Poincaré and Logarithmic Sobolev Inequality for Ginzburg-Landau Processes in Random Environment

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Abstract

We consider reversible, conservative Ginzburg–Landau processes in a random environment, whose potential are bounded perturbations of the Gaussian potential, evolving on a d-dimensional cube of length L. We prove in all dimensions that the spectral gap of the generator and the logarithmic Sobolev constant are of order L^{-2} almost surely with respect to the environment. We folow here the martingale approach introduced in [LY]. The main ideas are essentially the same but there are several tecnical difficulties coming from the unboundedness of the spins. The main ingredients for the Ginzburg-Landau process without environment are a local central limit theorem, uniform over the parameter and the environment from which follows the equivalence of ensembles, and sharp large deviations estimates.

Contents

0.1 Introduction

Poincaré and logarithmic Sobolev inequalities are powerful tools in the analysis of stochastic processes. A sharp estimate for the spectral gap, for instance, is of fundamental importance in the derivation of the hydrodynamic equation of nongradient systems [18], [16]. In the same way, the spectral gap and the logarithmic Sobolev inequality played a central role in the investigation of the decay to equilibrium of conservative systems in infinite volume [1], [2], [10], [8], [13]. More recently, [7] a logarithmic Sobolev inequality was one of the main tools in the derivation of the scaling limit of a non-attractive weakly asymmetric process whose hydrodynamic equation is given by a first order hyperbolic equation.

We continue in this article the investigation started in [11] and present a sharp estimate of the spectral gap and of the logarithmic Sobolev constant for Ginzburg-Landau processes in random environment whose potential is a bounded perturbation of the Gaussian potential. We believe, however, that the approach presented here extend to the case where we have a bounded perturbation of a convex potencial. In this case, it was recently observed by Caputo that whem the potencial is a purely convex function, the L^2 behavior of the inverse of the spectral gap and of the logatimic Sobolev constant cam be easily obtained by tecniques introduced for models with convex interactions. The precise assumptions are given in chapter 1. We follow here the martingale approach introduced in [15].

As for the Ginzburg–Landau process without environment, the main ingredients needed in the proof are a local central limit theorem, uniform over the parameter and the environment from which follows the equivalence of ensembles, and sharp large deviations estimates presented in chapter 5. In the presence of the environment these estimates are technically more demanding and some new arguments are needed. To bound the terms coming from the environment, we will need to require the environment to take bounded values and to impose a nice behavior of the variance and the mean as the chemical potential diverges.

The derivation of sharp estimates for the spectral gap and the logarithmic Sobolev inequality for conservative interacting particle systems has started with the pioneering work of Lu and Yau [15] and Yau [19], [20], where the martingale method was introduced. Landim, Sethuraman and Varadhan in [12] extended to unbounded spin systems the sharp estimate of the spectral gap, while Landim and Yau [13] proved the Poincaré and the logarithmic Sobolev inequality for Ginzburg-Landau processes where the potential is a bounded perturbation of the Gaussian potential. The estimate of the spectral gap was extended by Caputo in [4] for bounded perturbations of strictly convex potentials and was examined by Chafai [6] with an alternative approach. The martingale method was revisited recently by Cancrini and Martinelli in [3].

In the context of conservative systems in random environment, Quastel and Yau [17] proved a sharp estimate for the spectral gap of the symmetric exclusion process in random environment using the martingale approach. Caputo [5] presents a general method to derive Poincaré inequalities for conservative dynamics and deduces a sharp estimate for the spectral gap of symmetric exclusion processes in random environment.

The article in conceived as follows. In chapter 1, we present the main results. We prove the spectral gap in chapter 1 and the logarithmic Sobolev inequality in chapter 2. These chapter rely on estimates presented in chapter 3, 4 and 5. In chapter 3 we present some consequences of a local central limit theorem, uniform over the parameters. In chapter 4 we examine the assumptions made on the environment and in chapter 5 we prove some large deviations estimates.

Chapter 1

Spectral Gap

1.1 Notation and Results

For $L \geq 1$, denote by Λ_L the cube $\{1, \dots, L\}^d$. Configurations of the state space $\mathbb{R}^{\Lambda_L}$ are denoted by the Greek letters η, ξ, so that η_x indicates the value of the spin at $x \in \Lambda_L$ for the configuration η. The configuration η undergoes a diffusion on $\mathbb{R}^{\Lambda_L}$ whose infinitesimal generator $\mathcal{L}_{\Lambda_L}$ is given by

$$\mathcal{L}_{\Lambda_L} \;=\; \frac{1}{2} \sum_{\substack{x,y \in \Lambda_L \\ \|x,y\|=1}} (\partial_{\eta_x} - \partial_{\eta_y})^2 \;-\; \frac{1}{2} \sum_{\substack{x,y \in \Lambda_L \\ \|x,y\|=1}} (V_y'(\eta_y) - V_x'(\eta_x))(\partial_{\eta_y} - \partial_{\eta_x}) \; .$$

$V_x \colon \mathbb{R} \to \mathbb{R}$ represents the potential $V_x(a) = h_x a + V(a)$, where $V(a) = (1/2)a^2 + F(a)$, $F \colon \mathbb{R} \to \mathbb{R}$ is a smooth bounded function such that $\|F'\|_\infty < \infty$, $\mathbf{h} = \{h_x,\, x \in \mathbb{Z}^d\}$ is a collection of i.i.d. random variables to be specified later and

$$\int e^{-V(x)}\, dx \;=\; 1 \; .$$

Denote by $Z \colon \mathbb{R} \to \mathbb{R}$ the partition function

$$Z(\lambda) \;=\; \int_{-\infty}^{\infty} e^{\lambda a - V(a)}\, da \; , \tag{1.1}$$

by $R\colon \mathbb{R} \to \mathbb{R}$ the density function $\partial_\lambda \log Z(\lambda)$, which is smooth and strictly increasing, and by Φ the inverse of R so that

$$\alpha \;=\; \frac{1}{Z(\Phi(\alpha))} \int_{-\infty}^{\infty} a\, e^{\Phi(\alpha)a - V(a)}\, da$$

for each α in $\mathbb{R}$.

Denote by ν_λ the measure $Z(\lambda)^{-1}\exp\{\lambda a - V(a)\}da$, by $g_\lambda(a) = Z(\lambda)^{-1}\exp\{\lambda a - V(a)\}$ its density with respect to the Lebesgue measure and by $\sigma^2(\lambda)$ the variance of ν_λ: $\sigma^2(\lambda) = \partial_\lambda^2 \log Z(\lambda)$ or

$$\sigma^2(\lambda) \;=\; \int_{-\infty}^{\infty} a^2 g_\lambda(a)\, da \;-\; \Big(\int_{-\infty}^{\infty} a g_\lambda(a)\, da\Big)^2 .$$

We assume throughout this thesis that $\sigma^2(\cdot)$ has limits at $\pm\infty$: There exists $\sigma^2_\pm < \infty$ such that

$$\lim_{\lambda \to \pm\infty} \sigma^2(\lambda) \;=\; \sigma^2_\pm . \tag{1.2}$$

This is a new assumption with respect to the non-random case needed in order to estimate some terms which appears through the environment (cf. Lemma 4.1.2). We prove at the end of chapter 4 that (1.2) holds, if, for instance, F has limits at the boundary of $\mathbb{R}$:

$$\lim_{a \to \pm\infty} F(a) \;=\; F_\pm \tag{1.3}$$

for some finite values $F_\pm$.

For λ in $\mathbb{R}$, and a finite subset Λ of $\mathbb{Z}^d$, denote by $\nu^{\mathbf{h}}_{\Lambda,\lambda}$ the product measure on $\mathbb{R}^\Lambda$ defined by

$$\nu^{\mathbf{h}}_{\Lambda,\lambda}(d\eta) \;=\; \prod_{x\in\Lambda} \frac{1}{Z(\lambda - h_x)} e^{[\lambda - h_x]\eta_x - V(\eta_x)} d\eta_x$$

Most of the times we omit the superscript $\mathbf{h}$ in $\nu^{\mathbf{h}}_{\Lambda,\lambda}$. Notice that $E_{\nu^{\mathbf{h}}_{\Lambda,\Phi(\alpha)+h_x}}[\eta_x] = \alpha$ for all α in $\mathbb{R}$, x in Λ.

For each M in $\mathbb{R}$, denote by $\mu^{\mathbf{h}}_{\Lambda,M}$ the canonical measure on Λ with total spin equal to M :

$$\mu^{\mathbf{h}}_{\Lambda,M}\Big(\;\cdot\;\Big) \;=\; \nu^{\mathbf{h}}_{\Lambda,\lambda}\Big(\;\cdot\;\Big|\sum_{x\in\Lambda}\eta_x = M\Big)\,.$$

Expectation with respect to a measure m is denoted by E_m. Notice that the canonical measure $\mu^{\mathbf{h}}_{\Lambda,M}$ does not depend on λ but depends on $\mathbf{h}$.

An elementary computation shows that the measures $\{\nu^{\mathbf{h}}_{\Lambda_L,\lambda},\ \lambda\in\mathbb{R}\}$, $\{\mu^{\mathbf{h}}_{\Lambda_L,M},\ M\in\mathbb{R}\}$ are reversible for the Markov process with generator $\mathcal{L}_{\Lambda_L}$. In fact,

$$\begin{aligned}
&< \mathcal{L}_{\Lambda_L} f, g >_{\nu^{\mathbf{h}}_{\Lambda,\lambda}} = \frac{1}{2}\sum_{\substack{x,y\in\Lambda_L\\ \|x,y\|=1}} \int \Big[(\partial_{\eta_x} - \partial_{\eta_y})^2 f\Big] g \prod_{z\in\Lambda_L}\frac{1}{Z(\lambda_z)} e^{\lambda\eta_z - V_z(\eta_z)} d\eta_z \\
&-\frac{1}{2}\sum_{\substack{x,y\in\Lambda_L\\ \|x,y\|=1}} \int (V'_x(\eta_x) - V'_y(\eta_y))\Big[(\partial_{\eta_x} - \partial_{\eta_y}) f\Big] g \prod_{z\in\Lambda_L}\frac{1}{Z(\lambda_z)} e^{\lambda\eta_z - V_z(\eta_z)} d\eta_z \\
&= -\frac{1}{2}\sum_{\substack{x,y\in\Lambda_L\\ \|x,y\|=1}} \int \Big[(\partial_{\eta_x} - \partial_{\eta_y}) f\Big]\Big[(\partial_{\eta_x} - \partial_{\eta_y}) g\Big] \prod_{z\in\Lambda_L}\frac{1}{Z(\lambda_z)} e^{\lambda\eta_z - V_z(\eta_z)} d\eta_z
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{\substack{x,y\in\Lambda_L\\ \|x,y\|=1}}\int (V_x'(\eta_x)-V_y'(\eta_y))\Big[(\partial_{\eta_x}-\partial_{\eta_y})f\Big]g\prod_{z\in\Lambda_L}\frac{1}{Z(\lambda_z)}e^{\lambda\eta_z-V_z(\eta_z)}d\eta_z\\
&+\frac{1}{2}\sum_{\substack{x,y\in\Lambda_L\\ \|x,y\|=1}}\int (V_x'(\eta_x)-V_y'(\eta_y))\Big[(\partial_{\eta_x}-\partial_{\eta_y})f\Big]g\prod_{z\in\Lambda_L}\frac{1}{Z(\lambda_z)}e^{\lambda\eta_z-V_z(\eta_z)}d\eta_z\\
&=\frac{1}{2}\sum_{\substack{x,y\in\Lambda_L\\ \|x,y\|=1}}\int f(\partial_{\eta_x}-\partial_{\eta_y})^2 g\prod_{z\in\Lambda_L}\frac{1}{Z(\lambda_z)}e^{\lambda\eta_z-V_z(\eta_z)}d\eta_z\\
&+\frac{1}{2}\sum_{\substack{x,y\in\Lambda_L\\ \|x,y\|=1}}\int (V_x'(\eta_x)-V_y'(\eta_y))f\Big[(\partial_{\eta_x}-\partial_{\eta_y})g\Big]\prod_{z\in\Lambda_L}\frac{1}{Z(\lambda_z)}e^{\lambda\eta_z-V_z(\eta_z)}d\eta_z\\
&= \;<\mathcal{L}_{\Lambda_L}g,f>_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}
\end{aligned}
$$

The Dirichlet form D_{Λ_L} associated to $\mathcal{L}_{\Lambda_L}$ is given by

$$
\mathcal{D}_{\Lambda_L}(\nu^{\mathbf{h}}_{\Lambda,\lambda},f)=\frac{1}{2}\sum_{\substack{x,y\in\Lambda_L\\ \|x,y\|=1}}<(T^{x,y}f)^2>_{\nu^{\mathbf{h}}_{\Lambda,\lambda}}
$$

In this formula and below, for a probability measure μ, $<.>_\mu$ stands for the expectation with respect to μ. Furthemore, for x , $y\in\mathbb{Z}^d$, $T^{x,y}$ represents the operador which acts on smooth functions f as

$$
T^{x,y}f=\frac{\partial f}{\partial_{\eta_x}}-\frac{\partial f}{\partial_{\eta_y}}
$$

and μ stands for a invariant measures $\nu^{\mathbf{h}}_{\Lambda_L,\lambda}$ or $\mu^{\mathbf{h}}_{\Lambda_L,M}$.

For a positive integer L and $M\in\mathbb{R}$, denote by $W(L,M,h)$ the inverse of the spectral gap of the generator $\mathcal{L}_{\Lambda_L}$ with respect to the measure $\mu^{\mathbf{h}}_{\Lambda_L,M}$:

$$
W(L,M,\mathbf{h})\;=\;\sup_f\frac{<f;f>_{\mu^{\mathbf{h}}_{\Lambda_L,M}}}{D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M},f)}\;.
$$

In this formula the supremum is carried over all smooth functions f in $L^2(\mu^{\mathbf{h}}_{\Lambda_L,M})$ and $< f; f >_\mu$ stands for the variance of f with respect to μ. We also denote this variance by the symbol $\mathbf{Var}\,(\mu, f)$. Let

$$W(L, \mathbf{h}) \;=\; \sup_{M\in\mathbb{R}} W(L, M, \mathbf{h}) \;.$$

We assume that the environment of i.i.d. random variables $\mathbf{h} = \{h_x,\, x \in \mathbb{Z}^d\}$ take bounded values: There exists $A_0 < \infty$ such that $|h_x| \le A_0$. This is the only assumption needed on the environment. We denote by $\mathbb{P}$ the probability on $[-A_0, A_0]^{\mathbb{Z}^d}$ corresponding to the environment $\mathbf{h}$ and by $\mathbb{E}$, expectation with respect to $\mathbb{P}$.

Theorem 1.1.1 *Assume (1.2). There exists an almost sure event Ω_0 of $[-A_0, A_0]^{\mathbb{Z}^d}$ with the property that for all $\boldsymbol{h}$ in Ω_0, there exists a finite constant C_0 depending only on $\|F\|_\infty$, $\|F'\|_\infty$ and $\boldsymbol{h}$ such that*

$$W(L, \boldsymbol{h}) \;\le\; C_0(F, \boldsymbol{h}) L^2$$

for all $L \ge 2$.

A lower bound of the same order is easy to derive (cf. [11]). Fix a smooth function $H : [0,1]^d \to \mathbb{R}$ such that $\int H(u)du = 0$ and let $f_H(\eta) = \sum_{x\in\Lambda_L} H(\frac{x}{L})\eta_x$. An elementary computation shows that

$$\mathcal{D}_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f) = \frac{1}{2} \sum_{\substack{x,y\in\Lambda_L \\ \|x,y\|=1}} \Big[H\Big(\frac{y}{L}\Big) - H\Big(\frac{x}{L}\Big) \Big]^2 .$$

The basis $\{e_j, 1 \le j \le d\}$ stands for the canonical basis of $\mathbb{R}^d$. By Corollary 3.1.4, as $L \to \infty$, $\frac{M}{L^d} \to \alpha$, $\frac{<f_H, f_H>_{\nu_{\Lambda_L},M}}{L^2 \mathcal{D}_{\Lambda_L}(\nu_{\Lambda_L,M}, f_H)}$ converges to

$< \eta_{e_1}, \eta_{e_1} >_{\nu_\alpha^h} \frac{\int H(u)^2 du}{\int \|(\nabla H)\|^2 du}$. This proves that

$$\liminf_{L\to\infty} L^{-2} W(L,h) > 0.$$

For $L \geq 2$, a probability measure ν on $\mathbb{R}^{\Lambda_L}$ and a function f such that $< f^2 >_\nu = 1$, denote by $S_{\Lambda_L}(\nu, f)$ the entropy of $f^2 d\nu$ with respect to ν:

$$S_{\Lambda_L}(\nu, f) \;=\; \int f^2 \log f^2 d\nu \;;$$

and by $\theta(L, M, \mathbf{h})$ the inverse of the logarithmic Sobolev constant of the Ginzburg-Landau process on the cube Λ_L with respect to the measure $\mu^{\mathbf{h}}_{\Lambda_L,M}$:

$$\theta(L, M, \mathbf{h}) \;=\; \sup_f \frac{S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)}{D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)} \;.$$

In this formula, the supremum is carried over all smooth functions f in $L^2(\mu^{\mathbf{h}}_{\Lambda_L,M})$ such that $< f^2 >_{\mu^{\mathbf{h}}_{\Lambda_L,M}} = 1$. Let

$$\theta(L, \mathbf{h}) = \sup_{M\in\mathbb{R}} \theta(L, M, \mathbf{h}) \;.$$

To prove the logarithmic Sobolev inequality, we shall require that the function $\Gamma_1(\lambda) = R(\lambda) - \lambda$ has limits at the boundary of $\mathbb{R}$: There exists $\Gamma_\pm$ such that

$$\lim_{\lambda\to\pm\infty} \Gamma_1(\lambda) \;=\; \Gamma_\pm \;. \tag{1.4}$$

We prove at the end of chapter 4 that (1.4) holds, if, for instance, F satisfies (1.3).

Theorem 1.1.2 *Assume (1.2), (1.4) and that $\|F''\|_\infty < \infty$. There exists an almost sure event Ω_0 of $[-A_0, A_0]^{\mathbb{Z}^d}$ with the property that for all $\boldsymbol{h}$ in Ω_0, there exists a finite constant $C(\boldsymbol{h})$ depending only on $\|F\|_\infty$, $\|F'\|_\infty$, $\|F''\|_\infty$ and $\boldsymbol{h}$ such that*

$$\theta(L, \boldsymbol{h}) \ \leq \ C(F, \boldsymbol{h}) L^2$$

for all $L \geq 2$.

We follow here the martingale method developed by Lu and Yau to prove the Spectral Gap and a bound on the Logarithmic Sobolev constant for a conservative interacting particle system. This approach relies on a two a-priori estimates. First, a local central limit theorem for independent random variables with marginals equal to the marginals of the product measure ν_λ , uniform over the parameter $\lambda \in \mathbb{R}$. Second, a spectral gap or a logarithmic Sobolev inequality, uniform over the density, for a Glauber dynamics on one site which is reversible with respect to the one-site marginal of the canonical invariant measure.

1.2 One-site spectral gap

To fix ideas, we prove Theorem 1.1.1 in dimension 1. The reader can find in section A.3.3 of [KL] the arguments needed to extend the proof to higher dimensions. To detach the main ideas, we divide the proof in four steps. The proof goes by induction on the size of the cube. We start with $L = 2$.

In this section all constants denoted by C_0 depend only on $\|F\|_\infty$ and all constants denoted by C_1 depend only on $\|F\|_\infty$, $\|F'\|_\infty$. In the case they depend on some other parameter, the dependence is stated explicitly. These constants may change from line to line.

Consider a smooth function $f : \mathbb{R}^{\Lambda_2} \to \mathbb{R}$. We want to estimate $< f, f >_{\mu^{\mathbf{h}}_{\Lambda_2,M}}$ in terms of the Dirichlet form of f . Since for the measure $\mu^{\mathbf{h}}_{\Lambda_2,M}$ the total spin is fixed to be equal to M, let $g(a) = f(M-a, a)$ and notice that $< f, f >_{\mu^{\mathbf{h}}_{\Lambda_2,M}}$ is equal to $< g, g >_{\mu^{\mathbf{h},1}_{\Lambda_L,M}}$, where $\mu^{\mathbf{h},1}_{\Lambda_L,M}$ is the marginal distribution of η_1 with respect to $\mu^{\mathbf{h}}_{\Lambda_L,M}$.

The following result is helpful. Fix $L \geq 2$, M in $\mathbb{R}$ and an environment $\mathbf{h}$. The Glauber dynamics has a positive spectral gap which is uniform with respect to L, M and $\mathbf{h}$:

Lemma 1.2.1 *There is a finite constant C_0 depending only on $\|F\|_\infty$ such that*

$$\mathbf{Var}\,(\mu^{\boldsymbol{h},1}_{\Lambda_L,M}, f) \;\leq\; C_0\, E_{\mu^{\boldsymbol{h},1}_{\Lambda_L,M}}\Big[\Big(\frac{\partial f}{\partial \eta_1}\Big)^2\Big]$$

for every $L \geq 2$, M in $\mathbb{R}$, environment $\boldsymbol{h}$ and every smooth function $f : \mathbb{R} \to \mathbb{R}$ in $L^2(\mu^{\boldsymbol{h},1}_{\Lambda_L,M})$.

In the case of grand canonical measures, this result is true under the more general hypothesis of strict convexity at infinity of the potential ([Le] and references therein). In the case of canonical measures the main problem is to obtain a good approximation of the one-site marginal in terms of the one-site marginal of grand canonical measures.

Before proving this result, we conclude the first step. Applying this result to the function g defined above, we obtain that its variance is less than or equal to $C_0 E_{\mu^{\mathbf{h},1}_{\Lambda_2,M}}[(\partial g/\partial \eta_1)^2]$. Since $\partial g/\partial \eta_1 = (\partial f/\partial \eta_2 - \partial f/\partial \eta_1)$, we have that

$$
\begin{aligned}
< f; f >_{\mu^{\mathbf{h}}_{\Lambda_2,M}} &= < g; g >_{\mu^{\mathbf{h}}_{\Lambda_2,M}} = < g; g >_{\mu^{\mathbf{h},1}_{\Lambda_2,M}} \\
&\le C_0 \, E_{\mu^{\mathbf{h},1}_{\Lambda_2,M}} \Big[\Big(\frac{\partial g}{\partial \eta_1} \Big)^2 \Big] = C_0 \, E_{\mu^{\mathbf{h}}_{\Lambda_2,M}} \Big[\Big(\frac{\partial g}{\partial \eta_1} \Big)^2 \Big] \\
&= C_0 \, E_{\mu^{\mathbf{h}}_{\Lambda_2,M}} \Big[\Big(\frac{\partial f}{\partial \eta_2} - \frac{\partial f}{\partial \eta_1} \Big)^2 \Big] .
\end{aligned}
$$

This shows that $W(2, \mathbf{h}) \le C_0$, proving Theorem 1.1.1 in the case $L = 2$. We conclude this step with the

Proof of Lemma 1.2.1 We first prove the lemma for the grand canonical measure. Fix $\lambda \in \mathbb{R}$ and denote by $\nu^{\mathbf{h},1}_{\Lambda_L,\lambda}$ the one-site marginal of the product measure $\nu^{\mathbf{h}}_{\Lambda_L,\lambda}$. Fix $x_\lambda \in \mathbb{R}$, that will be specified later, and $f \in L^2(\nu^{\mathbf{h},1}_{\Lambda_L,\lambda})$. By Schwarz inequality, the variance of f is bounded above by

$$
\mathbf{Var}(\nu^{\mathbf{h}}_{\Lambda_L,\lambda}, f) \le \int_{\mathbb{R}} (f(x) - f(x_\lambda))^2 e^{-V^h_\lambda(x)} dx
$$

$$
\le \int_{x_\lambda}^{\infty} f'(y)^2 dy \int_y^{\infty} (x - x_\lambda) e^{-V^h_\lambda(x)} dx + \int_{-\infty}^{x_\lambda} f'(y)^2 dy \int_{-\infty}^{y} (x - x_\lambda) e^{-V^h_\lambda(x)} dx.
$$

where $V^h_\lambda(x) = -\lambda x + hx + \log Z(\lambda - h) + V(x)$. It remains to show that the expressions inside braces are uniformly bounded in x and λ for an apropriate choice of x_λ. Both expressions are handled in the same way and we consider, to fix ideas, the first one where we need to estimate

$$
\int_{x_\lambda}^{\infty} f'(y)^2 dy \int_y^{\infty} (x - x_\lambda) e^{-V^h_\lambda(x)} dx.
$$

Choose $x_\lambda = \lambda - h$ and change variables to reduce the previous expression to

$$\int_{\lambda-h}^{\infty} f'(y)^2 dy \int_{y-\lambda+h}^{\infty} x e^{-V_\lambda^h(x+\lambda-h)} dx$$

$$= \frac{e^{F(y)}}{e^{F(y)}} \int_{\lambda-h}^{\infty} f'(y)^2 dy \int_{y-\lambda+h}^{\infty} x e^{(\lambda-h)(x+\lambda-h)-\frac{1}{2}(x-\lambda+h)^2+\log Z(\lambda-h)-F(x+\lambda-h)} dx$$

$$\leq C_0 \int_{\lambda-h}^{\infty} f'(y)^2 e^{-V_\lambda^h(y)} dy$$

where $C_0 = exp\Big\{2\|F\|_\infty\Big\}$.

This conclude the proof of the lemma in the case of grand canonical measures. We now prove the Lemma for canonical measures.

For λ in $\mathbb{R}$, denote by $P_{\mathbf{h},\lambda}$ the probability measure on the product space $\mathbb{R}^{\mathbb{N}}$ that makes the coordinates $\{X_k,\, k \geq 1\}$ independent random variables with X_k having density $Z(\lambda - h_k)^{-1} \exp\{[\lambda - h_k]x - V(x)\}$. Denote by $E_{\mathbf{h},\lambda}$ expectation with respect to $P_{\mathbf{h},\lambda}$.

Denote by $\gamma_1(\lambda)$, $\sigma^2(\lambda)$, $\{\gamma_k(\lambda),\, k \geq 3\}$ the expectation, the variance and the k-th truncated moment of a random variable with density $Z(\lambda)^{-1} \exp\{\lambda x - V(x)\}$:

$$\begin{aligned} \gamma_1(\lambda) &= \tfrac{1}{Z(\lambda)} \textstyle\int_{\mathbb{R}} x\, e^{\lambda x - V(x)}\, dx \;, \\ \sigma^2(\lambda) &= \tfrac{1}{Z(\lambda)} \textstyle\int_{\mathbb{R}} [x - \gamma_1(\lambda)]^2\, e^{\lambda x - V(x)}\, dx \;, \\ \gamma_k(\lambda) &= \frac{1}{Z(\lambda)} \int_{\mathbb{R}} [x - \gamma_1(\lambda)]^k\, e^{\lambda x - V(x)}\, dx \;. \end{aligned} \tag{2.1}$$

For a finite subset Λ of $\mathbb{N}$, denote by $f_{\mathbf{h},\lambda,\Lambda}$, $g_{\mathbf{h},\lambda,\Lambda}$ the density of the random variable

$$\frac{1}{\sigma^2(\mathbf{h},\lambda,\Lambda)^{1/2}} \sum_{j\in\Lambda} [X_j - \gamma_1(\lambda - h_j)]$$

$$\sum_{j\in\Lambda} [X_j - \gamma_1(\lambda - h_j)] \;,$$

respectively. In this formula,

$$\sigma^2(\mathbf{h},\lambda,\Lambda) \;=\; \sum_{j\in\Lambda}\sigma^2(\lambda-h_j)\;.$$

We prove in Chapter 3 an Edgeworth expansion for $f_{\mathbf{h},\lambda,\Lambda_L}$ uniform over the parameter λ.

Let $R(x) = R_{\mathbf{h},L,M}(x)$ be the Radon-Nikodym derivative of $\mu^{\mathbf{h},1}_{\Lambda_L,M}(dx)$ with respect to the Lebesgue measure. Fix a smooth function f in $L^2(\mu^{\mathbf{h},1}_{\Lambda_L,M})$ and x_λ in $\mathbb{R}$ to be specified later. The variance of f with respect to $\mu^{\mathbf{h},1}_{\Lambda_L,M}$ is bounded above by

$$\int_{\mathbb{R}} \Big(f(x)-f(x_\lambda)\Big)^2 R(x)\,dx\;.$$

By Schwarz inequality, the previous expression is less than or equal to

$$\begin{aligned}&\int_{x_\lambda}^{\infty} dx\,[f'(x)]^2 R(x)\Big\{\frac{1}{R(x)}\int_x^\infty dy\,(y-x_\lambda)R(y)\Big\}\\ &\quad+\int_{-\infty}^{x_\lambda} dx\,[f'(x)]^2 R(x)\Big\{\frac{1}{R(x)}\int_{-\infty}^{x} dy\,(x_\lambda-y)R(y)\Big\}\;.\end{aligned}$$

It remains to show that the expressions inside braces are uniformly bounded in x, L, M and $\mathbf{h}$ for an appropriate choice of x_λ. Both expressions are handled in the same way and we consider, to fix ideas, the first one where we need to estimate

$$\sup_{x\ge x_\lambda}\Big\{\frac{1}{R(x)}\int_x^\infty dy\,(y-x_\lambda)R(y)\Big\}. \tag{2.2}$$

For a finite subset Λ of $\mathbb{N}$, denote by $\tilde{g}_{\mathbf{h},\lambda,\Lambda}$ the density of the random variable $\sum_{j\in\Lambda} X_j$.

We may write the density $R(\cdot)$ in terms of the densities $g_{\mathbf{h},\lambda,\Lambda}(\cdot)$ or $\tilde{g}_{\mathbf{h},\lambda,\Lambda}$ for appropriate sets Λ. Choose λ so that

$$M \;=\; E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[\sum_{x\in\Lambda_L}\eta_x\Big] \;=\; \sum_{x\in\Lambda_L}\gamma_1(\lambda-h_x)\ . \tag{2.3}$$

Remark 1.2.2 *Let X absolutely continuous random variable with density fuction f_X. Then, for every $a\neq 0$, $b\in\mathbb{R}$,*

$$f_{aX+b}(u)=f_X\Big(\frac{u-b}{a}\Big)\frac{1}{a}.$$

Indeed, if H is a continuous bounded function,

$$E[H(aX+b)]=\int H(u)f_{aX+b}(u)du=\int H(y)\frac{1}{a}f_X\Big(\frac{y-b}{a}\Big)dy.$$

Remark 1.2.3 *Let $X_1,...,X_L$ independents random variables with densities f_j. Then:*

$$\int_{x_1+...+x_L=M}\prod_{j=1}^{L}f_j(x_j)dx_1...dx_{L-1}$$

$$=\int f_1(x_1)...f_{L-1}(x_{L-1})f_L(M-x_1-...-x_{L-1})dx_1...dx_{L-1}$$

$$=f_{\sum_{j=1}^{L}X_j}(M)=f_{\sum_{j=1}^{L}X_j-\gamma_1(j)}(M-\sum_{j=1}^{L}\gamma_1(j))$$

We now compute the ratio $\frac{R(x)}{R(y)}$ explicitly:

Fix a test fuction $H : \mathbb{R} \to \mathbb{R}$. By definition of the canonical measure $\mu^{\mathbf{h}}_{\Lambda_L,M}$,

$$E^{h,1}_{\Lambda_L,M}[H(\eta_1)] = Z(\lambda - h_1)^{-1} \frac{\int H(x_1) 1_{\sum_{j=1}^L x_j = M} e^{\sum_{j=1}^L V_\lambda^{h_j}(x_j)} dx_1...dx_{L-1}}{\int 1_{\sum_{j=1}^L x_j = M} e^{\sum_{j=1}^L V_\lambda^h(x_j)} dx_1...dx_{L-1}}$$

$$= Z^{-1} \frac{\int dx_1 H(x_1) e^{(\lambda - h_1)x - V(x_1)} \int_{x_2+...+x_L = M - x_1} f_{\lambda,h,2}(x_2)...f_{\lambda,h,L}(x_L) dx_2...dx_{L-1}}{\int_{x_1+...+x_L=M} f_{\lambda,h,1}(x_1)...f_{\lambda,h,L}(x_L) dx_1...dx_{L-1}}$$

In view of Remark 1.2.3, the previous expression can be writte

$$= Z(\lambda - h_1)^{-1} \int dx_1 H(x_1) e^{(\lambda - h)x_1 - V(x_1)} \frac{f_{\sum_{j=2}^L X_j}(M - x_1)}{f_{\sum_{j=1}^L X_j}(M)}.$$

$$= Z(\lambda - h_1)^{-1} \int dx_1 H(x_1) e^{(\lambda - h_1)x_1 - V(x_1)} \frac{f_{\sum_{j=2}^L (X_j - \gamma_1(\lambda - h_j))}(\gamma_1(\lambda - h_1) - x)}{f_{\sum_{j=1}^L (X_j - \gamma_1(\lambda - h_j))}(0)}.$$

Thus, with the notation introduced just before Remark 1.2.2,

$$\begin{aligned} R(x) &= \frac{1}{Z(\lambda - h_1)} e^{[\lambda - h_1]x - V(x)} \frac{g_{\mathbf{h},\lambda,\Lambda_{2,L}}(\gamma_1(\lambda - h_1) - x)}{g_{\mathbf{h},\lambda,\Lambda_L}(0)} \\ &= \frac{1}{Z(\lambda - h_1)} e^{[\lambda - h_1]x - V(x)} \frac{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M - x)}{\tilde{g}_{\mathbf{h},\lambda,\Lambda_L}(M)} , \end{aligned}$$

where $\Lambda_{2,L} = \{2, \dots, L\}$. Choose $x_\lambda = \lambda - h_1$. Since $\|F\|_\infty$ is finite, then

$$\frac{R(y)}{R(x)} = \frac{e^{(\lambda - h_1)y - \frac{y^2}{2} - F(y)}}{e^{\lambda x - h_1 x - \frac{x^2}{2} - F(x)}} \frac{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M - y)}{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M - x)} ,$$

$$\leq C_0 \frac{e^{\frac{-(y-(\lambda-h_1))^2}{2}}}{e^{\frac{-(x-(\lambda-h_1))^2}{2}}} \frac{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M-y)}{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M-x)} \ .$$

Up to this point we proved that

$$\sup_{x\geq x_\lambda} \frac{1}{R(x)} \int_x^\infty (y-x_\lambda)R(y)dy$$
$$\leq C_0 \sup_{x\geq x_\lambda} \int_x^\infty (y-x_\lambda) \frac{e^{\frac{-(y-(\lambda-h_1))^2}{2}}}{e^{\frac{-(x-(\lambda-h_1))^2}{2}}} \frac{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M-y)}{\tilde{g}_{\mathbf{h},\lambda,\Lambda_L}(M-x)} dy \ ,$$
$$= C_0 \sup_{x\geq \lambda-h_1} \int_x^\infty (y-x_\lambda) \frac{e^{\frac{-(y-(\lambda-h_1))^2}{2}}}{e^{\frac{-(x-(\lambda-h_1))^2}{2}}} \frac{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M-y)}{\tilde{g}_{\mathbf{h},\lambda,\Lambda_L}(M-x)} dy \ .$$

Performing the change of variables

$$y' = y - (\lambda - h_1)$$
$$x' = x - (\lambda - h_1)$$

We obtain that the previous expression is equal to

$$C_0 \sup_{x\geq 0} \int_x^\infty y \frac{e^{\frac{-y^2}{2}}}{e^{\frac{-x^2}{2}}} \frac{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M-y-\lambda+h_1)}{\tilde{g}_{\mathbf{h},\lambda,\Lambda_L}(M-x-\lambda+h_1)} dy. \tag{2.4}$$

We need therefore to estimate

$$\exp\{x^2/2 - y^2/2\} \frac{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M-y-\lambda+h_1)}{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M-x-\lambda+h_1)} \ . \tag{2.5}$$

In view of the proof for grand canonical measure, which can be found in [11], to complete the proof, it remains to show that the ratio on the right hand side is uniformly bounded. To this end, we replace the parameter λ by an appropriate parameter μ making small the argument of the denominator. A computation shows that

$$\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(a) \;=\; \exp\{(\lambda-\mu)a\} \prod_{j=2}^{L} \frac{Z(\mu-h_j)}{Z(\lambda-h_j)}\, \tilde{g}_{\mathbf{h},\mu,\Lambda_{2,L}}(a).$$

In fact, for $b \in \mathbb{R}$

$$\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(b) = \int_{\sum_{j=2}^{L} x_j = b} \prod_{j=2}^{L} f_{\lambda,h,j}(x_j) \prod_{j=2}^{L-1} dx_j$$

$$= \int_{\sum_{j=2}^{L} x_j = b} e^{\sum_{j=2}^{L}([\lambda-h_j]x_j - V(x_j) - \log Z(\lambda-h_j))} \prod_{j=2}^{L-1} dx_j$$

$$= \int_{\sum_{j=2}^{L} x_j = b} e^{\sum_{j=2}^{L}(\lambda-\mu)x_j} \prod_{j=2}^{L} \frac{Z(\mu-h_j)}{Z(\lambda+h_j)} \prod_{j=2}^{L} f_{\mu,h,j}(x_j) dx_2 ... dx_{L-1}$$

$$= e^{(\lambda-\mu)b} \prod_{j=2}^{L} \frac{Z(\mu-h_j)}{Z(\lambda-h_j)} \tilde{g}_{\mathbf{h},\mu,\Lambda_{2,L}}(b).$$

Thus, the ratio appearing in (2.4) is equal to

$$\frac{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M-y-\lambda+h_1)}{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M-x-\lambda+h_1)} = e^{(\lambda-\mu)(x-y)} \frac{\tilde{g}_{\mathbf{h},\mu,\Lambda_{2,L}}(M-y-\lambda+h_1)}{\tilde{g}_{\mathbf{h},\mu,\Lambda_{2,L}}(M-x-\lambda+h_1)}$$

$$= e^{(\lambda-\mu)(x-y)} \frac{g_{\mathbf{h},\mu,\Lambda_{2,L}}(M-\sum_{j=2}^{L}\gamma_1(\mu-h_j)-y-\lambda+h_1)}{g_{\mathbf{h},\mu,\Lambda_{2,L}}(M-\sum_{j=2}^{L}\gamma_1(\mu-h_j)-x-\lambda+h_1)}$$

$$= e^{(\lambda-\mu)(x-y)} \frac{f_{\mathbf{h},\mu,\Lambda_{2,L}}\left(\frac{M-\sum_{j=2}^{L}\gamma_1(\mu-h_j)-y-\lambda+h_1}{\sigma^2(h,\mu,\Lambda_{2,L})^{\frac{1}{2}}}\right)}{f_{\mathbf{h},\mu,\Lambda_{2,L}}\left(\frac{M-\sum_{j=2}^{L}\gamma_1(\mu-h_j)-x-\lambda+h_1}{\sigma^2(h,\mu,\Lambda_{2,L})^{\frac{1}{2}}}\right)}.$$

Choose μ so that

$$\sum_{j=2}^{L}\gamma_1(\lambda-h_j) - \sum_{j=2}^{L}\gamma_1(\mu-h_j) \; = \; x$$

Then,

$$\begin{aligned} M-\sum_{j=2}^{L}\gamma_1(\mu-h_j)-x-\lambda+h_1 &= \gamma_1(\lambda-h_1)+\sum_{j=2}^{L}(\gamma_1(\lambda-h_j)-\gamma_1(\mu-h_j))-x-\lambda+h_1 \\ &= \Gamma_1(\lambda-h_1). \end{aligned}$$

Notice that $\mu \leq \lambda$ because $x \geq 0$ and $\gamma_1(\cdot)$ is an increasing function. With this choice, the ratio on the right hand side of (2.5) becomes

$$\begin{aligned} &\exp\{(\lambda-\mu)(x-y)\}\frac{g_{\mathbf{h},\mu,\Lambda_{2,L}}(\Gamma_1(\lambda-h_1)+x-y)}{g_{\mathbf{h},\mu,\Lambda_{2,L}}(\Gamma_1(\lambda-h_1))} \qquad (2.6) \\ &= \; \exp\{(\lambda-\mu)(x-y)\} \; \frac{f_{\mathbf{h},\mu,\Lambda_{2,L}}\Big(\sigma^2(\mathbf{h},\mu,\Lambda_{2,L})^{-1/2}\Big\{\Gamma_1(\lambda-h_1)+x-y\Big\}\Big)}{f_{\mathbf{h},\mu,\Lambda_{2,L}}\Big(\sigma^2(\mathbf{h},\mu,\Lambda_{2,L})^{-1/2}\Gamma_1(\lambda-h_1)\Big)} \end{aligned}$$

The exponential is bounded by 1 because $\mu \leq \lambda$ and $x \leq y$. To conclude the proof of the lemma it is therefore enough to show that the previous ratio is uniformly bounded.

We first consider the case in which L is **large**. We first consider the denominator.

$$f_{\mathbf{h},\mu,\Lambda_{2,L}}\Big(\sigma^2(\mathbf{h},\mu,\Lambda_{2,L})^{-1/2}\Gamma_1(\lambda-h_1)\Big).$$

By lemma 3.1.1.

$$\sigma^2(\mathbf{h},\mu,\Lambda_{2,L}) \geq C\sqrt{L-1}.$$

By the proof of lemma 3.1.1.

$$\|\gamma_1(\lambda-h_1)-\lambda+h_1\| = \|E_\lambda[X_1]-\lambda+h_1\| \leq C_0.$$

uniformly in λ.

In particular, by the Local Central Limit Theorem, there exist $L_0 = L(\|F\|_\infty)$, C_0, such that

$$f_{\mathbf{h},\mu,\Lambda_{2,L}}\Big(\sigma^2(\mathbf{h},\mu,\Lambda_{2,L})^{-1/2}\Gamma_1(\lambda-h_1)\Big) \geq C.$$

for $L \geq L_0$.

We now turn to the numerator of 2.6. By Local Central Limit Theorem

$$f_{\mathbf{h},\mu,\Lambda_{2,L}}\Big(\sigma^2(\mathbf{h},\mu,\Lambda_{2,L})^{-1/2}\Big\{\Gamma_1(\lambda-h_1)+x-y\Big\}\Big) \leq C_0.$$

In view of these estimate (2.4) is bounded by

$$\tilde{C}_0 \int y e^{\frac{x^2-y^2}{2}} dy \leq C_0$$

this proves the lemma for L large.

We now turn to the case in which L is **small**.

For $2 \leq L \leq L_0$, denote by $\tilde{f}_{\mathbf{h},\lambda,\Lambda}$ the density of the random variable

$$\frac{1}{\sqrt{|\Lambda|}} \sum_{j \in \Lambda} \{X_j - [\lambda - h_j]\} .$$

The ratio of the previous formula can be written as

$$\frac{\tilde{f}_{\mathbf{h},\mu,\Lambda_{2,L}}\Big(|\Lambda_{2,L}|^{-1/2}\Big\{ \sum_{j=2}^{L} \Gamma_1(\mu - h_j) + \Gamma_1(\lambda - h_1) + x - y\Big\}\Big)}{\tilde{f}_{\mathbf{h},\mu,\Lambda_{2,L}}\Big(|\Lambda_{2,L}|^{-1/2}\Big\{ \sum_{j=2}^{L} \Gamma_1(\mu - h_j) + \Gamma_1(\lambda - h_1)\Big\}\Big)} .$$

Since $\|\Gamma_1\|_\infty < \infty$, by Lemma 3.1.7, this ratio is bounded by $\exp\{C_0 L_0\}$. This concludes the proof of the lemma. □

1.3 Decomposition of the variance

We will obtain now a recursive equation for $W(L, \mathbf{h})$. Assume that we already estimated $W(K, \mathbf{h})$ for $2 \le K \le L-1$. Let us write the identity

$$f - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f] = \Big\{ f - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f|\eta_L] \Big\} + \Big\{ E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f|\eta_L] - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f] \Big\}.$$

Through this decomposition we way express the variance of f as

$$\begin{aligned} & E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big(f - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f] \Big)^2 \Big] \qquad (3.1) \\ = \; & E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big(f - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f|\eta_L] \Big)^2 \Big] \; + \; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big(E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f|\eta_L] - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f] \Big)^2 \Big] \; . \end{aligned}$$

The first term on the right-hand side is easily analyzed through the induction assumption and a simple computation on the Dirichlet form. We write

$$\begin{aligned} & E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big(f - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f|\eta_L] \Big)^2 \Big] \; = \; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big(f - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f|\eta_L] \Big)^2 \Big| \eta_L \Big] \Big] \\ & = \; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[\Big(f_{\eta_L} - E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[f_{\eta_L}] \Big)^2 \Big] \Big] \; . \end{aligned}$$

Here we used the fact that $E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[.|\eta_L] = E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[.]$. In this formula and below f_{η_L} stands for the real function defined on $\mathbb{R}^{\Lambda_{L-1}}$ whose value at $(\xi_1, ..., \xi_{L-1})$ is given by $f_{\eta_L}(\xi_1, ..., \xi_{L-1}) = f(\xi_1, ..., \xi_{L-1}, \eta_L)$. By the induction assumption this last expectation is bounded above by

$$W(L-1, \mathbf{h}) E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[D_{\Lambda_{L-1}}\Big(\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}, f_{\eta_L} \Big) \Big] \; \le \; W(L-1, \mathbf{h}) D_{\Lambda_L}\Big(\mu^{\mathbf{h}}_{\Lambda_L,M}, f \Big) \; .$$

Thus we proved that

$$E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\left[\left(f - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f|\eta_L]\right)^2\right] \;\le\; W(L-1,\mathbf{h})D_{\Lambda_L}\left(\mu^{\mathbf{h}}_{\Lambda_L,M}, f\right) . \qquad (3.2)$$

The second term in (5.4) is nothing more than the variance of $E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f|\eta_L]$, a function of one variable. Lemma 1.2.1 provides an estimate for this expression:

$$E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\left[\left(E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f|\eta_L] - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f]\right)^2\right] \;\le\; C_0 E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\left[\left(\frac{\partial}{\partial \eta_L} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f|\eta_L]\right)^2\right] \qquad (3.3)$$

for some constant C_0 depending only on $\|F\|_\infty$.

1.4 Bounds on Glauber Dynamics, small values of L

We now estimate the right hand side (3.3), which is the Glauber Dirichlet form of $E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f|\eta_L]$, in terms of the Kawasaki Dirichlet form of f . A straightforward computation gives that:

$$\frac{\partial}{\partial \eta_L} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f|\eta_L] = \frac{1}{L-1}\sum_{x=1}^{L-1} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{\partial f}{\partial \eta_L} - \frac{\partial f}{\partial \eta_x}\,\Big|\,\eta_L\Big] \qquad (4.1)$$
$$+ E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[f; \frac{1}{L-1}\sum_{x=1}^{L-1} V'(\eta_x)\,\Big|\,\eta_L\Big] .$$

In this formula $E[f;g|\mathcal{F}] = E[fg|\mathcal{F}] - E[f|\mathcal{F}]E[g|\mathcal{F}]$ stands for the conditional covariance of f and g. Notice that the variables h_x do not appear because we have covariance terms. We examine these two terms separately (4.1).

The first expression on the right hand side of (4.1) is easily estimated. Recall the definition of the operator $T^{x,y}f$. Since $T^{x,y}f = \sum_{x\le y\le L-1} T^{y+1,y}f$, by Schwarz inequality , we have that

$$\Big(\frac{1}{L-1}\sum_{x=1}^{L-1} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{\partial f}{\partial_{\eta_x}} - \frac{\partial f}{\partial_{\eta_y}}\Big|\eta_L\Big]\Big)^2 = \Big(\frac{1}{L-1}\sum_{x=1}^{L-1}\sum_{y=x}^{L-1} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{\partial f}{\partial_{\eta_{y+1}}} - \frac{\partial f}{\partial_{\eta_y}}\Big|\eta_L\Big]\Big)^2$$
$$\le \frac{1}{L-1}\sum_{x=1}^{L-1}(L-x)\sum_{y=x}^{L-1} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big(\frac{\partial f}{\partial_{\eta_{y+1}}} - \frac{\partial f}{\partial_{\eta_y}}\Big)^2\Big|\eta_L\Big]$$
$$\le \frac{1}{L-1}\sum_{y=1}^{L-1} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big(\frac{\partial f}{\partial_{\eta_{y+1}}} - \frac{\partial f}{\partial_{\eta_y}}\Big)^2\Big|\eta_L\Big]\sum_{x=y}^{L-1}(L-x)$$
$$\le L\sum_{y=1}^{L-1} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big(\frac{\partial f}{\partial_{\eta_{y+1}}} - \frac{\partial f}{\partial_{\eta_y}}\Big)^2\Big|\eta_L\Big]$$

Hence, we have that

$$E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big(\frac{1}{L-1}\sum_{x=1}^{L-1}E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{\partial f}{\partial_{\eta_x}}-\frac{\partial f}{\partial_{\eta_y}}\Big|\eta_L\Big]\Big)^2\Big]\le LD_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M},f). \quad (4.2)$$

The second term in (4.1) is also easy to handle for small values de L. Since $V(\phi)=\frac{1}{2}\phi^2+F(\phi)$ and since $\sum_{1\le x\le L-1}\eta_x$ is fixed for the measure $E_{\Lambda_L,M}[.|\eta_L]$. the square of the second term on the right hand side is equal to

$$E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[f;\frac{1}{L-1}\sum_{x=1}^{L-1}F'(\eta_x)\Big|\eta_L\Big]^2=\Big(E_{\Lambda_{L-1},M-\eta_L}\Big[f_{\eta_L};\frac{1}{L-1}\sum_{x=1}^{L-1}F'(\eta_x)\Big]\Big)^2$$

$$\le E_{\Lambda_{L-1},M-\eta_L}\Big[f_{\eta_L};f_{\eta_L}\Big]E_{\Lambda_{L-1},M-\eta_L}\Big[\Big(\frac{1}{L-1}\sum_{x=1}^{L-1}\tilde F(\eta_x)\Big)^2\Big].$$

In this formula, $\tilde F$ stands for $F'-E_{\nu_{\Lambda_{L-1}},M-\eta_L}[F']$.

By the induction assumption, the first variance is bounded above by $W(L-1,\mathbf{h})\,D_{\Lambda_{L-1}}(\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L},f_{\eta_L})$. For the second one, we divide the cube Λ_{L-1} in two cubes of the same size and use the inequality $(a+b)^2\le 2a^2+2b^2$ to separate the variance in two variances on cubes of size $L/2$. We now apply (3.1.5) to estimate the variance with respect to canonical measures by variances with respect to grand canonical measures. Since F' is bounded and the grand canonical measure is product, these last variances are clearly of order L^{-1}. Taking expectations with respect to $\mu^{\mathbf{h}}_{\Lambda_L,M}$, we finally obtain that

$$E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[f;\frac{1}{L-1}\sum_{x=1}^{L-1}V'(\eta_x)\Big|\eta_L\Big]\Big]^2\le\frac{C_1}{L}W(L-1)D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M},f).$$

for some finite constant C_1 depending on $\|F'\|_\infty$ only.

From this estimate and we get that the left hand side of, which is the second term of, is bounded above by

$$C_1\Big\{L + \frac{W(L-1,\mathbf{h})}{L}\Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f).$$

Putting together this estimate with (3.2), we obtain that

$$Var_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f] \leq \Big\{\Big(1+\frac{C_1}{L}\Big)W(L-1,\mathbf{h}) + C_1 L\Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)$$

or, taking a supremum over smooth functions f ,that

$$W(L,\mathbf{h}) \leq \Big(1+\frac{C_1}{L}\Big)W(L-1,\mathbf{h}) + C_1 L.$$

This inequality permits, together with the estimate $W(2,\mathbf{h}) \leq C_0$ obtained in section 1.1, to derive estimates of $W(L,\mathbf{h})$ for small values of L. We obtain by induction that $W(L,\mathbf{h}) \leq C_1 L^t$, *uniformly over the environment*, for some power t which depends only on $\|F\|_\infty$, $\|F'\|_\infty$.

Notice that we would obtain the right bound L^2 if we could prove that the constant C_1^*, which appears in the previous inequality, is strictly less than 2. Therefore, to prove the spectral gap, we have to improve our bounds on the covariance term to derive a factor of order εL^{-1} for $\varepsilon < 2$.

The bound $W(L,\mathbf{h}) \leq C_1 L^t$ will be used for small values of L. We now consider large values of L.

1.5 Bounds on Glauber Dynamics, large values of L

Here again we want to estimate the second term of (5.4). Applying Lemma 1.2.1, we bound this expression by the right hand side of (3.3). The first term of (4.1) is handled as before, giving (4.2). The second one requires a deeper analysis. Its square is equal to :

$$E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[f;\frac{1}{L-1}\sum_{x=1}^{L-1}F'(\eta_x)\,\Big|\,\eta_L\Big]^2 \;=\; E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[f;\frac{1}{L-1}\sum_{x=1}^{L-1}F'(\eta_x)\Big]^2 . \tag{5.1}$$

Here and below we omit the subscript η_L of f.

Fix $\varepsilon > 0$ small, a random environment $\mathbf{h}$ in some set Ω_0 and a positive integer $K = K(\varepsilon) \geq 2$. Both Ω_0 and K will be specified later. For each $L \geq K^2$, divide the interval $\{1, \ldots, L-1\}$ into $\ell = \lfloor (L-1)/K \rfloor$ adjacent intervals of length K or $K+1$, where $\lfloor a \rfloor$ represents the integer part of a. Keep in mind that K is large but fixed, while L (and thus ℓ) increase to $+\infty$.

Denote by I_j the j-th interval and by M_j the total spin on I_j: $M_j = \sum_{x\in I_j}\eta_x$. The right hand side of the previous formula is bounded above by

$$2\Big(E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[f;\sum_{j=1}^{\ell}a_j\Big\{\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)-E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)\Big]\Big\}\Big]\Big)^2 \tag{5.2}$$

$$+\,2\Big(E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[f;\sum_{j=1}^{\ell}a_jE_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)\Big]\Big]\Big)^2 ,$$

where $a_j = |I_j|/(L-1)$. Taking conditional expectation with respect to M_j, we have

$$2\Big(E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[f;\sum_{j=1}^{\ell}a_j\Big\{\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)\Big\}|\mathcal{F}_j\Big]\Big]\Big)^2$$

$$+ 2\Big(E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[f;\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\Big]\Big)^2$$

where $\mathcal{F}_j = \sigma(M_j;\eta_x \in M_j)$.We rewrite the first term as

$$2\Big(\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[f;\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\Big]\Big)^2$$

$$\leq 2\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[\mathbf{Var}(\mu^{\mathbf{h}}_{I_j,M_j},f)\mathbf{Var}(\mu^{\mathbf{h}}_{I_j,M_j},\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x))\Big]\ . \quad (5.3)$$

By the induction assumption, the variance $\mathbf{Var}(\mu^{\mathbf{h}}_{I_j,M_j},f)$ is bounded above by $W(|I_j|,\tau_j\mathbf{h})D_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j},f)$, where $\tau_j\mathbf{h}$ stands for the translation of the environment $\mathbf{h}$ for the origin to coincide with the left end of the interval I_j. By the a-priori estimate obtained in Step 3, this expression is bounded by $C_1K^tD_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j},f)$.

Now

$$E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\Big\{\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x) - E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\Big\}; \quad (5.4)$$

$$\Big\{\frac{1}{|B_j|}\sum_{x\in I_j} F'(\eta_x) - E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\Big\}\Big].$$

$$= \frac{1}{|I_j|^2}\sum_{x,y\in I_j} E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\tilde{F}'(\eta_x);\tilde{F}'(\eta_y)\Big]$$

$$\leq \frac{1}{|I_j|}4\big\|F'\big\|_\infty^2 + \frac{1}{|I_j|^2}\sum_{x\neq y} E_{\mu^{\mathbf{h}}_{I_j,M_j}}[\tilde{F}'(\eta_x);\tilde{F}'(\eta_y)] \quad (5.5)$$

where $\tilde{F}'(.) = F'(.) - < F' >_{\mu^{\mathbf{h}}_{I_j,M_j}}$.

If $|I_j|$ is large enough, by equivalence of ensembles

$$E_{\nu^{\mathbf{h}}_{I_j,\lambda}}[F'(\eta_x)] - E_{\mu^{\mathbf{h}}_{I_j,M_j}}[F'(\eta_x)] \leq \frac{C\,\|F'\|_\infty}{|I_j|}$$

The second term in (5.5), is bounded by

$$E_{\nu^{\mathbf{h}}_{I_j,\lambda}}\Big[\Big(F'(\eta_x) - E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[F'(\eta_x)\Big]\Big)\Big(F'(\eta_y) - E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[F'(\eta_y)\Big]\Big)\Big] + \frac{C\,\|F'\|_\infty}{|I_j|}. \tag{5.6}$$

Since the grand canonical measure is a product measure, (5.6) is bounded by

$$\Big(E_{\nu^{\mathbf{h}}_{I_j,\lambda}}\Big[F'(\eta_x)\Big] - E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[F'(\eta_x)\Big]\Big)\Big(E_{\nu^{\mathbf{h}}_{I_j,\lambda}}\Big[F'(\eta_y)\Big] - E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[F'(\eta_y)\Big]\Big) + \frac{C\,\|F'\|_\infty}{|I_j|}.$$
$$\leq \frac{C\,\|F'\|_\infty}{|I_j|}\frac{C\,\|F'\|_\infty}{|I_j|} + \frac{C\,\|F'\|_\infty}{|I_j|}$$

The second term of (5.5) is bounded by

$$\frac{C\,\|F'\|^2_\infty}{|I_j|}.$$

Then the variance of $|I_j|^{-1}\sum_{x\in I_j} F'(\eta_x)$ with respect to $\mu^{\mathbf{h}}_{I_j,M_j}$ is bounded above by $C_0|I_j|^{-1}\|F'\|^2_\infty$ uniformly over M_j, $\mathbf{h}$. (5.4) is thus less than or equal to

$$\frac{C_1 K^t}{L} \sum_{j=1}^{\ell} E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}} \Big[D_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j}, f) \Big] \ .$$

Since the previous sum is bounded by the global Dirichlet form $D_{\Lambda_{L-1}}(\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}, f)$, we proved that the first term of (5.2) is bounded above by

$$\frac{C_1 K^t}{L} D_{\Lambda_{L-1}}(\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}, f) \ . \tag{5.7}$$

The next result provides an estimate for the second covariance in (5.2). For λ in $\mathbb{R}$, let $\tilde{\sigma}^2(\lambda) = \mathbb{E}[\sigma^2(\lambda - h_1)]$.

Lemma 1.5.1 *Let $\lambda = \lambda(L, M - \eta_L, \boldsymbol{h})$ so that*

$$E_{\nu^{\boldsymbol{h}}_{\Lambda_{L-1},\lambda}} \Big[\sum_{x \in \Lambda_{L-1}} \eta_x \Big] \ = \ M - \eta_L \ . \tag{5.8}$$

There exist finite constants C_0, C_1 such that

$$\begin{aligned} &\Big(E_{\mu^{\boldsymbol{h}}_{\Lambda_{L-1},M-\eta_L}} \Big[f ; \sum_{j=1}^{\ell} a_j E_{\mu^{\boldsymbol{h}}_{I_j,M_j}} \Big[\frac{1}{|I_j|} \sum_{x \in I_j} F'(\eta_x) \Big] \Big] \Big)^2 \\ &\le \ \Big\{ \frac{C_1}{KL} \ + \ \frac{C_0}{L} \frac{1}{\ell} \sum_{j=1}^{\ell} \Big(\frac{1}{|I_j|} \sum_{x \in I_j} \sigma^2(\lambda - h_x) - \tilde{\sigma}^2(\lambda) \Big)^2 \Big\} \mathbf{Var}(\mu^{\boldsymbol{h}}_{\Lambda_{L-1},M-\eta_L}, f) \end{aligned}$$

for all $K \geq 2$, $L \geq K^2$.

It is now easy to conclude the proof of the lemma. Fix $0 < \varepsilon < 1/2$. Choose K_1 large enough for C_1/K_1 to be smaller that ε, where C_1 is the constant appearing in the previous lemma. Let $K_2 = K_2(\varepsilon C_0^{-1}, \sigma^2)$ be the positive integer given by (4.1.5) with σ^2 in place of U, where C_0 is the constant appearing in the statement of the previous lemma. Fix $K_0 = \max\{K_1, K_2\}$. For this fixed integer K_0, consider the sequence of disjoint intervals $\{I_1^L, \dots, I_\ell^L\}$ introduced above and an environment $\mathbf{h}$ in the set $\Omega_0(K_0, \sigma^2(\cdot), \varepsilon C_0^{-1})$ defined in (4.1.5).

By (4.1.5), there exists $\ell_0 = \ell_0(\mathbf{h}, \varepsilon C_0^{-1}, K_0, \sigma^2(\cdot))$ for which the right hand side of the statement of the previous lemma is bounded above by

$$\frac{2\varepsilon}{L} E_{\mu^{\mathbf{h}}_{\Lambda_{L-1}, M-\eta_L}}[f; f]$$

for all $\ell \geq \ell_0$. This bound together with (5.7) gives that (5.2), and therefore (5.1), is less than or equal to

$$\frac{C_1 K_0^t}{L} D_{\Lambda_{L-1}}(\mu_{\Lambda^{\mathbf{h}}_{L-1}, M-\eta_L}, f) \; + \; \frac{2\varepsilon}{L} E_{\mu^{\mathbf{h}}_{\Lambda_{L-1}, M-\eta_L}}[f_{\eta_L}; f_{\eta_L}] \, .$$

Since (5.1) is just the square of the second term of (4.1), taking expectation with respect to $\mu^{\mathbf{h}}_{\Lambda_L, M}$ in (5.1) and recalling (4.2), we have that (3.3) is bounded above by

$$C_1 \Big(L + \frac{K_0^t}{L} \Big) D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L, M}, f) \; + \; \frac{C_0 \varepsilon}{L} E_{\mu^{\mathbf{h}}_{\Lambda_L, M}}[f; f] \, .$$

Choose ε small enough for $C_0 \varepsilon \leq 1$. Adding this term to (3.2), in view of the decomposition (5.4), we deduce that

$$
\begin{aligned}
&E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big(f - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f]\Big)^2\Big] \\
&\quad\le\ \Big(1-\frac{1}{L}\Big)^{-1}\Big(W(L-1,\mathbf{h}) + C_1 L\Big) D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)
\end{aligned}
$$

for L large enough. Note that the integer L_0 from which this inequality holds depends on the environment $\mathbf{h}$ because ℓ_0 depends on $\mathbf{h}$.

Taking supremum over smooth functions $f\colon \mathbb{R}^{\Lambda_L} \to \mathbb{R}$ in $L^2(\mu^{\mathbf{h}}_{\Lambda_L,M})$, we obtain that

$$
W(L,\mathbf{h})\ \le\ \Big(1-\frac{1}{L}\Big)^{-1}\Big(W(L-1,\mathbf{h}) + C_1 L\Big)\ .
$$

It is not difficult to deduce from this recursive relation the existence of a constant $C = C(\|F\|_\infty, \|F'\|_\infty, \mathbf{h})$ such that $W(L,\mathbf{h}) \le CL^2$ for all $L \ge 2$. This concludes the proof of Theorem 1.1.1. □

We conclude this section with the

Proof of Lemma 1.5.1. For each finite subset Λ of $\mathbb{Z}$, denote by $R_\Lambda : \mathbb{R} \to \mathbb{R}$ the smooth strictly increasing function

$$
R_\Lambda(\lambda)\ =\ E_{\nu^{\mathbf{h}}_{\Lambda,\lambda}}\Big[\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\eta_x\Big]
$$

and denote by Φ_Λ the inverse of R_Λ. For $1 \le j \le \ell$, let $A_j : \mathbb{R} \to \mathbb{R}$ be given by

$$
A_j(m)\ =\ E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(m)}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\ , \tag{5.9}
$$

where λ is given by (5.8), let

$$m_j^* = E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}\eta_x\Big]$$

and rewrite the sum

$$\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[|I_j|^{-1}\sum_{x\in I_j}F'(\eta_x)\Big] \tag{5.10}$$

as

$$\begin{aligned}
&\sum_{j=1}^{\ell} a_j\Big\{E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)\Big] - E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(M_j/|I_j|)}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)\Big]\Big\} \qquad (5.11)\\
&\quad + \sum_{j=1}^{\ell} a_j\Big\{E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(M_j/|I_j|)}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)\Big] - A_j(m_j^*) - A_j'(m_j^*)(M_j/|I_j| - m_j^*)\Big\}\\
&\quad + \sum_{j=1}^{\ell} a_j\Big\{A_j(m_j^*) + A_j'(m_j^*)(M_j/|I_j| - m_j^*)\Big\}\ .
\end{aligned}$$

This decomposition is easy to understand. In the first term we compare an expectation with respect to a canonical measure with an expectation with respect to a grand canonical measure. In view of Corollary 3.1.4 for the difference to be small, the chemical potential should be $\Phi_{I_j}(M_j/|I_j|)$. The second term is a Taylor expansion up to the first order and the last one is what remains.

Notice that $A_j(m_j^*)$ is a function of L, $M-\eta_L$ and $\mathbf{h}$ only. It is therefore a constant with respect to the measure $\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}$. The same statement is true for $\sum_{j=1}^{\ell} a_j(M_j/|I_j| - m_j^*)$. Since we may add constants in covariances, by the previous observation, we may substitute (5.10) by (5.11) where the third term in (5.11) is replaced by

$$\sum_{j=1}^{\ell} a_j(A_j'(m_j^*) - c)(M_j/|I_j| - m_j^*) \ ,$$

where $c = \tilde{\sigma}^2(\lambda)^{-1} - 1$.

Up to this point, we have replaced the average of $E_{\mu^{\mathbf{h}}_{I_j,M_j}}[|I_j|^{-1}\sum_{x\in I_j} F'(\eta_x)]$ by the sum of three terms. We estimate separately the covariance of f with each term.

Set first

$$G_j^0 \ = \ E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big] - E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(M_j/|I_j|)}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big] \ .$$

Since F' is bounded, by the equivalence of ensembles, Corollary 3.1.4, $|G_j^0| \le C_1/K$. Hence, by Lemma 1.5.3, the contribution of the first term in (5.11) to the covariance appearing in the statement of the lemma is bounded by

$$\frac{C_1}{KL} E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[f;f] \ . \tag{5.12}$$

Consider now the second term in (5.11):

$$
\begin{aligned}
S_j^0 &= E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(M_j/|I_j|)}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big] - A_j(m_j^*) - A_j'(m_j^*)(M_j/|I_j| - m_j^*) \\
&= A_j(M_j/|I_j|) \;-\; A_j(m_j^*) \;-\; A_j'(m_j^*)(M_j/|I_j| - m_j^*)\;,
\end{aligned}
$$

We need estimate:

$$
E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[\sum_{j=1}^{l} a_j S_j^0; f\Big].
$$

Since we are allowed to add constants in covariances, by Schwarz inequality, the square of the covariance is bounded above by

$$
E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[f;f]\; E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[\Big(\sum_{j=1}^{\ell} a_j S_j\Big)^2\Big]\;, \tag{5.13}
$$

The second expectation in (5.13) is equal to

$$
\sum_{j=1}^{\ell} a_j^2 E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[S_j^2] \;+\; \sum_{j\neq k} a_j a_k E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[S_j S_k]\;. \tag{5.14}
$$

where

$$
S_j \;=\; S_j^0 \;-\; E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[S_j^0]
$$

and λ is given by (5.8).

We estimate separately the diagonal and the off diagonal terms. The sum of the diagonal term is less than or equal to

$$E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[S_j^2] + \frac{CK}{L}\sqrt{E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[S_j^4]}$$

for some constant C_0 which depends only on $\|F\|_\infty$.

We have

$$E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[S_j^2] \le E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[(S_j^0)^2].$$

By Lemma 4.1.1, $\|A_j''\|_\infty \le C_0$. In particular, S_j^0 is absolutely bounded by $C_0(M_j/|I_j| - m_j^*)^2$ and in this case

$$\begin{aligned}\sum_{j=1}^{\ell} a_j^2 E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[(S_j^0)^2] \; &\le \; C_0 \sum_{j=1}^{\ell} a_j^2 E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[(M_j/|I_j| - m_j^*)^4]\,. \\ &\le \frac{C}{K^2}.\end{aligned}$$

In the same way

$$\begin{aligned}E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[S_j^4] &\le C\Big(E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[S_j^4] + E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[S_j^4]\Big) \\ &\le C E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[S_j^4] \\ &\le \frac{C}{K^4}.\end{aligned}$$

In particular , first term in (5.8) is bounded by

$$\sum_{j=1}^{l} \frac{K^2}{L^2}\Big\{\frac{C}{K^2}+\frac{CK}{L}\frac{1}{K^2}\Big\}$$
$$\leq \frac{C}{KL}$$

by definition of l.

Second term in (5.12) is bounded by

$$E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[S_jS_k]$$
$$\leq E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[S_jS_k]+\frac{CK}{L}\sqrt{E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[S_j^2S_k^2]}.$$

Since the grand canonical measure $\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}$ is a product measure and since each term has mean zero with respect to $\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}$ the first term vanishes.

Second term

$$\frac{CK}{L}\sqrt{E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[S_j^2S_k^2]}\leq \frac{C}{LK}$$

because, we showed that

$$E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[S_j^2]\leq \frac{C}{K^2}.$$

Then the second term in (5.8) is bounded by $\frac{CK}{L}$.

By Lemma 3.1.1, each expectation is bounded by $C_0 K^{-2}$ so that the contribution to the covariance of the second term in (5.11) is bounded again by (5.12).

We turn now to the third term in (5.11) without $A_j(m_j^*)$ and with $A_j'(m_j^*)$ replaced by $A_j'(m_j^*) - c$, as explained above. We need to estimate

$$\Big(E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[f;\sum_{j=1}^{\ell} a_j\Big\{A_j(m_j^*) + A_j'(m_j^*)(M_j/|I_j| - m_j^*)\Big\}\Big]\Big)^2$$

$$= \Big(E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[f;\Big\{\sum_{j=1}^{l} a_j[A_j'(m_j^*) - c]\Big(\frac{M_j}{|I_j|} - m_j^*\Big)\Big\}]$$

Remark 1.5.2 *i)By Lemma 4.1.1,*

$$\begin{aligned} A_j'(m_j^*) &= \frac{1}{|I_j|^{-1}\sum_{x\in I_j}\sigma^2(\Phi_{I_j}(m_j^*) - h_x)} - 1 \\ &= \frac{1}{|I_j|^{-1}\sum_{x\in I_j}\sigma^2(\lambda - h_x)} - 1 \end{aligned}$$

because $m_j^* = R_{I_j}(\lambda)$ *and* $\Phi_{I_j} = R_{I_j}^{-1}$.

ii)

$$E_{\mu^{h}_{\Lambda_{L-1},\lambda}}[\sum_{j=1}^{l} a_j(c-1)\Big[\frac{M_j}{|B_j|} - m_j^*\Big]; f]$$

$$= (c-1)E_{\mu^{h}_{\Lambda_{L-1},\lambda}}\Big[\frac{1}{K}\sum_{j=1}^{l} M_j; f\Big] = 0$$

because $\sum_{j=1}^{L} M_j = M$ *is a constant.*

Since we are allowed to add constants in covariances, by Schwarz inequality, the square of the covariance is bounded above by

$$E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[f;f]\ \mathbf{Var}_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[\Big\{\sum_{j=1}^{l} a_j[A_j'(m_j^*)-c]\Big(\frac{M_j}{|I_j|}-m_j^*\Big)\Big\}\Big]\ , \tag{5.15}$$

the variance in (5.16) is equal to

$$\sum_{j=1}^{\ell} a_j^2 \mathbf{Var}_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[[A_j'(m_j^*)-c]\Big(\frac{M_j}{|I_j|}-m_j^*\Big)\Big]$$
$$+\sum_{j\neq k} a_j a_k \mathbf{Cov}_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[[A_j'(m_j^*)-c]\Big(\frac{M_j}{|I_j|}-m_j^*\Big);[A_k'(m_k^*)-c]\Big(\frac{M_k}{|I_k|}-m_k^*\Big)\Big]\ .$$

We estimate separately the diagonal and the off diagonal terms. The sum of the diagonal term is equal to

$$\sum_{j=1}^{\ell} a_j^2 (A_j'(m_j^*)-c)^2 E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[\Big(\frac{M_j}{|I_j|}-m_j^*\Big)^2\Big]\ .$$

By the equivalence of ensembles, the sum of the diagonal term is less than or equal to

$$\sum_{j=1}^{l} a_j^2 (A_j'(m_j^*)-c)^2 E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}\Big[\Big(\frac{M_j}{|I_j|}-m_j^*\Big)^2\Big]$$

$$+C\sum_{j=1}^{l} a_j^2 (A'_j(m_j^*) - c)^2 \frac{|I_j|}{L} \sqrt{E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}\Big[\Big(\frac{M_j}{|I_j|} - m_j^*\Big)^4\Big]}$$

By Lemma 3.1.1, this expression is less than or equal to

$$\frac{C_0}{L}\frac{1}{l}\sum_{j=1}^{\ell}(A'_j(m_j^*) - c)^2$$
$$+\frac{C_0 K}{L^2}\frac{1}{l}\sum_{j=1}^{\ell}(A'_j(m_j^*) - c)^2 \ .$$

Since $c = \mathbb{E}[\sigma^2(\lambda - h_1)]^{-1} - 1$, and since $\sigma^2(\cdot)$ is bounded above and below by finite, strictly positive constants, the previous expression is bounded by

$$\frac{C_0}{L}\frac{1}{\ell}\sum_{j=1}^{\ell}\Big(\frac{1}{|I_j|}\sum_{x\in I_j}\sigma^2(\lambda - h_x) - \tilde{\sigma}^2(\lambda)\Big)^2$$
$$+\frac{C_0 K}{L^2}\frac{1}{\ell}\sum_{j=1}^{\ell}\Big(\frac{1}{|I_j|}\sum_{x\in I_j}\sigma^2(\lambda - h_x) - \tilde{\sigma}^2(\lambda)\Big)^2 \ .$$
$$\leq \frac{C_0 K}{L^2}\sum_{j=1}^{l}\Big\{1 + \frac{K}{L}\Big\} \leq \frac{CK}{L}$$

On the other hand, by the equivalence of ensembles, Corollary 3.1.4, each off diagonal term is bounded by

$$\sum_{j\neq k} a_j a_k (A'_j(m_j^*) - c)(A'_k(m_k^*) - c) E_{L-1,\lambda}[(\frac{M_j}{|I_j|} - m_j^*)(\frac{M_k}{|I_k|} - m_k^*)]$$

$$+\sum_{j\neq k} a_j a_k (A'_j(m_j^*) - c)(A'_k(m_k^*) - c) C\frac{K}{L}\sqrt{E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[(\frac{M_j}{|I_j|} - m_j^*)^2(\frac{M_k}{|I_k|} - m_k^*)^2]}$$

for some finite constant C_0 depending only on $\|F\|_\infty$. Since the grand canonical measure $\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}$ is product and since each term has mean zero with respect to $\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}$, the first term vanishes. Since the measure is product, the contribution of the off diagonal terms is bounded by

$$\frac{CK^2}{L^2}\sum_{j\neq k}\frac{K}{L}\frac{1}{K}\leq\frac{CK}{L}.$$

This concludes the proof of the lemma. □

We now turn to a simple technical lemma needed in the proof of Lemma 1.5.1 above.

Lemma 1.5.3 *Fix λ given by (5.8). For $1\leq j\leq \ell$, let G^0_j be a family of functions in $L^2(\nu^{\boldsymbol{h}}_{\Lambda_{L-1},\lambda})$. Assume that each function G^0_j depends only on the variables $\{\eta_x\,,x\in I_j\}$. There exists a finite constant C_0, depending only on $\|F\|_\infty$, such that*

$$\Big(E_{\mu^{\boldsymbol{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[f;\sum_{j=1}^{\ell}a_jG^0_j\Big]\Big)^2\;\leq\;C_0E_{\mu^{\boldsymbol{h}}_{\Lambda_{L-1},M-\eta_L}}[f;f]\sum_{j=1}^{\ell}a_j^2\,E_{\nu^{\boldsymbol{h}}_{\Lambda_{L-1},\lambda}}[(G^0_j)^2]$$

for all L, $M-\eta_L$ and $\boldsymbol{h}$.

Proof: Since we are allowed to add constants in covariances, by Schwarz inequality, the square of the covariance is bounded above by

$$E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[f;f]\;E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[\Big(\sum_{j=1}^{\ell}a_jG_j\Big)^2\Big]\,,\tag{5.16}$$

where

$$G_j \;=\; G_j^0 \;-\; E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[G_j^0]$$

and λ is given by (5.8). The second expectation in (5.16) is equal to

$$\sum_{j=1}^{\ell} a_j^2 E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[G_j^2] \;+\; \sum_{j\neq k} a_j a_k E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[G_j G_k]\;. \tag{5.17}$$

We estimate separately the diagonal and the off diagonal terms. Since $2K \le L$, by Corollary 3.1.6 and since the variance is bounded by the L^2 norm, the sum of the diagonal term is less than or equal to

$$C_0 \sum_{j=1}^{\ell} a_j^2 E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[G_j^2] \;\le\; C_0 \sum_{j=1}^{\ell} a_j^2 E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[(G_j^0)^2]$$

for some constant C_0 which depends only on $\|F\|_\infty$.

On the other hand, by the equivalence of ensembles, Corollary 3.1.4, each off diagonal term is bounded by

$$E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[G_j G_k] \;+\; \frac{C_0 K}{L}\Big\{E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[G_j^2 G_k^2]\Big\}^{1/2}$$

for some finite constant C_0 depending only on $\|F\|_\infty$. Since the grand canonical measure $\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}$ is product and since each G_i has mean zero with respect to $\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}$, the first term vanishes. Since the measure is product, the contribution of the off diagonal terms is bounded by

$$
\begin{aligned}
&\frac{C_0 K}{L} \sum_{j \neq k} a_j a_k \Big\{ E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[G_j^2] E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[G_k^2] \Big\}^{1/2} \\
&\leq \; \frac{C_0 K}{L} \Big(\sum_{j=1}^{\ell} a_j \Big\{ E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[(G_j^0)^2] \Big\}^{1/2} \Big)^2
\end{aligned}
$$

because the variance is bounded by the L^2 norm. To conclude the proof, it remains to apply Schwarz inequality and to recall that $\ell \leq 2L/K$. □

Chapter 2

Logarithmic Sobolev Inequality

We prove in this section Theorem 1.1.2. The approach is similar to the one presented in last section for the spectral gap. We will derive a recursive formula for $\theta(L, \mathbf{h})$ in terms of $\theta(L-1, \mathbf{h})$ and L in four steps. As before, for $j = 0, 1, 2$, all constants C_j are allowed to depend on $\|F^{(i)}\|_\infty$ for $0 \le i \le j$ and may change from line to line. Here $F^{(i)}$ stands for the i-th derivative of F.

2.1 One-site logarithmic Sobolev inequality

We start our proof with the case $L = 2$. Let $f\colon \mathbb{R}^{\Lambda_2} \to \mathbb{R}$ be a smooth function such that $< f^2 >_{\mu^{\mathbf{h}}_{\Lambda_2,M}} = 1$. Let $g(\eta_1) = f(\eta_1, M - \eta_1)$. Since the total spin is fixed to be M, we have that $< g^2 >_{\mu^{\mathbf{h}}_{\Lambda_2,M}} =< f^2 >_{\mu^{\mathbf{h}}_{\Lambda_2,M}} = 1$ and that $S_{\Lambda_2}(\mu^{\mathbf{h}}_{\Lambda_2,M}, g) = S_{\Lambda_2}(\mu^{\mathbf{h}}_{\Lambda_2,M}, f)$. The next lemma permits to estimate the entropy of $S_{\Lambda_2}(\mu^{\mathbf{h}}_{\Lambda_2,M}, g)$ in terms of the Glauber Dirichlet form of g. This result is in fact a logarithmic Sobolev inequality for the Glauber dynamics obtained when restricting the Kawasaki exchange dynamics to one site. Recall that $\mu^{\mathbf{h},1}_{\Lambda_L,M}$ represents the one-site marginal of $\mu^{\mathbf{h}}_{\Lambda_L,M}$.

Lemma 2.1.1 *There exists a finite constant C_0 depending only on $\|F\|_\infty$ such that*

$$\int H(\eta_1)^2 \log H(\eta_1)^2 \, \mu^{\boldsymbol{h},1}_{\Lambda_L,M}(d\eta_1) \;\le\; C_0 E_{\mu^{\boldsymbol{h},1}_{\Lambda_L,M}}\Big[\Big(\frac{\partial H}{\partial \eta_1}\Big)^2\Big] \qquad (1.1)$$

for every $L \ge 2$, M in $\mathbb{R}$, environment $\boldsymbol{h}$ and smooth function $H\colon \mathbb{R} \to \mathbb{R}$ in $L^2(\mu^{\boldsymbol{h},1}_{\Lambda_L,M})$ such that $< H^2 >_{\mu^{\boldsymbol{h},1}_{\Lambda_L,M}} = 1$.

In the case of grand canonical measures, this result is true under the more general hypothesis of strict convexity at infinity of the potencial(cf (14) and references therein). In case of canonical measures the main problem is to obtain a good aproximation of the one-site marginal in terms of the one-site marginal of grand canonical measures.

We conclude the first step before proving the lemma. From the previous statement applied to $L = 2$ and $H = g$ we have that

$$S_{\Lambda_2}(\mu^{\mathbf{h}}_{\Lambda_2,M}, f) \;=\; S_{\Lambda_2}(\mu^{\mathbf{h}}_{\Lambda_2,M}, g) \;\le\; C_0 E_{\mu^{\mathbf{h},1}_{\Lambda_2,M}}\Big[\Big(\frac{\partial g}{\partial \eta_1}\Big)^2\Big]$$

$$=\; C_0 E_{\mu^{\mathbf{h}}_{\Lambda_2,M}}\Big[\Big(\frac{\partial g}{\partial \eta_1}\Big)^2\Big] \;=\; C_0 E_{\mu^{\mathbf{h}}_{\Lambda_2,M}}\Big[\Big(\frac{\partial f}{\partial \eta_2} - \frac{\partial f}{\partial \eta_1}\Big)^2\Big]$$

because $\partial g/\partial\eta_1 = \partial f/\partial\eta_1 - \partial f/\partial\eta_2$. This proves that $\theta(2, \mathbf{h}) \le C_0$ uniformly in $\mathbf{h}$, proving theorem 1.2 in the case $L = 2$. We conclude this step with the

Proof of Lemma 2.1.1

We first prove of lemma in the case of grand canonical measures. Recall that we denote by $\mu^{\mathbf{h},1}_{\Lambda_L,M}$ the one-site marginal of the measure $\mu^{\mathbf{h}}_{\Lambda_L,M}$. We want to show that there exists a constant C_0, independent of λ, such that

$$\int H(a)^2 \log H(a)^2 \mu^{\mathbf{h},1}_{\Lambda_L,M}(da) \leq C_0 \int [H'(a)]^2 \mu^{\mathbf{h},1}_{\Lambda_L,M}(da) \qquad (1.2)$$

for all smooth functions $H\colon \mathbb{R} \to \mathbb{R}$ such that $< H^2 >_{\mu^{\mathbf{h},1}_{\Lambda_L,M}} = 1$. Since the potencial V is a bounded pertubation of the Gaussian potencial, by Corollary 6.2.45 in [DS], the previous inequality holds with a constant C_0 that might depend on $\mathbf{h}$. All the matter here is to show that we may find a finite constant independent of $\mathbf{h}$. A change of variables permits to rewrite the left hand side of (1.2) as

$$\int H_\lambda(a)^2 \log H_\lambda(a)^2 e^{-F_\lambda(a)} \frac{1}{\sqrt{2}} e^{-\frac{a^2}{2}} da$$

where $H_\lambda(a) = H(a+\lambda)$, $F_\lambda(a) = F(a+\lambda) + \log Z(\lambda)$ and $Z(\lambda)$ is a normalizing constant. It is easy check that $\left\| \pm e^{F_\lambda} \right\|_\infty \leq e^{2\|F\|_\infty}$. In particular, by Corollary 6.2.45 in [DS], the previous expression is bounded above by

$$2e^{4\|F\|_\infty} \int H_\lambda(a)^2 e^{-F_\lambda(a)} \frac{1}{\sqrt{2}} e^{-\frac{a^2}{2}} da = 2e^{4\|F\|_\infty} \int [H'(a)]^2 \mu^1{}_\lambda(da)$$

This prove the lemma in the case of grand canonical measures with $C_0 = 2e^{4\|F\|_\infty}$.

For canonical measures, we just need to use the local central limit theorem for large values of L and explicit computations for small values of L. We start with the case of large values of L. Fix a smooth function $H\colon \mathbb{R} \to \mathbb{R}$ with $< H^2 >_{\mu^{\mathbf{h},1}_{\Lambda_L,M}} = 1$ and recall the notation introduced in the proof of Lemma 1.2.1.

We base our proof on two facts. First, that if a function W is strictly convex then the measure $\mu_W(dx) = Z^{-1} \exp\{-W(x)\} dx$ associated to the potential W satisfies a logarithmic Sobolev inequality. Secondly, if $\mu(dx)$ satisfies a logarithmic Sobolev inequality, and f is a density with respect to μ, which is bounded below and above ($0 < C_1 \leq f \leq C_1^{-1}$), then $f d\mu$

satisfies a logarithmic Sobolev inequality. The proof of these two well known sentences can be found, for instance, in [14].

In view of these statements, we just need to show that the above density is equivalent to the density of a measure associated to a convex potential. Here and below two functions g, f are said to be equivalent, $f \sim g$, if there exists a finite, strictly positive constant C_0 depending only on V (and not on $\mathbf{h}$, M, λ or L) such that $C_0 g \le f \le C_0^{-1} g$. We shall rely on the local central limit theorem to show the equivalence of the above density with some density associated to a convex potential.

For λ in $\mathbb{R}$, denote by $P_{\mathbf{h},\lambda}$ the probability measure on the product space $\mathbb{R}^{\mathbb{N}}$ that makes the coordinates $\{X_k,\, k \ge 1\}$ independent random variables with X_k having density $Z(\lambda - h_k)^{-1} \exp\{[\lambda - h_k]x - V(x)\}$. Denote by $E_{\mathbf{h},\lambda}$ expectation with respect to $P_{\mathbf{h},\lambda}$.

Denote by $\gamma_1(\lambda)$, $\sigma^2(\lambda)$, $\{\gamma_k(\lambda),\, k \ge 3\}$ the expectation, the variance and the k-th truncated moment of a random variable with density $Z(\lambda)^{-1} \exp\{\lambda x - V(x)\}$:

$$
\begin{aligned}
\gamma_1(\lambda) \;&=\; \frac{1}{Z(\lambda)} \int_{\mathbb{R}} x\, e^{\lambda x - V(x)}\, dx\ , \\
\sigma^2(\lambda) \;&=\; \frac{1}{Z(\lambda)} \int_{\mathbb{R}} [x - \gamma_1(\lambda)]^2\, e^{\lambda x - V(x)}\, dx\ , \\
\gamma_k(\lambda) \;&=\; \frac{1}{Z(\lambda)} \int_{\mathbb{R}} [x - \gamma_1(\lambda)]^k\, e^{\lambda x - V(x)}\, dx\ . \qquad (1.3)
\end{aligned}
$$

For a finite subset Λ of $\mathbb{N}$, denote by $f_{\mathbf{h},\lambda,\Lambda}$, $g_{\mathbf{h},\lambda,\Lambda}$ the density of the random variable

$$
\frac{1}{\sigma^2(\mathbf{h},\lambda,\Lambda)^{1/2}} \sum_{j\in\Lambda} [X_j - \gamma_1(\lambda - h_j)]\ , \quad \sum_{j\in\Lambda} [X_j - \gamma_1(\lambda - h_j)]\ ,
$$

respectively. In this formula,

$$\sigma^2(\mathbf{h}, \lambda, \Lambda) \ = \ \sum_{j\in\Lambda} \sigma^2(\lambda - h_j) \ .$$

Take $A^2 = H^2 \log H^2$. The left hand side of (1.1) can be written as

$$E_{\mu^{h,1}_{\Lambda_L,M}}[A(\eta_1)^2] = \frac{1}{Z(\lambda-h_1)} \frac{\int A(x_1)^2 1_{\sum_{j=1}^L x_j = M} e^{\sum_{j=1}^L V_\lambda^{h_j}(x_j)} dx_1...dx_{L-1}}{\int 1_{\sum_{j=1}^L x_j = M} e^{\sum_{j=1}^L V_\lambda^{h_j}(x_j)} dx_1...dx_{L-1}}$$

$$= \frac{1}{Z} \frac{\int dx_1 A(x_1)^2 e^{(\lambda-h_1)x - V(x_1)} \int_{x_2+...+x_L = M-x_1} f_{\lambda,h,2}(x_2)...f_{\lambda,h,L}(x_L) dx_2...dx_{L-1}}{\int_{x_1+...+x_L=M} f_{\lambda,h,1}(x_1)...f_{\lambda,h,L}(x_L) dx_1...dx_{L-1}}$$

In view of Remark 1.2.3, the previous expression can be writte

$$= \frac{1}{Z(\lambda-h_1)} \int dx_1 A(x_1)^2 e^{(\lambda-h_1)x_1 - V(x_1)} \frac{f_{\sum_{j=2}^L X_j}(M - x_1)}{f_{\sum_{j=1}^L X_j}(M)}.$$

$$= \frac{1}{Z(\lambda-h_1)} \int dx_1 A(x_1)^2 e^{(\lambda-h_1)x_1 - V(x_1)} \frac{f_{\sum_{j=2}^L (X_j - \gamma_1(\lambda-h_j))}(\gamma_1(\lambda-h_1) - x_1)}{f_{\sum_{j=1}^L (X_j - \gamma_1(\lambda-h_j))}(0)}.$$

$$= \frac{1}{Z(\lambda-h_1)} \int H(a)^2 \log H(a)^2 \, e^{[\lambda-h_1]a - V(a)} \frac{g_{\mathbf{h},\lambda,\Lambda_{2,L}}(\gamma_1(\lambda-h_1) - a)}{g_{\mathbf{h},\lambda,\Lambda_L}(0)} \, da \ , \tag{1.4}$$

where λ is chosen according to (1.2.3).

For L large enough $g_{\mathbf{h},\lambda,\Lambda_L}(0)$ is of order $L^{-1/2}$. In fact

$$g_{\mathbf{h},\lambda,\Lambda_L}(0) = \frac{1}{\sigma(\mathbf{h},\lambda,\Lambda_L)} f_{\mathbf{h},\lambda,\Lambda_L}(0) \geq C_0 L^{\frac{-1}{2}}$$

by Lemma 1.3.1.

We may therefore replace the denominator in the previous integral by $C_0 L^{-1/2}$. Choose μ according to

$$M \;=\; \sum_{j=1}^{L} \gamma_1(\lambda - h_j) \;=\; \sum_{j=2}^{L} \gamma_1(\mu - h_j) \;+\; a\,. \tag{1.5}$$

The numerator in (1.4) is equal to

$$\int 1_{\sum_{j=2}^{L} x_j = M-a} \prod_{j=2}^{L} \frac{1}{Z(\lambda - h_j)} e^{(\lambda - h_j)x_j - V_j(x_j)} dx_1 ... dx_{L-1}$$

$$= \int 1_{\sum_{j=2}^{L} x_j = M-a} \prod_{j=2}^{L} \frac{Z(\mu - h_j)}{Z(\lambda - h_j)} e^{(\lambda-\mu)x_j} e^{(\mu - h_j)x_j - V_j(x_j)} Z(\mu - h_j) dx_2 ... dx_L$$

$$= \prod_{j=2}^{L} \frac{Z(\mu - h_j)}{Z(\lambda - h_j)} e^{(\lambda-\mu)(M-a)} \tilde{g}_{\mathbf{h},\lambda,\Lambda_L}(M-a)$$

$$= \prod_{j=1}^{L-1} \frac{Z(\mu - h_j)}{Z(\lambda - h_j)} e^{(\lambda-\mu)(M-a)} g_{\mathbf{h},\lambda,\Lambda_L}(M - a - \sum_{j=1}^{L-1} \gamma_1(\mu - h_x))$$

$$= \prod_{j=2}^{L} \frac{Z(\mu - h_j)}{Z(\lambda - h_j)} e^{(\lambda-\mu)(M-a)} g_{\mathbf{h},\mu,\Lambda_{2,L}}(0)\,.$$

By Theorem 3.1.2, $g_{\mathbf{h},\mu,\Lambda_{2,L}}(0)$ is of order $L^{-1/2}$ for L large enough. We need therefore to show that

$$\exp\left\{(\lambda-h_1)a-V(a)+(\lambda-\mu)(M-a)+\sum_{j=2}^{L}\log Z(\mu-h_j)-\sum_{j=1}^{L}\log Z(\lambda-h_j)\right\}$$

is equivalent to a density associated to a strictly convex potential. The expression $\exp\{(\lambda-h_1)a-V(a)-\log Z(\lambda-h_1)\}$ is equivalent to a $\exp\{-(a-\lambda+h_1)^2/2\}$ because

$$\begin{aligned} f_{\mathbf{h},\lambda,1}(a) &= \frac{1}{Z(\lambda-h_1)}e^{(\lambda-h_1)a-\frac{a^2}{2}-F(a)} \\ &= \frac{e^{\frac{(\lambda-h_1)^2}{2}}}{Z(\lambda-h_1)}e^{\frac{-(a-(\lambda-h_1))^2}{2}-F(a)} \\ &= C_0 e^{\frac{-(a-\lambda-\lambda_L)^2}{2}} \end{aligned}$$

by Lemma 3.1.1.

By the estimates above, we may replace $\exp\{(\lambda-h_1)a-V(a)-\log Z(\lambda-h_1)\}$ by $\exp\{-(a-\lambda+h_1)^2/2\}$. Recall identity (1.5) and define Θ by

$$\begin{aligned} \Theta(a) &= \frac{1}{2}(a-\lambda+h_1)^2 \;-\; (\lambda-\mu)(M-a) \;-\; \sum_{j=2}^{L}\log\frac{Z(\mu-h_j)}{Z(\lambda-h_j)} \\ &= \frac{1}{2}(a-\lambda+h_1)^2 \;-\; (\lambda-\mu)\sum_{j=2}^{L}\gamma_1(\mu-h_j) \;-\; \sum_{j=2}^{L}\log\frac{Z(\mu-h_j)}{Z(\lambda-h_j)} \end{aligned}$$

$$= \frac{1}{2}(a-\lambda+h_1)^2-(\lambda-\mu)\sum_{j=1}^{L-1}\gamma_1(\mu-h_j)-\sum_{j=1}^{L-1}\Big[\log Z(\mu-h_j)-\log Z(\lambda-h_j)\Big].$$

It remains to shows that Θ is strictly convex. Since, $Z'(\lambda)/Z(\lambda) = \gamma_1(\lambda)$. We have,

$$(\partial\Theta)(a) = a-\lambda+h_1+\partial_a\mu\sum_{j=1}^{L-1}\gamma_1(\mu-h_j)-(\lambda-\mu)\sum_{j=1}^{L-1}\gamma'{}_1(\mu-h_j)\partial_a\mu-\partial_a\mu\sum_{j=1}^{L-1}\gamma_1(\mu-h_j).$$

$$= a-\lambda+h_1+\lambda-\mu = a-\mu+h_1.$$

because

$$\partial_a\mu\sum_{j=2}^{L}\gamma_1'(\mu-h_j) \;=\; -1$$

and

$$\partial_a\sum_{j=1}^{L-1}\log Z(\mu-h_j) = \sum_{j=1}^{L-1}\frac{1}{Z(\mu-h_j}Z'(\mu-h_j)\partial_a\mu$$
$$= \partial_a\mu\sum_{j=1}^{L-1}\gamma_1(\mu-h_1).$$

in view of (1.5). In particular, since $\gamma_1'(\lambda) = \sigma^2(\lambda)$,

$$(\partial_a^2\Theta)(a) \;=\; 1-\partial_a\mu$$

$$= 1 + \frac{1}{\sum_{j=1}^{L-1} {\gamma'}_1(\mu - h_j)}$$

$$= \ 1 \ + \ \frac{1}{\sum_{j=2}^{L} \sigma^2(\mu - h_j)} \ .$$

Therefore, Θ is strictly convex for L large enough. This proves the lemma in the canonical case for large values of L.

We now turn to the case of small values of L. Recall the notation introduced just before Lemma 3.1.7. Choose λ so that $M = \sum_{1\le j\le L}(\lambda - h_j)$. The Radon-Nikodym derivative of $\mu^{\mathbf{h},1}_{\Lambda_L,M}$ with respect to the Lebesgue measure can be written as

$$\frac{1}{Z(\lambda - h_1)} e^{[\lambda - h_1]a - V(a)} \frac{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M-a)}{\tilde{g}_{\mathbf{h},\lambda,\Lambda_{2,L}}(M)}$$

$$= \frac{\sqrt{L}}{\sqrt{L-1}} \frac{1}{Z(\lambda - h_1)} e^{[\lambda - h_1]a - V(a)} \frac{\tilde{f}_{\mathbf{h},\lambda,\Lambda_{2,L}}\Big(\frac{\lambda - h_1 - a}{\sqrt{L-1}}\Big)}{\tilde{f}_{\mathbf{h},\lambda,\Lambda_L}(0)} \ .$$

By Lemma 3.1.7, the denominator is bounded above by C_1^L for some finite constants C_1, while the numerator is bounded by $C_1^L \exp\{-(\lambda - h_1 - a)^2/2(L-1)\}$. Similar lower bounds can be obtained. Therefore, the measure $\mu^{\mathbf{h},1}_{\Lambda_L,M}$ is equivalent to the density associated to the strictly convex potential $(1/2)(L/L-1)(a - \lambda + h_1)^2$. This proves the lemma for small values of L. □

We now obtain a recursive formula for $\theta(L, \mathbf{h})$ in terms of $\theta(L-1, \mathbf{h})$, L. Assume that $\theta(K, \mathbf{h}) < \infty$ for $2 \le K \le L-1$.

2.2 Decomposition of the Entropy

Use an elementary property of the conditional expectation to decompose the entropy as

$$S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f) = \int f^2 \log \frac{f^2}{E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L]} \, d\mu^{\mathbf{h}}_{\Lambda_L,M} \qquad (2.1)$$

$$+ \int E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L] \log E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L] \, \mu^{\mathbf{h},L}_{\Lambda_L,M}(d\eta_L) \; .$$

Here $\mu^{\mathbf{h},L}_{\Lambda_L,M}$ stands for the one-site marginal of $\mu^{\mathbf{h}}_{\Lambda_L,M}$ at η_L.

The first term on the right hand side of (2.1) is estimated through the induction assumption. Indeed, taking conditional expectation with respect to η_L, we may rewrite this integral as

$$\int E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}} \Big[\frac{f^2}{E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L]} \log \frac{f^2}{E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L]} \Big] E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L] \, \mu^{\mathbf{h},L}_{\Lambda_L,M}(d\eta_L) \; .$$

Since the integral of $f^2/E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L]$ with respect to $\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}$ is equal to 1, the previous expression is bounded above by

$$\theta(L-1,\mathbf{h}) \int D_{\Lambda_{L-1}} \Big(\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}, f/E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L]^{1/2} \Big) E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L] \, d\mu^{\mathbf{h},L}_{\Lambda_L,M}(\eta_L) \; .$$

A direct computation shows that this expression is less than or equal to

$$\theta(L-1,\mathbf{h}) D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f) \; . \qquad (2.2)$$

The second term in (2.1) is estimated through Lemma 2.1.1. Let $H(\eta_L) = E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L]^{1/2}$. By Lemma 2.1.1, the second term on the right hand side of (2.1) is bounded above by

$$C_0 E_{\mu^{\mathbf{h},L}_{\Lambda_L,M}}\Big[\Big(\frac{\partial E_{\Lambda_L,M}[f^2|\eta_L]^{1/2}}{\partial\eta_L}\Big)^2\Big]\ .$$

A computation, similar to the one performed in (4.1), shows that $(\partial H/\partial\eta_L)^2$ is equal to

$$\frac{1}{4E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L]}\Big\{\frac{1}{L-1}\sum_{x=1}^{L-1} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{\partial f^2}{\partial\eta_L}-\frac{\partial f^2}{\partial\eta_x}\,\Big|\,\eta_L\Big] \tag{2.3}$$

$$+\, E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[f^2;\frac{1}{L-1}\sum_{x=1}^{L-1} V'(\eta_x)\,\Big|\,\eta_L\Big]\Big\}^2\ .$$

Following the computation presented just after (4.1), we obtain by Schwarz inequality, that

$$\frac{1}{4E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L]}\Big\{\frac{1}{L-1}\sum_{x=1}^{L-1} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{\partial f^2}{\partial\eta_L}-\frac{\partial f^2}{\partial\eta_x}\,\Big|\,\eta_L\Big]\Big\}^2 \tag{2.4}$$

$$\le\ C_0 L\sum_{x=1}^{L-1} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[(T^{x,x+1}f)^2\,\Big|\,\eta_L\Big]$$

for some finite universal constant C_0. We have thus a bound on the first term in (2.3).

The analysis of the second term on the right hand side of (2.3) is more demanding and is the main goal of section 3 and 4.

2.3 Bounds on the Glauber dynamics, small values of L

We first replace $V'(\eta_x)$ by $F'(\eta_x)$ because $\sum_{1\le y\le L-1}\eta_y$ is fixed for the measure $E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[\,\cdot\,|\,\eta_L]$. The following lemma will be particularly useful.

Lemma 2.3.1 *There exists a finite constant C_2 depending only on $\|F''\|_\infty$ such that*

$$E_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}}\Big[g^2;\frac{1}{L}\sum_{x=1}^{L}F'(\eta_x)\Big]^2 \;\le\; \frac{C_2\theta(L,\boldsymbol{h})}{L}\sum_{x=1}^{L-1}E_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}}\Big[(T^{x,x+1}g)^2\Big] \qquad (3.1)$$

for all $L\ge 2$, M in $\mathbb{R}$, environment $\boldsymbol{h}$ and smooth functions g in $L^2(\mu^{\boldsymbol{h}}_{\Lambda_L,M})$ such that $< g^2 >_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}} = 1$.

Proof: Denote by $\tilde F_{L,M}(\eta_x)$ the function $F'(\eta_x) - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[F'(\eta_x)]$. With this notation,

$$E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2;\frac{1}{L}\sum_{x=1}^{L}F'(\eta_x)\Big] \;=\; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2\frac{1}{L}\sum_{x=1}^{L}\tilde F_{L,M}(\eta_x)\Big]\,.$$

By the entropy inequality, this expression is bounded above by

$$\frac{1}{\beta L}\log\int \exp\Big\{\beta\sum_{x=1}^{L}\tilde F_{L,M}(\eta_x)\Big\}\, d\mu^{\mathbf{h}}_{\Lambda_L,M} \;+\; \frac{1}{\beta L} S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)$$

for every $\beta > 0$. By Proposition 5.1.1, the first term is bounded above by $C_2\beta$ for some finite constant C_2 that depends only on $\|F''\|_\infty$. Minimizing over $\beta > 0$ we obtain that the left hand side of (3.1) is bounded above by

$$C_2 L^{-1} S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)\ .$$

By definition of $\theta(L,\mathbf{h})$, this expression is less than or equal to the right hand side of (3.1). □

It follows from Lemma 2.3.1 applied to the measure $\mu^{\mathbf{h}}_{L-1,M-\eta_L}$ and to the function $g^2 = f^2/E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L]$ that the second term of (2.3) is bounded above by

$$\frac{C_2\theta(L-1,\mathbf{h})}{L}\sum_{x=1}^{L-2} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[(T^{x,x+1}f)^2\,\Big|\eta_L\Big]\ .$$

Taking expectation with respect to $\mu^{\mathbf{h}}_{\Lambda_L,M}$ in this formula and in (2.4), we obtain that the expectation of (2.3) is less than or equal to

$$C_2\Big\{L \;+\; L^{-1}\theta(L-1,\mathbf{h})\Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)\ .$$

The second term of (2.1), which is bounded by the expectation with respect to $\mu^{\mathbf{h}}_{\Lambda_L,M}$ of (2.3), is less than or equal to the same expression. Therefore, in view of (2.2),

$$S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f) \ \le\ \Big\{C_0 L + (1 + C_2 L^{-1})\theta(L-1, \mathbf{h})\Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)\ .$$

In particular, by definition of $\theta(L, \mathbf{h})$,

$$\theta(L, \mathbf{h}) \ \le\ C_0 L + (1 + C_2 L^{-1})\theta(L-1, \mathbf{h})\ .$$

This relation, together with the bound $\theta(2, \mathbf{h}) \le C_0$, proved in the first section, gives that $\theta(L, \mathbf{h}) < L^t$ for some finite t, which depends on C_2. Notice that this estimate is uniform over the environment $\mathbf{h}$.

2.4 Bounds on the Glauber dynamics, large values of L

We now give an alternative estimate of the second term of (2.3) that we shall use for large values of L.

Proposition 2.4.1 *Fix $\delta > 0$. There exists a finite constant C_2 and a set of environments Ω_*, which has $\mathbb{P}$ probability one, such that for any $\boldsymbol{h}$ in Ω_*, there exists $L_* = L_*(\boldsymbol{h})$ such that for all $L \ge L_*$,*

$$\Big(E_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}}\Big[g^2; \frac{1}{L}\sum_{x=1}^{L} F'(\eta_x)\Big]\Big)^2 \ \le\ \Big\{C_2 L + \frac{\delta\theta(L, \boldsymbol{h})}{L}\Big\} D_{\Lambda_L}(\mu^{\boldsymbol{h}}_{\Lambda_L,M}, g) \quad (4.1)$$

for all M in $\mathbb{R}$ and functions g in $L^2(\mu^{\boldsymbol{h}}_{\Lambda_L,M})$ such that $< g^2 >_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}} = 1$.

We first assume this result to conclude the proof of Theorem 1.1.2.

Proof of Theorem 1.1.2. Recall the decomposition (2.1) of the entropy and the estimate (2.2). The second term on the right hand side of (2.1) was estimated by Lemma 2.1.1, giving (2.3). The first term of (2.3) was bounded by (2.4). Fix $\delta < 2$. By Proposition 2.4.1 applied to the measure $\mu^{\mathbf{h}}_{L-1,M-\eta_L}$ and the function $g^2 = f^2/E_{\Lambda_L,M-\eta_L}[f^2|\eta_L]$, there exists a set Ω_* of $\mathbb{P}$ probability one with the following property. For each $\mathbf{h}$ in Ω_*, there exists $L_* = L_*(\mathbf{h})$ such that for $L \ge L_*$, the second term in (2.3) is bounded above by

$$\Big\{C_2 L + \frac{\delta\theta(L-1,\mathbf{h})}{L-1}\Big\} D_{\Lambda_{L-1}}\Big(\mu^{\mathbf{f}}_{\Lambda_{L-1},M-\eta_L}, f/E_{\mu^{\mathbf{h}}_{\Lambda_L,M-\eta_L}}[f^2|\eta_L]^{1/2}\Big)$$

for some finite constant C_2 and all M, η_L. Taking expectations with respect to $\mu^{\mathbf{h}}_{\Lambda_L,M}$ in (2.3), we obtain that the second term in (2.1) is less than or equal to

$$\Big\{C_2 L + \frac{\delta\theta(L-1,\mathbf{h})}{L-1}\Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)\ .$$

In particular, by (2.2) and (2.1),

$$S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f) \ \le\ \Big\{C_2 L + (1 + \frac{\delta}{L-1})\theta(L-1,\mathbf{h})\Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)$$

or, by definition of $\theta(L,\mathbf{h})$,

$$\theta(L,\mathbf{h}) \ \le\ \Big\{C_2 L + (1 + \frac{\delta}{L-1})\theta(L-1,\mathbf{h})\Big\}\ .$$

Since $\delta < 2$, it is easy to derive form this inequality the existence of a finite constant $C(\mathbf{h})$ such that $\theta(L,\mathbf{h}) \le C(\mathbf{h})L^2$ for all $L \ge 2$. The constant may depend on $\mathbf{h}$ because ℓ_* depends on the environment. This concludes the proof of Theorem 1.1.2. $\square$

We now turn to the proof of Proposition 2.4.1. For clarity reasons, we divide it in several lemmas. We first repeat the procedure presented in Step 4 of the previous section. Fix $K \geq 1$ and divide the interval $\{1, \dots, L\}$ into $\ell = \lfloor L/K \rfloor$ adjacent intervals of length K or $K+1$. Denote by I_j the j-th interval and by M_j the total spin on I_j: $M_j = \sum_{x \in I_j} \eta_x$. As for the proof of the spectral gap, K will be large but fixed, while ℓ will increase to infinity. The left hand side of (4.1) is bounded above by

$$2\Big(E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2; \frac{1}{L}\sum_{j=1}^{\ell}\Big\{\sum_{x\in I_j} F'(\eta_x) - E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\sum_{x\in I_j} F'(\eta_x)\Big]\Big\}\Big]\Big)^2 \qquad (4.2)$$

$$2\Big(E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2; \frac{1}{L}\sum_{j=1}^{\ell}|I_j| E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\Big]\Big)^2 .$$

Lemma 2.4.2 *There exists a finite constant C_2 such that*

$$\Big(\frac{1}{L}\sum_{j=1}^{\ell} E_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}}\Big[g^2; \Big\{\sum_{x\in I_j} F'(\eta_x) - E_{\mu^{\boldsymbol{h}}_{I_j,M_j}}\Big[\sum_{x\in I_j} F'(\eta_x)\Big]\Big\}\Big]\Big)^2$$
$$\leq \frac{C_2 K^t}{L} D_{\Lambda_L}(\mu^{\boldsymbol{h}}_{\Lambda_L,M}, g)$$

for all $K \geq 2$, $L \geq K^2$, M in $\mathbb{R}$, environment $\boldsymbol{h}$ and smooth functions g in $L^2(\mu^{\boldsymbol{h}}_{\Lambda_L,M})$ such that $< g^2 >_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}} = 1$.

Proof: Taking conditional expectation with respect to M_j, $\{\eta_x, x \notin I_j\}$, we rewrite the left hand side of the inequality presented in the statement of the lemma as

$$\Big(\frac{1}{L}\sum_{j=1}^{\ell} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2]\, E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[g_j^2; \sum_{x\in I_j} F'(\eta_x)\Big]\,\Big]\Big)^2 \tag{4.3}$$

$$\le \frac{\ell}{L^2}\sum_{j=1}^{\ell} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2]\Big(E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[g_j^2; \sum_{x\in I_j} F'(\eta_x)\Big]\Big)^2\,\Big]\;,$$

where $g_j^2 = g^2/E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2]$ has mean one with respect to $\mu^{\mathbf{h}}_{I_j,M_j}$. In the expression $E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2]$, it must be understood that only the variables η_x for x in I_j are integrated so that $E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2] = E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[g^2|M_j, \{\eta_x,\, x\notin I_j\}]$. In the last step we used Schwarz inequality and the fact $E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2]] = 1$. Fix $1 \le j \le \ell$. By the entropy inequality, $E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g_j^2; \sum_{x\in I_j} F'(\eta_x)]$ is bounded above by

$$\frac{1}{\beta}\log\int e^{\beta\sum_{x\in I_j} F_j(\eta_x)}\, d\mu^{\mathbf{h}}_{I_j,M_j} \;+\; \frac{1}{\beta} S_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j}, g_j)$$

for every $\beta > 0$. Here, $F_j(\eta_x) = F'(\eta_x) - E_{\mu^{\mathbf{h}}_{I_j,M_j}}[F']$. By definition of θ, the second term is bounded above by $\theta(|I_j|, \tau_j\mathbf{h})\beta^{-1}D_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j}, g_j)$, where $\tau_j\mathbf{h}$ stands for the translation of the environment $\mathbf{h}$ for the origin to be at the left end of the interval I_j. By the a-priori estimate on θ obtained in Step 3, this expression is less than or equal to $C_2|I_j|^t\beta^{-1}D_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j}, g_j)$. On the other hand, by Proposition 5.1.1, the first one is bounded above by $C_0\beta|I_j|$ for some finite constant C_0. Minimizing over β and summing over j, we get that (4.3) is less than or equal to

$$\frac{C_2}{L}\sum_{j=1}^{\ell} |I_j|^t E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2]\, D_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j}, g_j)\Big] \;\le\; \frac{C_2K^t}{L} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)\;.$$

This conclude the proof of lemma. □

We turn now to the second term of (4.2). Let $a_j = |I_j|/L$ and recall the definition of A_j given in (5.9). Since we may add constants in a covariance, the expectation in the second term of (4.2) is equal to

$$E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2; \sum_{j=1}^{\ell} a_j G_j\Big] \;+\; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2; \sum_{j=1}^{\ell} a_j A_j'(m_j)\Big(\frac{M_j}{|I_j|} - m_j\Big)\Big]\;.$$

To estimate this covariance we need to consider two cases. Let β_0, be the constant given by Proposition 5.1.6 and fix $0 < \delta < 2$. By Proposition 5.1.6, there exists $K_0 \geq 2$ for which the left hand side of (1.12) is bounded by $\delta\beta$ for all $\beta \leq \beta_0$ and all $K \geq K_0$, $L \geq K^2$, M in $\mathbb{R}$ and environment $\mathbf{h}$.

Lemma 2.4.3 *Fix $L \geq K^2 \geq K_0^2$, M in $\mathbb{R}$, an environment $\boldsymbol{h}$ and a smooth function g in $L^2(\mu^{\boldsymbol{h}}_{\Lambda_L,M})$ such that $< g^2 >_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}} = 1$. Assume that $\theta(L,\boldsymbol{h})L^{-1}D_{\Lambda_L}(\mu^{\boldsymbol{h}}_{\Lambda_L,M}, g) < \delta\beta_0^2$. Then,*

$$\Big(E_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}}\Big[g^2; \sum_{j=1}^{\ell} a_j G_j\Big]\Big)^2 \;\leq\; \frac{\delta\theta(L,\boldsymbol{h})}{L} D_{\Lambda_L}(\mu^{\boldsymbol{h}}_{\Lambda_L,M}, g)\;.$$

Proof: Fix a density g^2 satisfying the assumptions. By the entropy inequality, the expectation in the statement of the lemma is bounded by

$$\frac{1}{\beta L}\log E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\exp\Big\{\beta\sum_{j=1}^{\ell}|I_j|G_j\Big\}\Big] \;+\; \frac{1}{\beta L}S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) \tag{4.4}$$

for every $\beta > 0$. By Proposition 5.1.6 and our choice of K, L, the first term is bounded above by $\delta\beta$ for all $\beta < \beta_0$. The second one, by definition of θ, is bounded above by $(\theta(L,\mathbf{h})/\beta L)D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)$. Therefore, (4.4) is less than or equal to

$$\delta\beta \;+\; \frac{\theta(L,\mathbf{h})}{\beta L} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)$$

for all $\beta < \beta_0$. The value of β that minimizes this expression is

$$\beta_*^2 \;=\; \frac{\theta(L,\mathbf{h})}{\delta L} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)\;.$$

By hypothesis, $\beta_* < \beta_0$ and we may therefore minimize in $\beta < \beta_0$ to obtain that the square of (4.4) is bounded above by

$$\frac{\delta\theta(L,\mathbf{h})}{L} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)\;,$$

which concludes the proof of the lemma. □

Recall the explicit formula for $A'_j(m^*_j)$ given in Remark 1.5.2. Let $c = c(\lambda) = \mathbb{E}[\sigma^2(\lambda - h_x)]^{-1} - 1$. Since $\sigma^2(\cdot)$ is bounded above and below by strictly positive constants,

$$\frac{1}{\ell}\sum_{j=1}^{\ell} |I_j|(A'_j(m^*_j) - c)^2 \;\le\; \frac{C'_0}{\ell}\sum_{j=1}^{\ell}|I_j|\Big(\frac{1}{|I_j|}\sum_{x\in I_j}\sigma^2(\lambda - h_x) - \tilde\sigma^2(\lambda)\Big)^2\;.$$

Fix $\delta > 0$. Let C^* be the constant given by (4.1.6) with σ^2 in place of U. Let $K_1 \ge C_0 \exp\{C_0 C'_0 C^*\}\delta^{-1}$, where C_0 is the constant appearing in Lemma 5.1.7 and C'_0 the one in the previous formula. Fix $K \ge K_1$ and let Ω_1 be the set associated to $(K, \sigma^2(\cdot))$ through (4.1.6). Fix an environment $\mathbf{h}$ in Ω_1. By (4.1.6), there exists $\ell_0 = \ell_0(\mathbf{h})$, such that the right hand side of (5.1.20) is bounded by $\delta\beta$ for all $\ell \ge \ell_0$. The next result follows from these observations and the arguments presented in the proof of the previous lemma.

Lemma 2.4.4 *There exists a constant K_1 such that for all $K \geq K_1$, there exists a set $\Omega_1 = \Omega_1(K)$ with the following property. For all $\boldsymbol{h}$ in Ω_1, there exists $\ell_1 = \ell_1(K, \boldsymbol{h})$ such that for $\ell \geq \ell_1$, M in $\mathbb{R}$ and a smooth function g in $L^2(\mu^{\boldsymbol{h}}_{\Lambda_L,M})$ such that $< g^2 >_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}} = 1$,*

$$\Big(E_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}}\Big[g^2; \sum_{j=1}^{\ell} a_j A_j'(m_j)\Big(\frac{M_j}{|I_j|} - m_j\Big)\Big]\Big)^2 \;\leq\; \frac{\delta\theta(L,\boldsymbol{h})}{L} D_{\Lambda_L}(\mu^{\boldsymbol{h}}_{\Lambda_L,M}, g)$$

if $\theta(L,\boldsymbol{h})L^{-1}D_{\Lambda_L}(\mu^{\boldsymbol{h}}_{\Lambda_L,M}, g) < \delta\beta_0^2$.

It remains to consider the case where the Dirichlet form of g is large.

Lemma 2.4.5 *Fix $\delta > 0$. There exist $K_2 = K_2(\delta)$ and a finite constant C_2 with the following property. For each $K \geq K_2$, there exists a set of environments $\Omega_2 = \Omega_2(K)$ with $\mathbb{P}$ probability one such that for each $\boldsymbol{h}$ in Ω_2, there exists $\ell_2 = \ell_2(\boldsymbol{h})$, such that for all $\ell \geq \ell_2$,*

$$\Big(E_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}}\Big[g^2; \sum_{j=1}^{\ell} a_j G_j\Big]\Big)^2 \;\leq\; \Big\{\frac{\delta\theta(L,\boldsymbol{h})}{L} \;+\; C_2 K^t L\Big\} D_{\Lambda_L}(\mu^{\boldsymbol{h}}_{\Lambda_L,M}, g) \tag{4.5}$$

for all M in $\mathbb{R}$ and smooth function g in $L^2(\mu^{\boldsymbol{h}}_{\Lambda_L,M})$ satisfying $< g^2 >_{\mu^{\boldsymbol{h}}_{\Lambda_L,M}} =$ 1,

$$\theta(L,\boldsymbol{h})L^{-1}D_{\Lambda_L}(\mu^{\boldsymbol{h}}_{\Lambda_L,M}, g) \;\geq\; \delta\beta_0^2\ .$$

Proof: The covariance $E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[g^2; \sum_{1\leq j\leq \ell} a_j G_j]$ is equal to the covariance of g^2 and $\sum_{1\leq j\leq \ell} a_j H_j$, where $H_j = E_{\mu^{\mathbf{h}}_{\Lambda_{I_j},M_j}}[|I_j|^{-1}\sum_{x\in I_j} F'(\eta_x)]$. Since g^2

is a density with respect to $\mu^{\mathbf{h}}_{\Lambda_L,M}$, by Schwarz inequality, the left hand side of (4.5) is bounded above by

$$2\Big(\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2\Big(H_j - \tilde{H}_j\Big)\Big]\Big)^2 \;+\; 2\Big(\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[H_j - \tilde{H}_j\Big]\Big)^2 \,, \quad (4.6)$$

where $\tilde{H}_j = E_{\mu^{\mathbf{h}}_{\Lambda_{I_j},m|I_j|}}[|I_j|^{-1}\sum_{x\in I_j} F'(\eta_x)]$, $m = M/L$. Let $\bar{M}_j = M_j/|I_j|$.

Since g^2 is a density, and since by Lemma 5.1.8 $R(\theta) = E_{\mu^{\mathbf{h}}_{\Lambda_{I_j},\theta|I_j|}}[|I_j|^{-1}\sum_{x\in I_j} F'(\eta_x)]$ is Lipschitz continuous, uniformly in L and $\mathbf{h}$, by Schwarz inequality the first term is bounded above by

$$C_0 \sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2[\bar{M}_j - m]^2\Big] \;\le\; C_0 \sum_{1\le i\neq j\le \ell} a_j\, a_k\, E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2[\bar{M}_j - \bar{M}_i]^2\Big]$$

for some finite constant C_0 because m is just the average of the densities m_i.

By Lemma 2.4.6 below, each expectation is bounded by

$$\begin{aligned} 6\Big(\Gamma_{I_i}(\lambda,\mathbf{h}) - \tilde{\Gamma}_1(\lambda)\Big)^2 \;+\; 6\Big(\Gamma_{I_j}(\lambda,\mathbf{h}) - \tilde{\Gamma}_1(\lambda)\Big)^2 \;+\; \frac{C_0}{K} \\ +\, 6\Big(W_{I_i}(\mathbf{h}) - \mathbb{E}[h_1]\Big)^2 \;+\; 6\Big(W_{I_j}(\mathbf{h}) - \mathbb{E}[h_1]\Big)^2 \\ +\, C_2 K^t\Big\{D_{I_i}(\mu^{\mathbf{h}}_{I_i,M_i}, g) + D_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j}, g) + E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big\{\frac{\partial g}{\partial \eta_{y_i}} - \frac{\partial g}{\partial \eta_{x_j}}\Big\}^2\Big]\Big\}\,. \end{aligned}$$

In this formula, Γ_Λ, $\tilde{\Gamma}_1$ and W_Λ are defined in (4.7). Here we are assuming that the cubes I_j are ordered, that $i < j$ and that y_i is the rightmost site in I_i and x_j is the leftmost site in I_j. An elementary computation shows that the expectation in the previous formula is bounded above by $LD_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)$. Therefore, the first term in (4.6) is less than or equal to

$$\frac{C_0}{K} \;+\; \frac{C_0}{\ell}\sum_{j=1}^{\ell}\Big(\Gamma_{I_j}(\lambda,\mathbf{h})-\tilde\Gamma_1(\lambda)\Big)^2 \;+\; \frac{C_0}{\ell}\sum_{j=1}^{\ell}\Big(W_{I_j}(\mathbf{h})-\mathbb{E}[h_1]\Big)^2$$

$$+\, C_2 K^t L D_{\Lambda_L}(\mu_{\Lambda^{\mathbf{h}}_L,M}, g)\ .$$

There exists K_2 large enough for C_0/K to be smaller than $\delta^2\beta_0^2/3$ and for the constant given by (4.1.5), with $\Gamma_1(\cdot)$ and $Id(\cdot)$ in place of $U(\cdot)$ to be smaller than $\delta^2\beta_0^2/6C_0$ (cf. the remark stated in the penultimate paragraph of Chapter 4). Here I_d stands for the identity and C_0 for the constant which appears in the previous formula. Fix $K \geq K_2$ and let Ω_2 be the set of environments given by (4.1.5) associated to $(K, \Gamma_1(\cdot), \delta^2\beta_0^2/6C_0)$ and to $(K, Id, \delta^2\beta_0^2/6C_0)$. For any $\mathbf{h}$ in Ω_2, there exists $\ell_2 = \ell_2(\mathbf{h})$ such that for $\ell \geq \ell_2$, the previous expression is bounded by

$$\delta^2\beta_0^2 \;+\; C_2 K^t L D_{\Lambda_L}(\mu_{\Lambda^{\mathbf{h}}_L,M}, g)\ .$$

Since, by assumption,

$$\theta(L,\mathbf{h})L^{-1}D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) \;\geq\; \delta\beta_0^2\ ,$$

the previous expression is less than or equal to

$$\Big\{\frac{\delta\theta(L,\mathbf{h})}{L} + C_2K_1^tL\Big\}D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)\ .$$

The second term in (4.6) is easier to estimate since one need just to take $g = 1$ in the previous argument. Replacing δ by $\delta/2$, we conclude the proof of the lemma. □

Proof of Proposition 2.4.1. Let $K_* = \max\{K_0, K_1, K_2\}$, where K_i are the integers given by Lemmas 2.4.3, 2.4.4, 2.4.5. Let $\Omega_* = \Omega_1 \cap \Omega_2$, where Ω_i are the sets of environments given by Lemmas 2.4.4, 2.4.5 associated to K_*. Fix an environment $\mathbf{h}$ in Ω_* and let $\ell_*(\mathbf{h}) = \max\{\ell_1(\mathbf{h}), \ell_2(\mathbf{h})\}$, where $\ell_1(\mathbf{h})$ are the positive constants given by Lemmas 2.4.4, 2.4.5. For $\ell \geq \ell_*$, in view of Lemmas 2.4.2, 2.4.3, 2.4.4, 2.4.5 and the the decomposition (4.2), the left hand side of (4.1) is bounded above by

$$\Big\{\frac{C_2 K_*^{t-1}}{\ell} + C_2 K_*^{t-1} L + \frac{8\delta\,\theta(L,\mathbf{h})}{L}\Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)\ .$$

Since K^* is fixed, this concludes the proof. □

We conclude this section with a technical result needed above. For a finite subset Λ of $\mathbb{Z}$, let

$$\Gamma_\Lambda(\lambda,\mathbf{h}) \;=\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\Gamma_1(\lambda - h_x)\ , \quad W_\Lambda(\mathbf{h}) \;=\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda} h_x \tag{4.7}$$

and let $\tilde\Gamma_1(\lambda) = \mathbb{E}[\Gamma_1(\lambda - h_1)]$. Consider a cube Λ_{2K} as the union of two intervals of size K. For $i = 1,\, 2$, denote by $\bar M_i$, the average spin over the i-th cube: $\bar M_1 = K^{-1}\sum_{1\le x\le K}\eta_x$, $\bar M_2 = K^{-1}\sum_{K+1\le x\le 2K}\eta_x$. Let

$$U_\Lambda(\lambda,\mathbf{h}) \;=\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\gamma_1(\lambda - h_x)\ .$$

Lemma 2.4.6 *There exist finite constants C_0, C_2, such that*

$$\begin{aligned} E_{\mu^{\boldsymbol h}_{\Lambda_K,M}}\Big[g^2(\bar M_1 - \bar M_2)^2\Big] \;\le\; & 6\Big(W_{\Lambda_K}(\boldsymbol h) - \mathbb{E}[h_1]\Big)^2 + 6\Big(W_{\Lambda_{K+1,2K}}(\boldsymbol h) - \mathbb{E}[h_1]\Big)^2 \\ & + 6\Big(\Gamma_{\Lambda_K}(\lambda,\boldsymbol h) - \tilde\Gamma_1(\lambda)\Big)^2 \;+\; 6\Big(W_{\Lambda_{K+1,2K}}(\lambda,\boldsymbol h) - \tilde\Gamma_1(\lambda)\Big)^2 \end{aligned} \tag{4.8}$$

$$+\ \frac{C_0}{K}\ +\ C_2 K^t D_{\Lambda_{2K}}(\mu^{\boldsymbol{h}}_{\Lambda_{2K},M}, g)$$

for every $K \ge 1$, M in $\mathbb{R}$, environment $\boldsymbol{h}$ and function g in $L^2(\mu^{\boldsymbol{h}}_{\Lambda_{2K},M})$ such that $E_{\mu^{\mathbf{h}}_{\Lambda_{2K},M}}[g^2] = 1$.

Proof: Fix an environment $\mathbf{h}$ and a charge M. By the entropy inequality and by definition of θ, the left hand side of (4.8) is bounded above by

$$\frac{1}{\beta}\log E_{\mu^{\mathbf{h}}_{\Lambda_{2K},M}}\Big[\exp\Big\{\beta(\bar{M}_1 - \bar{M}_2)^2\Big\}\Big]\ +\ \frac{\theta(2K,\mathbf{h})}{\beta} D_{\Lambda_{2K}}(\mu_{\Lambda^{\mathbf{h}}_{2K},M}, g)\ .$$

By the a-priori estimate obtained in Step 2, the second term is less than or equal to $CK^t\beta^{-1}D_{\Lambda_{2K}}(\mu_{\Lambda^{\mathbf{h}}_{2K},M}, g)$. To estimate the first term, choose λ according to (2.3). Since $(\bar{M}_1 - \bar{M}_2)^2$ is bounded above by

$$6\Big(\bar{M}_1 - U_{\Lambda_K}(\lambda,\mathbf{h})\Big)^2\ +\ 6\Big(\Gamma_{\Lambda_K}(\lambda,\mathbf{h}) - \tilde{\Gamma}_1(\lambda)\Big)^2\ +\ 6\Big(W_{\Lambda_K}(\mathbf{h}) - \mathbb{E}[h_1]\Big)^2$$

$$+\,6\Big(W_{\Lambda_{K+1,2K}}(\mathbf{h}) - \mathbb{E}[h_1]\Big)^2\ +\ 6\Big(\Gamma_{\Lambda_{K+1,2K}}(\lambda,\mathbf{h}) - \tilde{\Gamma}_1(\lambda)\Big)^2$$

$$+\,6\Big(\bar{M}_2 - U_{\Lambda_{K+1,2K}}(\lambda,\mathbf{h})\Big)^2\ ,$$

by Schwarz inequality, the first term is bounded by the sum of the previous four expressions which are constants with respect to $\mu^{\mathbf{h}}_{\Lambda_{2K},M}$ with

$$\frac{1}{2\beta}\log E_{\mu^{\mathbf{h}}_{\Lambda_{2K},M}}\Big[\exp\Big\{12\beta\Big(\bar{M}_1 - U_{\Lambda_K}(\lambda,\mathbf{h})\Big)^2\Big\}\Big]$$

$$+\ \frac{1}{2\beta}\log E_{\mu^{\mathbf{h}}_{\Lambda_{2K},M}}\Big[\exp\Big\{12\beta\Big(\bar{M}_2 - U_{\Lambda_{K+1,2K}}(\lambda,\mathbf{h})\Big)^2\Big\}\Big]\ .$$

Recall that $e^x \le 1 + xe^x$ for $x > 0$ and that $\log(1+x) \le x$ to estimate the first term by

$$6E_{\mu^{\mathbf{h}}_{\Lambda_{2K},M}}\Big[\Big(\bar{M}_1 - U_{\Lambda_K}(\lambda, \mathbf{h})\Big)^2 \exp\Big\{12\beta\Big(\bar{M}_1 - U_{\Lambda_K}(\lambda, \mathbf{h})\Big)^2\Big\}\Big]\Big\} .$$

By Corollary 3.1.6, we may replace the expectation with respect to the canonical measure by an expectation with respect to a grand canonical measure, paying the price of a finite constant. Since the grand canonical measures are product, by Schwarz inequality, by the definition of $U_{\Lambda_K}(\lambda, \mathbf{h})$ and by the uniform estimates in Lemma 3.1.1, the previous expression is bounded above by

$$\frac{C_0}{K} E_{\nu^{\mathbf{h}}_{\Lambda_{2K},\lambda}}\Big[\exp\Big\{24\beta\Big(\bar{M}_1 - U_{\Lambda_K}(\lambda, \mathbf{h})\Big)^2\Big\}\Big]^{1/2} .$$

Since $\exp\{ax^2\}$ is a convex function for $a > 0$, this expression is less than or equal to

$$\frac{C_0}{K} \sup_{\lambda} E_{\nu_\lambda}\Big[\exp\Big\{24\beta\{\eta_1 - \gamma_1(\lambda)\}^2\Big\}\Big]^{1/2} .$$

For β small enough, the previous expectation is bounded, uniformly in λ. This proves the lemma. □

Chapter 3

Local Central Limit Theorem

3.1 Notation and Results

We prove in this chapter some estimates which follow from the local central limit theorem and which play a central role in the proof of the spectral gap and the logarithmic Sobolev inequality. All constants in this section depend only on $\|F\|_\infty$.

Fix a sequence $\mathbf{b} = \{b_k, , k \geq 1\}$ of real numbers. For λ in $\mathbb{R}$, denote by $P_{\mathbf{b},\lambda}$ the probability measure on the product space $\mathbb{R}^{\mathbb{N}}$ that makes the coordinates $\{X_k,\ k \geq 1\}$ independent random variables with X_k having density $Z(\lambda - b_k)^{-1} \exp\{[\lambda - b_k]x - V(x)\}$. Denote by $E_{\mathbf{b},\lambda}$ expectation with respect to $P_{\mathbf{b},\lambda}$.

Recall from (1.3) that $\gamma_1(\lambda)$, $\sigma^2(\lambda)$, $\{\gamma_k(\lambda),\ k \geq 3\}$ stand for the expectation, the variance and the k-th truncated moment of a random variable with density $Z(\lambda)^{-1} \exp\{\lambda x - V(x)\}$.

Lemma 3.1.1 *Assume that $\|F\|_\infty < \infty$. Then , there exist finite constant $\{C_k, k \geq 1\}$ depending only on$k$ and $\|F\|_\infty$, such that*

$$0 < C_2^{-1} < \sigma(\lambda)^2 < C_2$$

$$0 < C_k^{-1} < \gamma_k(\lambda) < C_k$$

for all λ *in* $\mathbb{R}$

proof:
We first clain that $Z(\lambda)e^{\frac{-\lambda^2}{2}}$ is bounded above and below by finite positive constants. Indeed, by definition,

$$Z(\lambda) = e^{\frac{\lambda^2}{2}} \int dae^{-\frac{1}{2}(a-\lambda)^2 - F(a)} = e^{\frac{\lambda^2}{2}} \int dae^{-\frac{1}{2}a^2 - F_\lambda(a)}$$

where $F_\lambda(a) = F(a+\lambda)$. Since F is absolutely bounded, this expression is bounded below and above by $\sqrt{2}e^{\frac{-\lambda^2}{2}}e^{\pm\|F\|_\infty}$, proving the clain.

We now claim that $|\gamma_1(\lambda) - \lambda|$ is bounded by $\|F\|_\infty e^{2\|F\|_\infty}$. Indeed , by definition, the difference $\gamma_1(\lambda) - \lambda$ is equal to

$$\frac{1}{Z(\lambda)} \int_{\mathbb{R}} (x-\lambda)e^{\lambda x - \frac{x^2}{2} - F_\lambda(x)} dx.$$

Changing variables, we may rewrite this integral as

$$\frac{\int_{\mathbb{R}} xe^{-\frac{x^2}{2} - F_\lambda(x)} dx}{\int_{\mathbb{R}} e^{-\frac{x^2}{2} - F_\lambda(x)} dx}.$$

Since $\int_{\mathbb{R}} xe^{-\frac{x^2}{2}} dx$ vanishes, by Schwartz inequality, the absolute value of this expression is bounded above by

$$e^{\|F\|_\infty} \frac{1}{\sqrt{2}} \Big| \int_{\mathbb{R}} xe^{-\frac{x^2}{2}} \Big(e^{-F_\lambda(x)} - 1\Big) dx \Big|.$$

In the case $F_\lambda(x) \geq 0$, the previous expression is bounded by

$$e^{\|F\|_\infty} \frac{1}{\sqrt{2}} \int_{\mathbb{R}} |x| e^{-\frac{x^2}{2}} \|F\|_\infty dx$$
$$\leq \|F\|_\infty e^{2e^{\|F\|_\infty}}$$

since $e^{-x} - 1 \leq x$, $x \geq 0$.

In the case $F_\lambda(x) \leq 0$, this expression is bounded by

$$e^{\|F\|_\infty} \frac{1}{\sqrt{2}} \int_{\mathbb{R}} |x| e^{-\frac{x^2}{2}} e^{-F_\lambda(x)} |1 - e^{-F_\lambda(x)}| dx$$
$$\leq e^{\|F\|_\infty} \frac{1}{\sqrt{2}} \int_{\mathbb{R}} |x| e^{-\frac{x^2}{2}} e^{\|F\|_\infty} \|F\|_\infty dx$$
$$\leq \|F\|_\infty e^{2e^{\|F\|_\infty}}$$

which proves the claim.

We now prove a lower bound for $\sigma(\lambda)^2$. The same ideas permit to derive an upper bound for $\sigma(\lambda)^2$ or upper and lower bounds for the moments $\{\gamma_{2j}(\lambda), j \geq 2\}$. A change of variables and the estimate on $Z(\lambda)e^{\frac{-\lambda^2}{2}}$ gives that

$$\sigma(\lambda)^2 \geq e^{-2\|F\|_\infty} \frac{1}{\sqrt{2}} \int_{\mathbb{R}} [x + \lambda - \gamma_1(\lambda_j)]^2 e^{-\frac{x^2}{2}} dx$$
$$\geq e^{-2\|F\|_\infty} \inf_{\beta, |\beta| \leq \|\gamma_1(\lambda) - \lambda\|_\infty} \frac{1}{\sqrt{2}} \int_{\mathbb{R}} [x + \beta]^2 e^{-\frac{x^2}{2}} dx \geq C_1 > 0.$$

This concludes the proof of the lemma.

It follows from this lemma that

$$\sup_{\lambda\in\mathbb{R}} \Big| \frac{\gamma_j(\lambda)}{\sigma^2(\lambda)^{j/2}} \Big| \;\le\; \tilde{C}_j \tag{1.1}$$

for all $j \ge 3$, which is the estimate needed in order to prove the uniform local central limit theorem.

For a finite subset Λ of $\mathbb{N}$, denote by $f_{\mathbf{b},\lambda,\Lambda}$ the density of the random variable

$$\frac{1}{\sigma^2(\mathbf{b},\lambda,\Lambda)^{1/2}} \sum_{j\in\Lambda} [X_j - \gamma_1(\lambda - b_j)] \ ,$$

where, for $k \ge 3$,

$$\sigma^2(\mathbf{b},\lambda,\Lambda) \;=\; \sum_{j\in\Lambda} \sigma^2(\lambda - b_j) \ , \quad \gamma_k(\mathbf{b},\lambda,\Lambda) \;=\; \sum_{j\in\Lambda} \gamma_k(\lambda - b_j) \ . \tag{1.2}$$

Theorem 3.1.2 *Assume that $\|F\|_\infty < \infty$. There exists $N_0 \ge 1$ and a finite constant C_0 depending only on $\|F\|_\infty$ such that*

$$\Big| f_{\boldsymbol{b},\lambda,\Lambda_N}(x) \;-\; \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \Big\{ 1 - \frac{\gamma_3(\boldsymbol{b},\lambda,\Lambda_N) x}{6\sigma^2(\boldsymbol{b},\lambda,\Lambda_N)^{3/2}} \Big\} \Big| \;\le\; \frac{C_0}{N}$$

for all $N \ge N_0$, x in $\mathbb{R}$, environment $\boldsymbol{b}$ and λ in $\mathbb{R}$.

The proof of this result is in [9].

Remark 3.1.3 *For a fixed parameter λ and b this is just the usual statement of local central limit theorem for independents random variables with finite fourth moments. The important point here is the uniformily over the parameter λ and the environment **b**. this uniformity can be obtained in virtue of (1.1) and the estimates presented in the Lemma 3.1 .*

In virtue of (1.1) and (1.2), both $\sigma^2(\mathbf{b},\lambda,\Lambda_N)$ and $\gamma_3(\mathbf{b},\lambda,\Lambda_N)$ are of order N. The second term in the expansion is therefore of order $N^{-1/2}$. For a fixed parameter λ this is just the usual statement of the local central limit theorem for independent random variables with finite fourth moments. The important point here is the uniformity over the parameter λ. This uniformity can be obtained in virtue of (1.1) and the estimates of Lemma 3.1.1.

The local central limit theorem gives asymptotic expansions of the expectation of a function with respect to a canonical measure. This is the content of the next result. Recall that $E_\nu[G;G]$ stands for the variance of G with respect to ν.

Corollary 3.1.4 *Fix $\ell \geq 1$ and fix a function $G\colon \mathbb{R}^\ell \to \mathbb{R}$. There exist $N_0 \geq 1$ and a finite constant C_0, depending only on $\|F\|_\infty$, such that for all $N \geq N_0$, environment **b**, and M in $\mathbb{R}$*

$$\Big| E_{\mu^{\boldsymbol{b}}_{\Lambda_N,M}}[G] \;-\; E_{\nu^{\boldsymbol{b}}_{\Lambda_N,\lambda}}[G] \Big| \;\leq\; \frac{C_0 \ell}{|\Lambda_N|}\, \|G\|_\infty \quad \textit{if } G \textit{ is bounded and}$$

$$\Big| E_{\mu^{\boldsymbol{b}}_{\Lambda_N,M}}[G] \;-\; E_{\nu^{\boldsymbol{b}}_{\Lambda_N,\lambda}}[G] \Big| \;\leq\; \frac{C_0 \ell}{|\Lambda_N|} \sqrt{E_{\nu^{\boldsymbol{b}}_{\Lambda_N,\lambda}}[G;G]}\,\cdot$$

*In these formulas, the chemical potential λ, which depends on N, M and **b**, is chosen so that*

$$M \;=\; E_{\nu^{\boldsymbol{b}}_{\Lambda_N,\lambda}}\Big[\sum_{x\in\Lambda_N} \eta_x \Big]\,. \tag{1.3}$$

The proof of this result is elementary (cf. Corollary A2.1.4 in [9]). Of course, by changing the value of the constant C, the first inequality remains valid for all values of $N \geq \ell$.

Corollary 3.1.5 *Let $G\colon \mathbb{R} \to \mathbb{R}$ be a smooth bounded function and let $G_{\boldsymbol{b},N,M} = G - E_{\mu^{\boldsymbol{b}}_{\Lambda_{N},M}}[G]$. There exists a finite constant C_0, depending only on $\|F\|_\infty$, such that*

$$E_{\mu^{\boldsymbol{b}}_{\Lambda_{N},M}}\Big[\Big(\frac{1}{N}\sum_{x=1}^{N} G_{\boldsymbol{b},N,M}(\eta_x)\Big)^2\Big] \;\leq\; C_0\frac{\|G\|_\infty^2}{N}$$

for all $N \geq 1$, environment $\boldsymbol{b}$ and M in $\mathbb{R}$.

Proof:
The variance is equal a to

$$\frac{1}{N}E_{\mu^{\mathbf{b}}_{\Lambda_{N},M}}[(G_{\mathbf{b},L,M}(\eta_1))^2] + \Big(1-\frac{1}{N}\Big)E_{\mu^{\mathbf{b}}_{\Lambda_{N},M}}[G_{\mathbf{b},L,M}(\eta_1)G_{\mathbf{b},L,M}(\eta_2)]$$

The first expression is bounded by $4\|G\|_\infty{}^2 N^{-1}$ for all $L \geq 1$ and $M \in \mathbb{R}$. The second one, by definition of $G_{\mathbf{b},L,M}$ is equal to

$$\Big(1-\frac{1}{N}\Big)\Big\{E_{\mu^{\mathbf{b}}_{\Lambda_{N},M}}[G(\eta_1)G(\eta_2)] - E_{\mu^{\mathbf{b}}_{\Lambda_{N},M}}[G(\eta_L)]\Big\}.$$

By Corollary 3.3 , since $\nu^{\mathbf{b}}_{\alpha}$ is a product measure, the first term of the previos expression is equal to $E_{\nu^{\mathbf{b}}_{\alpha}}[G(\eta_L)]^2 \pm CL^{-1}\|G\|_\infty$, where C is finite constant depending only on$\|F\|_\infty$. By the same result, the second term is equal to $E_{\nu^{\mathbf{b}}_{\alpha}}[G(\eta_L)]^2 \pm CL^{-1}\|G\|_\infty{}^2$, which concludes the proof.

Recall the definition of the variance $\sigma^2(\mathbf{b}, \lambda, \Lambda)$ and of the density $f_{\mathbf{b},\lambda,\Lambda}$ given in (1.2). For $1 \le K < N$, denote by $\mu^{\mathbf{b}}_{\Lambda_N,K,M}$ the marginal on $\mathbb{R}^{\Lambda_K}$ of the canonical measure $\mu^{\mathbf{b}}_{\Lambda_N,M}$. An elementary computation shows that $\mu^{\mathbf{b}}_{\Lambda_N,K,M}$ is absolutely continuous with respect to the Lebesgue measure and that its Radon-Nikodym derivative $R^{\mathbf{b}}_{N,K,M}(\mathbf{x}_K)$ is given by

$$\frac{R^{\mathbf{b}}_{N,K,M}(\mathbf{x}_K)}{g^{\mathbf{b}}_{\lambda,\Lambda_K}(\mathbf{x}_K)} \;=\; \frac{\sigma^2(\mathbf{b},\lambda,\Lambda_N)}{\sigma^2(\mathbf{b},\lambda,\Lambda_{K,N})} \frac{f_{\mathbf{b},\lambda,\Lambda_{K,N}}\left(\frac{\sum_{j=1}^K [\gamma_1(\lambda-b_j)-x_j]}{\sigma^2(\mathbf{b},\lambda,\Lambda_{K,N})^{1/2}}\right)}{f_{\mathbf{b},\lambda,\Lambda_N}(0)} \;,$$

where $\mathbf{x}_K = (x_1, \dots, x_K)$, $\Lambda_{K,N} = \{K+1, \dots, N\}$, for a finite set Λ,

$$g^{\mathbf{b}}_{\lambda,\Lambda}(\mathbf{x}_K) \;=\; \prod_{j\in\Lambda} \frac{1}{Z(\lambda - b_j)} e^{[\lambda-b_j]x_j - V(x_j)}$$

and λ is chosen according to (1.3). The next result shows that the ratio $R^{\mathbf{b}}_{N,K,M}/\, g^{\mathbf{b}}_{\lambda,\Lambda_K}$ is bounded above, uniformly over λ, provided K/N is bounded away from 1.

Corollary 3.1.6 *There exists a finite constant C_0, depending only on $\|F\|_\infty$, such that*

$$\frac{R^{\boldsymbol{b}}_{N,K,M}(\mathbf{x}_K)}{g^{\boldsymbol{b}}_{\lambda,\Lambda_K}(\mathbf{x}_K)} \;\le\; C_0 \;.$$

for all $N/2 \ge K \ge 1$, M in $\mathbb{R}$, environment $\boldsymbol{b}$ and $\mathbf{x}_K$ in $\mathbb{R}^{\Lambda_K}$. In this formula, λ is chosen for (1.3) to hold. In particular, if $K \le N/2$, for any local function $H\colon \mathbb{R}^{\Lambda_K} \to \mathbb{R}$,

$$E_{\mu^{\boldsymbol{b}}_{\Lambda_N,M}}[H(\eta_1,\dots,\eta_K)] \;\le\; C_0 E_{\nu^{\boldsymbol{b}}_{\Lambda_K,\lambda}}[\,|H(\eta_1,\dots,\eta_K)|\,] \qquad (1.4)$$

for λ satisfying (1.3).

For N large, the proof is an elementary consequence of the explicit formula for the ratio $R^{\mathbf{b}}_{N,K,M}/g^{\mathbf{b}}_{\lambda,\Lambda_K}$, the estimates for the variance presented in Lemma 3.1.1 and the local central limited theorem stated in Theorem 3.1.2. For N small, the local central limited theorem is replaced by a direct inspection. More details can be found in [11].

This Corollary provides an estimate on the variance of functions with respect to canonical measures in terms of the variance of the same function with respect to grand canonical measures. Indeed, fix a function $H : \mathbb{R}^{\Lambda_K} \to \mathbb{R}$, assume that $N \geq 2K$ and choose λ according to (1.3). By Corollary 3.1.6,

$$E_{\mu^{\mathbf{b}}_{\Lambda_N,M}}[H;H] \;\leq\; E_{\mu^{\mathbf{b}}_{\Lambda_N,M}}\Big[\Big(H - E_{\nu^{\mathbf{b}}_{\Lambda_N,\lambda}}[H]\Big)^2\Big] \;\leq\; C_0 E_{\nu^{\mathbf{b}}_{\Lambda_N,\lambda}}[H;H]\;. \quad (1.5)$$

Lemma 3.1.2 and its corollaries permit to estimate expectations with respect to a canonical measure $\mu^{\mathbf{b}}_{\Lambda_L,M}$, provided L is large. The next result provides an estimate for small values of L. The important point in this result is once again the uniformity over the parameter λ and the environment $\mathbf{b}$. For a finite set Λ of $\mathbb{N}$, denote by $\tilde{f}_{\mathbf{b},\lambda,\Lambda}$ the density of the random variable $|\Lambda|^{-1/2}\sum_{j\in\Lambda}\{X_j - (\lambda - b_j)\}$ under the measure $P_{\mathbf{b},\lambda}$. Note that we are not renormalizing by $\sigma^2(\mathbf{b},\lambda,\Lambda)$ and that we are subtracting $\lambda - b_j$ instead of $\gamma_1(\lambda - b_j)$.

Lemma 3.1.7 *There exists a positive and finite constant C_0, depending only on $\|F\|_\infty$, such that*

$$C_0^{-N}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \;\leq\; \tilde{f}_{\boldsymbol{b},\lambda,\Lambda_N}(x) \;\leq\; C_0^N\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

for every λ in $\mathbb{R}$, $N \geq 1$ and environment $\boldsymbol{b}$.

The proof is similar to the one of Lemma 5.6 in [11] and therefore omitted. The same argument shows that $g_\lambda(x) = Z(\lambda)^{-1}\exp\{\lambda x - V(x)\}$ is bounded above and below by a Gaussian density. More precisely, there exists a finite, strictly positive constant C_0 depending only on $\|F\|_\infty$, such that

$$C_0 \frac{1}{\sqrt{2\pi}} e^{-(x-\lambda)^2/2} \ \le \ g_\lambda(x) \ \le \ C_0^{-1} \frac{1}{\sqrt{2\pi}} e^{-(x-\lambda)^2/2} \tag{1.6}$$

for every λ in $\mathbb{R}$.

We conclude this section with an important estimate. The proof of this lemma follows closely the one of Lemma 5.7 in [11].

Lemma 3.1.8 *There exists $\beta_1 > 0$ and a finite constant C_0 such that*

$$E_{\nu^{\boldsymbol{b}}_{\Lambda_N,\lambda}} \Big[\exp \Big\{ \beta_1 |\Lambda_N| \{ m_{\Lambda_N} - R_{\boldsymbol{b},\Lambda_N}(\lambda) \}^2 \Big\} \Big] \ \le \ C_0$$

for every λ in $\mathbb{R}$, environment $\boldsymbol{b}$ and $N \ge 1$. In this formula, m_{Λ_N} stands for the charge average in Λ_N: $m_{\Lambda_N} = |\Lambda_N|^{-1} \sum_{x \in \Lambda_N} \eta_x$ and $R_{\boldsymbol{b},\Lambda_N}(\lambda) = E_{\nu^{\boldsymbol{b}}_{\Lambda_N,\lambda}}[m_{\Lambda_N}]$.

Proof:
For small values of L this statement is a straightforward consequence of the previous lemma, the fact that $\gamma_1(\lambda) - \lambda$ is absolutely bounded in Lemma*, and the fact that the statement holds for Gaussian distributions.
For large values of L, with the notation introduced in the beginning of this section, the expectation can be written as

$$\int_{\mathbb{R}} e^{\beta_0 \sigma(\lambda)^2 x^2} f_{\lambda,L}(x) dx.$$

for some appropriate choice of λ. Notice that the local central limit theorem, stated in Theorem 3.2 , gives a good bound only for small values of x. The idea is therefore to replace in the previos formula λ by a variable μ which makes x a typical value. By a direct computation.

$$f_{\mathbf{b},\lambda,\Lambda_L}(x) = \frac{\sigma_\lambda}{\sigma_\mu}(\frac{Z_\mu}{Z_\lambda})^L e^{(\lambda-\mu)[x\sigma_\lambda\sqrt{L}+L\gamma_1(\lambda)]} f_{\mu,L}(\frac{x\sigma_\lambda}{\sigma_\mu} + \frac{\sqrt{L}(\gamma_1(\lambda)-\gamma_1(\mu))}{\sigma_\mu}).$$

Choose μ for the expression inside $f_{\mathbf{b},\mu,\Lambda_L}$ to be small (in order to be able to use the local central limit estimate):

$$x\sigma_\lambda\sqrt{L} = L[\gamma_1(\lambda) - \gamma_1(\mu)].$$

With this choice, since by Theorem 3.2 $C_1^{-1} \leq f_{\mathbf{h},\mu,\Lambda_L}(0) \leq C_1$ for some universal constant C_1, and since by lemma 3.1 σ_λ is bounded,

$$f_{\mathbf{b},\lambda,\Lambda_L}(x) \approx exp\{L\log\{\frac{Z_\mu}{Z_\lambda}\} + (\lambda-\mu)[x\sigma_\lambda\sqrt{L} + L\gamma_1(\lambda)]\}.$$

where $\approx$ means that the left hand side is bounded above and below by the right hand side multiplied by finite positive constants. The expression inside the exponencial vanishes at $x = 0$. It is also not difficult to show that it is strictly concave in x (cf. [11]). In particular,

$$f_{\mathbf{b},\lambda,\Lambda_L}(x) \approx e^{-C_2x^2}.$$

for some finite constant C_2 and we are back to the Gaussian case.

Chapter 4

Environment

4.1 Notation and Results

We present in this chapter all results needed on the environment in the proof of the spectral gap and the logarithmic Sobolev inequality. We start with simple bounds on the derivative of a function. For a finite subset Λ of $\mathbb{Z}$, and an environment $\mathbf{b}$, denote by $R_\Lambda : \mathbb{R} \to \mathbb{R}$ the smooth strictly increasing function

$$R_\Lambda(\lambda) \;=\; E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}\Big[\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\eta_x\Big]$$

and denote by Φ_Λ the inverse of R_Λ. Let $A_\Lambda : \mathbb{R} \to \mathbb{R}$ be given by

$$A_\Lambda(m) \;=\; E_{\nu^{\mathbf{b}}_{\Lambda,\Phi_\Lambda(m)}}\Big[\frac{1}{|\Lambda|}\sum_{x\in\Lambda}F'(\eta_x)\Big]\;.$$

Lemma 4.1.1 *Fix a finite subset Λ and an environment $\boldsymbol{b}$. We claim that*

$$A'_\Lambda(m) \;=\; \frac{1}{|\Lambda|^{-1}\sum_{x\in\Lambda}\sigma^2(\Phi_\Lambda(m)-b_x)} - 1\;\cdot$$

*Moreover, there exists a constant C_0, depending only on $\|F\|_\infty$, such that $\|A''_\Lambda\|_\infty \leq C_0$ for all environment **b**.*

Proof: We claim that

$$E_{\nu^{\mathbf{b}}_{\Lambda,\Phi_\Lambda(\theta)}}\Big[\frac{1}{|\Lambda|}\sum_{x\in\Lambda} F'(\eta_x)\Big] \;=\; \Phi_\Lambda(\theta) \;-\; \theta \;-\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda} b_x\ . \qquad (1.1)$$

Indeed, by definition of the product measure $\nu^{\mathbf{b}}_{\Lambda,\lambda}$,

$$\begin{aligned}
E_{\nu^{\mathbf{m}}_{\Lambda,\lambda}}\Big[\frac{1}{|\Lambda|}\sum_{x\in\Lambda} F'(\eta_x)\Big] \;&=\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\frac{1}{Z(\lambda-b_x)}\int F'(a)e^{(\lambda-b_x)a-V(a)}\,da \\
&= \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\int\frac{1}{Z(\lambda-b_x)}e^{(\lambda-b_x)a-\frac{a^2}{2}-F(a)}da \\
= \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\int(a+F'(a)-\lambda+b_x)&\frac{1}{Z(\lambda-b_x)}e^{(\lambda-b_x)a-\frac{a^2}{2}-F(a)}da \\
+\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\int(\lambda-b_x-a)&\frac{1}{Z(\lambda-b_x)}e^{(\lambda-b_x)\eta-\frac{a^2}{2}-F(a)}da \\
= \frac{1}{|\Lambda|}\sum_{x\in\Lambda}&\{(\lambda-b_x)-R_{\{x\}}(\lambda)\} \\
-\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\frac{1}{Z(\lambda-b_x)}&\int\partial_a\{(\lambda-b_x)a-V(a)\}e^{(\lambda-b_x)a-V(a)}\,da\ .
\end{aligned}$$

Since $R_\Lambda(\lambda)=|\Lambda|^{-1}\sum_{x\in\Lambda}R_{\{x\}}(\lambda)$ and since the each term in the last sum vanishes, the previous expression is equal to

$$\lambda \;-\; R_\Lambda(\lambda) \;-\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda} b_x \;,$$

where $\lambda = \Phi_\Lambda(\theta)$, which proves (1.1).

Thus

$$A'_\Lambda(\theta) = \Phi'_\Lambda(\lambda) - 1$$

and

$$A''_\Lambda(\theta) = \Phi''_\Lambda(\lambda)$$

Since $\Phi_\Lambda = R_\Lambda^{-1}$,

$$\Phi'(\theta) = \frac{1}{R'_\Lambda(\theta)}$$

and

$$\Phi''(\theta) = \frac{-1}{[R'_\Lambda(\theta)]^3} R''(\Phi(\theta)) = \frac{-R''_\Lambda(\theta)}{[R'_\Lambda(\theta)]^3}$$

and since

$$\frac{d}{d\lambda}R_\Lambda(\lambda) \;=\; \frac{1}{\Lambda}\sum_{x\in\Lambda}\partial_\lambda \int \frac{1}{Z(\lambda - b_x)}\eta_x e^{(\lambda-b_x)\eta_x - \frac{{\eta_x}^2}{2} - F(\eta_x)} d\eta_x$$

$$= \frac{1}{\Lambda} \sum_{x \in \Lambda} \Big(\int \frac{1}{Z} a^2 e^{(\lambda - b_x)a - \frac{a^2}{2} - F(a)} da - \Big[\int \frac{1}{Z} a e^{(\lambda - b_x)a - \frac{a^2}{2} - F(a)} da \Big]^2 \Big)$$

$$= \frac{1}{|\Lambda|} \sum_{x \in \Lambda} E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[\eta_x; \eta_x] \;=\; \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma^2(\lambda - b_x)$$

by lemma 3.1.1, is in $\left[\frac{1}{C}, C\right]$

$$\frac{d^2}{d\lambda^2} R_\Lambda(\lambda) \;=\; \frac{1}{\Lambda} \sum_{x \in \Lambda} \partial_\lambda \Big(\int \frac{1}{Z} {\eta_x}^2 e^{(\lambda - b_x)\eta_x - \frac{{\eta_x}^2}{2} - F(\eta_x)} d\eta_x$$

$$- \Big[\int \frac{1}{Z} \eta_x e^{(\lambda - b_x)\eta_x - \frac{{\eta_x}^2}{2} - F(\eta_x)} d\eta_x \Big]^2 \Big)$$

$$= \frac{1}{\Lambda} \sum_{x \in \Lambda} \Big(E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[a^3] - E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[a^2] E^{\mathbf{b}}_{\Lambda,\lambda}[a] - 2E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[a] \{ E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[a^2] - E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[a]^2 \} \Big)$$

$$= \frac{1}{\Lambda} \sum_{x \in \Lambda} \Big(E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[a^3 - 3E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[a^2] E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[a] + 2E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[a]^3 \Big)$$

$$= \frac{1}{\Lambda} \sum_{x \in \Lambda} E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[(\eta_x - E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[\eta_x])^3]$$

$$= \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \gamma_3(\lambda - b_x)$$

by (1.1),

$$\frac{d}{d\theta} E_{\nu^{\mathbf{b}}_{\Lambda,\Phi_\Lambda(\theta)}} \Big[\frac{1}{|\Lambda|} \sum_{x \in \Lambda} F'(\eta_x) \Big] \;=\; \frac{1}{|\Lambda|^{-1} \sum_{x \in \Lambda} \sigma^2(\Phi_\Lambda(\theta) - b_x)} - 1 \,,$$

$$\frac{d^2}{d\theta^2} E_{\nu^{\mathbf{b}}_{\Lambda,\Phi_\Lambda(\theta)}} \Big[\frac{1}{|\Lambda|} \sum_{x \in \Lambda} F'(\eta_x) \Big] \;=\; - \frac{|\Lambda|^{-1} \sum_{x \in \Lambda} \gamma_3(\Phi_\Lambda(\theta) - b_x)}{\Big(|\Lambda|^{-1} \sum_{x \in \Lambda} \sigma^2(\Phi_\Lambda(\theta) - b_x) \Big)^3} \,.$$

The statement of the lemma follows from these identities (3.1.2) and from Lemma 3.1.1. □

We proceed with a lemma which explains the assumption made on $\sigma^2(\cdot)$, $\Gamma_1(\cdot)$.

Let $U : \mathbb{R} \to \mathbb{R}$ be a continuous functions with limits at the boundary:

$$\lim_{\lambda \to \infty} U(\lambda) = U_+ \ , \quad \lim_{\lambda \to -\infty} U(\lambda) = U_- \tag{1.2}$$

for some finite values U_-, U_+. Let $\tilde{U}(\lambda) = \mathbb{E}[U(\lambda - h_1)]$.

Recall that for $K \geq 2$, $L \geq K^2$, we decompose the cube Λ_L in ℓ disjoint cubes $\{I_1, \dots, I_\ell\}$ of length K or $K+1$. One should think that K is large but fixed. The key point in the next result is the uniformity in λ.

Lemma 4.1.2 *Fix a continuous function U satisfying (1.2), $\varepsilon > 0$, $K \geq 2$ and an increasing function t. There exists a measurable set Ω_0 of sequences $\boldsymbol{h}$ such that*

- $\mathbb{P}[\Omega_0] = 1$,
- *For any $\boldsymbol{h}$ in Ω_0, there exists $\ell_0 = \ell_0(\boldsymbol{h})$ such that*

$$\frac{1}{\ell} \sum_{j=1}^{\ell} t(|I_j|) \Big(\frac{1}{|I_j|} \sum_{x \in I_j} U(\lambda - h_x) - \tilde{U}(\lambda) \Big)^2 \ \leq \ c + \varepsilon \tag{1.3}$$

for all $\ell \geq \ell_0$ and all λ in $\mathbb{R}$. In this formula,

$$c \;=\; \max_{n=K,K+1} \Big\{ t(n)\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{x\in\Lambda_n} U(\lambda-h_x) - \tilde{U}(\lambda)\Big)^2\Big]\Big\}$$

Proof: Fix $\delta_0 > 0$. Since $U(\cdot)$ is a continuous bounded function with limits at the boundaries $\lambda = \pm\infty$, for each $\delta_0 > 0$, there exists a finite set of chemical potentials $\Gamma_{\delta_0} = \{\lambda_1, \dots, \lambda_r\}$ with the property that for every λ in $\mathbb{R}$, there exists λ_i in Γ_{δ_0} such that

$$\sup_{b,|b|\leq A_0} \Big| U(\lambda - b) - U(\lambda_i - b)\Big| \;\leq\; \frac{\delta_0}{t(K+1)^{1/2}}\ .$$

In other words,

$$\sup_{\lambda\in\mathbb{R}} \min_{\lambda_i\in\Gamma_{\delta_0}} \sup_{b,|b|\leq A_0} \Big| U(\lambda - b) - U(\lambda_i - b)\Big| \;\leq\; \frac{\delta_0}{t(K+1)^{1/2}}\ . \tag{1.4}$$

It is here and only here that we need the environment **h** to assume bounded values and the assumption (1.2) on the asymptotic behavior of $U(\cdot)$.

In view of (1.4), the left hand side of (1.3) is less than or equal to

$$6\delta_0^2 \;+\; 4 \max_{\lambda_i\in\Gamma_{\delta_0}} \frac{1}{\ell}\sum_{j=1}^{\ell} t(|I_j|)\Big(\frac{1}{|I_j|}\sum_{x\in I_j} U(\lambda_i - h_x) - \tilde{U}(\lambda_i)\Big)^2\ .$$

Fix $\delta_1 > 0$ and λ_i in Γ_{δ_0}. Since the intervals $\{I_j\,, 1 \le j \le \ell\}$ are disjoints, it is not difficult to show that the sequence

$$\mathbb{P}\Big[\frac{1}{\ell}\sum_{j=1}^{\ell}\Big\{t(|I_j|)\Big(\frac{1}{|I_j|}\sum_{x\in I_j} U(\lambda_i - h_x) - \tilde{U}(\lambda_i)\Big)^2 - c_j\Big\} \ge \delta_1\Big]$$

is summable in ℓ if

$$c_j \;=\; t(|I_j|)\mathbb{E}\Big[\Big(\frac{1}{|I_j|}\sum_{x\in I_j} U(\lambda_i - h_x) - \tilde{U}(\lambda_i)\Big)^2\Big]\;.$$

In particular, since $c_j \le c$, by the Borel-Cantelli lemma, there exists a set $\Omega_0 = \Omega_0(K, \delta_1, U)$ with the following properties: $\mathbb{P}[\Omega_0] = 1$ and for any $\mathbf{h}$ in Ω_0, there exists $\ell_0 = \ell_0(\mathbf{h})$ such that

$$\frac{1}{\ell}\sum_{j=1}^{\ell} t(|I_j|)\Big(\frac{1}{|I_j|}\sum_{x\in I_j} U(\lambda_i - h_x) - U(\lambda_i)\Big)^2 \;\le\; c + \delta_1\;.$$

for $\ell \ge \ell_0$. Since Γ_{δ_0} is a finite set, we can make the previous inequality to be uniform in λ_i.

In conclusion, for any sequence $\mathbf{h}$ in Ω_0, λ in $\mathbb{R}$ and $\ell \ge \ell_0$, the expression on the left hand side of (1.3) is bounded by $c + 6\delta_0^2 + 4\delta_1$. This concludes the proof of the lemma. □

This lemma is used in two different situations: with $t(n) = 1$ and with $t(n) = n$. In the first case, since U is bounded,

$$\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{x\in\Lambda_n} U(\lambda - h_x) - \tilde{U}(\lambda)\Big)^2\Big]$$

vanishes as $n \uparrow \infty$, uniformly in λ. In particular, taking K large enough, we may turn the constant c appearing in the right hand side of (1.3) as small as we wish. Hence, for every $\varepsilon > 0$, there exists $K_0 \geq 2$ and a $\mathbb{P}$-measure one set Ω_0 such that for any $\mathbf{h}$ in Ω_0, there exists $\ell_0 = \ell_0(\mathbf{h})$ such that

$$\frac{1}{\ell} \sum_{j=1}^{\ell} \Big(\frac{1}{|I_j|} \sum_{x \in I_j} U(\lambda - h_x) - \tilde{U}(\lambda) \Big)^2 \ \leq \ \varepsilon \tag{1.5}$$

for all $\ell \geq \ell_0$ and all λ in $\mathbb{R}$.

On the other hand, if $t(n) = n$, since U is bounded,

$$n\mathbb{E}\Big[\Big(\frac{1}{n} \sum_{x \in \Lambda_n} U(\lambda - h_x) - \tilde{U}(\lambda) \Big)^2 \Big] \ \leq \ 4\|U\|_\infty^2 \ .$$

Thus, we may take the constant c in the right hand side of (1.3) to be equal to $4\|U\|_\infty^2$ and ε to be 1. In this case, the statement of the lemma becomes: There exists $C = C(\|U\|_\infty)$ such that for all $K \geq 2$, there exists a $\mathbb{P}$-measure one set $\Omega_0 = \Omega_0(K, U)$ such that for any $\mathbf{h}$ in Ω_0, there exists $\ell_0 = \ell_0(\mathbf{h})$ such that

$$\frac{1}{\ell} \sum_{j=1}^{\ell} |I_j| \Big(\frac{1}{|I_j|} \sum_{x \in I_j} U(\lambda - h_x) - \tilde{U}(\lambda) \Big)^2 \ \leq \ C \tag{1.6}$$

for all $\ell \geq \ell_0$ and all λ in $\mathbb{R}$.

Notice that the previous results (1.5), (1.6) hold for $U = Id$, where Id stands for the identity. Of course, the identity does not satisfy (1.2) but we need this assumption only to guarantee uniformity over λ, a parameter which disappears in the case of the identity.

We conclude this chapter proving that (1.2), (1.4) hold if F converges at the boundary of $\mathbb{R}$. Indeed, assume that there exists $F_\pm$ such that

$$\lim_{a\to\pm\infty} F(a) \;=\; F_\pm \;.$$

In this case, by the dominated convergence theorem, $Z(\lambda)\exp\{-\lambda^2/2\}$ converges to $\sqrt{2\pi}\exp\{-F_\pm\}$ at $\pm\infty$. The same argument shows that Γ_1 converge to 0 at the boundary and that σ^2 converges to 1.

Chapter 5

Large Deviations Estimates

We obtain in this Chapter some estimates which play a central role in the proof of the logarithmic Sobolev inequality.

5.1 Notation and Results

In this section, for $j = 0, 1, 2$, all constants C_j are allowed to depend on $\|F^{(i)}\|_\infty$ for $0 \le i \le j$ and may change from line to line. Here $F^{(i)}$ stands for the i-th derivative of F.

Fix a differentiable function $H \colon \mathbb{R} \to \mathbb{R}$ with bounded derivative: $\|H'\|_\infty < \infty$. For each $L \ge 1$, x in Λ_L, λ, M in $\mathbb{R}$ and environment $\mathbf{h} = \{h_x, x \in \mathbb{Z}\}$, let

$$H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x) = H(\eta_x) - E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[H(\eta_x)] \ ,$$

$$H_{\mathbf{h},\Lambda_L,M}(\eta_x) = H(\eta_x) - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[H(\eta_x)] \ .$$

Although the notation is slightly ambiguous, the context will always clarify if we are subtracting the average with respect to the canonical measure or with respect to the grand canonical one. Notice that $H_{\mathbf{h},\Lambda_L,M}(\eta_x) - H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x) = E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[H(\eta_x)] - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[H(\eta_x)]$. In particular, if λ is chosen according to

(1.3), by Corollary 3.1.4,

$$
\begin{aligned}
&\Big| H_{\mathbf{h},\Lambda_L,M}(\eta_x) - H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x) \Big| \\
&= \Big| E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[H(\eta_x)] - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[H(\eta_x)] \Big| \;\le\; \frac{C_0 \|H'\|_\infty}{|\Lambda_L|} \qquad (1.1)
\end{aligned}
$$

for some finite constant C_0 depending only on $\|F\|_\infty$ because

$$
E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[H(\eta_x); H(\eta_x)] \le E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[\{H(\eta_x) - H(\theta_x)\}^2] \le \|H'\|_\infty^2 \sigma^2(\lambda - h_x).
$$

Here, θ_x stands for $E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[\eta_x] = \gamma_1(\lambda - h_x)$.

We claim that there exists a finite constant C_0 depending only on $\|F\|_\infty$ for which

$$
\begin{aligned}
|H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)| \;&\le\; C_0 \|H'\|_\infty \Big(1 + \Big| \eta_x - E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[\eta_x] \Big| \Big) \, , \qquad (1.2) \\
|H_{\mathbf{h},\Lambda_L,M}(\eta_x)| \;&\le\; C_0 \|H'\|_\infty \Big(1 + \Big| \eta_x - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[\eta_x] \Big| \Big)
\end{aligned}
$$

for all $L \ge 1$, M, λ in $\mathbb{R}$ and environment $\mathbf{h}$. Consider first the grand canonical case. By definition of $H_{\mathbf{h},\Lambda_L,\lambda}$ and by Schwarz inequality,

$$
\begin{aligned}
|H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)| \;&\le\; E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[\big| H(\eta_x) - H(\xi_x) \big| \Big] \;\le\; \|H'\|_\infty E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[\big| \eta_x - \xi_x \big| \Big] \\
&\le\; \|H'\|_\infty \Big\{ \big| \eta_x - E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[\eta_x] \big| \;+\; \sigma^2(\lambda - h_x)^{1/2} \Big\} \, .
\end{aligned}
$$

Of course, in the previous formulas, only the variable ξ_x is integrated. By Lemma 3.1.1, the second term inside braces in the last expression is bounded above by some finite constant C_0 that depends on $\|F\|_\infty$ only. This proves the claim in the grand canonical case. The same arguments apply to the canonical case provide we show that $E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[\eta_x;\eta_x]$ is uniformly bounded. But this is follows from estimate (3.1.5) of Corollary 3.1.6.

Proposition 5.1.1 *Fix a differentiable function $H\colon \mathbb{R}\to\mathbb{R}$ with bounded derivative. There exists a constant C_0, depending only on $\|F\|_\infty$, such that*

$$\frac{1}{\beta|\Lambda_L|}\log\int \exp\Big\{\beta\sum_{x\in\Lambda_L} H_{\mathbf{h},\Lambda_L,M}(\eta_x)\Big\}\, d\mu^{\mathbf{h}}_{\Lambda_L,M} \;\le\; C_0\|H'\|^2_\infty\beta \tag{1.3}$$

for all $\beta>0$, $L\ge 2$, M in $\mathbb{R}$ and environment $\mathbf{h}$.

Proof: We first prove this result for the grand canonical measure in place of the canonical measure. In this case we replace $H_{\mathbf{h},\Lambda_L,M}$ by $H_{\mathbf{h},\Lambda_L,\lambda}$ and, since $\nu^{\mathbf{h}}_{\Lambda_L,\lambda}$ is a product measure, we only need to show that

$$\frac{1}{\beta}\log\int \exp\beta\Big\{H(\eta_1)-E_{\nu_\lambda}[H(\eta_1)]\Big\}\, d\nu_\lambda \;\le\; C_0\|H'\|^2_\infty\beta \tag{1.4}$$

for all $\beta>0$ and some constant C_0 which depends only on $\|F\|_\infty$

We consider first the case of β small. By the spectral gap for the the Glauber Dynamics (Lemma 1.2.1), there exists a universal constant C_0, such that

$$< f^2 >_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} - < f >^2_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \le C_0 < (\partial_{\eta_1} f)^2 >_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} .$$

for all smooth functions f in $L^2(\nu^{\mathbf{h},1}_{\Lambda_L,\lambda})$. Let $C_1 = C_0\|H'\|^2_\infty$ and assume that $\beta\|H'\|_\infty < |\Lambda_L|^{-1/2}$.

Aplying this inequality to the function $f = exp\{\frac{\beta}{2}H_{\mathbf{h},\Lambda_L,\lambda}\}$, we obtain that

$$E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{\beta H_{\mathbf{h},\Lambda_L,\lambda}}\Big] \le \Big\{E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{(\frac{\beta}{2})H_{\mathbf{h},\Lambda_L,\lambda}}\Big]\Big\}^2 + C_0\Big(\frac{\beta}{2}\Big)^2\|H'\|^2_\infty E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{\beta H_{\mathbf{h},\Lambda_L,\lambda}}\Big].$$

so that

$$E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{\beta H_{\mathbf{h},\Lambda_L,\lambda}}\Big] \le \frac{1}{1 - C_0\|R'\|^2_\infty(\frac{\beta}{2})^2}\Big\{E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{(\frac{\beta}{2})H_{\mathbf{h},\Lambda_L,\lambda}}\Big]\Big\}^2$$

$$\le e^{(\frac{1}{2})C_0\|R'\|^2_\infty\beta^2}\Big\{E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{(\frac{\beta}{2})H_{\mathbf{h},\Lambda_L,\lambda}}\Big]\Big\}^2$$

because $(1-x)^{-1} \le e^{2x}$ for $0 \le x < \frac{1}{2}$. Iterating this estimate $n-1$ times we obtain that

$$E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{\beta H_{\mathbf{h},\Lambda_L,\lambda}}\Big] \le exp\Big\{C_1\beta^2\sum_{j=1}^{n}2^{-j}\Big\}\Big\{E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{(\frac{\beta}{2^n})H_{\mathbf{h},\Lambda_L,\lambda}}\Big]\Big\}^2$$

The exponential is obviously bounded by $exp\{C_1\beta^2\}$. On the other hand, we claim that

$$\lim_{n\to\infty} n\log E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{\frac{1}{n}H_{\mathbf{h},\Lambda_L,\lambda}}\Big] = 0 \qquad (1.5)$$

showing that the left hand side of 1.4 is bounded above by $C_1\beta = C_0\beta\|H'\|_\infty^2$ provided $\beta < C_1^{-\frac{1}{2}}$.

To prove 1.5, just notice that $exp\{\frac{1}{n}H_{\mathbf{h},\Lambda_L,\lambda}\}$ is bounded above by $1 + (\frac{1}{n})H_{\mathbf{h},\Lambda_L,\lambda} + (\frac{1}{n^2})H_{\mathbf{h},\Lambda_L,\lambda}exp\{(\frac{1}{n})|H_{\mathbf{h},\Lambda_L,\lambda}|\}$. Since $\log(1+x) \le x$ and since $H_{\mathbf{h},\Lambda_L,\lambda}$ has mean zero with with respect to $\nu^{\mathbf{h}}_{\Lambda_L,\lambda}$, we obtain that

$$n \log E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{\frac{1}{n}H_{\mathbf{h},\Lambda_L,\lambda}}\Big] \le \frac{1}{n}E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[H^2_{\mathbf{h},\Lambda_L,\lambda}exp\Big\{\frac{1}{n}|H_{\mathbf{h},\Lambda_L,\lambda}|\Big\}\Big]$$

By 1.2, the right hand side is bounded above by

$$\frac{C}{n}E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[\Big\{1 + (\eta_1 - \lambda + h_1)^2\Big\}exp\Big\{\frac{C|\eta_1 - \lambda + h_1|}{n}\Big\}\Big]$$

for some finite constant C depending only on $\|F\|_\infty^2$, $\|H'\|_\infty^2$. The expectation is bounded for all $n \ge 1$ because $\nu^{\mathbf{h}}_{\Lambda_L,\lambda}$ has Gaussian tails. This prove 1.3 for $\beta < C_1^{-\frac{1}{2}}$.

We now turn to the case of large β, which is simpler. assume that $\beta\|H'\|_\infty \ge |\Lambda_L|^{-1/2}$. It follows from 1.2 that the left hand side of 1.3 is bounded above by

$$C_2\big\|H'\big\|_\infty + \beta^{-1}\log E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{\beta\|H'\|_\infty C_2|\eta_1-\lambda+h_1|}\Big]. \tag{1.6}$$

Since $e^{|x|} \le e^x + e^{-x}$, we need only to estimate $E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[\exp\{\beta\|H'\|_\infty C_2(\eta_1 - \lambda + h_1)\}]$ for β and $-\beta$. Recall the definition of the partition function Z given in 1.1.1. The logarithm of the previous expectation is equal to $\log Z(\Phi(\lambda - h_1) + \beta\|H'\|_\infty C_2) - \log Z(\Phi(\lambda - h_1) - \beta\|H'\|_\infty C_2(\lambda - h_1)$. An elementary computation gives that $(\log Z)'(\Phi(\lambda - h_1)) = \lambda - h_1$ so that the previous difference can bewritten as $logZ(\Phi(\lambda - h_1) + \beta\|H'\|_\infty C_2) - \log Z(\Phi(\lambda - h_1) - (\log Z)'(\Phi(\lambda - h_1))\beta\|H'\|_\infty C_2$. By Taylor's expansion , this difference is bounded by $(\frac{1}{2})(\beta\|H'\|_\infty C_2)^2(\log Z)''(\lambda)$ for some λ between $\Phi(\lambda - h_1)$ and $\Phi(\lambda - h_1) + \beta\|H'\|_\infty C_2$. Since $(\log Z)''(\lambda) = \sigma^2(\lambda)$ and since, by lemma 3.1.1, $\sigma^2(\lambda)$ is bounded uniformly in λ, we have that

$$\log E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[exp\Big\{\beta\big\|H'\big\|_\infty C_2(\eta_1 - \alpha)\Big\}\Big] \le C\big\|H'\big\|^2_\infty \beta^2.$$

for some constant depending only $\|F\|_\infty$. Since $\log(a+b) \le \log 2 + max\{\log a, \log b\}$, 1.6 is bounded above by

$$C_2\big\|H'\big\|_\infty + \frac{\log 2}{\beta} = C_3 \left\|H'\right\|^2_\infty \beta.$$

which is obviously bounded above by $C_4\|H'\|^2_\infty\beta$ because $\beta\|H'\|_\infty \ge |\Lambda_L|^{-1/2}$. This conclude the proof of lemma in the case of the grand canonical measure.

We now turn to the canonical measure. We need to consider two cases. Assume first that $\beta\|H'\|_\infty \le |\Lambda_L|^{-1/2}$. By Schwarz inequality, the left hand side of (1.3) is bounded above by the sum of two terms. The second one is similar to the first which is equal to

$$\frac{1}{2\beta|\Lambda_L|}\log\int \exp\Big\{2\beta\sum_{x=1}^{L/2} H_{\mathbf{h},\Lambda_L,M}(\eta_x)\Big\}\,d\mu^{\mathbf{h}}_{\Lambda_L,M}\ .$$

The difference is that we are now summing only over one half of the cube and that we had to pay a factor 2 in the exponential to do it. Since $e^x \le 1 + x + x^2 e^{|x|}$, since $\log(1 + x) \le x$ and since $H_{\mathbf{h},\Lambda_L,M}(\eta_x)$ has mean zero with respect to $\mu^{\mathbf{h}}_{\Lambda_L,M}$, the previous expression is bounded above by

$$\frac{4\beta}{|\Lambda_L|} \int \Big\{ \sum_{x=1}^{L/2} H_{\mathbf{h},\Lambda_L,M}(\eta_x) \Big\}^2 \exp \Big\{ 2\beta \Big| \sum_{x=1}^{L/2} H_{\mathbf{h},\Lambda_L,M}(\eta_x) \Big| \Big\} \, d\mu^{\mathbf{h}}_{\Lambda_L,M} \ .$$

Since $e^{|x|} \le e^x + e^{-x}$, we may remove the absolute value in the exponential, provide we estimate the expression for $H_{\mathbf{h},\Lambda_L,M}$, as well as for $-H_{\mathbf{h},\Lambda_L,M}$.

Fix λ given by (1.3). By Corollary 3.1.6, we may replace the canonical measure by the grand canonical one paying the price of a finite constant and turning $H_{\mathbf{h},\Lambda_L,M}$ into a *non* mean-zero function. At this point, we need to estimate

$$\frac{C_0\beta}{|\Lambda_L|} \int \Big\{ \sum_{x=1}^{L/2} H_{\mathbf{h},\Lambda_L,M}(\eta_x) \Big\}^2 \exp \Big\{ 2\beta \sum_{x=1}^{L/2} H_{\mathbf{h},\Lambda_L,M}(\eta_x) \Big\} \, d\nu^{\mathbf{h}}_{\Lambda_L,\lambda} \ .$$

Since $\nu^{\mathbf{h}}_{\Lambda_L,M}$ is a product measure, expanding the square, we get that the previous integral is less than or equal to

$$\frac{C_0\beta}{|\Lambda_L|} \sum_{x=1}^{L/2} A_2(x) \prod_{z\neq x} A_0(z) \ + \ \frac{C_0\beta}{|\Lambda_L|} \sum_{x\neq y} A_1(x) A_1(y) \prod_{z\neq x,y} A_0(z) \ ,$$

where, for $1 \le x \le L/2$ and $j = 0, 1, 2$,

$$A_j(x) \ = \ E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[H_{\mathbf{h},\Lambda_L,M}(\eta_x)^j e^{2\beta H_{\mathbf{h},\Lambda_L,M}(\eta_x)} \Big] \ .$$

We claim that

$$A_0(x) \ \le \ \exp\{C_0\beta\|H'\|_\infty L^{-1/2}\} \tag{1.7}$$

$$\big|A_1(x)\big| \ \le \ \frac{C_0\|H'\|_\infty}{L^{1/2}} \tag{1.8}$$

$$A_2(x) \ \le \ C_0\|H'\|_\infty^2 \tag{1.9}$$

uniformly in $1 \le x \le L/2$ and for some constant C_0 which depends only on $\|F\|_\infty$.

Since the lemma follows from these estimates in the case $\beta\|H'\|_\infty \le L^{-1/2}$, we only need to prove them. We first examine the exponential $A_0(x)$. By (1.1),

$$E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{2\beta H_{\mathbf{h},\Lambda_L,M}(\eta_x)}\Big] \ \le \ e^{C_0\beta L^{-1}\|H'\|_\infty} E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{2\beta H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)}\Big] \ .$$

Since $H_{\mathbf{h},\Lambda_L,\lambda}$ has mean zero with respect to $\nu^{\mathbf{h}}_{\Lambda_L,\lambda}$, since $e^x \le 1+x+x^2e^{|x|}$, since by (1.2) $|H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)| \le C_0\|H'\|_\infty[1+|\eta_x - E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[\eta_x]|\,]$ and since $\beta\|H'\|_\infty \le L^{-1/2} \le 1$,

$$E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{2\beta H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)}\Big] \le 1 + E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[(2\beta H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x))^2 \exp\{2\beta H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)|\}]$$

$$\le 1+C\beta^2\big\|H'\big\|_\infty^2 E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[(1+|\eta_z-\gamma_1(\lambda-h_z)|)^2\exp\{C\beta\big\|H'\big\|_\infty(1+|\eta_z-\gamma_1(\lambda-h_z)|)\}]$$

$$\le 1+C\beta^2\big\|H'\big\|_\infty^2 e^{C\beta\|H'\|_\infty} E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[(1+|\eta_z-\gamma_1(\lambda-h_z)|)^2\exp\{C\beta\big\|H'\big\|_\infty|\eta_z-\gamma_1(\lambda-h_z)|\}]$$

$$\le \ 1 + C_0\beta^2\|H'\|_\infty^2 \sup_{\lambda\in\mathbb{R}} E_{\nu_\lambda}\Big[\{1+|\eta-\gamma_1(\lambda)|^2\}e^{2C_0|\eta-\gamma_1(\lambda)|}\Big] \ .$$

There exists some finite constant C_0', depending only on $\|F\|_\infty$, such that

$$\sup_{\lambda\in\mathbb{R}} E_{\nu_\lambda}\Big[\{1+|\eta-\gamma_1(\lambda)|^2\}e^{2C_0|\eta-\gamma_1(\lambda)|}\Big] \;\le\; C_0'$$

because ν_λ has uniform Gaussian tails. Since $1+x\le e^x$, we conclude that

$$\begin{aligned} E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{2\beta H_{\mathbf{h},\Lambda_L,M}(\eta_x)}\Big] \;&\le\; e^{C_0\beta\|H'\|_\infty}(1+C\beta^2\big\|R'\big\|_\infty^2)\\ &\le e^{c\beta\|R'\|_\infty}(1+C\beta\big\|R'\big\|_\infty)\\ &\le \exp\{C_0\beta\|H'\|_\infty L^{-1/2}\} \end{aligned} \tag{1.10}$$

because $\beta^2\|H'\|_\infty^2\le L^{-1}$.

We now turn to $A_2(x)$ in (1.7). As before, we may replace $H_{\mathbf{h},\Lambda_L,M}(\eta_x)$ by $H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)$ in the exponential. After this replacement, applying (1.1),

$$E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[H_{\mathbf{h},\Lambda_L,M}(\eta_x)^2e^{2\beta H_{\mathbf{h},\Lambda_L,M}(\eta_x)}\Big] \;\le\; C_0E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[H_{\mathbf{h},\Lambda_L,M}(\eta_x)^2e^{2\beta H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)}\Big]$$

because $\beta\|H'\|_\infty\le 1$. The same estimate (1.1) gives that the previous expression is less than or equal to

$$\frac{C_0\|H'\|_\infty^2}{|\Lambda_L|^2}E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{2\beta H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)}\Big] \;+\; C_0E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)^2e^{2\beta H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)}\Big]\,.$$

By (1.2), $|H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)|\le C_0\|H'\|_\infty(1+|\eta_x-\gamma_1(\lambda-h_x)|)$. The previous sum is thus bounded by

$$C_0 \|H'\|_\infty^2 \sup_{\lambda\in\mathbb{R}} E_{\nu_\lambda}\Big[\{1+|\eta-\alpha|^2\} e^{2C_0|\eta-\gamma_1(\lambda)|}\Big] \; .$$

This expression is less than $C_0\|H'\|_\infty^2$ because ν_λ has uniform exponential tails.

It remains to estimate $|A_1(x)|$ which is given by

$$\Big| E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[H_{\mathbf{h},\Lambda_L,M}(\eta_x) e^{2\beta H_{\mathbf{h},\Lambda_L,M}(\eta_x)} \Big] \Big| \; .$$

As before, we may replace $H_{\mathbf{h},\Lambda_L,M}(\eta_x)$ by $H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)$ in the exponential. After this replacement, applying (1.1), we bound the previous expression by

$$C_0 \Big| E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x) e^{2\beta H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)} \Big] \Big| \; + \; \frac{C_0\|H'\|_\infty}{|\Lambda_L|} E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[e^{2\beta H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)} \Big] \; .$$

By (1.9),the second term is seen to be less than or equal to

$$\frac{C_0\|H'\|_\infty}{|\Lambda_L|} e^{2\beta \frac{C_0\|H'\|_\infty}{|\Lambda_L|^{\frac{1}{2}}}} \le \frac{C_0\|H'\|_\infty}{|\Lambda_L|}$$

because $\beta\|H'\|_\infty \le \frac{\beta\|H'\|_\infty}{|\Lambda_L|^{\frac{1}{2}}}$.

The first term, since $|e^b - 1| \le |b| e^{|b|}$ and since $H_{\mathbf{h},\Lambda_L,\lambda}$ has mean zero, is bounded by

$$C_0 E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x) \Big(1 + |H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)| \beta e^{2\beta H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)} \Big) \Big]$$

$$\le C_0 \beta E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)^2 e^{2\beta |H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)|} \Big]$$

because $E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x) \Big] = 0$, then the previous expression is bounded by

$$C_0 \beta \|H'\|_\infty^2 \sup_{\lambda\in\mathbb{R}} E_{\nu_\lambda} \Big[\{1 + |\eta - \gamma_1(\lambda)|^2\} e^{2C_0|\eta - \gamma_1(\lambda)|} \Big]$$

because $\beta \|H'\|_\infty \le 1$ in the range considered. This expression is bounded by $C_0\beta\|H'\|_\infty^2$ because ν_λ has uniform exponential tails. Adding the two previous estimates we obtain that $A_1(x)$ is absolutely bounded by $C_0\|H'\|_\infty L^{-1/2}$ in the range considered. This concludes the proof of Claim (1.7) and therefore the proof of the lemma in the case of small values of β.

We now turn to the case of large β. Assume that $\beta^2\|H'\|_\infty^2 > |\Lambda_L|^{-1}$. We first replace $H_{\mathbf{h},\Lambda_L,M}$ by $H_{\mathbf{h},\Lambda_L,\lambda}$. By (1.1), the left hand side of (1.3) is bounded above by

$$\frac{1}{\beta|\Lambda_L|} \log \int \exp\Big\{ \beta \sum_{x\in\Lambda_L} H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x) \Big\} \, d\mu^{\mathbf{h}}_{\Lambda_L,M} \; + \; \frac{C_0\|H'\|_\infty}{|\Lambda_L|} \; .$$

Since $|\Lambda_L|^{-2} \le |\Lambda_L|^{-1} < \beta^2\|H'\|_\infty^2$, $|\Lambda_L|^{-1} \le \beta\|H'\|_\infty$. In particular, the second term is less than or equal to $C_0\beta\|H'\|_\infty^2$.

It remains to estimate the first term. By Schwarz inequality, this expression is bounded above by the sum of two terms,

$$\frac{1}{2\beta|\Lambda_L|}\log\int\exp\Big\{2\beta\sum_{1\le x\le L/2} H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)\Big\}\,d\mu^{\mathbf{h}}_{\Lambda_L,M}$$

$$+\frac{1}{2\beta|\Lambda_L|}\log\int\exp\Big\{2\beta\sum_{1+L/2\le x\le L} H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)\Big\}\,d\mu^{\mathbf{h}}_{\Lambda_L,M}\ .$$

Both terms are estimated in the same way. We consider only the first term:

$$\frac{1}{2\beta|\Lambda_L|}\log\int\exp\Big\{2\beta\sum_{1\le x\le L/2} H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)\Big\}\,d\mu^{\mathbf{h}}_{\Lambda_L,M}\ .$$

By Corollary 3.1.6, this expression is less than or equal to

$$\frac{C_0}{\beta|\Lambda_L|}\ +\ \frac{1}{2\beta|\Lambda_L|}\log\int\exp\Big\{2\beta\sum_{1\le x\le L/2} H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)\Big\}\,d\nu^{\mathbf{h}}_{\Lambda_L,\lambda}\ ,$$

where λ is chosen according to (1.3). Since $|\Lambda_L|^{-1}\le\beta^2\|H'\|^2_\infty$, the first term is bounded by $C\beta\|H'\|^2_\infty$. It remains to consider the second one which is equal to

$$\frac{1}{2\beta|\Lambda_L|}\sum_{1\le x\le L/2}\log\int\exp\Big\{2\beta H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)\Big\}\,d\nu^{\mathbf{h}}_{\Lambda_L,\lambda}$$

because $\nu^{\mathbf{h}}_{\Lambda_L,\lambda}$ is a product measure. In view of (1.4), this expression is bounded by $C\beta\|H'\|^2_\infty$. This concludes the proof. □

The same proof gives the following estimate that we state for further use.

Lemma 5.1.2 *Fix a differentiable function $H\colon \mathbb{R} \to \mathbb{R}$ with bounded derivative: $\|H'\|_\infty < \infty$. There exists a constant C_0, depending only on $\|F\|_\infty$, such that*

$$\frac{1}{\beta} \log \int \exp\Big\{\beta H_{\mathbf{h},\Lambda_L,M}(\eta_x)\Big\}\, d\mu^{\mathbf{h}}_{\Lambda_L,M} \ \le\ C_0 \|H'\|^2_\infty \beta$$

for all $\beta > 0$, $L \ge 2$, x in Λ_L, environment $\mathbf{h}$ and M in $\mathbb{R}$.

Proposition 5.1.1 provides an estimate, uniform over the charge M, on the expectation of $|\Lambda_L|^{-1} \sum_{x\in\Lambda_L} H_{\mathbf{h},\Lambda_L,M}(\eta_x)$ with respect to some measure $f d\mu^{\mathbf{h}}_{\Lambda_L,M}$ in terms of the entropy of this measure.

Corollary 5.1.3 *Fix $L \ge 2$, M in $\mathbb{R}$, an environment $\mathbf{h}$, a differentiable function $H\colon \mathbb{R} \to \mathbb{R}$ with bounded derivative and a density f with respect to $\mu^{\mathbf{h}}_{\Lambda_L,M}$. There exists a constant C_0, depending only on $\|F\|_\infty$, such that*

$$\Big(\int \Big\{ \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} H_{\mathbf{h},\Lambda_L,M}(\eta_x) \Big\}\, f\, d\mu^{\mathbf{h}}_{\Lambda_L,M} \Big)^2 \ \le\ C_0 \frac{\|H'\|^2_\infty}{|\Lambda_L|} S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, \sqrt{f})\,.$$

Proof: By the entropy inequality, the integral on the left hand side of the statement of the lemma is bounded above by

$$\frac{1}{\beta|\Lambda_L|} \log \int \exp\Big\{\beta \sum_{x\in\Lambda_L} H_{\mathbf{h},\Lambda_L,M}(\eta_x)\Big\}\, d\mu^{\mathbf{h}}_{\Lambda_L,M} \ +\ \frac{S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, \sqrt{f})}{\beta|\Lambda_L|}$$

for all $\beta > 0$. By Proposition 5.1.1, the first term is bounded above by $C_0\|H'\|^2_\infty \beta$ for some finite constant depending only on $\|F\|_\infty$. Minimizing in β we conclude the proof of the lemma. □

Lemma 5.1.2 provides a similar estimate in the case of a one site function.

Lemma 5.1.4 *Fix $L \geq 2$, M in $\mathbb{R}$, an environment $\mathbf{h}$, a differentiable function $H\colon \mathbb{R} \to \mathbb{R}$ with bounded derivative and a density f with respect to $\mu^{\mathbf{h},1}_{\Lambda_L,M}$. There exists a constant C_0, depending only on $\|F\|_\infty$, such that*

$$\Big(\int H_{\mathbf{h},\Lambda_L,M}(\eta_1)\, f(\eta_1)\, d\mu^{\mathbf{h},1}_{\Lambda_L,M} \Big)^2 \;\leq\; C_0 \|H'\|^2_\infty S_{\{1\}}(\mu^{\mathbf{h},1}_{\Lambda_L,M}, \sqrt{f})\; .$$

The proof is the same as the one of Corollary 5.1.3.

Fix $K \geq 1$, $L \geq K^2$ and divide the interval $\{1, \dots, L\}$ into $\ell = \lfloor L/K \rfloor$ adjacent intervals of length K or $K+1$, where $\lfloor a \rfloor$ represents the integer part of a. For $1 \leq j \leq \ell$, denote by I_j the j-th interval, by M_j the total spin on I_j: $M_j = \sum_{x \in I_j} \eta_x$ and let

$$\begin{aligned} m_j \;&=\; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}} \Big[\frac{1}{|I_j|} \sum_{x\in I_j} \eta_x \Big]\; , \\ m_j^* \;&=\; E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[\frac{1}{|I_j|} \sum_{x\in I_j} \eta_x \Big]\; , \\ A_j(m) \;&=\; E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(m)}} \Big[\frac{1}{|I_j|} \sum_{x\in I_j} F'(\eta_x) \Big]\; , \end{aligned}$$

where λ is given by (1.3). Notice that

$$|m_j - m_j^*| \;\leq\; C_0 L^{-1} \tag{1.11}$$

in virtue of Lemma 3.1.4 and Lemma (3.**??**).

Lemma 5.1.5 *For* $1 \le j \le \ell$, *let* $G_j = G_j(M_j; M, L, \mathbf{h})$ *be given by*

$$G_j \;=\; E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big] - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big] - A'_j(m_j)\Big(\frac{M_j}{|I_j|} - m_j\Big)\;.$$

There exists a finite constant C_0^*, *depending only on* $\|F\|_\infty$, *and a finite constant* C_1, *depending only on* $\|F\|_\infty$, $\|F'\|_\infty$, *such that*

$$|G_j| \;\le\; C_0^*(M_j/|I_j| - m_j^*)^2 + \frac{C_1}{|I_j|}\;.$$

Proof: Fix j and rewrite G_j as

$$\begin{aligned}
&E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big] \;-\; E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(M_j/|I_j|)}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\\
&+ E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(m_j)}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big] \;-\; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\\
&+ A_j(M_j/|I_j|) \;-\; A_j(m_j) \;-\; A'_j(m_j)(M_j/|I_j| - m_j)\;.
\end{aligned}$$

The first two lines will be estimated by the equivalence of ensembles, while the third is a Taylor expansion up to the second order.

By Corollary 3.1.4, the first line is bounded above by $C_1|I_j|^{-1}$ for some finite constant C_1. By Lemma 4.1.1, $\|A''_j\|_\infty$ is finite. In particular, in view of (1.11), the third term is bounded above by $C_0(M_j/|I_j| - m_j^*)^2 + C_0L^{-2}$. Taking conditional expectation with respect to $\sum_{x\in I_j}\eta_x$, $\{\eta_z,\, z\notin I_j\}$ in the second expectation, the second line can be rewritten as

$$E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(m_j)}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)\Big]\;-\;E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)\Big]\Big]\;.$$

Using the equivalence of ensembles, Corollary 3.1.4, we may replace the second term inside the expectation by an expectation with respect to the grand canonical measure with chemical potential given by $\Phi_{I_j}(M_j/|I_j|)$. We may also add inside the expectation $A'_j(m_j)(M_j/|I_j|-m_j)$, because this expression has mean zero. We recover in this way the same Taylor expansion up to the second order, which was shown above to be bounded by $C_0(M_j/|I_j|-m^*_j)^2+C_0L^{-2}$. Applying Corollary 3.1.6 to replace the canonical measure $\mu^{\mathbf{h}}_{\Lambda_L,M}$ by the grand canonical, we obtain that the second term in the decomposition of G_j is bounded above by $C_0|I_j|^{-1}$. □

Proposition 5.1.6 *There exist $\beta_0>0$ and a finite constant C_1 depending only on $\|F\|_\infty$, $\|F'\|_\infty$ such that*

$$\frac{1}{\beta L}\log E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\exp\Big\{\beta\sum_{j=1}^{\ell}|I_j|G_j\Big\}\Big]\;\le\;\frac{C_1\beta}{K} \tag{1.12}$$

for all $\beta\le\beta_0$, all $L\ge K^2$, all M in $\mathbb{R}$ and all environment $\mathbf{h}$.

Proof: We first prove the lemma in the grand canonical case with G replaced by the mean-zero function $\mathcal{G}$. For $1\le j\le\ell$, let $\mathcal{G}_j=\mathcal{G}_j(M_j;M,L,\mathbf{h})$ be given by

$$\mathcal{G}_j\;=\;E_{\mu^{\mathbf{h}}_{I_j,\lambda}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)\Big]-E_{\mu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)\Big]-A'_j(m^*_j)\Big(\frac{M_j}{|I_j|}-m^*_j\Big)\;.$$

To keep notation simple, assume that all cubes I_j has the same length K. Since $\mu^{\mathbf{h}}_{\Lambda_L,\lambda}$ is a product measure , the left hand side of (1.12) is equal to

$$\frac{1}{\beta K}\log E_{\mu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[exp\Big\{\beta K\mathcal{G}\Big\}\Big]$$

Since $e^x \leq 1 + x + x^2 e^{|x|}$, since $\log(1+x) \leq x$ and since $E_{\mu^{\mathbf{h}}_{\Lambda_L,\lambda}}[\mathcal{G}] = 0$, the previous expression is less than or equal a to

$$\frac{\beta}{K}\log E_{\mu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[(K\mathcal{G}(M_1))^2 exp\Big\{\beta K|\mathcal{G}(M_1)|\Big\}\Big]$$

We claim that there exists β_1 and a finite constant C_0 such that

$$\frac{\beta}{K}\log E_{\mu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[(K\mathcal{G}(M_1))^2 exp\Big\{\beta K|\mathcal{G}(M_1)|\Big\}\Big] \leq C_0 \qquad (1.13)$$

for all m in $\mathbb{R}$, all $K \geq 1$ and $\beta \leq \beta_1$. Since $\mathcal{G}(M_1) = E^{\mathbf{h}}_{I_1,M_1,\lambda}[F'] - A\Big(\frac{M_1}{|I_1|}\Big) + A\Big(\frac{M_1}{|I_1|}\Big) - A(m_1*) - A'(m_1^*)\Big[\frac{M_1}{|I_1|} - m_1^*\Big]$, by lemma 4.1.1 and Corollary 3.1.3 , $\mathcal{G}$ is bounded in absolute value by $CK^{-1} + C\Big(\frac{M_1}{|I_1|} - m_1^*\Big)^2$ for some constant C. In particular, the left hand side of 1.10 is bounded above by

$$Ce^{C\beta}E_{\mu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[\Big\{1 + K^2\Big(\frac{M_1}{|I_1|} - m_1^*\Big)^4 exp\Big\{C\beta K\Big(\frac{M_1}{|I_1|} - m_1^*\Big)^2\Big\}\Big]$$

$$\leq CE_{\mu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[exp\Big\{C'\beta K\Big(\frac{M_1}{|I_1|} - m_1^*\Big)^2\Big\}\Big]$$

By Lemma 3.1.8, there exists $\beta_1 > 0$ such that for $\beta < \beta_1$, the expectation is bounded uniformly in K and m. this prove claim (1.10) and that the left hand side of (1.9) is bounded by $\frac{C\beta}{K}$ for $\beta \leq \beta_1$, wich concludes the proof of the lemma in the grand canonical case.

We now turn to the canonical measure.

Fix $\beta_1 > 0$ given by Lemma 3.1.8 and set $\beta_0 = \beta_1/4C_0^*$, where C_0^* is the finite constant introduced in Lemma 5.1.5. By Schwarz inequality, the left hand side of (1.12) is bounded by the sum of two terms. The first one is equal to

$$\frac{1}{2\beta L} \log E_{\mu^{\mathbf{h}}_{\Lambda_L,M}} \Big[\exp \Big\{ 2\beta \sum_{j=1}^{\ell/2} |I_j| G_j \Big\} \Big] . \tag{1.14}$$

The difference between the second one and the first is that we sum over $\ell/2+1 \leq j \leq \ell$ instead of $1 \leq j \leq \ell/2$. We are now in a position to estimate the expectation with respect to a canonical measure by the expectation with respect to a grand canonical measure through Corollary 3.1.6.

Assume first that $\beta^2 \leq \min\{\ell^{-1}, \beta_0^2\}$. In this case, since $\exp\{x\} \leq 1 + x + x^2 \exp\{|x|\}$, since $\log(1+x) \leq x$ and since the sum that appears in the exponential of (1.12) has mean zero, the left hand side in (1.12) is bounded above by

$$\frac{2\beta}{L} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}} \Big[\Big(\sum_{j=1}^{\ell/2} |I_j| G_j \Big)^2 \exp \Big\{ 2\beta \Big| \sum_{j=1}^{\ell/2} |I_j| G_j \Big| \Big\} \Big] .$$

Since $e^{|x|} \leq e^x + e^{-x}$, we may remove the absolute value in the exponential provide we estimate the previous expression with $-\beta$ in place of β in the exponential. Consider the case with β. Fix λ given by (1.3). By Corollary 3.1.6, the previous expression without the absolute value in the exponential is less than or equal to

$$\frac{C_0\beta}{L} E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[\Big(\sum_{j=1}^{\ell/2} |I_j| G_j \Big)^2 \exp \Big\{ 2\beta \sum_{j=1}^{\ell/2} |I_j| G_j \Big\} \Big] .$$

Since $\nu^{\mathbf{h}}_{\Lambda_L,\lambda}$ is a product measure, expanding the square we obtain that this term is equal to

$$\frac{C_0\beta}{L} \sum_{j=1}^{\ell/2} |I_j|^2 E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[G_j^2 \, e^{2\beta |I_j| G_j} \Big] \prod_{k\neq j} E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[e^{2\beta |I_k| G_k} \Big] \qquad (1.15)$$

$$+ \frac{C_0\beta}{L} \sum_{j\neq k} |I_j||I_k| E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[G_j \, e^{2\beta |I_j| G_j} \Big] E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[G_k \, e^{2\beta |I_k| G_k} \Big] \prod_{i\neq j,k} E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[e^{2\beta |I_i| G_i} \Big] .$$

There are three different types of terms in the previous formula and we estimate them separately.

We claim that

$$E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[e^{2\beta |I_j| G_j} \Big] \;\le\; \exp C_1 \beta \Big\{ \ell^{-1} + \beta \Big\} , \qquad (1.16)$$

$$\Big| E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[G_j e^{2\beta |I_j| G_j} \Big] \Big| \;\le\; C_1 \Big\{ \frac{1}{L} + \frac{\beta}{K} \Big\} ,$$

$$E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[G_j^2 e^{2\beta |I_j| G_j} \Big] \;\le\; \frac{C_1}{K^2} .$$

Notice that, since $\beta^2 \le \min\{1, \ell^{-1}\}$, in view of the previous bounds, it is easy to show that (1.15) is less than or equal to $C_0 \beta K^{-1}$, which is what we wanted to prove. Therefore, to conclude the proof in the case $\beta^2 \le \min\{\ell^{-1}, \beta_0^2\}$, we need only to check (1.16). We start examining the exponential terms. Since $e^x \le 1 + x + x^2 e^{|x|}$, the expectation is bounded above by

$$1 \;+\; 2\beta |I_j| E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[G_j] \;+\; 4\beta^2 |I_j|^2 E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[G_j^2 e^{2\beta |I_j||G_j|} \Big] \, . \tag{1.17}$$

The linear term is easy to handle. By definition of G_j, the linear expectation is equal to

$$E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|} \sum_{x\in I_j} F'(\eta_x) \Big] \Big] \;-\; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{1}{|I_j|} \sum_{x\in I_j} F'(\eta_x) \Big]$$

$$-\; A_j'(m_j) \Big(E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[\frac{M_j}{|I_j|} \Big] \;-\; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{M_j}{|I_j|} \Big] \Big) \, .$$

Since the expectation with respect to the canonical measure $\mu^{\mathbf{h}}_{I_j,M_j}$ can be understood as a $\nu^{\mathbf{h}}_{\Lambda_L,\lambda}$ conditional expectation with respect to the σ-algebra generated by M_j, $\{\eta_x,\, x \notin I_j\}$, the first line is equal to

$$E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[\frac{1}{|I_j|} \sum_{x\in I_j} F'(\eta_x) \Big] \;-\; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{1}{|I_j|} \sum_{x\in I_j} F'(\eta_x) \Big] \, .$$

By Corollary 3.1.4, this expression is bounded by $C_0 \|F'\|_\infty L^{-1}$. On the other hand, since by Lemma 4.1.1, $\|A_j'(\cdot)\|_\infty$ is uniformly bounded in j, by Corollary 3.1.4 and by Lemma 1.3.1, the second line is also bounded by $C_0 L^{-1}$. The linear term in (1.17) is thus less than or equal to $C_1 \beta |I_j|/L$.

In view of Lemma 5.1.5, the quadratic term in (1.17) is bounded by

$$C_1 \beta^2 e^{C_1 \beta} E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[\Big\{ |I_j|^2 \Big(\frac{M_j}{|I_j|} - m_j^* \Big)^4 + 1 \Big\} e^{2 C_0^* \beta |I_j| (M_j/|I_j| - m_j^*)^2} \Big] \, .$$

Since $\beta \le \beta_0$ and since $x^2 e^{ax} \le C(a) e^{2ax}$, the previous expression is bounded above by

$$C_1 \beta^2 E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[e^{4C_0^* \beta_0 |I_j| (M_j/|I_j| - m_j^*)^2} \Big] .$$

Since $4C_0^* \beta_0 = \beta_1$, this expression is bounded by $C_1 \beta^2$ in virtue of Lemma 3.1.8.

To conclude the proof of the estimate of the exponential term in (1.16) it remain to recollect the previous estimate and to recall that $1 + x \le e^x$.

We turn now to the second expectation in (1.16). Since $|e^x - 1| \le |x| e^{|x|}$, this expectation is bounded above by

$$\Big| E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} [G_j] \Big| \; + \; 2\beta |I_j| E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[G_j^2 e^{2\beta |I_j| |G_j|} \Big] .$$

We have seen in the estimate of the linear term in (1.17) that the first expression in the previous formula is bounded above by $C_1 L^{-1}$ and we have seen in the estimate of the quadratic term that the second one is bounded above by $C_1 \beta K^{-1}$. This proves the second bound in (1.16).

The third estimate in (1.16) follows from the estimate of the quadratic term in (1.17). This concludes the proof of (1.16) and therefore of the lemma in the case $\beta^2 \le \min\{\ell^{-1}, \beta_0^2\}$.

Assume now that $\ell^{-1} \le \beta^2 \le \beta_0^2$. In this case, by Corollary 3.1.6, (1.14) is bounded above by

$$\frac{C_0}{\beta L} \; + \; \frac{1}{2\beta L} \sum_{j=1}^{\ell/2} \log E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[\exp \Big\{ 2\beta |I_j| G_j \Big\} \Big] . \qquad (1.18)$$

Since $e^x \le 1 + x + x^2 e^{|x|}$ and since $\log(1+x) \le x$, the logarithm is less than or equal to

$$2\beta |I_j| E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[G_j] \;+\; 4\beta^2 |I_j|^2 E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[G_j^2 \exp\Big\{2\beta|I_j||G_j|\Big\}\Big] \,. \tag{1.19}$$

We have seen in the estimate of (1.17) that the linear term is bounded by $C_1\beta\ell^{-1}$ and that the quadratic term is bounded by $C_1\beta^2$. In view of these estimates, (1.18) is less than or equal to

$$\frac{C_0}{\beta L} \;+\; \frac{C_1}{\beta K}\Big\{\frac{\beta}{\ell} + \beta\Big\}\,.$$

Since $\ell^{-1} \le \beta^2$, this expression is less than or equal to $C_1\beta/K$, which proves the lemma. □

Lemma 5.1.7 *Fix $K \ge 2$, $L \ge K^2$, an environment $\mathbf{h}$, c, M in $\mathbb{R}$ and a smooth function g in $L^2(\mu^{\mathbf{h}}_{\Lambda_L,M})$ such that $< g^2 >_{\mu^{\mathbf{h}}_{\Lambda_L,M}} = 1$. There exists a finite constant C_0 depending on $\|F\|_\infty$ such that*

$$\frac{1}{\beta L}\log E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\exp\Big\{\sum_{j=1}^{\ell} |I_j|[A'_j(m_j) - c]\Big(\frac{M_j}{|I_j|} - m_j\Big)\Big\}\Big]$$

$$\le\; \frac{C_0\beta}{K}\exp\Big\{\frac{C_0}{\ell}\sum_{j=1}^{\ell}|I_j|(A'_j(m_j) - c)^2\Big\}\,. \tag{1.20}$$

for every $\beta > 0$.

Proof: Assume first that $\beta^2\ell \le 1$. Let $H_j = [A'_j(m_j) - c](M_j/|I_j| - m_j)$. The beginning of the proof is identical to the one of Proposition 5.1.6 up to formula (1.15) with H_j in place of G_j. We claim that there exists a finite constant depending only on F such that

$$E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{2\beta|I_j|H_j}\Big] \;\le\; \exp C_0\beta\Big\{\ell^{-1}+\beta|I_j|X_j^2\Big\}\;, \tag{1.21}$$

$$\Big|E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[H_j e^{2\beta|I_j|H_j}\Big]\Big| \;\le\; C_0\Big\{\frac{1}{L}+\beta X_j^2\Big\}\;,$$

$$E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[H_j^2 e^{2\beta|I_j|H_j}\Big] \;\le\; \frac{C_0X_j^2}{K}\;,$$

where $X_j = A_j'(m_j) - c$. It follows from Lemma 4.1.1 that X_j is absolutely bounded.

We start estimating the exponential. Recall that m_j^* stands for the expectation of the density of particles in I_j for the grand canonical measure and that $|m_j - m_j^*| \le C_0L^{-1}$ according to (1.11). In particular, the exponential term is less than or equal to

$$e^{C_0\beta\ell^{-1}|X_j|}E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[e^{2\beta|I_j|X_j(M_j/|I_j|-m_j^*)}\Big]\;.$$

Since X_j is absolutely bounded, the first exponential is bounded by $\exp\{C_0\beta\ell^{-1}\}$. It remains to estimate the expectation. Since the expression in the exponential has zero mean, expanding the exponential up to the second order, we obtain that the expectation is bounded above by

$$1\;+\;4\beta^2|I_j|X_j^2E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[|I_j|(M_j/|I_j|-m_j^*)^2e^{2\beta|I_j|\,|X_j(M_j/|I_j|-m_j^*)|}\Big]\;.$$

Recall the definition of β_1 introduced in Lemma 3.1.8. Since $2ab \le Aa^2 + A^{-1}b^2$ for any $A>0$, the previous expression is less than or equal to

$$1 + 4\beta^2 |I_j| X_j^2 e^{2\beta^2 \beta_1^{-1} |I_j| X_j^2} E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[|I_j| (M_j/|I_j| - m_j^*)^2 e^{(\beta_1/2)|I_j|(M_j/|I_j| - m_j^*)^2} \Big] \, .$$

Since $ae^{(\beta_1/2)a} \le C(\beta_1) e^a$ for $a > 0$, since $1 + x \le e^x$, since X_j is bounded and since $\beta^2 \le \ell^{-1} = K/L \le K^{-1}$, by Lemma 3.1.8, the previous expression is bounded above by $\exp\{C_0 \beta^2 |I_j| X_j^2\}$, which proves the first estimate in (1.21).

To estimate the linear term, add and subtract H_j in the expectation and recall that $|\exp\{x\} - 1| \le |x| \exp\{|x|\}$, $|m_j - m_j^*| \le C_0/L$ to deduce that the linear term is absolutely bounded by

$$\frac{C_0 |X_j|}{L} \; + \; 2\beta X_j^2 E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[|I_j| (M_j/|I_j| - m_j)^2 e^{2\beta |I_j| \, |X_j (M_j/|I_j| - m_j)|} \Big] \, .$$

Replace m_j by m_j^* in the expressions above and recall that X_j is absolutely bounded to estimate the sum by

$$\frac{C_0}{L} \; + \; C_0 \beta X_j^2 E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[e^{2\beta |I_j| \, |X_j (M_j/|I_j| - m_j^*)|} \Big]$$
$$+ \, C_0 \beta X_j^2 E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}} \Big[|I_j| (M_j/|I_j| - m_j^*)^2 e^{2\beta |I_j| \, |X_j (M_j/|I_j| - m_j^*)|} \Big]$$

because β and X_j are bounded and $|m_j - m_j^*| \le C_0/L$. It remains to repeat the arguments presented for the exponential term to obtain that this expression is less than or equal to $C_0 L^{-1} + C_0 \beta X_j^2$, proving the second claim in (1.21).

By similar reasons, the quadratic term is bounded by $C_0 K^{-1} X_j^2$. This concludes the proof of (1.21).

With the estimates (1.15) it is not difficult to prove lemma in the case $\beta^2 \ell \le 1$.

We turn now to the case $\ell^{-1} \le \beta^2 \le \beta_0^2$ and follow the second part of the proof of Proposition 5.1.6 up to formula (1.19) with H_j in place of G_j. Since $|m_j - m_j^*| \le C_0 L^{-1}$ and since X_j is bounded, the linear term of (1.19) with H_j in place of G_j is bounded by $C \beta \ell^{-1}$. On the other hand, the quadratic term is less than or equal to $C_0\beta^2|I_j|X_j^2$. to derive this estimate, we need to keep in mind that X_j is absolutely bounded and that $K^2 \leq L$. Is is easy to conclude the proof of the lemma with these estimates. □

We conclude this section with a technical result needed in the proof of the logarithmic Sobolev inequality.

Lemma 5.1.8 *Fix a bounded function $H : \mathbb{R} \to \mathbb{R}$, an environment $\mathbf{h}$ and $L \geq 2$. The function $\tilde{H}_{L,\mathbf{h}} : \mathbb{R} \to \mathbb{R}$ defined by $\tilde{H}_{L,\mathbf{h}}(m) = E_{\mu^{\mathbf{h}}_{\Lambda_L, m|\Lambda_L|}}[H(\eta_1)]$ is Lipschitz continuous on $\mathbb{R}$ and the Lipschitz constant does not depend on L.*

Proof: An elementary computation shows that

$$\partial_M E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[H(\eta_1)] \;=\; -E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[F'(\eta_2); H(\eta_1)]$$

$$=\; -E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[H(\eta_1)\big\{F'(\eta_2) - < F'(\eta_2) >_{\mu^{\mathbf{h}}_{\Lambda_L,M}} \big\}\Big]\,.$$

By Corollary 3.1.4, the absolute value of the previous expression is bounded above by $C_1 L^{-1}$ for some finite constant C_0 depending on $\|H\|_\infty$, $\|F\|_\infty$ and $\|F'\|_\infty$ because the grand canonical measures are product. Since $\tilde{H}'_{L,\mathbf{h}} = L\partial_M E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[H(\eta_1)]$ the lemma is proved. □

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