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Limit Weierstrass Points on Nodal Curves

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aos meus pais

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Contents

LIMIT WEIERSTRASS POINTS ON NODAL CURVES

PARHAM SALEHYAN

Abstract. In the 1980's D. Eisenbud and J. Harris posed the following question: "What are the limits of Weierstrass points in families of curves degenerating to stable curves *not* of compact type?" In the present article, we show how to apply the results obtained by E. Esteves in 1996 to compute limits of canonical systems and Weierstrass points. We also deal with higher Weierstrass points. We restrict ourselves to the case where the limit curve has components intersecting at points in general position and the degeneration occurs along a general direction. When we deal with the usual Weierstrass points, of order one, we also suppose that all of the components of the limit curve intersect each other.

Introduction

Limits of ramification points and linear systems were studied by Eisenbud and Harris in the 1980's, when they developed the theory of limit linear series for curves of compact type; see [EH86]. Many important applications of their theory were found; a survey is given in [EH86].

Basically speaking, in [EH86] Eisenbud and Harris developed a theory to identify limits of linear systems and ramification points as smooth curves degenerate to curves of compact type, whereas in [EH87] they applied their theory to the study of limits of Weierstrass points.

The results in [E96] allow us to identify limits of linear systems and ramification points as smooth curves degenerate to reducible nodal curves of any type. In the present article, we show how to apply the results in [E96] to the study of limits of canonical systems and Weierstrass points. We also deal with higher Weierstrass points.

Computing limits of canonical systems can be difficult without further hypotheses. In the present work we compute them when the components of the limit curve intersect in general position and the degeneration occurs along a general direction. When we deal with Weierstrass points of order one, we also suppose that all of the components of the limit curve intersect each other. We describe our results below.

Let C be a projective, connected and nodal curve of arithmetic genus $g > 0$ defined over an algebraically closed field k of characteristic zero, and $C_1, \dots, C_t$ its irreducible components.

For each $i = 1, \dots, t$, let g_i be the arithmetic genus of C_i. To avoid known or special cases, we will always assume that $t > 1$ and $g_i > 0$ for each $i = 1, \dots, t$.

Let S be the spectrum of a discrete valuation ring A whose residue field is k. Let s (resp. η) denote the special (resp. generic) point of S. A *smoothing* of C is a flat, projective morphism $f : \mathcal{X} \to S$ such that $\mathcal{X}(\eta)$ is smooth and $C \cong \mathcal{X}(s)$. The smoothing is called *regular* if $\mathcal{X}$ is regular. The smoothing will also be denoted by $\mathcal{X}/S$.

Let $\mathcal{X}/S$ be a regular smoothing of C. Let β be a positive integer and $\mathcal{K}$ the relative canonical sheaf on $\mathcal{X}$ over S. Let $\mathcal{K}^\beta := \mathcal{K}^{\otimes\beta}$ denote the relative β-canonical sheaf. Let (V_η, L_η) be the β-canonical system on $\mathcal{X}(\eta)$, that is, V_η is the total system of sections of $L_\eta := \mathcal{K}^\beta(\eta)$. Let $\kappa_\beta := \dim V_\eta$, so $\kappa_1 = g$ and $\kappa_\beta = (2\beta - 1)(g - 1)$ if $\beta > 1$. Let $\mathcal{L}$ be an extension of L_η to $\mathcal{X}$. Let $\overline{V}_{\mathcal{L}} \subseteq H^0(C, \mathcal{L}(s))$ be subvectorspace of sections that extend to sections of $\mathcal{L}$. Then $\dim \overline{V}_{\mathcal{L}} = \kappa_\beta$ as well. We say that $(\overline{V}_{\mathcal{L}}, \mathcal{L}(s))$ is a *limit β-canonical system*. By Proposition 1.1, for each $i = 1, \dots, t$, there is a unique extension $\mathcal{L}_i$ of L_η meeting the following two conditions:

(1) The natural map $\rho_i : \overline{V}_{\mathcal{L}_i} \to H^0(C_i, \mathcal{L}_i(s)|_{C_i})$ is injective.
(2) The natural map $\rho_{i,j} : \overline{V}_{\mathcal{L}_i} \to H^0(C_j, \mathcal{L}_i(s)|_{C_j})$ is not zero for all $j \neq i$.

We say that $(\overline{V}_{\mathcal{L}_i}, \mathcal{L}_i(s))$ is the *limit β-canonical system associated to C_i*.

Let $\mathcal{W}$ be the relative Cartier divisor on $\mathcal{X}$ over S whose generic fibre $\mathcal{W}(\eta)$ is the Weierstrass divisor of (V_η, L_η). We call $W := \mathcal{W}(s)$ the *limit Weierstrass divisor*.

For each $i = 1, \dots, t$, let W_i be the Weierstrass divisor of $(\overline{V}_{\mathcal{L}_i}, \mathcal{L}_i(s)|_{C_i})$, viewed as a Weil divisor on C. Then, by Theorem 1.3,

$$W = \sum_{i=1}^{t} W_i + D$$

as Weil divisors on C, where D is a certain divisor supported in the set of points of intersection between distinct components of C.

Thus, in order to compute the limit Weierstrass divisor it is enough to compute the limit β-canonical systems associated to the components of C.

Our principal result is a characterization, under certain conditions, of the limit β-canonical system associated to C_1. Obviously, for other components of C the characterization is analogous. Our characterization is contained in the Theorem below.

Let K^β denote the β-canonical sheaf of C. For each nonempty subset I of $\{1, \dots, t\}$, let $C_I := \bigcup_{i \in I} C_i$, and denote by K_I^β the β-canonical sheaf of C_I. For each t-uple $\underline{n} := (n_1, \dots, n_t)$ of integers, I will be called *$\underline{n}$-balanced* if n_i is constant for $i \in I$.

For each distinct $i, j \in \{1, \dots, t\}$, let $\Delta_{i,j} := C_i \cap C_j$ and $\delta_{i,j} := \#\Delta_{i,j}$.

Our Lemma 3.1 claims that there is a unique t-uple $\underline{n}$ of integers satisfying certain inequalities, depending only on β, the $\delta_{i,j}$ and the genera of the components $C_1, \dots, C_t$ of C. The proof of the lemma yields an algorithm for finding this $\underline{n}$.

Theorem. *Assume that the components of C intersect at points in general position. For $\beta = 1$, assume $\Delta_{i,j} \neq \emptyset$ for all i and j; otherwise assume $g_i \geq 2$ for each i. Let $\underline{n} = (n_1, \dots, n_t)$ be the unique t-uple of integers given by* Lemma 3.1*. Then the complete system of sections of the general invertible sheaf L on C satisfying*

$$(1) \qquad L|_{C_I} \cong K_I^\beta\Big(\sum_{\substack{i \in I \\ j \notin I}} (\beta + n_j - n_i)\Delta_{i,j}\Big) \text{ for each } \underline{n}\text{-balanced } I \subseteq \{1, \dots, t\}$$

is a limit β-canonical system associated to C_1. Conversely, the limit β-canonical system associated to C_1 of a smoothing of C along the general direction is the complete system of sections of an invertible sheaf L on C satisfying (1)*.*

Now we explain briefly how we prove the Theorem.

Let $\mathcal{X}/S$ be a regular smoothing of C. For the t-uple $\underline{n} = (n_1, \dots, n_t)$ satisfying Lemma 3.1, put

$$\mathcal{L}^{\underline{n}} := \mathcal{K}^\beta \otimes \mathcal{O}_{\mathcal{X}}(n_1 C_1 + \cdots + n_t C_t),$$

and let $(V^{\underline{n}}, L^{\underline{n}}) := (\overline{V}_{\mathcal{L}^{\underline{n}}}, \mathcal{L}^{\underline{n}}(s))$ be the corresponding limit β-canonical system. The sheaf $L^{\underline{n}}$ satisfies conditions (1) in place of L.

In order to characterize all possible sheaves $L^{\underline{n}}$, we study the deformations of C in Section 4. Our Proposition 4.3 shows that all invertible sheaves L on C satisfying (30) are isomorphic to $L^{\underline{n}}$ for some regular smoothing $\mathcal{X}/S$ of C.

For each nonempty $I \subseteq \{1, \dots, t\}$, let $L_I^{\underline{n}} := L^{\underline{n}}|_{C_I}$. For each $i = 1, \dots, t$, let $I_i := \{1, \dots, t\} \setminus \{i\}$ and $\Delta_i := \sum_{j \neq i} \Delta_{i,j}$. The limit system $(V^{\underline{n}}, L^{\underline{n}})$ is the limit β-canonical system associated to C_1 if the following two conditions hold:

(1) $h^0(C_{I_1}, L_{I_1}^{\underline{n}}(-\Delta_1)) = 0$,
(2) $h^0(C_{I_i}, L_{I_i}^{\underline{n}}(-\Delta_i)) < \kappa_\beta$ for each $i = 2, \dots, t$.

The first condition states the injectivity of the natural map
$$H^0(C, L^{\underline{n}}) \to H^0(C_1, L^{\underline{n}}|_{C_1}),$$
whereas the second condition states that for each $i = 2, \dots, t$ the map
$$H^0(C, L^{\underline{n}}) \to H^0(C_i, L^{\underline{n}}|_{C_i})$$
has kernel of dimension less than κ_β, and hence not containing $V^{\underline{n}}$.

The first two statements of Proposition 6.2 claim that Conditions 1 and 2 are met, under certain hypotheses. Indeed, to compute the dimension of $H^0(C_{I_i}, L_{I_i}^{\underline{n}}(-\Delta_i))$ for each $i = 1, \dots, t$ we need to assume that the smoothing occurs along the general direction or, equivalently according to Proposition 4.3, that $L^{\underline{n}}$ is the general invertible sheaf on C satisfying conditions (1) in place of L. Then, also assuming that the components of C intersect at points in general position, our Lemma 5.1 allows us to compute $h^0(C_{I_i}, L_{I_i}^{\underline{n}}(-\Delta_i))$ in terms of the dimensions of
$$H^0(C_J, L_J^{\underline{n}}(-\textstyle\sum_{j \in J} \Delta_{i,j}))$$
for the $\underline{n}$-balanced subsets $J \subseteq I_i$. Since
$$L_J^{\underline{n}} = K_J^\beta\Big(\sum_{\substack{i \in J \\ j \notin J}} (\beta + n_j - n_i)\Delta_{i,j}\Big),$$
and the points in $\Delta_{i,j}$ for $i \in J$ and $j \notin J$ are in general position, these dimensions can be computed. These computations are mostly done in the proof of Lemma 6.1. Finally, after computing $h^0(C_{I_i}, L_{I_i}^{\underline{n}}(-\Delta_i))$ in terms of $\underline{n}$, the $\delta_{i,j}$ and the g_i, the inequalities of Lemma 3.1 are used to show that Conditions 1 and 2 are met.

The final step is to show that $V^{\underline{n}} = H^0(C, L^{\underline{n}})$. Our Proposition 6.2 also establishes this, by computing $h^0(C, L^{\underline{n}})$ with the same method described above, and showing that $h^0(C, L^{\underline{n}}) \leq \kappa_\beta$.

1. **Limit linear systems and ramification points**

Let C be a projective, connected and nodal curve defined over an algebraically closed field k of characteristic zero, and $C_1, \dots, C_t$ its irreducible components. Let S be the spectrum of a discrete valuation ring A whose residue field is k. Let s (resp. η) denote the special (resp. generic) point of S. A *smoothing* of C is a flat, projective morphism $f : \mathcal{X} \to S$ such that $\mathcal{X}(\eta)$ is smooth and $C \cong \mathcal{X}(s)$. If $\mathcal{X}$ is regular, the smoothing is called *regular*.

Fix a regular smoothing of $f : \mathcal{X} \to S$ of C. Fix an invertible sheaf L_η on $\mathcal{X}(\eta)$ and a nonzero vector space V_η of sections of L_η of dimension $r+1$. Since $\mathcal{X}$ is regular, there is an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}(\eta) \cong L_\eta$. We call such $\mathcal{L}$ an *extension of L_η to $\mathcal{X}$*. Since $\mathcal{X}$ is regular, $C_1, \dots, C_t$ are Cartier divisors on $\mathcal{X}$, and any Cartier divisor on $\mathcal{X}$ supported in C is a linear combination of $C_1, \dots, C_t$. It follows that the sheaves $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(n_1C_1 + \cdots + n_tC_t)$ are all the extensions of L_η to $\mathcal{X}$.

If $\mathcal{L}$ is an extension of L_η, put

$$V_{\mathcal{L}} := V_\eta \cap H^0(\mathcal{X}, \mathcal{L}),$$

where the above intersection is taken inside $H^0(\mathcal{X}(\eta), L_\eta)$. Since A is a discrete valuation ring, $V_{\mathcal{L}}$ is a free A-module of rank $r+1$, and $V_{\mathcal{L}} \otimes k(\eta) = V_\eta$. In addition, the induced homomorphism,

$$\overline{V}_{\mathcal{L}} = V_{\mathcal{L}} \otimes k \to H^0(\mathcal{X}, \mathcal{L}) \otimes k \to H^0(C, \mathcal{L}(s)),$$

is injective. To summarize, given an extension $\mathcal{L}$ of L_η to $\mathcal{X}$, the linear system (V_η, L_η) extends to a "linear system" $(V_{\mathcal{L}}, \mathcal{L})$ on $\mathcal{X}$, whose restriction $(\overline{V}_{\mathcal{L}}, \mathcal{L}(s))$ to C is also a linear system, all of the same rank.

If $V_\eta = H^0(\mathcal{X}(\eta), L_\eta)$ then $V_{\mathcal{L}} = H^0(\mathcal{X}, \mathcal{L})$. But $\overline{V}_{\mathcal{L}} = H^0(C, \mathcal{L}(s))$ only if the base-change map $H^0(\mathcal{X}, \mathcal{L}) \otimes k \to H^0(C, \mathcal{L}(s))$ is surjective, and hence an isomorphism.

We say that $(\overline{V}_{\mathcal{L}}, \mathcal{L}(s))$ is a *limit linear system*. (If (V_η, L_η) is the β-canonical system, we say that $(\overline{V}_{\mathcal{L}}, \mathcal{L}(s))$ is a *limit β-canonical system*.)

Proposition 1.1. [Es96, Theorem 1] *For each irreducible component $C_i \subseteq C$, there is a unique extension $\mathcal{L}_i$ of L_η to $\mathcal{X}$ with the following properties:*

(1) *the canonically induced homomorphism,*
$$\overline{V}_{\mathcal{L}_i} \to H^0(C_i, \mathcal{L}_i(s)|_{C_i}),$$
is injective;

(2) *for each irreducible component $C_j \subseteq C$ with $j \neq i$, the canonically induced homomorphism,*
$$\overline{V}_{\mathcal{L}_i} \to H^0(C_j, \mathcal{L}_i(s)|_{C_j}),$$
is not identically zero.

We say that $\mathcal{L}_i$ is the extension of L_η *associated to C_i* (and to V_η). We say that $(V_i, L_i) := (\overline{V}_{\mathcal{L}_i}, \mathcal{L}_i(s))$ is the *limit linear system associated to C_i*. (If (V_η, L_η) is the

β-canonical system, we say that (V_i, L_i) is the *limit β-canonical system associated to C_i.*)

Proposition 1.2. [Es96, Proposition 4] *Fix i, j with $i \neq j$. Let $l_{i,m}$, for $m \in \{1, \ldots, t\} \setminus \{i\}$, be the unique integers such that*

$$\mathcal{L}_i \cong \mathcal{L}_j(\sum_{m \neq i} l_{i,m} C_m).$$

Then $0 \leq l_{i,m} \leq l_{i,j}$ for each m.

The integers $l_{i,m}$ depend only on the specializations $\mathcal{L}_i(s)$ and $\mathcal{L}_j(s)$.

Let $\mathcal{R}$ denote the relative Cartier divisor on $\mathcal{X}$ over S whose generic fibre $\mathcal{R}(\eta)$ is the ramification divisor of (V_η, L_η). We call $R := \mathcal{R}(s)$ the *limit ramification divisor*. (If (V_η, L_η) is the β-canonical system of $\mathcal{X}(\eta)$, *limit β-Weierstrass divisor*, and denote $\mathcal{W} := \mathcal{R}$ and $W := R$.)

Let $\Delta_{i,j}$ denote the reduced Weil divisor on C whose support is $C_i \cap C_j$, for $i \neq j$.

Theorem 1.3. [Es96, Theorem 7] *For each $i = 1, \ldots, t$, let R_i be the ramification divisor of $(V_i, L_i|_{C_i})$, viewed as a Weil divisor on C. Then*

$$R = \sum_{i=1}^{t} R_i + \sum_{i<j} (r - l_{i,j})(r+1)\Delta_{i,j}$$

as Weil divisors on C.

2. Limit β-canonical systems: notation

Fix a curve C as in Section 1. Call $C_1, \dots, C_t$ its irreducible components. For each pair (i, j) of distinct integers in $\{1, \dots, t\}$, let $\Delta_{i,j}$ be the reduced Weil divisor with support $C_i \cap C_j$. Assume that $t > 1$. Let $\delta_{i,j} := \#\Delta_{i,j}$ for all i and j. Let

$$\delta := \sum_{i<j} \delta_{i,j}.$$

For each $i = 1, \dots, t$, let $\Delta_i := \sum_{j \neq i} \Delta_{i,j}$ and $\delta_i := \#\Delta_i$.

Let β be a positive integer. For each nonempty subset $I \subseteq \{1, \dots, t\}$ let

$$C_I := \bigcup_{i \in I} C_i,$$

let K_I^β be the β-canonical sheaf of C_I, and g_I the arithmetic genus. For simplicity, when appearing as an index, the subset $\{i\}$ will be replaced by i. We say that I is *connected* if C_I is connected. Let g denote the arithmetic genus and K^β the β-canonical sheaf of C. Since C is assumed connected, $\delta \geq t - 1$ and

$$g = g_1 + \cdots + g_t + \delta - t + 1.$$

Assume $g_i > 0$ for each $i = 1, \dots, t$. For each pair $I, J \subset \{1, \dots, t\}$ of disjoint nonempty subsets, let

$$\Delta_{I,J} := \sum_{i \in I;\, j \in J} \Delta_{i,j}$$

and $\delta_{I,J} := \#\Delta_{I,J}$. For each nonempty, proper subset $I \subset \{1, \dots, t\}$, let $\Delta_I := \Delta_{I,I^c}$, where $I^c := \{1, \dots, t\} \setminus I$ and $\delta_I := \#\Delta_I$.

Let $\underline{n} = (n_1, \dots, n_t)$ be a t-uple of integers. A nonempty $I \subseteq \{1, \dots, t\}$ will be called *$\underline{n}$-balanced* if n_i is constant for $i \in I$; if so, let $n_I := n_i$ for any $i \in I$. The integer n_I is called the *$\underline{n}$-weight* of I. Any nonempty subset $I \subseteq \{1, \dots, t\}$ is uniquely decomposed into maximal connected $\underline{n}$-balanced subsets. We call these subsets the *$\underline{n}$-components* of I.

Let $\mathcal{X}/S$ be a regular smoothing of C, as in Section 1. Let $\mathcal{K}^\beta := \mathcal{K}^{\otimes \beta}$, the relative β-canonical sheaf on $\mathcal{X}$ over S. Let (V_η, L_η) be the β-canonical system on $\mathcal{X}(\eta)$, that is,

$$L_\eta = \mathcal{K}^\beta(\eta) \text{ and } V_\eta = H^0(\mathcal{X}(\eta), \mathcal{K}^\beta(\eta)).$$

For any t-uple of integers $\underline{n} := (n_1, \dots, n_t)$, put

$$\mathcal{L}^{\underline{n}} := \mathcal{K}^\beta \otimes \mathcal{O}_{\mathcal{X}}(n_1 C_1 + \cdots + n_t C_t),$$

and let $(V^{\underline{n}}, L^{\underline{n}}) := (\overline{V}_{\mathcal{L}^{\underline{n}}}, \mathcal{L}^{\underline{n}}(s))$ denote the limit linear system associated to the β-canonical system on $\mathcal{X}(\eta)$ and the extension $\mathcal{L}^{\underline{n}}$ of $\mathcal{K}^\beta(\eta)$. (If $\underline{n}$ and $\underline{n}'$ are t-uples differing by a multiple of $(1,\dots,1)$, then $\mathcal{L}^{\underline{n}} \cong \mathcal{L}^{\underline{n}'}$.) For each nonempty subset $I \subseteq \{1,\dots,t\}$, let $L_I^{\underline{n}} := L^{\underline{n}}|_{C_I}$. If I is $\underline{n}$-balanced, then

$$L_I^{\underline{n}} \cong K_I^{\beta}\Big(\sum_{\substack{i\in I\\ j\notin I}}(\beta+n_j-n_i)\Delta_{i,j}\Big).$$

We will now focus on the component C_1. Let

$$\epsilon_i^{\underline{n}} := e_i + \textstyle\sum_{j\neq i}(n_j-n_i)\delta_{i,j} \text{ for each } i=1,\dots,t,$$

where $e_1 := (2\beta-1)(g_1-g)+\beta\delta_1$, and $e_i := (2\beta-1)(g_i-1)+\beta\delta_i$ for each $i>1$. Note that

$$h^0(C_i,L_i^{\underline{n}}) - h^1(C_i,L_i^{\underline{n}}) = \begin{cases} (2\beta-1)(g-1)+\epsilon_1^{\underline{n}}, & \text{if } i=1;\\ \epsilon_i^{\underline{n}}, & \text{for each } i>1.\end{cases}$$

More generally, for each nonempty subset $I \subseteq \{1,\dots,t\}$, let

$$\epsilon_I^{\underline{n}} := \sum_{i\in I}\epsilon_i^{\underline{n}} - \sum_{\substack{i,j\in I\\ j\neq i}}\delta_{i,j}.$$

If I is $\underline{n}$-balanced, we get

$$h^0(C_I,L_I^{\underline{n}}) - h^1(C_I,L_I^{\underline{n}}) = \begin{cases} (2\beta-1)(g-1)+\epsilon_I^{\underline{n}}, & \text{if } 1\in I;\\ \epsilon_I^{\underline{n}}, & \text{if } 1\notin I.\end{cases}$$

Observe that $\sum_{i=1}^t \epsilon_i^{\underline{n}} = \delta$, and if I is the disjoint union of $I_1,\dots,I_p$ then

$$\epsilon_I^{\underline{n}} = \sum_{i=1}^{p}\epsilon_{I_i}^{\underline{n}} - \sum_{1\le i<j\le p}\delta_{I_i,I_j}. \tag{2}$$

For each t-uple $\underline{n}=(n_1,\dots,n_t)$ and each nonempty, proper subset I of $\{1,\dots,t\}$ let $m_I^{\underline{n}} := \max\{n_i \,|\, i\in I\}$. Set $\gamma_I^{\underline{n}} := 0$ if $m_{I^c}^{\underline{n}} \ge m_I^{\underline{n}}$ and $\gamma_I^{\underline{n}} := 1$ otherwise. Define

$$\gamma_I^{\beta,\underline{n}} := \begin{cases} \gamma_I^{\underline{n}}, & \text{if } \beta=1,\\ 1, & \text{if } \beta>1 \text{ and } 1\in I,\\ 0, & \text{if } \beta>1 \text{ and } 1\notin I.\end{cases}$$

For each subset $I \subseteq \{1,\dots,t\}$, let $\underline{h}^I := (h_1^I,\dots,h_t^I)$ be the t-uple defined by $h_i^I := 1$ if $i\in I$ and $h_i^I := 0$ otherwise.

3. A numerical lemma

Recall the notation in Section 2.

Lemma 3.1. *Let β be a positive integer. For $\beta = 1$, assume $\delta_{i,j} \neq 0$ for all i and j. Then there is a unique t-uple $\underline{n} = (n_1, \dots, n_t)$ such that $n_1 = 0$ and for each proper, nonempty subset $I \subset \{1, \dots, t\}$,*

$$\gamma_{I^c}^{\beta,\underline{n}+\underline{h}^{I^c}} \leq \epsilon_I^{\underline{n}} \leq \delta_{I,I^c} - \gamma_I^{\beta,\underline{n}+\underline{h}^I},$$

where $I^c := \{1, \dots, t\} \setminus I$.

Proof. Let's first make a simple observation regarding the statement of the lemma: Since $\epsilon_I^{\underline{n}} + \epsilon_{I^c}^{\underline{n}} = \delta_{I,I^c}$, the inequality

$$\epsilon_I^{\underline{n}} \leq \delta_{I,I^c} - \gamma_I^{\beta,\underline{n}+\underline{h}^I}$$

is equivalent to

$$\epsilon_{I^c}^{\underline{n}} \geq \gamma_I^{\beta,\underline{n}+\underline{h}^I}.$$

Thus, we need only prove the lemma for all nonempty subsets $I \subseteq \{2, \dots, t\}$.

We divide the proof in several parts:

Part 1: We claim there is a t-uple $\underline{n}$ such that $\epsilon_i^{\underline{n}} >> 0$ for all $i = 2, \dots, t$. To prove our claim, define inductively a partition $\{2, \dots, t\} = I_1 \cup \cdots \cup I_q$ as follows: Let $I_1 := \{i \in \{2, \dots, t\} \,|\, \delta_{1,i} > 0\}$; assuming that I_r is defined, let $I_{r+1} \subseteq \{2, \dots, t\}$ be the subset consisting of those $i \in I \setminus (I_1 \cup \cdots \cup I_r)$ such that $\delta_{i,j} > 0$ for some $j \in I_r$. Since C is connected, if $I_1 \cup \cdots \cup I_r \neq \{2, \dots, t\}$, then I_{r+1} is nonempty. So, after finitely many steps, say q steps, we obtain a partition of $\{2, \dots, t\}$.

For each integer s, and each $r = 1, \dots, q$, let $\underline{n}^{r,s} = (0, n_2^{r,s}, \dots, n_t^{r,s})$ be the t-uple of integers defined by letting $n_i^{r,s} := s$ if $i \in I_r \cup \cdots \cup I_q$, and $n_i^{r,s} := 0$ otherwise. For each q-uple $\underline{s} := (s_1, \dots, s_q)$ of integers, let

$$\underline{n}^{\underline{s}} := \sum_{r=1}^{q} \underline{n}^{r,s_r}.$$

Then, for each $r = 1, \dots, q$ and each $i \in I_r$,

$$\epsilon_i^{\underline{n}^{\underline{s}}} = e_i - s_r(\sum_{j \in I_{r-1}} \delta_{i,j}) + s_{r+1}(\sum_{j \in I_{r+1}} \delta_{i,j}),$$

where $I_0 := \{1\}$, $I_{q+1} := \emptyset$ and $s_{q+1} := 0$. Note that, by construction, $\sum_{j \in I_{r-1}} \delta_{i,j} > 0$ for all $i \in I_r$, for $r = 1, \dots, q$. In particular, we can make $\epsilon_i^{\underline{n}^{\underline{s}}} >> 0$ for all $i \in I_q$, by letting $s_q << 0$. Assuming that $s_{r+1}, \dots, s_q$ were chosen such that $\epsilon_i^{\underline{n}^{\underline{s}}} >> 0$ for all $i \in I_{r+1} \cup \cdots \cup I_q$, we let $s_r << 0$ to make $\epsilon_i^{\underline{n}^{\underline{s}}} >> 0$ for all $i \in I_r$. By induction, there are integers $s_i << 0$ for $i = 1, \dots, q$ such that $\epsilon_i^{\underline{n}^{\underline{s}}} >> 0$ for each $i = 2, \dots, t$. The proof of our first claim is complete.

Part 2: For each integer $s \geq 0$, let $\underline{n}^s = (0, n_2^s, \dots, n_t^s)$ be a t-uple of integers. Assume that $n_i^s \leq n_i^{s+1}$ for each $i = 2, \dots, t$ and all $s \geq 0$. In addition, assume that

$$\lim_{s\to\infty} n_i^s = \infty$$

for some $i \in \{2, \dots, t\}$. We claim that there is $j \in \{2, \dots, t\}$ such that

$$\liminf_{s\to\infty} \epsilon_j^{\underline{n}^s} = -\infty. \tag{3}$$

In fact, up to reordering $\{2, \dots, t\}$, there is an integer u with $1 \leq u < t$ such that $n_i^s \to \infty$ if and only if $i > u$. Since

$$\sum_{i=2}^{t} \epsilon_i^{\underline{n}^s} = \sum_{i=2}^{t} e_i - \sum_{i=2}^{t} \delta_{1,i} n_i^s,$$

there is $j \in \{2, \dots, t\}$ such that (3) holds, unless $\delta_{1,i} = 0$ for each $i > u$. Suppose $\delta_{1,i} = 0$ for each $i > u$. Then $u > 1$ because C is connected. For each $i = 2, \dots, u$, either $\epsilon_i^{\underline{n}^s} \to \infty$ or $\delta_{i,u+1} = \cdots = \delta_{i,t} = 0$. However, since C is connected, we cannot have $\delta_{i,j} = 0$ for each $i = 1, \dots, u$ and each $j = u+1, \dots, t$. Thus $\epsilon_i^{\underline{n}^s} \to \infty$ for some $i \in \{2, \dots, u\}$. However, since

$$\sum_{i=2}^{t} \epsilon_i^{\underline{n}^s} \leq \sum_{i=2}^{t} e_i - \sum_{i=2}^{t} \delta_{1,i} n_i^0$$

for each $s \geq 0$, then (3) holds for some $j \neq i$. The proof of our claim is complete.

Part 3: It follows from Part 1 that there is a t-uple $\underline{n} = (n_1, \dots, n_t)$ such that $n_1 = 0$ and

$$\epsilon_R^{\underline{n}} \geq \gamma_{R^c}^{\beta, \underline{n}+\underline{h}^{R^c}} \quad \text{for each nonempty } R \subseteq \{2, \dots, t\}. \tag{4}$$

Assume that there is a nonempty subset $I \subseteq \{2, \dots, t\}$ such that

$$\epsilon_I^{\underline{n}} \geq \delta_{I,I^c} - \gamma_I^{\beta, \underline{n}+\underline{h}^I} + 1. \tag{5}$$

Suppose I is minimal with the above property. Let $\underline{n}' := \underline{n} + \underline{h}^I$. We claim that (4) holds for $\underline{n}'$ in place of $\underline{n}$.

Indeed, let $R \subseteq \{2, \dots, t\}$ be a nonempty subset. We will show that

$$\epsilon_R^{\underline{n}'} \geq \gamma_{R^c}^{\beta, \underline{n}'+\underline{h}^{R^c}}. \tag{6}$$

We divide the proof according to three cases:

Case 1: Suppose first that $R \subseteq I$. Suppose for the moment that $R \neq I$, and let $S := I \setminus R$. Since I is minimal for property (5),

$$\epsilon_S^{\underline{n}} \leq \delta_{S,S^c} - \gamma_S^{\beta, \underline{n}+\underline{h}^S}. \tag{7}$$

Since I is a disjoint union of R and S, from (2) we get

$$\epsilon_I^{\underline{n}} = \epsilon_R^{\underline{n}} + \epsilon_S^{\underline{n}} - \delta_{R,S},$$

and thus, from (5) and (7),

$$\epsilon_R^{\underline{n}} \geq \delta_{R,I^c} + \gamma_S^{\beta,\underline{n}+\underline{h}^S} - \gamma_I^{\beta,\underline{n}+\underline{h}^I} + 1. \tag{8}$$

Note now that

$$\gamma_I^{\beta,\underline{n}+\underline{h}^I} \leq \gamma_R^{\beta,\underline{n}+\underline{h}^I} + \gamma_S^{\beta,\underline{n}+\underline{h}^S}. \tag{9}$$

In fact, if $\beta > 1$, since $1 \notin I$, all terms in (9) are zero. So suppose $\beta = 1$. Since all the terms in (9) are either 0 or 1, we may also suppose that $\gamma_R^{\underline{n}+\underline{h}^I} = 0$ and $\gamma_S^{\underline{n}+\underline{h}^S} = 0$, and try to prove that $\gamma_I^{\underline{n}+\underline{h}^I} = 0$. In equivalent terms, suppose

$$m_R^{\underline{n}} + 1 \leq \max(m_S^{\underline{n}} + 1, m_{I^c}^{\underline{n}}) \text{ and } m_S^{\underline{n}} + 1 \leq \max(m_R^{\underline{n}}, m_{I^c}^{\underline{n}}).$$

Coupling the two inequalities, we get $m_R^{\underline{n}} + 1 \leq \max(m_R^{\underline{n}}, m_{I^c}^{\underline{n}})$. This is only possible if $m_R^{\underline{n}} + 1 \leq m_{I^c}^{\underline{n}}$. Then $m_S^{\underline{n}} + 1 \leq m_{I^c}^{\underline{n}}$ as well, and thus

$$m_I^{\underline{n}} + 1 = \max(m_R^{\underline{n}}, m_S^{\underline{n}}) + 1 \leq m_{I^c}^{\underline{n}}.$$

The latter inequality means $\gamma_I^{\underline{n}+\underline{h}^I} = 0$.

Using (9) in (8), we get

$$\epsilon_R^{\underline{n}} \geq \delta_{R,I^c} - \gamma_R^{\beta,\underline{n}+\underline{h}^I} + 1. \tag{10}$$

Note that the above inequality holds for I instead of R, as we recover inequality (5).

Finally, drop the hypothesis that $R \neq I$. Then, by (10),

$$\epsilon_R^{\underline{n}'} = \epsilon_R^{\underline{n}} - \delta_{R,I^c} \geq 1 - \gamma_R^{\beta,\underline{n}+\underline{h}^I}.$$

To get (6), we need only show that

$$\gamma_R^{\beta,\underline{n}+\underline{h}^I} + \gamma_{R^c}^{\beta,\underline{n}'+\underline{h}^{R^c}} \leq 1.$$

Now, the inequality holds unless both terms on the left are 1. However, if $\beta > 1$ then $\gamma_R^{\beta,\underline{n}+\underline{h}^I} = 0$ because $1 \notin R$. And if $\beta = 1$, either $\gamma_{R^c}^{\underline{n}'+\underline{h}^{R^c}} = 0$ because $m_{R^c}^{\underline{n}'} + 1 \leq m_R^{\underline{n}'}$, or $m_R^{\underline{n}'} \leq m_{R^c}^{\underline{n}'}$, and hence $\gamma_R^{\underline{n}+\underline{h}^I} = 0$.

Case 2: Suppose now that $R \subseteq I^c$. Then, by (4),

$$\epsilon_R^{\underline{n}'} = \epsilon_R^{\underline{n}} + \delta_{R,I} \geq \gamma_{R^c}^{\beta,\underline{n}+\underline{h}^{R^c}} + \delta_{R,I}.$$

Now,

$$\gamma_{R^c}^{\beta,\underline{n}+\underline{h}^{R^c}} + \delta_{R,I} \geq 1,$$

because $\delta_{R,I} \geq 1$ for $\beta = 1$, by hypothesis, and $\gamma_{R^c}^{\beta,\underline{n}+\underline{h}^{R^c}} = 1$ for $\beta > 1$, since $1 \in R^c$. Thus $\epsilon_R^{\underline{n}'} \geq 1 \geq \gamma_{R^c}^{\beta,\underline{n}'+\underline{h}^{R^c}}$.

Case 3: Let $R_1 := R \cap I$ and $R_2 := R \cap I^c$. Suppose R_1 and R_2 are nonempty. Then, using (10) with R_1 in place of R, and (4) with R_2 in place of R, we get

$$\begin{aligned}\epsilon_R^{\underline{n}'} =& \epsilon_{R_1}^{\underline{n}'} + \epsilon_{R_2}^{\underline{n}'} - \delta_{R_1,R_2}\\ =& \epsilon_{R_1}^{\underline{n}} - \delta_{R_1,I^c} + \epsilon_{R_2}^{\underline{n}} + \delta_{R_2,I} - \delta_{R_1,R_2}\\ \geq& \delta_{R_1,I^c} - \gamma_{R_1}^{\beta,\underline{n}+\underline{h}^I} + 1 - \delta_{R_1,I^c} + \gamma_{R_2^c}^{\beta,\underline{n}+\underline{h}^{R_2^c}} + \delta_{R_2,I} - \delta_{R_1,R_2}\\ =& \delta_{R_2,I\setminus R_1} + 1 + \gamma_{R_2^c}^{\beta,\underline{n}+\underline{h}^{R_2^c}} - \gamma_{R_1}^{\beta,\underline{n}+\underline{h}^I}.\end{aligned}$$

To get (6), we need only show that

$$\gamma_{R_1}^{\beta,\underline{n}'} + \gamma_{R^c}^{\beta,\underline{n}'+\underline{h}^{R^c}} \leq \gamma_{R_2^c}^{\beta,\underline{n}+\underline{h}^{R_2^c}} + 1.$$

If $\beta > 1$ then $\gamma_{R_1}^{\beta,\underline{n}'} = 0$ because $1 \notin R_1$, and hence the inequality holds. So suppose $\beta = 1$. We may also suppose $\gamma_{R_1}^{\beta,\underline{n}'} = 1$ and $\gamma_{R^c}^{\beta,\underline{n}'+\underline{h}^{R^c}} = 1$, and try to prove that $\gamma_{R_2^c}^{\beta,\underline{n}+\underline{h}^{R_2^c}} = 1$. In equivalent terms, suppose $m_{R_1}^{\underline{n}'} > m_{R_1^c}^{\underline{n}'}$ and $m_{R^c}^{\underline{n}'} + 1 > m_R^{\underline{n}'}$. Using the first inequality, we get $m_{I\setminus R_1}^{\underline{n}} < m_{R_1}^{\underline{n}}$, and hence

$$m_{I\setminus R_1}^{\underline{n}} + 1 \leq m_I^{\underline{n}}.$$

And using the second inequality,

$$m_{R_2}^{\underline{n}} = m_{R_2}^{\underline{n}'} \leq m_R^{\underline{n}'} \leq m_{R^c}^{\underline{n}'} = \max(m_{I^c\setminus R_2}^{\underline{n}}, m_{I\setminus R_1}^{\underline{n}} + 1).$$

Combining the last two displayed inequalities, we get

$$m_{R_2}^{\underline{n}} \leq \max(m_{I^c\setminus R_2}^{\underline{n}}, m_I^{\underline{n}}) = m_{R_2^c}^{\underline{n}},$$

which is equivalent to $\gamma_{R_2^c}^{\beta,\underline{n}+\underline{h}^{R_2^c}} = 1$.

Part 4: In Part 3, we started with a t-uple $\underline{n} = (n_1, \dots, n_t)$ satisfying $n_1 = 0$ and inequalities (4), and produced another t-uple $\underline{n}' = (n_1', \dots, n_t')$ satisfying $n_1' = 0$ and the same inequalities, (6), as $\underline{n}$. We may thus apply Part 3 again, with $\underline{n}'$ instead of $\underline{n}$, and so on, as long as (5) holds for some nonempty subset $I \subseteq \{2, \dots, t\}$. However, $n_i' \geq n_i$ for each i, with strict inequality for some i. Hence, according to Part 2, this process must come to an end. When it does, we obtain a t-uple $\underline{n}$ satisfying all the inequalities stated in the lemma.

Part 5: Finally, suppose there are two t-uples $\underline{n}$ and $\underline{n}'$ with $n_1 = n_1' = 0$ and satisfying the inequalities in the statement of the lemma. Let $\underline{x} := \underline{n}' - \underline{n}$. Set $x := \max(x_i)$. Assume that $\underline{x} \neq 0$. Then, by exchanging $\underline{n}$ and $\underline{n}'$ if necessary, we may assume that $x > 0$. Let $I \subseteq \{2, \dots, t\}$ be the maximal $\underline{x}$-balanced subset of weight x. Then, using the stated left-hand inequalities for $\underline{n}'$,

$$\epsilon_I^{\underline{n}} = \epsilon_I^{\underline{n}'} + \sum_{\substack{i\in I\\ j\in I^c}} \delta_{I,\{j\}}(x - x_j) \geq \gamma_{I^c}^{\beta,\underline{n}'+\underline{h}^{I^c}} + \delta_{I,I^c} + \sum_{\substack{i\in I\\ j\in I^c}} \delta_{I,\{j\}}(x - 1 - x_j).$$

We claim that $\epsilon_I^{\underline{n}} > \delta_{I,I^c} - \gamma_I^{\beta,\underline{n}+\underline{h}^I}$, thus reaching a contradiction with the stated right-hand inequalities for $\underline{n}$. Indeed, we need only show that

$$\gamma_{I^c}^{\beta,\underline{n}'+\underline{h}^{I^c}} + \gamma_I^{\beta,\underline{n}+\underline{h}^I} + \sum_{\substack{i\in I\\ j\in I^c}} \delta_{I,\{j\}}(x-1-x_j) > 0.$$

However, if the last inequality does not hold, then $\gamma_I^{\beta,\underline{n}+\underline{h}^I} = \gamma_{I^c}^{\beta,\underline{n}'+\underline{h}^{I^c}} = 0$ and $\delta_{I,\{j\}}(x-1-x_j) = 0$ for each $j \in I^c$. Since $1 \in I^c$, the equality $\gamma_{I^c}^{\beta,\underline{n}'+\underline{h}^{I^c}} = 0$ implies $\beta = 1$. Then, by hypothesis, $\delta_{I,\{j\}} > 0$ for each $j \in I^c$. So $x_j = x-1$ for each $j \in I^c$. Since $1 \in I_c$ and $x_1 = 0$, we have $\underline{x} = \underline{h}^I$. So $\gamma_{I^c}^{\underline{n}} = \gamma_{I^c}^{\underline{n}'+\underline{h}^{I^c}}$, and hence $\gamma_{I^c}^{\underline{n}} = 0$. This is equivalent to

$$m_{I^c}^{\underline{n}} \leq m_I^{\underline{n}}.$$

On the other hand, $\gamma_I^{\beta,\underline{n}+\underline{h}^I} = 0$ is equivalent to

$$m_I^{\underline{n}} + 1 \leq m_{I^c}^{\underline{n}}.$$

We have a contradiction. $\square$

Example. We will show that the condition "$\delta_{i,j} \neq 0$ for all i and j" is necessary for the *uniqueness* of $\underline{n}$, when $\beta = 1$, in Lemma 3.1.

Let C be a curve with three components, namely C_1, C_2 and C_3, such that $g_1 > 0$, $g_2 = g_3 = g_0 > 1$, $\delta_{1,2} = \delta_{1,3} = 1$ and $\delta_{2,3} = 0$. So $\delta_1 = 2$ and $\delta_2 = \delta_3 = 1$. We verify that $\underline{n} = (0, n_2, n_3) = (0, g_0, g_0 - 1)$ and $\underline{n} = (0, n_2, n_3) = (0, g_0 - 1, g_0)$ satisfy the inequalities in Lemma 3.1.

If $\underline{n} = (0, g_0, g_0 - 1)$ then $\epsilon_1^{\underline{n}} = \epsilon_3^{\underline{n}} = 1$ and $\epsilon_2^{\underline{n}} = 0$. If $I = \{1,2\}$ then $\epsilon_I^{\underline{n}} = 0$, $\gamma_{\{3\}}^{1,(0,g_0,g_0)} = 0$, $\gamma_{\{1,2\}}^{1,(1,g_0+1,g_0-1)} = 1$ and $\delta_{I,I^c} = 1$. If $I = \{2,3\}$ then $\epsilon_I^{\underline{n}} = 1$, $\gamma_{\{1\}}^{1,(1,g_0,g_0-1)} = 0$, $\gamma_{\{2,3\}}^{1,(0,g_0+1,g_0)} = 1$ and $\delta_{I,I^c} = 2$. If $I = \{1,3\}$ then $\epsilon_I^{\underline{n}} = 1$, $\gamma_{\{2\}}^{1,(0,g_0+1,g_0-1)} = 1$, $\gamma_{\{1,3\}}^{1,(1,g_0,g_0)} = 0$ and $\delta_{I,I^c} = 1$. In either of the three cases, the inequalities of the lemma are satisfied.

By the symmetry of the numerical invariants of C, also for $\underline{n} = (0, g_0 - 1, g_0)$ the inequalities of the lemma are satisfied.

4. **Deformation theory**

Let C be a curve, as in Section 1. Call $C_1, \dots, C_t$ the irreducible components of C. Let $\mathrm{Div}(C)$ denote the group of Cartier divisors and $\mathrm{Pic}(C)$ the Picard group of C. For each $c \in (k^*)^t$ and $D \in \mathrm{Div}(C)$, denote by $c \cdot D$ the result of the action of c on D.

Proposition 4.1. [EM02, Proposition 6.3] *Let $D_1, \dots, D_t$ be Cartier divisors on $C_1, \dots, C_t$.*

(1) *There exists a Cartier divisor E on C such that $E|_{C_i} \equiv D_i$ for each $i = 1, \dots, t$.*
(2) *For each Cartier divisor E on C such that $E|_{C_i} \equiv D_i$ for each $i = 1, \dots, t$, there is a Cartier divisor D on C such that $D \equiv E$ and $D|_{C_i} = D_i$ for each $i = 1, \dots, t$.*
(3) *If D and D' are Cartier divisors on C such that $D|_{C_i} = D_i = D'|_{C_i}$ for each $i = 1, \dots, t$, then $D \equiv D'$ if and only if there is $c \in (k^*)^t$ such that $D = c \cdot D'$.*

Let Δ be the set of points of intersection between distinct components of C and $\delta := \#\Delta$. For each $p \in \Delta$, let $i_p, j_p \in \{1, \dots, t\}$ be the unique integers such that $i_p < j_p$ and $p \in C_{i_p} \cap C_{j_p}$. We say that p is a *reducible node* of every subcurve of C containing $C_{i_p} \cup C_{j_p}$. For each t-uple of integers $\underline{n} := (n_1, \dots, n_t)$, put $\underline{n}_p := n_{j_p} - n_{i_p}$. Let u_p and v_p be the local parameters of C_{i_p} and C_{j_p} at p.

Let $k^*_\Delta := \prod_{p \in \Delta} k^*$. For each $a \in k^*_\Delta$, define a map $E_a : \mathbb{Z}^t \to \mathrm{Div}(C)$ as follows: For each $\underline{n} \in \mathbb{Z}^t$, the image $E_a(\underline{n})$ is the Cartier divisor on C, trivial off Δ, and defined at each $p \in \Delta$ by the local equation $a_p{}^{\underline{n}_p} u_p{}^{\underline{n}_p} + v_p{}^{-\underline{n}_p}$. Then E_a is a homomorphism of groups. We call E_a a *pre-enriched structure on C*.

It is easy to prove that the set of pre-enriched structures on C does not depend on the choice of the local parameters; see [EM02, 6.5].

A map $L : \mathbb{Z}^t \to \mathrm{Pic}(C)$ is called an *enriched structure on C* if there is a pre-enriched structure $E : \mathbb{Z}^t \to \mathrm{Div}(C)$ such that $L(\underline{n}) \cong \mathcal{O}_C(E(\underline{n}))$ for each $\underline{n} \in \mathbb{Z}^t$.

Theorem 4.2. [EM02, Theorem 6.6] *Let C be a nodal curve and $C_1, \dots, C_t$ its irreducible components. Then, for each regular smoothing $\mathcal{X}/S$ of C there is an enriched structure L on C such that*

$$L(\underline{n}) \cong \mathcal{O}_{\mathcal{X}}(n_1 C_1 + \cdots + n_t C_t)|_C \text{ for each } \underline{n} \in \mathbb{Z}^t. \tag{11}$$

Conversely, for each enriched structure L on C there is a regular smoothing $\mathcal{X}/S$ of C satisfying (11).

Recall the notation in Section 2.

Proposition 4.3. *Let N be an invertible sheaf on C and $\underline{\tau} = (\tau_1, \dots, \tau_t)$ a t-uple of integers. Let $\mathcal{P}$ be the partition of $\{1, \dots, t\}$ in maximal $\underline{\tau}$-balanced subsets. Then there is a regular smoothing $\mathcal{X}/S$ of C such that*

$$N \cong \mathcal{O}_{\mathcal{X}}(\tau_1 C_1 + \cdots + \tau_t C_t)|_C \tag{12}$$

if and only if

$$N|_{C_I} \cong \mathcal{O}_{C_I}\Big(\sum_{\substack{i\in I\\ j\notin I}}(\tau_j-\tau_i)\Delta_{i,j}\Big) \text{ for each } I\in\mathcal{P}. \tag{13}$$

Proof. If either (12) or (13) holds, then

$$N|_{C_i} \cong \mathcal{O}_{C_i}\Big(\sum_{j\neq i}(\tau_j-\tau_i)\Delta_{i,j}\Big). \tag{14}$$

By Statement 2 of Proposition 4.1, there is a Cartier divisor D on C such that $N\cong\mathcal{O}_C(D)$ and

$$D|_{C_i} = \sum_{j\neq i}(\tau_j-\tau_i)\Delta_{i,j} \text{ for each } i=1,\dots,t. \tag{15}$$

As each enriched structure arises from a pre-enriched structure, there is an enriched structure $L:\mathbb{Z}^t\to\mathrm{Pic}(C)$ such that $N\cong L(\underline{\tau})$ if and only if there is a pre-enriched structure $E:\mathbb{Z}^t\to\mathrm{Div}(C)$ such that

$$N\cong\mathcal{O}_C(E(\underline{\tau})). \tag{16}$$

Now, for any pre-enriched structure $E:\mathbb{Z}^t\to\mathrm{Div}(C)$, Condition (15) holds with $E(\underline{\tau})$ in place of D. So, by Statement 3 of Proposition 4.1, given a pre-enriched structure $E:\mathbb{Z}^t\to\mathrm{Div}(C)$, Condition (16) holds if and only if there is a $c\in(k^*)^t$ such that

$$E(\underline{\tau}) = c\cdot D. \tag{17}$$

For each subset I of $\{1,\dots,n\}$, denote by $k_I^*\subseteq(k^*)^t$ the subgroup of t-uples $c=(c_1,\dots,c_t)$ such that $c_j=1$ for all $j\notin I$. We will view k_I^* acting on $\mathrm{Div}(C_I)$ in the natural way. Again by Statement 3 of Proposition 4.1, Condition (13) is equivalent to the existence of a $c_I\in k_I^*$ for each $I\in\mathcal{P}$ such that

$$\sum_{\substack{i\in I\\ j\notin I}}(\tau_j-\tau_i)\Delta_{i,j} = c_I\cdot D|_{C_I}. \tag{18}$$

More concretely, since (15) holds, there is a unique $\alpha\in k_\Delta^*$ such that $\alpha_p u_p^{\tau_p}+v_p^{-\tau_p}$ is a local equation of D at p for each $p\in\Delta$. Then, there are a $c\in(k^*)^t$ and a pre-enriched structure $E:\mathbb{Z}^t\to\mathrm{Div}(C)$ such that (17) holds if and only if there are $a\in k_\Delta^*$ and a $c\in(k^*)^t$ such that

$$a_p^{\tau_p} = (c_{i_p}/c_{j_p})\alpha_p \text{ for each } p\in\Delta. \tag{19}$$

In addition, for each $I\in\mathcal{P}$ and $c_I\in k_I^*$, Condition (18) holds if and only if

$$(c_{I,i_p}/c_{I,j_p})\alpha_p = 1 \text{ for each reducible node } p \text{ of } C_I. \tag{20}$$

Now, if (19) holds for certain $a\in k_\Delta^*$ and $c\in(k^*)^t$, then

$$(c_{i_p}/c_{j_p})\alpha_p = 1 \text{ for each reducible node } p \text{ of } C_I, \text{ for each } I\in\mathcal{P}. \tag{21}$$

Conversely, suppose (21) holds for a certain $c \in (k^*)^t$. Then (19) holds for the same c and a certain $a \in k^*_\Delta$. Indeed, choose $a_p = 1$ if p is a reducible node of C_I for $I \in \mathcal{P}$. Otherwise, $\underline{\tau}_p \neq 0$, and we can freely choose $a_p \in k^*$ such that $a_p^{\underline{\tau}_p} = (c_{i_p}/c_{j_p})\alpha_p$.

Finally, suppose there is $c \in (k^*)^t$ such that (21) holds. Write $c = \prod_{I\in\mathcal{P}} c_I$ for $c_I \in k^*_I$. Then (20) holds. Conversely, if there are $c_I \in k^*_I$ for all $I \in \mathcal{P}$ such that (20) holds for each $I \in \mathcal{P}$, then (21) holds for $c := \prod_{I\in\mathcal{P}} c_I$.

So we have just seen that (13) is equivalent to the existence of an enriched structure L on C such that $N \cong L(\underline{\tau})$. Now, apply Theorem 4.2. $\square$

5. General invertible sheaves

Lemma 5.1. *Let X be a projective, reduced curve defined over an algebraically closed field k. Let $D \subset X$ be a collection of nodes and $\pi : \widetilde{X} \to X$ the partial normalization of X along D. Let $\widetilde{L}$ be an invertible sheaf on $\widetilde{X}$ and L the general invertible sheaf on X such that $\pi^* L \cong \widetilde{L}$. If, for each $p \in D$, there is $s \in H^0(X, L)$ such that $s(p) \neq 0$, then*

$$h^0(X, L) = h^0(\widetilde{X}, \widetilde{L}) - \#D.$$

Proof. We claim first that the natural bilinear map,

$$j : H^0(X, L) \times T \to H^1(X, L),$$

is zero, where

$$T := \ker(H^1(X, \mathcal{O}_X) \to H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})).$$

In fact (cf. [**?**]), let $\Phi : \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$ denote the pullback map and put $S := \Phi^{-1}(\widetilde{L})$. Then $L \in S$. We can identify T as the tangent space to S at L, which is also the set $M := \mathrm{Mor}((B, 0), (S, L))$ of pointed maps, where $B := \mathrm{Spec}(k[t]/(t^2))$ and 0 is the unique point of B.

Let $\mathcal{U}$ be the universal invertible sheaf on $X \times S$. Let $v \in M$. So we have the natural exact sequence

$$0 \to L \to v^*\mathcal{U} \to L \to 0,$$

where $v^*\mathcal{U}$ is the pullback of $\mathcal{U}$ to $X \times B$ under $v : B \to S$. The restriction of j to $H^0(X, L) \times \{v\}$ can be identified with the connecting map $\partial_v : H^0(X, L) \to H^1(X, L)$ in the associated long exact sequence in cohomology. Now, there is an open dense subset $S_0 \subseteq S$ such that $h^0(X, N)$ is constant for $N \in S_0$. Since S_0 is reduced, for each $N \in S_0$, and each affine open neighborhood A of N in S_0, the restriction map $H^0(X \times A, \mathcal{U}|_{X\times A}) \to H^0(X, N)$ is surjective. Now, since L is general, $L \in S_0$. The map $v : B \to S$ factors through an affine open neighborhood of L in S_0. So the restriction map

$$H^0(X \times B, v^*\mathcal{U}) \to H^0(X, L)$$

is surjective, and hence the connecting homomorphism ∂_v is zero. Since we have chosen v arbitrary, j is zero.

By Serre duality, the vanishing of j implies that the natural bilinear map,

$$H^0(X, L) \times \mathrm{Hom}(L, \omega) \to H^0(X, \omega),$$

factors through $H^0(X, \pi_*\widetilde{\omega})$, where ω is the dualizing sheaf on X, and $\widetilde{\omega}$ that on $\widetilde{X}$.

Let $\phi \in \mathrm{Hom}(L, \omega)$. Then $\phi(s)$ is a section of $\pi_*\widetilde{\omega}$ for each $s \in H^0(X, L)$. Equivalently, $\phi(s)(p) = 0$ for each $p \in D$, since D is a collection of nodes of X, and $\widetilde{X}$ is the partial normalization of X along D. Now, by hypothesis, for each $p \in D$ there is $s \in H^0(X, L)$ such that $s(p) \neq 0$. Then $\phi(p) = 0$ for each $p \in D$ or, equivalently, ϕ factors through $\pi_*\widetilde{\omega}$. Since ϕ was arbitrary,

$$\mathrm{Hom}(L, \omega) = \mathrm{Hom}(L, \pi_*\widetilde{\omega}) = \mathrm{Hom}(\pi^*L, \widetilde{\omega}).$$

By Serre duality,

$$H^1(X, L) = H^1(X, \pi_*\pi^*L) = H^1(\widetilde{X}, \pi^*L).$$

Taking cohomology on the natural exact sequence,

$$0 \to L \to \pi_*\pi^*L \to \bigoplus_{p \in D} k(p) \to 0,$$

we get

$$h^0(X, L) = h^0(X, \pi_*\pi^*L) - \#D = h^0(\widetilde{X}, \widetilde{L}) - \#D,$$

as we wished to show. □

Example. The generality of L is a necessary condition for Lemma 5.1. Indeed, if X is an irreducible curve with a node p, and $\widetilde{X}$ is the partial normalization of X at p, then $h^0(X, \mathcal{O}_X) = h^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 1$.

6. **Limit β-canonical systems: conclusion**

Recall the notation in Sections 1, 2 and 3. From now on $\underline{n}$ will stand for the unique t-uple mentioned in the statement of Lemma 3.1. Wherever we assume that *the components of C intersect at points in general position*, it is to be understood that, for each effective divisor $D=\sum_{i<j}D_{i,j}$, with $0\leq D_{i,j}\leq \Delta_{i,j}$ for all i and j, and each connected $\underline{n}$-balanced subset $I\subseteq\{1,\dots,t\}$,

$$h^0\big(C_I, K_I^\beta\big(\sum_{i\in I;\, j\notin I}((\beta+n_j-n_i)\Delta_{i,j}-D_{i,j})\big)\big)$$

is the minimum possible, given the components $C_1,\dots,C_t$ and the numerical invariants $\delta_{i,j}$.

More precisely, for each distinct $i,j\in\{1,\dots,t\}$ let Ξ^i_j be an ordered subset of $\delta_{i,j}$ simple points of C_i. Choose the Ξ^i_j in such a way that $\Xi^i_j\cap\Xi^i_\ell=\emptyset$ for all distinct $i,j,\ell\in\{1,\dots,t\}$. Let $\widetilde{C}$ be the nodal curve obtained as the union of $C_1,\dots,C_t$ with the ordered identification of Ξ^i_j and Ξ^j_i for all distinct i and j. Call $\Xi_{i,j}$ the reduced Weil divisor of $\widetilde{C}$ with support $C_i\cap C_j$. For each nonempty $I\subseteq\{1,\dots,t\}$, let $\widetilde{C}_I\subseteq\widetilde{C}$ be the union of the components C_i for $i\in I$, and let $\widetilde{K}^\beta_I$ denote the β-canonical sheaf of $\widetilde{C}_I$.

The only nondiscrete invariants of $\widetilde{C}$ are $C_1,\dots,C_t$ and the points in Ξ^i_j for all i and j. Let the Ξ^i_j vary. Had we replaced C by $\widetilde{C}$, the t-uple $\underline{n}$ mentioned in the statement of Lemma 3.1 would not change.

We say that *the components of C intersect at points in general position* if, for each effective divisor $D=\sum_{i<j}D_{i,j}$, with $0\leq D_{i,j}\leq\Delta_{i,j}$ for all i and j, and each connected $\underline{n}$-balanced subset $I\subseteq\{1,\dots,t\}$,

$$h^0\big(C_I,K_I^\beta\big(\sum_{\substack{i\in I\\ j\notin I}}(\beta+n_j-n_i)\Delta_{i,j}-D_{i,j}\big)\big)\leq h^0\big(\widetilde{C}_I,\widetilde{K}_I^\beta\big(\sum_{\substack{i\in I\\ j\notin I}}(\beta+n_j-n_i)\Xi_{i,j}-E_{i,j}\big)\big)$$

for all possible choices of Ξ^i_j and of effective divisors $E_{i,j}$ such that $E_{i,j}\leq\Xi_{i,j}$ and $\deg E_{i,j}=\deg D_{i,j}$ for all i and j.

By semicontinuity, if the Ξ^i_j are chosen generically, also the components of $\widetilde{C}$ intersect at points in general position. So the condition is general.

Let $I\subseteq\{1,\dots,t\}$ be a connected $\underline{n}$-balanced subset. Since C_I is connected, $h^0(K_I)=g_I$ and $h^1(K_I)=1$. In addition, $h^0(K_I(E))=g_I+\deg E-1$ for each effective nonzero divisor E of C_I supported in the nonsingular locus.

If $\beta>1$ and $g_i\geq 2$ for each $i=1,\dots,t$, then $h^0(K_I^\beta)=(2\beta-1)(g_I-1)$ and $h^1(K_I^\beta)=0$. Moreover, $h^0(K_I^\beta(E))=(2\beta-1)(g_I-1)+\deg E$ for each effective divisor E of C_I supported in the nonsingular locus.

Lemma 6.1. *Assume that the components of C intersect at points in general position. Let $\mathcal{X}/S$ be a smoothing of C along the general direction. For $\beta=1$, assume $\delta_{i,j}\neq 0$*

for all i and j; otherwise assume $g_i \geq 2$ for all i. For each i and j, let $D_{i,j} \leq \Delta_{i,j}$ be an effective divisor. Let $\underline{n}$ be the unique t-uple of integers mentioned in Lemma 3.1*. Let $I \subset \{1, \dots, t\}$ be a proper nonempty subset, and put*

$$L := L_I^{\underline{n}}(-\sum_{\substack{i \in I \\ j \notin I}} D_{i,j}).$$

If, for each $i \in I$, the restriction map $H^0(C_I, L) \to H^0(C_i, L|_{C_i})$ is nonzero, then

$$\begin{cases} h^1(C_I, L) = 0 & \textit{for } \beta > 1, \\ h^1(C_I, L) \leq \gamma_I^{\underline{n}+\underline{h}^I} & \textit{for } \beta = 1. \end{cases}$$

Proof. Note that the smoothing $\mathcal{X}/S$ of C is regular because it is taken along the general direction. We will first treat the special case where I is $\underline{n}$-balanced and connected. In this case, we need only that the smoothing be regular.

In fact, we have

$$L_I^{\underline{n}} \cong K_I^{\beta}\big(\sum_{\substack{i \in I \\ j \notin I}} (\beta + n_j - n_i)\Delta_{i,j}\big).$$

Let P and N be effective divisors of C_I with disjoint supports such that

$$\sum_{\substack{i \in I \\ j \notin I}} (\beta + n_j - n_i)\Delta_{i,j} - \sum_{\substack{i \in I \\ j \notin I}} D_{i,j} = P - N. \tag{22}$$

Then $L \cong K_I^{\beta}(P - N)$.

Clearly,

$$h^1(C_I, K_I^{\beta}(P)) \leq h^1(C_I, K_I^{\beta}).$$

If $\beta > 1$ then $h^1(C_I, K_I^{\beta}) = 0$, and thus also $h^1(C_I, K_I^{\beta}(P)) = 0$. Now, suppose for the moment that $\beta = 1$. Then $h^1(C_I, K_I^{\beta}) = 1$, and thus $h^1(C_I, K_I^{\beta}(P)) \leq 1$ with equality only if $P = 0$. However, from the expression (22), and the hypothesis that $\Delta_{i,j} \neq 0$ for all i and j, we see that $P = 0$ only if $n_j \leq n_i$ for all $i \in I$ and $j \notin I$, and hence only if $\gamma_I^{\underline{n}+\underline{h}^I} = 1$. Thus

$$\begin{cases} h^1(C_I, K_I^{\beta}(P)) = 0 & \text{for } \beta > 1, \\ h^1(C_I, K_I^{\beta}(P)) \leq \gamma_I^{\underline{n}+\underline{h}^I} & \text{for } \beta = 1. \end{cases}$$

It is thus enough to show that

$$h^1(C_I, L) = h^1(C_I, K_I^{\beta}(P)). \tag{23}$$

By Riemann–Roch, (23) is equivalent to $h^0(C_I, L) = h^0(C_I, K_I^{\beta}(P)) - \deg N$.

Write $|N| = \{p_1, \dots, p_s\}$, so $N = \sum_\ell w_\ell p_\ell$ for $w_\ell > 0$. For each $\ell = 1, \dots, s$, let $i_\ell \in I$ such that $p_\ell \in C_{i_\ell}$, and pick a general point $q_\ell \in C_{i_\ell}$ (contained in the nonsingular locus of C_I). Let $\widetilde{N} = \sum_\ell w_\ell q_\ell$ and $\widetilde{L} := K_I^{\beta}(P - \widetilde{N})$.

Since the irreducible components of C intersect at points in general position, $h^0(C_I, L) \leq h^0(C_I, \widetilde{L})$. By semicontinuity, the reverse inequality holds. Thus

$$h^0(C_I, \widetilde{L}) = h^0(C_I, L). \tag{24}$$

For each $i \in I$, since $I \setminus \{i\}$ is $\underline{n}$-balanced, by the same reason,

$$h^0(C_{I\setminus\{i\}}, \widetilde{L}|_{C_{I\setminus\{i\}}}(-\Delta_{I\setminus\{i\},i})) = h^0(C_{I\setminus\{i\}}, L|_{C_{I\setminus\{i\}}}(-\Delta_{I\setminus\{i\},i})).$$

By hypothesis, the restriction maps $H^0(C_I, L) \to H^0(C_i, L|_{C_i})$ are nonzero for each $i \in I$. Equivalently,

$$h^0(C_{I\setminus\{i\}}, L|_{C_{I\setminus\{i\}}}(-\Delta_{I\setminus\{i\},i})) < h^0(C_I, L).$$

Thus, also the restriction maps

$$H^0(C_I, \widetilde{L}) \to H^0(C_i, \widetilde{L}|_{C_i}) \text{ are nonzero for each } i \in I. \tag{25}$$

So, there is an open dense subset $U \subseteq C_I$ contained in the nonsingular locus of C_I such that, for $q_\ell \in U \cap C_{i_\ell}$ for $\ell = 1, \dots, s$, both (24) and (25) hold for $\widetilde{L} := K_I^\beta(P - \widetilde{N})$, where $\widetilde{N} = \sum_\ell w_\ell q_\ell$. It is now enough to show that $h^0(C_I, \widetilde{L}) = h^0(C_I, K_I^\beta(P)) - \deg \widetilde{N}$ for one such $\widetilde{N}$.

Let $b \leq s$ be a positive integer. Suppose we have already chosen $q_\ell \in U \cap C_{i_\ell}$ for all $\ell < b$ such that

$$h^0(C_I, K_I^\beta(P - E)) = h^0(C_I, K_I^\beta(P)) - \deg E,$$

where $E := \sum_{\ell<b} w_\ell q_\ell$. It is enough to show that there is $q_b \in U \cap C_{i_b}$ such that

$$h^0(C_I, K_I^\beta(P - E - w_b q_b)) = h^0(C_I, K_I^\beta(P - E)) - w_b. \tag{26}$$

Choose $q_b \in U \cap C_{i_b}$ such that q_b is not a ramification point of the linear system of sections of $K_I^\beta(P - E)|_{C_{i_b}}$ spanned by $H^0(C_I, K_I^\beta(P - E))$. Now, if $r_\ell \in U \cap C_{i_\ell}$ for $\ell = b+1, \dots, s$, and $F := \sum_{\ell>b} w_\ell r_\ell$, then (25) holds for $\widetilde{L} := K_I^\beta(P - E - w_b q_b - F)$. In particular, the restriction map

$$H^0(C_I, K_I^\beta(P - E - w_b q_b)) \to H^0(C_{i_b}, K_I^\beta(P - E - w_b q_b)|_{C_{i_b}})$$

is nonzero. Thus (26) holds.

For the general case, let $I_1, \dots, I_p$ be the $\underline{n}$-components of I. For each distinct $i, j \in \{1, \dots, p\}$, let $E_{i,j} \leq \Delta_{I_i, I_j}$ be the effective divisor supported on the set of nodes where all sections in $H^0(C_I, L)$ vanish. Let $e_{i,j} := \deg E_{i,j}$ for each i and j. Let $E_i := \sum_{j \neq i} E_{i,j}$ and $e_i := \deg E_i$ for each $i = 1, \dots, p$. Let $E := \sum_{i<j} E_{i,j}$ and $e := \deg E$. Note that $2e = e_1 + \cdots + e_p$.

Let $\psi : X \to C_I$ be the partial normalization of C_I along $|E|$. Let $M := \psi^* L(-F)$ where F is the reduced divisor of X with support $\psi^{-1}(|E|)$. Let $\widetilde{X}$ be the disjoint union of the C_{I_i} for $i = 1, \dots, p$, and $\pi : \widetilde{X} \to X$ the natural map.

Since the smoothing occurs along a general direction, Proposition 4.3 tells us that L is the general invertible sheaf on C_I such that

$$L|_{C_{I_i}} \cong K_{I_i}^{\beta}\Big(\sum_{\substack{j\in I_i\\ \ell\notin I_i}}(\beta+n_\ell-n_j)\Delta_{j,\ell}-\sum_{\substack{j\in I_i\\ \ell\notin I}}D_{j,\ell}\Big)$$

for each $i=1,\dots,p$. Since the pullback $\psi^*:\mathrm{Pic}(C_I)\to\mathrm{Pic}(X)$ is surjective, it follows that M is the general invertible sheaf on X such that

$$\pi^*M \cong \bigoplus_{i=1}^{p} K_{I_i}^{\beta}\Big(\sum_{\substack{j\in I_i\\ \ell\notin I_i}}(\beta+n_\ell-n_j)\Delta_{j,\ell}-\sum_{\substack{j\in I_i\\ \ell\notin I}}D_{j,\ell}-E_i\Big)$$

By the projection formula,

$$H^0(C_I,L)=H^0(X,M).$$

Thus, for each $p\in X$ with $\psi(p)\in\bigcup_{j\neq i}|\Delta_{I_i,I_j}|\setminus|E|$, there is a section of $H^0(X,M)$ not vanishing at p. By Lemma 5.1,

$$h^0(C_I,L)=\sum_{i=1}^{p}h^0(C_{I_i},L|_{C_{I_i}}(-E_i))-\sum_{i<j}\delta_{I_i,I_j}+e. \tag{27}$$

Suppose first that $\beta>1$. Applying the statement of the lemma to $L|_{C_{I_i}}(-E_i)$, which falls in the special case proved first, we get

$$h^1(C_{I_i},L|_{C_{I_i}})=h^1(C_{I_i},L|_{C_{I_i}}(-E_i))=0;$$

so, by Riemann–Roch,

$$h^0(C_{I_i},L|_{C_{I_i}}(-E_i))=h^0(C_{I_i},L|_{C_{I_i}})-e_i$$

for each $i=1,\dots,p$. Using (27), we get

$$h^0(C_I,L)=\sum_{i=1}^{p}h^0(C_{I_i},L|_{C_{I_i}})-\sum_{i<j}\delta_{I_i,I_j}-e.$$

However, it follows from the long exact sequence in cohomology associated to the short exact sequence

$$0\to L\to\bigoplus_{i=1}^{p}L|_{C_{I_i}}\to\bigoplus_{i<j}\ \bigoplus_{p\in\Delta_{I_i,I_j}}k(p)\to 0$$

that

$$h^0(C_I,L)\geq\sum_{i=1}^{p}h^0(C_{I_i},L|_{C_{I_i}})-\sum_{i<j}\delta_{I_i,I_j},$$

with equality if and only if

$$h^1(C_I, L) = \sum_{i=1}^{p} h^1(C_{I_i}, L|_{C_{I_i}}).$$

So equality holds, and hence $h^1(C_I, L) = 0$, as we wished to show.

Now, suppose $\beta = 1$. Again, apply the statement of the lemma to $L|_{C_{I_i}}(-E_i)$, this time to get

$$h^1(C_{I_i}, L|_{C_{I_i}}(-E_i)) \leq \gamma_{I_i}^{\underline{n}+\underline{h}^{I_i}};$$

so, by Riemann–Roch,

$$h^0(C_{I_i}, L|_{C_{I_i}}(-E_i)) \leq h^0(C_{I_i}, L|_{C_{I_i}}) - e_i + \gamma_{I_i}^{\underline{n}+\underline{h}^{I_i}}$$

for each $i = 1, \dots, p$. From (27) we obtain

$$h^0(C_I, L) \leq \sum_{i=1}^{p} h^0(C_{I_i}, L|_{C_{I_i}}) - \sum_{i<j} \delta_{I_i, I_j} - e + \sum_{i=1}^{p} \gamma_{I_i}^{\underline{n}+\underline{h}^{I_i}}. \tag{28}$$

Observe now that

$$\sum_{i=1}^{p} \gamma_{I_i}^{\underline{n}+\underline{h}^{I_i}} = \gamma_I^{\underline{n}+\underline{h}^I}. \tag{29}$$

Indeed, since each two components of C_I do intersect, the $\underline{n}$-weights of the I_i are different. So, there is $j \in \{1, \dots, p\}$ such that $m_{I_j}^{\underline{n}} > m_{I_i}^{\underline{n}}$ for each $i \neq j$. Hence $\gamma_{I_i}^{\underline{n}+\underline{h}^{I_i}} = 0$ for each $i \neq j$. In addition, $\gamma_{I_j}^{\underline{n}+\underline{h}^{I_j}} = 0$ if and only if $m_{I^c}^{\underline{n}} \geq m_{I_j}^{\underline{n}} + 1$. Now,

$$m_I^{\underline{n}} = \max(m_{I_1}^{\underline{n}}, \dots, m_{I_p}^{\underline{n}}) = m_{I_j}^{\underline{n}},$$

and $\gamma_I^{\underline{n}+\underline{h}^I} = 0$ if and only if $m_{I^c}^{\underline{n}} \geq m_I^{\underline{n}} + 1$. So

$$\gamma_I^{\underline{n}+\underline{h}^I} = \gamma_{I_j}^{\underline{n}+\underline{h}^{I_j}} = \sum_{i=1}^{p} \gamma_{I_i}^{\underline{n}+\underline{h}^{I_i}},$$

as we wished to show.

Combining (29) with (28), we get

$$h^0(C_I, L) \leq \sum_{i=1}^{p} h^0(C_{I_i}, L|_{C_{I_i}}) - \sum_{i<j} \delta_{I_i, I_j} - e + \gamma_I^{\underline{n}+\underline{h}^I}.$$

Since $\gamma_I^{\underline{n}+\underline{h}^I} \leq 1$, if $e > 0$ we get

$$h^0(C_I, L) \leq \sum_{i=1}^{p} h^0(C_{I_i}, L|_{C_{I_i}}) - \sum_{i<j} \delta_{I_i, I_j}.$$

The same inequality can be obtained directly from (27) if $e = 0$. So, as in the case $\beta > 1$, we get

$$h^1(C_I, L) = \sum_{i=1}^{p} h^1(C_{I_i}, L|_{C_{I_i}}).$$

But

$$\sum_{i=1}^{p} h^1(C_{I_i}, L|_{C_{I_i}}) \leq \sum_{i=1}^{p} h^1(C_{I_i}, L|_{C_{I_i}}(-E_i)) \leq \sum_{i=1}^{p} \gamma_{I_i}^{\underline{n}+\underline{h}^{I_i}} = \gamma_I^{\underline{n}+\underline{h}^I},$$

finishing the proof. ☐

For each β define,

$$\kappa_\beta := \begin{cases} g, \text{ if } \beta = 1, \\ (2\beta-1)(g-1), \text{ if } \beta > 1. \end{cases}$$

Proposition 6.2. *Assume that the components of C intersect at points in general position. Let $\mathcal{X}/S$ be a smoothing of C along the general direction. For $\beta = 1$, assume $\delta_{i,j} \neq 0$ for all i and j; otherwise assume $g_i \geq 2$ for all i. Let $\underline{n}$ be the unique t-uple of integers mentioned in* Lemma 3.1. *Then*

(1) $h^0(C_{\{2,\dots,t\}}, L^{\underline{n}}_{\{2,\dots,t\}}(-\Delta_1)) = 0.$
(2) $h^0(C_I, L^{\underline{n}}_I(-\Delta_i)) < \kappa_\beta$ *for each* $i = 2, \dots, t$, *where* $I := \{1, \dots, t\} \setminus \{i\}$.
(3) $h^0(C, L^{\underline{n}}) = \kappa_\beta$.

In particular, $(H^0(C, L^{\underline{n}}), L^{\underline{n}})$ *is the limit β-canonical system associated to C_1.*

Proof. We shall prove the first two statements simultaneously. Let $T := \{1, \dots, t\}$. Fix $l \in T$ and let $T' := T \setminus \{l\}$. Let $I \subseteq T'$ be a minimal subset such that

$$H^0(C_I, L^{\underline{n}}_I(-\Delta_{I,J})) = H^0(C_{T'}, L^{\underline{n}}_{T'}(-\Delta_l)),$$

where $J := T \setminus I$. We need only show that $h^0(C_I, L^{\underline{n}}_I(-\Delta_{I,J})) = 0$ if $1 \notin I$, and that $h^0(C_I, L^{\underline{n}}_I(-\Delta_{I,J})) < \kappa_\beta$ if $1 \in I$.

If $I = \emptyset$, we are done. Suppose I is nonempty. By the minimality of I, the restriction map

$$H^0(C_I, L^{\underline{n}}_I(-\Delta_{I,J})) \to H^0(C_i, L^{\underline{n}}_i(-\Delta_{i,J}))$$

is nonzero for each $i \in I$. So, Lemma 6.1 yields

$$\begin{cases} h^1(C_I, L^{\underline{n}}_I(-\Delta_{I,J})) = 0 & \text{for } \beta > 1, \\ h^1(C_I, L^{\underline{n}}_I(-\Delta_{I,J})) \leq \gamma_I^{\underline{n}+\underline{h}^I} & \text{for } \beta = 1. \end{cases}$$

Let $\rho := (2\beta-1)(g-1)$ if $1 \in I$, and $\rho := 0$ otherwise. Using Riemann–Roch, we get

$$h^0(C_I, L^{\underline{n}}_I(-\Delta_{I,J})) \leq \epsilon^{\underline{n}}_I - \delta_{I,J} + \rho + \gamma_I^{\underline{n}+\underline{h}^I}$$

for $\beta = 1$, whereas

$$h^0(C_I, L^{\underline{n}}_I(-\Delta_{I,J})) = \epsilon^{\underline{n}}_I - \delta_{I,J} + \rho$$

for $\beta > 1$. So, by Lemma 3.1,

$$h^0(C_I, L_I^{\underline{n}}(-\Delta_{I,J})) \leq \rho$$

for $\beta = 1$, while

$$h^0(C_I, L_I^{\underline{n}}(-\Delta_{I,J})) \leq \rho - \gamma_I^{\beta,\underline{n}+\underline{h}^I}.$$

for $\beta > 1$. Thus $h^0(C_I, L_I^{\underline{n}}(-\Delta_{I,J})) = 0$ if $1 \not\in I$, whereas $h^0(C_I, L_I^{\underline{n}}(-\Delta_{I,J})) < \kappa_\beta$ if $1 \in I$, as we wished to show.

The first two statements imply that $(V^{\underline{n}}, L^{\underline{n}})$ is the limit β-canonical system associated to C_1. Now, $\dim V^{\underline{n}} = \kappa_\beta$. So, for the third statement, it will be enough to show that $h^0(C, L^{\underline{n}}) \leq \kappa_\beta$. By Riemann–Roch, it is enough to show that $h^1(C, L^{\underline{n}}) \leq 1$ for $\beta = 1$, and $h^1(C, L^{\underline{n}}) = 0$ for $\beta > 1$.

Taking cohomology on the natural exact sequence

$$0 \to L_1^{\underline{n}}(-\Delta_1) \to L^{\underline{n}} \to L_{\{2,\dots,t\}}^{\underline{n}} \to 0,$$

we get

$$h^1(C, L^{\underline{n}}) \leq h^1(C_1, L_1^{\underline{n}}(-\Delta_1)) + h^1(C_{\{2,\dots,t\}}, L_{\{2,\dots,t\}}^{\underline{n}}).$$

Now, since $(V^{\underline{n}}, L^{\underline{n}})$ is the limit β-canonical system associated to C_1, the restriction maps

$$H^0(C_{\{2,\dots,t\}}, L_{\{2,\dots,t\}}^{\underline{n}}) \to H^0(C_i, L_i^{\underline{n}})$$

are nonzero for all $i \geq 2$. So, Lemma 6.1 yields

$$h^1(C_{\{2,\dots,t\}}, L_{\{2,\dots,t\}}^{\underline{n}}) \leq 1,$$

with equality only if $\beta = 1$.

Thus, we need only show that $h^1(C_1, L_1^{\underline{n}}(-\Delta_1)) = 0$. Note that, since $\kappa_\beta > \delta_1$ and $\kappa_\beta \leq h^0(C_1, L_1^{\underline{n}})$, we have $h^0(C_1, L_1^{\underline{n}}(-\Delta_1)) \neq 0$. So, Lemma 6.1 yields either $h^1(C_1, L_1^{\underline{n}}(-\Delta_1)) = 0$, for $\beta > 1$, or $h^1(C_1, L_1^{\underline{n}}(-\Delta_1)) \leq \gamma_1^{\underline{n}+\underline{h}^1}$, for $\beta = 1$.

The case $\beta > 1$ is finished. So, suppose $\beta = 1$. We need only show that $\gamma_1^{\underline{n}+\underline{h}^1} = 0$. By contradiction, suppose $\gamma_1^{\underline{n}+\underline{h}^1} = 1$. Since $n_1 = 0$, we have $n_i \leq 0$ for each $i \geq 2$. So $\epsilon_1^{\underline{n}} \leq e_1$. In addition, since each two components of C do intersect, equality $\epsilon_1^{\underline{n}} = e_1$ holds only if $\underline{n} = 0$. However, $\epsilon_1^{\underline{n}} \geq 0$ by Lemma 3.1. Since $e_1 = (g_1 - g) + \delta_1$, we have that $(g - g_1) \leq \delta_1$, with equality only if $\underline{n} = 0$ and $\epsilon_1^{\underline{n}} = 0$.

Since each two components of C do intersect, $(g - g_1) \leq \delta_1$ yields $t = 2$ and $g_2 = 1$. In particular, equality $(g - g_1) = \delta_1$ holds, and thus $\underline{n} = 0$ and $\epsilon_1^{\underline{n}} = 0$. However, since $\underline{n} = 0$, Lemma 3.1 yields $\epsilon_1^{\underline{n}} \geq 1$, a contradiction. $\square$

Now we prove our principal result.

Theorem. *Assume that the components of C intersect at points in general position. For $\beta = 1$, assume $\Delta_{i,j} \neq \emptyset$ for all i and j; otherwise assume $g_i \geq 2$ for each i. Let $\underline{n} = (n_1, \dots, n_t)$ be the unique t-uple of integers given by* Lemma 3.1. *Then the complete system of sections of the general invertible sheaf L on C satisfying*

$$L|_{C_I} \cong K_I^{\beta}\Big(\sum_{\substack{i\in I\\ j\notin I}}(\beta + n_j - n_i)\Delta_{i,j}\Big) \text{ for each } \underline{n}\text{-balanced } I \subseteq \{1, \dots, t\} \tag{30}$$

is a limit β-canonical system associated to C_1. Conversely, the limit β-canonical system associated to C_1 of a smoothing of C along the general direction is the complete system of sections of an invertible sheaf L on C satisfying (30)*.*

Proof. We prove the first statement. Let L be the general invertible sheaf on C satisfying (30). It follows from Proposition 4.3 that there is a regular smoothing $\mathcal{X}/S$ of C such that

$$L \cong \mathcal{K}^{\beta}(n_1C_1 + \cdots + n_tC_t)|_C,$$

where $\mathcal{K}^{\beta}$ is the relative β-canonical sheaf on $\mathcal{X}$ over S. Since L is general, the smoothing $\mathcal{X}/S$ occurs along the general direction. Applying Proposition 6.2, we conclude that $(H^0(C, L), L)$ is the limit β-canonical system associated to C_1 and $\mathcal{X}/S$. The converse statement follows directly from Proposition 6.2.

□

References

[EH86] D. Eisenbud and J. Harris. *Limit linear series: basic theory.* Inventiones Mathematicae **85** (1986), 337–371.

[EH87] D. Eisenbud and J. Harris. *Existence, decomposition and limits of certain Weierstrass points.* Inventiones Mathematicae **87** (1987), 495–515.

[Es96] E. Esteves. *Limit linear systems and ramification points on reducible nodal curves.* Matemática Contemporânea **16** (1996), 1–18.

[EM02] E. Esteves and N. Medeiros. *Limit canonical systems on curves with two components.* Inventiones Mathematicae **149** (2002), 267–338.

[S93] C. S. Seshadri. *Vector bundles on curves.* Contemporary Mathematics **153** (1993), 163–200.