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On the Choices Under Ambiguity

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ON THE CHOICES UNDER AMBIGUITY

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Abstract

This thesis proposes models of choices under ambiguity and presents some studies about the connections between ambiguity and incomplete markets

In ambiguity theory, this thesis presents two new axiomatizations: the first takes the notion of *confidence function*, which generalize the multiple priors model when there exists a worst consequence; in the second approuch, the existence of referential consequence is supposed and the sign-dependents confidence functions generalizes cumulative prospect theory with ambiguity aversion on gains and losses.

The last part presents some preliminaries results about cost function and incompleteness of financial markets. Some cases of incompletness entails interesting formulas for the cost functions using Choquet integral.

Contents

Part I

Choices under Ambiguity.

Chapter 1

Choice Theory and Ambiguity: a general review.

1.1 Introduction

The human decision-making process may well be one of the most complicated systematic phenomena studied by sciences. Decision Theory is a branch of human sciences concerning to it where the axiomatic method is the main methodology and Functional Analysis, Measure Theory and Convex Analysis are very useful. In Economic Theory this branch presents a very important influence in areas such as general equilibrium and game theory.

There are two main aspects in a model of individual choice: the decision maker and the avaliable *object* of choice. For example, in the pure general equilibrium theory proposed by Arrow and Debreu (1954) we have as decision makers I consumers, and the avaliable objects of choice are commodity bundles $x_i = (x_{1i}, ..., x_{ni}) \in \mathbb{R}^n_+$ for each consumer i. Other important example is in game theory: in a game as proposed by Nash (1951) the decision makers are players and the objects of choices are sets of individual strategies.

A decision maker is characterized by a binary relation on the choice set or, in a more restritive setting, by a real valued object function on the choice set. For example, the consumers in the Arrow and Debreu's model are summarized by binary relations or utility functions.

Formally, let X be a nonempty set, called *choice set*, a decision maker is characterized by a binary relation $\succsim$ on X, which we call a *preference relation*. Given two elements x, y belonging to X, the expression $x \succsim y$ (or, $(x, y) \in \succsim$) means that x is at least as good as y. From $\succsim \subseteq X \times X$, we can derive two other relations on X: the strict preference relation or asymetric component $\succ$ is defined by

$$\succ := \{(x, y) \in \succsim : (y, x) \notin \succsim\},$$

and the indifference relation or symetric component $\sim$ is defined by

$$\sim := \{(x, y) \in \succsim : (y, x) \in \succsim\}.$$

1.2 General Representation

For a preference relation $\succsim$ on X, a crucial property is

Definition 1 *A function $u : X \to \mathbb{R}$ is a utility function representing preference relation $\succsim \subseteq X \times X$ if, for any $x, y \in X$,*

$$x \succsim y \Leftrightarrow u(x) \geq u(y).$$

The representability of preferences by a utility function is closely linked to the assumption of the following two properties below for $\succsim$:

Definition 2 *A preference relation $\succsim \subseteq X \times X$ is complete if for any $x, y \in X$ we have $x \succsim y$ or $y \succsim x$.*

The completeness of preference relation says that the decision maker has a well-defined preference between every pair of possible alternatives.

Definition 3 *A preference relation* $\succsim \subseteq X \times X$ *is transitive when for any* $x, y, z \in X$*, if* $x \succsim y$ *and* $y \succsim z$*, then* $x \succsim z$*.*

An immediate result say that $\succsim \subseteq X \times X$ can be represented by a utility function only if it is complete and transitive. The converse is true when X is finite or countable infinite, but it is false in general; *e.g.*, take the well known example where $X = \mathbb{R}^2_+$ and $\succsim_l$ is the lexicographic preference: $x \succsim_l y \Leftrightarrow x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. Another trivial result says that any arbitrary increasing transformation of a given utility function woud result on other utility function for the same preference.

Now, we consider the choice set X as a topological space and always assume that $\succsim \subseteq X \times X$ is complete and transitive. The first proof of existence of a continuous utility function representing a preference relation was given by Wold[1] in the case where $X = \mathbb{R}^n_+$ and under the restritive class of strictly monotone preferences. For the most classical result on the existence of a continuous representation we need the following property for a preference relation:

Definition 4 *A preference relation* $\succsim$ *on the topological space* X *is continuous when for any* $x \in X$ *the sets* $L_x = \{y \in X : x \succsim y\}$ *and* $U_x = \{y \in X : y \succsim x\}$ *are closed.*

It is well known that if X is a connected and separable topological space, then every continuous preference relation given on X admits a continous utility representation, this is a result due to Eilenberg (1941). Moreover, by Debreu's representation theorem, the assumption of connectedness can be replaced by another topological condition: the topological space X has a countable basis[2]. Therefore, if X is a Banach space, Eilenberg's theorem implies that the norm separability is a sufficient condition for a norm continuous preference to have a norm continuous representation, and by a result of Estévez and Hervés (1995) norm separability is also a necessary condition.

[1] See, for instance, Bearton and Mehta (1994).

[2] We note that in the context of connected metric spaces, the Eilenberg and Debreu theorems are equivalent. This is because a metric space X is separable iff X has a countable basis.

However, it is well known that in many important economic applications, the choice set to deal with are typically nonseparable[3]. Monteiro (1987) shows that, for a path connected topogical space X, a continuous preference $\succsim$ has a continuous utility representation iff it is countably bounded, *i.e.*, there is some countable subset $\widehat{X}$ of X such that for all $x \in X$ there exist $y, z \in \widehat{X}$ such that $y \succsim x \succsim z$. For a Banach space X, Campión, Candeal and Induráin (2005) proved that every weakly continuous preference on X can be represented by a weakly continuous utility function.

1.3 Risk, Uncertainty and Ambiguity.

Decision theory with risk or uncertainty describes a class of models designed to formalize the manner in which a decision maker chooses among alternative courses of action that implies in consequences that are not known at the time the choice is made.

1.3.1 Choice under Risk

One of the most well known models in the modern economic theory is the von Neumann and Morgenstern expected utility theory with risk. The essence of the von Neumann-Morgenstern theory is a set of restrictions imposed on the preference relations over lotteries that allows their representation by the mathematical expectation of a real function on the set of outcomes. A main aspect of the model is the specific functional form of the representation, namely, the linearity in the probabilities. This functional is known as the von Neumann-Morgenstern utility function.

Formally, the first primitive of the theory is a nonempty convex subset X of a linear space. An special case of particular importance is when X is a set of distributions with

[3] See, for instance, section 4 of Mas-Colell and Zame (1991).

finite supports over a arbitrary set W of prizes or outcomes,

$$X = \left\{ x : W \to [0,1] / \, x(w) \neq 0 \text{ for finitely many } w's \text{ in } W \text{ and } \sum_{w \in W} x(w) = 1 \right\}.$$

The elements of X are random outcomes or (roulette) *lotteries*. Denote by δ_w the element of X that assigns the unit probability to $w \in W$. Let $x, y \in X$, since X is a convex subset of the linear space of measures on W, the mixed lottery $\alpha x + (1-\alpha)y$, is a lottery in X yielding the outcome $w \in W$ with probability $\alpha x(w) + (1-\alpha)y(w)$. It is customary to interpret it as a compound lottery in order to assign the behavioral meaning of the mixture operation.

The other primitive of theory is a binary (preference) relation over X to be denoted by $\succsim$. Now, we present the axioms:

Axiom (vN-M1): $\succsim$ is a weak order, *i.e.*, $\succsim$ is complete and transitive.

Axiom (vN-M2): For all $x, y, z \in X$, if $x \succ y$ and $y \succ z$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha x + (1-\alpha)z \succ y$ and $y \succ \beta x + (1-\beta)z$.

This axiom is known as the *Archmedean Axiom* and it is equivalent to the *Mixture Condition*:

$$\begin{aligned} \text{For any } x, y, z &\in X, \text{ the sets } \{\alpha \in [0,1] : \alpha x + (1-\alpha)y \succsim z\} \\ \text{and } \{\alpha &\in [0,1] : z \succsim \alpha x + (1-\alpha)y\} \text{ are closed.} \end{aligned}$$

Axiom (vN-M3): For any $x, y, z \in X$, and $\alpha \in [0,1]$

$$x \sim y \Rightarrow \alpha x + (1-\alpha)z \sim \alpha y + (1-\alpha)z.$$

This last axiom is known as *Independence Axiom*. This axiom says that the preference between the compound lotteries $\alpha x + (1-\alpha)z$ and $\alpha y + (1-\alpha)z$ is determined by the preference between x and y, for any lottery z and weight α.

A real valued function u on X is affine if $u(\alpha x + (1-\alpha)y) = \alpha u(x) + (1-\alpha)u(y)$

for any $\alpha \in (0,1)$. Hence, if X is the set of lotteries over W, the affinity of u implies $u(x) = \sum\limits_{w \in W} x(w)u(\delta_w)$.

The most common version of von Neumann-Morgenstern expected utility theorem is given by:

Theorem 5 *Let X be a convex subset of some linear space, with a binary relation $\succsim$ on it. A necessary and sufficient condition for the relation $\succsim$ to satisfy axiom (vN-M1), axiom (vN-M2) and axiom (vN-M3) is the existence of an affine real valued function u on X such that for all x and y in X: $x \succsim y$ iff $u(x) \geq u(y)$. Furthermore, an affine real valued function v on X can replace u in the above statement iff there exist a positive number α and a real number β such that $v(x) = \alpha u(x) + \beta$ on X (*i.e.*, u is unique up to positive affine transformation).*

For the proof of this theorem the reader is referred to Fishburn (1970).

The function $u : X \to \mathbb{R}$ is usually referred to as the von Neumann-Morgenster utility. The original theorem differs from this version in several aspects, in particular, instead of the operation of convex combination in linear space, von Neumann and Morgenstern introduce an abstract mixture operation that satisfies almost all the conditions of mixture sets as presented in Herstein and Milnor (1953). A minor variant of the previous theorem has been proved by Herstein and Milnor (1953) for the more general framework in which the set of consequences is a mixture set. In their version, the independence axiom is replaced by the weaker condition: if $x, y \in X$ and $x \sim y$, then for any $z \in X$, $\frac{1}{2}x + \frac{1}{2}z \sim \frac{1}{2}y + \frac{1}{2}z$, and the Archmedean axiom is replaced by the mixture condition as mentioned earlier.

Experimental studies of decision making and risk reveal systematic violations of independence axiom, *e.g.*, Allais (1953) and Kahneman and Tversky (1979). Prompted by some experimental results, several alternatives theories were proposed by weakening the independence axiom. For a nice survey of these theories see section 3 of Karni and Schmeidler (1991).

1.3.2 Choice under Uncertainty and Ambiguity

Most economic problems involve decision making under uncertainty rather than risk. The usual argument is that risk implies the existence of *objective probabilities*, as is implicit in von Neumann-Morgenstern theory. Savage's theory (1954) of decision making under uncertainty takes the notions of consequences, states of nature and acts as primitives. Acts are functions assigning consequence to states. Savage (1954, Chapters1-5) proposed a set of axioms for a preference relation on acts that allows the representation of it as the mathematical expectation of a real function on the set of outcomes with respect to a unique probability measure on the set of states. As in von Neumann-Morgenstern model, an essential aspect of Savage's theory is the linearity of the preference functional. However, unlike the von Neumann-Morgenstern model, in Savage's theory the existence of probabilies is established jointly with that of the utility function. In this fact, Savage's theory differs from the usual statistical models where the existence of a family of probability laws is postulated. Hence, Savage's work resolved the conceptual problem of the existence of *purely* subjective probability and it is well known as Subjective Expected Utility (SEU) theory.

Anscombe and Aumann (1963) suggested a model of preference relation over acts which allows the derivation of a unique subjective probability over a finite set of states of nature[4]. To do this they extended the set of acts by enlarging the set of consequences to include all lotteries over the set of outcomes, *i.e.*, they assumed that the set of consequences is the choice set X as in the von Neumann-Morgenstern theory. Hence an act is a mapping from the set of states of nature S to a convex subset X of a linear space. This setting is much more ameable to mathematical treatment than Savage's. This is specially aparent in Fishburn's (1970) reformulation and extention of Anscombe-Aumann analysis for an arbitrary state space.

[4] In Savage's approach there is an axiom, namely non-atomicity axiom, which implies that there are infinitely many states of nature. For an approach of purely subjective probability with a finite set of states of nature see Gul (1992).

The Bayesian paradigm has as the main tenet the assumption that whenever a fact is not kown, one should have probabilistic beliefs about it. Hence, SEU theory from Savage or Anscombe and Aumann provided a behaviorial foundation for this feature of Bayesianism. But, it was shown by Ellsberg (1961) to be an inaccurate description of people's behavior (see Chapter 2 for details). Ever since the seminal thought experiment of Ellsberg, it has been acknowledged that the awareness of missing information, *ambiguity* in Ellsberg's terminology, affects the subject's willingness to bet. Moreover, Ellsberg emphasized the relevance of *aversion* to ambiguity. Below, we present the basic framework that entails the Anscombe-Aumann result, as well the most popular models of ambiguity aversion: Choquet Expected Utility of Schmeidler (1989) and Maxmin Expected Utility of Gilboa and Schmeidler (1989).

Mathematical preliminaries

Consider a set S of *states of nature (world)*, endowed with a σ-algebra Σ of subsets called *events*, and a non-empty set X of *consequences*. We denote by $\mathcal{F}$ the set of all the (simple) *acts*: finite-valued functions $f : S \to X$ which are Σ-measurable[5]. Moreover, we denote by $B_0(S, \Sigma)$ the set of all real-valued Σ-measurable simple functions $a : S \to \mathbb{R}$. The norm in $B_0(S, \Sigma)$ is given by $\|a\|_\infty = \sup_{s \in S} |a(s)|$ (called *sup norm*) and we can define the space of all bounded and Σ-measurable functions by $B(S, \Sigma) := \overline{B_0(S, \Sigma)}^{\|\cdot\|_\infty}$, i.e., $B(S, \Sigma)$ consists of all uniform limits of finite linear combinations of characteristic functions of sets in Σ (see Dunfort and Schwartz 1988, page 240).

Clearly, we note that $u(f) \in B_0(S, \Sigma)$ whenever $u : X \to \mathbb{R}$ and f belongs to $\mathcal{F}$, where the function $u(f) : S \to \mathbb{R}$ is the mapping defined by $u(f)(s) = u(f(s))$, for all $s \in S$.

Let x belong to X, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. Hence, we can identify X with the set $\mathcal{F}_c$ of the constant acts in $\mathcal{F}$. Given $f, g \in \mathcal{F}$ and $E \in \Sigma$, we denote by $fEg \in \mathcal{F}$ the act that yields the consequence $f(s)$ if $s \in E$

[5] Let $\succsim_0$ be a binary relation on X, we say that a function $f : S \to X$ is Σ-measurable if, for all $x \in X$, the sets $\{s \in S : f(s) \succsim_0 x\}$ and $\{s \in S : f(s) \succ_0 x\}$ belong to Σ.

and the consequence $g(s)$, otherwise.

Additionally, we assume that X is a convex subset of a vector space. For instance, this is the case if X is the set of all finite-support lotteries on a set of prizes Z, as it happens in the classic setting of Anscombe and Aumann (1963).

Using the linear structure of X we can define as usual for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$ the act:

$$\begin{aligned} \alpha f + (1-\alpha)g &: S \to X \\ (\alpha f + (1-\alpha)g)(s) &= \alpha f(s) + (1-\alpha)g(s) \end{aligned}$$

The decision maker's preferences are given by a binary relation $\succsim$ on $\mathcal{F}$, whose symmetric and asymmetric components are denoted by $\sim$ and $\succ$:

$$\begin{aligned} \sim &:= \{(x, y) \in \succsim \wedge (y, x) \in \succsim\} \\ \succ &:= \{(x, y) \in \succsim \wedge (y, x) \notin \succsim\} \end{aligned}$$

If $f \in \mathcal{F}$, an element $c_f \in X$ is a *certainty equivalent* of f if $c_f \in \{x \in X : x \sim f\}$.

By $ba(S, \Sigma)$ will be undertood the family of all bounded finitely additive set functions with domain Σ and range $\mathbb{R}$. The set $ba(S, \Sigma)$ endowed with the norm $\|\lambda\| = \sup_{E \in \Sigma} |\lambda(E)|$ is a Banach space, where by definition, given $E \subset S$:

$$v(\lambda, E) = sup\{\sum |\lambda(E_i)| : \{E_i\} \text{ is any finite sequence of disjoint sets in } \Sigma \text{ with } E_i \subset E\}.$$

From the well known inequality $v(\lambda, S) \leq 2 \sup_{E \in \Sigma} |\lambda(E)|$, the total variation $v(\lambda, S)$ is a norm in $ba(S, \Sigma)$ which is equivalent to the norm $\|\lambda\|$. Moreover, $(ba(S, \Sigma), v(\cdot, S))$ is isometrically isomorphic to the norm dual of the Banach space $(B(S, \Sigma), \|\cdot\|_\infty)$ (Dunford and Schwartz(1988), page 258), where the duality being[6]

$$\langle \lambda, a \rangle = \int_S a(s)\lambda(ds),$$

for any $a \in B(S,\Sigma)$ and $\lambda \in ba(S,\Sigma)$. Hence, the weak* topology $\sigma(ba, B)$ on $ba(S,\Sigma)$ is the weakest topology for which all functionals $ba(S,\Sigma) \ni \lambda \mapsto \langle \lambda, a \rangle$ are continuous, where $a \in B(S,\Sigma)$.

Given a functional $I : B(S,\Sigma) \to \mathbb{R}$, we say that I is: *monotonic* if $I(a) \geq I(b)$ for all $a, b \in B(S,\Sigma)$ such that $a(s) \geq b(s)$ for all $s \in S$; *constant additive* if $I(a + k\mathbf{1}_S) = I(a) + k$ for all $a \in B(S,\Sigma)$ and $k \in \mathbb{R}$; *positively homogeneous* if $I(\lambda a) = \lambda I(a)$ for all $a \in B(S,\Sigma)$ and $\lambda \geq 0$; *constant linear* if it is constant additive and positively homogeneous.

Representation theorem

The decision makers prefences are given by a binary relation $\succsim$ on $\mathcal{F}$, whose symmetric and asymmetric components are denoted by $\sim$ and $\succ$.

We introduce the basic preference model that entails the Anscombe-Aumann theorem as a particular case, and some popular models of ambiguity-sensitive preferences of Schmeidler (1989) and Gilboa and Schmeidler (1989).

The model is characterized by the following four axioms:

Axiom 1 (weak order nondegenered). For all $f, g, h \in \mathcal{F}$: (1) either $f \succsim g$ or $g \succsim f$, (2) if $f \succsim g$ and $g \succsim h$, then $f \succsim h$, (3) there are $f, g \in \mathcal{F}$ such that $f \succ g$.

[6] The integral is in the sense of Dunford and Schwartz (1988) (see page 112). But, in the importante case where λ is such that $\lambda \geq 0$ and $\lambda(S) = 1$, an equilivant definition is given by the using of Lebesgue integration:

$$\int_S a(s)\lambda(ds) = \int_{-\infty}^{0} [\lambda(\{s : a(s) \geq \alpha\}) - 1]\, d\alpha + \int_0^{+\infty} \lambda(\{s : a(s) \geq \alpha\}) d\alpha.$$

As is usual, we adopt the notion $\int_S a(s)\lambda(ds) = \int a d\lambda$.

Axiom 2 (Certainty independence). If $f, g \in \mathcal{F}$, $x \in X$, and $\lambda \in (0,1]$ then

$$f \sim g \Leftrightarrow \lambda f + (1-\lambda)x \sim \lambda g + (1-\lambda)x.$$

Axiom 3 (Mixture continuity). For all $f, g, h \in \mathcal{F}$ the sets:

$$\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\},\ \{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\} \text{ are closed.}$$

Axiom 4 (Monotonicity). For all $f, g \in \mathcal{F}$:

$$\text{if } f(s) \succsim g(s) \text{ for all } s \in S \text{ then } f \succsim g.$$

The following representation result is easily proved by mimicking the arguments of Gilboa and Schmeidler (1989, Lemmas 3.1-3.3).

Lemma 6 *A binary relation $\succsim$ on $\mathcal{F}$ satisfies axioms 1-4 if and only if there exists a monotonic, constant linear functional $I : B_0(S, \Sigma) \to \mathbb{R}$ and a nonconstant affine function $u : X \to \mathbb{R}$ such that*

$$f \succsim g \Leftrightarrow I(u(f)) \geq I(u(g)).$$

Moreover, I is unique and u unique up to positive affine transformation.

Ghirardato and Marinacci (2001) called a preference $\succsim$ satisfying axioms 1-4 an *invariant biseparable preference*. *Invariant* refers to the mentioned invariance of I to utility normalization. *Biseparable* means that the representation on binary acts of such preferences satisfies the following separability condition: Let $\mu : \Sigma \to [0,1]$ be defined by $\mu(E) := I(\mathbf{1}_E)$. Then, μ is a normalized and monotone set-function (a *capacity*) and for all $x, y \in X$ such that $x \succsim y$ and $E \in \Sigma$

$$I(u(xEy)) = u(x)\mu(E) + u(y)(1-\mu(E)).$$

Some of the best-known models of decision making in the presence of ambiguity employ invariant biseparable preferences. However, these models incorporate some additional assumptions on how the decision maker reacts to ambiguity, *i.e.*, whether he exploits hedging opportunits or not. These assumptions are summarized in the following axiom:

Axiom 5. For all $f, g \in \mathcal{F}$ such that $f \sim g$:

(a) (Ambiguity neutrality) $\frac{1}{2}f + \frac{1}{2}g \sim f$.

(b) (Comonotonic ambiguity neutrality) $\frac{1}{2}f + \frac{1}{2}g \sim f$ if f and g are comonotonic[7].

(c) (Uncertainty Aversion) $\frac{1}{2}f + \frac{1}{2}g \succsim f$.

Axiom 5(c) is due to Schmeidler (1989), and it says that the decision maker will in general prefer the mixture, possibly a hedge, to its components[8]. The others are simple variations on that property.

Proposition 7 *Let $\succsim$ be a preference satisfying axioms 1-4. Then*

*(i) (**Anscombe-Aumann representation**): $\succsim$ satisfies axiom 6(a) iff there exists a probability $p \in ba_{+}^{1}(S,\Sigma) = \{p \in ba(S,\Sigma) : p \geq 0$ and $p(S) = 1\}$[9] and $I(f) = \int u(f)dp$ for any $f \in \mathcal{F}$. In this case $p = \mu$.*

*(ii) (**Schmeildler representation**) $\succsim$ satisfies axiom 6(b) iff there exists a capacity $\mu : \Sigma \to [0,1]$ and $I(f) = \int u(f)d\mu$ for any $f \in \mathcal{F}$, where the integral is taken in the sense of Choquet (for the definition of Choquet integral see Chapter 2, Section 2.5.2).*

*(iii) (**Gilboa-Schmeidler representation**) $\succsim$ satisfies axiom 6(c) iff there exists a nonempty, weakly* compact and convex set $C \subset ba_{+}^{1}(S,\Sigma)$ such that $I(f) = min_{p \in C}\left(\int u(f)dp\right)$ for all $f \in \mathcal{F}$. Moreover, C is unique and $\mu(E) = min_{p \in C}p(E)$ for all $E \in \Sigma$.*

Thus, a decision maker who satisfies axioms 1-4 and is indifferent to hedging opportunities satisfies the SEU model. We note that we can replace axioms 2 and 6(a) by the classical independence axiom of Anscombe-Aumann: if $f, g, h \in \mathcal{F}$ and $\beta \in (0,1]$ then

$$f \sim g \Rightarrow \beta f + (1-\beta)h \sim \beta g + (1-\beta)h.$$

[7] A pair of acts $f, g \in \mathcal{F}$ is comonotonic when there are no states $s, s' \in S$ such that $f(s) \succ f(s')$ and $g(s') \succ g(s)$.

[8] Another termilogy for this property is *ambiguity hedging.*

[9] *We note that if $p \in ba_{+}^{1}(S,\Sigma)$ then $\sup\limits_{E\in\Sigma} |p(E)| = v(p,S) = 1$.*

A decision maker who is indifferent to hedging opportunities when they involve comonotonic acts, but may care otherwise, satisfies the CEU model of Schmeidler (1989), with beliefs given by the capacity μ. A decision maker who uniformly likes ambiguity hedging oppportunities chooses according to a maxmin decision rule. Indeed, axioms 1-4 and 6(c) are the axioms proposed by Gilboa and Schmeidler (1989) to characterizes maxmin expected utility preferences.

Chapter 2

Ambiguity through Confidence Functions

2.1 Introduction

The presence of vagueness in probability judgements is an important issue in decision making, as Frank Knight (1921, page 227) commented: *The action which follows upon an opinion depends as much upon the amount of confidence in that opinion as it does upon the favorableness of the opinion itself.* Here, we may understand an *opinion* as some probability judgement and following Knight's argument we think that a decision maker may have different degrees of confidence in his own probability assignments, and that this is a crucial factor in the decision making process.

In order to make the preceding discussion more concrete we consider the Ellsberg's seminal article (Ellsberg, 1961) that presented the following *mind experiments*: there are two urns A and B, each containing one hundred balls. Each ball is either red or black. In urn A there are fifty balls of each color and there is no additional information about urn B. One ball is chosen at random from each urn. There are four states of nature, denoted by $S = \{(r,r),(r,b),(b,r),(b,b)\}$ where (r,r) denotes the event that the ball chosen from urn A is red and the ball chosen from urn B is red, etc. We can construct four bets denoted by A^r, A^b, B^r, B^b, where the bet A^r yields \$100 if the state (r, r) or (r, b) occurs and *zero* if it does not, *i.e.*, A^r is a bet on a red ball in urn A. According to Ellsberg most decision makers are indifferent between betting on a red ball in urn A and betting on a black ball in urn A and are similary indifferent between bets on a red ball in urn B or a black ball in urn B. However, there is a nonnegligible proportion of decision makers who prefer every bet from urn A (red or black) to every bet from urn B (red or black).

A *confidence function* describes the degree of confidence on the alternative probabilistic model governing the relevant phenomenon. If we assume the existence of a confidence function over probabilities concerning urn A, it is plausible to take φ_A such that

$$\varphi_A((\alpha, 1-\alpha)) = 0 \text{ if } \alpha \neq 1/2 \text{ and } \varphi_A((1/2, 1/2)) = 1$$

where $(\beta, 1-\beta)$ denotes the lotteries that assign weight β for a red ball and $1-\beta$ for a black ball. On the other hand, in urn B the situation is less simple due to the lack of information about the proportion of balls. For example, taking considerations of symmetry into account we may assume a confidence function φ_B, such that, $\varphi_B((\alpha, 1-\alpha)) = 4(\alpha - \alpha^2)$ is the degree of confidence in distribution $(\alpha, 1-\alpha)$. This latter example illustrates a situation where a decision maker has a subjective judgement that reflects a better amount of confidence in distributions closer to the symmetrical case $(\frac{1}{2}, \frac{1}{2})$.

Savage (1964) proposed a theory that relies solely upon behavioral data and gave a set of axioms upon preferences amongst *acts* (*i.e.*, maps from states to consequences) under which choice under uncertainty reduces to choice under risk, *i.e.*, the decision maker's preference can be represented by a pair u and p, where u is a utility function over the consequences and p is a probability over the states of nature. In a setting where objective probabilities are embedded in the consequence space, Anscombe and Aumann (1963) gave an alternative and simpler axiomatic treatment. This treatment is especially apparent in Fishburn's (1970) well-known reformulation and extension of Anscombe and

Aumann's approach which employs usual linear-space arguments[1] and entails the same representation. Hence, if we assume the axiomatizations of subjective expected utility (SEU) and consider the Ellsberg's preceding experiment, the decision maker's confidence function in urn B would have assigned the value one to a unique probability p. This view can be interpreted as the extreme binary assignment of confidence degree over the priors, that is, one decision maker *à la* SEU may have total confidence in a unique prior[2]. However, as it is well known, a decision maker consistent with the observations from Ellsberg's experiment is not consistent with the SEU characterization. Such an example illustrates the fact that in situations where some events come with probabilistic information and some events do not, subjective probabilities do not always suffice to fully encode all aspects of an individual's uncertain beliefs.

Ellsberg Paradox and some normative failures of SEU[3] have inspired the development of non-probabilistic models of preferences over subjectively uncertain acts. One important line of research replaces the subjective expected utility function with a more general functional, such as the Choquet expected utility (CEU) of Schmeidler (1989) or the maxmin expected utility (MEU) of Gilboa and Schmeidler (1989). Decision makers with MEU preferences evaluate an act using the minimun expected utility over a non-empty, convex and (weakly*) compact set of probabilities, while decision makers with CEU preferences evaluate an act using its expected utility computed according to a capacity (a non additive probability). Although these models are not the same in general, they coincide in the case of ambiguity aversion, that is, CEU with a convex capacity. In this case, the Choquet expected utility with respect to a capacity v reduces to the

[1] Ghirardato, Maccheroni, Marinacci and Siniscalchi (2003) provide a simple definition of *subjective mixture* of acts that makes it possible to exploit all the advantages of the set-up pioneered by Anscombe and Aumann and Fishburn relying solely on behavioral data, and hence retaining the conceptual appeal of Savage's approach.

[2] One stronger example in economics is the *rational expectation hypothesis*: under this assumption all agents share the same probability on some relevant economic phenomenon. But it is important to highlight that the axioms from Savage or Anscombe and Aumann imply no restricitions on the form of the probabilistic expectations, in particular, they do not imply that expectations are rational.

[3] For a stimulant discussion see Gilboa, Postlewaite and Schmeidler (2004), section 3.2.

minimum expected value over the set of probability distribution given by the core of the capacity v (definitions can be found in the section 2.5.2). Formally, a decision maker with MEU preferences ranks acts according to the following criterion

$$J(f) = \min_{p \in C} \int u(f)dp$$

An ambiguity aversion decision maker *a la* CEU or MEU exibits a behavior compatible with the Ellsberg Paradox. In this case, we may think that the decision-maker's confidence function is the characteristic function of some nonempty, convex and (weakly*) closed set C of probabilties: if the prior belongs to C the (normalized) confidence is one, otherwise the confidence is null. Hence, the MEU criterion can be written as follows

$$J(f) = \min_{\{p:\mathbf{1}_C(p) \geq \alpha\}} \frac{1}{\mathbf{1}_C(p)} \int u(f)dp$$

for any $\alpha \in (0, 1]$, where

$$\mathbf{1}_A(p) = \begin{cases} 1, \ p \in C \\ 0, \ p \notin C \end{cases}$$

However, in general, it is not reasonable that the decision-maker has null confidence on priors close to set C: e.g., consider the Ellsberg Paradox and an ambiguity aversion decision maker with a set of priors given by $C = \{(\beta, 1 - \beta) : \beta \in [0.4, 0.6]\}$, which implies in a confidence function given by

$$\varphi((\beta, 1 - \beta)) = \begin{cases} 1, \ if \ \beta \in [0.4, 0.6] \\ 0, \ if \ \beta \notin [0.4, 0.6] \end{cases}.$$

We think that it is questionable to associate null confidence on priors such as $(0.39, 0.61)$. Fortunately, our setting shows that the decision maker *à la* MEU may have a non-zero degree of confidence on priors that does not belong to C.

In a setting where we consider the case of Anscombe and Aumann's bounded below acts, our representation has as its main component a mapping φ from the set $\Delta \equiv$

$ba^1_+(S,\Sigma)$ of all finitely additive probabilities on the unit interval $[0,1]$, a minimal level of confidence $\alpha_0 \in (0,1]$, and a real-valued affine function u on X, such that

$$J(f) = \min_{\{p:\varphi(p)\geq\alpha_0\}} \frac{1}{\varphi(p)} \int u(f)dp.$$

Function φ belongs to the class of *fuzzy set* on Δ (mappings from Δ to $[0,1]$) that presents the following properties: quasi-concavity, normality, and (weakly*) upper semi-continuity[4]. A mapping in this class is called a *confidence function,* and models the ambiguity. Hence, the term *ambiguity* refers purely to the vague perception of the likelihood subjectively associated with an event by a decision maker, e.g., when asked about his subjective estimate of the probability of an event, the decision maker replies: *It is more or less 70 percent.* The number $\alpha_0 \in (0,1]$ is the minimal level of confidence that matters for the decision maker: a prior with a confidence of less than α_0 is not relevant, and if $\varphi(p_1) \geq \varphi(p_2) \geq \alpha_0$ then the decision maker presents a greater confidence on p_1 than p_2.

We axiomatize the previous representation by showing how it rests on a simple set of axioms that generalizes the MEU axiomatization of Gilboa and Schmeidler (1989).

2.2 Axioms

We assume there exists $x_* \in X$ such that $f \succsim x_*$ for every f belonging to $\mathcal{F}$, x_* is called the *worst consequence.*

(**Axiom 1**) Weak order non-degenerate. If $f, g, h \in \mathcal{F}$:

(complete) either $f \succsim g$ or $g \succsim f$

(transitivity) $f \succsim g$ and $g \succsim h$ imply $f \succsim h$

there exists $(f, g) \in \mathcal{F}^2$ such that $(f, g) \in \succ$

[4] We recall that $C \subset \Delta$ is weakly* compact iff $\mathbf{1}_C$ is weakly* upper semicontinous. Normality says that $\{p : \varphi(p) = 1\}$ is nonempty.

(**Axiom 2**) Continuity. For all $f, g, h \in \mathcal{F}$ the sets:

$$\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\},\ \{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\} \text{ are closed.}$$

(**Axiom 3**) Monotonicity. For all $f, g \in \mathcal{F}$:

$$\text{if } f(s) \succsim g(s) \text{ for all } s \in S \text{ then } f \succsim g.$$

(**Axiom 4**) Uncertainty aversion. If $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$:

$$f \sim g \Rightarrow \alpha f + (1-\alpha)g \succsim f$$

(**Axiom 5**) Worst independence. For all $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$:

$$f \sim g \Rightarrow \alpha f + (1-\alpha)x_* \sim \alpha g + (1-\alpha)x_*.$$

(**Axiom 6**) Independence on X. For all $x, y, z \in X$:

$$x \sim y \Rightarrow \tfrac{1}{2}x + \tfrac{1}{2}z \sim \tfrac{1}{2}y + \tfrac{1}{2}z.$$

(**Axiom 7**) Bounded attraction for certainty. There exists $\delta \geq 1$ such that for all $f \in \mathcal{F}$ and $x, y \in X$:

$$f \sim x \Rightarrow \tfrac{1}{2}x + \tfrac{1}{2}y \succsim \tfrac{1}{2}f + \tfrac{1}{2}(\tfrac{1}{\delta}y + (1-\tfrac{1}{\delta})x_*).$$

Axioms 1, 2, 3 and 6 are standard and well understood[5]. We note that these axioms imply that the restriction of $\succsim \subset \mathcal{F}\times\mathcal{F}$ to $X\times X$, denoted by $\succsim|_{X\times X}$, has a von Neumann-Morgenstein representation (Lemma 10). Moreover, it is well known that if a preference relation $\succsim$ satisfies axioms $1, 2$ and 3 then each act $f \in \mathcal{F}$ admits a certainty equivalent $c_f \in X$.

Axiom 4 is due to Schmeidler (1989) and it says that the decision maker will, in general, prefer the mixture to its components.

[5] Axiom 3 says that the preference is monotone and is essentially a state-independent condition saying that the decision maker always weakly prefers acts delivering statewise weakly better payoffs, regardless of the state where better payoffs occur.

The classical independent axiom among acts is due to Anscombe and Aumann and it imposes the idea that if $f, g, h \in F$ and $\beta \in (0, 1]$ then

$$f \sim g \Rightarrow \beta f + (1-\beta)h \sim \beta g + (1-\beta)h,$$

and it says that the preference among mixtures $\beta f + (1-\beta)h$ and $\beta g + (1-\beta)h$ is completely determined by the preference between f and g. An important weakening of this axiom, called certainty independence, was introduced by Gilboa and Schmeidler (1989) in their characterization of MEU preferences: it imposes only that h belongs to the set of constant acts X. Our axiom 5 requires that independence holds whenever acts are mixed with the worst consequence x_*.

Axiom 7 says that the decision maker, despite of uncertainty aversion, presents a bounded attraction to certainty. For concreteness, consider the following example where we suppose that consequences are monetary payoffs: There are two states of nature s_1 and s_2, consider two acts $f \equiv 1\{s_1\}4$ and $g \equiv x = 2$ and a decision maker (risk neutral) who has a preference where $f \sim x$. The certainty independence axiom implies that $3/2\{s_1\}3 \equiv \frac{1}{2}f + \frac{1}{2}x \sim x = 2$ but, if we impose only our axiom 7, we may obtain that $3/2\{s_1\}3 \succ 2$ and that (note that $y = x$ and $x_* = 0$)

$$2 \succsim \frac{1}{2}f + \frac{1}{2}\left(\frac{1}{\delta}2 + \left(1 - \frac{1}{\delta}\right)0\right) \equiv \left(\frac{1}{2} + \delta^{-1}\right)\{s_1\}\left(2 + \delta^{-1}\right)$$

Hence, the set $\{f \in \mathcal{F}/X : x \succsim f\}$ becomes small when we drop the certainty independence axiom for axiom 7. This axiom should be viewed as a behavioral feature of bounded attraction of certain acts, if we compare it to the certain independece axiom of Gilboa and Schmeidler.

2.3 Main Theorem

We can now state our main theorem, which characterizes preferences satisfying axioms A.1-A7.

Let $\mathfrak{X}$ be an arbitrary set, a fuzzy set in $\mathfrak{X}$ is any function $\varphi : \mathfrak{X} \to [0,1]$, which generalizes characteristic functions $\mathbf{1}_A : \mathfrak{X} \to [0,1]$ where $A \subset \mathfrak{X}$, $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ if $x \notin A$. This notion is due to Zadeh (1965).

Here we take $\mathfrak{X} = ba^1_+(S, \Sigma)$, the set of all finitely additive probabilities on Σ endowed with the natural restriction of the weak* topology on $ba(S, \Sigma)$.

Let $\mathcal{C}_c(ba^1_+(S, \Sigma))$ denote the collection of all nonempty convex weakly* closed subsets of $ba^1_+(S, \Sigma)$. As an extension of $\mathcal{C}_c(ba^1_+(S, \Sigma))$ we define:

Definition 8 *The set* $\mathrm{F}_{\mathcal{R}^*}(ba^1_+(S, \Sigma))$ *of regular* fuzzy sets consists of all mappings* $\varphi : ba^1_+(S, \Sigma) \to [0,1]$ *with the properties*[6]*:*

(a) φ *is normal;*

$$\{p \in ba^1_+(S, \Sigma) : \varphi(p) = 1\} \neq \emptyset$$

(b) φ *is weakly* upper semicontinuous;*

$$\{p \in ba^1_+(S, \Sigma) : \varphi(p) \geq \alpha\} \textit{ is weakly* closed for any } \alpha \in [0,1]$$

(c) φ *is quasi-concave;*

$$\varphi(\beta p_1 + (1-\beta) p_2) \geq \min\{\varphi(p_1), \varphi(p_2)\} \textit{ for any } \beta \in [0,1].$$

Remark 1 *We note that the weak* support of* φ*, denoted by*

$$supp^* \varphi := \overline{\{p \in ba^1_+(S, \Sigma) : \varphi(p) > 0\}}^{\sigma(ba, B)}$$

[6] *For an exposition of the concept of regular fuzzy sets over* $\mathbb{R}^n$ *see Puri and Ralescu (1985), page 1374.*

is weakly compact by Alaoglu's theorem (see Dunford and Schwartz (1988), page 424).*

Remark 2 *We can embed $\mathcal{C}_c(ba^1_+(S,\Sigma))$ into $\mathrm{F}_{\mathcal{R}^*}(ba^1_+(S,\Sigma))$ by the natural mapping $P \mapsto \mathbf{1}_P$. We will use the notation for level sets*

$$L\alpha\varphi = \{p \in ba^1_+(S,\Sigma) : \varphi(p) \geq \alpha\} \text{ for any } \alpha \in (0,1].$$

Moreover, we note that $\varphi \in \mathrm{F}_{\mathcal{R}^*}(ba^1_+(S,\Sigma))$ if and only if the correspondence

$$\alpha \mapsto L\alpha\varphi$$

takes values only on $\mathcal{C}_c(ba^1_+(S,\Sigma))$. Because of this previous result, a quasi-concave fuzzy set φ is called fuzzy convex *(all level sets $L\alpha\varphi$ are convex).*

The main theorem is:

Theorem 9 *Let $\succsim$ be a binary relation on $\mathcal{F}$, the following conditions are equivalent:*

(i) $\succsim$ satisfies conditions A.1-A.7;

(ii) there exists a non-constant affine function $u : X \to \mathbb{R}_+$, a $\alpha_0 \in (0,1]$ and a regular fuzzy set $\varphi : ba^1_+(S,\Sigma) \to [0,1]$ such that, for all $f, g \in \mathcal{F}$*

$$f \succsim g \Leftrightarrow \inf_{p \in L_{\alpha_0}\varphi} \frac{1}{\varphi(p)} \int_S u(f)dp \geq \inf_{p \in L_{\alpha_0}\varphi} \frac{1}{\varphi(p)} \int_S u(g)dp$$

Futhermore, given u in (ii) then for all $\lambda > 0$ we can adopt the tranformation λu in our representation.

Remark 3 *As soon as $\delta > 1$ or equivalently $\alpha_0 < 1$, our preference is not invariant, that is, we cannot change u by a positive affine transformation $v = \lambda u + \beta$ when $\beta \neq 0$; but it is not a surprise. For the Anscombe and Aumann framework Ghirardato, Maccheroni and Marinacci (2005) proved that invariance is equivalent to the axiom introduced by Gilboa and Schmeidler (1989) in their characterization of MEU preferences, as we remarked in the previous section:*

Certainty Independence: If $f, g \in \mathcal{F}$, $x \in X$, and $\lambda \in (0,1]$ then

$$f \sim g \Leftrightarrow \lambda f + (1-\lambda)x \sim \lambda g + (1-\lambda)x.$$

It requires that independence holds whenever acts are mixed with a constant act. This axiom implies that $\delta = 1$. For more details see Remark 4.

We interpret the regular* fuzzy set φ (which we called *confidence function*), in our representation, as the degree of confidence over each probability and α_0 as the minimal level of confidence held by the decision maker. The confidence function models the ambiguity, hence the term *ambiguity* refers purely to the vague perception of the likelihood subjectively associated with an event by a decision maker.

We note that if $\varphi = \mathbf{1}_P$, where $P \in \mathcal{C}_c(ba_+^1(S,\Sigma))$, we obtain the representation of Gilboa and Schmeidler (1989) under the existence of a worst consequence.

We turn now to the proof of Theorem 9.

Part (ii)$\Rightarrow$(i) is straightforward. The (i)$\Rightarrow$(ii) will result from Lemma 10 to Lemma 15 and its Corollary 19.

Lemma 10 *There exists an affine $u : X \to \mathbb{R}$ non-constant function such that for all $x, y \in X : x \succsim y$ iff $u(x) \geq u(y)$. Moreover, we can choose u such that $u(x_*) = 0$.*

Proof: By axioms 1,2 and 6 the premises of the von Neumann-Morgenstern theorem are satisfied [see Schmeidler (1989), page 577] and there exist an afine function $u : X \to \mathbb{R}$ such that for all $x, y \in X : x \succsim y$ iff $u(x) \geq u(y)$. Therefore, we can choose $u(x_*) = 0$. By axiom 1, there exists $f, g \in \mathcal{F}$ s.t. $f \succ g$; given $x, y \in X$ such that $x \succsim f(s)$ and $g(s) \succsim y$ for all $s \in S$, then by monotonicity (axiom 3) we have that $x \succ y$, then u cannot be constant. Finally, we can suppose that there exists $x \in X$ s.t. $u(x) = 1$.

Lemma 11 *For any $u : X \to \mathbb{R}$ satisfying Lemma 10 there exists a unique $J : \mathcal{F} \to \mathbb{R}$ such that*

(i) $f \succeq g$ iff $J(f) \geq J(g)$ for all $f, g \in \mathcal{F}$.

(ii) If $f = x\mathbf{1}_S \in \mathcal{F}_c \equiv X$ (the set of constant functions) then $J(f) = u(x)$.

Proof: On $\mathcal{F}_c$ the functional J is uniquely determined by (ii). Since for all $f \in \mathcal{F}$ there exists a $c_f \in \mathcal{F}_c$ such that $f \sim c_f$, we set $J(f) = u(c_f)$ and by construction J satisfies (i), hence it is also unique.

We denote by $B_0(S, \Sigma, K)$ the functions in $B_0(S, \Sigma)$ that assume finitely many values in $K \subset \mathbb{R}$ and by $B_0^+(S, \Sigma) = B_0(S, \Sigma, \mathbb{R}_+)$. For $k \in \mathbb{R}$, let $k\mathbf{1}_S \in B_0(S, \Sigma)$ be the constant function on S such that $k\mathbf{1}_S(S) = \{k\}$.

Lemma 12 *Let u and J be defined as in Lemmas 10 and 11, then there exists a functional*

$$I : B_0^+(S, \Sigma) \to \mathbb{R}$$

where for every $f \in \mathcal{F}$ $I(uof) = J(f)$, such that:

(i) *I is superadditive, i.e., for $a, b \in B_0^+(S, \Sigma) : I(a + b) \geq I(a) + I(b)$;*

(ii) *I is positively homogeneous,i.e., for $a \in B_0^+(S, \Sigma), \lambda \geq 0 : I(\lambda a) = \lambda I(a)$;*

(iii) *I is monotonic, i.e., for $a, b \in B_0^+(S, \Sigma) : a \geq b \Rightarrow I(a) \geq I(b)$;*

(iv) *I is normalized, i.e., $I(\mathbf{1}_S) = 1$;*

(v) *For every $a \in B_0^+(S, \Sigma)$ and $\xi \geq 0$*

$$I(a + \xi\mathbf{1}_S) \leq I(a) + \delta\xi.$$

Proof: We begin with $B_0(S, \Sigma, u(X))$ and then extend I to all $B_0^+(S, \Sigma)$. If $f \in \mathcal{F}$ then $u(f) \in B_0(S, \Sigma, u(X))$. Now, if $a \in B_0(S, \Sigma, u(X))$ we have that there exists $\{E_i\}_{i=1}^n \subset \Sigma$ a partition of S and $\{x_i\}_{i=1}^n \subset X$ such that

$$a := \sum_{i=1}^n u(x_i)\mathbf{1}_{E_i},$$

hence, we can choose $f \in \mathcal{F}$ such that $f(s) = x_i$ when $s \in E_i$ and we conclude that $a = u(f)$.

From this, we can write $B_0(S,\Sigma,u(X)) = \{u(f) : f \in \mathcal{F}\}$; *therefore,* $u(f) = u(g) \Leftrightarrow u(f(s)) = u(g(s)),\ \forall s \in S \Leftrightarrow f(s) \sim g(s),\ \forall s \in S$; *and, by axiom 3 (monotonicity),* $f \sim g$, i.e., $u(f) = u(g) \Leftrightarrow J(f) = J(g)$.

Define $I(a) = J(f)$ *whenever* $a = u(f)$. *Hence, we have it that* I *is well defined over* $B_0(S,\Sigma,u(X))$.

Now, if $a = u(f)$ *and* $b = u(f) \in B_0(S,\Sigma,u(X))$ *and* $a \geq b$, *then* $u(f(s)) \geq u(g(s))$ *for any* $s \in S$ *and, by axiom 3 (monotonicity), we have it that* $f \succsim g$, i.e., $J(f) \geq J(g)$ *and* $I(a) = I(u(f)) = J(f) \geq J(g) = I(u(g)) = I(b)$; *which proves that* I *is monotonic.*

Set $k \in u(X)$, *then there exists some* $x \in X$ *such that* $k = u(x)$ *and* $I(k\mathbf{1}_S) = I(u(x)\mathbf{1}_S) = J(x) = u(x) = k$, i.e., I *is normalized. In particular, since* $1 \in u(X), I(\mathbf{1}_S) = 1$.

We now show that I *is positively homogeneous. Assume* $a = \alpha b$, *where* $a, b \in B_0(S,\Sigma,u(X))$ *and* $0 < \alpha \leq 1$. *Let* $g \in \mathcal{F}$ *satisfy* $u(g) = b$ *and define* $f = \alpha g + (1-\alpha)x_*$. *Hence* $u(f) = \alpha u(g) + (1-\alpha)u(x_*) = \alpha b = a$, *so* $I(a) = J(f)$. *We have it that* $J(c_g) = J(g) = I(b)$. *By axiom 5 (worst independence),* $\alpha c_g + (1-\alpha)x_* \sim \alpha g + (1-\alpha)x_* = f$, *hence* $J(f) = J(\alpha c_g + (1-\alpha)x_*) = \alpha J(c_g) + (1-\alpha)J(x_*) = \alpha J(c_g)$ *and we can write*

$$I(\alpha b) = I(a) = J(f) = \alpha J(c_g) = \alpha I(b).$$

Furthermore, this implies positive homogeneity for $\alpha > 1 : a = \alpha b \Rightarrow b = \alpha^{-1}a \Rightarrow I(b) = \alpha^{-1}I(a) \Rightarrow I(a) = \alpha I(b)$.

Now, by positive homogeneity we can extend I *to all* $B_0^+(S,\Sigma)$, *since* $u(X)$ *is a non-empty interval of* $\mathbb{R}_+$ *containing* 0.

Next, we show that (v) *is satisfied. Let there be given* $a \in B_0^+(S,\Sigma)$ *and* $\xi \geq 0$. *By homogeneity we may assume without loss of generality that* $2a$ *and* $2\delta\xi\mathbf{1}_S \in B_0(S,\Sigma,u(X))$. *Now we define* $\beta = I(2a) = 2I(a)$. *Let* $f \in \mathcal{F}$ *such that* $u(f) = 2a$ *and* $y, z \in X$ *satisfy* $u(y) = \beta$ *and* $u(z) = 2\delta\xi$, *then* $J(f) = I(u(f)) = 2I(a) = \beta = I(\beta\mathbf{1}_S) = I(u(y)) = J(y)$, i.e., $f \sim y$. *By axiom 7 (bounded attraction for certainty), there exists* $\delta \geq 1$ *such that*

$$\frac{1}{2}y + \frac{1}{2}z \succeq \frac{1}{2}f + \frac{1}{2}(\frac{1}{\delta}z + (1 - \frac{1}{\delta})x_*)$$

hence

$$\frac{1}{2}J(y) + \frac{1}{2}J(z) \geq J(\frac{1}{2}f + \frac{1}{2}(\frac{1}{\delta}z + (1 - \frac{1}{\delta})x_*))$$

then

$$\frac{1}{2}I(u(y)) + \frac{1}{2}I(u(z)) \geq I(\frac{1}{2}u(f) + \frac{1}{2}u(\frac{1}{\delta}z + (1 - \frac{1}{\delta})x_*))$$

from the facts above

$$\frac{1}{2}I(\beta\mathbf{1}_S) + \frac{1}{2}I(2\delta\xi\mathbf{1}_S) \geq I(\frac{1}{2}2a + \frac{1}{2}(\frac{1}{\delta}u(z) + (1 - \frac{1}{\delta})u(x_*)))$$

we obtain

$$I(a) + \delta\xi \geq I(a + \frac{1}{\delta}\delta\xi\mathbf{1}_S) = I(a + \xi\mathbf{1}_S)$$

It remains to show that I *is superadditive. Let there be given* $a, b \in B_0^+(S, \Sigma)$ *and, once again, by homogeneity we assume that* $a, b \in B_0(S, \Sigma, u(X))$. *First, we note that axiom 4 (ambiguity aversion) implies that* I *is quasi-concave, in fact:*

Since $a, b \in B_0(S, \Sigma, u(X))$ *we can choose* $f, g \in \mathcal{F}$ *such that* $a = u(f)$ *and* $b = u(g)$, *since* $\alpha a + (1-\alpha)b = \alpha u(f) + (1-\alpha)u(g) = u(\alpha f + (1-\alpha)g)$, *we obtain* $I(\alpha a + (1-\alpha)b) = J(\alpha f + (1-\alpha)g)$ *and, by axiom 4 (uncertainty aversion),* $\alpha f + (1-\alpha)g \succsim g$ *if* $f \succsim g$, *hence* $J(\alpha f + (1-\alpha)g) \geq min\{J(f), J(g)\}$, *i.e.,* $I(\alpha a + (1-\alpha)b) \geq min\{I(a), I(b)\}$.

Now, since I *is positively homogeneous it follows that* I *is concave (see Berge, 1965), then* $\frac{1}{2}I(a+b) = I(\frac{1}{2}a + \frac{1}{2}b) \geq \frac{1}{2}I(a) + \frac{1}{2}I(b)$, *that is,* $I(a+b) \geq I(a) + I(b)$.

Remark 4 *We note that* $I : B_0^+(S, \Sigma) \to \mathbb{R}$ *as in the previous Lemma is constant additive, i.e., for any* $a \in B_0^+(S, \Sigma)$ *and* $\xi \geq 0$

$$I(a + \xi\mathbf{1}_S) = I(a) + \xi$$

if $\delta = 1$. In fact,

$$I(a) + \xi = I(a) + I(\xi \mathbf{1}_S) \leq I(a + \xi \mathbf{1}_S) \leq I(a) + \delta\xi.$$

Lemma 13 *There exists a unique continuous extension of I to $B^+(S,\Sigma)$. Clearly, this extension satisfies $(i),(ii),(iii)$ and (v) from the last lemma on $B^+(S,\Sigma)$.*

Proof:

Since $a = b + a - b \leq b + \|a-b\|_\infty$, by monotonicity:

$$I(a) \leq I(b + \|a-b\|_\infty)$$

and by (v):

$$I(a) \leq I(b) + \delta \|a-b\|_\infty$$

that is

$$I(a) - I(b) \leq \delta \|a-b\|_\infty,$$

therefore

$$|I(a) - I(b)| \leq \delta \|a-b\|_\infty$$

and by equality $B^+(S,\Sigma) = \overline{B_0^+(S,\Sigma)}^{\|\cdot\|_\infty}$, there exist a unique continuous extension of I.

Building upon Fan's Theorem 14 below, we give in the next Lemma 15, the key result for our representation theorem 9. This Lemma can be seen as a generalization of the representation Theorem proposed by Chateauneuf (1991) for Gilboa and Schmeidler's model. In fact, both models coincide if $\delta = 1$.

Consider a real Banach space E and denote by E^* the dual space of E:

Theorem 14 *(Fan, 1956; page 126) Given an arbitrary set Λ, let the system*

$$\langle f, x_i \rangle \geq \alpha_i, i \in \Lambda \qquad (\pounds)$$

of linear inequalities; where $\{x_i\}_{i\in\Lambda}$ *be a family of elements, not all* 0*, in real normed linear space* E*, and* $\{\alpha_i\}_{i\in\Lambda}$ *be a corresponding family of real numbers.*

Let $\sigma := sup \sum_{j=1}^{n} r_j \alpha_{i_j}$ *when* $n \in \mathbb{N}$*, and* r_j *vary under conditions:* $r_j \geq 0$, $\forall j \in \{1, ..., n\}$ *and* $\left\| \sum_{j=1}^{n} r_j x_{i_j} \right\|_E = 1$*. Then the system* $(\pounds)$ *has a solution* $f \in E^*$ *if and only if* σ *is finite. Moreover, if the system* $(\pounds)$ *has a solution* $f \in X^*$*, and if the zero-functional is not a solution of* $(\pounds)$*, then* $\sigma = min \{\|f\|_{E^*} : f$ *is a solution of* $(\pounds)\}$*.*

Lemma 15 *Let* Σ *be a* σ*-algebra of subsets of a set* S *and let* I *be a functional on the set* $B^+(S, \Sigma)$*. The following two assertions are equivalent:*

Assertion 1*:* *I satisfies the properties:*

1) I *is superadditive: for* $a, b \in B^+(S, \Sigma)$

$$I(a + b) \geq I(a) + I(b);$$

2) I *is positively homogeneous: for* $a, b \in B^+(S, \Sigma)$, $\lambda \geq 0$:

$$I(\lambda a) = \lambda I(a);$$

3) I *is monotonic: for* $a, b \in B^+(S, \Sigma)$:

$$a \geq b \Rightarrow I(a) \geq I(b);$$

4) I *is normalized:*

$$I(\mathbf{1}_S) = 1;$$

5) *There exists* $\delta \geq 1$ *such that for all* $a \in B^+(S, \Sigma)$ *and* $k \geq 0$*:*

$$I(a + k\mathbf{1}_S) \leq I(a) + \delta k.$$

Assertion 2*: there exists* $\alpha_0 \in (0, 1]$ *and a normal fuzzy set* $\varphi : ba^1_+(S, \Sigma) \to [0, 1]$ *such* *that for any $a \in B^+(S,\Sigma)$:*

$$I(a) = \underset{p \in L_{\alpha_0}\varphi}{inf} \frac{1}{\varphi(p)} \int_S adp$$

Proof: In order to simplify the notation we set $ba_+^1(S,\Sigma) = \Delta$, $B^+(S,\Sigma) = B^+$, and $\int_S adp = E_p(a)$ for every $(a,p) \in B^+ \times \Delta$.

Assertion 2 implies Assertion 1 is straighforward.

In order to prove that Assertion 1 implies Assertion 2 we need the following lemma:

Lemma 16 *The mapping*

$$\begin{aligned} \varphi^* \quad &: \quad \Delta \to \mathbb{R} \\ p \quad &\mapsto \quad \varphi^*(p) = \underset{a \in B^+}{inf} \frac{E_p(a)}{I(a)} \end{aligned}$$

is a normal fuzzy set[7]*. Moreover, the functional*

$$\begin{aligned} I^* \quad &: \quad B^+ \to \mathbb{R} \\ a \quad &\mapsto \quad I^*(a) = \underset{p \in \Delta}{inf} \frac{E_p(a)}{\varphi^*(p)} \end{aligned}$$

satisfies $I^*(a) = I(a)$, for any $a \in B^+$.

Proof: Since for all $a \in B^+$, $E_p(a) \geq 0$ and $I(a) \geq 0$, clearly $\varphi^*(p) \geq 0$ and $\frac{E_p(\mathbf{1}_S)}{I(\mathbf{1}_S)} = 1$ implies that $\varphi^*(p) \in [0,1]$ for all $p \in \Delta$.

Let us show that φ^* is normal, i.e., that there exists a $p_0 \in \Delta$ such that $\varphi^*(p_0) = 1$, since $\varphi^*(p_0) \leq 1$ it is enough to show that there exists $p_0 \in \Delta$ such that

$$E_{p_0}(a) \geq I(a) \ \forall a \in B^+$$

Setting $E = B$, we need to show that there exists $f \in E^*$ such that $f(\mathbf{1}_S) \geq 1$, $f(-\mathbf{1}_S) \geq$

[7] *Note that we adopt the usual convention $0/0 = 1$ and $r/0^+ = +\infty$ if $r > 0$.*

-1 *and* $f(a) \geq I(a)$ *for all* $a \in B^+$. *Then we have a system of linear inequalities and can now use Fan's theorem:*

Let us consider $\lambda_1, \lambda_2, ..., \lambda_n \geq 0$ *and* $\mathbf{1}_S$, $-\mathbf{1}_S$, $a_j \in B^+$, $3 \leq j \leq n$ *such that:*

$$\left\| \lambda_1 \mathbf{1}_S + \lambda_2(-\mathbf{1}_S) + \sum_{j=3}^{n} \lambda_j \, a_j \right\|_{\infty} = 1$$

it follows that

$$\lambda_1 \mathbf{1}_S - \lambda_2 \mathbf{1}_S + \sum_{j=3}^{n} \lambda_j \, a_j \leq \mathbf{1}_S$$

hence

$$\lambda_1 \mathbf{1}_S + \sum_{j=3}^{n} \lambda_j \, a_j \leq (\lambda_2 + 1)\mathbf{1}_S$$

from (1),(3),(4) *and* (2) *it comes that:*

$$\lambda_1 + \sum_{j=3}^{n} \lambda_j I(\, a_j) \leq \lambda_2 + 1$$

therefore

$$\lambda_1 - \lambda_2 + \sum_{j=3}^{n} \lambda_j I(\, a_j) \leq 1$$

i.e., $\sum\limits_{j=1}^{n} \lambda_j \alpha_j \leq 1$; *where* $\alpha_1 = 1$, $\alpha_2 = -1$, *and* $\alpha_j = I(a_j), 3 \leq j \leq n$. *Hence* σ *is finite and from Fan's theorem there exists* $p_0 \in \Delta$ *such that* $E_{p_0}(a) \geq I(a)$ *for all* $a \in B^+$.

Now, we have that for any $a \in B^+$, $I^*(a) = \inf\limits_{p \in \Delta} \frac{E_p(a)}{\varphi_+(p)} \in \mathbb{R}_+$. *It remains to prove that* $I^*(a) = I(a)$, *for any* $a \in B^+$.

Let a_0 *be chosen in* B^+, *and first prove that* $I^*(a_0) \geq I(a_0)$ *: If* $I(a_0) = 0$ *this is immediate. Assume, now,* $I(a_0) > 0$. *Note that it is enough to prove* $I^*(a_0) \geq I(a_0)$ *if* $1 \geq I(a_0) > 0$. *Actually, let* a_0 *such that* $I(a_0) > 1$ *and choose* $\lambda > 0$ *such that* $\lambda I(a_0) \leq 1$, *since* I^* *and* I *are positively homogeneous, one obtains:*

$$\lambda I(a_0) = I(\lambda a_0) \leq I^*(\lambda a_0) = \lambda I^*(a_0)$$

hence $I(a_0) \leq I^*(a_0)$. *Considering* $a_0 \in B^+$ *such that* $1 \geq I(a_0) > 0,$ *we have that*

$$\varphi^*(p) = \inf_{a \in B^+} \frac{E_p(a)}{I(a)} \leq \frac{E_p(a_0)}{I(a_0)},\ \forall p \in \Delta,$$

hence,

$$I(a_0) \leq \frac{E_p(a_0)}{\varphi^*(p)},\ \forall p \in \Delta$$

and from the definition of $I^* : I^*(a_0) \geq I(a_0)$.

Let us now prove that $I^*(a_0) \leq I(a_0)$ *for any chosen* $a_0 \in B^+$. *Clearly, it is enough to prove this inequality when* $I^*(a_0) > 0$. *Since* $I^*(a_0)$ *is the greatest lower bound of the set of real numbers given by* $\left\{\frac{Eq(a_0)}{\varphi^*(q)} : q \in \Delta\right\}$ *if we find* $p \in \Delta$ *such that* $\frac{Ep(a_0)}{\varphi^*(p)} \leq I(a_0)$ *then the result will be proved:*

Let us first show that there exists $f \in E^*$ *such that* $\delta \geq f(\mathbf{1}_S) \geq 1$, $f(a_0) = I(a_0)$ *and* $f(a) \geq I(a)$ *for all* $a \in B^+$*., i.e.,* $f \in E^*$ *such that*

$$\begin{aligned} f(\mathbf{1}_S) &\geq 1,\ f(-\mathbf{1}_S) \geq -\delta,\ f(a_0) \geq I(a_0) \\ f(-a_0) &\geq -I(a_0) \text{ and } f(a) \geq I(a)\ \forall a \in B^+ \end{aligned}$$

Again, we use Fan's theorem:

Let us consider $\lambda_1, ..., \lambda_n \geq 0$ *and* $\mathbf{1}_S$, $-\mathbf{1}_S$, a_0, $-a_0$, $a_j \in B^+$, $5 \leq j \leq n$ *such that:*

$$\left\| \lambda_1 \mathbf{1}_S + \lambda_2(-\mathbf{1}_S) + \lambda_3 a_0 + \lambda_4(-a_0) + \sum_{j=5}^{n} \lambda_j\, a_j \right\|_\infty = 1$$

it follows that

$$\lambda_1 \mathbf{1}_S - \lambda_2 \mathbf{1}_S + \lambda_3 a_0 - \lambda_4 a_0 + \sum_{j=5}^{n} \lambda_j\, a_j \leq \mathbf{1}_S$$

hence

$$\lambda_1 \mathbf{1}_S + \lambda_3 a_0 + \sum_{j=5}^{n} \lambda_j\, a_j \leq \lambda_4 a_0 + (\lambda_2 + 1)\mathbf{1}_S$$

By properties of I in assertion 1 it comes that:

$$\lambda_1 + \lambda_3 I(a_0) + \sum_{j=5}^{n} \lambda_j I(\, a_j) \leq \lambda_4 I(a_0) + (\lambda_2 + 1)\delta$$

therefore

$$\lambda_1 + \lambda_2 + \lambda_3 I(a_0) - \lambda_4 I(a_0) + \sum_{j=5}^{n} \lambda_j I(\, a_j) \leq \delta$$

By Fan's theorem, it comes that there exists $\eta \in [1, \delta]$, $p \in \Delta$ such that:

$$\begin{aligned} (1)\, \eta E_p(a_0) &= I(a_0), \text{ and} \\ (2)\, \eta E_p(a) &\geq I(a) \text{ for all } a \in B^+ \end{aligned}$$

From (2) it comes that $E_p(a)/I(a) \geq \eta^{-1}$, for all $a \in B^+$. Actually, by the initial convention, $E_p(a) = 0$ implies $I(a) = 0$ and then $E_p(a)/I(a) = 1 \geq \eta^{-1}$. Moreover, if $E_p(a) > 0$ and $I(a) = 0$ then $E_p(a)/I(a) = +\infty \geq \eta^{-1}$.

Consequentely, $\varphi^*(p) \geq \eta^{-1}$, and therefore $\varphi^*(p) > 0$.

Let us show that this entails $E_p(a_0) > 0$. In fact, $0 < I^*(a_0) \leq E_p(a_0)/\varphi^*(p)$, so we get $E_p(a_0) > 0$. Hence, (1) entails $E_p(a_0) > 0$. Consequently,

$$\frac{E_p(a_0)}{I(a_0)} = \frac{1}{\eta} \leq \varphi^*(p),$$

that is,

$$\frac{E_p(a_0)}{\varphi^*(p)} \leq I(a_0),$$

as desired.

Corollary 17 *Set $\alpha_0 = 1/\delta$ and $L_{\alpha_0}\varphi^* = \{p \in \Delta : \varphi^*(p) \geq \alpha_0\}$, then for every $a \in B^+$*

$$I(a) = \underset{p \in L_{\alpha_0}\varphi^*}{inf} \frac{E_p(a)}{\varphi^*(p)} := I'(a)$$

Proof:

First, it is immediate that $I'(a) \geq I^*(a) = I(a)$ for any $a \in B^+$. In order to show that $I'(a) = I(a)$ for every $a \in B^+$, it is enough to show that for a given a_0 belonging to B^+ such that $I'(a_0) > 0$, there exists $p_0 \in L_{\alpha_0}\varphi^*$ such that $E_{p_0}(a_0)/\varphi^*(p_0) \leq I(a_0)$. However, we know, by the previous lemma, that there exists $p_0 \in \Delta$ such that $E_{p_0}(a)/I(a) \geq 1/\delta$ for every $a \in B^+$, i.e., $p_0 \in L_{\alpha_0}\varphi^*$. Since $I'(a) > 0$, it follows that $E_{p_0}(a_0) > 0$ and, again by the previous lemma, $E_{p_0}(a_0)/I(a_0) = \varphi^*(p_0)$, and then $I(a_0) = E_{p_0}(a_0)/\varphi^*(p_0)$.

Remark 5 *Let φ be a normal fuzzy set satisfying the model, i.e.*

$$I(a) = \inf_{p \in L_{\alpha_0}\varphi} \frac{1}{\varphi(p)} \int a dp \text{ for all } a \in B^+,$$

and let φ^* be defined by

$$\varphi^*(p) = \inf_{a \in B^+} \frac{E_p(a)}{I(a)},$$

then for any $p \in L_{\alpha_0}\varphi$ one obtains $\varphi^*(p) \geq \varphi(p)$.

Proof:

Let $p \in L_{\alpha_0}\varphi$ then for all $a \in B^+$, $I(a) \leq E_p(a)/\varphi(p)$. Hence, $\varphi(p)I(a) \leq E_p(a)$ for all $a \in B^+$. Since $\varphi(p) > 0$, if $E_p(a) = 0$ then $I(a) = 0$ and in this case $E_p(a)/I(a) = 1 \geq \varphi(p)$. If $E_p(a) > 0$ in any case, due to the convention $r/0 = +\infty$ if $r > 0$, one obtains that $E_p(a)/I(a) \geq \varphi(p)$. Hence, $\varphi(p) \leq E_p(a)/I(a)$ for all $a \in B^+$ and, therefore, $\varphi^*(p) \geq \varphi(p)$.

Remark 6 *From Lemma 16, there exists a normal fuzzy set φ such that*

$$I(a) = \inf_{p \in \Delta} \frac{E_p(a)}{\varphi(p)} \text{ for all } a \in B^+.$$

In fact, for any such φ, one obtains that $\varphi^*(p) \geq \varphi(p)$ for all $p \in \Delta$.

Proof:

Take $p \in \Delta$, then for all $a \in B^+$, $I(a) \leq E_p(a)/\varphi(p)$; if $\varphi(p) = 0$, clearly $E_p(a)/I(a) \geq \varphi(p)$ for all $a \in B^+$ and $\varphi^*(p) \geq \varphi(p)$. If $\varphi(p) > 0$, the same proof as for Remark 5 applies.

Corollary 18 *The mapping $\varphi^* : \Delta \to \mathbb{R}$ is a regular fuzzy set.*

Proof:

We know that φ^* is a normal fuzzy set. Now, let us show that φ^* is fuzzy convex. In fact, we have it that φ^* is concave: taking $p_1, p_2 \in \Delta$ and $r \in [0,1]$, denote by $p^r = rp_1 + (1-r)p_2$. Hence for every $a \in B^+$ $E_{p^r}(a) = rE_{p_1}(a) + (1-r)E_{p_2}(a)$ and

$$\begin{aligned} \varphi^*(p^r) &= \inf_{a \in B^+} \frac{rE_{p_1}(a) + (1-r)E_{p_2}(a)}{I(a)} \\ &\geq r \inf_{a \in B^+} \frac{E_{p_1}(a)}{I(a)} + (1-r) \inf_{a \in B^+} \frac{E_{p_2}(a)}{I(a)} \\ &= r\varphi^*(p_1) + (1-r)\varphi^*(p_2). \end{aligned}$$

in particular, φ^* is quasiconcave.

Finally, let us show that φ^* is weakly upper semicontinuous. For each $a \in \{b \in B^+ : I(b) > 0\} := \{I > 0\}$, define*

$$\begin{aligned} \psi_a &: \Delta \to \mathbb{R} \\ p &\mapsto \psi_a(p) = E_p(a)/I(a). \end{aligned}$$

By the definition of weak topology we have it that ψ_a is weakly* upper semicontinuous for any $a \in \{I > 0\}$. Note that*

$$\varphi^*(p) = \inf_{a \in B^+} \frac{E_p(a)}{I(a)} = \inf_{\{I>0\}} \frac{E_p(a)}{I(a)}$$

Since the sets $\{p \in \Delta : \psi_a(p) \geq \alpha\}$ are weakly closed for any $a \in \{I > 0\}$ and for any $\alpha \in [0,1]$, we obtain that*

$$\{p \in \Delta : \varphi^*(p) \geq \alpha\} = \bigcap_{\{I>0\}} \{p \in \Delta : \psi_a(p) \geq \alpha\}$$

is weakly closed as desired (in fact, we have an infimun over continuous functions and* *is well known that it is upper semicontinuous).*

Using the maximal confidence function φ^* we can write:

Corollary 19 *Under the conditions on the Main Theorem for each u, there is a (unique) maximal confidence function* $\varphi^* : \Delta \to [0,1]$ *such that:*

$$J(f) = \min_{p\in\Delta} \frac{1}{\varphi^*(p)} \int u(f)dp$$

given by

$$\varphi^*(p) = \inf_{f\in\mathcal{F}} \left(\frac{\int u(f)dp}{u(c_f)} \right)$$

This corollary completes the proof of the Main Theorem.

Function φ^* should be viewed as the *upper confidence function*, specifying maximal confidence among priors that the decision maker may face in order to be consistent with our main representation.

In contrast with the MEU model, a decision maker that presents a behavior consistent with our set of axioms, in general, does not evaluate the acts by their minimal expected utility on the set of priors that matters, i.e., on the set $L_{\alpha_0}\varphi^*$. Hence, we obtain a non-extremely pessimistic behavior over $L_{\alpha_0}\varphi^*$ and this is true because in general we do not have a decision maker with a uniform confidence on the priors. For instance, we may observe two decision makers who share the same sets of priors but one is more cautious than the other. This can occur when there are differents personal confidence functions[8].

Another preference representation that has the Gilboa and Schmeidler model as a particular case and uses a mapping on the set of probabilities is the variational preferences[9] proposed by Maccheroni, Marinacci and Rustichini (2004): variational preferences

[8]Another model of decision making under ambiguity that has a similar *non-extremely pessimistic behavior interpretation* is the smooth model proposed by Klibanoff, Marinacci and Mukerji (2005). In their representation the doubt about the *right* probability is given by a subjective probability over Δ. In our case, we recall that this doubt is given by the confidence function (a regular* fuzzy set).

[9]The multiplier preferences of Hansen and Sargent (2001) and the mean-variance preference of Markovitz (1952) and Tobin (1958) are also special cases. The worst consequence is not required in the axiomatization of the variational preferences.

have the following representation

$$V(f) = \min_{p\in\Delta} \left(\int u(f)dp + c^*(p) \right)$$

where $c^* : \Delta \to \mathbb{R} \cup \{+\infty\}$ is a convex, weakly* lower semicontinuous function, such that $\{p \in \Delta : c^*(p) = 0\}$ is nonempty.

We note that function c^* generalizes the indicator functions from Convex Analysis: $\delta_P : \Delta \to \mathbb{R} \cup \{+\infty\}$ where $\delta_P(p) = 0$ if $p \in P$ and $\delta_P(p) = +\infty$ if $p \notin P$. In our representation, we saw that φ^* generalizes the characteristic function from Measure Theory. Function c^* can be interpreted as the index of ambiguity aversion and has a nice expression for the minimal index of ambiguity aversion:

$$c^*(p) = \sup_{f\in\mathcal{F}} \left(u(c_f) - \int u(f)dp \right).$$

Moreover, if $u(X)$ is unbounded then the index of ambiguity aversion c^* is unique.

We note that variational preferences are not positively homogeneous. Hence, our preference and the variational preference captures different designs of behavior under uncertainty. It follows from the different axiomatic foundations.

An alternative interpretation of variational preference, where the function $c^* : \Delta \to \mathbb{R} \cup \{+\infty\}$ should be viewed as a cost function of a malevolent Nature[10], can be translated for our preference: function $\varphi^* : \Delta \to [0, 1]$ should be viewed as a *plausibility function* of a malevolent nature. Each number $\varphi^*(p)$ captures the decision maker's perception of the relative plausibility of the different models p that Nature can choose in order to make the decision maker the most possible worst off; if $\varphi^*(p_1) \geq \varphi^*(p_2)$ then model p_1 is weakly

[10] In this interpretation, decision makers ranking payoff profiles according to maxmin rules can be viewed as believing they are playing a zero-sum game againt Nature. For a interesting discussion see pages 5-7 of Maccheroni et. al. (2004).

more plausible than model p_2. Hence, the decision maker's play follows the rule

$$\underset{f\in\mathcal{F}}{max}\,\underset{p\in\Delta}{min}\left\{\frac{1}{\varphi^*(p)}\int u(f)dp\right\}$$

where the strategies are pairs $(f,p)\in\mathcal{F}\times\Delta$, and $\mathcal{F}$ is the decision maker's set of pure strategies and Δ is the Nature's set of pure strategies.

2.4 Ambiguity Attitudes

We now analyse ambiguity attitude features for the class of preferences that satisfies the list of axioms $A.1$ to $A.7$. By our mean theorem, each preference in this class is represented by a utility functional J on $\mathcal{F}$, such that:

$$J(f) = \underset{p\in\Delta}{min}\,\frac{1}{\varphi^*(p)}\int u(f)dp$$

where $u : X \rightarrow \mathbb{R}_+$ is an affine function such that $u(x_*) = 0$ and φ^* is a confidence funtion, i.e., it belongs to $F_{\mathcal{R}^*}(ba^1_+(S,\Sigma))$. As the maximal confidence function, φ^* has the expression:

$$\varphi^*(p) = \underset{f\in\mathcal{F}}{inf}\left(\frac{\int u(f)dp}{u(c_f)}\right)$$

Ghirardato and Marinacci (2002) proposed a notion of absolute ambiguity aversion by building on a notion of comparative ambiguity. The comparative ambiguity attitude says: Given two preferences $\succsim_1$ and $\succsim_2$, the preference relation $\succsim_1$ is *more ambiguity averse than* $\succsim_2$ if for all $f\in\mathcal{F}$ and

$$x\in X,\ f\succsim_1 x \Rightarrow f\succsim_2 x.$$

The absolute notion of ambiguity aversion defined by Ghirardato and Marinacci (2002) considers SEU preferences as benchmarks for ambiguity neutrality: We say that a preference relation $\succsim$ is *ambiguity averse* if it is more ambiguity averse than some SEU preference.

Now, if we consider the behavioral assumptions present in our main theorem, which includes the preference for randomization of Schmeidler (1989) described by the uncertainty aversion axiom, it is not surprising that we obtain in a precise sense:

Proposition 20 *The preference $\succsim$ as in our main theorem is ambiguity averse.*

Proof:

We have it that $J(f) = min_{p\in\Delta}\{\varphi^*(p)^{-1}E_p(u(f))\}$ with $\varphi^* \in \mathrm{F}_{\mathcal{R}^*}(ba_+^1(S,\Sigma))$, in particular the normality of φ^* says that we can take some $p' \in \Delta$ such that $\varphi^*(p') = 1$. Now, we define $V(f) = E_{p'}(u(f))$ and obtain a SEU preference $\succsim_V$. Furthermore, inequality $V(f) \geq J(f)$ entails that $f \succsim_J x \Rightarrow f \succsim_V x$.

For any pair of preferences, we have an affine utility index on consequences that assigns null utility for the worst consequence x_*. Hence, given two preferences we can suppose the same index u on the set of consequences.

Proposition 21 *Consider φ_1^*, $\varphi_2^* \in \mathrm{F}_{\mathcal{R}^*}(ba_+^1(S,\Sigma))$ and let be $u : X \to \mathbb{R}_+$ an affine function such that $u(x_*) = 0$ where the pair (u, φ_i^*) represents $\succsim_i$, $i = 1, 2$. Are equivalent:*

(1) $\succsim_1$ is more ambiguity averse than $\succsim_2$;

(2) $\varphi_1^* \geq \varphi_2^*$.

Proof:

(1) $\Rightarrow$ (2)*: For any $f \in \mathcal{F}$, if $f \sim_1 x$ then $f \succsim_2 x$, moreover:*

$$J_1(u(f)) = u(x) \leq J_2(u(f))$$

i.e., $J_1 \leq J_2$. Hence,

$$\varphi_1^*(p) = \inf_{f\in\mathcal{F}} \frac{E_p(u(f))}{J_1(u(f))} \geq \inf_{f\in\mathcal{F}} \frac{E_p(u(f))}{J_2(u(f))} = \varphi_2^*(p)$$

as desired.

$(2) \Rightarrow (1)$: *For any* $f \in \mathcal{F}$ *and* $x \in X$, *if* $f \succsim_1 x$ *then*

$$\underset{p\in\Delta}{inf}\left(\frac{\int u(f)dp}{\varphi_1^*(p)}\right) \geq u(x),$$

since $\varphi_1^* \geq \varphi_2^*$ *implies that*

$$\underset{p\in\Delta}{inf}\left(\frac{\int u(f)dp}{\varphi_2^*(p)}\right) \geq \underset{p\in\Delta}{inf}\left(\frac{\int u(f)dp}{\varphi_1^*(p)}\right)$$

we conclude that $f \succsim_2 x$.

This proposition says that more ambiguity averse preference relations are characterized, up to index normalization, by greater functions φ^*. If $\varphi_1^* \geq \varphi_2^*$, we may think that decision maker 1 has a greater doubt about the likehood of the events than decision maker 2.

Example 22 *The maximal ambiguity aversion behavior is characterized by* $\varphi^*(p) = 1$ *for any* $p \in \Delta$. *In this case*

$$J(f) = min_{p\in\Delta}\left(E_p(u(f))\right) = \underset{s\in S}{min}\, u(f(s))$$

is an expression that reflects extreme ambiguity aversion.

Example 23 *The minimal ambiguity aversion corresponds here to ambiguity neutrality, as we know that our preferences are ambiguity averse. The least ambiguity averse functions* φ^* *are associated with SEU preferences. In this case we obtain that*

$$\varphi^*(p) = \underset{\{E\in\Sigma:q(E)>0\}}{inf} \frac{p(E)}{q(E)}$$

where $q \in \Delta$ *is the subjective probability of the decision maker. For details see Propositions 26 and 27. Note that* $\varphi^*(p) = 0$ *if and only if there exists an event* E_0 *such that* $q(E_0) > p(E_0) = 0$, *that is,* p *and* q *disagree about some miracle.*

Example 24 *Consider*

$$J_v(f) = \int u(f)dv,$$

where $v : \Sigma \to [0,1]$ *is a capacity (see Section 2.5.2) such that there exist* $\lambda \in (0,1)$ *and* $q \in \Delta$

$$\begin{aligned} v(E) &= \lambda q(E), \textit{ if } \Sigma \ni E \neq S, \\ v(S) &= 1. \end{aligned}$$

The functional $J_{\lambda q}$ is the well known ε-contaminated model. Denote by φ_q^* the maximal confidence function of a SEU preference with subjective probability q, and $\varphi_{\lambda q}^*$ the maximal confidence function of ε-contamined model for $\lambda = 1 - \varepsilon$. We then obtain that,

$$\varphi_{\lambda q}^*(p) = \underset{\{E \in \Sigma : q(E) > 0\}}{inf} \left\{ \frac{p(E)}{\lambda q(E)} \wedge 1 \right\} = \frac{\varphi_q^*(p)}{\lambda} \wedge 1.$$

Hence, $\varphi_{\lambda q}^*(p) = 1$ iff $\varphi_q^*(p) \geq \lambda$. That is, the set of full confidence for a preference in accordance with $J_{\lambda q}$ is the same as the level set $L\lambda\varphi_q^*$, which is the set of priors with a confidence level greater than λ for the decision maker that presents subjective probability q. On the other hand, we can obtain that

$$J_v(f) = \underset{p \in \lambda q + (1-\lambda)\Delta}{min} \int u(f)dp,$$

and, by Section 2.5.1, $\varphi_v^*(p) = 1$ iff $p = \lambda q + (1-\lambda)p'$ for some $p' \in \Delta$.

2.5 Special Cases

Suitably choosing the confidence function, we can obtain well known cases in the literature:

2.5.1 Maxmin Expected Utility

Gilboa and Schmeidler (1989) characterized preference relations over acts, which has a numerical representation by a functional I on $B(S, \Sigma)$ that satisfies the formula

$$I(a) = \min_{p \in C} E_p(a)$$

where $C \subset \Delta$ is non empty, convex and $\sigma(ba, B)$-compact set.

Here we focus on non negative functions. Recall that the confidence function $\varphi^* : \Delta \to \mathbb{R}$ is given by

$$p \mapsto \varphi^*(p) = \inf_{a \in B^+} \frac{E_p(a)}{I(a)}$$

By our main result, φ^* is the maximal confidence function that satisfies the Gilboa and Schmeidler representation.

We note that for all $p \in C$, $I(a) \leq E_p(a)$ for any $a \in B^+$, since $I(\mathbf{1}_S) = p(S) = 1$ we obtain that $\varphi^*(p) = 1$, $\forall p \in C$.

If $p \notin C$ by a separation theorem for locally convex linear topological space (Dunford and Schwartz (1988), page 418) there exists $a_0 \in B^+$ such that

$$E_p(a_0) < \min\{E_q(a_0) : q \in C\} = I(a_0)$$

therefore

$$\varphi^*(p) = \inf_{a \in B^+} \frac{E_p(a)}{I(a)} \leq \frac{E_p(a_0)}{I(a_0)} < 1$$

and we conclude that $\varphi^*(p) = 1$ if and only if $p \in C$. However, it is not true that if $p \notin C$ then $\varphi^*(p) = 0$.

Example 25 *Taking the Gilboa and Schmeidler's functional with* $S = \{s_1, s_2\}$ *and* $C =$

$\{(\lambda, 1-\lambda) : \lambda \in [0.4, 0.6]\}$ *we obtain that:*

$$\varphi^*(\lambda) = \begin{cases} 1, \text{ if } \lambda \in [0.4, 0.6] \\ \lambda/0.4, \text{ if } \lambda \in [0, 0.4) \\ (1-\lambda)/0.4, \text{ if } \lambda \in (0.6, 1] \end{cases}$$

that is $\varphi^* \neq \mathbf{1}_C$.

It is worth noting that the confidence function decreases while the probability moves away from the full confidence set of priors C and, in a sufficiently fast way, in order to keep our decision maker a maxmin expected utility agent with respect to C. We know that φ^* is maximal by Remark 6. Consider, for example, a distortion φ_r^* of φ^* given by

$$\varphi_r^*(\lambda) = \begin{cases} 1, \text{ if } \lambda \in [0.4, 0.6] \\ (1-\lambda)/0.4, \text{ if } \lambda \in (0.6, 1] \\ \left(\frac{0.5(\lambda-0.4)}{0.2+r}\right) + 1, \text{ if } \lambda \in [0.2-r, 0.4) \\ \left(\frac{0.5}{0.2-r}\right)\lambda, \text{ if } \lambda \in [0, 0.2-r) \end{cases}$$

where $r \in (0, 0.2)$ *and note that* $\lim_{r \searrow 0} \varphi_r^*(\lambda) = \varphi^*(\lambda)$ *for any* $\lambda \in [0,1]$. *Define for any* $a \in \mathbb{R}_+^2$

$$I^*(a) = \min_{\lambda \in [0,1]} \left(\frac{\lambda a_1 + (1-\lambda) a_2}{\varphi_r^*(\lambda)} \right),$$

in this case we obtain that $I^*((1,0)) = 0.4 - 2r < 0.4 = I((1,0))$.

2.5.2 Choquet Expected Utility

Let a functional $I : B^+(S, \Sigma) \to \mathbb{R}$ be defined by

$$I(a) = \int a dv = \int_0^{+\infty} v(\{a \geq \lambda\}) \, d\lambda$$

where the set-function $v : \Sigma \to [0,1]$ is a capacity, i.e:

(i) $v(\emptyset) = 0$, $v(S) = 1$

(ii) $E, F \in \Sigma$ such that $E \subset F \Rightarrow v(E) \leq v(F)$

Moreover, we assume that v is a convex capacity, i.e:

(ii) For all events $E, F \in \Sigma : v(E \cup F) + v(E \cap F) \geq v(E) + v(F)$.

When v is convex, the well known result of Schmeidler (1986) says that the core of v

$$\mathcal{C}(v) = \{p \in \Delta : p(E) \geq v(E),\ \forall E \in \Sigma\}$$

is nonempty (convex and weakly* compact). Moreover,

$$\int a dv = \min_{p \in \mathcal{C}(v)} E_p(a).$$

Define the application

$$\begin{array}{ccc} \widetilde{\varphi} & : & \Delta \to \mathbb{R} \\ p & \mapsto & \widetilde{\varphi}(p) = \inf\limits_{E \in \Sigma} \dfrac{p(E)}{v(E)} \end{array}$$

Proposition 26 *The mapping $\widetilde{\varphi}$ is a normal fuzzy set and, for every $a \in B(S, \Sigma)$,*

$$(1) \qquad I(a) = \inf_{p \in L_{\alpha_0}\varphi} \frac{E_p(a)}{\widetilde{\varphi}(p)} := I'(a)$$

for any level of minimal confidence $\alpha_0 \in (0, 1]$.

Proof:

Let us first prove that $\widetilde{\varphi}$ is a normal fuzzy set. Take $p \in \Delta$, clearly $\widetilde{\varphi}(p) \in \overline{\mathbb{R}}_+$, and since $p(S) = v(S) = 1$ it turns out that $\widetilde{\varphi}(p) \in [0, 1]$.

Finally $\widetilde{\varphi}$ is normal: since v is convex we saw that $\mathcal{C}(v)$ is nonempty. Note that $\widetilde{\varphi}(p) = 1$ if and only if $p \in \mathcal{C}(v)$.

Let us first prove equality (1), when a belongs to $B_0^+(S, \Sigma)$, the set of real-valued

Σ*-measurable, non-negative simple functions. Then*

$$a = \sum_{i=1}^{m} x_i \mathbf{1}_{E_i}$$

where $\{E_i\}_{i=1}^m \subset \Sigma$ *is a partition of* S *where* $x_1 > x_2 > ... > x_m \geq 0 = x_{m+1}$.

First, let us prove that

$$I'(a) = \underset{p \in L_{\alpha_0}\varphi}{inf} \frac{E_p(a)}{\widetilde{\varphi}(p)} \geq I(a).$$

It is enough to show that for any given $p \in L_{\alpha_0}\widetilde{\varphi}$ *we have:*

$$(2) \quad \frac{E_p(a)}{\widetilde{\varphi}(p)} \geq \int adv.$$

Set $d_1(p) = E_p(a) - \widetilde{\varphi}(p) \int adv$*; hence* (2) *is equivalent to* $d_1(p) \geq 0$*. We note that*

$$d_1(p) = \sum_{i=1}^{m} (x_i - x_{i+1}) \left[p\left(\bigcup_{j=1}^{i} E_i \right) - \widetilde{\varphi}(p) v \left(\bigcup_{j=1}^{i} E_i \right) \right],$$

since for all $i \in \{1, ..., m\}$

$$\widetilde{\varphi}(p) \leq \frac{p\left(\bigcup_{j=1}^{i} E_i \right)}{v\left(\bigcup_{j=1}^{i} E_i \right)}$$

and $(x_i - x_{i+1}) \geq 0$*, we obtain* $d_1(p) \geq 0$.

It remains to show that $I'(a) \geq I(a)$ *: taking* $p_0 \in \mathcal{C}(v)$ *such that*

$$\int adv = \min_{p \in \mathcal{C}(v)} E_p(a) = E_{p_0}(a).$$

Since $\widetilde{\varphi}(p_0) = 1$*, we obtain*

$$I^*(a) \leq E_{p_0}(a) = I(a).$$

Therefore, $I'(a) = I(a)$ for all $a \in B_0^+$.

Let us take now a belongs to B^+: We know that there exists $a_n \in B_0^+$, $a_n \to a$ uniformly. From the previous case, $I'(a_n) = I(a_n)$ for all $n \geq 1$. From Lemma 13 $I'(a_n) \to I'(a)$, but $I(a_n) \to I(a)$. Hence, $I'(a) = I(a)$.

In fact, it is true that

Proposition 27 *For any probability $p \in \Delta$ we have that $\widetilde{\varphi}(p) = \varphi^*(p)$.*

Proof:

Note first that $E \in \Sigma$ implies that $\mathbf{1}_E \in B$, then $0 \leq \varphi^*(p) \leq \widetilde{\varphi}(p)$, so the proof has only to be done if $\widetilde{\varphi}(p) > 0$.

In the previous proposition, when restricting to B_0^+, we obtain that

$$\frac{E_p(a)}{\widetilde{\varphi}(p)} \geq \int adv, \text{ for every } a \in B_0^+$$

so, by continuity,

$$\frac{E_p(a)}{\widetilde{\varphi}(p)} \geq \int adv, \text{ for every } a \in B^+.$$

Hence, $E_p(a) \geq \widetilde{\varphi}(p)I(a)$ for any $a \in B^+$. If $I(a) = 0$ either $E_p(a) = 0$ and $E_p(a)/I(a) = 1 \geq \widetilde{\varphi}(p)$, or $E_p(a) > 0$ and $E_p(a)/I(a) = +\infty \geq \widetilde{\varphi}(p)$. Finally, if $I(a) > 0$, clearly $E_p(a)/I(a) \geq \widetilde{\varphi}(p)$ and therefore $\varphi^*(p) \geq \widetilde{\varphi}(p)$.

Remark 7 *An interesting result is that the minimal confidence function, in accordance with our main representation, related with the CEU model is given by*

$$\varphi_*(p) = \inf_{a \in B^+} \frac{e^{\int adp}}{e^{\int adv}} = \inf_{a \in B^+} e^{\left(\int adp - \int adv\right)}, \text{ for any } p \in \Delta.$$

If $p \in core(v)$, since $\int adp - \int adv \geq 0$ for any $a \in B^+$, we obtain that $\varphi_*(p) = e^0 = 1$. Now, if $q \notin core(v)$, then there exists some event E_0 such that $v(E_0) > q(E_0)$. Consider the sequence of simple functions $a_n = n 1_{E_0}$:

$$\varphi_*(q) \leq \inf_{n \in \mathbb{N}} e^{n(q(E_0) - v(E_0))} = 0,$$

that is,

$$\varphi_* \equiv 1_{core(v)}.$$

Moreover, we note that $\varphi_*(p) = e^{-c^*(p)}$*, where* c^* *is the ambiguity index of Variational Preference related to the MEU model (see, for instance, Proposition 12 of Maccheroni et al. (2005))*[11].

Removing the restriction of non-negativity, one obtains in the general case the following result:

Proposition 28 *If we define the confidence function* φ *for any* $p \in \Delta$ *by:*

$$\varphi(p) = \inf_{E\in\Sigma} \left\{ \frac{p(E)}{v(E)} \wedge \frac{1-v(E)}{1-p(E)} \right\}$$

then, for every function $a \in B(S, \Sigma)$*, we have that*

$$(2) \qquad \int adv = \inf_{p \in L_{a_0}\varphi} \left\{ \frac{E_p(a^+)}{\varphi(p)} + \varphi(p) E_p(a^-) \right\},$$

for any level $\alpha_0 \in (0,1]$*; where* $a^+ = a \vee 0$*,* $a^- = a \wedge 0$*, and the Choquet integral of a with respect to* v *is given by*

$$\int adv = \int_{-\infty}^{0} [v(\{a \geq \lambda\}) - 1] d\lambda + \int_{0}^{+\infty} v(\{a \geq \lambda\})\, d\lambda.$$

Proof:

The proof is similar to the proof of Proposition 26: Take $p \in \Delta$*, clearly* $\varphi(p) \in [0,1]$*.*

In order to prove that φ *is normal, note that since* v *is convex then* $\mathcal{C}(v)$ *is non empty. Moreover,* $p \in \mathcal{C}(v)$ *if and only if* $p(E) \geq v(E)$*, or equivalently,* $1 - v(E) \geq 1 - p(E)$ *for every* $E \in \Sigma$*, and then* $\varphi(p)$*.*

Let us prove now equality (2), where a belongs to $B_0(S,\Sigma)$ *the set of real-valued* Σ*-*

[11] *Using separation argument, this result holds for the MEU model.*

measurable simple functions. Then

$$a = \sum_{i=1}^{m} x_i \mathbf{1}_{E_i} + \sum_{k=1}^{n} y_i \mathbf{1}_{A_k} = a^+ + a^-$$

where $\{E_i\}_{i=1}^m$, $\{A_k\}_{k=1}^n \subset \Sigma$ *are partitions of* S *where* $x_1 > x_2 > ... > x_m \geq 0 = x_{m+1}$ *and* $0 \geq y_1 > y_2 > ... > y_n$ *with* $y_{n+1} = 0$.

First let us prove that

$$I'(a) := \inf_{p \in L_{\alpha_0}\varphi} \left(\frac{E_p(a^+)}{\varphi(p)} + \varphi(p) E_p(a^-) \right) \geq I(a) = I(a^+) + I(a^-)$$

It is enough to show that for a given $p \in L_{\alpha_0}\widetilde{\varphi}$ *we have:*

$$(2) \quad \frac{E_p(a^+)}{\widetilde{\varphi}(p)} \geq \int a^+ dv$$

$$(3) \varphi(p) E_p(a^-) \geq \int a^- dv$$

Set $d_1(p) = E_p(a) - \varphi(p) \int a dv$; *hence* (2) *is equivalent to* $d_1(p) \geq 0$, *which we proved in Proposition 26.*

Setting now $d_2(p) = \varphi(p) E_p(a^-) - \int a^- dv$, (3) *is equivalent to* $d_2(p) \geq 0$.

We note that

$$\varphi(p) \leq \frac{1 - v(A)}{1 - p(A)}, \forall A \in \Sigma$$

implies $\varphi(p)p(A) - v(A) \geq \varphi(p) - 1$ *for every event* $A \in \Sigma$. *Now, since* $y_k - y_{k+1} \geq 0$ *for every* $k < n$:

$$\begin{aligned} d_2(p) &= (y_1 - y_2) \left[\varphi(p)p(A_1) - v(A_1) \right] \\ &\quad + (y_2 - y_3) \left[\varphi(p)p(A_1 \cup A_2) - v(A_1 \cup A_2) \right] \\ &\quad + ... + \\ &\quad + (y_{n-1} - y_n) \left[\varphi(p)p \left(\bigcup_{k=1}^{n-1} A_k \right) - v \left(\bigcup_{k=1}^{n-1} A_k \right) \right] \end{aligned}$$

$$
\begin{aligned}
& +y_n\,[\varphi(p)-1] \\
\geq\ & (y_1-y_2)\,[\varphi(p)-1] \\
& +(y_2-y_3)\,[\varphi(p)-1] \\
& +\ldots+ \\
& +(y_{n-1}-y_n)\,[\varphi(p)-1] \\
& +y_n\,[\varphi(p)-1] \\
=\ & (y_1-y_n)(\varphi(p)-1)\geq 0
\end{aligned}
$$

It remains to show that $I'(a)\geq I(a)$: taking $p_0\in\mathcal{C}(v)$ such that

$$\int a dv = \min_{p\in\mathcal{C}(v)} E_p(a) = E_{p_0}(a).$$

Since $\varphi(p_0)=1$, we obtain

$$I'(a)\leq E_{p_0}(a^+)+E_{p_0}(a^-)=I(a).$$

Therefore, $I'(a)=I(a)$ for all $a\in B_0(S,\Sigma)$.

Let us take now a belonging to $B(S,\Sigma)$: We know that there exists $a_n\in B_0(S,\Sigma)$, $a_n\to a$ uniformly. From the previous case $I'(a_n)=I(a_n)$ for all $n\geq 1$. From Lemma 13, $I'(a_n)\to I'(a)$, but $I(a_n)\to I(a)$. Hence, $I'(a)=I(a)$.

2.6 Concluding Remarks

1. Proposition 28 suggests a more general functional that defines a preference without the bounded below assumption: functional J on $\mathcal{F}$ is determined by a confidence function $\varphi:\Delta\to[0,1]$, a non constant affine function $u:X\to\mathbb{R}$, and a minimal

level of confidence $\alpha_0 \in (0,1]$, where for any act $f \in \mathcal{F}$:

$$J(f) = \underset{p \in L_{\alpha_0}\varphi}{min} \int_S u(f)\varphi(p)^{-sgn\{u(f)\}} dp,$$

which is the subject of future research.

2. An important property for a functional $I : B^+(S,\Sigma) \to \mathbb{R}$ is to be *continuous at certainty*. This property says that if a sequence of events $\{E_n\}_{n\geq 1} \subset \Sigma$ is such that $E_n \nearrow S$ then $I(1_{E_n}) \nearrow 1$. Obviously, $E_n \nearrow S \nRightarrow \mathbf{1}_{E_n} \nearrow \mathbf{1}_S$ uniformly, *e.g.*, $S = \mathbb{R}$ and $E_n = [-n, n]$, $n \in \mathbb{N}$.

   Let $\Sigma \ni E_n \nearrow S$, consider the functional I obtained in Lemma 16 and suppose that it satisfies the continuity at certainty. Consider an arbitrary prior $p \in L\alpha\varphi^*$ for some $\alpha \in (0,1]$, by the formula of φ^* we obtain that,

$$\frac{p\left(E_n\right)}{I(1_{E_n})} \geq \varphi^*(p) \geq \alpha,$$

   and,

$$1 \geq \lim_n p\left(E_n\right) \geq \lim_n \alpha I(1_{E_n}) = \alpha,$$

   that is,

$$\lim_n p\left(E_n\right) \in [\alpha, 1].$$

   Hence, when we consider $\alpha = 1$ the prior p is a countably additive probability. Since φ^* is weakly* upper semicontinuous, $L_1\varphi^*$ is weakly* closed subset of Δ, it follows that $L\alpha\varphi^*$ is weakly compact because a subset of $ca^1_+(S,\Sigma) := \{q \in ba^1_+(S,\Sigma) : q$ is countable additive$\}$ is weakly* compact iff it is weakly compact. Moreover, there exist a probability $q \in L_1\varphi^*$ such that for any $p \in L_1\varphi^*$, $p \ll q$ (p is absolutely continuous with respect to q). In the CEU case (Section 2.5.2) this fact entails the well konwn result of Schmeidler (1972): the core of a convex capacity $v : \Sigma \to [0,1]$ that satisfies the continuity at certainty is a weak compact subset of $ca^1_+(S,\Sigma)$.

Moreover, for multiple prior model (Section 2.5.1) we obtain the same result for the set of priors C, which agree with results from Epstein and Wang (1995) and Chateauneuf, Maccheroni, Marinacci and Tallon (2005).

Chapter 3

Sign-Dependent Confidence Functions

3.1 Introduction

The subjective expected utility (SEU) theory of Savage (1954) or Anscombe and Aumann (1963) is the most well known model of preference under uncertainty in economic theory. The main feature of this model is that the decision maker's beliefs are represented by a probability measure over states of nature. However, a substancial body of evidence showed that decision makers systematically violated SEU's basic tenet: decision makers's behavior is not consistent with a probability measure when the likelihoods of alternatives states are not objective. A gamble with known payoffs over states with probabilities that are not well defined is termed *ambiguous*. Motivated by evidences, some alternatives models have been proposed in order to model choices under ambiguity. The most well known models that study the belief's attitudes about the likelihood of states include Choquet expected utility (CEU) of Schmeidler (1989) and maxmin expected utility of Gilboa and Schmeidler (1989).

Other important issue in decision making is the attitude about payoffs (gains and losses). A model where we can identify the attitudes towards likelihood and payoffs is given by Tversky and Kahneman's (1992) cumulative prospect theory (CPT). They called an act $a : S \to X$ an *uncertain prospect* (henceforth *prospect*) and argued that what matters for utility are gains and losses, not final assets. To represent a preference relation $\succsim$ amongst prospect, CPT generalized CEU model in a setting where we have a finite set of states S, and the set of consequences X are monetary outcomes (real numbers): the outcome 0 is interpreted as the *status quo*, and all positive number is a gain and all negative number is a loss. CPT makes use of two normalized and monotone set-functions (capacities) $\rho^+, \rho^- : 2^S \to [0,1]$, as well as a value function $v : X \to \mathbb{R}$, which is continous strictly increasing with $v(0) = 0$.

For a prospect a, the gains of a is given by $a^+ = a \vee 0$, and the losses of a is given by $a^- = a \wedge 0$. A CPT's preference if represented by a functional on $\mathbb{R}^S$, such that

$$V(f) = \int v(a^+)d\rho^+ + \int v(a^-)d\rho^-,$$

where the integrals are in the sense of Choquet. CPT model coincides with CEU model when the Choquet integral of a prospect is calculed with respect to a capacity ρ such that, for any $E \in 2^S$, $\rho(E) = \rho^+(E) = 1 - \rho^-(E^c)$. We note that the value function is unique up to positive affine transformation if and only if V satisfies the CEU model, that is, the functional V is not sign-dependent iff V is the Choquet integral with respect to ρ above.

Some recent experimental results demonstrated that attitudes about payoffs and beliefs about the likelihood of states exhibit interaction effects both behaviorally an neurally. Decision-making research on choice under uncertainty finds experimental evidences where the participants are ambiguity-seeking in neither gains and losses[1]. In the CEU model an ambiguity averse preference is characterized by a convex capacity. Following this fact and the experimental evidence we may suppose that both capacities ρ^+ and ρ^-

[1]For example, Smith, Dickhaut, McCabe and Pardo (2002) agree with this conclusion and their experimental results showed that the brain does not honor a classical assumption of economics: the separation of payoffs and beliefs. For risk setting, we recall that the decision-making research on choice behavior finds risk-avoiding in gains and risk-seeking in losses, *e.g.*, Kahneman and Tversky (1979).

are convex. But, in general they are different.

In a setting where we consider the case of Anscombe and Aumann's acts (prospects) under the assumption of a *referential consequence* $\overline{x} \in X$, our representation has as main component two mappings φ^+ and φ^- from the set $\Delta \equiv ba^1_+(S,\Sigma)$ of all finitely additive probabilities on the unit interval $[0,1]$, a minimal level of confidence $\alpha_0 \in (0,1]$, and a real-valued affine function u on X such that $u(\overline{x}) = 0$, such that

$$J(f) = \underset{p \in L_{\alpha_0}\varphi_1}{inf} \frac{1}{\varphi_1(p)} \int_S u(f)^+ dp + \underset{p \in L_{\alpha_0}\varphi_2}{inf} \varphi_2(p) \int_S u(f)^- dp$$

The functions φ^+ and φ^- belong to the class of *fuzzy set* on Δ (mappings from Δ to $[0,1]$) that presents the following properties: quasi-concavity, normality, and (weakly*) upper semicontinuity. A mapping in this class is called a *confidence function* and it models the ambiguity. Hence, the term *ambiguity* refers purely to the vague perception of the likelihood subjectively associated with an event by a decision maker. Moreover, since in general confidence functions for gains and losses are different, its captures the notion of dependence between payoffs (gains or losses) and beliefs.

We axiomatize the previous representation by showing how it rests on a simple set of axioms that generalizes the CEU axiomatization of Schmeidler(1989) under additional requirement of ambiguity aversion. In particular, we obtain the CPT model of Tversky and Kahneman (1992) with ambiguity aversion behavior in the Anscombe and Aumann's setting. Indeed, we impose only ambiguity aversion on gains and on losses for the general case proposed here.

3.2 Axioms

We take a binary relation $\succsim$ on $\mathcal{F}$ and assume that there exists a referential consequence $\overline{x} \in X$ with respect to $\succsim$: a consequence $y \in X$ such that $y \succ \overline{x}$ is a gain and a consequence $z \in X$ such that $\overline{x} \succ z$ is a loss. Hence the status-quo $\overline{x}$ describe if an consequence is perceived to be a gain or a loss. A *value function* is an affine mapping $u : X \to \mathbb{R}$ such that $u(\overline{x}) = 0$.

Given an act $f \in \mathcal{F}$ we define the gains and the losses of f respectively by:

$$\begin{aligned} f^{+} &: S \to X \\ s &\mapsto f^{+}(s) = \begin{cases} f(s), \text{ if } f(s) \succsim \overline{x} \\ \overline{x}, \text{ if } \overline{x} \succ f(s) \end{cases} \end{aligned}$$

and

$$\begin{aligned} f^{-} &: S \to X \\ s &\mapsto f^{-}(s) = \begin{cases} \overline{x}, \text{ if } f(s) \succsim \overline{x} \\ f(s), \text{ if } \overline{x} \succ f(s) \end{cases} \end{aligned}$$

(**Axiom 1**) Weak order non-degenerate. If $f, g, h \in \mathcal{F}$:

(complete) either $f \succsim g$ or $g \succsim f$

(transitivity) $f \succsim g$ and $g \succsim h$ imply $f \succsim h$

there exist $(f, g) \in \mathcal{F}^2$ such that $(f, g) \in \succ$

(**Axiom 2**) Continuity. For all $f, g, h \in \mathcal{F}$ the sets:

$\{\alpha \in [0, 1] : \alpha f + (1 - \alpha)g \succsim h\}$, $\{\alpha \in [0, 1] : h \succsim \alpha f + (1 - \alpha)g\}$ are closed.

(**Axiom 3**) Monotonicity. For all $f, g \in \mathcal{F}$:

$$\text{if } f(s) \succsim g(s) \text{ for all } s \in S \text{ then } f \succsim g.$$

(**Axiom 4**) Weak Uncertainty aversion. If $f, g \in \mathcal{F}$ and $\alpha \in (0, 1)$:

$$\begin{gathered} 4.1)\ f^{+} \sim g^{+} \Rightarrow \alpha f^{+} + (1 - \alpha)g^{+} \succsim f^{+} \\ 4.2)\ f^{-} \sim g^{-} \Rightarrow \alpha f^{-} + (1 - \alpha)g^{-} \succsim f^{-} \\ 4.3)\ f^{+} \sim x \text{ and } f^{-} \sim y \Rightarrow \tfrac{1}{2}f + \tfrac{1}{2}\overline{x} \sim \tfrac{1}{2}x + \tfrac{1}{2}y \\ 4.4) \text{ If } \tfrac{1}{2}f(s) + \tfrac{1}{2}g(s) \sim \overline{x} \text{ for any } s \in S \text{ then} \\ \overline{x} \succsim \tfrac{1}{2}(\tfrac{1}{2}f^{+} + \tfrac{1}{2}g^{+}) + \tfrac{1}{2}(\tfrac{1}{2}f^{-} + \tfrac{1}{2}g^{-}) \end{gathered}$$

(**Axiom 5**) Referential independence. For all $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$:

$$f \sim g \Rightarrow \alpha f + (1-\alpha)\overline{x} \sim \alpha g + (1-\alpha)\overline{x}.$$

(**Axiom 6**) Independence on X. For all $x, y, z \in X$:

$$x \sim y \Rightarrow \tfrac{1}{2}x + \tfrac{1}{2}z \sim \tfrac{1}{2}y + \tfrac{1}{2}z.$$

(**Axiom 7**) Bounded attraction for certainty. There exist $\delta \geq 1$ such that for all $f \in \mathcal{F}$ and $x, y \in X$:

$$\begin{aligned} &f^{+} \sim x \text{ and } y \succsim \overline{x} \Rightarrow \tfrac{1}{2}x + \tfrac{1}{2}y \succsim \tfrac{1}{2}f^{+} + \tfrac{1}{2}(\tfrac{1}{\delta}y + (1-\tfrac{1}{\delta})\overline{x}) \\ &f^{-} \sim x \text{ and } \overline{x} \succsim y \Rightarrow \tfrac{1}{2}x + \tfrac{1}{2}(\tfrac{1}{\delta}y + (1-\tfrac{1}{\delta})\overline{x}) \succsim \tfrac{1}{2}f^{-} + \tfrac{1}{2}y \end{aligned}$$

3.3 Main Theorem

We can now state our main theorem, which characterizes preferences satisfying axioms A.1-A7.

Let $\mathcal{C}_c(ba_+^1(S,\Sigma))$ denote the collection of all nonempty convex weakly* closed subsets of $ba_+^1(S,\Sigma)$. As an extension of $\mathcal{C}_c(ba_+^1(S,\Sigma))$ we define

Definition 29 *The set $\mathrm{F}_{\mathcal{R}^*}(ba_+^1(S,\Sigma))$ of regular* fuzzy sets consist of all mappins $\varphi : ba_+^1(S,\Sigma) \to [0,1]$ with the properties:*

(a) φ is normal;

$$\{p \in ba_+^1(S,\Sigma) : \varphi(p) = 1\} \neq \emptyset$$

(b) φ is weakly upper semicontinuous;*

$$\{p \in ba_+^1(S,\Sigma) : \varphi(p) \geq \alpha\} \text{ is weakly* closed for any } \alpha \in [0,1]$$

(c) φ is quasi-concave;

$$\varphi(\beta p_1 + (1-\beta)p_2) \geq min\{p_1, p_2\} \text{ for any } \beta \in [0,1].$$

Remark 8 *We note that the weak* support of φ, denoted by*

$$supp^*\varphi := \overline{\{p \in ba^1_+(S,\Sigma) : \varphi(p) > 0\}}^{\sigma(ba,B)}$$

is weakly compact by the Alaoglu's theorem (see Dunford and Schwartz (1988), page 424).*

Remark 9 *We can embedding $\mathcal{C}_c(ba^1_+(S,\Sigma))$ into $\mathrm{F}_{\mathcal{R}^*}(ba^1_+(S,\Sigma))$ by the natural mapping $P \mapsto \mathbf{1}_P$. We will use the notation for level sets*

$$L\alpha\varphi = \{p \in ba^1_+(S,\Sigma) : \varphi(p) \geq \alpha\} \text{ for any } \alpha \in (0,1].$$

Moreover, we note that $\varphi \in \mathrm{F}_{\mathcal{R}^*}(ba^1_+(S,\Sigma))$ if only if the correspondence

$$\alpha \mapsto L\alpha\varphi$$

take values only on $\mathcal{C}_c(ba^1_+(S,\Sigma))$. Because this previous result a quasi-concave fuzzy set φ is called fuzzy convex *(all level set $L\alpha\varphi$ is convex).*

The main theorem says that:

Theorem 30 *Let $\succsim$ be a relation on $\mathcal{F}$, the following conditions are equivalent:*

(i) $\succsim$ satisfies conditions A.1-A.7;

(ii) there exist a non-constant value function $u : X \to \mathbb{R}$, a $\alpha_0 \in (0,1]$ and two regular fuzzy sets $\varphi_1, \varphi_2 : ba^1_+(S,\Sigma) \to [0,1]$ such that, for all $f, g \in \mathcal{F}$*

$$f \succsim g \Leftrightarrow J(f) \geq J(g)$$

where

$$J(f) = \inf_{p \in L_{\alpha_0}\varphi_1} \frac{1}{\varphi_1(p)} \int_S u(f)^+ dp + \inf_{p \in L_{\alpha_0}\varphi_2} \varphi_2(p) \int_S u(f)^- dp$$

Futhermore, given u from (ii) then for all $\lambda > 0$ we can adopted the tranformation λu in our representation.

Lemma 31 *There exists an affine $u : X \to \mathbb{R}$ non-constant function such that for all $x, y \in X : x \succsim y$ iff $u(x) \geq u(y)$. Moreover, we can choose u such that $u(\overline{x}) = 0$.*

proof: By axioms 1,2 and 6 the premises of the the von Neumann-Morgenstern theorem are satisfied and there exist an afinne function $u : X \to \mathbb{R}$ such that for all $x, y \in X : x \succsim y$ iff $u(x) \geq u(y)$. Moreover, we can choose $u(\overline{x}) = 0$. Therefore, there exists $f, g \in \mathcal{F}$ s.t. $f \succ g$; given $x, y \in X$ such that $x \succsim f(s)$ and $g(s) \succsim y$ for all $s \in S$, then by monotonicity (axiom 3) we have that $x \succ y$, and u can not be constant. We note that we can suppose that $[-1, 1] \subset u(X)$.

Lemma 32 *Given a $u : X \to \mathbb{R}$ from lema above there exists a unique $J : \mathcal{F} \to \mathbb{R}$ such that*

(i) $f \succeq g$ iff $J(f) \geq J(g)$ for all $f, g \in F$.

(ii) If $f = x\mathbf{1}_S \in \mathcal{F}_c$ then $J(f) = u(x)$.

proof: On $\mathcal{F}_c$ the functional J is uniquely determined by (ii). Since for all $f \in \mathcal{F}$ there exists a $c_f \in \mathcal{F}_c$ such that $f \sim c_f$ we set $J(f) = u(c_f)$ and by construction J satisfies (i), hence it is also unique.

Lemma 33 *There exists a functional*

$$I : B_0(S, \Sigma) \to \mathbb{R}$$

where for all $f \in \mathcal{F}$, $I(uof) = J(f)$ such that:I satisfies the properties:

1) *I is weak-superadditive: for $a, b \in B_0(S, \Sigma)$*

$$1.1)\ I(a^+ + b^+) \geq I(a^+) + I(b^+);$$

$$1.2)\ I(a^- + b^-) \geq I(a^-) + I(b^-);$$

$$1.3)\ I(a) = I(a^+) + I(a^-);$$

$$1.4)\ 0 \geq I(a) + I(-a);$$

2) *I is positively homogeneous: for* $a, b \in B_0(S, \Sigma), \lambda \geq 0 : I(\lambda a) = \lambda I(a)$;

3) *I is monotonic: for* $a, b \in B_0(S, \Sigma) : a \geq b \Rightarrow I(a) \geq I(b)$;

4) *I is normalized:* $I(\mathbf{1}_S) = 1$;

5) *There exists a* $\delta \geq 1$ *such that for all* $a \in B_0(S, \Sigma)$:

$$5.1) I(a^+ + k\mathbf{1}_S) \leq I(a^+) + \delta k \text{ if } k \geq 0;$$
$$5.1) I(a^- + k\mathbf{1}_S) \leq I(a^-) + \delta^{-1} k \text{ if } k \leq 0.$$

proof: We begin with $B_0(S, \Sigma, u(X))$ *domain and extend to all* $B_0(S, \Sigma)$. *If* $f \in \mathcal{F}$ *then* $u(f) \in B_0(S, \Sigma, u(X))$. *Now, if* $a \in B_0(S, \Sigma, u(X))$ *we have that there exists* $\{E_i\}_{i=1}^n \subset \Sigma$ *a partition of* S *and* $\{x_i\}_{i=1}^n \subset X$ *such that*

$$a := \sum_{i=1}^{n} u(x_i)\mathbf{1}_{E_i}$$

hence, we can choose $f \in \mathcal{F}$ *such that* $f(s) = x_i$ *when* $s \in E_i$ *and we conclude that* $a = u(f)$.

From this, we can write $B_0(S, \Sigma, u(X)) = \{u(f) : f \in \mathcal{F}\}$; *therefore,* $u(f) = u(g) \Leftrightarrow u(f(s) = u(g(s)), \forall s \in S \Leftrightarrow f(s) \sim g(s), \forall s \in S$; *and by axiom 3 (monotonicity)* $f \sim g$, i.e., $u(f) = u(g) \Leftrightarrow J(f) = J(g)$.

Define $I(a) = J(f)$ *whenever* $a = u(f)$; *hence we have that* I *is well defined over* $B_0(S, \Sigma, u(X))$.

Now, if $a = u(f)$ *and* $b = u(f) \in B_0(S, \Sigma, u(X))$ *and* $a \geq b$ *then* $u(f(s)) \geq u(g(s))$ *for all* $s \in S$ *and by axiom 3 (monotonicity) we have that* $f \succsim g$, i.e., $J(f) \geq J(g)$ *and* $I(a) = I(u(f)) = J(f) \geq J(g) = I(u(g)) = I(b)$; *this prove that* I *is monotonic.*

Let be $k \in u(X)$ *then there exists a* $x \in X$ *such that* $k = u(x)$ *and* $I(k\mathbf{1}_S) = I(u(x)\mathbf{1}_S) = J(x) = u(x) = k$, i.e., I *is normalized. In particular, since* $1 \in u(X), I(\mathbf{1}_S) = 1$.

We now show that I *is homogeneous; assume* $a = \alpha b$ *where* $a, b \in B_0(S, \Sigma, u(X))$ *and* $0 < \alpha \leq 1$. *Let* $g \in \mathcal{F}$ *satisfy* $u(g) = b$ *and define* $f = \alpha g + (1-\alpha)\overline{x}$.*Hence* $u(f) = \alpha u(g) + (1-\alpha)u(\overline{x}) = \alpha b = a$, *so* $I(a) = J(f)$. *We have that* $J(c_g) = J(g) = I(b)$.

By axiom 5 (referential independence), $\alpha c_g + (1-\alpha)\overline{x} \sim \alpha g + (1-\alpha)\overline{x} = f$*, hence* $J(f) = J(\alpha c_g + (1-\alpha)\overline{x}) = \alpha J(c_g) + (1-\alpha)J(x) = \alpha J(c_g)$ *and we can write*

$$I(\alpha b) = I(a) = J(f) = \alpha J(c_g) = \alpha I(b).$$

Futhermore, this imply equality for $\alpha > 1 : a = \alpha b \Rightarrow b = \alpha^{-1}a \Rightarrow I(b) = \alpha^{-1}I(a) \Rightarrow I(a) = \alpha I(b)$.

Now, by homogeneity we can extend I *to all* $B_0(S,\Sigma)$*: For each* $b \in B_0(S,\Sigma)$ *we have that there exists a partition* $\{E_k\}_{k=1}^m$ *of* S *and real numbers* $\{\beta_k\}_{k=1}^m$ *such that* $b = \sum_{k=1}^m \beta_k \mathbf{1}_{E_k}$*. Hence is enough to define* $G(b) = \|b\|_\infty I(\widetilde{b})$*, where* $\widetilde{b} = \sum_{k=1}^m (\beta_k / \|b\|_\infty)\mathbf{1}_{E_k}$*. Clearly* G *extend* I*, is monotone, homogeneous and by abuse of notation we set* $G = I$*.*

Next we show that satisfies (v)*; let there be given* $a \in B_0(S,\Sigma)$ *,* $\xi_1 \geq 0$ *and* $\xi_2 \leq 0$*. By homogeneity we may assume without loss of generality that* $2a$*,* $2\xi_1\mathbf{1}_S$*,* $2\xi_2\mathbf{1}_S \in B_0(S,\Sigma,u(X))$*.*

Let Let $f \in \mathcal{F}$ *such that* $u(f) = 2a$*, hence* $u(f^+) = u(f)^+ = 2a^+$ *and* $u(f^-) = u(f)^- = 2a^-$*.*

Now we define $\beta_1 = I(2a^+) = 2I(a^+)$ *and taking* $y_1, z_1 \in X$ *satisfy* $u(y_1) = \beta_1$ *and* $u(z_1) = 2\xi_1 \geq 0 = u(\overline{x})$*. Then* $J(f^+) = I(u(f^+)) = 2I(a^+) = \beta_1 = I(\beta_1\mathbf{1}_S) = I(u(y_1)) = J(y_1)$*, i.e.,* $f^+ \sim y_1$*. By axiom 7 (bounded atrraction for certanty) there exists* $\delta \geq 1$ *such that*

$$\frac{1}{2}y_1 + \frac{1}{2}z_1 \succeq \frac{1}{2}f^+ + \frac{1}{2}(\frac{1}{\delta}z_1 + (1-\frac{1}{\delta})\overline{x})$$

hence

$$\frac{1}{2}J(y_1) + \frac{1}{2}J(z_1) \geq J(\frac{1}{2}f^+ + \frac{1}{2}(\frac{1}{\delta}z_1 + (1-\frac{1}{\delta})\overline{x}))$$

then

$$\frac{1}{2}I(u(y_1)) + \frac{1}{2}I(u(z_1)) \geq I(\frac{1}{2}u(f^+) + \frac{1}{2}u(\frac{1}{\delta}z_1 + (1-\frac{1}{\delta})\overline{x}))$$

$\therefore$

$$\frac{1}{2}I(\beta_1\mathbf{1}_S) + \frac{1}{2}I(2\xi_1\mathbf{1}_S) \geq I(\frac{1}{2}2a^+ + \frac{1}{2}(\frac{1}{\delta}u(z_1) + (1-\frac{1}{\delta})u(\overline{x})))$$

$\therefore$

$$I(a^+) + \xi_1 \mathbf{1}_S \geq I(a^+ + \frac{1}{\delta}\xi_1 \mathbf{1}_S)$$

taking $k = \delta^{-1}\xi_1$, *we obtain that:*

$$I(a^+) + \delta k \geq I(a^+ + k)$$

Now the other case: We define $\beta_2 = I(2a^-) = 2I(a^-)$. *Recall that we have that* $u(f^-) = u(f)^- = 2a^-$, *taking* $y_2, z_2 \in X$ *satisfy* $u(y_2) = \beta_2$ *and* $u(z_2) = 2\xi_2 < 0 = u(\overline{x})$, *then* $J(f^-) = I(u(f^-)) = 2I(a^-) = \beta_2 = I(\beta_2 \mathbf{1}_S) = I(u(y_2)) = J(y_2)$, i.e., $f^- \sim y_2$. *By axiom 7 there exists* $\delta \geq 1$ *such that*

$$\frac{1}{2}y_2 + \frac{1}{2}(\frac{1}{\delta}z_2 + (1 - \frac{1}{\delta})\overline{x}) \succeq \frac{1}{2}f^- + \frac{1}{2}z_2$$

hence

$$I(\frac{1}{2}u(y_2) + \frac{1}{2}(\frac{1}{\delta}u(z_2) + (1 - \frac{1}{\delta})u(\overline{x})) \geq I(\frac{1}{2}u(f^-) + \frac{1}{2}u(z_2))$$

$\therefore$

$$\frac{1}{2}I(u(y_2)) + \frac{1}{2}I(\frac{1}{\delta}u(z_2)) \geq I(a^- + \xi_2 \mathbf{1}_S)$$

$\therefore$

$$\frac{1}{2}I(\beta_2 \mathbf{1}_S) + \frac{1}{2}I(\frac{1}{\delta}2\xi_2 \mathbf{1}_S) \geq I(a^- + \xi_2 \mathbf{1}_S)$$

and finally:

$$I(a^-) + \frac{1}{\delta}\xi_2 \geq I(a^- + \xi_2 \mathbf{1}_S)$$

It is left to show that I *is weak-superadditive; let there be given* $a, b \in B_0(S, \Sigma)$ *and assume that* $a, b \in B_0(S, \Sigma, u(X))$. *First, we note that axiom 4, itens 4.1) and 4.2), imply that* $I \mid_{B^+(S,\Sigma)}$ *and* $I \mid_{B^-(S,\Sigma)}$*is quasi-concave. Now, since* I *is positively homogeneous follows that* $I \mid_{B^+(S,\Sigma)}$ *and* $I \mid_{B^-(S,\Sigma)}$ *are concave (see Berge (1965)), hence* $I(a^+ + b^+) \geq I(a^+) + I(b^+)$ *and* $I(a^- + b^-) \geq I(a^-) + I(b^-)$.

To show 1.3) let be w.l.g. $a \in B_0(S, \Sigma, u(X))$. *Let* $f \in \mathcal{F}$ *such that* $u(f) = a$. *Now* *taking $x, y \in X$ such that $f^+ \sim x$ and $f^- \sim y$ we obtain by axiom 4, item 4.3),*

$$J(\frac{1}{2}f + \frac{1}{2}\overline{x}) = J(\frac{1}{2}x + \frac{1}{2}y)$$

$\therefore$

$$I(\frac{1}{2}u(f) + \frac{1}{2}u(\overline{x})) = \frac{1}{2}I(u(x)) + \frac{1}{2}I(u(y))$$

$\therefore$

$$I(a) = I(u(f^+)) + I(u(f^-)) = I(a^+) + I(a^-)$$

Finally, we will show that $0 \geq I(a) + I(-a)$: Suppose w.l.g. we can write $a = u(f)$ and let be an act $g \in \mathcal{F}$ such that

$$\frac{1}{2}f(s) + \frac{1}{2}g(s) \sim \overline{x} \text{ for every } s \in S,$$

then $u(f(s)) + u(g(s)) = 0$ for any $s \in S$, that is, $-a = u(g)$. Now by axiom 4, item 4.4, we have that

$$J(\overline{x}) \geq J\left(\frac{1}{2}(\frac{1}{2}f^+ + \frac{1}{2}g^+) + \frac{1}{2}(\frac{1}{2}f^- + \frac{1}{2}g^-)\right)$$

$$0 \geq I\left(\frac{1}{2}(\frac{1}{2}u(f^+) + \frac{1}{2}u(g^+)) + \frac{1}{2}(\frac{1}{2}u(f^-) + \frac{1}{2}u(g^-))\right)$$

by 1.3) and homogeity

$$0 \geq \frac{1}{2}I(\frac{1}{2}u(f^+) + \frac{1}{2}u(g^+)) + \frac{1}{2}I(\frac{1}{2}u(f^-) + \frac{1}{2}u(g^-))$$

by 1.1) and 1.2) we obtain:

$$0 \geq \frac{1}{2}(\frac{1}{2}I(u(f^+)) + \frac{1}{2}I(u(g^+))) + \frac{1}{2}(\frac{1}{2}I(u(f^-)) + \frac{1}{2}I(u(g^-)))$$

and again by 1.3)

$$0 \geq \frac{1}{4}\left(I(u(f)) + I(u(g))\right)$$

since $a = u(f)$ and $-a = u(g)$

$$0 \geq I(a) + I(-a)$$

Finally, by the fundamental Lemma 37 in appendix, we obtain the desirade result. Taking the confidence functions φ_+ and φ_- we obtain the corollary:

Corollary 34 *Under the conditions on the Main Theorem, there is a two confidence functions $\varphi_+, \varphi_- : \Delta \to [0,1]$ such that:*

$$J(f) = \min_{p \in \Delta} \frac{1}{\varphi_+(p)} \int_S u(f^+(s))p(ds) + \min_{p \in \Delta} \varphi_-(p) \int_S u(f^-(s))p(ds)$$

where

$$\varphi_+(p) = \inf_{f \succsim \overline{x}} \left(\frac{\int u(f)dp}{u(c_f)} \right)$$

and

$$\varphi_-(p) = \inf_{\overline{x} \succsim f} \left(\frac{u(c_f)}{\int u(f)dp} \right)$$

3.3.1 Cumulative Prospect Theory with Ambiguity-avoiding in Gains and Losses

Let a functional $I : B(S, \Sigma) \to \mathbb{R}$ be defined by

$$\begin{aligned} I(a) &= \int a^+ dv^+ + \int a^- dv^- \\ &= \int_{-\infty}^{0} \left[v^-\left(\{a \geq \lambda\}\right) - 1 \right] d\lambda + \int_0^{+\infty} v^+\left(\{a \geq \lambda\}\right) d\lambda \end{aligned}$$

where the set-functions $v^+, v^- : \Sigma \to [0,1]$ are capacities.

Moreover we suppose that v^+ and v^- are convex capacity such that $\mathcal{C}(v^+) \cap \mathcal{C}(v^-) \neq \emptyset$.

Define the mappings:

$$\begin{aligned} \widetilde{\varphi}_+ &: \Delta \to \mathbb{R} \\ p &\mapsto \widetilde{\varphi}_+(p) = \inf_{E\in\Sigma} \frac{p(E)}{v^+(E)} \end{aligned}$$

$$\begin{aligned} \widetilde{\varphi}_- &: \Delta \to \mathbb{R} \\ p &\mapsto \widetilde{\varphi}_-(p) = \inf_{E\in\Sigma} \frac{1-v^-(E)}{1-p(E)} \end{aligned}$$

Proposition 35 *The mapping $\widetilde{\varphi}$ is a normal fuzzy set and for every $a \in B(S,\Sigma)$*

$$(1) \qquad I(a) = \inf_{p\in L_{\alpha_0}\widetilde{\varphi}_+} \left(\frac{E_p(a^+)}{\widetilde{\varphi}_+(p)} \right) + \inf_{p\in L_{\alpha_0}\widetilde{\varphi}_-} \left(\widetilde{\varphi}_-(p) E_p(a^-) \right)$$

for any level of minimal confidence $\alpha_0 \in (0,1]$.

Proof:

Similar to the proof of Proposition 28.

In fact, it is true that

Proposition 36 *For any probability $p \in \Delta$ we have that $\widetilde{\varphi}_+(p) = \varphi_+(p)$ and $\widetilde{\varphi}_-(p) = \varphi_-(p)$*

Proof:

Note that for $\widetilde{\varphi}_+$ the proof is the same as in Proposition 27.

We note that, for every event $E \in \Sigma$

$$I(-\mathbf{1}_E) = \int_{-\infty}^{0} \left[v^-\left(\{-\mathbf{1}_E \geq \lambda\}\right) - 1 \right] d\lambda = \int_{-1}^{0} \left[v^-\left(E\right) - 1 \right] d\lambda = 1 - v^-(E).$$

Since $E \in \Sigma$ implies that $-\mathbf{1}_E \in B(S,\Sigma)$ we obtain

$$\widetilde{\varphi}_-(p) = \inf_{E\in\Sigma} \frac{1-v^-(E)}{1-p(E)} = \inf_{E\in\Sigma} \frac{I(-\mathbf{1}_E)}{E_p(-\mathbf{1}_E)} \geq \varphi_-(p).$$

Now, it is enough to show when $\widetilde{\varphi}_-(p) > 0$. We know that for every $a \in B^-$

$$0 \geq \widetilde{\varphi}_-(p) E_p(a) \geq \int a dv^- = I(a)$$

hence, if $I(a) = 0$ then $E_p(a) = 0$ and $I(a)/E_p(a) = 1 \geq \widetilde{\varphi}_-(p)$.

Now, when $I(a) < 0$ the case $E_p(a) = 0$ is trivial by convention $r/0^- = +\infty$ if $r < 0$. If $E_p(a) < 0$ then by previous inequality, $\widetilde{\varphi}_-(p) > 0$ and

$$\frac{I(a)}{E_p(a)} \geq \widetilde{\varphi}_-(p)$$

which complete the prove that $\varphi_-(p) \geq \widetilde{\varphi}_-(p)$.

3.4 Appendix

For $a \in B(S, \Sigma)$, one denotes the positive part $a^+ = a \vee 0$ of a and the negative part $a^- = a \wedge 0$ of a.

Building upon Fan's Theorem (Fan, 1956), we give in the next Lemma 37, the key result for our representation theorem of Chapter 3. This Lemma can be seen as a extension of the Lemma 15 from the Chapter 2.

Lemma 37 *The following two assertions are equivalent:*

Assertion 1*: I satisfies the properties:*

1) *I is weak-superadditive: for $a, b \in B(S, \Sigma)$*

$$1.1)\ I(a^+ + b^+) \geq I(a^+) + I(b^+);$$
$$1.2)\ I(a^- + b^-) \geq I(a^-) + I(b^-);$$
$$1.3)\ I(a) = I(a^+) + I(a^-);$$
$$1.4)\ 0 \geq I(a) + I(-a);$$

2) *I is positively homogeneous: for $a, b \in B(S, \Sigma), \lambda \geq 0 : I(\lambda a) = \lambda I(a)$;*

3) *I is monotonic: for $a, b \in B(S, \Sigma) : a \geq b \Rightarrow I(a) \geq I(b)$;*

4) I *is normalized:* $I(k\mathbf{1}_S) = k$ *for any* $k \in \mathbb{R}$;

5) *There exists a* $\delta \geq 1$ *such that for all* $a \in B(S,\Sigma)$*:*

$$5.1) I(a^+ + k\mathbf{1}_S) \leq I(a^+) + \delta k \text{ if } k \geq 0;$$
$$5.1) I(a^- + k\mathbf{1}_S) \leq I(a^-) + \delta^{-1} k \text{ if } k \leq 0.$$

Assertion 2*: There exists* $\alpha_0 \in (0,1]$ *and two normal fuzzy sets* $\varphi_1, \varphi_2 : ba_+^1(S,\Sigma) \to [0,1]$ *where* $L_1\varphi_1 \cap L_1\varphi_2 \neq \emptyset$*, such that for any* $a \in B(S,\Sigma)$:

$$I(a) = \inf_{p \in L_{\alpha_0}\varphi_1} \frac{1}{\varphi_1(p)} \int_S a^+(s)p(ds) + \inf_{p \in L_{\alpha_0}\varphi_2} \varphi_2(p) \int_S a^-(s)p(ds)$$

Proof: In order to simplify the notation we set $ba_+^1(S,\Sigma) = \Delta$ *and* $B(S,\Sigma) = B$.

Assertion 2 implies Assertion 1:

We have that 1.1), 1.2) and 1.3) are immediate.

Let $\widehat{p}$ *belongs to* $L_1\varphi_1 \cap L_1\varphi_2$*, it comes that:*

$$I(a) \leq E_{\widehat{p}}(a^+) + E_{\widehat{p}}(a^-)$$

and

$$I(-a) \leq E_{\widehat{p}}(-a^-) + E_{\widehat{p}}(-a^+)$$

hence $I(a) + I(-a) \leq 0$ *and 1.3) is satisfied.*

We note that 2), 3) and 4) are immediate too. Moreover, 5) is straightforward.

In order to prove that Assertion 1 implies Assertion 2 we need several lemmas:

Lemma 38 *The mapping*

$$\begin{aligned} \varphi_+ \quad &: \quad \Delta \to \mathbb{R} \\ p \quad &\mapsto \quad \varphi_+(p) = \inf_{a \in B^+} \frac{E_p(a)}{I(a)} \end{aligned}$$

is a normal fuzzy set[2]. *Moreover, the functional*

$$\begin{aligned} I_+ \quad &: \quad B^+ \to \mathbb{R} \\ a \quad &\mapsto \quad I_+(a) = \inf_{p\in\Delta} \frac{E_p(a)}{\varphi_+(p)} \end{aligned}$$

satisfies $I_+(a) = I(a)$, *for all* $a \in B^+$.

Proof:

It is same as in Lemma 16 from Chapter 2, as well the colloraries and remaks below.

Corollary 39 *Set* $\alpha_0 = 1/\delta$ *and* $L_{\alpha_0}\varphi_+ = \{p \in \Delta : \varphi_+(p) \geq \alpha_0\}$, *then for every* $a \in B^+$

$$I(a) = \inf_{p\in L_{\alpha_0}\varphi_+} \frac{E_p(a)}{\varphi_+(p)} := I_+^*(a)$$

Remark 10 *Let be a normal fuzzy set satisfying the model, i.e.*

$$I(a) = \inf_{p\in L_{\alpha_0}\varphi} \frac{1}{\varphi(p)} \int a dp \, \text{for all } a \in B^+,$$

and let φ_+ *defined by*

$$\varphi_+(p) = \inf_{a\in B^+} \frac{E_p(a)}{I(a)}$$

then for any $p \in L_{\alpha_0}\varphi$ *one obtains* $\varphi_+(p) \geq \varphi(p)$.

Remark 11 *From the mean affirmation in Lemma 37 there exists a normal fuzzy set* φ *such that*

$$I(a) = \inf_{p\in\Delta} \frac{E_p(a)}{\varphi(p)} \text{ for all } a \in B^+.$$

It turns out that for any such φ, *on obtains* $\varphi_+(p) \geq \varphi(p)$ *for all* $p \in \Delta$.

Lemma 40 *Let*

$$\varphi_- \quad : \quad \Delta \to \mathbb{R}$$

[2] *Note that we adopt the usual convention* $0/0 = 1$ *and* $r/0 = +\infty$ *if* $r > 0$.

$$p \quad \mapsto \quad \varphi_-(p) = \underset{a \in B^-}{inf} \frac{I(a)}{E_p(a)}$$

then φ_- *is a normal fuzzy set. Moreover,*

$$\begin{aligned} I_- \quad &: \quad B^- \to \mathbb{R} \\ a \quad &\mapsto \quad I_-(a) = \underset{p \in \Delta}{inf} \, \varphi_-(p) E_p(a) \end{aligned}$$

satisfies $I_-(a) = I(a)$ *for every* $a \in B^-$.

Proof:

Since for any $a \in B^-$, $E_p(a) \leq 0$ *and* $I(a) \leq 0$, *clearly* $\varphi_-(p) \geq 0$ *and* $I(-\mathbf{1}_S)/E_p(-\mathbf{1}_S) = 1$ *implies that* $\varphi_-(p) \in [0,1]$.

Let us show that there exists $p_0 \in \Delta$ *such that* $\varphi_-(p_0) = 1$. *It is enough to show that there exists* $p_0 \in \Delta$ *such that* $I(a) \leq E_{p_0}(a)$ *for every* $a \in B^-$. *Setting* $E = B(S, \Sigma)$ *it is equivalent to show that there exists* $f \in E^*$ *such that* $f(\mathbf{1}_S) \geq 1$, $f(-\mathbf{1}_S) \geq -1$ *and* $f(a) \geq I(a)$ *for any* $a \in B^-$. *Let us consider* $\lambda_1, \lambda_2, ..., \lambda_n \geq 0$ *and* $\mathbf{1}_S$, $-\mathbf{1}_S$, $a_j \in B^-$, $3 \leq j \leq n$ *such that:*

$$\left\| \lambda_1 \mathbf{1}_S + \lambda_2(-\mathbf{1}_S) + \sum_{j=3}^{n} \lambda_j \, a_j \right\|_\infty = 1$$

it follows that

$$\lambda_1 \mathbf{1}_S - \lambda_2 \mathbf{1}_S + \sum_{j=3}^{n} \lambda_j \, a_j \leq \mathbf{1}_S$$

hence

$$\lambda_1 \mathbf{1}_S + \sum_{j=3}^{n} \lambda_j \, a_j \leq (\lambda_2 + 1) \mathbf{1}_S$$

hence 1.2), 2), 3) *and* 4) *give:*

$$\lambda_1 + \sum_{j=3}^{n} \lambda_j I(\, a_j) \leq \lambda_2 + 1$$

therefore

$$\lambda_1 - \lambda_2 + \sum_{j=3}^{n} \lambda_j I(a_j) \leq 1$$

and by Fan's theorem there exists $p_0 \in \Delta$ *such that* $E_{p_0}(a) \geq I(a)$ *for all* $a \in B^-$.

It remains now to prove that $I_-(a) = I(a)$ *for every* $a \in B^-$. *Let* a_0 *be chosen in* B^- *and first prove that* $I_-(a_0) \geq I(a_0)$. *If* $I_-(a_0) = 0$ *the result is obvious. If* $I_-(a_0) < 0$ *then there exists* $p_0 \in \Delta$ *such that* $E_{p_0}(a_0) < 0$ *and* $\varphi_-(p_0) > 0$. *We have that* $\varphi_-(p_0) \leq I(a_0)/E_{p_0}(a_0)$, *hence* $\varphi_-(p_0)E_{p_0}(a_0) \geq I(a_0)$. *Since this is true for any* p_0 *such that* $E_{p_0}(a_0) < 0$ *and* $\varphi_-(p_0) > 0$, *this entails* $I_-(a_0) \geq I(a_0)$.

Let us now prove that $I_-(a_0) \leq I(a_0)$ *for any* $a_0 \in B^-$. *Clearly it is enough to prove this inequality when* $I(a_0) < 0$.

First let us prove that there exists $f \in E^*$ *such that* $1/\delta \leq f(\mathbf{1}_S) \leq 1$, $f(a_0) = I(a_0)$ *and* $f(a) \geq I(a)$ *for every* $a \in B^-$, *i.e., there exists* $f \in E^*$ *such that* $f(\mathbf{1}_S) \geq 1/\delta$, $f(-\mathbf{1}_S) \geq -1$, $f(a_0) \geq I(a_0)$, $f(-a_0) \geq -I(a_0)$ *and* $f(a) \geq I(a)$ *for every* $a \in B^-$.

Let us consider $\lambda_1, ..., \lambda_n \geq 0$ *and* $\mathbf{1}_S$, $-\mathbf{1}_S$, a_0, $-a_0$, $a_j \in B^-$, $5 \leq j \leq n$ *such that:*

$$\left\| \lambda_1 \mathbf{1}_S + \lambda_2(-\mathbf{1}_S) + \lambda_3 a_0 + \lambda_4(-a_0) + \sum_{j=5}^{n} \lambda_j \, a_j \right\|_\infty = 1$$

it follows that

$$\lambda_1 \mathbf{1}_S - \lambda_2 \mathbf{1}_S + \lambda_3 a_0 - \lambda_4 a_0 + \sum_{j=5}^{n} \lambda_j \, a_j \leq \mathbf{1}_S$$

hence

$$(1 + \lambda_2)(-\mathbf{1}_S) + \lambda_3 a_0 + \sum_{j=5}^{n} \lambda_j \, a_j \leq (-\lambda_1)\mathbf{1}_S + \lambda_4 a_0$$

By properties of I *in assertion 1 it comes that:*

$$(1 + \lambda_2)(-1) + \lambda_3 I(a_0) + \sum_{j=5}^{n} \lambda_j \, I(a_j) \leq -\lambda_1 \delta^{-1} + \lambda_4 I(a_0)$$

therefore

$$\lambda_1\delta^{-1} - \lambda_2 + \lambda_3 I(a_0) - \lambda_4 I(a_0) + \sum_{j=5}^{n} \lambda_j I(\,a_j) \leq 1$$

By Fan's theorem, it comes that there exists $\zeta \in [\delta^{-1}, 1]$, $\widetilde{p} \in \Delta$ *such that:*

$$\begin{aligned} (1)\,\zeta E_{\widetilde{p}}(a_0) &= I(a_0), \text{ and} \\ (2)\,\zeta E_{\widetilde{p}}(a) &\geq I(a) \text{ for all } a \in B^- \end{aligned}$$

From (2) *it come that* $I(a)/E_{\widetilde{p}}(a) \geq \zeta$, *for all* $a \in B^-$. *Actually, by the initial convention,* $E_{\widetilde{p}}(a) = 0$ *implies* $I(a) = 0$ *and then* $E_{\widetilde{p}}(a)/I(a) = 1 \geq \zeta$, *moreover if* $E_{\widetilde{p}}(a) < 0$ *implies from* (2) *that* $I(a)/E_p(a) \geq \zeta$.

Since $I(a_0) < 0$, (1) *implies that* $E_{\widetilde{p}}(a_0) < 0$ *and* $\zeta = I(a_0)/E_{\widetilde{p}}(a_0)$, *therefore:*

$$\varphi_-(\widetilde{p}) = I(a_0)/E_{\widetilde{p}}(a_0) > 0$$

consequenty $I(a_0) = \varphi_-(\widetilde{p})E_{\widetilde{p}}(a_0) \geq I_-(a_0)$, *which completes the proof.*

Corollary 41 *Set* $\alpha_0 = 1/\delta$ *and* $L_{\alpha_0}\varphi_+ = \{p \in \Delta : \varphi_+(p) \geq \alpha_0\}$, *then for every* $a \in B^-$

$$I(a) = \inf_{p \in L_{\alpha_0}\varphi_-} \varphi_-(p)E_p(a) := I_-^*(a)$$

Proposition 42 *The mappins* φ_+ *and* φ_- *are regular* fuzzy sets.*

Proof:

We know that φ_+ *and* φ_- *are normal fuzzy sets. It is simple to show that* φ_+ *is concave, in particular,* φ_+ *is fuzzy convex. The weakly* upper semicontinuity is consequence of a result saying that infimun of continuous mapping is a upper semicontiuos mapping. For* φ_- *is similar with some additional work, but it is easy.*

Lemma 43 *The fuzzy set* $\varphi = \varphi_+ \wedge \varphi_-$ *is normal.*

Since for every $p \in \Delta$ *the values* $\varphi_+(p)$ *and* $\varphi_-(p)$ *belongs to* $[0, 1]$, *it is enough to* *show that there exists* $p' \in \Delta$ *such that*

$$\frac{E_{p'}(a)}{I(a)} \geq 1 \text{ and } \frac{I(-a)}{E_{p'}(-a)} \geq 1 \text{ for every } a \in B^+,$$

that is,

$$E_{p'}(a) \geq I(a) \text{ and } E_{p'}(-a) \geq I(-a) \text{ for every } a \in B^+.$$

This is equivalent to show that there exists $f \in E^*$ *such that*

$$\begin{aligned} f(\mathbf{1}_S) &\geq 1,\ f(-\mathbf{1}_S) \geq -1 \\ f(a) &\geq I(a),\ \forall a \in B^+ \\ f(-a) &\geq I(-a),\ \forall a \in B^+ \end{aligned}$$

Let us consider $\lambda_1, \lambda_2 \geq 0$, $\beta_1, ..., \beta_m \geq 0$, $\gamma_1, ..., \gamma_n \geq 0$ *and* $\mathbf{1}_S$, $-\mathbf{1}_S$, $a_1, ..., a_m \in B^+$, $b_1, ..., b_n \in B^+$ *such that*

$$\lambda_1 \mathbf{1}_S + \lambda_2(-\mathbf{1}_S) + \sum_{i=1}^{m} \beta_i a_i + \sum_{j=1}^{n} \gamma_j(-b_j) \leq \mathbf{1}_S$$

i.e.

$$\lambda_1 \mathbf{1}_S + \sum_{i=1}^{m} \beta_i a_i \leq \mathbf{1}_S + \left(\lambda_2 \mathbf{1}_S + \sum_{j=1}^{n} \gamma_j(b_j)\right)$$

setting $c = \lambda_2 \mathbf{1}_S + \sum_{j=1}^{n} \gamma_j(b_j)$, *from monotonicity 3), 1.1), 2), 4) and 5.1) it comes that:*

$$\lambda_1 + \sum_{i=1}^{m} \beta_i I(a_i) \leq \delta + I(c)$$

we have that 1.4) implies $I(c) \leq -I(-c)$, *then applying 1.2) one obtains:*

$$\lambda_1 + \sum_{i=1}^{m} \beta_i I(a_i) \leq \boldsymbol{\delta} - I(-\lambda_2 \mathbf{1}_S) - \sum_{j=1}^{n} I\left(\gamma_j(-b_j)\right)$$

note that 1.3), 2) and 4) imply $-I(-\lambda_2 \mathbf{1}_S) = \lambda_2$*, finally applying again 2) it comes:*

$$\lambda_1 - \lambda_2 + \sum_{i=1}^{m} \beta_i I(a_i) + \sum_{j=1}^{n} \gamma_j I(-b_j) + \leq \delta$$

By Fan's theorem it comes the desired result, which completes the proof that the fuzzy set $\varphi = \varphi_+ \wedge \varphi_-$ *is normal.*

Lemma 44 *The functional*

$$\begin{aligned} I^* \quad &: \quad B(S, \Sigma) \to \mathbb{R} \\ a \quad &\mapsto \quad I^*(a) = I_+(a^+) + I_-(a^-) \end{aligned}$$

satisfies $I^*(a) = I(a)$ *for every* $a \in B$*. Moreover, it turns out that:*

$$I(a) = \inf_{p \in \Delta} \frac{E_p(a)}{\varphi_+(p)} + \inf_{p \in \Delta} \varphi_-(p) E_p(a)$$

Proof:

We know that $I \mid_{B^+} = I_+$ *and* $I \mid_{B^-} = I_-$*, hence*

$$I(a) \stackrel{(1.3)}{=} I(a^+) + I(a^-) = I_+(a^+) + I_-(a^-) = I^*(a).$$

which completes the proof of fundamental Lemma 37.

Part II

Equilibrium Theory and Ambiguity.

Chapter 4

Multiple Priors, Prices and Incomplete Markets

4.1 Cost Functions and Incomplete Markets

Since the Arrow's *Role of Securities* paper the theory of equilibrium for markets in which both spot commodities and securities are traded is the source of lot important issues such as equilibrium existence and asset pricing under complete or incomplete markets. Arrow (1953) proposed this approuch and used the results from Arrow and Debreu (1954) as well as McKenzie (1954) for existence of equilibrium.

We consider cost functions *(hedging prices)* in the presence of a competitive and possibly incomplete market for financial securities without arbitrage opportunities. As is well-known, non-arbitrage principle and the assumption of complete markets enforce linear princing rule: the cost of replication of the asset is given by the mathematical expectation of their payoffs under the unique equivalent martigale measure. On the other hand, the market incompleteness imply that, while some securities can be replicated by financial portfolios feasibles on the market, in general it is not possible. Hence, while in a complete market market every asset can be hedged perfectly, in an incomplete case it is possible to stay on the safe side by superhedging. This concepty define a cost function or hedging price.

Our framework consider a market with one period of uncertainty and a finite set S of state of nature. Given a finite set $\{X_0, X_1, ..., X_m\}$ of assets $X_j : S \to \mathbb{R}$, our goal is to identify and study the consequences of non-arbitrage principle on the cost function $C : \mathbb{R}^S \to \mathbb{R}$ that satisfy a set of natural properties arising from the interaction of price-taker agents on the market[1].

4.1.1 Framework

Let $S = \{s_1, ..., s_n\}$ a finite set of states of nature. At date one, one and only one state s will occur, and an asset $X \in \mathbb{R}^S$ bought at date $t = 0$ will deliver payoff $X(s)$ at date 1 if s occurs.

Assume that at date 0 consumers can trade a finite number $m + 1$ of assets $X_j \in \mathbb{R}^S$, $0 \leq j \leq m$, with respective prices p_j.

We suppose that $X_0 = 1_S$ is the riskless bond and for sake of simplicity that $p_0 = 1$.

A portfolio of an agent is idenfied with a vector $\theta = (\theta_0, \theta_1, ...\theta_m) \in \mathbb{R}^{m+1}$, where θ_j denotes the quantities of assets X_j possessed by the agent.

The market $\mathcal{M} = (X_j, p_j, 0 \leq j \leq m)$ is assumed to offer *no-arbitrage opportunity* (NAO):

Definition 45 *(NAO): For any portfolio* $\theta \in \mathbb{R}^{m+1}$,

$$\sum_{j=0}^{m} \theta_j X_j > 0 \Rightarrow \sum_{j=0}^{m} \theta_j p_j > 0,$$
$$\sum_{j=0}^{m} \theta_j X_j = 0 \Rightarrow \sum_{j=0}^{m} \theta_j p_j = 0.$$

Let $\mathcal{A}$ be the field of all subset of S and Δ the set of probability measures on $(S, \mathcal{A})$.

[1] An axiomatic study of (insurance) prices was proposed by Castagnoli, Maccheroni and Marinacci (2002) by imposing some normative restriction on the price functional. But our approach is not the same because we take in a explicit away the non-arbitrage assumption and our goal is the study of cost functions and the *kind* of incompleteness.

The set

$$\mathcal{Q} = \{P \in \Delta : E_P(X_j) = p_j, \; \forall j \in \{0, ..., m\}\}$$

is called the set of *risk neutral probabilities* or *martingale measures.*

Denote by $F := span\,(X_0, X_1, ..., X_m)$ the *subspace of income transfers* or the *set of attainable claims.* As is well known, if we define the functional

$$\begin{aligned} C &: F \to \mathbb{R} \\ \sum_{j=0}^{m} \theta_j X_j &= X \mapsto C(X) = \sum_{j=0}^{m} \theta_j p_j, \end{aligned}$$

by the NAO assumption, we have that C is a strictly positive linear funtional on the linear subspace F, which is represented by some $P \in \Delta$ in the form

$$C(X) = E_P(X) \text{ for any } X \in F.$$

Moreover, P is uniquely determined if and only if $F = \mathbb{R}^S$. The set

$$\mathcal{Q} = \{P \in \Delta : E_P(X_j) = p_j, \; \forall j \in \{0, ..., m\}\}$$

is called the set of *risk neutral probabilities* or *martingale measures.*

By the facts above we call p_j the cost of X_j, *i.e.*, $p_j = C(X_j)$. Let now X be arbitrary contingent claim in $\mathbb{R}^S$, as an extension of C on F, we define the cost of X as the infimun of the costs of portfolios θ such that the payoffs of these portfolios are greater or equal to X:

$$C(X) = inf \left\{ \sum_j \theta_j p_j : \sum_j \theta_j X_j \geq X \right\}.$$

Jouini and Kallal (1995) proved that the minimum cost at which a contingent claim can be obtained through securities trading is its largest expected value with respect to the risk neutral probabilities[2], *i.e,*

$$C(X) = \sup_{P \in \mathcal{Q}} E_P(X)$$

and, of course, if the market is complete $C(X) = E_{P_0}(X)$, with $\{P_0\} = \mathcal{Q}$.

A simple and important fact says that X is attainable if and only if the mapping $\Phi : P \mapsto E_P(X)$ is constant over all $P \in \mathcal{Q}$ (see, for instance, El Karoui and Quenez (1995), proposition 1.7.1). Hence, if X is attainable then the price for X can be derived by an absence of arbitrage and $E_P(X)$ does not depend on $P \in \mathcal{Q}$. Now, if X is not attainable, X cannot be priced by arbitrage because the set of probabilities $\mathcal{Q}$ does not agree on X, i.e., there exists $P, P' \in Q$ such that $E_P(X) \neq E_{P'}(X)$. In particular, note that if we can choose a non attainable asset such as $X = \mathbf{1}_E$ then we can take two probabilities $P, P' \in Q$ such that $P(E) \neq P'(E)$, i.e., the event E is *ambiguous.*

4.1.2 Main Result

Clearly the cost function *(hedging price)* of a market $\mathcal{M}$ with the NAO property satisfies:

1) subadditivity:

$$C(X+Y) \leq C(X) + C(Y),\ \forall X, Y \in \mathbb{R}^S$$

2)Positive homogeneity:

$$C(\lambda X) = \lambda C(X),\ \forall X \in \mathbb{R}^S,\ \forall \lambda \geq 0$$

3) Monotonicity:

$$X \geq Y \Rightarrow C(X) \geq C(Y), \forall X, Y \in \mathbb{R}^S$$

[2] Another classical references on *hedging prices* is El Karoui and Quenez (1995) and Cvitanic and Karatzas (1993). In this works we can found some analoguos results such as Jouini and Kallal (1995).

4) Constant additivity:

$$C(X + k1_S) = C(X) + k, \, \forall X \in \mathbb{R}^S, \, \forall k \in \mathbb{R}$$

but does this suffice to characterize the cost functions.

Definition 46 *A mapping* $C : \mathbb{R}^S \to \mathbb{R}$ *will be a cost function* (hedging price) *if only if there exists a subspace* F *of* $\mathbb{R}^S$ *such that there exists a probability* P *on* $(S, \mathcal{A})$*:*

$$C(X) = E_P(X) \text{ for any } X \in F$$

and

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X) \text{ for any } X \in \mathbb{R}^S$$

where $\mathcal{Q} = \{P \in \Delta : \; E_P(X) = C(X), \, \forall X \in F\}$ *is a nonempty, convex, and closed set of probabilities.*

The existence of multiple risk neutral probabilities may reflects the ambiguity due to the limited information that agents have in dealing with assets that are not replicable by the market.

Before giving a characterization of a some particular cost functions let us introduce a definition:

Definition 47 *Let* $C : \mathbb{R}^S \to \mathbb{R}$*, denote* $\mathcal{E}$ *the set of unambiguous events, i.e.:*

$$\mathcal{E} = \{B \in \mathcal{A} : C(\mathbf{1}_B) + C(\mathbf{1}_{B^c}) = 1\}$$

Remark 12 *For* $C : \mathbb{R}^S \to \mathbb{R}$ *we will use the notation* $C(A)$ *instead of* $C(\mathbf{1}_A)$ *for any* $A \in \mathcal{A}$*, and we will talk of the set-function* C *on* $\mathcal{A}$*.*

Definition 48 $C : \mathcal{A} \to [0, 1]$ *is a capacity if,*

(i) $C(\emptyset) = 0$ *and* $C(S) = 1$

(ii) $A \supseteq B \Rightarrow C(A) \geq C(B)$

Moreover, C *is concave if for any* $A, B \in \mathcal{A}$*,*

$$C(A \cup B) + C(A \cap B) \leq C(A) + C(B).$$

Definition 49 *A collection* $\mathcal{D} \subset \mathcal{A}$ *is a* λ*-system if it has the following properties:*

(i) $\emptyset, S \in \mathcal{D}$;

(ii) $A, B \in \mathcal{D}$*,* $A \cap B = \emptyset \Rightarrow A \cup B \in \mathcal{D}$*;*

(iii) $A \in \mathcal{D} \Rightarrow A^c \in \mathcal{D}$*.*

Moreover, $\mathcal{D} \subset \mathcal{A}$ *is an algebra if it satisfies in addition:*

(iv) $A, B \in \mathcal{D} \Rightarrow A \cap B \in \mathcal{D}$*.*

Lemma 50 *If* C *is a concave capacity then*

$$\mathcal{E} = \{B \in \mathcal{A} : C(B) + C(B^c) = 1\}$$

is an algebra.

proof:

See, for instance, Nehring (1999).

Remark 13 *It is well known that any concave capacity* C *on* $\mathcal{A}$ *has the following representation:*

$$C(E) = max_{P \in \mathcal{K}} P(E),$$

see, for example, Chateauneuf and Jaffray (1989). But the conversely is not true (examples can be found in Schmeidler (1972) or Huber and Strassen (1973)).

Definition 51 *For a set-function* C *on* $\mathcal{A}$ *and* $\mathcal{E} \subset \mathcal{A}$ *such that* $S \in \mathcal{E}$*, denote* $\widetilde{C}$ *the outer set-function of* C *on* $\mathcal{A}$ *with respect to* $\mathcal{E}$*, where* $\widetilde{C}$ *is defined by:*

$$\forall A \in \mathcal{A},\ \widetilde{C}(A) = inf\left\{C(B) : A \subset B \in \mathcal{E}\right\}.$$

Definition 52 *Let C be a set-function on $\mathcal{A}$, the anti-core of C, denoted by $acore(C)$, is the subset of Δ :*

$$acore(C) = \{P \in \Delta : P(A) \leq C(A), \forall A \in \mathcal{A}\}$$

A market $\mathcal{M} = (X_j, p_j, 0 \leq j \leq m)$ is a market with $\{0,1\}$-securities if for each $j = 1, ..., m$ there exists an event $E_j \in \mathcal{A}$ such that $X_j \equiv \mathbf{1}_{E_j}$, *i.e.*, any security X_j delivery zero or one in each contingent state.

Theorem 53 *Let $C : \mathbb{R}^S \rightarrow \mathbb{R}$ be the cost function of a market $\mathcal{M}$ with $\{0,1\}$-securities without arbitrage opportunities. Then, C satisfies 1) subadditivity, 2)Positive homogeneity, 3) Monotonicity, 4) Constant additivity and 5) $acore(C) = acore(\widetilde{C})$.*

Proof:

By hypothesis, $\mathcal{M} = \left(\mathbf{1}_{E_j}, p_j, 0 \leq j \leq m\right)$ where $E_0 = S$ and $E_j \in \mathcal{A}$ for any $j = 1, ..., m$. Moreover,

$$\mathcal{Q} = \{P \in \Delta : P(E_j) = C(E_j);\ j = 1, ..., m\} \neq \emptyset,$$

by the NAO property and

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X).$$

Since for any $E \in \mathcal{A}$, $C(E) = max_{P \in \mathcal{Q}} P(E)$, we obtain that $C(\emptyset) = 0$ and $C(S) = 1$, therefore $\mathcal{E} \neq \emptyset$.

Let us denote

$$\mathcal{Q}' = \{P \in \Delta : P(B) = C(B), \text{ for any } B \in \mathcal{E}\},$$

we intent to show that $\mathcal{Q} = \mathcal{Q}'$:

Fix $P \in \mathcal{Q}'$, and note that $E_j \in \mathcal{E}$, for any $j = 1, ..., m$. In fact, we know that if $Q \in \mathcal{Q}$ then $Q(E_j) = C(E_j)$, $j = 1, ..., m$. Hence, $Q(E_j^c) = 1 - Q(E_j) = 1 - C(E_j)\, \forall Q \in \mathcal{Q}$, and *then* $C(E_j^c) = \max_{P\in\mathcal{Q}} (1 - Q(E_j)) = 1 - C(E_j)$ *as desired. Now, by definion of* $\mathcal{Q}'$ *and* $\mathcal{E}$, $P(E_j) = C(E_j)$ *for any* $j = 0, 1, ..., m$.

Now, take $P \in \mathcal{Q}$. *For* $B \in \mathcal{E}$, $P(B) \leq \max_{Q\in\mathcal{Q}} Q(B) = C(B)$, *and* $P(B^c) \leq C(B^c)$, *hence* $P(B) = C(B)$, i.e., $P \in \mathcal{Q}'$.

We note that for any $A \subset S$, $\widetilde{C}(A) = \inf_{A\subset B\in\mathcal{E}} C(B) \geq C(A)$, *hence if* P *is such that* $P(A) \leq C(A)$ *for any* $A \subset S$, *then* $P(A) \leq \widetilde{C}(A)$ *for any* $A \subset S$, i.e., *acore*$(C) \subset$ *acore*$(\widetilde{C})$. *Remains to prove acore*$(\widetilde{C}) \subset$ *acore*(C). *Note that* $\mathcal{Q} =$*acore*$(\widetilde{C})$, *in fact: If* $P \in \mathcal{Q}$, $P(A) \leq C(A)$ *for any* $A \subset S$, *hence* $P \in$ *acore*$(C) \subset$ *acore*$(\widetilde{C})$; *Let now* $Q \in$ *acore*$(\widetilde{C})$. *We will show that* $Q \in \mathcal{Q}' (= \mathcal{Q})$, i.e., $Q(B) = C(B)$ *for any* $B \subset S$ *such that* $C(B) + C(B^c) = 1$. *Since* $Q \in$ *acore*$(\widetilde{C})$, *we have that* $Q(B) \leq \widetilde{C}(B) = C(B)$ *and* $Q(B^c) \leq \widetilde{C}(B^c) = C(B^c)$, *and then* $Q(B) = C(B)$.

Now we have acore$(C) \subset$ *acore*$(\widetilde{C}) = \mathcal{Q} \subset$ *acore*(C), i.e., *acore*$(C) =$ *acore*$(\widetilde{C})$.

Note that the family of unambiguous events $\mathcal{E}$ is large than $\{E_j\}_{j=0}^m$. Then if we denote by $G = span\,\{1_B : B \in \mathcal{E}\}$, we obtain that $F \subset G$. But, is the conversely true?

Proposition 54 *$B \in \mathcal{E}$ if and only if $1_B \in F$.*

If $1_B \in F$ *we have that* $C(B) = P(B)$ *for any* $P \in \mathcal{Q}$. *Hence,* $max_{P\in\mathcal{Q}}P(B) + max_{P\in\mathcal{Q}}P(B^c) = P'(B) + P'(B^c) = 1$ *for any* $P' \in \mathcal{Q}$, *i.e.,* $B \in \mathcal{E}$.

If $B \in \mathcal{E}$ *and* $1_B \notin F$, *there exist* $P_1, P_2 \in \mathcal{Q}$ *such that* $P_1(B) > P_2(B)$. *Hence,*

$$1 = max_{P\in\mathcal{Q}}P(B) + max_{P\in\mathcal{Q}}P(B^c) > P_2(B) + max_{P\in\mathcal{Q}}P(B^c),$$

that is,

$$max_{P\in\mathcal{Q}}P(B^c) < 1 - P_2(B) = P_2(B^c),$$

but we taken $P_2 \in \mathcal{Q}$.

By this previous Proposition we obtain that $G = F$. We note that the last results shows that the set of primitive assets $\{1_{S,}1_{E1},...,1_{E_m}\}$ revels all unambiguous events:

$$\begin{aligned} B &\in \mathcal{E} \Rightarrow 1_B = \sum_{k\in\Xi} 1_{E_k} \text{ or } 1_B = 1_S - \sum_{k\in\Xi} 1_{E_k}, \\ \text{for some } \Xi &\subset \{1,...,m\}, \text{ where } E_k \cap E_{k'} = \emptyset, \, \Xi \ni k \neq k' \in \Xi. \end{aligned}$$

We say that X, Y are comonotonic when

$$(X(s) - X(s'))\,(Y(s) - Y(s')) \geq 0 \text{ for any } s, s' \in S.$$

A cost function C is comonotonic additive if $C(X+Y) = C(X) + C(Y)$ for any pair of comonotonic assets X and Y. This property is stronger than property 4 present in the previous theorem. By impose this stronger requirement we obtain the following corrolary:

Corollary 55 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be the cost function of a market $\mathcal{M}$ with $\{0,1\}$-securities without arbitrage opportunities. Then, C satisfies 1) subadditivity, 2)Positive homogeneity, 3) Monotonicity, 4') Comonotonic additive then for every $A \in \mathcal{A}$,*

$$C(A) = \widetilde{C}(A) := \underset{\{B\in\mathcal{E}:A\subset B\}}{inf} C(B).$$

Proof:

We have that for any $A \in \mathcal{A}$,

$$C(A) = \underset{P\in\mathcal{Q}}{max}\, P(A),$$

hence, clearly C is concave, i.e., *for any $E, F \in \mathcal{A}$,*

$$C(E \cup F) + C(E \cap F) \leq C(E) + C(F).$$

From 1), 2), 3) and 4') we have that $\mathcal{E} = \{B \in A : C(B) + C(B^c) = 1\}$ is an algebra, hence simple computation shows that $\widetilde{C}$ is also concave. By definition of $\widetilde{C}$ we saw that

$\widetilde{C} \geq C$ *over* $\mathcal{A}$. *It remains to prove that* $\widetilde{C}(A) \leq C(A)$ *for every* $A \in \mathcal{A}$. *Let* $A \in \mathcal{A}$, *since* $\widetilde{C}$ *is concave, there exists* $P \in acore(\widetilde{C})$ *such that* $P(A) = \widetilde{C}(A)$, *by the Theorem 53, item 5),* $P \in acore(C)$, *hence* $P(A) \leq C(A)$, *that is,* $\widetilde{C}(A) \leq C(A)$.

Remark 14 *In the last proof, we saw that if the hedging price is given by a subadditive Choquet capacity then the set of unambiguous events is an algebra of subsets. In particular, if two events* E_1 *and* E_2 *are unambiguous then* $E_1 \cap E_2$ *belongs to* $\mathcal{E}$. *This fact entails some restrictions on the incompletness of financial market,* e.g., *consider the case where we have four states of nature and* $\{s_1, s_2\}$ *and* $\{s_2, s_3\}$ *are unambiguous. Hence,* $\{s_2\}$ *is unambiguous what implies that if we have as primitive assets* $(1,1,0,0)$ *and* $(0,1,1,0)$ *then* $(0,1,0,0)$ *is unambiguous, i.e.,* $(0,1,0,0) \in F$.

Remark 15 *The conversely of Theorem 53 says that:*

Let $C : \mathbb{R}^S \to \mathbb{R}$ *be a mapping that satisfies 1), 2), 3), 4) and* C *and* $\widetilde{C}$ *are such that* $acore(C) = acore(\widetilde{C})$. *Then* C *is the cost function of a market* $\mathcal{M}$ *with* $\{0,1\}$*-securities without arbitrage opportunities.*

Futhermore, the conversely of Corrolary 55 can be statement in a similar away.

Proof: This is a subject of ongoing research.

4.2 Special Cases

Now we will discusse some exemplos and give a more specific formula for cost functions

Example 56 *Consider a market* $\mathcal{M}$ *with only the riskless asset* 1_S *(a bond). In this case, as is easy to see, we obtain a cost function given by*

$$C(X) = \max_{s \in S} X(s),$$

and the set of unambiguos events is given by $\mathcal{E} = \{\emptyset, S\}$.

Example 57 *Suppose a market where we have $S = \{s_1, s_2, s_3\}$ and a market*

$$\mathcal{M} = \left\{1_S, 1_{\{s_1\}}; 1, p_1\right\}$$

that presents the bond and only one Arrow's security. In this case we obtain a cost function given by,

$$C(X) = \int_{\{s_1\}} X dP + P\left(\{s_2, s_3\}\right) \underset{s \in \{s_2, s_3\}}{max} X(s),$$

where, $P \in \Delta$ is such that $P(\{s_1\}) = p_1$ and $P\left(\{s_2, s_3\}\right) = 1 - p_1$. We note that

$$\mathcal{E} = \{\emptyset, S, \{s_1\}, \{s_2, s_3\}\}.$$

For example,

$$\begin{aligned} C((1,2,2)) &= p_1 + (1-p_1)2 = 2 - p_1, (\theta_0 = 2 \text{ and } \theta_1 = -1), \\ C((4,2,3)) &= 4p_1 + (1-p_1)3 = 3 + p_1, \ (\theta_0 = 3 \text{ and } \theta_1 = 1). \end{aligned}$$

4.2.1 Economies with Arrow's securities and one bond.

Now we will consider markets where there are only Arrow securities and one bond.

Definition 58 *We say that a state $s^* \in S$ is a unambiguos Arrow's state if $\{s^*\}$ belongs to $\mathcal{E}$. We denote by E_0 the union of all unambiguous Arrow's states,*

$$E_0 = \bigcup_{\{s^*\} \in \mathcal{E}} \{s^*\},$$

and we will call this part of state space by the set of Arrow's states.

We note that in this case the set of unambiguous events is an algebra.

Proposition 59 *Consider the market with Arrow's securities and one bond:*

$$\mathcal{M} = \left\{ 1_S, \left(1_{\{s_k\}}\right)_{k=1,...,K}; 1, (p_k)_{k=1,..,K} \right\},$$

then there are a set E_0 and a martingale measure P, such that, for any contingent claim $X \in \mathbb{R}^S$,

$$C(X) = \int_{E_0} X dP + P(E_o^c) \max_{s \in E_o^c} X(s).$$

Proof:

Main steps;

Define $E_0 \in \mathcal{E}$ as above and note that for any $P \in \mathcal{Q}$ we have that

$$P(E_0) = \sum_{k=1}^{K} p_k$$

and we set $P(E \cap E_0) = \sum_{k \in \{j : s_j \in E\}} p_k$ for any $E \subset S$. Now we define the capacity v on the field of all subsets of S by:

$$\begin{aligned} v(E) &= P(E \cap E_0), \text{ if } E \subset E_0, \text{ and} \\ v(E) &= P(E \cap E_0) + P(E_0^c), \text{ if } E \cap E_0^c \neq \emptyset. \end{aligned}$$

Now, is simple to proof that $v = C = \widetilde{C}$.

Example 60 *Consider $S = \{s_1, ..., s_5\}$ where*

$$\mathcal{M} = \left\{ 1_S, \left(1_{\{s_k\}}\right)_{k=1,...,3}; 1, (p_k)_{k=1,..,3} \right\},$$

in this case

$$C(X) = \sum_{k=1}^{3} X(s_k) p_k + (1 - p_1 - p_2 - p_3) max \left\{ X(s) : s \in \{s_4, s_5\} \right\},$$

and, for example,

$$\begin{aligned} C\left((2,0,4,3,2)\right) &= 2p_1 + 4p_3 + (1 - p_1 - p_2 - p_3)3 \\ &= 3 - p_1 - 3p_2 + p_3. \end{aligned}$$

4.2.2 Economies with ***disjoint assets*** and one bond.

In this subsection we will consider a market where the existence of Arrow's securities is not necessary, but the securities does not delivery promisses in the same state.

We consider a market

$$\mathcal{M} = \left\{1_S, (1_{E_k})_{k=1,\ldots,K}\,; 1, (p_k)_{k=1,..,K}\right\}$$

where $E_j \cap E_i = \emptyset$ for any $j \neq i$.

Proposition 61 *For a market $\mathcal{M}$ with disjoint assets as above, there exists disjoint family unambiguous events $\{F_k\}_{k=1}^{L} \subset \mathcal{E}$,such that $\bigcup_{k=1}^{L} F_k = S$, and for any contingente claim $X \in \mathbb{R}^S$:*

$$C(X) = \sum_{k=1}^{L} P(F_k) max_{s \in F_k} \{X(s)\}$$

Proof:

The proof follows by take the capacity

$$v(E) = \sum_{k \in \{j : F_j \cap E \neq \emptyset\}} P(F_k), \textit{ for any } E \subset S,$$

where $\{F_k\}_{k=1}^{L}$ is a partition of S by unambiguous events, and

$$P(F_k) = \sum_j p_j \textit{ if } F_k = \bigcup_j E_j$$

for some $\{E_j\} \subset \{E_k\}_{k=1}^{K}$*, and*

$$P(F_k) = 1 - \sum_j p_j \text{ if } F_k = \left(\bigcup_j E_j\right)^c$$

for some $\{E_j\} \subset \{E_k\}_{k=1}^{K}$*. We note that* $v = C = \widetilde{C}$*.*

Corollary 62 *Consider a market* $\mathcal{M}$ *where there are only disjoint securities, and some these securities are possibly Arrow's securities. Define* $\Lambda = \{k : \#F_k = 1\}$*, then*

$$E_0 = \bigcup_{k \in \Lambda} \{s_k^*\},$$

where $F_k = \{s_k^*\}$*, and we obtain the following formula:*

$$C(X) = \int_{E_0} X dP + \sum_{k \in \Lambda^c} P(F_k) max_{s \in F_k} \{X(s)\}.$$

Example 63 *Suppose a market with the state space* $S = \{s_1, s_2, s_3, s_4\}$ *where:*

$$\mathcal{M} = \left\{1_S, 1_{\{s_1\}}, 1_{\{s_2,s_3\}}; 1, p_1, p_2\right\},$$

in this case we obtain that

$$\mathcal{E} = \{\emptyset, S, \{s_1\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \{s_4\}\}$$

is an algebra, and

$$\begin{aligned} C(X) &= p_1 X(s_1) + (1 - p_1 - p_2) X(s_4) + p_1 max_{s \in \{s_2,s_3\}} X(s) \\ &= \int_{E_0} X(s) dP + P(E_0^c) max_{s \in E_0} \{X(s)\}, \text{ where } E_0 = \{s_1, s_4\}. \end{aligned}$$

Example 64 *Now, we will take a case where we have a more complex financial market.*

Suppose a market with four states of nature, where:

$$\mathcal{M} = \left\{1_S, 1_{\{s_1,s_2\}}, 1_{\{s_2,s_3\}}; 1, p_1, p_2\right\}, \text{ where } p_1 \neq p_2 \text{ and } p_1 + p_2 < 1.$$

Simple computations shows that:

$$\begin{aligned}
C(\emptyset) &= 0, C(S) = 1, C(\{s_1, s_2\}) = p_1, C(\{s_2, s_3\}) = p_2, \\
C(\{s_3, s_4\}) &= 1 - p_1, C(\{s_1, s_4\}) = 1 - p_2, \\
C(\{s_1\}) &= p_1 \wedge (1 - p_2), C(\{s_2\}) = p_1 \wedge p_2, \\
C(\{s_3\}) &= p_2 \wedge (1 - p_1), C(\{s_4\}) = (1 - p_1) \wedge (1 - p_2), \\
C(\{s_1, s_3\}) &= p_1 + p_2, C(\{s_2, s_4\}) = 1, \\
C(\{s_1, s_2, s_3\}) &= p_1 + p_2, C(\{s_1, s_3, s_4\}) = 1 \wedge (2 - p_1 - p_2) = 1, \\
C(\{s_1, s_2, s_4\}) &= 1, C(\{s_2, s_3, s_4\}) = 1.
\end{aligned}$$

We note that

$$C(\{s_1, s_2, s_3\}) + C(\{s_2\}) = p_1 + p_2 + p_1 \wedge p_2,$$

and

$$C(\{s_1, s_2\}) + \{s_2, s_3\} = p_1 + p_2,$$

i.e., *C is not concave.*

Moreover, the set of unambiguous events is given by

$$\mathcal{E} = \{\emptyset, S, \{s_1, s_2\}, \{s_3, s_4\}, \{s_2, s_3\}, \{s_1, s_4\}\},$$

and we have that $\mathcal{E}$ is a λ-system but it is not an algebra (the event $\{s_2\} = \{s_1, s_2\} \cap$

$\{s_2, s_3\}$ *do not belongs to* $\mathcal{E}$*). Another important fact is that:*

$$\widetilde{C}\left(\{s_1, s_3\}\right) = 1 > p_1 + p_2 = C\left(\{s_1, s_3\}\right),$$

i.e.*, we have* $\widetilde{C} \neq C$*.*

4.3 Ambiguity aversion and Incomplete Markets: some comments about known results.

A financial market is said to be complete if the contingent payoffs from different marketed financial contracts are varied enough to span all contingences. However, in almost every financial market in the real world the span is less than the full set of contingencies, *i.e.*, the markets are incomplete. A coherent argument given by Keynes (1936, chapter 16) for the missing markets is that *when agents face substancial uncertainty they are reluctant to make more than limited contractual commitments for the future*, that is, the attitudes towards uncertainty can explain the endogeneous failure of some insurance and asset markets.

Dow and Werlang (1992) studied an optimal portfolio choice problem with one risky asset and one safe asset in a static model with ambiguity aversion agents modeled by Choquet expect utility (CEU) functions. They proved that *there is a range of prices at which the agent has no position in the risky asset.* This constitutes a striking difference with a subjective expected utility (SEU) decision maker, for whom this price interval is reduced to a point, as is well known.

The inertia interval presents in Dow and Werlang (1992) was simply a statement about the optimal portfolio choice corresponding to exogenously determined prices, given an initially riskless position. However, as Mukerji and Tallon (2001) remarked, *it does not follow from this result at the individual level that no-trade is an equilibrium when closing the model by allowing agents to trade their risks*, as is simple to see in an Edgeworth box, because the area of mutually advantageous trade is nonempty. Indeed, no-trade is an equilibrium outcome in this economy if and only if endowment is Pareto optimal to begin with. Introduction of ambiguity aversion in an economy through Choquet functionals, in general, does not impede the trade in risk sharing contracts and would not be a reason for incomplete risk sharing. For example, Chateauneuf, Dana and Tallon (2000) proved, under common convex capacity, that *risk sharing proceeds just as in an economy with SEU agents.*

Mukerji and Tallon (2001) studed if uncertainty aversion in a heterogeneous agent CEU model might lead to an endogeneous breakdown in markets for some assets. By the previous observation about an Edgeworth box economy, they saw that more assumption would be needed, which is the introduction of a component in asset payoffs that is independent of the endowments of both the endowments and the payoff of any other asset as well, *i.e.*, we have idiosyncratic components. Mukerji and Tallon (2001) proved that *when the assets available to trade risk among agents are affected by idiosyncratic risk, and if agents perceive this idyosyncratic component as being ambiguous and the ambiguity is high enough, then every equilibrium involves no trade over these assets*[3]. Hence, Mukerji and Tallon (2001) shows how ambiguity aversion may endogenously limit the scope of risk sharing obtainable through the bonds traded in an economy, and therefore, explain why the actual behavior of such an economy is better described by the assumption of incomplete markets, rather than complete markets. Another useful fact presents in the work of Mukerji and Tallon (2001), is that the theory of ambiguity aversion *a la* CEU provides an endogenously generated natural explanation of why only some class of assets, *e.g.*, bonds issued in *emerging markets*, and not all assets, will be affected by the increase in uncertainty.

On the other hand, since the result of Mukerji and Tallon (2001) only holds if asset payoffs vary across states over which endowments are constant, and the ambiguity aver-

[3]Note that this is to be contrasted with the situation in which agents are SEU, in which standart replication and diversification arguments ensure that full risk sharing may be obtained and the equilibrium is Pareto optimal, *e.g.*, see Werner (1997).

sion on it is sufficiently large, some cristicisms appear: in general, there are no observable distinctions between models *a la* CEU and the standart SEU model[4].

Another setting where conditions under which endogenous market incompleteness can arise is given by Rigotti and Shannon (2005). They consider a standart Arrow-Debreu exchange economy with a complete set of state-contingent security markets in which the distinction between uncertainty and risk is formalized by assuming agents have *incomplete preferences* over state-contingent consumption bundles, as in Bewley (2002). Without completeness, individual decision making depends on a set of probabilities over the states of nature. In this case, a bundle is preferred to another iff it has larger expected utility for all probabilities in this set. In contrast with the result of Mukerji and Tallon (2001), any initial endowments can be a non-trade equilibrium; it depends on the degree of uncertainty, *i.e.*, the size of the individual set of priors. In an intermediary situation, Rigotti and Shannon (2005) divide the state space into two sets: risky states, in which each agent assigns a precise probability, and uncertainty states, in which some agents assign multiple probabilities. They proved that there may be equilibria in which securities contingent on uncertainty states are not traded, while securities contingent on the risky events are traded. Moreover, it can explain the endogenous failure of some insurance and asset markets. As suggested by Rigotti and Shannon (2005), further analysis about the connection between uncertainty and incomplete markets are a promising area for further exploration.

[4]Rigott and Shannon (2005) presented a similar argument about the indeterminance of equilibrium.

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