Instituto de Matemática Pura e Aplicada

Doctoral Thesis

NONNEGATIVELY CURVED FIVE-MANIFOLDS WITH NON-ABELIAN SYMMETRY

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Rio de Janeiro
September 28th, 2012

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Thesis presented to the Post-graduate Program in Mathematics at Instituto de Matemática Pura e Aplicada as partial fulfillment of the requirements for the degree of Doctor in Philosophy in Mathematics.

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Rio de Janeiro
2012

Ao meu filho, Gael.

Agradecimentos

AGRADEÇO...
aos meus orientadores Luis Adrián Florit e Wolfgang Ziller por terem sido muito mais que dois, por desempenharem os papéis de orientadores, colaboradores, revisores e também de amigos fundamentais para estes dias difíceis que se passaram;
ao Impa e a todos os seus funcionários e pesquisadores pelo acolhimento, pela competência para lidar com minhas necessidades acadêmicas. Pela determinação e eficiência de todas essas pessoas em cumprir os nobres objetivos da instituição;
ao CNPq por viabilizar esta jornada através do auxílio financeiro na maior parte da minha vida acadêmica;
à Unirio por me fornecer condições de trabalho adequadas para realizar minha pesquisa em um ambiente acolhedor e amistoso;
aos amigos e familiares que mais ou menos presentes durante este tempo foram fundamentais no meu equilíbrio emocional. Não ouso nomeá-los porque seria impossível fazê-lo sem esquecer alguém certamente inesquescível;
às garotas que fizeram parte deste último ano por me servirem de inspiração para a conclusão deste trabalho. Por motivos óbvios estes nomes não constarão nestes Agradecimentos.
Finalmente, agradeço à vida por me ter feito amadurecer nestes anos, por me ter confiado um filho que me proporciona tantos momentos lindos e também por me ensinar novas maneiras de viver e amar.

Resumo

Sejam G um grupo de Lie conexo não-abeliano e M uma G-variedade compacta e simplesmente-conexa de dimensão 5. Mostramos que M tem que ser difeomorfa à esfera, a somas conexas de cópias de $\mathbb{S}^3 \times \mathbb{S}^2$, ao fibrado não-trivial de $\mathbb{S}^3$ sobre $\mathbb{S}^2$, ou a somas conexas arbitrárias de cópias da variedade de Wu, $\mathrm{SU}(3)/\mathrm{SO}(3)$, e cópias da variedade de Brieskorn do tipo $(2, 3, 3, 3)$. Esta descrição é baseada na classificação que fazemos de todas estas ações para $G = \mathrm{SU}(2)$ ou $\mathrm{SO}(3)$.

Como consequência destes resultados puramente topológicos, provamos que as variedades M que admitem métricas invariantes pela ação de G com curvatura (seccional) não-negativa são precisamente a esfera, o produto $\mathbb{S}^3 \times \mathbb{S}^2$, o fibrado não-trivial de $\mathbb{S}^3$ sobre $\mathbb{S}^2$ e a variedade de Wu. Descrevemos completamente as ações de G nestas variedades.

A partir desta classificação equivariante fazemos uma classificação parcial das G-variedades de dimensão 5 compactas e simplesmente-conexas com curvatura positiva.

Nonnegatively curved five-manifolds with non-abelian symmetry

Contents

1 Introduction

Known examples of manifolds which admit metrics of positive (sectional) curvature are rare when compared with nonnegatively curved examples. In fact, besides rank one symmetric spaces, compact manifolds with positive curvature are known to exist only in dimensions below 25, while to generate new nonnegatively curved manifolds from known ones it is enough, for example, to take products, quotients or biquotients (see [34] for a survey). It is also known that cohomogeneity one manifolds with codimension two singular orbits admit invariant metric of nonnegative curvature, which gives rise to nonnegatively curved metrics on exotic 7-spheres (c.f., [16]).

By the Soul Theorem non-compact nonnegatively curved manifolds are diffeomorphic to a vector bundle over a compact manifold with nonnegative curvature. In positive curvature Bonnet-Myers implies that the fundamental group is finite and in nonnegative curvature a finite cover is diffeomorphic to the product of a torus with a compact simply-connected manifold with nonnegative curvature (see [5]). We will hence only consider compact simply-connected manifolds.

In dimensions 2 and 3 the classification is purely topological since such simply-connected compact manifolds are diffeomorphic to a sphere. The first follows from the topological classification of compact surfaces and the latter from Perelman's work on Poincaré's Conjecture [27].

More recently, positively and nonnegatively curved manifolds were studied under the additional assumption of having a "large" isometry group (see e.g. the surveys [12] and [32]). The beginning of this subject was the result by Hsiang and Kleiner [18] that a simply-connected 4-dimensional Riemannian manifold with positive curvature and $\mathbb{S}^1$-symmetry must be either $\mathbb{S}^4$ or $\mathbb{CP}^2$. As a consequence, any metric with positive curvature on $\mathbb{S}^2\times\mathbb{S}^2$ must have discrete isometry group, providing a partial answer to the Hopf Conjecture. The isometric circle-actions on 4-dimensional simply-connected compact manifolds with nonnegative (and positive) curvature were classified in [11], [18] and [15]. Kleiner [21] also classified isometric $\mathrm{SU}(2)$-actions with discrete kernel on simply-connected 4-manifolds with nonnegative curvature.

In 2002, Rong [28] showed that a positively curved compact simply-connected 5-dimensional manifold with a 2-torus acting by isometries has to be diffeomorphic to a 5-sphere, although the actions are not yet classified. In 2009 Galaz-Garcia and Searle [10], only assuming nonnegative curvature, showed that a 5-manifold which admits an action of a 2-torus is diffeomorphic to either $\mathbb{S}^5$, $\mathbb{S}^3\times\mathbb{S}^2$, the nontrivial $\mathbb{S}^3$-bundle over $\mathbb{S}^2$ denoted by $\mathbb{S}^3\tilde{\times}\mathbb{S}^2$, or the Wu-manifold $\mathcal{W}=\mathrm{SU}(3)/\mathrm{SO}(3)$. The description of the actions is not yet solved.

The classification of isometric circle actions on positively curved 5-manifolds is a very difficult problem and at the moment seems out of reach. Thus, the question that arises is which

5-manifolds admit a metric of nonnegative (or positive) curvature with symmetry containing a connected non-abelian group G. We will be able to classify such manifolds with nonnegative curvature and obtain a partial classification in positive curvature. For this purpose we first classify all five-dimensional compact simply-connected manifolds which admit an action of a connected non-abelian Lie group without any geometric assumptions. They are either $\mathbb{S}^5$, $\mathbb{S}^3 \times \mathbb{S}^2$, $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$, connected sums of $\mathbb{S}^3 \times \mathbb{S}^2$, or connected sums $k\,\mathcal{W} \# l\,\mathcal{B}$ of copies of the Wu-manifold $\mathcal{W}$ and the Brieskorn variety $\mathcal{B}$ of type $(2,3,3,3)$. Since any non-abelian connected Lie group contains SO(3) or SU(2) as a subgroup, it is natural to classify in addition the actions by these groups up to equivariant diffeomorphisms.

To describe the actions we introduce the following key construction.

Main example. Let $m \leq n$ and l be nonnegative integers and consider the $\mathbb{S}^1$-action on $\mathrm{SU}(2) \times \mathbb{S}^3 = \mathbb{S}^3 \times \mathbb{S}^3$ given by

$$x \cdot (p,(z,w)) = (px^l,(x^m z, x^n w)),$$

where we regard SU(2) as the group of unit quaternions, $p \in \mathrm{SU}(2)$, $x \in \mathbb{S}^1 = \{e^{i\theta} \in \mathrm{SU}(2)\}$ and $(z,w) \in \mathbb{S}^3 \subset \mathbb{C}^2$. This action is free whenever $\gcd(l,m) = \gcd(l,n) = 1$. Notice that $l = 1$ if either m or both are zero. As we will see, the quotient $\mathcal{N}^l_{m,n} := (\mathrm{SU}(2) \times \mathbb{S}^3)/\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^2$ if $m+n$ is even and diffeomorphic to $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$ otherwise. Consider $\mathcal{N}^l_{m,n}$ as an SU(2)-manifold by defining

$$g \cdot [(p,(z,w))] = [(gp,(z,w))],$$

for $g \in \mathrm{SU}(2)$. This action has isotropy groups isomorphic to $\mathbb{Z}_m$, $\mathbb{Z}_n$ and $\mathbb{Z}_{\gcd(m,n)}$ if m and n are both positive, $\mathbb{Z}_n$ and SO(2) if $n > m = 0$ and only one isotropy type (SO(2)) if $m = n = 0$. If $\gcd(m,n)$ is even, the action has ineffective kernel $\mathbb{Z}_2$ and hence is an effective action by SO(3). Notice that the actions on $\mathcal{N}^1_{1,1}$ and $\mathcal{N}^1_{2,2}$ are free, the actions on $\mathcal{N}^1_{1,1}$, $\mathcal{N}^1_{0,2}$ and $\mathcal{N}^1_{0,0}$ are linear, and that, to complete all the linear actions on $\mathbb{S}^3 \times \mathbb{S}^2$ we should include the SO(3)-action induced by the embedding of $\mathrm{SO}(3) \subset \mathrm{SO}(4)$ in the first factor. Finally observe that by O'Neill's formula the standard product metric on $\mathbb{S}^3 \times \mathbb{S}^3$ induces a G-invariant metric of nonnegative curvature on $\mathcal{N}^l_{m,n}$.

Throughout this work unless otherwise stated, G will denote SO(3) or SU(2) and M a simply-connected compact G-manifold of dimension 5. Our first result is a complete classification of all such nonnegatively curved G-manifolds.

Theorem A. *Let M be a G-manifold which admits an invariant metric with nonnegative curvature. Then M is equivariantly diffeomorphic to either $\mathbb{S}^5$, $\mathbb{S}^3 \times \mathbb{S}^2$, or $\mathcal{W} = \mathrm{SU}(3)/\mathrm{SO}(3)$ with the natural linear G-actions, or $\mathcal{N}^l_{m,n}$.*

Notice that the natural metrics on these manifolds are G-invariant with nonnegative curvature. For positive curvature we have the following partial classification.

Theorem B. *If the G-manifold M admits an invariant metric with positive curvature, then it is either equivariantly diffeomorphic to $\mathbb{S}^5$ with a linear action, or possibly $\mathcal{W}$ with the linear $\mathrm{SU}(2)$-action, or $\mathcal{N}^l_{m,n}$ with trivial principal isotropy group, i.e., $\gcd(m,n) = 1$ or 2.*

In the context of Theorem B it is natural to conjecture that only the linear actions on $\mathbb{S}^5$ admit invariant metrics of positive curvature.

Theorems A and B will be a consequence of a general equivariant classification of SO(3) and SU(2)-actions in dimension five. We begin with the case without singular orbits.

Theorem C. *The G-manifolds without singular orbits are equivalent to either $\mathcal{N}^1_{0,0}$ or $\mathcal{N}^l_{m,n}$ for some choice of positive integers m, n and l. The $\mathrm{SO}(3)$-manifolds correspond to $\mathcal{N}^l_{m,n}$ with m and n even.*

We will see that these actions are pairwise non-equivalent for different choices of the parameter l when $\gcd(m,n) \geq 3$ by showing that the fundamental group of the fixed point set of the principal isotropy group is isomorphic to $\mathbb{Z}_l$. For $\gcd(m,n) = 1$ or 2, the SU(2), respectively SO(3)-actions $\mathcal{N}^l_{m,n}$ and $\mathcal{N}^{l'}_{m,n}$ are equivalent precisely when $l = l'$ modulo $mn/\gcd(m,n)$.

Theorem C easily implies the following.

Corollary 1. *A G-action without singular orbits on M extends to $\mathrm{U}(2) \times \mathbb{S}^1$ if $G = \mathrm{SU}(2)$ and to $\mathrm{SO}(3) \times T^2$ if $G = \mathrm{SO}(3)$. In particular, M admits an effective T^3-action.*

Finally, for actions with singular orbits we have

Theorem D. *If the G-manifold M has singular orbits, M is equivariantly diffeomorphic to either a linear action on $\mathbb{S}^5$ or:*

(a) The $\mathrm{SO}(3)$-action on either $\mathcal{N}^1_{0,2m} = \mathbb{S}^3 \times \mathbb{S}^2$, connected sums of k copies of the Wu-manifold and l copies of the Brieskorn variety of type $(2,3,3,3)$, or connected sums of copies of $\mathbb{S}^3 \times \mathbb{S}^2$ with the linear action by $\mathrm{SO}(3) \subset \mathrm{SO}(4)$ on the first factor;

(b) The $\mathrm{SU}(2)$-action on either $\mathcal{N}^1_{0,2m+1} = \mathbb{S}^3 \tilde{\times} \mathbb{S}^2$, or the Wu-manifold with the left action by $\mathrm{SU}(2) \subset \mathrm{SU}(3)$.

The case of $G = \mathrm{SO}(3)$ was studied in [19], although the author missed the equivariant connected sums of $\mathbb{S}^3 \times \mathbb{S}^2$ with the above SO(3)-action, and did not describe some of the actions explicitly. For partial results about differentiable classifications, see [24], [25], [23] and [26].

In Section 2 we discuss preliminaries and describe the basic examples. In Section 3 we introduce the SU(2)-manifolds $\mathcal{N}^l_{m,n}$, prove Corollary 1 assuming Theorem C and prove some results needed for the proof of Theorem C. Sections 4 and 5 are devoted to the proofs of Theorems C and D. In Section 6 we prove Theorems A and B.

2 Preliminaries

In this section we fix our notation, define some of the objects we will work with and state the main basic results we need. An advanced reader can start in Section 2 and come back to this section whenever necessary (see [4]).

An *action*, $\alpha : G \circlearrowright M$, of the Lie group G on the smooth manifold M is a morphism $\alpha : G \to \mathrm{Diff}(M)$ whose image $(\alpha g)(p)$ we represent by $g \cdot p$ or gp, for $g \in G$ and $p \in M$. A manifold endowed with a G-action is called a *G-manifold*. We say that two G-manifolds M and N are *equivalent* if there is an equivariant diffeomorphism $f : M \to N$ between them, that is $f(gp) = gf(p)$ for all $p \in M$ and $g \in G$. In this work, we usually consider actions up to equivalence, thus when we refer to the "number of actions" it means the "the number of actions up to equivariant diffeomorphisms", etc. The action is said to be *effective* if the kernel of α is trivial. An ineffective G-action is equivalent to the induced action of the Lie group $G/\ker(\alpha)$ that is effective.

Given a point p of M, the *isotropy group of* p is the subgroup G_p of G given by

$$G_p = \{g \in G \,:\, gp = p\}.$$

The actions that have all the isotropy groups conjugated to some fixed subgroup have special properties some of which are discussed in Section 2.1. For example, for those actions the space of orbits has a natural manifold structure. In the particular case when the isotropy group of any point is trivial the action is called *free*.

Given a point p of M, the *orbit of* p is the set

$$G(p) = \{gp \in M \,:\, g \in G\}.$$

It is well known that $G(p)$ is an embedded submanifold of M diffeomorphic to G/G_p. The isotropy groups of points at the same orbit are conjugated, more precisely, $G_{gp} = gG_pg^{-1}$.

Given a subgroup K of G we denote by (K) the set of subgroups of G which are conjugated to K, i.e.

$$(K) = \{gKg^{-1} : g \in G\}.$$

When a subgroup K' of G belongs to (K) we say that K' *is of type* (K). This notion clearly defines an equivalence relation on the set of subgroups of G. If H is a subgroup of gKg^{-1}

for some $g \in G$, we write $(H) \leq (K)$ and say that H *has type smaller than* K. For isotropy subgroups of a group action we have the Principal Orbit Theorem ([4], p.179) which claims:

(a) There exists a unique minimal isotropy type (H), the elements of which are called *principal isotropy groups*;

(b) The set $M_{(H)}$ of points with isotropy group in (H) is open, dense in M.

If H is a principal isotropy group then, $\dim H \leq \dim K$ for any isotropy subgroup K of the action. If K is an isotropy subgroup with $\dim H < \dim K$ then we call K a *singular isotropy group*. The remaining possibility is when an isotropy group K with the same dimension as H has "strictly bigger isotropy type" than H. This happens when H is a proper open component (not necessarily connected) of K. In this case the group K is called an *exceptional isotropy group*.

We say that the orbit type of $G(p)$ is bigger than $G(q)$, and write $(G(q)) \leq (G(p))$, when the isotropy types satisfy $(G_p) \leq (G_q)$. The orbit $G(p)$ is called principal, singular or exceptional according to the associated isotropy group G_p.

As previously observed, it is well known that the orbit space admits a manifold structure if the action has only one isotropy (orbit) type. In general, the quotient M/G of the manifold by the action with the induced topology is a topological manifold with boundary. If M is simply-connected and G is connected, then M/G is also simply-connected since the projection $M \to M/G$ has the path lifting property (see [4] p. 91). The dimension of the orbit space M/G is called the *cohomogeneity* of the action. Clearly $\dim(M/G) = \dim M - \dim G + \dim H$, where H is a principal isotropy group.

From the differentiable point of view, the orbit space M/G has the structure of a stratified space. For this work it is enough to know that when the action has isotropy types of the same dimension the orbit space has the structure of an orbifold. Roughly speaking an *orbifold* is a topological space locally modeled on $\mathbb{R}^n$ modulo a finite group action. If M is a manifold and Γ is a group acting on M with only finite isotropy groups, then M/Γ has the structure of an orbifold ([31], p. 302). When M is simply-connected, then M is a covering space in the sense of orbifolds and Γ is called the *orbifold fundamental group* of M/Γ. It is important to observe that the underlying space of the orbifold M/Γ, the space from the topological point of view, may even be simply-connected. For precise results and definitions about orbifolds we refer to Chapter 13 of [31].

A *representation* of a Lie group G is a morphism $G \to \mathrm{Gl}(n, \mathbb{R})$. It determines a linear action of G on $\mathbb{R}^n$. A representation is called *irreducible* if the unique linear subspaces of $\mathbb{R}^n$ which are invariant by the action are $\{0\}$ and the whole $\mathbb{R}^n$. It is useful to note that any representation of a compact Lie group G can be seen as an *orthogonal representation*, $G \to \mathrm{O}(n)$, on $\mathbb{R}^n$ with a suitable inner product. In the same way one shows that given a smooth action of a compact Lie group G on a manifold M there is a Riemannian metric g on M for which the action is by isometries. In this case, G is seen as a subgroup of the isometry group $\mathrm{Iso}(M, g)$ of the metric g.

Let S be an embedded submanifold through $p \in M$. Suppose that S is invariant under the action of G_p, i.e., $G_p(S) = S$. Let $G \times_{G_p} S$ be the quotient of $G \times S$ by the action $h \cdot (g, s) = (gh^{-1}, h(s))$. Then S is called a *slice through* p, if the map

$$G \times_{G_p} S \to M, \qquad [(g, s)] \mapsto gs,$$

is a G-equivariant diffeomorphism onto an open neighborhood of $G(p)$. The Slice Theorem claims that there is a slice through each point p of M. The action of the isotropy group G_p on the slice S is called *isotropy action*. This action is commonly identified with the action of derivatives at p of the diffeomorphisms in G_p. This is a linear action of G_p on the tangent space of S at the point p and it is called the *isotropy representation of* G_p. In general, this representation can be seen as a morphism $G_p \to \mathrm{O}(l)$, where l is the codimension of the orbit $G(p)$ in M.

An exceptional orbit $G(p)$ is called *special exceptional* if the isotropy action of G_p on S, the slice through p, has a codimension one fixed point set in S. Geometrically it means that the action $G_p \circlearrowright S$ is a reflection in the fixed hypersurface $S^{G_p} \subset S$.

It is well known that a compact connected non-abelian Lie group G contains either $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$, to see this, take a subgroup corresponding to the subalgebra generated by a root space and a root vector in the root system of the Lie algebra of G. Thus, in order to classify 5-dimensional manifolds with an action of a non-abelian connected Lie group G it is enough to classify the actions of $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$.

For the next results we refer to Chapter IV Sections 4 and 8 of [4]. Let M be a compact, connected and simply-connected G-manifold where G is compact and connected.

PROPOSITION 2. *The orbits of maximal dimension are orientable.*

PROPOSITION 3. *There are no special exceptional orbits.*

PROPOSITION 4. *If the action has cohomogeneity 2, then the space of orbits, M/G, is a 2-dimensional topological manifold with (or without) boundary. The boundary, if not empty, consists of the singular orbits and in this case there are no exceptional orbits.*

PROPOSITION 5. *If the action has cohomogeneity 3, then M/G is a simply-connected (smooth) 3-manifold possibly with boundary.*

It is known that a compact simply-connected 3-manifold is diffeomorphic to $\mathbb{S}^3$, (see [27]). The following completes the possible orbit spaces for a cohomogeneity 3 action.

PROPOSITION 6. *A simply-connected compact 3-dimensional manifold with boundary is diffeomorphic to a 3-sphere with k open 3-disks removed.*

Proof. Let X be a simply-connected compact 3-manifold with boundary. We claim that the boundary of X is a disjoint union of 2-spheres. In fact, Poincaré duality to the pair $(X, \partial X)$ guarantees that $H_2(X, \partial X) \simeq H^1(X)$, so from the exactness of the relative homology sequence,

$$\cdots \to H_2(X, \partial X) \to H_1(\partial X) \to H_1(X) \to \cdots$$

we conclude that $H_1(\partial X) = 0$. So each connected component of ∂X is homeomorphic to $\mathbb{S}^2$ by the classification of compact surfaces.

The manifold obtained from X by covering each connected component with a 3-disk is simply-connected compact without boundary, so it is a 3-sphere and the proposition is proved. □

THEOREM 7 (Barden-Smale [2] and [29]). *If M and N are simply-connected, compact, 5-dimensional manifolds with isomorphic second homology groups with integer coefficients and the same second Stiefel-Whitney classes, then M and N are diffeomorphic.*

THEOREM 8 ([4] p. 93). *Let G be a compact Lie group and X a topological manifold. If N is a G-manifold with orbit space $X \times I$ whose orbit types are constant along $\{x\} \times I$ for all $x \in X$, then there is a G-manifold M satisfying:*

(i) The quotient $M/G \simeq X$ and each element in M/G corresponds to the same orbit type as in N/G;

(ii) The G-manifold $M \times I$, with G acting trivially on I, is equivalent to N;

(iii) If $f : N \to M \times I$ is a G-equivalence and $\pi_N : N \to X \times I$ and $\pi_M : M \to X$ are the orbit maps, then the diagram

$$\begin{tikzcd} N \arrow[rr, "f"] \arrow[dr, "\pi_N"'] & & M \times I \arrow[dl, "\pi_M \times Id_I"] \\ & X \times I & \end{tikzcd}$$

commutes.

Moreover, M can be taken to be $\pi_N^{-1}(X \times \{0\})$ and $f|_{\pi_N^{-1}(X \times \{0\})} : M \to M \times I$ as the inclusion $p \mapsto (p, 0)$.

2.1 Generalities about actions with one orbit type

The following results are well known from the theory of group actions; we refer to Bredon [4] Chapter II Section 2 for the first and Section 5 for the second proposition.

Proposition 9. *Let G be a Lie group acting on M with only one isotropy type (H). The group $\Gamma_H := N(H)/H$ acts freely on the fixed point set $M^H = \{p \in M\,;\, hp = p \text{ for all } h \in H\}$ and the orbit space $M^H/N(H)$ is diffeomorphic to M/G. In particular, $\Gamma_H \to M^H \to M/G$ is a principal bundle.*

Proposition 10. *If G acts on M with unique isotropy type (H), then the manifold M is the total space of the bundle associated to the principal bundle in Proposition 9. More precisely, the map*

$$\Phi : M^H \times_{\Gamma_H} (G/H) \to M$$

defined by $\Phi[(p, gH)] = gp$ is a G-equivariant diffeomorphism.

Proposition 11. *There are as many equivalence classes of G-manifolds with unique isotropy type (H) and orbit space $M/G \simeq \mathbb{S}^n$ as elements in $\pi_{n-1}(\Gamma_H)$.*

Proof. It is known (c.f. A. Borel [3]) that given a closed subgroup $H \subset G$, there is a bijective correspondence between the set of isomorphism classes of principal bundles $\Gamma_H \to P \to B$, where $\Gamma_H = N(H)/H$, and the set of equivalence classes of G-manifolds M with unique orbit type (G/H). This correspondence, described in Proposition 10, associates the principal bundle $\Gamma_H \to P \to B$ to the G-manifold $M := P \times_N (G/H)$ with G acting only on G/H by left multiplication on the cosets. It is known (see [30], Corollary 18.6) that the set of isomorphism classes of F-bundles over $\mathbb{S}^n$ are in bijection with $\pi_{n-1}(F)$ whenever F is arcwise connected. Now, the proposition follows from the fact that for a Lie group K, the principal K-bundle is determined by the K_o-fiber bundle over the same basis, where K_o is the identity component of K. Therefore, the bijection holds for K-principal bundles over $\mathbb{S}^n$ even if the Lie group K is not connected. □

Hereafter in this work unless explicit stated G will denote SU(2) or SO(3). If M is simply-connected of dimension 5 with one orbit type, then the orbit space M/G is a 2 or 3 dimensional compact simply-connected manifold without boundary. Hence, M/G is diffeomorphic to either $\mathbb{S}^2$ or $\mathbb{S}^3$, as a consequence of the classification of the compact surfaces and Perelman's proof of Poincaré's Conjecture [27] . In this case we can count the number of G-manifolds using Proposition 11: it coincides with the order of the first or the second homotopy group of Γ_H if $M/G = \mathbb{S}^2$ or $\mathbb{S}^3$ respectively.

2.2 SO(3) and SU(2) group structure and examples

The Lie group SO(3) is the group of rotational symmetries of $\mathbb{R}^3$. Every nontrivial $A \in \mathrm{SO}(3)$ fixes a unique direction on $\mathbb{R}^3$, called its *axis of rotation*. Choosing a special orthonormal basis on $\mathbb{R}^3$ the map A can be expressed as $A = \mathrm{diag}(R(\theta), 1)$, where $R(\theta)$ is the rotation of angle θ on the plane orthogonal to the axis of rotation.

Every nontrivial subgroup of SO(3) is isomorphic to either the cyclic group $\mathbb{Z}_k$, the dihedral group D_m, the tetrahedral group T, the octahedral group O, the icosahedral group I, the circle SO(2) or the normalizer of SO(2) in SO(3), which is O(2) (see [33] Section 7.1).

The special unitary group SU(2) is the Lie group of linear operators of $\mathbb{C}^2$ that preserve the hermitian inner product and have determinant one. Alternatively, SU(2) can be seen as $\mathbb{S}^3 \subset \mathbb{C}^2$ or the unit quaternions, $\mathbb{S}^3 \subset \mathbb{H}$. For α and $\beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$ we have the following correspondence between the three expressions of an element in SU(2)

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in \mathbb{C}^{2\times 2} \quad \sim \quad (\alpha, \beta) \in \mathbb{C}^2 \quad \sim \quad \alpha + \beta j \in \mathbb{H}.$$

The quaternion notation will be generally used for the group SU(2), while the $\mathbb{S}^3 \subset \mathbb{C}^2$ notation when considering SU(2) just as a manifold. The circle $\theta \mapsto e^{i\theta} \in \mathbb{H}$ will be considered as a canonical choice of a subgroup of SU(2) isomorphic to $\mathbb{S}^1$.

Identifying $\mathbb{R}^3$ with the subspace $\{\mathrm{Re}\,\mathbb{H} = 0\} \subset \mathbb{H}$, consider the linear action $\phi : \mathrm{SU}(2) \circlearrowright \mathbb{R}^3$ given by $q \cdot v = q\, v\, q^{-1}$. It is easy to check that $\ker(\phi) = \mathbb{Z}_2$, that the action is by isometries and preserves orientation. Thus, it gives rise to a map $\phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ which is the universal 2-fold cover. The subgroups of SU(2) are isomorphic to $\mathbb{Z}_{2k+1}$ or the pre-images by ϕ of the subgroups of SO(3) (see [1] §2). A more geometric description of the subgroups of SU(2) is given by considering it as the unit quaternions. Any closed nontrivial subgroup of SU(2) is then isomorphic to one of the following:

(i) The cyclic group $\mathbb{Z}_k = \langle e^{2\pi i/k} \in \mathbb{H}\rangle$ of order k;

(ii) The dicyclic group, $\mathrm{Dic}_n = \langle e^{i\pi/n},\, j \in \mathbb{H}\rangle$ of order $4n$;

(iii) The binary tetrahedral group, $\mathrm{T}^* = \langle \frac{1}{2}(1+i+j+k), \frac{1}{2}(1+i+j-k) \in \mathbb{H}\rangle$ of order 24;

(iv) The binary octahedral group, $\mathrm{O}^* = \langle \frac{1}{2}(1+i+j+k), \frac{\sqrt{2}}{2}(1+i) \in \mathbb{H}\rangle$ of order 48;

(v) The binary icosahedral group, $\mathrm{I}^* = \langle \frac{1}{2}(1+i+j+k), \frac{1}{2}(\varphi + \varphi^{-1} i + j) \in \mathbb{H}\rangle$ where φ is the golden ratio $\varphi = \frac{1}{2}\left(1+\sqrt{5}\right)$, which has 120 elements;

(vi) The circle $\mathrm{SO}(2) = \{e^{i\theta} \in \mathbb{H}\}$;

(vii) The pin group, $\mathrm{Pin}(2) = \langle \mathrm{SO}(2)\,,\, j \in \mathbb{H}\rangle = N_{\mathrm{SU}(2)}(\mathrm{SO}(2))$.

We introduce here some of the G-actions with $G = \mathrm{SO}(3)$ or $\mathrm{SU}(2)$ that will appear in our classification.

Example 1. *The* $\mathrm{SO}(3)$*-action on* $\mathbb{S}^5$ *with isotropy types* $(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $(\mathrm{O}(2))$ *and* $\mathrm{SO}(3)$*.*
It has exactly two fixed points, and corresponds to the linear action on $\mathbb{R}^6$ which arises from the unique irreducible representation of $\mathrm{SO}(3)$ in $\mathrm{SO}(5)$ by adding a fixed direction to $\mathbb{R}^5$. Identifying $\mathbb{R}^6$ with the set of 3×3 symmetric matrices

$$\mathbb{R}^6 \simeq \{X \in \mathbb{R}^{3\times 3}\,;\, X^T = X\},$$

this $\mathrm{SO}(3)$-action is given by $A \cdot X = AXA^{-1}$. Moreover, defining the inner product on $\mathbb{R}^6$ by $\langle A, B\rangle = \operatorname{tr} AB$, the action is by isometries. The corresponding $\mathrm{SO}(3)$-linear action on $\mathbb{S}^5$ is this action on the unit vectors of $\mathbb{R}^6$. The Spectral Theorem guarantees that in each orbit there is a diagonal matrix, thus each orbit is uniquely determined by the eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ of the operator associated to the matrix. To avoid ambiguity we write them in decreasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Thus, the orbit space is

$$\mathbb{S}^5/\,\mathrm{SO}(3) = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 : \lambda_1 \geq \lambda_2 \geq \lambda_3 \text{ and } \Sigma\lambda_i^2 = 1\},$$

which is homeomorphic to a closed 2-disk with two fixed points, $\pm(\sqrt{3}/3)\,\mathrm{Id}$ in the boundary. The matrices with exactly two equal eigenvalues have isotropy either $\mathrm{S}(\mathrm{O}(2)\,\mathrm{O}(1))$ or $\mathrm{S}(\mathrm{O}(1)\,\mathrm{O}(2))$ corresponding to the *boundary faces* of the quotient and the matrices with three distinct eigenvalues have isotropy group $\mathbb{Z}_2 \times \mathbb{Z}_2$, corresponding to the interior points of the disk. From the metric point of view the angle between the boundary faces of the quotient is $\pi/3$ and has vertices in the fixed points.

Example 2. *The* $\mathrm{SO}(3)$*-action on* $\mathbb{S}^5$ *with isotropy types* $\{1\}$ *and* $(\mathrm{SO}(2))$*.*
Consider $\mathbb{S}^5 \subset \mathbb{R}^3 \times \mathbb{R}^3$ and let the action be $A \cdot (x, y) = (Ax, Ay)$. The points $(x, y) \in \mathbb{S}^5$ with x and y linearly independent have trivial isotropy group. Any other point has isotropy group $\mathrm{SO}(2)$. The quotient is a 2-disk.

Example 3. *The* $\mathrm{SO}(3)$*-action on* $\mathbb{S}^5$ *with isotropy types* $(\mathrm{SO}(2))$ *and* $\mathrm{SO}(3)$*.*
This corresponds to the natural inclusion of $\mathrm{SO}(3)$ in $\mathrm{SO}(6)$, given by $\mathrm{diag}(A, 1, 1, 1) \in \mathrm{SO}(6)$ for $A \in \mathrm{SO}(3)$. It is easy to verify that the quotient is a 3-disk with the boundary corresponding to the fixed points.

Example 4. *The* $\mathrm{SU}(2)$*-action on* $\mathbb{S}^5$ *with isotropy types* $\{1\}$ *and* $\mathrm{SU}(2)$*.*
This is the action determined by the unique $\mathrm{SU}(2)$ linear action on $\mathbb{C}^3$ given by the embedding

$B \in \mathrm{SU}(2) \mapsto \mathrm{diag}(B, 1) \in \mathrm{SU}(3)$. The quotient is the 2-disk with the boundary corresponding to the circle of fixed points $\{(0, 0, z) \in \mathbb{C}^3 : |z| = 1\}$. We will see that this is the unique SU(2)-action with fixed points.

Example 5. *The* SO(3)*-linear on* $\mathbb{S}^3 \times \mathbb{S}^2$ *with isotropy types* (SO(2)) *and* SO(3)*.*
The action is defined by the embedding $\mathrm{SO}(3) \subset \mathrm{SO}(4) \times \mathrm{SO}(3)$ in the first coordinate. If N and $S \in \mathbb{S}^3$ are the north and south pole, the 2-spheres $\{N\} \times \mathbb{S}^2$ and $\{S\} \times \mathbb{S}^2$ are fixed and any other point have isotropy SO(2). The orbit space is diffeomorphic to $\mathbb{S}^2 \times [-1, 1]$.

Example 6. *The* SU(2)*-action on the Wu-manifold* $\mathcal{W} := \mathrm{SU}(3)/\,\mathrm{SO}(3)$*, with isotropy types* $\{1\}$ *and* (SO(2))*.*
The Wu-manifold is the quotient of SU(3) by right multiplication of $\mathrm{SO}(3) \subset \mathrm{SU}(3)$. Given $B \in \mathrm{SU}(2)$ and $[C] \in \mathcal{W}$, the action is $B \cdot [C] = [(\mathrm{diag}(B, 1)C)]$. Thus, if $B \in \mathrm{diag}(\mathrm{SU}(2), 1)$ is in the isotropy group of the point $[C] \in \mathcal{W}$, there exists an $A \in \mathrm{SO}(3)$ such that $BC = CA$. This means that the isotropy groups of $\mathrm{SU}(2) \circlearrowright \mathcal{W}$ are isomorphic to the isotropy groups of the SO(3)-action on $\mathrm{SU}(2) \setminus \mathrm{SU}(3) \simeq \mathbb{S}^5$ by right multiplication on the cosets. This is a linear SO(3)-action on $\mathbb{S}^5$, and hence corresponds to the representation of SO(3) in SO(6) in Example 2. This implies that the isotropy types of the $\mathrm{SU}(2) \circlearrowright \mathcal{W}$ are $\{1\}$ and (SO(2)), while the quotient is the 2-disk.

Example 7. *The* SO(3)*-action on* $\mathcal{W}$ *with isotropy types* $(\mathbb{Z}_2 \times \mathbb{Z}_2)$*,* (O(2)) *and* SO(3)*.*
The group SO(3) acts on $\mathcal{W}$ as $A \cdot [B] = [AB]$. The isotropy types are $(\mathbb{Z}_2 \times \mathbb{Z}_2)$, (O(2)) and SO(3) corresponding respectively to the orbits that have diagonal matrices with three, two or one distinct eigenvalues, up to signs.

This action has exactly three fixed points, they are [Id], $[\mathrm{diag}(e^{(\pi i)/3}, e^{(\pi i)/3}, -e^{(\pi i)/3})]$ and $[\mathrm{diag}(e^{(2\pi i)/3}, e^{(2\pi i)/3}, -e^{(2\pi i)/3})]$. The isotropy type (O(2)) occurs in three different ways each one as a *boundary face* of the quotient and has the form of either $\mathrm{S}(\mathrm{O}(2)\,\mathrm{O}(1)) = \mathrm{diag}(A, \det A)$, $\mathrm{S}(\mathrm{O}(1)\,\mathrm{O}(2)) = \mathrm{diag}(\det A, A)$ or $B \cdot \mathrm{S}(\mathrm{O}(2)\,\mathrm{O}(1)) \cdot B^{-1}$, where $A \in \mathrm{O}(2)$ and B is the 3 by 3 matrix with column vectors respectively e_3, e_1 and e_2. The quotient is a topological two disk. If the action is by isometries, then the quotient is a flat triangle. Each edge corresponds to some embedding of the isotropy subgroup $\mathrm{O}(2) \subset \mathrm{SO}(3)$. It was proved in [19] that this is the unique SO(3)-action on the Wu-manifold up to conjugacy.

Example 8. *The* SO(3)*-action on the Brieskorn variety, $\mathcal{B}$ of type* $(2,3,3,3)$ *with isotropy types* $(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $(\mathrm{O}(2))$ *and four fixed points.*
The Brieskorn variety of type $(2,3,3,3)$ can be defined as

$$\mathcal{B} = \left\{(z_o, z_1, z_2, z_3) \in \mathbb{C}^4\,;\, z_o^2 + z_1^3 + z_2^3 + z_3^3 = 0 \text{ and } |z_o|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\right\}.$$

In [19] this action is constructed by a process of gluing four open sets and the manifold is identified computing topological invariants. Each of these four open sets corresponds to an isolated fixed point. As far as we know an explicit description of this action is not known.

Given two n-dimensional G-manifolds with fixed points, choose Riemannian metrics invariant under the G-actions and consider a small ball of radius r around a fixed point in each manifold. If the isotropy actions of G on the slices of those fixed points are the same, then the actions on the boundaries of the balls are equivalent and we can to make a connected sum of the two G-manifolds gluing along those spheres to obtain a new smooth G-manifold.

The next SO(3)-manifold is missing in the classification in [19].

Example 9. *The* SO(3)*-action on the connected sum of k copies of* $\mathbb{S}^3 \times \mathbb{S}^2$ *with isotropy group* SO(2) *and* $k+1$ *two-spheres consisting of set of fixed points.*
We start with the SO(3)-action in $\mathbb{S}^\times\mathbb{S}^2$ from Example 5. At a fixed point the isotropy representation is given by SO(3)-action on $\mathbb{R}^3 \times \mathbb{R}^2$ that is standard in the first coordinate and trivial on the second. We can now take the connected sum of 2 copies of $\mathbb{S}^3 \times \mathbb{S}^2$ at the fixed points. This provides a connected sum of two fixed 2-spheres, so if we do this for k copies of $\mathbb{S}^3 \times \mathbb{S}^2$, we obtain $k+1$ fixed 2-spheres. The orbit space of the action is diffeomorphic to a 3-sphere with $k+1$ three-disks removed.

Notice that the manifolds are not diffeomorphic. In fact, for n-dimensional manifolds M and N, it is well known that $H_r(M\#N) \simeq H_r(M) \oplus H_r(N)$ for $0 < r < n$. So, denoting $\mathbb{S}^3 \times \mathbb{S}^2$ by M we have that

$$H_2(M\# \cdots \#M\ ;\ \mathbb{Z})\ \simeq\ \mathbb{Z}^k.$$

Example 10. *The* SO(3)*-actions on* $k\,\mathcal{W}\#\,l\,\mathcal{B}$.
The isotropy representation around an isolated fixed point of an SO(3)-manifold of dimension 5 must be the unique irreducible one (see Example 1). The SO(3)-action on $\mathbb{S}^4$ where the connected sum takes place has quotient an interval, so there are exactly two ways to connect the manifolds. In [19], it is shown that depending on the way that $\mathcal{W}$ and $\mathcal{B}$ are connected we get distinct SO(3)-manifolds.

We will see that these are precisely the simply-connected 5-dimensional SO(3)-manifolds with isolated fixed points.

3 The main example

The following construction is crucial since it generates all 5-dimensional G-manifolds without singular orbits and most actions with singular orbits for $G = \mathrm{SO}(3)$ or $\mathrm{SU}(2)$.

Let G be a Lie group, $\mu : H \to G$ a morphism of a Lie group H in G and N an H-manifold. Consider the action of H on $G \times N$ defined by

$$\Theta(h, (g, x)) = (g\mu(h^{-1}), hx). \tag{1}$$

If for each point $x \in N$ the intersection of $\mathrm{Ker}(\mu)$ with the isotropy group H_x is trivial, then the action Θ is free. So the quotient, $G \times_H N$, is a G-manifold where we define the G-action by

$$k \cdot [(g, x)] = [(kg, x)].$$

The isotropy group at the point $[(g, x)]$ is isomorphic to the subgroup $\mu(H_x) \simeq H_x$ of G.

When the rank of the group G is one and $H = \mathbb{S}^1$, instead of considering a morphism μ as above, we can fix a monomorphism of $\mathbb{S}^1$ in G and take an integer power of it. This is what is done in the next example.

Example 11 (Main example). Let m, n and l be nonnegative integer numbers and, to avoid ambiguity, we assume that $m \le n$ and set $l = 1$ whenever $m = 0$. Consider the $\mathbb{S}^1$-action on $\mathrm{SU}(2) \times \mathbb{S}^3$ given by

$$x * (p, (z, w)) = (px^l, (x^m z, x^n w)), \tag{2}$$

where $p \in \mathrm{SU}(2)$ and $(z, w) \in \mathbb{S}^3 \subset \mathbb{C}^2$. We regard the $\mathrm{SU}(2)$ factor as the group of unit quaternions. We fix $\mathbb{S}^1 \subset \mathbb{C} \subset \mathbb{H}$ for the monomorphism from the circle to $\mathrm{SU}(2) \subset \mathbb{H}$.

As explained in the general construction, the quotient

$$\mathcal{N}^l_{m,n} = \mathrm{SU}(2) \times_{\mathbb{S}^1} \mathbb{S}^3$$

is a manifold whenever $\gcd(l, m) = \gcd(l, n) = 1$. We see from the sequence of homotopies of the principal bundle

$$\mathbb{S}^1 \to \mathrm{SU}(2) \times \mathbb{S}^3 \to \mathcal{N}^l_{m,n}$$

that the manifold $\mathcal{N}^l_{m,n}$ is simply-connected.

Proposition 12. *The manifolds $\mathcal{N}^l_{m,n}$ are diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^2$ if $m+n$ is even, otherwise they are diffeomorphic to the unique nontrivial $\mathbb{S}^3$-bundle over $\mathbb{S}^2$, denoted by $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$.*

Proof. The exact homotopy sequence of the principal bundle $\mathbb{S}^1 \to \mathbb{S}^3 \times \mathbb{S}^3 \to M$ shows that the second homotopy group is $\pi_2(M) \simeq \mathbb{Z}$. Since M is simply-connected the Hurewicz morphism $h_* : \pi_2(M) \to H_2(M)$ is an isomorphism. Thus $H_2(M) \simeq \mathbb{Z}$.

The second Stiefel-Whitney class for these quotients is computed in detail in [8] p. 77. It is shown that $w_2 = 0$ if $m+n$ is even and $w_2 = 1$ otherwise. So, the result follows from Barden-Smale classification, c.f. Theorem 7. $\square$

Now we define the SU(2)-action on $\mathcal{N}^l_{m,n}$ by

$$g \cdot [(p,(z,w))] = [(gp,(z,w))].$$

We will also denote this SU(2)-manifold by $\mathcal{N}^l_{m,n}$. This action has the same isotropy structure as the $\mathbb{S}^1$-action on the second factor $\mathbb{S}^3$. If the integers l, m and n are nonzero, the isotropy group of the point $[(p,(z,w))]$ is respectively $\mathbb{Z}_m$, $\mathbb{Z}_n$ or $\mathbb{Z}_{\gcd(m,n)}$ according $w = 0$, $z = 0$ or both z and w are non-zero. In particular, m and n are invariants of the SU(2)-manifold $\mathcal{N}^l_{m,n}$. For convenience we set $\gcd(0,0) = 1$.

In general, for SU(2)-actions, if the principal isotropy group contains $\mathbb{Z}_2$ as a subgroup, the action is ineffective since $\mathbb{Z}_2$ is normal in SU(2). Therefore, $\mathcal{N}^l_{2m,2n}$ becomes an effective SO(3)-manifold with isotropy groups $\mathbb{Z}_m$, $\mathbb{Z}_n$ and $\mathbb{Z}_{\gcd(m,n)}$. Observe that the underlying manifolds are all diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^2$ and l has to be odd to be coprime with $2m$ and $2n$.

Remark 1. The $\mathbb{S}^1$-action (2) is a restriction of the $\mathrm{SU}(2) \times T^3$-action on $\mathrm{SU}(2) \times \mathbb{S}^3$ given by

$$(g,(r,s,t)) \cdot (p,(z,w)) = (gpr,(sz,tw))$$

where $r,s,t \in \mathbb{S}^1$. In our example $\Delta\mathbb{S}^1 = \{(1,(x^l,x^m,x^n))\} \subset \mathrm{SU}(2) \times T^3$ is the $\mathbb{S}^1$ group that acts on $\mathrm{SU}(2) \times \mathbb{S}^3$. Hence, the quotient $\mathcal{N}^l_{m,n}$ admits an action by $\mathrm{SU}(2) \times (T^3/\Delta\mathbb{S}^1)$ with kernel generated by $(-1,[(-1,1,1)])$. Notice that if m and n are even then $(1,(-1,1,1)) \in \Delta\mathbb{S}^1$, so the kernel is $\mathbb{Z}_2 \subset \mathrm{SU}(2)$ and $\mathrm{SO}(3) \times T^2$ acts effectively on $\mathcal{N}^l_{m,n}$. Otherwise, the ineffective kernel is the diagonal $\mathbb{Z}_2$ in $\mathrm{SU}(2) \times \mathbb{S}^1 \subset \mathrm{SU}(2) \times T^2$, thus $\mathrm{U}(2) \times \mathbb{S}^1$ acts effectively on $\mathcal{N}^l_{m,n}$. In any case, there is an effective action of a 3-torus on $\mathcal{N}^l_{m,n}$.

This proves Corollary 1, assuming Theorem C.

As we pointed out, if $lmn \neq 0$, the isotropy types of $\mathcal{N}^l_{m,n}$ are $(\mathbb{Z}_m)$, $(\mathbb{Z}_n)$ and $(\mathbb{Z}_{\gcd(m,n)})$. Hence, if $m = n$, the action has only one isotropy type $(\mathbb{Z}_n)$. In particular, we get free actions on $\mathbb{S}^3 \times \mathbb{S}^2$ when m and n are both equal to either 1 or 2. The SU(2)-manifolds $\mathcal{N}^l_{1,1}$ are all equivalent since there is only one isomorphism class of SU(2)-principal bundles over $\mathbb{S}^2$ (c.f. Corollary 18.6 in [30]). The same result shows that there are two non-equivalent SO(3)-principal bundles over $\mathbb{S}^2$, but just one of them is simply-connected, $\mathcal{N}^1_{2,2}$, with the free SO(3)-action. Notice that the free SU(2)-manifold $\mathcal{N}^0_{1,1}$ corresponds to the left multiplication on the first factor of $\mathrm{SU}(2) \times \mathbb{S}^2$.

If $m = n = 0$ then $l = 1$ and the $\mathbb{S}^1$-action on the first factor reduces to the Hopf action with quotient diffeomorphic to $\mathbb{S}^2 \simeq \mathrm{SU}(2)/\,\mathrm{SO}(2)$. Thus the SU(2)-action is the natural product on the cosets and has unique isotropy type equal to (SO(2)). On the other hand, if $m = 0 < n$, then $l = 1$ again and the isotropy types are $(\mathbb{Z}_n)$ and (SO(2)).

Remark 2. We do not obtain new G-manifolds by taking negative integer parameters. In fact, if we regard the $\mathbb{S}^1$-action on the first factor of $\mathrm{SU}(2) \times \mathbb{S}^3$ considering $\mathrm{SU}(2) \subset \mathbb{C}^2$ rather than the unit quaternions, for $x \in \mathbb{S}^1$ and $(u, v) \in \mathrm{SU}(2)$ we obtain the action $x \cdot (u, v) = (ux^l, v\overline{x}^l)$. Therefore, the SU(2)-manifolds $\mathcal{N}^l_{m,n}$ and $\mathcal{N}^{-l}_{m,n}$ are equivalent by switching (u, v) to (v, u). In the same way we can consider the SU(2)-equivariant diffeomorphism $f : \mathcal{N}^l_{m,n} \to \mathcal{N}^l_{-m,n}$ which takes $[(p,(z,w))]$ to $[(p,(\overline{z},w))]$. The equivalence for n negative is analogous.

Hereafter in this chapter $\mathcal{N}^l_{m,n}$ will be denoted by $\mathcal{N}^l_{n_1,n_2}$ and its elements are now represented by $[(p,(z_1,z_2))]$. Let n_1 and n_2 be positive integers. We will describe the isotropy representations in the neighborhoods of the exceptional orbits of $\mathcal{N}^l_{n_1,n_2}$ and later we compute the clutching function of the decomposition in this representation. This is important since the isotropy representations and the homotopy class of the clutching function arise as invariants to be used in the proof of Theorem C. In the remainder we will use in addition to l, n_1 and n_2 the following integers:

$$d = \gcd(n_1, n_2),\ n_j = dq_j \text{ and } u_j l + v_j n_j = 1 \text{ for some integers } u_j \text{ and } v_j.$$

First consider $\mathbb{S}^3 = B_1 \cup B_2$ where $B_j = \{(z_1, z_2) \in \mathbb{S}^3 : |z_j| \geq 1/\sqrt{2}\}$ for $j = 1$ or 2, and the identification of the boundaries is the trivial one (the identity map). Note that

$$\mathcal{N}^l_{n_1,n_2} = \mathrm{SU}(2) \times_{\mathbb{S}^1} B_1 \bigcup_{\mathrm{Id}} \mathrm{SU}(2) \times_{\mathbb{S}^1} B_2.$$

Now we describe an equivalence between $\mathrm{SU}(2) \times_{\mathbb{S}^1} B_j$ and a certain quotient of $\mathrm{SU}(2) \times D^2$ by $\mathbb{Z}_{n_j}$. Assume $j = 2$, the other case being analogous. For $[(p,(z_1,z_2))] \in \mathrm{SU}(2) \times_{\mathbb{S}^1} B_2$ we have $z_2 \neq 0$ therefore we can write $[(p,(z_1,z_2))] = [(\,p,(\,z_1, |z_2|\, z_2/|z_2|\,))]$. Take $x = \zeta\eta_2 \in \mathbb{S}^1$ with $\eta_2^{n_2} = 1$ and $\zeta^{n_2} = \overline{z_2}/|z_2|$ where $\arg(\zeta) < 2\pi/n_2$ in order to obtain $x^{n_2} = \overline{z_2}/|z_2|$.

Define $\hat{p} = p\zeta^l$ for each $p \in \mathrm{SU}(2)$ and $\hat{z}_1 = \zeta^{n_1} z_1$. Then

$$[(\, p, (\, z_1, z_2\,))] = [(\, px^l, (\, x^{n_1} z_1, x^{n_2} z_2\,))] = [(\, \hat{p}\eta_2^l, (\, \eta_2^{n_1}\hat{z}_1\,,\, \sqrt{1-|\hat{z}_1|^2}\,))],$$

where $\eta_2 \in \mathbb{Z}_{n_2} \subset \mathbb{S}^1 \subset \mathbb{C}$.

So, for some equivariant diffeomorphism $\varphi : \mathrm{SU}(2) \times_{\mathbb{Z}_{n_1}} \mathbb{S}^1 \to \mathrm{SU}(2) \times_{\mathbb{Z}_{n_2}} \mathbb{S}^1$ we have

$$\mathcal{N}^l_{n_1,n_2} = \mathrm{SU}(2) \times_{\mathbb{Z}_{n_1}} D^2 \bigcup_{\varphi} \mathrm{SU}(2) \times_{\mathbb{Z}_{n_2}} D^2, \tag{3}$$

with the actions $\mathbb{Z}_{n_j} \circlearrowright \mathrm{SU}(2) \times D^2$ given by $\eta_j \cdot (p,z) = (p\eta_j^l, \eta_j^{n_i} z)$, $1 \le i \ne j \le 2$. Now, $\xi_j = \eta_j^l$ is a generator of $\mathbb{Z}_{n_j}$, and if we write $n_j = dq_j$, where $d = \gcd(n_1,n_2)$, this last action becomes $\xi_j \cdot (p,z) = (p\xi_j, \xi_j^{dq_i u_j} z)$, for some integer v_j with $u_j l + v_j n_j = 1$. This is the expression provided by the Slice Theorem in a neighborhood of the isolated orbit of type $(\mathrm{SU}(2)/\mathbb{Z}_{n_j})$ and the number $a_j := q_i u_j$ modulo q_j determines the action of the isotropy group $\mathbb{Z}_{n_j}$ in the slice, i.e. the slice representation. As a consequence we have the following result:

Proposition 13. *If $\mathcal{N}^l_{n_1,n_2}$ and $\mathcal{N}^{l'}_{n_1,n_2}$ are equivalent, then $l \equiv \pm l' \mod q_1 q_2$.*

Since $\gcd(a_j, q_j) = 1$, there are integers b_j and r_j such that $a_j b_j + q_j r_j = 1$ and $0 \le b_j < q_j$. Recall that u_j is the inverse of l in $\mathbb{Z}_{q_j}$, so from $a_j = q_i u_j$ we obtain $l \equiv b_j q_i \mod q_j$. Hence there is an integer k such that

$$l = b_1 q_2 + b_2 q_1 + k q_1 q_2. \tag{4}$$

The number l determines the isotropy action in the neighborhood of the exceptional orbits $\mathrm{SU}(2)/\mathbb{Z}_{n_j}$, for $j = 1$ or 2, since it determines b_j. Therefore, it describes each $\mathrm{SU}(2) \times_{\mathbb{Z}_{n_j}} D^2$ in the expression (3).

Proposition 14. *The map $l(b_1,b_2,k) = b_1 q_2 + b_2 q_1 + k q_1 q_2$ is a bijection between the sets $\mathcal{P} = \{(b_1,b_2,k) \in \mathbb{Z}^3 : 0 \le b_j < q_j, (b_j,q_j) = 1, j = 1,2\}$ and $\mathcal{Q} = \{l \in \mathbb{Z} : (l,q_j) = 1, j=1,\ 2\}$.*

Proof. For any $l \in \mathcal{Q}$ given, there is a unique solution (x_o, y_o) for the diofantine equation $xq_1 + yq_2 = l$ satisfying $0 \le x_o < q_2$. Call $x_o = b_2$ and note that b_2 is coprime with q_2 since l is. On the other hand, there are integers k and b_1 both uniquely determined by the properties $y_o = kq_2 + b_1$ and $0 \le b_1 < q_2$. Thus we have $l = b_1 q_2 + q_1(b_2 + kq_2)$ that clearly implies that b_1 and q_1 are coprime since $l \in \mathcal{Q}$. This establishes the bijection between $\mathcal{P}$ and $\mathcal{Q}$. $\square$

For $d = \gcd(n_1,n_2) \ge 3$, the next result guarantees that the parameter $l \ge 0$ itself is an invariant of the action, or equivalently, the $\mathrm{SU}(2)$-manifolds $\mathcal{N}^l_{n_1,n_2}$ and $\mathcal{N}^{l'}_{n_1,n_2}$ are equivalent if and only if $l = \pm l'$.

Proposition 15. *If the principal isotropy group H of $\mathcal{N}^l_{n_1,n_2}$ is isomorphic to $\mathbb{Z}_d$ for $d \geq 3$, the fixed point set $(\mathcal{N}^l_{n_1,n_2})^H$ is a disjoint union of two copies of lens spaces $\mathbb{S}^3/\mathbb{Z}_l$.*

Proof. Let $H \simeq \mathbb{Z}_d$ be the subgroup of $\mathrm{SU}(2)$ generated by $e^{2\pi i/d}$ and notice that $N(H) = N(\mathbb{S}^1)$, where $H \subset \mathbb{S}^1$ and $N(K)$ is the normalizer of the subgroup $K \subset G$ in G. This easily implies that an element $[(p,(z_1,z_2))]$ belongs to M^H if, and only if, $p \in N(H)$. Thus $(\mathcal{N}^l_{n_1,n_2})^H = N(H) \times_{\mathbb{S}^1} \mathbb{S}^3$. Therefore

$$(\mathcal{N}^l_{n_1,n_2})^H = (\mathbb{S}^1 \times_{\mathbb{S}^1} \mathbb{S}^3) \bigcup (\mathbb{S}^1 \times_{\mathbb{S}^1} \mathbb{S}^3),$$

since $N(\mathbb{Z}_d) \simeq \mathrm{Pin}(2)$.

Notice that every $[(y,(z_1,z_2))] \in \mathbb{S}^1 \times_{\mathbb{S}^1} \mathbb{S}^3$ has a representative with $y = 1$. In fact, $[(y,(z_1,z_2))] = [(1,(\xi^{n_1}\zeta^{n_1}z_1, \xi^{n_2}\zeta^{n_2}z_2))]$ where $\zeta^l = y$ with $\arg(\zeta) < 2\pi/l$ and $\xi^l = 1$. Now define $\hat{z_1} = \zeta^{n_1}z_1$ and $\hat{z_2} = \zeta^{n_2}z_2$, as a new parametrization for the 3-sphere. Thus $\mathbb{S}^1 \times_{\mathbb{S}^1} \mathbb{S}^3$ is diffeomorphic to the quotient of $\mathbb{S}^3$ by the $\mathbb{Z}_l$-action

$$\xi \cdot (z_1,z_2) = (\xi^{n_1}z_1, \xi^{n_2}z_2).$$

Hence $\mathbb{S}^1 \times_{\mathbb{S}^1} \mathbb{S}^3$ is a Lens space $\mathbb{S}^3/\mathbb{Z}_l$. $\square$

Clearly Proposition 15 only has an assumption on the principal isotropy group, so the fixed point set M^H when M is either $\mathcal{N}^1_{0,0}$ or $\mathcal{N}^1_{0,n}$ is the disjoint union of two copies of a 3-sphere.

Notice that the principal isotropy group $\mathbb{Z}_d \subset \mathbb{Z}_{n_j}$ acts trivially on the slice D^2, so $\mathrm{SU}(2) \times_{\mathbb{Z}_{n_j}} D^2$ is equivalent to $\mathrm{SU}(2)/\mathbb{Z}_d \times_{\mathbb{Z}_{q_j}} D^2$ and the equivalence is just given by $[(p,z)] \sim [(p\mathbb{Z}_d, z)]$. Therefore the clutching function φ is an equivariant map defined from $\mathrm{SU}(2)/\mathbb{Z}_d \times_{\mathbb{Z}_{q_1}} \mathbb{S}^1$ to $\mathrm{SU}(2)/\mathbb{Z}_d \times_{\mathbb{Z}_{q_2}} \mathbb{S}^1$. So, it is sufficient to compute φ along the path $t \mapsto [(\mathbb{Z}_d, \mu(t)^{n_2})]$ where

$$\mu(t) = e^{2\pi i t/dq_1q_2} \in \mathbb{C} \subset \mathbb{H}.$$

Whenever necessary to make arguments more clear, we will denote by $[\![.\,,.]\!]$ the classes in $\mathrm{SU}(2)/\mathbb{Z}_d \times_{\mathbb{Z}_{q_2}} D^2$. We claim that

$$\varphi([(\mathbb{Z}_d, \mu(t)^{n_2})]) = [\![(\overline{\mu(t)}^l \mathbb{Z}_d, \overline{\mu(t)}^{n_1})]\!]. \tag{5}$$

In fact, we identified $[(\mathbb{Z}_d, \mu(t)^{n_2})] \in \mathrm{SU}(2)/\mathbb{Z}_d \times_{\mathbb{Z}_{q_1}} D^2$ with $[(1,(1/\sqrt{2}, \mu(t)^{n_2}/\sqrt{2})]$ that lies in the boundary of $\mathrm{SU}(2) \times_{\mathbb{S}^1} \mathbb{S}^3_1$. Let $x(t) = \overline{\mu(t)}\eta_2 \in \mathbb{S}^1$ and use the equivalence by $\mathbb{S}^1$ to get $[(1,(1/\sqrt{2}, \mu(t)^{n_2}/\sqrt{2})] = [(\overline{\mu(t)}^l \eta_2^l, (\overline{\mu(t)}^{n_1}\eta_2^{n_1}/\sqrt{2}, 1/\sqrt{2}))]$. If we see this last element in $\mathrm{SU}(2) \times_{\mathbb{S}^1} \mathbb{S}^3_2$, then we use the previous identification yields $[\![(\overline{\mu(t)}^l \mathbb{Z}_d, \overline{\mu(t)}^{n_1})]\!] \in \mathrm{SU}(2)/\mathbb{Z}_d \times_{\mathbb{Z}_{q_2}} D^2$ and the claim follows.

The discussion above can be summarized as follows.

Proposition 16. *The* $\mathrm{SU}(2)$*-manifold* $\mathcal{N}^l_{n_1,n_2}$ *with* $n_1n_2 \neq 0$*, when written as a union of the slice representations at the two exceptional orbits has the form*

$$\mathcal{N}^l_{n_1,n_2} = \mathrm{SU}(2) \times_{\mathbb{Z}_{n_1}} D^2 \bigcup_{\varphi} \mathrm{SU}(2) \times_{\mathbb{Z}_{n_2}} D^2,$$

where the actions $\mathbb{Z}_{n_j} \circlearrowright \mathrm{SU}(2) \times D^2$ *are given by* $\eta_j \cdot (p, z) = (p\eta_j^l, \eta_j^{n_i} z)$*,* $1 \leq i \neq j \leq 2$*, and the clutching function* $\varphi : \mathrm{SU}(2)/\mathbb{Z}_d \times_{\mathbb{Z}_{q_1}} \mathbb{S}^1 \to \mathrm{SU}(2)/\mathbb{Z}_d \times_{\mathbb{Z}_{q_2}} \mathbb{S}^1$ *is*

$$\varphi([(\mathbb{Z}_d, \mu(t)^{n_2})]) = [\![(\overline{\mu(t)}^l \mathbb{Z}_d, \overline{\mu(t)}^{n_1})]\!].$$

It will be seen in Section 4 that the number k in (4) represents the homotopy class of the clutching function φ when $d \geq 2$.

Remark 3. In order to construct $\mathrm{SO}(3)$-manifolds one could take $G = \mathrm{SO}(3)$, define the monomorphism $\mathbb{S}^1 \to \mathrm{SO}(3)$ by

$$e^{i\theta} \overset{\mu}{\mapsto} \mathrm{diag}(R(\theta), 1)$$

and consider the $\mathbb{S}^1$-action on $\mathrm{SO}(3) \times \mathbb{S}^3$ by

$$\overline{\Theta}_{(l,n_1,n_2)}(x, (A, (z_1, z_2))) = (A\mu(x)^l, (x^{n_1} z_1, x^{n_2} z_2)),$$

just as in (1). We write $A\mu(x)^l = Ax^l$ for short and denote by $\mathcal{M}^l_{n_1,n_2}$ the orbit space $(\mathrm{SO}(3) \times \mathbb{S}^3)/\overline{\Theta}_{(l,n_1,n_2)}$. This manifolds are turned into $\mathrm{SO}(3)$-manifolds by the left multiplication in the first coordinate as before. We claim that the $\mathrm{SO}(3)$-manifold $\mathcal{M}^l_{n_1,n_2}$ is equivalent to the $\mathrm{SO}(3)$-action on $\mathcal{N}^l_{2n_1,2n_2}$. To see this, first observe that it is a consequence of the exact sequence of homotopies for the bundle associated to the circle action that $\mathcal{M}^l_{n_1,n_2}$ is not simply-connected when the parameter l is even. Now, the universal 2-fold covering homomorphism $\phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ (see Section 2.2) takes the Hopf circle $\{e^{i\theta}\} \subset \mathbb{S}^3 \subset \mathbb{H}$ to the circle $\{\mathrm{diag}(R(2\theta), 1)\} \subset \mathrm{SO}(3)$. So we write $\phi(x) = x^2$, for $x \in \mathbb{S}^1$. Call $\Theta_{(l,n_1,n_2)}$, the $\mathbb{S}^1$-action on $\mathrm{SU}(2) \times \mathbb{S}^3$ defined in (2). Thus

$$(\phi \times \mathrm{Id}) \circ \Theta_{(l,2n_1,2n_2)} = \overline{\Theta}_{(2l,2n_1,2n_2)} \circ (\phi \times \mathrm{Id}),$$

and the map $\phi \times \mathrm{Id}$ induces the equivalence

$$[(p, (z_1, z_2))] \mapsto [(\phi(p), (z_1, z_2))]$$

between the $\mathrm{SO}(3)$-actions on $\mathcal{N}^l_{2n_1,2n_2}$ and $\mathcal{M}^l_{n_1,n_2}$.

4 Actions without singular orbits and proof of Theorem C

This section is devoted to prove Theorem C and is organized as follows. We first verify that the quotient is homeomorphic to a two or three-dimensional sphere. Then we show that, if the action has only one isotropy type, the circle and cyclic groups are the only possible isotropy subgroups of the action and classify the actions with exactly one isotropy type equal to $(\mathrm{SO}(2))$. After that, we prove that the action has at most two exceptional orbits and that the pair (H, K) of principal and exceptional isotropy groups must be $(\mathbb{Z}_d, \mathbb{Z}_n)$, $(\mathrm{D}_2, \mathrm{T})$ or $(\mathrm{Dic}_2, \mathrm{T}^*)$. Then, we use the Slice Theorem to construct M as a union of two neighborhoods of the exceptional orbits and compute the fundamental group of the union to conclude that only cyclic isotropies can occur for simply-connected G-manifolds. The actions constructed in this way depend on three integer parameters: one comes from the clutching function, while the other two correspond to the isotropy representations around the exceptional orbits. We finish the proof by establishing a one-to-one correspondence between the distinct general actions constructed and the $\mathrm{SU}(2)$-manifolds $\mathcal{N}^l_{n_1,n_2}$ for $n_1 n_2 \neq 0$.

Hereafter, in this chapter G denotes $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$. Let us understand the quotient space.

Lemma 17. *Let $G \circlearrowright M^5$ be an action without singular orbits. Then either it has only one isotropy type* $(\mathrm{SO}(2))$ *and $M/G \simeq \mathbb{S}^3$ or the isotropy groups are finite and $M/G \simeq \mathbb{S}^2$.*

Proof. By Propositions 4 and 5, the orbit space is homeomorphic to a compact, simply-connected 2 or 3-dimensional manifold, possibly with boundary. We claim that if the action has exceptional orbits the quotient is two dimensional without boundary. In fact, if the principal orbits have codimension 3, the principal and exceptional isotropy groups have to be $H = \mathrm{SO}(2)$ and $K = \mathrm{O}(2)$ (or $K = \mathrm{Pin}(2)$ if $G = \mathrm{SU}(2)$). Then the orbits are $\mathbb{S}^2$ and $\mathbb{RP}^2$. But since the orbits of maximal dimension are orientable by Proposition 2, we obtain that there is no cohomogeneity 3 action with exceptional orbits. On the other hand, in the cohomogeneity two case the boundary of the orbit space is the set of singular orbits, by Proposition 4. But there are no singular orbits since, by Proposition 4, they cannot co-exist with exceptional orbits in cohomogeneity two G-manifolds if G is connected. So the quotient does not have boundary, $M/G \simeq \mathbb{S}^2$ topologically and the claim follows. If the action has only one isotropy type, it was observed in Section 2.1 that the quotient is a base space of a fiber bundle, thus it is a simply-connected topological two or three manifold without boundary. In any case M/G is a topological sphere. □

4.1 Actions with only one (noncyclic) isotropy type

In this section we show that a simply-connected G-manifold with only one orbit type has isotropy either $\mathbb{Z}_m$ or SO(2) and classify the actions with isotropy SO(2). The classification of the actions with only one isotropy type ($\mathbb{Z}_m$) will be done later together with the classification of G-manifolds with exceptional orbits.

By Proposition 11 the number of G-manifolds with only one orbit type equal to (G/H) and quotient an n-sphere is the order of the $(n-1)$-th homotopy group of $N(H)/H$, where $N(H)$ is the normalizer of H on G. These homotopy groups are presented in Table 1 (see [19] for SO(3) and [1] for SU(2)).

G = SO(3)

H	$\{1\}$	$\mathbb{Z}_2$	$\mathbb{Z}_m$	D_2	D_m	T	I	O	SO(2)	O(2)
N(H)	SO(3)	O(2)	O(2)	O	D_{2m}	O	I	O	O(2)	O(2)
N(H)/H	SO(3)	SO(2)	O(2)	D_3	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$	$\mathbb{Z}_2$	$\{1\}$
π_{n-1}**(N(H)/H)**	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

G = SU(2)

H	$\{1\}$	$\mathbb{Z}_2$	$\mathbb{Z}_m$	Dic_2	Dic_m	T^*	I^*	O^*	SO(2)	Pin(2)
N(H)	SU(2)	SU(2)	Pin(2)	O^*	Dic_{2m}	O^*	I^*	O^*	Pin(2)	Pin(2)
N(H)/H	SU(2)	SO(3)	Pin(2)	D_3	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$	$\mathbb{Z}_2$	$\{1\}$
π_{n-1}**(N(H)/H)**	$\{1\}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

Table 1: Here $m \geq 3$ and n is the cohomogeneity of the action, i.e. $n = 2 + \dim H$

The following are the simplest examples.

Example 12. Let $H \subset G$ be a Lie subgroup and X be a manifold. Define a G-action on $(G/H) \times X$ by $g \cdot (kH, x) = (gkH, x)$. This action has unique isotropy type (H) and its orbit space is X.

Looking at Table 1 we see that there are two distinct free G-manifolds for $G = \mathrm{SO}(3)$ but only one if $G = \mathrm{SU}(2)$. One of the two free SO(3)-manifolds is $\mathrm{SO}(3) \times \mathbb{S}^2$ which is not simply-connected, the other is $\mathcal{N}^1_{2,2}$, hence simply-connected. The free SU(2)-action is $\mathcal{N}^0_{1,1}$, which is the left multiplication on the first coordinate of $\mathrm{SU}(2) \times \mathbb{S}^2$.

The SO(3)-action with unique isotropy type (SO(2)) is $\mathcal{N}^1_{0,0}$. It is the linear SO(3)-action on the first factor of $\mathbb{S}^2 \times \mathbb{S}^3$. Notice that SU(2) is a rank one Lie group whose center is $\mathbb{Z}_2$, thus all the circle subgroups of SU(2) contain $\mathbb{Z}_2$ and the action is ineffective, hence the SU(2)-action with unique isotropy (SO(2)) also corresponds to $\mathcal{N}^1_{0,0}$. Finally, all the G-manifolds with unique isotropy H equal to D_m with $m \geq 2$, T, I, O or O(2), if $G = \mathrm{SO}(3)$ and for H isomorphic to Dic_m with $m \geq 2$, T^*, I^*, O^* or Pin(2) if $G = \mathrm{SU}(2)$ are described by Example 12. But none of them is simply-connected since in all these cases the fundamental group of G/H is nontrivial.

For each $m \geq 3$ there are infinitely many examples of G-manifolds with unique isotropy $\mathbb{Z}_m$, either for $G = \mathrm{SO}(3)$ or SU(2). The same holds for $m = 2$ and $G = \mathrm{SO}(3)$, but for $G = \mathrm{SU}(2)$ there are exactly two such actions, they coincide with the free SO(3)-manifolds since the SU(2)-actions with principal isotropy $\mathbb{Z}_2$ are ineffective. We will see in the next section that $\mathcal{N}^l_{m,m}$ are precisely the examples with isotropy $\mathbb{Z}_m$.

4.2 Actions with exceptional orbits or unique cyclic isotropy type

In this section we conclude the proof of Theorem C. We will classify the G-manifolds with exceptional orbits and actions with only one isotropy type ($\mathbb{Z}_m$). The latter can be seen as a particular case of the former when the isotropy groups are $\mathbb{Z}_m$, $\mathbb{Z}_n$ and $\mathbb{Z}_{\gcd(m,n)}$ for $m = n$. The condition of simply-connectedness imposes strong restrictions on the isotropies (c.f., Proposition 18) and limits the number of exceptional orbits to two (c.f., Lemma 20). In this situation we can construct M as a union of the neighborhoods of the exceptional orbits using the Slice Theorem.

Proposition 18. *If the G-manifold M has exceptional orbits, then the pair of principal and exceptional isotropy types, (H, K), is either $(\mathbb{Z}_d, \mathbb{Z}_m)$, $(\mathrm{D}_2, \mathrm{T})$ or $(\mathrm{Dic}_2, \mathrm{T}^*)$. Moreover, exceptional orbits are isolated and H coincides with the kernel of the slice representation of K.*

In order to prove Proposition 18 the following will be essential.

Lemma 19. *For $G = \mathrm{SO}(3)$ or $\mathrm{SU}(2)$, let $K \subset G$ be a finite subgroup with a normal subgroup $N \triangleleft K$ such that $K/N \simeq \mathbb{Z}_n$ for some $n \geq 3$. Then the pair (N, K) is one of the following: $(\mathbb{Z}_d, \mathbb{Z}_m)$, $(\mathrm{D}_2, \mathrm{T})$ or $(\mathrm{Dic}_2, \mathrm{T}^*)$.*

Proof. The exceptional isotropy K is a finite subgroup of G, thus if $G = \mathrm{SO}(3)$ it is isomorphic to one of the following $\mathbb{Z}_m$, D_k, T, O and I (see, Section 2.2). The icosahedral group I is isomorphic to A_5, so it is simple. The octahedral group O is isomorphic to S_4, so it has two normal subgroups: A_4 that is an index two subgroup and the Klein group $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$, with quotient $S_4/D_2 \simeq S_3$. The tetrahedral group T is isomorphic to A_4, its unique nontrivial normal subgroup is the Klein group D_2, which has index 3. So the quotient T/D_2 is $\mathbb{Z}_3$. The quotients of the dihedral groups are dihedral groups (including $\mathrm{D}_1 = \mathbb{Z}_2$). So, (N, K) is either $(\mathbb{Z}_d, \mathbb{Z}_m)$ or $(\mathrm{D}_2, \mathrm{T})$ for $G = \mathrm{SO}(3)$.

The finite subgroups of SU(2) are $\mathbb{Z}_m$, Dic_k, T^*, O^* and I^*. The unique nontrivial normal subgroup of I^* is $\mathbb{Z}_2$ and the quotient is the icosahedral group. The binary octahedral group O^* has three nontrivial normal subgroups, they are: the Quaternion group Dic_2, with quotient $\mathrm{O}^* / \mathrm{Dic}_2 \simeq \mathrm{D}_3$, the binary tetrahedral group with index two and $\mathbb{Z}_2$, that has quotient $\mathrm{O}^* / \mathbb{Z}_2$ isomorphic to the octahedral group. The binary tetrahedral group has two nontrivial normal subgroups, they are Dic_2, with index three (so, $\mathrm{T}^* / \mathrm{Dic}_2 \simeq \mathbb{Z}_3$) and $\mathbb{Z}_2$ with quotient T. The normal subgroups of the binary dihedral groups Dic_k have quotient either dihedral groups or binary dihedral groups (for the normal subgroups of the binary dihedral groups we refer to Coxeter [7] p.75). So, if $G = \mathrm{SU}(2)$, the pair (N, K) is either $(\mathbb{Z}_d, \mathbb{Z}_m)$ or $(\mathrm{Dic}_2, \mathrm{T}^*)$ and this concludes the proof of the lemma. $\square$

Proof of Proposition 18. As observed in Lemma 17 the isotropy groups are finite. Let $K \subset G$ be an exceptional isotropy group and consider the slice representation $\rho : K \to \mathrm{O}(2)$ of K with $N = \ker(\rho)$. The quotient $\mathbb{R}^2/(K/N)$ is homeomorphic to $\mathbb{R}^2$ since it is a parametrization of a neighborhood of the orbit space, which is homeomorphic to $\mathbb{S}^2$. Therefore $K/N \simeq \mathbb{Z}_n \subset \mathrm{SO}(2)$ with $n \geq 3$. The possibilities for N and K are determined by Lemma 19.

We claim that $H = N$. This is clear if K is cyclic. If it is not cyclic then N is an index three subgroup of K and since $N \subset H \subsetneq K$, we get $H = N$. The slice action of K on $\mathbb{R}^2$, represented by ρ, only fixes the origin, thus the exceptional orbits of the G-action on M are isolated. $\square$

The action of the tetrahedral group and the binary tetrahedral group on the linear slice $\mathbb{R}^2$ have kernel D_2 and Dic_4 respectively, since in our cases the principal isotropy groups are normal subgroups of K. So, the effective actions to be considered on $\mathbb{R}^2$ in theses cases are by $\mathbb{Z}_3 \simeq \mathrm{T} / \mathrm{D}_2 = \mathrm{T}^* / \mathrm{Dic}_2$.

Lemma 20. *There are at most two exceptional orbits.*

Proof. The exceptional orbits in the quotient $M/G \simeq \mathbb{S}^2$ (topologically) represent orbifold singularities, in fact, a neighborhood of an exceptional orbit is parametrized by $\mathbb{R}^2/\mathbb{Z}_m$ with $m \geq 3$. It is known that a 2-dimensional orbifold with underlying space $\mathbb{S}^2$ with more than two singularities has nontrivial orbifold fundamental group (c.f. Thurston [31], p. 304). On the other hand, in our situation, there is an onto map from $\pi_1(M)$ to the orbifold fundamental group of the quotient M/G (c.f. Molino [22], p. 273 and 274). Therefore, there are at most two exceptional orbits since M is simply-connected. $\square$

In order to construct a simply-connected 5-dimensional G-manifold without singular orbits with orbit space homeomorphic to $\mathbb{S}^2$ and at most two exceptional orbits, we use the Slice Theorem to understand the neighborhoods of the exceptional orbits and the G-action in each one. Then we glue the two neighborhoods with an equivariant diffeomorphism along their boundaries.

From now on in this section, unless explicitly mentioned, $G = \mathrm{SU}(2)$. A neighborhood of an exceptional orbit G/K is given by $A = G \times_K D^2$, where $D^2 \subset \mathbb{C}$ is the 2-disk and the linear slice action of K on D^2 has kernel equal to the principal isotropy group $H \subset K$ of the G-action. The action of G on A is given by $g \cdot [(p, z)] = [(gp, z)]$. The isotropy action in the slice only fixes the origin in D^2, since the exceptional orbits are isolated. The manifold M with at most two exceptional orbits can be written as

$$M = A_1 \bigcup_{\varphi} A_2 \quad , \quad A_j = G \times_{K_j} D^2, \tag{6}$$

where $\varphi : G \times_{K_1} \mathbb{S}^1 \to G \times_{K_2} \mathbb{S}^1$ is a G-equivariant diffeomorphism. Since H is only acting on the first factor of the product $G \times D^2$, we can write $G \times_{K_j} D^2 = (G/H) \times_{K_j/H} D^2$. Recall that in all our cases K_j/H is a cyclic group $\mathbb{Z}_{q_j}$. Since φ is G-equivariant, it is a bundle map between the fiber bundles

$$G/H \to (G/H) \times_{K_j/H} \mathbb{S}^1 \to \mathbb{S}^1/\mathbb{Z}_{q_j}, \tag{7}$$

for $j = 1, 2$. Also here, φ is completely determined by the image of the path $t \mapsto [(H, e^{2\pi i t/q_1})]$ in ∂A_1 for $0 \leq t \leq 1$. Notice that if we take $t \in [0, 1]$ fixed, the map φ becomes a G-equivariant diffeomorphism of G/H on itself, so it is identified with an element $\kappa(t)$ of $N(H)/H$, where we can assume that $\kappa(0) = H$. Therefore,

$$\varphi[(H, e^{2\pi i t/q_1})] = [\![(\kappa(t), e^{2\pi i t/q_2})]\!]. \tag{8}$$

Before applying this construction to the cyclic and noncyclic isotropy types we compute the fundamental groups of A_j and $A_1 \cap A_2 \simeq \partial(A_1)$. Each component A_j deformation retracts to G/K_j, so the fundamental group of A_j is isomorphic to K_j. Observe that $\mathbb{Z}_{q_j} \to G/H \times \mathbb{S}^1 \to G/H \times_{\mathbb{Z}_{q_j}} \mathbb{S}^1$ is a principal bundle. So, considering the sequence of homotopies of this bundle and of the bundle in (7) we obtain that H and $q_j\mathbb{Z}$ are normal subgroups of the fundamental group of $A_1 \cap A_2$ and that $\pi_1(A_1 \cap A_2) \simeq H \rtimes \mathbb{Z}$.

We claim that the elements of H and $\mathbb{Z}$ in $\pi_1(A_1 \cap A_2)$ commute if $\gcd(q_1, q_2) = 1$. In fact, in this case the whole subgroup $\mathbb{Z}$ is normal in the fundamental group. Let α and β be elements in $\pi_1(A_1 \cap A_2)$ such that α is a generator of the component $\mathbb{Z}$ and β is an element of H. Since both subgroups are normal there is an integer k and an element $\gamma \in H$ such that $\beta\alpha\beta^{-1} = \alpha^k$

and $\alpha^{-1}\beta\alpha = \gamma$. Then $\alpha^k\beta = \beta\alpha = \alpha\gamma$, and thus $\alpha^{k-1} = \gamma\beta^{-1} \in H$, so $k = 1$ and the claim follows. Therefore, $\pi_1(A_1 \cap A_2) \simeq H \times \mathbb{Z}$, if $H = \mathbb{Z}_d$.

If the groups K_j have the same type as the principal isotropy type (H), then the action has only one isotropy type. This case is considered in Section 4.1 where it is shown that the possible simply-connected G-manifolds with unique finite isotropy type are those with cyclic isotropy. So, to complete the classification we are also constructing the G-manifolds with unique isotropy type $\mathbb{Z}_m$ and not just those with exceptional orbits. In other words, we allow $K_1 = H = K_2$ in the cyclic case.

We first show that the isotropy groups are cyclic.

Lemma 21. *The 5-dimensional compact simply-connected G-manifolds with exceptional orbits only have cyclic isotropy groups.*

Proof. It was shown in Proposition 18 that we only need to consider actions with isotropies $H = \mathrm{D}_2$ and $K = \mathrm{T}$ if $G = \mathrm{SO}(3)$ and isotropies $H = \mathrm{Dic}_2$ and $K = \mathrm{T}^*$ if $G = \mathrm{SU}(2)$. We assume that $G = \mathrm{SU}(2)$ since $\mathrm{SO}(3)/\mathrm{D}_2 = \mathrm{SU}(2)/\mathrm{Dic}_2$. In any case we have $M = G/H \times_{\mathbb{Z}_3} D^2 \cup_\varphi G/H \times_{\mathbb{Z}_3} D^2$ where $\mathbb{Z}_3$ is the quotient K/H. Note that there are exactly two non-equivalent $\mathbb{Z}_3$-actions on $G/H \times D^2$, namely $\xi \cdot (pH, z) = (p\bar{\xi}H, \xi^{c_j} z)$ for $c_j = 1$ or 2 and $\xi = e^{2\pi i/3}$. Recall that $A_j = G/H \times_{(K_j/N)} D^2$. Moreover, the clutching function φ is trivial since the normalizer $N(H)/H$ is discrete (see Table 1).

It is known that $\mathrm{T}^* \simeq \mathrm{Dic}_2 \rtimes \mathbb{Z}_3$ where the $\mathbb{Z}_3$ is generated by $w = -1/2(1 + i + j + k)$ and the $\mathbb{Z}_3$-action on Dic_2 is the automorphism that cyclically rotates i, j and k (see [7] p. 76). The isomorphism between $\mathrm{Dic}_2 \rtimes \mathbb{Z}_3$ and T^* takes (x, w) to $xw \in \mathrm{T}^*$. So, the action of T^* on $\mathrm{SU}(2) \times D^2$, which has quotient $G/H \times_{\mathbb{Z}_3} D^2$ is defined by $(x, w) \cdot (p, z) = (pw^{-1}x^{-1}, w^{c_j} z)$.

Also, the action of $\mathrm{Dic}_2 \rtimes \mathbb{Z}$ on $\mathrm{SU}(2) \times \mathbb{R}$, which has quotient $G/H \times_{\mathbb{Z}_3} \mathbb{S}^1$ is given by $(x, a) \cdot (g, s) = (gw^{-a}x^{-1}, s + 2\pi c_j a/3)$. Since φ is trivial, the induced maps $i_j* : \mathrm{Dic}_2 \rtimes \mathbb{Z} \to \mathrm{T}^*$ take (x, a) to (x, w^{ac_j}) and it is clear that $\pi_1(M) \simeq (\mathrm{T}^* * \mathrm{T}^*)/\mathrm{T}^*$ is nontrivial when the quotient is provided by the amalgamation property $i_1(x, a) = i_2(x, a)$ for all $(x, a) \in \mathrm{Dic}_2 \rtimes \mathbb{Z}$ and any choice of c_1 and c_2. $\square$

By Lemmas 20 and 21, we only need to consider $\mathrm{SU}(2)$-actions with isotropy types $(\mathbb{Z}_{n_1})$, $(\mathbb{Z}_{n_2})$ and $(\mathbb{Z}_d)$ where $d \mid \gcd(n_1, n_2)$. To avoid ambiguity assume $n_1 \le n_2$. The $\mathrm{SO}(3)$-manifolds will be just the $\mathrm{SU}(2)$-actions with ineffective kernel $\mathbb{Z}_2$.

Consider $\mathbb{Z}_{n_1}$ and $\mathbb{Z}_{n_2}$ as subgroups of the same circle parametrized by $t \mapsto e^{2\pi i t} \in \mathrm{SU}(2)$, using quaternion notation. Let $n_j = dq_j$ and $\xi_j = e^{2\pi i/dq_j}$ be a generator of $\mathbb{Z}_{n_j}$.

Lemma 22. *The $\mathbb{Z}_{n_j}$-actions on* $\mathrm{SU}(2) \times D^2$ *are given by*

$$\xi_j \cdot (p, z) = (p\overline{\xi_j}, \xi_j^{da_j} z),$$

for some a_j with $\gcd(a_j, q_j) = 1$ *and* $0 \leq a_j < q_j$ *for* $j = 1, 2$.

Proof. The action in the first coordinate is just right multiplication by the inverse since the action in the product is provided by the Slice Theorem. So it depends on the choice of the $\mathrm{SO}(2)$ inside $\mathrm{SU}(2)$ that contains $\mathbb{Z}_{n_j}$. Since in $\mathrm{SU}(2)$ any two circles are conjugated, we choose the circle $t \mapsto e^{2\pi i t}$. The lemma follows from the fact that on the second coordinate the action is the slice representation, so it is a linear $\mathbb{Z}_{n_j}$-action on D^2 with kernel $\mathbb{Z}_d$. $\square$

Remark 4. The isotropy representation in a point with isotropy $\mathbb{Z}_{n_j}$ is determined by the number a_j. It is well known that two representations of ρ and $\rho' : \mathbb{Z}_m \to \mathrm{O}(2)$ are equivalent if and only if $\rho' = \overline{\rho}$. Notice that they rotate the 2-plane in opposite directions, so A_j is unchanged if we consider $q_j - a_j$ instead of a_j in the isotropy representation. However, the orientation of the slice in A_j is reversed.

A simply-connected $\mathrm{SU}(2)$-manifold M with exceptional orbits, or with only one cyclic isotropy type, is determined by the isotropy types, the parameters a_1 and a_2, and the clutching function φ. The elements $a_j \in \mathbb{Z}_{q_j}$ have multiplicative inverse, say b_j, so we write $M = M(b_1, b_2, \varphi)$.

For $d \geq 3$ if we consider an orientation on the manifold and on G, it naturally defines an orientation on $G/\mathbb{Z}_d$ and on the slices through the exceptional orbits. Therefore, by Remark 4, the G-manifolds $M(b_1, b_2, \varphi)$ and $M(q_1 - b_1, b_2, \varphi)$ cannot be equivalent, although they have equivalent slice representations.

The proof of the next result is inspired by Theorem 5.1 of [4].

Proposition 23. *The* $\mathrm{SU}(2)$*-manifolds* $M(b_1, b_2, \varphi_o)$ *and* $M(b_1, b_2, \varphi_1)$ *are equivalent if and only if the clutching functions φ_o and φ_1 are homotopic.*

Proof. Call $M_o = M(b_1, b_2, \varphi_o)$ and $M_1 = M(b_1, b_2, \varphi_1)$. Let H be a homotopy between φ_o and φ_1. Define $F : \partial A_1 \times I \to \partial A_2 \times I$ by $F(p, t) = (H(p, t), t)$ and the G-manifold

$$N = A_1 \times I \bigcup_F A_2 \times I,$$

with trivial action on the intervals. This makes N a cylinder with M_o on the bottom and M_1 on the top. Observe that N/G is homeomorphic to $\mathbb{S}^2 \times I$ and that if π is the projection of N in $\mathbb{S}^2 \times I$, then $\pi^{-1}(\mathbb{S}^2 \times \{0\}) = M_o$, thus Theorem 8 asserts that N is equivalent to $M_o \times I$

(the product of G-manifolds with trivial G-action on the interval) and, therefore, M_o and M_1 are equivalent.

Conversely, let $f : M_o \to M_1$ be a G-equivariant diffeomorphism. Assume that the manifolds are both written as above using the Slice Theorem and that f restricted to A_1 is the identity map. Considering $\xi_j(t) = e^{2\pi i t/n_j}$, the clutching functions are

$$\varphi_i[(\mathbb{Z}_d, \xi_1(t)^d)] = [\![(\kappa_i(t), \xi_2(t)^d)]\!],$$

for $i = 0$ or 1, as in (8). So, $f\,|_{\mathrm{SU}(2)\times_{\mathbb{Z}_{n_2}}\mathbb{S}^1}$ is an SU(2)-equivariant diffeomorphism of $\mathrm{SU}(2)\times_{\mathbb{Z}_{n_2}} \mathbb{S}^1$ given by

$$[\![(\mathbb{Z}_d, \xi_2(t)^d)]\!] \mapsto [\![(\kappa_o(t)^{-1}\kappa_1(t), \xi_2(t)^d)]\!], \tag{9}$$

that extends equivariantly to $\mathrm{SU}(2)\times_{\mathbb{Z}_{n_2}} D^2$. Since the slice representation is a parametrization of the orbit space this extension must take the slice $[\![(\mathbb{Z}_d, s\xi_2(t)^d)]\!]$ to $[\![(h_s(t), s\xi_2(t)^d)]\!]$, where $s \in [0,1]$, $h_1(t) = \kappa_o(t)^{-1}\kappa_1(t)$ and observe that $h_o(t)$ does not depend on t since $[\![(h_o(t), 0)]\!] = f([\![(\mathbb{Z}_d, 0)]\!])$. So, the path $\kappa_o^{-1}\kappa_1$ on $N(\mathbb{Z}_d)/\mathbb{Z}_d$ is homotopically trivial, and therefore the clutching functions φ_o and φ_1 are homotopic. □

For $d = 1$ the clutching function φ has only one homotopy class, since $N(H) = \mathrm{SU}(2)$ is simply-connected, so it can be represented by $M(b_1, b_2)$. We have seen in Remark 4 that the SU(2)-manifolds $M(b_1, b_2)$ and $M(b_1', b_2')$ are equivalent if and only if $b_j' = b_j$ or $b_j' = n_j - b_j$ for both $j = 1$ and 2 simultaneously. The manifold $M(b_1, b_2)$ is equivalent to $\mathcal{N}^l_{n_1,n_2}$ when $l = b_1 n_2 + b_2 n_1$ since the isotropy representations coincide as one can see from Proposition 16. Thus Theorem C is proved if $\gcd(n_1, n_2) = 1$.

For $d \geq 2$, let us analyze the clutching function φ in more detail. The path κ defined in (8) must satisfy $\kappa(1) = \xi_1^{b_1}\xi_2^{b_2}\kappa(0)$, since φ is SU(2)-equivariant. This follows from

$$[\![(\xi_1^{b_1}\kappa(0), 1)]\!] = \varphi[(\xi_1^{b_1}\mathbb{Z}_d, 1)] = \varphi[(\mathbb{Z}_d, \xi_1^d)] = [\![(\kappa(1), \overline{\xi_2}^d)]\!] = [\![(\overline{\xi_2}^{b_2}\kappa(1), 1)]\!]. \tag{10}$$

Recall our notation $\xi_j(t) = e^{2\pi i t/n_j}$. By Proposition 23 we can assume that the path κ is given by $\kappa(t) = \xi_1(t)^{b_1}\xi_2(t)^{b_2+kq_2}\mathbb{Z}_d$ with $k \in \mathbb{Z}$. Therefore

$$\varphi[(\overline{\xi_1(t)}^{b_1}\mathbb{Z}_d\,,\ \xi_1(t)^d)] = [\![(\xi_2(t)^{b_2}e^{2\pi i k/d}\mathbb{Z}_d\,,\ \overline{\xi_2(t)}^d)]\!],$$

Thus the homotopy class of φ is precisely represented by k.

Notice that for $\mu(t) = e^{2\pi i t/dq_1q_2} \in \mathrm{SU}(2)$ and $l = b_1q_2 + b_2q_1 + kq_1q_2$ the clutching function has the form

$$\varphi[(\mathbb{Z}_d\,,\ \mu(t)^{n_2})] = [\![(\mu(t)^l\mathbb{Z}_d\,,\ \overline{\mu(t)}^{n_1})]\!], \tag{11}$$

which is the same expression as in Proposition 16 by changing l by $-l$. The sign does not matter for our purposes since $\mathcal{N}^l_{n_1,n_2} = \mathcal{N}^{-l}_{n_1,n_2}$ as observed in Remark 2.

The map φ has two homotopy classes if $d = 2$ and depends on the number $k \in \mathbb{Z}$ if $d \geq 3$. Thus M can be represented by $M(b_1, b_2, \epsilon)$ or $M(b_1, b_2, k)$ respectively, for $\epsilon \in \{0, 1\}$ and $k \in \mathbb{Z}$.

PROPOSITION 24. *The fundamental group of the manifold* $M(b_1, b_2, k)$ *is a cyclic group of order* $\gcd(n_1, n_2, l)$.

Proof. To compute the fundamental group of M, we describe the action of $\pi_1(\mathrm{SU}(2) \times_{\mathbb{Z}_{n_1}} \mathbb{S}^1)$ on the universal covering $\mathrm{SU}(2) \times \mathbb{R}$, such that the quotient is $A_1 \cap A_2 = \mathrm{SU}(2) \times_{\mathbb{Z}_{n_1}} \mathbb{S}^1$. Take curves α and β in $A_1 \cap A_2$ that are generators of the fundamental group. For each $j = 1, 2$, we include $A_1 \cap A_2$ in the component A_j by the inclusion i_j and use the $\pi_1(A_j)$-action on the universal covering $\mathrm{SU}(2) \times D^2$ of A_j to regard the loops α and β as elements of $\pi_1(A_j)$. Van Kampen's Theorem asserts that the fundamental group of M is the free product $\pi_1(A_1) * \pi_1(A_2)$ with relations $i_{1*}[\alpha] = i_{2*}[\alpha]$ and $i_{1*}[\beta] = i_{2*}[\beta]$, where the maps $i_{j*} : \pi_1(A_1 \cap A_2) \to \pi_1(A_j)$, for $j = 1, 2$ are induced by i_j on the fundamental groups.

The action of the fundamental group $\pi_1(A_1 \cap A_2) \simeq \mathbb{Z}_d \times \mathbb{Z}$ on $\mathrm{SU}(2) \times \mathbb{R}$ with quotient $\mathrm{SU}(2) \times_{\mathbb{Z}_{n_1}} \mathbb{S}^1$ is given by

$$(\overline{u}, k) \cdot (p, s) = (p\xi_1^{-(uq_1 + kb_1)}, s + 2\pi k/q_1),$$

where $\mathbb{Z}_d = \{\overline{0}, \overline{1}, \cdots, \overline{d-1}\}$ and $a_1 b_1 + u_1 q_1 = 1$. Observe that the space of orbits $(\mathrm{SU}(2) \times \mathbb{R})/(\mathbb{Z}_d \times \mathbb{Z})$ is exactly the quotient $(\mathrm{SU}(2) \times \mathbb{S}^1)/\mathbb{Z}_{n_j}$. Indeed, since $\gcd(b_1, q_1) = 1$, for any $0 \leq l < n_1$ there are integers u and k with $0 \leq u < d$ such that $l = uq_1 + kb_1$. So, the $\mathbb{Z}_{n_1}$-action can be written as

$$\xi_1^l \cdot (p, e^{is}) = (p\xi_1^{-(uq_1+kb_1)}, \exp(s + 2\pi u a_1 + 2\pi k a_1 b_1/q_1)) = (p\xi_1^{-(uq_1+kb_1)}, \exp(s + 2\pi k/q_1)),$$

that clearly defines the same quotient as $(\mathrm{SU}(2) \times \mathbb{R})/(\mathbb{Z}_d \times \mathbb{Z})$.

It is convenient to define the loops in $\mathrm{SU}(2)/\mathbb{Z}_d \times_{\mathbb{Z}_{q_1}} \mathbb{S}^1$ that generate its fundamental group, $\mathbb{Z}_d \times \mathbb{Z}$. The loop $\alpha : [0, 1] \to A_1 \cap A_2$ defined by $\alpha(t) = [(\xi_1(t)^{-q_1}\mathbb{Z}_d, 1)]$ corresponds to $(\overline{1}, 0)$ in $\mathbb{Z}_d \times \mathbb{Z}$. In fact, let $\widetilde{\alpha}(t) = (\xi_1(t)^{-q_1}, 0)$ be a lifting of α by $(1, 0) \in \mathrm{SU}(2) \times \mathbb{R}$. So, $\widetilde{\alpha}(1) = (\xi_1^{-q_1}, 0) = (\overline{1}, 0)_{\mathbb{Z}_d \times \mathbb{Z}} \cdot (1, 0) = (\overline{1}, 0)_{\mathbb{Z}_d \times \mathbb{Z}} \cdot \widetilde{\alpha}(0)$.

On the other hand, the loop $\beta : [0, 1] \to A_1 \cap A_2$ defined by $\beta(t) = [(\xi_1(t)^{-b_1}\mathbb{Z}_d, \xi_1(t)^d)]$ corresponds to $(\overline{0}, 1) \in \mathbb{Z}_d \times \mathbb{Z}$. In fact, consider $\widetilde{\beta}(t) = (\xi_1(t)^{-b_1}, 2\pi t/q_1)$, a lifting of β to $\mathrm{SU}(2) \times \mathbb{R}$ by $\widetilde{\beta}(0) = (1, 0)$. Then $\widetilde{\beta}(1) = (\xi_1^{-b_1}, 2\pi/q_1) = (\overline{0}, 1)_{\mathbb{Z}_d \times \mathbb{Z}} \cdot \widetilde{\beta}(0)$. So, β corresponds to $(\overline{0}, 1) \in \mathbb{Z}_d \times \mathbb{Z}$.

If e_j is a generator of $\mathbb{Z}_{n_j}$ in the free product $\mathbb{Z}_{n_1} * \mathbb{Z}_{n_2}$, the induced loop $(i_1 \circ \alpha)(t)$ corresponds to $e_1^{q_1}$. The loop $(i_1 \circ \beta)$ in A_1 corresponds to $e_1^{b_1} \in \mathbb{Z}_{n_1}$. In fact, the lifting of $i_1 \circ \beta$ by the point $(1, 1)$ of $\mathrm{SU}(2) \times D^2$ is the curve $t \mapsto (\xi_1(t)^{-b_1}, \xi_1(t)^d)$. Then,

$$\xi_1^{b_1} \cdot \widetilde{(i_1 \circ \beta)}(0) = \xi_1^{b_1} \cdot (1, 1) = (\overline{\xi_1}^{b_1}, \xi_1^d) = \widetilde{(i_1 \circ \beta)}(1).$$

That is, $i_{1*} : \mathbb{Z}_d \times \mathbb{Z} \to \mathbb{Z}_{n_1}$ takes (α^u, β^n) to $e_1^{uq_1+nb_1}$.

We need to include the loops α and β in A_2. To do this we simply use the composition, i_2, of the clutching function φ with the inclusion of $\mathrm{SU}(2)/\mathbb{Z}_d \times_{\mathbb{Z}_{q_2}} \mathbb{S}^1$ in A_2. The induced loop $(i_2 \circ \alpha)(t) = [\![(\xi_2(t)^{-q_2}\mathbb{Z}_d, 1)]\!]$ has lifting by $(1,1) \in \mathrm{SU}(2) \times D^2$, with end point $(\xi^{-q_2}, 1) = \xi_2^{q_2} \cdot (1,1)$. So, $(i_2 \circ \alpha)$ corresponds to $e_2^{q_2} \in \mathbb{Z}_{n_2}$. The loop $(i_2 \circ \beta)(t) = [\![(\xi_2(t)^{(b_2+kq_2)}\mathbb{Z}_d, \xi_2(t)^{-d})]\!]$ has lifting $t \mapsto (\xi_2(t)^{(b_2+kq_2)}, \xi_2(t)^{-d})$ that starts at $(1,1) \in \mathrm{SU}(2) \times D^2$ and that ends at $(\xi_2^{(b_2+kq_2)}, \xi_2^{-d}) = \xi_2^{(-b_2-kq_2)} \cdot (1,1)$, so the loop corresponds to $e_2^{-(b_2+q_2k)} \in \mathbb{Z}_{n_2}$. Therefore the map i_{2*} takes (α^u, β^n) to $e_2^{(u-kn)q_2-nb_2}$. We conclude that the fundamental group of M is generated by e_1 and e_2 with the relations $e_1^{n_1} = e_2^{n_2} = 1$, $e_1^{q_1} = e_2^{q_2}$ and $e_1^{b_1} = e_2^{-b_2-kq_2}$. From these three identities and using that $\gcd(b_j, q_j) = 1$ we see that $\pi_1(M) \simeq \mathbb{Z}_{\gcd(n_1,n_2,b_1q_2+b_2q_1+kq_1q_2)}$. □

Remark 5. As a consequence of Proposition 24, $d = \gcd(n_1, n_2)$ when the $\mathrm{SU}(2)$-manifold is simply-connected. In fact, $\gcd(n_1, n_2, l) = 1$ and $\gcd(b_j, q_j) = 1$ imply that $\gcd(q_j, l) = 1$, therefore $\gcd(q_1, q_2) = 1$. So, the $\mathrm{SU}(2)$-action depends on a triple of integer parameters that belong to $\mathcal{P} = \{(b_1, b_2, k) \in \mathbb{Z}^3 \,:\, 0 \le b_j < q_j \,,\, (b_j, q_j) = 1 \,,\, j = 1, 2\}$.

We will show now that the G-manifold $M(b_1, b_2, k)$ is equivariantly diffeomorphic to $\mathcal{N}^l_{n_1,n_2}$ where $l = b_1q_2 + b_2q_1 + kq_1q_2$. It will also be proved (c.f., Proposition 15) that l is the order of the fundamental group of the fixed point set of the principal isotropy group $\mathbb{Z}_d$, whenever $d \ge 3$. For the next result recall that we have defined $\gcd(0, 0) = 1$.

Theorem 25. *Let $n_1 \le n_2$ be positive integers with $d = \gcd(n_1, n_2) \ge 2$, set $q_j = n_j/d$ and take $b_j \in \mathbb{Z}$ coprime with q_j satisfying $0 \le b_j < q_j$, for $j = 1, 2$. Let k be an integer for $d \ge 3$ or $k \in \mathbb{Z}_2$ for $d = 2$. Then, the $\mathrm{SU}(2)$-manifold $M(b_1, b_2, k)$ is equivariantly diffeomorphic to $\mathcal{N}^l_{n_1,n_2}$, where $l = b_1q_2 + b_2q_1 + kq_1q_2$. Moreover, these $\mathrm{SU}(2)$-manifolds are pairwise nonequivalent except for $M(b_1, b_2, k) = M(q_1 - b_1, q_2 - b_2, -k - 2)$.*

Proof. It is clear that $M(b_1, b_2, k)$ is determined by the isotropy representations around the exceptional orbits and the homotopy class of the clutching function φ. So, $M(b_1, b_2, k) = \mathcal{N}^l_{n_1,n_2}$ with $l = b_1q_2 + b_2q_1 + kq_1q_2$ since by Proposition 16 they coincide in both representations and also have the same clutching function, up to homotopy. Moreover, Proposition 14 and Proposition 24 show that each l is reached exactly once by that formula.

For $d = 2$ we know that $k \in \{0, 1\}$ since the homotopy class of the clutching function is defined modulo 2. We use the identity $M(b_1, b_2, k) = \mathcal{N}^l_{n_1,n_2}$ and $\mathcal{N}^{-l}_{n_1,n_2} = \mathcal{N}^l_{n_1,n_2}$ (see, Remark 2) to conclude that $M(b_1, b_2, k) = M(q_1 - b_1, q_2 - b_2, -k - 2) = M(q_1 - b_1, q_2 - b_2, k)$ for $k = 0$ or 1. Remark 4 shows that otherwise these $\mathrm{SU}(2)$-manifolds are pairwise distinct.

For $d \ge 3$ the G-manifolds $M(b_1, b_2, k)$ are nonequivalent exactly when the numbers

$l = b_1q_2 + b_2q_1 + kq_1q_2$ are different since by Proposition 15 the parameter l is an invariant of the action. This, and Remark 4, imply that $M(b_1, b_2, k)$ is equivalent to $M(b_1', b_2', k')$ if and only if the parameters are exactly the same, or $b_j' = q_j - b_j$ and $k' = -k - 2$. This corresponds to replacing l by $-l$ in $\mathcal{N}^l_{n_1,n_2}$. □

Observe that it is a consequence of the discussion in the proof of Theorem 25 that the manifolds $\mathcal{N}^l_{n_1,n_2}$ and $\mathcal{N}^{l'}_{n_1,n_2}$ are equivalent if, and only if, $l \equiv l' \mod 2q_1q_2$ for $d = 2$. This concludes the proof of Theorem C.

5 Actions with singular orbits and proof of Theorem D

In this section we classify the 5-dimensional compact simply-connected G-manifolds with singular orbits. In these cases the number of orbit types cannot exceed 3, c.f., Lemma 28, and the quotient is homeomorphic to a 2-disk or it is a compact simply-connected 3-manifold with (or without boundary), c.f., Propositions 4 and 5.

The classification of actions with two or three orbit types containing singular orbits, when the quotient is a given manifold with boundary, is a classical problem studied by Bredon [4], Hsiang and Hsiang [17] and Janich [20].

The classification of SO(3)-actions on simply-connected 5-manifolds with singular orbits and cohomogeneity 2 was carried out in 1979 by Hudson [19]. In the Appendix of [19] the classification of cohomogeneity 3 actions was discussed but the SO(3)-manifolds in Example 9 were overlooked. For the sake of completeness we include her classification of SO(3)-actions with singular orbits but without fixed points. We also prove again the classification of SO(3)-manifolds without fixed points since it can be obtained in the same way as in SU(2) case. The classification presented here is simpler than Hudson's since after counting how many actions exist she constructed the actions by gluing pieces and later identifying the manifolds by computing topological invariants. Here we just see how many distinct actions exist and check that all of them have been described in Section 2.2 and Example 11.

In Section 5.1 we use a classical result to classify the actions with exactly two orbit types. In Section 5.2 we make a few comments about Hudson's classification of SO(3)-manifolds with three orbit types.

The following two results are the main goal of this section. The geometry of the actions in Theorems 26 and 27 were discussed in Section 2.2.

Theorem 26. *Let M be a compact simply-connected 5-dimensional G-manifold without fixed points. If M has singular orbits, then it is equivariantly diffeomorphic to either:*

(i) The $\mathrm{SO}(3)$*-linear action on* $\mathbb{S}^5 \subset \mathbb{R}^3 \times \mathbb{R}^3$ *given by* $A \cdot (x, y) = (Ax, Ay)$*;*

(ii) The $\mathrm{SU}(2)$*-manifold* $\mathcal{N}^1_{0,n}$ *for* $n > 0$*, which is an* $\mathrm{SO}(3)$*-manifold if* n *is even;*

(iii) The $\mathrm{SU}(2)$*-action on the Wu-manifold given by* $B \cdot [C] = [\mathrm{diag}(B, 1)C]$ *for* $B \in \mathrm{SU}(2)$ *and* $[C] \in \mathrm{SU}(3)/\mathrm{SO}(3)$*.*

On the other hand, for G-manifolds with fixed points we have the following.

THEOREM 27. *Any compact simply-connected 5-dimensional* G*-manifold with fixed points is equivalent to either:*

(i) The $\mathrm{SU}(2)$*-linear action on* $\mathbb{S}^5 \subset \mathbb{C}^3$ *given by* $B \cdot (u, v, w) = (B(u, v), w)$*;*

(ii) The $\mathrm{SO}(3)$*-linear action on* $\mathbb{S}^5$*, given by the suspension of the irreducible representation of* $\mathrm{SO}(3)$ *on* $\mathbb{R}^5$*;*

(iii) The $\mathrm{SO}(3)$*-action on connected sums of copies of* $\mathbb{S}^3 \times \mathbb{S}^2$ *with linear action in the first coordinate of each copy;*

(iv) The $\mathrm{SO}(3)$*-action on connected sums of* m *copies of the Brieskorn variety of type* $(2, 3, 3, 3)$ *and* n *copies of the Wu-manifold.*

Remark 6. Theorems 26 and 27 show (see Section 5.2) that there are no G-manifolds with only one fixed point and that if there are respectively two or three fixed points, the manifold is diffeomorphic to $\mathbb{S}^5$ or $\mathcal{W}$. So, if the G-action has at most three fixed points, then it is diffeomorphic to $\mathbb{S}^5$, is a quotient of either $\mathrm{SU}(3)$ or $\mathbb{S}^3 \times \mathbb{S}^2$. Therefore, the metric induced in the quotient is G-invariant and has nonnegative curvature.

The following lemma gives strong restrictions to the possible chains of isotropy subgroups. It is inspired by Lemma 1A in [19].

LEMMA 28. *Let* M *be a 5-dimensional simply-connected* G*-manifold with singular orbits. Then:*

(a) If the action has exactly two orbit types, say $(H) \leq (K)$*, then the pair of principal and singular isotropy groups* (H, K) *is* $(\mathbb{Z}_m, \mathrm{SO}(2))$*,* $(\mathrm{D}_m, \mathrm{O}(2))$ *or* $(\mathrm{SO}(2), \mathrm{SO}(3))$ *if* $G = \mathrm{SO}(3)$ *and* $(\mathbb{Z}_m, \mathrm{SO}(2))$ *or* $(\{1\}, \mathrm{SU}(2))$*, if* $G = \mathrm{SU}(2)$*, for* $m \geq 1$*;*

(b) If the action has three orbit types then the isotropy types are $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{O}(2) \subset \mathrm{SO}(3)$*. There is no* $\mathrm{SU}(2)$*-action with three isotropy types;*

(c) Neither SO(3) *nor* SU(2) *acts on M with more than three orbit types.*

Proof. For a singular point $p \in M$, if $K = G_p$, it is known that the slice action $K \circlearrowright \mathbb{R}^k$, of the isotropy group K on the tangent space of a slice at p, is a linear action which has the same isotropy structure as the action $G \circlearrowright M$ in a neighborhood of p. So the chain of isotropy types $(H) \leq \cdots \leq (K)$ is possible for a G-action on M only if there is a representation $\rho : K \to \mathrm{O}(k)$ such that the action $\rho(K) \circlearrowright \mathbb{R}^k$ has the same chain of isotropy types.

The dimension of the quotient is either 2 or 3. Suppose that the action has cohomogeneity 3 ($\dim M/G = 3$). Then the principal isotropy group must be one-dimensional, actually it must be $H = \mathrm{SO}(2)$ since by Proposition 2 the orbits of maximal dimension are orientable. Then, the singular isotropy group must be G and the action has a fixed point $p \in M$. The unique nontrivial linear action of SU(2) on $T_pM \simeq \mathbb{R}^5$ has a fixed direction and trivial principal isotropy group (c.f., Example 4), thus $H = \{1\}$ and not SO(2). Therefore, if the G-manifold M has cohomogeneity 3, the isotropies are $\mathrm{SO}(2) \subset \mathrm{SO}(3)$. The natural inclusion $\mathrm{SO}(3) \subset \mathrm{SO}(5)$ induces an action of SO(3) on $\mathbb{R}^5$ with isotropy types $(\mathrm{SO}(2)) \leq (\mathrm{SO}(3))$.

If the action has cohomogeneity 2, then the quotient is a 2-disk and there are no exceptional orbits since we assume there is a singular orbit (see Proposition 4). Moreover, the principal isotropy group is finite. Let $p \in M$ be a point in a singular orbit. If there are no fixed points, then the isotropy group $K = G_p$ is 1-dimensional, so $\dim G/K = 2$ and the codimension of the singular orbit is $k = 3$. Now we proceed with a case-by-case analysis of the representations of the subgroups of G in O(3).

If $K \simeq \mathrm{SO}(2)$, then $H = \{1\}$ or $\mathbb{Z}_m$ since the representations of SO(2) in O(3) are

$$\begin{array}{rccc} \rho_m: & \mathrm{SO}(2) & \longrightarrow & \mathrm{O}(3) \\ & R(\theta) & \mapsto & \mathrm{diag}(1, R(m\theta)), \end{array}$$

where $R(\theta)$ is the rotation by θ.

If $K \simeq \mathrm{O}(2)$ (and thus $G = \mathrm{SO}(3)$), then $H \simeq \mathrm{D}_m$ (recall that $\mathrm{D}_1 \simeq \mathbb{Z}_2$) since the representations of O(2) in O(3) are equivalent to $\overline{\rho_m} : \mathrm{O}(2) \to \mathrm{O}(3)$, such that $\overline{\rho_m}|_{\mathrm{SO}(2)} = \rho_m$ and

$$\overline{\rho_m}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore the isotropy groups are D_m and O(2).

The unique representation of Pin(2) in O(3) is the trivial one, so it cannot be isotropy group of a singular point of an SU(2)-manifold of dimension five.

If $K = \mathrm{SO}(3)$ and the action has cohomogeneity 2, then the isotropy representation at a fixed point is the unique irreducible representation of SO(3) on $\mathbb{R}^5$ since this is the unique such representation of SO(3) with finite principal isotropy group. This is the action described on Example 1 and the chain of isotropies is $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{O}(2) \subset \mathrm{SO}(3)$.

If $K = \mathrm{SU}(2)$, then the representation in SO(5) is $\mathrm{SU}(2) \subset \mathrm{SO}(4) \subset \mathrm{SO}(5)$. It has one fixed direction and the principal isotropy group is the trivial group. So we have $(\{1\}, \mathrm{SU}(2))$.

The subgroups of G have dimensions either zero, one or three. If there are three isotropy types and (G) is not one of them, two of them have the same dimension. Since there are no exceptional orbits, there are two one-dimensional isotropy types, say (K) and (K'). Let $M_{(K)}$ and $M_{(K')}$ be the set of points in M whose isotropy groups belong to (K) and (K'), respectively. Both sets $M_{(K)}$ and $M_{(K')}$ are submanifolds of M and are projected to the boundary of the disk in the quotient. So there is a point $p \in M$ such that in any neighborhood of p there are orbits of type (G/K) and (G/K'), but there is no representation neither of O(2) nor Pin(2) on O(3) with isotropy SO(2). This shows that if the action has more than two distinct orbit types, then there is a fixed point. It also shows that there is no 5-dimensional G-action with more than three orbit types and the lemma follows. □

5.1 Actions with singular orbits and exactly two orbit types

The class of G-manifolds with two orbit types and quotient a given manifold with boundary follows from the general classification in [4] Chapter V. We will apply this classification to our case, first assuming that the orbit space is a disk and then that it is a 3-disk with small disks removed from inside (see Propositions 4, 5 and 6).

We will now briefly describe a one-to-one correspondence between the set of equivalence classes of actions with singular orbits and two orbit types with quotient a disk, and the set of orbits of an specific action, that in our case will have only finitely many orbits. This will allow us to classify the G-actions on 5-manifolds with singular orbits, two orbit types and quotient a 2 or 3-disk, for $G = \mathrm{SO}(3)$ or $\mathrm{SU}(2)$.

Let G be a compact Lie group and $H \subset K$ closed subgroups of G. Let $D(2)$ be the n-disk of radius 2. Assume that $D(2)$ is the orbit space of a G-action on M which assigns type (G/H) to the open disk and type (G/K) to the boundary $\mathbb{S}^{n-1}(2)$ of the disk. Let $p : M \to D(2)$ be the projection to the orbit space. There is a principal bundle $N(H)/H = \Gamma_H \to P \xrightarrow{\rho} D(1)$, corresponding to the disk $D(1)$ of radius 1 since over $D(1)$ the action has only principal orbits. So $p^{-1}(D(1)) \simeq G/H \times_{\Gamma_H} P$, see Section 2.1. Let $S = (N(H) \cap N(K))/H$ and $\pi : G/H \to G/K$ be the standard projection. In our situation, it is known (see, Tube Theorem in [4], p. 242) that there is a principal bundle $S \to Q \to \mathbb{S}^{n-1}$ such that $p^{-1}(I \times \mathbb{S}^{n-1}(2)) \simeq M_\pi \times_S Q$, where M_π denotes the mapping cylinder for the map π, i.e., the space obtained from the disjoint union $(G/H \times I) \cup G/K$ with the points $(gH, 1)$ identified to gK for all $g \in G$.

The mapping cylinder M_π is a G-manifold with the action on cosets. In our cases, it is a manifold with boundary $G/H \times \{0\}$ and thus, $M_\pi \times_S Q$ has boundary $G/H \times_S Q$. Let us denote by $P_1 \subset P$ the pre-image of $\mathbb{S}^{n-1}(1)$ by the bundle ρ. Then, there is a G-equivariant diffeomorphism

$$f : G/H \times_S Q \to G/H \times_{\Gamma_H} P_1,$$

such that the G-manifold M is given by

$$M(f) = (M_\pi \times_S Q) \bigcup_f (G/H \times_{\Gamma_H} P)$$

over D(2).

The equivalence class of $M(f)$ as a G-manifold is the same as $M(\varphi_1 f)$ where $\varphi_1 := \varphi \mid_{P_1}$, and $\varphi \in \mathrm{Aut}^G(\rho, \rho)$ is a self-equivalence of the principal bundle $\Gamma_H \to P \xrightarrow{\rho} D(1)$. So, the classes of G-manifolds are determined by the orbits of the action

$$\mathrm{Aut}^G(\rho, \rho) \circlearrowright \mathrm{Diff}^G(G/H \times_S Q, G/H \times_{\Gamma_H} P_1).$$

Using some identifications, homotopy invariance and the fact that the orbit space is contractible the action becomes $\pi_o(\Gamma_H) \circlearrowright \pi_{n-1}(\Gamma_H/S)$ where π_j is the j-th homotopy group. We have outlined the theorem below.

Theorem 29. *(Bredon [4], p. 257 and p. 331) Given a Lie group G and a pair of closed subgroups $H \subset K$ of G, there is a one-to-one correspondence between the actions $G \circlearrowright M$, up to equivariant diffeomorphism, with exactly two isotropy types (H) and (K) such that the quotient is an n-disk, and the orbits of an action*

$$\pi_o(\Gamma_H) \circlearrowright \pi_{n-1}(N(H)/(N(H) \cap N(K))). \tag{12}$$

Remark 7. For our purposes an explicit expression for the $\pi_o(\Gamma_H)$-action in Theorem 29 is not needed since the groups involved in (12) are quite simple as one can see in Table 2.

Due to Lemma 28 and Theorem 29 we obtain an upper bound l (see Table 2) for the number of G-manifolds with exactly two orbit types, hence the classification of this kind of G-actions is complete by showing that we have as many examples as possible. The remaining part of the proof of Theorems 26 and 27 follows from [19] since only SO(3)-manifolds can have three isotropy types.

The examples below represent the corresponding numeration in Table 2. Notice that there are as many non-equivalent actions in the examples as the upper bound l in the table.

Examples.

	H	K	G	$\pi_{n-1}(N(H)/(N(H)\cap N(K))$	$\pi_o(\Gamma_H)$	l
(a)	$\{1\}$	SO(2)	SO(3)	$\pi_1(\mathbb{RP}^2)$	1	2
(b)	$\{1\}$	SO(2)	SU(2)	$\pi_1(\mathbb{RP}^2)$	1	2
(c)	$\{1\}$	SU(2)	SU(2)	$\{0\}$	1	1
(d)	$\mathbb{Z}_2$	SO(2)	SO(3)	$\{0\}$	1	1
(e)	$\mathbb{Z}_2$	SO(2)	SU(2)	$\pi_1(\mathbb{RP}^2)$	1	2
(f)	$\mathbb{Z}_m$	SO(2)	SO(3)	$\{0\}$	1	1
(g)	$\mathbb{Z}_m$	SO(2)	SU(2)	$\{0\}$	1	1
(h)	$\mathbb{Z}_2$	O(2)	SO(3)	$\pi_1(\mathbb{S}^1)$	$\mathbb{Z}_2$	$\mathbb{Z}$
(i)	D_m	O(2)	SO(3)	$\{0\}$	$\mathbb{Z}_2$	1
(j)	SO(2)	SO(3)	SO(3)	$\{0\}$	1	1

Table 2: l is an upper bound for the number of actions with 2 isotropy types $H \subset K$

(a) *The* SO(3)*-actions with isotropy types* $\{1\}$ *and* SO(2)*.* There are exactly two such actions, the linear action on $\mathbb{S}^5$ of Example 2 and $\mathcal{N}^1_{0,2}$, which is an SO(3)-action on $\mathbb{S}^3 \times \mathbb{S}^2$.

(b) *The* SU(2)*-actions with isotropy types* $\{1\}$ *and* SO(2)*.* There are two such actions, one is $\mathcal{N}^1_{0,1}$ and the other is SU(2) $\circlearrowleft$ $\mathcal{W} = \mathrm{SU}(3)/\mathrm{SO}(3)$ given by $B \cdot [C] = [\mathrm{diag}(B,1)C]$, described in Example 6.

(c) *The* SU(2)*-action on* $\mathbb{S}^5$ *with isotropy types* $\{1\}$ *and* SU(2)*.* This is the linear action described in Example 4.

(d), (e), (f) and (g) *The* G*-action with isotropy types* $\mathbb{Z}_m$ *and* SO(2)*, where* $G = \mathrm{SO}(3)$ *or* SU(2) *and* $m \geq 2$*.* Notice that an SU(2)-action with principal isotropy $\mathbb{Z}_2$ is ineffective since $\mathbb{Z}_2 \subset \mathrm{SU}(2)$ is a normal subgroup. So the action is an SO(3)-action with trivial principal isotropy. In our case, the two examples of SU(2)-actions with isotropies $\mathbb{Z}_2$ and $\mathbb{S}^1$ are the two SO(3)-actions with isotropies $\{1\}$ and $\mathbb{S}^1$. In the same way the SU(2)-actions with principal isotropy $\mathbb{Z}_{2m}$ are ineffective with kernel $\mathbb{Z}_2$, thus they are SO(3)-actions with principal isotropy $\mathbb{Z}_m$. For all the other cases there is one example for each integer m, $\mathcal{N}^1_{0,2m}$ and $\mathcal{N}^1_{0,2m+1}$. The SO(3)-manifolds are $\mathbb{S}^3 \times \mathbb{S}^2$, the SU(2)-manifolds are always the nontrivial $\mathbb{S}^3$-bundle over $\mathbb{S}^2$ by Proposition 12.

(h) and (i) *The* SO(3)*-actions with isotropy types* $\mathbb{Z}_2$ *and* O(2) *or* D_m *and* O(2)*.* Remark 3C in [19] asserts that these examples are not simply-connected for any m.

(j) *The* SO(3)*-action on* $\mathbb{S}^5$ *with isotropy types* SO(2) *and* SO(3)*.* This is the linear action described in Example 3.

We have seen (c.f., Propositions 5 and 6 and Example 5) that the orbit space may not to be a disk. Now, we consider the other possible quotients for an action with singular orbits. The following proposition is a trivial consequence of Theorem 6.1 in [4], when the singular isotropy group is the whole group.

Proposition 30. *There is precisely one* SO(3)*-action (up to equivalence) with isotropy types* (SO(2)) *and* SO(3) *with quotient a 3-sphere with* k *three-disks removed,* $k > 0$*.*

These actions are precisely those described in Example 9 and this concludes the classification of actions with singular orbits and two orbit types.

5.2 Actions with three orbit types

By Lemma 28, there are no SU(2)-actions with three orbit types. The same lemma shows that the SO(3)-actions with 3 orbit types have isotropy groups $\mathbb{Z}_2 \times \mathbb{Z}_2$, O(2) and SO(3). So, all these actions have fixed points. Hudson showed in [19] that these SO(3)-manifolds have at least two fixed points and classified the actions. If there are exactly two fixed points, then the action is equivalent to the linear SO(3)-action on the 5-sphere described in Example 1. If it has three fixed points then it is equivalent to left multiplication on the cosets of the Wu-manifold $\mathcal{W} = \mathrm{SU}(3)/\mathrm{SO}(3)$ (see Example 7). Moreover, there are two 5-dimensional SO(3)-manifolds with exactly four fixed points: the Brieskorn variety, $\mathcal{B}$ of type $(2, 3, 3, 3)$, and the connected sum of two Wu-manifolds with the above actions. All other examples of SO(3)-manifolds with three orbit types have more than four fixed points. It is also a consequence of Lemma 28 that the fixed point set of an SO(3)-manifold with 3 orbit types is finite. As observed at the end of Section 2.2 and proved in [19] the SO(3)-manifolds with more than two isolated fixed points are connected sums of copies of $\mathcal{B}$ and $\mathcal{W}$.

6 Five-manifolds with nonnegative curvature

In this section we prove Theorems A and B. Theorem B is a consequence of Theorems C and D, by using Frankel's Lemma [9] and the classification of the G-manifolds M with fixed point set with codimension one or two in M/G (c.f., [13] and [14]). In our context, the following lemma provides a more elementary proof.

Lemma 31. *Let M be a simply-connected compact* $\mathrm{SO}(3)$*-manifold of dimension 5. If M admits an invariant metric of nonnegative (resp. positive) curvature, then the number of isolated fixed points cannot exceed 3 (resp. 2).*

Proof. If the action has isolated fixed points, then the quotient is a topological 2-disk since the isotropy action in a neighborhood of an isolated fixed point is equivalent to the irreducible $\mathrm{SO}(3)$-representation in $\mathrm{SO}(5)$, and thus, the isotropy types are $(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $(\mathrm{O}(2))$ and $\mathrm{SO}(3)$. The orbit strata of a group action is well known to be totally geodesic (see e.g. [12]). Hence the boundary of $M/\mathrm{SO}(3)$ consists of a geodesic polygon with n edges and n vertices. The edges correspond to the singular isotropies $\mathrm{O}(2)$ and vertices are the fixed points. From the isotropy representation it follows that the angles between the edges are all equal to $\pi/3$ (see Example 1).

By O'Neill's formula the (interior of the) quotient inherits a metric of nonnegative (resp. positive) curvature if M has an invariant metric of nonnegative (resp. positive) curvature. Thus, by the Gauss-Bonnet Theorem, the sum of inner angles of the n-polygon $M/\mathrm{SO}(3)$ is equal to, or bigger (resp. strictly bigger) than $\pi(n-2)$. So, $n = 2$ or 3 if the curvature is nonnegative and $n = 2$ if the curvature is positive. □

Proof of Theorem A. By Theorem D and Section 5.2 the G-manifolds with more than 3 isolated fixed points are $k\,\mathcal{W} \# l\,\mathcal{B}$ with $(k, l) \neq (1, 0)$ (since $\mathrm{SO}(3) \circlearrowright \mathcal{W}$ has exactly 3 fixed points). So, by Lemma 31 they do not admit invariant metrics with nonnegative curvature.

The connected sum M of k copies of $\mathbb{S}^3 \times \mathbb{S}^2$ (see Example 9) has quotient X diffeomorphic to $\mathbb{S}^3$ with $k+1$ three-disks removed. If M admits a metric of nonnegative curvature, then X with the induced metric also has nonnegative curvature. As follows from the proof of the Soul Theorem [6], a compact nonnegatively curved manifold X with non-empty convex boundary contains a totally geodesic compact submanifold Σ without boundary and Σ is a deformation retracts of X. In our case $\dim \Sigma \neq 0$ since X is not a disk. Also, $\dim \Sigma \neq 1$ since X is simply-connected. Thus Σ is a simply-connected surface and a neighborhood of Σ is diffeomorphic to $\mathbb{S}^2 \times (-1, 1)$. Using the flow of the gradient like vector field in the proof of the Soul Theorem it follows that ∂X has two connected components. Therefore, $k = 1$ and only $\mathbb{S}^3 \times \mathbb{S}^2$ with the linear $\mathrm{SO}(3)$-action on the first factor admits an invariant metric of nonnegative curvature.

On the other hand, all the other actions in Theorems C and D, i.e., the linear actions on $\mathbb{S}^5$ and $\mathbb{S}^3 \times \mathbb{S}^2$, the $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$ left multiplication on the cosets in $\mathcal{W}$, and $\mathcal{N}^l_{m,n}$ clearly admit an invariant metric of nonnegative curvature. □

Proof of Theorem B. We restrict ourselves to the actions in Theorem A. By Theorems C and D, the G-manifolds diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^2$ are $\mathcal{N}^l_{m,n}$ and the $\mathrm{SO}(3)$-manifold in Example 5. This last one has quotient X diffeomorphic to a 3-sphere with two 3-disks removed, thus its soul is homeomorphic to a 2-sphere and by the Soul Theorem X cannot be positively curved. So neither $\mathbb{S}^3 \times \mathbb{S}^2$ admits an invariant metric of positive curvature. Also, the SO(3)-action on $\mathcal{W}$ has three fixed points (see Example 7), and therefore does not admit an invariant metric of positive curvature by Lemma 31.

We finally observe that $\mathcal{N}^l_{m,n}$ with $\gcd(m,n) \geq 3$ does not admit a metric of positive curvature. In fact, by Proposition 15 the fixed point set of the principal isotropy group $\mathbb{Z}_{\gcd(m,n)}$ has two connected components of dimension three and by Frankel's Lemma [9] in a positively curved manifold M^n the sum of the dimensions of two totally geodesic submanifolds cannot exceed n. It is also clear that the fixed point set of the SO(3)-manifold $\mathcal{N}^1_{0,0}$ is the disjoint union of two 3-spheres, thus also cannot admit metric of positive curvature. □

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