A variational principle for dimension for a class of non-conformal repellers

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Tese de doutoramento

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À memória de
minha avó Matilde

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Introduction

It is an open problem to determine for which maps f a compact invariant set Λ carries an ergodic invariant measure of full Hausdorff dimension. This is known when f is conformal and expanding, and is a consequence of the thermodynamic formalism introduced by Ruelle and Bowen (see [B1], [B2] and [R]).

This kind of questions in the non-conformal setting were first considered by Bedford [Be] and McMullen [Mu]. They showed, independently, that, for the class of transformations called *general Sierpinski Carpets*, there exists an ergodic measure of full Hausdorff dimension. Moreover they showed that generically in that class the Hausdorff dimension and the box dimension are not equal, a phenomenon that does not occur in the conformal setting. Following [Mu], Gatzouras and Lalley [GL] generalized these results for certain self-affine transformations. Then, Gatzouras and Peres [GP] considered non-linear maps of the form $f(x, y) = (f_1(x), f_2(y))$, where f_1 and f_2 are conformal and expanding maps satisfying $\inf |Df_1| \geq \sup |Df_2|$, and showed that, for a large class of invariant sets Λ, there exist ergodic invariant measures with dimension arbitrarily close to the dimension of Λ. They used arguments of bounded distortion for approximating these non-linear maps by self-affine transformations.

In this work we further extend these results to a class of skew-product expanding maps of the 2-torus of the form $f(x, y) = (a(x, y), b(y))$ satisfying a domination condition – in rough terms, the y-direction is less expanding than the x-direction – and consider invariant sets which possess a simple Markov structure. Combining a mixture of techniques of [Be] and [Mu] with a bounded distortion argument, we show that there exist ergodic invariant measures with dimension arbitrarily close to the dimension of the invariant set. As a starting step, we use our methods to give a new treatment of the self-affine case considered in [GL].

As an application of these results we are able to show that certain exceptional sets that appear in Ergodic Theory, e.g. in connection with the Ergodic Theorem, have full Hausdorff dimension. The latter kind of problem had been treated, e.g. in [BS], in the conformal setting.

1. Main results

We begin by describing what we mean by a *general Sierpinski carpet*. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus and $f_0 \colon \mathbb{T}^2 \to \mathbb{T}^2$ be given by $f_0(x, y) = (lx, my)$ where $l > m > 1$ are integers. The grid of lines $[0, 1] \times \{i/m\}$, $i = 0, ..., m - 1$, and $\{j/l\} \times [0, 1]$, $j = 0, ..., l - 1$, form a set of rectangles each of which is mapped by f_0 onto the entire torus (these rectangles are the domains of invertibility of f_0). Now choose some of these rectangles and consider the fractal set Λ_0 consisting of those points that always remain in these chosen rectangles when iterating f_0. Geometrically, Λ_0 is the limit (in the Hausdorff metric) of *n-approximations*: the 1-approximation consists of the chosen rectangles, the 2-approximation consists in dividing each rectangle of the 1-aproximation into $l \times m$ subrectangles and selecting those with the same pattern as in the begining, and so on (see Figure 1). We say that (f_0, Λ_0) is a *general Sierpinski carpet.*

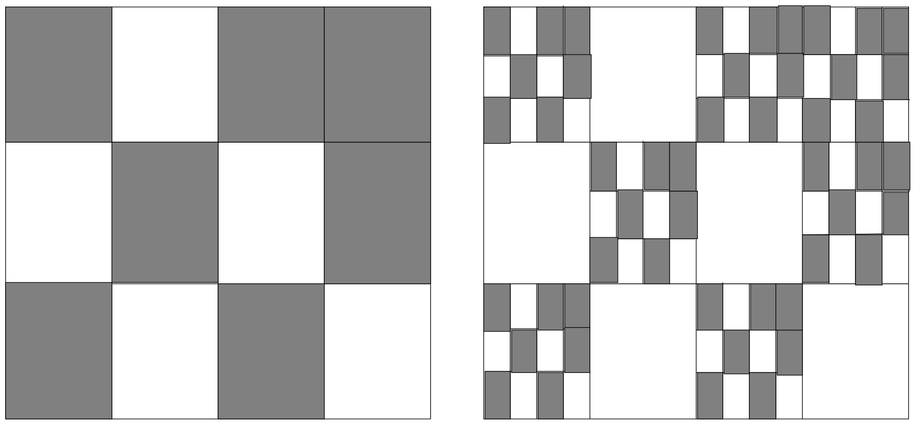

FIGURE 1. $l = 4$, $m = 3$; 1-approximation and 2-approximation

DEFINITION 1. Let (f_0, Λ_0) be a general Sierpinski carpet. There exists $\varepsilon > 0$ such that if f is $\varepsilon-\mathrm{C}^1$ close to f_0 then there is a unique homeomorphism $h\colon \mathbb{T}^2 \to \mathbb{T}^2$ close to the identity which conjugates f and f_0, i.e, $f \circ h = h \circ f_0$ (see [S]). The f-invariant set $\Lambda = h(\Lambda_0)$ is called the *f-continuation* of Λ_0.

DEFINITION 2. Let $\mathcal{S}$ be the class of C^2 maps $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ of the form

$$f(x, y) = (a(x, y), b(y)).$$

Notation: $\dim_{\mathrm{H}} \Lambda$ and $\dim_{\mathrm{H}} \mu$ stand for the Hausdorff dimension of a set Λ and a measure μ, respectively.

THEOREM A. *Let (f_0, Λ_0) be a general Sierpinski carpet. There exists $\varepsilon > 0$ such that if $f \in \mathcal{S}$ is $\varepsilon - \mathrm{C}^2$ close to f_0, and Λ is the f-continuation of Λ_0, then*

$$\dim_{\mathrm{H}} \Lambda = \sup\{\dim_{\mathrm{H}} \mu : \mu(\Lambda) = 1,\ \mu \text{ is } f\text{-invariant and ergodic}\}.$$

We also give a description of the maximizing measures in terms of equilibrium measures for the relativised variational principle. By the proof of Theorem A we obtain the following

COROLLARY A. *Let (f_0, Λ_0) be a general Sierpinski carpet and $\varepsilon > 0$ be given by Theorem A. Then*

$$B_{\mathrm{C}^2}(f_0, \varepsilon) \cap \mathcal{S} \ni f \mapsto \dim_{\mathrm{H}} \Lambda_f$$

is a continuous function. Here $B_{\mathrm{C}^2}(f_0, \varepsilon)$ is the ball in the C^2 topology, centered at f_0 of radius ε, and Λ_f is the f-continuation of Λ_0.

Let M be a compact metric space and $f\colon M \to M$ a continuous map. Denote by $\mathrm{C}(M)$ the space of all continuous functions $\varphi\colon M \to \mathbb{R}$. We define

$$\mathcal{B}_f = \left\{ x \in M : \frac{1}{n} \sum_{i=1}^{n-1} \varphi(f^i(x)) \text{ converges, for all } \varphi \in \mathrm{C}(M) \right\}.$$

Then (see [M]),

Birkhoff's Ergodic Theorem: $\mu(\mathcal{B}_f) = 1$ for any f-invariant Borel probability measure μ on M.

We define the *Birkhoff exceptional set* by $\mathcal{E}_f = M \backslash \mathcal{B}_f$. Then

THEOREM B. *With the same hypothesis of Theorem A,*

$$\dim_{\mathrm{H}} \mathcal{E}_{f|\Lambda} = \dim_{\mathrm{H}} \Lambda.$$

2. Preliminaries

Let us mention some basic results about fractal geometry and pontwise dimension. For proofs we refer the reader to the books [F] and [P].

We are going to define the Hausdorff dimension of a set $F \subset \mathbb{R}^n$. The diameter of a set $U \subset \mathbb{R}^n$ is denoted by $|U|$. If $\{U_i\}$ is a countable collection of sets of diameter at most δ that cover F, i.e. $F \subset \bigcup_{i=1}^{\infty} U_i$ with $|U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of F. Given $t \geq 0$, we define the *t-dimensional Hausdorff measure of F* as

$$\mathcal{H}^t(F) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^t : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

It is not difficult to see that there is a critical value t_0 such that

$$\mathcal{H}^t(F) = \begin{cases} \infty & \text{if } t < t_0 \\ 0 & \text{if } t > t_0. \end{cases}$$

We define the *Hausdorff dimension* of F, written $\dim_{\mathrm{H}} F$, as being this critical value t_0.

Now we are going to define another type of dimension which is very usefull in practice, as we shall see. Let F be a non-empty bounded subset of $\mathbb{R}^n$ and let $N(F, \delta)$ be the smallest number of balls of radius δ needded to cover F. We define the *box dimension* of F as

$$\overline{\dim}_{\mathrm{B}} F = \overline{\lim_{\delta \to 0}} \frac{\log N(F, \delta)}{-\log \delta}.$$

We always have $\dim_{\mathrm{H}} F \leq \overline{\dim}_{\mathrm{B}} F$, and we have equality at least for one-dimensional *dynamically defined Cantor sets* (see [PT]).

PROPOSITION 1. *Let $E \subset \mathbb{R}^m$ and $F, F_i \subset \mathbb{R}^n$. Then*
(1) $\dim_{\mathrm{H}}(E \times F) \geq \dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F$*;*
(2) $\dim_{\mathrm{H}}(E \times F) \leq \dim_{\mathrm{H}} E + \overline{\dim}_{\mathrm{B}} F$*;*
(3) $\dim_{\mathrm{H}}(\bigcup_{i=1}^{\infty} F_i) = \sup_i \dim_{\mathrm{H}} F_i$*;*
(4) $\overline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} \overline{F}$*.*

The first two items show the importance of considering the box dimension. The last two items show, essentially, the difference between the two dimensions considered.

PROPOSITION 2. *Let $F \subset \mathbb{R}^n$ and suppose that $f \colon F \to \mathbb{R}^m$ satisfies a Hölder condition*

$$|f(x) - f(y)| \leq C|x - y|^{\alpha} \quad (x, y \in F).$$

Then $\dim_{\mathrm{H}} f(F) \leq (1/\alpha) \dim_{\mathrm{H}} F$*.*

In particular, this proposition shows that *Hausdorff dimension is invariant under bi-Lipschitz transformations*.

Let μ be a Borel probability measure on $\mathbb{R}^n$. The Hausdorff dimension of the measure μ was defined by L.-S. Young as

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} F : \mu(F) = 1\}.$$

So, by definition, one has

$$\dim_{\mathrm{H}} F \geq \sup\{\dim_{\mathrm{H}} \mu : \mu(F) = 1\}.$$

In practice, to calculate the Hausdorff dimension of a measure, it is usefull to compute its *upper pointwise dimension*:

$$\overline{d}_\mu(x) = \overline{\lim_{r\to 0}} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ stands for the open ball of radius r centered at the point x. The *lower pointwise dimension* $\underline{d}_\mu(x)$ is defined similarly using $\underline{\lim}$. The relations between these dimensions are given by

Proposition 3.

(1) *If* $\underline{d}_\mu(x) \geq d$ *for* μ*-a.e.* x *then* $\dim_{\mathrm{H}} \mu \geq d$.
(2) *If* $\overline{d}_\mu(x) \leq d$ *for* μ*-a.e.* x *then* $\dim_{\mathrm{H}} \mu \leq d$.
(3) *If* $\underline{d}_\mu(x) = \overline{d}_\mu(x) = d$ *for* μ*-a.e.* x *then* $\dim_{\mathrm{H}} \mu = d$.

Proposition 4. *If* $\underline{d}_\mu(x) \leq d$ *for* ***every*** $x \in F$ *then* $\dim_{\mathrm{H}} F \leq d$.

CHAPTER 1

Variational principle

The goal of this chapter is to prove Theorem A. As a preparation, we describe how our methods apply to the situation treated in [GL]. In particular we obtain a new proof of their main result. Later, when dealing with our more general skew-product maps, we shall focus on the bounded distortion argument and refer to the earlier situation for steps that are similar to the self-affine case.

1. The self-affine case

Let $S_1, S_2, ..., S_r$ be contractions of $\mathbb{R}^2$. Then there is a unique nonempty compact set Λ of $\mathbb{R}^2$ such that

$$\Lambda = \bigcup_{i=0}^{r} S_i(\Lambda).$$

This set is constructed like the general Sierpinski carpets (there, the contractions are the inverse branches of f_0 corresponding to the chosen rectangles, and the equation above simply means that the set is f_0-invariant). We will refer to Λ as the limit set of the semigroup generated by $S_1, S_2, ..., S_r$.

We shall consider the class of self-affine sets Λ that are the limit sets of the semigroup generated by the mappings A_{ij} given by

$$A_{ij} = \begin{pmatrix} a_{ij} & 0 \\ 0 & b_i \end{pmatrix} x + \begin{pmatrix} c_{ij} \\ d_i \end{pmatrix}, \quad (i,j) \in \mathcal{I}.$$

Here $\mathcal{I} = \{(i,j) : 1 \leq i \leq m \text{ and } 1 \leq j \leq n_i\}$ is a finite index set. We assume $0 < a_{ij} \leq b_i < 1$, for each pair (i,j), $\sum_{i=1}^m b_i \leq 1$, and $\sum_{j=1}^{n_i} a_{ij} \leq 1$ for each i. Also, $0 \leq d_1 < d_2 < ... < d_m < 1$ with $d_{i+1} - d_i \geq b_i$ and $1 - d_m \geq b_m$ and, for each i, $0 \leq c_{i1} < c_{i2} < ... < c_{in_i} < 1$ with $c_{i(j+1)} - c_{ij} \geq a_{in_i}$. These hypothesis guarantee that the rectangles

$$R_{ij} = A_{ij}([0,1] \times [0,1])$$

have interiors that are pairwise disjoint, with edges parallel to the x- and y-axes, are arranged in "rows" of height b_i, and have height $\geq$ width (see Figure 1).

Within the setting described above we have

THEOREM 1.

$$\dim_{\mathrm{H}} \Lambda = \sup_{\mathbf{p}} \left\{ \frac{\sum_{i=1}^m p_i \log p_i}{\sum_{i=1}^m p_i \log b_i} + t(\mathbf{p}) \right\} \tag{1}$$

where $\mathbf{p} = (p_1, p_2, ..., p_m)$ *is a probability vector, and* $t(\mathbf{p})$ *is the unique real in* $[0,1]$ *satisfying*

$$\sum_{i=1}^m p_i \log \left(\sum_{j=1}^{n_i} a_{ij}^{t(\mathbf{p})} \right) = 0.$$

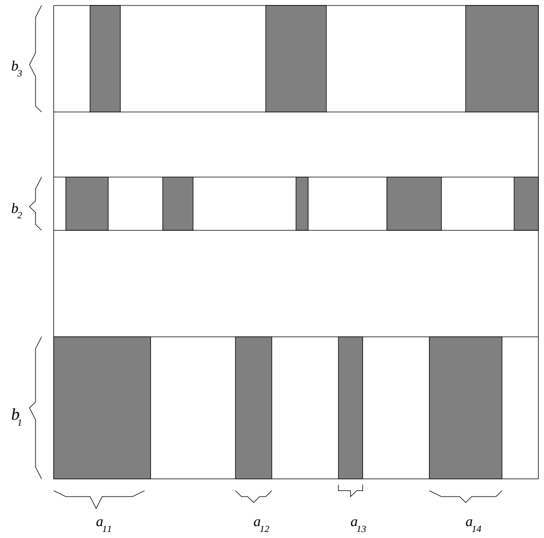

FIGURE 1

REMARK 1.

(1) The expression between brackets in (1) is the Hausdorff dimension of a Bernoulli measure $\mu_{\mathbf{p}}$.
(2) It is easy to see that the function $t \mapsto \sum_i p_i \log(\sum_j a_{ij}^t)$ is strictly decreasing, so the number $t(\mathbf{p})$ is well defined.
(3) The formula obtained by [GL] is slightly different from ours: they take the supremum over all Bernoulli distributions $\{p_{ij}\}$; we take the supremum over the "vertical" Bernoulli distributions $\{p_i\}$.
(4) The number $t(\mathbf{p})$ is given by a random Moran formula relative to the distribution $\mathbf{p}$.

PROOF. There is a natural symbolic representation associated with our system that we shall describe now. Consider the sequence space $\Omega = \mathcal{I}^{\mathbb{N}}$. Elements of Ω will be represented by $\omega = (\omega_1, \omega_2, ...)$ where $\omega_n = (i_n, j_n) \in \mathcal{I}$. Given $\omega \in \Omega$ and $n \in \mathbb{N}$, let $\omega(n) = (\omega_1, \omega_2, ..., \omega_n)$ and define the *cylinder of order* n,

$$C_{\omega(n)} = \{\omega' \in \Omega : \omega_i' = \omega_i, \ i = 1, ..., n\},$$

and the *basic rectangle of order* n,

$$R_{\omega(n)} = A_{i_1 j_1} \circ A_{i_2 j_2} \circ \cdots \circ A_{i_n j_n}([0,1] \times [0,1]).$$

We have that $(R_{\omega(n)})_n$ is a decreasing sequence of closed rectangles of height $\prod_{l=1}^n b_{i_l}$ and width $\prod_{l=1}^n a_{i_l j_l}$. Thus $\bigcap_{n=1}^\infty R_{\omega(n)}$ consists of a single point which belongs to Λ that we denote by $\chi(\omega)$. This defines a continuous and surjective map $\chi \colon \Omega \to \Lambda$ which is at most 4 to 1, and only fails to be a homeomorphism when some of the rectangles R_{ij} have nonempty intersection. Nevertheless, this lack of injectivity will present no problem when dealing with Hausdorff dimension.

Let d be the expression in the right hand side of (1).

Part 1: $\dim_{\mathrm{H}} \Lambda \geq d$

This part is based on [Mu, GL]. Define

$$d_{\mathbf{p}} = \frac{\sum_{i=1}^m p_i \log p_i}{\sum_{i=1}^m p_i \log b_i} + t(\mathbf{p}).$$

Let $\tilde{\mu}_{\mathbf{p}}$ be the Bernoulli measure on Ω that assigns to each symbol $(i,j) \in \mathcal{I}$ the probability

$$p_{ij} = p_i \frac{a_{ij}^{t(\mathbf{p})}}{\sum_{k=1}^{n_i} a_{ik}^{t(\mathbf{p})}}.$$

In other words, we have

$$\tilde{\mu}_{\mathbf{p}}(C_{\omega(n)}) = \prod_{l=1}^{n} p_{i_l j_l}.$$

The existence of such a measure is guaranted by Kolmogorov's Existence Theorem (see [Bi]). Let $\mu_{\mathbf{p}}$ be the probability measure on Λ which is the pushforward of $\tilde{\mu}_{\mathbf{p}}$ by χ, i.e., $\mu_{\mathbf{p}} = \tilde{\mu}_{\mathbf{p}} \circ \chi^{-1}$. We will see that $\dim_{\mathrm{H}} \mu_{\mathbf{p}} = d_{\mathbf{p}}$. This proves Part 1, because $\dim_{\mathrm{H}} \Lambda \geq \dim_{\mathrm{H}} \mu_{\mathbf{p}}$.

For calculating the Hausdorff dimension of $\mu_{\mathbf{p}}$, we shall consider some special sets called *approximate squares*. Given $\omega \in \Omega$ and $n \in \mathbb{N}$, define

$$L_n(\omega) = \max \left\{ k \geq 1 : \prod_{l=1}^{n} b_{i_l} \leq \prod_{l=1}^{k} a_{i_l j_l} \right\} \tag{2}$$

and the *approximate square*

$$B_n(\omega) = \{\omega' \in \Omega : i'_l = i_l,\, l = 1, ..., n \text{ and } j'_l = j_l,\, l = 1, ..., L_n(\omega)\}.$$

We have that each approximate square $B_n(\omega)$ is a finite union of cylinder sets, and that approximate squares are *nested*, i.e., given two, say $B_n(\omega)$ and $B_{n'}(\omega')$, either $B_n(\omega) \cap B_{n'}(\omega') = \emptyset$ or $B_n(\omega) \subset B_{n'}(\omega')$ or $B_{n'}(\omega') \subset B_n(\omega)$. Moreover,

$$\mathrm{int}(\tilde{B}_n(\omega)) \cap \Lambda \subset \chi(B_n(\omega)) \subset \tilde{B}_n(\omega) \cap \Lambda$$

where $\tilde{B}_n(\omega)$ is a closed rectangle in $\mathbb{R}^2$ with sides parallel to the $x-$ and $y-$axes, height $\prod_{l=1}^{n} b_{i_l}$, and width $\prod_{l=1}^{L_n(\omega)} a_{i_l j_l}$. (The rectangle $\tilde{B}_n(\omega)$ is the intersection of the rectangle $R_{\omega(L_n(\omega))}$ with the horizontal strip of height $\prod_{l=1}^{n} b_{i_l}$ containing $R_{\omega(n)}$.) By (2),

$$1 \leq \frac{\prod_{l=1}^{L_n(\omega)} a_{i_l j_l}}{\prod_{l=1}^{n} b_{i_l}} \leq \max a_{ij}^{-1}, \tag{3}$$

hence the term "approximate square".

The importance of approximate squares is that they allow us to construct *Moran covers*, as we shall explain now (see [PW or P]). Fix $0 < r < 1$. Given $\omega \in \Omega$, let $n(\omega)$ denote the unique positive integer such that

$$\prod_{l=1}^{n(\omega)} b_{i_l} > r, \qquad \prod_{l=1}^{n(\omega)+1} b_{i_l} \leq r. \tag{4}$$

We have that $\omega \in B_{n(\omega)}(\omega)$, and if $\omega' \in B_{n(\omega)}(\omega)$ and $n(\omega') \leq n(\omega)$ then $B_{n(\omega)}(\omega) \subset B_{n(\omega')}(\omega')$. Let $B(\omega)$ be the largest approximate square containing ω with the property that $B(\omega) = B_{n(\omega')}(\omega')$ for some $\omega' \in B(\omega)$ and $B_{n(\omega'')}(\omega'') \subset B(\omega)$ for any $\omega'' \in B(\omega)$. The sets $B(\omega)$ corresponding to different $\omega \in \Omega$ either coincide or are disjoint. We denote these sets by $B^{(k)}$, $k = 1, ..., N_r$. Then the sets $\tilde{B}^{(k)} = \chi(B^{(k)})$, $k = 1, ..., N_r$ comprise a cover of Λ which we denote by $\mathcal{U}_r$ and refer to as a *Moran cover*.

The fundamental property of Moran covers, beside being constructed using basic cylinders, is that, given $z \in \Lambda$ and $r > 0$, there exists a number $M > 0$ which does *not* depend on z, r with the following property: the number of sets $\tilde{B}^{(k)}$ in a Moran cover $\mathcal{U}_r$ that have non-empty intersection with the ball $B(z,r)$ is bounded from above by M. We call M a *Moran multiplicity factor*. This property will be crucial in calculating Hausdorff dimension and box dimension. With this end we define

$$\overline{d}_{\mu,\mathcal{M}}(z) = \inf_{\omega\colon z=\chi(\omega)} \overline{\lim_{n\to\infty}} \frac{\log \mu(\chi(B_n(\omega)))}{\log |\chi(B_n(\omega))|},$$

where μ is a probability measure supported on Λ and $z \in \Lambda$. Similarly, we define $\underline{d}_{\mu,\mathcal{M}}(z)$ using $\underline{\lim}$. Then, it is proven in [PW, Theorem 7] or [P, Theorem 15.3]:

LEMMA 1.
(1) $\overline{d}_\mu(z) \le \overline{d}_{\mu,\mathcal{M}}(z)$ *for all* $z \in \Lambda$*;*
(2) $\underline{d}_{\mu,\mathcal{M}}(z) \le \underline{d}_\mu(z)$ *for* μ*-a.e.* $z \in \Lambda$*;*
(3) *if* $\underline{d}_{\mu,\mathcal{M}}(z) = \overline{d}_{\mu,\mathcal{M}}(z) \overset{\text{def}}{=} d(z)$ *for* μ*-a.e.* $z \in \Lambda$*, then* $\underline{d}_\mu(z) = \overline{d}_\mu(z) = d(z)$ *for* μ*-a.e.* $z \in \Lambda$*.*

We are now ready to prove

LEMMA 2. $\dim_{\mathrm{H}} \mu_{\mathbf{p}} = d_{\mathbf{p}}$.

PROOF. If there exists some i such that $p_i = 1$, then the problem is essentially one-dimensional and its solution is known (we leave the details to the reader). So, we assume that $p_i < 1$ for all i. Then, if Δ is the union of the borders of all basic rectangles, one has $\mu_{\mathbf{p}}(\Delta) = 0$. According to Lemma 1 together with Proposition 3, one is left to prove that

$$\lim_{n\to\infty} \frac{\log \tilde{\mu}_{\mathbf{p}}(B_n(\omega))}{\sum_{l=1}^n \log b_{i_l}} = d_{\mathbf{p}} \text{ for } \tilde{\mu}_{\mathbf{p}}\text{-a.e. } \omega.$$

Let

$$P(\omega, n, i) = \sharp\{1 \le l \le n : i_l = i\} \text{ for } i = 1, ..., m.$$

By Kolmogorov's Strong Law of Large Numbers (KSLLN) (see [Bi]),

$$\frac{P(\omega, n, i)}{n} \underset{n\to\infty}{\longrightarrow} p_i \text{ for } \tilde{\mu}_{\mathbf{p}}\text{-a.e. } \omega.$$

We have that

$$\tilde{\mu}_{\mathbf{p}}(B_n(\omega)) = \prod_{l=1}^n p_{i_l} \cdot \prod_{l=1}^{L_n(\omega)} \frac{a_{i_l j_l}^{t(\mathbf{p})}}{\sum_{j=1}^{n_{i_l}} a_{i_l j}^{t(\mathbf{p})}},$$

thus

$$\frac{\log \tilde{\mu}_{\mathbf{p}}(B_n(\omega))}{\sum_{l=1}^n \log b_{i_l}} = \frac{\frac{1}{n}\sum_{l=1}^n \log p_{i_l}}{\frac{1}{n}\sum_{l=1}^n \log b_{i_l}} + t(\mathbf{p}) \cdot \frac{\sum_{l=1}^{L_n(\omega)} \log a_{i_l j_l}}{\sum_{l=1}^n \log b_{i_l}}$$

$$- \frac{\frac{1}{L_n(\omega)} \sum_{l=1}^{L_n(\omega)} \log\Big(\sum_{j=1}^{n_{i_l}} a_{i_l j}^{t(\mathbf{p})}\Big)}{\frac{n}{L_n(\omega)} \frac{1}{n} \sum_{l=1}^n \log b_{i_l}} = \alpha + t(\mathbf{p}) \cdot \beta - \frac{\gamma}{\delta}.$$

By (KSLLN),

$$\alpha \to \frac{\sum_{i=1}^m p_i \log p_i}{\sum_{i=1}^m p_i \log b_i} \text{ for } \tilde{\mu}_{\mathbf{p}}\text{-a.e. } \omega.$$

Write

$$\beta = 1 + \frac{1}{n} \cdot \frac{\sum_{l=1}^{L_n(\omega)} \log a_{i_l j_l} - \sum_{l=1}^n \log b_{i_l}}{\frac{1}{n}\sum_{l=1}^n \log b_{i_l}} = 1 + \frac{1}{n} \cdot \frac{\eta}{\theta}.$$

By (3), we have that $0 \le \eta \le \log(\max a_{ij}^{-1})$. Also, $|\theta| \ge \log(\min b_i^{-1}) > 0$. So, $\beta \to 1$. By (KSLLN) and definition of $t(\mathbf{p})$,

$$\gamma = \sum_{i=1}^m \frac{P(\omega, L_n(\omega), i)}{L_n(\omega)} \log\left(\sum_{j=1}^{n_i} a_{ij}^{t(\mathbf{p})}\right) \to 0 \text{ for } \tilde{\mu}_{\mathbf{p}}\text{-a.e. } \omega.$$

Since $n/L_n(\omega) \geq 1$, we have that $|\delta| \geq \log(\min b_i^{-1}) > 0$, and so, $\gamma/\delta \to 0$, thus completing the proof. □

Part 2: $\dim_{\mathrm{H}} \Lambda \leq d$

This part is based on [Be] and inspired by Formula 2 in Proposition 1. First, we introduce some notation. Let $\pi\colon [0,1]^2 \to [0,1]$ be the vertical projection given by $\pi(x,y) = y$. Then, $\pi(\Lambda)$ has a natural symbolic representation which we denote by $\chi_V\colon \{1,...,m\}^{\mathbb{N}} \to \pi(\Lambda)$. There may be some exceptional points in $\pi(\Lambda)$ that have two representations. Now fix $y \in \pi(\Lambda)$, and assume that $y = \chi_V(i_1, i_2, ...)$ has unique representation. Then $\Lambda_y = \Lambda \cap \{(x,y) : x \in [0,1]\}$ also has a symbolic representation that we denote by $\chi_y\colon \Sigma_y \to \Lambda_y$, where $\Sigma_y = \prod_{l=1}^{\infty}\{1,...,n_{i_l}\}$. As before, given a sequence $\theta = (j_1, j_2, ...) \in \Sigma_y$ and $n \in \mathbb{N}$, define the *horizontal cilinder of order* n,

$$C^y_{j_1 j_2 \cdots j_n} = \{\theta' \in \Sigma_y : j'_l = j_l,\, l = 1,...,n\},$$

and the *basic horizontal interval of order* n,

$$\Delta^y_{j_1 j_2 \cdots j_n} = R_{(i_1 j_1)(i_2 j_2)\dots(i_n j_n)} \cap \{(x,y) : x \in [0,1]\}.$$

Then, $\chi_y(\theta) = \bigcap_{n=1}^{\infty} \Delta^y_{j_1 j_2 \cdots j_n}$.

If $y \in \pi(\Lambda)$ has unique representation $(i_1, i_2, ...)$ then we define

$$P(y,n,i) = \sharp\{1 \leq l \leq n : i_l = i\} \text{ for } i = 1,...,m,$$

and $\bar{t}_y$ as being the unique real in $[0,1]$ satisfying

$$\overline{\lim_{n\to\infty}} \sum_{i=1}^{m} \frac{P(y,n,i)}{n} \log\left(\sum_{j=1}^{n_i} a_{ij}^{\bar{t}_y}\right) = 0.$$

If $y \in \pi(\Lambda)$ has two representations, say θ_1 and θ_2, then procceding analogously, we get two numbers $\bar{t}_{1y}$ and $\bar{t}_{2y}$. Define $\bar{t}_y = \max\{\bar{t}_{1y}, \bar{t}_{2y}\}$. For defineteness, if $\bar{t}_y = \bar{t}_{*y}$, we also define $P(y,n,i)$ as before using the representation θ_*. Then we have

Lemma 3. $\overline{\dim}_{\mathrm{B}}\Lambda_y \leq \bar{t}_y$ *for all* $y \in \pi(\Lambda)$.

Proof. This lemma is essentially contained in [PW, Theorem 3] or [P, Theorem 13.1]. Fix $y \in \pi(\Lambda)$. We assume that $y = \chi_V(i_1, i_2, ...)$ has unique representation (if not, we leave the details to the reader). Let $\bar{d} = \overline{\dim}_{\mathrm{B}}\Lambda_y$. Then, given $\epsilon > 0$, there exists $r_0(\epsilon) > 0$ such that $N(\Lambda_y, r) \geq r^{\epsilon - \bar{d}}$ for $0 < r \leq r_0(\epsilon)$. For such an r, we consider the Moran cover $\mathcal{U}_{y,r}$ of Λ_y by basic intervals $\Delta_{j_1 \cdots j_{n(\theta_k)}}$, for some $\theta_k \in \Sigma_y$, $k = 1,...,N_r$. This cover is defined as before but instead of (4) the number $n(\theta)$ is defined by

$$\prod_{l=1}^{n(\theta)} a_{i_l j_l} > r, \qquad \prod_{l=1}^{n(\theta)+1} a_{i_l j_l} \leq r. \tag{5}$$

Since this cover need not be optimal, we have $N_r \geq N(\Lambda_y, r)$. By (5), if $A = \max a_{ij}^{-1}$ then we have for $k = 1,...,N_r$,

$$\frac{r}{A} \leq \prod_{l=1}^{n(\theta_k)+1} a_{i_l j_l} \leq r$$

and hence
$$C_1 \log \frac{1}{r} - 1 \leq n(\theta_k) \leq C_2 \log \frac{A}{r} + 1$$
for some constants $C_1, C_2 > 0$. This implies that $n(\theta_k)$ can take on at most $B = C_2 \log \frac{A}{r} - C_1 \log \frac{1}{r} + 2$ possible values. Then there exists a value that is repeated at least $\frac{N_r}{B}$ times, i.e., there exists a positive integer $N \in [C_1 \log \frac{1}{r} - 1, C_2 \log \frac{A}{r} + 1]$ such that
$$\sharp\{k : n(\theta_k) = N\} \geq \frac{N_r}{B} \geq \frac{N(\Lambda_y, r)}{B} \geq \frac{r^{\epsilon - \bar{d}}}{C_2 \log \frac{A}{r}} \geq r^{2\epsilon - \bar{d}}$$
if r is sufficiently small. It follows that
$$\sum_{i=1}^{m} \frac{P(y,N,i)}{N} \log \left(\sum_{j=1}^{n_i} a_{ij}^{\bar{d}-2\epsilon} \right) = \frac{1}{N} \log \left(\sum_{j_1,\ldots,j_N} \prod_{l=1}^{N} a_{i_l j_l}^{\bar{d}-2\epsilon} \right)$$
$$\geq \frac{1}{N} \log \left(\sum_{k \colon n(\theta_k)=N} \prod_{l=1}^{n(\theta_k)} a_{i_l j_l}^{\bar{d}-2\epsilon} \right) \geq \frac{1}{N} \log \left(\sum_{k \colon n(\theta_k)=N} r^{\bar{d}-2\epsilon} \right) \geq 0.$$
Since N can be taken arbitrarily large, this implies that $\bar{t}_y \geq \bar{d} - 2\epsilon$. Since ϵ can be taken arbitrarily small, this finishes the proof. $\square$

Just by taking sublimits we get
$$\forall_{y \in \pi(\Lambda)} \, \exists_{\mathbf{p}} : \bar{t}_y = t(\mathbf{p}). \tag{6}$$
This property will allow us to cover Λ by appropriate sets for calculating the Hausdorff dimension, and they are given by
$$\Lambda_{\mathbf{p},\epsilon} = \{(x,y) \in \Lambda : \bar{t}_y \in B_\epsilon(t(\mathbf{p})) \text{ and }$$
$$P_n(y) \text{ has an accumulation point in } B_\epsilon(\mathbf{p})\},$$
where
$$P_n(y) = \left(\frac{P(y,n,i)}{n} \right)_{i=1}^{m} \quad \text{and} \quad B_\epsilon(\mathbf{p}) = \{\mathbf{q} : |q_i - p_i| < \epsilon, \, i = 1, \ldots, m\}.$$
We have that, for every ϵ, $\Lambda = \bigcup_{\mathbf{p}} \Lambda_{\mathbf{p},\epsilon}$. The use of ϵ is just to get a finite union by compacteness arguments.

First, we calculate the dimension of the vertical part. Let
$$G_{\mathbf{p},\epsilon} = \{y \in \pi(\Lambda) : P_n(y) \text{ has an accumulation point in } B_\epsilon(\mathbf{p})\}.$$
Then

Lemma 4.
$$\dim_{\mathrm{H}} G_{\mathbf{p},\epsilon} = \sup \left\{ \frac{\sum_{i=1}^{m} q_i \log q_i}{\sum_{i=1}^{m} q_i \log b_i} : \mathbf{q} \in B_\epsilon(\mathbf{p}) \right\}. \tag{7}$$

Proof. Let s be the expression on the right hand side of (7).

($\geqslant$) Let
$$G_{\mathbf{p}} = \{y \in \pi(\Lambda) : P_n(y) \to \mathbf{p}\}.$$
Let $\nu_{\mathbf{p}}$ be the pushforward of $\mu_{\mathbf{p}}$ by π, i.e., $\nu_{\mathbf{p}}$ is the Bernoulli measure associated to $\mathbf{p}$. It is well known that (see Lemma 2)
$$\dim_{\mathrm{H}} \nu_{\mathbf{p}} = \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log b_i}.$$
By Kolmogorov's Strong Law of Large Numbers we have that $\nu_{\mathbf{p}}(G_{\mathbf{p}}) = 1$, so
$$\dim_{\mathrm{H}} G_{\mathbf{p}} \geq \dim_{\mathrm{H}} \nu_{\mathbf{p}}.$$

The result follows because $G_{\mathbf{q}} \subset G_{\mathbf{p},\epsilon}$ for all $\mathbf{q} \in B_\epsilon(\mathbf{p})$.

($\leqslant$) Let

$$\tilde{G}_{\mathbf{p}} = \{y \in \pi(\Lambda) : P_n(y) \text{ accumulates in } \mathbf{p}\}.$$

We have that

$$\forall_{y \in G_{\mathbf{p},\epsilon}} \ \exists_{\mathbf{q} \in B_\epsilon(\mathbf{p})} : y \in \tilde{G}_{\mathbf{q}}$$

and

$$y \in \tilde{G}_{\mathbf{q}} \Rightarrow \underline{\lim}_{n\to\infty} \frac{\log \nu_{\mathbf{q}}(I_n(y))}{\log |I_n(y)|} \le \frac{\sum_{i=1}^m q_i \log q_i}{\sum_{i=1}^m q_i \log b_i},$$

where $I_n(y)$ is the *basic vertical interval of order* n containing y (obtained by projecting a basic rectangle of order n of a point in $\pi^{-1}(y)$). Thus

$$\forall_{y \in G_{\mathbf{p},\epsilon}} \ \exists_{\mathbf{q} \in B_\epsilon(\mathbf{p})} : \underline{\lim}_{n\to\infty} \frac{\log \nu_{\mathbf{q}}(I_n(y))}{\log |I_n(y)|} \le s. \tag{8}$$

We will see that (8) $\Rightarrow \dim_{\mathrm{H}} G_{\mathbf{p},\epsilon} \le s$ (see also Proposition 4, [Mu], [GL]).

Given $\delta, \eta > 0$, we shall build a cover $\mathcal{U}_{\delta,\eta}$ of $G_{\mathbf{p},\epsilon}$ of diameter $< \eta$ such that

$$\sum_{U \in \mathcal{U}_{\delta,\eta}} |U|^{s+2\delta} \le M_\delta$$

where M_δ is an integer depending on δ but not on η. This implies the result. Let $b = \max b_i < 1$. It is clear that there exists a finite number of Bernoulli measures $\nu_1, ..., \nu_{M_\delta}$ such that

$$\forall_{\mathbf{q} \in B_\epsilon(\mathbf{p})} \ \exists_{k \in \{1,...,M_\delta\}} : \frac{\nu_{\mathbf{q}}(I_n)}{\nu_k(I_n)} \le b^{-\delta n}$$

for all basic vertical intervals of order n, I_n. By (8), we can build a cover of $G_{\mathbf{p},\epsilon}$ by basic vertical intervals $I_{n(y_i)}$, $i = 1, 2, ...$ that are disjoint and have diameters $< \eta$, such that

$$\nu_{\mathbf{q}^{\mathbf{i}}}(I_{n(y_i)}) \ge |I_{n(y_i)}|^{s+\delta}$$

for some $\mathbf{q}^{\mathbf{i}} \in B_\epsilon(\mathbf{p})$. It follows that

$$\begin{aligned}
\sum_i |I_{n(y_i)}|^{s+2\delta} &\le \sum_i \nu_{\mathbf{q}^{\mathbf{i}}}(I_{n(y_i)}) \cdot |I_{n(y_i)}|^\delta \\
&\le \sum_i \nu_{k_i}(I_{n(y_i)})\, b^{-\delta n(y_i)} \cdot b^{\delta n(y_i)} \\
&\le \sum_{k=1}^{M_\delta} \sum_i \nu_k(I_{n(y_i)}) \le M_\delta
\end{aligned}$$

as we wish. $\square$

Lemma 5.

$$\dim_{\mathrm{H}} \Lambda_{\mathbf{p},\epsilon} \le \sup \left\{ \frac{\sum_{i=1}^m q_i \log q_i}{\sum_{i=1}^m q_i \log b_i} + t(\mathbf{p}) + \epsilon : \mathbf{q} \in B_\epsilon(\mathbf{p}) \right\}. \tag{9}$$

Proof. Let $\tilde{d}$ be the expression on the right hand side of (9).

Note: We can use Moran covers for calculating the box dimension. I.e., if $\mathcal{U}_{y,r}$ is the Moran cover of radius r of Λ_y, then

$$\overline{\dim}_{\mathrm{B}} \Lambda_y = \overline{\lim_{r\to 0}} \frac{\log \sharp \mathcal{U}_{y,r}}{-\log r}.$$

Using Lemma 3 and this note we get that, if y is such that $\bar{t}_y \in B_\epsilon(t(\mathbf{p}))$ then there exists a positive integer N such that

$$r < \frac{1}{N} \Rightarrow \sharp\mathcal{U}_{y,r} < r^{-t(\mathbf{p})-\epsilon}. \tag{10}$$

Let

$$A_N = \{y : (x,y) \in \Lambda_{\mathbf{p},\epsilon} \text{ for some } x \text{ and (10) holds}\},$$

and

$$\Lambda_N = \{(x,y) \in \Lambda_{\mathbf{p},\epsilon} : y \in A_N\}.$$

Then $\Lambda_{\mathbf{p},\epsilon} = \bigcup_{N=1}^{\infty} \Lambda_N$, and we are left to prove that $\dim_{\mathrm{H}} \Lambda_N \leq \tilde{d}$ for all N.

Since $A_N \subset G_{\mathbf{p},\epsilon}$, we have $\dim_{\mathrm{H}} A_N \leq \dim_{\mathrm{H}} G_{\mathbf{p},\epsilon}$. So, if we take $a > \dim_{\mathrm{H}} G_{\mathbf{p},\epsilon}$, there exists a cover $\mathcal{U}$ of A_N formed by basic vertical intervals of diameters $< \frac{1}{N}$ such that

$$\sum_{u\in\mathcal{U}} |u|^a < 1.$$

Let $u \in \mathcal{U}$, $r = |u|$ and $y \in u \cap A_N$. We are going to obtain an adequate cover of Λ_N in the horizontal strip $H = u \times [0,1]$. Consider the Moran cover $\mathcal{U}_{y,r}$ of Λ_y. To each element of $\mathcal{U}_{y,r}$ there corresponds a basic rectangle R_{n_k} of Λ, of a certain order n_k, $k = 1, ..., N_r$. If n is the order of u then, by the domination property $b_i \geq a_{ij}$, we have that $n_k \leq n$. This implies that the rectangles R_{n_k} cross the strip H. Then $\mathcal{U}(u) = \{R_{n_k} \cap H : k = 1, ..., N_r\}$ is a cover of $\Lambda_N \cap H$ (because the basic rectangles of order n form such a cover, and these must be contained in the R_{n_k}) (see Figure 2).

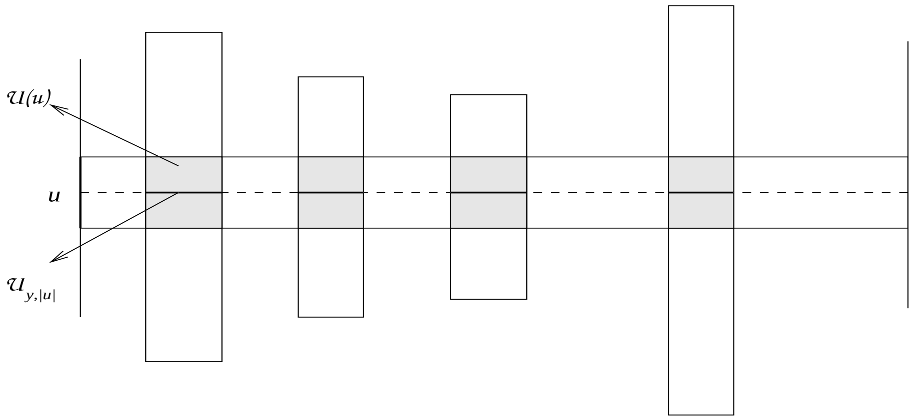

FIGURE 2. Construction of $\mathcal{U}(u)$

By the property of a Moran cover, if $v \in \mathcal{U}(u)$ then

$$r \leq |v| \leq \sqrt{1+A^2}\, r. \tag{11}$$

By (10), $\sharp\mathcal{U}(u) \leq r^{-t(\mathbf{p})-\epsilon}$. So $\mathcal{U}_* = \bigcup_{u\in\mathcal{U}} \mathcal{U}(u)$ is a cover of Λ_N such that

$$\begin{aligned}\sum_{v\in\mathcal{U}_*} |v|^{a+t(\mathbf{p})+\epsilon} &\leq (\sqrt{1+A^2}\,)^{a+t(\mathbf{p})+\epsilon} \sum_{u\in\mathcal{U}} r^a \sum_{v\in\mathcal{U}(u)} r^{t(\mathbf{p})+\epsilon} \\ &\leq (\sqrt{1+A^2}\,)^{a+t(\mathbf{p})+\epsilon} \sum_{u\in\mathcal{U}} |u|^a \leq (\sqrt{1+A^2}\,)^{a+t(\mathbf{p})+\epsilon}.\end{aligned}$$

This implies that $\dim_{\mathrm{H}} \Lambda_N \leq a + t(\mathbf{p}) + \epsilon$, as we wish.

$\square$

Conclusion of Part 2

Consider the subspace A of $\mathbb{R}^{m+1}$ given by the set of points $(\mathbf{p}, t)$ such that $t = t(\mathbf{p})$. Let $B = A \cap (\{(p_1, ..., p_m) : \sum_{i=1}^m p_i = 1, p_i \geq 0\} \times [0,1])$. By property (6), for every point $(x, y) \in \Lambda$ there exists $(\mathbf{p}, t(\mathbf{p})) \in B$ such that $\bar{t}_y = t(\mathbf{p})$ and $P_n(y)$ accumulates in $\mathbf{p}$. The fact that B is compact is a consequence of the continuity of the function $\mathbf{p} \mapsto t(\mathbf{p})$, which, in turn, is a consequence of the function $\psi(t) = \sum_{i=1}^m p_i \log\left(\sum_{j=1}^{n_i} a_{ij}^t\right)$ satisfying $\psi'(t) \leq \log(\max a_{ij}) < 0$. Thus, given $\epsilon > 0$, there exists a finite number of points $\mathbf{p}^1, ..., \mathbf{p}^N$ such that

$$B \subset \bigcup_{k=1}^N B_\epsilon(\mathbf{p}^k) \times B_\epsilon(t(\mathbf{p}^k)).$$

It follows from Lemma 5 that

$$\dim_{\mathrm{H}} \Lambda \leq \sup_k \sup \left\{ \frac{\sum_{i=1}^m q_i \log q_i}{\sum_{i=1}^m q_i \log b_i} + t(\mathbf{p}^k) + \epsilon : \mathbf{q} \in B_\epsilon(\mathbf{p}^k) \right\}.$$

Letting $\epsilon \to 0$, this concludes Part 2 and ends the proof of the theorem.

□

COROLLARY 1. *There exists* $\mathbf{p}^*$ *such that*

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{H}} \mu_{\mathbf{p}^*}.$$

PROOF. It follows from Theorem 1, Lemma 2, and the continuity of the function $\mathbf{p} \mapsto t(\mathbf{p})$. □

2. The skew-product case

As in the introduction, $f_0 \colon \mathbb{T}^2 \to \mathbb{T}^2$ is the map given by $f_0(x, y) = (lx, my)$ where $l > m > 1$ are integers, and Λ_0 is a general Sierpinski carpet for f_0. Given $\varepsilon > 0$ small (depending only on l and m, and to be specified along the proof), we consider an $\varepsilon - \mathrm{C}^2$ perturbation f of f_0, of the form $f(x, y) = (a(x, y), b(y))$. Then f preserves the horizontal lines and we say that f posesses a strong unstable foliation. Following [S], there is a unique homeomorphism $h \colon \mathbb{T}^2 \to \mathbb{T}^2$ close to the identity which conjugates f and f_0, i.e, $f \circ h = h \circ f_0$. We consider the f-invariant set Λ which is the *continuation* of Λ_0, i.e., $\Lambda = h(\Lambda_0)$. In general, h and its inverse are only Hölder-continuous with exponent $\frac{\log m - \varepsilon}{\log l + \varepsilon}$, so this case cannot be trivially reduced to the self-affine case. Geometrically, Λ is also constructed as the general Sierpinski carpets but using a *distorted* grid of lines, as we shall explain now. By imposition, f preserves the horizontal lines. There is also an f-invariant foliation $\mathcal{F}^c$ by nearly vertical C^2 curves. This is a direct consequence of the fact that the vertical lines constitute a *normally expanding invariant foliation* for f_0, see [HPS]. In general, $\mathcal{F}^c$ is not a smoth foliation (i.e., its holonomy maps are not differentiable) but is at least Hölder-continuous. Then, there are m horizontal lines and l leaves of $\mathcal{F}^c$ that divide $\mathbb{T}^2$ into $l \times m$ *distorted rectangles* each of which is mapped by f onto the entire $\mathbb{T}^2$. As before, Λ is constructed by choosing some of these *basic rectangles of order 1* and then refining inductively by *basic rectangles of order n*.

The idea for proving the variational principle in this case is fixing a big n, think of f^n acting affinely on each basic rectangle of order n, and then use the techniques of the self-affine case. Of course we get an error, but this error will aproach zero as n increases, due to the *bounded distortion property* (see [PT]):

Notation: $\partial_x a^n(z) = \prod_{j=0}^{n-1} \partial_x a(f^j z)$.

Bounded Distortion Lemma. *There exists $C > 0$ such that, given $n \in \mathbb{N}$ and points z, w belonging to a same basic rectangle of order n,*

$$\left| \log \frac{\partial_x a^n(z)}{\partial_x a^n(w)} \right| \leq C.$$

Proof. From the fact that f is locally expanding it follows that, for some $\sigma > 1$,

$$d(f^i z, f^i w) \leq \sigma^{i-n+1} \text{ for } 0 \leq i \leq n-1.$$

Since $\partial_x a$ is Lipschitz-continuous and is bounded away from 0, we have that $\log \partial_x a$ is also Lipschitz-continuous. It follows that, for some constant $\tilde{C} > 0$,

$$\begin{aligned} \left| \log \frac{\partial_x a^n(z)}{\partial_x a^n(w)} \right| &\leq \sum_{i=0}^{n-1} |\log \partial_x a(f^i z) - \log \partial_x a(f^i w)| \\ &\leq \sum_{i=0}^{n-1} \tilde{C} d(f^i z, f^i w) \leq \sum_{i=0}^{n-1} \tilde{C} \sigma^{i-n+1} \\ &\leq \frac{\tilde{C}}{1 - \sigma^{-1}} =: C. \end{aligned}$$

□

Remark 2.

(1) It is important to notice that the constant C is *universal*, meaning that it does not depend on the map f (in a bounded domain). That is why we use the C^2 topology. In fact, given any $\theta > 0$, we could have used the $\mathrm{C}^{1+\theta}$ topology.
(2) We have a similar result using the map b instead of a. By taking maximum, we can assume that the constant C is the same.

From this we get the following

Corollary 2. *If H_1 and H_2 are two horizontal segments of a same basic rectangle then*

$$\text{length}(H_1) \leq e^C \text{length}(H_2).$$

Proof. If H_1 and H_2 belongs to a basic rectangle of order n, then $f^n H_1$ and $f^n H_2$ are two horizontal segments with length 1. So, by the intermediate value theorem, there exists $z_i \in H_i, i = 1, 2$ such that $\text{lenght}(H_i) = (\partial_x a^n(z_i))^{-1}$. The result follows by the Bounded Distortion Lemma. □

Note that if the ε-perturbation is small then we still have the following domination property

$$\min_x \partial_x a(x, y) > b'(y) \text{ for all } y. \tag{12}$$

Proof of Theorem A. Let us introduce a new notation adequate to our setting. Given $n \in \mathbb{N}$, and $(i_k, j_k) \in \mathcal{I}, k = 1, ..., n$, consider the n-tuples $\mathbf{i} = (i_1, ..., i_n)$, $\mathbf{j} = (j_1, ..., j_n)$, and the basic rectangle and interval of order n

$$R^n_{\mathbf{ij}} = R_{(i_1 j_1)(i_2 j_2)\dots(i_n j_n)} \text{ and } R^n_{\mathbf{i}} = \pi(R_{(i_1 j_1)(i_2 j_2)\dots(i_n j_n)}).$$

Let

$$a_{\mathbf{ij},n} = \max_{R^n_{\mathbf{ij}}} (\partial_x a^n)^{-1} \text{ and } b_{\mathbf{i},n} = \max_{R^n_{\mathbf{i}}} ((b^n)')^{-1}.$$

Let $\mathbf{p}^n = (p^n_{\mathbf{i}})$ be a probability vector in $\mathbb{R}^{nm}$. We define $t_n(\mathbf{p}^n)$ as being the unique real in $[0, 1]$ satisfying

$$\sum_{\mathbf{i}} p^n_{\mathbf{i}} \log \left(\sum_{\mathbf{j}} a^{t_n(\mathbf{p}^n)}_{\mathbf{ij},n} \right) = 0. \tag{13}$$

With this notation, what we shall prove is that

$$\dim_{\mathrm{H}} \Lambda = \lim_{n\to\infty} \sup_{\mathbf{p}^n} \left\{ \frac{\sum_{\mathbf{i}} p_{\mathbf{i}}^n \log p_{\mathbf{i}}^n}{\sum_{\mathbf{i}} p_{\mathbf{i}}^n \log b_{\mathbf{i},n}} + t_n(\mathbf{p}^n) \right\}.$$

Put

$$d_{\mathbf{p}^n} = \frac{\sum_{\mathbf{i}} p_{\mathbf{i}}^n \log p_{\mathbf{i}}^n}{\sum_{\mathbf{i}} p_{\mathbf{i}}^n \log b_{\mathbf{i},n}} + t_n(\mathbf{p}^n).$$

Part 1: $\dim_{\mathrm{H}} \Lambda \geq \overline{\lim_{n\to\infty}} \sup_{\mathbf{p}^n} d_{\mathbf{p}^n}$

Fix n. Let $\mathbf{p}^n$ be a probability vector such that $p_{\mathbf{i}}^n < 1$ for all $\mathbf{i}$ (we leave the other case to the reader). Let $\mu_{\mathbf{p}^n}$ be the Bernoulli measure for f^n on Λ given by

$$\mu_{\mathbf{p}^n}(R_{\mathbf{ij}}^n) = p_{\mathbf{i}}^n \frac{a_{\mathbf{ij},n}^{t_n(\mathbf{p}^n)}}{\sum_{\mathbf{k}} a_{\mathbf{ik},n}^{t_n(\mathbf{p}^n)}}.$$

We will see that the Hausdorff dimension of $\mu_{\mathbf{p}^n}$ is "approximately" $d_{\mathbf{p}^n}$. As before, we shall use *approximate squares.* Given $\omega \equiv ((\mathbf{i_1 j_1})(\mathbf{i_2 j_2})...) \in \Omega$ and $k \in \mathbb{N}$, define

$$L_k^n(\omega) = \max\left\{ l \geq 1 : \partial_x a^{nl}(z) \leq (b^{nk})'(y) \right\}, \tag{14}$$

where $z = \chi(\omega)$ and $y = \pi(z)$, and the *n-approximate square*

$$B_k^n(\omega) = \{\omega' \in \Omega : \mathbf{i_l}' = \mathbf{i_l},\ l = 1, ..., k \text{ and } \mathbf{j_l}' = \mathbf{j_l},\ l = 1, ..., L_k^n(\omega)\}.$$

We have that

$$\mathrm{int}(\tilde{B}_k^n(\omega)) \cap \Lambda \subset \chi(B_k^n(\omega)) \subset \tilde{B}_k^n(\omega) \cap \Lambda$$

where $\tilde{B}_k^n(\omega)$ is the intersection of the basic rectangle $R_{\omega(nL_k^n(\omega))}$ with the horizontal strip containing $R_{\omega(nk)}$ of the same height. If H is a horizontal segment of $\tilde{B}_k^n(\omega)$ then, by the bounded distortion property, we have that

$$e^{-C} \leq \frac{\mathrm{length}(H)}{(\partial_x a^{nL_k^n(\omega)}(z))^{-1}} \leq e^C.$$

By (14),

$$1 \leq \frac{(b^{nk})'(y)}{\partial_x a^{nL_k^n(\omega)}(z)} \leq A_n, \tag{15}$$

where $A_n = \max \partial_x a^n$, so

$$e^{-C} \leq \frac{\mathrm{length}(H)}{((b^{nk})'(y))^{-1}} \leq A_n e^C.$$

Also, by bounded distortion,

$$e^{-C} \leq \frac{\mathrm{height}(\tilde{B}_k^n(\omega))}{((b^{nk})'(y))^{-1}} \leq e^C.$$

Let $\alpha \geq 0$ be the maximum variation of the leaves of $\mathcal{F}^c$. We have that $\alpha \to 0$ when $\varepsilon \to 0$, so we can take $\alpha < \frac{e^{-2C}}{2}$ (remember that C is universal). Then, if $r = ((b^{nk})'(y))^{-1}$, it is easy to see that $\tilde{B}_k^n(\omega)$ is contained in a ball of radius $(2A_n + \sqrt{1+\alpha^2})e^C r$, and contains a ball of radius $(e^{-C} - 2e^C\alpha)r$. Hence the term "n-approximate square". We use the following notation

$$\alpha = \beta \pm \epsilon \text{ means } |\alpha - \beta| \leq \epsilon.$$

Let $B = (\min \log b')^{-1}$. Then

Lemma 6. $\dim_{\mathrm{H}} \mu_{\mathbf{p}^n} = d_{\mathbf{p}^n} \pm \frac{2BC}{n}$.

Proof. As in Lemma 2, one is left to prove that

$$\lim_{k\to\infty} \frac{\log \mu_{\mathbf{p}^n}(\chi(B_k^n(\omega)))}{-\sum_{l=1}^k \log(b^n)'(b^{n(l-1)}y)} = d_{\mathbf{p}^n} \pm \frac{2BC}{n} \quad \text{for } \mu_{\mathbf{p}^n}\text{-a.e. } z.$$

We have that

$$\frac{\log \mu_{\mathbf{p}^n}(\chi(B_k^n(\omega)))}{-\sum_{l=1}^k \log(b^n)'(b^{n(l-1)}y)} = \frac{\sum_{l=1}^k \log p_{\mathbf{i_l}}^n}{\sum_{l=1}^k \log b_{\mathbf{i_l},n}} \cdot \frac{\sum_{l=1}^k \log b_{\mathbf{i_l},n}}{-\sum_{l=1}^k \log(b^n)'(b^{n(l-1)}y)}$$

$$+\ t_n(\mathbf{p}^n) \cdot \frac{\sum_{l=1}^{L_k^n(\omega)} \log a_{\mathbf{i_l j_l},n}}{-\sum_{l=1}^k \log(b^n)'(b^{n(l-1)}y)} + \frac{\sum_{l=1}^{L_k^n(\omega)} \log\Big(\sum_{\mathbf{j}} a_{\mathbf{i_l j},n}^{t_n(\mathbf{p}^n)}\Big)}{\sum_{l=1}^k \log(b^n)'(b^{n(l-1)}y)}$$

$$= \alpha\cdot\beta + t_n(\mathbf{p}^n)\cdot\gamma + \delta\,.$$

That

$$\alpha \to \frac{\sum_{\mathbf{i}} p_{\mathbf{i}}^n \log p_{\mathbf{i}}^n}{\sum_{\mathbf{i}} p_{\mathbf{i}}^n \log b_{\mathbf{i},n}} \quad\text{and}\quad \delta\to 0 \quad \text{for } \mu_{\mathbf{p}^n}\text{-a.e. } z$$

is as in Lemma 2. Now we see that β and γ are “approximately” 1. Write

$$\beta = 1 - \frac{1}{nk}\cdot \frac{\sum_{l=1}^k \log b_{\mathbf{i_l},n} + \sum_{l=1}^k \log(b^n)'(b^{n(l-1)}y)}{\frac{1}{nk}\sum_{l=0}^{nk-1} \log b'(b^l y)} = 1 - \frac{1}{nk}\cdot\frac{\eta}{\theta}\,.$$

Since $b^{n(l-1)}y \in R_{\mathbf{i_l}}^n$, using bounded distortion we get $|\eta| \le kC$. Clearly, $\theta \ge \min\log b'$. So $|\beta - 1| \le \frac{BC}{n}$. Write

$$\gamma = 1 - \frac{1}{nk}\cdot\frac{1}{\theta}\cdot\left(\sum_{l=1}^{L_k^n(\omega)} \log a_{\mathbf{i_l j_l},n} + \sum_{l=1}^{L_k^n(\omega)} \log \partial_x a^n(f^{n(l-1)}z)\right)$$

$$- \frac{1}{nk}\cdot\frac{1}{\theta}\cdot\left(\sum_{l=1}^{k} \log(b^n)'(b^{n(l-1)}y) - \sum_{l=1}^{L_k^n(\omega)} \log \partial_x a^n(f^{n(l-1)}z)\right)$$

$$= 1 - \frac{1}{nk}\cdot\frac{1}{\theta}\cdot\iota - \frac{1}{nk}\cdot\frac{1}{\theta}\cdot\kappa\,.$$

By bounded distortion, $|\iota| \le kC$, and by (15), $|\kappa| \le \log A_n$. So $|\lim\gamma - 1| \le \frac{BC}{n}$. The result follows. $\square$

This ends Part 1.

Part 2: $\dim_{\mathrm{H}}\Lambda \le \underline{\lim}_{n\to\infty} \sup_{\mathbf{p}^n} d_{\mathbf{p}^n}$

Fix n. If $y\in\pi(\Lambda)$ has unique representation $(\mathbf{i_1}, \mathbf{i_2}, ...)$ then we define

$$P^n(y,k,\mathbf{i}) = \sharp\{1\le l\le k : \mathbf{i_l} = \mathbf{i}\} \quad \text{for all } \mathbf{i},$$

and $\bar t_{y,n}$ as being the unique real in $[0,1]$ satisfying

$$\overline{\lim_{k\to\infty}} \sum_{\mathbf{i}} \frac{P^n(y,k,\mathbf{i})}{k} \log\left(\sum_{\mathbf{j}} a_{\mathbf{ij},n}^{\bar t_{y,n}}\right) = 0.$$

If $y\in\pi(\Lambda)$ has two representations we proceed as in the previous chapter. Then

Lemma 7. $\forall_{\epsilon>0}\ \exists_{n_\epsilon\in\mathbb{N}}\ \forall_{n\ge n_\epsilon}\ \forall_{y\in\pi(\Lambda)}\ \overline{\dim}_{\mathrm{B}}\Lambda_y \le \bar t_{y,n} + \epsilon.$

Proof. This follows closely Lemma 3. Fix n and y. We assume that $y = \chi_V(\mathbf{i_1}, \mathbf{i_2}, ...)$ has unique representation (if not, we leave the details to the reader). Let $\bar d = \overline{\dim}_{\mathrm{B}}\Lambda_y$. Then, given $\epsilon>0$, there exists $r_0(\epsilon)>0$ such that $N(\Lambda_y, r) \ge r^{\epsilon/3-\bar d}$ for $0<r\le r_0(\epsilon)$. For such an r, consider the Moran cover $\mathcal{U}_{y,r}^n$ of Λ_y by

basic intervals $\Delta_{\mathbf{j_1}\cdots\mathbf{j_{k(\theta_l)}}}$, for some $\theta_l \in \Sigma_y$, $l = 1, ..., N_r$. This cover is defined as before but instead of (5) the number $k(\theta)$ is defined by

$$\partial_x a^{nk(\theta)}(z) < r^{-1}, \quad \partial_x a^{n(k(\theta)+1)}(z) \geq r^{-1},$$

where $z = \chi_V(\theta)$. Since this cover need not be optimal, we have $N_r \geq N(\Lambda_y, r)$. It follows that

$$\frac{1}{r} \leq \prod_{j=0}^{k(\theta_l)} \partial_x a^n(f^{nj} z_l) \leq \frac{A_n}{r},$$

and hence

$$C_n \log \frac{1}{r} - 1 \leq k(\theta_l) \leq D_n \log \frac{A_n}{r} + 1$$

for some constants $C_n, D_n > 0$. This implies that $k(\theta_l)$ can take on at most $B_n = D_n \log \frac{A_n}{r} - C_n \log \frac{1}{r} + 2$ possible values. Then there exists a value that is repeated at least $\frac{N_r}{B_n}$ times, i.e., there exists a positive integer $N \in [C_n \log \frac{1}{r} - 1, D_n \log \frac{A_n}{r} + 1]$ such that

$$\sharp\{l : k(\theta_l) = N\} \geq \frac{N_r}{B_n} \geq \frac{N(\Lambda_y, r)}{B_n} \geq \frac{r^{\epsilon/3 - \bar{d}}}{D_n \log \frac{A_n}{r}} \geq r^{\epsilon/2 - \bar{d}}$$

if $r \leq r_1(\epsilon, n)$ is sufficiently small. It follows that

$$\begin{aligned}
0 &\leq \frac{1}{N} \log \left(\sum_{l:\, k(\theta_l) = N} r^{\bar{d} - \epsilon/2} \right) \\
&\leq \frac{1}{N} \log \left(\sum_{l:\, k(\theta_l) = N} \prod_{j=1}^{k(\theta_l)} (\partial_x a^n (f^{n(j-1)} z_l))^{\epsilon/2 - \bar{d}} \right) \\
&\leq \frac{1}{N} \log \left(\sum_{\mathbf{j_1}, \ldots, \mathbf{j_N}} \prod_{l=1}^{N} (e^C a_{\mathbf{i_l j_l}, n})^{\bar{d} - \epsilon/2} \right) \quad \text{by bounded distortion} \\
&\leq \sum_{\mathbf{i}} \frac{P^n(y, N, \mathbf{i})}{N} \log \left(\sum_{\mathbf{j}} a_{\mathbf{ij}, n}^{\bar{d} - \epsilon/2} \right) + C \\
&\leq \sum_{\mathbf{i}} \frac{P^n(y, N, \mathbf{i})}{N} \log \left(\sum_{\mathbf{j}} a_{\mathbf{ij}, n}^{\bar{d} - \epsilon} \right) + \frac{\epsilon}{2} \log \lambda_n + C,
\end{aligned}$$

where $\lambda_n = \max a_{\mathbf{ij}, n}$. If we take n_ϵ such that $\lambda_{n_\epsilon} \leq e^{-2C\epsilon^{-1}}$, say

$$n_\epsilon = \left[\frac{2C}{\log \min \partial_x a} \, \epsilon^{-1} \right] + 1, \tag{16}$$

then

$$n \geq n_\epsilon \Rightarrow \frac{\epsilon}{2} \log \lambda_n + C \leq 0,$$

and thus

$$\sum_{\mathbf{i}} \frac{P^n(y, N, \mathbf{i})}{N} \log \left(\sum_{\mathbf{j}} a_{\mathbf{ij}, n}^{\bar{d} - \epsilon} \right) \geq 0.$$

Since N can be taken arbitrarily large, this implies that $\bar{t}_{y,n} \geq \bar{d} - \epsilon$. $\square$

Again, just by taking sublimits we get

$$\forall_{n \in \mathbb{N}} \, \forall_{y \in \pi(\Lambda)} \, \exists_{\mathbf{p}^n} : \bar{t}_{y,n} = t_n(\mathbf{p}^n).$$

Then we define appropriate sets for calculating the Hausdorff dimension that will cover Λ by

$$\Lambda^n_{\mathbf{p}^n,\epsilon} = \{(x,y) \in \Lambda : \bar{t}_{y,n} \in B_\epsilon(t_n(\mathbf{p}^n)) \text{ and } P^n_k(y) \text{ has an accumulation point in } B_{\delta_{n,\epsilon}}(\mathbf{p}^n)\},$$

where

$$P^n_k(y) = \left(\frac{P^n(y,k,\mathbf{i})}{k}\right)_{\mathbf{i}}, \; B_\delta(\mathbf{p}^n) = \{\mathbf{q}^n : |q^n_{\mathbf{i}} - p^n_{\mathbf{i}}| < \delta, \text{ for all } \mathbf{i}\},$$

and $\delta_{n,\epsilon} > 0$ is chosen in such a way that

$$\forall_{\mathbf{p}^n} \, \forall_{\mathbf{q}^n \in B_{\delta_{n,\epsilon}}(\mathbf{p}^n)} \frac{\sum_{\mathbf{i}} q^n_{\mathbf{i}} \log q^n_{\mathbf{i}}}{\sum_{\mathbf{i}} q^n_{\mathbf{i}} \log b_{\mathbf{i},n}} \le \frac{\sum_{\mathbf{i}} p^n_{\mathbf{i}} \log p^n_{\mathbf{i}}}{\sum_{\mathbf{i}} p^n_{\mathbf{i}} \log b_{\mathbf{i},n}} + \epsilon.$$

We have that, for every ϵ and n, $\Lambda = \bigcup_{\mathbf{p}^n} \Lambda^n_{\mathbf{p}^n,\epsilon}$.

Let

$$G^n_{\mathbf{p}^n,\delta} = \{y \in \pi(\Lambda) : P^n_k(y) \text{ has an accumulation point in } B_\delta(\mathbf{p}^n)\}.$$

Then

Lemma 8.

$$\dim_{\mathrm{H}} G_{\mathbf{p}^n,\delta} = \sup\left\{\frac{\sum_{\mathbf{i}} q^n_{\mathbf{i}} \log q^n_{\mathbf{i}}}{\sum_{\mathbf{i}} q^n_{\mathbf{i}} \log b_{\mathbf{i},n}} : \mathbf{q}^n \in B_\delta(\mathbf{p}^n)\right\} \pm \frac{BC}{n}.$$

Proof. Similar to the proof of Lemma 4 using the bounded distortion arguments used in the proof of Lemma 6. □

Lemma 9. *For every $\epsilon > 0$ and $n \ge n_\epsilon$,*

$$\dim_{\mathrm{H}} \Lambda^n_{\mathbf{p}^n,\epsilon} \le \frac{\sum_{\mathbf{i}} p^n_{\mathbf{i}} \log p^n_{\mathbf{i}}}{\sum_{\mathbf{i}} p^n_{\mathbf{i}} \log b_{\mathbf{i},n}} + t_n(\mathbf{p}^n) + \frac{BC}{n} + 2\epsilon.$$

Proof. Similar to the proof of Lemma 5, using Moran covers $\mathcal{U}^n_{y,r}$. We just note that, according to Corollary 2, instead of (11) we have

$$e^{-C} r \le |v| \le e^C \sqrt{1 + A_n^2},$$

which is also suitable. We got rid of the "sup" using $\delta_{n,\epsilon}$. □

Using compacteness arguments as in the previous chapter we get

$$\dim_{\mathrm{H}} \Lambda \le \sup_{\mathbf{p^n}} \left\{\frac{\sum_{\mathbf{i}} p^n_{\mathbf{i}} \log p^n_{\mathbf{i}}}{\sum_{\mathbf{i}} p^n_{\mathbf{i}} \log b_{\mathbf{i},n}} + t_n(\mathbf{p}^n)\right\} + \frac{BC}{n} + 2\epsilon. \tag{17}$$

Letting $n \to \infty$ and $\epsilon \to 0$, this proves Part 2.

Conclusion

By Part 1, Part 2 and Lemma 6 we have

$$\dim_{\mathrm{H}} \Lambda = \lim_{n\to\infty} \sup_{\mathbf{p}^n} \dim_{\mathrm{H}} \mu_{\mathbf{p}^n}.$$

As in the previous section, for each n, the supremum is attained. Thus

$$\dim_{\mathrm{H}} \Lambda = \lim_{n\to\infty} \dim_{\mathrm{H}} \mu_n,$$

where μ_n is a Bernoulli measure for f^n. Define the *ergodic mean of* μ_n by

$$\tilde{\mu}_n = \frac{1}{n} \sum_{j=0}^{n-1} \mu_n \circ f^{-j}.$$

Then $\tilde{\mu}_n$ is an f-invariant measure on Λ. It is easy to see that $\tilde{\mu}_n$ is ergodic. Since $\tilde{\mu}_n(F) = 1 \Rightarrow \mu_n(F) = 1$, we have $\dim_{\mathrm{H}} \tilde{\mu}_n \geq \dim_{\mathrm{H}} \mu_n$, so

$$\dim_{\mathrm{H}} \Lambda = \lim_{n\to\infty} \dim_{\mathrm{H}} \tilde{\mu}_n.$$

□

REMARK 3. By Lemma 6, (17) and (16), there exists a universal constant $D > 0$ such that, for all $n > 0$,

$$|\dim_{\mathrm{H}} \Lambda - \sup_{\mathbf{p}^n} d_{\mathbf{p}^n}| \leq \frac{D}{n}.$$

REMARK 4. The existence of an invariant measure of full dimension is not trivial, because the function $\mu \mapsto \dim_{\mathrm{H}} \mu$ is not, in general, upper-semicontinuous.

REMARK 5. According to the proof of Theorem A, we have that

$$\dim_{\mathrm{H}} \Lambda = \lim_{n\to\infty} \sup_{\mathbf{p}^n} \{\lambda_n(\mathbf{p}^n) + t_n(\mathbf{p}^n)\},$$

where

$$\lambda_n(\mathbf{p}^n) = \frac{\sum_{\mathbf{i}} p_{\mathbf{i}}^n \log p_{\mathbf{i}}^n}{\sum_{\mathbf{i}} p_{\mathbf{i}}^n \log b_{\mathbf{i},n}},$$

and $t_n(\mathbf{p}^n)$ is defined implicitly by (13). Using the Implicit Function Theorem and Lagrange Multipliers we get that, for each n, the supremum above is attained at a probability vector $\bar{\mathbf{p}}^n$ satisfying

$$\bar{p}_{\mathbf{i}}^n = b_{\mathbf{i},n}^{\lambda_n(\bar{\mathbf{p}}^n)} \Big(\sum_{\mathbf{j}} a_{\mathbf{ij},n}^{t_n(\bar{\mathbf{p}}^n)}\Big)^{\rho_n(\bar{\mathbf{p}}^n)},$$

where

$$\rho_n(\bar{\mathbf{p}}^n) = \frac{\sum_{\mathbf{i}} \bar{p}_{\mathbf{i}}^n \log b_{\mathbf{i},n}}{\sum_{i,j} \bar{q}_{\mathbf{ij}}^n \log a_{\mathbf{ij},n}}, \quad \bar{q}_{\mathbf{ij}}^n = \bar{p}_{\mathbf{i}}^n \frac{a_{\mathbf{ij},n}^{t_n(\bar{\mathbf{p}}^n)}}{\sum_k a_{\mathbf{ik},n}^{t_n(\bar{\mathbf{p}}^n)}}.$$

When defining the number $a_{\mathbf{ij},n}$, we took the maximum over the basic rectangle $R_{\mathbf{ij}}^n$ of the function $(\partial_x a^n)^{-1}$. Because of the bounded distortion property we could have used any other point in this rectangle. Similarly for $b_{\mathbf{i},n}$. So, we can use the n-periodic points $z_{\mathbf{ij},n} \in R_{\mathbf{ij}}^n$ and $z_{\mathbf{i},n} = \pi(z_{\mathbf{ij},n})$ (note that the symbolic dynamics is a Bernoulli shift). In this way, we get that the measures

$$\nu_n = \sum_{\mathbf{i}} \bar{p}_{\mathbf{i}}^n \delta_{z_{\mathbf{i},n}},$$

where δ_z denotes the Dirac measure concentrated at the point z, are b-invariant. Take a sequence (n_k) such that $\nu_{n_k} \rightharpoonup \nu$ (limit on the weak* topology).

Problem: Can we take ν ergodic?

If the answer to this question is yes then we believe that there is an ergodic measure of full Hausdorff dimension.

PROOF OF COROLLARY A. We begin by introducing some notation. Since now we are varying the map f, we explicit the dependence of the objects on the map. Consider the numbers $\underline{\lambda}_{n,f}(\mathbf{p}^n)$ and $\underline{t}_{n,f}(\mathbf{p}^n)$ defined by using minimum instead of maximum when defining the numbers $a_{\mathbf{ij},n,f}$ and $b_{\mathbf{i},n,f}$. If we use points $z_{\mathbf{ij}} \in R_{\mathbf{ij},f}^n$ and $z_{\mathbf{i}} \in R_{\mathbf{i},f}^n$, then we get the numbers $\lambda_{n,f,\{z_{\mathbf{i}}\}}(\mathbf{p}^n)$ and $t_{n,f,\{z_{\mathbf{ij}}\}}(\mathbf{p}^n)$. Note that $\lambda_{n,f}(\mathbf{p}^n)$ and $t_{n,f}(\mathbf{p}^n)$ are increasing functions of $a_{\mathbf{ij},n,f}$ and $b_{\mathbf{i},n,f}$.

In the proof of Theorem A we could have used any of these points so, see Remark 3, we get a constant $D > 0$ such that, for all $f \in B_{\mathrm{C}^2}(f_0, \varepsilon) \cap \mathcal{S}$ and $n > 0$,

$$\sup_{\mathbf{p}^n}\{\lambda_{n,f}(\mathbf{p}^n) + t_{n,f}(\mathbf{p}^n)\} - \frac{D}{n} \leq \dim_{\mathrm{H}} \Lambda_f \leq \sup_{\mathbf{p}^n}\{\underline{\lambda}_{n,f}(\mathbf{p}^n) + \underline{t}_{n,f}(\mathbf{p}^n)\} + \frac{D}{n}. \quad (18)$$

Let us see continuity at f. Let $\delta > 0$. Fix n such that $\frac{D}{n} < \frac{\delta}{3}$. It is easy to see that there exists $\rho > 0$ such that if $g \in B_{\mathrm{C}^2}(f, \rho) \cap \mathcal{S}$ then

$$|\lambda_{n,g,\{z_{\mathbf{i}}\}}(\mathbf{p}^n) - \lambda_{n,f,\{z_{\mathbf{i}}\}}(\mathbf{p}^n)| < \frac{\delta}{3} \quad \text{and} \quad |t_{n,g,\{z_{\mathbf{ij}}\}}(\mathbf{p}^n) - t_{n,f,\{z_{\mathbf{ij}}\}}(\mathbf{p}^n)| < \frac{\delta}{3}$$

for all $\mathbf{p}^n$ and $\{z_{\mathbf{i}}\} \in R^n_{\mathbf{i},f} \cap R^n_{\mathbf{i},g}$, $\{z_{\mathbf{ij}}\} \in R^n_{\mathbf{ij},f} \cap R^n_{\mathbf{ij},g}$. So

$$\begin{aligned}
\underline{\lambda}_{n,f}(\mathbf{p}^n) + \underline{t}_{n,f}(\mathbf{p}^n) &\leq \lambda_{n,f,\{z_{\mathbf{i}}\}}(\mathbf{p}^n) + t_{n,f,\{z_{\mathbf{ij}}\}}(\mathbf{p}^n) \\
&\leq \lambda_{n,g,\{z_{\mathbf{i}}\}}(\mathbf{p}^n) + t_{n,g,\{z_{\mathbf{ij}}\}}(\mathbf{p}^n) + \frac{\delta}{3} \\
&\leq \lambda_{n,g}(\mathbf{p}^n) + t_{n,g}(\mathbf{p}^n) + \frac{\delta}{3}.
\end{aligned}$$

Taking supremum in $\mathbf{p}^n$ and using (18) we get

$$\dim_{\mathrm{H}} \Lambda_f \leq \dim_{\mathrm{H}} \Lambda_g + \delta.$$

Changing the roles of f and g we get the desired result.

□

3. Connection with the relativised variational principle

Let X, Y be compact metric spaces and let $T\colon X \to X$, $S\colon Y \to Y$, $\pi\colon X \to Y$ be continuous maps such that π is surjective and $\pi \circ T = S \circ \pi$. Let $\mathcal{M}(T)$ be the set of all T-invariant probability measures on X. Given $\varphi \in \mathrm{C}(X)$ and $\nu \in \mathcal{M}(S)$, the *relativised variational principle* (see [LW]) says that

$$\sup_{\substack{\mu\in\mathcal{M}(T)\\ \mu\circ\pi^{-1}=\nu}} \left\{h_\mu(T|S) + \int_X \varphi\, d\mu\right\} = \int_Y P(T, \varphi, \pi^{-1}(y))\, d\nu(y), \quad (19)$$

where $h_\mu(T|S)$ denotes the relative metric entropy of T with respect to S, and $P(T, \varphi, Z)$ denotes the relative pressure of T with respect to φ and a compact set $Z \subset X$.

We say that μ is an *equilibrium state* for (19) if the supremum is attained at μ. In [DG, DGH], sufficient conditions on (X, T, Y, S, π) (concerning expansion) are given for the existence of a unique equilibrium state for (19) relative to any $\nu \in \mathcal{M}(S)$ and any Hölder-continuous φ. Moreover, they show that the unique equilibrium state has a *Gibbs property* which will be usefull when calculating its Hausdorff dimension. We are in these conditions with $X = \Lambda$, $Y = \pi(\Lambda)$, $T = f|\Lambda$ and $S = b|\pi(\Lambda)$.

Let $\varphi\colon \mathbb{T}^2 \to \mathbb{R}$ be given by $\varphi = -\log \partial_x a$. Given $\nu \in \mathcal{M}(b|\pi(\Lambda))$, there is a unique real $t(\nu) \in [0, 1]$ such that

$$\int_Y P(f|\Lambda, t(\nu)\varphi, \pi^{-1}(y))\, d\nu(y) = 0.$$

The reason for this is that the function $\psi(t) = P(f|\Lambda, t\varphi, \pi^{-1}(y))$ is continuous, strictly decreasing because $\varphi \leq -\log \min \partial_x a < 0$, $\psi(0) = h(T, \pi^{-1}(y)) \geq 0$, and $\psi(1) \leq 0$. Denote by μ_ν the unique equilibrium state for (19) relative to ν and $t(\nu)\varphi$.

Let $\mathcal{M}_{ep}(b|\pi(\Lambda)) = \mathcal{M}(b|\pi(\Lambda)) \cap \{\text{ergodic and non-atomic}\}$.

LEMMA 10. *If* $\nu \in \mathcal{M}_{ep}(b|\pi(\Lambda))$ *then*

$$\dim_{\mathrm{H}} \mu_\nu = \dim_{\mathrm{H}} \nu + t(\nu).$$

PROOF. It follows from [DG, DGH] that the conditional measures of μ_ν on the fibers $\pi^{-1}(y)$ – $\mu_{\nu,y}$ – have the following *Gibbs property*:

There are positive constants c_1, c_2 and a Hölder-continuous function $A_\nu \colon \mathbb{T}^1 \to \mathbb{R}$ satisfying $\int \log A_\nu \, d\nu = 0$ such that, for **all** y,

$$c_1 \le \frac{\mu_{\nu,y}(R_n \cap \pi^{-1}(y))}{\exp\{t(\nu)S_n\varphi(z) + S_n(\log A_\nu)(y)\}} \le c_2 \tag{20}$$

for all $n \in \mathbb{N}$, n-basic rectangle R_n, and $z \in R_n \cap \pi^{-1}(y)$. As usual, $S_n\varphi(z) = \sum_{j=0}^{n-1} \varphi(f^j(z))$ and $S_n(\log A_\nu)(y) = \sum_{j=0}^{n-1} \log A_\nu \, (b^j(y))$. Then, integrating and using the bounded distortion property for φ and $\log A_\nu$, we find positive constants $\tilde{c_1}$ and $\tilde{c_2}$ such that

$$\tilde{c_1} \le \frac{\mu_\nu(H_n)}{\exp\{t(\nu)S_n\varphi(z) + S_n(\log A_\nu)(y)\}\, \nu(\pi(H_n))} \le \tilde{c_2} \tag{21}$$

for all $n \in \mathbb{N}$, horizontal strip H_n of an n-basic rectangle, and $z \in H_n$, $y = \pi(z)$.

To compute the Hausdorff dimension of μ_ν we shall use the approximate squares defined before (see (14)). We note that Δ, the union of the borders of all basic rectangles, has μ_ν-measure 0. This is because $\nu \in \mathcal{M}_{ep}(b|\pi(\Lambda))$ and (20). Let $z \in \Lambda - \Delta$, $R_n(z)$ be the basic rectangle of order n containing z, and $B_n(z)$ be the approximate square containing z, which is the intersection of the basic rectangle $R_{L_n(z)}(z)$ with the horizontal strip containing $R_n(z)$ of the same height. Then $B_n(z)$ is "approximately" a square with side of length $e^{S_{L_n(z)}\varphi(z)}$. By (21) we get, for some $c > 0$,

$$\frac{\log \mu_\nu(B_n(z))}{S_{L_n(z)}\varphi(z)} = \frac{\log \nu(\pi(R_n(z)))}{S_{L_n(z)}\varphi(z)} + t(\nu) + \underbrace{\frac{\frac{1}{L_n(z)} S_{L_n(z)}(\log A_\nu)(\pi(z))}{\frac{1}{L_n(z)} S_{L_n(z)}\varphi(z)}}_{\alpha} \pm \frac{c}{n}.$$

Using Birkhoff's Ergodic Theorem and $\int \log A_\nu \, d\nu = 0$, we see that $\alpha \to 0$ for μ_ν-a.e. z. Using Birkhoff's Ergodic Theorem, Shannon-Mc Millan-Breiman's Theorem and

$$\frac{S_{L_n(z)}\varphi(z)}{S_n(-\log b')(\pi(z))} = 1 + \frac{1}{n}\frac{S_{L_n(z)}\varphi(z) - S_n(-\log b')(\pi(z))}{\frac{1}{n}S_n(-\log b')(\pi(z))} \to 1,$$

we see that

$$\frac{\log \nu(\pi(R_n(z)))}{S_{L_n(z)}\varphi(z)} \to \frac{h_\nu(b)}{\int \log b' \, d\nu} \quad \text{for } \mu_\nu\text{-a.e. } z.$$

By Lemma 1 and Proposition 3, we conclude that

$$\dim_{\mathrm{H}} \nu = \frac{h_\nu(b)}{\int \log b' \, d\nu}.$$

So,

$$\frac{\log \mu_\nu(B_n(z))}{S_{L_n(z)}\varphi(z)} \to \dim_{\mathrm{H}} \nu + t(\nu) \quad \text{for } \mu_\nu\text{-a.e. } z.$$

Again, by Lemma 1 and Proposition 3, we conclude that

$$\dim_{\mathrm{H}} \mu_\nu = \dim_{\mathrm{H}} \nu + t(\nu).$$

□

THEOREM 2.

$$\dim_{\mathrm{H}} \Lambda = \sup_{\nu \in \mathcal{M}_{ep}(b|\pi(\Lambda))} \dim_{\mathrm{H}} \mu_\nu.$$

PROOF. By the lemma above, we only have to see that

$$\dim_{\mathrm{H}} \Lambda \leq \sup_{\nu \in \mathcal{M}_{ep}(b|\pi(\Lambda))} \{\dim_{\mathrm{H}} \nu + t(\nu)\}. \tag{22}$$

Given $\nu \in \mathcal{M}_{ep}(b|\pi(\Lambda))$ and $n \in \mathbb{N}$, let $t_n(\nu)$ be the unique real satisfying

$$\sum_{\mathbf{i}} \nu(R^n_{\mathbf{i}}) \log\Big(\sum_{\mathbf{j}} a^{t_n(\nu)}_{\mathbf{ij},n}\Big) = 0.$$

We want to compare $t_n(\nu)$ with $t(\nu)$. Let $y \in \pi(\Lambda)$ with unique representation $(\mathbf{i_1}, \mathbf{i_2}, \ldots)$. Consider the n-Bernoulli measure μ_y on Λ_y given by

$$\mu_y(\Delta_{\mathbf{j_1}\ldots\mathbf{j_k}}) = \prod_{l=1}^{k} \frac{a^{t_n(\nu)}_{\mathbf{i_l j_l},n}}{\sum_{\mathbf{j}} a^{t_n(\nu)}_{\mathbf{i_l j},n}}.$$

LEMMA 11. *$\dim_{\mathrm{H}} \mu_y = t_n(\nu) \pm \frac{BC}{n}$ for ν-a.e. y.*

PROOF. The proof is similar to that of Lemma 6:

$$\begin{aligned}\frac{\log \mu_y(\Delta_{\mathbf{j_1}\ldots\mathbf{j_k}})}{\log|\Delta_{\mathbf{j_1}\ldots\mathbf{j_k}}|} &= t_n(\nu) \cdot \frac{\sum_{l=1}^{k} \log a_{\mathbf{i_l j_l},n}}{\log|\Delta_{\mathbf{j_1}\ldots\mathbf{j_k}}|} - \frac{\sum_{\mathbf{i}} \frac{P^n(y,k,\mathbf{i})}{k} \log\Big(\sum_{\mathbf{j}} a^{t_n(\nu)}_{\mathbf{ij},n}\Big)}{\frac{1}{k}\log|\Delta_{\mathbf{j_1}\ldots\mathbf{j_k}}|} \\ &= t_n(\nu)\cdot\alpha - \beta.\end{aligned}$$

Using the bounded distortion property we see that $|\alpha - 1| \leq \frac{BC}{n}$. By Birkhoff's Ergodic Theorem and definition of $t_n(\nu)$ we see that $\beta \to 0$ for ν-a.e. y. The result follows. □

Then

$$\dim_{\mathrm{H}} \Lambda_y \geq t_n(\nu) - \frac{BC}{n} \text{ for } \nu\text{-a.e. } y. \tag{23}$$

On the other hand, by [K] we have that

$$\dim_{\mathrm{H}} \Lambda_y = t(\nu) \text{ for } \nu\text{-a.e. } y, \tag{24}$$

so

$$t_n(\nu) \leq t(\nu) + \frac{BC}{n}.$$

Let us now return to the proof of (22). By the proof of Theorem A (see Remarks 3 and 5), there exists $\nu_n \in \mathcal{M}(b^n|\pi(\Lambda))$, which is Bernoulli for b^n with positive weights, such that

$$\dim_{\mathrm{H}} \Lambda \leq \dim_{\mathrm{H}} \nu_n + t_n(\nu_n) + \frac{D}{n}.$$

Let $\tilde{\nu}_n \in \mathcal{M}_{ep}(b|\pi(\Lambda))$ be the ergodic mean of ν_n, as defined before. Then, $\dim_{\mathrm{H}} \tilde{\nu}_n \geq \dim_{\mathrm{H}} \nu_n$. Since $\tilde{\nu}_n(R^n_{\mathbf{i}}) \geq \nu_n(R^n_{\mathbf{i}})/n$, we have

$$\sum_{\mathbf{i}} \tilde{\nu}_n(R^n_{\mathbf{i}}) \log\Big(\sum_{\mathbf{j}} a^{t_n(\nu_n)}_{\mathbf{ij},n}\Big) \geq 0,$$

which implies $t_n(\tilde{\nu}_n) \geq t_n(\nu_n)$. Thus

$$\dim_{\mathrm{H}} \Lambda \leq \dim_{\mathrm{H}} \tilde{\nu}_n + t_n(\tilde{\nu}_n) + \frac{D}{n} \leq \dim_{\mathrm{H}} \tilde{\nu}_n + t(\tilde{\nu}_n) + \frac{D+BC}{n}.$$

Since n is arbitrarily big, this concludes the proof. □

PROPOSITION 5. *For all $\nu \in \mathcal{M}_{ep}(b|\pi(\Lambda))$,*

$$\dim_{\mathrm{H}} \Lambda_y = \overline{\dim}_{\mathrm{B}} \Lambda_y = t(\nu) \ \textit{ for } \nu\textit{-a.e. } y.$$

PROOF. It follows from Lemma 7 that

$$\overline{\dim}_{\mathrm{B}} \Lambda_y \leq t_n(\nu) + \frac{D}{n} \quad \text{for } \nu\text{-a.e. } y \tag{25}$$

(note that $\bar{t}_{y,n} = t_n(\nu)$ for ν-a.e. y). By (23) and (25) we get

$$\dim_{\mathrm{H}} \Lambda_y = \overline{\dim}_{\mathrm{B}} \Lambda_y \quad \text{for } \nu\text{-a.e. } y.$$

The equality with $t(\nu)$ follows from (24). □

REMARK 6. The map

$$\mathcal{M}_{ep}(b|\pi(\Lambda)) \ni \nu \mapsto \dim_{\mathrm{H}} \nu = \frac{h_\nu(b)}{\int \log b' \, d\nu}$$

is upper-semicontinuous. By (23), (24) and (25) we see that the map $\mathcal{M}_{ep}(b|\pi(\Lambda)) \ni \nu \mapsto t(\nu)$ is continuous. So the map $\mathcal{M}_{ep}(b|\pi(\Lambda)) \ni \nu \mapsto \dim_{\mathrm{H}} \mu_\nu$ is upper-semicontinuous. However, we cannot conclude there is an invariant measure of full dimension because the subset $\mathcal{M}_{ep}(b|\pi(\Lambda)) \subset \mathcal{M}(b|\pi(\Lambda))$ is not closed.

CHAPTER 2

Exceptional sets

The Birkhoff exceptional set $\mathcal{E}_f$ defined in the introduction has 0-measure with respect to every f-invariant probability measure. We would like to caracterize this set from the topological point of view. For instance, if f is *uniquely ergodic*, i.e., has only one invariant probability measure, then $\mathcal{E}_f = \emptyset$. On the other hand, if f has *plenty* invariant probability measures, then $\mathcal{E}_f$ might have full Hausdorff dimension. The goal of this chapter is to prove Theorem B whose heart relies on Theorem A. We begin by describing the conformal setting considered in [BS]. Then, using these ideas and our methods, we extend these results, first for the self-affine case which is simpler, and then for the skew-product case.

1. Revisiting the Bernoulli shift

The results already known, see [BS], concern the hyperbolic and conformal setting which, essentialy, reduces to studying a symbolic dynamics with an appropriate metric. We give a brief description of these results. Consider the topological space $\Sigma = \{1, ..., m\}^{\mathbb{N}}$ and the *Bernoulli shift* $\sigma\colon \Sigma \to \Sigma$ defined by $\sigma(x)_n = x_{n+1}$ for $x = (x_1, x_2, ...) \in \Sigma$. As usual, we define the *cylinder of order* n as

$$C_n(x) = \{y \in \Sigma : y_i = x_i,\ i = 1, ..., n\}.$$

Given a positive function $u \in \mathrm{C}(\Sigma)$, constant on cylinders of order 1, we consider the metric on Σ defined by

$$d(x, y) = e^{-S_N u(x)}$$

where

$$S_n u(x) = \sum_{j=0}^{n-1} u(\sigma^j x) \quad \text{and} \quad N = \inf\{n \geq 1 : x_n \neq y_n\} - 1$$

(if $N = 0$ we put $d(x, y) = 1$). Then $\operatorname{diam}(C_n(x)) = e^{-S_n u(x)}$. Here $|C_n(x)| = n$. In what follows, the function u is fixed and we consider the corresponding metric. The next theorem is taken from [BS].

THEOREM 3. $\dim_{\mathrm{H}} \mathcal{E}_\sigma = \dim_{\mathrm{H}} \Sigma$.

SKETCH OF PROOF. The following statement is the heart of the result:

Given $\epsilon > 0$, there exist distinct ergodic measures μ_1 and μ_2 such that

$$\dim_{\mathrm{H}} \mu_i \geq \dim_{\mathrm{H}} \Sigma - \epsilon,\ i = 1, 2.$$

We assume this statement, which is a consequence of the thermodynamic formalism. Say, $\dim_{\mathrm{H}} \mu_1 \geq \dim_{\mathrm{H}} \mu_2$. Then the argument goes as follows:

(1) Construct a set $\Omega \subset \Sigma$ by *mixing* generic sets for μ_1 and μ_2 in such a way that $\Omega \subset \mathcal{E}_\sigma$.
(2) Construct a probability μ on Ω by *mixing* μ_1 and μ_2 in such a way that $\dim_{\mathrm{H}} \mu \geq \dim_{\mathrm{H}} \mu_2 - 2\epsilon$.

(3) Then,

$$\dim_{\mathrm{H}} \mathcal{E}_\sigma \geq \dim_{\mathrm{H}} \Omega \geq \dim_{\mathrm{H}} \mu \geq \dim_{\mathrm{H}} \Sigma - 3\epsilon,$$

and the result follows because ϵ can be taken arbitrarily small.

Since $\mu_1 \neq \mu_2$, there exists $g \in \mathrm{C}(\Sigma)$ such that $\int g\,d\mu_1 \neq \int g\,d\mu_2$. Let $\delta \in (0, \epsilon)$ be such that $|\int g\,d\mu_1 - \int g\,d\mu_2| > 4\delta$. For each $i = 1, 2$ and $l \in \mathbb{N}$, let Γ_i^l be the set of points x such that for $n \geq l$ one has

$$\left|\frac{1}{n} S_n g(x) - \int g\,d\mu_i\right| < \delta \quad \text{and} \quad -\frac{\log \mu_i(C_n(x))}{S_n u(x)} > \dim_{\mathrm{H}} \mu_i - \delta.$$

By Birkhoff's Ergodic Theorem and Shannon-Mc Millan-Breiman's Theorem, there exists an increasing sequence of positive integers l_s such that, for every s,

$$\mu_{p_s}(\Gamma_{p_s}^{l_s}) > 1 - \frac{\epsilon}{2^s},$$

where $p_s = s \,(\mathrm{mod}\, 2)$. Let m_s be the increasing sequence of positive integers defined inductively by $m_1 = l_1$, $m_s = (m_{s-1} + l_{s+1})!$ (so that $\frac{l_{s+1}}{m_s} \to 0$ and $\frac{m_{s-1}}{m_s} \to 0$). Define the families of cylinder sets by

$$\mathcal{C}_s = \{C_{m_s}(x) : x \in \Gamma_{p_s}^{l_s}\},$$

and

$$\mathcal{D}_1 = \mathcal{C}_1, \ \mathcal{D}_s = \{\underline{C}\overline{C} : \underline{C} \in \mathcal{D}_{s-1}, \, \overline{C} \in \mathcal{C}_s\}.$$

Set

$$\Omega = \bigcap_{s \geq 1} \bigcup_{C \in \mathcal{D}_s} C.$$

Thus, if $x \in \Omega$ then for every $s \geq 1$ there exists $C \in \mathcal{D}_s$ such that $x \in C$, and it is not difficult to see that, for sufficiently large s,

$$\left|\frac{1}{|C|} \sum_{j=0}^{|C|} g(\sigma^j x) - \int g\,d\mu_{p_s}\right| < 2\delta.$$

This implies that $\Omega \subset \mathcal{E}_\sigma$.

We now construct a measure μ on Ω by

$$\begin{cases} \mu(C) = \mu_1(C) & \text{if } C \in \mathcal{D}_1 \\ \mu(\underline{C}\overline{C}) = \mu(\underline{C})\mu_{p_s}(\overline{C}) & \text{if } \underline{C}\overline{C} \in \mathcal{D}_s, \, s > 1. \end{cases}$$

Then

$$\mu(\Omega) \geq \prod_{s=1}^{\infty} \left(1 - \frac{\epsilon}{2^s}\right) > 0,$$

and we can normalize μ to make it a probability measure. In order to estimate the dimension of μ, we shall prove that

$$\varliminf_{n \to \infty} -\frac{\log \mu(C_n(x)}{S_n u(x)} \geq \dim_{\mathrm{H}} \mu_2 - 2\delta, \quad \text{for every } \ x \in \Omega.$$

Given $x \in \Omega$ and $q \in \mathbb{N}$, let s_q be such that $|C^{s_q}| \leq q \leq |C^{s_q+1}|$ where

$$\mathcal{D}_{s_q+1} \ni C^{s_q+1} \subset C_q(x) \subset C^{s_q} \in \mathcal{D}_{s_q}.$$

Let us consider two cases. First, suppose that

$$|C^{s_q}| \leq q \leq |C^{s_q}| + l_{s_q+1}. \tag{26}$$

Then $\frac{q}{|C^{s_q}|} \to 1$ when $q \to \infty$, and

$$-\frac{\log \mu(C_q(x))}{S_q u(x)} \geq -\frac{\log \mu(C^{s_q})}{S_{|C^{s_q}|} u(x)} \cdot \overbrace{\frac{S_{|C^{s_q}|} u(x)}{S_q u(x)}}^{\to 1} \geq \dim_{\mathrm{H}} \mu_2 - 2\delta,$$

for suficiently large q. If (26) does not hold, then

$$\mu(C_q(x)) = \mu(C^{s_q})\mu_{p_{s_q+1}}(C),$$

where $C_q(x) = C^{s_q}C$, C contains an element of $\mathcal{C}_{s_q+1}$ and $|C| > l_{s_q+1}$. Then

$$\begin{aligned}
-\frac{\log\mu(C_q(x))}{S_q u(x)} &= -\frac{\log\mu(C^{s_q})}{S_{|C^{s_q}|}u(x)}\cdot\frac{S_{|C^{s_q}|}u(x)}{S_q u(x)}\\
&\quad -\frac{\log\mu_{p_{s_q+1}}(C)}{S_{|C|}u(\sigma^{q-|C|}x)}\cdot\frac{S_{|C|}u(\sigma^{q-|C|}x)}{S_q u(x)}\\
&\geq(\dim_{\mathrm{H}}\mu_2 - 2\delta)\cdot\underbrace{\frac{S_{|C^{s_q}|}u(x) + S_{|C|}u(\sigma^{q-|C|}x)}{S_q u(x)}}_{=1},
\end{aligned}$$

for suficiently large q. By Proposition 3, this implies that $\dim_{\mathrm{H}}\mu \geq \dim_{\mathrm{H}}\mu_2 - 2\delta$, as we wish.

□

2. The self-affine case

Here we consider the transformations described in Section 1 of Chapter 1. Our dynamics $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ is given by $f(z) = A_{ij}^{-1}z$ if $z \in R_{ij}$.

THEOREM 4. $\dim_{\mathrm{H}}\mathcal{E}_{f|\Lambda} = \dim_{\mathrm{H}}\Lambda$.

PROOF. By Theorem 1 and its corollary, we can find a probability vector $\mathbf{p}^*$ such that

$$\dim_{\mathrm{H}}\Lambda = \dim_{\mathrm{H}}\mu_{\mathbf{p}^*} = \dim_{\mathrm{H}}\nu_{\mathbf{p}^*} + t(\mathbf{p}^*).$$

If there exists some i such that $p_i^* = 1$, then the problem is essentially one-dimensional and it reduces to the previous section. So, we assume that $p_i^* < 1$ for all i, which avoids technicalities envolving the borders of basic rectangles. Given $0 < \delta < \dim_{\mathrm{H}}\nu_{\mathbf{p}^*}$, we can find $\mathbf{p}$ (with $p_i < 1$ for all i) such that

- $\nu_{\mathbf{p}} \neq \nu_{\mathbf{p}^*}$,
- $\dim_{\mathrm{H}}\nu_{\mathbf{p}} > \dim_{\mathrm{H}}\nu_{\mathbf{p}^*} - \delta$,
- $\left|\sum_{i=1}^m p_i \log\left(\sum_{j=1}^{n_i} a_{ij}^{t(\mathbf{p}^*)}\right)\right| < \delta\cdot\log(\min b_i^{-1})$.

Let $b\colon \mathbb{T}^1 \to \mathbb{T}^1$ be the dynamics induced by f on the vertical, i.e, $b\circ\pi = \pi\circ f$ (we say that b is a factor of f). By the techniques of the previous section, there exist a set $\Upsilon \subset \mathcal{E}_{b|\pi(\Lambda)}$ and a measure ν on $\pi(\Lambda)$ with $\nu(\Upsilon) = 1$, such that, for all $y = \chi_V(i_1, i_2, ...) \in \Upsilon$ and all sufficiently large n,

- $\dfrac{\log\nu(I_n(y))}{\sum_{l=1}^n \log b_{i_l}} \geq \dim_{\mathrm{H}}\nu_{\mathbf{p}^*} - \delta,$ (27)
- $\left|\sum_{i=1}^m \dfrac{P(y,n,i)}{n}\log\left(\sum_{j=1}^{n_i} a_{ij}^{t(\mathbf{p}^*)}\right)\right| \leq \delta\cdot\log(\min b_i^{-1}).$ (28)

We construct a probability measure μ on Λ by

$$\mu(R_{(i_1j_1)(i_2j_2)\dots(i_nj_n)}) = \nu(I_n(y))\cdot\prod_{l=1}^n \frac{a_{i_lj_l}^{t(\mathbf{p}^*)}}{\sum_{j=1}^{n_{i_l}} a_{i_lj}^{t(\mathbf{p}^*)}}.$$

Then $\pi^{-1}(\Upsilon) \subset \mathcal{E}_{f|\Lambda}$ and $\mu(\pi^{-1}(\Upsilon)) = 1$. As in the proof of Lemma 2, we have that

$$\frac{\log \mu(\chi(B_n(\omega)))}{\sum_{l=1}^{n} \log b_{i_l}} = \frac{\log \nu(I_n(y))}{\sum_{l=1}^{n} \log b_{i_l}} + t(\mathbf{p}^*) \cdot \frac{\sum_{l=1}^{L_n(\omega)} \log a_{i_l j_l}}{\sum_{l=1}^{n} \log b_{i_l}} - \frac{\sum_{i=1}^{m} \frac{P(y, L_n(\omega), i)}{L_n(\omega)} \log\Big(\sum_{j=1}^{n_{i_l}} a_{i_l j}^{t(\mathbf{p}^*)}\Big)}{\frac{n}{L_n(\omega)} \frac{1}{n} \sum_{l=1}^{n} \log b_{i_l}}.$$

Attending to (27), (28) and

$$\frac{\sum_{l=1}^{L_n(\omega)} \log a_{i_l j_l}}{\sum_{l=1}^{n} \log b_{i_l}} \to 1,$$

we get that, for all $\chi(\omega) \in \pi^{-1}(\Upsilon)$ and all sufficiently large n,

$$\frac{\log \mu(\chi(B_n(\omega)))}{\sum_{l=1}^{n} \log b_{i_l}} \geq \dim_{\mathrm{H}} \nu_{\mathbf{p}^*} + t(\mathbf{p}^*) - 3\delta.$$

By Lemma 1 and Proposition 3, we get that $\dim_{\mathrm{H}} \mu \geq \dim_{\mathrm{H}} \Lambda - 3\delta$, thus

$$\dim_{\mathrm{H}} \mathcal{E}_{f|\Lambda} \geq \dim_{\mathrm{H}} \pi^{-1}(\Upsilon) \geq \dim_{\mathrm{H}} \mu \geq \dim_{\mathrm{H}} \Lambda - 3\delta.$$

Letting $\delta \to 0$, we get the desired result.

$\square$

3. The skew-product case

Now f and Λ are like in Theorem A.

Remark 7. It follows easily from Theorem 4 and the proof of Theorem A that,

$$\dim_{\mathrm{H}}\Big(\bigcup_{n=1}^{\infty} \mathcal{E}_{f^n|\Lambda}\Big) = \dim_{\mathrm{H}} \Lambda.$$

But, using Section 3 of Chapter 1, we can say more.

Proof of Theorem B. Given $\epsilon > 0$, by Theorem 2 there exist $n \in \mathbb{N}$ and a Bernoulli measure ν_n for $b^n|\pi(\Lambda)$ with positive weights such that

$$\dim_{\mathrm{H}} \tilde{\nu}_n + t(\tilde{\nu}_n) > \dim_{\mathrm{H}} \Lambda - \epsilon,$$

where, as before, $\tilde{\nu}_n$ is the ergodic mean of ν_n. As in Lemma 10, we consider the Hölder-continuous function $A_{\tilde{\nu_n}}$ satisfying

$$\int \log A_{\tilde{\nu_n}} \, d\tilde{\nu_n} = 0.$$

It is easy to see that there exists another Bernoulli measure η_n for $b^n|\pi(\Lambda)$ with positive weights such that, for $0 < \delta < \min\{\epsilon, \dim_{\mathrm{H}} \tilde{\nu}_n\}$,

$$\tilde{\eta}_n \neq \tilde{\nu}_n, \tag{29}$$

$$\dim_{\mathrm{H}} \tilde{\eta}_n > \dim_{\mathrm{H}} \tilde{\nu}_n - \delta, \tag{30}$$

$$\Big| \int \log A_{\tilde{\nu}_n} \, d\tilde{\eta}_n \Big| < \delta. \tag{31}$$

In fact, if $p_{i_1 i_2 \ldots i_n}^n$ are the weights of ν_n, choose two indexes, i_+ and i_-, and construct η_n with weights given by

$$\begin{aligned}
q_{i_1 i_2 \ldots i_n}^n &= p_{i_1 i_2 \ldots i_n}^n \quad \text{if } i_1 \neq i_+ \text{ and } i_1 \neq i_-, \\
q_{i_+ i_2 \ldots i_n}^n &= p_{i_+ i_2 \ldots i_n}^n + \rho, \\
q_{i_- i_2 \ldots i_n}^n &= p_{i_- i_2 \ldots i_n}^n - \rho,
\end{aligned}$$

where $\rho > 0$ is small. So, $\tilde{\eta}_n(R_{i_+ i_2 \dots i_n}) \neq \tilde{\nu}_n(R_{i_+ i_2 \dots i_n})$ and (29) follows. Since η_n is arbitrarly close to ν_n, (30) and (31) follow by continuity. Then, as in Theorems 3 and 4, we construct a set $\Upsilon \subset \mathcal{E}_{b|\pi(\Lambda)}$ and a measure ν on $\pi(\Lambda)$ with $\nu(\Upsilon) = 1$, such that, for all $y \in \Upsilon$ and all sufficiently large k,

$$\frac{\log \nu(I_k(y))}{S_k(-\log b')(y)} \geq \dim_{\mathrm{H}} \tilde{\nu}_n - \delta, \tag{32}$$

$$\left| \frac{1}{k} S_k(\log A_{\tilde{\nu}_n})(y) \right| \leq \delta \cdot \min \log b'. \tag{33}$$

Now, we consider the probability measure μ on Λ given by

$$\mu = \nu \times \mu_{\tilde{\nu}_n, y},$$

where $\mu_{\tilde{\nu}_n, y}$ are the conditional measures of $\mu_{\tilde{\nu}_n}$ on the fibers $\pi^{-1}(y)$ (see Lemma 10). Then $\pi^{-1}(\Upsilon) \subset \mathcal{E}_{f|\Lambda}$ and $\mu(\pi^{-1}(\Upsilon)) = 1$. Using (21) we get, for some $c > 0$ and all $z \in \pi^{-1}(\Upsilon)$, $y = \pi(z)$,

$$\begin{aligned} \frac{\log \mu(B_k(z))}{S_k(-\log b')(y)} =& \frac{\log \nu(I_k(y))}{S_k(-\log b')(y)} + t(\tilde{\nu}_n) \cdot \frac{S_{L_k(z)}\varphi(z)}{S_k(-\log b')(y)} \\ &+ \frac{\frac{1}{L_k(z)} S_{L_k(z)}(\log A_{\tilde{\nu}_n})(y)}{\frac{k}{L_k(z)} \frac{1}{k} S_k(-\log b')(y)} \pm \frac{c}{k}. \end{aligned}$$

Using (32), (33) and

$$\frac{S_{L_k(z)}\varphi(z)}{S_k(-\log b')(y)} \to 1,$$

we get that, for all $z \in \pi^{-1}(\Upsilon)$ and all sufficiently large k,

$$\frac{\log \mu(B_k(z))}{S_k(-\log b')(y)} \geq \dim_{\mathrm{H}} \tilde{\nu}_n + t(\tilde{\nu}_n) - 3\delta.$$

This implies that $\dim_{\mathrm{H}} \mu \geq \dim_{\mathrm{H}} \Lambda - 4\epsilon$ and

$$\dim_{\mathrm{H}} \mathcal{E}_{f|\Lambda} \geq \dim_{\mathrm{H}} \pi^{-1}(\Upsilon) \geq \dim_{\mathrm{H}} \mu \geq \dim_{\mathrm{H}} \Lambda - 4\epsilon.$$

Letting $\epsilon \to 0$, we get the desired result.

□

Commentaries

In Theorems A and B we took an ε-perturbation of a general Sierpinski carpet. The reason for doing that is to inherit from the linear system a domination condition plus a Markov structure. Taking as reference any other linear system and noting that, by [S], any expandig map $f\colon \mathbb{T}^n \to \mathbb{T}^n$ is topologically conjugate to a linear expanding endomorphism (not necessarly close to each other), perhaps we could consider more general systems.

In Theorems A and B we considered invariant sets which are *continuations* of general Sierpinski carpets, thus posessing a Bernoulli shift as symbolic dynamics. Using the techniques of [GP] perhaps we could consider more general invariant sets, such as subshifts of finite type.

A non-trivial problem related to the variational principle is the existence (and uniqueness) of an ergodic invariant measure of full Hausdorff dimension, see Remark 4. As mentioned is Remarks 5 and 6, this problem is related to the problem of determining whether a limiting measure is ergodic or not. See the problem proposed in Remark 5.

In this work we restricted ourselfs to skew-product maps. More general systems present a new phenomenon: the horizontal lines turn into an invariant *fractal foliation*. The leaves of this fractal foliation are graphs of Hölder-continuous maps of the x-axis, which are the images by the conjugation of the horizontal lines.

Example. Let us consider a perturbation of the form

$$f(x,y) = (lx, m(y-\beta(x))),$$

where $\beta\colon \mathbb{T}^1 \to \mathbb{T}^1$ is C^1-close to 0. Solving the equation $h \circ f_0 = f \circ h$ with $h(x,y) = (x, y+\phi(x))$, we come to

$$\phi(x) = \frac{1}{m}\phi(lx) + \beta(x).$$

By the contracting mapping principle this equation has a unique continuous solution ϕ, which in this case is given explicitly by

$$\phi(x) = \sum_{n=0}^{\infty} m^{-n}\beta(l^n x).$$

This is an example of a *Weierstrass function*, which were introduced (by Weierstrass), with $\beta(x) = \cos(2\pi x)$, to give examples of Hölder-continuous functions that are not differentiable at any point. In this case, the leaves of the fractal foliation are just vertical translations of the graph of ϕ.

More generally, let $\psi\colon [0,1] \to \mathbb{R}$ be given by

$$\psi(x) = \sum_{n=0}^{\infty} \lambda^{-sn}\beta(\lambda^n x),$$

where $\lambda > 1$, $0 < s < 1$ and $\beta\colon \mathbb{R} \to \mathbb{R}$ is a C^1 periodic function with period 1. Denote by $\mathcal{G}(\psi)$ the graph of ψ. A major problem in Dimension Theory is

Conjecture: $\dim_{\mathrm{H}} \mathcal{G}(\psi) = 2 - s$ "for most λ and β".

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