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Stable projections of cartesian products of regular Cantor sets.

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Contents

1. Introduction

Regular Cantor sets on the line play a fundamental role in dynamical systems and notably also in some problems in number theory. They are defined as the maximal invariant for a one-dimensional expanding map of class C^{1+} and have some kind of self-similarity property: small parts of them are diffeomorphic to big parts with uniformly bounded distortion (see precise definition in Section 2). In both settings, dynamics and number theory, a key question is whether the arithmetic difference of two such sets contains an interval when the sum of their Hausdorff dimensions is bigger than one. Some background on regular Cantor sets which are relevant to our work can be found in [10] and [11].

From the dynamics side, the transverse geometry of the stable foliation of a horseshoe for a diffeomorphism of a surface is described by a regular Cantor set. In 1983, J. Palis and F. Takens ([9], [8]) proved a theorem about homoclinic bifurcations associated to a basic set that assures full density of hyperbolicity in the parameter family provided that the Hausdorff dimension of the basic set is smaller than one. A central fact used in the proof is that: If K_1 and K_2 are regular Cantor sets on the real line such that the sum of their Hausdorff dimensions is smaller than one, then $K_1 - K_2 = \{x - y | x \in K_1, y \in K_2\}$ (the arithmetic difference between K_1 and K_2) is a set of zero Lebesgue measure (indeed of Hausdorff dimension smaller than 1). In the same year, looking for some kind of converse of this result Palis conjectured (see [7]) that: for generic pairs of regular Cantor sets (K_1, K_2) of the real line either:
(i) $K_1 - K_2$ has zero measure, or else;
(ii) $K_1 - K_2$ contains an interval.
The statement (ii) should correspond in homoclinic bifurcations to open sets of tangencies. A slightly stronger statement is that, if K_1 and K_2 are generic regular Cantor sets and the sum of their Hausdorff dimensions is bigger than 1, then $K_1 - K_2$ contains intervals.

From the number theory side, the set of the real numbers whose coefficients of the continued fraction (of positive index) belong to some finite fixed set of possible values is a regular Cantor set defined by the Gauss's map. In 1947, M. Hall ([1]) proved that $C(4)+C(4) = [\sqrt{2}-1, 4(\sqrt{2}-1)]$ where $C(4)$ is the set of real numbers whose continued fraction coefficients are at most 4.

In 1993, the concept of stable intersection of two regular Cantor sets was introduced (see [4]): two regular Cantor sets K_1 and K_2 have *stable intersection* if $\widetilde{K}_1 \cap \widetilde{K}_2 \neq \emptyset$ for any $(\widetilde{K}_1, \widetilde{K}_2)$ perturbations of (K_1, K_2) in C^{1+}-topology of regular Cantor sets (see for a definition of the topology). C.G Moreira and J.C. Yoccoz solved a strong version of Palis's conjecture (see [5])

Theorem 1.1 (Moreira-Yoccoz, 2001). *There exist an open and dense set*

$$\mathcal{U} \subset \{(K_1, K_2), K_1, K_2\ C^{\infty}\text{-regular Cantor sets } | HD(K_1) + HD(K_2) > 1\}$$

such that $(K_1, K_2) \in \mathcal{U} \Rightarrow I_s(K_1, K_2)$ *is dense in* $K_1 - K_2$ *and*

$$HD((K_1 - K_2)/I_s(K_1, K_2)) < 1$$

where $I_s(K_1, K_2) := \{t \in \mathbb{R} | (K_1, K_2 + t) \text{ has stable intersection}\}$.

The same authors ([6]) proved the following fact concerning generic homoclinic bifurcations associated to two dimensional horseshoes with Hausdorff dimension bigger than one: they yield open sets of stable tangencies in the parameter line with positive density at the initial bifurcation value. Moreover, the unions of this set with the hyperbolicity set in the parameter line generically have full density at the initial bifurcation value.

We are interested in the following more general question in the setting of geometric measure theory:

Question. *Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be a surjective linear map. Under which conditions on $K_1, \ldots, K_n$ regular Cantor sets, the set $\pi(K_1 \times \ldots \times K_n)$ contains a non-empty open set of $\mathbb{R}^k$?*

The Moreira-Yoccoz's theorem gives a complete answer for $(n, k) = (2, 1)$.

Firstly, some natural conditions related to $HD(K_1), \ldots, HD(K_n)$ are needed, indeed: let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{R}^n$. Then for all $I \subset \{1, \ldots, n\}$

$$HD(\pi(K_1 \times \ldots \times K_n)) \leq \sum_{i \in I} HD(K_i) + \dim\Big(\operatorname{span}\{\pi(e_i), i \in I^c\}\Big).$$

We say that $t \in \mathbb{R}^k$ is a stable projection value for $K_1, \ldots, K_n$ if $t \in \pi(\widetilde{K}_1 \times \ldots \times \widetilde{K}_n)$ for any $(\widetilde{K}_1, \ldots, \widetilde{K}_n)$ perturbation of $(K_1, \ldots, K_n)$ in C^{1+}-topology of regular Cantor sets (see Section 2 for a definition of the topology). $P_s(K_1, \ldots, K_n)$ denotes the set of such stable projection values t.

In the present work we will provide an answer to the Question, by proving the following:

Theorem 1.2. *There is an open and dense subset $\mathcal{U}$ of the set*

$$\Big\{(K_1, \ldots, K_n), K_1, \ldots, K_n, \text{ are } C^\infty\text{-regular Cantor sets with}$$
$$\sum_{i \in I} HD(K_i) + \dim\Big(\operatorname{span}\{\pi(e_i), i \in I^c\}\Big) > k, \text{ for all } I \subset \{1, \ldots, n\}, I \neq \emptyset\Big\},$$

such that, if $(K_1, \ldots, K_n) \in \mathcal{U}$, then $P_s(K_1, \ldots, K_n)$ is dense in $\pi(K_1 \times \ldots \times K_n)$ and

$$HD(\pi(K_1 \times \ldots \times K_n) \backslash P_s(K_1, \ldots, K_n)) < k.$$

For $\pi : \mathbb{R}^2 \to \mathbb{R}$ given by $\pi(x, y) = x - y$, our result becomes the Moreira-Yoccoz's theorem.

For $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ given by $\pi(x_1, \ldots, x_n) = (x_1 - x_2, \ldots, x_1 - x_n)$, our result talks about simultaneous stable intersection of n regular Cantor sets with sum of Hausdorff dimension bigger than $n - 1$.

For $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ given by $\pi(x_1, x_2, x_3, x_4) = (x_1 - x_2, x_3 - x_4)$, our result talks about simultaneous stable intersection of two independent pairs of regular Cantor sets.

The main reference to our work is [5]. Some new ideas were needed in the proof, for instance, a new Marstrand type theorem (see Section 8).

2. Preliminaries

2.1. **Regular Cantor sets.** Let $\mathbb{A}$ be a finite alphabet, $\mathcal{B}$ a subset of $\mathbb{A}^2$, and Σ the bilateral subshift of finite type of $\mathbb{A}^{\mathbb{Z}}$ with allowed transitions $\mathcal{B}$.

We will always assume that Σ is topologically mixing, and that every letter in $\mathbb{A}$ occurs in Σ.

Definition 2.1. An *expansive map of type* Σ is a map g with the following properties:

(i) The domain of g is a disjoint union $\bigcup_{\mathcal{B}} I(a,b)$, where, for each (a,b), $I(a,b)$ is a compact subinterval of $I(a) := [0,1] \times a$;

(ii) For each $(a,b) \in \mathcal{B}$, the restriction of g to $I(a,b)$ is a smooth diffeomorphism onto $I(b)$ satisfying $|Dg(t)| > 1$ for all t.

The *regular Cantor set* associated to g is the maximal invariant set

$$K = \bigcap_{n \geq 0} g^{-n} \Big(\bigcup_{\mathcal{B}} I(a,b) \Big).$$

Let Σ^+ be the forward unilateral subshift associated to Σ. There exist a unique homeomorphism $h : \Sigma^+ \to K$ such that

$$h(\underline{a}) \in I(a_0), \text{ for } \underline{a} = (a_0, a_1, \ldots) \in \Sigma^+,$$
$$h \circ \sigma = g \circ h.$$

For each $(a,b) \in \mathcal{B}$, let

$$f_{a,b} := [g|_{I(a,b)}]^{-1};$$

this is a contracting diffeomorphism from $I(b)$ onto $I(a,b)$. If $\underline{a} = (a_0, \ldots, a_n)$ is a word of Σ, we put

$$f_{\underline{a}} := f_{a_0,a_1} \circ \ldots \circ f_{a_{n-1},a_n};$$

this is a contracting diffeomorphism from $I(a_n)$ onto a subinterval of $I(a_0)$ that we denote by $I(\underline{a})$.

Remark 2.2. If $(a_0, a_1, \ldots) \in \Sigma^+$, then the size of $I(a_0, \ldots, a_n)$ decrease exponentially, and the conjugation h is given by $h(\underline{a}) = \cap_{n=1}^{\infty} I(a_0, \ldots, a_n)$.

We put

$$K(\underline{a}) := K \cap I(\underline{a}) = f_{\underline{a}}(K).$$

Let r be a real number > 1, or $r = +\infty$. The space of C^r expansive maps of type Σ, endowed with the C^r topology, will be denoted by Ω_Σ^r. The union $\Omega_\Sigma = \bigcup_{r>1} \Omega_\Sigma^r$ is endowed with the inductive limit topology.

We have the following well-known result (see [10]):

Proposition 2.3 (Bounded Distortion Property)**.** *Let* $r \in (1, +\infty)$, $g \in \Omega_\Sigma^r$. *Then, there exist a constant* $c > 0$ *such that: for any word* $\underline{a} = (a_0, \ldots, a_n)$ *in* Σ *and any* $x, x' \in I(a_n)$*, we have*

$$|\log|f'_{\underline{a}}(x)| - \log|f'_{\underline{a}}(x')|| \leq C|x - x'|^{r-1}.$$

The same C *is also valid in a neighborhood of* g *in* Ω_Σ^r.

Given two sequences $\underline{a} = (\ldots, a_{-1}, a_0)$ and $\underline{b} = (b_0, b_1, \ldots)$ of Σ (finite or infinite), we say that $\underline{b}$ is *compatible* with $\underline{a}$ if $a_0 = b_0$, and then denote by $\underline{a} \vee \underline{b}$ the new word obtained by concatenation of $\underline{a}$ and $\underline{b}$

$$(\ldots, a_{-1}, a_0 = b_0, b_1, \ldots).$$

In the finite case we has the identity $f_{\underline{a} \vee \underline{b}} = f_{\underline{a}} \circ f_{\underline{b}}$, and also the relation

$$|I(\underline{a} \vee \underline{b})| \asymp |I(\underline{a})||I(\underline{b})|. \tag{2.1}$$

2.2. **Limit geometries.** Let $\Sigma^- = \{(\theta_n)_{n\leq 0}, (\theta_i,\theta_{i+1}) \in \mathcal{B} \text{ for } i<0\}$. We equip Σ^- with the following ultrametric distance: for $\underline{\theta} \neq \underline{\widetilde{\theta}} \in \Sigma^-$, set

$$d(\underline{\theta},\underline{\widetilde{\theta}}) = \begin{cases} 1 & \text{if } \theta_0 \neq \widetilde{\theta}_0, \\ |I(\underline{\theta}\wedge\underline{\widetilde{\theta}})| & \text{otherwise;} \end{cases}$$

where $\underline{\theta}\wedge\underline{\widetilde{\theta}} = (\theta_{-n},\dots,\theta_0)$ if $\widetilde{\theta}_{-j} = \theta_{-j}$ for $0\leq j\leq n$ and $\widetilde{\theta}_{-n-1} \neq \theta_{-n-1}$.

Now, let $\underline{\theta} \in \Sigma^-$; for $n>0$, let $\underline{\theta}^{(n)} = (\theta_{-n},\dots,\theta_0)$, and let $B(\underline{\theta}^{(n)})$ be the affine map from $I(\underline{\theta}^{(n)})$ onto $I(\theta_0)$ such that the diffeomorphism $k_n^{\underline{\theta}} = B(\underline{\theta}^{(n)}) \circ f_{\underline{\theta}^{(n)}}$ is orientation-preserving.

We have the following result (see [11] and [5]):

Proposition 2.4. *Let $r\in(1,+\infty)$, $g\in\Omega_\Sigma^r$.*

1. *For any $\underline{\theta}\in\Sigma^-$, there is a diffeomorphism $k^{\underline{\theta}} \in \mathrm{Diff}_+^r(I(\theta_0))$ such that the $k_n^{\underline{\theta}}$ converge to $k^{\underline{\theta}}$ in $\mathrm{Diff}_+^{r'}$, for any $r'<r$, uniformly in $\underline{\theta}$. The convergence is also uniform in a neighborhood of g in Ω_Σ^r. It follows that $(\underline{\theta},g)\mapsto k_g^{\underline{\theta}}$ is continuous.*
2. *If r is an integer, or $r=+\infty$, then $k_n^{\underline{\theta}}$ converge to $k^{\underline{\theta}}$ in Diff_+^r. More precisely, for every $0\leq j\leq r-1$, there is a constant C_j (independent on $\underline{\theta}$ and on a neighborhood of g in Ω_Σ^r) such that*

$$\left|D^j \log D[k_n^{\underline{\theta}} \circ (k^{\underline{\theta}})^{-1}](x)\right| \leq C_j |I(\underline{\theta}^{(n)})|.$$

It follows that $\underline{\theta}\to k^{\underline{\theta}}$ is Lipschitz in the following sense: for $\theta_0=\widetilde{\theta}_0$, we have

$$|D^j \log D[k^{\underline{\widetilde{\theta}}} \circ (k^{\underline{\theta}})^{-1}](x)| \leq C_j d(\underline{\theta},\underline{\widetilde{\theta}}).$$

The *limit geometric* of K associate to $\underline{\theta}$ is the Cantor set

$$K^{\underline{\theta}} = k^{\underline{\theta}}(K(\theta_0)).$$

For $\underline{\theta}\in\Sigma^-$ and $\underline{a}$ a word in Σ starting with $a_0=\theta_0$, we denote:

$$I^{\underline{\theta}}(\underline{a}) = k^{\underline{\theta}}(I(\underline{a})),$$
$$K^{\underline{\theta}}(\underline{a}) = k^{\underline{\theta}}(K(\underline{a})).$$

The $k^{\underline{\theta}}, \underline{\theta}\in\Sigma^-$ are related in the following useful way:

$$k^{\underline{\theta}} \circ f_{\underline{a}} = F^{\underline{\theta}}(\underline{a}) \circ k^{\underline{\theta}\vee\underline{a}}, \tag{2.2}$$

for every $\underline{a}$ compatible with $\underline{\theta}$, where $F^{\underline{\theta}}(\underline{a})$ is the affine map from $I(a_n)$ onto $I^{\underline{\theta}}(\underline{a})$ with the same orientation as $f_{\underline{a}}$. Therefore, the homothety part of $F^{\underline{\theta}}(\underline{a})$ is $\varepsilon(\underline{a})|I^{\underline{\theta}}(\underline{a})|$ where $\varepsilon(\underline{a})=+1$ (resp. -1) if $f_{\underline{a}}$ is orientation-preserving (resp. -reversing), and the translation part is $k^{\underline{\theta}}(f_{\underline{a}}(0))$ (understanding the 0 from $I(a_n)$).

By the Proposition 2.4.2, for any $\underline{\theta},\underline{\widetilde{\theta}}\in\Sigma^-$ and $\underline{a}$ a word in Σ with $a_0=\theta_0=\widetilde{\theta}_0$, we have

$$\begin{aligned} \left| |I^{\underline{\widetilde{\theta}}}(\underline{a})||I^{\underline{\theta}}(\underline{a})|^{-1} - 1 \right| &\leq C d(\underline{\theta},\underline{\widetilde{\theta}}), \\ |k^{\underline{\widetilde{\theta}}}(x) - k^{\underline{\theta}}(x)| &\leq C d(\underline{\theta},\underline{\widetilde{\theta}}). \end{aligned} \tag{2.3}$$

The constant C is independent of $\underline{\theta},\underline{\widetilde{\theta}},\underline{a}$, and some neighborhood of g in Ω_Σ^r.

2.3. **Renormalization operators.** Let $r \in (1, +\infty]$. For $a \in \mathbb{A}$, denote by $\mathcal{P}^r(a)$ the space of C^r-embeddings of $I(a)$ into $\mathbb{R}$, endowed with the C^r topology. $\overline{\mathcal{P}}^r(a)$ denote the subset of $\mathcal{P}^r(a)$ of embeddings h with $h(0) = 0, h(1) = 1$. For each $h \in \mathcal{P}^r(a)$, there exits a unique affine map over $\mathbb{R}$ such that, after left composition, h become to $\overline{\mathcal{P}}^r(a)$; this composition we denote by $[h]$. We also consider $\mathcal{P}(a) = \bigcup_{r>1} \mathcal{P}^r(a)$ and $\overline{\mathcal{P}}(a) = \bigcup_{r>1} \overline{\mathcal{P}}^r(a)$ endowed with the inductive limit topologies.

Let $\underline{a} = (a_0, \ldots, a_n)$ a word of Σ, and $g \in \Omega^r_\Sigma$. We define the renormalization operator

$$T^g_{\underline{a}} : \mathcal{P}^r(a_n) \to \mathcal{P}^r(a_0)$$
$$h \mapsto h \circ f_{\underline{a}}.$$

Obviously, $(h, g) \mapsto T^g_{\underline{a}}(h)$ is a continuous map from $\mathcal{P}(a_n) \times \Omega_\Sigma$ to $\mathcal{P}(a_0)$.

We can reinterpret the Proposition 2.4.1 as follows: For $\underline{\theta} \in \Sigma^-$ and any bounded sequence $h_n \in \mathcal{P}^r(\theta_{-n})$, the sequence $[T^g_{\underline{\theta}^{(n)}}(h_n)]$ converges (in the $C^{r'}$ topology, $r' < r$) to a limit in $\mathcal{P}^r(\theta_0)$. The limit is independent of the h_n and the convergence is uniform in $\underline{\theta}$ and bounded subsets of the $\mathcal{P}^r(a)$.

On the limit set of the renormalization operators, we can also view the renormalization dynamics as follows. Let $\mathcal{A} = \{(\underline{\theta}, A)\}$, where $\underline{\theta} \in \Sigma^-$ and A is now an *affine* embedding of $I(\theta_0)$ into $\mathbb{R}$. We have a canonical map

$$\mathcal{A} \to \mathcal{P}^r = \bigcup_{\mathbb{A}} \mathcal{P}^r(a)$$
$$(\underline{\theta}, A) \mapsto A \circ k^{\underline{\theta}} \quad (\in \mathcal{P}^r(\theta_0)).$$

By the equation (2.2) we can lift the action of the renormalization operators to $\mathcal{A}$ as

$$T_{\underline{a}}(\underline{\theta}, A) = (\underline{\theta} \vee \underline{a}, A \circ F^{\underline{\theta}}(\underline{a}))$$

The map $((\underline{\theta}, A), g) \mapsto T^g_{\underline{a}}(\underline{\theta}, A)$ is also continuous.

2.4. **The nonlinearity condition.** In [5], Moreira and Yoccoz introduced the following notion:

Definition 2.5. We say that K is *nonlinear*, if there exist $\underline{\theta}^0, \underline{\theta}^1 \in \Sigma^-$, with $\theta^0_0 = \theta^1_0$, and $x_0 \in K^{\underline{\theta}^0}(\theta^0_0)$ such that

$$|D \log D[k^{\underline{\theta}^1} \circ (k^{\underline{\theta}^0})^{-1}](x_0)| \neq 0.$$

Notice that there are neighborhoods $V, \widehat{V}$ of $\underline{\theta}^0, \underline{\theta}^1$ in Σ^-, respectively, and a neighborhood J of x_0, such that

$$|D \log D[k^{\widehat{\underline{\theta}}} \circ (k^{\underline{\theta}})^{-1}](x)| \geq \gamma > 0,$$

for all $x \in J, \underline{\theta} \in V, \widehat{\underline{\theta}} \in \widehat{V}$.

Fix a conveniently large constant c_0. The *size* of a word $\underline{a}$ in Σ is the length $|I(\underline{a})|$; we say that $\underline{a}$ has approximate size ρ if

$$c_0^{-1}\rho \leq |I(\underline{a})| \leq c_0\rho$$

and denote by $\Sigma(\rho)$ the set of those words. It cardinality is of order ρ^{-d}, where $d = HD(K)$.

In [6] was proven that the nonlinear condition implies the following:

Property 2.6. There exist $\eta > 0$, $\rho_1 \in (0,1)$ such that: for all $0 < \rho < \rho_1$, $1 \le \xi \le \rho^{-1}$, $\Phi \in \mathbb{R}$ and $\underline{\theta} \in V, \widehat{\underline{\theta}} \in \widehat{V}$, we have

$$\#\left\{ \underline{b} \in \Sigma(\rho), \left| \sin\left(\frac{1}{2}\xi \log \frac{|I^{\widehat{\underline{\theta}}}(\underline{b})|}{|I^{\underline{\theta}}(\underline{b})|} + \Phi \right) \right| \ge \eta \right\} \ge \eta \rho^{-d}.$$

2.5. **The family of random perturbations.** Fix an integer $r \ge 2$. We construct a family of random perturbations of g, depending on the scale parameter ρ; the perturbations will become close to g in C^r topology when ρ is small. The scale ρ will be assumed to be small.

We first pick a subset Σ^0 of $\Sigma(\rho^{1/r})$ such that

$$K = \bigcup_{\underline{a} \in \Sigma^0} K(\underline{a})$$

is a partition of K into disjoint cylinders.

We then define Σ^1 as the subset of Σ^0 formed of the words $\underline{a} \in \Sigma^0$ such that no words in $\Sigma(\rho^{1/3r})$ appears twice in $\underline{a}$.

Let $\tau > 1$ be a constant sufficiently close to 1 to have the following: let $\widehat{I}(\underline{a})$, for $\underline{a} \in \Sigma^0$, be the interval with the same center as $I(\underline{a})$ and τ^2 times the size of $I(\underline{a})$; then the $\widehat{I}(\underline{a})$, $\underline{a} \in \Sigma^0$, are pairwise disjoint.

We then choose, once and for all, a smooth even function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying

$$\chi(x) = 1 \text{ for } |x| \le \tau$$
$$\chi(x) = 0 \text{ for } |x| \ge \tau^2.$$

For $\underline{a} \in \Sigma^1$, we define the vector field $X_{\underline{a}}$, with support $\subset \widehat{I}(\underline{a})$ by

$$X_{\underline{a}}(x) = \sigma \rho^{1+1/2r} \chi(B_{\underline{a}}(x)) \frac{\partial}{\partial x}$$

where σ is a conveniently large constant, and $B_{\underline{a}}$ is an affine map sending $I(\underline{a})$ onto $[-1,+1]$; there are two such maps, but as χ is even, they give the same $X_{\underline{a}}$.

The probability space underlying the family of random perturbations is $\Omega = [-1,+1]^{\Sigma^1}$, equipped with the normalized Lebesgue measure.

For $\underline{\omega} = (\omega(\underline{a}))_{\underline{a} \in \Sigma^1} \in \Omega$, we define $\Phi_{\underline{\omega}}$ to be the time-one of the vector field $X_{\underline{\omega}} = -\sum_{\underline{a}} \omega(\underline{a}) X_{\underline{a}}$, and $g^{\underline{\omega}} = g \circ \Phi_{\underline{\omega}}$.

Remark 2.7. By construction, the vector fields $X_{\underline{a}}$, $\underline{a} \in \Sigma^1$, have disjoint supports. The size of the intervals $I(\underline{a})$, $\underline{a} \in \Sigma^1$, have order $\rho^{1/r}$, therefore the affine map $B_{\underline{a}}$ entering the definition of $X_{\underline{a}}$ has derivative of the order of $\rho^{-1/r}$. It follows that, when ρ is small, $X_{\underline{a}}$ is small in the C^r topology. Then, the $\Phi_{\underline{\omega}}$ for any $\underline{\omega} \in \Omega$, are close to the identity in the C^r topology.

In particular, the estimates in Proposition 2.4 for the $k^{\underline{\theta}}$ are valid for $g^{\underline{\omega}}$ uniformly in $\underline{\omega}$, if we consider no more than r derivatives.

Remark 2.8. The finite set $\bigcup_{a \in \mathbb{A}} \{\inf K(a), \sup K(a)\}$ of the extreme points of the Cantor set K are invariant under g. Let x_0 be a point in this set; if $\underline{a}_0$ is the element of Σ^0 such that $x_0 \in K(\underline{a}_0)$, then $\underline{a}_0$ is the initial part of an eventually periodic word in Σ^+. Therefore $\underline{a}_0 \in \Sigma^0 - \Sigma^1$ if ρ is small enough. It follows that $g^{\underline{\omega}}$ coincides with g in $I(\underline{a}_0)$ (for any $\underline{\omega} \in \Omega$), in particular $g^{\underline{\omega}}$ is expansive of type Σ, and that $\inf K(a), \sup K(a)$ are still the extremes points of the perturbed Cantor set $K_{\underline{\omega}}(a)$.

Remark 2.9. For $\underline{a} \in \Sigma^0$, let $\widetilde{I}(\underline{a})$ be the ρ-neighborhood of $I(\underline{a})$.

Let $\underline{a} \in \Sigma^0$, $a_{-1} \in \mathbb{A}$ such that $(a_{-1}, a_0) \in \mathcal{B}$; let $\underline{a}'$ be the inicial part of $a_{-1}\underline{a}$ that belongs to Σ^0. We have $f_{a_{-1}a_0}(I(\underline{a})) \subset I(\underline{a}')$; because the $f_{a,b}$ are contractions, there exists $\alpha > 0$ such that the $\alpha\rho$-neighborhood of $f_{a_{-1}a_0}(\widetilde{I}(\underline{a}))$ is contained in $\widetilde{I}(\underline{a}')$.

On the other hand, for the perturbed inverse branch $f^{\underline{\omega}}_{a_{-1}a_0}, \underline{\omega} \in \Omega$, we have, for any $x \in \widetilde{I}(\underline{a})$

$$f^{\underline{\omega}}_{a_{-1}a_0}(x) = \begin{cases} f_{a_{-1}a_0}(x) & \text{if } \underline{a}' \in \Sigma^0 - \Sigma^1 \\ f_{a_{-1}a_0}(x) + \sigma\rho^{1+1/2r}\omega(\underline{a}') & \text{if } \underline{a}' \in \Sigma^1 \end{cases}$$

(because $X_{\underline{a}'}$ is constant on $\widetilde{I}(\underline{a}')$). This allows us to conclude that $f^{\underline{\omega}}_{a_{-1}a_0}(\widetilde{I}(\underline{a})) \subset \widetilde{I}(\underline{a}')$ (if ρ is small enough), and therefore that the perturbed Cantor set $K_{\underline{\omega}}$ will be contained in $\bigcup_{\Sigma^0} \widetilde{I}(\underline{a})$.

We have the following result (see [5]):

Lemma 2.10. *Let $\underline{\theta} \in \Sigma^-, \underline{\omega} \in \Omega$ and $\underline{a}$ a word with $a_0 = \theta_0$.*

1. *If $|I(\underline{a})| > c_0'^{-1}\rho$, then*

$$\big||I^{\underline{\theta},\underline{\omega}}(\underline{a})||I^{\underline{\theta}}(\underline{a})|^{-1} - 1\big| \leq C\sigma\rho^{1-\frac{1}{2r}};$$

2. $$|k^{\underline{\theta},\underline{\omega}}(x) - k^{\underline{\theta}}(x)| \leq C\sigma\rho^{1-\frac{1}{2r}}.$$

   It follows that:

$$|k^{\underline{\theta},\underline{\omega}}(f^{\underline{\omega}}_{\underline{a}}(x)) - k^{\underline{\theta}}(f_{\underline{a}}(x))| \leq C\sigma\rho^{1-\frac{1}{2r}}.$$

The constant C is independent of $\underline{\theta}, \underline{\omega}, \underline{a}, \rho$, and the size σ of the perturbation.

We finish with a decomposition of the space Ω, that we use later. For $\underline{a} \in \Sigma(\rho^{1/2r})$, let $\Sigma^-(\underline{a})$ be the open and closed subset of Σ^- formed by the $\underline{\theta}$ ending with $\underline{a}$. Choose a subset Σ^{2-} of $\Sigma(\rho^{1/2r})$ such that

$$\Sigma^- = \bigcup_{\Sigma^{2-}} \Sigma^-(\underline{a})$$

is a partition of Σ^-.

For $\underline{a} \in \Sigma^{2-}$, define $\Sigma^1(\underline{a})$ as the set of words in Σ^1 starting with $\underline{a}$. For $\underline{\theta} \in \Sigma^-(\underline{a})$, we also define $\Sigma^1(\underline{\theta}) = \Sigma^1(\underline{a})$.

Letting $\underline{\theta} \in \Sigma$, we write

$$\Omega = [-1,+1]^{\Sigma^1(\underline{\theta})} \times [-1,+1]^{\Sigma^1 - \Sigma^1(\underline{\theta})},$$
$$\underline{\omega} = (\underline{\omega}', \underline{\omega}'')$$

and for each such an $\underline{\omega}$, we set

$$\underline{\omega}^* = (0, \underline{\omega}'').$$

This depends on $\underline{\theta}$, but, nearby, $\widehat{\underline{\theta}}$ (with $d(\underline{\theta}, \widehat{\underline{\theta}}) < c_0^{-1}\rho^{1/2r}$) will belong to the same $\Sigma^{2-}(\underline{a})$ and give the same projection $\underline{\omega}^*$ of $\underline{\omega}$.

3. Stable projection of cartesian product of regular Cantor sets: Recurrent compact criterion and statement of the main result

In this section, assume we are given n sets of data $(\mathbb{A}_1, \mathcal{B}_1, \Sigma_1, g_1), \ldots, (\mathbb{A}_n, \mathcal{B}_n, \Sigma_n, g_n)$ defining regular Cantor sets $K_1, \ldots, K_n$, and the surjective linear map $\pi : \mathbb{R}^n \to \mathbb{R}^k$.

We define as in Section 2 the spaces $\mathcal{P}_i = \bigcup_{\mathbb{A}_i} \mathcal{P}_i(a^i)$, $i = 1, \ldots, n$.

A n-uple $(h_1, \ldots, h_n)$, $(h_1 \in \mathcal{P}_1(a^1), \ldots, h_n \in \mathcal{P}_n(a^n))$ is called a *smooth configuration* for $K_1(a^1), \ldots, K_n(a^n)$.

Definition 3.1. We say that t is a *projection value* for the smooth configuration $(h_1, \ldots, h_n) \in \mathcal{P}_1(a^1) \times \ldots \times \mathcal{P}_n(a^n)$ or *projection value* for $(h_1, \ldots, h_n)$ and $(g_1, \ldots, g_n)$ if

$$t \in \pi(h_1(K_1(a^1)) \times \ldots \times h_n(K_n(a^n)));$$

and *stable projection value* if t is a projection value for all $(\widetilde{h}_1, \ldots, \widetilde{h}_n)$ perturbation of $(h_1, \ldots, h_n)$ in $\mathcal{P}_1(a^1) \times \ldots \times \mathcal{P}_n(a^n)$ and $(\widetilde{g}_1, \ldots, \widetilde{g}_n)$ perturbation of $(g_1, \ldots, g_n)$ in $\Omega_{\Sigma_1} \times \ldots \times \Omega_{\Sigma_n}$.

Obviously, if t is a stable projection value for $(h_1, \ldots, h_n)$ and $(g_1, \ldots, g_n)$, then $\widetilde{t}$ is a stable projection value for $(\widetilde{h}_1, \ldots, \widetilde{h}_n)$ and $(\widetilde{g}_1, \ldots, \widetilde{g}_n)$, for all $\widetilde{t}$ perturbation of t, $(\widetilde{h}_1, \ldots, \widetilde{h}_n)$ perturbation of $(h_1, \ldots, h_n)$ and $(\widetilde{g}_1, \ldots, \widetilde{g}_n)$ perturbation of $(g_1, \ldots, g_n)$.

Actually, rather than working in the product space $\mathcal{P}_1 \times \ldots \times \mathcal{P}_n$, it is better to go the quotient $\mathcal{Q}$ by the left action by composition of the group $\{G : \mathbb{R}^n \to \mathbb{R}^n, G(x) = \lambda x + v, \lambda \in \mathbb{R}^*, v \in \ker \pi\}$, endowed with the quotient topology.

The renormalization operators

$$T^1_{\underline{a}^1} \ldots T^n_{\underline{a}^n}(h_1, \ldots, h_n) := (T^1_{\underline{a}^1}(h_1), \ldots, T^n_{\underline{a}^n}(h_n))$$

are invariant under of the about action group; hence they are defined on the quotient space $\mathcal{Q}$.

Notice we have the following *topological equivalence:*

(3.1)

$$\mathcal{Q} \cong \overline{\mathcal{P}}_1 \times \ldots \times \overline{\mathcal{P}}_n \times (\mathbb{R}^*)^{n-1} \times \mathbb{R}^k$$
$$[(h_1, \ldots, h_n)] \to$$
$$\Big([h_1], \ldots, [h_n], \frac{h_1(1) - h_1(0)}{h_n(1) - h_n(0)}, \ldots, \frac{h_{n-1}(1) - h_{n-1}(0)}{h_n(1) - h_n(0)}, \frac{\pi(h_1(0), \ldots, h_n(0))}{h_n(1) - h_n(0)}\Big).$$

The following remark underline the fundamental role played by the renormalization operators.

Remark 3.2. 0 is a projection value for the smooth configuration $(h_1^0, \ldots, h_n^0)$ if and only if there exists a sequence $(h_1^m, \ldots, h_n^m)_{m \geq 0}$, with $(h_1^{m+1}, \ldots, h_n^{m+1}) = T^1_{\underline{a}^1} \ldots T^n_{\underline{a}^n}(h_1^m, \ldots, h_n^m)$ for some renormalization operator (depending on m) with at least one word $\underline{a}^i$ being nontrivial (i.e. of two or more letters), such that $[(h_1^m, \ldots, h_n^m)]$ is *relatively compact in* $\mathcal{Q}$.

As Section 2, we can introduce the spaces $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of the affine embedding. We denote by $\mathcal{C}$ the quotient of $\mathcal{A}_1 \times \ldots \times \mathcal{A}_n$ by the left action by composition of the group $\{G : \mathbb{R}^n \to \mathbb{R}^n, G(x) = \lambda x + v, \lambda \in \mathbb{R}^*, v \in \ker \pi\}$. Elements in $\mathcal{C}$ are called *affine relative configurations.*

We have *canonical maps*

$$\mathcal{A}_1 \times \ldots \times \mathcal{A}_n \to \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$$
$$\mathcal{C} \to \mathcal{Q}$$

which allow us to define renormalization operators on the spaces $\mathcal{A}_1 \times \ldots \times \mathcal{A}_n$ or $\mathcal{C}$.

When one is looking for stable projection values, the following notion is crucial.

Definition 3.3. A nonempty compact set $\mathcal{L}$ in $\mathcal{C}$ is *recurrent* if for every $u \in \mathcal{L}$ (suppose represented by $[(\underline{\theta}^1, A_1), \ldots, (\underline{\theta}^n, A_n)]$), we can find words $\underline{a}^1, \ldots, \underline{a}^n$ compatibles with $\underline{\theta}^1, \ldots, \underline{\theta}^n$, respectively, with at least one $\underline{a}^i$ nontrivial, such that $T^1_{\underline{a}^1} \ldots T^n_{\underline{a}^n} u \in \mathrm{int}\mathcal{L}$.

Let $\mathcal{L}$ be a recurrent compact set. The actions $\mathcal{C} \times \Omega_{\Sigma_1} \times \ldots \times \Omega_{\Sigma_n} \to \mathcal{C}$ via renormalization operators are continuous, then there are finitely many compact sets $\mathcal{L}_1, \ldots, \mathcal{L}_N$, and words $\underline{a}^1_j, \ldots, \underline{a}^n_j$, $1 \le j \le N$ (with, for every j, at least one word is nontrivial) such that

(i) $\bigcup_{1 \le j \le N} \mathcal{L}_j$ is a neighborhood of $\mathcal{L}$;
(ii) $T^{g_1}_{\underline{a}^1_j} \ldots T^{g_n}_{\underline{a}^n_j}$ is defined on $\mathcal{L}_j$ and sends $\mathcal{L}_j$ into $\mathrm{int}\mathcal{L}$.

From this, we deduce immediately that any recurrent compact set for $g_1, \ldots, g_n$ is still recurrent for $\widetilde{g}_1, \ldots, \widetilde{g}_n$ in a neighborhood of $(g_1, \ldots, g_n)$ in $\Omega_{\Sigma_1} \times \ldots \times \Omega_{\Sigma_n}$.

Remark 3.4. The image of $\mathcal{L}$ under the canonical map $\mathcal{C} \to \mathcal{Q}$ is a compact set $\mathcal{R}$ *recurrent* in the following sentence:

there exist a neighborhood V of the Id in the affine maps space $\mathrm{Aff}(\mathbb{R})$, such that, for all $[(h_1, \ldots, h_n)] \in \mathcal{R}$ there exist a compatible words $(\underline{a}^1, \ldots, \underline{a}^n) \in \{(\underline{a}^1, \ldots, \underline{a}^n_j), 1 \le j \le N\}$, such that $[T^1_{\underline{a}^1} \ldots T^n_{\underline{a}^n}(A_1 \circ h_1, \ldots, A_n \circ h_n)] \in \mathcal{R}$ for all $A_1, \ldots, A_n \in V$.

Via the equivalence 3.1: there exist $\delta > 0$, such that, for all $(k_1, \ldots, k_n, s, t) \in \mathcal{R}$ there exist a compatible words $(\underline{a}^1, \ldots, \underline{a}^n) \in \{(\underline{a}^1, \ldots, \underline{a}^n_j), 1 \le j \le N\}$, such that if $(k'_1, \ldots, k'_n, s', t') = T^1_{\underline{a}^1} \ldots T^n_{\underline{a}^n}(k_1, \ldots, k_n, s, t)$, then $(k'_1, \ldots, k'_n, \widetilde{s}', \widetilde{t}') \in \mathcal{R}$ for any $\widetilde{s}', \widetilde{t}'$ with $|\widetilde{s}' - s'| \le \delta, |\widetilde{t}' - t'| \le \delta$.

Proposition 3.5. *If* $[(\underline{\theta}^1, A_1), \ldots, (\underline{\theta}^n, A_n)]$ *is contained in a recurrent compact set, then* 0 *is a stable projection value for* $(A_1 \circ k^{\underline{\theta}^1}_{g_1}, \ldots, A_n \circ k^{\underline{\theta}^n}_{g_n})$.

Proof. Suppose $[(\underline{\theta}^1, A_1), \ldots, (\underline{\theta}^n, A_n)]$ is contained in a recurrent compact set $\mathcal{L}$ for $(g_1, \ldots, g_n)$. Let $(\widetilde{h}_1, \ldots, \widetilde{h}_n)$ perturbation of $(A_1 \circ k^{\underline{\theta}^1}_{g_1}, \ldots, A_n \circ k^{\underline{\theta}^n}_{g_n})$ in $\mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ and $(\widetilde{g}_1, \ldots, \widetilde{g}_n)$ perturbation of $(g_1, \ldots, g_n)$.

Then $\mathcal{L}$ is still recurrent for $(\widetilde{g}_1, \ldots, \widetilde{g}_n)$; and $(A_1 \circ k^{\underline{\theta}^1}_{\widetilde{g}_1}, \ldots, A_n \circ k^{\underline{\theta}^n}_{\widetilde{g}_n})$ is close to $(A_1 \circ k^{\underline{\theta}^1}_{g_1}, \ldots, A_n \circ k^{\underline{\theta}^n}_{g_n})$ in $\mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ (see Proposition 2.4.1), therefore $(\widetilde{h}_1, \ldots, \widetilde{h}_n)$ is a perturbation for $(A_1 \circ k^{\underline{\theta}^1}_{\widetilde{g}_1}, \ldots, A_n \circ k^{\underline{\theta}^n}_{\widetilde{g}_n})$. Hence in the initial assumption, we may assume $(\widetilde{g}_1, \ldots, \widetilde{g}_n) = (g_1, \ldots, g_n)$.

Let $[(\underline{\theta}^1, A_1), \ldots, (\underline{\theta}^n, A_n)] \in \mathrm{int}\mathcal{L}$. Set $(k^0_1, \ldots, k^0_n, s^0, t^0) := [(A_1 \circ k^{\underline{\theta}^1}_{g_1}, \ldots, A_n \circ k^{\underline{\theta}^n}_{g_n})]$ and $(\widetilde{k}^0_1, \ldots, \widetilde{k}^0_n, \widetilde{s}^0, \widetilde{t}^0) := [(\widetilde{h}_1, \ldots, \widetilde{h}_n)]$, then $(k^0_1, \ldots, k^0_n, \widetilde{s}^0, \widetilde{t}^0) \in \mathcal{R}$. If we have $(k^m_1, \ldots, k^m_n, \widetilde{s}^m, \widetilde{t}^m) \in \mathcal{R}$ and $(\widetilde{k}^m_1, \ldots, \widetilde{k}^m_n, \widetilde{s}^m, \widetilde{t}^m) \in \mathcal{Q}$, with k^m_i closer to $\widetilde{k}^m_i$, then, by the Remark 3.4, after some renormalization operator we have

$(k_1^{m+1},\ldots,k_n^{m+1},s^{m+1},t^{m+1}) \in \mathcal{R}$ and $(\widetilde{k}_1^{m+1},\ldots,\widetilde{k}_n^{m+1},\widetilde{s}^{m+1},\widetilde{t}^{m+1}) \in \mathcal{Q}$, in fact $(k_1^{m+1},\ldots,k_n^{m+1},\widetilde{s}^{m+1},\widetilde{t}^{m+1}) \in \mathcal{R}$, with k_i^{m+1} more closer to $\widetilde{k}_i^{m+1}$.

By the Remark 3.2, 0 is a stable projection value for the smooth configuration $(\widetilde{h}_1,\ldots,\widetilde{h}_n)$. □

We consider the subset $\mathcal{V} \subset \Omega_{\Sigma_1} \times \ldots \times \Omega_{\Sigma_n}$ of $(g_1,\ldots,g_n)$ such that, for all $\big((\underline{\theta}^1,A_1),\ldots,(\underline{\theta}^n,A_n)\big) \in A_1 \times \ldots \times A_n$, there exist $t \in \mathbb{R}^k$ such that t is a stable projection value for the configuration $(A_1 \circ k^{\underline{\theta}^1},\ldots,A_n \circ k^{\underline{\theta}^n})$.

Theorem 3.6. *1. $\mathcal{V}$ is a open subset of*

$$\left\{(g_1,\ldots,g_n), \sum_{i\in I} HD(K_i) + \dim\Big(\operatorname{span}\{\pi(e_i), i \in I^c\}\Big) \geq k, \text{ for all } I \subset \{1,\ldots,n\}\right\}.$$

Also, given $(g_1,\ldots,g_n) \in \mathcal{V}$, there exist $d^* < k$ such that for any smooth configuration $(h_1,\ldots,h_n) \in \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$, the set

$$P_s = \big\{t \in \mathbb{R}^k, t \text{ is a stable projection value for } (h_1,\ldots,h_n)\big\}$$

is (open and) dense in

$$P = \big\{t \in \mathbb{R}^k, t \text{ is a projection value for } (h_1,\ldots,h_n)\big\}$$

moreover, $HD(P - P_s) \leq d^*$. The same d^* is also valid for $\widetilde{g}_1,\ldots,\widetilde{g}_n$ in a neighborhood of $g_1,\ldots,g_n$ in $\Omega_{\Sigma_1} \times \ldots \times \Omega_{\Sigma_n}$.

2. Suppose $g_1,\ldots,g_n$ have a nonempty recurrent compact set of affine relative configurations, which meets all possible relative orientations; and have periodic points $x_1,\ldots,x_n$ (of period $l_1,\ldots,l_n$ respectively), such that $\log|Dg_1^{l_1}(x_1)|$, …, $\log|Dg_n^{l_n}(x_n)|$ are rationally independent. Then $(g_1,\ldots,g_n) \in \mathcal{V}$.

Proof. Let $g_1,\ldots,g_n \in \Omega_{\Sigma_1} \times \ldots \times \Omega_{\Sigma_n}$, and $R > 1$ be larger than the supreme of the derivatives of the expansive maps $g_1,\ldots,g_n$. Then $(g_1,\ldots,g_n) \in \mathcal{V}$ if and only if for every $(\underline{\theta}^1,\ldots,\underline{\theta}^n) \in \Sigma_1^- \times \ldots \times \Sigma_n^-$, $\lambda_1,\ldots,\lambda_{n-1} \in J_R = [-R,-R^{-1}] \cup [R^{-1},R]$, there exist $t \in \mathbb{R}^k$ such that t is a stable projection value for the configuration $(\lambda_1 k^{\underline{\theta}^1},\ldots,\lambda_{n-1}k^{\underline{\theta}^{n-1}},k^{\underline{\theta}^n})$. We denote by $\mathcal{K}$ the following compact subset of $\mathcal{P}_1 \times \ldots \times \mathcal{P}_n$:

$$\left\{(\lambda_1 k^{\underline{\theta}^1},\ldots,\lambda_{n-1}k^{\underline{\theta}^{n-1}},k^{\underline{\theta}^n}),(\underline{\theta}^1,\ldots,\underline{\theta}^n) \in \Sigma_1^- \times \ldots \times \Sigma_n^-,(\lambda_1,\ldots,\lambda_{n-1}) \in J_R^{n-1}\right\}.$$

If $(g_1,\ldots,g_n) \in \mathcal{V}$, notice that the following is true: there exist $\delta > 0$, a neighborhood $\mathcal{Z}$ of $\mathcal{K}$ in $\mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ and a neighborhood $\mathcal{W}$ of $(g_1,\ldots,g_n)$ in $\Omega_{\Sigma_1} \times \ldots \times \Omega_{\Sigma_n}$ such that for each $(h_1,\ldots,h_n) \in \mathcal{Z}$ and $(\widetilde{g}_1,\ldots,\widetilde{g}_n) \in \mathcal{W}$, there exist a δ-ball of stable projections values for $(h_1,\ldots,h_n)$ and $(\widetilde{g}_1,\ldots,\widetilde{g}_n)$. In particular, $\mathcal{V}$ is open.

The final assertion of the part 1. in the theorem is now a easy consequence of the last statement. Suppose t is a projection value for $(h_1,\ldots,h_n)$. Let ε be small; we consider words $\underline{a}^1,\ldots,\underline{a}^n$ in $\Sigma_1,\ldots,\Sigma_n$ respectively, such that $I(\underline{a}^1),\ldots,I(\underline{a}^n)$ have size of order ε, and such that t is a projection value for $(h_1 \circ f_{\underline{a}^1},\ldots,h_n \circ f_{\underline{a}^n})$. Then $|t - \pi(h_1 \circ f_{\underline{a}^1}(0),\ldots,h_n \circ f_{\underline{a}^n}(0))| \leq c\varepsilon$; and

$$\left(\frac{h_1 \circ f_{\underline{a}^1} - h_1 \circ f_{\underline{a}^1}(0)}{h_n \circ f_{\underline{a}^n}(1) - h_n \circ f_{\underline{a}^n}(0)},\ldots,\frac{h_n \circ f_{\underline{a}^n} - h_n \circ f_{\underline{a}^n}(0)}{h_n \circ f_{\underline{a}^n}(1) - h_n \circ f_{\underline{a}^n}(0)}\right) \in \mathcal{Z},$$

that mean, $(h_1,\ldots,h_n)$ is stable $\widetilde{t}$-projecting for all $\widetilde{t}$ in some $c'\varepsilon\delta$-ball contains in a $c''\varepsilon$-ball of $\pi(h_1 \circ f_{\underline{a}^1}(0),\ldots,h_n \circ f_{\underline{a}^n}(0))$. In other words, for all $t \in P$ and ε small, the set $P_s \cap B_\varepsilon(t)$ contains a $c\delta\varepsilon$-ball (for some $c > 0$ fixed). But this last property at the same time guarantees that P_s is dense in P, and that $HD(P-P_s)$ is at most $d^* < k$, where $!d^*$ depends only on $c\delta$.

2. Let $\underline{a}^i$ the word in Σ_i of length l_i+1 such that $f_{\underline{a}^i}(x_i) = x_i$. Set $\overline{\underline{a}}^i = (\dots, \underline{a}^i, \underline{a}^i, \underline{a}^i)$. Deriving the relation $k^{\overline{\underline{a}}^i} \circ f_{\underline{a}^i} = F^{\overline{\underline{a}}^i}(\underline{a}^i) \circ k^{\overline{\underline{a}}^i}$, we have $DF^{\overline{\underline{a}}^i}(\underline{a}^i) = f'_{\underline{a}^i}(x_i)$.

Let N be the set of $(\underline{\theta}^1, \dots, \underline{\theta}^n, \lambda_1, \dots, \lambda_{n-1}) \in \Sigma_1^- \times \dots \times \Sigma_n^- \times (\mathbb{R}^*)^{n-1}$ such that $(\lambda_1 k^{\underline{\theta}^1}, \dots, \lambda_{n-1} k^{\underline{\theta}^{n-1}}, k^{\underline{\theta}^n})$ has a stable projection value.

From the existence of such compact recurrence set, N is a nonempty open subset of $\Sigma_1^- \times \dots \times \Sigma_n^- \times (\mathbb{R}^*)^{n-1}$. Using renormalization operator going to that open set, we deduce that for any given $(\underline{\theta}^1, \dots, \underline{\theta}^n) \in \Sigma_1^- \times \dots \times \Sigma_n^-$, there are $\lambda_1^j, \dots, \lambda_{n-1}^j$, $j = 1, \dots, 2^{n-1}$, of all possible $n-1$-sings, such that $(\underline{\theta}^1, \dots, \underline{\theta}^n, \lambda_1^j, \dots, \lambda_{n-1}^j) \in N$ for $j = 1, \dots, 2^{n-1}$. The hypotheses on the periodic points implies that $(\overline{\underline{a}}^1, \dots, \overline{\underline{a}}^n, \lambda_1, \dots, \lambda_{n-1}) \in N$ for all $(\lambda_1, \dots, \lambda_{n-1}) \in (\mathbb{R}^*)^{n-1}$. Using again the renormalization operators, we get that $N = \Sigma_1^- \times \dots \times \Sigma_n^- \times (\mathbb{R}^*)^{n-1}$. □

Our main result is now the following.

Theorem 3.7. *Let $g_1, \dots, g_n$ be C^∞ expansive maps as above. Assume that the associated Cantor sets satisfy*

$$\sum_{i \in I} HD(K_i) + \dim\Big(\text{span}\left\{\pi(e_i), i \in I^c\right\}\Big) \geq k, \text{ for all } I \subset \{1, \dots, n\}.$$

Then we can find, arbitrarily close to $g_1, \dots, g_n$ (in the C^∞-topology), perturbations $\widetilde{g}_1, \dots, \widetilde{g}_n$ having a nonempty recurrent compact set *of affine relative configurations, which meets all possible relative orientations.*

4. Outline of the proof of the main theorem

The first step in the proof is perturbe if necessary to make sure that at least $n-1$ of the n regular Cantor sets are nonlinear, say $K_1, \dots, K_{n-1}$ (see Definition 2.5, for precise definition of nonlinearity); and that

$$\sum_{i \in I} HD(K_i) + \dim\Big(\text{span}\left\{\pi(e_i), i \in I^c\right\}\Big) > k, \text{ for all } I \subset \{1, \dots, n\}, I \neq \emptyset.$$

Notice that $\dim(\text{span}\{\pi(e_1), \dots, \pi(e_{n-1})\}) = k$, hence there exist $I \subset \{1, \dots, n-1\}$ with $\#I = k$, such that $\dim(\text{span}\{\pi(e_i), i \in I\}) = k$; say $I = \{1, \dots, k\}$.

The equivalence 3.1 gives us a canonical identification $\mathcal{C} \cong \Sigma_1^-, \dots, \Sigma_n^- \times (\mathbb{R}^*)^{n-1} \times \mathbb{R}^k$. In this coordinates, applying the equation (2.2), the renormalization operators over $\mathcal{C}$ are:

$$T^1_{\underline{a}^1} \dots T^n_{\underline{a}^n}(\underline{\theta}^1, \dots, \underline{\theta}^n, s_1, \dots, s_{n-1}, t) =$$
$$\left(\underline{\theta}^1 \vee \underline{a}^1, \dots, \underline{\theta}^n \vee \underline{a}^n, \varepsilon(\underline{a}^1, \underline{a}^n)\frac{|I^{\underline{\theta}^1}(\underline{a}^1)|}{|I^{\underline{\theta}^n}(\underline{a}^n)|} s_1, \dots, \varepsilon(\underline{a}^{n-1}, \underline{a}^n)\frac{|I^{\underline{\theta}^{n-1}}(\underline{a}^{n-1})|}{|I^{\underline{\theta}^n}(\underline{a}^n)|} s_{n-1}, t'\right)$$

where $\varepsilon(\underline{a}^i, \underline{a}^n) = \varepsilon(\underline{a}^i)\varepsilon(\underline{a}^n)$ and

$$t' = \frac{t + \pi\big(s_1 k^{\underline{\theta}^1}(f_{\underline{a}^1}(0)), \dots, s_{n-1} k^{\underline{\theta}^{n-1}}(f_{\underline{a}^{n-1}}(0)), k^{\underline{\theta}^n}(f_{\underline{a}^n}(0))\big)}{\varepsilon(\underline{a}^n)|I^{\underline{\theta}^n}(\underline{a}^n)|}.$$

We want to perturb $g_1,\dots,g_n$ (actually it will be sufficient to perturb $g_1,\dots,g_k$) in the C^∞-topology in order to create a nonempty recurrent compact set of affine relative configurations.

But a neighborhood in the C^∞ topology is a neighborhood in some C^r topology for finite r. We now fix such an integer $r\geq 2$.

The required perturbation for $g_1,\dots,g_k$ will be picked by a probabilistic argument out of the family of random perturbation $g_1^{\underline{\omega}_1},\dots,g_k^{\underline{\omega}_k}$ construed in subsection 2.5 (with the same constants τ near to 1, σ conveniently large). Recall $\underline{\omega}_i\in\Omega_i=[-1,+1]^{\Sigma_i^1}$ where $\Sigma_i^1\subset\Sigma_i(\rho^{1/r})$, $i=1,\dots,k$.

4.1. **The Multidimensional Scale Recurrence Lemma.** We may consider the renormalization operators acting on the space $\mathcal{S}=\Sigma_1^-\times\dots\times\Sigma_n^-\times(\mathbb{R}^*)^{n-1}$ of the relative scales, as

$$T^1_{\underline{a}^1}\dots T^n_{\underline{a}^n}(\underline{\theta}^1,\dots,\underline{\theta}^n,s_1,\dots,s_{n-1})=$$
$$=\left(\underline{\theta}^1\vee\underline{a}^1,\dots,\underline{\theta}^n\vee\underline{a}^n,\varepsilon(\underline{a}^1,\underline{a}^n)\frac{|I^{\underline{\theta}^1}(\underline{a}^1)|}{|I^{\underline{\theta}^n}(\underline{a}^n)|}s_1,\dots,\varepsilon(\underline{a}^{n-1},\underline{a}^n)\frac{|I^{\underline{\theta}^{n-1}}(\underline{a}^{n-1})|}{|I^{\underline{\theta}^n}(\underline{a}^n)|}s_{n-1}\right).$$

The first main ingredient in the proof of the main theorem is a recurrence result for the action of these renormalization operators on the space $\mathcal{S}$ of relative scales; we now present this result, which we call the Multidimensional Scale Recurrence Lemma. Its proof is deferred to Section 7.

We will always restrict our attention to a compact subset

$$\mathcal{S}_R=\Sigma_1^-\times\dots\times\Sigma_n^-\times([-R,-R^{-1}]\cup[R^{-1},R])^{n-1}$$

of $\mathcal{S}$, with R conveniently large.

Consider conveniently large constants $c_0\ll\widetilde{c}_0$. According to subsection 2.4, $\Sigma_i(\rho)$ is the set of words $\underline{a}^i$ in Σ_i such that

$$c_0^{-1}\rho\leq\left|I(\underline{a}^i)\right|\leq c_0\rho.$$

Note its cardinality are of order ρ^{-d_i}, where $d_i=HD(K_i)$. We also denote by $\widetilde{\Sigma}_i(\rho)$ the set of word $\widetilde{\underline{a}}^i$ in Σ_i such that $\widetilde{c}_0^{-1}\rho\leq\left|I(\underline{a}^i)\right|\leq c_0\rho$. We say that $\widetilde{\underline{a}}^i$ is an *extension* of $\underline{a}^i$, if $\widetilde{\underline{a}}^i$ ends with $\underline{a}^i$. Then, denoting $J_R=[-R,-R^{-1}]\cup[R^{-1},R]$ and $d=d_1+\dots+d_n$, we have:

Multidimensional Scale Recurrence Lemma

Let $K_1,\dots,K_{n-1}$ be nonlinear regular Cantor sets. For c_0 and R conveniently large, there exist $\widetilde{c}_0,c_1,c_2,c_3>0$, $\rho_0\in(0,1)$ such that, for all $0<\rho<\rho_0$, and for all collection of sets $(E(\underline{a}^1,\dots,\underline{a}^n))_{(\underline{a}^1,\dots,\underline{a}^n)\in\Sigma_1(\rho)\times\dots\times\Sigma_n(\rho)}$ satisfying

$$E(\underline{a}^1,\dots,\underline{a}^n)\subset J_R^{n-1},$$
$$\mathrm{Leb}(J_R^{n-1}-E(\underline{a}^1,\dots,\underline{a}^n))<c_1,$$

there is another collection $(E^*(\underline{a}^1,\dots,\underline{a}^n))_{(\underline{a}^1,\dots,\underline{a}^n)\in\Sigma_1(\rho)\times\dots\times\Sigma_n(\rho)}$ of compact subsets of J_R^{n-1} satisfying

(i) $E^*(\underline{a}^1,\dots,\underline{a}^n)$ is contained in a $c_2\rho$-neighborhood of $E(\underline{a}^1,\dots,\underline{a}^n)$;
(ii) for more than half of the $(\underline{a}^1,\dots,\underline{a}^n)$, the Lebesgue measure of $E^*(\underline{a}^1,\dots,\underline{a}^n)$ is greater than $1-1/2^{n-1}$ the volume of J_R^{n-1};
(iii) for each $(\underline{a}^1,\dots,\underline{a}^n)\in\Sigma_1(\rho)\times\dots\times\Sigma_n(\rho)$, and each $s\in E^*(\underline{a}^1,\dots,\underline{a}^n)$, there are at least $c_3\rho^{-d}$ n-uples $(\underline{b}^1,\dots,\underline{b}^n)\in\Sigma_1(\rho)\times\dots\times\Sigma_n(\rho)$, each one

with extension $(\underline{\widetilde{b}}^1,\dots,\underline{\widetilde{b}}^n)\in\widetilde{\Sigma}_1(\rho)\times\dots\times\widetilde{\Sigma}_n(\rho)$ (with $\underline{\widetilde{b}}^1,\dots,\underline{\widetilde{b}}^n$ starting respectively with the last letter of $\underline{a}^1,\dots,\underline{a}^n$) such that, for each $\underline{\theta}^1\in\Sigma_1,\dots,\underline{\theta}^n\in\Sigma_n$ ending respectively with $\underline{a}^1,\dots,\underline{a}^n$ and

$$T^1_{\underline{\widetilde{b}}^1}\dots T^n_{\underline{\widetilde{b}}^n}(\underline{\theta}^1,\dots,\underline{\theta}^n,s)=(\underline{\theta}^1\vee\underline{\widetilde{b}}^1,\dots,\underline{\theta}^n\vee\underline{\widetilde{b}}^n,s')$$

the ρ-neighborhood of s' is contained in $E^*(\underline{b}^1,\dots,\underline{b}^n)$.

Let $\mathcal{S}^*(\underline{a}^1,\dots,\underline{a}^n)$ be the set of $(\underline{\theta}^1,\dots,\underline{\theta}^n,s_1,\dots,s_{n-1})\in\mathcal{S}_R$ such that $\underline{\theta}^i$ ends with $\underline{a}^i$, $i=1,\dots,n$ and $s\in E^*(\underline{a}^1,\dots,\underline{a}^n)$; set $\mathcal{S}^*=\bigcup_{\underline{a}^1,\dots,\underline{a}^n}\mathcal{S}^*(\underline{a}^1,\dots,\underline{a}^n)$. Property (iii) says that there is a positive proportion of compatibles $(\underline{\widetilde{b}}^1,\dots,\underline{\widetilde{b}}^n)\in\widetilde{\Sigma}_1(\rho)\times\dots\times\widetilde{\Sigma}_n(\rho)$ such that the image by $T^1_{\underline{\widetilde{b}}^1}\dots T^n_{\underline{\widetilde{b}}^n}$ is "well inside" $\mathcal{S}^*$.

4.2. **Construction of the candidates for recurrent compact set.** We construct the set $\mathcal{L}=\mathcal{L}_{\underline{\omega}_1,\dots,\underline{\omega}_k}$ of relative configurations for with we want to prove recurrence for at least one $(\underline{\omega}_1,\dots,\underline{\omega}_k)\in\Omega_1\times\dots\times\Omega_k$.

As indicated, this set will depend on $\underline{\omega}_1,\dots,\underline{\omega}_k$, but only as far as the coordinate t is concerned. The image of $\mathcal{L}_{\underline{\omega}_1,\dots,\underline{\omega}_k}$ under the projection map: $\mathcal{C}\to\mathcal{S}$ will be a subset $\widetilde{\mathcal{L}}$ of $\mathcal{S}$ independent of $\underline{\omega}_1,\dots,\underline{\omega}_k$.

Let $\widehat{\Sigma}_i(\rho^3)$ be the subset of $\Sigma_i(\rho^3)$ formed by words $\underline{a}$ such that:

- (i) no word $\underline{b}\in\Sigma_i(\rho^{1/3r})$ appears twice in $\underline{a}$;
- (ii) if $\underline{c}\in\Sigma_i(\rho^{1/6r})$ appears at the end of $\underline{a}$, then it does not appear elsewhere in $\underline{a}$.

We next define $\widehat{\Sigma}_i^-$ as the subset of Σ_i^- formed by $\underline{\theta}^i$ which ends with a word in $\widehat{\Sigma}_i(\rho^3)$. This is an open and closed subset in Σ_i^-.

To define $\widetilde{\mathcal{L}}$, we will apply the Multidimensional Scale Recurrence Lemma (with $\rho^{1/2}$ instead of ρ). A family of subsets $E(\underline{a}^1,\dots,\underline{a}^n)$ of J_R^{n-1}, for $(\underline{a}^1,\dots,\underline{a}^n)\in\Sigma_1(\rho^{1/2})\times\dots\times\Sigma_n(\rho^{1/2})$ will be carefully constructed in Section 5, in relation to the Marstrand-Kaufman's type theorem (see Theorem 8.2), and it will satisfy the hypothesis

$$\mathrm{Leb}(J_R^{n-1}-E(\underline{a}^1,\dots,\underline{a}^n))<c_1,\ \text{for all } (\underline{a}^1,\dots,\underline{a}^n).$$

Then, the Multidimensional Scale Recurrence Lemma gives us another family $E^*(\underline{a}^1,\dots,\underline{a}^n)\subset J_R^{n-1}$, for $(\underline{a}^1,\dots,\underline{a}^n)\in\Sigma_1(\rho^{1/2})\times\dots\times\Sigma_n(\rho^{1/2})$, with the properties (i), (ii), (iii) indicated in the statement of the lemma.

The set $\widetilde{\mathcal{L}}$ is defined to be the subset of $\mathcal{S}_R$ formed by the $(\underline{\theta}^1,\dots,\underline{\theta}^n,s)$ such that $\underline{\theta}^i\in\widehat{\Sigma}_i^-$, $i=1,\dots,k$, and there exist $(\underline{a}^1,\dots,\underline{a}^n)\in\Sigma_1(\rho^{1/2})\times\dots\times\Sigma_n(\rho^{1/2})$ with $s\in E^*(\underline{a}^1,\dots,\underline{a}^n)$ and $\underline{\theta}^1,\dots,\underline{\theta}^n$ ending with $\underline{a}^1,\dots,\underline{a}^n$ respectively.

$\widetilde{\mathcal{L}}$ is a compact set, and by the property (ii) of Lemma, it is also nonempty and meets all possible signs for the scales!.

We remark that, if $(\underline{\theta}^1,\dots,\underline{\theta}^n,s)\in\widetilde{\mathcal{L}}$ and $(\underline{\widetilde{\theta}}^1,\dots,\underline{\widetilde{\theta}}^n)\in\Sigma_1^-\times\dots\times\Sigma_n^-$ with $d(\underline{\theta}^1,\underline{\widetilde{\theta}}^1)<c_0^{-1}\rho^3,\dots,d(\underline{\theta}^k,\underline{\widetilde{\theta}}^k)<c_0^{-1}\rho^3$ and $d(\underline{\theta}^{k+1},\underline{\widetilde{\theta}}^{k+1})<c_0^{-1}\rho^{1/2},\dots,d(\underline{\theta}^n,\underline{\widetilde{\theta}}^n)<c_0^{-1}\rho^{1/2}$, then $(\underline{\widetilde{\theta}}^1,\dots,\underline{\widetilde{\theta}}^n,s)\in\widetilde{\mathcal{L}}$.

Now, for every $(\underline{\theta}^1,\dots,\underline{\theta}^n,s)\in\widetilde{\mathcal{L}}$ and every $(\underline{\omega}_1,\dots,\underline{\omega}_k)\in\Omega_1\times\dots\times\Omega_k$ we define in Section 5 a nonempty compact subset $L^0_{\underline{\omega}_1,\dots,\underline{\omega}_k}(\underline{\theta}^1,\dots,\underline{\theta}^n,s)$ of $\mathbb{R}^k$. In fact, the set $L^0_{\underline{\omega}_1,\dots,\underline{\omega}_k}(\underline{\theta}^1,\dots,\underline{\theta}^n,s)$ will only depend (as far as $\underline{\omega}_1,\dots,\underline{\omega}_k$ are concerned) on the projections $\underline{\omega}_1^*,\dots,\underline{\omega}_k^*$ of $\underline{\omega}_1,\dots,\underline{\omega}_k$ associated to $\underline{\theta}^1,\dots,\underline{\theta}^{n-1}$ (see the definitions in the Subsection 2.5).

Finally, notice that for every $(\underline{\omega}_1,\dots,\underline{\omega}_k)\in\Omega_1\times\ldots\times\Omega_k$, the set

$$\mathcal{L}^0_{\underline{\omega}_1,\dots,\underline{\omega}_k}=\Big\{(\underline{\theta}^1,\dots,\underline{\theta}^n,s,t),(\underline{\theta}^1,\dots,\underline{\theta}^n,s)\in\widetilde{\mathcal{L}},t\in L^0_{\underline{\omega}_1,\dots,\underline{\omega}_k}(\underline{\theta}^1,\dots,\underline{\theta}^n,s)\Big\}$$

is a nonempty compact subset of $\mathcal{C}$. Then we define the candidate as follows:

$$\begin{aligned}\mathcal{L}_{\underline{\omega}_1,\dots,\underline{\omega}_k}=&\Big\{(\underline{\theta}^1,\dots,\underline{\theta}^n,s,t)\in\widetilde{\mathcal{L}}\times\mathbb{R}^k,\exists(\widetilde{\underline{\theta}}^1,\dots,\widetilde{\underline{\theta}}^n,\widetilde{s},\widetilde{t})\in\mathcal{L}^0_{\underline{\omega}_1,\dots,\underline{\omega}_k}\\&\text{with } d(\underline{\theta}^1,\widetilde{\underline{\theta}}^1)\le\rho^{5/2},\dots,d(\underline{\theta}^n,\widetilde{\underline{\theta}}^n)\le\rho^{5/2},|s-\widetilde{s}|\le\rho,|t-\widetilde{t}|\le\rho\Big\}.\end{aligned}$$

4.3. **The Probabilistic argument: Proof of the main theorem.** Consider the neighborhood $\mathcal{L}^1_{\underline{\omega}_1,\dots,\underline{\omega}_k}$ of $\mathcal{L}^0_{\underline{\omega}_1,\dots,\underline{\omega}_k}$ in $\widetilde{\mathcal{L}}\times\mathbb{R}^k$:

$$\begin{aligned}\mathcal{L}^1_{\underline{\omega}_1,\dots,\underline{\omega}_k}=&\Big\{(\underline{\theta}^1,\dots,\underline{\theta}^n,s,t)\in\widetilde{\mathcal{L}}\times\mathbb{R}^k,\exists(\underline{\theta}^1_0,\dots,\underline{\theta}^n_0,s_0,t_0)\in\mathcal{L}^0_{\underline{\omega}_1,\dots,\underline{\omega}_k}\\&\text{with } d(\underline{\theta}^1,\underline{\theta}^1_0)<2\rho^{5/2},\dots,d(\underline{\theta}^n,\underline{\theta}^n_0)<2\rho^{5/2},|s-s_0|<2\rho,|t-t_0|<2\rho\Big\}.\end{aligned}$$

Fix $u=(\underline{\theta}^1,\dots,\underline{\theta}^n,s,t)\in\widetilde{\mathcal{L}}\times\mathbb{R}^k$. We define two subsets $\overline{\Omega}^0(u),\overline{\Omega}^1(u)$ of $\overline{\Omega}:=\Omega_1\times\ldots\times\Omega_k$. First,

$$\overline{\Omega}^1(u)=\Big\{(\underline{\omega}_1,\dots,\underline{\omega}_k)\in\overline{\Omega},u\in\mathcal{L}^1_{\underline{\omega}_1,\dots,\underline{\omega}_k}\Big\}.$$

Second, $\overline{\Omega}^0(u)$ is the set of $(\underline{\omega}_1,\dots,\underline{\omega}_k)\in\overline{\Omega}$ such that there exist $(\underline{b}^1,\dots,\underline{b}^n)\in\Sigma'_1(\rho)\times\ldots\times\Sigma'_n(\rho)$ with $b^1_0=\theta^1_0,\dots,b^n_0=\theta^n_0$ and the image $T^1_{\underline{b}^1}\dots T^n_{\underline{b}^n}(u)=(\underline{\theta}'^1,\dots,\underline{\theta}'^n,s',t')$ satisfies:

(i) for any $\widetilde{s}'$ with $|\widetilde{s}'-s'|<\frac{1}{2}\rho^{1/2}$, we have $(\underline{\theta}'^1,\dots,\underline{\theta}'^n,\widetilde{s}')\in\widetilde{\mathcal{L}}$;
(ii) $t'\in L^0_{\underline{\omega}_1,\dots,\underline{\omega}_k}(\underline{\theta}'^1,\dots,\underline{\theta}'^n,s')$.

(The renormlization operators $T^1_{\underline{b}^1}\dots T^n_{\underline{b}^n}$ depend on $\underline{\omega}_1,\dots,\underline{\omega}_k$; the dependence is not indicated in the notation to keep the text readable.)

The following crucial estimative will be proved in Section 6.

Proposition 4.1. *Assume that σ is chosen conveniently large. Then there exists $c_4>0$ and $l>0$, such that, for any $u\in\widetilde{\mathcal{L}}\times\mathbb{R}^k$,*

$$\mathbb{P}(\overline{\Omega}^1(u)-\overline{\Omega}^0(u))\le\exp(-c_4\rho^{-l}).$$

With this estimate, the end of the proof of the main theorem is not difficult. Assume that ρ is small enough.

The sets $L^0_{\underline{\omega}_1,\dots,\underline{\omega}_k}(\underline{\theta}^1,\dots,\underline{\theta}^n,s)$ will always satisfy

$$t\in L^0_{\underline{\omega}_1,\dots,\underline{\omega}_k}(\underline{\theta}^1,\dots,\underline{\theta}^n,s)\Rightarrow|t|\le 2c_R$$

We choose finite subsets $\Delta^0_1,\dots,\Delta^0_n,\Delta^2$ of $\widehat{\Sigma}^-_1,\dots,\widehat{\Sigma}^-_k,\Sigma^-_{k+1},\dots,\Sigma^-_n$, $[-(2c_R+2\rho),(2c_R+2\rho)]^k$ respectively such that:

- Δ^0_i is $\rho^{5/2}$ dense in $\widehat{\Sigma}^-_i$, $\#\Delta^0_i\le c_5\rho^{-\frac{5}{2}d_i}$, $i=1,\dots,k$;
- Δ^0_i is $\rho^{5/2}$ dense in Σ^-_i, $\#\Delta^0_i\le c_5\rho^{-\frac{5}{2}d_i}$, $i=k+1,\dots,n$;
- Δ^2 is $\rho^{5/2}$ dense in $[-(2c_R+2\rho),(2c_R+2\rho)]^k$, $\#\Delta^2\le c_5\rho^{-\frac{5}{2}k}$.

Then, for each $(\underline{\theta}^1,\dots,\underline{\theta}^n)\in\Delta_1^0\times\ldots\times\Delta_n^0$, we choose a finite subset $\Delta^1(\underline{\theta}^1,\dots,\underline{\theta}^n)$ of the fiber of $\widetilde{\mathcal{L}}$ over $(\underline{\theta}^1,\dots,\underline{\theta}^n)$, which is $\rho^{5/2}$ dense in this fiber, and has cardinality $\le c_5\rho^{-\frac{5}{2}(n-1)}$. Let then

$$\Delta=\left\{u=(\underline{\theta}^1,\dots,\underline{\theta}^n,s,t),\underline{\theta}^1\in\Delta_1^0,\underline{\theta}^n\in\Delta_n^0,t\in\Delta^2,s\in\Delta^1(\underline{\theta}^1,\dots,\underline{\theta}^n)\right\}.$$

One has

$$\#\Delta\le c_5^{n+2}\rho^{-\frac{5}{2}(d+k+n-1)}\le c_5^{n+2}\rho^{-\frac{5}{2}(k+2n-1)}.$$

Now, if ρ is small enough,

$$c_5^{n+2}\rho^{-\frac{5}{2}(k+2n-1)}\exp\left(-c_4\rho^{-l}\right)<1,$$

and therefore we can find $(\underline{\omega}_1^0,\dots,\underline{\omega}_k^0)\in\Omega_1\times\ldots\times\Omega_{n-1}$ such that, for any $u\in\Delta$, either $(\underline{\omega}_1^0,\dots,\underline{\omega}_k^0)\notin\overline{\Omega}^1(u)$ or $(\underline{\omega}_1^0,\dots,\underline{\omega}_k^0)\in\overline{\Omega}^0(u)$ (or both).

Fix $(\underline{\omega}_1^0,\dots,\underline{\omega}_k^0)$ as above.

Claim. The nonempty compact set $\mathcal{L}=\mathcal{L}_{\underline{\omega}_1^0,\dots,\underline{\omega}_k^0}$ of affine relative configurations, is recurrent for $g_1^{\underline{\omega}_1^0},\dots,g_k^{\underline{\omega}_k^0},g_{k+1},\dots,g_n$.

Let $u=(\underline{\theta}^1,\dots,\underline{\theta}^n,s,t)\in\mathcal{L}$.

We can find $\underline{\theta}_0^1\in\Delta_1^0,\dots,\underline{\theta}_0^n\in\Delta_n^0,t_0\in\Delta^2$ with $d(\underline{\theta}^1,\underline{\theta}_0^1)<\rho^{5/2},\dots,d(\underline{\theta}^n,\underline{\theta}_0^n)<\rho^{5/2},|t-t_0|<\rho^{5/2}$. Then, by construction of $\widetilde{\mathcal{L}}$, $(\underline{\theta}_0^1,\dots,\underline{\theta}_0^n,s)\in\widetilde{\mathcal{L}}$; hence we can find $s_0\in\Delta^1(\underline{\theta}_0^1,\dots,\underline{\theta}_0^n)$ with $|s-s_0|<\rho^{5/2}$. Let $u_0=(\underline{\theta}_0^1,\dots,\underline{\theta}_0^n,s_0,t_0)\in\Delta$. Notice that $u_0\in\mathcal{L}^1_{\underline{\omega}_1^0,\dots,\underline{\omega}_k^0}$; in other words, $(\underline{\omega}_1^0,\dots,\underline{\omega}_k^0)\in\overline{\Omega}^1(u_0)$.

Then, by the choice of $(\underline{\omega}_1^0,\dots,\underline{\omega}_k^0)$, we have $(\underline{\omega}_1^0,\dots,\underline{\omega}_k^0)\in\overline{\Omega}^0(u_0)$. This means that there exist $(\underline{b}^1,\dots,\underline{b}^n)\in\Sigma_1'(\rho)\times\ldots\times\Sigma_n'(\rho)$ compatible with $(\underline{\theta}^1,\dots,\underline{\theta}^n)$ such that, if

$$u_0':=T_{\underline{b}^1}^{\underline{\omega}_1^0}\dots T_{\underline{b}^k}^{\underline{\omega}_k^0}T_{\underline{b}^{k+1}}^{k+1}\dots T_{\underline{b}^n}^n(u_0)=(\underline{\theta}'^1_0,\dots,\underline{\theta}'^n_0,s_0',t_0'),$$

we have

$$(\underline{\theta}'^1_0,\dots,\underline{\theta}'^n_0,\widetilde{s}_0')\in\widetilde{\mathcal{L}}\text{ for }|\widetilde{s}_0'-s_0'|<\frac{1}{2}\rho^{1/2}$$

and $t_0'\in L^0_{\underline{\omega}_1^0,\dots,\underline{\omega}_k^0}(\underline{\theta}'^1_0,\dots,\underline{\theta}'^n_0,s_0')$ (i.e. $u_0'\in\mathcal{L}^0_{\underline{\omega}_1^0,\dots,\underline{\omega}_k^0}$).

Let

$$u':=T_{\underline{b}^1}^{\underline{\omega}_1^0}\dots T_{\underline{b}^k}^{\underline{\omega}_k^0}T_{\underline{b}^{k+1}}^{k+1}\dots T_{\underline{b}^n}^n(u)=(\underline{\theta}'^1,\dots,\underline{\theta}'^n,s',t').$$

We want to prove that $u'\in\operatorname{int}\mathcal{L}$, but

$$\begin{aligned}\operatorname{int}\mathcal{L}=\Big\{&(\underline{\theta}^1,\dots,\underline{\theta}^n,s,t)\in\operatorname{int}\widetilde{\mathcal{L}}\times\mathbb{R}^k,\exists(\widetilde{\underline{\theta}}^1,\dots,\widetilde{\underline{\theta}}^n,\widetilde{s},\widetilde{t})\in\mathcal{L}^0_{\underline{\omega}_1^0,\dots,\underline{\omega}_k^0}\\ &\text{with }d(\underline{\theta}^1,\widetilde{\underline{\theta}}^1)<\rho^{5/2},\dots,d(\underline{\theta}^n,\widetilde{\underline{\theta}}^n)<\rho^{5/2},|s-\widetilde{s}|<\rho,|t-\widetilde{t}|<\rho\Big\}.\end{aligned}$$

The desired result follows from the following lemma:

Lemma 4.2. *The following estimates hold:*

$$d(\underline{\theta}'^1,\underline{\theta}'^1_0)\le c_6\rho^{7/2},\dots,d(\underline{\theta}'^n,\underline{\theta}'^n_0)\le c_6\rho^{7/2},\ |s'-s_0'|\le c_6\rho^{5/2},\ |t'-t_0'|\le c_6\rho^{3/2}.$$

Proof. By the equation (2.1) we have $d(\underline{\theta}'^i,\underline{\theta}'^i_0)\asymp d(\underline{\theta}^i,\underline{\theta}_0^i)|I(\underline{b}^i)|$, $i=1,\dots,n$, and by the relation (2.3) we have

$$|s'-s_0'|\le(cc_0')^2|s-s_0|+|s_0'|c\max_i d(\underline{\theta}^i,\underline{\theta}_0^i),$$

and

$$|t'-t'_0| \le cc'_0\rho^{-1}\big(|t-t_0|+|s-s_0|+(|s_0|+1)\max_i d(\underline{\theta}^i,\underline{\theta}^i_0)\big)+c|t'_0|d(\underline{\theta}^n,\underline{\theta}^n_0).$$

□

5. Application of Marstrand-Kaufman's type theorem

5.1. **Construction of the sets E.** Given a point $(\underline{\theta}^1,\dots,\underline{\theta}^n,s)\in\mathcal{S}$, and points $x_1\in K_1(\theta^1_0)$, …, $x_n\in K_n(\theta^n_0)$, we define

$$\pi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}(x_1,\dots,x_n)=-\pi(s_1k^{\underline{\theta}^1}(x_1),\dots,s_{n-1}k^{\underline{\theta}^{n-1}}(x_{n-1}),k^{\underline{\theta}^n}(x_n)).$$

We equip each set $K_i(\theta^i_0)$ with the d_i-dimensional Hausdorff measure μ_{d_i}, $i=1,\dots,n$. It is well-know that there exist constants $c_8>c_7>0$ such that, for $\theta^1_0\in\mathbb{A}_1,\dots,\theta^n_0\in\mathbb{A}_n$:

$$c_7<\mu_{d_1}\times\ldots\times\mu_{d_n}(K_1(\theta^1_0)\times\ldots\times K_n(\theta^n_0))<c_8.$$

For $(\underline{\theta}^1,\dots,\underline{\theta}^n,s)\in\mathcal{S}$, we denote by $\mu(\underline{\theta}^1,\dots,\underline{\theta}^n,s)$ the image under $\pi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}$ of $\mu_{d_1}\times\ldots\times\mu_{d_n}$ on $K_1(\theta^1_0)\times\ldots\times K_n(\theta^n_0)$. The Marstrand-Kaufman's type theorem tell us that, for fixed $(\underline{\theta}^1,\dots,\underline{\theta}^n)$, the measure $\mu(\underline{\theta}^1,\dots,\underline{\theta}^n,s)$ is absolutely continuous with respect to k-dimensional Lebesgue measure for Lebesgue-almost-every s, with L^2-density $\chi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}$ satisfying

$$\int_{J_R^{n-1}}\left\|\chi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}\right\|^2_{L^2}ds\le c_9(R),$$

where $c_9(R)$ is independent on $\underline{\theta}^1,\dots,\underline{\theta}^n$.

Remark 5.1. When one controls $\left\|\chi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}\right\|_{L^2}$, this gives, by the Cauchy-Schwarz inequality, a lower bound for the Lebesgue measure of $\pi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}(X)$, where X is a subset of $K_1(\theta^1_0)\times\ldots\times K_n(\theta^n_0)$ with positive $\mu_{d_1}\times\ldots\times\mu_{d_n}$-measure; indeed:

$$\mu_{d_1}\times\ldots\times\mu_{d_n}(X)\le\int_{\pi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}(X)}\chi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}(t)dt\le\left|\pi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}(X)\right|^{1/2}\left\|\chi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}\right\|_{L^2}$$

and therefore

$$\left|\pi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}(X)\right|\ge(\mu_{d_1}\times\ldots\times\mu_{d_n}(X))^2\left\|\chi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}\right\|^{-2}_{L^2}$$

From now on we suppose $k<n-1$. In the end of the section we indicate how to modify the arguments in the case $k=n-1$.

For $i=1,\dots,k$, let $P_i:\mathbb{R}^{n-1}\to\mathbb{R}^n$ be given by

$$P_i(x_1,\dots,x_{n-1})=(x_1,\dots,x_{i-1},0,x_i,\dots,x_{n-1}).$$

We define $\pi_i=\pi\circ P_i$, $\pi^i_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}=\pi_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}\circ P_i$ and denote by $\mu_i(\underline{\theta}^1,\dots,\underline{\theta}^n,s)$ the image under $\pi^i_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}$ of $\mu_{d_1}\times\ldots\times\mu_{d_{i-1}}\times\mu_{d_{i+1}}\times\ldots\times\mu_{d_n}$.

Let $\mathfrak{m}_i:=\mathfrak{m}(\pi_i,d_1,\dots,d_{i-1},d_{i+1},d_n)$. We note that $\mathfrak{m}_i+d_i\ge\mathfrak{m}(\pi,d_1,\dots,d_n)>k$. We define m_i as follow: if $\mathfrak{m}_i>k$, then $m_i=k$; if $\mathfrak{m}^i\le k$, then $m_i<\mathfrak{m}_i$ such that $m_i+d_i>k$. The Marstrand-Kaufman's type theorem tell us that: if $m_i=k$, the measure $\mu_i(\underline{\theta}^1,\dots,\underline{\theta}^n,s)$ is absolutely continuous with respect to k-dimensional Lebesgue measure for Lebesgue-almost-every $(s_1,\dots,s_{i-1},s_i,\dots,s_n)$, with L^2-density $\chi^i_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}$ satisfying

$$\int_{J_R^{n-1}}\left\|\chi^i_{\underline{\theta}^1,\dots,\underline{\theta}^n,s}\right\|^2_{L^2}ds\le c_9(R);$$

otherwise, if $m_i < k$, then

$$\int_{J_R^{n-1}} I_{m_i}(\mu_i(\underline{\theta}^1, \ldots, \underline{\theta}^n, s))ds \le c_9(R).$$

Fix $(\underline{\theta}^1, \ldots, \underline{\theta}^n) \in \Sigma_1^- \times \ldots \times \Sigma_n^-$. Let $(\underline{a}^1, \ldots, \underline{a}^n) \in \Sigma_1(\rho^{1/2r}) \times \ldots \times \Sigma_n(\rho^{1/2r})$ with $a_0^1 = \theta_0^1, \ldots, a_0^n = \theta_0^n$. Then

$$c_{10}^{-1} \le |I^{\underline{\theta}^i}(\underline{a}^i)||I^{\underline{\theta}^n}(\underline{a}^n)|^{-1} \le c_{10}, i = 1, \ldots, n-1,$$

therefore we have

$$\int_{J_R^{n-1}} \left\| \chi_{T^1_{\underline{a}^1} \ldots T^n_{\underline{a}^n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)} \right\|_{L^2}^2 ds \le c_9'(R),$$

with $c_9'(R)$ independent of $\underline{\theta}^1, \ldots, \underline{\theta}^n, \underline{a}^1, \ldots, \underline{a}^n$. On the other hand, one has

$$\#\Sigma_i(\rho^{1/2r}) \le c_{11}\rho^{-d_i/2r}, i = 1, \ldots, n.$$

We conclude that

$$\int_{J_R^{n-1}} \sum_{\underline{a}^1, \ldots, \underline{a}^n} \left\| \chi_{T^1_{\underline{a}^1} \ldots T^n_{\underline{a}^n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)} \right\|_{L^2}^2 ds \le c_{11}^n c_9'(R) \rho^{-\frac{d}{2r}}.$$

We define, with c_{12} conveniently large to be determined later:

$$E_0(\underline{\theta}^1, \ldots, \underline{\theta}^n) = \Big\{ s \in J_R^{n-1}, \left\|\chi_{\underline{\theta}^1, \ldots, \underline{\theta}^n, s}\right\|_{L^2}^2 \le c_{12} \\ \text{and} \sum_{\underline{a}^1, \ldots, \underline{a}^n} \left\| \chi_{T^1_{\underline{a}^1} \ldots T^n_{\underline{a}^n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)} \right\|_{L^2}^2 \le c_{12}\rho^{-\frac{d}{2r}} \Big\},$$

$E_i(\underline{\theta}^1, \ldots, \underline{\theta}^n) = \left\{ s \in J_R^{n-1}, \left\|\chi^i_{\underline{\theta}^1, \ldots, \underline{\theta}^n, s}\right\|_{L^2}^2 \le c_{12} \right\}$ if $m_i = k$, otherwise, if $m_i < k$ $E_i(\underline{\theta}^1, \ldots, \underline{\theta}^n) = \left\{ s \in J_R^{n-1}, I_{m_i}(\mu_i(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)) \le c_{12} \right\}$.

We now define $E(\underline{\theta}^1, \ldots, \underline{\theta}^n) = E_0(\underline{\theta}^1, \ldots, \underline{\theta}^n) \cap E_1(\underline{\theta}^1, \ldots, \underline{\theta}^n) \cap \ldots \cap E_k(\underline{\theta}^1, \ldots, \underline{\theta}^n)$.

One has, for any $(\underline{\theta}^1, \ldots, \underline{\theta}^n) \in \Sigma_1^- \times \ldots \times \Sigma_n^-$:

$$m(J_R^{n-1} - E(\underline{\theta}^1, \ldots, \underline{\theta}^n)) \le c_{12}^{-1}((k+1)c_9(R) + c_{11}^n c_9'(R)) < c_1,$$

if $c_{12} > c_1^{-1}((k+1)c_9(R) + c_{11}^n c_9'(R))$.

Then, for $(\underline{c}^1, \ldots, \underline{c}^n) \in \Sigma_1(\rho^{1/2}) \times \ldots \times \Sigma_n(\rho^{1/2})$, we define $E(\underline{c}^1, \ldots, \underline{c}^n)$ as the set of $s \in J_R^{n-1}$ such that there exist $\underline{\theta}^1, \ldots, \underline{\theta}^n$ ending respectively with $\underline{c}^1, \ldots, \underline{c}^n$ such that $s \in E(\underline{\theta}^1, \ldots, \underline{\theta}^n)$.

5.2. **Construction of the sets L^0.** For some $c_0' > \tilde{c}_0^2$, we denote by $\Sigma_i'(\rho)$ the set of words $\underline{a}^i$ in Σ_i such that ${c_0'}^{-1}\rho \le \left|I(\underline{a}^i)\right| \le c_0'\rho$. We will say that two words $\underline{b}_0^i, \underline{b}_1^i \in \Sigma_i'(\rho)$ are *independent* if there is no word $\underline{b}^i \in \Sigma_i(\rho^{1/2r})$ such that both $\underline{b}_0^i$ and $\underline{b}_1^i$ start with $\underline{b}^i$.

Let $0 < l < (m_i + d_i - k)/4r$ for all $i = 1, \ldots, k$ and define $N = [\rho^{-l}]$.

Let $(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) \in \widetilde{\mathcal{L}}$, and $(\underline{\omega}_1, \ldots, \underline{\omega}_k) \in \Omega_1 \times \ldots \times \Omega_k$. We define $L^0_{\underline{\omega}_1, \ldots, \underline{\omega}_k}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)$ to be the set of $t \in \mathbb{R}^k$ for which there exist n-uples $(\underline{b}_j^1, \ldots, \underline{b}_j^n)$, $1 \le j \le N$, in $\Sigma_1'(\rho) \times \ldots \times \Sigma_n'(\rho)$, compatibles with $\underline{\theta}^1, \ldots, \underline{\theta}^n$ such that, if we set

$$T^{\underline{\omega}_1^*}_{\underline{b}_j^1} \ldots T^{\underline{\omega}_k^*}_{\underline{b}_j^k} T^{k+1}_{\underline{b}_j^{k+1}} \ldots T^n_{\underline{b}_j^n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s, t) = (\underline{\theta}_j^1, \ldots, \underline{\theta}_j^n, s_j, t_j),$$

the following hold:

(i) for $i = 1, \ldots, k$, the words $\underline{b}_1^i, \ldots, \underline{b}_N^i$ are pairwise independent;
(ii) for $1 \le j \le N$, and $|\widetilde{s} - s_j| \le \frac{2}{3}\rho^{1/2}$, $(\underline{\theta}_j^1, \ldots, \underline{\theta}_j^n, \widetilde{s}) \in \widetilde{\mathcal{L}}$;
(iii) for $1 \le j \le N$, $|t_j| \le 2c_R$,

where $c_R = \sup\left\{|\pi(x)|, x \in [-R, R]^{n-1} \times [-1, 1]\right\}$.

We will use also a slightly smaller set $L^{-1}_{\underline{\omega}_1,\ldots,\underline{\omega}_k}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)$; it is define in the same way as $L^0_{\underline{\omega}_1,\ldots,\underline{\omega}_k}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)$, but with (ii), (iii) replaced by:

(ii)′ for $1 \le j \le N$, and $|\widetilde{s} - s_j| \le \frac{3}{4}\rho^{1/2}$, $(\underline{\theta}_j^1, \ldots, \underline{\theta}_j^n, \widetilde{s}) \in \widetilde{\mathcal{L}}$;
(iii)′ for $1 \le j \le N$, $|t_j| \le c_R$.

We will prove the following estimative.

Proposition 5.2. *There exist $c_{14} > 0$ such that, for any $(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) \in \widetilde{\mathcal{L}}$ and any $(\underline{\omega}_1, \ldots, \underline{\omega}_k) \in \Omega_1 \times \ldots \times \Omega_k$ the Lebesgue measure of $L^{-1}_{\underline{\omega}_1,\ldots,\underline{\omega}_k}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)$ is $> c_{14}$.*

5.3. **Proof of the Proposition 5.2.** Let$(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) \in \widetilde{\mathcal{L}}$ be given. By construction of $\widetilde{\mathcal{L}}$, there exist $(\widetilde{\underline{\theta}}^1, \ldots, \widetilde{\underline{\theta}}^n, \widetilde{s})$ with $d(\underline{\theta}^1, \widetilde{\underline{\theta}}^1) \le c_0\rho^{1/2}, \ldots, d(\underline{\theta}^n, \widetilde{\underline{\theta}}^n) \le c_0\rho^{1/2}$, $|s - \widetilde{s}| \le c_2\rho^{1/2}$ such that

$$\widetilde{s} \in E(\widetilde{\underline{\theta}}^1, \ldots, \widetilde{\underline{\theta}}^n).$$

For $i = 1, \ldots, n$ choose a subfamily Σ_i^2 of $\Sigma_i(\rho^{1/2r})$ of words starting with θ_0^i such that

$$K_i(\theta_0^i) = \bigcup_{\Sigma_i^2} K_i(\underline{a}^i)$$

is a partition of $K_i(\theta_0^i)$.

For each $(\underline{a}^1, \ldots, \underline{a}^n) \in \Sigma_1^2 \times \ldots \times \Sigma_n^2$,

$$c_{15}^{-1}\rho^{\frac{d}{2r}} \le \mu_{d_1} \times \ldots \times \mu_{d_n}(K_1(\underline{a}^1) \times \ldots \times K_n(\underline{a}^n)) \le c_{15}\rho^{\frac{d}{2r}}.$$

Let $J(\underline{a}^1, \ldots, \underline{a}^n) := \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(I(\underline{a}^1) \times \ldots \times I(\underline{a}^n))$ for general $(\underline{a}^1, \ldots, \underline{a}^n) \in \Sigma_1 \times \ldots \times \Sigma_n$. If we consider the random perturbations $g_1^{\underline{\omega}_1^*}, \ldots, g_k^{\underline{\omega}_k^*}$ and denote $\overline{\underline{\omega}}^* := (\underline{\omega}_1^*, \ldots, \underline{\omega}_k^*)$, by 2.10, the vertices of each $J^{\overline{\underline{\omega}}^*}(\underline{a}^1, \ldots, \underline{a}^n)$ are moved by a distance of order at most $\sigma\rho^{1-\frac{1}{2r}}$. If we replace $\underline{\theta}^1, \ldots, \underline{\theta}^n, s$ by $\widetilde{\underline{\theta}}^1, \ldots, \widetilde{\underline{\theta}}^n, \widetilde{s}$ the vertices of $\widetilde{J}(\underline{a}^1, \ldots, \underline{a}^n)$ only move by a distance of order at most $\rho^{1/2}$. We note that

$$\pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(I(\underline{a}^1 \vee \underline{b}^1) \times \ldots \times I(\underline{a}^n \vee \underline{b}^n)) = |I^{\underline{\theta}^n}(\underline{a}^n)|^{-1}\pi_{T^1_{\underline{a}^1}\cdots T^n_{\underline{a}^n}(\underline{\theta}^1,\ldots,\underline{\theta}^n,s)}(I(\underline{a}^1) \times \ldots \times I(\underline{a}^n)).$$

For each $(\underline{a}^1, \ldots, \underline{a}^n) \in \Sigma_1^2 \times \ldots \times \Sigma_n^2$, there is a point $x(\underline{a}^1, \ldots, \underline{a}^n)$ in $\mathbb{R}^k$ and a constant $c_{15} > 0$ such that all previously considered $J(\underline{a}^1, \ldots, \underline{a}^n)$ are contained in the closed ball $B(\underline{a}^1, \ldots, \underline{a}^n)$ with center $x(\underline{a}^1, \ldots, \underline{a}^n)$ and radius $c_{15}\rho^{1/2r}$.

For $i = 1, \ldots, k$, let $J^i(\underline{a}^1, \ldots, \underline{a}^n) := \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s} \circ P_i(I(\underline{a}^1) \times \ldots \times I(\underline{a}^n))$. Similarly, there is a point $x(\underline{a}^1, \ldots, \underline{a}^n)$ in $\mathbb{R}^k$ such that all the $J^i(\underline{a}^1, \ldots, \underline{a}^n)$, also considering the random perturbations $g_1^{\underline{\omega}_1^*}, \ldots, g_k^{\underline{\omega}_k^*}$ and the $(\widetilde{\underline{\theta}}^1, \ldots, \widetilde{\underline{\theta}}^n, \widetilde{s})$, are contained in the closed ball $B^i(\underline{a}^1, \ldots, \underline{a}^n)$ with center $x^i(\underline{a}^1, \ldots, \underline{a}^n)$ and radius $c_{15}\rho^{1/2r}$.

Say $(\underline{a}^1, \ldots, \underline{a}^n)$ is *good* if:

(1) $|x(\widetilde{\underline{a}}^1, \ldots, \widetilde{\underline{a}}^n) - x(\underline{a}^1, \ldots, \underline{a}^n)| \le 2c_{15}\rho^{1/2r}$ for no more than $c_{13}^{-1}\rho^{-(d-k)/2r}$ $(\widetilde{\underline{a}}^1, \ldots, \widetilde{\underline{a}}^n)$.

(2) for each $i = 1, \dots, k$, $|x^i(\underline{\widetilde{a}}^1, \dots, \underline{\widetilde{a}}^n) - x^i(\underline{a}^1, \dots, \underline{a}^n)| \le 3c_{15}\rho^{1/2r}$ for no more than $c_{13}^{-1}\rho^{-(d-m_i-d_i)/2r}$ $(\underline{\widetilde{a}}^1, \dots, \underline{\widetilde{a}}^n)$, all with the same $\underline{\widetilde{a}}^i$.

Lemma 5.3. *The number of bad* $(\underline{a}^1, \dots, \underline{a}^n)$ *is less than*

$$C_k c_{15}^{(1+k)} c_{13} c_{12} \rho^{-d/2r}$$

Proof. Let $(\underline{a}^1, \dots, \underline{a}^n)$ with the Property1: there are at least $c_{13}^{-1}\rho^{-(d-k)/2r}$ $(\underline{\widetilde{a}}^1, \dots, \underline{\widetilde{a}}^n)$ such that $|x(\underline{\widetilde{a}}^1, \dots, \underline{\widetilde{a}}^n) - x(\underline{a}^1, \dots, \underline{a}^n)| \le 2c_{15}\rho^{1/2r}$.

Then $\widehat{B}(\underline{a}^1, \dots, \underline{a}^n)$, the ball with the same center as $B(\underline{a}^1, \dots, \underline{a}^n)$ but with three times the radius, contains all the $\widetilde{J}(\underline{\widetilde{a}}^1, \dots, \underline{\widetilde{a}}^n)$. This mean that

$$\int_{\widehat{B}(\underline{a}^1,\dots,\underline{a}^n)} \chi_{\underline{\widetilde{\theta}}^1,\dots,\underline{\widetilde{\theta}}^n,\widetilde{s}} \ge c_{15}^{-1} c_{13}^{-1} \rho^{k/2r} = c_{15}^{-(1+k)} c_{13}^{-1} C_k |\widehat{B}(\underline{a}^1, \dots, \underline{a}^n)|.$$

Let B the union over all $(\underline{a}^1, \dots, \underline{a}^n)$ satisfying the Property1 of the balls $\widehat{B}(\underline{a}^1, \dots, \underline{a}^n)$. By the Vitali covering lemma we obtain

$$\int_B \chi_{\underline{\widetilde{\theta}}^1,\dots,\underline{\widetilde{\theta}}^n,\widetilde{s}} \ge c_{15}^{-(1+k)} c_{13}^{-1} C'_k |B|.$$

But then, by Cauchy-Schwarz,

$$c_{15}^{-(1+k)} c_{13}^{-1} C'_k |B| \int_B \chi_{\underline{\widetilde{\theta}}^1,\dots,\underline{\widetilde{\theta}}^n,\widetilde{s}} \le \left(\int_B \chi_{\underline{\widetilde{\theta}}^1,\dots,\underline{\widetilde{\theta}}^n \widetilde{s}} \right)^2 \le |B| \int_B \chi^2_{\underline{\widetilde{\theta}}^1,\dots,\underline{\widetilde{\theta}}^n,\widetilde{s}}$$

and thus $\int_B \chi_{\underline{\widetilde{\theta}}^1,\dots,\underline{\widetilde{\theta}}^n,\widetilde{s}} \le C_k c_{15}^{1+k} c_{13} c_{12}$.

As B contains $\widetilde{J}(\underline{a}^1, \dots, \underline{a}^n)$ for all $(\underline{a}^1, \dots, \underline{a}^n)$ with the Property1, then the number of such $(\underline{a}^1, \dots, \underline{a}^n)$ is at most $C_k c_{15}^{2+k} c_{13} c_{12} \rho^{-d/2r}$.

Let $(\underline{a}^1, \dots, \underline{a}^n)$ with the Property2: there are at least $c_{13}^{-1}\rho^{-(d-m'_1-d_1)/2r}$ $(\underline{\widetilde{a}}^1, \dots, \underline{\widetilde{a}}^n)$, all with the same $\underline{\widetilde{a}}^1$ such that $|x^1(\underline{\widetilde{a}}^1, \dots, \underline{\widetilde{a}}^n) - x^1(\underline{a}^1, \dots, \underline{a}^n)| \le 3c_{15}\rho^{1/2r}$.

If $m'_1 = k$, the previous case gives a estimative over the possible $(\underline{a}^2, \dots, \underline{a}^n)$. We give the same estimative when $m'_1 < k$. The ball $\widehat{B}^1(\underline{a}^1, \dots, \underline{a}^n)$ with the same center as $B^1(\underline{a}^1, \dots, \underline{a}^n)$ but with four times the radius, contains all the $\widetilde{J}^1(\underline{\widetilde{a}}^1, \dots, \underline{\widetilde{a}}^n)$. This mean that

$$\mu^1(\underline{\widetilde{\theta}}^1, \dots, \underline{\widetilde{\theta}}^n, \widetilde{s})(\widehat{B}^1(\underline{a}^1, \dots, \underline{a}^n)) \ge c_{15}^{-1} c_{13}^{-1} \rho^{m'_1/2r}.$$

Let B^1 the union over all $(\underline{a}^2, \dots, \underline{a}^n)$ of the $(\underline{a}^1, \dots, \underline{a}^n)$ satisfying the Property2 of the balls $\widehat{B}^1(\underline{a}^1, \dots, \underline{a}^n)$. The Vitali covering lemma say that we have a subcolection $\widehat{B}^1_1, \dots, \widehat{B}^1_l$ of the $\widehat{B}^1(\underline{a}^1, \dots, \underline{a}^n)$ of disjoint balls such that $B^1 \subset \widetilde{B}_1 \cup \dots \cup \widetilde{B}_l$, where $\widetilde{B}_j$ is the ball with the same center as $\widehat{B}^1_j$ but with three times the radius. Then

$$\begin{aligned} c_{12} \ge I_{m'_1}(\mu_1(\underline{\widetilde{\theta}}^1, \dots, \underline{\widetilde{\theta}}^n, \widetilde{s})) &\ge \sum_{j=1}^{l} \int_{\widehat{B}^1_j \times \widetilde{B}_j} \frac{d\mu_1(\underline{\widetilde{\theta}}^1, \dots, \underline{\widetilde{\theta}}^n, \widetilde{s}) d\mu_1(\underline{\widetilde{\theta}}^1, \dots, \underline{\widetilde{\theta}}^n, \widetilde{s})}{|x-y|^{m'_1}} \\ &\ge \sum_{j=1}^{l} \frac{\mu_1(\underline{\widetilde{\theta}}^1, \dots, \underline{\widetilde{\theta}}^n, \widetilde{s})(\widehat{B}^1_j)}{(24c_{15}\rho^{1/2r})^{m'_1}} \mu_1(\underline{\widetilde{\theta}}^1, \dots, \underline{\widetilde{\theta}}^n, \widetilde{s})(\widetilde{B}_j) \end{aligned}$$

and thus $\mu_1(\underline{\widetilde{\theta}}^1, \dots, \underline{\widetilde{\theta}}^n, \widetilde{s})(B^1) \le 24^{m'_1} c_{15}^{(1+m'_1)} c_{13} c_{12}$. □

Now, we construct the n-uples $(b^1, \dots, b^n)$ amongst which the n-uples $(b^1_j, \dots, b^n_j)$ of definition of $L^0_{\underline{\omega}_1,\dots,\underline{\omega}_k}(\underline{\theta}^1, \dots, \underline{\theta}^n, s)$ must be looked for. We make the following easy observation:

Lemma 5.4. *Let $\underline{\theta}^i \in \widehat{\Sigma}_i^-$. The number of words $\underline{c}^i \in \widetilde{\Sigma}_i(\rho^{1/2})$ compatible with $\underline{\theta}^i$ such that $\underline{\theta}^i \vee \underline{c}^i \notin \widehat{\Sigma}_i^-$ is $o(\rho^{-1/2d_i})$ as $\rho \to 0$, uniformly in $\underline{\theta}^i$.*

It follows from conclusion (iii) of the Multidimensional Scale Recurrence Lemma and the last observation that we can find at least $\frac{1}{2}c_3\rho^{-d/2}$ n-uples $(\underline{c}_j^1, \ldots, \underline{c}_j^n) \in \widetilde{\Sigma}_1(\rho^{1/2}) \times \ldots \times \widetilde{\Sigma}_n(\rho^{1/2})$ such that

$$T^1_{\underline{c}_j^1} \ldots T^n_{\underline{c}_j^n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) = (\underline{\theta}_j^1, \ldots, \underline{\theta}_j^n, s_j) \in \widetilde{\mathcal{L}}.$$

We can for each j find at least $\frac{1}{2}c_3\rho^{-d/2}$ n-uples $(\underline{d}_{jl}^1, \ldots, \underline{d}_{jl}^n) \in \widetilde{\Sigma}_1(\rho^{1/2}) \times \ldots \times \widetilde{\Sigma}_n(\rho^{1/2})$ such that writing $T^1_{\underline{d}_{jl}^1} \ldots T^n_{\underline{d}_{jl}^n}(\underline{\theta}^1, \ldots, \underline{\theta}_j^n, s_j) = (\underline{\theta}_{jl}^1, \ldots, \underline{\theta}_{jl}^n, s_{jl})$, we have

$$(\underline{\theta}_{jl}^1, \ldots, \underline{\theta}_{jl}^n, s') \in \widetilde{\mathcal{L}} \text{ if } |s' - s_{jl}| \leq \rho^{1/2}.$$

Concatenation of the $\underline{c}_j^1, \ldots, \underline{c}_j^n$ and $\underline{d}_{jl}^1, \ldots, \underline{d}_{jl}^n$ give a family of words $\underline{b}_{jl}^1, \ldots, \underline{b}_{jl}^n$ in $\Sigma_1'(\rho) \times \ldots \times \Sigma_n'(\rho)$ with at least $\frac{1}{4}c_3^2\rho^{-d}$ elements.

Lemma 5.5. *If c_{13} has been chosen sufficiently small, there are at least $\frac{1}{6}c_3c_{15}^{-2}\rho^{-\frac{d}{2r}}$ n-uples $(\underline{a}^1, \ldots, \underline{a}^n) \in \Sigma_1^2 \times \ldots \times \Sigma_n^2$ which are good and which satisfy*

$$\left\| \chi_{T^1_{\underline{a}^1} \ldots T^n_{\underline{a}^n}(\underline{\widetilde{\theta}}^1, \ldots, \underline{\widetilde{\theta}}^n, \widetilde{s})} \right\|_{L^2}^2 \leq c_{13}^{-1}$$

and such that at least $\frac{1}{6}c_3c_{15}^{-1}\rho^{-d(\frac{1}{2}-\frac{1}{2r})}$ n-uples $(\underline{c}_j^1, \ldots, \underline{c}_j^n)$ start with $(\underline{a}^1, \ldots, \underline{a}^n)$.

The proof is exactly as [5]. Let us call excellent those n-uples $(\underline{a}^1, \ldots, \underline{a}^n)$ given by the last lemma.

For each $(\underline{c}^1, \ldots, \underline{c}^n) \in \widetilde{\Sigma}_1(\rho^{1/2}) \times \ldots \times \widetilde{\Sigma}_n(\rho^{1/2})$, there are points $x(\underline{c}^1, \ldots, \underline{c}^n)$ and $\widetilde{x}(\underline{c}^1, \ldots, \underline{c}^n)$ in $\mathbb{R}^k$ with distance at most $c_{15}\rho^{1/2}$ such that $J^{\underline{\omega}^*}(\underline{c}^1, \ldots, \underline{c}^n)$ is contained in the closed ball $B(\underline{c}^1, \ldots, \underline{c}^n)$ with center $x(\underline{c}^1, \ldots, \underline{c}^n)$ and radius $c_{15}\rho^{1/2r}$ for all $\underline{\omega}^*$; and $\widetilde{J}(\underline{c}^1, \ldots, \underline{c}^n)$ is contained in the closed ball $\widetilde{B}(\underline{c}^1, \ldots, \underline{c}^n)$ with center $\widetilde{x}(\underline{c}^1, \ldots, \underline{c}^n)$ and radius $c_{15}\rho^{1/2r}$. Then $\widetilde{B}'(\underline{c}^1, \ldots, \underline{c}^n)$, the ball with the same center as $\widetilde{B}(\underline{c}^1, \ldots, \underline{c}^n)$ but with two times the radius, contains $B(\underline{c}^1, \ldots, \underline{c}^n)$.

Let $(\underline{a}^1, \ldots, \underline{a}^n)$ be an excellent n-uple. Let $\widetilde{J}_1(\underline{a}^1, \ldots, \underline{a}^n)$ be the union over the n-uples $(\underline{c}_j^1, \ldots, \underline{c}_j^n)$ start with $(\underline{a}^1, \ldots, \underline{a}^n)$ of the $\widetilde{J}(\underline{c}_j^1, \ldots, \underline{c}_j^n)$. Cause $(\underline{a}^1, \ldots, \underline{a}^n)$ is excellent, it follows from Remark 5.1 that

$$|\widetilde{J}_1(\underline{a}^1, \ldots, \underline{a}^n)| \geq c_{16}c_{13}\rho^{k/2r}.$$

Let $J_1^{\underline{\omega}^*}(\underline{a}^1, \ldots, \underline{a}^n)$ be the union over the n-uples $(\underline{c}_j^1, \ldots, \underline{c}_j^n)$ start with $(\underline{a}^1, \ldots, \underline{a}^n)$ of the $J^{\underline{\omega}^*}(\underline{c}_j^1, \ldots, \underline{c}_j^n)$. We note that $|J^{\underline{\omega}^*}(\underline{c}_j^1, \ldots, \underline{c}_j^n)| \geq c_{15}^{-1}\rho^{\frac{k}{2}}$ for all $\underline{\omega}^*$. Applying the Vitali covering lemma to the collection of balls $\widetilde{B}'(\underline{c}_j^1, \ldots, \underline{c}_j^n)$ for those $(\underline{c}_j^1, \ldots, \underline{c}_j^n)$, we get

$$|J_1^{\underline{\omega}^*}(\underline{a}^1, \ldots, \underline{a}^n)| \geq C_k c_{15}^{-(1+k)}|\widetilde{J}_1(\underline{a}^1, \ldots, \underline{a}^n)| \geq c_{17}c_{13}\rho^{k/2r}.$$

Similarly, for each $(\underline{c}_j^1, \ldots, \underline{c}_j^n)$ (starting with $(\underline{a}^1, \ldots, \underline{a}^n)$), let $J_1^{\underline{\omega}^*}(\underline{c}_j^1, \ldots, \underline{c}_j^n)$ be the union of the $J^{\underline{\omega}^*}(\underline{b}_{jl}^1, \ldots, \underline{b}_{jl}^n)$ for $\underline{b}_{jl}^1, \ldots, \underline{b}_{jl}^n$ starting with $(\underline{c}_j^1, \ldots, \underline{c}_j^n)$. There are at least $\frac{1}{2}c_3\rho^{-d/2}$ such $(\underline{b}_{jl}^1, \ldots, \underline{b}_{jl}^n)$, and $T^1_{\underline{c}_j^1} \ldots T^n_{\underline{c}_j^n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) = (\underline{\theta}_j^1, \ldots, \underline{\theta}_j^n, s_j) \in \widetilde{\mathcal{L}}$.

Therefore, again using Remark 5.1, we conclude that

$$|J_1^{\overline{\omega}^*}(\underline{c}_j^1, \ldots, \underline{c}_j^n)| \geq c_{18}|J^{\overline{\omega}^*}(\underline{c}_j^1, \ldots, \underline{c}_j^n)|.$$

(The argument is the same as that above: we first consider $(\widetilde{\underline{\theta}}_j^1, \ldots, \widetilde{\underline{\theta}}_j^n, \widetilde{s}_j)$ with $d(\underline{\theta}_j^1, \widetilde{\underline{\theta}}_j^1), \ldots, d(\underline{\theta}_j^n, \widetilde{\underline{\theta}}_j^n), |s_j - \widetilde{s}_j|$ of order $\rho^{1/2}$, such that $\left\| \chi_{\widetilde{\underline{\theta}}_j^1, \ldots, \widetilde{\underline{\theta}}_j^n, \widetilde{s}_j} \right\|_{L^2}^2 \leq c_{12}$)

Finally, let $J_2^{\overline{\omega}^*}(\underline{a}^1, \ldots, \underline{a}^n)$ be the union over those $(\underline{c}_j^1, \ldots, \underline{c}_j^n)$ starting with $(\underline{a}^1, \ldots, \underline{a}^n)$ of the sets $J_1^{\overline{\omega}^*}(\underline{c}_j^1, \ldots, \underline{c}_j^n)$. Note that $J_1^{\overline{\omega}^*}(\underline{c}_j^1, \ldots, \underline{c}_j^n)$ is subset of $J^{\overline{\omega}^*}(\underline{c}_j^1, \ldots, \underline{c}_j^n)$. Applying the Vitali covering lemma to the collection of balls $B(\underline{c}_j^1, \ldots, \underline{c}_j^n)$ for those $(\underline{c}_j^1, \ldots, \underline{c}_j^n)$, we get

$$|J_2^{\overline{\omega}^*}(\underline{a}^1, \ldots, \underline{a}^n)| \geq c_{19}c_{13}\rho^{k/2r}.$$

Let $\phi^{\overline{\omega}^*}$ be the sum, over excellent n-uples $(\underline{a}^1, \ldots, \underline{a}^n)$, of the characteristic functions of $J_2^{\overline{\omega}^*}(\underline{a}^1, \ldots, \underline{a}^n)$.

The number of excellent n-uples $(\underline{a}^1, \ldots, \underline{a}^n)$ is $\geq \frac{1}{6}c_3c_{15}^{-2}\rho^{-\frac{d}{2r}}$, and therefore

$$\int \phi^{\overline{\omega}^*} \geq c_{19}c_3c_{15^{-2}}c_{13}\rho^{-\frac{d-k}{2r}}.$$

On the other hand, because excellent pairs are good, one has

$$\phi^{\overline{\omega}^*} \leq c_{13}^{-1}\rho^{-(d-k)/2r}.$$

We conclude that

$$m(\phi^{\overline{\omega}^*} \geq c_{13}^2\rho^{-(d-k)/2r}) \geq c_{20}c_{13}^2 := c_{14}$$

if $c_{13} > 0$ is small enough (recall that the support of $\phi^{\overline{\omega}^*}$ is contained in $[-c_R, c_R]^k$)

Remark 5.6. If $k = n-1$ then, from $\mathfrak{m} = \mathfrak{m}(\pi, d_1, \ldots, d_n) \geq k$ we get $\dim(\text{span}\{\pi(e_i), i \in I\}) = k$ for all $I \subset \{1, \ldots, n\}$ with $\#I = k$ and so $\mathfrak{m} = d_1 + \ldots + d_n$.

This case is simpler. For instance $E(\underline{\theta}^1, \ldots, \underline{\theta}^n) := E_0(\underline{\theta}^1, \ldots, \underline{\theta}^n)$, $N = \left[c_{13}^2\rho^{-\frac{d-k}{2r}}\right]$ and the definition of *good* n-uple just need the part (1). All arguments are the same adapted to these changes.

6. Proof of the Proposition 4.1

We first recall the setting where $u = (\underline{\theta}^1, \ldots, \underline{\theta}^n, s, t) \in \widetilde{\mathcal{L}} \times \mathbb{R}^k$.

The set $\overline{\Omega}^1(u)$ is the set of parameters $(\underline{\omega}_1, \ldots, \underline{\omega}_k) \in \Omega_1 \times \ldots \times \Omega_k =: \overline{\Omega}$, such that there exists $\widetilde{u} = (\widetilde{\underline{\theta}}^1, \ldots, \widetilde{\underline{\theta}}^n, \widetilde{s}, \widetilde{t}) \in \widetilde{\mathcal{L}} \times \mathbb{R}^k$ with

$$d(\underline{\theta}^1, \widetilde{\underline{\theta}}^1) < 2\rho^{5/2}, \ldots, d(\underline{\theta}^n, \widetilde{\underline{\theta}}^n) < 2\rho^{5/2}, |s - \widetilde{s}| < 2\rho, |t - \widetilde{t}| < 2\rho,$$

and

$$\widetilde{t} \in L^0_{\underline{\omega}_1, \ldots, \underline{\omega}_k}(\widetilde{\underline{\theta}}^1, \ldots, \widetilde{\underline{\theta}}^n, \widetilde{s}).$$

The set $\overline{\Omega}^0(u)$ is the set os parameters $(\underline{\omega}_1, \ldots, \underline{\omega}_k) \in \overline{\Omega}$ for which there exist $(\underline{b}^1, \ldots, \underline{b}^n) \in \Sigma'_1(\rho) \times \ldots \times \Sigma'_n(\rho)$ (with $b_0^1 = \theta_0^1, \ldots, b_0^n = \theta_0^n$) such that the image

$$u' = T^{\underline{\omega}_1}_{\underline{b}_j^1} \ldots T^{\underline{\omega}_k}_{\underline{b}_j^k} T^{k+1}_{\underline{b}_j^{k+1}} \ldots T^n_{\underline{b}_j^n}(u) = (\underline{\theta}'^1, \ldots, \underline{\theta}'^n, s', t')$$

satisfies:

$$(\underline{\theta}'^1, \ldots, \underline{\theta}'^n, \widetilde{s}') \in \widetilde{\mathcal{L}} \text{ for } |\widetilde{s}' - s'| < \frac{1}{2}\rho^{1/2}$$

$$t' \in L^0_{\underline{\omega}_1,\ldots,\underline{\omega}_k}(\underline{\theta}'^1,\ldots,\underline{\theta}'^n,s').$$

We have to prove that, provided σ is large enough,

$$\mathbb{P}(\overline{\Omega}^1(u) - \overline{\Omega}^0(u)) \leq \exp(-c_4\rho^{-l})$$

where $\mathbb{P}$ is normalized Lebesgue measure on $\overline{\Omega}$.

Recall the decomposition of Subsection 2.5, associated to $\underline{\theta}^i, i = 1,\ldots,k$:

$$\begin{aligned}\Omega_i &= [-1,+1]^{\Sigma_i^1(\underline{\theta}^i)} \times [-1,+1]^{\Sigma_i^1 - \Sigma_i^1(\underline{\theta}^i)},\\ &= \Omega_i' \times \Omega_i''\\ \underline{\omega}_i &= (\underline{\omega}_i', \underline{\omega}_i''),\end{aligned}$$

which only depends on $\underline{\theta}^i$ through an endword in $\Sigma_i(\rho^{1/2r})$. We have set $\underline{\omega}_i^* = (0,\underline{\omega}_i'')$ in Subsection 2.5, and $L^0_{\underline{\omega}_1,\ldots,\underline{\omega}_k}(\underline{\theta}^1,\ldots,\underline{\theta}^n,s)$ actually only depend on $\underline{\omega}_1^*,\ldots,\underline{\omega}_k^*$ (or $\underline{\omega}_1'',\ldots,\underline{\omega}_k''$), *not* on $\underline{\omega}_1',\ldots,\underline{\omega}_k'$.

Therefore, the property $(\underline{\omega}_1,\ldots,\underline{\omega}_k) \in \overline{\Omega}^1(u)$ depends only on $\underline{\omega}_1'',\ldots,\underline{\omega}_k''$ (one has $\Sigma_i^1(\underline{\theta}^i) = \Sigma_i^1(\widetilde{\underline{\theta}}^i)$ as $d(\underline{\theta}^i,\widetilde{\underline{\theta}}^i) < 2\rho^{5/2}$).

We will fix $(\underline{\omega}_1'',\ldots,\underline{\omega}_k'') \in \Omega_1''\times\ldots\times\Omega_k'' = \overline{\Omega}''$. Then either $(\underline{\omega}_1^*,\ldots,\underline{\omega}_k^*) \notin \overline{\Omega}^1(u)$, and then $(\Omega_1'\times\{\underline{\omega}_1''\})\times\ldots\times(\Omega_k'\times\{\underline{\omega}_k''\})\cap\overline{\Omega}_1(u)$ is empty, or $(\underline{\omega}_1^*,\ldots,\underline{\omega}_k^*) \in \overline{\Omega}^1(u)$. In this last case, we will prove that

$$\mathbb{P}_{\overline{\Omega}'}(\overline{\Omega}' - \overline{\Omega}'^0(u)) \leq \exp(-c_4\rho^{-l}),$$

where

$$\begin{aligned}\overline{\Omega}' &= \Omega_1' \times \ldots \times \Omega_k',\\ \overline{\Omega}'^0(u) &= \left\{(\underline{\omega}_1',\ldots,\underline{\omega}_k') \in \overline{\Omega}', (\underline{\omega}_1',\underline{\omega}_1'',\ldots,\underline{\omega}_k',\underline{\omega}_k'') \in \overline{\Omega}^0(u)\right\},\end{aligned}$$

and $\mathbb{P}_{\overline{\Omega}'}$ is Lebesgue measure normalized on $\overline{\Omega}'$. The desired result will then follow by Fubini's theorem.

From now on, $(\underline{\omega}_1'',\ldots,\underline{\omega}_k'') \in \overline{\Omega}''$ is fixed, with $(\underline{\omega}_1^*,\ldots,\underline{\omega}_k^*) \in \overline{\Omega}^1(u)$. This means that there exist $\widetilde{u} = (\widetilde{\underline{\theta}}^1,\ldots,\widetilde{\underline{\theta}}^n,\widetilde{s},\widetilde{t}) \in \widetilde{\mathcal{L}} \times \mathbb{R}^k$ with

$$d(\underline{\theta}^1,\widetilde{\underline{\theta}}^1) < 2\rho^{5/2},\ldots,d(\underline{\theta}^n,\widetilde{\underline{\theta}}^n) < 2\rho^{5/2}, |s-\widetilde{s}| < 2\rho, |t-\widetilde{t}| < 2\rho,$$

and

$$\widetilde{t} \in L^0_{\underline{\omega}_1^*,\ldots,\underline{\omega}_k^*}(\widetilde{\underline{\theta}}^1,\ldots,\widetilde{\underline{\theta}}^n,\widetilde{s}).$$

By definition of $L^0_{\underline{\omega}_1^*,\ldots,\underline{\omega}_k^*}$, there exits n-uples $(\underline{b}_j^1,\ldots,\underline{b}_j^n)$, $1 \leq j \leq N$, in $\Sigma_1'(\rho)\times\ldots\times\Sigma_n'(\rho)$, compatibles with $\underline{\theta}^1,\ldots,\underline{\theta}^n$, with $N = \left[\rho^{-l}\right]$, such that, with

$$\widetilde{u}_j := T^{\underline{\omega}_1^*}_{\underline{b}_j^1}\ldots T^{\underline{\omega}_k^*}_{\underline{b}_j^k} T^{k+1}_{\underline{b}_j^{k+1}}\ldots T^n_{\underline{b}_j^n}(\widetilde{\underline{\theta}}^1,\ldots,\widetilde{\underline{\theta}}^n,\widetilde{s},\widetilde{t}) = (\widetilde{\underline{\theta}}_j^1,\ldots,\widetilde{\underline{\theta}}_j^n,\widetilde{s}_j,\widetilde{t}_j),$$

we have:

(i) for $i = 1,\ldots,k$, the words $\underline{b}_1^i,\ldots,\underline{b}_N^i$ are pairwise independent;
(ii) for $1 \leq j \leq N$, and $|\widetilde{\widetilde{s}}_j - \widetilde{s}_j| \leq \frac{2}{3}\rho^{1/2}$, $(\widetilde{\underline{\theta}}_j^1,\ldots,\widetilde{\underline{\theta}}_j^n,\widetilde{\widetilde{s}}_j) \in \widetilde{\mathcal{L}}$;
(iii) for $1 \leq j \leq N$, $|\widetilde{t}_j| \leq 2c_R$.

For $1 \leq j \leq N$, consider

$$u_j := T^{\underline{\omega}_1^*}_{\underline{b}_j^1} \dots T^{\underline{\omega}_k^*}_{\underline{b}_j^k} T^{k+1}_{\underline{b}_j^{k+1}} \dots T^n_{\underline{b}_j^n}(\underline{\theta}^1, \dots, \underline{\theta}^n, s, t) = (\underline{\theta}_j^1, \dots, \underline{\theta}_j^n, s_j, t_j),$$

and more generally, for $(\underline{\omega}'_1, \dots, \underline{\omega}'_k) \in \overline{\Omega}'$, $(\underline{\omega}_1, \dots, \underline{\omega}_k) = (\underline{\omega}'_1, \underline{\omega}''_1, \dots, \underline{\omega}'_k, \underline{\omega}''_k)$:

$$\begin{aligned} u_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k) :=& T^{\underline{\omega}_1}_{\underline{b}_j^1} \dots T^{\underline{\omega}_k}_{\underline{b}_j^k} T^{k+1}_{\underline{b}_j^{k+1}} \dots T^n_{\underline{b}_j^n}(\underline{\theta}^1, \dots, \underline{\theta}^n, s, t) \\ =& (\underline{\theta}_j^1, \dots, \underline{\theta}_j^n, s_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k), t_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k)). \end{aligned}$$

Lemma. *For $1 \leq j \leq N$, and all $(\underline{\omega}'_1, \dots, \underline{\omega}'_k) \in \overline{\Omega}'$, $(\underline{\theta}_j^1, \dots, \underline{\theta}_j^n, \widehat{s}_j) \in \widetilde{\mathcal{L}}$ if $|\widehat{s}_j - s_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k)| < \frac{1}{2}\rho^{1/2}$.*

Proof. By the equation (2.1) we have $d(\underline{\theta}_j^1, \widetilde{\underline{\theta}}_j^1) < c\rho^{7/2}, \dots, d(\underline{\theta}_j^n, \widetilde{\underline{\theta}}_j^n) < c\rho^{7/2}$, for $1 \leq j \leq N$, then $\widetilde{\mathcal{L}}$ has the same fiber over $(\underline{\theta}_j^1, \dots, \underline{\theta}_j^n)$ and over $(\widetilde{\underline{\theta}}_j^1, \dots, \widetilde{\underline{\theta}}_j^n)$. By the relation (2.3) we have

$$|\widetilde{s}_j - s_j| \leq c|\widetilde{s} - s| + c\rho^{5/2} \leq 3c\rho \leq \frac{1}{12}\rho^{1/2},$$

if ρ is small enough. Also, by Lemma 2.10.1:

$$|s_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k) - s_j| \leq c\sigma\rho^{1-1/2r} \leq \frac{1}{12}\rho^{1/2}.$$

The result follows by property (ii) above. □

Let $(\underline{\omega}'_1, \dots, \underline{\omega}'_k) \in \overline{\Omega}'$, in view of the lemma above, if there exists $1 \leq j \leq N$ with

$$t_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k) \in L^0_{\underline{\omega}_1, \dots, \underline{\omega}_k}(\underline{\theta}_j^1, \dots, \underline{\theta}_j^n, s_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k))$$

then $(\underline{\omega}'_1, \dots, \underline{\omega}'_k) \in \overline{\Omega}'^0$.

For $i = 1, \dots, k$; let $\underline{a}^i \in \Sigma_i^{2-}$ be the endword of $\underline{\theta}^i$ in $\Sigma_i^{2-} (\subset \Sigma_i(\rho^{1/2r}))$. Now, for $1 \leq j \leq N$, let $\underline{a}_j^i$ be the beginning of $\underline{a}^i \vee \underline{b}_j^i$ in $\Sigma_i^1(\underline{\theta}^i)$ (we recall that $\underline{\theta}_j^i \in \widehat{\Sigma}_i^-$). Because $\underline{b}_1^i, \dots, \underline{b}_N^i$ are pairwise independent, the elements $\underline{a}_1^i, \dots, \underline{a}_N^i$ of $\Sigma_i^1(\underline{\theta}^i)$ are distinct.

Let $\Gamma_i = \{\underline{a}_1^i, \dots, \underline{a}_N^i\}$; we will denote by ω_i^j $(1 \leq j \leq N)$ the coordinate of $\underline{\omega}'_i$ corresponding to $\underline{a}_j^i$. Consider the decomposition

$$\begin{aligned} \Omega'_i &= [-1, +1]^{\Gamma_i} \times [-1, +1]^{\Sigma_i^1(\underline{\theta}^i) - \Gamma_i}, \\ &= \widehat{\Omega}'_i \times \widetilde{\Omega}'_i \\ \underline{\omega}'_i &= (\widehat{\underline{\omega}}'_i, \widetilde{\underline{\omega}}'_i) = ((\omega_i^1, \dots, \omega_i^N), \widetilde{\underline{\omega}}'_i). \end{aligned}$$

We will prove that for any $(\widetilde{\underline{\omega}}'_1, \dots, \widetilde{\underline{\omega}}'_k) \in \widetilde{\Omega}'_1 \times \dots \times \widetilde{\Omega}'_i = \overline{\widetilde{\Omega}}'$

$$\mathbb{P}_{\overline{\widehat{\Omega}}'}(\overline{\widehat{\Omega}}' - \overline{\widehat{\Omega}}'^0) \leq \exp(-c_4\rho^{-l}),$$

where

$$\overline{\widehat{\Omega}}' = \widehat{\Omega}'_1 \times \dots \times \widehat{\Omega}'_i,$$

$$\overline{\widehat{\Omega}}'^0 = \left\{ (\widehat{\underline{\omega}}'_1, \dots, \widehat{\underline{\omega}}'_k) \in \overline{\widehat{\Omega}}', \exists j, t_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k) \in L^0_{\underline{\omega}_1, \dots, \underline{\omega}_k}(\underline{\theta}_j^1, \dots, \underline{\theta}_j^n, s_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k)) \right\}$$

and $\mathbb{P}_{\overline{\widehat{\Omega}}'}$ is Lebesgue measure normalized on $\overline{\widehat{\Omega}}'$. By Fubini's theorem, this will imply our statement.

For $\underline{\widetilde{\omega}}'_i \in \widetilde{\Omega}'_i$, set $\underline{\widehat{\omega}}_i = ((\underline{0}, \underline{\widetilde{\omega}}_i), \underline{\omega}''_i)$. For $1 \le j \le N$, define

$$L_j := L^{-1}_{\underline{\widehat{\omega}}_1, \dots, \underline{\widehat{\omega}}_k}(\underline{\theta}^1_j, \dots, \underline{\theta}^n_j, s_j(\underline{\widehat{\omega}}_1, \dots, \underline{\widehat{\omega}}_k)).$$

Choose $\overline{\underline{\theta}}^i \in \Sigma^-_i$ with $d(\overline{\underline{\theta}}^i, \underline{\theta}^i) < \rho^{5/2}$, such that the endword $\underline{a}^i \in \Sigma^{2-}_i$ of $\underline{\theta}^i$ (and $\overline{\underline{\theta}}^i$) does not appear elsewhere in $\overline{\underline{\theta}}^i$. Consider

$$T^{\omega_1}_{\underline{b}^1_j} \dots T^{\omega_k}_{\underline{b}^k_j} T^{k+1}_{\underline{b}^{k+1}_j} \dots T^n_{\underline{b}^n_j}(\overline{\underline{\theta}}^1, \dots, \overline{\underline{\theta}}^k, \underline{\theta}^{k+1}, \dots, \underline{\theta}^n, s, t)$$
$$= (\overline{\underline{\theta}}^1_j, \dots, \overline{\underline{\theta}}^k_j, \underline{\theta}^{k+1}_j, \dots, \underline{\theta}^n_j, \bar{s}_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k), \bar{t}_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k)).$$

Lemma 6.1. *$\bar{t}_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k)$ depends only (for fixed $\underline{\omega}''_1, \dots, \underline{\omega}''_k$) on $\omega^j_1, \dots, \omega^j_k$. Moreover, there exist $c'_{11} > 0$ such that*

$$m\left\{(\omega^j_1, \dots, \omega^j_k) \in [-1, +1]^k, \bar{t}_j(\omega^j_1, \dots, \omega^j_k) \in L_j\right\} \ge c'_{11}.$$

Lemma 6.2. *If $\bar{t}_j(\omega^j_1, \dots, \omega^j_k) \in L_j$, then*

$$t_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k) \in L^0_{\underline{\omega}_1, \dots, \underline{\omega}_k}(\underline{\theta}^1_j, \dots, \underline{\theta}^n_j, s_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k)).$$

The two lemmas imply Proposition 4.1, since $\mathbb{P}_{\overline{\widehat{\Omega}}'}(\overline{\widehat{\Omega}}' - \overline{\widehat{\Omega}}'^0) \le \left(1 - \frac{c'_{11}}{2^k}\right)^N$, and we recall that $N = \left[\rho^{-l}\right]$.

Proof of the Lemma 6.1. In [5] it is proved that the endpoints of $I^{\overline{\underline{\theta}}^i, \underline{\omega}^i}(\underline{b}^i_j)$ satisfied

$$k^{\overline{\underline{\theta}}^i, \underline{\omega}_i}(f^{\omega_i}_{\underline{b}^i_j}(0)) = B_i \circ k^{\overline{\overline{\underline{\theta}}}^i, \underline{\widehat{\omega}}_i}(f^{\omega^*_i}_{\underline{a}^i \vee \underline{b}^i_j}(0) + \sigma \omega^j_i \rho^{1+1/2r}),$$
$$k^{\overline{\underline{\theta}}^i, \underline{\omega}_i}(f^{\omega_i}_{\underline{b}^i_j}(1)) = B_i \circ k^{\overline{\overline{\underline{\theta}}}^i, \underline{\widehat{\omega}}_i}(f^{\omega^*_i}_{\underline{a}^i \vee \underline{b}^i_j}(1) + \sigma \omega^j_i \rho^{1+1/2r}),$$

where B_i is the affine map (with the appropriate orientation) sending $I(\underline{a})$ onto $I(\theta^i_0)$ and $\overline{\overline{\underline{\theta}}}^i$ is such that $\overline{\overline{\underline{\theta}}}^i \vee \underline{a}^i = \overline{\underline{\theta}}^i$. Neither B_i nor $k^{\overline{\overline{\underline{\theta}}}^i, \underline{\widehat{\omega}}_i}$ depends on $\underline{\omega}'_i$. Note that $c^{-1}\sigma\rho < \left|\partial_{\omega^j_i} k^{\overline{\underline{\theta}}^i, \underline{\omega}_i}(f^{\omega_i}_{\underline{b}^i_j}(0))\right| < c\sigma\rho$.

Then $\bar{t}_j$ as a function of $\underline{\omega}'_1, \dots, \underline{\omega}'_k$, only depends on $\omega^j_1, \dots, \omega^j_k$. On the other hand, we have $|L_j| > c_{11}$ by the Proposition 5.2, and $L_j \subset [-2c_R, 2c_R]^k$. So to end the proof of the lemma, we have just to control $\bar{t}_j(\underline{0})$ (independently of $\underline{\omega}''_1, \dots, \underline{\omega}''_k$). Let

$$T^{\omega^*_1}_{\underline{b}^1_j} \dots T^{\omega^*_k}_{\underline{b}^k_j} T^{k+1}_{\underline{b}^{k+1}_j} \dots T^n_{\underline{b}^n_j}(\overline{\underline{\theta}}^1, \dots, \overline{\underline{\theta}}^k, \underline{\theta}^{k+1}, \dots, \underline{\theta}^n, s, t) = (\overline{\underline{\theta}}^1_j, \dots, \overline{\underline{\theta}}^k_j, \underline{\theta}^{k+1}_j, \dots, \underline{\theta}^n_j, s^*_j, t^*_j)$$

We have just seen that in fact $\bar{t}_j(\underline{0}) = t^*_j$. On the other hand,

$$T^{\omega^*_1}_{\underline{b}^1_j} \dots T^{\omega^*_k}_{\underline{b}^k_j} T^{k+1}_{\underline{b}^{k+1}_j} \dots T^n_{\underline{b}^n_j}(\widetilde{\underline{\theta}}^1, \dots, \widetilde{\underline{\theta}}^n, \widetilde{s}, \widetilde{t}) = (\widetilde{\underline{\theta}}^1_j, \dots, \widetilde{\underline{\theta}}^n_j, \widetilde{s}_j, \widetilde{t}_j),$$

with $d(\underline{\theta}^1, \widetilde{\underline{\theta}}^1) < 3\rho^{5/2}, \dots, d(\underline{\theta}^k, \widetilde{\underline{\theta}}^k) < 3\rho^{5/2}$, $d(\underline{\theta}^{k+1}, \widetilde{\underline{\theta}}^{k+1}) < 2\rho^{5/2}, \dots, d(\underline{\theta}^n, \widetilde{\underline{\theta}}^n) < 2\rho^{5/2}$, $|s - \widetilde{s}| < 2\rho$, $|t - \widetilde{t}| < 2\rho$ and $|\widetilde{t}_j| \le 2c_R$.

Then $|t^*_j - \widetilde{t}_j| \le c'$, which implies $|\bar{t}_j(\underline{0})| \le 2c_R + c'$. □

Proof of the Lemma 6.2. By definitions of $L^{-1}_{\underline{\widehat{\omega}}_1,\dots,\underline{\widehat{\omega}}_k}(\underline{\theta}^1_j,\dots,\underline{\theta}^n_j,s_j(\underline{\widehat{\omega}}_1,\dots,\underline{\widehat{\omega}}_k))$ and $L^0_{\underline{\omega}_1,\dots,\underline{\omega}_k}(\underline{\theta}^1_j,\dots,\underline{\theta}^n_j,s_j(\underline{\omega}'_1,\dots,\underline{\omega}'_k))$, is sufficient to show that for all $(\underline{c}^1,\dots,\underline{c}^n)\in\Sigma'_1(\rho)\times\dots\times\Sigma'_n(\rho)$, if

$$T^{\underline{\widehat{\omega}}_1(j)}_{\underline{c}^1}\dots T^{\underline{\widehat{\omega}}_k(j)}_{\underline{c}^k}T^{k+1}_{\underline{c}^{k+1}}\dots T^n_{\underline{c}^n}(\underline{\theta}^1_j,\dots,\underline{\theta}^n_j,s_j(\underline{\widehat{\omega}}_1,\dots,\underline{\widehat{\omega}}_k),\bar{t}_j(\omega^j_1,\dots,\omega^j_k))=(\underline{\theta}'^1,\dots,\underline{\theta}'^n,\widehat{s}',\widehat{t}')$$

and

$$T^{\underline{\omega}_1(j)}_{\underline{c}^1}\dots T^{\underline{\omega}_k(j)}_{\underline{c}^k}T^{k+1}_{\underline{c}^{k+1}}\dots T^n_{\underline{c}^n}(\underline{\theta}^1_j,\dots,\underline{\theta}^n_j,s_j(\underline{\omega}'_1,\dots,\underline{\omega}'_k),t_j(\underline{\omega}'_1,\dots,\underline{\omega}'_k))=(\underline{\theta}'^1,\dots,\underline{\theta}'^n,s',t')$$

(where $\underline{\widehat{\omega}}_i(j)$ (resp. $\underline{\omega}_i(j)$) is obtained from $\underline{\widehat{\omega}}_i$ (resp. $\underline{\omega}_i$) by setting the coordinates in $\Sigma^1_i(\underline{\theta}^i_j)$ equal to 0), then

$$|s'-\widehat{s}'|\le\frac{1}{12}\rho^{1/2},$$
$$|t'-\widehat{t}'|\le c_R.$$

The first inequality is easy: by Lemma 2.10.1, we have

$$|s_j(\underline{\omega}'_1,\dots,\underline{\omega}'_k)-s_j(\underline{\widehat{\omega}}_1,\dots,\underline{\widehat{\omega}}_k)|\le c\sigma\rho^{1-1/2r}$$

and applying a second time the same estimate from this lemma (to compare $T^{\underline{\widehat{\omega}}_1(j)}_{\underline{c}^1}\dots T^{\underline{\widehat{\omega}}_k(j)}_{\underline{c}^k}$ and $T^{\underline{\omega}_1(j)}_{\underline{c}^1}\dots T^{\underline{\omega}_k(j)}_{\underline{c}^k}$) will give

$$|s'-\widehat{s}'|\le c\sigma\rho^{1-1/2r}$$

which is $\le\frac{1}{12}\rho^{1/2}$ for small enough ρ.

To prove the second inequality, we first compare $\bar{t}_j(\omega^j_1,\dots,\omega^j_k)$ and $t_j(\underline{\omega}'_1,\dots,\underline{\omega}'_k)$. As $d(\underline{\theta}^i,\overline{\underline{\theta}}^i)<\rho^{5/2}$, by (2.3) we have

$$|\bar{t}_j(\omega^j_1,\dots,\omega^j_k)-t_j(\underline{\omega}'_1,\dots,\underline{\omega}'_k)|\le c\rho^{3/2}.$$

We next prove the

Claim. $|s_j(\underline{\widehat{\omega}}_1,\dots,\underline{\widehat{\omega}}_k)-s_j(\underline{\omega}'_1,\dots,\underline{\omega}'_k)|\le c\sigma\rho^{1+1/2r}$.

Proof. As $d(\underline{\theta}^i,\overline{\underline{\theta}}^i)<\rho^{5/2}$, by (2.3) we have

$$|s_j(\underline{\widehat{\omega}}_1,\dots,\underline{\widehat{\omega}}_k)-\overline{s}_j(\underline{\widehat{\omega}}_1,\dots,\underline{\widehat{\omega}}_k)|\le c\rho^{5/2},$$
$$|s_j\underline{\omega}'_1,\dots,\underline{\omega}'_k)-\overline{s}_j(\underline{\omega}'_1,\dots,\underline{\omega}'_k)|\le c\rho^{5/2}.$$

Now we compare $\overline{s}_j(\underline{\widehat{\omega}}_1,\dots,\underline{\widehat{\omega}}_k),\overline{s}_j(\underline{\omega}'_1,\dots,\underline{\omega}'_k)$. The endpoints of $I^{\overline{\underline{\theta}}^i,\underline{\omega}^i}(\underline{b}^i_j)$ are

$$k^{\overline{\underline{\theta}}^i,\underline{\omega}_i}(f^{\underline{\omega}_i}_{\underline{b}^i_j}(0))=B_i\circ k^{\overline{\overline{\underline{\theta}}}^i,\underline{\widehat{\omega}}_i}(f^{\underline{\omega}^*_i}_{\underline{a}^i\vee\underline{b}^i_j}(0)+\sigma\omega^j_i\rho^{1+1/2r}),$$
$$k^{\overline{\underline{\theta}}^i,\underline{\omega}_i}(f^{\underline{\omega}_i}_{\underline{b}^i_j}(1))=B_i\circ k^{\overline{\overline{\underline{\theta}}}^i,\underline{\widehat{\omega}}_i}(f^{\underline{\omega}^*_i}_{\underline{a}^i\vee\underline{b}^i_j}(1)+\sigma\omega^j_i\rho^{1+1/2r}).$$

As $k^{\overline{\overline{\underline{\theta}}}^i,\underline{\widehat{\omega}}_i}$ are C^2-bounded, we obtain $\Big||I^{\overline{\underline{\theta}}^i,\underline{\widehat{\omega}}^i}(\underline{b}^i_j)||I^{\overline{\underline{\theta}}^i,\underline{\omega}^i}(\underline{b}^i_j)|^{-1}-1\Big|\le C\sigma\rho^{1+1/2r}$ (a reinforcement of Lemma 2.10.1) then

$$|\overline{s}_j(\underline{\widehat{\omega}}_1,\dots,\underline{\widehat{\omega}}_k)-\overline{s}_j(\underline{\omega}'_1,\dots,\underline{\omega}'_k)|\le c\sigma\rho^{1+1/2r},$$

proving the claim. □

Finally, let

$$T^{\underline{\widehat{\omega}}_1(j)}_{\underline{c}^1}\dots T^{\underline{\widehat{\omega}}_k(j)}_{\underline{c}^k}T^{k+1}_{\underline{c}^{k+1}}\dots T^n_{\underline{c}^n}(\underline{\theta}^1_j,\dots,\underline{\theta}^n_j,s_j(\underline{\omega}'_1,\dots,\underline{\omega}'_k),t_j(\underline{\omega}'_1,\dots,\underline{\omega}'_k))=(\underline{\theta}'^1,\dots,\underline{\theta}'^n,\widetilde{s}',\widetilde{t}').$$

Lemma 6.3. $|t' - \widetilde{t}'| \leq c\sigma\rho^{1/2r}$.

The lemma is proved bellow. Using it, we finish the proof of Lemma 6.2. Because $|\bar{t}_j(\omega_1^j, \dots, \omega_k^j) - t_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k)| \leq c\rho^{3/2}$ and $|s_j(\widehat{\underline{\omega}}_1, \dots, \widehat{\underline{\omega}}_k) - s_j(\underline{\omega}'_1, \dots, \underline{\omega}'_k)| \leq c\sigma\rho^{1+1/2r}$, we have

$$|\widetilde{t}' - \widehat{t}'| \leq c\sigma\rho^{1/2r},$$

and therefore

$$|t' - \widehat{t}'| \leq c\sigma\rho^{1/2r} \leq 1 + R,$$

for ρ small enough, as was required. □

Proof of the Lemma 6.3. Notice that

$$|t' - \widetilde{t}'| \leq c\rho^{-1} \max_{1 \leq i \leq k} |k^{\underline{\theta}_j^i, \underline{\omega}_i(j)}(f_{\underline{c}_i}^{\underline{\omega}_i(j)}(0)) - k^{\underline{\theta}_j^i, \widehat{\underline{\omega}}_i(j)}(f_{\underline{c}_i}^{\widehat{\underline{\omega}}_i(j)}(0))|.$$

We replace $\underline{\theta}_j^i$ by $\overline{\underline{\theta}}_j^i$; as now $d(\underline{\theta}_j^i, \overline{\underline{\theta}}_j^i) < c\rho^{7/2}$, we have

$$|k^{\underline{\theta}_j^i, \underline{\omega}_i(j)}(f_{\underline{c}_i}^{\underline{\omega}_i(j)}(0)) - k^{\overline{\underline{\theta}}_j^i, \underline{\omega}_i(j)}(f_{\underline{c}_i}^{\underline{\omega}_i(j)}(0))| \leq c\rho^{7/2},$$

and

$$|k^{\underline{\theta}_j^i, \widehat{\underline{\omega}}_i(j)}(f_{\underline{c}_i}^{\widehat{\underline{\omega}}_i(j)}(0)) - k^{\overline{\underline{\theta}}_j^i, \widehat{\underline{\omega}}_i(j)}(f_{\underline{c}_i}^{\widehat{\underline{\omega}}_i(j)}(0))| \leq c\rho^{7/2}.$$

Finally, in [5] it is proved that $k^{\overline{\underline{\theta}}_j^i, \underline{\omega}_i(j)}(f_{\underline{c}_i}^{\underline{\omega}_i(j)}(0))$ do not depend on $\omega_i^l, l \neq j$, and that the dependence on ω_i^j satisfies

$$\left| \frac{\partial k^{\overline{\underline{\theta}}_j^i, \underline{\omega}_i(j)}(f_{\underline{c}_i}^{\underline{\omega}_i(j)}(0))}{\partial \omega_i^j} \right| \leq c\sigma\rho^{1+1/2r}.$$

Therefore $|k^{\overline{\underline{\theta}}_j^i, \underline{\omega}_i(j)}(f_{\underline{c}_i}^{\underline{\omega}_i(j)}(0)) - k^{\overline{\underline{\theta}}_j^i, \widehat{\underline{\omega}}_i(j)}(f_{\underline{c}_i}^{\widehat{\underline{\omega}}_i(j)}(0))| \leq c\sigma\rho^{1+1/2r}$ which guarantees the estimate of the lemma. □

7. Proof of the Multidimensional Scale Recurrence Lemma

7.1. **A General setting.** We consider a finite alphabet A and a finite set Z with a maps

$$\begin{aligned} \alpha : Z &\to A & \qquad \omega : Z &\to A \\ \lambda &\mapsto \alpha(\lambda) & \qquad \lambda &\mapsto \omega(\lambda). \end{aligned}$$

Define

$$\begin{aligned} N_i^j &= \#\left\{\lambda \in Z, \alpha(\lambda) = i, \omega(\lambda) = j\right\}, \\ N_i &= \#\left\{\lambda \in Z, \alpha(\lambda) = i\right\}, \\ p_i^j &= N_i^{-1} N_i^j. \end{aligned}$$

The stochastic matrix (p_i^j) has a left eigenvector $(p^i)_{i \in A}$ satisfying

$$\sum_i p^i p_i^j = p^j, \quad \sum_i p^i = 1, \quad p^i \geq 0.$$

Remark 7.1. If $0 < c \leq p_i^j \leq c'$, then $c \leq p^i \leq c'$.

Now, if we set

$$p_\lambda^{\lambda'} = \begin{cases} 0 & \text{if } \omega(\lambda) \neq \alpha(\lambda') \\ N_{\alpha(\lambda')}^{-1} & \text{if } \omega(\lambda) = \alpha(\lambda'), \end{cases}$$
$$p^\lambda = N_{\alpha(\lambda)}^{-1} p^{\alpha(\lambda)},$$

then $(p_\lambda^{\lambda'})$ is again a stochastic matrix with left eigenvector (p^λ). Indeed we have (with $\lambda' \in A, \alpha(\lambda') = j$)

$$\begin{aligned} \sum_\lambda p^\lambda p_\lambda^{\lambda'} &= \sum_i \sum_{\{\lambda, \alpha(\lambda)=i, \omega(\lambda)=j\}} N_i^{-1} p^i N_j^{-1} \\ &= \sum_i p_i^j p^i N_j^{-1} = p^j N_j^{-1} = p^{\lambda'}. \end{aligned}$$

For $z = (z_\lambda)_{\lambda \in Z} \in \mathbb{C}^Z$, define

$$\|z\|^2 = \sum_\lambda p^\lambda |z_\lambda|^2 .$$

Remark 7.2. If $z = (z_\lambda)_{\lambda \in Z}, w = (w_\lambda)_{\lambda \in Z}$ are satisfying

$$|w_\lambda| \leq \sum_{\lambda'} p_\lambda^{\lambda'} |z_{\lambda'}| ,$$

then, using Cauchy-Schwarz inequality

$$|w_\lambda|^2 \leq \sum_{\lambda'} p_\lambda^{\lambda'} \sum_{\lambda'} p_\lambda^{\lambda'} |z_{\lambda'}|^2 = \sum_{\lambda'} p_\lambda^{\lambda'} |z_{\lambda'}|^2 ;$$

and therefore

$$\begin{aligned} \|w\|^2 = \sum_\lambda p^\lambda |w_\lambda|^2 &\leq \sum_\lambda \sum_{\lambda'} p^\lambda p_\lambda^{\lambda'} |z_{\lambda'}|^2 \\ &= \sum_{\lambda'} p^{\lambda'} |z_{\lambda'}|^2 = \|z\|^2 . \end{aligned}$$

Suppose we have a family $(a_\lambda^{\lambda'})$ of vectors of $\mathbb{R}^n$. By Remark 7.2, for $\xi \in \mathbb{R}^n$, the linear operator $w = U_\xi(z)$ defined by

$$w_\lambda = \sum_{\lambda'} p_\lambda^{\lambda'} e^{i\xi \cdot a_\lambda^{\lambda'}} z_{\lambda'},$$

acting on $(\mathbb{C}^Z, \|.\|)$ with norm ≤ 1.

Assumption 7.3. There is $0 < \kappa_0 < 1$ such that the operator U_ξ has norm $\|U_\xi\| \leq \kappa_0$, for any $\xi \in \mathbb{R}^n$ with $1 \leq |\xi|_\infty \leq \rho^{-1}$.

Proposition 7.4. *Under Assumption 7.3, there exist* $0 < \kappa_1 < 1, \varepsilon > 0, 0 < \tau < 1$*, depending only on* κ_0 *such that, for all family* $(E_\lambda)_{\lambda \in Z}$ *of bounded measurable subsets of* $\mathbb{R}^n$*, with measure* $|E_\lambda| \leq \varepsilon$*, we have*

$$\sum p^\lambda |E_\lambda^*| \leq \kappa_1 \sum p^\lambda |E_\lambda|$$

where $E_\lambda^* = \left\{x, \# \left\{\lambda', \alpha(\lambda') = \omega(\lambda), B_\rho(x) \subset E_{\lambda'} - a_\lambda^{\lambda'}\right\} > \tau N_{\omega(\lambda)}\right\}$.

Proof. For $\lambda \in Z$, let:

$$\begin{aligned} X_\lambda =& \chi_{E_\lambda}, \\ Y_\lambda(x) =& \frac{1}{N_{\omega(\lambda)}} \sum_{\alpha(\lambda')=\omega(\lambda)} X_{\lambda'}(x + a_\lambda^{\lambda'}), \\ Z_\lambda(x) =& \frac{1}{2^n \rho^n} \int_{(-\rho,\rho)^n} Y_\lambda(x+y)dy. \end{aligned}$$

Note that $E_\lambda^* \subset \{x, Z_\lambda(x) > \tau\}$

Claim. Existe $0 < \kappa_2 < 1$ depending only on κ_0 such that

$$\sum_\lambda p^\lambda \left|Z_\lambda\right|_{L^2}^2 \leq \kappa_2 \sum_\lambda p^\lambda \left|X_\lambda\right|_{L^2}^2 = \kappa_2 \sum_\lambda p^\lambda \left|E_\lambda\right|.$$

Proof of the Claim. By Plancherel theorem, it's equivalent to

$$\sum_\lambda p^\lambda |\widehat{Z}_\lambda|_{L^2}^2 \leq \kappa_2 \sum_\lambda p^\lambda |\widehat{X}_\lambda|_{L^2}^2,$$

considering the normalize Fourier transform as

$$\widehat{X}_\lambda(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} X_\lambda(x)dx.$$

Note that

$$\begin{aligned} \widehat{Y}_\lambda(\xi) =& \frac{1}{N_{\omega(\lambda)}} \sum_{\alpha(\lambda')=\omega(\lambda)} e^{ia_\lambda^{\lambda'}\cdot\xi} \widehat{X}_{\lambda'}(\xi) \\ =& \sum_{\lambda'} p_\lambda^{\lambda'} e^{ia_\lambda^{\lambda'}\cdot\xi} \widehat{X}_{\lambda'}(\xi), \end{aligned}$$

and

$$\widehat{Z}_\lambda(\xi) = \frac{\sin\rho\xi_1}{\rho\xi_1} \cdots \frac{\sin\rho\xi_n}{\rho\xi_n} \widehat{Y}_\lambda(\xi).$$

We estimated $\sum p^\lambda |\widehat{Z}_\lambda(\xi)|^2$ in various ways, depending on ξ:

a) If $|\xi|_\infty \leq 1$, we use

$$|\widehat{Z}_\lambda(\xi)| \leq |\widehat{Y}_\lambda(\xi)| \leq \sum_{\lambda'} p_\lambda^{\lambda'} |\widehat{X}_{\lambda'}(\xi)| \leq \sum_{\lambda'} p_\lambda^{\lambda'} \left|E_{\lambda'}\right|$$

to get, by Remark 7.2 and $|E_\lambda| = |X_\lambda|_{L^2}^2 = (2\pi)^{-n} |\widehat{X}_\lambda|_{L^2}^2$

$$\sum_\lambda p^\lambda |\widehat{Z}_\lambda(\xi)|^2 \leq \sum_\lambda p^\lambda \left|E_\lambda\right|^2 \leq \varepsilon(2\pi)^{-n} \sum_\lambda p^\lambda |\widehat{X}_\lambda|_{L^2}^2;$$

b) If $1 \leq |\xi|_\infty \leq \rho^{-1}$, we use the Assumption 7.3 in

$$|\widehat{Z}_\lambda(\xi)| \leq |\widehat{Y}_\lambda(\xi)| = \left(U_\xi(\widehat{X})\right)_\lambda$$

to get

$$\sum_\lambda p^\lambda |\widehat{Z}_\lambda(\xi)|^2 \leq \kappa_0^2 \sum_\lambda p^\lambda |\widehat{X}_\lambda(\xi)|^2;$$

c) If $|\xi|_\infty \geq \rho^{-1}$, we have $|\widehat{Z}_\lambda(\xi)| \leq \kappa_3 |\widehat{Y}_\lambda(\xi)|$, where

$$\kappa_3 = \max_{t \geq 1} \left| \frac{\sin t}{t} \right|;$$

hence, by $\|U_\xi\| \leq 1$

$$\sum_\lambda p^\lambda |\widehat{Z}_\lambda(\xi)|^2 \leq \kappa_3^2 \sum_\lambda p^\lambda |\widehat{X}_\lambda(\xi)|^2.$$

Putting these estimates together gives

$$\int \sum_\lambda p^\lambda |\widehat{Z}_\lambda(\xi)|^2 \, d\xi \leq \varepsilon \pi^{-n} \sum_\lambda p^\lambda |\widehat{X}_\lambda|_{L^2}^2 + [\max(\kappa_0, \kappa_3)]^2 \int \sum_\lambda p^\lambda |\widehat{X}_\lambda(\xi)|^2 \, d\xi,$$

or

$$\sum_\lambda p^\lambda |\widehat{Z}_\lambda|_{L^2}^2 \leq \kappa_2 \sum_\lambda p^\lambda |\widehat{X}_\lambda|_{L^2}^2,$$

with $\kappa_2 = [\max(\kappa_0, \kappa_3)]^2 + \pi^{-n}\varepsilon$. If ε is small enough, $\kappa_2 < 1$, wish concludes the proof of the claim. □

To finish the proof of proposition, note

$$\sum_\lambda p^\lambda \left| E_\lambda^* \right| \leq \frac{1}{\tau^2} \sum_\lambda p^\lambda |Z_\lambda|_{L^2}^2 \leq \frac{\kappa_2}{\tau^2} \sum_\lambda p^\lambda \left| E_\lambda \right|,$$

and put $\kappa_1 = \tau = \kappa_2^{\frac{1}{3}}$. □

Remark 7.5. The main point of the proposition is that $\tau, \varepsilon, \kappa_1$ depend neither on the ρ nor on the E_λ nor on the $a_\lambda^{\lambda'}$ or the combinatorics.

We denote by $V_\delta(E)$ the δ-neighborhood of a subset $E \subset (\mathbb{R}^n, |.|_\infty)$ (i.e. the set of points at distance $< \delta$ from E)

Corollary 7.6. *Under Assumption 7.3, there exist* $0 < \kappa_4 < 1, \varepsilon_0 > 0, 0 < \tau < 1$, *depending only on* κ_0, *and there exist* $\Delta > 0$, *depending only on* Δ_1, κ_0 *such that, for all family* $(E_\lambda)_{\lambda \in Z}$ *of bounded measurable subsets of* $\mathbb{R}^n$, *with* $|V_{\Delta\rho}(E_\lambda)| \leq \varepsilon_0$, *we have*

$$\sum_\lambda p^\lambda |V_{\Delta\rho}(\widetilde{E}_\lambda)| \leq \kappa_4 \sum_\lambda p^\lambda \left| V_{\Delta\rho}(E_\lambda) \right|$$

where $\widetilde{E}_\lambda = \left\{ x, \# \left\{ \lambda', \alpha(\lambda') = \omega(\lambda), B_{\Delta_1\rho}(x) \cap (E_{\lambda'} - a_\lambda^{\lambda'}) \neq \emptyset \right\} > \tau N_{\omega(\lambda)} \right\}$.

Proof. We first observe that, for any bounded subset E,

$$\left| V_{(\Delta+\Delta_1+1)\rho}(E) \right| \leq \left(1 + \frac{\Delta_1 + 1}{\Delta} \right)^n |V_{\Delta\rho}(E)|$$

On the other hand, if we consider the new family $\overline{E}_\lambda := V_{(\Delta+\Delta_1+1)\rho}(E_\lambda)$, then

$$V_{\Delta\rho}(\widetilde{E}_\lambda) \subset \overline{E}_\lambda^*.$$

We thus take Δ large enough to have

$$\kappa_4 := \left(1 + \frac{\Delta_1 + 1}{\Delta} \right)^n \kappa_1 < 1, \quad \varepsilon_0 := \left(1 + \frac{\Delta_1 + 1}{\Delta} \right)^{-n} \varepsilon,$$

and apply the Proposition 7.4 to the family $\overline{E}_\lambda$. □

Corollary 7.7. *Under Assumption 7.3, and assuming $N_i^j \asymp_c \#Z$, there exist $0 < \kappa_4 < 1, \Delta > 0$ as above, and $\varepsilon_1 > 0, 0 < \tau_1 < 1$, depending only on $\kappa_0, c, \#A$, such that, for all family $(E_\lambda)_{\lambda \in Z}$ of bounded measurable subsets of $\mathbb{R}^n$, with $\sum_\lambda p^\lambda \, |V_{\Delta\rho}(E_\lambda)| \le \varepsilon_1$, we have*

$$\sum_\lambda p^\lambda |V_{\Delta\rho}(\widehat{E}_\lambda)| \le \kappa_4 \sum_\lambda p^\lambda \, |V_{\Delta\rho}(E_\lambda)|$$

where $\widehat{E}_\lambda = \left\{ x, \# \left\{ \lambda', \alpha(\lambda') = \omega(\lambda), B_{\Delta_1\rho}(x) \cap (E_{\lambda'} - a_\lambda^{\lambda'}) \neq \emptyset \right\} > \tau_1 N_{\omega(\lambda)} \right\}$.

Proof. Note that $N_i \asymp_{c'} \#Z$, $p^\lambda \asymp_{c''} \#Z^{-1}$ for $c' = c\#A$, $c'' = c(c')^2$. Let ε_0, τ from Corollary 7.6 and take ε_1 small enough to have $\tau_1 := \tau + c'c''\varepsilon_1/\varepsilon_0 < 1$.

Suppose $\sum_\lambda p^\lambda \, |V_{\Delta\rho}(E_\lambda)| \le \varepsilon_1$. Defining $\Lambda := \{\lambda, |V_{\Delta\rho}(E_\lambda)| > \varepsilon_0\}$ we have

$$\#\Lambda \le c'' \frac{\varepsilon_1}{\varepsilon_0} \#Z.$$

Now consider the new family $E'_\lambda := E_\lambda$ if $\lambda \in \Lambda$, and $E'_\lambda = \emptyset$ otherwise, then

$$\widehat{E}_\lambda \subset \widetilde{E'_\lambda}.$$

The result follow applying Corollary 7.6 to the family $(E'_\lambda)_\lambda$. □

7.2. **Setting of the Multidimensional Scale Recurrence Lemma.** Recall that $\Sigma_i(\rho)$ is the set of words $\underline{a}^i$ in Σ_i such that

$$c_0^{-1}\rho \le \left| I(\underline{a}^i) \right| \le c_0\rho$$

and that

$$J_R = [-R, -R^{-1}] \cup [R^{-1}, R].$$

We define

$$\mathcal{I}_0 = \left\{ (\varepsilon(\underline{a}^1, \underline{a}^n), \ldots, \varepsilon(\underline{a}^{n-1}, \underline{a}^n)), (\underline{a}^1, \ldots, \underline{a}^n) \in \Sigma_1(\rho) \times \ldots \times \Sigma_n(\rho) \right\}.$$

If c_0 is large enough, $\mathcal{I}_0$ is a subgroup of the multiplicative group $\mathcal{I} = \{-1, +1\}^{n-1}$, with multiplication $u * v = (u_1 v_1, \ldots, u_{n-1} v_{n-1})$ for $u = (u_1, \ldots, u_{n-1})$, $v = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1}$.

In relation to the abstract setting, we set

$$A = \mathbb{A}_1 \times \ldots \times \mathbb{A}_n \times \mathcal{I}_0$$
$$Z = \Sigma_1(\rho) \times \ldots \times \Sigma_n(\rho) \times \mathcal{I}_0.$$

The maps $\alpha : Z \to A$, $\omega : Z \to A$ are defined as follows: for $(\underline{a}^1, \ldots, \underline{a}^n) \in \Sigma_1(\rho) \times \ldots \times \Sigma_n(\rho)$, $u = (u_1, \ldots, u_{n-1}) \in \mathcal{I}_0$,

$$\alpha(\underline{a}^1, \ldots, \underline{a}^n, u) = (a_0^1, \ldots, a_0^n, u_1\varepsilon(\underline{a}^1, \underline{a}^n), \ldots, u_{n-1}\varepsilon(\underline{a}^{n-1}, \underline{a}^n))$$
$$\omega(\underline{a}^1, \ldots, \underline{a}^n, u) = (a_{l_1}^1, \ldots, a_{l_n}^n, u),$$

where $a_0^1, \ldots, a_0^n$ are the first letters of $\underline{a}^1, \ldots, \underline{a}^n$, and $a_{l_1}^1, \ldots, a_{l_n}^n$ are the last letters.

It follows from the hypothesis that $\Sigma_1, \ldots, \Sigma_n$ are topologically mixing, that if c_0 has been chosen large enough, for all $i, j \in A$,

$$c^{-1}\rho^{-d} \le N_i^j \le c\rho^{-d},$$

where $d = d_1 + \ldots + d_n$. Hence, p^λ and $p_\lambda^{\lambda'}$ have order ρ^d.

Finally, we define the family of translations $(a_\lambda^{\lambda'})_{\alpha(\lambda')=\omega(\lambda)}$. To do this, we choose, for each $\underline{a}^i \in \Sigma_i(\rho)$ an element $\underline{\theta}^i \in \Sigma_i{}^-$ ending with $\underline{a}^i$. Then, if $\lambda = (\underline{a}^1, \ldots, \underline{a}^n, u)$ $\lambda' = (\underline{b}^1, \ldots, \underline{b}^n, v)$ satisfy $\alpha(\lambda') = \omega(\lambda)$, we set

$$a_\lambda^{\lambda'} = \left(\log|I^{\underline{\theta}^1}(\underline{b}^1)| - \log|I^{\underline{\theta}^n}(\underline{b}^n)|, \ldots, \log|I^{\underline{\theta}^{n-1}}(\underline{b}^{n-1})| - \log|I^{\underline{\theta}^n}(\underline{b}^n)|\right).$$

Lemma 7.8. *The hypothesis of the Multidimensional Scale Recurrence Lemma implies that the Assumption 7.3 holds, namely there exist* $0 < \kappa_0 < 1$ *such that for* $1 \le |\xi|_\infty \le \rho^{-1}$ *the operator* $w = U_\xi(z)$ *defined by*

$$w_\lambda = \sum_{\lambda'} p_\lambda^{\lambda'} e^{i\xi\cdot a_\lambda^{\lambda'}} z_{\lambda'}$$

has norm $\le \kappa_0$.

Proof. Let $\eta_0 \ll 1$. Suppose there are $1 \le |\xi|_\infty \le \rho^{-1}$ and $z \in \mathbb{C}^Z$ such that, with $w = U_\xi(z)$:

$$1 = \|z\|^2 = \sum p^\lambda |z_\lambda|^2,$$
$$1 - \eta_0 \le \|w\|^2 = \sum p^\lambda |w_\lambda|^2.$$

As [5],[6], we conclude that there are $\eta_1 = \eta_1(\eta_0) \ll 1$, $\eta_5 \ll 1$ and $\widetilde{Z} \subset Z$ with $\#(Z - \widetilde{Z}) \le \eta_1 \rho^{-d}$ such that: for all $\lambda, \widehat{\lambda} \in \widetilde{Z}$ with $\omega(\lambda) = \omega(\widehat{\lambda})$, there exist $\Phi \in \mathbb{R}$ such that

$$\left|\sin\left(\frac{1}{2}\xi\cdot(a_\lambda^{\lambda'} - a_{\widehat{\lambda}}^{\lambda'}) + \Phi\right)\right| < \eta_5$$

for all λ' with $\alpha(\lambda') = \omega(\lambda)$ but at most $\eta_5\rho^{-d}$ elements.

We pick $i \in \{1, \ldots, n-1\}$ such that $|\xi_i| = |\xi|_\infty$. To derive a contradiction with the Property 2.6 for K_i, we choose $\lambda = (\underline{a}^1, \ldots, \underline{a}^n, u)$, $\widehat{\lambda} = (\widehat{\underline{a}}^1, \ldots, \widehat{\underline{a}}^n, u)$ in $\widetilde{Z}$ with $\widehat{\underline{a}}^j = \underline{a}^j$ if $j \ne i$, such that the choosing $\underline{\theta}^i$ ending with $\underline{a}^i$ is in V_i and the choosing $\widehat{\underline{\theta}}^i$ ending with $\widehat{\underline{a}}^i$ is in $\widehat{V}_i$. In particular $\omega(\lambda) = \omega(\widehat{\lambda})$. Finally, note that for $\lambda' = (\underline{b}^1, \ldots, \underline{b}^n, v)$ we have

$$\left|\sin\left(\frac{1}{2}\xi\cdot(a_\lambda^{\lambda'} - a_{\widehat{\lambda}}^{\lambda'}) + \Phi\right)\right| = \left|\sin\left(\frac{1}{2}\xi_i \log\frac{I^{\widehat{\underline{\theta}}^i}(\underline{b}^i)}{I^{\underline{\theta}^i}(\underline{b}^i)} + \Phi\right)\right|.$$

□

We set $\widetilde{Z} = \widetilde{\Sigma}_1(\rho) \times \ldots \times \widetilde{\Sigma}_n(\rho) \times \mathcal{I}_0$, and for $\widetilde{\lambda}' \in \widetilde{Z}$ we extended the definitions of $\alpha(\widetilde{\lambda}')$ and $(a_\lambda^{\widetilde{\lambda}'})_{\alpha(\widetilde{\lambda}')=\omega(\lambda)}$ naturally. We call $\widetilde{\lambda} = (\widetilde{\underline{a}}^1, \ldots, \widetilde{\underline{a}}^n, \widetilde{u}) \in \widetilde{Z}$ an *extension* of $\lambda = (\underline{a}^1, \ldots, \underline{a}^n, u) \in Z$ if $\widetilde{\underline{a}}^i$ ends with $\underline{a}^i$, for $i = 1, \ldots, n$, and $\widetilde{u} = u$.

Now we will use the notations from Corollary 7.7. Denote $r = \log R$. Given a family $(E(\lambda))_\lambda$ of subset of $[-r, r]^{n-1}$, we define $\widehat{\widehat{E}}(\lambda)$ as the set of $x \in [-r, r]^{n-1}$ such that

$$\#\left\{\lambda' \text{ with extension } \widetilde{\lambda}', \alpha(\widetilde{\lambda}') = \omega(\lambda), B_{\Delta_1\rho}(x + a_\lambda^{\widetilde{\lambda}'}) \subset [-r, r]^{n-1} - E(\lambda')\right\}$$

is less than $(1 - \tau_1)N_{\omega(\lambda)}$. ($\widehat{\widehat{E}}(\lambda)$ are a version -with boundary- of the $\widehat{E}(\lambda)$.)

Fixed κ_5 such that $\kappa_4 < \kappa_5 < 1$.

Lemma 7.9. *There exist $c_0, r, \widetilde{c}_0$ conveniently large, such that, for any sufficiently small ρ, and $\sum_\lambda p^\lambda |V_{\Delta\rho}(E(\lambda))| \leq \varepsilon_1$, we have*

$$\sum_\lambda p^\lambda |V_{\Delta\rho}(\widehat{\widehat{E}}(\lambda))| \leq \kappa_5 \sum_\lambda p^\lambda |V_{\Delta\rho}(E(\lambda))|.$$

Proof. Notice that $|a_\lambda^{\lambda'}| \leq 2\log cc_0, \forall \lambda, \lambda' \in Z$ with $\alpha(\lambda') = \omega(\lambda)$, where $c > 0$ is such that $c^{-1} \leq |Dk^{\underline{\theta}^i}| \leq c$ for all $\underline{\theta}^i \in \Sigma_i^-$, $i = 1, \ldots, n$. Then we will have

$$\widehat{\widehat{E}}(\lambda) \cap [-r + 2\log cc_0 + \Delta_1\rho, r - 2\log cc_0 - \Delta_1\rho]^{n-1} \subset \widehat{E}(\lambda).$$

We will assume $\Delta_1\rho \leq \frac{1}{2}\log cc_0, \Delta\rho \leq \frac{1}{2}\log cc_0$. We put $R = 10L\log cc_0$ for some L large to be determined. We can find a cube D in $[-r, r]^{n-1}$ of side $9\log cc_0$, such that

$$\sum_\lambda p^\lambda |V_{\Delta\rho}(E(\lambda) \cap D)| \leq \frac{1}{L^{n-1}} \sum_\lambda p^\lambda |V_{\Delta\rho}(E(\lambda))|.$$

The set $[-r, r]^{n-1} - [-r + \frac{5}{2}\log cc_0, r - \frac{5}{2}\log cc_0,]^{n-1}$ is union of $(4L)^{n-1} - (4L-2)^{n-1}$ cubes of side $\frac{5}{2}\log cc_0$. Let B any of this cubes (with center x_B) and let $\lambda = (\underline{a}^1, \ldots, \underline{a}^n, u) \in Z$, then there are positive periodic words $\underline{c}^1 \in \Sigma_1, \ldots, \underline{c}^n \in \Sigma_n$, starting with $\underline{a}^1, \ldots, \underline{a}^n$, respectively, such that, for $t = (\log|I(\underline{c}^1)| - \log|I(\underline{c}^n)|, \ldots, \log|I(\underline{c}^{n-1})| - \log|I(\underline{c}^n)|)$, $x_B + t$ belongs to the cube D_1 of the same centers as D and side $\log cc_0$. Now, if $\lambda' = (\underline{b}^1, \ldots, \underline{b}^n, v) \in Z$ with $\alpha(\lambda') = \omega(\lambda)$, then $\widehat{\lambda}' = (\underline{c}^1 \vee \underline{b}^1, \ldots, \underline{c}^n \vee \underline{b}^n, v)$ is a extension of λ' with $\alpha(\widehat{\lambda}') = \omega(\lambda)$; and for any $y \in B$, applying (2.1), we have

$$\begin{aligned} |y + a_\lambda^{\widehat{\lambda}'} - (x_B + t)| &\leq |y - x_B| + |a_\lambda^{\lambda'}| + |a_\lambda^{\widehat{\lambda}'} - a_\lambda^{\lambda'} - t| \\ &\leq \frac{5}{4}\log cc_0 + 2\log cc_0 + c' \\ &\leq \frac{7}{2}\log cc_0, \end{aligned}$$

therefore $B_{\Delta_1\rho}(y + a_\lambda^{\widehat{\lambda}'}) \in D$. This mean that $\widehat{\widehat{E}}(\lambda) \cap B$ is contained in

$$\left\{x, \#\left\{\lambda', \alpha(\lambda') = \omega(\lambda), x \in V_{\Delta_1\rho}(E(\lambda') \cap D) - a_\lambda^{\widehat{\lambda}'}\right\} > \tau_1 N_{\omega(\lambda)}\right\},$$

hence $V_{\Delta\rho}(\widehat{\widehat{E}}(\lambda) \cap B) \subset \left\{\sum_{\lambda' \in N_{\omega(\lambda)}} \chi_{V_{(\Delta+\Delta_1)\rho}(E(\lambda') \cap D) - a_\lambda^{\widehat{\lambda}'}} > \tau_1 N_{\omega(\lambda)}\right\}$, to get

$$\begin{aligned} |V_{\Delta\rho}(\widehat{\widehat{E}}(\lambda) \cap B)| &\leq \frac{1}{\tau_1 N_{\omega(\lambda)}} \sum_{\lambda' \in N_{\omega(\lambda)}} |V_{(\Delta+\Delta_1)\rho}(E(\lambda') \cap D) - a_\lambda^{\widehat{\lambda}'}| \\ &\leq \frac{1}{\tau_1}\left(1 + \frac{\Delta_1}{\Delta}\right)^{n-1} \sum_{\lambda'} p_\lambda^{\lambda'} |V_{\Delta\rho}(E(\lambda') \cap D)|, \end{aligned}$$

therefore $\sum_\lambda |V_{\Delta\rho}(\widehat{\widehat{E}}(\lambda) \cap B)| \leq 2(\tau_1 L^{n-1})^{-1} \sum_\lambda p^\lambda |V_{\Delta\rho}(E(\lambda))|$, for $\Delta \gg \Delta_1$.

Finally, take L large enough to have $\kappa_4 + 2\frac{[(4L)^{n-1} - (4L-2)^{n-1}]}{\tau_1 L^{n-1}} < \kappa_5$. □

7.3. **The proof.** We are given a family $E(\underline{a}^1, \ldots, \underline{a}^n)$ of subset of J_R^{n-1}, for $(\underline{a}^1, \ldots, \underline{a}^n) \in \Sigma_1(\rho) \times \ldots \times \Sigma_n(\rho)$. Fixed $w \in \mathcal{I}$. Define, for each $\lambda = (\underline{a}^1, \ldots, \underline{a}^n, u) \in Z$

$$F(\lambda) = F_w(\lambda) = \left\{(x_1, \ldots, x_{n-1}), u * w * (e^{x_1}, \ldots, e^{x_{n-1}}) \in E(\underline{a}^1, \ldots, \underline{a}^n)\right\},$$

which is a subset of $[-r,r]^{n-1}$. To say that all $J_R^{n-1} - E(\underline{a}^1, \ldots, \underline{a}^n)$ have small measure amounts is to say all $[-r,r] - F(\lambda), \lambda \in Z$, have small measure:

$$\left|[-r,r]^{n-1} - F(\lambda)\right| \leq \bar{c}_1.$$

For $\lambda \in Z$, we set

$$E_0(\lambda) = [-r,r]^{n-1} - \overline{V_{\Delta\rho}(F(\lambda))}.$$

Starting from E_0, we define for $k \geq 0$ sets $E_k(\lambda)$ in the following way; if $E_k(\lambda)$ has already been defined, we then set $E_{k+1}(\lambda) = E_0(\lambda) \cup \widehat{\widehat{E}}_k(\lambda)$.

It is clear, by induction, that for each $\lambda \in Z$, $(E_k(\lambda))_{k\geq 0}$ is a sequence of increasing open subsets of $[-r,r]^{n-1}$. On the other hand, by Lemma 7.9, we have

$$\sum p^\lambda \left|V_{\Delta\rho}(E_{k+1}(\lambda))\right| \leq \kappa_5 \sum p^\lambda \left|V_{\Delta\rho}(E_k(\lambda))\right| + \max_\lambda \left|V_{\Delta\rho}(E_0(\lambda))\right|,$$

and therefore

$$\sum p^\lambda \left|V_{\Delta\rho}(E_k(\lambda))\right| \leq \frac{1}{1-\kappa_5} \max_\lambda \left|V_{\Delta\rho}(E_0(\lambda))\right|, \forall k \geq 0;$$

whenever $\max_\lambda |V_{\Delta\rho}(E_0(\lambda))|$ be sufficiently small. However, $V_{\Delta\rho}(E_0(\lambda))$ is contained in $[-r-\Delta\rho, r+\Delta\rho] - F(\lambda)$ and therefore its measure is less than $2^{n-1}(n-1)\Delta\rho(r+\Delta\rho)^{n-2} + \bar{c}_1$.

Defining $E_\infty(\lambda) = \bigcup_{k\geq 0} E_k(\lambda)$, we set

$$F^*(\lambda) = F^*_w(\lambda) = [-r,r]^{n-1} - E_\infty(\lambda).$$

By construction of $F^*(\lambda)$, if $x \in F^*(\lambda)$, then

$$\#\left\{\lambda' \text{ with extension } \widetilde{\lambda}', \alpha(\widetilde{\lambda}') = \omega(\lambda), B_{\Delta_1\rho}(x + a_\lambda^{\widetilde{\lambda}'}) \subset F^*(\lambda')\right\} \geq (1-\tau_1)N_{\omega(\lambda)}.$$

Now we come back to J_R^{n-1}, setting, for $(\underline{a}^1, \ldots, \underline{a}^n) \in \Sigma_1(\rho) \times \ldots \times \Sigma_n(\rho)$,

$$E^*(\underline{a}^1, \ldots, \underline{a}^n) = \bigsqcup_{[w]\in\mathcal{I}/\mathcal{I}_0} \bigsqcup_{u\in\mathcal{I}_0} \left\{u * w * (e^{x_1}, \ldots, e^{x_{n-1}}), (x_1, \ldots, x_{n-1}) \in F^*_w(\underline{a}^1, \ldots, \underline{a}^n, u)\right\}.$$

Firstly, $E^*(\underline{a}^1, \ldots, \underline{a}^n)$ are compact subsets of J_R^{n-1}. We put $c_1 := \bar{c}_1 r^{-(n-1)}$.

The part (i) of the Lemma follows from

$$F^*(\lambda) \subset [-r,r]^{n-1} - E_0 \subset \overline{V_{\Delta\rho}(F(\lambda))},$$

taking for instant $c_2 := 2\Delta R$, assuming ρ sufficiently small.

For $(\underline{a}^1, \ldots, \underline{a}^n) \in \Sigma_1(\rho) \times \ldots \times \Sigma_n(\rho)$, we have

$$|J_R^{n-1} - E^*(\underline{a}^1, \ldots, \underline{a}^n)| \leq r^{n-1} \sum_{[w]\in\mathcal{I}/\mathcal{I}_0} \sum_{u\in\mathcal{I}_0} |[-r,r]^{n-1} - F^*_w(\underline{a}^1, \ldots, \underline{a}^n, u)|.$$

As p^λ have orders ρ^d, then

$$\sum_{(\underline{a}^1,\ldots,\underline{a}^n)} |J_R^{n-1} - E^*(\underline{a}^1, \ldots, \underline{a}^n)| \leq r^{n-1} \frac{\#(\mathcal{I}/\mathcal{I}_0)}{1-\kappa_5} (2^{n-1}(n-1)\Delta\rho(r+\Delta\rho)^{n-2} + \bar{c}_1)\rho^{-d},$$

and the part (ii) follows for ρ and $\bar{c}_1$ sufficiently small.

By the relation (2.3), there exits $C > 0$ such that if $\lambda = (\underline{a}^1, \ldots, \underline{a}^n, u) \in Z$, $\widetilde{\lambda}' = (\widetilde{\underline{b}}^1, \ldots, \widetilde{\underline{b}}^n, v) \in \widetilde{Z}$ satisfy $\alpha(\widetilde{\lambda}') = \omega(\lambda)$ and any $\underline{\theta}^1 \in \Sigma_1^-, \ldots, \underline{\theta}^n \in \Sigma_n^-$ ending with $\underline{a}^1, \ldots, \underline{a}^n$ respectively, then

$$\left|\left(\log|I^{\underline{\theta}^1}(\widetilde{\underline{b}}^1)| - \log|I^{\underline{\theta}^n}(\widetilde{\underline{b}}^n)|, \ldots, \log|I^{\underline{\theta}^{n-1}}(\widetilde{\underline{b}}^{n-1})| - \log|I^{\underline{\theta}^n}(\widetilde{\underline{b}}^n)|\right) - a_\lambda^{\widetilde{\lambda}'}\right| \leq C\rho.$$

The property (iii) follows with c_3 small enough to have $c_3\rho^{-d} < (1-\tau_1)N_{\omega(\lambda)}$, and with $\Delta_1 > R + C$.

8. The Marstrand-Kaufman's type theorem

Let μ be a finite Borel measure on $\mathbb{R}^k$. The *s-energy* of μ is

$$I_s(\mu) = \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^s},$$

and the *Fourier transform* of μ is denoted by $\widehat{\mu}$ and defined as

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^m} e^{-i\xi\cdot x} d\mu(x).$$

Is well known that if μ is compactly supported and $\widehat{\mu} \in L^2(\mathbb{R}^k)$, then μ is absolutely continuous with respect to k-dimensional Lebesgue measure, with L^2-density χ satisfying $\|\chi\|_{L^2} = (2\pi)^{-\frac{k}{2}} \|\widehat{\mu}\|_{L^2}$.

Energy and Fourier transform are related as follow (see [3], Lemma 12.12)

$$I_s(\mu) = (2\pi)^{-k} c(s,k) \int |\xi|^{s-k} |\widehat{\mu}(\xi)|^2 d\xi,$$

for $0 < s < k$ and μ with compact support.

Definition 8.1. For $\pi : \mathbb{R}^n \to \mathbb{R}^k$ a surjective linear map and $d_1, \ldots, d_n$ nonnegative real numbers, we define $\mathfrak{m} = \mathfrak{m}(\pi, d_1, \ldots, d_n)$ as

$$\mathfrak{m} = \min\left\{ \sum_{i\in I} d_i + \dim\Big(span\left\{\pi(e_i), i \in I^c\right\}\Big), I \subset \{1, \ldots, n\}, I \neq \emptyset \right\},$$

with the convention $\dim\emptyset = 0$, where $e_1, \ldots e_n$ is the canonical basis of $\mathbb{R}^n$.

For every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, define $D_t(x) = (t_1x_1, \ldots, t_nx_n)$.

Theorem 8.2. *Let π and $d_1, \ldots, d_n$ be as in definition 8.1 with $\mathfrak{m} = \mathfrak{m}(\pi, d_1, \ldots, d_n) \neq 0, 1, \ldots, k-1$. Then, there exist $d_1' \le d_1, \ldots, d_n' \le d_n$ such that for every finite Borel measures $\mu_1, \ldots, \mu_n$ on $\mathbb{R}$, denoting $\mu_s = (\pi \circ D_{(s,1)})_*(\mu_1 \times \ldots \times \mu_n)$ for $s \in \mathbb{R}^{n-1}$, we have*

$$\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} |\widehat{\mu_s}(\xi)|^2 e^{-|s|^2} d\xi ds \le C_{\mathfrak{m}} I_{d_1'}(\mu_1) \ldots I_{d_n'}(\mu_n),$$

where $C_{\mathfrak{m}} > 0$ is some constant depending only on π, n, k and $\mathfrak{m}$.

Proof. We denote $\nu_t = (\pi \circ D_t)_*(\mu_1 \times \ldots \times \mu_n)$. In [2] we proof that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} |\widehat{\nu_t}(\xi)|^2 e^{-\frac{1}{2}|t|^2} d\xi dt \le C_{\mathfrak{m}} I_{d_1'}(\mu_1) \ldots I_{d_n'}(\mu_n).$$

Finally, notice that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} |\widehat{\nu_t}(\xi)|^2 e^{-\frac{1}{2}|t|^2} d\xi dt = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} |\widehat{\mu_s}(\xi)|^2 e^{-\frac{1}{2}r^2|s|^2} e^{-\frac{1}{2}r^2} |r|^{n-1-\mathfrak{m}} d\xi ds dr,$$

hence $\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} |\widehat{\mu_s}(\xi)|^2 e^{-|s|^2} d\xi ds \le (1+\sqrt{2}) e^{\frac{1}{2}r^2} |r|^{\mathfrak{m}+1-n} C_{\mathfrak{m}} I_{d_1'}(\mu_1) \ldots I_{d_n'}(\mu_n)$ for some $r \in [1, \sqrt{2}]$. □

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A GENERALIZATION OF MARSTRAND'S THEOREM FOR PROJECTIONS OF CARTESIAN PRODUCTS

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Abstract. We prove the following variant of Marstrand's theorem about projections of cartesian products of sets:

Let $K_1, \dots, K_n$ Borel subsets of $\mathbb{R}^{m_1}, \dots, \mathbb{R}^{m_n}$ respectively, and $\pi : \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n} \to \mathbb{R}^k$ be a surjective linear map. We set

$$\mathfrak{m} := \min \left\{ \sum_{i \in I} \dim_H(K_i) + \dim \pi(\bigoplus_{i \in I^c} \mathbb{R}^{m_i}), I \subset \{1, \dots, n\}, I \neq \emptyset \right\}.$$

Consider the space $\Lambda_m = \{(t, O), t \in \mathbb{R}, O \in SO(m)\}$ with the natural measure and set $\Lambda = \Lambda_{m_1} \times \dots \times \Lambda_{m_n}$. For every $\lambda = (t_1, O_1, \dots, t_n, O_n) \in \Lambda$ and every $x = (x^1, \dots, x^n) \in \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n}$ we define $\pi_\lambda(x) = \pi(t_1 O_1 x^1, \dots, t_n O_n x^n)$. Then we have

Theorem. (i) *If $\mathfrak{m} > k$, then $\pi_\lambda(K_1 \times \dots \times K_n)$ has positive k-dimensional Lebesgue measure for almost every $\lambda \in \Lambda$.*

(ii) *If $\mathfrak{m} \leq k$ and $\dim_H(K_1 \times \dots \times K_n) = \dim_H(K_1) + \dots + \dim_H(K_n)$, then $\dim_H(\pi_\lambda(K_1 \times \dots \times K_n)) = \mathfrak{m}$ for almost every $\lambda \in \Lambda$.*

1. Introduction

The behavior of dimensions of projections of subsets of euclidean spaces has been studied for decades.

Let us denote by $\dim_H(X)$ the Hausdorff dimension of the set X. For n and k integers with $0 < k < n$, $G(n, k)$ denotes the Grassmann manifold of all k-dimensional subspaces of $\mathbb{R}^n$, with the natural measure. For $V \in G(n, k)$, $P_V : \mathbb{R}^n \to V$ is the orthogonal projection onto V. The following is a fundamental result on dimensions of projections:

Theorem (Marstrand-Kaufman-Mattila)**.** *Let $E \subset \mathbb{R}^n$ a Borel set. Then:*

(i) *If $\dim_H(E) > k$, then $P_V(E)$ has positive k-dimensional Lebesgue measure for almost every $V \in Gr(n, k)$.*

(ii) *If $\dim_H(E) \leq k$, then $\dim_H(P_V(E)) = \dim_H(E)$ for almost every $V \in Gr(n, k)$.*

This theorem was first proven by Marstrand [3] in 1954 for planar sets. Marstrand's proof used geometric methods. Later, Kaufman [2] gave an alternative proof of the same result using potential-theoretic methods. Finally, Mattila [4] generalized it to higher dimensions; his proof combined the methods of Marstrand and Kaufman.

There are other variants of the Marstrand-Mattila's theorem. They were unified in a more general result due to Peres and Schlag [7]. They studied general smooth families of projections, using some methods from harmonic analysis. The crucial characteristic that is common to all families of projections considered in Peres-Schlag's result is a transversality property (see [7], Definition 7.2).

We are interested in a Marstrand's projection result that actually is outside of the Peres-Schlag's scheme (the families of projections considered here, in general, are not transversal). This result was motivated by the problem of understanding the behavior of projections of cartesian products of sets product of sets, by a fixed projection map.

Let $K_1, \ldots, K_n$ Borel subsets of $\mathbb{R}^{m_1}, \ldots, \mathbb{R}^{m_n}$ respectively, and $\pi : \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n} \to \mathbb{R}^k$ be a linear map. Then

(1.1)

$$\dim_H(\pi(K_1 \times \ldots \times K_n)) \leq \min \left\{ \sum_{i \in I} \dim_H(K_i) + \dim \pi(\bigoplus_{i \in I^c} \mathbb{R}^{m_i}), I \subset \{1, \ldots, n\} \right\},$$

with the conventions $\sum_{i \in \emptyset} \dim_H(K_i) = 0, \dim \emptyset = 0$.

Consider the space $\Lambda_m = \{(t, O), t \in \mathbb{R}, O \in SO(m)\}$ with the natural measure and set $\Lambda = \Lambda_{m_1} \times \ldots \times \Lambda_{m_n}$. For every $x = (x^1, \ldots, x^n) \in \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$ and every $\lambda = (t_1, O_1, \ldots, t_n, O_n) \in \Lambda$ we define $\pi_\lambda(x) = \pi(t_1 O_1 x^1, \ldots, t_n O_n x^n)$. Suppose that π is surjective and set

$$\mathfrak{m} := \min \left\{ \sum_{i \in I} \dim_H(K_i) + \dim \pi(\bigoplus_{i \in I^c} \mathbb{R}^{m_i}), I \subset \{1, \ldots, n\}, I \neq \emptyset \right\}.$$

Then we have

Theorem 1.1. (i) *If $\mathfrak{m} > k$, then $\pi_\lambda(K_1 \times \ldots \times K_n)$ has positive k-dimensional Lebesgue measure for almost every $\lambda \in \Lambda$.*

(ii) *If $\mathfrak{m} \leq k$ and $\dim_H(K_1 \times \ldots \times K_n) = \dim_H(K_1) + \ldots + \dim_H(K_n)$, then $\dim_H(\pi_\lambda(K_1 \times \ldots \times K_n)) = \mathfrak{m}$ for almost every $\lambda \in \Lambda$.*

In a work in progress, we plan use the Theorem 2.3 to generalize the result of Moreira and Yoccoz [6] about stable intersections of two regular Cantor sets for projections of cartesian products of several regular Cantor sets. Our goal is to prove the following result: for any given surjective linear map $\pi : \mathbb{R}^n \to \mathbb{R}^k$, typically for regular Cantor sets on the real line $K_1, \ldots, K_n$ with $\mathfrak{m} > k$, the set $\pi(K_1 \times \ldots \times K_n)$ persistently contains non-empty open sets of $\mathbb{R}^k$. Such a result would in particular imply an analogous result for simultaneous stable intersections of several regular Cantor sets on the real line.

In another work in progress, in collaboration with Pablo Shmerkin, we plan to use the results of this paper combined with the techniques in [1] in order to obtain exact formulas for the Hausdorff dimensions of projections of cartesian products of (real or complex) regular Cantor sets under explicit irrationality conditions.

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2. Statement the main results

Let μ be a finite Borel measure on $\mathbb{R}^m$. The *s-energy* of μ is

$$I_s(\mu) = \int \int \frac{d\mu(x) d\mu(y)}{|x - y|^s}.$$

We know (see [5], Theorem 8.9(3)) that for a Borel set $K \subset \mathbb{R}^m$

$$\dim_H(K) = \sup\{s \in \mathbb{R}, \text{there is a compactly supported measure } \mu \text{ on } K \text{ which } 0 < \mu(\mathbb{R}^m) < \infty \text{ and } I_s(\mu) < \infty\}. \tag{2.1}$$

The *Fourier transform* of μ is denoted by $\widehat{\mu}$ and defined as

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^m} e^{-i\xi\cdot x} d\mu(x).$$

It is well-know that if $\widehat{\mu} \in L^2(\mathbb{R}^m)$, then μ is absolutely continuous with L^2-density. Energy and Fourier transform are related as follow (see [5], Lemma 12.12)

$$I_s(\mu) = (2\pi)^{-m} c(s,m) \int |\xi|^{s-m} |\widehat{\mu}(\xi)|^2 \, d\xi,$$

for $0 < s < m$ and μ with compact support.

We summarize the above observations as the following result:
Let $F \subset \mathbb{R}^k$ a Borel set supporting a probability measure ν with $\int |\xi|^{s-k} |\widehat{\nu}(\xi)|^2 \, d\xi < \infty$. If $s \geq k$, then F has positive k-dimensional Lebesgue measure. Otherwise, if $0 < s < k$, then $\dim_H(F) \geq s$.

Let $\pi : \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n} \to \mathbb{R}^k$ be a linear map. For each $I \subset \{1, \ldots, n\}$, let $P_I : \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n} \to \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$ the orthogonal projection onto the subspace $\bigoplus_{i\in I} \mathbb{R}^{m_i}$, where $\mathbb{R}^{m_i}$ is as a canonical subspace of $\mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$. Then $\pi = \pi \circ P_I + \pi \circ P_{I^c}$ so, for $K_1, \ldots, K_n$ Borel subsets of $\mathbb{R}^{m_1}, \ldots, \mathbb{R}^{m_n}$ respectively we have

$$\begin{aligned}
&\dim_H(\pi(K_1 \times \ldots \times K_n)) \\
&\leq \dim_H \Big(\pi P_I(K_1 \times \ldots \times K_n) \times \pi P_{I^c}(K_1 \times \ldots \times K_n)\Big) \\
&\leq \dim_H \Big(\pi P_I(K_1 \times \ldots \times K_n) \times \pi(\bigoplus_{i\in I^c} \mathbb{R}^{m_i})\Big) \\
&\leq \sum_{i\in I} \dim_H(K_i) + \dim \pi(\bigoplus_{i\in I^c} \mathbb{R}^{m_i}).
\end{aligned}$$

(In the last inequality, we assume that $\dim_H(K_1 \times \ldots \times K_n) = \dim_H(K_1) + \ldots + \dim_H(K_n)$) This prove the inequality (1.1) and also motivates us to define:

Definition 2.1. For $\pi : \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n} \to \mathbb{R}^k$ a surjective linear map and $d_1, \ldots, d_n$ nonnegative real numbers, we define $\mathfrak{m} = \mathfrak{m}(\pi, d_1, \ldots, d_n)$ as

$$\mathfrak{m} = \min \left\{ \sum_{i\in I} d_i + \dim \pi(\bigoplus_{i\in I^c} \mathbb{R}^{m_i}), I \subset \{1, \ldots, n\}, I \neq \emptyset \right\}.$$

Remark 2.2. If in addition $d_i \leq m_i$ (which holds for dimensions of subsets of $\mathbb{R}^{m_i}$), then, for the open and total measure family of linear maps π with the following transversality property:

$$\dim \pi(\bigoplus_{i\in I} \mathbb{R}^{m_i}) = \min\big(k, \dim(\bigoplus_{i\in I} \mathbb{R}^{m_i})\big), \text{ for all } I \subset \{1, \ldots, n\},$$

the equivalence $\mathfrak{m}(\pi, d_1, \ldots, d_n) > k \Leftrightarrow d_1 + \ldots + d_n > k$ holds. However, in general we must check more than one of the $2^n - 1$ conditions appearing in the definition of $\mathfrak{m}$.

Consider the space $\Lambda_m = \{(t, O), t \in \mathbb{R}, O \in SO(m)\}$, with the product measure $\mathcal{L}^1 \times \Theta^m$, where $\mathcal{L}^1$ denotes the one dimensional Lebesgue measure and Θ^m denotes the left-right invariant Haar probability measure on $SO(m)$. Notice that the set $C(m) = \{tO, t \in \mathbb{R}, O \in SO(m)\}$ represents essentially the family of linear conformal maps on $\mathbb{R}^m$. $C(2) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a, b \in \mathbb{R} \right\}$, which can be viewed as the set of multiplications by a complex number.

We set $\Lambda = \Lambda_{m_1} \times \ldots \times \Lambda_{m_n}$. For every $x = (x^1, \ldots, x^n) \in \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$, and every $\lambda = (t_1, O_1, \ldots, t_n, O_n) \in \Lambda$ we define $\pi_\lambda(x) = \pi(t_1 O_1 x^1, \ldots, t_n O_n x^n)$. Also, given any finite measure μ on $\mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$, let $\nu_\lambda = (\pi_\lambda)_* \mu$. We also define

$$I_{d_1,\ldots,d_n}(\mu) = \int\int \frac{d\mu(x) d\mu(y)}{|x^1 - y^1|^{d_1} \ldots |x^n - y^n|^{d_n}}.$$

Our main result is now the following:

Theorem 2.3. *Let π and $d_1, \ldots, d_n$ be as in definition 2.1 with $\mathfrak{m} = \mathfrak{m}(\pi, d_1, \ldots, d_n) \neq 0, 1, \ldots, k-1$. Then, there exist $d_1' \le d_1, \ldots, d_n' \le d_n$ such that for every Borel measure μ on $\mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$ we have*

$$\int_\Lambda \int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} \left|\widehat{\nu_\lambda}(\xi)\right|^2 \rho(\lambda) d\xi d\lambda \le C_{\mathfrak{m}} I_{d_1', \ldots, d_n'}(\mu),$$

where $\rho(\lambda) = |t_1|^{m_1 - 1} \ldots |t_n|^{m_n - 1} e^{-\frac{1}{2}(|t_1|^2 + \ldots + |t_n|^2)}$ and $C_{\mathfrak{m}} > 0$ is some constant depending only on $\pi, n, k, m_1, \ldots, m_n$ and $\mathfrak{m}$.

In the proof of Theorem 2.3 the key tool will be the following combinatorial lemma.

Lemma 2.4 (Weights Lemma)**.** *Let $s, d_1, \ldots, d_n \ge 0$ and $V_1, \ldots, V_n$ vector subspaces of a same finite dimension vector space satisfying the following 2^n conditions*

$$\sum_{i \in I} d_i + \dim\big(\sum_{i \in I^c} V_i\big) \ge s, \text{ for every } I \subset \{1, \ldots, n\}$$

(with the conventions $\sum_{i \in \emptyset} d_i = 0$, $\dim \emptyset = 0$).

Fixed a generating set $\{v_1^i, \ldots, v_{m_i}^i\}$ of V_i for each $i \in \{1, \ldots, n\}$. Consider the family $\mathbb{J}$ of all possible $J = (\mathrm{J}_1, \ldots, \mathrm{J}_n)$, $\mathrm{J}_i \subset \{v_1^i, \ldots, v_{m_i}^i\}$ such that $\mathrm{J}_1 \cup \ldots \cup \mathrm{J}_n$ is a linearly independent system with dimension greater than or equal to s. Define

$$\overline{\mathbb{J}} = \left\{ (J, i) \in \mathbb{J} \times \{1, \ldots, n\}, \widehat{J}(i) := (\#\mathrm{J}_1, \ldots, \#\mathrm{J}_n) + (s - (\#\mathrm{J}_1 + \ldots + \#\mathrm{J}_n)) e_i \ge 0 \right\},$$

where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$ and $\ge$ means that the inequality is coordinate to coordinate.

Then, there exist non-negative real numbers $(\alpha_{(J,i)})_{(J,i) \in \overline{\mathbb{J}}}$ with sum equal to 1 such that

$$\sum_{(J,i) \in \mathbb{J}} \alpha_{(J,i)} \widehat{J}(i) \le d.$$

Proof of Theorem 1.1. The theorem follows immediately from the Theorem 2.3 applied to $\mu = \mu_1 \times \ldots \times \mu_n$ for suitable measures μ_i compactly supported in K_i coming from the equation (2.1). Noting that in the part (i), the condition $\dim_H(K_1) > 0, \ldots, \dim_H(K_n) > 0$ follows from the hypotheses; and in the part (ii), we may assume the same condition by reduction to some cartesian product if necessary. $\square$

Remark 2.5. We can derive the part (ii) of the Theorem 1.1 from the part (i). Assume $\dim_H(K_i) > 0$. Let $k' < \mathfrak{m} \leq k'+1 \leq k$ and consider any $k' < s < \mathfrak{m}$, and set $\Lambda^s = \{\lambda \in \Lambda, \dim_H(\pi_\lambda(K_1 \times \ldots \times K_n)) < s\}$. The idea is to add another factor to the cartesian product: Let $m_0 := k - k'$ and consider K_0 a sufficiently regular subset of $\mathbb{R}^{m_0}$ with $\dim_H(K_0) = k - s$, and $\widetilde{\pi} : \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n} \to \mathbb{R}^k$ with

$$\widetilde{\pi} \circ P_{I^n} = \pi, \text{ where } I^n = \{1, \ldots, n\},$$

$$\dim \widetilde{\pi}\Big(\bigoplus_{i \in I \cup \{0\}} \mathbb{R}^{m_i}\Big) = \min\big(k, m_0 + \dim \pi(\bigoplus_{i \in I} \mathbb{R}^{m_i})\big), \text{ for all } I \subset \{1, \ldots, n\}.$$

In particular $\widetilde{\pi}$ is surjective. Notice that

$$\sum_{i \in I} \dim_H(K_i) + \dim \widetilde{\pi}(\bigoplus_{i \in I^c} \mathbb{R}^{m_i}) > k, \text{ for all } I \subset \{0, 1, \ldots, n\}, I \neq \emptyset,$$

and also that $\dim_H(\widetilde{\pi}_{(\lambda_0,\lambda)}(K_1 \times \ldots \times K_n)) < k$ for all $(\lambda_0, \lambda) \in \Lambda_{m_0} \times \Lambda^s$. Applying the Theorem 1.1.(i) in this new setting, we conclude that Λ^s is a zero measure subset of Λ.

Remark 2.6. Theorem 2.3, when combined with Proposition 7.5 of [7], also gives us a result on exceptional sets:

In the setting of the Theorem 1.1, part (i), we have

$$\dim_H\Big(\big\{\lambda \in \Lambda, t_i \neq 0 \text{ } if \text{ } m_i > 1, \mathcal{L}^k(\pi_\lambda(K_1 \times \ldots \times K_n)) = 0\big\}\Big) \leq l + k - \mathfrak{m},$$

where $l = \dim \Lambda_{m_1} \times \ldots \times \Lambda_{m_n} = n + \sum_{i=1}^n m_i(m_i - 1)/2$.

3. Proof of the main results

Proof of Theorem 2.3. Notice that

$$\begin{aligned}|\widehat{\nu_\lambda}(\xi)|^2 &= \int\int e^{i\xi\cdot\pi_\lambda(y-x)} d\mu(x)d\mu(y),\\ &= \int\int e^{i\pi^T\xi\cdot(t_1O_1(y^1-x^1),\ldots,t_nO_n(y^n-x^n))} d\mu(x)d\mu(y),\end{aligned}$$

and that, for all $z \in \mathbb{R}^m$, $\eta \in \mathbb{R}^m$,

$$\begin{aligned}\int_{\mathbb{R}}\int_{SO(m)} e^{i\eta\cdot tOz}|t|^{m-1}e^{-\frac{1}{2}|t|^2}d\Theta^m dt &= \int_{\mathbb{R}}\int_{S^{m-1}} e^{i|z|\eta\cdot t\theta}|t|^{m-1}e^{-\frac{1}{2}|t|^2}d\sigma^{m-1}dt\\ &= 2\int_{\mathbb{R}^m} e^{i|z|\eta\cdot x}e^{-\frac{1}{2}|x|^2}dx\\ &= 2\pi^{\frac{m}{2}}e^{-\frac{1}{2}(|z||\eta|)^2},\end{aligned}$$

where σ^{m-1} denotes the normalized Lebesgue measure on S^{m-1}. Therefore by Fubini's theorem

$$\begin{aligned}&\int_\Lambda\int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k}|\widehat{\nu_\lambda}(\xi)|^2\rho(\lambda)d\xi d\lambda\\ &= \lim_{a\to\infty}\int_{|\xi|\leq a}\int_\Lambda |\xi|^{\mathfrak{m}-k}|\widehat{\nu_\lambda}(\xi)|^2\rho(\lambda)d\lambda d\xi\\ &= c\lim_{a\to\infty}\int\int\left(\int_{|\xi|\leq a}|\xi|^{\mathfrak{m}-k}e^{-\frac{1}{2}|D_{x,y}(\xi)|^2}d\xi\right)d\mu(x)d\mu(y)\\ &= c\int\int\left(\int_{\mathbb{R}^k}|\xi|^{\mathfrak{m}-k}e^{-\frac{1}{2}|D_{x,y}(\xi)|^2}d\xi\right)d\mu(x)d\mu(y),\end{aligned}$$

where $D_{x,y} = \left(D^1(\left|y^1 - x^1\right|), \ldots, D^n(|y^n - x^n|)\right) \circ \pi^T$, and $D^i(t) : \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$ is the diagonal transformation, $D^i(t) = t.Id$, for $t \in \mathbb{R}$.

We fixed x, y assuming that $y^i - x^i \neq 0$ for all $i = 1, \ldots, n$. We estimate $\int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi$ separately, when $\mathfrak{m} \geq k$ and $\mathfrak{m} < k$. In both case we apply the Lemma 2.4 for $V_i = \pi(\mathbb{R}^{m_i})$, taking $v^i_j = \pi(e^i_j)$, where e^i_j, $j = 1, \ldots, m_i$ is the canonical basis of $\mathbb{R}^{m_i}$ as subspace of $\mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$.

We use the notation $z^I = (z_1^{i_1}, \ldots, z_n^{i_n})$ if $z = (z_1, \ldots, z_n) \in \mathbb{R}^n_+$ and $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, for $z = (\left|y^1 - x^1\right|, \ldots, |y^n - x^n|)$.

Suppose $\mathfrak{m} \geq k$. Let i_0 such that $z_{i_0} \leq z_i$ for all $i = 1, \ldots, n$. Notice that $\mathfrak{m}(\pi, d - (\mathfrak{m} - k)e_{i_0}) \geq k$ and in particular $d - (\mathfrak{m} - k)e_{i_0} \geq 0$. We apply the Lemma 2.4 to $d - (\mathfrak{m} - k)e_{i_0}$ and $s = k$. For each $J \in \mathbb{J}$, just looking for the sums in $\frac{1}{2}\left|D_{x,y}(\xi)\right|^2$ related to J and using the change of variables formula to an appropriated linear isomorphs of $\mathbb{R}^k$, we have

$$\int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi \leq c' z_{i_0}^{k-\mathfrak{m}} z^{-\widehat{J}} \int_{\mathbb{R}^k} |\eta|^{\mathfrak{m}-k} e^{-\frac{1}{2}|\eta|^2} d\eta,$$

for some constant $c' > 0$ depending only on π and $\mathfrak{m}-k$, where $\widehat{J} := (\#\mathrm{J}_1, \ldots, \#\mathrm{J}_n)$. Therefore

$$\int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi \leq c'' z_{i_0}^{k-\mathfrak{m}} \prod_{J \in \mathbb{J}} z^{-\alpha_J \widehat{J}} = c'' z^{-(\sum_J \alpha_J \widehat{J} + (\mathfrak{m}-k)e_{i_0})} = c'' z^{-d'}.$$

Suppose $k' - 1 < \mathfrak{m} < k'$, where $1 \leq k' \leq k$. We apply the Lemma 2.4 to d and $s = \mathfrak{m}$. Let $(J, i) \in \overline{\mathbb{J}}$ with $\#\mathrm{J}_1 + \ldots + \#\mathrm{J}_n = k'$. From $\mathfrak{m} < k'$ we have $\mathrm{J}_i \neq \emptyset$. In the same way as in the previous case, notice that

$$\int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi \leq \widetilde{c} z^{-\widehat{J}} \int_{\mathbb{R}^{k'}} \int_{\mathbb{R}^{k-k'}} \left(|\eta'_1|/z_i + |\eta''|\right)^{\mathfrak{m}-k} e^{-\frac{1}{2}|\eta'|^2} d\eta' d\eta'',$$

for some constant $\widetilde{c} > 0$ depending only on π and $\mathfrak{m} - k$. We affirm that

$$\int_{\mathbb{R}^{k'}} \int_{\mathbb{R}^{k-k'}} \left(|\eta'_1|/z_i + |\eta''|\right)^{\mathfrak{m}-k} e^{-\frac{1}{2}|\eta'|^2} d\eta' d\eta'' \leq \widetilde{c}' z_i^{k'-\mathfrak{m}},$$

for some constant $\widetilde{c}' > 0$ depending only on $\mathfrak{m}, k, k'$. If $k' = k$ the affirmation is true, since $\mathfrak{m} - k > -1$. If $k' < k$, applying polar coordinates in $\mathbb{R}^{k-k'}$ we have

$$\begin{aligned}\int_{\mathbb{R}^{k'}} \int_{\mathbb{R}^{k-k'}} \left(|\eta'_1|/z_i + |\eta''|\right)^{\mathfrak{m}-k} e^{-\frac{1}{2}|\eta'|^2} d\eta' d\eta'' &\leq C \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (t/z_i + r)^{\mathfrak{m}-k'-1} e^{-\frac{1}{2}t^2} dr dt \\ &= C(k' - m)^{-1} \int_{\mathbb{R}_+} (t/z_i)^{\mathfrak{m}-k'} e^{-\frac{1}{2}t^2} dt.\end{aligned}$$

Then $\int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi \leq \widetilde{c}'' z^{-\widehat{J}(i)}$, and therefore

$$\int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi \leq \widetilde{c}'' \prod_{(J,i) \in \overline{\mathbb{J}}} z^{-\alpha_{(J,i)} \widehat{J}(i)} = \widetilde{c}'' z^{-\sum_{(J,i)\in\overline{\mathbb{J}}} \alpha_{(J,i)} \widehat{J}(i)} = \widetilde{c}'' z^{-d'}.$$

□

Proof of Lemma 2.4. Claim: The vertices of the polyhedron

$$P = \Big\{ (d_1, \dots, d_n) \in \mathbb{R}^n, d_1 \geq 0, \dots, d_n \geq 0 \\ \sum_{i \in I} d_i + \dim \big(\sum_{i \in I^c} V_i \big) \geq s, \text{ for all } I \subset \{1, \dots, n\} \Big\}$$

have all the form $\widehat{J}(i)$ for some $(J, i) \in \mathbb{J}$.

$P \subset \overline{\mathbb{R}}^n_+$, therefore P is a pointed polyhedron (i.e. it does not contain any non trivial affine subspace). We proceed by induction on n. For $n = 1$ it is trivial. Let $x = (x_1, \dots, x_n)$ any vertex of the polyhedron. Then, there are n independent inequalities from the definition of P that become equality in x (see [8], page 104).

If $x_n = 0$, notice that $x' = (x_1, \dots, x_{n-1})$ is now a vertex of the polyhedron

$$P' = \Big\{ (d_1, \dots, d_{n-1}) \in \mathbb{R}^{n-1}, d_1 \geq 0, \dots, d_{n-1} \geq 0 \\ \sum_{i \in I} d_i + \dim \big(\sum_{i \in I^c} V_i \big) \geq s, \text{ for all } I \subset \{1, \dots, n-1\} \Big\}$$

(i.e. $x' \in P'$ and x' satisfies $n-1$ independent equalities). By induction hypothesis, there exist some $J' = (\mathrm{J}'_1, \dots, \mathrm{J}'_{n-1}) \in \mathbb{J}'$ and $i' \in \{1, \dots, n-1\}$ such that $x' = \widehat{J'}(i')$. Then, $J = (\mathrm{J}'_1, \dots, \mathrm{J}'_{n-1}, \emptyset) \in \mathbb{J}$ and $i = i'$ are such that $x = \widehat{J}(i)$.

Suppose $x_1 \neq 0, \dots, x_n \neq 0$. By simplicity, we denote $\sum_{i \in I} V_i$ by V_I. Consider

$$\mathcal{I} = \left\{ I \subset \{1, \dots, n\}, I \neq \emptyset, \sum_{i \in I} x_i + \dim V_{I^c} = s \right\}.$$

By the assumption on x, there are $I_1, \dots, I_n \in \mathcal{I}$ such that the associated $0, 1$ row vectors $\widetilde{I}_1, \dots, \widetilde{I}_n$ defining the equalities, are independent.

If $I, J \in \mathcal{I}$, then

$$\begin{aligned} \dim V_{I^c} + \dim V_{J^c} &= 2s - \sum_{i \in I} x_i - \sum_{i \in J} x_i \\ &= 2s - \sum_{i \in I \cup J} x_i - \sum_{i \in I \cap J} x_i \\ &\leq \dim V_{I^c \cap J^c} + \dim V_{I^c \cup J^c} \\ &\leq \dim (V_{I^c} \cap V_{J^c}) + \dim (V_{I^c} + V_{J^c}) \\ &= \dim V_{I^c} + \dim V_{J^c}, \end{aligned}$$

therefore, $I \cup J \in \mathcal{I}$ and $I \cap J \in \mathcal{I}$. Let $I_0 \in \mathcal{I}$ a minimal element by inclusion. Then, for any $J \in \mathcal{I}$, we have

$$I_0 \subset J \text{ or } I_0 \cap J = \emptyset.$$

This means the invertible matrix of rows $\widetilde{I}_1, \dots, \widetilde{I}_n$ has $\#I_0$ identical columns, and therefore $\#I_0 = 1$, say $I_0 = \{n\}$, or, equivalently, $x_n = s - \dim(V_1 + \ldots + V_{n-1})$.

Notice that now $\widetilde{x} = (x_1, \dots, x_{n-1})$ is a vertex of the polyhedron

$$\widetilde{P} = \Big\{ (d_1, \dots, d_{n-1}) \in \mathbb{R}^{n-1}, d_1 \geq 0, \dots, d_{n-1} \geq 0 \\ \sum_{i \in I} d_i + \dim \big(\sum_{i \in I^c} V_i \big) \geq \dim(V_1 + \ldots + V_{n-1}), \text{ for all } I \subset \{1, \dots, n-1\} \Big\}$$

By induction hypothesis, there exist some appropriate $\widetilde{J} = (\widetilde{\mathrm{J}}_1, \ldots, \widetilde{\mathrm{J}}_{n-1}) \in \widetilde{\mathbb{J}}$ such that $\widetilde{x} = (\#\widetilde{\mathrm{J}}_1, \ldots, \#\widetilde{\mathrm{J}}_{n-1})$. We can take $\mathrm{J}_n \subset \left\{v_1^n, \ldots, v_{m_n}^n\right\}$ such that $V_1 + \ldots + V_{n-1} + \langle \mathrm{J}_n \rangle = V_1 + \ldots + V_n$ and $J = (\widetilde{\mathrm{J}}_1, \ldots, \widetilde{\mathrm{J}}_{n-1}, \mathrm{J}_n) \in \mathbb{J}$. Notice that $x = \widehat{J}(n)$. This finishes the proof of the claim.

To finish the prove of the lemma, notice that for a pointed polyhedron P (see [8], page 108), we have

$$P = \text{conv.hull}\left\{x^1, \ldots, x^r\right\} + \text{cone}\left\{y^1, \ldots, y^t\right\}$$

where x^i are the vertices of P and y^i are its extremal rays; and we have necessary $y^i \geq 0$ since $P \subset \overline{\mathbb{R}}_+^n$. □

Remark 3.1. Notice that $\widehat{J}(i) \in P$ for all $(J, i) \in \mathbb{J}$, hence we conclude from Lemma 2.4 that

$$P = \text{conv.hull}\{\widehat{J}(i), (J, i) \in \mathbb{J}\} + \text{cone}\left\{e_1, \ldots, e_n\right\}.$$

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