EXISTENCE AND STABILITY OF EQUILIBRIUM STATES FOR ROBUST CLASSES OF NON-UNIFORMLY HYPERBOLIC MAPS

DOCTORAL THESIS OF PAULO VARANDAS
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Abstract. We prove existence of finitely many ergodic equilibrium states for a large class of non-uniformly expanding local diffeomorphisms on compact manifolds and Hölder continuous potentials with not very large oscillation. No Markov structure is assumed. If the transformation is topologically mixing there is a unique equilibrium state, it is exact and satisfies a non-uniform Gibbs property. Under mild additional assumptions we also prove that the equilibrium states vary continuously with the dynamics and the potentials (statistical stability) and are also stable under stochastic perturbations of the transformation.

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À memória de
meu avô Marcelo

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1. Introduction

The theory of equilibrium states of smooth dynamical systems was initiated by the pioneer works of Sinai, Ruelle, Bowen [Sin72, BR75, Bow75, Rue76]. For uniformly hyperbolic diffeomorphisms and flows they proved that equilibrium states exist and are unique for every Hölder continuous potential, restricted to every basic piece of the non-wandering set. The basic strategy to prove this remarkable fact was to (semi)conjugate the dynamics to a subshift of finite type, via a Markov partition.

Several important difficulties arise when trying to extend this theory beyond the uniformly hyperbolic setting and, despite significant progress by several authors, a global picture is still very far from complete. For one thing, existence of generating Markov partitions is known only in a few cases and, often, such partitions can not be finite. Moreover, equilibrium states may actually fail to exist if the system exhibits critical points or singularities (see Buzzi [Buz01]).

A natural starting point is to try and develop the theory first for smooth systems which are hyperbolic in the sense of Pesin theory: Lyapunov exponents are non-zero "almost everywhere". This approach was advocated in Alves, Bonatti, Viana [ABV00] where the authors assume non-uniform hyperbolicity at *Lebesgue* almost every point and deduce existence and finiteness of physical (Sinai-Ruelle-Bowen) measures. In this setting, physical measures are absolutely continuous with respect to Lebesgue measure, along expanding directions. However, it was not immediately clear how this kind of hypothesis may be useful in general, since one expects most equilibrium states to be singular with respect to Lebesgue measure.

Nevertheless, in a series of recent works, Oliveira, Viana [OV07] managed to push this idea ahead and prove existence and prove existence and uniqueness of equilibrium states for a fairly large class of smooth transformations on compact manifolds, inspired by [ABV00]. Roughly speaking, they assume the transformation is expands on most of the phase space, possibly with some mild contracting behavior on the complement. Moreover, they assume the potential is Hölder continuous and its oscillation $\sup \phi - \inf \phi$ is not too big. However, their approach is limited by a number of additional conditions that do not seem natural, including the existence of a Markov partition. In a similar context, Arbieto, Matheus [AM] prove the equilibrium states exhibit exponential decay of correlations for all Hölder continuous observables. More recently, Pinheiro [Pin07] constructed general inducing schemes for a large class of non-invertible transformations that contain the setting of [ABV00] and every measure satisfying very weak expanding conditions.

Other important contributions outside of the uniformly hyperbolic setting include works by Leplaideur, Rios [LR06] on horsehoes with tangencies at the boundary of hyperbolic systems, Bruin, Keller, Todd [BK98, BT06, BT07], Pesin, Senti [PS05, PS06, PSZ07], Denker, Przytycki, Urbanski [DU91a, DU91b, DPU96] on one-dimensional maps, inducing schemes and rational functions on the Riemann sphere, Buzzi, Maume, Paccaut, Sarig, Schmitt [Buz99, BPS01, BMD02, BS03, Sar99, Sar03] and Yuri [Yur99, Yur03] on piecewise expanding maps in higher dimensions, countable Markov shifts and maps with indifferent periodic points, just to refer to some of the most recent advances.

In this work we build a unified and rather complete theory of existence and finiteness of equilibrium states for the class of non-uniformly expanding transformations originally proposed in [ABV00, Appendix], that strictly contains the setting of

[OV07] and includes, for instance, transformations derived from expanding ones via deformation by isotopy.

One important point is that we completely remove the need for a Markov partition (generating or not). In fact, one of the technical novelties with respect to previous recent works in this area is that we prove, in an abstract way inspired by Ledrappier [Led84], that every equilibrium state must be absolutely continuous with respect to a certain conformal measure. When the map is topologically mixing, the equilibrium state is unique, and a non-lacunary Gibbs measure. We also prove stability of the equilibrium states under random noise (stochastic stability) and continuity under variations of the dynamics (statistical stability).

Our strategy to prove these results combines notions from the qualitative theory of non-uniformly hyperbolic dynamical systems (e.g. Pesin's local unstable leaves) with the quantitative notion of hyperbolic times introduced in [Alv00, ABV00] and goes as follows. First we construct an expanding conformal measure ν as a special eigenmeasure of the dual of the Ruelle-Perron-Frobenius operator. Then we show that every accumulation point μ of the Cèsaro sum of the push-forwards $f_*^n\nu$ is an invariant probability measure that is absolutely continuous with respect to ν with density bounded away from infinity, and that there are finitely many distinct such ergodic measures. In addition, we prove that these absolutely continuous invariant measures are equilibrium states, and that any equilibrium state is necessarily an expanding measure. Finally, we establish an abstract version of Ledrappier's theorem [Led84] and characterize equilibrium states as invariant measures absolutely continuous with respect to ν.

This paper is organized as follows. The precise statement of our results is given in Section 2. We included in Section 3 preparatory material that will be necessary for the proofs. Following the approach described above, we construct an expanding conformal measure and prove that there are finitely many invariant and ergodic measures absolutely continuous with respect it through Sections 4 and 5. In Section 6 we prove Theorems A and B. In Section 7 we prove the stochastic and statistical stability results stated in Theorems C and D. Some examples are presented in Section 8. Finally, in Section 9 we discuss our assumptions, interesting problems and some perspectives.

2. Statement of the results

2.1. **Hypotheses.** We consider M to be a connected, compact Riemannian manifold of dimension m without boundary and d be the distance induced by the Riemann metric. For all our results we assume that f and ϕ satisfy conditions (H1), (H2), and (P) stated in what follows.

Let $f : M \to M$ denote a $C^{1+\alpha}$ local diffeomorphism, for some $\alpha \in (0,1)$, and assume that there exist constants $\sigma > 1$ and $L > 0$, and an open region $\mathcal{A} \subset M$ such that

- (H1) $\|Df(x)^{-1}\| \le L$ for every $x \in \mathcal{A}$ and $\|Df(x)^{-1}\| \le \sigma^{-1}$ for all $x \in M \backslash \mathcal{A}$, and L is close to 1: the precise conditions are given in (3.2) and (3.3) below.
- (H2) There exists $k_0 \ge 1$ and a covering $\mathcal{P} = \{P_1, \dots, P_{k_0}\}$ of M by domains of injectivity for f such that $\mathcal{A}$ can be covered by $q < \deg(f)$ elements of $\mathcal{P}$.

The first condition means that we allow expanding and contracting behavior to coexist in M: f is uniformly expanding outside $\mathcal{A}$ and not too contracting inside

$\mathcal{A}$. The second one requires that every point has at least one preimage in the expanding region.

In addition we assume that $\phi : M \to \mathbb{R}$ is Hölder continuous and that its variation is not too big. More precisely, assume that:

(P) $\sup \phi < \inf \phi + \log \deg(f) - \log q$.

Notice this is an open condition on the potential, relative to the uniform norm, and it is satisfied by constant functions. It can be weakened somewhat. For one thing, all we need for our estimates is the supremum of ϕ over the union of the elements of $\mathcal{P}$ that intersect $\mathcal{A}$. With some extra effort (replacing the q elements of $\mathcal{P}$ that intersect $\mathcal{A}$ by the same number of smaller domains), one may even consider the supremum over $\mathcal{A}$. However, we do not use nor prove this fact here.

Let us comment on this hypothesis. A related condition, $P_{\text{top}}(f, \phi) > \sup \phi$, was introduced by Denker, Urbański [DU91a] in the context of rational maps on the sphere. Another related condition, $P(f, \phi, \partial\mathcal{Z}) < P(f, \phi)$, is used by Buzzi, Paccaut, Schmitt [BPS01], in the context of piecewise expanding multidimensional maps, to control the map's behavior at the boundary $\partial\mathcal{Z}$ of the domains of smoothness: without such a control, equilibrium states may fail to exist [Buz01]. Condition (P) seems to play a similar role in our setting.

2.2. **Existence of equilibrium states.** We say that f is *topologically mixing* if, for each open set U there is a positive integer N so that $f^N(U) = M$. Let $\mathcal{B}$ denote the Borel σ-algebra of M. An f-invariant probability measure η is *exact* if the σ-algebra $\mathcal{B}_\infty = \cap_{n\geq 0} f^{-n}\mathcal{B}$ is η-trivial, meaning that it contains only zero and full η-measure sets. Given a continuous map $f : M \to M$ and a potential $\phi : M \to \mathbb{R}$, the variational principle for the pressure asserts that

$$P_{\text{top}}(f, \phi) = \sup \left\{ h_\mu(f) + \int \phi \, d\mu : \mu \text{ is } f\text{-invariant} \right\}$$

where $P_{\text{top}}(f, \phi)$ denotes the topological pressure of f with respect to ϕ and $h_\mu(f)$ denotes the metric entropy. An *equilibrium state* for f with respect to ϕ is an invariant measure that attains the supremum in the right hand side above.

Theorem A. *Let $f : M \to M$ be a local diffeomorphism and $\phi : M \to \mathbb{R}$ a Hölder continuous potential satisfying (H1), (H2), and (P). Then, there is a finite number of ergodic equilibrium states $\mu_1, \mu_2, \ldots, \mu_k$ for f with respect to ϕ such that any equilibrium state μ is a convex linear combination of $\mu_1, \mu_2, \ldots, \mu_k$. In addition, if the map f is topologically mixing then the equilibrium state is unique and exact.*

Our strategy for the construction of equilibrium states is, first to construct a certain conformal measure ν which is expanding and a non-lacunary Gibbs measure. Then we construct the equilibrium states, which are absolutely continuous with respect to this reference measure ν. Both steps explore a weak hyperbolicity property of the system. In what follows we give precise definitions of the notions involved.

A probability measure ν, not necessarily invariant, is *conformal* if there exists some function $\psi : M \to \mathbb{R}$ such that

$$\nu(f(A)) = \int_A e^{-\psi} d\nu$$

for every measurable set A such that $f \mid A$ is injective.

Let $S_n\phi = \sum_{j=0}^{n-1} \phi \circ f^j$ denote the nth Birkhoff sum of a function ϕ. The *dynamical ball* of center $x \in M$, radius $\delta > 0$, and length $n \geq 1$ is defined by

$$B(x,n,\delta) = \{y \in M : d(f^j(y), f^j(x)) \leq \delta, \ \forall\, 0 \leq j \leq n\}.$$

An integer sequence $(n_k)_{k\geq 1}$ is *non-lacunary* if it is increasing and $n_{k+1}/n_k \to 1$ when $k \to \infty$.

Definition 2.1. A probability measure ν is a *non-lacunary Gibbs measure* if there exist uniform constants $K > 0$, $P \in \mathbb{R}$ and $\delta > 0$ so that, for ν-almost every $x \in M$ there exists some non-lacunary sequence $(n_k)_{k\geq 1}$ such that

$$K^{-1} \leq \frac{\nu(B(x,n_k,\delta))}{\exp(-P\, n_k + S_{n_k}\phi(y))} \leq K$$

for every $y \in B(x, n_k, \delta)$ and every $k \geq 1$.

The weak hyperbolicity property of f is expressed through the notion of hyperbolic times, which was introduced in [Alv00, ABV00]. We say that n is a *c-hyperbolic time* for $x \in M$ if

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| < e^{-ck} \quad \text{for every } 1 \leq k \leq n. \tag{2.1}$$

Often we just call them hyperbolic times, since the constant c will be fixed, as in (3.2). We denote by H the set of points $x \in M$ with infinitely many hyperbolic times and by H_j the set of points having $j \geq 1$ as hyperbolic time. A probability measure ν, not necessarily invariant, is *expanding* if $\nu(H) = 1$.

The *basin of attraction* of an f-invariant probability measure μ is the set $B(\mu)$ of points $x \in M$ such that

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \text{ converges weakly to } \mu \text{ when } n \to \infty.$$

Theorem B. *Let $f : M \to M$ be a local diffeomorphism and $\phi : M \to \mathbb{R}$ be a Hölder continuous potential satisfying (H1), (H2), and (P). Let $\mu_1, \mu_2, \ldots, \mu_k$ be the ergodic equilibrium states of f for ϕ. Then every μ_i is absolutely continuous with respect to some conformal, expanding, non-lacunary Gibbs measure ν. The union of all basins of attraction $B(\mu_i)$ contains ν-almost every point $x \in M$. If, in addition, f is topologically mixing then the unique absolutely continuous invariant measure μ is a non-lacunary Gibbs measure.*

As a byproduct of the previous results we can obtain the existence of equilibrium states for *continuous* potentials satisfying (P). Without some extra condition no finitude of equilibrium states is expected to hold.

Corollary 1. *Let $f : M \to M$ be a local diffeomorphism satisfying (H1) and (H2). If $\phi : M \to \mathbb{R}$ is a continuous potential satisfying (P) then there exists an equilibrium state for f with respect to ϕ.*

2.3. **Stability of equilibrium states.** Let $\mathcal{F}$ be a family of local diffeomorphisms and $\mathcal{W}$ be some family of continuous potentials ϕ. A pair $(f, \phi) \in \mathcal{F} \times \mathcal{W}$ is *statistically stable* (relative to $\mathcal{F} \times \mathcal{W}$) if, for any sequences $f_n \in \mathcal{F}$ converging to f in the $C^{1+\alpha}$-topology and $\phi_n \in \mathcal{W}$ converging to ϕ in the uniform topology, and for any choice of an equilibrium state μ_n of f_n for ϕ_n, every weak* accumulation point of the sequence $(\mu_n)_{n\geq 1}$ is an equilibrium state of f for ϕ. In particular, when the equilibrium state is unique, statistical stability means that it depends continuously on the data (f, ϕ).

Theorem C. *Suppose every $(f, \phi) \in \mathcal{F} \times \mathcal{W}$ satisfies (H1), (H2), and (P), with uniform constants (including the Hölder constants of ϕ). Assume that the topological pressure $P_{\mathrm{top}}(f, \phi)$ varies continuously in the parameters $(f, \phi) \in \mathcal{F} \times \mathcal{W}$. Then every pair $(f, \phi) \in \mathcal{F} \times \mathcal{W}$ is statistically stable relative to $\mathcal{F} \times \mathcal{W}$.*

The assumption on continuous variation of the topological pressure might hold in great generality in this setting. See the comment at the end of Subsection 7.1 for a discussion.

Now let $\mathcal{F}$ be a family of local diffeomorphisms satisfying (H1) and (H2) with uniform constants. A *random perturbation* of $f \in \mathcal{F}$ is a family θ_ε, $0 < \varepsilon \leq 1$ of probability measures in $\mathcal{F}$ such that there exists a family $V_\varepsilon(f)$, $0 < \varepsilon \leq 1$ of neighborhoods of f, depending monotonically on ε and satisfying

$$\operatorname{supp} \theta_\varepsilon \subset V_\varepsilon(f) \quad \text{and} \quad \bigcap_{0<\varepsilon\leq 1} V_\varepsilon(f) = \{f\}.$$

Consider the skew product map

$$\begin{array}{rcl} F : \mathcal{F}^{\mathbb{N}} \times M & \to & \mathcal{F} \times M \\ (\underline{\mathrm{f}}, x) & \mapsto & (\sigma(\underline{\mathrm{f}}), f_1(x)) \end{array}$$

where $\underline{\mathrm{f}} = (f_1, f_2, \ldots)$ and $\sigma : \mathcal{F}^{\mathbb{N}} \to \mathcal{F}^{\mathbb{N}}$ is the shift to the left. For each $\varepsilon > 0$, a measure μ^ε on M is *stationary* (respectively, *ergodic*) for the random perturbation if the measure $\theta_\varepsilon^{\mathbb{N}} \times \mu^\varepsilon$ on $\mathcal{F}^{\mathbb{N}} \times M$ is invariant (respectively, ergodic) for F.

We assume the random-perturbation to be *non-degenerate*, meaning that, for every $\varepsilon > 0$, the push-forward of the measure θ_ε under any map

$$\mathcal{F} \ni g \mapsto g(x)$$

is absolutely continuous with respect to some probability measure ν, with density uniformly (on x) bounded from above, and its support contains a ball around $f(x)$ with radius uniformly (on x) bounded from below. The first condition implies that any stationary measure is absolutely continuous with respect to ν. In Theorem 7.3 we shall use also the second condition to conclude that, assuming ν is expanding and conformal, for any $\varepsilon > 0$ there exists a finite number of ergodic stationary measures μ_1^ε, μ_2^ε, $\ldots$, μ_l^ε. We say that f is *stochastically stable* under random perturbation if every accumulation point, as $\varepsilon \to 0$, of stationary measures $(\mu^\varepsilon)_{\varepsilon>0}$ absolutely continuous with respect to ν is a convex combination of the ergodic equilibrium states μ_1, μ_2, $\ldots$, μ_k of f for ϕ.

A *Jacobian* of f with respect to a probability measure η is a measurable function $J_\eta f$ such that

$$\eta(f(A)) = \int_A J_\eta f \, d\eta \tag{2.2}$$

for every measurable set A (in some full measure subset) such that $f \mid A$ is injective. A Jacobian may fail to exist, in general, and it is essentially unique when it exists. If f is at most countable-to-one and the measure η is invariant, then Jacobians do exist (see [Par69]).

Theorem D. *Let $(\theta_\varepsilon)_\varepsilon$ be a non-degenerate random perturbation of $f \in \mathcal{F}$ and ν be the reference measure in Theorem B. Assume ν admits a Jacobian for every*

$g \in \mathcal{F}$, and the Jacobian varies continuously with g in the uniform norm. Then f is stochastically stable under the random perturbation $(\theta_\varepsilon)_\varepsilon$.

The conditions on the Jacobian are automatically satisfied in some interesting cases, for instance when ν is the Riemannian volume or f is an expanding map. This is usually associated to the potential $\phi = -\log|\det(Df)|$. Example 8.1 describes a situation where this potential satisfies the condition (P).

3. Preliminary results

Here, we give a few preparatory results needed for the proof of the main results. The content of this section may be omitted in a first reading and the reader may choose to return here only when necessary.

3.1. **Combinatorics of orbits.**

Since the region $\mathcal{A}$ is contained in q elements of the partition $\mathcal{P}$ we can assume without any loss of generality that $\mathcal{A}$ is contained in the first q elements of $\mathcal{P}$. Given $\gamma \in (0,1)$ and $n \geq 1$, let us consider the set $I(\gamma, n)$ of all itineraries $(i_0, \ldots, i_{n-1}) \in \{1, \ldots, k_0\}^n$ such that $\#\{0 \leq j \leq n-1 : i_j \leq q\} > \gamma n$. Then let

$$c_\gamma = \limsup_{n\to\infty} \frac{1}{n} \log \# I(\gamma, n). \tag{3.1}$$

Lemma 3.1. *Given $\varepsilon > 0$ there exists $\gamma_0 \in (0,1)$ such that $c_\gamma < \log q + \varepsilon$ for every $\gamma \in (\gamma_0, 1)$.*

Proof. It is clear that

$$\# I(\gamma, n) \leq \sum_{k=[\gamma n]}^{n} \binom{n}{k} p^{n-k} q^k \leq \sum_{k=[\gamma n]}^{n} \binom{n}{k} p^{(1-\gamma)n} q^n,$$

where $p = k_0 - q$ denotes the number of elements in $\mathcal{P}$ that do not intersect $\mathcal{A}$. Assume that $\gamma > 1/2$. A standard computation using Stirling's formula implies that

$$\sum_{k=[\gamma n]}^{n} \binom{n}{k} \leq \frac{n}{2} \binom{n}{[\gamma n]} \leq C_1 e^{2t(1-\gamma)n}$$

for some uniform constants $C_1 > 0$ and $t > 0$. Hence $c_\gamma < \log q + \varepsilon$ provided that γ is sufficiently close to 1, which proves the lemma. □

We are in a position to state our precise condition on the constant L in assumption (H1) and the constant c in the definition of hyperbolic time. By (P), we may find $\varepsilon_0 > 0$ small such that $\sup\phi - \inf\phi + \varepsilon_0 < \log\deg(f) - \log q$. By Lemma 3.1, we may find $\gamma < 1$ such that $c_\gamma < \log q + \varepsilon_0/4$. Assume L is close enough to 1 and c is close enough to zero so that

$$\sigma^{-(1-\gamma)} L^\gamma < e^{-2c} < 1 \tag{3.2}$$

and

$$\sup\phi - \inf\phi < \log\deg(f) - \log q - \varepsilon_0 - m\log L \tag{3.3}$$

3.2. **Hyperbolic times.** The next lemma, whose proof is based on a lemma due to Pliss (see e.g. [Mañ87]), asserts that, for points satisfying a certain condition of asymptotic expansion, there are infinitely many hyperbolic times: even more, the set of hyperbolic times has positive density at infinity.

Lemma 3.2 ([ABV00] Corollary 3.2)**.** *Let $x \in M$ and $n \geq 1$ be such that*

$$\frac{1}{n}\sum_{j=1}^{n} \log \|Df(f^j(x))^{-1}\| \leq -2c < 0.$$

Then, there is $\theta > 0$, depending only on f and c, and a sequence of hyperbolic times $1 \leq n_1(x) < n_2(x) < \cdots < n_l(x) \leq n$ for x, with $l \geq \theta n$.

Corollary 3.3. *Let η be a probability measure relative to which*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log \|Df(f^j(x))^{-1}\| \leq -2c < 0$$

holds almost everywhere. If A is a positive measure set then

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \frac{\eta(A \cap H_j)}{\eta(A)} \geq \frac{\theta}{2}.$$

Proof. By Lemma 3.2, for η-almost every point $x \in M$ there is $N(x) \in \mathbb{N}$ so that $n^{-1}\sum_{j=0}^{n-1} \chi_{H_j}(x) \geq \theta$ for every $n \geq N(x)$. Fix an integer $N \geq 1$ and choose $\tilde{A} \subset A$ so that $\eta(\tilde{A}) \geq \eta(A)/2$ and $N(x) \geq N$ for every $x \in \tilde{A}$. If we integrate the expression above with respect to η on A we obtain that

$$\frac{1}{n}\sum_{j=0}^{n-1} \eta(H_j \cap A) \geq \theta\eta(\tilde{A}) \geq \frac{\theta}{2}\eta(A)$$

for every integer n larger than N, completing the proof of the lemma. □

Lemma 3.4 ([ABV00] Lemma 2.7)**.** *There exists $\delta = \delta(c, f) > 0$ such that, whenever n is a hyperbolic time for a point x, the dynamical ball $V_n(x) = B(x, n, \delta)$ is mapped diffeomorphically by f^n onto the ball $B(f^n(x), \delta)$, with*

$$d(f^{n-j}(y), f^{n-j}(z)) \leq e^{-\frac{c}{2}j} d(f^n(y), f^n(z))$$

for every $1 \leq j \leq n$ and every $y, z \in V_n(x)$.

If n is a hyperbolic time for a point $x \in M$, the neighborhood $V_n(x)$ given by the lemma above is called *hyperbolic pre-ball.* As a consequence of the previous lemma we obtain the following property of bounded distortion on pre-balls.

Corollary 3.5. *Assume $J_\eta f = e^\psi$ for some Hölder continuous function ψ. There exist a constant $K_0 > 0$ so that, if n is a hyperbolic time for x then*

$$K_0^{-1} \leq \frac{J_\eta f^n(y)}{J_\eta f^n(z)} \leq K_0$$

for every $y, z \in V_n(x)$.

Proof. Let n a hyperbolic time for a point x in M and (C,α) be the Hölder constants of ψ. Then

$$|S_n\psi(y) - S_n\psi(z)| \leq \sum_{j=0}^{n-1} |\psi(f^j(y)) - \psi(f^j(z))| \leq C\sum_{j=0}^{n-1} d(f^j(y), f^j(z))^\alpha$$

for any given $y, z \in V_n(x)$. Using Lemma 3.4 we deduce that

$$|S_n\psi(y) - S_n\psi(z)| \leq C\sum_{j=0}^{+\infty} e^{-c\alpha/2j} d(f^n(x), f^n(y))^\alpha \leq C\delta^\alpha \sum_{j=0}^{+\infty} e^{-c\alpha j/2}.$$

Choosing K_0 as the exponential of this last term and noting $J_\eta f^n$ is the exponential of $S_n\psi$, the result follows immediately. □

3.3. **Non-lacunary sequences.** The set H of points with infinitely many hyperbolic times plays a central role in our strategy. We are going to see that for such a point the sequence of hyperbolic times has some special properties. The first one is described in the following remark:

Remark 3.6. If n is a hyperbolic time for x then, clearly, $n-s$ is a hyperbolic time for $f^s(x)$, for any $1 \leq s < n$. The following converse is a simple consequence of (2.1): if $k < n$ is a hyperbolic time for x and there exists $1 \leq s \leq k$ such that $n-s$ is a hyperbolic time for $f^s(x)$ then n is a hyperbolic time for x. Thus, if $n_j(x)$, $j \geq 1$ denotes the sequence of values of n for which x belongs to H_n then, for every j and l

$$n_j(x) + n_l(f^{n_j(x)}(x)) = n_{j+l}(x)$$

We will refer to this property as *concatenation* of hyperbolic times. Moreover, if n is a hyperbolic time for x and k is a hyperbolic time for $f^n(x)$, the intersection $V_n(x) \cap f^{-k}(V_k(f^k(x)))$ coincides with the hyperbolic pre-ball $V_{n+k}(x)$.

The next lemma, that is borrowed from [OV07], provides an abstract criterium for non-lacunarity *at almost every point* of certain sequences of functions.

Lemma 3.7. *Let $T : M \to \mathbb{N}$ and $T_i : M \to \mathbb{N}$, $i \in \mathbb{N}$ be measurable functions and η be a probability measure such that*

$$T(f^{T_i(x)}(x)) \geq T_{i+1}(x) - T_i(x)$$

at η-almost every $x \in M$. Assume η is invariant under f and T is integrable for η. Then $(T_i(x))_i$ is non-lacunary for η-almost every x.

Proof. Fix $x \in M$. By definition, $(T_i(x))_i$ is non-lacunary if and only if for every positive rational number β there exists $i_0 \geq 1$ such that $T_{i+1}(x) - T_i(x) \leq \beta T_i(x)$ for every $i \geq i_0$:

$$\Big\{x : \{T_i(x)\} \text{ is lacunary}\Big\} = \bigcup_{\beta\in\mathbb{Q}_+} \{x : T_{i+1}(x) - T_i(x) \geq \beta T_i(x) \text{ i.o.}\}$$

(i.o. stands for 'infinitely often'). By hypothesis, this set is contained in

$$\bigcup_{\beta\in\mathbb{Q}_+} \{x : T(f^{T_i(x)}(x)) \geq \beta T_i(x) \text{ i.o.}\} \subset \bigcup_{\beta\in\mathbb{Q}_+} \{x : T(f^n(x)) \geq \beta n \text{ i.o.}\}$$

and so it suffices to show that the last set has zero measure for any $\beta \in \mathbb{Q}_+$. This follows directly from Borel-Cantelli, together with the observation that, since η is f-invariant,

$$\sum_n \eta(\{x : T(f^n(x)) \geq \beta n\}) = \sum_n \eta(\{x : T(x) \geq \beta n\})$$

and this is bounded above by $\sum_j j\eta(\{T = j\}) = \int T\, d\eta < \infty$. □

The application we have in mind is when $T_i = n_i$ is the sequence of hyperbolic times, with $T = n_1$. In this case the assumption of the lemma follows from the concatenation property in Remark 3.6. In this way we obtain

Corollary 3.8. *If η is an invariant expanding measure and $n_1(\cdot)$ is η-integrable then the sequence $n_j(\cdot)$ is non-lacunary at η-almost every point.*

3.4. **Relative pressure.** We recall the notion of topological pressure on non necessarily compact invariant sets, and quote some useful properties. In fact, we present two alternative characterizations of the relative pressure, both from a dimensional point of view. See Chapter 4 §11 and Appendix II of [Pes97] for proofs and more details.

Let M be a compact metric space, $f : M \to M$ be a continuous transformation, $\phi : M \to \mathbb{R}$ be a continuous function, and Λ be an f-invariant set.

Relative pressure using partitions: Given any finite open covering $\mathcal{U}$ of Λ, denote by $\mathcal{I}_n$ the space of all n-strings $\underline{\mathrm{i}} = \{(U_0, \ldots, U_{n-1}) : U_i \in \mathcal{U}\}$ and put $n(\underline{\mathrm{i}}) = n$. For a given string $\underline{\mathrm{i}}$ set

$$\underline{\mathrm{U}} = \underline{\mathrm{U}}(\underline{\mathrm{i}}) = \{x \in M : f^j(x) \in U_{i_j}, \text{ for } j = 0 \ldots n(\underline{\mathrm{i}})\}$$

to be the cylinder associated to $\underline{\mathrm{i}}$ and $n(\underline{\mathrm{U}}) = n$ to be its depth. Furthermore, for every integer $N \geq 1$, let $\mathcal{S}_N\mathcal{U}$ be the space of all cylinders of depth at least N. Given $\alpha \in \mathbb{R}$ define

$$m_\alpha(f, \phi, \Lambda, \mathcal{U}, N) = \inf_{\mathcal{G}} \Big\{ \sum_{\underline{\mathrm{U}} \in \mathcal{G}} e^{-\alpha n(\underline{\mathrm{U}}) + S_{n(\underline{\mathrm{U}})}\phi(\underline{\mathrm{U}})} \Big\},$$

where the infimum is taken over all families $\mathcal{G} \subset \mathcal{S}_N\mathcal{U}$ that cover Λ and we write $S_n\phi(\underline{\mathrm{U}}) = \sup_{x \in \underline{\mathrm{U}}} S_n\phi(x)$. Let

$$m_\alpha(f, \phi, \Lambda, \mathcal{U}) = \lim_{N \to \infty} m_\alpha(f, \phi, \Lambda, \mathcal{U}, N)$$

(the sequence is monotone increasing) and

$$P_\Lambda(f, \phi, \mathcal{U}) = \inf\{\alpha : m_\alpha(f, \phi, \Lambda, \mathcal{U}) = 0\}.$$

Definition 3.9. The *pressure of (f, ϕ) relative to* Λ is

$$P_\Lambda(f, \phi) = \lim_{\mathrm{diam}(\mathcal{U}) \to 0} P_\Lambda(f, \phi, \mathcal{U}).$$

Theorem 11.1 in [Pes97] states that the limit does exist, that is, given any sequence of coverings $\mathcal{U}_k$ of L with diameter going to zero, $P_L(f, \phi, \mathcal{U}_k)$ converges and the limit does not depend on the choice of the sequence.

Relative pressure using dynamical balls:

Fix $\varepsilon > 0$. Set $\mathcal{I}_n = M \times \{n\}$ and $\mathcal{I} = M \times \mathbb{N}$. For every $\alpha \in \mathbb{R}$ and $N \geq 1$, define

$$m_\alpha(f, \phi, \Lambda, \varepsilon, N) = \inf_{\mathcal{G}} \Big\{ \sum_{(x,n) \in \mathcal{G}} e^{-\alpha n + S_n \phi(\mathrm{B}(x,n,\varepsilon))} \Big\}, \tag{3.4}$$

where the infimum is taken over all finite or countable families $\mathcal{G} \subset \cup_{n \geq N} \mathcal{I}_n$ such that the collection of sets $\{B(x, n, \varepsilon) : (x, n) \in \mathcal{G}\}$ cover Λ. Then let

$$m_\alpha(f, \phi, \Lambda, \varepsilon) = \lim_{N \to \infty} m_\alpha(f, \phi, \Lambda, \mathcal{U}, N)$$

(once more, the sequence is monotone increasing) and

$$P_\Lambda(f, \phi, \varepsilon) = \inf \{\alpha : m_\alpha(f, \phi, \Lambda, \varepsilon) = 0\}.$$

According to Remark 1 in [Pes97, Page 74] there is a limit when $\varepsilon \to 0$ and it coincides with the relative pressure:

$$P_\Lambda(f, \phi) = \lim_{\varepsilon \to 0} P_\Lambda(f, \phi, \varepsilon).$$

Remark 3.10. Since ϕ is uniformly continuous, the definition of the relative pressure is not affected if one replaces, in (3.4), the supremum $S_n\phi(B(x, n, \varepsilon))$ by the value $S_n\phi(x)$ at the center point.

The following properties on relative pressure, will be very useful later. See Theorem 11.2 and Theorem A2.1 in [Pes97], and also [Wal82, Theorem 9.10].

Proposition 3.11. *Let M be a compact metric space, $f : M \to M$ be a continuous transformation, $\phi : M \to \mathbb{R}$ be a continuous function, and Λ be an f-invariant set. Then*

1. $P_\Lambda(f, \phi) \geq \sup \left\{h_\mu(f) + \int \phi d\mu\right\}$ *where the supremum is over all invariant measures μ such that $\mu(\Lambda) = 1$. If Λ is compact, the equality holds.*
2. $P_{\mathrm{top}}(f, \phi) = \sup\{P_\Lambda(f, \phi), P_{M \setminus \Lambda}(f, \phi)\}$.

The proof of the next proposition, which is probably well-known but that we could not find in the literature, was obtained jointly with Marcelo Viana.

Proposition 3.12. *Let M be a compact metric space, $f : M \to M$ be a continuous transformation, $\phi : M \to \mathbb{R}$ be a continuous function, and Λ be an f-invariant set. Then $P_\Lambda(f^\ell, S_\ell \phi) = \ell P_\Lambda(f, \phi)$ for every $\ell \geq 1$.*

Proof. Fix $\ell \geq 1$. By uniform continuity of f, given any $\rho > 0$ there exists $\varepsilon > 0$ such that $d(x, y) < \varepsilon$ implies $d(f^j(x), f^j(y)) < \rho$ for all $0 \leq j < \ell$. It follows that

$$B_f(x, \ell n, \varepsilon) \subset B_{f^\ell}(x, n, \varepsilon) \subset B_f(x, \ell n, \rho), \tag{3.5}$$

where $B_g(x, n, \varepsilon)$ denotes the dynamical ball for a map g. This is the crucial observation for the proof.

First, we prove the $\geq$ inequality. Given $N \geq 1$ and any family $\mathcal{G}_\ell \subset \cup_{n \geq N} \mathcal{I}_n$ such that the balls $B_{f^\ell}(x, j, \varepsilon)$ with $(x, j) \in \mathcal{G}_\ell$ cover Λ, denote

$$\mathcal{G} = \{(x, j\ell) : (x, j) \in \mathcal{G}_\ell\}.$$

The second inclusion in (3.5) ensures that the balls $B_f(x, k, \rho)$ with $(x, k) \in \mathcal{G}$ cover Λ. Clearly,

$$\sum_{(x,j) \in \mathcal{G}_\ell} e^{-\alpha \ell j + \sum_{i=0}^{j-1} S_\ell \phi(f^{i\ell}(x))} = \sum_{(x,k) \in \mathcal{G}} e^{-\alpha k + \sum_{i=0}^{k-1} \phi(f^i(x))}.$$

Since $\mathcal{G}_\ell$ is arbitrary, and recalling Remark 3.10, this proves that

$$m_{\alpha\ell}(f^\ell, S_\ell\phi, \Lambda, \varepsilon, N) \geq m_\alpha(f, \phi, \Lambda, \rho, N\ell).$$

Therefore, $m_{\alpha\ell}(f^\ell, S_\ell\phi, \Lambda, \varepsilon) \geq m_\alpha(f, \phi, \Lambda, \rho)$. Then $P_\Lambda(f^\ell, S_\ell\phi, \varepsilon) \geq \ell P_\Lambda(f, \phi, \rho)$. Since $\varepsilon \to 0$ when $\rho \to 0$, it follows that $P_\Lambda(f^\ell, S_\ell\phi) \geq \ell P_\Lambda(f, \phi)$.

For the $\leq$ inequality, we observe that the definition of the relative pressure is not affected if one restricts the infimum in (3.4) to families $\mathcal{G}$ of pairs (x, k) such that k is always a multiple of ℓ. More precisely, let $m_\alpha^\ell(f, \phi, \Lambda, \varepsilon, N)$ be the infimum over this subclass of families, and let $m_\alpha^\ell(f, \phi, \Lambda, \varepsilon)$ be its limit as $N \to \infty$.

Lemma 3.13. *We have $m_\alpha^\ell(f, \phi, \Lambda, \varepsilon) \leq m_{\alpha-\rho}(f, \phi, \Lambda, \varepsilon)$ for every $\rho > 0$.*

Proof. We only have to show that, given any $\rho > 0$,

$$m_\alpha^\ell(f, \phi, \Lambda, \varepsilon, N) \leq m_{\alpha-\rho}(f, \phi, \Lambda, \varepsilon, N) \tag{3.6}$$

for every large N. Let ρ be fixed and N be large enough so that $N\rho > \ell(\alpha + \sup|\phi|)$. Given any $\mathcal{G} \subset \cup_{n \geq N} \mathcal{I}_n$ such that the balls $B_f(x, k, \varepsilon)$ with $(x, k) \in \mathcal{G}$ cover Λ, define $\mathcal{G}'$ to be the family of all (x, k'), $k' = \ell[k/\ell]$ such that $(x, k) \in \mathcal{G}$. Notice that

$$-\alpha k' + S_{k'}\phi(x) \leq -\alpha k + \alpha\ell + S_k\phi(x) + \ell \sup|\phi| \leq (-\alpha + \rho)k + S_k\phi(x)$$

given that $k \geq N$. The claim follows immediately. □

Let $\mathcal{G}'$ be any family of pairs (x, k) with $k \geq N\ell$ and such that every k is a multiple of ℓ. Define $\mathcal{G}_\ell$ to be the family of pairs (x, j) such that $(x, j\ell) \in \mathcal{G}'$. The first inclusion in (3.5) ensures that if the balls $B_f(x, k, \varepsilon)$ with $(x, k) \in \mathcal{G}'$ cover Λ then so do the balls $B_{f^\ell}(x, j, \varepsilon)$ with $(x, j) \in \mathcal{G}_\ell$. Clearly,

$$\sum_{(x,k)\in\mathcal{G}'} e^{-\alpha k + \sum_{i=0}^{k-1} \phi(f^i(x))} = \sum_{(x,j)\in\mathcal{G}_\ell} e^{-\alpha\ell j + \sum_{i=0}^{j-1} S_\ell\phi(f^{i\ell}(x))}.$$

Since $\mathcal{G}_\ell$ is arbitrary, and recalling Remark 3.10, this proves that

$$m_\alpha^\ell(f, \phi, \Lambda, \varepsilon, N\ell) \geq m_{\alpha\ell}(f^\ell, S_\ell\phi, \Lambda, \varepsilon, N).$$

Taking the limit when $N \to \infty$ and using Lemma 3.13,

$$m_{\alpha-\rho}(f, \phi, \Lambda, \varepsilon) \geq m_\alpha^\ell(f, \phi, \Lambda, \varepsilon) \geq m_{\alpha\ell}(f^\ell, S_\ell\phi, \Lambda, \varepsilon).$$

It follows that $\ell\big(P_\Lambda(f, \phi, \varepsilon) + \rho\big) \geq P_\Lambda(f^\ell, S_\ell\phi, \varepsilon)$. Since ρ is arbitrary, we conclude that $\ell P_\Lambda(f, \phi, \varepsilon) \geq P_\Lambda(f^\ell, S_\ell\phi, \varepsilon)$ and so $P_\Lambda(f^\ell, S_\ell\phi) \geq \ell P_\Lambda(f, \phi)$. □

The next lemma will be used later to reduce some estimates for the relative pressure to the case when $\phi \equiv 0$. Denote $h_\Lambda(f) = P_\Lambda(f, 0)$ for any invariant set Λ.

Lemma 3.14. $P_\Lambda(f, \phi) \leq h_\Lambda(f) + \sup\phi$.

Proof. Let $\mathcal{U}$ be any open covering of M and $N \geq 1$. By definition,

$$m_\alpha(f, \phi, \Lambda, \mathcal{U}, N) = \inf_{\mathcal{G}} \Big\{ \sum_{\underline{\mathrm{U}} \in \mathcal{G}} e^{-\alpha n(\underline{\mathrm{U}}) + S_{n(\underline{\mathrm{U}})}\phi(\underline{\mathrm{U}})} \Big\},$$

where the infimum is taken over all families $\mathcal{G} \subset \mathcal{S}_N\mathcal{U}$ that cover Λ. Therefore,

$$m_\alpha(f, \phi, \Lambda, \mathcal{U}, N) \leq \inf_{\mathcal{G}} \Big\{ \sum_{\underline{\mathrm{U}} \in \mathcal{G}} e^{(-\alpha + \sup\phi)n(\underline{\mathrm{U}})} \Big\} = m_{\alpha - \sup\phi}(f, 0, \Lambda, \mathcal{U}, N).$$

Since N and $\mathcal{U}$ are arbitrary, this gives that $P_\Lambda(f, \phi) \leq h_\Lambda(f) + \sup\phi$, as we wanted to prove. □

3.5. **Natural extension and Pesin's theory.** Here we present the natural extension associated to a non-invertible transformation and recall some results on the existence of (local) stable and unstable manifolds in the context of non-uniform hyperbolicity.

Let $(M, \mathcal{B}, \eta)$ be a probability space and let f denote a measurable non-invertible transformation. Consider the space

$$\hat{M} = \Big\{(\dots, x_2, x_1, x_0) \in M^{\mathbb{N}} : f(x_{i+1}) = x_i, \ \forall i \geq 0\Big\},$$

endowed with the metric $\hat{d}(\underline{\mathrm{x}}, \underline{\mathrm{y}}) = \sum_{i\geq 0} 2^{-i} d(x_i, y_i)$, $\underline{\mathrm{x}}, \underline{\mathrm{y}} \in \hat{M}$ and with the sigma-algebra $\hat{\mathcal{B}}$ that we now describe. Let $\pi_i : \hat{M} \to M$ denote the projection in the ith coordinate. Note also that $f^{-i}(\mathcal{B}) \subset \mathcal{B}$ for every $i \geq 0$, because f^i is a measurable transformation. Let $\hat{\mathcal{B}}_0$ be the smallest sigma-algebra that contain the elements $\pi_i^{-1}(f^{-i}(\mathcal{B}))$. The measure $\hat{\eta}$ defined on the algebra $\bigcup_{i=0}^{\infty} \pi_i^{-1}(f^{-i}\mathcal{B})$ by

$$\hat{\eta}(E_i) = \eta(\pi_i(E_i)) \quad \text{for every } E_i \in \pi_i^{-1}(f^{-i}(\mathcal{B})),$$

admits an extension to the sigma-algebra $\hat{\mathcal{B}}_0$. Let $\hat{\mathcal{B}}$ denote the completion of $\hat{\mathcal{B}}_0$ with respect to $\hat{\eta}$. The *natural extension* of f is the transformation

$$\hat{f} : \hat{M} \to \hat{M}, \quad \hat{f}(\dots, x_2, x_1, x_0) = (\dots, x_2, x_1, x_0, f(x_0)),$$

on the probability space $(\hat{M}, \hat{\mathcal{B}}, \hat{\eta})$. The measure $\hat{\eta}$ is the unique $\hat{f}$-invariant probability measure such that $\pi_*\hat{\eta} = \eta$. Furthermore, $\hat{\eta}$ is ergodic if and only if η is ergodic, and its entropy $h_{\hat{\eta}}(\hat{f})$ coincides with $h_\eta(f)$. We refer the reader to [Rok64] for more details and proofs. For simplicity reasons, when no confusion is possible we denote by π the projection in the zeroth coordinate and by x_0 the point $\pi(\hat{x})$.

If the transformation f is C^1-differentiable then the dynamical cocycle Df on the tangent bundle TM induces a cocycle $\hat{A} : \mathcal{E} \to \mathcal{E}$ over $\hat{f}$ defined on the fiber bundle $\mathcal{E}$ such that $\mathcal{E}_{\hat{x}} = T_{x_0}M$ for every $\hat{x} \in \hat{M}$. In fact, the action of the cocycle in each fiber $\mathcal{E}_{\hat{x}}$ is given by the map $\hat{A}(\hat{x}) : \mathcal{E}_{\hat{x}} \to \mathcal{E}_{\hat{f}(\hat{x})}$, where

$$\hat{A}(\hat{x})v = Df(\pi(\hat{x}))\, v$$

for every $\hat{x} \in \hat{M}$ and every $v \in \mathcal{E}_{\hat{x}}$. Given $n \geq 1$ and $\hat{x} \in \hat{M}$, set $\hat{A}^n(\hat{x}) := \hat{A}(\hat{f}^{n-1}(\hat{x})) \circ \dots \circ \hat{A}(\hat{x})$ and $\hat{A}^{-n}(\hat{x}) := [\hat{A}^n(\hat{f}^{-n}(\hat{x}))]^{-1}$. Oseledets' Theorem (see e.g. [Ose68] and [KS86, Appendix 2]) asserts the following:

Proposition 3.15. *Assume* $\log\|\hat{A}\|, \log\|\hat{A}^{-1}\| \in L^1(\hat{\eta})$. *There exists a full $\hat{\eta}$-measure set $\hat{R}$ such that, for any $\hat{x} \in \hat{R}$ there is an invariant decomposition*

$$T_{x_0}M = E^1_{\hat{x}} \oplus \dots \oplus E^{k(\hat{x})}_{\hat{x}},$$

and real numbers $\lambda^1(\hat{x}) \geq \dots \geq \lambda^{k(\hat{x})}(\hat{x})$ (Lyapunov exponents) satisfying

$$\lambda^i(\hat{x}) = \lim_{n\to\pm\infty} \frac{1}{n} \log \|\hat{A}^n(\hat{x})v\|, \text{ for every } v \in E^i_{\hat{x}} \setminus \{0\}$$

and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \sphericalangle(E^i_{\hat{f}^n(\hat{x})}, E^j_{\hat{f}^n(\hat{x})}) = 0$$

for every $i \neq j$. Moreover, $s(\hat{x})$, $\lambda^i(\hat{x})$ and $E^i(\hat{x})$ are invariant and vary measurably with $\hat{x}$.

In the presence of positive Lyapunov exponents, Pesin's theory guarantees the existence of local unstable manifolds passing through almost every point and varying measurably. Given $\hat{x} \in \hat{R}$, consider the subbundles $E^u_{\hat{x}} = \bigoplus_{\lambda^i(\hat{x})>0} E^i_{\hat{x}}$ and $E^{cs}_{\hat{x}} = \bigoplus_{\lambda^i(\hat{x})\le 0} E^i_{\hat{x}}$. Set also $\lambda^u(\hat{x}) = \inf\{\lambda^i(\hat{x}) : \lambda^i(\hat{x}) > 0\}$ and consider the set $\hat{B}_\lambda = \{\hat{x} \in \hat{R} : \lambda^u(\hat{x}) > \lambda\}$.

Proposition 3.16. *Given $0 < \alpha' < \alpha$, $\varepsilon > 0$ and $2\varepsilon < \lambda_1 < \lambda - \varepsilon$ there are measurable functions δ_ε and γ_ε from $\hat{B}_\lambda$ to $\mathbb{R}_+$ and, for every $\hat{x} \in \hat{B}_\lambda$, there exists a Lipschitz transformation $\psi_{\hat{x}}$ from the ball $B^u(\hat{x}, \delta_\varepsilon(\hat{x}))$, of radius $\delta_\varepsilon(\hat{x})$ in $E^u_{\hat{x}}$, to $E^{cs}_{\hat{x}}$ such that*

(1) *$\psi_{\hat{x}}$ varies measurably with $\hat{x}$, $W^u_{loc}(\hat{x}) = \exp_{x_0}(graph\, \psi_{\hat{x}})$ is a $C^{1+\alpha'}$ embedded manifold of dimension $\dim E^u_{\hat{x}}$ in M tangent to $E^u_{\hat{x}}$ at x_0, and*
$$Lip_{\alpha'}(\exp_{x_0} \psi_{\hat{x}}) \le \gamma_\varepsilon(\hat{x});$$
(2) *For every $y_0 \in W^u_{loc}(\hat{x})$ there is a unique $\hat{y} \in \hat{M}$ such that $\pi(\hat{y}) = y_0$ and*
$$d(x_{-n}, y_{-n}) \le \gamma_\varepsilon(\hat{x})\, e^{-\lambda_1 n}$$
for every $n \ge 0$;

(3) *If $\hat{z} \in \hat{M}$ and for every $n \ge 0$ it holds*
$$d(x_0, z_0) \le \delta_\varepsilon(\hat{x})/\gamma_\varepsilon(\hat{f}^{-1}(\hat{x})) \quad \text{and} \quad d(x_{-n}, z_{-n}) \le e^{-\lambda_1 n}\delta_\varepsilon(\hat{x})$$
then $z_0 \in W^u_{loc}(\hat{x})$;

(4) *There is a sequence of embedded disks $(W_{-n}(\hat{x}))_{n\ge 0}$ of dimension $\dim E^u_{\hat{x}}$ in M satisfying $W_0(\hat{x}) = W^u_{loc}(\hat{x})$, $f(W_{-n}(\hat{x})) \supset W_{-n+1}(\hat{x})$ and such that*
$$W^u(\hat{x}) = \bigcup_{n\ge 0} f^n(W_{-n}(\hat{x})));$$
is an immersed manifold and coincides with the set of points $y_0 \in M$ for which there exists $\hat{y} \in \hat{M}$ for which $y_0 = \pi(\hat{y})$ such that
$$\limsup_{n\to\infty} \frac{1}{n} \log d(x_{-n}, y_{-n}) < 0;$$
(5) *If d^u denotes the restriction of the Riemannian metric to the disks $W_{-n}(\hat{x})$ then $d^u(y_0, z_0) \le \gamma^2_\varepsilon(\hat{x})\, d(y_0, z_0)$ for every $y_0, z_0 \in W^u_{loc}(\hat{x})$;*

(6) *If $\hat{W}^u_{loc}(\hat{x})$ is the set of points $\hat{y} \in \hat{M}$ given by (2) above then it holds that*
$$d^u(y_{-n}, z_{-n}) \le \gamma_\varepsilon(\hat{x})\, e^{-\lambda_1 n}$$
for every $\hat{y}, \hat{z} \in W^u_{loc}(\hat{x})$ and every $n \ge 0$; and

(7) *$\gamma_\varepsilon(\hat{f}^n(\hat{x})) \le \gamma_\varepsilon(\hat{x})e^{\varepsilon|n|}$ and $\delta_\varepsilon(\hat{f}^n(\hat{x})) \le \delta_\varepsilon(\hat{x})e^{\varepsilon|n|}$ for every $n \in \mathbb{Z}$.*

In view of the this proposition, the *global unstable manifold* of $\hat{x}$ in $\hat{M}$ is defined as the set
$$\hat{W}^u_{\text{loc}}(\hat{x}) = \Big\{\hat{y} \in \hat{M} : \overline{\lim}_{n\to\infty} \frac{1}{n} \log d^u(y_{-n}, x_{-n}) \le -\lambda^u(\hat{x})\Big\}.$$

The proof of this result in the endomorphism case follows the original one due to Pesin for invertible transformations. We refer the reader to [FHY83, KS86, LQ95, QZ02] for detailed presentations and proofs. A presentation of the unstable manifold theorem for non-uniformly hyperbolic endomorphisms can be found in [Zhu98].

We shall omit the dependence of $W^u_{\mathrm{loc}}(\hat{x})$ on λ_1 and ε for notational simplicity. Since local unstable leaves vary measurably with the point then there are compact sets of arbitrary large measure, referred as *hyperbolic blocks*, restricted to which the local unstable leaves passing through those points vary continuously. More precisely,

Corollary 3.17. *There are countably many compact sets $(\hat{\Lambda}_i)_{i\in\mathbb{N}}$ whose union is a $\hat{\eta}$-full measure set and such that the following holds: for every $i \geq 1$ there are positive numbers $\varepsilon_i \ll 1$, λ_i, r_i, , γ_i and R_i such that for every $\hat{x} \in \hat{\Lambda}_i$ there exists an embedded submanifold $W^u_{loc}(\hat{x})$ in M of dimension $\dim E^u(\hat{x})$, tangent to $E^u(\hat{x})$ at $\hat{x}$ and*

(1) *If $y_0 \in W^u_{loc}(\hat{x})$ then there is a unique $\hat{y} \in \hat{M}$ such that for every $n \geq 1$*
$$d(x_{-n}, y_{-n}) \leq r_i e^{-\varepsilon_i n} \quad \text{and} \quad d(x_{-n}, y_{-n}) \leq \gamma_i e^{-\lambda_i n};$$
(2) *For every $0 < r \leq r_i$ the set $W^u_{loc}(\hat{y}) \cap B(x_0, r)$ is connected and the map*
$$B(\hat{x}, \varepsilon_i r) \cap \hat{\Lambda}_i \ni \hat{y} \mapsto W^u_{loc}(\hat{y}) \cap B(x_0, r)$$
is continuous (in the Hausdorff topology);

(3) *If $\hat{y}$ and $\hat{z}$ belong to $\hat{\Lambda}_i$ then either $W^u_{loc}(\hat{y}) \cap B(x_0, r)$ and $W^u_{loc}(\hat{z}) \cap B(x_0, r)$ coincide or are disjoint; if these sets are disjoint and, in addition, $\hat{y} \in \hat{W}^u(\hat{z})$ then $d^u(y_0, z_0) > 2r_i$;*

(4) *If $\hat{y} \in \hat{\Lambda}_i \cap B(\hat{x}, \varepsilon_i r)$ then $W^u_{loc}(\hat{y})$ contains the ball of d^u-radius R_i around $W^u_{loc}(\hat{y}) \cap B(x_0, r)$.*

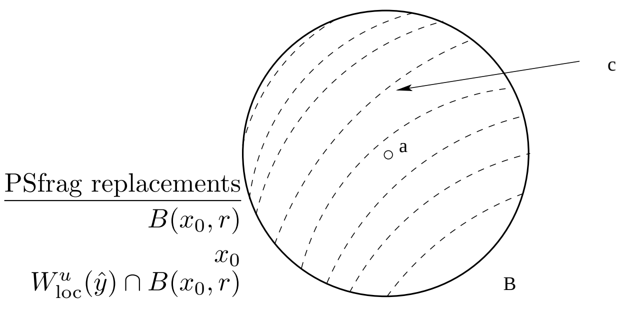

Figure 1. Continuous dependence of local unstable leaves in M.

4. Conformal measures

The *Ruelle-Perron-Fröbenius transfer operator* $\mathcal{L}_\phi : C(M) \to C(M)$ associated to $f : M \to M$ and $\phi : M \to \mathbb{R}$ is the linear operator defined on the space $C(M)$ of continuous functions $g : M \to \mathbb{R}$ by
$$\mathcal{L}_\phi g(x) = \sum_{f(y)=x} e^{\phi(y)} g(y).$$
Notice that $\mathcal{L}_\phi g$ is indeed continuous if g is continuous, because f is a local homeomorphism. It is also easy to see that $\mathcal{L}_\phi$ is a bounded operator,relative to the norm of uniform convergence in $C(M)$:
$$\|\mathcal{L}_\phi\| \leq \deg(f) e^{\sup|\phi|}.$$

The dual operator $\mathcal{L}_\phi^*$ acts on the Borel measures of M by Consider the dual operator $\mathcal{L}_\phi^* : \mathcal{M}(M) \to \mathcal{M}(M)$ acting on the space $\mathcal{M}(M)$ of Borel measures in M by

$$\int g\, d(\mathcal{L}_\phi^* \eta) = \int (\mathcal{L}_\phi g)\, d\eta$$

for every $g \in C(M)$. Let $\lambda_0 = r(\mathcal{L}_\phi)$ be the spectral radius of $\mathcal{L}_\phi$. In this section we prove the following:

Theorem 4.1. *There exists $k \geq 1$, $r(\mathcal{L}_\phi) = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_k \geq \deg(f)e^{\inf \phi}$ real numbers and expanding conformal probability measures $\nu_0, \nu_1, \dots, \nu_k$ such that*

$$\mathcal{L}_\phi^* \nu_i = \lambda_i \nu_i, \ \forall\, 0 \leq i \leq k, \quad \textit{and} \quad \bigcup_{i=0}^{k} \operatorname{supp}(\nu_i) = \overline{H}.$$

Moreover, each ν_i is a non-lacunary Gibbs measure and has a Jacobian with respect to f given by $J_{\nu_i} f = \lambda_i e^{-\phi}$. If f is topologically mixing then ν_0 is an expanding conformal measure such that $\operatorname{supp} \nu_0 = \overline{H} = M$.

4.1. **Eigenmeasures of the transfer operator.**

Lemma 4.2. *Suppose ν is a Borel probability such that $\mathcal{L}_\phi^* \nu = \lambda \nu$ for some $\lambda > 0$. Then the Jacobian of ν with respect to f exists and is given by $J_\nu f = \lambda e^{-\phi}$.*

Proof. We will sketch the proof of this standard lemma for completeness. Let A be any measurable set such that $f \mid A$ is injective. Take a sequence $(g_n)_n$ of continuous functions on M such that $g_n \to \chi_A$ at ν-almost every point and $\sup |g_n| \leq 2$ for all n. Then,

$$\mathcal{L}_\phi(e^{-\phi} g_n)(x) = \sum_{f(y)=x} e^{\phi(y)}\, e^{-\phi(y)} g_n(y) = \sum_{f(y)=x} g_n(y).$$

The last expression converges to $\chi_{f(A)}(x)$ at ν-almost every point, because $f \mid A$ is injective. Hence, by the dominated convergence theorem,

$$\int \lambda e^{-\phi} g_n\, d\nu = \int e^{-\phi} g_n\, d(\mathcal{L}_\phi^* \nu) = \int \mathcal{L}_\phi(e^{-\phi} g_n)\, d\nu \to \nu(f(A)).$$

Since the left hand side also converges to $\int_A \lambda e^{-\phi} d\nu$, we conclude that

$$\nu(f(A)) = \int_A \lambda e^{-\phi} d\nu,$$

which proves the lemma. □

Lemma 4.3. *The spectral radius λ_0 of the operator $\mathcal{L}_\phi$ is at least* $\deg(f)\, e^{\inf \phi}$ *and it is an eigenvalue for the dual operator $\mathcal{L}_\phi^*$.*

Proof. Observe that, for every positive integer n and every $x \in M$,

$$\mathcal{L}_\phi^n 1(x) = \sum_{f^n(y)=x} e^{S_n \phi(y)} \geq \deg(f)^n\, e^{n \inf \phi}.$$

So, the spectral radius is at least $\deg(f)\, e^{\inf \phi}$, as claimed in the first part of the lemma. The second part follows from general results in functional analysis. Let C^+ be the open convex cone of positive continuous functions on M and consider the linear subspace

$$N = \{\mathcal{L}_\phi g - \lambda_0 g : g \in C(M)\}.$$

Notice that these sets are disjoint. Indeed, assuming otherwise then there exists some continuous function $g \in C(M)$ such that $\mathcal{L}_\phi g - \lambda_0 g$ is a strictly positive continuous function. By compactness and continuity, there is $\varepsilon > 0$ such that $\mathcal{L}_\phi g \geq (\lambda_0 + \varepsilon) g$. Since $\mathcal{L}_\phi$ is a positive operator, it is clear that

$$\mathcal{L}_\phi^n g \geq (\lambda_0 + \varepsilon)^n g \quad \text{for every } n \geq 1.$$

This shows that the spectral radius of $\mathcal{L}_\phi$ is at least $\lambda_0 + \varepsilon$, contradicting the definition of λ_0. This contradiction proves that $C^+ \cap N = \emptyset$, as we claimed. Then, by Masur's theorem (see [Dei85, Proposition 7.2]), there exists some continuous linear functional $\nu_0 : C(M) \to \mathbb{R}$ such that

$$\int g \, d\nu_0 > 0 \text{ for every } g \in C^+ \quad \text{and} \quad \int g \, d\nu_0 = 0 \text{ for every } g \in N.$$

The first property means that ν_0 is a measure and so, up to normalization, we may suppose it is a probability. The second property means that

$$\int g \, d(\mathcal{L}_\phi^* \nu_0) = \int \mathcal{L}_\phi g \, d\nu_0 = \lambda_0 \int g \, d\nu_0 \quad \text{for every } g \in C(M),$$

that is, $\mathcal{L}_\phi^* \nu = \lambda_0 \nu_0$. Thus, λ_0 is indeed an eigenvalue for the dual operator $\mathcal{L}_\phi^*$. □

Throughout, let λ denote a fixed eigenvalue of $\mathcal{L}_\phi^*$ larger than $\deg(f) e^{\inf \phi}$**, let ν be any eigenmeasure of $\mathcal{L}_\phi^*$ associated to λ and set** $P = \log \lambda$. The only property of λ that we shall use is that $\lambda > e^{\log q + \sup \phi + \varepsilon_0}$. From Lemma 4.2 we get that

$$J_\nu f(x) = \lambda_0 e^{-\phi(x)} > e^{\log q + \varepsilon_0} > q \quad \text{for all } x \in M. \tag{4.1}$$

This property will allow us to prove that ν-almost every point spends at most a fraction γ of time inside the domain $\mathcal{A}$ where f may fail to be expanding. As we will see later, in Lemma 6.5, $\log \lambda = P_{\text{top}}(f, \phi)$. This determines completely the spectral radius of $\mathcal{L}_\phi$ as the *unique* eigenvalue of $\mathcal{L}_\phi^*$ larger than the lower bound above. Consequently all the eigenvalues λ_i given by Theorem 4.1 are equal and coincide with $\lambda_0 = r(\mathcal{L}_\phi)$ and $\frac{1}{k} \sum_{j=0}^{k} \nu_i$ is an expanding conformal measure whose support coincides with the closure of the set H. The later is the conformal measure referred at Theorem B.

4.2. **Expanding structure.** Here we prove that any eigenmeasure ν as above is expanding and has integrable first hyperbolic time. Given $n \geq 1$, let $B(n)$ denote the set of points $x \in M$ whose frequency of visits to $\mathcal{A}$ up to time n is at least γ, that is,

$$B(n) = \Big\{ x \in M : \frac{1}{n} \#\{0 \leq j \leq n-1 : f^j(x) \in \mathcal{A}\} \geq \gamma \Big\}.$$

Proposition 4.4. *The measure $\nu(B(n))$ decreases exponentially fast as n goes to infinity. Consequently, ν-almost every point belongs to $B(n)$ for at most finitely many values of n.*

Proof. The strategy is to cover $B(n)$ by elements of the covering $\mathcal{P}^{(n)} = \bigvee_{j=0}^{n-1} f^{-j} \mathcal{P}$ which, for convenience, will be referred to as cylinders. Then, the estimate relies on an upper bound for the measure of each cylinder, together with an upper bound on the number of cylinders corresponding to large frequency of visits to $\mathcal{A}$.

Since f^n is injective on every $P \in \mathcal{P}^{(n)}$ then we may use (4.1) to conclude that

$$1 \geq \nu(f^n(P)) = \int_P J_\nu f^n \, d\nu = \int_P \prod_{j=0}^{n-1} (J_\nu f \circ f^j) d\nu \geq e^{(\log q + \varepsilon_0)n} \nu(P).$$

This proves that $\nu(P) \leq e^{-(\log q + \varepsilon_0)n}$ for every $P \in \mathcal{P}^n$. Since $B(n)$ is contained in the union of cylinders $P \in \mathcal{P}^n$ associated to itineraries in $I(\gamma, n)$, we deduce from our choice of γ after Lemma 3.1 that

$$\nu(B(n)) \leq \# I(\gamma, n) e^{-(\log q + \varepsilon_0)n} \leq e^{-\varepsilon_0 n/2},$$

for every large n. This proves the first statement in the lemma. The second one is a direct consequence, using the Borel-Cantelli lemma. □

Corollary 4.5. *The measure ν is expanding and satisfies $\int n_1 \, d\nu < \infty$.*

Proof. By Proposition 4.4, almost every point x is outside $B(n)$ for all but finitely many values of n. Then, in view of our choice (3.2),

$$\sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq \gamma \log L + (1-\gamma) \log \sigma^{-1} \leq -2c$$

if n is large enough. In view of Lemma 3.2, this proves that ν-almost every point has infinitely many hyperbolic times (positive density at infinity). In other words, ν is expanding. Moreover, using Proposition 4.4 once more,

$$\int n_1 d\nu = \sum_{n=0}^{\infty} \nu(\{x : n_1(x) > n\}) \leq 1 + \sum_{n=1}^{\infty} \nu(B(n)) < \infty,$$

as we claimed. □

4.3. **Gibbs property.** Now we prove that ν satisfies a Gibbs property at hyperbolic times. Later we shall see that hyperbolic times form a non-lacunary sequence, almost everywhere, and then it will follow that ν is a non-lacunary Gibbs measure.

Lemma 4.6. *The support of ν is an f-invariant set contained in the closure of H. For any $\rho > 0$ there exists $\xi > 0$ such that $\nu(B(x, \rho)) \geq \xi$ for every $x \in \operatorname{supp}(\nu)$.*

Proof. Since ν is expanding, it is clear $\operatorname{supp}(\nu) \subset \overline{H}$. Let $x \in M$. Since f is a local homeomorphism, the relation $V = f(W)$ is a one-to-one correspondence between small neighborhoods W of x and small neighborhood V of $f(x)$. Moreover,

$$\nu(V) = \int_W J_\nu f \, d\nu.$$

is positive if and only if $\nu(W) > 0$, because the Jacobian is bounded away from zero and infinity. This proves that the support is invariant by f. The second claim in the lemma is standard. Assume, by contradiction, that there exists $\rho > 0$ and a sequence $(x_n)_{n \geq 1}$ in $\operatorname{supp}(\nu)$ such that $\nu(B(x_n, \rho)) \to 0$ as $n \to \infty$. Since $\operatorname{supp}(\nu)$ is compact set, the sequence must accumulate at some point $z \in \operatorname{supp}(\nu)$. Then

$$\nu(B(z, \rho)) \leq \liminf_{n \to \infty} \nu(B(x_n, \rho)) = 0,$$

which contradicts $z \in \operatorname{supp}(\nu)$. This completes the proof of the lemma. □

Lemma 4.7. *There exists $K > 0$ such that, if n is a hyperbolic time for $x \in \operatorname{supp}(\nu)$ then*

$$K^{-1} \le \frac{\nu(B(x,n,\delta))}{e^{-Pn+S_n\phi(y)}} \le K,$$

for every $y \in B(x,n,\delta)$.

Proof. Since $f^n \mid B(x,n,\delta)$ is injective, we get from the previous lemma that

$$\xi(\delta) \le \nu(B(f^n(x),\delta)) = \int_{B(x,n,\delta)} J_\nu f^n \, d\nu \le 1$$

for every $x \in \operatorname{supp}(\nu)$. Then, the bounded distortion property in Corollary 3.5 applied to the Hölder continuous function $J_\nu f = \lambda e^{-\phi}$ gives that

$$K_0^{-1}\xi(\delta) \le \nu(B(x,n,\delta))\lambda^n e^{-S_n\phi(y)} \le K_0$$

for every $y \in B(x,n,\delta)$. Recalling that $P = \log\lambda$, this gives the claim with $K = K_0\, \xi(\delta)^{-1}$. □

Remark 4.8. The same proof gives a somewhat stronger result: for ν-almost every x and any $0 < \varepsilon \le \delta$, there exists $K(\varepsilon) > 0$ such that

$$K^{-1}(\varepsilon) \le \frac{\nu(B(x,n,\varepsilon))}{e^{-Pn+S_n\phi(x)}} \le K(\varepsilon).$$

if n is a hyperbolic time for x. It suffices to take $K(\varepsilon) = K_0\xi(\varepsilon)^{-1}$.

We proceed with the proof of Theorem 4.1. We have proven that any eigenmeasure ν for $\mathcal{L}_\phi$ associated to an eigenvalue $\lambda \ge \deg(f)e^{\inf\phi}$ is necessarily expanding, satisfies the Gibbs property at hyperbolic times and has a Jacobian $J_\nu f = \lambda e^{-\phi}$. Furthermore, Lemma 4.3 guarantees that the spectral radius λ_0 is an eigenvalue of the operator $\mathcal{L}_\phi$. Let ν_0 denote any such eigenmeasure. If f is topologically mixing then $\operatorname{supp}\nu_0 = \overline{H} = M$. Indeed, given an open set U there exists $N \ge 1$ such that $f^N(U) = M$. Since $J_{\nu_0} f$ is bounded from zero and infinity then clearly $\nu_0(U) > 0$, which proves our claim. Hence, to prove Theorem 4.1 we are left to show that there are finitely many eigenmeasures of $\mathcal{L}_\phi^*$ associated to eigenvalues greater or equal to $\deg(f)e^{\inf\phi}$ whose union of their supports coincide with $\overline{H}$. Given an f-invariant compact set Λ we denote by $\mathcal{L}_\Lambda : C(\Lambda) \to C(\Lambda)$ the restriction of the operator $\mathcal{L}_\phi$ to the space of continuous functions $C(\Lambda)$.

Lemma 4.9. *There are finitely many $\lambda_0 \ge \lambda_1 \ge \cdots \ge \lambda_k \ge \deg(f)e^{\inf\phi}$ and probability measures $\nu_0, \nu_1, \dots, \nu_k$ such that $\mathcal{L}_\phi^*\nu_i = \lambda_i\nu_i$, for every $0 \le i \le k$, and that the union of their supports coincides with the closure of the set H.*

Proof. We obtain the desired finite sequence of conformal measures using the ideas involved in the proof of Lemma 4.3 recursively. Indeed, Lemma 4.3, Corollary 4.5 and Lemma 4.7 assert that there exists an expanding conformal measure ν_0 such that $\mathcal{L}_\phi^*\nu_0 = \lambda_0\nu_0$ and satisfies the Gibbs property at hyperbolic times. Clearly $\operatorname{supp}(\nu_0)$ is an invariant set contained in $\overline{H}$.

If $\operatorname{supp}(\nu_0) = \overline{H}$ then we are done. Otherwise we proceed as follows. As we shall see in Lemma 5.3, the interior of the support of any expanding conformal measure ν is non-empty and contains almost every point in a ball of radius δ (depending only on f and c). Consider the non-empty compact invariant set $K_1 = M \setminus \mathit{interior}(\operatorname{supp}(\nu_0))$ and set $\lambda_1 = r(\mathcal{L}_{K_1}) \le \lambda_0$. It is easy to check that $\lambda_1 \ge \deg(f)e^{\inf\phi}$. Then we may argue as in the proof of Lemma 4.3: the cone of strictly positive functions in K_1 is disjoint from the subspace $\{\mathcal{L}_\phi g - \lambda g : g \in C(K_1)\}$ and so there exists a probability measure ν_1 such that $\mathcal{L}_\phi^* \nu_1 = \lambda_1 \nu_1$ whose support $\operatorname{supp}(\nu_1)$ is contained in K_1. Since $\lambda_1 \geq \deg(f) e^{\inf \phi}$ then ν_1 is also expanding and its support must also contain a ball of radius δ in its interior.

Since M is compact this procedure will finish after a finite number of times. Hence there are finitely many compact sets $K_0, \dots, K_k$ and expanding measures $\nu_0, \dots, \nu_k$ such that $\operatorname{supp}(\nu_i) \subset K_i$ and $\overline{H} = \bigcup_i \operatorname{supp}(\nu_i)$. This completes the proof of the lemma. $\square$

For any conformal measure ν_i as above, we prove in Proposition 5.1) that there are finitely many invariant ergodic measures that are absolutely continuous with respect to ν_i, that their densities are bounded from above and that their basins cover ν_i-almost every point. Hence, the non-lacunarity of the sequence of hyperbolic times will be a consequence of Lemma 3.7. So, up to the proof of Proposition 5.1, this shows that each ν_i is a non-lacunary Gibbs measure and completes the proof of Theorem 4.1.

5. Absolutely continuous invariant measures

In this section we analyze carefully the Cesaro averages

$$\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j \nu,$$

and prove that every weak* accumulation point is absolutely continuous with respect to ν. It is well known, and easy to check, that the accumulation points are invariant probabilities. In the topologically mixing setting we also prove that there is a unique absolutely continuous invariant measure and that it satisfies the non-lacunar Gibbs property. The precise statement is

Proposition 5.1. *There are finitely many invariant, ergodic probability measures $\mu_1, \mu_2, \dots, \mu_k$ that are absolutely continuous with respect to ν and such any absolutely continuous invariant measure is a convex linear combination of $\mu_1, \mu_2, \dots, \mu_k$. In addition, the measures μ_i are expanding and the densities $d\mu_i/d\nu$ are bounded away from infinity. Moreover, the union of the basins $B(\mu_i)$ cover ν-almost every point in M. If f is topologically mixing then there is a unique absolutely continuous invariant measure and it is a non-lacunary Gibbs measure.*

5.1. **Existence and finitude.**

First we prove that every accumulation point of $(\nu_n)_{n \geq 1}$ is absolutely continuous invariant measure with bounded density. For every $n \in \mathbb{N}$ it holds that

$$H_n^c \subset \Big\{ n_1(\cdot) > n \Big\} \bigcup \Big[\bigcup_{k=0}^{n-1} H_k \cap f^{-k}(\{n_1(\cdot) > n - k\}) \Big].$$

In particular, we can use the inclusion above to write

$$\nu_n \leq \mu_n + \frac{1}{n} \sum_{j=0}^{n-1} \eta_j,$$

where

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j(\nu \mid H_j) \quad \text{and} \quad \eta_j = \sum_{l=0}^{\infty} f_*^l\big(f_*^j(\nu \mid H_j)|\{n_1 > l\}\big).$$

Lemma 5.2. *There exists $C_2 > 0$ such that for every positive integer n the measures $f_*^n(\nu \mid H_n)$, μ_n and ν_n are absolutely continuous with respect to ν with densities bounded from above by C_2. Moreover, the same holds for every weak* accumulation point μ of $(\nu_n)_{n\geq 1}$.*

Proof. Let A be any measurable set of small diameter, say $\operatorname{diam}(A) < \delta/2$, and such that $\nu(A) > 0$. First we claim that there is $C_2 > 0$ such that

$$f_*^n(\nu \mid H_n)(A) \leq C_2\, \nu(A), \quad \forall n \geq 1.$$

Observe that either $f_*^n(\nu|H_n)(A) = 0$, or A is contained in a ball $B = B(f^n(x), \delta)$ of radius δ for some $x \in H_n$. In the first case we are done. In the later situation we compute

$$f_*^n(\nu \mid H_n)(A) = \nu(f^{-n}(A) \cap H_n) = \sum_i \nu(f_i^{-n}(A \cap B)),$$

where the sum is over all hyperbolic inverse branches $f_i^{-n} : B \to V_i$ for f^n. Recall that the ν-measure of any positive measure ball of radius δ is at least $\xi(\delta) > 0$ by Lemma 4.6. Thus, by bounded distortion

$$f_*^n(\nu \mid H_n)(A) \leq K_0 \sum_i \frac{\nu(A)}{\nu(B)}\, \nu(V_i) \leq K_0\, \xi(\delta)^{-1}\, \nu(A),$$

which proves our claim with $C_2 = K_0\, \xi(\delta)^{-1}$. It follows from the arbitrariness of A that both $f_*^n(\nu \mid H_n)$ and μ_n are absolutely continuous with respect to ν with density bounded from above by C_2.

Similar estimates on the density of η_n hold using that $\{n_1 > n\} \subset B(n)$, there are at most $e^{c_\gamma n}$ cylinders in $B(n)$, and that $J_\nu f^n > e^{(\log q + \varepsilon_0)n}$ on each of one of them. Indeed,

$$((f_*^l \nu)|\{n_1 > l\})(A) \leq \sum_{\substack{P \in \mathcal{P}^{(l)} \\ P \cap B(l) \neq \emptyset}} \nu(f^{-l}(A) \cap P) \leq \#B(l)\; e^{-(\log q + \varepsilon_0)l} \nu(A)$$

for every $l \geq 1$ and every measurable set A. Using that $df_*^n(\nu \mid H_n)/d\nu \leq K_0\, \xi(\delta)^{-1}$ and summing up the previous terms one concludes that

$$\eta_j(A) \leq K_0\, \xi(\delta)^{-1} \sum_{l=0}^{\infty} e^{-\frac{\varepsilon_0}{4} l}\, \nu(A), \forall j \geq 1.$$

This shows that (up to replace C_2 by a larger constant) the measures ν_n are also absolutely continuous with respect to ν and that $d\nu_n/d\nu$ is bounded from above by C_2. The second assertion in the lemma is an immediate consequence by weak* convergence. □

The following lemma, whose proof explores the generating property of hyperbolic pre-balls, plays a key role in proving finitude of equilibrium states.

Lemma 5.3. *If G is an f-invariant set such that $\nu(G) > 0$ then there is a disk Δ of radius $\delta/4$ so that $\nu(\Delta \backslash G) = 0$.*

Proof. In the case that ν coincides with the Lebesgue measure this corresponds to [ABV00, Lemma 5.6]. Since the argument will be used later on we give a brief sketch of the proof.

Let $\varepsilon > 0$ be small. Take a compact K and an open set O such that $K \subset G \cap H \subset O$ and $\nu(O \setminus K) < \varepsilon \nu(K)$. Set $n_0 \in \mathbb{N}$ such that $B(x, n, \delta) \subset O$ for any $x \in K \cap H_n$. If $n(x)$ denotes the first hyperbolic time of x larger than n_0 then

$$K \subset \bigcup_{x \in K} B(x, n(x), \delta/4) \subset O.$$

Set $V(x) = B(x, n(x), \delta)$ and $W(x) = B(x, n(x), \delta/4)$. Since K is compact it is covered by finite open sets $(W(x))_{x \in X}$ for some family $X = \{x_1, \dots, x_k\}$. Now we proceed recursively and define

$$n_1 = \inf\{n(x) : x \in X\} \quad \text{and} \quad X_1 = \{x \in X : n(x) = n_1\}$$

and, assuming that n_i and X_i are well defined for $1 \le i \le m-1$, set

$$n_m = \inf\{n(x) : x \in X \setminus (X_1 \cup \dots \cup X_{m-1})\} \quad \text{and} \quad X_m = \{x \in X : n(x) = n_m\}$$

up to some finite positive integer s. Let $\tilde{X}_1 \subset X_1$ be a maximal family of points with pairwise disjoint $W(\cdot)$ elements. Moreover, given $\tilde{X}_i \subset X_i$ for $1 \le i \le m-1$ let $\tilde{X}_m \subset X_m$ maximal such that every $W(x)$, $x \in \tilde{X}_m$, does not intersect any element $W(y)$ for some $y \in \tilde{X}_1 \cup \dots \tilde{X}_m$. If $\tilde{X} = \cup\{\tilde{X}_i : 1 \le i \le s\}$ then the dynamical balls $W(x)$, $x \in \tilde{X}$, are pairwise disjoint (by construction). It is also easy to see that for every $y \in X$ there exists $x \in \tilde{X}$ such that $W(y) \subset V(x)$. Hence

$$\nu\Big(\bigcup_{x \in \tilde{X}} W(x) \setminus K \Big) \le \nu(O \setminus K) < \varepsilon \nu(K)$$

and, by the bounded distortion property,

$$\nu\Big(\bigcup_{x \in \tilde{X}} W(x) \Big) \ge \tau \nu\Big(\bigcup_{x \in \tilde{X}} V(x) \Big)$$

for some $\tau > 0$. We conclude immediately that there exists $x \in \tilde{X}$ such that

$$\frac{\nu(W(x) \setminus G)}{\nu(W(x))} \le \frac{\nu(W(x) \setminus K)}{\nu(W(x))} < \tau^{-1} \varepsilon.$$

Using the bounded distortion of f^n restricted to the dynamical ball $W(x)$ once more it follows that

$$\nu(B \setminus f^n(G)) < \tau^{-1} K_0 \varepsilon,$$

where B is a ball of radius $\delta/4$ around $f^n(x)$. Since ε was arbitrary and G is invariant then there exists a sequence Δ_n of balls of radius $\delta/4$ such that $\nu(\Delta_n \setminus G) \to 0$ as $n \to \infty$. By compactness, the sequence $(\Delta_n)_n$ accumulate on a ball Δ that satisfies the requirements of the lemma. □

We are now in a position to show that there are finitely many distinct ergodic measures $\mu_1, \mu_2, \dots, \mu_k$ absolutely continuous with respect to ν. Indeed, let μ be any invariant measure that is absolutely continuous. Then, either μ is ergodic or there are disjoint invariant sets I_1 and I_2 of positive ν-measure such that $\mu(\cdot) = a_1 \mu(\cdot \cap I_1)/\mu(I_1) + a_2 \mu(\cdot \cap I_2)/\mu(I_2)$, where $a_i = \mu(I_i)$. In the later case it is also clear that each of the measures involved in the sum is absolutely continuous with respect to ν. Repeating the process one obtains that μ can be written as linear convex combination of ergodic absolutely continuous invariant measures

$\mu_1, \mu_2, \dots, \mu_k$. Indeed, since M is compact the previous lemma implies that this process will stop after a finite number of steps (depending only on δ) with each μ_i ergodic. It is also clear from the construction that each μ_i is expanding and that their basins cover almost every point.

5.2. **Invariant non-lacunary Gibbs measure.** Through the rest of this section assume that f is topologically mixing. Here we prove that there is a unique invariant measure μ absolutely continuous with respect to ν and that it is a non-lacunary Gibbs measure. This will complete the proof of Proposition 5.1. We begin with a couple of auxiliary lemmas. Let $\theta > 0$ and $\delta > 0$ be given by Lemmas 3.2 and 3.4.

Lemma 5.4. *$f^n(H_n) = M$ for every $n \geq 1$.*

Proof. Fix any $x \in M$. By (H2) there exists a preimage $y \in M$ of the point x by f that y does not belong to $\mathcal{A}$. In consequence, $\|Df(y)^{-1}\| \leq \sigma^{-1} < e^{-c}$ which proves that $y \in H_1$ and $x \in f(H_1)$. The full statement of the lemma is obtained by repeating this argument recursively. Indeed, for any given $x \in M$ and $n \geq 1$ there are $y_1, y_2, \dots, y_n \in M$ such that $f(y_1) = x$, $f(y_{j+1}) = y_j$ and $y_j \notin \mathcal{A}$ for every $1 \leq j \leq n$. Hence,

$$\prod_{j=n-i}^{n-1} \|Df(f^j(y_n))^{-1}\| \leq \sigma^{-i} \leq e^{-ci} \quad \text{for every } 1 \leq i \leq n,$$

which proves that $y_n \in H_n$ and $x \in f^n(H_n)$ and finishes the proof of the lemma. $\square$

Lemma 5.5. *There exists a constant $\tau_0 > 0$, and for any n there is a finite subset $\hat{H}_n$ of H_n such that the dynamical balls $B(x, n, \delta/4)$, $x \in \hat{H}_n$, are pairwise disjoint and their union W_n satisfies $\nu(W_n) \geq \tau_0 \nu(H_n)$.*

Proof. This lemma is a direct consequence of Lemma 3.4 in [ABV00]. Indeed, if $\omega = f^n_*(\nu \mid \cup\{B(n, x, \delta/4) : x \in H_n\})$, $\Omega = f^n(H_n) = M$ and $r = \delta$ in that lemma then there exists a finite set $I \subset f^n(H_n)$ such that the pairwise disjoint union Δ_n of balls of radius $\delta/4$ around points in I satisfies

$$\omega\big(\Delta_n \cap f^n(H_n)\big) \geq \tau_0\, \omega(f^n(H_n)).$$

Set $\hat{H}_n = H_n \cap f^{-n}(I)$. Since the restriction of f^n to any dynamical ball $B(x, n, \delta/4)$, $x \in \hat{H}_n$ is a bijection it is easy to see that these dynamical balls are pairwise disjoint. Furthermore, their union W_n satisfies $\nu\big(W_n\big) \geq \tau_0\, \nu(H_n)$. This completes the proof of the lemma. $\square$

In the remaining of the section, let μ be an arbitrary accumulation point of the sequence $(\nu_n)_n$ and $(n_k)_k$ be a subsequence of the integers such that

$$\mu = \lim_{k\to\infty} \nu_{n_k}.$$

In the next lemmas we prove that the density $d\mu/d\nu$ is bounded away from zero in some small disk and use this to deduce the uniqueness of the equilibrium state and the non-lacunar Gibbs property.

Lemma 5.6. *There exists $C_1 > 0$ and a small disk $D(x)$ around a point x in M such that the density $d\mu/d\nu$ in the disk $D(x)$ is bounded from below by C_1.*

Proof. Given a small $\varepsilon > 0$ we construct a disk $D(x)$ of radius smaller than ε where the assertion above holds. Let W_j and $\hat{H}_j$ be given by the previous lemma and let $W_{j,\varepsilon} \subset W_j$ denote the preimages by f^j of the disks $\Delta_{j,\varepsilon}$ of radius $\delta/4 - \varepsilon$ around points in $f^j(\hat{H}_j)$. Lemma 3.5 implies that

$$\frac{\nu(W_{j,\varepsilon})}{\nu(W_j)} \geq K_0^{-1} \frac{\nu(\Delta_{j,\varepsilon})}{\nu(\Delta_j)},$$

where the right hand side is larger than some uniform positive constant τ_1 that depends only on the radius of the disks $\Delta_{j,\varepsilon}$ (recall Lemma 4.6). Observe also that Corollary 3.3 with $A = M$ implies that

$$\frac{1}{n} \sum_{j=0}^{n-1} \nu(H_j) \geq \theta/2$$

for every large n. This shows that there is a positive constant τ_2 such that the measures ν_n^ε satisfy $\nu_n^\varepsilon(M) \geq \tau_2$ for every large n, where

$$\nu_n^\varepsilon = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j(\nu | W_{j,\varepsilon}).$$

Thus, there exists a subsequence of $(\nu_{n_k}^\varepsilon)_k$ that converge to some measure ν_∞^ε and

$$\operatorname{supp}(\nu_\infty^\varepsilon) \subset \bigcap_{n \geq 1} (\overline{\bigcup_{j \geq n} \Delta_{j,\varepsilon}}).$$

Choose $x \in \operatorname{supp}(\nu_\infty^\varepsilon)$ and a disk $D(x)$ of radius smaller than ε around x such that $\nu_\infty^\varepsilon(\partial D(x)) = 0$. By construction, $D(x)$ is contained in every disk of Δ_j such that the corresponding disk of $\Delta_{j,\varepsilon}$ intersects $D(x)$. Let $\tilde{\Delta}_j$ denote the pairwise disjoint union of disks in Δ_j that contain $D(x)$ and $\tilde{W}_j$ be defined accordingly as the preimages of $\tilde{\Delta}_j$. It is clear that $\nu_n \geq \nu_n^0$, where

$$\nu_n^0 = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j(\nu | \tilde{W}_j).$$

Moreover, since $d\ f_*^j(\nu \mid \tilde{W}_j)/d\nu$ is Hölder continuous, the bounded distortion at Lemma 3.5 implies that it is bounded from below by its L^1 norm up to the multiplicative constant K_0^{-1}. So,

$$\frac{d\nu_n^0}{d\nu}(y) = \frac{1}{n} \sum_{j=0}^{n-1} \frac{d\ f_*^j(\nu \mid \tilde{W}_j)}{d\nu}(y) = \frac{1}{n} \sum_{j=0}^{n-1} \Big[\sum_{\substack{f^j(z)=y \\ z \in \tilde{W}_j}} \lambda^{-j} e^{S_j \phi(z)} \Big] \geq K_0^{-1} \frac{1}{n} \sum_{j=0}^{n-1} \nu(\tilde{W}_j)$$

for every $y \in D(x)$. Furthermore, by construction the set $W_{j,\varepsilon} \cap f^{-j}(D(x))$ is contained in $\tilde{W}_j$. This guarantees that

$$\frac{d\nu_n^0}{d\nu}(y) \geq K_0^{-1} \frac{1}{n} \sum_{j=0}^{n-1} \nu(\tilde{W}_j) \geq K_0^{-1} \nu_n^\varepsilon(D(x)) \geq K_0^{-1} \frac{\nu_\infty^\varepsilon(D(x))}{2}$$

for every large $n \geq 1$ in the subsequence of $(n_k)_k$ chosen above. By weak* convergence it holds that $d\mu/d\nu \geq C_1$ in the disk $D(x)$. $\square$

We finish this section by proving the uniqueness of the equilibrium state, which completes the proof of Proposition 5.1.

Lemma 5.7. *If f is topologically mixing there is a unique invariant measure μ absolutely continuous with respect to ν. Moreover, the density $d\mu/d\nu$ is bounded away from zero and infinity and the sequences of hyperbolic times $\{n_j(x)\}$ are non-lacunary μ-almost everywhere. Furthermore, μ is a non-lacunary Gibbs measure.*

Proof. We have proven that any accumulation point μ of $(\nu_n)_n$ is absolutely continuous with respect to ν and that the density $h = d\mu/d\nu$ is bounded from above by C_2 and is bounded from below by C_1 on some disk $D(x)$. Since f is topologically mixing there is $N \geq 1$ be such that $f^N(D(x)) = M$, that is, any point has some preimage by f^N in $D(x)$. It is not difficult to check that $h \in L^1(\nu)$ satisfies $\mathcal{L}_\phi h = \lambda h$. Then

$$h(y) = \lambda^{-N} \sum_{f^N(z)=y} e^{S_N\phi(z)} h(z) \geq C_1 \lambda^{-N} e^{N \inf \phi}$$

for almost every $y \in M$, which allows to deduce that the measures μ and ν are equivalent.

We claim that μ is ergodic. Indeed, if G is any f-invariant set such that $\mu(G) > 0$ then it follows from Lemma 5.3 that there is a disk Δ of radius $\delta/4$ such that $\nu(\Delta \setminus G) = 0$. Furthermore, using that $J_\nu f$ is bounded from above and from below, the invariance of G and that there is $\tilde{N} \geq 1$ such that $f^{\tilde{N}}(\Delta) = M$ it follows that $\nu(M \setminus G) = 0$, or equivalently, that $\mu(G) = 1$, proving our claim. So, if $\mu_1 \ll \nu$ is any f-invariant probability measure then $\mu_1 \ll \mu$. By invariance of $d\mu_1/d\mu$ and ergodicity of μ it follows that $d\mu_1/d\mu$ is almost everywhere constant and that $\mu_1 = \mu$. This proves the uniqueness of the absolutely continuous invariant measure. Lemma 5.2 also implies that

$$C_3 \nu(B(x, n, \delta)) \leq \mu(B(x, n, \delta)) \leq C_2 \nu(B(x, n, \delta))$$

for ν-almost every x and every $n \geq 1$, where $C_3 = C_1 \lambda^{-N} e^{N \inf \phi}$. In particular μ is expanding and, if n is a hyperbolic time for x and $y \in B(x, n, \delta)$ then

$$K^{-1} C_3 \leq \frac{\mu(B(x, n, \delta))}{e^{-Pn + S_n\phi(y)}} \leq K C_2.$$

Corollary 4.5 implies that the first hyperbolic time map n_1 is μ_i-integrable. Hence, the sequence of hyperbolic times is almost everywhere non-lacunary (see Corollary 3.8) and both μ and ν are non-lacunary Gibbs measures. This completes the proof of the lemma. □

6. Proof of Theorems A and B

In this section we manage to estimate the topological entropy of f for the potential ϕ using the characterizations of relative pressure given in Section 3.4: $P_{H^c}(f, \phi) < \log \lambda$ and $P_H(f, \phi) \leq \log \lambda$. Then, using that the measure theoretical pressure $P_\mu(f, \phi) = h_\mu(f) + \int \phi \, d\mu$ of every absolutely continuous invariant measure given by Proposition 5.1 is at least $\log \lambda$, we deduce that $P_{\text{top}}(f, \phi) = \log \lambda$ and that equilibrium states do exist. Finally, the variational property of equilibrium states yields that they coincide with the absolutely continuous invariant measures. This will complete the proofs of Theorems A and B.

6.1. **Existence of equilibrium states.** We give two estimates on the relative pressure and deduce the existence of equilibrium states for f with respect to ϕ. The following result was obtained jointly with Marcelo Viana and Krerley Oliveira.

Proposition 6.1. $P_{H^c}(f, \phi) < \log \lambda$.

Since we deal with a potential ϕ whose oscillation is not very large, the main point in the proof of Proposition 6.1 is to control the relative pressure $h_{H^c}(f)$. The key idea is that $h_{H^c}(f)$ can be bounded above using the maximal distortion and growth rate of the inverse branches that cover H^c. We will begin with some preparatory lemmas.

Lemma 6.2. *Let M be a compact manifold of dimension m. There exists $C > 0$ and a sequence of finite open coverings $(\mathcal{Q}_k)_{k\geq 1}$ of M such that* $\operatorname{diam}(\mathcal{Q}_k) \to 0$ *as $k \to \infty$, and every set $E \subset M$ satisfying* $\operatorname{diam}(E) \leq D \operatorname{diam} \mathcal{Q}_k$ *intersects at most CD^m elements of $\mathcal{Q}_k$.*

Proof. First we construct a special family of triangulations in M. Choose a finite triangulation $\mathcal{T}_0$ in M. Then, for any $T \in \mathcal{T}_0$ there is a diffeomorphism ϕ_T from a neighborhood of $T \subset M$ on $\mathbb{R}^m$ such that T is mapped diffeomorphically onto the standard unitary m-dimensional simplex $T_0 \subset \mathbb{R}^m$. Given $k \geq 1$, denote by T_k the triangulation in $T_0 \subset \mathbb{R}^m$ by regular m-simplices of size 2^{-k} and let $\mathcal{T}_k$ be the triangulation of M obtained by the pulling back the elements in T_k by the diffeomorphisms ϕ_T. Clearly, there exists $C' > 0$ (depending only on the finite set of diffeomorphisms) such that $\operatorname{diam}(\mathcal{T}_k) \leq C' 2^{-k}$ for every $k \geq 1$.

Fix a sequence of positive numbers $(\varepsilon_k)_{k\geq 1}$ such that $0 < \varepsilon_k \ll 2^{-k}$ for every $k \geq 1$. Let Q_k be the family of open neighborhoods of size ε_k around elements of T_k in $\mathbb{R}^m$ and $\mathcal{Q}_k$ be the finite open cover obtained by diffeomorphic preimages of elements in Q_k by the diffeomorphisms ϕ_T.

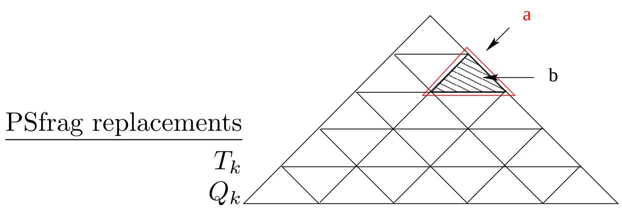

FIGURE 2. Family of Coverings in $\mathbb{R}^m$.

We claim that the family of finite open coverings $(\mathcal{Q}_k)_k$ satisfies the requirements of the lemma. Indeed, it is immediate that $\operatorname{diam}(\mathcal{Q}_k) \leq C' \operatorname{diam}(Q_k) \to 0$ as $k \to \infty$. On the other hand, given $k \geq 1$ and a diffeomorphism ϕ_T, if $E \subset M$ is such that $\operatorname{diam}(E) < D \operatorname{diam}(\mathcal{Q}_k)$ then the diameter of its image $\phi_T(E) \subset \mathbb{R}^m$ is at most $C' D\, (1 + \varepsilon_k/2^{-k})\, 2^{-k}$. Thus $\phi_T(E)$ can clearly intersect at most $[C' D\, (1 + \varepsilon_k/2^{-k})]^m$ elements of T_k. In addition, since $\varepsilon_k \ll 2^{-k}$ then every point in T_0 is covered by at most C'' elements of Q_k for some uniform constant $C'' > 0$ (depending only on the dimension m). This shows that E can intersect at most CD^m elements of $\mathcal{Q}_k$ with $C = [2C']^m\, C''\, \#\mathcal{T}_0$, and completes the proof of the lemma. □

The next result is the most technical lemma in the article and provides the key estimate to prove Proposition 6.1.

Lemma 6.3. *Given any $\ell \geq 1$ the following property holds:*

$$h_{H^c}(f^\ell) \leq (\log q + m \log L + \varepsilon_0/2)\, \ell + \log C.$$

Proof. Fix $\ell \geq 1$ and let $(\mathcal{Q}_k)_k$ be the family of finite open coverings given by the previous lemma. Since $\operatorname{diam}(\mathcal{Q}_k) \to 0$ as $k \to \infty$ then

$$P_{H^c}(f, \phi) = \lim_{k \to \infty} P_{H^c}(f, \phi, \mathcal{Q}_k),$$

by Definition 3.9. Recall $\mathcal{P}$ is the finite covering given by $(H2)$ and $B(n, \gamma)$ is the set of points whose frequency of visits to $\mathcal{A}$ up to time n is at least γ. The starting point is the next observation:

Claim 1: *For every $0 < \varepsilon < \gamma$ there exists $j_0 \geq 1$ such that for every $j \geq j_0$ the following holds:*

$$B(n, \gamma) \subset B(\ell j, \gamma - \varepsilon) \quad \textit{for every } j\,\ell \leq n < (j+1)\ell.$$

Proof of Claim 1: Given $\varepsilon > 0$, let j_0 be a positive integer larger than $(1-\gamma)/\varepsilon$. Given an arbitrary large n we can write $n = \ell j + r$, where $0 \leq r < \ell$ and $j \geq j_0$. Moreover, if x belongs to $B(n, \gamma)$ then $\#\{0 \leq i \leq n-1 : f^i(x) \in \mathcal{A}\} \geq \gamma\, n$ and consequently

$$\frac{1}{\ell j}\, \#\{0 \leq i \leq \ell j - 1 : f^i(x) \in \mathcal{A}\} \geq \gamma + \frac{\gamma r - r}{\ell j}.$$

Our choice of j_0 implies that the right hand side above is bounded from below by $\gamma - \varepsilon$. This shows that x belongs to $B(\ell j, \gamma - \varepsilon)$ and proves the claim. □

We proceed with the proof of the lemma. Observe that the set H^c is covered by

$$\bigcup_{n \geq N} \bigcup_{P \in \mathcal{P}^{(n)}} \Big\{ P \in \mathcal{P}^{(n)} : P \cap B(n, \gamma) \neq \emptyset \Big\}$$

for every $N \geq 1$. Let $\varepsilon > 0$ be small such that $\# I(n, \gamma - \varepsilon) \leq \exp(\log q + \varepsilon_0/2) n$ for every large n. Such an ε do exist because c_γ varies monotonically on γ (see the proof of Lemma 3.1). Then, the previous claim allow us to cover H^c using only cylinders whose depth is a multiple of ℓ: for any $N \geq 1$

$$H^c \subset \bigcup_{j \geq \frac{N}{\ell}} \bigcup_{P \in \mathcal{P}^{(\ell j)}} \Big\{ P \in \mathcal{P}^{(\ell j)} : P \cap B(\ell j, \gamma - \varepsilon) \neq \emptyset \Big\}. \tag{6.1}$$

Thus, from this moment on we will only consider iterates $n = j\,\ell$. Denote by $\mathcal{R}^{(n)}$ the collection of cylinders in $\mathcal{P}^{(n)}$ that intersect $B(n, \gamma - \varepsilon)$. Our aim is now to cover any element in $\mathcal{R}^{(n)}$ by cylinders relatively to the transformation f^ℓ. Given $k \geq 1$, denote by $\mathcal{S}_{f^\ell, j} \mathcal{Q}_k$ the set of j-cylinders of f^ℓ by elements in $\mathcal{Q}_k$, that is

$$\mathcal{S}_{f^\ell, j} \mathcal{Q}_k = \Big\{ Q_0 \cap f^{-\ell}(Q_1) \cap \cdots \cap f^{-\ell(j-1)}(Q_{j-1}) : Q_i \in \mathcal{Q}_k, i = 0, \ldots, j-1 \Big\}.$$

Furthermore, let $\mathcal{G}_{n,k}$ be the set of cylinders in $\mathcal{S}_{f^\ell, j} \mathcal{Q}_k$ that intersect any element of $\mathcal{R}^{(n)}$.

Claim 2: *Let $k \geq 1$ be large and fixed. Then*

$$\# \mathcal{G}_{j\ell, k} \leq \#\, \mathcal{Q}_k \times [C L^{\ell m}]^j \times e^{(\log q + \varepsilon_0/2) j \ell}$$

for every large j.

Proof of Claim 2: Recall $n = j\ell$ and fix $P_n \in \mathcal{R}^{(n)}$. Since f is a local diffeomorphism then the inverse branch $f^{-n} : f^n(P_n) \to P_n$ extends to the union of all $Q \in \mathcal{Q}_k$ so that $Q \cap f^n(P_n) \neq \emptyset$, provided that k is large. Notice that

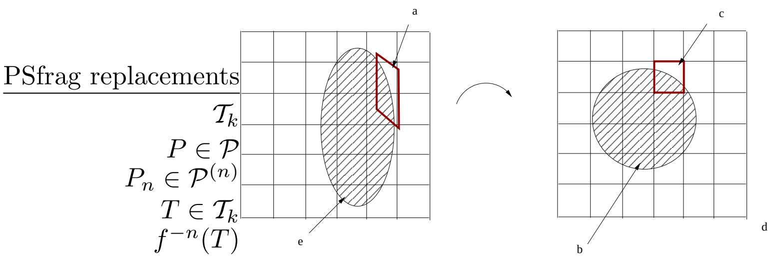

FIGURE 3. Covering elements of $\mathcal{R}^{(n)}$.

$\operatorname{diam}(f^{-\ell}(Q)) \leq L^\ell \operatorname{diam}(Q)$ for every $Q \in \mathcal{Q}_k$ because $\log \|Df(x)^{-1}\| \leq L$ for every $x \in M$. By Lemma 6.2, $f^{-\ell}(Q)$ intersects at most $CL^{\ell m}$ elements of the covering $\mathcal{Q}_k$. This proves that there are at most $\# \mathcal{Q}_k \times [CL^{\ell m}]^j$ cylinders in $\mathcal{S}_{f^\ell,j}\mathcal{Q}_k$ that intersect P_n. The claim is a direct consequence of our choice of ε since $\#\mathcal{R}^{(n)} \leq e^{(\log q + \varepsilon_0/2)n}$ for large n. □

Finally we complete the proof of the lemma. Indeed, it is immediate from (6.1) that

$$m_\alpha(f^\ell, 0, H^c, \mathcal{Q}_k, N) \leq \sum_{j \geq N/\ell} \sum_{\underline{\mathrm{U}} \in \mathcal{G}_{\ell j,k}} e^{-\alpha n(\underline{\mathrm{U}})} = \sum_{j \geq N/\ell} e^{-\alpha j} \# \mathcal{G}_{\ell j,k}$$

for every large k. Moreover, Claim 2 implies that the sum in the right hand side above converges to zero as $N \to \infty$ (independently of k) whenever $\alpha > (\log q + \varepsilon_0/2 + m \log L)\,\ell + \log C$. This shows that $h_{H^c}(f^\ell) \leq (\log q + m \log L + \varepsilon_0/2)\,\ell + \log C$ and completes the proof of the lemma. □

Proof of Proposition 6.1. Recall that $h_{H^c}(f^\ell) = \ell\, h_{H^c}(f)$, by Proposition 3.12. Then, as a consequence of the previous lemma we obtain

$$h_{H^c}(f) \leq \log q + m \log L + \varepsilon_0/2 + \frac{\log C}{\ell}$$

for every $\ell \geq 1$. Finally, it follows from (3.3) and Lemma 3.14 that

$$P_{H^c}(f, \phi) \leq \log q + m \log L + \sup \phi + \varepsilon_0 < \log \deg f + \inf \phi \leq \log \lambda.$$

□

In the present lemma we give an upper bound on the relative pressure of ϕ relative to the set H. More precisely,

Lemma 6.4. $P_H(f, \phi) \leq \log \lambda$.

Proof. Recall the characterization of relative pressure using dynamical balls in Subsection 3.4. Pick $\alpha > \log \lambda$. For any given $N \geq 1$, H is contained in the union of the sets H_n over $n \geq N$. Thus, given $0 < \varepsilon \leq \delta$

$$H \subset \bigcup_{n \geq N} \bigcup_{x \in H_n} B(x, n, \varepsilon).$$

Now we claim that there exists $D > 0$ (depending only on $m = \dim(M)$) so that for every $n \geq N$ there is a family $\mathcal{G}_n \subset H_n$ in such a way that every point in H_n is covered by at most D dynamical balls $B(x, n, \varepsilon)$ with $x \in \mathcal{G}_n$. In fact, Besicovitch's covering lemma asserts that there is a constant $D > 0$ (depending on m) and an at most countable family $\mathcal{G}_n \subset H_n$ such that every point of $f^n(H_n)$ is contained in at most D elements of the family $\{B(f^n(x), \varepsilon) : x \in \mathcal{G}_n\}$. Using that each dynamical ball $B(x, n, \varepsilon)$, $x \in H_n$, is mapped diffeomorphically onto $B(f^n(x), \varepsilon)$, it follows that every point in H_n is contained in at most D dynamical balls $B(x, n, \varepsilon)$ with $x \in \mathcal{G}_n$, proving our claim. Given any positive integer $N \geq 1$, it follows by bounded distortion and the Gibbs property of ν at hyperbolic times that

$$m_\alpha(f, \phi, H, \varepsilon, N) \leq K(\varepsilon) \sum_{n \geq N} e^{-(\alpha - P)n} \Big\{ \sum_{x \in \mathcal{G}_n} \nu(B(x, n, \varepsilon)) \Big\}.$$

Consequently $m_\alpha(f, \phi, H, \varepsilon, N) \leq K(\varepsilon) \, \frac{D}{1 - e^{-(\alpha - P)}} \, e^{-(\alpha - P)N}$, which tends to zero as $N \to \infty$ independently of ε. This shows that $P_H(f, \phi) \leq \log \lambda$ and completes the proof of the lemma. □

We know that every ergodic component of an absolutely continuous invariant measure is also absolutely continuous. Now we prove that the absolutely continuous invariant measures are indeed an equilibrium states.

Lemma 6.5. *If μ is an ergodic measure absolutely continuous with respect to ν then $P_\mu(f, \phi) \geq \log \lambda$. Moreover, μ is an equilibrium state for f with respect to ϕ and the following equalities hold*

$$P_{\text{top}}(f, \phi) = P_H(f, \phi) = \log \lambda.$$

Proof. The previous estimates and Proposition 3.11 guarantee that

$$P_{\text{top}}(f, \phi) = \sup\{P_H(f, \phi), P_{H^c}(f, \phi)\} \leq \log \lambda.$$

Using that $d\mu/d\nu \leq C_2$, that ν satisfies the Gibbs property at hyperbolic times and μ-almost every point x admits a sequence $\{n_k(x)\}$ of hyperbolic times then

$$\mu(B(x, n_k, \varepsilon)) \leq C_2 \, K(\varepsilon) \, e^{-P n_k + S_{n_k} \phi(y)}$$

for every $0 < \varepsilon \leq \delta$, every $k \geq 1$ and every $y \in B(x, n_k, \varepsilon)$. Thus, Brin-Katok's local entropy formula for ergodic measures and Birkhoff's ergodic theorem (see e.g. [Mañ87]) immediately imply that

$$h_\mu(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu(B(x, n, \varepsilon)) \geq P - \int \phi \, d\mu,$$

where the first equality holds μ-almost everywhere. In particular

$$\log \lambda \geq P_{\text{top}}(f, \phi) \geq P_H(f, \phi) \geq \sup_{\mu(H)=1} \Big\{ h_\mu(f) + \int \phi \, d\mu \Big\} \geq \log \lambda,$$

which proves that μ is an equilibrium state and that the three quantities in the statement of the lemma do coincide. This completes the proof of the lemma. □

6.2. **Finitude of ergodic equilibrium states.** In this subsection we will complete the proof of Theorems A and B. First we combine that every equilibrium state is an expanding measure with some ideas involved in the proof of the variational properties of SRB measures in [Led84] to deduce that every equilibrium state is absolutely continuous with respect to some conformal measure supported in the closure of the set H, and to obtain finitude of ergodic equilibrium states. Finally, we show that under the topologically mixing assumption there is a unique equilibrium state, and that it is exact and a non-lacunary Gibbs measure. We begin with the following abstract result:

Theorem 6.6. *Let $f : M \to M$ be a $C^{1+\alpha}$ local diffeomorphism, $\phi : M \to \mathbb{R}$ be a Hölder continuous potential and ν be a conformal measure such that $J_\nu f = \lambda e^{-\phi}$, where $\lambda = \exp(P_{\text{top}}(f, \phi))$. Assume that η is an equilibrium state for f with respect to ϕ gives full weight to* $\text{supp}(\nu)$ *and such that all the Lyapunov exponents are positive. Then η is absolutely continuous with respect to ν.*

Let us stress out that this theorem holds in a more general setting. Since this fact will not be used here, we will postpone the discussion to Remark 6.15 near the end of the section. The finitude of equilibrium states is a direct consequence of the previous result. Indeed,

Corollary 6.7. *Let f be a $C^{1+\alpha}$ local diffeomorphism and let ϕ be a Hölder continuous potential satisfying (H1), (H2) and (P). There exists an expanding conformal probability measure ν such that every equilibrium state for f with respect to ϕ is absolutely continuous with respect to ν with density bounded from above. If, in addition, f is topologically mixing then there is unique equilibrium state and it is a non-lacunary Gibbs measure.*

Proof. Let ν be the expanding conformal measure given by Theorem 4.1 and η be an ergodic equilibrium state for f with respect to ϕ. We claim that η is an expanding measure. Indeed, assume by contradiction that one can decompose η as a linear convex combination of two measures $\eta = t\eta_1 + (1-t)\eta_2$ with $\eta_2(H^c) = 1$ for some $0 \le t < 1$. But Lemma 6.5, the first part of Proposition 3.11 and the convexity of the pressure yield

$$P_\eta(f, \phi) = tP_{\eta_1}(f, \phi) + (1-t)P_{\eta_2}(f, \phi) \le t\, P_{\text{top}}(f, \phi) + (1-t)\, P_{H^c}(f, \phi) < P_{\text{top}}(f, \phi),$$

which contradicts that η is an equilibrium state and proves our claim. Moreover, $\eta(\text{supp}(\nu)) = 1$ because the support of ν coincides with the closure of H. Finally, using that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \le -2c < 0$$

at η-almost every point (Corollary 6.7) and $\|Df^n(x)v\| \ge \prod_{j=0}^{n-1} \|Df(f^j(x))^{-1}\|^{-1}$ for every vector v such that $\|v\| = 1$, every $x \in M$ and $n \ge 1$, it follows that

$$\liminf_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| \ge 2c > 0$$

for η-almost every $x \in M$ and every $v \in T_x M$. This shows that η has only positive Lyapunov exponents and that all the assumptions of Theorem 6.6 are verified. Therefore, this result is a direct consequence of the previous theorem and Proposition 5.1. □

In the sequel we prove Theorem 6.6. Since f is a non-invertible transformation we use the natural extension, introduced in Subsection 3.5, to deal with unstable manifolds.

Proof of Theorem 6.6. It is easy to check, using the variational principle, that almost every ergodic component of an equilibrium state is an equilibrium state. Thus, by ergodic decomposition it is enough to prove the result for ergodic measures.

Let η be an ergodic equilibrium state and $(\hat{f}, \hat{\eta})$ be the natural extension of η introduced in Subsection 3.5. Recall that $\hat{\eta}$ is an $\hat{f}$-invariant probability measure in $\hat{M}$ such that $\pi_*\hat{\eta} = \eta$ and that the Lyapunov exponents of the induced cocycle $\hat{A}$ with respect to the measure $\hat{\eta}$ coincide with the Lyapunov exponents of f with respect to η (see Proposition 3.15). Hence, $\hat{\eta}$ has only positive Lyapunov exponents.

We proceed with the construction of a special partition $\hat{\mathcal{Q}}$ of $\hat{M}$ that is closely related with Ledrappier's geometric construction in Proposition 3.1 of [Led84] and provides a key ingredient for the proof of Theorem 6.6. The main differences from the original result due to Ledrappier are that the natural extension $\hat{M}$ is not in general a manifold and that there is no well defined unstable foliation in M. The arguments involved in the construction of a partition satisfying the properties above follow well known ideas that can be traced back to [Led84, LS86]. See also [QZ02] for a construction in a related context. First we set some notations. Given a partition $\hat{\mathcal{Q}}$ denote by $\hat{\mathcal{Q}}(\hat{x})$ the element of $\hat{\mathcal{Q}}$ that contains $\hat{x} \in \hat{M}$. We say that $\hat{Q}$ is an *increasing partition* if $(\hat{f}^{-1}\hat{\mathcal{Q}})(\hat{x}) \subset \hat{\mathcal{Q}}(\hat{x})$ for $\hat{\eta}$-almost every $\hat{x}$. In this case, we write $\hat{f}^{-1}\hat{\mathcal{Q}} \succ \hat{\mathcal{Q}}$.

Proposition 6.8. *There exists an invariant and full $\hat{\eta}$-measure set $\hat{S} \subset \hat{M}$, and a measurable partition $\hat{\mathcal{Q}}$ of $\hat{S}$ such that:*

1. $\hat{f}^{-1}\hat{\mathcal{Q}} \succ \hat{\mathcal{Q}}$,
2. $\bigvee_{j=0}^{+\infty} \hat{f}^{-j}\hat{\mathcal{Q}}$ *is the partition into points,*
3. *The sigma-algebras $\mathcal{M}_n$ generated by the partitions $\hat{f}^{-n}\hat{\mathcal{Q}}$, $n \geq 1$, generate the σ-algebra in $\hat{S}$, and*
4. *For almost every $\hat{x}$ the element $\hat{\mathcal{Q}}(\hat{x}) \subset \hat{W}^u(\hat{x})$ contains a neighborhood of $\hat{x}$ in $\hat{W}^u(\hat{x})$ and the projection $\pi(\hat{\mathcal{Q}}(\hat{x}))$ contains a neighborhood of x_0 in M.*

Proof. Since $\hat{\eta}$ is an hyperbolic measure, Pesin's theory applied to the natural extension $\hat{f}$ guarantees the existence of local unstable manifolds at almost every point. Take $i \geq 1$ such that $\hat{\eta}(\hat{\Lambda}_i) > 0$ and let r_i, ε_i, γ_i and R_i be given by Corollary 3.17. Fix also $0 < r \leq r_i$ and $\hat{x} \in \operatorname{supp}(\hat{\eta} \mid_{\hat{\Lambda}_i})$. Recall that $\hat{y} \mapsto W^u_{\text{loc}}(\hat{y}) \cap B(x_0, r)$ is a continuous function on $B(\hat{x}, \varepsilon_i r) \cap \hat{\Lambda}_i$. Consider the sets

$$\hat{V}(\hat{y}, r) = \{\hat{z} \in \hat{W}^u_{\text{loc}}(\hat{y}) : z_0 \in B(x_0, r)\},$$

defined for any $\hat{y} \in B(\hat{x}, \varepsilon_i r) \cap \hat{\Lambda}_i$. Define also

$$\hat{S}(\hat{x}, r) = \bigcup \{\hat{V}(\hat{y}, r) : \hat{y} \in B(\hat{x}, \varepsilon_i r) \cap \hat{\Lambda}_i\}$$

and the partition $\hat{\mathcal{Q}}_0(r)$ of $\hat{M}$ whose elements are the connected components $\hat{V}(\hat{y}, r)$ of unstable manifolds just constructed and their complement $\hat{M} \setminus \hat{S}(\hat{x}, r)$. Furthermore, consider the set $\hat{S}_r$ and the partition $\hat{\mathcal{Q}}(r)$ given by

$$\hat{S}_r = \bigcup_{n=0}^{+\infty} \hat{f}^n(\hat{S}(\hat{x}, r)) \quad \text{and} \quad \hat{Q}(r) = \bigvee_{n=0}^{+\infty} \hat{f}^n(\hat{Q}_0(r)).$$

Then, the partition $\hat{\mathcal{Q}}$ coincides with the partition $\hat{\mathcal{Q}}(r)$ and the set $\hat{S}$ is given by $\bigcap_{j\geq 0} \hat{f}^{-j}(\hat{S}_r)$ for a particular choice of the parameter r. In what follows, for notational convenience and when no confusion is possible we shall omit the dependence of the partition $\hat{\mathcal{Q}}$ on r.

It is clear from the construction that every partition $\hat{\mathcal{Q}}(r)$ is increasing, that is the content of (1). In addition, since $\hat{\eta}$ is ergodic and $\hat{\eta}(\hat{S}(\hat{x}, r)) > 0$ then the set of points that return infinitely often to $\hat{S}(\hat{x}, r)$, which we called $\hat{S}_r$, is a full measure set by Poincaré's Recurrence Theorem. In other words, if a point $\hat{y}$ belongs to $\hat{S}_r$ there are positive integers $(n_j)_j$ such that $\hat{f}^{n_j}(\hat{y}) \in \hat{V}(\hat{f}^{n_j}(\hat{y}), r)$. Hence, the backward distance contraction along unstable leaves guarantees that the d^u-diameter of the partition $\bigvee_{n=0}^{n} \hat{f}^{-j}\hat{\mathcal{Q}}$ tend to zero as $n \to \infty$, proving (2). By construction, there is a full measure set such that any two distinct points $\hat{y}$ and $\hat{z}$ lie in different elements of $\hat{f}^{-n}\hat{\mathcal{Q}}$ for some $n \in \mathbb{N}$. Indeed, if $\hat{f}^{-n}\hat{\mathcal{Q}}(\hat{y}) = \hat{f}^{-n}\hat{\mathcal{Q}}(\hat{z})$ for every $n \geq 0$ then $\hat{f}^n(\hat{y})$ and $\hat{f}^n(\hat{z})$ lie infinitely often in the same local unstable manifold. But (2) implies that $\hat{y}$ and $\hat{z}$ should coincide, which is a contradiction and proves our claim. In particular, the decreasing family of σ-algebras $\mathcal{M}_n$, $n \geq 1$, generate the σ-algebra in $\hat{S}_r$, which proves (3).

We proceed to show that the partition $\hat{\mathcal{Q}}(r)$ satisfies (4) for Lebesgue almost every parameter r. Given $0 < r \leq r_i$ and $\hat{y} \in \hat{S}_r$ define

$$\beta_r(\hat{y}) = \inf_{n\geq 0} \Big\{ R_i, \frac{r}{\gamma_i}, \frac{1}{2\gamma_i} e^{\lambda_i n} d(y_{-n}, \partial B(x_0, r)) \Big\},$$

that it clearly non-negative. It is enough to obtain the following:

(a) If $z_0 \in W^u_{\text{loc}}(\hat{y})$ and $d^u(y_0, z_0) < \beta_r(\hat{y})$ then there exists $\hat{z} \in \hat{\mathcal{Q}}(\hat{y})$ such that $\pi(\hat{z}) = z_0$;

(b) There exists a full Lebesgue measure set of parameters $0 < r \leq r_i$ such that the function $\beta_r(\cdot)$ is strictly positive almost everywhere and $\hat{\eta}(\partial \hat{\mathcal{Q}}(r)) = 0$.

Take $\hat{y} \in \hat{S}_r$ and assume that $z_0 \in W^u_{\text{loc}}(\hat{y})$ is such that $d^u(y_0, z_0) < \beta_r(\hat{y})$. If $\hat{y} \in \hat{S}(\hat{x}, r)$ then there exists $\hat{w} \in B(\hat{x}, \varepsilon_i r)$ such that $\hat{y} \in \hat{W}^u_{\text{loc}}(\hat{w})$. Furthermore, since $d^u(y_0, z_0) < \beta_r(\hat{y}) < R_i$ then there exists $\hat{z} \in \hat{W}^u_{\text{loc}}(\hat{w})$ such that $\pi(\hat{z}) = z_0$. Hence

$$d^u(y_{-n}, z_{-n}) \leq \gamma_i e^{-n\lambda_i} d^u(y_0, z_0), \ \forall n \in \mathbb{N},$$

which implies that $d^u(y_{-n}, z_{-n}) \leq r$ and $d^u(y_{-n}, z_{-n}) \leq 1/2\, d(y_{-n}, \partial B(x_0, r))$ for every $n \in \mathbb{N}$. Together with Corollary 3.17, this shows that y_{-n} and z_{-n} belong to the same element of the partition $\hat{\mathcal{Q}}_0$ for every $n \geq 1$ and, assuming (b) for the moment, that $\pi(\mathcal{Q}(\hat{y}))$ contains a neighborhood of y_0 in $W^u_{\text{loc}}(\hat{y})$. On the other hand, if $\hat{y} \in \hat{S}_r \setminus \hat{S}(\hat{x}, r)$ then there exists $k \geq 1$ such that $\hat{f}^{-k}(\hat{y}) \in \hat{S}(\hat{x}, r)$ and consequently the projection of the set

$$\hat{\mathcal{Q}}(\hat{y}) = \hat{f}^k(\hat{\mathcal{Q}}(\hat{f}^{-k}(\hat{y}))$$

contains an open neighborhood of y_0 in $W^u_{\text{loc}}(\hat{y})$. This completes the proof of (a).

The proof of (b) is slightly more involving. We begin with the following remark from measure theory: if $r_0 > 0$, ϑ is a Borel measure in $[0, r_0]$ and $0 < a < 1$ then Lebesgue almost every $r \in [0, r_0]$ satisfies

$$\sum_{k=0}^{\infty} \vartheta\big([r - a^k, r + a^k]\big) < \infty. \tag{6.2}$$

Indeed, the set

$$B_{a,k} = \Big\{ r \in [0, r_0] : \vartheta\big([r - a^k, r + a^k]\big) > \frac{\vartheta\big([0, r_0]\big)}{k^2} \Big\}$$

can be covered by a family I_k of balls of radius a^k centered at points of $B_{a,k}$ in such a way that any point is contained in at most two intervals of I_k. Since

$$\# I_k \, \frac{\vartheta\big([0, r_0]\big)}{k^2} \leq \sum_{I \in I_k} \vartheta(I) \leq 2\, \vartheta([0, r_0])$$

then $\# I_k \leq 2k^2$ and it is clear that $\text{Leb}(B_{a,k}) \leq 2a^k \# I_k$ is summable. Borel-Cantelli's lemma implies that Lebesgue almost every $r \in [0, r_0]$ belongs to finitely many sets $B_{a,k}$, which proves the summability condition in (6.2).

Back to the proof of (b), let ϑ be the measure of the interval $[0, r_i]$ defined by $\vartheta(E) = \hat{\eta}\big(\, \hat{y} \in \hat{M} : d(x_0, y_0) \in E \,\big)$. The previous assertion guarantees that for Lebesgue almost every $r \in [0, r_i]$ it holds

$$\sum_{k=0}^{\infty} \hat{\eta}\big(\, \hat{y} \in \hat{M} : |d(x_0, y_0) - r| < e^{-\lambda_i k} \,\big) < \infty. \tag{6.3}$$

On the other hand, there exists $D > 0$ such that $|d(z_0, x_0) - r| < D\tau$ whenever $d(z_0, \partial B(x_0, r)) < \tau$ and $0 < \tau < r \leq r_i$. Therefore

$$\sum_{k=0}^{\infty} \hat{\eta}\big(\, \hat{y} \in \hat{M} : |d(y_{-n}, \partial B(x_0, r))| < D^{-1} e^{-\lambda_i k} \,\big) \leq$$

$$\leq \sum_{k=0}^{\infty} \hat{\eta}\big(\, \hat{y} \in \hat{M} : |d(x_0, y_{-n}) - r| < e^{-\lambda_i k} \,\big),$$

which is finite because of the invariance of $\hat{\eta}$ and the former condition (6.3). Using Borel-Cantelli's lemma once more it follows that $\hat{\eta}$-almost every $\hat{y}$ satisfies

$$|d(y_{-n}, \partial B(x_0, r))| < D^{-1} e^{-\lambda_i k}$$

for at most finitely many positive integers k, proving that $\beta_r(\hat{y}) > 0$. Furthermore, since $\eta(\cup_{n \geq 0} f^n(\partial B(x_0, r))) = 0$ for all but a countable set of parameters $0 < r \leq r_i$ then $\hat{\mathcal{Q}}(\hat{y})$ contains a neighborhood of $\hat{y}$ in $\hat{W}^u_{\text{loc}}(\hat{y})$ for $\hat{\eta}$-almost every $\hat{y} \in \hat{M}$. This shows that (b) holds and, in consequence, for Lebesgue almost every $r \in [0, r_i]$ the partition $\hat{\mathcal{Q}}(r)$ satisfies the requirements of the proposition. $\square$

Let $(\hat{\eta}_x)_x$ be the disintegration of the measure $\hat{\eta}$ on the measurable partition $\hat{\mathcal{Q}}$, given by Rokhlin's theorem. Recall that for $\hat{\eta}$-almost every $\hat{x}$ the map $\pi \mid_{\hat{W}^u_{\text{loc}}(\hat{x})}$: $\hat{W}^u_{\text{loc}}(\hat{x}) \to W^u_{\text{loc}}(\hat{x})$ is a bijection. For any such $\hat{x}$ let $\hat{\nu}_x$ be the measure on $\hat{W}^u_{\text{loc}}(\hat{x})$

obtained as the pull-back of $\nu \mid_{W^u_{\text{loc}}(\hat{x})}$ by the bijection $\pi \mid_{\hat{W}^u_{\text{loc}}(\hat{x})}$. Let $\hat{\nu}$ denote the measure defined on $\hat{M}$ by the disintegration $(\hat{\nu}_{\hat{x}})_{\hat{x}}$, that is to say that

$$\hat{\nu}(\hat{E}) = \int \hat{\nu}_{\hat{x}}(\hat{E})\, d\hat{\eta}(\hat{x})$$

for every measurable set $\hat{E}$ in $\hat{M}$. As a byproduct of the previous result we obtain

Corollary 6.9. $0 < \hat{\nu}_{\hat{x}}(\hat{\mathcal{Q}}(\hat{x})) < \infty$*, for* $\hat{\eta}$*-almost every* $\hat{x}$*.*

Proof. For every $\hat{x}$ in a full $\hat{\eta}$-measure set one has that

$$\hat{\nu}_{\hat{x}}(\hat{\mathcal{Q}}(\hat{x})) = \nu\big(\pi(\hat{\mathcal{Q}}(\hat{x})) \cap W^u_{\text{loc}}(\hat{x})\big).$$

Since $\hat{\eta}$ is an expanding measure then $\hat{W}^u_{\text{loc}}(\hat{x})$ is a neighborhood $\hat{x}$ and $W^u_{\text{loc}}(\hat{x}) \cap \pi(\hat{\mathcal{Q}}(\hat{x}))$ contains a neighborhood of x_0 in M. In addition, since $\eta(\operatorname{supp}\nu) = 1$, for every $\hat{x}$ in a full $\hat{\eta}$-measure set it holds that $x_0 \in \operatorname{supp}(\nu)$. Then it is clear that $0 < \hat{\nu}_{\hat{x}}(\hat{\mathcal{Q}}(\hat{x})) < \infty$, $\hat{\eta}$-almost everywhere, which proves the corollary. □

The next preparatory lemma shows that $\hat{\nu}$ has a Jacobian with respect to $\hat{f}$ and establishes Rokhlin's formula for the natural extension.

Lemma 6.10. *The measure* $\hat{\nu}$ *has a Jacobian* $J_{\hat{\nu}}\hat{f} = J_\nu f \circ \pi$ *with respect to* $\hat{f}$*. In addition,*

$$h_{\hat{\eta}}(\hat{f}) = \int \log J_{\hat{\nu}}\hat{f}\; d\hat{\eta}.$$

Furthermore, for $\hat{\eta}$*-almost every* $\hat{x}$ *and every* $\hat{y} \in \hat{\mathcal{Q}}(\hat{x})$ *the product*

$$\Delta(\hat{x}, \hat{y}) = \prod_{j=1}^{\infty} \frac{J_{\hat{\nu}}\hat{f}(\hat{f}^{-j}(\hat{x}))}{J_{\hat{\nu}}\hat{f}(\hat{f}^{-j}(\hat{y}))}$$

is positive and finite.

Proof. Since the sigma-algebra $\hat{\mathcal{B}}$ is the completion of the sigma-algebra generated by the cylinders $\pi_i^{-1}(f^{-i}\mathcal{B})$, $i \geq 1$, then the first claim in the lemma is a consequence from the fact that

$$\hat{\nu}_{\hat{f}(\hat{x})}(\hat{f}(\hat{E})) = \int_{\hat{E}\cap(\hat{f}^{-1}\hat{\mathcal{Q}})(\hat{x})} J_\nu f \circ \pi \, d\hat{\nu}_{\hat{x}} \tag{6.4}$$

for almost every $\hat{x}$ and every small cylinder $\hat{E} = \pi^{-1}(E)$. Indeed, if $\hat{E}$ is a small cylinder then it is clear that

$$\hat{\nu}(\hat{f}(\hat{E})) = \int \hat{\nu}_{\hat{f}(\hat{x})}(\hat{f}(\hat{E}))\, d\hat{\eta}(\hat{x}) = \int\int_{\hat{E}\cap(\hat{f}^{-1}\hat{\mathcal{Q}})(\hat{x})} J_\nu f \circ \pi \, d\hat{\nu}_{\hat{x}}\, d\hat{\eta}(\hat{x}). \tag{6.5}$$

Let $\tilde{\nu}_{\hat{x}}$ denote the restriction of the measure $\hat{\nu}_{\hat{x}}$ to the set $(\hat{f}^{-1}\hat{\mathcal{Q}})(\hat{x}) \subset \hat{\mathcal{Q}}(\hat{x})$. Then $\hat{\nu}$ has a disintegration $\hat{\nu} = \int \tilde{\nu}_{\hat{x}}\, d\hat{\eta}$ with respect to the measurable partition $\hat{f}^{-1}\hat{\mathcal{Q}}$. Together with (6.5) this gives

$$\hat{\nu}(\hat{f}(\hat{E})) = \int\int_{\hat{E}} J_\nu f \circ \pi \, d\tilde{\nu}_{\hat{x}}\, d\hat{\eta}(\hat{x}) = \int_{\hat{E}} J_\nu f \circ \pi \, d\hat{\nu},$$

which proves that $\hat{\nu}$ has a Jacobian and $J_{\hat{\nu}}\hat{f} = J_\nu f \circ \pi$. Hence, to prove the first assertion in the lemma we are reduced to prove (6.4) above. If $f \mid E$ is injective and $\hat{E} = \pi^{-1}(E)$ then

$$\hat{\nu}_{\hat{f}(\hat{x})}(\hat{f}(\hat{E})) = \hat{\nu}_{\hat{f}(\hat{x})}\big(\hat{f}[\hat{E} \cap (\hat{f}^{-1}\hat{\mathcal{Q}})(\hat{x})]\big) = \nu\big(f(E \cap \pi((\hat{f}^{-1}\hat{\mathcal{Q}})(\hat{x}))\big)$$

$$= \int_{E\cap\pi((\hat{f}^{-1}\hat{\mathcal{Q}})(\hat{x})} J_\nu f \, d\nu = \int_{\hat{E}\cap(\hat{f}^{-1}\hat{\mathcal{Q}})(\hat{x})} J_\nu f \circ \pi \, d\hat{\nu}_{\hat{x}},$$

which proves (6.4). On the other hand, $h_\eta(f) = \int J_\nu f \, d\eta$ because η is an equilibrium state, $P_{\text{top}}(f, \phi) = \log \lambda$ and $J_\nu f = \lambda e^{-\phi}$. So, using $\pi_* \hat{\eta} = \eta$ we obtain

$$h_{\hat{\eta}}(\hat{f}) = h_\eta(f) = \int \log J_\nu f \, d\eta = \int \log(J_\nu f \circ \pi) \, d\hat{\eta} = \int \log J_{\hat{\nu}} \hat{f} \, d\hat{\eta},$$

which proves the second assertion in the lemma. Finally, the Hölder continuity of the Jacobian $J_{\hat{\nu}}\hat{f} = J_\nu f \circ \pi$, the fact that $\hat{\mathcal{Q}}$ is subordinated to unstable leaves and the backward distance contraction for points in the same unstable leaf yield that the product

$$\Delta(\hat{x}, \hat{y}) = \prod_{j=1}^{\infty} \frac{J_{\hat{\nu}}\hat{f}(\hat{f}^{-j}(\hat{x}))}{J_{\hat{\nu}}\hat{f}(\hat{f}^{-j}(\hat{y}))}$$

is convergent for almost every $\hat{x}$ and every $\hat{y} \in \hat{\mathcal{Q}}(\hat{x})$. The proof of the lemma is now complete. □

The last main ingredient to the proof of Theorem 6.6 is the following generating property of the partition $\hat{\mathcal{Q}}$.

Proposition 6.11. $h_{\hat{\eta}}(\hat{f}) = H_{\hat{\eta}}(\hat{f}^{-1}\hat{\mathcal{Q}} \mid \hat{\mathcal{Q}})$.

The proof of this result involves two preliminary lemmas. Let $i \geq 1$ and $\hat{\Lambda}_i$ be given as in the proof of Proposition 6.8 and r_i given by Corollary 3.17. The following lemma gives a dynamical characterization of the local unstable manifolds.

Lemma 6.12. *Given $\varepsilon > 0$ there is a measurable function $\hat{D}_\varepsilon : \hat{B}_\lambda \to \mathbb{R}_+$ satisfying $\log \hat{D}_\varepsilon \in L^1(\hat{\eta})$ and such that, if $d(x_{-n}, y_{-n}) \leq \hat{D}_\varepsilon(\hat{f}^{-n}(\hat{x}))$ $\forall n \geq 0$ then $\hat{y} \in \hat{W}^u_{loc}(\hat{x})$ and $d(x_0, y_0) < 2r_i$.*

Proof. Since $\hat{\eta}(\hat{\Lambda}_i) > 0$ and $\hat{\eta}$ is assumed to be ergodic then some iterate of almost every point will eventually belong to $\hat{\Lambda}_i$ by Poincaré's recurrence theorem. So, the first hitting time $R(\hat{x})$ is well defined almost everywhere in $\hat{\Lambda}_i$ and $\int_{\hat{\Lambda}_i} R \, d\hat{\eta} = 1/\hat{\eta}(\hat{\Lambda}_i)$, by Kac's lemma. This proves that the logarithm of the function $\hat{D}_\varepsilon : \hat{M} \to \mathbb{R}$ given by

$$\hat{D}_\varepsilon(\hat{x}) = \begin{cases} \min\{2r_i \,,\, \delta_i \,,\, \delta_i/\gamma_i\} \, e^{-(\lambda+\varepsilon)R(\hat{x})} & \text{, if } \hat{x} \in \hat{\Lambda}_i \\ \min\{2r_i \,,\, \delta_i \,,\, \delta_i/\gamma_i\} & \text{, otherwise} \end{cases}$$

is $\hat{\eta}$-integrable. On the other hand, if $\hat{x} \in \hat{\Lambda}_i$ then $R(\hat{f}^{-n}(\hat{x})) = n$. Any $\hat{y} \in \hat{M}$ such that $d(x_{-n}, y_{-n}) \leq \hat{D}_\varepsilon(\hat{f}^{-n}(\hat{x}))$ for every $n \geq 0$ clearly satisfies $d(x_0, y_0) < 2r_i$ and, by Proposition 3.16(2), belongs to $W^u_{\text{loc}}(\hat{x})$. This concludes the proof of the lemma. □

This result allow us to construct an auxiliary measurable partition of finite entropy that will be useful to compute the metric entropy $h_{\hat{\eta}}(\hat{f})$.

Lemma 6.13. *There exists a measurable partition $\hat{\mathcal{P}}$ of $\hat{S}$ such that $H_{\hat{\eta}}(\hat{\mathcal{P}}) < \infty$, $\operatorname{diam}(\hat{\mathcal{P}}(\hat{x})) \leq \hat{D}_\varepsilon(\hat{x})$ at $\hat{\eta}$-almost every $\hat{x}$, and that the partition*

$$\hat{\mathcal{P}}^{(\infty)} = \bigvee_{n=0}^{+\infty} \hat{f}^n \hat{\mathcal{P}}$$

is finer than $\hat{\mathcal{Q}}$.

Proof. Let $\hat{D}_\varepsilon$ be the measurable function given by the previous lemma. By Lemma 2 in [Mañ81], there exists a measurable and countable partition $\hat{\mathcal{P}}_0$ such that $H_{\hat{\eta}}(\hat{\mathcal{P}}_0) < \infty$ and $\operatorname{diam} \hat{\mathcal{P}}(\hat{x}) \leq \hat{D}_\varepsilon(\hat{x})$ for a.e. $\hat{x} \in \hat{M}$. Let $\hat{\mathcal{P}}$ be the finite entropy partition obtained as the refinement of $\hat{\mathcal{P}}_0$ and $\{\hat{M} \setminus \hat{S}(\hat{x}, r), \hat{S}(\hat{x}, r)\}$. Notice that there is a full measure set where any two points $\hat{x}$ and $\hat{y}$ belong to the same element of $\hat{f}^n \hat{\mathcal{P}}$ for every $n \geq 0$ if and only

$$d(x_{-n}, y_{-n}) \leq \hat{D}_\varepsilon(\hat{f}^{-n}\hat{x}) \quad \text{for every } n \geq 0.$$

In particular, Lemma 6.12 above implies that each element of $\hat{\mathcal{P}}$ is a piece of some local unstable manifold. Hence, since $\hat{\mathcal{P}}$ was chosen to refine $\{\hat{M} \setminus \hat{S}(\hat{x}, r), \hat{S}(\hat{x}, r)\}$ then it is easy to see that

$$\bigcap_{n \geq 0} \hat{f}^n \hat{\mathcal{P}}(\hat{f}^{-n}(\hat{x})) \subset \hat{\mathcal{Q}}(\hat{x}).$$

for almost every $\hat{x}$. So, the partition $\hat{\mathcal{P}}$ just constructed satisfies the conclusions of the lemma. □

Proof of Proposition 6.11. Let $\varepsilon > 0$ be arbitrary small. Up to a refinement of the partition $\hat{\mathcal{P}}$ we may assume without loss of generality that $h_{\hat{\eta}}(\hat{f}, \hat{\mathcal{P}}) \geq h_{\hat{\eta}}(\hat{f}) - \varepsilon$. Since the partition $\hat{\mathcal{P}}^{(\infty)}$ is finer than $\hat{\mathcal{Q}}$ then

$$h_{\hat{\eta}}(\hat{f}, \hat{\mathcal{P}}) = h_{\hat{\eta}}(\hat{f}, \hat{\mathcal{P}}^{(\infty)}) = h_{\hat{\eta}}(\hat{f}, \hat{\mathcal{P}}^{(\infty)} \vee \hat{\mathcal{Q}}) = h_{\hat{\eta}}(\hat{f}, \hat{f}^n \hat{\mathcal{P}}^{(\infty)} \vee \hat{\mathcal{Q}})$$

for every $n \geq 1$. Using that $h_{\hat{\eta}}(\hat{f}, \hat{\xi}) = H_{\hat{\eta}}(\hat{f}^{-1}\hat{\xi}, \hat{\xi})$ for every increasing partition $\hat{\xi}$, the right hand side term in the previous equalities coincides with the relative entropy $H_{\hat{\eta}}(\hat{f}^n \hat{\mathcal{P}}^{(\infty)} \vee \hat{\mathcal{Q}} \mid \hat{f}^{n+1} \hat{\mathcal{P}}^{(\infty)} \vee \hat{f}\hat{\mathcal{Q}})$ and, consequently,

$$h_{\hat{\eta}}(\hat{f}, \hat{\mathcal{P}}) = H_{\hat{\eta}}(\hat{\mathcal{Q}} \mid \hat{f}\hat{\mathcal{Q}} \vee \hat{f}^n \hat{\mathcal{P}}^{(\infty)}) + H_{\hat{\eta}}(\hat{\mathcal{P}}^{(\infty)} \mid \hat{f}^{-n}\hat{\mathcal{Q}} \vee \hat{f}\hat{\mathcal{P}}^{(\infty)}).$$

The second term in the right hand side above is bounded by $H_{\hat{\eta}}(\hat{\mathcal{P}})$, which is finite. Then Proposition 6.8(3) implies that it tends to zero as $n \to \infty$. On the other hand, the diameter of almost every element in $\hat{f}^{-n+1}\hat{\mathcal{Q}}$ tend to zero as $n \to \infty$, proving that there exists a sequence of sets $(\hat{D}_n)_{n \geq 1}$ in $\hat{M}$ satisfying $\lim_n \hat{\eta}(\hat{D}_n) = 1$ and such that $\hat{f}\mathcal{Q}(\hat{x}) \subset \hat{f}^n \hat{\mathcal{P}}^{(\infty)}(\hat{x})$ for every $\hat{x} \in \hat{D}_n$. Then

$$H_{\hat{\eta}}(\hat{\mathcal{Q}} \mid \hat{f}\hat{\mathcal{Q}} \vee \hat{f}^n \hat{\mathcal{P}}^{(\infty)}) = \int -\log \hat{\eta}_{(\hat{f}\hat{\mathcal{Q}} \vee \hat{f}^n \hat{\mathcal{P}}^{(\infty)})(\hat{x})}(\hat{\mathcal{Q}}(\hat{x}))\, d\hat{\eta}(\hat{x}) \geq$$

$$\geq \int_{\hat{D}_n(\hat{x})} -\log \hat{\eta}_{(\hat{f}\hat{\mathcal{Q}})(\hat{x})}(\hat{\mathcal{Q}}(\hat{x}))\, d\hat{\eta}(\hat{x}),$$

where the measures $\hat{\eta}_{\hat{f}\hat{\mathcal{Q}} \vee \hat{f}^n \hat{\mathcal{P}}^{(\infty)}}$ and $\hat{\eta}_{\hat{f}\hat{\mathcal{Q}}}$ denote respectively the conditional measures of η with respect to the partitions $\hat{f}\hat{\mathcal{Q}} \vee \hat{f}^n \hat{\mathcal{P}}^{(\infty)}$ and $\hat{f}\hat{\mathcal{Q}}$. This proves that $\lim_n H_{\hat{\eta}}(\hat{\mathcal{Q}} \mid \hat{f}\hat{\mathcal{Q}} \vee \hat{f}^n \hat{\mathcal{P}}^{(\infty)}) \geq H_{\hat{\eta}}(\hat{\mathcal{Q}} \mid \hat{f}\hat{\mathcal{Q}})$. Since the other inequality is always true we deduce that $h_{\hat\eta}(\hat f, \hat{\mathcal P}) = H_{\hat\eta}(\hat{\mathcal Q} \mid \hat f \hat{\mathcal Q})$. Since $\varepsilon > 0$ was chosen arbitrary this proves that $h_{\hat\eta}(\hat f) = H_{\hat\eta}(\hat{\mathcal Q} \mid \hat f \mathcal Q)$, as claimed. $\square$

It follows from Lemma 6.10 and Proposition 6.11 that

$$H_{\hat\eta}(\hat f^{-1}\hat{\mathcal Q} \mid \hat{\mathcal Q}) = \int \log J_{\hat\nu}\hat f \, d\hat\eta. \tag{6.6}$$

With this in mind we obtain the following

Lemma 6.14. *$\hat\eta$ admits a disintegration $(\hat\eta_{\hat x})_{\hat x}$ along the measurable partition $\hat{\mathcal Q}$ such that*

$$\hat\eta_{\hat x}(B) = \frac{1}{Z(\hat x)} \int_{\hat{\mathcal Q}(\hat x)\cap B} \Delta(\hat x, \hat y)\, d\hat\nu_{\hat x}(\hat y), \quad \textit{where} \quad Z(\hat x) = \int_{\hat{\mathcal Q}(\hat x)} \Delta(\hat x, \hat y)\, d\hat\nu_{\hat x}(\hat y) \tag{6.7}$$

for every measurable set B and $\hat\eta$-almost every $\hat x$. In consequence $\hat\eta_{\hat x}$ is absolutely continuous with respect to $\hat\nu_{\hat x}$ for almost every $\hat x$.

Proof. Recall that $\Delta(\hat x, \hat y)$ is well defined for almost every $\hat x$ and every $\hat y \in \hat{\mathcal Q}(\hat x)$ according to Lemma 6.10. In particular Corollary 6.9 implies that $0 < Z(\hat x) < \infty$ almost everywhere. Let $\rho_{\hat x}$ denote the measure in the right hand side of the first equality in (6.7). Since $\hat f^{-1}\hat{\mathcal Q} \succ \hat{\mathcal Q}$ a simple computation involving a change of coordinates gives that

$$\rho_{\hat x}((\hat f^{-1}\hat{\mathcal Q})(\hat x)) = \frac{1}{Z(\hat x)} \int_{(\hat f^{-1}\hat{\mathcal Q})(\hat x)} \Delta(\hat x, \hat y)\, d\hat\nu_{\hat x}(\hat y) = \frac{Z(\hat f(\hat x))}{Z(\hat x)\; J_{\hat\nu}\hat f(\hat x)}.$$

We claim that

$$-\int \log \rho_{\hat x}((\hat f^{-1}\hat{\mathcal Q})(\hat x))\, d\hat\eta = \int \log J_{\hat\nu}\hat f \, d\hat\eta.$$

Since $\rho_{\hat x}$ is a probability measure then $-\log \rho_{\hat x}((\hat f^{-1}\hat{\mathcal Q})(\hat x))$ is a positive function and clearly the negative part of this function belongs to $L^1(\hat\eta)$. Using that $J_\nu f$ is bounded away from zero and infinity the same is obviously true also for $\log \frac{Z(\hat f(\hat x))}{Z(\hat x)}$. So, Birkhoff's ergodic theorem yields that the limit

$$\omega(\hat x) := \lim_{n\to\infty} \frac{1}{n} \log Z(\hat f^n(\hat x)) = \lim_{n\to\infty} \frac{1}{n} \log \frac{Z(\hat f^n(\hat x))}{Z(\hat x)} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \frac{Z \circ \hat f(\hat f^j(\hat x))}{Z(\hat f^j(\hat x))}$$

do exist (although possibly infinite) and that

$$\int \omega(\hat x) d\hat\eta(\hat x) = \int \log \frac{Z(\hat f(\hat x))}{Z(\hat x)} d\hat\eta(\hat x).$$

Since Z is almost everywhere positive and finite, the sequence $1/n \log Z(\hat f^n(\hat x))$ converge to zero in probability and, consequently, it is almost everywhere convergent to zero along some subsequence $(n_j)_j$. This shows that $\omega(\hat x) = 0$ for $\hat\eta$-almost every $\hat x$ and proves our claim. On the other hand using relation (6.6) and the equality

$$H_{\hat\eta}(\hat f^{-1}\hat{\mathcal Q} \mid \hat{\mathcal Q}) = -\int \log \hat\eta_{\hat x}(\hat f^{-1}\hat{\mathcal Q}(\hat x))\, d\hat\eta(\hat x)$$

we obtain

$$\int \log \Big(\frac{d\hat\rho}{d\hat\eta}\Big|_{\hat f^{-1}\hat{\mathcal Q}}\Big) d\hat\eta = 0.$$

Since the logarithm is a strictly concave function then

$$0 = \int \log\Big(\frac{d\hat{\rho}_{\hat{x}}}{d\hat{\eta}_{\hat{x}}}\Big|_{\hat{f}^{-1}\hat{\mathcal{Q}}}\Big) d\hat{\eta} \le \log\Big(\int \frac{d\hat{\rho}_{\hat{x}}}{d\hat{\eta}_{\hat{x}}}\Big|_{\hat{f}^{-1}\hat{\mathcal{Q}}} d\hat{\eta}\Big) = 0,$$

and the equality holds if and only if the Radon-Nykodym derivative $\frac{d\hat{\rho}_{\hat{x}}}{d\hat{\eta}_{\hat{x}}}$ restricted to the sigma-algebra generated by $\hat{f}^{-1}\hat{\mathcal{Q}}$ is almost everywhere constant and equal to one. Replacing $\hat{f}$ by any power $\hat{f}^n$ in the previous computations it is not difficult to check that $\hat{\eta}_{\hat{x}}$ and $\hat{\rho}_{\hat{x}}$ coincide in the increasing family of sigma-algebras generated by the partitions $\hat{f}^{-n}(\hat{\mathcal{Q}})$, $n \ge 1$. Proposition 6.8(3) readily implies that $\hat{\eta}_{\hat{x}} = \rho_{\hat{x}}$ at $\hat{\eta}$-almost every $\hat{x}$, which completes the proof of the lemma. □

We know from the previous lemma that $\hat{\eta}_{\hat{x}} \ll \hat{\nu}_{\hat{x}}$ almost everywhere. Then, using that $W^u_{\text{loc}}(\hat{x})$ is a neighborhood of x_0 in M and the bijection

$$\pi\mid_{\hat{W}^u_{\text{loc}}(\hat{x})} : \hat{W}^u_{\text{loc}}(\hat{x}) \to W^u_{\text{loc}}(\hat{x})$$

it follows that $\pi_*\hat{\eta}_{\hat{x}} \ll \nu$ for $\hat{\eta}$-almost every $\hat{x}$. Since $(\hat{\eta}_{\hat{x}})$ is a disintegration of $\hat{\eta}$ and $\pi_*\hat{\eta} = \eta$ it is immediate that $\eta \ll \nu$. This completes proof of the Theorem 6.6. □

Remark 6.15. We point out that Theorem 6.6 holds for general endomorphisms that admit critical or singular behavior. Let us comment on the necessary modifications in the proof. Assume that f is a general endomorphism, ϕ is an Hölder continuous potential and ν is an expanding conformal measure such that $J_\nu f = \lambda e^{-\phi}$, where $\lambda = \exp P_{\text{top}}(f, \phi)$. This is the case e.g when $\phi = -\log|\det Df|$ and ν is the Lebesgue measure. Assume also that η is an equilibrium state for f with respect to ϕ such that $\log|\det Df| \in L^1(\eta)$ and $\eta(\operatorname{supp}\nu) = 1$. The integrability condition allows us to obtain Pesin's local unstable leaves through almost every point (see e.g. [QZ02] for a precise statement). Then, the construction of an increasing partition as in Proposition 6.8 and the proof of the absolute continuity of η with respect to ν remains unaltered. In the case that $\phi = -\log|\det Df|$ this corresponds to the proof of the necessity condition in Pesin's formula for the entropy, which is the converse of the result in [QZ02].

Through the remaining of the section assume that f is topologically mixing. Since equilibrium states coincide with the invariant measures that are absolutely continuous with respect to ν then there is only one equilibrium state μ for f with respect to ϕ. Thus, Theorem B is a direct consequence of Proposition 5.1 and the previous statement. To finish the proof of Theorem A it remains only to show exactness of the equilibrium state:

Lemma 6.16. *μ is exact.*

Proof. Let $E \in \mathcal{B}_\infty$ be such that $\mu(E) > 0$ and let $\varepsilon > 0$ be arbitrary. There are measurable sets $E_n \in \mathcal{B}$ such that $E = f^{-n}(E_n)$. On the other hand, since μ is regular there exists a compact set K and an open set O such that $K \subset E \cap H \subset O$ and $\mu(O \setminus K) < \varepsilon\mu(K)$, where H denotes as before the set of points with infinitely many hyperbolic times and $\varepsilon > 0$ is small. The same argument used in the proof of Lemma 5.3 shows that there exists $\tau > 0$ $n \ge 1$ and $x \in H_n$ such that

$$\frac{\mu(B(x, n, \delta/4) \setminus E)}{\mu(B(x, n, \delta/4))} < \tau^{-1}\varepsilon.$$

Since n is a hyperbolic time then $f^n\mid_{B(x,n,\delta)}$ is a diffeomorphism that satisfies the bounded distortion property. Hence

$$\frac{\mu(B(f^n(x),\delta/4)\setminus f^n(E))}{\mu(B(f^n(x),\delta/4))} < K_0\tau^{-1}\varepsilon.$$

The topologically mixing assumption guarantees the existence of a uniform $N \geq 1$ (depending only on δ) such that every ball of radius $\delta/4$ is mapped onto M by f^N. Furthermore, since $\mu \ll \nu$ with density $h = \frac{d\mu}{d\nu}$ bounded away from zero and infinity then $J_\mu f = J_\nu f\ (h\circ f)/h$ satisfies $C^{-1} \leq J_\mu f \leq C$ for some constant $C > 1$. In particular, since d^N is an upper bound for the number of inverse branches of f^N, C bounds the maximal distortion of the Jacobian at each iterate and μ is f-invariant we obtain that

$$\mu(M\setminus E) = \mu(M\setminus E_{n+N}) < K_0 d^N C^N \tau^{-1}\varepsilon.$$

The arbitrariness of $\varepsilon > 0$ shows that $\mu(E) = 1$. This proves that μ is exact. □

We finish this section with the following:

Proof of Corollary 1. Let ϕ be any continuous potential satisfying (P). The existence of equilibrium states for f with respect to ϕ follows from upper semi-continuity of the metric entropy. Consider a sequence $\{\phi_n\}$ of Hölder continuous potentials satisfying (P) such that $\|\phi_n - \phi\|_0 \to 0$ as $n\to\infty$. Let μ be an accumulation point of the sequence $\{\mu_n\}$, where μ_n is an equilibrium state for f with respect to ϕ_n given by Theorem B. Note that the constants c and δ given by Lemma 3.4 are uniform for every μ_n. So, given a partition $\mathcal{R}$ of diameter smaller than δ such that $\mu(\partial\mathcal{R}) = 0$, it is generating with respect to μ_n and satisfies that

$$\mu_n \mapsto h_{\mu_n}(f,\mathcal{R})$$

is upper semi-continuous. It follows from the continuity of $\phi \mapsto P_{\text{top}}(f,\phi)$ and $\phi \mapsto \int \phi\, d\mu$ that

$$\begin{aligned} h_\mu(f,\mathcal{R}) \geq \limsup_{n\to\infty} h_{\mu_n}(f) &= \limsup_{n\to\infty}\left[P_{\text{top}}(f,\phi_n) - \int \phi_n\, d\mu_n\right] \\ &= P_{\text{top}}(f,\phi) - \int \phi\, d\mu \geq h_\mu(f). \end{aligned}$$

This proves that μ is an equilibrium state for f with respect to ϕ and completes the proof of the corollary. □

7. Stability of equilibrium states

7.1. **Statistical stability.** Here we prove upper semi-continuity of the metric entropy and use the continuity assumption on the topological pressure to prove that the equilibrium states vary continuously with respect to the data f and ϕ.

Proof of Theorem C. Let $\mathcal{W}$ be the set of Hölder continuous potentials and $\mathcal{F}$ the set of $C^{1+\alpha}$-local diffeomorphisms introduced in Subsection 2.3. The strategy is to construct a generating partition for *all* maps in $\mathcal{F}$. A similar argument was considered in [Ara07]. Fix $(f,\phi)\in\mathcal{F}\times\mathcal{W}$ and arbitrary sequences $\mathcal{F}\ni f_n\to f$ in the $C^{1+\alpha}$-topology and $\mathcal{W}\ni\phi_n\to\phi$ in the uniform topology, let μ_n be an equilibrium state for f_n with respect to ϕ_n and η be an f-invariant measure obtained as an accumulation point of the sequence $(\mu_n)_n$.

We begin with the following observation. Since the constants c and δ given by Lemma 3.4 are uniform in $\mathcal{F}$, any partition $\mathcal{R}$ of diameter smaller than $\delta/2$ satisfying $\eta(\partial\mathcal{R}) = 0$ generates the Borel sigma-algebra for every $g \in \mathcal{F}$. Then, Kolmogorov-Sinai theorem implies that $h_{\mu_n}(f_n) = h_{\mu_n}(f_n, \mathcal{R})$ and $h_\eta(f) = h_\eta(f, \mathcal{R})$, that is,

$$h_{\mu_n}(f_n) = \inf_{k\geq 1} \frac{1}{k} H_{\mu_n}(\mathcal{R}_n^{(k)}) \quad \text{and} \quad h_\eta(f) = \inf_{k\geq 1} \frac{1}{k} H_\eta(\mathcal{R}^{(k)}),$$

where $H_\eta(\mathcal{R}) = \sum_{R\in\mathcal{R}} -\eta(R)\log\eta(R)$ and we considered the dynamically defined partitions

$$\mathcal{R}_n^{(k)} = \bigvee_{j=0}^{k-1} f_n^{-j}(\mathcal{R}) \quad \text{and} \quad \mathcal{R}^{(k)} = \bigvee_{j=0}^{k-1} f^{-j}(\mathcal{R}).$$

Since η gives zero measure to the boundary of $\mathcal{R}$ then $H_{\mu_n}(\mathcal{R}_n^{(k)})$ converge to $H_\eta(\mathcal{R}^{(k)})$ as $n \to \infty$ by weak* convergence. Furthermore, for every $\varepsilon > 0$ there is $N \geq 1$ such that

$$h_{\mu_n}(f_n) \leq \frac{1}{N} H_{\mu_n}(\mathcal{R}_n^{(N)}) \leq \frac{1}{N} H_\eta(\mathcal{R}^{(N)}) + \varepsilon \leq h_\eta(f) + 2\varepsilon.$$

Recalling the continuity assumption of the topological pressure $P_{\text{top}}(f,\phi)$ on the data (f,ϕ), that μ_n is an equilibrium state for $f_n$with respect to ϕ_n, and that $\int \phi_n d\mu_n \to \int \phi\, d\eta$ as $n \to \infty$, it follows that

$$h_\eta(f) + \int \phi\, d\eta \geq P_{\text{top}}(f,\phi).$$

This shows that η is an equilibrium state for f with respect to ϕ. Since every equilibrium state belongs to the convex hull of ergodic equilibrium states and these coincide with finitely many ergodic measures absolutely continuous with respect to ν (recall Theorem B), this completes the proof of Theorem C. □

We finish this subsection with some comments on the assumption involving the continuity of the topological pressure. The map $\phi \mapsto P(f,\phi)$ varies continuously, provided that f is a continuous transformation (see for instance [Wal82, Theorem 9.5]). On the other hand, in this setting the topological pressure $P_{\text{top}}(f,\phi)$ coincides with $\log\lambda_{f,\phi}$, where $\lambda_{f,\phi}$ is the spectral radius of the transfer operator $\mathcal{L}_{f,\phi}$, *for every $f \in \mathcal{F}$ and every $\phi \in \mathcal{W}$*. Moreover, the operators $\mathcal{L}_{f,\phi}$ vary continuously with the data (f,ϕ). So, the continuous variation of the topological pressure should be a consequence of the most likely spectral gap for the transfer operator $\mathcal{L}_{f,\phi}$ in the space of Hölder continuous observables. Such a spectral gap property was obtained in [AM06] in a related context.

7.2. **Stochastic stability.** The results in this section are inspired by some analogous in [AA03]. First we introduce some definitions and notations. Given $\underline{f} \in \mathcal{F}^{\mathbb{N}}$, define $\underline{f}^j = f_j \circ \ldots f_2 \circ f_1$. Let $(\theta_\varepsilon)_{0<\varepsilon\leq 1}$ be a family of probability measures in $\mathcal{F}$. Given a (not necessarily invariant) probability measure ν, we say that (f,ν) is *non-uniformly expanding along random orbits* if there exists $c > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df(\underline{f}^j(x))^{-1}\| \leq -2c < 0$$

for $(\theta_\varepsilon^{\mathbb{N}} \times \nu)$-almost every $(\underline{f}, x) \in \mathcal{F}^{\mathbb{N}} \times M$. If this is the case, Pliss's lemma guarantees the existence of infinitely many hyperbolic times for almost every point where, in this setting, $n \in \mathbb{N}$ is a *c-hyperbolic time for* $(\underline{f}, x) \in \mathcal{F}^{\mathbb{N}} \times M$ if

$$\prod_{j=n-k}^{n-1} \|Df(\underline{f}^j(x))^{-1}\| < e^{-ck} \quad \text{for every } 0 \le k \le n-1.$$

We refer the reader to [AA03, Proposition 2.3] for the proof. Given $\varepsilon > 0$, let $n_1^\varepsilon : \mathcal{F}^{\mathbb{N}} \times M \to \mathbb{N}$ denote the first hyperbolic time map. Set also $H_n(\underline{f}) = \{x \in M : n \text{ is a } c\text{-hyperbolic time for } (\underline{f}, x)\}$. In the remaining of the section let $f \in \mathcal{F}$ and ν be an expanding conformal measure such that $\operatorname{supp} \nu = H$. The next result shows that f has random non-uniform expansion. More precisely,

Lemma 7.1. *Let $(\theta_\varepsilon)_{0<\varepsilon\le 1}$ be a family of probability measures in $\mathcal{F}$ such that $\operatorname{supp}\theta_\varepsilon$ is contained in a small neighborhood $V_\varepsilon(f)$ of f and $\bigcap_\varepsilon V_\varepsilon(f) = \{f\}$. If $\mathcal{F} \ni g \mapsto J_\nu g$ is a continuous function and ε is small enough then (f, ν) is non-uniformly expanding along every random orbit of $(\hat{f}, \theta_\varepsilon)$. Furthermore,*

$$(\theta_\varepsilon^{\mathbb{N}} \times \nu)(\{(\underline{f}, x) \in \mathcal{F}^{\mathbb{N}} \times M : n_1^\varepsilon(\underline{f}, x) > k\})$$

decays exponentially fast and, consequently, $\int n_1^\varepsilon \, d\,(\theta_\varepsilon^{\mathbb{N}} \times \nu) < \infty$.

Proof. Given $g \in \mathcal{F}$, let $\mathcal{A}_g \subset M$ be the region described in (H1) and (H2). Denote by $\tilde{\mathcal{A}}$ the enlarged set obtained as the union of the regions $\mathcal{A}_g$ taken over all $g \in \operatorname{supp}\theta_\varepsilon$. If $\varepsilon > 0$ is small enough then we can assume that $\tilde{\mathcal{A}}$ is contained in the same q elements of the covering $\mathcal{P}$ as the set $\mathcal{A}_f$.

Now we claim that, if γ is chosen as before and $\underline{f} \in \mathcal{F}^{\mathbb{N}}$ the measure of the set

$$B(n, \underline{f}) = \Big\{x \in M : \frac{1}{n}\#\{0 \le j \le n-1 : \underline{f}^j(x) \in \tilde{\mathcal{A}}\} \ge \gamma\Big\}$$

decays exponentially fast. Indeed, the same proof of Lemma 3.1 yields that $B(n, \underline{f})$ is covered by at most $e^{(\log q + \varepsilon_0/2)n}$ elements of $\mathcal{P}^{(n)}(\underline{f}) = \bigvee_{j=0}^{n-1} \underline{f}^{-j}(\mathcal{P})$, for every large n. On the other hand, since $\operatorname{supp}(\theta_\varepsilon)$ is compact the function $\operatorname{supp}\theta_\varepsilon \ni g \mapsto J_\nu g$ is uniformly continuous: for every $\varepsilon > 0$ there exists $a(\varepsilon) > 0$ (that tends to zero as $\varepsilon \to 0$) such that

$$e^{-a(\varepsilon)} \le \frac{J_\nu f(x)}{J_\nu g(x)} \le e^{a(\varepsilon)}$$

for every $g \in \operatorname{supp}(\theta_\varepsilon)$ and every $x \in M$. As in the proof of Proposition 4.4, this implies that

$$1 \ge \nu(\underline{f}^n(P)) = \int_P \prod_{j=0}^{n-1} J_\nu f_j \circ \underline{f}^j \, d\nu \ge e^{-a(\varepsilon)n} \int_P J_\nu f^n \, d\nu > e^{(\log q + \varepsilon_0 - a(\varepsilon))n} \, \nu(P)$$

and, consequently, $\nu(P) \le e^{-(\log q + \varepsilon_0 - a(\varepsilon))n}$ for every $P \in \mathcal{P}^{(n)}(\underline{f})$ and every large n. Hence

$$\nu(B(n, \underline{f})) \le \#\{P \in \mathcal{P}^{(n)}(\underline{f}) : P \cap B(n, \underline{f}) \ne \emptyset\} \times e^{-(\log q + \varepsilon_0 - a(\varepsilon))n}$$

which decays exponentially fast and proves the claim. Then, the set

$$\underline{B}(n) = \Big\{(\underline{f}, x) \in \mathcal{F}^{\mathbb{N}} \times M : \frac{1}{n}\#\{0 \le j \le n-1 : \underline{f}^j(x) \in \tilde{\mathcal{A}}\} \ge \gamma\Big\}$$

is such that $(\theta_\varepsilon \times \nu)(B(n)) = \int \nu\,(B(n, \underline{f})) \, d\theta_\varepsilon^{\mathbb{N}}(\underline{f})$ also decays exponentially fast. Borel-Cantelli guarantee that the frequency of visits of the random orbit $\{\underline{f}^j(x)\}$

to $\tilde{\mathcal{A}}$ is smaller than γ for $\theta_\varepsilon^{\mathbb{N}} \times \nu$-almost every $(\underline{\mathrm{f}}, x)$. Moreover, since every $g \in \mathcal{F}$ satisfy (H1) and (H2) with uniform constants this proves that f is non-uniformly expanding along random orbits. Moreover, the first hyperbolic time map n_1^ε is integrable because

$$\int n_1 \, d(\theta_\varepsilon^{\mathbb{N}} \times \nu) = \sum_{n\geq 0} (\theta_\varepsilon^{\mathbb{N}} \times \nu)(\{n_1 > n\}) \leq \sum_{n \geq 0} (\theta_\varepsilon^{\mathbb{N}} \times \nu)(B(n)) < \infty.$$

This completes the proof of the lemma. □

Remark 7.2. Before proceeding with the proof, let us discuss briefly the continuity assumption on $\mathcal{F} \ni g \to J_\nu g$. First notice that in our setting this is automatically satisfied when ν coincides with the Lebesgue measure since it reduces to the continuity of $g \mapsto \log |\det Dg|$. Given $g \in \mathcal{F}$, let ν_g denote the expanding conformal measure and set $P_g = P_{\text{top}}(f, \phi)$. Observe that if k is a c-hyperbolic time for x with respect to f then it is a $c/2$-hyperbolic time for x with respect to every g sufficiently close to f. Consequently

$$K(c/2,\delta)^{-2} e^{-|P_f - P_g| k} \leq \frac{\nu_g(B(x,k,\delta))}{\nu_f(B(x,k,\delta))} \leq K(c/2,\delta)^2 e^{|P_f - P_g|k},$$

which proves that the conformal measures ν_f and ν_g are comparable at hyperbolic times and that $J_\nu g = d(g_*^{-1}\nu)/d\nu$ is a well defined object in the domain of each inverse branch g^{-1}. So, in general, the relation above indicates that the continuity of the topological pressure should play a crucial role to obtain the continuity of the Jacobian $\mathcal{F} \ni g \to J_\nu g$.

Given $n \geq 1$ define $f_x^n : \mathcal{F}^{\mathbb{N}} \to M$ given by $f_x^n(\mathrm{g}) := \mathrm{g}^n(x)$. Since f is non-uniformly expanding and non-uniformly expanding along random orbits then there are finitely many ergodic stationary measures absolutely continuous with respect to ν. More precisely,

Theorem 7.3. *Let $(\theta_\varepsilon)_\varepsilon$ be a non-degenerate random perturbation of $f \in \mathcal{F}$. Given $\varepsilon > 0$ there are finitely many ergodic stationary measures $\mu_1^\varepsilon, \mu_2^\varepsilon, \dots, \mu_l^\varepsilon$ that are absolutely continuous with respect to the conformal measure ν and*

$$\mu_i^\varepsilon = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int \underline{\mathrm{f}}_*^j (\nu | B(\mu_i^\varepsilon)) \, d\theta_\varepsilon^{\mathbb{N}}(\underline{\mathrm{f}}), \tag{7.1}$$

for every $1 \leq i \leq l$. In addition, $l \geq 1$ can be taken constant for every sufficiently small ε.

Proof. This proof follows closely the one of Theorem C in [AA03]. For that reason we give a brief sketch of the proof and refer the reader to [AA03] for details. It is easy to check that any accumulation point μ^ε of the sequence of probability measures

$$\frac{1}{n} \sum_{j=0}^{n-1} (f_x^j)_* \theta_\varepsilon^{\mathbb{N}} \tag{7.2}$$

on M is a stationary measure. Moreover, any stationary measure μ^ε is absolutely continuous with respect to ν because of the non-degeneracy of the random perturbation and

$$\mu^\varepsilon(E) = \int \mu^\varepsilon(g^{-1}(E)) \, d\theta_\varepsilon(g) = \int 1_E(g(x)) \, d\theta_\varepsilon(g) \, d\mu^\varepsilon(x) = \int ((f_x)_* \theta_\varepsilon^{\mathbb{N}})(E) \, d\mu^\varepsilon$$

for every measurable set E.

On the other hand, by the ergodic decomposition of the F-invariant probability measure $\theta_\varepsilon^{\mathbb{N}} \times \mu^\varepsilon$ there are ergodic stationary measures. We prove that there can be at most finitely many of them. Indeed, a point x belongs to the basin of attraction $B(\mu^\varepsilon)$ of an ergodic stationary measure μ^ε if and only if

$$\frac{1}{n}\sum_{j=0}^{n-1} \psi(\underline{f}^j(x)) \to \int \psi \, d\mu^\varepsilon \tag{7.3}$$

for every $\psi \in C(M)$ and $\theta_\varepsilon^{\mathbb{N}}$-almost every $\underline{f} \in \mathcal{F}^{\mathbb{N}}$. In addition, if $x \in B(\mu^\varepsilon)$ then $g(x) \in B(\mu^\varepsilon)$ for every $g \in \operatorname{supp}(\theta_\varepsilon)$. Furthermore, the non-degeneracy of the random perturbation implies that $B(\mu^\varepsilon)$ contains the ball of radius r_ε centered at $f(x)$. Then, the compactness of M implies that there are finitely many ergodic absolutely continuous stationary measures $\mu_1^\varepsilon, \dots, \mu_l^\varepsilon$, with $1 \le l \le l(\varepsilon)$. Since $\nu(B(\mu_i^\varepsilon)) > 0$, integrating (7.3) with respect to ν and using the dominated convergence theorem one obtains

$$\int \psi \, d\mu_i^\varepsilon = \lim_n \frac{1}{n}\sum_{j=0}^{n-1} \int_{B(\mu_i^\varepsilon)} \psi \circ \underline{f}^j \, d\nu = \lim_n \frac{1}{n}\sum_{j=0}^{n-1} \int \psi \, d\, \underline{f}_*^j(\nu|B(\mu_i^\varepsilon))$$

for every $\psi \in C(M)$ and $\theta_\varepsilon^{\mathbb{N}}$-almost every $\underline{f} \in \mathcal{F}$. This proves the first statement of the theorem.

It remains to show that $l = l(\varepsilon)$ can be chosen constant for every sufficiently small ε. The support of each stationary measure μ_i^ε is an invariant set with non-empty interior (see [AA03]). Since f is non-uniformly expanding then $\operatorname{supp}(\mu_i^\varepsilon)$ contains some hyperbolic pre-ball $V_n(x)$ associated to f and, by invariance, a ball of radius δ. This proves that $l(\varepsilon) \le l_0$ for every small $\varepsilon > 0$. On the other direction, since the set $\operatorname{supp}(\mu_i^\varepsilon)$ has positive ν-measure and is forward invariant by f it must be contained in the support of some ergodic stationary measure $\mu_i^{\varepsilon'}$ for every ε' smaller than ε. This proves the l can be taken constant for small ε and completes sketch of the proof of the theorem. □

Now we are in a position to prove that the equilibrium states constructed in Theorem A are stochastically stable.

Proof of Theorem D. Let $(\mu^\varepsilon)_{\varepsilon>0}$ be a sequence of stationary measures absolutely continuous with respect to ν and let η be any weak* accumulation point. Theorem 7.3 implies that there is $l \ge 1$ such that there are exactly l ergodic stationary measures $\mu_1^\varepsilon, \dots, \mu_l^\varepsilon$ that are absolutely continuous with respect to ν, for every sufficiently small ε. Furthermore,

$$\mu_i^\varepsilon = \lim_{n\to\infty} \nu_{n,i}^\varepsilon \quad \text{where} \quad \nu_{n,i}^\varepsilon = \frac{1}{n}\sum_{j=0}^{n-1} \int \underline{f}_*^j(\nu \mid B(\mu_i^\varepsilon)) \, d\theta_\varepsilon^{\mathbb{N}}(\underline{f}).$$

Proceed as in the beginning of Subsection 5.1 and write $\nu_n^\varepsilon \le \xi_n^\varepsilon + \frac{1}{n}\sum_{j=0}^{n-1} \eta_j^\varepsilon$ with

$$\xi_{n,i}^\varepsilon = \frac{1}{n}\sum_{j=0}^{n-1} \int_{B(\mu_i^\varepsilon)} \underline{f}_*^j(\nu \mid H_j(\underline{f})) \, d\theta_\varepsilon^{\mathbb{N}}(\underline{f})$$

and

$$\eta_{n,j}^\varepsilon = \sum_{k>0} \int_{B(\mu_i^\varepsilon)} \underline{f}_*^k \left([\underline{f}_*^j(\nu \mid H_j(\underline{f}))] \mid \{n_1^\varepsilon(\cdot, \sigma^j(\underline{f})) > k\} \right) d\theta_\varepsilon^{\mathbb{N}}(\underline{f}).$$

The arguments from Section 5 and the uniform integrability of $\varepsilon \mapsto n_1^\varepsilon \in L^1(\theta_\varepsilon^{\mathbb{N}} \times \nu)$ yield that each measure $\nu_{n,i}^\varepsilon$ is absolutely continuous with respect to ν with density bounded from above by a constant depending only on ε. By weak* convergence it follows that η is also absolutely continuous with respect to ν and, consequently, η belongs to the convex hull of finitely many ergodic equilibrium states $\mu_1, \dots, \mu_k$ for f with respect to ϕ. This completes the proof of the theorem. □

8. Some Examples

It is worthwhile including some remarks on the role of the hypotheses (H1),(H2) and (P), specially in connection with the supports of the measures we construct, the existence and finitude of equilibrium states.

Example 8.1. Let $f_0 : \mathbb{T}^d \to \mathbb{T}^d$ be a linear expanding map. Fix some covering $\mathcal{P}$ for f_0 and some $P_1 \in \mathcal{P}$ containing a fixed (or periodic) point p. Then deform f_0 on a small neighborhood of p inside P_1 by a pitchfork bifurcation in such a way that p becomes a saddle for the perturbed local diffeomorphism f. By construction, f coincides with f_0 in the complement of P_1, where uniform expansion holds. Observe that we may take the deformation in such a way that f is never too contracting in P_1, which guarantees that (H1) holds, and that f is still topologically mixing. Condition (P) is clearly satisfied by $\phi \equiv 0$. Hence, Theorems A and B imply that there exists a unique measure of maximal entropy, it is supported in the whole manifold $\mathbb{T}^d$ and it is a non-lacunary Gibbs measure.

Now, we give an example where the union of the supports of the equilibrium states does not coincide with the whole manifold.

Example 8.2. Let f_0 be an expanding map in $\mathbb{T}^2$ and assume that f_0 has a periodic point p with two complex conjugate eigenvalues $\tilde{\sigma} e^{i\varpi}$, with $\tilde{\sigma} > 3$ and $k\varpi \notin 2\pi\mathbb{Z}$ for every $1 \le k \le 4$. It is possible to perturb f_0 through an Hopf bifurcation at p to obtain a local diffeomorphism f, C^5-close to f_0 and such that p becomes a periodic attractor for f (see e.g. [HV05] for details). Moreover, if the perturbation is small then (H1) and (H2) hold for f. Thus, there are finitely many ergodic measures of maximal entropy for f. Since these measures are expanding their support do not intersect the basin of attraction the periodic attractor p.

An interesting question concerns the restrictions on f imposed by (P). For instance, if $\phi = -\log|\det Df|$ satisfies (P) then there can be no periodic attractors. In fact, if this is the case then the expanding conformal measure ν coincides with the Lebesgue measure. Since ν is an expanding measure and positive on open sets there can not exist periodic attractors. There are examples where the potential $\phi = -\log|\det Df|$ does satisfy (P). In fact, if f is as in Example 8.1 above, condition (P) can be rewritten as

$$\frac{\sup_{x\in\mathbb{T}^2} |\det Df(x)|}{\inf_{x\in\mathbb{T}^2} |\det Df(x)|} < \deg(f), \tag{8.1}$$

which is clearly satisfied if the perturbation is small enough.

The next example shows that some control on the potential ϕ is needed to have uniqueness of the equilibrium state: in absence of the hypothesis (P), uniqueness may fail even if we assume (H1) and (H2). Recall the following well known example of intermittency.

Example 8.3. *(Manneville-Pomeau map)* If $\alpha \in (0,1)$, let $f : [0,1] \to [0,1]$ be the local diffeomorphism given by

$$f_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } 0 \le x \le \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

Observe that conditions (H1) and (H2) are satisfied. It is well known that f has a finite invariant probability measure μ absolutely continuous with respect to Lebesgue. It is not difficult to see using by Pesin formula and Ruelle inequality, that μ is an equilibrium state for the potential $\phi = -\log|\det Df|$. On the other hand, since $h_{\delta_0}(f) = 0$ and $Df(0) = 1$, the Dirac mass δ_0 is also an equilibrium measure for ϕ. Thus, *uniqueness fails* in this topologically mixing context. For the sake of completeness, let us mention that in this example f is not a local diffeomorphism, but one can modify it to a local diffeomorphisms in $S^1 = [0,1]/\sim$ by

$$f_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } 0 \le x \le \frac{1}{2} \\ x - 2^\alpha(1-x)^{1+\alpha} & \text{if } \frac{1}{2} < x \le 1, \end{cases}$$

where $\sim$ means that the extremal points in the interval are identified. Note that the potential ϕ is not (Hölder) continuous.

The previous phenomenon concerning the lack of uniqueness of equilibrium states can appear near the boundary of the class of maps and potentials satisfying (H1) and (H2) and (P).

Example 8.4. Let f_α be the map given by the previous example and let $(\phi_\beta)_{\beta>0}$ be the family of Hölder continuous potentials given by $\phi_\beta = -\log(\det|Df| + \beta)$. On the one hand, observe that ϕ_β converge to $\phi = -\log(|\det Df|)$ as $\beta \to 0$. On the other hand, similarly to (8.1), one can write condition (P) as

$$\frac{\beta+2+\alpha}{\beta+1} < 2, \quad \text{or} \quad \beta > \alpha.$$

For every $\alpha > 0$, since f_α is topologically mixing and satisfies (H1),(H2) and $\phi_{2\alpha}$ satisfies (P) for every $\alpha > 0$ there is a unique equilibrium state μ_α for f_α with respect to $\phi_{2\alpha}$. Moreover, $\phi_{2\alpha}$ approaches ϕ, which seems to indicate that the condition (P) on the potential should be close to optimal in order to get uniqueness of equilibrium states.

9. Questions and perspectives

In this final section we will discuss several open questions, comment our assumptions and discuss possible future perspectives.

Weak Gibbs property: Recall that in the topologically mixing case the unique equilibrium state is a non-lacunary Gibbs measure. This holds because this measure is equivalent to the expanding conformal measure ν. A natural question is wether the equilibrium states also enjoy this weak Gibbs property in general. To obtain this it would be enough to show that the density of any absolutely continuous invariant measure is bounded away from zero in its support.

Decay of correlations and Limit Theorems: Other question is to analyze the velocity of mixing or, in other words, the rate of decay of correlations. It would also be interesting to establish some other statistical properties and limit theorems as the Central Limit Theorem (CLT) and the Local Limit Theorem (LLT) in a suitable Banach subspace of the space of Hölder continuous potentials.

Regularity of the transformation and potentials: It is well known that C^1 generic maps are not $C^{1+\alpha}$, failing to satisfy bounded distortion properties. In general, there exists an expanding conformal measure ν_ϕ and a positive real number $\lambda_\phi = r(\mathcal{L}_\phi)$ such that $\mathcal{L}_\phi^* \nu_\phi = \lambda_\phi \nu_\phi$ for every *continuous* covering map f and potential ϕ. We point out that the $C^{1+\alpha}$ assumption was only used to obtain Pesin's local unstable leaves in the proof of the uniqueness of the equilibrium state. For some applications this regularity is often necessary: e.g. in order to apply the results to the family of potentials $\phi_t = -t \log|\det Df|$. Nevertheless, it would be interesting to obtain a direct proof of the uniqueness in the context of C^1 transformations. We note that in the presence of Markov partitions, which is the setting of [OV07], Oliveira and Viana prove the uniqueness of equilibrium states using a different method that does not require the Hölder regularity of the derivative of the transformation.

Random transformations: A question inspired by [AMO04] and our stochastic stability result is to prove the existence and uniqueness of equilibrium states for some open classes of non-uniformly expanding random maps including the ones in [AMO04]. Indeed, it seams reasonable to construct an invariant measure which is absolutely continuous with respect to $\nu_{\mathbb{P}}$ in the same spirit of [AA03]. Moreover, following [AMO04] it might be that such measure is an equilibrium state. An important step and possible difficulty in that direction lies in obtain a random version of Brin-Katok's local entropy formula.

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