Theorem 2.1.2

Use a triangular inequality similar to the one used in the Proof of Theorem 2.1.1 and Propositions 3.2.1, 3.1.2 and 3.2.2. □

We now prove the propositions used in the proof of Theorem 2.1.2. The following proposition generalizes Proposition 2.3 from [41], where it is assumed $a_{ij} \in C^{1,\beta}_{per}(Y)$, $u_0 \in H^3(\Omega)$ and $\Omega \subset \mathbb{R}^2$. We note here that Theorem 1.1 from [37] gives conditions concerning the discontinuities of the functions a_{ij} such that χ^j and $\chi^{ij} \in W^{1,\infty}_{per}(Y)$.

Proposition 3.2.1 *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a convex domain, u_ϵ be the solution of Problem (1.2), and χ^j, u_0, u_1, θ_ϵ and χ^{ij} be defined by Equations (2.8), (2.15), (2.10), (2.16) and (2.17), respectively. Assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{3,p}(\Omega)$, χ^j and $\chi^{ij} \in W^{1,q}_{per}(Y)$, for $p, q > d$ and $1/p + 1/q \leq 1/2$. Then there exists a constant c independent of u_0 and ϵ such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\theta_\epsilon(\cdot)\|_0 \leq C\epsilon^2 \|u_0\|_{3,p} (\max_j \|\chi^j\|_{0,q} + \max_{kj} \|\chi^{kj}\|_{1,q}).$$

Proof: Define the field v_1 by

$$(v_1(x, y))_k = -a_{ki}(y)\chi^j \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) + a_{kl}(y) \frac{\partial \chi^{ij}}{\partial y_l} \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x), \tag{3.27}$$

hence

$$a(y)\nabla_x u_1(x, y) + a(y)\nabla_y u_2(x, y) = v_1(x, y). \tag{3.28}$$

Let $q(y) = \phi(y)$, ϕ defined by Equation (3.3) and let $\psi_{ij} \in W^{1,q}_{\text{per}}(Y)$ such that

$$curl_y \psi_{1j} = \tilde{\psi}_{1j} = \begin{pmatrix} -a_{11}\chi^j + a_{1l}\partial_l \chi^{1,j} - c^1_{1j} \\ -a_{21}\chi^j + a_{2l}\partial_l \chi^{1,j} - \phi^{(3)}_j - c^2_{1j} \\ -a_{31}\chi^j + a_{3l}\partial_l \chi^{1,j} + \phi^{(2)}_j - c^3_{1j} \end{pmatrix},$$

$$curl_y \psi_{2j} = \tilde{\psi}_{2j} = \begin{pmatrix} -a_{12}\chi^j + a_{1l}\partial_l \chi^{2,j} + \phi^{(3)}_j - c^1_{2j} \\ -a_{22}\chi^j + a_{2l}\partial_l \chi^{2,j} - c^2_{2j} \\ -a_{32}\chi^j + a_{3l}\partial_l \chi^{2,j} - \phi^{(1)}_j - c^3_{2j} \end{pmatrix}$$

and

$$curl_y \psi_{1j} = \tilde{\psi}_{3j} = \begin{pmatrix} -a_{13}\chi^j + a_{1l}\partial_l \chi^{3,j} - \phi^{(2)}_j - c^1_{3j} \\ -a_{23}\chi^j + a_{2l}\partial_l \chi^{3,j} + \phi^{(1)}_j - c^2_{3j} \\ -a_{33}\chi^j + a_{3l}\partial_l \chi^{3,j} - c^3_{3j} \end{pmatrix}.$$

where the constants c^l_{ij} are chosen such that each entry of the vectors $\tilde{\psi}_{ij}$ has integral zero over Y, e.g. $c^1_{1j} = \int_Y -a_{11}\chi^j + a_{1l}\partial_l \chi^{1,j} dy$. It is easy to check that $\nabla_y \cdot \tilde{\psi}_{kj} = 0$, what guarantees by Lemma 1.3.1 the existence of such functions ψ_{kj}, and by Remark 3.11 in [30] we have

$$\|\psi_{kj}\|_{1,q} \leq c(\|\chi^j\|_{0,q} + \|\chi^{kj}\|_{1,q}). \tag{3.29}$$

Define

$$p(x,y) = \psi_{kj}(y)\frac{\partial^2 u_0}{\partial x_k \partial x_j}(x) \tag{3.30}$$

and let

$$v_2(x,y) = -curl_x p(x,y),$$

a simple calculation give

$$\nabla_y \cdot v_2 = -\nabla_x \cdot v_1, \quad \nabla_x \cdot v_2 = 0 \tag{3.31}$$

and

$$\begin{aligned} \|v_2(\cdot,\cdot/\epsilon)\|_0 &\leq c\|u_0\|_{3,p} \max_{kj} \|\psi_{kj}\|_{1,q,Y} \\ &\leq c\|u_0\|_{3,p}(\|\chi^j\|_{0,q} + \|\chi^{kj}\|_{1,q}), \quad \text{by} \quad (3.29) \end{aligned} \tag{3.32}$$

Define

$$\psi_\epsilon(x) = u_\epsilon(x) - u_0(x) - \epsilon u_1(x, x/\epsilon) - \epsilon^2 u_2(x, x/\epsilon)$$

and

$$\xi_\epsilon(x) = a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x, x/\epsilon) - \epsilon v_1(x, x/\epsilon) - \epsilon^2 v_2(x, x/\epsilon),$$

where v_0 is defined by (3.1). Then

$$\begin{aligned} &a(x/\epsilon)\nabla\psi_\epsilon - \xi_\epsilon(x) \\ &= a(x/\epsilon)\nabla u_\epsilon(x) - a(x/\epsilon)\nabla u_0(x) - \epsilon a(x/\epsilon)\nabla u_1(x, x/\epsilon) \\ &\quad -\epsilon^2 a(x/\epsilon)\nabla u_2(x, x/\epsilon) - a(x/\epsilon)\nabla u_\epsilon(x) \\ &\quad +v_0(x, x/\epsilon) + \epsilon v_1(x, x/\epsilon) + \epsilon^2 v_2(x, x/\epsilon) \\ &= -a(x/\epsilon)\nabla_x u_0(x) - \epsilon a(x/\epsilon)\nabla_x u_1(x, x/\epsilon) - a(x/\epsilon)\nabla_y u_1(x, x/\epsilon) \\ &\quad -\epsilon^2 a(x/\epsilon)\nabla_x u_2(x, x/\epsilon) - \epsilon a(x/\epsilon)\nabla_y u_2(x, x/\epsilon) \\ &\quad +v_0(x, x/\epsilon) + \epsilon v_1(x, x/\epsilon) + \epsilon^2 v_2(x, x/\epsilon) \\ &= \epsilon^2(v_2(x, x/\epsilon) - a(x/\epsilon)\nabla_x u_2(x, x/\epsilon)), \quad \text{by (3.1), and (3.28).} \end{aligned}$$

From the definition of u_2 and (3.32) we obtain

$$\|a(x/\epsilon)\nabla\psi_\epsilon - \xi_\epsilon\|_0 \le c\epsilon^2\|u_0\|_{3,p} \max_{kj}(\|\chi^j\|_{0,q} + \|\chi^{kj}\|_{1,q}). \tag{3.33}$$

Define $\varphi_\epsilon \in H^1(\Omega)$ as the weak solution of

$$-\nabla \cdot a(x/\epsilon)\nabla\varphi_\epsilon = 0 \ \text{ in } \Omega, \quad \text{and} \quad \varphi_\epsilon(x) = u_2(x, x/\epsilon) \ \text{ on } \ \partial\Omega. \tag{3.34}$$

We observe that the Sobolev embedding theorem and the hypothesis $p, q > d$, implies the function u_2 is continuous. Therefore, we use the maximum principle to obtain

$$\begin{aligned} \|\varphi_\epsilon\|_0 &\le c\|\varphi_\epsilon\|_{0,\infty} \\ &\le c\max_{ij}\|\chi^{ij}\|_{0,\infty,Y}\|\partial_{x_i x_j}u_0\|_{0,\infty} \\ &\le c\max_{ij}\|\chi^{ij}\|_{1,q,Y}\|u_0\|_{3,p}. \end{aligned} \tag{3.35}$$

Given $g \in L^2(\Omega)$, let $w_\epsilon \in H^1(\Omega)$ denote the solution of

$$\int_\Omega a(x/\epsilon)\nabla w_\epsilon(x)\nabla\psi(x)dx = \int_\Omega g(x)\psi(x)dx, \quad \forall\psi \in H_0^1(\Omega). \tag{3.36}$$

Since $\psi_\epsilon + \epsilon\theta_\epsilon + \epsilon^2\varphi_\epsilon \in H_0^1(\Omega)$ we obtain

$$\begin{aligned} \int_\Omega g(\psi_\epsilon + \epsilon\theta_\epsilon + \epsilon^2\varphi_\epsilon)dx &= \int_\Omega a(x/\epsilon)(\nabla\psi_\epsilon + \epsilon\nabla\theta_\epsilon + \epsilon^2\nabla\varphi_\epsilon)\nabla w_\epsilon(x)dx \\ &= \int_\Omega a(x/\epsilon)\nabla\psi_\epsilon\nabla w_\epsilon(x)dx, \end{aligned} \tag{3.37}$$

where we have used the definition of θ_ϵ and φ_ϵ to obtain (3.37). We observe

$$\int_\Omega a^\epsilon\nabla\psi_\epsilon\nabla w_\epsilon dx = \int_\Omega (a^\epsilon\nabla\psi_\epsilon - \xi_\epsilon)\cdot\nabla w_\epsilon dx + \int_\Omega \xi_\epsilon\cdot\nabla w_\epsilon dx. \tag{3.38}$$

We now estimate the second term on the right hand side of (3.38)

$$\begin{aligned} \int_\Omega \xi_\epsilon\cdot\nabla w_\epsilon dx &= \int_\Omega (a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x, x/\epsilon) - \epsilon v_1(x, x/\epsilon) \\ &\quad -\epsilon^2 v_2(x, x/\epsilon))\cdot\nabla w_\epsilon(x)dx \\ &= \int_\Omega f w_\epsilon(x) + \nabla_x\cdot v_0(x, x/\epsilon)w_\epsilon(x) \\ &\quad -\epsilon v_1(x, x/\epsilon)\cdot\nabla w_\epsilon(x) + \epsilon\nabla_x v_1(x, x/\epsilon)w_\epsilon(x)dx, \end{aligned} \tag{3.39}$$

here we used the definition of u_ϵ, (3.15), integration by parts and (3.31) to obtain (3.39). Using (3.27)

$$\begin{aligned}\int_\Omega v_1(x, x/\epsilon)\cdot\nabla w_\epsilon(x) &= \int_\Omega\left(-a^\epsilon_{ki}\chi^j_\epsilon\frac{\partial^2 u_0}{\partial x_j\partial x_i}(x)\right.\\ &\left.+\, a^\epsilon_{kl}\frac{\partial\chi^{ij}_\epsilon}{\partial y_l}\frac{\partial^2 u_0}{\partial x_j\partial x_i}(x)\right)\frac{\partial w_\epsilon}{\partial x_k}(x)dx. \quad (3.40)\end{aligned}$$

Consider the partition of unit, ρ_i, defined in the proof of Proposition 3.1.1, then

$$\begin{aligned}\int_\Omega a^\epsilon_{kl}\frac{\partial\chi^{ij}_\epsilon}{\partial y_l}\frac{\partial^2 u_0}{\partial x_j\partial x_i}\frac{\partial w_\epsilon}{\partial x_k}(x)dx &= \sum_1^{i_m}\int_{\Omega^i_\epsilon} a^\epsilon_{kl}\frac{\partial\chi^{ij}_\epsilon}{\partial y_l}\rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}\frac{\partial w_\epsilon}{\partial x_k}dx\\
&= \sum_1^{i_m}\int_{\Omega^i_\epsilon} a^\epsilon_{kl}\frac{\partial\chi^{ij}_\epsilon}{\partial y_l}\frac{\partial}{\partial x_k}\left(\rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}w_\epsilon(x)\right)\\
&\quad -a^\epsilon_{kl}\frac{\partial\chi^{ij}_\epsilon}{\partial y_l}w_\epsilon(x)\frac{\partial}{\partial x_k}\left(\rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}\right)dx\\
&= \sum_1^{i_m}\int_{\Omega^i_\epsilon}\epsilon^{-1}\left(a^\epsilon_{ij}-a^\epsilon_{ik}\frac{\partial\chi^j_\epsilon}{\partial_{y_k}}+A_{ij}\right)\rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}w_\epsilon \quad (3.41)\\
&\quad +a^\epsilon_{ki}\chi^j_\epsilon\left(\frac{\partial}{\partial_{x_k}}\left(\rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}(x)\right)w_\epsilon(x)+\rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}(x)\frac{\partial w_\epsilon}{\partial_{x_k}}(x)\right)dx\\
&\quad -\int_\Omega a_{kl}\frac{\partial\chi^{ij}_\epsilon}{\partial y_l}w_\epsilon(x)\frac{\partial}{\partial x_k}\left(\frac{\partial^2 u_0}{\partial x_j\partial x_i}\right)dx\\
&= \int_\Omega\epsilon^{-1}\left(\nabla_x v_0\frac{\partial^2 u_0}{\partial x_j\partial x_i}(x)-f\right)w_\epsilon(x)dx \quad (3.42)\\
&\quad -\int_\Omega a^\epsilon_{ki}\chi^j_\epsilon\frac{\partial^2 u_0}{\partial x_j\partial x_i}\frac{\partial w_\epsilon}{\partial_{x_k}}(x)dx-\int_\Omega\nabla_x\cdot v_1 dx,\end{aligned}$$

where we have used the definition of χ^{ij} to arrive in (3.41). From (3.39), (3.40) and (3.42) we obtain

$$\int_\Omega \xi_\epsilon\cdot\nabla w_\epsilon(x)dx = 0,$$

and hence from (3.33) and (3.38)

$$\begin{aligned}\left|\int_\Omega g(\psi_\epsilon+\epsilon\theta_\epsilon+\epsilon^2\varphi_\epsilon)dx\right| &\leq \|a^\epsilon\nabla\psi_\epsilon-\xi_\epsilon)\|_0\|w_\epsilon\|_1\\ &\leq c\epsilon^2\|u_0\|_{3,p}(\|\chi^j\|_{0,q,Y}+\|\chi^{kj}\|_{1,q,Y})\|g\|_{-1}.\end{aligned}$$

Dividing by g and taking the supremum over g, we have

$$\|u_\epsilon - u_0 - \epsilon u_1 - \epsilon\theta_\epsilon - \epsilon^2 u_2 - \epsilon^2\varphi_\epsilon\| \le c\epsilon^2\|u_0\|_{3,p}\max_{kj}(\|\chi^j\|_{0,q} + \|\chi^{kj}\|_{1,q})$$

Observe that $u_2(x, x/\epsilon)$ and $\varphi_\epsilon(x)$ are bounded in $L^2(\Omega)$ independent of ϵ by $\|u_0\|_{3,p}\max_{kj}\|\chi^{kj}\|_{1,q}$, see (3.35). Hence

$$\|u_\epsilon - u_0 - \epsilon u_1 - \epsilon\theta_\epsilon\| \le c\epsilon^2\|u_0\|_{3,p}(\max_j\|\chi^j\|_{0,q} + \max_{kj}\|\chi^{kj}\|_{1,q}).$$

□

The following proposition estimates the L^2 norm of $\bar\phi - \bar\theta_\epsilon$, and it is used in the proof of Theorem 2.1.2

Proposition 3.2.2 *Let u_0, χ^j, $\bar\theta_\epsilon$ and $\bar\phi$ be defined by (2.15), (2.8), (2.20) and (2.24), respectively. Assume that $u_0 \in W^{3,p}(\Omega)$, $\bar\phi \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(Y)$, for $1/p + 1/q \le 1/2$. Then we have*

$$\|\bar\theta_\epsilon - \bar\phi\|_0 \le c\epsilon\|u_0\|_{3,p}.$$

Proof: Observe that $\bar\phi \in W^{2,p}(\Omega)$ and $p \ge 2$, hence from Corollary 3.1.1 and Remark 3.1.1 we obtain

$$\|\bar\theta_\epsilon - \bar\phi\|_0 \le c\epsilon\|\bar\phi\|_{2,p}.$$

Since

$$\bar\phi|_{\partial\Omega} = \sum_k \varphi_k\chi_k^*\nabla u_0\cdot\eta_k|_{\partial\Omega},$$

by regularity theory, see Theorem 1.3.2, $\|\bar\phi\|_{2,p} \le c(\chi^*)\|u_0\|_{3,p}$, and the proposition follows. □

3.3 Proof of Theorem 2.1.3

Proof: Use a triangular inequality similar to the one used in the Proof of Theorem 2.1.1 and Propositions 3.3.1, 3.1.2, and 3.2.2. Observe that if $a_{ij} \in C^{1,\beta}_{per}(Y), \beta > 0$, by regularity theory $\chi^j \in C^{1,\beta}_{per}$, $v_e \in C^{1,\beta}$ and $\nabla(v_e - \chi_e^*)\exp(-\gamma y_1) \in L^\infty(G_e)$; see Theorems 1.3.4 and 1.3.8, and Remark 6.4 in [41]. By the Sobolev embedding theorem, $u_0 \in W^{2,\infty}(\Omega)$, hence Proposition 3.1.2 holds for $\delta = 1/2$. □

The following proposition is used in the proof of Theorem 2.1.3. Proposition 3.3.1 generalizes Proposition 2.3 from [41] to the case $\Omega \subset \mathbb{R}^3$.

Proposition 3.3.1 *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a convex domain, u_ϵ be the solution of Problem (1.2), and u_0, u_1, and θ_ϵ be defined by Equations (2.15),(2.10), and (2.16), respectively. Assume $a_{ij} \in C^{1,\beta}(Y), \beta > 0$ and $u_0 \in H^3(\Omega)$. Then there exists a constant c independent of u_0 and ϵ, such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\theta_\epsilon(\cdot)\|_0 \leq C\epsilon^2 \|u_0\|_3.$$

Proof: Since $a_{ij} \in C^{1,\beta}(Y)$ by regularity theory $\chi^i \in C^{2,\beta}(Y)$, $\chi^{ij} \in C^1(Y)$ and by Theorem 3 in [7] we obtain

$$\|\varphi_\epsilon\|_0 \leq c\|u_2(\cdot, \cdot/\epsilon)\|_{0,\partial\Omega} \leq c\|u_0\|_3 \|\chi^i j\|_{0,\infty}$$

where the function φ_ϵ is defined by (3.34) and we have used the trace theorem in the last inequality. The rest of the proof of follows exactly as the proof of Proposition 3.2.1. □

Chapter 4

Numerical Methods

4.1 Method I:

We now give the algorithm to obtain the numerical approximation for u_ϵ, the following method works in the case Ω is a rectangular region and bilinear finite elements are used to approximate u_0.

Step 1: Solve the cell problem (2.8) with a second order accurate conforming finite element in a partition $\mathcal{T}_{\hat{h}}(Y)$. Call these solutions $\chi^j_{\hat{h}}$.

Step 2: Obtain $A^{\hat{h}}$ by

$$A^{\hat{h}}_{ij} = \frac{1}{|Y|}\int_Y a_{lm}(y)\frac{\partial}{\partial y_l}(y_i - \chi^i_{\hat{h}})\frac{\partial}{\partial y_m}(y_j - \chi^j_{\hat{h}})dy.$$

Step 3: Let $V^h(\Omega) = \{v \in C^0(\Omega);\ v|_K \in \mathcal{Q}_1(K),\ K \in \mathcal{T}_h(\Omega),\ K \text{ rectangular}\}$ and $V^h_0(\Omega) = V^h(\Omega) \cap H^1_0(\Omega)$. Let $u^{h,\hat{h}}_0 \in V^h_0$ satisfying

$$\int_\Omega (A^{\hat{h}}\nabla u^{h,\hat{h}}_0, \nabla v^h)dx = \int_\Omega f v^h dx, \qquad \forall v^h \in V^h_0.$$

The justification for using a rectangular mesh is postponed to Remark 4.1.1.

Step 4: Define $u^{h,\hat{h}}_1$ as

$$u^{h,\hat{h}}_1(x) = -\chi^j_{\hat{h}}\left(\frac{x}{\epsilon}\right)\frac{\partial u^{h,\hat{h}}_0}{\partial x_j}(x). \tag{4.1}$$

Note that this leads to a nonconforming approximation for u_1 in the partition $\mathcal{T}_h(\Omega)$.

Step 5: Let τ be a positive integer and $G_e^\tau = (-\tau, 0) \times (0, 1)$. Define $\tilde{v}_e \in H^1(G_e^\tau)$ the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y \tilde{v}_e = 0 \ \text{ in } \ G_e^\tau, \\
&\tilde{v}_e(0, y_2) = \chi_{\hat{h}}^1(1, y_2), \quad 0 \le y_2 \le 1, \\
&\partial_\eta \tilde{v}_e = 0, \quad \text{on } \{y \in G_e^\tau;\ y_1 = -\tau\}, \\
&\text{and } \tilde{v}_e(y_1, 0) = \tilde{v}_e(y_1, 1), \quad -\tau \le y_1 \le 0.
\end{aligned}$$

Let $v_e^{\hat{h},\tau}$ be a numerical approximation of $\tilde{v}_e$ using a second order accurate conforming finite element on a mesh $\mathcal{T}_{\hat{h}}(G_e^\tau)$.

Step 6: Define

$$\chi_e^{*,\hat{h},\tau} \;=\; \int_0^1 v_e(-\tau, y_2)dy_2$$

The other cases $k \in \{w, n, s\}$ are treated similarly.

Step 7: Let $\bar{\phi}^{h,\hat{h},\tau}$ be a second order accurate finite element approximation in a mesh of size h for the following equation

$$(4.2) \qquad -\nabla A^{\hat{h}}\nabla\psi = 0, \qquad \psi = \chi^{*,\hat{h},\tau}\partial_\eta u_0^{\hat{h},h} \ \text{ on } \ \partial\Omega.$$

Remark 4.1.1 *Since $u_0^{\hat{h},h} \in H_0^1(\Omega)$, the domain Ω is rectangular, and bilinear rectangular elements are considered to obtain $u_0^{\hat{h},h}$, is easy to see that $\partial_\eta u_0^{\hat{h},h}$ is continuous on $\partial\Omega$ and linear in every edge of $\mathcal{T}_h(\partial\Omega)$. Observe also that the zero Dirichlet boundary condition implies $\partial_\eta u_0^{\hat{h},h} = 0$ at the corners of Ω. Therefore $\chi^{*,\hat{h},\tau}\partial_\eta u_0^{\hat{h},h} \in H^{1/2}(\partial\Omega)$ and Equation (4.10) is well posed. Taking $\bar{\phi}^{h,\hat{h},\tau} \in V^h(\Omega)$ allows us to use the same stiffness matrix for obtaining $u_0^{\hat{h},h}$ and $\bar{\phi}^{h,\hat{h},\tau}$.*

Step 8: Observe that in Equation. (2.22) the term $v_e(\frac{x_1-1}{\epsilon}, \frac{x_2}{\epsilon})$ appears. Since the approximation $v_e^{\hat{h},\tau}$ is defined in G_e^τ, it is possible to calculate

$v_e^{\hat{h},\tau}(\frac{x_1-1}{\epsilon},\frac{x_2}{\epsilon})$ only if $x_1 \geq 1-\epsilon\tau$. Since the functions $v_k - \chi_k^*$ decays exponentially to zero away from the boundary its is natural to consider the following approximation

$$\tilde{\phi}_\epsilon^{e,h,\hat{h},\tau}(x_1,x_2) = \begin{cases} \varphi_e(x_1)(v_e^{\hat{h},\tau}(\frac{x_1-1}{\epsilon},\frac{x_2}{\epsilon}) - \chi_e^{*,\hat{h},\tau})\frac{\partial u_0^{h,\hat{h}}}{\partial x_1} & \text{if } x_1 > 1-\epsilon\tau, \\ 0 & \text{if } x_1 \leq 1-\epsilon\tau, \end{cases} \tag{4.3}$$

and

$$\tilde{\phi}_\epsilon^{h,\hat{h},\tau} = \sum_{k\in\{e,w,n,s\}} \tilde{\phi}_\epsilon^{k,h,\hat{h},\tau}. \tag{4.4}$$

Step 9: Approximate θ_ϵ by $\phi_\epsilon^{h,\hat{h},\tau} = \tilde{\phi}_\epsilon^{h,\hat{h},\tau} + \bar{\phi}^{h,\hat{h},\tau}$ and finally construct the numerical approximation for u_ϵ as

$$u_\epsilon^{h,\hat{h},\tau} = u_0^{h,\hat{h}} + \epsilon u_1^{h,\hat{h}} + \epsilon\phi_\epsilon^{h,\hat{h},\tau}. \tag{4.5}$$

Remark 4.1.2 *Only two stiffness matrices are need to be formed: one for Steps 3 and 7, and another one for Steps 1 and 5. In Step 5, an iterative method based on vector-matrix multiplication together with the periodicity of the matrix on Step 1 is explored.*

4.2 Method II:

Step 7 in Method I uses the trace of the normal derivative of $u_0^{\hat{h},h}$ as the boundary condition in Equation (4.10), as we have seen in Remark 4.1.1 this is the reason that we have to consider bilinear elements to approximate u_0 in Method I. For the method presented in this section we then introduce the Lagrange multiplier space to approximate $\partial_\eta u_0$ on $\partial\Omega$. This modification allows the use of a linear finite element space to approximate u_0 and the generalization of the method to the case Ω a convex polygonal domain with rational edges, see the Appendix A. It also allows to prove error estimates for the L^2 norm under weak assumptions on u_0.

Step 1: Approximate the solution of Problem (2.8) with a second order accurate conforming finite element on a partition $\mathcal{T}_{\hat{h}}(Y)$. Denote these solutions by $\chi_{\hat{h}}^j$.

Step 2: Define $A^{\hat{h}}_{ij} = \frac{1}{|Y|}\int_Y a_{lm}(y)\frac{\partial}{\partial y_l}(y_i - \chi^i_{\hat{h}})\frac{\partial}{\partial y_m}(y_j - \chi^j_{\hat{h}})dy$.

Step 3: Let $V^h(\Omega)$ be a conforming second order accurate finite element space on a mesh $\mathcal{T}_h(\Omega)$, and $V^h_0(\Omega) = V^h(\Omega)\cap H^1_0(\Omega)$. Define $u^{h,\hat{h}}_0 \in V^h_0(\Omega)$ as the solution of

$$\int_\Omega (A^{\hat{h}}\nabla u^{h,\hat{h}}_0, \nabla v^h)dx = \int_\Omega f v^h dx, \qquad \forall v^h \in V^h_0(\Omega).$$

Step 4: Since $\partial_\eta u_0$ appears as boundary condition imposed in Equation (2.24), it is important to obtain a good discrete approximation for it. In oder to approximate $\partial_\eta u_0$ we define $Y^h = V^h(\Omega)|_{\partial\Omega}$, $Y^h_k = Y^h|_{\Gamma_k}$ and $Y^h_{0,k} = \{\lambda^h \in Y^h_k; \lambda^h = 0 \text{ at } \partial\Gamma_k\}$. Let $\lambda^{h,\hat{h}}_k \in Y^h_{0,k}$ be the solution of

$$\int_{\Gamma_k} \lambda^{h,\hat{h}}_k \phi^h d\sigma = \int_\Omega A^{\hat{h}}_{ij}\partial_i u^{h,\hat{h}}_0 \partial_j \phi^h dx - \int_\Omega f\phi^h dx, \tag{4.6}$$

$\forall\ \phi^h \in V^h(\Omega)$, such that $\phi^h|_{\partial\Omega\setminus\Gamma_k} = 0$. We later show that $\lambda^{h,\hat{h}}_k$ is a good approximation for $A\nabla u_0 \cdot \eta_k$ on Γ_k, hence we approximate $\partial_\eta u_0$ by $\mu^{h,\hat{h}}$ where

$$\mu^{h,\hat{h}}|_{\Gamma_k} = \lambda^{h,\hat{h}}_k / A^{\hat{h}}_{l_k l_k},\ l_k = \begin{cases} 1 & \text{if} \quad k = e, w \\ 2 & \text{if} \quad k = n, s. \end{cases}$$

Step 5: We observe that we use $\mu^{h,\hat{h}}$ as the approximation for $\partial_\eta u_0$ in Equation (4.10), hence in order to guarantees that the final numerical approximation for u_ϵ satisfies the zero Dirichlet boundary condition we define the approximation for ∇u_0 as

$$\Psi^{h,\hat{h}} = \nabla u^{h,\hat{h}}_0 + \sum_{k\in\{e,w,n,s\}} E_k(\mu^{h,\hat{h}} - \nabla u^{h,\hat{h}}_0 \cdot \eta^k)\eta^k. \tag{4.7}$$

Here $E_k(\cdot)$ denotes a discrete extension by zero on Ω. More specifically, $E_k(\mu^{h,\hat{h}} - \nabla u^{h,\hat{h}}_0 \cdot \eta^k)(z) = 0$, where z is a vertex of $\mathcal{T}_h(\Omega)\setminus\Gamma_k$, and $E_k(\mu^{h,\hat{h}} - \nabla u^{h,\hat{h}}_0 \cdot \eta^k)|_{K_i} \in V^h(\Omega)|_{K_i}$, $\forall\ K_i \in \mathcal{T}_h(\Omega)$.

Step 6: Define

$$u^{h,\hat{h}}_1(x, x/\epsilon) = -\Psi^{h,\hat{h}}_j(x)\chi^j_{\hat{h}}(x/\epsilon). \tag{4.8}$$

Note that this leads to a nonconforming approximation for u_1 in the partition $\mathcal{T}_h(\Omega)$.

Step 7: Let τ be a positive integer and $G_e^\tau = \{y \in R^2;\ -\tau \leq y_1 \leq 0 \text{ and } 0 \leq y_2 \leq 1\}$. Define $\tilde{v}_e \in H^1(G_e^\tau)$ as the weak solution of

$$\begin{aligned}&-\nabla_y \cdot a(y)\nabla_y \tilde{v}_e = 0 \ \text{ in } \ G_e^\tau,\\ &\tilde{v}_e(y) = \chi_{\hat{h}}^1(1/\epsilon, y_2) \ \text{ on } \ \{y \in G_e^\tau, y_1 = 0\},\\ &\partial_\eta \tilde{v}_e = 0 \ \text{ on } \{y \in G_e^\tau;\ y_1 = -\tau\},\\ &\text{and } v_e(y_1, 0) = v_k(y_1, 1) \ \text{ for } \ -\tau \leq y_1 \leq 0.\end{aligned}$$

Let $v_e^{\hat{h},\tau}$ be a numerical approximation of $\tilde{v}_e$ using a second order accurate conforming finite element on a mesh $\mathcal{T}_{\hat{h}}(G_e^\tau)$, and define

$$\chi_e^{*,\hat{h},\tau} \ = \ \int_0^1 v_e^{\hat{h},\tau}(-\tau, y_2)dy_2.$$

The other cases $k \in \{w, n, s\}$ are treated similarly.

Step 8: Observe that the term $v_e(\frac{x_1-1}{\epsilon}, \frac{x_2}{\epsilon})$ appears in Equation (2.22). The approximation $v_e^{\hat{h},\tau}$ is defined in G_e^τ, hence we have defined $v_e^{\hat{h},\tau}(\frac{x_1-1}{\epsilon}, \frac{x_2}{\epsilon})$ only when $x_1 \geq 1 - \epsilon\tau$. Since the functions $v_e - \chi_e^*$ decays exponentially to zero in the $-\eta_e$ direction, its is natural to consider the following approximation

$$\tilde{\phi}_\epsilon^{e,h,\hat{h},\tau}(x_1, x_2) = \begin{cases} \left(v_e^{\hat{h},\tau}\left(\frac{x_1-1}{\epsilon}, \frac{x_2}{\epsilon}\right) - \chi_e^{*,\hat{h},\tau}\right)\Psi^{h,\hat{h}} & \text{if } \ x_1 > 1 - \epsilon\tau \\ 0 & \text{otherwise.}\end{cases}$$

Step 9: Let

$$\tilde{\phi}_\epsilon^{h,\hat{h},\tau} = \sum_{k \in \{e,w,n,s\}} \tilde{\phi}_\epsilon^{k,h,\hat{h},\tau}, \tag{4.9}$$

where the others terms $\tilde{\phi}_\epsilon^{k,h,\hat{h},\tau}$ are defined in a similar way.

Step 10: Let $\bar{\phi}^{h,\hat{h},\tau}$ be a second order accurate finite element approximation on a mesh of size h for the following equation (see Remark 4.2.1)

$$-\nabla \cdot A^{\hat{h}}\nabla\psi = 0 \ \text{ in } \ \Omega, \quad \text{and} \quad \psi = \chi^{*,\hat{h},\tau}\mu^{h,\hat{h}} \ \text{ on } \ \partial\Omega. \tag{4.10}$$

Step 11: Approximate θ_ϵ by $\phi_\epsilon^{h,\hat{h},\tau} = \tilde{\phi}_\epsilon^{h,\hat{h},\tau} + \bar{\phi}^{h,\hat{h},\tau}$ and finally construct the numerical solution for Equation (1.2),

$$u_\epsilon^{h,\hat{h},\tau} = u_0^{h,\hat{h}} + \epsilon u_1^{h,\hat{h}} + \epsilon \phi_\epsilon^{h,\hat{h},\tau}. \tag{4.11}$$

Remark 4.2.1 *By construction* $\mu^{h,\hat{h}} = 0$ *at the corners of* Ω, *therefore* $\chi^{*,\hat{h},\tau}\mu^{h,\hat{h}} \in H^{1/2}(\partial\Omega)$. *This implies that Equation (4.10) is well posed. In addition* $\chi^{*,\hat{h},\tau}\mu^{h,\hat{h}} \in V^h|_{\partial\Omega}$ *hence we can look for a numerical solution of Equation (4.10) in* $V^h(\Omega)$.

Chapter 5

Finite element error analysis

This chapter is devoted to the error analysis due to the numerical approximation introduced in methods I and II.For the discrete error analysis we assume $\hat{h} = 0$ and $\tau = \infty$, i.e. $v_k^{\hat{h},\tau} = v_k$, $\chi_{\hat{h}}^j = \chi^j$ and $A^{\hat{h}} = A$, and for this reason we will note make reference to the index τ and $\hat{h}$ when we make reference to the the numerical approximation for u_0, ∇u_0, $\bar{\phi}$, $\tilde{\phi}_\epsilon$ and u_ϵ, i.e. $u_\epsilon^h = u_\epsilon^{h,\hat{h},\tau}$ and similar for the other terms; an error analysis including the error due to the numerical approximation of the functions v_k and χ^j, and the matrix A is currently work under progress.

The main results of this chapter are Theorems 5.1.1, 5.1.2 that provide error estimates for the L^2 norm and the broken H^1 norms for the method II, and Theorems 5.2.1 that provide error estimates for the L^2 norm and the H^1 broken norms for the method I. Here in this chapter we choosed to present first the error analysis for the Method II. We also observe that the error estimates given by Theorems 5.1.1 and 5.1.2 are better then the ones obtainned in [4, 24, 45].

5.1 Error analysis for Method II:

In this section assume we use linear or bilinear finite elements to approximate u_0.

Theorem 5.1.1 *Let u_ϵ be the solution of the Problem (1.2), u_0, χ^j and u_h be defined by Equations (2.8), (2.15) and (4.11), respectively, and the functions v_k and the constants χ_k^* be defined as in Subsection 2.1.1. Assume $a_{ij} \in$*

$L^\infty_{per}(Y)$, $u_0 \in W^{2,p}(\Omega)$, $\chi^j \in W^{1,q}_{per}(Y)$, and v_e and $\nabla(v_e - \chi^*_e) exp(-\gamma y_1) \in L^s(G_e)$, for $1/p + 1/q \leq 1/2$ and $1/s + 3/p \leq 1$. We also assume similar hypothesis for the other functions v_k. Then there exists a constant c independent of ϵ and h such that

$$\|u_\epsilon - u_h\|_{1,h} \leq c(h + \epsilon)\|u_0\|_{2,p}$$

and

$$\|u_\epsilon - u_h\|_0 \leq c(h^2 + \epsilon + \epsilon h)\|u_0\|_{2,p}.$$

Proof: By the triangular inequality we have

$$\begin{aligned} |u_\epsilon - u_h|_{1,h} \leq & \ |u_\epsilon - u_0 - u_1 - \phi_\epsilon|_1 + |u_0 - u_0^h|_{1,h} + \epsilon|u_1 - u_1^h|_{1,h} \\ & + \epsilon|\bar{\phi} - \bar{\phi}^h|_{1,h} + \epsilon|\tilde{\phi}_\epsilon - \tilde{\phi}_\epsilon^h|_{1,h}, \end{aligned}$$

and the theorem follows from Theorem 2.1.1, the approximation error (5.1), and Propositions 5.1.2, 5.1.3 and 5.1.4. □

Theorem 5.1.2 *Let u_ϵ be the solution of the Problem (1.2), χ^j, u_0, χ^{ij}, $\bar{\phi}$ and u_h be defined by Equations (2.8), (2.15), (2.17), (2.24) and (4.11), respectively, and the functions v_k and the constants χ^*_k be defined as in Subsection 2.1.1. Assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{3,p}(\Omega)$, $\bar{\phi} \in W^{2,p}(\Omega)$ and $\chi^{ij} \in W^{1,q}_{per}(Y)$, for $p > 2$ and $1/p + 1/q \leq 1/2$. Also assume $\chi^j \in W^{1,\infty}(Y)$, and v_e and $\nabla(v_e - \chi^*_e) exp(-\gamma y_1) \in L^\infty(G_e)$. We also assume similar hypothesis for the other functions v_k. Then there exists a constant c independent of ϵ and h such that*

$$\|u_\epsilon - u_h\|_0 \leq c(h^2 + \epsilon^{\frac{3}{2}} + \epsilon h)\|u_0\|_{3,p}.$$

Furthermore, if $a_{ij} \in C^{1,\beta}_{per}(Y)$ and $u_0 \in H^3(\Omega)$, then

$$\|u_\epsilon - u_h\|_0 \leq c(h^2 + \epsilon^{\frac{3}{2}} + \epsilon h)\|u_0\|_3.$$

Proof: The same proof of Theorem 5.1.1 holds here, except that Theorem 2.1.1 is replaced by Theorems 2.1.3 and 2.1.2. □

We now prove the propositions used in the proofs of Theorems 5.1.1 and 5.1.2.

For the approximation error of the term u_0 we use standard finite element analysis to obtain

$$\|u_0 - u_0^h\|_{1,p} \leq ch\|u_0\|_{2,p} \tag{5.1}$$

and

$$\|u_0 - u_0^h\|_{0,p} \leq ch^2\|u_0\|_{2,p} \tag{5.2}$$

for $2 \leq p < \infty$; see Theorem 1.4.4. Let $\mathcal{I}^h$ be the usual local pointwise $\mathcal{P}_1$ or $\mathcal{Q}_1$ interpolate and $K \in \mathcal{T}_h(\Omega)$, then

$$|u_0 - u_0^h|_{2,p,K} \leq |u_0 - \mathcal{I}^h u_0|_{2,p,K} + |\mathcal{I}^h u_0 - u_0^h|_{2,p,K}.$$

Using an interpolation error estimate, see Theorem 1.4.1, we obtain

$$|u_0 - \mathcal{I}^h u_0|_{s,p,h} \leq ch^{m-s}|u_0|_{m,p,h}, \text{ for } 0 \leq s \leq m, \tag{5.3}$$

and from an inverse inequality, see Theorem 1.4.3, we have

$$|\mathcal{I}^h u_0 - u_0^h|_{2,p,K} \leq ch^{-1}\|\mathcal{I}^h u_0 - u_0^h\|_{1,p,K}. \tag{5.4}$$

Finally from (5.3), (5.4) and (5.1) we obtain

$$\|u_0 - u_0^h\|_{2,p,h} \leq c\|u_0\|_{2,p}. \tag{5.5}$$

In order to estimate the L^2 and the broken H^1 semi-norm of $u_1 - u_1^h$, (see Proposition 5.1.2) we note that $u_1 - u_1^h = (\partial_{x_j} u_0 - \Psi_j^h)\chi^j$ hence by a Cauchy inequality and the Sobolev embedding Theorem we obtain $\|u_1 - u_1^h\|_0 \leq c\|\partial_{x_j} u_0 - \Psi_j^h\|_{0,p}\|\chi^j\|_{0,q}$ for $1/p + 1/q \leq 1/2$. Therefore we have to estimate the error between Ψ^h and ∇u_0 on the L^p and on the broken $W^{1,p}$ semi-norm, (see Proposition 5.1.1) this is done by first estimating the error between $A\nabla u_0 \cdot \eta$ and λ^h in the trace space of $W^{1,p}(\Omega)$ over Γ_k in different norms; see Lemma 5.1.3. Lemmas 5.1.1 and 5.1.2 are auxiliary results used for obtaining Lemma 5.1.3.

Consider the following spaces:

Case $2 < p < \infty$**:** Since $W^{1-1/p,p}(\Gamma_k) \hookrightarrow C^0(\Gamma_k)$, we define the spaces $W_{00}^{1-1/p,p}(\Gamma_k) = \{\varphi \in W^{1-1/p,p}(\Gamma_k);\ \varphi = 0 \text{ on } \partial\Gamma_k\}$ equipped with the norm $\|\cdot\|_{W_{00}^{1-1/p,p}(\Gamma_k)} = \|\cdot\|_{W^{1-1/p,p}(\Gamma_k)}$.

Case $p = 2$**:** We set $W_{00}^{1-1/p,p}(\Gamma_k) = H_{00}^{1/2}(\Gamma_k)$ and $\|\cdot\|_{W_{00}^{1-1/p,p}(\Gamma_k)} = \|\cdot\|_{H_{00}^{1/2}(\Gamma_k)}$; see [39] for the definition of $H_{00}^{1/2}(\Gamma_k)$.

Case $1 < p < 2$**:** We define $W_{00}^{1-1/p,p}(\Gamma_k) = W^{1-1/p,p}(\Gamma_k)$ equipped with the norm $\|\cdot\|_{W_{00}^{1-1/p,p}(\Gamma_k)} = \|\cdot\|_{W^{1-1/p,p}(\Gamma_k)}$.

These spaces have the following important feature. Denote by $\tilde{\varphi}$ the extension by zero to $\partial\Omega \setminus \Gamma_k$ of a given function $\varphi \in W_{00}^{1-1/p,p}(\Gamma_k)$. Then by the Trace Theorem 1.5.2.3 [29] there exists a function $\psi_\varphi \in W^{1,p}(\Omega)$ such that $\psi_\varphi|_{\partial\Omega} = \tilde{\varphi}$ and

$$c_1\|\varphi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \leq \|\psi_\varphi\|_{1,p} \leq c\|\tilde{\varphi}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq c_2\|\varphi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}. \tag{5.6}$$

We also introduce the dual space of $W_{00}^{1-1/p,p}(\Gamma_k)$, denoted by $W^{-1+1/p,p'}(\Gamma_k)$, where $1/p + 1/p' = 1$.

The following inverse inequality is required in the proof of Lemma 5.1.3.

Lemma 5.1.1 *Let* $1 < p < \infty$ *and* $v^h \in Y_{0,k}^h$. *Then*

$$\|v^h\|_{W_{00}^{1-1/p,p}(\Gamma_k)} \leq ch^{-1}\|v^h\|_{W^{-1+1/p',p}(\Gamma_k)}. \tag{5.7}$$

Proof: Consider the following inverse inequality (see Theorem 1.4.3)

$$\|v^h\|_{s,q,\partial\Omega} \leq ch^{-s}\|v^h\|_{0,q,\partial\Omega}, \quad \forall\ v^h \in Y^h,\ 1 \leq q \leq \infty \text{ and } 0 \leq s \leq 1. \tag{5.8}$$

Given $v^h \in Y_{0,k}^h$ let $\tilde{v}^h \in Y^h$ be the extension of v^h to $\partial\Omega \setminus \Gamma_k$ by zero. Using (5.6) and (5.8) we obtain

$$\begin{aligned} \|v^h\|_{W_{00}^{1-1/p,p}(\Gamma_k)} &\leq c\|\tilde{v}^h\|_{W^{1-1/p,p}(\partial\Omega)} \\ &\leq ch^{-1+1/p}\|\tilde{v}^h\|_{L^p(\partial\Omega)} = ch^{-1+1/p}\|v^h\|_{L^p(\Gamma_k)}. \end{aligned} \tag{5.9}$$

Let $\mathcal{P}_{0,k}$ denote the L^2 projector to $Y_{0,k}^h$ and assume that $v^h \in Y_{0,k}^h$. Then

$$\|v^h\|_{L^p(\Gamma_k)} = \sup_{\phi \in L^{p'}(\Gamma_k)} \frac{\langle v^h, \phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}} = \sup_{\phi \in L^{p'}(\Gamma_k)} \frac{\langle v^h, \mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}}.$$

By Theorem 1 in [18] we have

$$\|\mathcal{P}_{0,k}\phi\|_{L^{p'}(\Gamma_k)} \leq c\|\phi\|_{L^{p'}(\Gamma_k)} \quad 1 \leq p' \leq \infty. \tag{5.10}$$

Hence

$$\begin{aligned} \|v^h\|_{L^p(\Gamma_k)} &\leq c \sup_{\phi \in L^{p'}(\Gamma_k)} \frac{\|v^h\|_{W^{-1+\frac{1}{p'},p}(\Gamma_k)}\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)}}{\|\mathcal{P}_{0,k}\phi\|_{L^{p'}(\Gamma_k)}} \\ &\leq ch^{-1+\frac{1}{p'}}\|v^h\|_{W^{-1+\frac{1}{p'},p}(\Gamma_k)}, \end{aligned} \tag{5.11}$$

where on the last inequality we have used (5.9) for $\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-1/p',p'}(\Gamma_k)}$. Combining inequalities (5.9) and (5.11) we obtain (5.7). □

The following lemma provide stability and error estimates concerning $\mathcal{P}_{0,k}$, the L^2 projector to $Y_{0,k}^h$. These results are required in the proof of Lemma 5.1.3.

Lemma 5.1.2 *Let $2 \leq p < \infty$ and $\mathcal{P}_{0,k} : W^{-1+\frac{1}{p},p'}(\Gamma_k) \to Y_{0,k}^h$ be the L^2 projector to $Y_{0,k}^h$. Then we have*

$$\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \leq c\|\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \quad \forall\ \phi \in W_{00}^{1-\frac{1}{p},p}(\Gamma_k), \tag{5.12}$$

$$\|\phi - \mathcal{P}_{0,k}\phi\|_{L^p(\Gamma_k)} \leq ch^{1-\frac{1}{p}}\|\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \quad \forall\ \phi \in W_{00}^{1-\frac{1}{p},p}(\Gamma_k), \tag{5.13}$$

$$\|\phi - \mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} \leq ch^{1-\frac{1}{p}}\|\phi\|_{L^{p'}(\Gamma_k)} \quad \forall\ \phi \in L^{p'}(\Gamma_k) \tag{5.14}$$

and

$$\|\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} \leq c\|\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} \quad \forall\ \phi \in W^{-1+\frac{1}{p},p'}(\Gamma_k). \tag{5.15}$$

Proof of (5.12):

Case $p > 2$: Observe that $\mathcal{P}_{0,k} : L^p(\Gamma_k) \to Y_{0,k}^h$ is stable in L^p and $W^{1,p}$, i.e. $\|\mathcal{P}_{0,k}\phi\|_{L^p(\Gamma_k)} \leq c\|\phi\|_{L^p(\Gamma_k)} \ \ \forall\ \phi \in L^p(\Gamma_k)$, and $\|\mathcal{P}_{0,k}\phi\|_{W^{1,p}(\Gamma_k)} \leq c\|\phi\|_{W^{1,p}(\Gamma_k)} \ \ \forall\ \phi \in W^{1,p}(\Gamma_k)$, respectively; see Theorems 1 and 2 in [18]. Since $W^{1-\frac{1}{p},p}(\Gamma_k) = [L^p(\Gamma_k), W^{1,p}(\Gamma_k)]_{1-1/p,p}$; see Theorem 1.2.3, we obtain the stability of $\mathcal{P}_{0,k}$ in $W^{1-\frac{1}{p},p}(\Gamma_k)$ by interpolation, see Theorem 1.2.4, and inequality (5.12) holds for $p > 2$.

Case $p = 2$: By definition $H_{00}^{1/2}(\Gamma_k) = [L^2(\Gamma_k), H_0^1(\Gamma_k)]_{1/2}$ and the proof is analogue to the case $p > 2$.

Proof of (5.13):

Case $p > 2$: Let $\mathcal{I}^h : L^p(\Gamma_k) \to V^h(\Omega)|_{\Gamma_k}$ denote the standard $\mathcal{P}_1$ or $\mathcal{Q}_1$ interpolation operator. Then we have

$$\begin{aligned}
\|\phi - \mathcal{P}_{0,k}\phi\|_{L^p(\Gamma_k)} &\leq \|\phi - \mathcal{I}^h\phi\|_{L^p(\Gamma_k)} + \|\mathcal{P}_{0,k}(\phi - \mathcal{I}^h\phi)\|_{L^p(\Gamma_k)} \\
&\leq c\|\phi - \mathcal{I}^h\phi\|_{L^p(\Gamma_k)}, \text{ by (5.10)} \\
&\leq ch^{1-\frac{1}{p}}\|\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}, \text{ by (5.3).} \qquad (5.16)
\end{aligned}$$

Case $p = 2$: Follows similarly to the case $p > 2$ by replacing $\mathcal{I}^h$ by the Clement interpolation operator defined by (2.13) [47] and using Theorem 1.2.4.

Proof of (5.14):

$$\begin{aligned}
\|\phi - \mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} &= \sup_{v \in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\langle \phi - \mathcal{P}_{0,k}\phi, v\rangle}{\|v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \\
&= \sup_{v \in W_{00}^{1-\frac{1}{p},p}} \frac{\langle \phi - \mathcal{P}_{0,k}\phi, v - \mathcal{P}_{0,k}v\rangle}{\|v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \\
&= \sup_{v \in W_{00}^{1-\frac{1}{p},p}} \frac{\langle \phi, v - \mathcal{P}_{0,k}v\rangle}{\|v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \\
&\leq \sup_{v \in W_{00}^{1-\frac{1}{p},p}} \frac{\|\phi\|_{L^{p'}(\Gamma_k)}\|v - \mathcal{P}_{0,k}v\|_{L^p(\Gamma_k)}}{\|v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \\
&\leq ch^{1-\frac{1}{p}}\|\phi\|_{L^{p'}(\Gamma_k)}, \qquad (5.17)
\end{aligned}$$

where we have used (5.13) to obtain the last inequality.

Proof of (5.15):

$$\begin{aligned}
\|\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} &= \sup_{v \in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\langle \mathcal{P}_{0,k}\phi, v\rangle}{\|v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \\
&\leq c \sup_{v \in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\langle \mathcal{P}_{0,k}\phi, \mathcal{P}_{0,k}v\rangle}{\|\mathcal{P}_{0,k}v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}}, \quad \text{by} \quad (5.12) \\
&\leq c \sup_{v \in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\langle \phi, \mathcal{P}_{0,k}v\rangle}{\|\mathcal{P}_{0,k}v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \leq c\|\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)}.
\end{aligned}$$

□

The following lemma estimate the error between $A\nabla u_0 \cdot \eta$ and its numerical approximation λ^h. This lemma is used in the proof of Proposition 5.1.1.

Lemma 5.1.3 *Let λ^h be defined by Equation (4.6) and $\lambda = \partial_{\eta_A} u_0 = A_{ij}\partial_j u_0 \eta_i$, where η_i is the ith component of the normal vector to Γ_k. Assume that*

$u_0 \in W^{2,p}(\Omega)$. *Then we have*

$$\|\lambda - \lambda^h\|_{W_{00}^{1-1/p,p}(\Gamma_k)} \le c\|u_0\|_{2,p} \quad \textit{for} \quad 2 \le p < \infty, \tag{5.18}$$

$$\|\lambda - \lambda^h\|_{L^p(\Gamma_k)} \le ch^{1-\frac{1}{p}}\|u_0\|_{2,p} \quad \textit{for} \quad 2 \le p \le \infty \tag{5.19}$$

and

$$\|\lambda - \lambda^h\|_{W^{-1+1/p,p'}(\Gamma_k)} \le ch\|u_0\|_{2,p} \quad \textit{for} \quad 2 \le p < \infty. \tag{5.20}$$

Proof of (5.18): From Remark 2.1.1 if $p = 2$, or from the Sobolev embedding theorem if $p > 2$, we have

$$\|\lambda\|_{W_{00}^{1-1/p,p}(\Gamma_k)} \le c\|u_0\|_{2,p}. \tag{5.21}$$

In order to prove inequality (5.18) observe that

$$\|\lambda - \lambda^h\|_{W_{00}^{1-1/p,p}(\Gamma_k)} \le \|\lambda\|_{W_{00}^{1-1/p,p}(\Gamma_k)} + \|\lambda^h\|_{W_{00}^{1-1/p,p}(\Gamma_k)},$$

and

$$\|\lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} = \sup_{\phi \in W^{-1+\frac{1}{p},p'}(\Gamma_k)} \frac{\langle \lambda^h, \phi \rangle}{\|\phi\|_{W^{-1+1/p,p'}(\Gamma_k)}}.$$

Since $\lambda^h \in Y_{0,k}^h$ then $\langle \lambda^h, \phi \rangle = \langle \lambda^h, \mathcal{P}_{0,k}\phi \rangle$, and using (5.15) we obtain

$$\|\lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \le c \sup_{\phi \in W^{-1+\frac{1}{p},p'}(\Gamma_k)} \frac{\langle \lambda^h, \mathcal{P}_{0,k}\phi \rangle}{\|\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)}}. \tag{5.22}$$

Now we introduce the A-discrete harmonic extension operator $\mathcal{H}^h : Y^h \to V^h(\Omega)$ defined as the solution of

$$\int_\Omega A_{ij}\partial_i \mathcal{H}^h g^h \partial_j v^h dx = 0 \ \ \forall \ v^h \in V_0^h(\Omega), \quad \text{and} \quad \mathcal{H}^h g^h|_{\partial\Omega} = g^h.$$

The A-harmonic extension operator $\mathcal{H} : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ is defined similarly. By Theorem 5.4 in [48] (a generalization of Lax-Milgram theorem for Banach spaces) we have

$$\|\mathcal{H}g\|_{W^{1,p}(\Omega)} \le c\|g\|_{W^{1-1/p,p}(\partial\Omega)}, \quad \text{for } 1 < p < \infty. \tag{5.23}$$

Hence if $g^h \in Y^h_{0,k}$, $\tilde{g}^h$ denotes the extension of g^h by zero to $\partial\Omega \setminus \Gamma_k$, from Theorem 1.4.5 it follows

$$
\begin{aligned}
\|\mathcal{H}^h \tilde{g}^h\|_{1,p} &\leq c\|\mathcal{H}\tilde{g}^h\|_{W^{1,p}(\Omega)} \\
&\leq \|g^h\|_{W^{1-1/p,p}_{00}(\Gamma_k)}, \text{ by (5.23)}.
\end{aligned} \tag{5.24}
$$

Let $\tilde{\mathcal{P}}_{0,k}\phi$ denote the extension of $\mathcal{P}_{0,k}\phi$ to $\partial\Omega\backslash\Gamma_k$ by zero. From the definition of λ^h, the stability of the A-discrete harmonic extension, (5.24) and (5.1), we obtain

$$
\begin{aligned}
\langle \lambda^h, \mathcal{P}_{0,k}\phi\rangle &= \langle \lambda, \mathcal{P}_{0,k}\phi\rangle + a(u^h_0 - u_0, \mathcal{H}^h\tilde{\mathcal{P}}_{0,k}\phi) \\
&\leq \|\lambda\|_{W^{1-1/p,p}_{00}(\Gamma_k)}\|\mathcal{P}_{0,k}\phi\|_{W^{-1+1/p,p'}(\Gamma_k)} + ch\|u_0\|_{2,p}\|\mathcal{P}_{0,k}\phi\|_{W^{1-1/p',p'}_{00}(\Gamma_k)} \\
&\leq c\left(\|\lambda\|_{W^{1-1/p,p}_{00}(\Gamma_k)} + c\|u_0\|_{2,p}\right)\|\mathcal{P}_{0,k}\phi\|_{W^{-1+1/p,p'}(\Gamma_k)}.
\end{aligned} \tag{5.25}
$$

Here we used the inverse estimate (5.7) applied to $\mathcal{P}_{0,k}\phi$ to obtain (5.25) . Inequality (5.18) follows from (5.25), (5.22) and (5.21).

Proof of (5.20): We observe that

$$
\begin{aligned}
\|\lambda - \lambda^h\|_{W^{-1+1/p,p'}(\Gamma_k)} &= \sup_{\phi\in W^{1-\frac{1}{p},p}_{00}(\Gamma_k)} \frac{\langle \lambda - \lambda^h, \phi\rangle}{\|\phi\|_{W^{1-\frac{1}{p},p}_{00}(\Gamma_k)}} \\
&\leq c \sup_{\phi\in W^{1-\frac{1}{p},p}_{00}(\Gamma_k)} \frac{\langle \lambda - \lambda^h, \phi - \mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{W^{1-\frac{1}{p},p}_{00}(\Gamma_k)}} \\
&\quad + c \sup_{\phi\in W^{1-\frac{1}{p},p}_{00}(\Gamma_k)} \frac{\langle \lambda - \lambda^h, \mathcal{P}_{0,k}\phi\rangle}{\|\mathcal{P}_{0,k}\phi\|_{W^{1-\frac{1}{p},p}_{00}(\Gamma_k)}}.
\end{aligned} \tag{5.26}
$$

In order to estimate the second term on the right hand side of (5.26) we use the definition of λ and λ^h, and the inequality (5.24) to obtain

$$
\begin{aligned}
\langle \lambda - \lambda^h, \mathcal{P}_{0,k}\phi\rangle &= a(u^h_0 - u_0, \mathcal{H}^h\tilde{\mathcal{P}}_{0,k}\phi) \\
&\leq ch\|u_0\|_{2,p}\|\mathcal{P}_{0,k}\phi\|_{W^{1-1/p',p'}_{00}(\Gamma_k)} \\
&\leq ch\|u_0\|_{2,p}\|\mathcal{P}_{0,k}\phi\|_{W^{1-1/p,p}_{00}(\Gamma_k)} \text{ since } p > p'.
\end{aligned} \tag{5.27}
$$

For estimating the first term on the right hand side of (5.26) we note that

$$
\|\phi - \mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} = \sup_{v\in W^{1-\frac{1}{p},p}_{00}(\Gamma_k)} \frac{\langle \phi - \mathcal{P}_{0,k}\phi, v - \mathcal{P}_{0,k}v\rangle}{\|v\|_{W^{1-\frac{1}{p},p}_{00}(\Gamma_k)}}
$$

$$
\begin{aligned}
&\leq \sup_{v\in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\|\phi-\mathcal{P}_{0,k}\phi\|_{L^2(\Gamma_k)}\|v-\mathcal{P}_{0,k}v\|_{L^2(\Gamma_k)}}{\|v\|_{W_{00}^{1-1/p,p}(\Gamma_k)}}\\
&\leq ch\|\phi\|_{W_{00}^{1-1/p,p}(\Gamma_k)}. \qquad (5.28)
\end{aligned}
$$

In the last inequality we used (5.13) and the fact that $W_{00}^{1-1/p,p}(\Gamma_k) \hookrightarrow H_{00}^{1/2}(\Gamma_k)$ for $p>2$. Hence,

$$
\begin{aligned}
\langle \lambda-\lambda^h, \phi-\mathcal{P}_{0,k}\phi\rangle &\leq \|\lambda-\lambda^h\|_{W_{00}^{1-1/p,p}(\Gamma_k)}\|\phi-\mathcal{P}_{0,k}\phi\|_{W^{-1+1/p,p'}(\Gamma_k)}\\
&\leq ch\|u_0\|_{2,p}\|\phi\|_{W_{00}^{1-1/p,p}(\Gamma_k)}, \text{ by (5.18) and (5.28)}, \qquad (5.29)
\end{aligned}
$$

and the inequality (5.20) follows from (5.26), (5.27) and (5.29).

Proof of (5.19):

Case $2\leq p<\infty$: We have

$$
\begin{aligned}
\|\lambda-\lambda^h\|_{L^p(\Gamma_k)} &\leq \sup_{\phi\in L^{p'}(\Gamma_k)} \frac{\langle \lambda-\lambda^h, \phi-\mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}}\\
&+ \sup_{\phi\in L^{p'}(\Gamma_k)} \frac{\langle \lambda-\lambda^h, \mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}}. \qquad (5.30)
\end{aligned}
$$

The first term on the right hand side of (5.30) is bounded as follows:

$$
\begin{aligned}
\sup_{\phi\in L^{p'}(\Gamma_k)} \frac{\langle \lambda-\lambda^h, \phi-\mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}} &\leq \sup_{\phi\in L^{p'}(\Gamma_k)} \frac{\|\lambda-\lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}\|\phi-\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)}}{\|\phi\|_{L^{p'}(\Gamma_k)}}\\
&\leq ch^{1-\frac{1}{p}}\|u_0\|_{2,p}. \qquad (5.31)
\end{aligned}
$$

Here we have used (5.14) and (5.18) to arrive in (5.31). In order to estimate the second term on the right hand side of (5.30) we use the definition of λ and λ^h to obtain

$$
\begin{aligned}
\sup_{\phi\in L^{p'}(\Gamma_k)} \frac{\langle \lambda-\lambda^h, \mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}} &\leq \frac{\int_Y a_{ij}\partial_i(u_0-u_0^h)\partial_j(\mathcal{H}^h\tilde{\mathcal{P}}_{0,k}\phi)dy}{\|\mathcal{P}_{0,k}\phi\|_{L^{p'}(\Gamma_k)}}\\
&\leq ch\frac{\|u_0\|_{2,p}\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p'},p'}}}{\|\mathcal{P}_{0,k}\phi\|_{L^{p'}(\Gamma_k)}}\\
&\leq ch^{1-\frac{1}{p}}\|u_0\|_{2,p}, \text{ by (5.9)}.
\end{aligned}
$$

Case $p = \infty$: Let $z \in \Gamma_k$, then

$$|\lambda(z) - \lambda^h(z)| \leq |\lambda(z) - \mathcal{P}_{0,k}\lambda(z)| + |\lambda^h(z) - \mathcal{P}_{0,k}\lambda(z)|. \tag{5.32}$$

For the first term of (5.32), by Theorem 3.1 [51] there exists a positive constant c such that

$$\begin{aligned} |\lambda(z) - \mathcal{P}_{0,k}\lambda(z)| &\leq c\|\lambda - v^h\|_{0,\infty,\Gamma_k} \\ &\quad + c\ \exp(-ch)\|\lambda - v^h\|_{0,1,\Gamma_k},\ \forall\ v^h \in Y_{0,k}. \end{aligned} \tag{5.33}$$

The use of $\mathcal{Q}_1$ elements to approximate u_0 implies $A\nabla u_0^h \cdot \eta_k|_{\Gamma_k} \in Y_{0,k}$, therefore we can take $v^h = A\nabla u_0^h \cdot \eta_k$ in (5.33) and use (5.1) to obtain

$$\|\lambda - \mathcal{P}_{0,k}\lambda\|_{0,\infty} \leq ch\|u_0\|_{2,\infty}. \tag{5.34}$$

When $\mathcal{P}_1$ elements are used to approximate u_0, although $A\nabla u_0^h \cdot \eta_k|_{\Gamma_k} \notin Y_{0,k}$, it is easy to see that there exists $v^h \in Y_{0,k}$ such that $\|A\nabla u_0^h \cdot \eta_k - v^h\|_{0,\infty,\Gamma_k} \leq ch\|A\nabla u_0^h \cdot \eta_k\|_{0,\infty,\Gamma_k}$, hence (5.34) follows from (5.1) and (5.33).

To estimate the second term on the right hand side of (5.32), following [51] let $E_z \subset \Gamma_k$ denotes an edge of an element $K_z \in \mathcal{T}^h(\Omega)$ such that $z \in E_z$, and define δ_z as the polynomial of degree 1 on E_z such that

$$\int_{E_z} \delta_z(s)v(s)ds = v(z), \text{ for any } v \text{ polynomial of degree 1.}$$

Regard δ_z as extended by zero to $\Gamma_k \setminus E_z$ and denote by $\tilde{\delta}_z^h \in V^h(\Omega)$ the extension by zero of $\mathcal{P}_{0,k}\delta_z$ to Ω. Then we have

$$\begin{aligned} \lambda^h(z) - \mathcal{P}_{0,k}\lambda(z) &= \int_{\Gamma_k} \mathcal{P}_{0,k}(\lambda^h - \lambda)\delta_z ds \\ &= \int_{\Gamma_k} (\lambda^h - \lambda)\mathcal{P}_{0,k}\delta_z ds \\ &= \int_{\Omega} A_{ij}\partial_i(u_0 - u_0^h)\partial_j(\tilde{\delta}_z^h)dx \end{aligned} \tag{5.35}$$

where we have used the definition of λ^h to obtain (5.35). From (5.1) and (5.35) follows

$$|\lambda^h(z) - \mathcal{P}_{0,k}\lambda(z)| \leq ch\|u_0\|_{2,\infty}\|\tilde{\delta}_z^h\|_{1,1}.$$

Using an inverse estimate followed by a Poincare inequality we have

$$\|\tilde{\delta}_z^h\|_{1,1} \le ch^{-1}\|\tilde{\delta}_z^h\|_{0,1} \le c\|\mathcal{P}_{0,k}\delta_z\|_{0,1,\Gamma_k}.$$

Finally, we use the fact that $\|\mathcal{P}_{0,k}\delta_z\|_{0,1,\Gamma_k} \le c$, see Lemma 3.5 in [51], and (5.19) follows. □

Proposition 5.1.1 estimates the error between ∇u_0 and its proposed numerical approximation Ψ^h. This Proposition is required in the proof of Proposition 5.1.2.

Proposition 5.1.1 *Let u_0 and Ψ^h be defined by Equations (2.15) and (4.7), respectively. Assume $u_0 \in W^{2,p}(\Omega)$ and that linear or bilinear finite elements are used to approximate u_0. Then for $2 \le p \le \infty$ we have*

$$\|(\nabla u_0 - \Psi^h)\cdot\nu\|_{0,p} \le ch\|u_0\|_{2,p}, \quad \forall\ \nu \in \mathbb{R}^2 \text{ with } |\nu| = 1 \tag{5.36}$$

and

$$\|(\nabla u_0 - \Psi^h)\cdot\nu\|_{1,p,h} \le c\|u_0\|_{2,p}, \quad \forall\ \nu \in \mathbb{R}^2 \text{ with } |\nu| = 1. \tag{5.37}$$

Proof of (5.36): From the triangular inequality we have

$$\|(\nabla u_0 - \Psi^h)\cdot\nu\|_{0,p} \le \|(\nabla u_0 - \nabla u_0^h)\cdot\nu\|_{0,p} + \|(\nabla u_0^h - \Psi^h)\cdot\nu\|_{0,p}. \tag{5.38}$$

Use (5.1) to estimate the first term on the right hand side of (5.38). For the second term, by the definition of Ψ^h, we have

$$\|(\nabla u_0^h - \Psi^h)\cdot\nu\|_{0,p} \le c \sum_{k\in\{e,w,n,s\}} \|E_k(\mu^h - \nabla u_0^h\cdot\eta^k)\|_{0,p}.$$

Consider $k = e$ and that bilinear elements are used to approximate u_0; the other cases, $k \in \{w,n,s\}$ or when $\mathcal{P}_1$ elements are used, follow in a similar way. From definition, the function $E_e\left(\mu^h - \frac{\partial u_0^h}{\partial x_1}\right)$ is linear in the x_1 direction and equal to zero in $x_1 \le 1-h$, hence

$$\|E_e(\mu^h - \nabla u_0^h\cdot\eta^k)\|_{0,p} \le h^{1/p}\left\|\partial_{x_1}u_0^h - \mu^h\right\|_{0,p,\Gamma_e}, \quad \text{if } 2 \le p < \infty$$

or

$$\|E_e(\mu^h - \nabla u_0^h\cdot\eta^k)\|_{0,\infty} \le \left\|\partial_{x_1}u_0^h - \mu^h\right\|_{0,\infty,\Gamma_e}, \quad \text{if } p = \infty.$$

Case $2 \le p < \infty$: The triangular inequality gives

$$\left\|\partial_{x_1}u_0^h - \mu^h\right\|_{0,p,\Gamma_e} \le \left\|\partial_{x_1}u_0^h - \partial_{x_1}u_0\right\|_{0,p,\Gamma_e} + \left\|\partial_{x_1}u_0 - \mu^h\right\|_{0,p,\Gamma_e}. \tag{5.39}$$

In order to estimate the first term on the right hand side of (5.39), let $K \in \mathcal{T}_h(\Omega)$ containing an edge $E \subset \Gamma_k$. Applying a Trace Theorem we have

$$
\begin{aligned}
\left\|\partial_{x_1} u_0^h - \partial_{x_1} u_0\right\|_{0,p,E} &\leq c\left(h^{-1}\left\|\partial_{x_1} u_0^h - \partial_{x_1} u_0\right\|_{0,p,K}^p\right. \\
&\quad \left. + h^{p-1}\left\|\partial_{x_1} u_0^h - \partial_{x_1} u_0\right\|_{1,p,K}^p\right)^{1/p}. \qquad (5.40)
\end{aligned}
$$

From (5.1), (5.5) and (5.40) we obtain

$$\left\|\partial_{x_1} u_0^h - \partial_{x_1} u_0\right\|_{0,p,\Gamma_e} \leq ch^{1-1/p}\|u_0\|_{2,p}. \qquad (5.41)$$

For second term on the right hand side of (5.39), we apply the definition of λ and λ^h to obtain $\left\|\partial_{x_1} u_0 - \mu^h\right\|_{0,p,\Gamma_e} = A_{11}\left\|\lambda - \lambda^h\right\|_{0,p,\Gamma_e}$, and therefore from (5.19) we have

$$\left\|\partial_{x_1} u_0 - \mu^h\right\|_{0,p,\Gamma_e} \leq ch^{1-1/p}\|u_0\|_{2,p}. \qquad (5.42)$$

From (5.39), (5.41) and (5.42) we obtain

$$\|E_e(\mu^h - \nabla u_0^h \cdot \eta^e)\|_{0,p} \leq ch\|u_0\|_{2,p},$$

and hence estimate (5.36) holds for $p < \infty$.

Case $2 = \infty$: We have

$$\left\|\partial_{x_1} u_0^h - \mu^h\right\|_{0,\infty,\Gamma_e} \leq \|\partial_{x_1} u_0^h - \partial_{x_1} u_0\|_{0,\infty,\Gamma_e} + \|\partial_{x_1} u_0 - \mu^h\|_{0,\infty,\Gamma_e},$$

and applying (5.19) and (5.1) we have

$$\left\|\partial_{x_1} u_0 - \mu^h\right\|_{0,\infty,\Gamma_e} \leq ch\|u_0\|_{2,\infty},$$

and hence estimate (5.36) follows for $p = \infty$.

Proof of (5.37): We have

$$
\begin{aligned}
\|(\nabla u_0^h - \Psi^h)\cdot\nu\|_{0,p} &\leq c\|(\nabla u_0 - \Psi^h)\cdot\nu\|_{0,p} + \|(\nabla u_0 - \nabla u_0^h)\cdot\nu\|_{0,p} \\
&\leq ch\|u_0\|_{2,p}, \quad \text{by (5.1) and (5.36)} \qquad (5.43)
\end{aligned}
$$

and from an inverse inequality, Theorem 1.4.3, follows that

$$\|(\nabla u_0^h - \Psi^h)\cdot\nu\|_{1,p,h} \leq c\|u_0\|_{2,p}.$$

Since

$$\|(\nabla u_0 - \Psi^h)\cdot\nu\|_{1,p,h} \le c\left(\|(\nabla u_0^h - \nabla u_0)\cdot\nu\|_{1,p,h} + \|(\nabla u_0^h - \Psi^h)\cdot\nu\|_{1,p,h}\right),$$

we obtain (5.37) from (5.5). □

The following proposition estimates the error between u_1 and u_1^h. These estimates are required in the proof of Theorems 5.1.1 and 5.1.2.

Proposition 5.1.2 *Let u_1 and u_1^h be defined by (2.10) and (4.8), respectively. Assume that $u_0 \in W^{2,p}(\Omega)$ and $\chi^i \in W^{1,q}_{per}(Y)$, for $1/p + 1/q \le 1/2$. Then there exists a constant c independent of ϵ and h such that*

$$|u_1 - u_1^h|_{1,h} \le c\|u_0\|_{2,p}\|\chi\|_{1,q,Y}\left(\frac{h^2}{\epsilon^2}+1\right)^{1/2} \tag{5.44}$$

and

$$\|u_1 - u_1^h\|_0 \le ch\|u_0\|_{2,p}\|\chi\|_{1,q,Y}, \tag{5.45}$$

where $\|\chi\|_{1,q,Y} = \sum_i \|\chi^i\|_{1,q,Y}$.

Proof of (5.44): We have

$$\begin{aligned} |u_1 - u_1^h|^2_{1,h} \le\ & 2\sum_{K_j\in\mathcal{T}_h(\Omega)}\int_{K_j}\sum_{j\in 1,2}((\partial_{x_i}u_0 - \Psi_i^h)\partial_{x_j}\chi^i(\cdot/\epsilon))^2 \\ & +(\chi^i(\cdot/\epsilon)\cdot\partial_{x_j}(\partial_{x_i}u_0 - \Psi_i^h))^2dx. \end{aligned} \tag{5.46}$$

For the first term on the right hand side of (5.46) we have

$$\begin{aligned} \sum_{K_j\in\mathcal{T}_h(\Omega)}\int_{K_j}\sum_{j\in 1,2}((\partial_{x_i}u_0 - \Psi_i^h)\partial_{x_j}\chi^i(\cdot/\epsilon))^2dx &\le |\partial_{x_i}u_0 - \Psi_i^h|^2_{0,p}\|\partial_{x_j}\chi^i(\cdot/\epsilon)\|^2_{0,q} \\ &\le \epsilon^{-2}|\partial_{x_i}u_0 - \Psi_i^h|^2_{0,p}\|\chi\|^2_{1,q,Y} \\ &\le c\epsilon^{-2}h^2\|u_0\|^2_{2,p}\|\chi\|^2_{1,q,Y}, \end{aligned} \tag{5.47}$$

where we have used (5.36) to obtain (5.47).

The second term on the right hand side of (5.46) is bounded by a Cauchy inequality, $\|\chi^i\partial_j(\partial_i u_0 - \Psi_i^h)\|_0^2 \le \|\chi\|^2_{0,q}|\partial_i u_0 - \Psi_i^h|^2_{1,p,h}$.

Proof of (5.45): Direct application of Cauchy inequality and the approximation error estimate (5.1). □

The following proposition estimates the error between $\tilde{\phi}_\epsilon$ and $\tilde{\phi}_\epsilon^h$. This Proposition is required in the proof of Theorems 5.1.1 and 5.1.2.

Proposition 5.1.3 *Let $\tilde{\phi}_\epsilon$ and $\tilde{\phi}_\epsilon^h$ be defined by (2.23) and (4.9), respectively. Assume that $u_0 \in W^{2,p}(\Omega)$ and $v_k \in W^{1,q}(G_k)$, for $1/p + 1/q \le 1/2$. Then*

$$|\tilde{\phi}_\epsilon - \tilde{\phi}_\epsilon^h|_{1,h} \le c\left(\frac{h^2}{\epsilon^2}+1\right)^{1/2} \max_k \|v_k\|_{1,q,G_k}\|u_0\|_{2,p} \tag{5.48}$$

and

$$\|\tilde{\phi}_\epsilon - \tilde{\phi}_\epsilon^h\|_0 \le ch \max_k \|v_k - \chi_k^*\|_{0,q,G_k}\|u_0\|_{2,p}. \tag{5.49}$$

Proof: From definition of $\tilde{\phi}_\epsilon$ and $\tilde{\phi}_\epsilon^h$ we have

$$|\tilde{\phi}_\epsilon - \tilde{\phi}_\epsilon^h|_{1,h} \le \sum_{k\in\{e,w,n,s\}} |\tilde{\phi}_\epsilon^k - \tilde{\phi}_\epsilon^{k,h}|_{1,h},$$

and the proposition follows from arguments similar to the ones given in the proof of Proposition 5.1.2. □

Finally, we prove the last proposition used in the proof of Theorems 5.1.1 and 5.1.2. Proposition 5.1.4 estimates the error between $\bar{\phi}$ and $\bar{\phi}^h$.

Proposition 5.1.4 *Let $\bar{\phi}$ be defined by Equation (2.24), $\bar{\phi}^h$ be the finite element approximation to the Equation (4.10), and assume that $u_0 \in H^2(\Omega)$. Then we have*

$$\|\bar{\phi} - \bar{\phi}^h\|_1 \le c\|u_0\|_2 \tag{5.50}$$

and

$$\|\bar{\phi} - \bar{\phi}^h\|_0 \le ch\|u_0\|_2. \tag{5.51}$$

Proof of (5.50): We note that $\chi^*\mu^h \in H^{1/2}(\partial\Omega)$, see Remark 4.2.1, hence we define $\psi \in H^1(\Omega)$ as the solution of

$$\nabla \cdot A\nabla\psi = 0 \;\text{ in }\; \Omega \quad \psi = \chi^*\mu^h \;\text{ on }\; \partial\Omega. \tag{5.52}$$

From regularity theory and (5.18) we have

$$\|\psi\|_1 \le \sum_k \|\chi^*\mu^h\|_{H_{00}^{1/2}(\Gamma_k)} \le c\|u_0\|_2, \tag{5.53}$$

and from triangular inequality

$$\|\bar{\phi} - \bar{\phi}^h\|_1 \le \|\bar{\phi} - \psi\|_1 + \|\bar{\phi}^h - \psi\|_1.$$

Since $\chi^*\mu^h \in V^h(\Omega)$, the problem of finding $\bar{\phi}$ reduces to a conforming finite element problem, hence the standard finite element analysis and (5.53) gives

$$|\bar{\phi}^h - \psi|_1 \leq c\|u_0\|_2.$$

Finally, from regularity theory and Lemma 5.1.3 we obtain

$$\begin{aligned} |\bar{\phi} - \psi|_1 &\leq \|\chi^*\mu^h - \chi^*\partial_\eta u_0\|_{H^{1/2}(\partial\Omega)} \\ &\leq \sum_k \|\chi^*\mu^h - \chi^*\partial_\eta u_0\|_{H^{1/2}_{00}(\Gamma_k)} \leq c\|u_0\|_2. \end{aligned}$$

Proof of (5.51): From the triangular inequality

$$\|\bar{\phi} - \bar{\phi}^h\|_0 \leq c\|\bar{\phi} - \psi\|_0 + \|\bar{\phi}^h - \psi\|_0,$$

and from standard finite element analysis and (5.53) we obtain

$$\|\bar{\phi}^h - \psi\|_0 \leq ch\|\psi\|_1 \leq ch\|u_0\|_2.$$

Applying Theorem 1.3.6 we have

$$\begin{aligned} \|\bar{\phi} - \psi\|_0 &\leq c(\sum_k \|\chi^*\partial_\eta u_0 - \chi^*\mu^h\|^2_{H^{-1/2}(\Gamma_k)})^{1/2} \\ &\leq ch\|u_0\|_2 \ \text{ by (5.20).} \end{aligned}$$

□

5.2 Error analysis for Method I:

In this section assume that bilinear finite elements are used to approximate u_0.

Theorem 5.2.1 *Let u_ϵ be the solution of the Problem (1.2), u_0, χ^j and u_h be defined by Equations (2.8), (2.15) and (4.5), respectively, and the functions v_k and the constants χ^*_k be defined as in Subsection 2.1.1. Assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{2,p}(\Omega)$, $\chi^j \in W^{1,q}_{per}(Y)$, v_e and $\nabla(v_e - \chi^*_e)exp(-\gamma y_1) \in L^s(G_e)$, for $1/p + 1/q \leq 1/2$ and $1/s + 3/p \leq 1$. We also assume similar hypothesis for the other functions v_k. Then there exists a constant c independent of ϵ and h such that*

$$\|u_\epsilon - u_h\|_{1,h} \leq c(h + \epsilon)\|u_0\|_{2,p}.$$

Furthermore, if $u_0 \in W^{3,p}(\Omega),\ p > 2$ *then*

$$\|u_\epsilon - u_h\|_0 \le c(h^2 + \epsilon^{3/2} + \epsilon h)(\|u_0\|_{2,\infty} + \|u_0\|_3).$$

Proof: By the triangular inequality we have

$$\begin{aligned} |u_\epsilon - u_h|_{1,h} \;\le\; & |u_\epsilon - u_0 - u_1 - \phi_\epsilon|_1 + |u_0 - u_0^h|_{1,h} + \epsilon|u_1 - u_1^h|_{1,h} \\ & + \epsilon|\bar{\phi} - \bar{\phi}^h|_{1,h} + \epsilon|\tilde{\phi}_\epsilon - \tilde{\phi}_\epsilon^h|_{1,h}, \end{aligned}$$

and the theorem follows from Theorem 2.1.1, the approximation error (5.1), and Propositions 5.2.1, 5.2.2 and 5.2.3. □

Proposition 5.2.1 *Let* u_1 *and* u_1^h *be defined by (2.10) and (4.1), respectively. Assume that* $u_0 \in W^{2,p}(\Omega)$ *and* $\chi^i \in W^{1,q}_{per}(Y)$, *for* $1/p + 1/q \le 1/2$. *Then there exists a constant* c *independent of* ϵ *and* h *such that*

$$|u_1 - u_1^h|_{1,h} \le c\|u_0\|_{2,p}\|\chi\|_{1,q,Y}\left(\frac{h^2}{\epsilon^2} + 1\right)^{1/2} \tag{5.54}$$

and

$$\|u_1 - u_1^h\|_0 \le ch\|u_0\|_{2,p}\|\chi\|_{1,q,Y}, \tag{5.55}$$

where $\|\chi\|_{1,q,Y} = \sum_i \|\chi^i\|_{1,q,Y}$.

Proof: Similar to the proof of Proposition 5.1.2. □

Proposition 5.2.2 *Let* $\tilde{\phi}_\epsilon$ *and* $\tilde{\phi}_\epsilon^h$ *be defined by (2.23) and (4.4), respectively. Assume that* $u_0 \in W^{2,p}(\Omega)$ *and* $v_k \in W^{1,q}(G_k)$, *for* $1/p + 1/q \le 1/2$. *Then*

$$|\tilde{\phi}_\epsilon - \tilde{\phi}_\epsilon^h|_{1,h} \le c\left(\frac{h^2}{\epsilon^2} + 1\right)^{1/2} \max_k \|v_k\|_{1,q,G_k}\|u_0\|_{2,p}, \tag{5.56}$$

and

$$\|\tilde{\phi}_\epsilon - \tilde{\phi}_\epsilon^h\|_0 \le ch \max_k \|v_k - \chi_k^*\|_{0,q,G_k}\|u_0\|_{2,p}. \tag{5.57}$$

Proof: Similar to the proof of Proposition 5.1.3. □

Proposition 5.2.3 *Let $\bar{\phi}$ be defined by Equation (4.2), $\bar{\phi}^h$ be the finite element approximation to the Equation (4.2), and assume that $u_0 \in H^2(\Omega)$. Then we have*

$$\|\bar{\phi} - \bar{\phi}^h\|_1 \leq c\|u_0\|_2. \tag{5.58}$$

In addition, if $u_0 \in W^{2,\infty}(\Omega)$ then

$$\|\bar{\phi} - \bar{\phi}^h\|_0 \leq ch\|u_0\|_{2,\infty,\Omega}. \tag{5.59}$$

<u>*Proof of (5.58)*</u>: Let $\psi \in H^1(\Omega)$ be defined by equation (4.2), then

$$\|\bar{\phi} - \bar{\phi}^h\|_1 \leq \|\bar{\phi} - \psi\|_1 + \|\bar{\phi}^h - \psi\|_1. \tag{5.60}$$

Since we use bilinear elements to approximate u_0^h, $\chi^*\partial_\eta u_0^h$ is continuous along $\partial\Omega$ and the problem of finding $\bar{\phi}$ is a conforming finite element problem. From the standard finite element analysis we obtain

$$\|\bar{\phi}^h - \psi\|_1 \leq c\|\psi\|_1.$$

In order to estimate the second term on the right hand side of (5.60), we observe that it is sufficient to bound $\|\chi^*\partial_\eta u_0^h - \chi^*\partial_\eta u_0\|_{H^{1/2}(\partial\Omega)}$. By remarks 2.1.1 and 4.1.1, we have

$$\chi^*\partial_\eta u_0 - \chi^*\partial_\eta u_0^h = \sum_k \varphi_k \chi_k^* \left(\frac{\partial u_0}{\partial x_{i_k}} - \frac{\partial u_0^h}{\partial x_{i_k}} \right)\Bigg|_{\partial\Omega},$$

where $i_k = 1$ if $k \in \{e, w\}$ and $i_k = 2$ if $k \in \{n, s\}$. Although the trace of $\partial_{x_i} u_0^h$ is continuous and belongs to $H^{1/2}(\partial\Omega)$, $\partial_{x_i} u_0^h$ is not in $H^1(\Omega)$, this implies that we can not use the trace theorem to estimate $\|\partial_{x_i} u_0 - \partial_{x_i} u_0^h\|_{H^{1/2}(\partial\Omega)}$. Then we use the interpolation operator Π, from Lemma 1.4.1, to obtain

$$\begin{aligned} \left\|\partial_{x_i} u_0 - \partial_{x_i} u_0^h\right\|_{H^{1/2}(\partial\Omega)} &= \left\|\partial_{x_i} u_0 - \Pi\partial_{x_i} u_0^h\right\|_{H^{1/2}(\partial\Omega)} \\ &\leq c\left\|\partial_{x_i} u_0 - \Pi\partial_{x_i} u_0^h\right\|_1 \\ &\leq c\left\|\partial_{x_i} u_0 - \Pi\partial_{x_i} u_0\right\|_1 + \left\|\Pi\partial_{x_i} u_0 - \Pi\partial_{x_i} u_0^h\right\|_1. \end{aligned} \tag{5.61}$$

For estimating the first term on the right hand side of (5.61), we use Lemma 1.4.1 to obtain

$$\left\|\partial_{x_i} u_0 - \Pi\partial_{x_i} u_0\right\|_1 \leq c\left\|\partial_{x_i} u_0\right\|_1,$$

and from the inverse inequality and Lemma 1.4.1

$$\begin{aligned}\left\|\Pi\partial_{x_i}u_0 - \Pi\partial_{x_i}u_0^h\right\|_1 &\leq \frac{c}{h}\left\|\Pi\partial_{x_i}u_0 - \Pi\partial_{x_i}u_0^h\right\|_0 \\ &\leq \frac{c}{h}\left\|\partial_{x_i}u_0 - \partial_{x_i}u_0^h\right\|_0 \\ &\leq \frac{c}{h}\|u_0 - u_0^h\|_1 \\ &\leq c\|u_0\|_2\end{aligned}$$

Proof of (5.59): From a triangular inequality

$$\|\bar{\phi} - \bar{\phi}^h\|_0 \leq c\|\bar{\phi} - \psi\|_0 + \|\bar{\phi}^h - \psi\|_0. \tag{5.62}$$

From Remark 4.1.1 $\psi \in H^1(\Omega)$ and from the standard finite element analysis we obtain

$$\|\bar{\phi}^h - \psi\|_0 \leq ch\|\psi\|_1$$

In order to estimate the first term on the right hand side of (5.62) we apply Theorem 1.4.4 to obtain

$$\|\chi^*\partial_\eta u_0 - \chi^*\partial_\eta u_0^h\|_{L^\infty(\partial\Omega)} \leq ch\|u_0\|_{2,\infty,},$$

and from maximum principle it follows

$$\|\bar{\phi} - \psi\|_0 \leq ch\|u_0\|_{2,\infty,\Omega}.$$

□

Chapter 6

Numerical Experiments and Conclusions

In this section, we present some numerical results for solving our model problem with

$$a(x/\epsilon) = \left(\frac{2 + P\sin(2\pi x_1/\epsilon)}{2 + P\cos(2\pi x_2/\epsilon)} + \frac{2 + \sin(2\pi x_2/\epsilon)}{2 + P\sin(2\pi x_1/\epsilon)} \right) I_{2\times 2}$$

$$f(x) = -1 \quad , u = 0 \quad \text{on} \quad \partial\Omega, \quad \text{and} \quad P = 1.8.$$

The above example was considered in [31]. We compare the solution obtained by our method with the solution obtained by a second order accurate finite element method in a fine mesh of size h_f, which we call u^*_ϵ.

6.1 Numerical results for Method I:

Tables 6.1 and 6.2 provide absolute errors estimates for $u^*_\epsilon - u^{h,\hat{h},\tau}_\epsilon$, on the $\| \cdot \|_0$ norm and $| \cdot |_{1,h}$ semi norm for different values of h and ϵ. We have used $\tau = 2$, $\hat{h} = 1/128$, $h_f = 1/2048$, and a triangular mesh with continuous piecewise linear functions to approximate $\chi^j_{\hat{h}}$ and $v^{\hat{h},\tau}_e$.

From Tables 6.1 and 6.2, we see that for $\epsilon << h$ we have errors of order $O(h^2)$ and $O(h)$ for the L^2 norm and semi norm H^1 respectively. We observe that when we fix $h, \hat{h}$ and p, and decrease ϵ, the errors almost do not change, hence the dominant error term is $O(h)$. Also looking the diagonal values in these tables we see clearly that the numerical error agrees with the theoretical rates from Theorem 5.2.1.

Table 6.1: $\|\cdot\|_0$ error

$\epsilon\downarrow \quad h\rightarrow$	1/8	1/16	1/32	1/64
1/16	2.7085e-04	7.7993e-05		
1/32	2.6300e-04	6.6246e-05	1.7773e-05	
1/64	2.5388e-04	5.9446e-05	1.4414e-05	1.2137e-05

Table 6.2: $|\cdot|_{1,h}$ error

$\epsilon\downarrow \quad h\rightarrow$	1/8	1/16	1/32	1/64
1/16	0.0097	0.0066		
1/32	0.0089	0.0051	0.0036	
1/64	0.0086	0.0045	0.0026	0.0018

Table 6.3 shows the improvement obtained in the final approximation by considering the numerical approximation for the boundary corrector. We observe a better improvement on the $\|\cdot\|_0$ norm rather then on $|\cdot|_{1,h}$ semi norm. The reason for this is that $\bar{\phi}$ is obtained through the homogenized equation associated to Problem (2.20), therefore it is a good approximation for $\bar{\theta}_\epsilon$ on $L^2(\Omega)$ norm but not on $|\cdot|_1$ semi norm. The term $\tilde{\phi}_\epsilon$ is defined in a thin boundary layer that mostly force the approximation to satisfies the zero Dirichlet boundary condition.

Table 6.4 compares the L^2 error between the proposed method and the multi-scale finite element presented in [31]. We used $h_f = 1/3200$ for $\epsilon = 1/50, 1/100$, and $h_f = 1/1600$ for $\epsilon = 1/25$. Observe that a factor 4 is obtained on our method for $\|u_\epsilon^* - u_\epsilon^{h,\hat{h},\tau}\|_0$ when u_ϵ^* is computed very accurately. We note that we do not obtain factor 4 from the $\epsilon = 1/50$, $h = 32$ to $\epsilon = 1/100$, $h = 64$ because h_f is not small enough to capture the fast scale. This is an explicit evidence that our method is more accurate than standard finite element methods on a very fine mesh.

In our numerical tests we observed a very fast convergence of $v_e^{\hat{h},\tau}$ to the constant $\chi_e^{*,\hat{h},\tau}$ as $y_1 \rightarrow -\tau$. Considering $\tau_1 < \tau_2 \in \{1, 2...8\}$ we obtained that $\sup_{\{y_2\in[0,1], y_1\in[-\tau_2,-\tau_1]\}} |v_e^{\hat{h},\tau_1}(-\tau_1, y_2) - v_e^{\hat{h},\tau_2}(y_1, y_2)| \leq 10^{-14}$. That confirms the numerical approximation for $\tilde{\phi}_\epsilon^e$ given by Formula (4.3) is reasonable.

Table 6.3:

$\epsilon = 1/64,\ h = 1/32,\ h_f = 1/1024$

	$\|\cdot\|_0$	$\vert\cdot\vert_{1,h}$
$u_\epsilon^* - u_0^{h,\hat{h}}$	0.0287	0.0215
$u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}}$	0.0213	0.0026
$u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}} - \epsilon\bar{\phi}^{h,\hat{h},\tau}$	6.1557e-05	0.0026
$u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}} - \epsilon(\bar{\phi}^{h,\hat{h},\tau} + \tilde{\phi}_\epsilon^{h,\hat{h},\tau})$	6.1557e-05	0.0024

Table 6.4:

$\tau = 2,\ \hat{h} = 64,$

h	ϵ	$\|u_\epsilon^* - u_\epsilon^{h,\hat{h},\tau}\|_0$	MsFEM-O L^2 Error
1/16	1/25	6.92e-05	6.23 e-05
1/32	1/50	1.77e-05	8.43 e-05
1/64	1/100	1.24e-05	9.32 e-05

The Figures bellow show the error evolution as we include the asymptotic expansion terms in our numerical approximation, for $h_f = 1/100$, $h = 1/10$, $\hat{h} = 1/50$, $\tau = 2$ and $\epsilon = 1/20$; Figure 1 is the plot of the "exact" solution u_ϵ^*. In Figure 2 from left to right we see that amplitude of error oscillations decreases when we include the approximation for u_1. Its is possible to see an overall improvement in the error from Figure 2 (left) to Figure 3 (right) when the approximation for $\bar{\phi}$ is included, and finally in Figure 3 (left) we see that the zero boundary condition is satisfied when the complete approximation $u_0^{h,\hat{h}} + \epsilon u_1^{h,\hat{h}} + \epsilon(\bar{\phi}^{h,\hat{h},\tau} + \tilde{\phi}^{h,\hat{h},\tau})$ is considered.

6.2 Numerical results for Method II:

Table 6.5 provides absolute errors estimates for $u_\epsilon^* - u_\epsilon^{h,\hat{h},\tau}$, on the $\|\cdot\|_0$ norm and $|\cdot|_{1,h}$ semi norm for different values of h and ϵ. We have used $\tau = 2$, $\hat{h} = 1/128$, $h_f = 1/2048$, and a triangular mesh with continuous piecewise linear functions to approximate $\chi_{\hat{h}}^j$ and $v_\epsilon^{\hat{h},\tau}$.

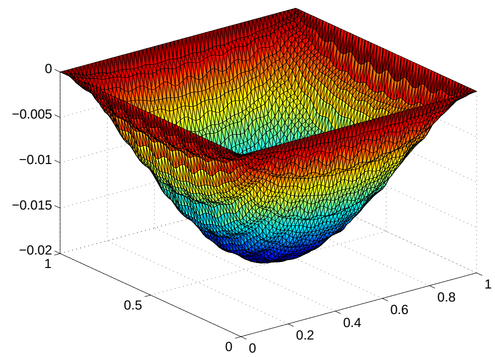

Figure 6.1: u_ϵ^*

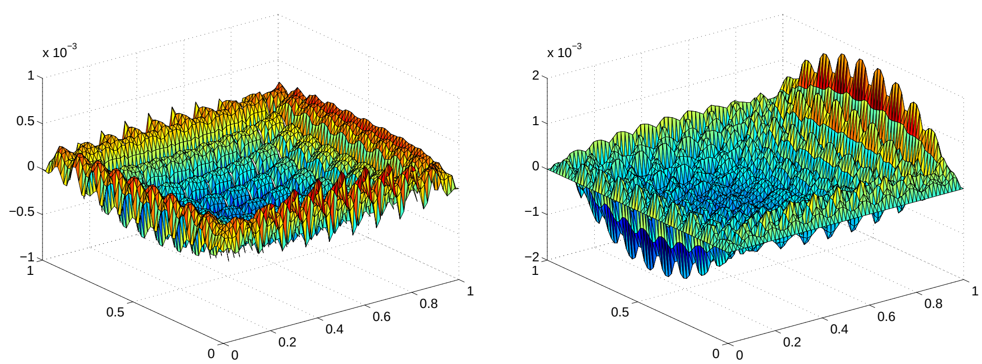

Figure 6.2: $u_\epsilon^* - u_0^{h,\hat{h}}$ (left), and $u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}}$ (right)

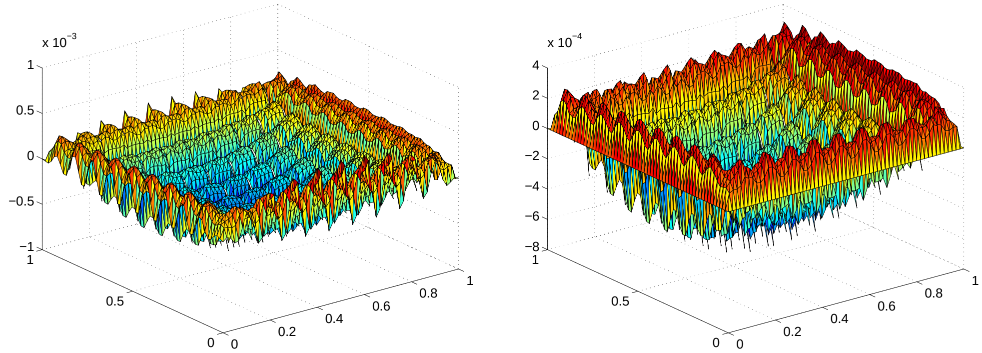

Figure 6.3: $u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}} - \epsilon \bar{\phi}^{h,\hat{h},\tau}$ (left), and $u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}} - \epsilon(\bar{\phi}^{h,\hat{h},\tau} + \tilde{\phi}^{h,\hat{h},\tau})$ (right)

Table 6.5: $u_\epsilon^* - u_\epsilon^{h,\hat{h},\tau}$ error

$\|\cdot\|_0$ error

$\epsilon\downarrow \quad h\rightarrow$	1/8	1/16	1/32	1/64
1/16	2.7085e-04	7.7993e-05		
1/32	2.6300e-04	6.6246e-05	1.7773e-05	
1/64	2.5388e-04	5.8069e-05	1.6020e-05	1.2137e-05
		$\|\cdot\|_{1,h}$ error		
1/16	0.0097	0.0067		
1/32	0.0086	0.0051	0.0036	
1/64	0.0086	0.0044	0.0025	0.0018

From Table 6.5, we see that for $\epsilon << h$ we have errors of order $O(h^2)$ and $O(h)$ for the L^2 norm and H^1 semi norm , respectively. We observe that when we fix h and decrease ϵ the errors almost do not change. This is evidence that in this case the dominant error term is $O(h)$. Also looking at the diagonal values in this table we see clearly that the numerical error agrees with the theoretical rates from Theorems 5.1.1 and 5.1.2. We also note that the numerical results for Methods I and II are quite similar.

We also consider the following example:

$$a(y) = \begin{cases} 2 & \text{if } 2/5 < y_1 < 3/5 \text{ or } 2/5 < y_2 < 3/5 \\ 1 & \text{otherwise.} \end{cases} \qquad \text{and } f = -1$$

Table 6.6: $u_\epsilon^* - u_\epsilon^{h,\hat{h},\tau}$ error

$\|\cdot\|_0$ error, $h_f = 1/2000$

$\epsilon\downarrow \quad h\rightarrow$	1/10	1/20	1/40
1/20	4.8318e-04	1.3043e-04	
1/40	4.7578e-04	1.1954e-04	3.0805e-05
1/64	2.5388e-04	5.9446e-05	1.4414e-05

We compare the solution obtained by our method with the solution obtained by a second order accurate finite element method in a fine mesh of size

Table 6.7: $u_\epsilon^* - u_\epsilon^{h,\hat{h},\tau}$ error

$|\cdot|_{1,h}$ error, $h_f = 1/2000$

$\epsilon \downarrow \quad h \rightarrow$	1/10	1/20	1/40
1/20	0.0180	0.0092	
1/40	0.0179	0.0090	0.0046
1/64	0.0086	0.0045	0.0026

h_f, which we call u_ϵ^*. Tables 6.6 and 6.7 provide absolute errors estimates for $u_\epsilon^* - u_\epsilon^{h,\hat{h},\tau}$, on the $\|\cdot\|_0$ norm and $|\cdot|_{1,h}$ semi norm for different values of h and ϵ. We have used $\tau = 2$, $\hat{h} = 1/128$, and a triangular mesh with continuous piecewise linear functions to approximate $\chi_{\hat{h}}^j$ and $v_e^{\hat{h},\tau}$.

6.3 Conclusions

We propose new methods for approximating numerically the solution of Equation (1.2). These methods are strongly based on the periodicity of the coefficients a_{ij}, and for this reason they have relative low computational cost with optimal error convergence rate.

Although the convergence analysis presented here does not work for the quasi periodic case $a_{ij}(x, x/\epsilon)$, we believe that the numerical approximation presented here can be generalized for this case. This would be done by approximating matrix $a(x, x/\epsilon)$ by $\sum_j a^j(x/\epsilon) I_{K_j}(x)$, where I_{K_J} is the characteristic function for $K_j \in \mathcal{T}_k(\Omega)$, and then solving cell problem in each sub-domain K_j. Also it is important to note that the method presented in Appendix A extends directly to the case of non-homogeneous Dirichlet boundary condition, i.e. u_ϵ and u_0 defined as in Remark 3.1.1, as far as $\partial_\eta u_0 \in H_{00}^{1/2}(\Gamma_k)$ and $u_0 \in W^{2,\infty}(\Omega)$. The generalization of the method to more general boundary conditions, including the Neumann case, is currently research under progress.

The possible applications of the methods are the simulation of fluids in porous media flow and on the field of composite materials. We observe that the use multiscale methods have been successfully applied to two phase flow simulations; see [22, 34] and references there in.

Appendix A

Convex Polygonal Domain Case

We assume that $Y = (0,1)^2$ and Ω is a bounded open convex polygonal region in $\mathbb{R}^2$. More specifically, we assume that $\partial\Omega = \cup_{k=1}^{N}\Gamma^k$, where each Γ^k is a line segment with outward normal denoted by $N_k = (p_k, q_k)^t$, with p_k and q_k integers and relative prime. This hypothesis is required to guarantee periodicity of $a(x/\epsilon)$ on the line containing Γ_k.

A.1 Theoretical Approximation

A.1.1 Boundary Corrector Approximation

Observe that $u_0 = 0$ along $\partial\Omega$ implies $\nabla u_\epsilon|_{\Gamma_k} = \eta^k \partial_{\eta^k} u_0$, where $\eta^k = N_k/|N_k|$. We then decompose $\theta_\epsilon = \tilde{\theta}_\epsilon + \bar{\theta}_\epsilon$ where

$$-\nabla \cdot a(x/\epsilon)\nabla\tilde{\theta}_\epsilon = 0 \ \text{ in } \ \Omega, \quad \tilde{\theta}_\epsilon = -u_1 - \chi^* \partial_\eta u_0 \ \text{ on } \ \partial\Omega, \tag{A.1}$$

$$-\nabla \cdot a(x/\epsilon)\nabla\bar{\theta}_\epsilon = 0 \ \text{ in } \ \Omega, \quad \bar{\theta}_\epsilon = \chi^* \partial_\eta u_0 \ \text{ on } \ \partial\Omega, \tag{A.2}$$

and $\chi^*|_{\Gamma_k} = \chi^*_k$ are properly chosen constants . By Remark 2.1.1 Problems (A.1) and (A.2) are well posed. The approximation ϕ_ϵ for θ_ϵ is defined later as $\tilde{\phi}_\epsilon + \bar{\phi}_\epsilon$, where $\tilde{\phi}_\epsilon \approx \tilde{\theta}_\epsilon$ and $\bar{\phi}_\epsilon \approx \bar{\theta}_\epsilon$. Next we define constants χ^*_k for which the approximation $\tilde{\phi}_\epsilon$ decays exponentially to zero away from the boundary.

Let $\tau^k = (\eta^k)^\perp$ be the $\pi/2$ rotation counterclockwise of η^k. We introduce the following normal and tangential coordinate system

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = - \begin{pmatrix} \eta^k \cdot y \\ \tau^k \cdot y \end{pmatrix} \tag{A.3}$$

We observe that a function periodic in y with period 1 is periodic in y' with period $T_k = (p_k^2 + q_k^2)^{1/2}$. Associated to each side Γ_k of $\partial\Omega$, let $G_k = \{y \in \mathbb{R}^2;\ y_1' \leq 0;\ \text{and}\ 0 \leq y_2' \leq T_k\}$; $v_k \in H^1(G_k)$ is the weak solution of

$$\text{(A.4)} \qquad \begin{aligned} -\nabla_y \cdot a(y + \delta_\epsilon \eta^k)\nabla_y v_k = 0 \text{ in } G_k \\ v_k(y) = \chi^j(y + \delta_\epsilon \eta^k)\eta_j^k \quad \text{on}\{y \in G_k, y_1' = 0\} \\ v_k|_{y_1'=c} \text{ is } [0, T_k]\text{-periodic in } y_2' \text{ direction for } -\infty < c < 0 \\ \text{and } \frac{\partial v_k}{\partial y_i}\exp(-\gamma y_1') \in L^2(G_k), \quad i = 1, 2, \end{aligned}$$

where $\delta_\epsilon = T_k\left(s_k/(\epsilon T_k) - \lfloor s_k/(\epsilon T_k)\rfloor\right)$, and s_k is such that $\Gamma_k \subset \{x \in \mathbb{R}^2;\ x \cdot \eta^k = s_k\}$; ($\lfloor\cdot\rfloor$ denotes the integer part). By Theorem 1.3.10 there exists a unique solution to (A.4) and a constant χ_k^* such that $v_k - \chi_k^*$ decays exponentially to zero when $y_1' \to -\infty$, i.e.

$$(v_k - \chi_k^*)\exp(-\gamma y_1') \in L^2(G_k).$$

We note by Remark 2.1.1 that $(u_1(x, \frac{x}{\epsilon}) - \chi^* \partial_\eta u_0)|_{\Gamma_k} \in H_{00}^{1/2}(\Gamma_k)$. Thus we can split $\tilde{\theta}_\epsilon = \sum_{k=1}^N \tilde{\theta}_\epsilon^k$, where

$$L_\epsilon \tilde{\theta}_\epsilon^k = 0 \ \text{ in } \ \Omega, \qquad \tilde{\theta}_\epsilon^k = \begin{cases} -u_1(x, \frac{x}{\epsilon}) - \chi^* \partial_\eta u_0 & \text{on } \Gamma_k \\ 0 & \text{on } \partial\Omega \setminus \Gamma_k. \end{cases}$$

We approximate $\tilde{\theta}_\epsilon^k$ by $\tilde{\phi}_\epsilon^k$ given by

$$\text{(A.5)} \qquad \tilde{\phi}_\epsilon^k(x_1, x_2) = \varphi_k(x)\left(v_k\left(\frac{x - s_k \eta^k}{\epsilon}\right) - \chi_k^*\right)\nabla u_0 \cdot \eta^k,$$

where $\varphi_k(x)$ is a cut-off function such that $\varphi_k|_{\Gamma_k} = 1$, $\varphi_k|_{\partial\Omega\setminus\Gamma_k} = 0$, and $\varphi_k \nabla u_0 \cdot \eta^k \in W^{1,\infty}(\Omega)$ if $u_0 \in W^{2,\infty}(\Omega)$. For example, assume $\Gamma_k = \{x \in \mathbb{R}^2;\ x_1 = 0,\ 0 \leq x_2 \leq c\}$ and that x_1^+ is the inner normal direction. Let Γ_{k-1} and Γ_{k+1} be the edges with vertexes at the point $(0, a)$ and $(0, 0)$, respectively, and let $\alpha_k > 0$ and $\alpha_{k+1} < 0$ be the angles between the x_1 axis and Γ_{k-1} and Γ_{k+1}, respectively. Then we define

$$\text{(A.6)} \quad \varphi_k(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x_1 \leq \delta;\ 0 \leq x_2 \leq a \\ 1 - (x_2 - a)/(x_1 \tan\alpha_k) & \text{if} \quad 0 \leq x_1 \leq \delta;\ x_2 > a \\ 1 + x_2/(x_1 \tan\alpha_{k+1}) & \text{if} \quad 0 \leq x_1 \leq \delta;\ x_2 < 0 \\ \text{smooth} & \text{if} \quad \delta \leq x_1 \leq 2\delta \\ 0 & \text{if} \quad x_1 \geq 2\delta. \end{cases}$$

A simple calculation give

$$(A.7)|\nabla\varphi_k(x)| \le \begin{cases} c/\mathrm{dist}(x,(0,a)) & \text{if} \quad x \in \{z \in \Omega, \mathrm{dist}(x,(0,a)) \le \delta\} \\ c/\mathrm{dist}(x,(0,0)) & \text{if} \quad x \in \{z \in \Omega, \mathrm{dist}(x,(0,0)) \le \delta\} \\ c & \quad\text{otherwise} \end{cases}$$

where δ and c are fixed constants, and hence $\varphi_k \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Since $\partial_{\eta^k} u_0 \in H^{1/2}_{00}(\Gamma_k)$, if we assume $u_0 \in W^{2,\infty}(\Omega)$ we obtain

$$\|\varphi_k \nabla u_0 \cdot \eta^k\|_{1,\infty} \le c\|u_0\|_{2,\infty}. \tag{A.8}$$

Hence $\tilde{\phi}_\epsilon = \sum_{k=1}^N \tilde{\phi}_\epsilon^k$ approximates $\tilde{\theta}_\epsilon$, and $\tilde{\phi}_\epsilon = \tilde{\theta}_\epsilon$ on the boundary of Ω.

The boundary condition imposed in Equation (A.2) does not depend on ϵ. An effective approximation for $\bar{\theta}_\epsilon$ is given by $\bar{\phi} \in H^1(\Omega)$ the solution of

$$-\nabla \cdot A\nabla\bar{\phi} = 0 \;\text{ in }\; \Omega, \qquad \bar{\phi} = \chi^* \partial_\eta u_0 \;\text{ on }\; \partial\Omega.$$

We define our theoretical approximation for u_ϵ as $u_0 + \epsilon u_1 + \epsilon\phi_\epsilon$, where $\phi_\epsilon = \tilde{\phi}_\epsilon + \bar{\phi}$. Note that $\phi_\epsilon|_{\partial\Omega} = \theta_\epsilon|_{\partial\Omega}$, therefore $u_0 + \epsilon u_1 + \epsilon\phi_\epsilon = 0$ on $\partial\Omega$. We have the following error bounds

Theorem A.1.1 *Let u_ϵ be the solution of the Problem (1.2), u_0, u_1 and ϕ_ϵ defined by Equations (2.15), (2.10) and (2.25), respectively. Assume $a_{ij} \in L^\infty_{per}(Y)$ and $u_0 \in W^{2,\infty}(\Omega)$. Then there exists a constant c, such that*

$$\|u_\epsilon - u_0 - \epsilon u_1 - \epsilon\phi_\epsilon\|_1 \le c\epsilon\|u_0\|_{2,\infty}.$$

Proof: The proof of Theorem 2.1.1 extends immediately to the case of a general polygonal domain with rational normals to obtain

$$\|u_\epsilon - u_0 - \epsilon u_1 - \epsilon\phi_\epsilon\|_1 \le c\epsilon \max_k \|\varphi_k \nabla u_0 \eta_k\|_{1,\infty},$$

and the proposition follows from (A.8) □

A.2 Finite Element Approximation

We now describe how to approximate the terms u_0, u_1, $\tilde{\phi}_\epsilon$ and $\bar{\phi}$ numerically.

- Let $\chi^j_{\hat{h}}$ be a numerical approximation of χ^j using a second order accurate conforming finite element on a mesh $\mathcal{T}_{\hat{h}}(Y)$.

- Define $A^{\hat{h}}_{ij} = \frac{1}{|Y|}\int_Y a_{lm}(y)\frac{\partial}{\partial y_l}(y_i - \chi^i_{\hat{h}})\frac{\partial}{\partial y_m}(y_j - \chi^j_{\hat{h}})dy$.

- Let $V^h(\Omega)$ be the space of $\mathcal{P}_1$ finite elements associated to a triangular mesh $\mathcal{T}_h(\Omega)$, and $V^h_0(\Omega) = V^h(\Omega) \cap H^1_0(\Omega)$. Define $u_0^{h,\hat{h}} \in V_0^h(\Omega)$ as the solution of

$$\int_\Omega (A^{\hat{h}}\nabla u_0^{h,\hat{h}}, \nabla v^h)dx = \int_\Omega f v^h dx, \qquad \forall v^h \in V_0^h(\Omega).$$

- Let $Y_k^h = \{\lambda^h \in L^2(\Gamma_k);\ \lambda^h = \phi^h|_{\Gamma_k},\ \phi^h \in V^h(\Omega)\}$, and $Y_{0,k}^h = \{\lambda^h \in Y_k^h;\ \lambda^h = 0 \text{ at } \partial\Gamma_k\}$. Define $\lambda_k^h \in Y_{0,k}^h$, as the solution of

$$\int_{\Gamma_k} \lambda_k^h \phi^h d\sigma = \int_\Omega A^{\hat{h}}_{ij}\partial_i u_0^{h,\hat{h}}\partial_j\phi^h dx - \int_\Omega f\phi^h dx, \tag{A.9}$$

$\forall \phi^h \in V^h(\Omega);\ \phi^h|_{\partial\Omega\setminus\Gamma_k} = 0$. Observe that λ_k^h approximates $A\nabla u_0 \cdot \eta^k$ and that $u_0 \in H^1_0(\Omega) \cap H^2(\Omega)$ implies $\nabla u_0 \cdot \tau^k|_{\Gamma_k} = 0$. Hence define $\nu^{h,\hat{h}}$ by

$$\begin{aligned} A^{\hat{h}}\nu^{h,\hat{h}} \cdot \eta^k &= \lambda_k^h, \\ \nu^{h,\hat{h}} \cdot \tau^k &= 0, \end{aligned}$$

and then approximate $\partial_{\eta^k} u_0$ by $\mu^{h,\hat{h}} = \nu^{h,\hat{h}} \cdot \eta^k$.

- Define the approximation for ∇u_0 as

$$\Psi^{h,\hat{h}} = \nabla u_0^{h,\hat{h}} + \sum_{k=1}^{N} E_k(\mu^{h,\hat{h}} - \nabla u_0^{h,\hat{h}} \cdot \eta^k)\eta^k.$$

Here $E_k(\cdot)$ denotes a discrete extension by zero on Ω. More specifically, $E_k(\mu^{h,\hat{h}} - \nabla u_0^{h,\hat{h}} \cdot \eta^k)(z) = 0$, where z is a vertex of $\mathcal{T}_h(\Omega) \setminus \Gamma_k$, and $E_k(\mu^{h,\hat{h}} - \nabla u_0^{h,\hat{h}} \cdot \eta^k)|_{K_i} \in V^h(\Omega)|_{K_i},\ \forall\ K_i \in \mathcal{T}_h(\Omega)$.

- Define $u_1^{h,\hat{h}}(x, x/\epsilon) = -\Psi_j^{h,\hat{h}}(x)\chi^j_{\hat{h}}(x/\epsilon)$. Note that this leads to a non-conforming approximation for u_1 on the partition $\mathcal{T}_h(\Omega)$.

- Let τ be a positive integer and $G_k^\tau = \{y \in \mathbb{R}^2; y_1' \leq 0,\ |y_1'| < \tau;\ \text{and } 0 < y_2' < T_k\}$. Define $\tilde{v}_k \in H^1(G_k^\tau)$ as the weak solution of

$$
\begin{aligned}
&-\nabla_y \cdot a(y+\delta_\epsilon \eta^k)\nabla_y \tilde{v}_k = 0 \quad \text{in} \quad G_k^\tau \\
&\tilde{v}_k(y) = \chi_{\hat{h}}^j(y+\delta_\epsilon \eta^k)\eta_j^k, \ \text{ on } \{y \in G_k, y_1' = 0\} \\
&\partial_\eta \tilde{v}_k = 0, \quad \text{on } \{y \in G_k^\tau;\ |y_1'| = \tau\} \\
&\text{and } v_k|_{y_2'=0} = v_k|_{y_2'=T_k}, \quad \text{for} \quad |y_1'| < \tau.
\end{aligned}
$$

  Let $v_k^{\hat{h},\tau}$ be a numerical approximation of $\tilde{v}_k$ using a second order accurate conforming finite element on a mesh $\mathcal{T}_{\hat{h}}(G_e^\tau)$.

- Define

$$
\chi_k^{*,\hat{h},\tau} \quad = \quad \frac{1}{T_k}\int_0^{T_k} v_k|_{\{|y_1'|=\tau\}\cap \partial G_k^\tau} dy_2'
$$

- Observe that the term $v_k\big((x - s_k\eta^k)/\epsilon\big)$ appears in Equation (A.5). Since the approximation $v_k^{\hat{h},\tau}$ is defined on G_k^τ, it is possible to calculate $v_k^{\hat{h},\tau}\big((x - s_k\eta^k)/\epsilon\big)$ only if $|x_1' - s_k| \leq \epsilon\tau$. The functions $v_k - \chi_k^*$ decays exponentially to zero in the $-\eta^k$ direction, hence its is natural to consider the following approximation

$$
\tilde{\phi}_\epsilon^{k,h,\hat{h},\tau}(x_1,x_2) =
\begin{cases}
\left(v_k^{\hat{h},\tau}\left(\frac{x-s_k\eta^k}{\epsilon}\right) - \chi_k^{*,\hat{h},\tau}\right)\varphi_k \Psi^{h,\hat{h}} \cdot \eta^k & \text{if} \ \ |x_1' - s_k| < \epsilon\tau \\
0 & \text{if} \ \ |x_1' - s_k| \geq \epsilon\tau.
\end{cases}
$$

  Let $\tilde{\phi}_\epsilon^{h,\hat{h},\tau} = \sum_{k=1}^N \tilde{\phi}_\epsilon^{k,h,\hat{h},\tau}$.

- Let $\bar{\phi}^{h,\hat{h},\tau}$ be a second order accurate finite element approximation on a mesh of size h for the following equation

$$
\text{(A.10)} \qquad -\nabla A^{\hat{h}} \nabla \varrho = 0, \qquad \varrho = \chi^{*,\hat{h},\tau}\mu^{h,\hat{h}} \ \text{ on } \ \partial\Omega.
$$

- Approximate θ_ϵ by $\phi_\epsilon^{h,\hat{h},\tau} = \tilde{\phi}_\epsilon^{h,\hat{h},\tau} + \bar{\phi}^{h,\hat{h},\tau}$ and finally define the numerical solution for Equation (1.2) by

$$
\text{(A.11)} \qquad u_\epsilon^{h,\hat{h},\tau} = u_0^{h,\hat{h}} + \epsilon u_1^{h,\hat{h}} + \epsilon \phi_\epsilon^{h,\hat{h},\tau}.
$$

A.3 Error Analysis

For the discrete error analysis we assume $\hat{h} = 0$ and $\tau = \infty$, i.e. $v_k^{\hat{h},\tau} = v_k$, $\chi_{\hat{h}}^j = \chi^j$ and $A^{\hat{h}} = A$.

Theorem A.3.1 *Let u_ϵ be the solution of the Problem (1.2), u_0 and u^h be defined by Equations (2.15) and (A.11), respectively. Assume $a_{ij} \in L^\infty_{per}(Y)$ and $u_0 \in W^{2,\infty}(\Omega)$. Then there exists a constant c, such that*

$$\|u_\epsilon - u_h\|_{1,h} \leq c(h+\epsilon)\|u_0\|_{2,\infty}$$

and

$$\|u_\epsilon - u_h\|_0 \leq c(h^2 + \epsilon + \epsilon h)\|u_0\|_{2,\infty}.$$

Proof: Similar to the proof of Theorem 5.1.1, replacing Theorem 2.1.1 by Theorem A.1.1 and Proposition 5.1.1 by estimates (A.14) and (A.15).

Given Γ_k, define $\{z_k^1, z_k^2\} = \partial\Gamma_k$, and let $\Omega_{k,h}^i = \{K \in \mathcal{T}^h(\Omega),\ \overline{K} \cap z_k^i \neq \emptyset\}$ and $\Omega_{k,h}^c = \Omega \setminus \cup_i \Omega_{k,h}^i$ then

$$\begin{aligned} \|\varphi_k(\nabla u_0 - \Psi^h)\cdot\eta_k\|_{1,\infty,\Omega_{k,h}^i} &\leq \|\varphi_k\|_{0,\infty}\|(\nabla u_0^h - \Psi^h)\cdot\eta_k\|_{1,\infty,\Omega_{k,h}^i} \\ &\quad + \|(\nabla u_0 - \Psi^h)\cdot\eta_k\partial_{x_j}\varphi_k\|_{0,\infty,\Omega_{k,h}^i} \\ &\leq c\|u_0\|_{2,\infty} + \|(\nabla u_0 - \Psi^h)\cdot\eta_k\partial_{x_j}\varphi_k\|_{0,\infty,\Omega_{k,h}^i} \qquad \text{(A.12)} \end{aligned}$$

But from (A.7) we have that $|\partial_{x_j}\varphi_k(x)| \leq c/\text{dist}(x, z_k^i)$, $(\nabla u_0 - \Psi^h)\cdot\eta_k(z_k^i) = 0$ and $\|(\nabla u_0 - \Psi^h)\cdot\eta_k\|_{0,\infty,\Omega_{k,h}^i} \leq c\|u_0\|_{2,\infty}\text{dist}(x, z_k^i)$, hence

$$\|(\nabla u_0 - \Psi^h)\cdot\eta_k\partial_{x_j}\varphi_k\|_{0,\infty,\Omega_{k,h}^i} \leq c\|u_0\|_{2,\infty},$$

and therefore

$$\|\varphi_k(\nabla u_0 - \Psi^h)\cdot\eta_k\|_{1,\infty,\Omega_{k,h}^i} \leq c\|u_0\|_{2,\infty}.$$

We now estimate the second term on the right hand side of (A.12),

$$\begin{aligned} \|\varphi_k(\nabla u_0^h - \Psi^h)\cdot\eta_k\|_{1,p,\Omega_{k,h}^c} &\leq c\|\varphi_k\|_{0,\infty,\Omega_{k,h}^c}\|u_0\|_{2,\infty} \\ &\quad + \|\varphi_k\|_{1,\infty,\Omega_{k,h}^c}\|(\nabla u_0 - \Psi^h)\cdot\eta_k\|_{0,\infty,\Omega_{k,h}^c} \\ &\leq c\|u_0\|_{2,\infty} \qquad \text{(A.13)} \end{aligned}$$

Here we used (A.7) and Proposition 5.1.1 to arrive in A.13. Finally we obtain

$$\|\varphi_k(\nabla u_0 - \Psi^h) \cdot \eta_k\|_{1,p,\Omega} \leq c\|u_0\|_{2,\infty}. \tag{A.14}$$

From the proof of estimate (5.37) we obtain immediately

$$\|\varphi_k(\nabla u_0 - \Psi^h) \cdot \eta_k\|_{0,p,\Omega} \leq ch\|u_0\|_{2,\infty}. \tag{A.15}$$

□

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