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Pure Strategy Equilibria in Auctions Under More General Assumptions

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RESUMO

Esta tese trata da existência de Equilíbrio de Nash em estratégias puras no contexto de leilões. Iniciamos com uma discussão do assunto, mostrando seus fundamentos e principais resultados. Em seguida, estabelecemos um lema básico para caracterização do comportamento ótimo em leilões, que é usado como base no estudo de leilões não-monótonos. Para esse tipo de leilões, apresentamos novos resultados de existência, compreendendo vários tipos de leilões simétricos com informação privada independente e multidimensional. Mostramos que uma regra simples de desempate é capaz de garantir a existência de equilíbrio para uma grande classe de leilões. Após aplicarmos tais resultados para leilões com lances multidimensionais, estudamos leilões assimétricos monótonos, e damos uma prova curta de existência que engloba casos não cobertos por resultados anteriores. Em seguida, criticamos a hipótese de afiliação, usualmente empregada em Teoria de Leilões. Por fim, apresentamos um teorema de ponto fixo para correspondências que não requer a hipótese de convexidade de seus valores.

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ABSTRACT

This thesis deals with Pure Strategy Equilibria in Auctions. We begin by introducing the subject through a presentation of its fundamentals and a survey of the main results. After a basic result about the bidding behavior, we study auctions with multidimensional types and without monotonic assumptions. We are able to give new existence results for this kind of symmetric auctions, with independent types. We also show that a simple tie-breaking rule is sufficient to ensure the existence of pure strategy equilibrium. We apply these results to the study of single object auctions with multidimensional bids. We also give a new proof of equilibrium existence for monotonic asymmetric auctions, in a setting that includes new results, e.g., the pure strategy equilibrium existence for asymmetric double auctions. All these results are under the assumption of independence. We argue that affiliation, the assumption normally used as a generalization of independence, is not convenient. Finally, we present a fixed point theorem for set-valued maps that does not need the assumption of convex values.

SUMÁRIO

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APRESENTAÇÃO

Esta tese trata da existência de equilíbrio de Nash em estratégias puras no contexto de leilões. Está composta por sete capítulos, que correspondem aos artigos desenvolvidos durante sua preparação.

O Capítulo 1, "Equilibria in Auctions: From Fundamentals to Applications", constitui uma introdução a todo o assunto de equilíbrio de Nash em leilões. É um artigo escrito com o fim de oferecer uma rápida visão dos principais aspectos do problema da existência de equilíbrio em leilões, mesmo para não-economistas ou não matemáticos. É, portanto, um artigo de divulgação (e pesquisa) do campo, servindo perfeitamente como a introdução de nosso trabalho. Neste capítulo, por exemplo, introduzimos os principais conceitos que serão usados nos capítulos seguintes, e também discutimos os fundamentos e escolhas metodológicas adotadas pela Teoria de Leilões.

O Capítulo 2, "The Basic Principle of Bidding", apresenta o primeiro resultado que obtive em minha pesquisa. O princípio é bastante simples: todos os lances de um participante de um leilão são para igualar o benefício marginal do lance ao seu custo marginal. Embora bastante intuitivo, é um resultado novo em Teoria de Leilões e vale em condições bastante gerais. Sua generalidade pode ser usada para a obtenção de conclusões da observação de lances reais ou experimentais, mesmo durante o períodos iniciais, em que os jogadores ainda estão aprendendo. Isso pode abrir a possibilidade de investigar como mudam as crenças dos agentes ao longo das iterações e se eventualmente elas convergem para uma Common Prior, como usualmente se postula.

O Capítulo 3, "Pure Strategy Equilibria of Multidimensional Non-Monotonic Auctions", provavelmente contém os resultados teóricos mais relevantes desta tese. Generalizamos os resultados de existência de equilíbrio em leilões simétricos com tipos independentes até agora disponíveis. Enfraquecemos as hipóteses de tipos unidimensionais com utilidades monótonas, para qualquer dimensão de tipos e sem hipóteses sobre a monotonicidade das funções de utilidade. Também explicitamos quais são as condições em que não são necessárias regras especiais de desempate. No caso em que tais condições não são satisfeitas, oferecemos uma simples regra de desempate que garante a existência de equilíbrio em qualquer dos casos analisados.

O Capítulo 4, "Single Object Auctions with Multidimensional Bids" é na verdade uma extensão curta da análise do Capítulo 3. O problema abordado, no entanto, pode ter uma importância que a literatura parece não haver ainda considerado suficientemente. De fato, vários leilões do mundo real são leilões com lances multidimensionais. Há provavelmente razões importantes para isto. Este capítulo é uma abordagem inicial para alguns desses tipos de leilões.

O Capítulo 5, "Is Affiliation a Good Assumption?", mostra como é restritiva e insatisfatória a hipótese de afiliação, largamente utilizada na literatura sobre leilões como uma conveniente generalização de independência. O artigo, porém, não se limita a apresentar a crítica. Ao final, apresentamos uma proposta de metodologia que, uma vez desenvolvida, pode se mostrar mais geral e mais aplicável à realidade, ainda com conclusões interessantes.

O Capítulo 6, “Monotonic Equilibria of Auctions” dá uma demonstração relativamente curta para a existência de equilíbrios monótonos. Em particular, permite obter um resultado original para a existência de equilíbrio em leilões duplos com tipos independentes e utilidades assimétricas. Tal tipo de resultado pode ser considerado como uma contribuição à fundamentação microeconômica para equilíbrios com expectativas racionais.

O Capítulo 7, “A Fixed Point Theorem for Non-Convex Set-Valued Maps”, é, na verdade, um artigo puramente matemático, embora apresentemos, ao seu final, uma aplicação para Teoria de Jogos. O teorema de ponto fixo nele apresentado é uma espécie de Teorema de Kakutani sem a hipótese de convexidade nos valores da correspondência (“set-valued map”). Vale apenas para espaços de funções e usa como substituto para a convexidade o conceito de decomponibilidade. Em muitos casos, esta condição é bastante natural, mas o teorema ainda requer algumas condições de continuidade que podem ser restritivas. É importante frisar que a busca de um teorema de ponto fixo sem a hipótese de convexidade também foi motivada pela questão de equilíbrio em leilões, uma vez que esse tipo de jogo não goza de tal propriedade.

Optamos por apresentar os capítulos em formatos próximos aos dos artigos, de forma que gozam de suficiente autonomia. O leitor observará, no entanto, que todos obedecem a um único tema, que dá unidade a esta tese: a existência de Equilíbrio de Nash em jogos de informação incompleta e, em particular, em Leilões. Por que esse assunto é relevante e por que nos dedicamos a ele? Esta breve apresentação não comporta a resposta a tão importante questão. Mas se a curiosidade do leitor ficou despertada, permita-me transformá-la em estímulo para que vire a página e comece a buscar, no Capítulo 1, minha tentativa de respondê-la.

CHAPTER 1
EQUILIBRIA IN AUCTIONS: FROM FUNDAMENTALS TO APPLICATIONS

Abstract. This chapter presents the main results of equilibria existence in auctions, and it is equally an introduction and a guide for the literature. We departure from the fundamentals of the area, explaining its assumptions and methodology to attack the relevant questions. Then, we cite the most important applications of the theory, showing how it can be useful to auction practitioners.

1. Introduction

Auctions are intensively and widely used in the economy: in the internet, in the market of bonds and treasury bills, in privatization of public firms, in the allocation of spectrum, etc. Even some economic institutions not commonly thought as auctions, like R&D races and lobbying, can be well modelled as auctions.[1]

This intensive use creates a strong demand for theoretical results that could give guidance to auction practitioners. Among the main questions, there are some very complex issues: what is the expected revenue and how to increase it? How the rules of the auction attract bidders and affect their bidding behavior? How to prevent collusion? Is the auction efficient — in the economic sense of giving the object to whom value it the most? What is the value and impacts of information? How the seller has to treat the information of his own? And so on.

It is evident that in all of the above questions, the strategic aspects of the behavior of the seller and that of the bidders have to be considered. Indeed, one could say that the most important aspect of the study of auctions is to understand the strategic aspects of its design. This opens the door to understand all the other questions, as efficiency, revenue, collusion, etc. From the complete understanding of the strategic behavior of participants, the theorist can infer better founded conclusions about the phenomena observed in the real auctions and derive advices.

The scientific tool for treating strategic iterations is Game Theory, a well explored and fruitful area of Economics. Its most used and accepted concept is Nash equilibrium. In some sense, it is from the understanding of the properties of the equilibrium that the theory can provide useful conclusions.[2]

The purpose of this chapter is to describe the subject of Nash equilibria in auctions, being an introduction and a guide for the literature.[3] We begin in section 2, where we make a detailed and careful construction of the standard models of auctions. In section 3, we discuss the suitability of Nash equilibrium concept for Auction Theory. In section 4, we explain why the standard results in general games do not apply to auctions, leading to the necessity of special results. In particular, the observations of these two

[1] See footnote 8.

[2] As we discuss in section 3 it is still possible to say something without the use of the equilibrium concept.

[3] The reader interested in other issues of Auction Theory is referred to the survey of Klemperer (1999).

sections justify the number of papers dealing with the question of existence. We proceed in section 5 by discussing the results in mixed strategies and its limitations. The pure strategy results are briefly surveyed in section 6 and its assumptions discussed in section 7. In section 8 and 9 we discuss, respectively, the dimension and the dependence of the private information of the bidders. Section 10 approaches the applicability of the theory to the reality. Section 11 is a brief conclusion.

2. A Model for Auctions

Let restrict us to auctions of one indivisible object.[4] We begin by fixing the number of bidders, n.[5] In general, the object has uncertain value to the participants, and we can model it as random variables, V_1, ..., V_n, one for each bidder. Of course, the value is permitted (but not required) to be the same for all bidders. If they are equal, the auction is said to be a (pure) common value auction, as we define at the end of this section.

The case where the value of each bidder is common knowledge between the bidders is trivial for one object auctions: the bidder with the highest valuation will win any form of auctions and pay the second highest valuation. Indeed, a price slightly above the second highest valuation implies that only one bidder desires the object. So, the allocation problem is solved at that price and the opponents have no way to push upward the price that the winner will pay. Nevertheless, this argument uses some implicit assumptions. First, we are ignoring strategic behavior by the seller. In other words, he is passive and/or do not have information about the valuation of the bidders. Otherwise, he could design auctions to extract the rent from the bidders, as showed by an example in Myerson (1981) or by McAfee, McMillan and Reny (1989). Another point is that the bidders have no rights on the object (as is usually the case in auctions). If this is not the case, we turn to bargaining models. Finally, this simplicity is restricted to single object auctions. For auctions of many objects, even the complete information case is not trivial.[6]

Consider first the sealed bid auctions, that is, auctions where the only thing that the bidder has to do is to write a bid in a envelope and to delivery it to the auctioneer. The bidder with the highest bid receives the object but the payment depends on the auction rule or format. The more common formats are the following:

- first-price auctions: the winner pays his bid;
- second-price auctions: the winner pays the second highest bid (or the highest looser bid);
- All pay auction: all bidders pay their bids, but, as before, just the player with the highest bid receives the object.[7]

[4]This is the most treated case in the literature, although recent efforts have been made to treat multi-unit auctions. This class of auctions is very important, because it embraces the auctions for treasury bills and bonds, spectrum rights, energy contracts and so on. We treat some aspects of multi-unit auctions in section 10.

[5]Some papers are interested in analyze how the conditions and rules of the auction affect the number of participants. Then, its number is not assumed fixed in the beginning. See, for instance, Levin and Smith (1994).

[6]See, for instance, Menezes (1996).

[7]Although not very common in practice, the all pay auctions can model, for instance, the R&D races. Everybody pays the investment in the research, but the firm that invested the most reaches the patent first, and receives the prize. In lobbying, everyone gives a gift to the decision maker, but the favor goes to whom gave the most valuable gift.

- War of attrition: the losers pay their bids and the winner pays the highest losers' bid.[8]

A very popular auction format that are not included in the sealed bid class is the English or open auction. In this format, the bidders successively place bids in order to outbid the opponents. When no bidder wants to place a higher bid, the auction ends and the author of the last bidder is the winner, paying his bid.[9]

Another example is the Dutch auction, used to sell flowers in Holland: the auction starts displaying a very high price for the item and then such price continuously lowers, until some participant claims the object. At that moment, the price stops and the bidder pay it. Theoretically, Dutch auction is equivalent to a first-price auction, because no information is learned during the auction and the strategy is equivalent to decide at what price to buy the object, exactly the same decision of a first-price sealed-bid auction. Interestingly, empirical and experimental studies do not confirm this theoretical equivalence.[10]

We will mainly focus on the first-price auctions, one of the most treated in the literature. The problem of the bidder is simply to choose a bid in order to maximize his *expected* profit, where the profit is the difference between the value of the object and the payment, in case of winning. But, "expected" in which sense?

Each bidder has some information about V_i, that leads to conditional distribution of its value. Of course, this includes the case where the bidder knows with certainty the true value of V_i. Also, each bidder i forms beliefs about the values of V_j, for $j \neq i$ and about the beliefs that bidders $j \neq i$ have about V_i. Unfortunately, the things do not stop here: each bidder i has now to consider the beliefs of the bidders $j \neq i$ over the beliefs of i over the beliefs of j, and so on. This makes the problem rather complex and apparently intractable.

In order to address the issue, let us first observe that the situation described is a game of incomplete information. It is a game because each participant has to choose an action (the bid) and the payoff of each participant depends on the actions (bids) of everyone. It is of incomplete information because the players do not know what the payoffs of the participants are. The solution for these kind of games was given in the insightful and remarkable paper of Harsanyi (1967-8).[11] In the case of auctions with risk neutral bidders, we can describe his approach as follows.[12]

2.1. **Auctions as Incomplete Information Games.** The payoff of bidder i is 0 if he looses and $V_i - b_i$ if he is the winner. So, the uncertainty about the payoff is in V_i, the value of the object. If such object is, for instance, the right to explore a petroleum field, the value depends on the quantity and the quality of the oil in the field, the cost of drilling and extraction — that are affected by the technology used by the firm

[8] The war of attrition models conflicts among animals and the struggle for survival among firms.

[9] Some auction houses use the practice of maintain the reserve price unrevealed. So, if the last bid is not above the reserve price, there is no winner. In this situation, the object is said to be "bought in", which is a terminology consistent with the best way of model unrevealed reserve prices: it works like a bid of the seller.

[10] See Lucking-Reiley (1999) and Kagel, Harstad and Levin (1987).

[11] Without the Harsanyi's tools, Vickrey (1961) has achieved some of the most important conclusions in Auction Theory for a long period. His remarkable paper can be considered the foundation of the field.

[12] There is no difficult in treat other attitudes towards risk. Indeed, the original treatment in section 3 of Harsanyi (1967-8) includes this case.

—, the international prices of the petroleum and some other private characteristics of the firms. In other words, many variables usually are the determinants of the value of the object. Between these variables, some are known by the bidder himself, some are known by the other bidders, and some are unknown to all of them.

Thus, it is convenient and realistic to describe V_i as a function of a set of parameters, that is, to assume that

$$V_i = u_i\left(a_{0i}, a_{1i}, ..., a_{ni}\right), \tag{1}$$

where a_{0i} is a vector of parameters that are unknown to all players and a_{ki} is a vector of parameters that are unknown to some of the players but are known to player k, for $k = 1, ..., n$.[13] The subscript i in the above parameters is just to say that they are referred to the value of the object to player i. Without loss of generality, we can assume that the function u_i is common knowledge for all players.[14] Harsanyi assumes that the set of all possible values of the parameters a_{ki} is A_{ki}, (a subset of) an Euclidean space with the convenient dimension. The vector $a_k = (a_{k1}, ..., a_{kn})$, for $k = 1, ..., n$ describes the information that player k has about the functions u_i, for $i = 1, ..., n$. The vector $a_0 = (a_{01}, ..., a_{0n})$ summarizes the parameters that none players have about u_i. Of course, we can write V_i as function of the vector $a = (a_1, ..., a_n)$, where the unnecessary parameters does not influence the payoff. So, we can write

$$V_i = u_i\left(a_0, a\right), \tag{2}$$

where a is also denoted by (a_i, a_{-i}), with $a_{-i} \equiv (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$. The range of a is denoted by A and the range of a_{-i} by A_{-i}.

Harsanyi adheres to the Bayesian doctrine, that assumes that the players attribute subjective probabilities to the unknown parameters, that is, each player form a subjective probability over A. Of course, we could again parameterize the possible probabilities and say that s_{ki} is the parameter that player k knows about the subjective probability of player i. Repeating the previous procedure, s_i denotes the vector of parameters known to bidder i. We end up by saying that each player form a probability $\mu_i = \mu_i^{s_i}$, over the set of unknown parameters: $A_0 \times A_{-i} \times S_{-i}$. It is assumed that $\mu_i^{s_i}$ is a conditional probability $\mu_i(\cdot|s_i)$ and that there is a prior distribution $\mu_i(\cdot)$ over $A_0 \times A \times S$. Finally, it is assumed that the priors are equal across the players, that is, $\mu_1 = ... = \mu_n = \mu$.[15] So, the differences between the subjective probabilities are due to the parameters s_i.

The parameters that are unknown to all players do not influence their behavior in the auction. Thus, bidders can restrict attention to $\overline{V}_i$, the expected value of V_i with respect to a_0. That is, we can eliminate the vector a_0 in equation (2), so that we have

$$\overline{V}_i = \overline{u}_i\left(a|s_i\right) = \int_{A_0} u_i\left(a_0, a\right)\mu\left(da_0, a|s_i\right). \tag{3}$$

[13] The reader can think in the parameters as being random variables correlated to the random variable V_i. The utility function in equation (1) is, then, just a conditional expectation.

[14] This is valid if the parameters can be infinite dimensional, since the uncertainty over all functions can be parameterized by such kind of parameter. The hypothesis that the parameter lies in an Euclidean space is the restritive one.

[15] This is known as Common Prior Assumption.

Now, we embrace all the information that bidder i possess before choosing the bid in a unique vector

$$t_i = (a_i, s_i) = (a_{i1}, ..., a_{in}, s_{i1}, ..., s_{in}) \tag{4}$$

and call it the type (or the signal) of bidder i. It represents all information that he has about: his own payoff, the payoff of the others, his own beliefs and the beliefs of the others. Again we write $t = (t_1, ..., t_n)$ so that we can rewrite (3) as

$$\overline{V}_i = v_i(t) = \overline{u}_i(a|s_i). \tag{5}$$

Now, we can introduce some standard concepts of Auction Theory.

(1) **Private Values** - If $v_i(t)$ depends just on t_i, that is, if there is no parameter known to some player $j \neq i$ but unknown to player i, the auction is said to be a private value auction. In other words, player i knows everything that the entire set of players know about his payoff.
(2) **Common Value** - If $v_i(t) = v(t)$ for all i, that is, if the value of the object is the same for all bidders, it is a common value auction.[16]
(3) **Symmetric, Interdependent Values** - If $v_i(t) = v(t_i, t_{-i})$, where $v(t_i, t_{-i}) = v\left(t_i, t'_{-i}\right)$ for all t'_{-i} that is a permutation of t_{-i}, we are in a symmetric auction that can have (or not) private value parts. This is the case analyzed by Milgrom and Weber (1982) and it is a clear generalization of the common and the private value settings.[17]
(4) **Asymmetric, Interdependent Values** - If $v_i(t)$ does not assume any of the above special forms, we are in a general asymmetrical, interdependent value auction.

It is worth to highlight that common value is not the opposite of private values, as one may wrongly conclude from the terminology. Common value means that the value of the object is equal to all participants (but see footnote 17), while private value means that all information regarding the value of the object to a bidder is actually possessed by him.

3. Why Nash Equilibrium?

Despite the almost omnipresence of the concept of Nash equilibrium, it is worth to discuss whether it is convenient to apply it for auctions.

First of all, Nash equilibrium is concise, simple and indicates self stable behaviors by the players.[18] The stability of the outcome is, indeed, one of the more important requirement to an economist. Otherwise, he would study or derive properties of a phenomenon that is likely to disappear.

Another reason comes from the following argument: assume that a group of specialists is required to give (individual) advisories to participants how to play a game. If

[16]Some authors call this *pure* common value and use the term "common value" to what we call interdependent value.

[17]We are implicitly assuming that the space of types is the same.

[18]The concept, in its own, requires just this: the stability of the actions. Nevertheless, especially in extensive form games, there are good examples of Nash equilibrium that are not stable. This has led to some refinements of the concepts. See van Damme (1983).

the recommendations are not equilibrium, some player(s) can do better by choosing another action. In other words, the recommended strategies will be self defecting.

Of course, there are some objections to the above arguments and to the use of the concept. One of the most important is that Nash equilibrium strategies are, in general, very difficult to calculate, especially in auctions. So, how the players would compute their equilibrium strategies to even consider play them? More than that, if one player learns the equilibrium strategies but is not sure that the others will follow their strategies, then to play his equilibrium strategy may not be the best thing to do.

An answer to this point is that the players will learn the equilibrium in some time, and those who do not learn the equilibrium strategy will be rolled out of the market.[19] In auctions, it would mean that the experienced players, or the players that survive to many auctions, have learned to play the equilibrium strategies. Experimenters have confirmed this at least in private value auctions.[20]

Nevertheless, in the real world such learning process has some problems. It implicitly requires that the bidders participate in a number of similar auctions, with similar bidders, but preferably not the same. The reason is that if the same bidders meet repeatedly in the auctions, it is likely to arise some form of collusion.

Another point is that the "transitory" period of learning the equilibrium can prolong for a non-negligible time. This would depend of the uncertainty involved and the information that are revealed at the end of the auction, which is not much, in general. For example, the estimated values of the bidders are unobservable and, in some auctions, only the highest bid is known after its end.

Adding to the list of difficulties, some experimenters reported consistent bidding behavior above the Nash equilibrium prescription.[21] This leads to consider what can be said if the players does not follow Nash equilibrium strategies.

Indeed, it is worth to remember that, as Pearce (1984) and Bernheim (1984) pointed out, rationality of the players does not imply Nash equilibrium. It only implies the use of rationalizable strategies. In this setting, Battigalli and Siniscalhi (2003) derived interesting conclusions. Some of them can be explanations for the findings of experimentalists that are in contradiction with equilibrium behavior.

Even under weaker hypothesis, it is still possible to say something interesting. In the chapter 2, we prove the "Basic Principle of Bidding", that roughly says that any rational bidder with unitary demand always bid in order to equalize the marginal utility of bidding to its marginal cost. As stated, it is so simple that seems obvious, but it is a new interpretation. It does not depend on assumptions about the dimension of types, monotonicity or linear separability of the utility function. If one accepts the Becker doctrine,[22] we can derive conclusions about all bids observed, learning something even about inexperienced bidders.

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[19] This is the classic evolutionary argument to justify equilibrium.

[20] See, e.g., Kagel, Harstad and Levin (1987). The common value setting is more controversial. See Kagel (1995).

[21] See Kagel (1995) for a discussion and a survey of the corresponding literature.

[22] We call "Becker doctrine", after Gary Becker, the notion that we can better understand the human behavior by maintaining the assumption that they are rational and maximize utility conditioning in the information that they have. So, useful conclusions about beliefs can be derived from the observed behavior. In the case of auctions, this would mean that we can learn something about the beliefs and utility function of the bidders, even if they do not have a Common Prior. In that case, only their beliefs are inconsistent, but they are rationally following the Basic Principle of Bidding.

Concluding this discussion, we would remember that if there is something that an economist must understand is the concept of "trade-offs". In our case: weaker assumptions lead to weaker conclusions. As long as we interested in more substantive information about auctions, we have to accept the cost of obtaining it, and to work at least with the notion of Nash equilibrium.

We hope that it is now apparent to the reader that the notion that bidders follow Nash equilibrium strategies is, in fact, a "meta-assumption" for our methodology.[23] More clearly: we believe that the acceptance of this meta-assumption provides an optimum combination between the mathematical structure that we require in our model of the world and the conclusions that we derive from it.

Having justified the methodology, let us focus in the question of existence and characterization of Nash equilibrium.

4. Equilibria Existence in Games

The first general existence result for equilibrium in games, is that of Nash (1950).[24] The hypotheses of his theorem is that of continuity and quasiconcavity of the utility functions. With them, it is easy to prove that the correspondence (set-valued map) of the best-responses are upper semicontinuous with closed and convex values. An application of Kakutani's Theorem concludes the proof.

Unfortunately, auctions are neither continuous nor quasiconcave.

The utilities in auctions are discontinuous because of ties in winning bids that occur with positive probability. In those cases, it is necessary to break the tie by giving the object to one of the tying bidders. However a bidder that receives the object with probability strictly less than one can bid arbitrarily close of the tying bid and increasing discontinuously the probability of winning. This implies a discontinuous changing in the utility. In the appendix, we give counterexamples to show that standard auctions are not quasiconcave.[25]

Since discontinuous games are not so rare (indeed, they are very common), some papers have tried to weaken Nash's assumptions. Among them, we can cite Dasgupta and Maskin (1986), Simon (1987), Baye et. al. (1993) and Reny (1999). In the last (and more general) paper, the condition of continuity is weakened to the requirement of better reply security and the quasiconcavity is dispensed, at least for the mixed strategy result.[26] Reny (1999) also shows how his theorem proves the existence of equilibrium for multi-unit, private value, independent types, pay-your-bid auctions.[27] It is still unclear how far Reny's Theorem go to prove general existence results in auctions.

What is important to highlight here is that the problem of existence of equilibrium in auctions is more difficult than in other economic games. This factor and the interest

[23] Of course, it is a testable (meta-) assumption. We have to recognize, nevertheless, that it is hard to test it, even in experiments, where the conditions are controlled. Kagel (1995) describe an intense debate about the consistency between the observed behavior of individuals in experiments and the assumption that they play Nash equilibria. In empirical tests, the things are even more problematic. See Laffont (1997).

[24] See also Debreu (1952) and Glicksberg (1952).

[25] Because auctions are not quasiconcave, in chapter 7 we work with a theorem similar to Kakutani's, without the requirement of convexity of values. Nevertheless, it still needs continuity properties that are too restrictive for auctions.

[26] It is still necessary to pure strategy equilibrium.

[27] The reader unfamiliar with these terms can consult section 7, below.

in such results lead to many papers dealing with the question of equilibria existence in auctions.

5. Mixed Strategy Equilibrium and Tie-Breaking Rules

The most general available result that applies to auctions is that of Jackson, Simon, Swinkels and Zame (2002). They guarantee the existence of equilibrium in general discontinuous games and all kind of (atomless) distribution of types. Nevertheless, their result has at least two undesirable characteristics: first, it requires that the tie breaking rule were endogenously determined; second, it is in mixed strategy.

The standard way of solving ties in games is to appeal to a random device, like a coin, to decide the outcome of the game. Under this standard rule, many discontinuous games do not have equilibria. Simon and Zame (1990) are probably the first to point out that a special (and endogenous) tie-breaking rule can solve the problem of existence. The paper of Jackson, Simon, Swinkels and Zame (2002) is, indeed, a generalization of Simon and Zame (1990) for games with incomplete information.

Lebrun (1996) seems to be the first to mention the need of special tie-breaking rules in Auction Theory. Maskin and Riley (2000) also express its importance and adopt a special (non-endogenous) tie-breaking rule for auctions where the types of the bidders are finite: in case of a tie, conduct a second-price auction among the tying bidders.

The problem with endogenous rules is that the rules of the game cannot be previously specified. The theoretical result also does not offer guidance for the way of chosen the rule. It is just an existence result. Then, the players begin the game (or at least decide to participate) without knowing what is the rule.

At least to private value auctions, Jackson and Swinkels (2004) solve a part of the problem. They show that the equilibrium is invariant for the tie breaking rules, so that the problem becomes of minor importance. Nevertheless, their paper still provides the result in mixed strategy.[28]

The problem with mixed strategy is twofold: first, it is difficult to accept that in auctions, where it is involved sometimes considerable amount of money, a bidder chooses the bid through a random device (like a coin).[29] Second - and more important for Auction Theory - the mixed strategy equilibrium fail to provide useful information about the bidding behavior of the players.

In chapter 3, we offer the following contributions to this problem: first, we give necessary and sufficient conditions for the existence of equilibria without ties with positive probability. Up to now, the literature had not clarified when it is necessary and when it is not necessary the use of special tie breaking rules in auctions. Second, we propose an exogenous and simple tie-breaking rule for the use when the necessary and sufficient conditions do not hold. This result has the advantage of being in pure strategy. Also, it does not require the announcement of types (as Jackson et. al.), but it can be implemented through an all-pay auction.[30]

Before we describe their result, we would like to review some of the previous results in pure strategy.

[28] They offer a specialization of the result that leads to pure strategies equilibria.

[29] In fact, this is a general objection to mixed strategies in game theory.

[30] The important restrictions that they made are that the types are independent and the utilities are symmetric.

6. Pure Strategy Equilibrium

Roughly speaking, two methods have been used in the literature for establishing equilibrium in pure strategy. The first is to analyze the solution of a (set of) differential equation(s) derived from the first order condition of the bidder's best-reply problems, and then, under some assumptions, to prove that this solution satisfies sufficient conditions for an equilibrium.

The second method begins by restricting the feasible bids or types to discrete sets. Then, the auction is reduced to a finite game which is showed to have equilibrium. The properties of convergence guarantee the desired behavior at the limit as the grid of actions/types becomes finer.

The first method has the advantage of characterizing the equilibrium strategy and, in some cases, providing its analytical expression. On the other hand, the second method seems more general.

Papers that use the first approach includes Milgrom and Weber (1982), Lebrun (1999), Lizzeri and Persico (2000), and Krishna and Morgan (1997). Milgrom and Weber (1982) assume that types are affiliated and that each bidder's utility function is increasing in all types.[31] Under these conditions, they show that there exists a pure strategy equilibrium for which the bidding functions are monotonic. Their results cover the first-price, the second price and English auctions but require, as an important condition, symmetry. Krishna and Morgan (1997) use similar methods to the case of all-pay auction (first-price all-pay auction) and war of attrition (second-price all-pay auction). Lebrun (1999) extends the differential equation approach to the asymmetric first-price auction, but with private values and independent types. Lizzeri and Persico (2000) use the approach to the case of common values with reserve price, affiliated types, but only two bidders.

Maskin and Riley (2000), Athey (2001) and Reny and Zamir (2004) follow the second approach. Maskin and Riley (2000) establish existence of equilibrium for asymmetrical first-price auctions with either affiliated types and private values or independent types and common value. Athey (2001)'s model embraces a wider class of discontinuous games. She establishes the existence for several games and, in particular, she obtains existence for the 2 bidder first-price auction with affiliated types and common value. Reny and Zamir (2004) extend her result to the n bidders' case.

Table 1 below summarizes the findings for first-price auctions, the case more treated in the literature.

[31] Affiliation can be defined as follows. Consider n real random variables $t_1, ..., t_n$ and let $f(\cdot)$ be its joint density function. Then, the variables are affiliated if $f(t) \cdot f(t')$ 6 $f(t \vee t') \cdot f(t \wedge t')$, where $t \vee t' \equiv (\max\{t_1, t_1'\}, ..., \max\{t_n, t_n'\})$ and $t \wedge t' \equiv (\min\{t_1, t_1'\}, ..., \min\{t_n, t_n'\})$. For a definition without use of the density function and other discussions, see Milgrom and Weber (1982). In chapter 6, we argue that affiliation is a very restrictive assumption.

Assumptions	Symmetric, interdependent values	Asymmetric, private values	Asymmetric, interdependent values
Independent types, n bidders		Lebrun (1996, 1999)	Maskin and Riley (2000)
Affiliated types, 2 bidders			Lizzeri and Persico (2000), Athey (2001)
Affiliated types, n bidders	Milgrom and Weber (1982)	Maskin and Riley (2000)	Reny and Zamir (2004)

Table 1 - Summary of existence results in first-price auctions

We will not describe all the results in other formats of auctions, but just cite some of them.

The second-price auction is analyzed by Milgrom (1981). Many papers treat English auctions, but Milgrom and Weber (1982) is a good starting point. Williams (1991) treats double auctions with the simplification that the bid is determined depending just on the buyers' bid. His setting is a symmetric private value auction with independent types, risk neutrality and unitary demand. Fudenberg, Mobius and Szeidl (2003) and Perry and Reny (2003) provide more general results in double auctions, but for large number of bidders.

Amann and Leininger (1996) and Krishna and Morgan (1997) study the cases of all pay auctions and war of attrition.

Building in a general method developed by Milgrom and Shannon (1994), Athey (2001) provide general pure strategy equilibria results when a game satisfies a "non-primitive" condition, namely, "single crossing property". She applies her result to various unidimensional auctions. Extending Athey's theorem to the multidimensional framework, McAdams (2003) prove the existence of pure strategy equilibrium in multi-unit uniform price auctions.[32]

If we want to add to this list the results regarding other issues, but with equilibrium results, the list becomes interminable.[33] Even limiting ourselves to papers that treat the equilibrium (almost always) as a central question, we find that our list is still incomplete.

It is amazing that *all* of the above results share a common characteristic: the equilibrium bidding functions are monotonic non-decreasing.[34] This is a so universal characteristic that one could ask why this is so. Obviously, this comes from the assumptions

[32] Remember that Reny (1999) proves the existence just for "pay-your-bid" multi-unit auctions.

[33] Such issues are, e.g., risk aversion (Matthews (1983), Maskin and Riley (1984)), collusion (McAfee and McMillan (1992)), entry of bidders (Levin and Smith (1994), Campbell (1998)), financial constraints (Zheng (2001), Fang and Parreiras (2002)), etc.

[34] There exist some exceptions. In Zheng (2001), the private information is the budget constraint, and the bidding behavior can be non-monotonic. Nevertheless, there is also a monotonic equilibrium. McAdams (2003b) gives an example with three bidders and affiliated types, where a non-monotonic equilibrium can exist. Ewerhart and Fieseler (2003) and Athey and Levin (2001) study auctions with multidimensional bids that can exhibit non-monotonic equilibria.

usually made. Thus, it is worth to analyze such assumptions and discover whether they are restrictive or not.

7. The Standard Assumptions in Auction Theory

We can distinguish three classes of assumptions that lead to the monotonic bidding functions: 1) about the types (dimension and distribution); 2) about the dependence of the utility with the types; 3) about the cross dependence of the utility with the types and bids.

Under the first class of hypotheses, it's normal to suppose that types are unidimensional and independent or affiliated.[35]

In the second class, generally it is assumed that utilities (values) are strictly increasing with own type and constant (private values) or non-decreasing (interdependent values) with other's types.

The third class of hypotheses assumes that the second cross derivative of types and bids are positive ($\partial^2_{bt} u \geqslant 0$). Risk neutrality and risk aversion, for instance, imply this class of hypothesis. It is known as "single crossing conditions".[36,37]

Regarding the first class, we argue in subsection 7.1 that the simplification to unidimensional types is not harmful in private values settings, while it can be restrictive in interdependent values cases. Independence and affiliation is briefly discussed in subsection 9.

7.1. **Dimension of Types.**

As equation (4) makes clear, the type of a bidder is *per se* multidimensional. So, why unidimensional types are that used in auction theory?

The main reason is, of course, simplicity. When the types are real numbers, the real analysis can be used and it is clear how is easy to work with it in comparison with multidimensional analysis.

Although it seems restrictive, the unidimensionality of types is acceptable at least in the case of private values with independent types. Indeed, in this case, the bidders do not have information concerning the value of the object to the opponents. So, we can reparameterize the signals of each bidder to the unidimensional type

$$\tau_i \equiv E\left[V_i | t_i\right],$$

and this summarizes all the information that bidder i needs to know. In other words, τ_i is a sufficient statistic for the information of bidder i. This is the reason why the unidimensional signal is understood as the value for the bidder.[38] In the case of

[35] Of course, when dealing with continuous types, distributions are routinely assumed to be atomless.

[36] See Athey (2001) for a discussion on single crossing conditions. We will not discuss this class, since it has, at least in Auction Theory, the good justification of risk aversion or risk neutrality.

[37] The term Single Crossing Condition is also used for the property that $\partial_i u_i > \partial_i u_j$, for $j \neq i$. This is used for establishing efficiency. See Dasgupta and Maskin (2000) and Krishna (2002 and 2003).

[38] It is worth to see also footnote 14 of Milgrom and Weber (1982), p. 1097, that we reproduce here for reader's convenience:

> To represent a bidder's information by a single real-valued signal is to make two substantive assumptions. Not only must his signal be a sufficient statistic for all of the information he possesses concerning the value of the object to him, it must also adequately summarize his information concerning the signals received by the other bidders. The derivation of such a statistic from several separate pieces of information is in general a difficult task (see, for example, the discussion in Engelbrecht-Wiggans and Weber [17]). It is in the light of these difficulties that we choose to view each X_i

dependent types, the argument is a bit more problematic, since the bidder does not want to loose information when summarizing his information to a unidimensional variable. Nevertheless, the reduction can be done if there is a unidimensional sufficient statistic τ_i that summarizes all the relevant information for the bidder.

In the case of interdependent values, when each bidder has some valuable information regarding the values of the others, the existence of such sufficient statistics is less clear. At least when the types are independent, we show in chapter 3 that it there exists (under some assumptions on the utility function).

Of course, the case with interdependent values and correlated types is much more demanding and it cannot be reduced to the unidimensional case in general. This is problematic, since this setting is just the more likely to describe real situations. Indeed, in the real auctions the bidders generally base their behavior in many observations and variables, and their estimates might not be independent. But this is the subject of the next subsection.

7.2. **Distribution.** Concerning the distribution of the types, most of the equilibrium results assume independence. Of course this is a strong assumption and it is very valuable to try to relax it.

Milgrom and Weber (1982) do that through the concept of affiliation. Under this kind of dependence, they prove the existence in first-price, second-price and English auctions. Since their paper, any model that relaxes the independence hypothesis in Auction Theory uses the concept of affiliation.[39]

Despite such agreement — or because of it —, in chapter 6 we argue that affiliation is not a satisfactory or valuable generalization of independence, since it is very unstable and restrictive. More than that, we argue that the recommendations derived from affiliation are very unreliable and not suitable to apply directly in the real world. We also provide alternative explanations for the phenomena that affiliation is considered to successfully explain, as the predominant use of English auctions. That chapter also proposes an alternative approach to treat dependence.

7.3. **Monotonicity of the utilities.** The discussion about the dimension on types also revealed some of the main reasons to assume that the value function (5) is monotonic. Indeed, in the private values case, this is straightforward, but in the interdependent values case, the justificative is much less compeling. If the bidders' information is multidimensional, the assumption is clearly restrictive.

If we want a more general theory for multidimensional information, it is necessary to cope with non-monotonicity. In chapter 3, we present a method to deal with multidimensional and non-monotonic auctions.

8. Multidimensional Bids

Multidimensional bids can model the very important case of multi-unit auctions, as Treasury Bill auctions. Indeed, each bidder is required to submit a demand function, that is a vector $b_i = (b_{i1}, ..., b_{iL})$, where b_{ih} specifies the price that the bidder are

as a "value estimate," which may be correlated with the "estimates" of others but is the only piece of information available to bidder i.

[39]Jackson and Swinkels (2004) is an exception, but their result is in mixed strategy, as we have previously said.

prepared to pay for the $h - th$ unity of the object that he receive. The importance of such auctions are sufficient to justify the interest in them.

The most important sealed price formats for multi-unit auctions are the following:

- Discriminatory (or "pay-your-bid" auction) — each bidder pays exactly the bid given for the unity received. For instance, if bidder i wins 3 objects with the bid $b_i = (16, 12, 9, 7, 5, 4)$, then he pays $16 + 12 + 9 = 37$.
- Uniform Price Auction — all the units are sold by the same price, that is chosen between the lowest winning and the highest losing bid. In the example above, if the highest losing bid was 7 and the lowest winning bid was 9, the price can be anything between 7 and 9 (but it is specified before the auction itself). If it is, say, the lowest winning bid, bidder i in the example above has to pay $3 \cdot 7 = 21$.[40]
- Vickrey Auctions — for each unit bought, the seller pays only the minimum necessary to win such unit. For instance, suppose that the maximum bids of the opponents of bidder i are 18, 11, 10, 7, 4, 2. Then, for the first unit, bidder i has to pay 2, for the second, 4 and for the last, 7, that is, he pays 13.

As in the case of single unit auctions, there are open formats in close correspondence with the given above. The literature on these kind of auctions is growing at a fast pace. For a more complete and detailed introduction, see Krishna (2002).

Nevertheless, there are other cases where multidimensional bid can arise, even for single unit auctions. This can model situations like the buy of a service with many different characteristics. For instance, some companies and governments buy goods and specially services through a mechanism where the potential suppliers are requested to specify not only prices, but also warranty, quality, time to delivery and other characteristics. All these characteristics have to be taken in account in order to decide the winner in the contest.

An example is the auction for the B Band of mobile phones in Brazil. The government asked for bids that include not only the price for the license (p_i), but also the price to the consumers (c_i). The winner was the company with highest $B(p_i, c_i) = 0,6p_i - 0,4c_i$. We treat auctions of this kind in chapter 4.

9. Applications of Auction Theory

Auction Theory is useful for theoretical and practical purposes. In the following subsections, we try to briefly illustrate both kind of applications.

9.1. **Double Auctions.** Double auctions are market institutions where the sellers place offers (a price for which they accept sell their objects) and buyers place proposals (prices for which they buy the objects). Of course, there is a great variety of manners where the offers and proposals (bids) can be placed and the transactions made, but a reasonable and popular example of double auction is a "standard" financial market. In fact, double auction can be the correct model for many of the market institutions that

[40]If the price is chosen to be the lowest winning bid, this is a multi-unit version of the first-price auction, while if it is highest losing bid, this is a multi-unit version of the second-price auction. Nevertheless, observe that these are not the only multi-unit versions of such auctions. Indeed, the pay-your-bid auction is another multi-unit version of the first-price auction, and the Vickrey auction below is another version for the second price auctions.

are used in the real world. Thus, it is clear that the understand of such mechanisms are highly worth.

Among the real world applications, it is the study of market mechanism as the financial markets. In these institutions, bidders can use their private information in order to manipulate the price of the object. Obviously, as the number of bidders increase, the power to do so diminishes. Nevertheless, the correct theoretical construction to analyze such possibilities is Auction Theory.

On the other hand, some important questions in the General Equilibrium Theory and Rational Expectation Equilibrium Theory can be correctly and fruitfully addressed via double auctions, as price taking behavior, expectations' formation, efficiency, strategic behavior, informational asymmetries, etc. Milgrom (1981) address, for instance, the microeconomic foundation for Rational Expectation Theory through Auction Theory.

9.2. **Designing of Market Mechanisms.** Up to now, some very successful applications of Auction Theory to the design of real auctions were reported. The cases of spectrum allocation auctions seems to be the most famous.[41]

It is interesting to observe that the main concern in such auctions were the question of efficiency, although the revenue were also a concern. Fortunately, the theory seems well developed in the question of efficiency.[42] Also the question of revenue maximization were intensively explored. This can explain the participation of economists in such designs.

10. Conclusion

We have offered an introductory view of the subject of equilibria in auction. In doing that, we reviewed the literature and pointed out some applications. As important applications of Auction Theory, we distinguish the following: (1) multi-unit auctions, that model the selling mechanisms of Treasury Bills; (2) double auctions, that can serve as a basis for strategic foundations for general equilibrium theory and rational expectations equilibria; (3) the design of auction formats for specific situations as electricity and public resources as electromagnetic spectrum. These three wide range of applications are likely to maintain for a long time the interest in Auction Theory.

Appendix

Counterexamples for the Quasiconcavity of Auctions

We give counterexamples for independent private values auction with types uniformly distributed on $[0,1]$. An example with non-monotonic bidding function is easy. Consider that bidder 2 follows the strategy $b(t_2) = \frac{1}{2}$, and bidder 1 consider two strategies:

$$b^1(t_1) = \begin{cases} 5/8, & \text{if } t_1 \in \left[\frac{6}{8}, \frac{7}{8}\right] \\ 0, & \text{otherwise} \end{cases}$$

and

[41]See Binmore and Klemperer (2002), Milgrom (2004) and Klemperer (2004).

[42]See Maskin (1992), Dasgupta and Maskin (2000) Jehiel and Moldovanu (2001).

$$b^2(t_1) = \begin{cases} 5/8, & \text{if } t_1 \in \left[\frac{7}{8}, 1\right] \\ 0, & \text{otherwise} \end{cases}$$

Each strategy gives a positive payoff. Nevertheless, $b(t) = \frac{1}{2}b^1(t) + \frac{1}{2}b^2(t)$ never wins and, hence, has a zero payoff.

Now, we provide a counterexample with non-decreasing bidding functions. Assume that the bidder 2 follows the strategy

$$b(t_2) = \begin{cases} \frac{t_2}{3}, & \text{if } t_2 \in \left[0, \frac{3}{8}\right) \\ \frac{5}{32} + \frac{t_2}{4}, & \text{if } t_2 \in \left[\frac{3}{8}, \frac{7}{8}\right) \\ 5t_2 - 4 & \text{if } t_2 \in \left[\frac{7}{8}, 1\right] \end{cases}$$

Observe that there is a discontinuity in $t_2 = \frac{3}{8} : b\left(\frac{3}{8}^-\right) = \frac{1}{8} < \frac{1}{4} = b\left(\frac{3}{8}\right)$. Nevertheless, b is increasing. The probability that bidder 1 wins with a bid b is given by

$$G(b) = \begin{cases} 3b, & \text{if } b \in \left[0, \frac{1}{8}\right) \\ \frac{3}{8}, & \text{if } b \in \left[\frac{1}{8}, \frac{2}{8}\right) \\ 4b - \frac{5}{8} & \text{if } b \in \left[\frac{2}{8}, \frac{3}{8}\right) \\ \frac{b+4}{5} & \text{if } b \in \left[\frac{3}{8}, 1\right] \end{cases}$$

Now, consider two strategies for bidder 1: $b^1(t) = \frac{1}{8}$, $\forall t \in [0,1]$ and $b^1(t) = \frac{3}{8}$ $\forall t \in [0,1]$. Its payoffs are:

$$\int_0^1 \left(t - \frac{1}{8}\right)\frac{3}{8}dt = \frac{9}{64}, \text{ and}$$
$$\int_0^1 \left(t - \frac{3}{8}\right)\frac{7}{8}dt = \frac{7}{64},$$

Nevertheless, the bidding function $b(t) = \frac{1}{2}b^1(t) + \frac{1}{2}b^2(t) = \frac{2}{8}$ gives the payoff:

$$\int_0^1 \left(t - \frac{2}{8}\right)\frac{3}{8}dt = \frac{6}{64} < \min\left\{\frac{9}{64}, \frac{7}{64}\right\},$$

which shows that the auction is not quasiconcave even for monotonic strategies.

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CHAPTER 2
THE BASIC PRINCIPLE OF BIDDING

ABSTRACT. The Basic Principle of Bidding simply says that every bidder bids in order to equalize the marginal utility to the marginal cost of bidding. It holds for every kind of auctions, with monotonic or non-monotonic utilities, multidimensional dependent types, asymetries and any attitude towards risk. Moreover, it does not require the Common Prior Assumption. So, it can be used as a very reliable property of the bidding behavior, even for initial trials in experiments.

JEL Classification Numbers: C62, C72, D44, D82.

Keywords: auctions, pure strategy equilibria, non-monotonic bidding functions, tie-breaking rules

1. Introduction

Many experimental and empirical works suggest that the participants of auctions do (or at least may) not follow their equilibrium strategies.[1] Although there is a considerable debate about this point, it highlights the assumption that equilibrium behavior might be too strong. An alternative approach is to assume only that the players follow rationalizable strategies, instead of equilibrium strategies. Pursuing this idea, Battigalli and Siniscalchi (2003) show that some empirical and experimental findings can be explained. Nevertheless, they still assume what Harsanyi (1967-8) calls consistency of beliefs, that is, the subjective probability that players attribute to the distribution of types of the opponents is just a conditional distribution and the conditional distribution of all players comes from the same prior distribution.[2] This is almost always assumed in game theory, but does not need to be true, as Harsanyi stresses. Indeed, at the beginning of the iteration between players, they may have inconsistent beliefs. As a result, the first rounds of the game do not satisfy the consistency of beliefs and have to be discarded in order to use the received theory.

Of course, one may think that nothing can be said without this basic assumption. We show, on the contrary, that something interesting can be said. If we adhere to the even weaker assumption that bidders are rational, then we prove that they act in order to equalize their marginal utility to the marginal cost of bidding. This basic principle can provide insights for empirical and experimental studies, since every bid (even the initial or the apparently inconsistent ones) bears valuable information about the beliefs of the players. Also, the principle holds under fairly general conditions, which are given by the Characterization Lemma.

The Characterization Lemma is valid for dependent types (with arbitrary dimension), asymmetric utilities with any attitude towards risk and does not require assumptions as to monotonicity or separability of transfers. The model embraces all kind of sealed-bid auctions where each player is interested in just one object (to buy or sell).

[1]For a survey of experimental works, see Kagel (1995) and for the empirical literature on auction data, see Laffont (1997).

[2]This is also called common prior assumption.

When one introduces the additional hypotheses of risk neutrality, symmetry and monotonicity of the utility function, the characterization provided by the Lemma reduces to the first-order conditions obtained by Milgrom and Weber (1982) for first- and second-price auctions, by Krishna and Morgan (1997) for the all-pay auction and war of attrition, and by Williams (1991) for buyers'-bids double auctions.

In the next section we describe the model. Section 3 presents the Characterization Lemma, which is proved in section 4. Section 5 is a brief conclusion.

2. The Model

There are N players.[3] Player i ($i = 1, ..., N$) receives a private information, t_i, and chooses an action that is a real number (i.e., he submits a bid b_i). The "auction house" computes the bids and determines who "wins" and who "looses". If player i wins, he receives $\overline{u}_i(t, b)$ and if she looses, she receives $\underline{u}_i(t, b)$, where $t = (t_i, t_{-i})$ is the profile of all signals and $b = (b_i, b_{-i})$ is the profile of bids submitted.[4]

Information

We assume that the private signal of each player, t_i, lives in an arbitrary probabilistic space, $(T_i, \Im_i, \tau_i)$. We assume that the product space, $(T, \Im, \tau)$, is such that τ is absolutely continuous with respect to the product $\times_{i=1}^{N} \tau_i$ of its marginals.

Bidding

After receiving the private information, each player submits a sealed proposal, that is, a bid (or offer) that is a real number. There is a reserve price $b_{\min} \geqslant 0$, that correspond to the minimum valid bid.[5] In addition, the bidders can make a non-participation decision (-1).

Allocation and payoffs

We suppose that each bidder sees a number that depends only on the submitted bids by the opponents and that determines the threshold of the winning and losing events. We denote such number as $b_{(-i)}$. For instance, if the auction is an one-object auction where all players are buyers, $b_{(-i)}$ is the maximal bid of the opponents, that is, $b_{(-i)} \equiv \max_{j \neq i} b_j$, provided $b_j \geqslant b_{\min}$ for at least one player $j \neq i$. If there are K objects for selling and a reserve price $b_0 > 0$, then $b_{(-i)} \equiv \max \left\{ b_{\min}, b_{(K)}^{-i} \right\}$, where $b_{(m)}^{-i}$ is the m-th order statistic of $(b_1, ..., b_{i-1}, b_{i+1}, ..., b_N)$, that is, $b_{(1)}^{-i} \geqslant b_{(2)}^{-i} \geqslant ... \geqslant b_{(N-1)}^{-i}$.

In double auctions between m sellers and $n = I - m$ buyers, there are m objects for selling and the m highest bids are "winners" in the sense that they end the auction with one object, being the player a buyer or a seller. Then, for a player i (buyer or seller) $b_{(-i)} \equiv \max \left\{ b_{\min}, b_{(m)}^{-i} \right\}$.

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[3]Our model is inspired in auction games, although it can encompass a general class of discontinuous games. For convenience and easy understanding, we will use the terminology of auction theory, such as "bidding functions" and "bids" for strategies and actions, respectively.

[4]We consider the dependence on b instead of b_i because we want to include in our results auctions where the payoff depends on bids of the opponents, such as the second-price auction, for instance. Also, this allows the study of "exotic" auctions, i.e., auctions where the payment is an arbitrary function of all bids.

[5]If there is no reserve price (in the usual sense), let $b_{\min} = 0$.

If $b_i < b_{\min}$ (that is, player i does not participate), the payoff is 0. If $b_i > b_{(-i)}$, player i is "holder of an object" (and she has a ex-post payoff $\overline{u}_i(t,b)$ in this situation). If $b_{\min} \leqslant b_i < b_{(-i)}$, player i receives $\underline{u}_i(t,b)$.[6]

Observe that the model permits to treat buyers and sellers in the same manner. Only, if player i is a seller, she begins with a object and if $b_i < b_{(-i)}$, she sells her object. If she is a buyer, the situation $b_i < b_{(-i)}$ corresponds to maintain her previous situation: without the object. Also, the model allows for any specification of the price to be paid by the bidders.

If $b_i = b_{(-i)}$, there is a tie and a specific rule (that may include a random device and/or the requirement of a further action a_i) may determine if the player is a winner or a looser.[7] We model this by saying that the player receives $u_i^T(t,b,a)$, a value between $\overline{u}_i(t,b)$ and $\underline{u}_i(t,b)$.[8] We do not need to specify $u_i^T(t,b,a)$ for the two first results.

This setting is very general and applies to a broad class of discontinuous games. For example, $\overline{u}_i(t,b) = v_i(t) - b_i$ and $\underline{u}_i(t,b) = 0$ correspond to a first-price auction with risk neutrality.[9] If $\overline{u}_i(t,b) = v_i(t) - b_i$ and $\underline{u}_i(t,b) = -b_i$ we have the all-pay auction. If $\overline{u}_i(t,b) = v_i(t) - b_{(-i)}$ and $\underline{u}_i(t,b) = -b_i$, this is the war of attrition. As pointed out by Lizzeri and Persico (2000), we can have also combinations of these games. For example, $\overline{u}_i(t,b) = v_i(t) - \alpha b_i - (1-\alpha) b_{(-i)}$ and $\underline{u}_i(t,b) = 0$, with $\alpha \in (0,1)$, gives a combination of the first- and second-price auctions. Another possibility is the third-price auction or an auction where the payment is a general function of the others' bids. It is also useful to consider K-unit auctions with unitary demand, among N buyers, $1 < K < N$. Then, $b_{(-i)} = b_{(K)}^{-i}$. Then, a pay-your-bid auction is given by $\overline{u}_i(t,b) = v_i(t_i) - b_i$ and $\underline{u}_i(t,b) = 0$. If it is a uniform price with the price determined by the highest looser's bid, $\overline{u}_i(t,b) = v_i(t_i) - b_{(-i)}$ and $\underline{u}_i(t,b) = 0$. If it is a uniform price with the price determined by the lowest winner's bid, $\underline{u}_i(t,b) = 0$, $\overline{u}_i(t,b) = v_i(t_i) - b_{(-i)}$ if $b_i > b_{(K-1)}^{-i}$ and $\overline{u}_i(t,b) = v_i(t_i) - b_i$ otherwise. Observe that even in this last case, $\overline{u}_i(t,b)$ is continuous if $v_i(t_i)$ is.

Notation

In order to avoid confusion, we will use bold letters to denote bidding functions, i.e., $\mathbf{b} = (\mathbf{b}_i)_{i \in I} \in \times_{i \in I} \mathbb{L}^1(T_i, [-1, M])$. If we fix the other's strategies, $\mathbf{b}_{-i}$, let $F_{b_{(-i)}}(b_i|t_i) \equiv \tau_{-i}(\{t_{-i} : \mathbf{b}_{-i}(t_{-i}) < b_i\} \mid t_i)$ and $f_{b_{(-i)}}(\cdot|t_i)$ be its Radon-Nykodim derivative with respect to the Lebesgue measure, i.e., the density function.[10] We use the notation $F_{b_{(-i)}}^{\perp}(\cdot|t_i)$ for the distribution function of the singular part of the measure

[6] In most auctions, $\underline{u}_i$ is normalized as 0. However, in double and all-pay auctions or if there is an entry fee, this is not the case.

[7] The required action can be the submission of another bid for a Vickrey auction that will decide who will receive the object (as in Maskin and Riley (2000)) or the announcement of the type (as in Jackson et. al. (2002)). Since the only revealed information in the case of a tie is its occurrence, the action can be required together with the submission of the bid.

[8] The specification of a tie-breaking rule is important for the existence of equilibria, as shown by Jackson *et al.* (2002). With this terminology, the proposal of an "endogenous tie-breaking rule" of Simon and Zame (1990) corresponds to specify endogenously u_i^T in order to ensure the equilibrium existence.

[9] If we put $\overline{u}_i(t,b) = U_i(v_i(t) - b_i)$ we can have any attitude towards risk.

[10] Note that, with such convention, the cumulative distribution functions - c.d.f.'s - are left continuous.

$F_{b_{(-i)}}(\cdot|t_i)$, that is, the part that assigns positive measure to sets of bids with zero Lebesgue measure.

If the profile $\mathbf{b}_{-i}$ is fixed, the expected payoff of bidder i of type t_i, when bidding b_i, is:

$$\begin{aligned}\Pi_i(t_i, b_i, \mathbf{b}_{-i}) \equiv \int \Big[& \overline{u}_i\left(t, b_i, \mathbf{b}_{-i}\left(t_{-i}\right)\right) 1_{\left[b_i > \mathbf{b}_{(-i)}(t_{-i})\right]} \\ & + u_i^T\left(t, b_i, \mathbf{b}_{-i}\left(t_{-i}\right), a\right) 1_{\left[b_i = \mathbf{b}_{(-i)}(t_{-i})\right]} \\ & + \underline{u}_i\left(t, b_i, \mathbf{b}_{-i}\left(t_{-i}\right)\right) 1_{\left[b_i < \mathbf{b}_{(-i)}(t_{-i})\right]}\Big] \tau_{-i}\left(dt_{-i}|t_i\right).\end{aligned} \tag{1}$$

if $b_i \in [0, M]$ and $\Pi_i(t_i, b_i, \mathbf{b}_{-i}) = 0$ if $b_i < 0$. It is worth observing that if the probability of bid b_i being equal to $\mathbf{b}_{(-i)}$, conditional on t_i, is zero, the tie-breaking rule is not important and the second term in the integral may be omitted.

Again, when there is no possibility of confusion, we will write $\Pi_i(t_i, b_i)$ for $\Pi_i(t_i, b_i, \mathbf{b}_{-i})$ and omit the arguments and the measure. So, we have

$$\begin{aligned}&\Pi_i(t_i, b_i) \\ &= \int \left\{\overline{u}_i 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} + u_i^T 1_{\left[b_{(-i)} = \mathbf{b}_i\right]} + \underline{u}_i \left(1 - 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} - 1_{\left[b_{(-i)} = \mathbf{b}_i\right]}\right)\right\} \\ &= \int \left\{u_i 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} + \left(u_i^T - \underline{u}_i\right) 1_{\left[b_{(-i)} = \mathbf{b}_i\right]} + \underline{u}_i\right\} \\ &= \int u_i 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} + \int \left(u_i^T - \underline{u}_i\right) 1_{\left[b_{(-i)} = \mathbf{b}_i\right]} + \int \underline{u}_i.\end{aligned}$$

where $u_i \equiv \overline{u}_i - \underline{u}_i$ is the net payoff.

3. The Basic Principle of Bidding

Our first result is a characterization of the payoff through its derivative with respect to the bid given by an integral expression, i.e., a kind of fundamental theorem of calculus. For this, we will need the following assumption:

(H) *$\overline{u}_i$ and $\underline{u}_i$ are absolutely continuous on b_i and $\partial_{b_i}\overline{u}_i$ and $\partial_{b_i}\underline{u}_i$ are essentially bounded.*

Lemma 1 (Payoff Characterization) — Assume (H). Fix a profile of bidding functions $\mathbf{b}_{-i}$. The payoff can be expressed by

$$\begin{aligned}\Pi_i(t_i, b_i) = E&\left[\left(u_i^T - \underline{u}_i\right)\left(t_i, b_i, \cdot\right) 1_{\left[\mathbf{b}_{(-i)} = b_i\right]} | t_i\right] \\ & + \int_{[0, b_i)} E[u_i\left(t_i, b_i, \cdot\right) | t_i, \mathbf{b}_{(-i)} = \beta] dF^{\perp}_{b_{(-i)}}\left(\beta | t_i\right) + \int_{[0, b_i)} \partial_{b_i}\Pi_i(t_i, \beta) d\beta.\end{aligned}$$

where $\partial_{b_i}\Pi_i(t_i, \beta)$ exists for almost all β and in this case it is given by

$$\begin{aligned}\partial_{b_i}\Pi_i(t_i, \beta) = E&\left[\partial_{b_i}\overline{u}_i\left(t_i, \beta, \cdot\right) 1_{\left[\beta > \mathbf{b}_{(-i)}\right]} + \partial_{b_i}\underline{u}_i\left(t_i, \beta, \cdot\right) 1_{\left[\beta < \mathbf{b}_{(-i)}\right]} | t_i\right] \\ & + E[u_i\left(t_i, \beta, \cdot\right) | t_i, \mathbf{b}_{(-i)} = \beta] f_{b_{(-i)}}\left(\beta | t_i\right).\end{aligned} \tag{2}$$

Proof. The proof follows the demonstration of the Leibiniz rule. The main point is the use of a theorem of Rudin (1966) on the derivatives of measures and its integral expression. See the details in the next section.■

The most important part of Lemma 1 is the expression of $\partial_{b_i}\Pi_i(t_i,\beta)$. One of the best ways to understand Lemma 1 is through the following:

Corollary 2 (The Basic Principle of Bidding) — Under regularity assumptions, the optimum bid is such that the marginal cost of bidding is equal to the marginal utility from bidding. More formally: if $\Pi_i(t_i,\cdot)$ is differentiable at $b_i \in \arg\max_\beta \Pi_i(t_i,\beta)$ and there is no tie with positive probability at b_i, then

$$E[u_i|t_i,\mathbf{b}_{(-i)}=b_i]f_{b_{(-i)}}(b_i|t_i) = -E\left[\partial_{b_i}\overline{u}_i 1_{\left[b_i>\mathbf{b}_{(-i)}\right]} + \partial_{b_i}\underline{u}_i 1_{\left[b_i<\mathbf{b}_{(-i)}\right]}|t_i\right]. \tag{3}$$

Obverse that $E[u_i|t_i,\mathbf{b}_{(-i)}=b_i]f_{b_{(-i)}}(b_i|t_i)$ represents the marginal benefit of bidding, that is, the marginal utility that a bidder has from changing from losing to winning events. On the other hand, $E\left[-\partial_{b_i}\overline{u}_i 1_{\left[b_i>\mathbf{b}_{(-i)}\right]}|t_i\right]$ represents the marginal cost of changing the bid in all the events where a bidder is already winning. In the same manner, $E\left[-\partial_{b_i}\underline{u}_i 1_{\left[b_i<\mathbf{b}_{(-i)}\right]}|t_i\right]$ represents the marginal cost of changing the bid in the events where he is loosing. Thus, we can read the above condition in an intuitive and simple manner: at the optimum of the best-reply problem, the marginal benefit of bidding, $E[u_i|t_i,\mathbf{b}_{(-i)}=b_i]f_{b_{(-i)}}(b_i|t_i)$, must be equal to its marginal cost,

$$-E\left[\partial_{b_i}\overline{u}_i 1_{\left[b_i>\mathbf{b}_{(-i)}\right]} + \partial_{b_i}\underline{u}_i 1_{\left[b_i<\mathbf{b}_{(-i)}\right]}|t_i\right].$$

Note that we do not require separability in the monetary transfer (risk neutrality) to reach such an interpretation.

This interpretation is useful for understanding the bidding behavior. In first-price auctions, the marginal cost of bidding is what implies a decreasing in the way bidders bid. In second-price auctions, the marginal cost of bidding is zero (because $\partial_{b_i}\overline{u}_i = 0$), so that each bidder bids until its marginal utility of bidding became zero.

Corollary 1 is a generalization of the necessary conditions first-order for the first- and second-price auctions presented in Milgrom and Weber (1982), for the war of attrition and all-pay auctions presented in Krishna and Morgan (1997), as we show in Examples 1- 4 below. Example 5 shows how the Basic Principle of Bidding is concise. Such an example is the application of Corollary 1 for double auctions and it presents a comparison with the equivalent expression obtained by Williams (1991).

Example 1 — First-price auction

When we restrict ourselves to the case of the first-price auction with risk neutrality (i.e., $\underline{u}_i = 0$ and $\overline{u}_i = v_i - b_i$), then $\partial_{b_i}\overline{u}_i = -1$ and $\partial_{b_i}\underline{u}_i = 0$. The condition (3) becomes:

$$b_i = E[v_i|t_i,\mathbf{b}_{(-i)}=b_i] - \frac{F_{b_{(-i)}}(b_i|t_i)}{f_{b_{(-i)}}(b_i|t_i)}. \tag{4}$$

This (necessary) first-order condition provides a useful way to determine best-reply bids. Note that this expression admits non-monotonic bidding functions $\mathbf{b}_{(-i)}$, contrary to Milgrom and Weber's model. It also encompasses asymmetries in utilities and distribution of types. Assuming affiliation and monotonic utilities, Milgrom and Weber (1982) can restrict themselves to the space of non-decreasing symmetric bidding functions (i.e., $\mathbf{b}_i = \mathbf{b}^*$, for all $i \in I$). Thus,

$$\mathbf{b}_{(-i)}\left(t_{-i}\right) = \max_{j \neq i} \mathbf{b}^*\left(t_j\right) = x \Longleftrightarrow t_{(-i)} \equiv \max_{j \neq i} t_j = \left(\mathbf{b}^*\right)^{-1}(x),$$

i.e., conditioning on $\mathbf{b}_{(-i)} = b_i$ is the same to conditioning on $t_{(-i)} = t_i$. Also, $f_{b_{(-i)}}\left(b_i|t_i\right) = f_{t_{(-i)}}\left(t_i|t_i\right)/\left(\mathbf{b}^*\left(t_i\right)\right)'$ and $F_{b_{(-i)}}\left(b_i|t_i\right) = F_{t_{(-i)}}\left(t_i|t_i\right)$. With this, (4) becomes

$$\mathbf{b}^{*\prime}(s) = \left\{E\left[v|t_i = s, t_{(-i)} = s\right] - \mathbf{b}^*(s)\right\} \frac{f_{t_{(-i)}}(s|s)}{F_{t_{(-i)}}(s|s)}$$

whose solution is shown to be an equilibrium under affiliation.

Example 2 — Second price auction

In the second price auction, Milgrom and Weber's model is equivalent to $\overline{u}_i(t,b) = v_i(t) - b_{(-i)}$ and $\underline{u}_i(t,b) = 0$. Then, $\partial_{b_i}\overline{u}_i = \partial_{b_i}\underline{u}_i = 0$ and (3) reduces to $E[v_i - b_i|t_i, \mathbf{b}_{(-i)} = b_i]\; f_{b_{(-i)}}\left(b_i|t_i\right) = 0$ which can be simplified to

$$b_i = E[v_i|t_i, \mathbf{b}_{(-i)} = b_i].$$

Again with monotonicity and symmetry assumptions, Milgrom and Weber's expression for the equilibrium bid function can be obtained:

$$\mathbf{b}^*(s) = E\left[v|t_i = s, t_{(-i)} = s\right] \equiv \bar{v}(s,s).$$

Example 3 — All-pay auction

Krishna and Morgan (1997) extend the method of Milgrom and Weber (1982) to the cases of war of attrition and all-pay auctions. In the all-pay auction, their model is equivalent to $\overline{u}_i(t,b) = v_i(t) - b_{(-i)}$ and $\underline{u}_i(t,b) = -b_i$. Then, $\partial_{b_i}\overline{u}_i = 0$ and $\partial_{b_i}\underline{u}_i = -1$. So, (3) reduces to

$$E[v_i(t)|t_i, \mathbf{b}_{(-i)} = b_i] f_{b_{(-i)}}\left(b_i|t_i\right) = 1.$$

Under the same hypothesis of monotonicity and symmetry, they find the following differential equation:

$$\mathbf{b}^{*\prime}(s) = E\left[v|t_i = s, t_{(-i)} = s\right] f_{t_{(-i)}}(s|s),$$

whose solution they show to be an equilibrium under affiliation.

Example 4 — War of attrition

In the war of attrition, Krishna and Morgan (1997) model is equivalent to $\overline{u}_i(t,b) = v_i(t) - b_{(-i)}$ and $\underline{u}_i(t,b) = -b_i$. Then, $\partial_{b_i}\overline{u}_i = 0$ and $\partial_{b_i}\underline{u}_i = -1$. So, (3) reduces to

$$E[v_i(t)|t_i, \mathbf{b}_{(-i)} = b_i] f_{b_{(-i)}}\left(b_i|t_i\right) = 1 - F_{b_{(-i)}}\left(b_i|t_i\right).$$

Again, with monotonicity and symmetry, they derive the equation

$$\mathbf{b}^{*\prime}(s) = E\left[v|t_i = s, t_{(-i)} = s\right] \frac{1 - F_{t_{(-i)}}(s|s)}{f_{t_{(-i)}}(s|s)},$$

and the equilibrium is shown to exist under affiliation.

Example 5 — Double auction

In the analysis of a double auction with private values, risk neutrality, independent types and symmetry among buyers and sellers, Williams (1991) assumes that the payment is determined by the buyer's bid. So, it is optimum for the seller to bid her value. To analyze the behavior of the *buyer* i, Williams (1991) reaches the following expression:

$$\begin{aligned}(5)\qquad \partial_{b_i}\Pi_i(v,\beta) = &\Big[nf_1(\beta)\,K_{n,m}\left(\mathbf{b}^{-1}(\beta),\beta\right)\\ &+(m-1)\,\frac{f_2(v_b)}{b'(\beta)}L_{n,m}\left(\mathbf{b}^{-1}(\beta),\beta\right)\Big](v-\beta)\\ &-M_{n,m}\left(\mathbf{b}^{-1}(\beta),\beta\right)\end{aligned}$$

where $\mathbf{b}$ denotes here the symmetric bidding function followed by all buyers, f_1 is the common density function of sellers, f_2 is the common density function of buyers, n is the number of sellers, m is the number of buyers and $M_{n,m}(\cdot,\cdot)$ is given by:[11]

$$M_{n,m}(v,\beta) \equiv \sum_{\substack{i+j=m,\\ 0\leqslant i\leqslant m-1}} \binom{n}{j}\binom{m-1}{i} F_1(\beta)^j\, F_2(v)^i\,(1-F_1(\beta))^{n-j}\,(1-F_2(v))^{m-1-i}\,.$$

The expression (5) is just a special case of (3). In fact, the expression in brackets in (5) is just $f_{b_{(-i)}}(\beta)$ and $M_{n,m}\left(\mathbf{b}^{-1}(\beta),b\right)$ is $F_{b_{(-i)}}(\beta)$.

An important application of the Characterization Lemma will be given in the next section where we give necessary and sufficient conditions to the existence of equilibrium in common-value auctions with multidimensional independent types and non-monotonic utilities.

Another possibility is the investigation of how far auction theory can lead us under a weaker hypothesis. For instance, the Characterization Lemma can be understood as a general condition for bidding behavior, able to describe the behavior of rational bidders without assuming that bidders follow their equilibrium strategies. We have exposed such a possibility in the introduction (subsection 1.1).

4. Proof of Lemma 1

Let us first rewrite the expression of Π_i:

$$\Pi_i(t_i,b_i) = E\left[\underline{u}_i|t_i\right] + E\left[\left(u_i^T-\underline{u}_i\right)1_{\left[\mathbf{b}_{(-i)}=b_i\right]}|t_i\right] + E\left[u_i 1_{\left[b_i>\mathbf{b}_{(-i)}\right]}|t_i\right],$$

where $u_i = \overline{u}_i - \underline{u}_i$.

We consider each term above separately. The first one has a derivative with respect to b_i almost everywhere and is equal to $E\left[\partial_{b_i}\underline{u}_i|t_i\right]$. The derivative of the last term with respect to b_i is just $E\left[\partial_{b_i}\underline{u}_i|t_i\right]$. Also,

$$E\left[\underline{u}_i|t_i\right] = \int E\left[\partial_{b_i}\underline{u}_i|t_i\right]d\beta.$$

[11] To obtain $K_{n,m}(\cdot,\cdot)$ just substitute $n-1$ for n where it occurs in $M_{n,m}(\cdot,\cdot)$. To obtain $L_{n,m}(\cdot,\cdot)$, substitute $m-2$ for $m-1$ where it occurs in $M_{n,m}(\cdot,\cdot)$.

The second term is different from zero just where there is an atom in the distribution of $\mathbf{b}_{(-i)}$. Thus, it is equal to zero for almost all b_i, and its derivative is zero almost everywhere.

Now consider the last term in its original form, $\int u_i 1_{[b_i > \mathbf{b}_{(-i)}]}$. Let $a^n \to b_i^+$ (i.e., $a^n > b_i$; the other case is analogous). We have

$$\int \left\{u_i\left(t_i, a^n, \cdot\right) 1_{\left[a^n > \mathbf{b}_{(-i)}\right]}\right\} - \int \left\{u_i\left(t_i, b_i, \cdot\right) 1_{\left[b_i > \mathbf{b}_{(-i)}\right]}\right\}$$
$$= \int \left\{\left[u_i\left(t_i, a^n, \cdot\right) - u_i\left(t_i, b_i, \cdot\right)\right] 1_{\left[a^n > \mathbf{b}_{(-i)}\right]}\right\} + \int u_i\left(t_i, b_i, \cdot\right) 1_{\left[a^n > \mathbf{b}_{(-i)} > b_i\right]}.$$

Since u_i has bounded derivative with respect to almost all b_i, $\frac{u_i(t_i,a^n,\cdot)-u_i(t_i,b_i,\cdot)}{a^n-b_i} \to \partial_{b_i} u_i$, for almost all b_i. Also, $1_{\left[a^n > \mathbf{b}_{(-i)}\right]} \to 1_{\left[b_i > \mathbf{b}_{(-i)}\right]}$. These imply that

$$\frac{u_i\left(t_i, a^n, \cdot\right) - u_i\left(t_i, b_i, \cdot\right)}{a^n - b_i} 1_{\left[a^n > \mathbf{b}_{(-i)}\right]} \to \partial_{b_i} u_i 1_{\left[b_i > \mathbf{b}_{(-i)}\right]}$$

for almost all b_i and these functions are (almost everywhere) bounded. By the Lebesgue Theorem, the integral converges, that is, there exists

$$\lim_{a^n \to b_i} \int \frac{u_i\left(t_i, a^n, \cdot\right) - u_i\left(t_i, b_i, \cdot\right)}{a^n - b_i} 1_{\left[a^n > \mathbf{b}_{(-i)}\right]}$$

and it is equal to $E\left[\partial_{b_i} u_i\left(t_i, b_i, \cdot\right) 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} | t_i\right]$.

Now we want to determine the derivative of the other term. For this purpose, define for each $t_i \in T_i$ fixed, the measure ρ over $\mathbb{R}_+$ by

$$\rho\left(V\right) \equiv \int u_i\left(t_i, b_i, \cdot\right) 1_{\left[\mathbf{b}_{(-i)} \in V\right]}.$$

We have

$$\lim_{a^n \to b_i} \frac{1}{a^n - b_i} \int u_i\left(t_i, b_i, \cdot\right) 1_{\left[a^n > \mathbf{b}_{(-i)} > b_i\right]}$$
$$= \lim_{a^n \to b_i} \frac{1}{a^n - b_i} \rho\left(\left[b_i, a^n\right)\right)$$
$$= \lim_{a^n \to b_i} \left\{\frac{\rho\left(\left[b_i, a^n\right)\right)}{m\left(\left[b_i, a^n\right)\right)}\right\}$$
$$= D\rho\left(b_i\right)$$

where the existence of $\lim_{r \to 0} \frac{\rho(B(b_i,r))}{m(B(b_i,r))} = D\rho\left(b_i\right)$ is ensured by Theorem 8.6 of Rudin (1966) for almost all b_i, that is, $m\left(\left\{v : \nexists \lim_{r\to 0} \frac{\rho(B(b_i,r))}{m(B(b_i,r))}\right\}\right) = 0$. Theorem 8.6 of Rudin (1966) also says that $D\rho$ coincides almost everywhere with the Radon-Nikodym derivative $\frac{d\rho}{dm}\left(.\right)$ and that

$$\rho\left(V\right) = \rho^{\perp}\left(V\right) + \int_V \frac{d\rho}{dm}\left(\beta\right) d\beta.$$

where $\rho^{\perp}$ denotes the orthogonal part of ρ, and it has the property

$$\lim_{a^n \to b_i} \frac{1}{a^n - b_i} \rho^{\perp}\left(\left[b_i, a^n\right)\right) = 0,$$

by the same theorem.

It is easy to see that ρ is absolutely continuous with respect to $F_{b_{(-i)}}$. The Radon-Nikodym Theorem guarantees the existence of the Radon-Nikodym derivative of ρ with respect to $F_{b_{(-i)}}$, denoted by $E[u_i\left(t_i, b_i, \cdot\right)||t_i, \mathbf{b}_{(-i)}\left(t_{-i}\right) = \beta]$ such that

$$\rho\left(V\right) \equiv \int_V E[u_i\left(t_i, b_i, \cdot\right)|t_i, \mathbf{b}_{(-i)} = \beta] dF_{b_{(-i)}}\left(\beta|t_i\right).$$

Then, it is easy to see that the Radon-Nikodym derivative $\frac{d\rho}{dm}\left(b_i\right)$ is simply

$$E[u_i\left(t_i, b_i, \cdot\right)|t_i, \mathbf{b}_{(-i)}\left(t_{-i}\right) = b_i] f_{b_{(-i)}}\left(b_i|t_i\right),$$

where $f_{b_{(-i)}}\left(\cdot|t_i\right)$ is the Radon-Nikodym derivative of $F_{b_{(-i)}}\left(\cdot|t_i\right)$. Thus,

$$\begin{aligned}\partial_{b_i}\Pi_i(t_i, \beta) = E\left[\partial_{b_i}\overline{u}_i\left(t_i, \beta, \cdot\right) 1_{\left[\beta > \mathbf{b}_{(-i)}\right]} + \partial_{b_i}\underline{u}_i\left(t_i, \beta, \cdot\right) 1_{\left[\beta < \mathbf{b}_{(-i)}\right]}|t_i\right] \\ + E[u_i\left(t_i, \beta, \cdot\right)|t_i, \mathbf{b}_{(-i)} = \beta] f_{b_{(-i)}}\left(\beta|t_i\right),\end{aligned}$$

and, by the Lebesgue Theorem,

$$\begin{aligned}\Pi_i(t_i, b_i) = E\left[\left(u_i^T - \underline{u}_i\right)\left(t_i, b_i, \cdot\right) 1_{\left[\mathbf{b}_{(-i)} = b_i\right]}|t_i\right] \\ + \int_{[0,b_i)} E[u_i\left(t_i, b_i, \cdot\right)|t_i, \mathbf{b}_{(-i)} = \beta] dF^{\perp}_{b_{(-i)}}\left(\beta|t_i\right) + \int_{[0,b_i)} \partial_{b_i}\Pi_i(t_i, \beta) d\beta.\end{aligned}$$

This concludes the proof.■

5. Conclusion

We think the Basic Principle can be useful to the empirical and experimental literature, because of its generality.

For example, each bid in a experiment can be considered, even the initial ones. This would permit to observe how the agents learn and modify their priors. The fact that it does not assume the common prior assumptions seems a very attractive characteristic.

In the empirical literature, it can lead to generalizations of the identification results. Indeed, Athey and Haile (2002) use simpler versions of the Basic Principle in their results on identification.

CHAPTER 3
PURE STRATEGY EQUILIBRIA OF MULTIDIMENSIONAL AND NON-MONOTONIC AUCTIONS

ABSTRACT. We give necessary and sufficient conditions for the existence of symmetric equilibrium without ties in interdependent values auctions, with multidimensional independent types and no monotonic assumptions. In this case, non-monotonic equilibria might happen. When these conditions are not satisfied, we are still able to prove the existence of pure strategy equilibrium with an "all-pay auction tie-breaking rule" that consists in conducting an all-pay auction in case of tie. As a direct implication of these results, we obtain a generalization of the Revenue Equivalence Theorem.

JEL Classification Numbers: C62, C72, D44, D82.

Keywords: auctions, pure strategy equilibria, non-monotonic bidding functions, tie-breaking rules

1. Introduction

The received literature on pure strategy equilibria on auctions is mainly restricted to the setting of unidimensional types and monotonic utilities. Although recent efforts have been made to treat the case of multidimensional types (see McAdams (2003), for instance), the monotonicity assumption is usually maintained. In dealing with multidimensional types, this is obviously restrictive (see also our examples in section 5).

So, to develop a satisfactory theory of equilibria with multidimensional types, it is necessary to take in account the possibility of non-monotonic utility functions.

However, even in the unidimensional case, non-monotonic auctions are problematic. To see why, consider a symmetric first-price auction between two buyers, such that the value of the object is given by $v\left(t_i, t_{-i}\right) = \alpha + t_i + \beta t_{-i}$, with independent types distributed on $[0,1]$.

The received theory ensures the existence of a monotonic *pure* strategy equilibrium only if $\beta \geqslant 0$. See Milgrom and Weber (1982), Maskin and Riley (2000), Athey (2001). If $\beta < 0$, we only know that there exists a tie-breaking rule (endogenously defined) that guarantees the existence of *mixed* strategy equilibrium (see Jackson, Simon, Swinkels and Zame (2002), henceforth JSSZ).

That the case $\beta < 0$ is problematic can be seen through particular cases. Indeed, if $\alpha = 5$, $\beta = -4$ and the distribution is uniform on $[0,1]$, this is example 1 of JSSZ. If $\alpha = 3$, $\beta = -2$ and types assumes values 0 or 1 with probabilities $\frac{2}{3}$ and $\frac{1}{3}$, respectively, it is the example 3 of Maskin and Riley (2000). Both cases are counterexamples to the existence of equilibrium, even with special tie-breaking rules. Maskin and Riley (2000) show that there is no equilibrium for their example neither under the standard tie-breaking rule (that assigns the object randomly to the tying bidders), nor under the Vickrey auction tie-breaking rule, defined as "if a tie occurs for the high bid, a Vickrey auction is conduct among the high bidders". JSSZ make the claim, corrected in Jackson, Simon, Swinkels and Zame (2004), that there is no tie-breaking rule that is type-independent that ensures the existence of equilibrium for their example.

Some questions arise from the contrast of the theoretical results for $\beta \geqslant 0$ and $\beta < 0$: For which set of β the standard tie-breaking rule is sufficient to ensure the existence of equilibrium? Is it possible to define a specific tie-breaking rule for all β? For which set of β there is no equilibrium in pure strategy? The results are valid only for first-price auctions? Is there any $\beta < 0$ such that the obtained equilibria for different auctions obey the Revenue Equivalence Theorem? Is the equilibrium unique?

This paper provides the following answer to the above questions: If $\beta > -1$, there exists equilibrium in pure strategies under the standard tie-breaking rule. If we adopt an "all-pay auction tie-breaking rule", that consists in conducting an all-pay auction among the tying bidders in the case of a tie, then there exists a *pure* strategy equilibrium for all β (provided $\alpha \geqslant \max\{0, -\beta\}$, otherwise the object can have negative values). Moreover, the all-pay auction tie-breaking rule works for all standard type of auctions and the equilibria obtained under it obey the Revenue Equivalence Theorem. We also prove that the equilibrium is unique if $\beta > -1$, but are multiple otherwise.

It is important to note that the all-pay auction tie-breaking rule is "type-independent", in the sense that it does not require private information. So, our result contradicts the claim of JSSZ that there is no type-independent tie-breaking rule that supports the existence of equilibrium. Correcting this claim, Jackson et.al. (2004) gives another example that does not have equilibrium with type-independent tie-breaking rule. This new example does not contradict our theory. The reason is that it is not a standard auction: there is an uncertainty about the number of objects in the auction.

Our results hold for symmetric auctions with independent (non-atomic) types, but are valid for a wide class of auction formats where the bidders have unitary demands (first-price, second-price, all-pay, war of attrition). Moreover, we impose no restriction about the dimension of the set of types, nor make monotonic assumptions about the value of the object. All the answers provided above for the specific example are given in a general setting (that of weakly separable utilities, as defined by assumption H3 in section 5). Of course, the condition for the existence of equilibrium is something more complex for this general case, but it is still easy to verify.

Our results are based in what we call the "Indirect Auction Approach", which we describe in the subsection 1.1. In section 2, we describe the model. Section 3 formally presents the Indirect Auction Approach. Section 4 develops the theory for general auctions, obtaining necessary and sufficient conditions for the existence of equilibrium. Section 5 particularizes to the case of weakly separable utilities and gives a concise condition for the existence of equilibrium. Moreover, the all-pay auction tie-breaking rule is introduced and the equilibrium existence proved. As a corollary, we obtain the Revenue Equivalence Theorem. Section 6 concludes with a discussion about the limits of our results and reviews the contributions of the paper in light of the related literature. All the proofs are collected in appendices.

1.1. **The Indirect Auction Approach.** For standard auctions, higher bids correspond to higher probability of winning. If a bidding function $b(\cdot)$ is fixed and followed by all participants in a symmetric auction, we can associate to each bid (and so, to each type), the probability of winning. All types that bid the same bid under $b(\cdot)$ have the same probability of winning. This allows us to introduce the concept of conjugation.

If $b(t) = b(s)$, and hence, t and s have the same probability of winning, we say that t and s are conjugated.

The use of the probability of winning as analytical tool is not new in auction theory. Sometimes in the literature, what we call conjugation is named "reduced form": "The function relating a bidder's type to his probability of winning is the reduced form of the auction." (Border, 1991, p. 1175). See also Matthews (1984) and Chen (1986). Therefore, what we will call "indirect auction" can be also called "reduced form auction". These papers analyze problems related to the characterization and existence of optimal auctions. Hence, the auction is treated, as Myerson (1981) does, only by considering the probability of winning and the payments. In turn, our problem is to find the equilibrium for fixed auction rules. Moreover, our indirect auction is not "equivalent" to the direct one. So, it is not a merely "reduced form" of the auction. (See remarks after Theorem 1 in section 4). It is in the light of these differences and in the attempt to do not confuse terms that we decided to use a different terminology.

The terminology comes from the "Taxation Principle" which allows us to implement the optimal direct truthful mechanism through some convenient indirect one. In this case, we are implementing the equilibrium in the auction using an indirect auction obtained from the reparameterization of types through the probability of winning.

Returning to the description of the method, the main idea is to reparameterize the types and to associate to all conjugated types $s \in S$, the probability of winning the auction. As stated, this idea should seem unpromising since the probability of winning will be different for each different bidding function that we begin with. Moreover, if we do not previously fix a bidding function, no conjugation can be defined.

To overcome these problems, we define conjugations without needing to mention bidding functions, as a suitable reparameterization of the types. The definition comes from an insight acquired from the above notion of conjugation. Once we have defined conjugations (Definition 2 in Section 3), we can define in subsection 3.2 the Indirect Auction. For this, we simply integrate the utilities of the direct auction for all types that are conjugated. From our definition of conjugation, the indirect auction is now an auction with the same format of the direct auction (for instance, a first-price auction if the original auction is a first-price auction) between two players with independent signals, uniformly distributed on $[0, 1]$. This makes the analysis of equilibrium existence easier. An important result of the subsection 3.2 is the relationship between the payoffs of direct and indirect auctions, which is made in Proposition 1.

With these preparatory results, we can finally deal with the problem of equilibrium existence in section 4. First, we prove that the existence of a regular equilibrium implies nice properties for the conjugation that it defines. This is the content of Theorem 1. These properties are almost sufficient for the existence of the equilibrium, which is proved in Theorem 2: since we have defined the conjugation without mentioning a bidding function, then whenever we can find a conjugation that meets the conditions of Theorem 2, there exists a regular equilibrium of the direct auction. If we manage to find the correct conjugation, we are done. We show how to perform this task in two examples (1 and 2) at the end of section 4.

In section 5, we treat the case of weakly separable utilities that include the separable utilities as a special case, that is, $v(t_i, t_{-i}) = v^1(t_i) + v^2(t_{-i})$. In this setting, we are able to give necessary and sufficient conditions for the existence of regular equilibrium (Theorem 3). This is useful, but it raises the question: what can be done if the necessary and sufficient conditions of Theorem 3 are not met?

Theorem 4 provides the answer. If we conduct an all-pay auction in the case of ties, the equilibrium exists in pure strategies with ties of positive probability. Moreover, we generalize the Revenue Equivalence Theorem for this setting, even with the tie-breaking rule.

2. The Model

There are N bidders in an auction of $L < N$ homogenous objects, but each bidder is interested in just one object. Player i ($i = 1, ..., N$) receives a private information, t_i, possibly multidimensional, and chooses a bid $b_i \in B \equiv \{b_{OUT}\} \cup [b_{\min}, +\infty)$, where $b_{\min} > b_{OUT}$ is the minimal valid bid and if $b_i = b_{OUT}$, bidder i does not participate in the auction and gets a payoff of 0.

Let $t = (t_i, t_{-i})$ be the profile of all signals and $b = (b_i, b_{-i})$, the profile of submitted bids. Let $b^{-i}_{(m)}$ be the m-th order statistic of $(b_1, ..., b_{i-1}, b_{i+1}, ..., b_N)$, that is, $b^{-i}_{(1)} \geqslant b^{-i}_{(2)} \geqslant ... \geqslant b^{-i}_{(N-1)}$. Let $b_{(-i)} \equiv \max\left\{b_{\min}, b^{-i}_{(L)}\right\}$.

The auction is a standard one. That is, the bidder i receives an object if $b_i > b_{(-i)}$ and none if $b_i < b_{(-i)}$. Ties ($b_i = b_{(-i)}$) are broken by the standard tie-breaking rule, that is, the object is randomly divided between the tying bidders. More specifically, the payoff of bidder i is given by

$$u_i(t, b) = \begin{cases} v(t_i, t_{-i}) - p^W\left(b_i, b_{(-i)}\right), & \text{if } b_i > b_{(-i)} \\ -p^L\left(b_i, b_{(-i)}\right), & \text{if } b_i < b_{(-i)} \\ \frac{v(t_i, t_{-i}) - b_i}{m}, & \text{if } b_i = b_{(-i)} \end{cases}$$

where v is the value of the object for all bidders, p^W and p^L are the payments made in the events of winning and losing, respectively and m is the number of bidders tying.

Our setting is given by the following assumptions:

(H0) The types are independent and identically distributed in the same compact set S, according to a non-atomic probability measure μ on S. v is positive ($v > 0$), continuous and symmetric in its last $N-1$ arguments, that is, if t'_{-i} is a permutation of t_{-i}, $v\left(t_i, t'_{-i}\right) = v(t_i, t_{-i})$.

The restrictive aspect of (H0) is the symmetry. The others are very natural. For instance, the assumption that v is positive is not restrictive, since S is compact and, hence, v assumes a minimum m. Then, if we add $m + 1$ to the value of the object, v becomes positive.

The specific auction is determined by p^W and p^L. We will consider alternatively, two cases. The first one, embodied in (H1)-1 below, cover first-price auctions, for instance. The second case is defined by (H1)-2 and covers second price auctions, among other more exotic formats.

(H1) Over the domain $B \times B$, p^W and p^L are non-negative, differentiable, $p^L(b_{OUT}, \cdot) = 0$, $\partial_1 p^W \geqslant 0$, $\partial_1 p^L \geqslant 0$ and, alternatively:
(H1)-1: $\partial_1 p^W(\cdot) > 0$ or $\partial_1 p^L(\cdot) > 0$ or
(H1)-2: $\partial_1 p^W = \partial_1 p^L \equiv 0$ and $\partial_2\left(p^W - p^L\right) > 0$.

Observe that assumption (H1) is rather weak. It covers virtually all kind of standard single-objetc auctions or multi-unit auctions with unitary demands, and allows the use of entry fees. Some examples are:

(F) First-price auctions: $p^W\left(b_i, b_{(-i)}\right) = b_i$ and $p^L\left(b_i, b_{(-i)}\right) = 0$.
(S) Second-price auctions: $p^W\left(b_i, b_{(-i)}\right) = b_{(-i)}$ and $p^L\left(b_i, b_{(-i)}\right) = 0$.
(A) All-pay auctions: $p^W\left(b_i, b_{(-i)}\right) = b_i$ and $p^L\left(b_i, b_{(-i)}\right) = b_i$.
(W) War of attrition: $p^W\left(b_i, b_{(-i)}\right) = b_{(-i)}$ and $p^L\left(b_i, b_{(-i)}\right) = b_i$.

An active reserve price, that is, a $b_{\min}$ that excludes some bidders, makes the statement of our equilibrium results more complex. So, we will postpone the analysis of this case to the Appendix B and through the paper we will make use of the following assumption:

(H2) v, p^W, p^L and $b_{\min}$ are such that no bidder plays b_{OUT}, that is, no bidder prefers to stay out of the auction.

We denote the auction described above by (S, μ, v). Observe that we make no restriction about the dimension of S. Also, we are considering just symmetric auctions. Thus, throughout this section, when we talk about a strategy, we always mean a symmetric one. Under these assumptions we will introduce a new approach to prove existence of equilibria in auctions. We call it the "Indirect Auction Approach". This is the subject of the next section.

3. The Indirect Auction Approach

In the subsection 3.1, we describe the basic element of our method: the conjugation. In subsection 3.2, the indirect auction is defined and its basic properties derived. See the introduction for a description of the approach.

3.1. Conjugations.

We will be interested in regular bidding functions as defined below:

Definition 1. A bounded measurable function $b : S \to \mathbb{R}$ is regular if the c.d.f.

$$F_b(c) \equiv \Pr\{s \in S : b(s) < c\}$$

is absolutely continuous and strictly increasing in its support, $[b_*, b^*]$.

From the fact that $F_b(\cdot)$ is absolutely continuous, we conclude that $F_b(c) = \Pr\{s \in S : b(s) \leqslant c\}$. Let $\mathcal{S}$ denote the set of regular functions. Observe that $\mathcal{S}$ contains non-monotonic bidding functions. It is formed by functions b that do not induce ties with positive probability (because F_b is absolutely continuous) and that do not have gaps in the support of the bids (because F_b is increasing).

If a bidding function $b \in \mathcal{S}$ is fixed, let us call the c.d.f. of the maximum bid of the opponents $\tilde{P}^b$. That is, we define the transformation $\tilde{P}^b : \mathbb{R}_+ \to [0, 1]$ by:

$$\begin{aligned} \tilde{P}^b(c) &= \left(\Pr\{t_i \in S : b(t_i) < c\}\right)^{N-1} \\ &= \Pr\left\{t_{-i} \in S^{N-1} : b(t_j) < c, j \neq i\right\}. \end{aligned} \tag{1}$$

By the definition of $\mathcal{S}$, $\tilde{P}^b$ is strictly increasing and its image is the whole interval $[0, 1]$.

Now, we will denote by $P^b : S \to [0,1]$ the composition $P^b = \tilde{P}^b \circ b$. So, for a fixed $b \in \mathcal{S}$, followed by all players, $P^b(t_i)$ is the probability of player i of type t_i wins the auction:

$$
\begin{aligned}
P^b(t_i) &= \Pr\left\{t_{-i} \in S^{N-1} : b_{(-i)}(t_{-i}) < b(t_i)\right\} \\
&= \Pr\left\{t_{-i} \in S^{N-1} : b(t_j) \leqslant b(t_i), \forall j \neq i\right\}.
\end{aligned} \tag{2}
$$

The following observation is important: from the symmetry required by (H0), the above function does not depend on i and $P^b(t_i) \lesseqqgtr P^b(t_j)$ if and only if $b(t_i) \lesseqqgtr b(t_j)$. Obviously, two players have the same probability of winning if and only if they play the same bids. So, we have the following:

$$
\left\{t_{-i} \in S^{N-1} : b_{(-i)}(t_{-i}) < b(t_i)\right\} = \left\{t_{-i} \in S^{N-1} : P^b_{(-i)}(t_{-i}) < P^b(t_i)\right\},
$$

where $P^b_{(-i)}(t_{-i}) \equiv \max_{j \neq i} P^b(t_j)$. The equality of these events and (2) imply that

$$
P^b(t_i) = \Pr\left\{t_{-i} \in S^{N-1} : P^b_{(-i)}(t_{-i}) < P^b(t_i)\right\}.
$$

This observation is what will allow us to define conjugations without mentioning bidding functions. This will be very important in order to state our results. We have the following:

Definition 2. A conjugation for the auction (S, μ, v) is a measurable and surjective function $P : S \to [0,1]$ such that for each $i = 1, \ldots N$,
(3)
$$
P(t_i) = \Pr\{t_{-i} \in S^{N-1} : P_{(-i)}(t_{-i}) \leqslant P(t_i)\} = [\Pr\{t_j \in S : P(t_j) < P(t_i), j \neq i\}]^{N-1}.
$$

Observe that in the above definition, we do not need to mention the strategy $b \in \mathcal{S}$. It is also clear from the previous discussion that definition 2 is not empty, that is, for any regular function $b \in \mathcal{S}$ there exists a conjugation defined by (2) that satisfies the above definition.

Observe also that, since the range of P is $[0,1]$, we have, for all $c \in [0,1]$,

$$
\Pr\left\{t_{-i} \in S^{N-1} : P_{(-i)}(t_{-i}) < c\right\} = c. \tag{4}
$$

The above equation will be important in the sequel. It simply means that the distribution of $P_{(-i)}(t_{-i})$ is uniform on $[0,1]$.

Given $b \in \mathcal{S}$, equation (2) defines just one conjugation compatible with it. On the other hand, given a conjugation P, any function $b \in \mathcal{S}$ that is an increasing transformation of P is compatible with P. To see this, suppose that there is an increasing function $h : [0,1] \to \mathbb{R}_+$, such that $b(t_i) = h(P(t_i))$ for $\mu-$almost all $t_i \in S$. Then,

$$
\begin{aligned}
P(t_i) &= \Pr\{t_{-i} : P(t_j) < P(t_i), \forall j \neq i\} \\
&= \Pr\{t_{-i} : h(P(t_j)) < h(P(t_i)), \forall j \neq i\} \\
&= \Pr\{t_{-i} : b(t_j) < b(t_i), \forall j \neq i\}.
\end{aligned}
$$

That is, given a conjugation P, there are many functions $b \in \mathcal{S}$ compatible with it. In particular, $b = P$ is a bidding function compatible with P.

3.2. **Indirect Auctions.** We proceed to define the indirect auction $\left(\tilde{S}, \tilde{\mu}, \tilde{v}\right)$ related to the direct auction (S, μ, v). The relation between them is given by the conjugation $P : S \to [0, 1]$. If the direct type of a player is $t_i \in S$, the indirect type will be $P(t_i)$. So, $\tilde{S}$ is just $[0, 1]$. For each direct strategy $b : S \to \mathbb{R}$, it will correspond an indirect strategy $\tilde{b} : [0, 1] \to \mathbb{R}$, such that the direct strategy will be the composition of the indirect strategy and the conjugation, that is, $b = \tilde{b} \circ P$, where $P = P^b$. What is this indirect strategy? Remember that $P^b = \tilde{P}^b \circ b$ and $\tilde{P}^b$ is increasing. So, given b, if we take the indirect strategy as $\tilde{b} \equiv \left(\tilde{P}^b\right)^{-1}$ then $b = \tilde{b} \circ P$, as we want. On the other hand, if it is given an indirect strategy $\tilde{b}$ and a conjugation P, we have the associated direct strategy $b = \tilde{b} \circ P$. So we have just to define the indirect payoffs:

Definition 3. Fix a conjugation P for an auction (S, μ, v). The indirect utility function of bidder i associated to this conjugation is $\tilde{v} : [0, 1]^2 \to \mathbb{R}$, given by

$$\tilde{v}(x, y) \equiv E[v(t_i, t_{-i}) | P(t_i) = x, P_{(-i)}(t_{-i}) = y]. \tag{5}$$

Now, fix a conjugation P and define the following function:

$$\tilde{\Pi}(x, c) \equiv E\left[\Pi(t_i, c) \,|\, P(t_i) = x\right], \tag{6}$$

where, $\Pi(t_i, c)$ is the interim payoff of the direct auction. The notation should suggest to the reader that $\tilde{\Pi}_i(x, c)$ will be the interim payoff of the indirect auction. Indeed, we have the following:

Proposition 1. Assume (H0). Given $b \in \mathcal{S}$, consider the corresponding conjugation $P = P^b$ (as defined by (2)) and the indirect bidding function $\tilde{b} = \left(\tilde{P}^b\right)^{-1}$. Alternatively, given a conjugation P and an indirect bidding function $\tilde{b}$, let $b = \tilde{b} \circ P$ be the corresponding direct bidding function. Thus,

(i)

$$\tilde{\Pi}(x, c) = \int_0^{\tilde{b}^{-1}(c)} \left[\tilde{v}(x, \alpha) - p^W\left(c, \tilde{b}(\alpha)\right)\right] d\alpha - \int_{\tilde{b}^{-1}(c)}^1 p^L\left(c, \tilde{b}(\alpha)\right) d\alpha. \tag{7}$$

(ii) Assume that P is such that for all s with $P(s) = x$, and for all $x, y \in [0, 1]$,

$$\tilde{v}(x, y) = E[v(t) | P(t_i) = x, P_{(-i)}(t_{-i}) = y] = E[v(t) | t_i = s, P_{(-i)}(t_{-i}) = y]. \tag{8}$$

Then, for all t_i such that $P(t_i) = x$ and for all $c \in B$,

$$\tilde{\Pi}(x, c) = \Pi(t_i, c). \tag{9}$$

Proof. See Appendix A.■

Observe that, because of (7), $\tilde{\Pi}(x, c)$ is formally equivalent to the interim payoff of an auction between two bidders, with signals uniformly distributed on $[0, 1]$, where the opponent is following the strategy $\tilde{b}(\cdot)$ and the (common-value) utility function is given by $\tilde{v}(x, \alpha)$. So, we define the indirect auction as follows:

Definition 4. Given an auction (S, μ, v) and a conjugation P for it, the associated indirect auction is an auction between two players with independent types uniformly distributed on $[0,1]$ and where the utility function is $\tilde{v}$ defined by (5). The indirect auction is denoted by $\left(\tilde{S}, \tilde{\mu}, \tilde{v}\right)$ where $\tilde{\mu}$ is the Lebesgue measure in $\tilde{S} = [0,1]$.

The reader should keep in mind that the indirect auction is just an auxiliary and fictitious auction that will help in the analysis of the "direct" one. It is clear through definitions 1-4 how a conjugation relates the direct and the indirect auction. Obviously, a function $\tilde{b} : [0,1] \to \mathbb{R}_+$ is equilibrium of the indirect auction if for almost all $x \in [0,1]$, $\tilde{\Pi}\left(x, \tilde{b}(x)\right) \geqslant \tilde{\Pi}(x, c)$, $\forall c \in B = \{b_{OUT}\} \cup [b_{\min}, +\infty)$.

4. Characterization and Sufficient Conditions for Regular Equilibria

The results and definitions of the two previous subsections allow us to show that the existence of a direct equilibrium implies the existence of the indirect one (Theorem 1, below). Conversely, (with an extra relatively weak assumption of consistency of payoffs), the existence of equilibrium in indirect auctions implies the existence in direct ones (Theorem 2).

Theorem 1 *(Necessary Conditions).* Assume (H0)-(H2). If there is a pure strategy equilibrium $b \in \mathcal{S}$ for the direct auction (S, μ, v) and there exists $\partial_b \Pi(s, b(s))$ for all s, then:

(i) the associated conjugation $P = P^b$ (given by (2)) satisfies the following property: if $s \in S$ is such that $P(s) = x$, then[1]

(10)

$$\tilde{v}(x,x) = E[v(t_i, t_{-i}) | P(t_i) = x, P_{(-i)}(t_{-i}) = x] = E[v(t_i, t_{-i}) | t_i = s, P_{(-i)}(t_{-i}) = x];$$

(ii) the indirect bidding function $\tilde{b} = \left(\tilde{P}^b\right)^{-1}$, where $\tilde{P}^b$ is given by (1), is the increasing equilibrium of the indirect auction.

Moreover, if $\tilde{v}$ is continuous, that is, if it has a continuous representative, then:

(iii) (H1)-1 implies that $\tilde{b}$ is differentiable and

$$\tilde{b}'(x) = \frac{\tilde{v}(x,x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) > \tilde{b}(\alpha)\right]} + \partial_1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) < \tilde{b}(\alpha)\right]}\right]}, \tag{11}$$

and (H1)-2 implies that

$$\tilde{v}(x,x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right) = 0; \tag{12}$$

(iv) the expected payment of a bidder of type t_i is given by

$$p(t_i) = \int_0^{P(t_i)} \tilde{v}(\alpha, \alpha)\, d\alpha;$$

[1]This condition is related to an analogous condition of Araujo and Moreira (2001).

(v) for all x and $y \in [0,1]$,

$$\int_y^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha \geqslant 0. \tag{13}$$

Proof. See Appendix C.■

Theorem 1 says that if a multidimensional auction has a regular equilibrium, then it can be reduced to a unidimensional auction with two players (the indirect one). However, the reader should note that such reduction is non-trivial and that the indirect auction is not equivalent to the direct one. The indirect auction is a "fictitious" game, where each bidder is facing up a "fictitious" player, the "opponent", that does not correspond to a real player. So, the dimension reduction is meant in this particular sense.

Observe that the expression in condition (iv) does not depend on the specific format of the payment rules, p^W and p^L, but it depends on the conjugation. If the conjugation is different for different auction formats, then we do not have the Revenue Equivalence Theorem. Nevertheless, for the class of auctions considered in the next section, we are able to prove that the conjugation is unique and the Revenue Equivalence Theorem holds. On the other hand, condition (iv) plays an important role to prove the existence of equilibrium in the next result.

Theorem 2 is a kind of converse of Theorem 1. The main difference is that we do not require $\tilde{v}$ to be continuous and we need condition (i)', which is slightly stronger than condition (i) in Theorem 1.

Theorem 2 *(Sufficient Conditions).* Assume (H0)-(H2). Consider a direct auction (S,μ,v), a conjugation P and its associated indirect auction $\left(\tilde{S},\tilde{\mu},\tilde{v}\right)$. Assume that

(i)' for all $s \in S$ such that $P(s) = x$, and all $y \in [0,1]$,

(14)
$$\tilde{v}(x,y) = E[v(t_i,t_{-i})|P(t_i) = x, P_{(-i)}(t_{-i}) = y] = E[v(t_i,t_{-i})|t_i = s, P_{(-1)}(t_{-i}) = y];$$

(ii) for all x and $y \in [0,1]$,

$$\int_y^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha \geqslant 0;$$

(iii) there is an increasing function $\tilde{b}$, such that

$$\hat{p}(y) \equiv \int_0^y p^W\left(\tilde{b}(y),\tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L\left(\tilde{b}(y),\tilde{b}(\alpha)\right) d\alpha = \int_0^y \tilde{v}(\alpha,\alpha)\, d\alpha,$$

where $\hat{p}(P(t_i)) = p(t_i)$ is the expected payment of a bidder of type t_i.

Then, $\tilde{b}$ is the equilibrium of the indirect auction and $b = \tilde{b} \circ P$ is the equilibrium of the direct auction. Moreover, if $\tilde{v}$ is continuous, there exists $\partial_b \Pi(s, b(s))$ for all s (which implies that all conditions of Theorem 1 are satisfied).

Proof. See Appendix C.■

Remark 1. For the four specific formats, namely, the first-price auction (F), second-price auction (S), all-pay auction (A) and war of attrition (W), the function $\tilde{b}$ is given, respectively, by

$$\text{(15)} \qquad \text{(F)} \quad \tilde{b}(x) = \frac{1}{x}\int_0^x \tilde{v}(\alpha,\alpha)\,d\alpha$$

$$\text{(16)} \qquad \text{(S)} \quad \tilde{b}(x) = \tilde{v}(x,x)$$

$$\text{(17)} \qquad \text{(A)} \quad \tilde{b}(x) = \int_0^x \tilde{v}(\alpha,\alpha)\,d\alpha$$

$$\text{(18)} \qquad \text{(W)} \quad \tilde{b}(x) = \int_0^x \frac{\tilde{v}(\alpha,\alpha)}{1-\alpha}d\alpha$$

Conditions (iii) and (iv) reduce to the requirement that the function $\tilde{b}$ above is increasing. In particular, the equilibrium may exist for an all-pay auction, for instance, but not for a first-price auction.

Remark 2. Although natural, condition (i)' can be still too restrictive. We need it in order to apply Proposition 1 and reach the conclusion that for all t_i such that $P(t_i) = x$ and for all $c \in R$, we have: $\tilde{\Pi}(x,c) = \Pi(t_i,c)$ (see (9) in Proposition 1). In turn, this implies that the equilibrium of the indirect auction is equilibrium of the direct auction. So, instead of assuming condition (i)' above, it would be sufficient to require the (necessary) condition (i) of Theorem 1 and that (9) is valid.

Theorem 2 reduces the problem of equilibrium existence to find a conjugation that meets requirements (i)', (ii) and (iii). In the next section we treat a still general case (weakly separable auctions) where such conjugation can be easily defined. Nevertheless, we would like to give two examples where the assumptions of the next section are not satisfied. These examples illustrate a kind of heuristics for the existence problem. In example 1, we have a monotonic equilibrium and also a U-shaped one, which shows that the conjugation is not unique. In example 2, there is no monotonic equilibrium, but there is a bell-shaped equilibrium. Another example where Theorem 2 can be applied is an example provided by Jehiel, Meyer-ter-Vehn, Moldovanu and Zame (2004).

Example 1 — Consider a symmetric first-price auction with two bidders, types uniformly distributed on $[0,1]$ and utility function given by

$$v(t_i,t_{-i}) = t_i + \left(3 - 4t_i + 2t_i^2\right)t_{-i}.$$

Observe that $\partial_{t_i} v(t_i,t_{-i}) = 1 - 4t_{-i} + 4t_i t_{-i}$ can be negative. Thus, the received theory cannot be applied. Nevertheless, there exists a monotonic equilibrium. Indeed, in this case, the conjugation will be given by $P(t_i) = t_i$ and we obtain

$$\tilde{v}(x,y) = x + \left(3 - 4x + 2x^2\right)y.$$

This clearly satisfies condition (i)'. Condition (ii) follows from the fact that $x > y$ implies

$$\int_y^x \left[\tilde{v}(x,z) - \tilde{v}(z,z)\right]dz = \frac{(x-y)^2}{6}\left[3 + 3x^2 - 8y + 3y^2 + x(-4+6y)\right] \geqslant 0.$$

Condition (iii) is also satisfied, because the function

$$\tilde{b}(x) = \frac{1}{x}\int_0^x v(z,z)\,dz = \frac{x\left(24 - 16x + 3x^2\right)}{12}$$

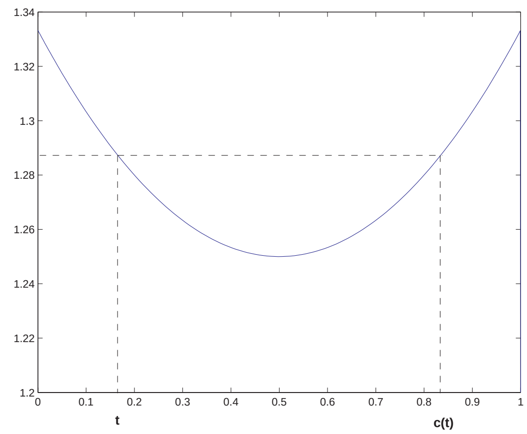

FIGURE 1. Equilibrium bidding function in Example 1.

is increasing. Clearly, the above function implies condition (iv). Thus, there exists a monotonic equilibrium by Theorem 2.

Nevertheless, this is not the unique equilibrium. If we assume that there exists a U-shaped equilibrium, the conjugation can be expressed by $P(t_i) = |c(t_i) - t_i|$, where $c(t_i)$ is the type that bid the same as t_i (see Figure 1). Observe that $c \circ c(t_i) = t_i$. Condition (i) of Theorem 1 requires that

$$s + \left(3 - 4s + 2s^2\right) \frac{s + c(s)}{2} = c(s) + \left(3 - 4c(s) + 2c(s)^2\right) \frac{s + c(s)}{2},$$

that is,

$$s - c(s) = [s - c(s)]\,[4 - 2c(s) - 2s]\, \frac{s + c(s)}{2},$$

which simplifies to $[s + c(s)]\,[2 - s - c(s)] = 1 \Rightarrow s + c(s) = 1$. Then, $c(s) = 1 - s$ and $P(s) = |1 - 2s|$. This gives the expression:

$$\tilde{v}(x, y) = \frac{1}{2} + \frac{1}{4}\left[3 - 4\left(\frac{1-x}{2}\right) + 2\left(\frac{1-x}{2}\right)^2 + 3 - 4\left(\frac{1+x}{2}\right) + 2\left(\frac{1+x}{2}\right)^2\right]$$

which simplifies to $\tilde{v}(x, y) = \left(5 + x^2\right)/4$ and condition (i)' and (ii) are easily seen to be satisfied. Also, condition (iii) and (iv) are satisfied, since

$$\tilde{b}(x) = \frac{1}{x}\int_0^x \tilde{v}(z, z)\, dz = \frac{5}{4} + \frac{x^2}{12}$$

is increasing. Then, $b(s) = \frac{5}{4} + \frac{(1-2s)^2}{12}$ is a direct equilibrium, plotted in Figure 1.

Observe that no tie rules are needed in this case, because ties occur with zero probability. However, for each equilibrium bid, exactly two types pool and have the same probability of winning.

Example 1 has a monotonic equilibrium, as is usual in auction theory, but there is another non-monotonic equilibrium. Example 2 below shows a case where there is no monotonic equilibrium, but there is a bell-shaped equilibrium (Figure 2).

Example 2 — Consider again a symmetric first-price auction with two bidders and signals uniformly distributed in $[1.5, 3]$ such that the value of the object is given

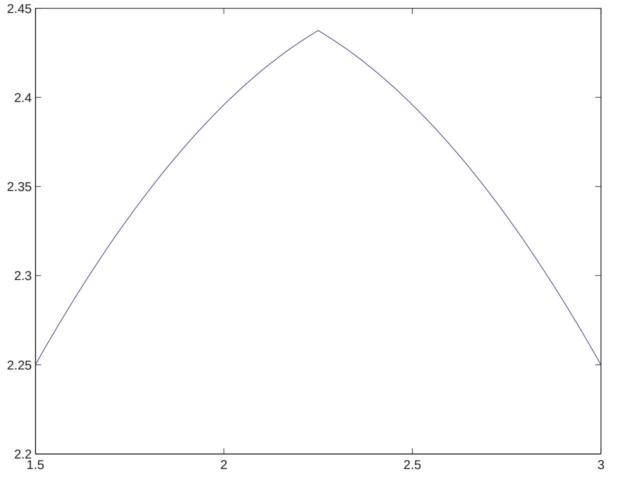

FIGURE 2. Equilibrium bidding function in Example 2.

by $v(t_i, t_{-i}) = t_i\left(t_{-i} - \frac{t_i}{2}\right)$. In Appendix D, we show that this auction does not have monotonic regular equilibria, but there is a bell-shaped equilibrium as shown in Figure 2.

Example 1 shows that it is possible for a standard auction to have multiple equilibria. Example 2 suggests that the correct conjugation can fail to exist — at least with a fixed shape (that we begin by assuming). Thus, one would be interested in cases where it is possible to ensure the uniqueness of the equilibrium and where it is possible to find explicitly the conjugation. We do this under the context of assumption H3, to be presented in the next subsection.

5. Equilibrium Existence of Weakly Separable Auctions

Theorem 2 teaches us that the question of equilibrium existence is solved if we are able to find the proper conjugation. In examples 1 and 2 of the previous section we have shown situations where the conjugations could be obtained. However, there we assumed some features of the conjugation that are not necessary and were able to find the correct conjugation for those settings. Now we will work in a setting where a conjugation always exists: the weakly separable auctions. These are the auctions that satisfy the following assumption:

(**H3**) (Weak Separability). $v(t_i, t_{-i})$ is such that if $v(t_i, t_{-i}) < v(t_i', t_{-i})$ for some t_{-i} then $v\left(t_i, t_{-i}'\right) < v\left(t_i', t_{-i}'\right)$ for all t_{-i}'. Moreover, if $C \subset \mathbb{R}$ has zero Lebesgue measure, then $\mu\{s \in S : v^1(s) \in C\} = 0$, where

$$v^1(s) \equiv E\left[v(t_i, t_{-i})\,|t_i = s\right].$$

Assumption (H3) is restrictive, but it is valid in many economic meaningful cases. For instance, for separable utilities such as $v(t_i, t_{-i}) = u^1(t_i) + u^2(t_{-i})$, it requires only that $u^1(t_i)$ does not assume any value with positive probability. The same is valid for utilities like $v(t_i, t_{-i}) = \{\left[u^1(t_i)\right]^\alpha + \left[u^2(t_{-i})\right]^\beta\}^\gamma$, or $v(t_i, t_{-i}) = \gamma\left[u^1(t_i)\right]^\alpha \left[u^2(t_{-i})\right]^\beta$, with α, β, $\gamma > 0$. Of course, private values are included in the separable utilities case. Of course, there are cases that do not satisfy it, such as the examples 1 and

2 above. It is also clear that (H3) can deal with even more complicated dependences, as examples 3 and 4 below illustrates.

Under (H3), we can define explicitly the conjugation:

$$P(t_i) \equiv \Pr\left\{t_{-i} \in S^{N-1} : v^1(t_j) < v^1(t_i), j \neq i\right\}. \tag{19}$$

Then, $\tilde{v}$ is given by

$$\tilde{v}(x,y) = E[v(t_i,t_{-i})|P(t_i) = x, P_{(-i)}(t_{-i}) = y]. \tag{20}$$

Under (H3), we can give a *necessary and sufficient* condition for the equilibrium existence of the direct auction: merely that the solution $\tilde{b}$ to the first-order condition of the indirect auction be increasing. This is the content of the following:

Theorem 3 *(Necessary and Sufficient Condition For Equilibrium Existence).* Assume (H0)-(H3). There exists an equilibrium $b \in \mathcal{S}$ if there exists an increasing function $\tilde{b}$ that satisfies

$$\int_0^y p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha = \int_0^y \tilde{v}(\alpha,\alpha)\, d\alpha. \tag{21}$$

If this is the case, the equilibrium of the direct auction is given by $b = \tilde{b} \circ P$ and the expected payment of a bidder of type s is given by

$$p(s) = \int_0^{P(s)} \tilde{v}(\alpha,\alpha)\, d\alpha. \tag{22}$$

Additionally, if $\tilde{v}$ is continuous, then there exists an equilibrium $b \in \mathcal{S}$ and there exists $\partial_b \Pi(s, b(s))$ for all s if and only if there exists an increasing function $\tilde{b}$ that satisfies the following:

- For (H1)-1, $\tilde{b}$ is differentiable and

$$\tilde{b}'(x) = \frac{\tilde{v}(x,x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) > \tilde{b}(\alpha)\right]} + \partial_1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) < \tilde{b}(\alpha)\right]}\right]};$$

  with initial condition $\int_0^1 p^L\left(\tilde{b}(0), \tilde{b}(\alpha)\right) d\alpha = 0$.
- For (H1)-2, $\tilde{b}$ is continuous and satisfies, for all $x \in (0,1)$,

$$\tilde{v}(x,x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right) = 0.$$

  Moreover, if there is a unique $\tilde{b}$ that satisfies such properties, the equilibrium of the direct auction in regular pure strategies is also unique.

Proof. See Appendix C.■

Remark 3. As explained in Remark 1, if a multidimensional auction has a regular equilibrium, it can always be reduced (in a non-trivial way) to a one dimension auction (the indirect auction). So, for obtaining equilibrium existence, we have to consider auctions that can be so "reduced". This is what assumption H3 allows us to explicitly do. It still encompasses cases where such reduction is not trivial, as we show in examples 3 and 4 below. The reduction of the dimension of types is not a novelty in auction theory. While studying the efficiency of auctions, Dasgupta and Maskin (2000) use a condition close to H3, while Jehiel, Moldovanu and Stacchetti (1996) made such reduction for the purpose of revenue maximization. Nevertheless, for the purpose of showing equilibrium existence in auctions, one cannot use only H3 or the Dasgupta and Maskin's condition, since the received theory would require the extra assumption of monotonicity of $\tilde{v}$ on the reparameterized types. As we show in examples 4 and 5, this is not always possible. So, an important feature of Theorem 3 is that it does not require $\tilde{v}$ to be monotonic.

Example 3 (*Spectrum Auction).*[2]

Consider a first-price auction of a spectrum license. The license covers two periods of time:

(1) In the first period, the regulator lets the winner explores its monopoly power. Let t_i^1 be the estimative of bidder i of the monopolist surplus in this first period. Of course, the true surplus will be better approximated by $\left(t_1^1+...+t_N^1\right)/\ N$. If the bidder i (a firm) wins the auction, it has to invest t_i^2, a privately known amount, to build the network that will support the service. So, in the first period, the license gives to the firm

$$\frac{t_1^1+...+t_N^1}{N}-t_i^2.$$

(2) In the second period, the regulator makes an estimate of the operational costs of the firm. The regulator cannot observe the true operational cost, t_i^3, which is a private information of the firm. Nevertheless, the regulator has a proxy that is a sufficient statistic for the mean operational cost of all participants in the auction, $\left(t_1^3+...+t_N^3\right)/\ N$. The regulator will fix a price that will give zero profit for a firm with the mean operational costs.[3] So, in the second period, the license gives to the winner

$$\frac{t_1^3+...+t_N^3}{N}-t_i^3.$$

So, the value of the object is given by

$$v\left(t_i,t_{-i}\right)=\frac{t_1^1+...+t_N^1}{N}-t_i^2+\frac{t_1^3+...+t_N^3}{N}-t_i^3.$$

Let the signals $t_i=\left(t_i^1,t_i^2,t_i^3\right)$, $i=1$, ..., N be independent. Observe that the problem cannot be reduced to a single dimension. Indeed, if we summarize the private information by, say, $s_i=t_i^1/N-t_i^2+t_i^3(1/N-1)$, we lose the information about t_i^1 and t_i^3 that are needed for the value function of bidders $j\neq i$. Also, the model cannot be reparameterized to an increasing one. If we try to put $-t_i^3$ in the place of t_i^3, then the dependence of $v\left(t_i,t_{-i}\right)$ on the signals t_j^3 will be decreasing. So, the received theory does not ensure the existence of pure strategy equilibrium for this case. Nevertheless, assumption (H3) is trivially satisfied. In Appendix D, we assume the $t_i=\left(t_i^1,t_i^2,t_i^3\right)$ are independent and uniformly distributed on $\left[\underline{s}^1,\overline{s}^1\right]\times\left[\underline{s}^2,\overline{s}^2\right]\times\left[\underline{s}^3,\overline{s}^3\right]$, with $\underline{s}^1$, $\underline{s}^2$, $\underline{s}^3\geqslant 0$ and we show that a sufficient condition for the existence of equilibrium in pure strategy is

$$\frac{\underline{s}^1}{N}-\overline{s}^2-\overline{s}^3\frac{N-1}{N}-1\geqslant 0.$$

[2]This example is more complex, but formally similar to example 5 of Dasgupta and Makin (2000).

[3]We assume that the regulator is institutionally constrained to follow such a procedure, so the optimality of this regulation is not an issue here.

The derivation in Appendix D indeed provides necessary and sufficient conditions for the existence of equilibrium. Above, we have only simplified it for a sufficient condition.■

Example 4 — Job Market

We can model the job market (for, say, a manager) as an auction between competing firms, where the object is the job contract with that manager. It is natural to assume that the manager has a multidimensional vector of characteristics, $m = (m^1,$ ..., $m^k)$. For the sake of simplicity, we assume that the firms learn such characteristics through interviews and curriculum analysis. Each firm also has a position to be filled by the manager, with specific requirements for each dimension of the characteristics. For instance, if dimension 1 is ability to communicate and the position is to be the manager of a production section, there is level of desirability of this ability. An overly communicative person may not be good. The same goes for the other characteristics. A bank may desire a sufficiently (but not exaggeratedly) high level of risk loving or audacity on the part of the manager, while a family business may desire a much lower level. Even efficiency or qualification can have a level of desirability. Sometimes, the rejection of a candidate is explained by over-qualification. Therefore, let $t_i = (t_i^1$, ..., $t_i^K)$ be the value of the characteristics desired by the firm.

Since the firms are competitors, then if one hires the employee, the other will remain with a vacant position, at least for a time.[4] In this way, the winning firm also benefits from the fact that its competitors have a vacant position — and, then, are not operating perfectly well. The higher the abilities required for the job, the more the competitor suffers.[5] So, the utility in this auction is as

$$v\left(t_i, t_{-i}\right) = \sum_{k=1}^{K} a^k m^k - \sum_{k=1}^{K} b^k \left(t_i^k - m^k\right)^2 + \sum_{j \neq i} \sum_{k=1}^{K} c^k t_j^k,$$

where a^k is the level of importance of characteristic k of the manager, $b^k > 0$ represents how important is the distance from the desired level t_i^k of the characteristic k, and c^k is the weight of the benefit that firm i receives from the fact that the competitors are lacking $\sum_{j \neq i} t_j^k$ of the ability k. As in the previous example, we cannot simplify this model to a unidimensional monotonic model. In Appendix D we analyze the case where there is just one dimension ($K = 1$), 2 players ($N = 2$) and types are uniformly distributed on $[0, 1]$, $b = b^1 \geqslant 0$. We show that if $m^1 = m > 1/2$, there exists a pure strategy equilibrium in regular strategies if and only if

$$c \equiv c^1 \geqslant \max\left\{\frac{2b\,(m-2)}{3}, \frac{2b\,(1-2m)\,(1+m)}{3}\right\}$$

[4] Of course, this model works only for non-competitive job markets. In other words, the buyers (the contracting firms) have no access to a market with many homogenous employees to hire. This is implicit when we model it as an auction. So, this is the reason why a firm that does not contract the manager suffers — it is not possible to find a suitable substitute instantaneously. It is possible that this also occurs in other kinds of auctions.

[5] If firms act in a oligopolistic market, it is possible to justify such externality through the fact that the vacant position influences the quality of the product delivered by the firms and, hence, the equilibrium in this market.

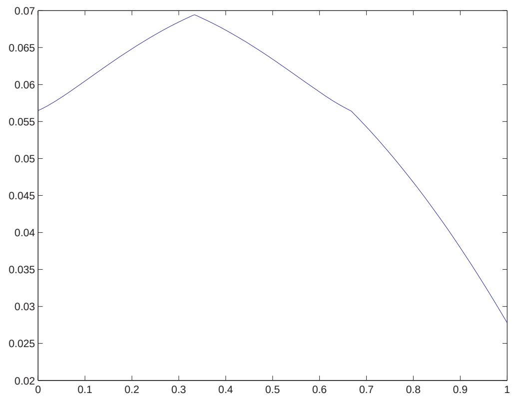

FIGURE 3. Equilibrium bidding function in Example 4.

and if $m < 1/2$, there exists a pure strategy equilibrium in regular strategies if and only if

$$c \leqslant \min\left\{\frac{2b\,(m+1)}{3}, \frac{2b\,(1-2m)\,(1+m)}{3}\right\}.$$

Observe that for both cases the value $c = 0$ ensures the existence of equilibrium. This is expected, since it corresponds to a private value auction.

For $a = b = 1/5$, $c = 1/20$ and $m = 1/3$, the equilibrium bidding function is shown in Figure 3.

■

Now, we can return to the example given in the introduction. Theorem 3 gives the conditions for the equilibrium existence.

Example 5 (JSSZ, example 1, Maskin and Riley, example 3) — Let us consider a first price auction with two bidders, independent types uniformly distributed on $[0, 1]$. Let $v^1(t_i) = t_i$ and $v^2(t_{-i}) = \alpha + \beta t_{-i}$. It is clear that $P(t_i) = t_i$ in this case and $\tilde{v}(x, y) = \alpha + x + \beta y$. So, $\tilde{v}(x, x) = \alpha + (1+\beta)\,x$. So,

$$\tilde{b}(x) = \frac{1}{x}\int_0^x \tilde{v}(z, z)\,dz = \frac{1}{x}\left[\alpha x + \frac{(1+\beta)\,x^2}{2}\right] = \alpha + \frac{(1+\beta)\,x}{2},$$

which is increasing only if $\beta > -1$. Observe that for $\tilde{b}(\cdot) \geqslant 0$, it is necessary $\alpha \geqslant -(1+\beta)\,x/2$, otherwise negative bids have to be allowed.■

The example above is used by JSSZ to show that equilibrium may fail to exist under the standard tie-breaking rule. They then provide a general existence result based on endogenous tie-breaking rules. Nevertheless, their result has some undesirable properties. First, it is in mixed strategies. Second, the tie-breaking rule is endogenous, so it is not possible to know what rule has to be applied in order to guarantee the existence. Third, the rule requires that the players announce their types, which is theoretically convenient but it is unfeasible in the real world.

Instead, consider the following rule: if a tie occurs, conduct an all-pay auction among the tying bidders. If another tie occurs, split randomly the object.[6]

We show now that this "all-pay auction tie-breaking rule" ensures the existence of equilibrium for all auctions hat we are considering. Therefore, the generality and simplicity of the rule can be counted as its last advantage.

Theorem 4 (General Existence). Assume (H0) -(H3) and that the all-pay auction tie-breaking rule is adopted. If there is a continuous function $\tilde{b}$, not necessarily increasing, such that

$$\int_0^y p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha = \int_0^y \tilde{v}(\alpha, \alpha)\, d\alpha,$$

then there exists a pure strategy equilibrium.

Proof. See Appendix C.■

Remark 4. The main ingredients in the proof of Theorem 4 are the payment expression and the fact the bidding function of an all-pay auction is increasing. This is so because in the all-pay auction, the bid

$$\tilde{b}(x) = \int_0^x \tilde{v}(\alpha, \alpha)\, d\alpha$$

is exactly the expect payment, which implies (21), and it is increasing, because $\tilde{v}$ is positive, since v is positive by (H0).

The reader should note that Theorem 4 does not claim the uniqueness of the equilibrium. Indeed, if $\tilde{b}$ is not increasing, there are many equilibria. There are two sources for this multiplicity.

The first source is that under the all-pay auction tie-breaking rule, any level of the bid in the range where $\tilde{b}$ is not increasing can be chosen to be the level of the tie. This is shown in the Figure 4. For instance, any a_0 can be chosen between x_0 and x_1. Once one of the three elements a_k, b_k or c_k is determined, so are the other two. However, these possibilities lead to the same expected payment and payoff for each bidder in the auction.

Another point is that the tie-breaking rule is not unique, in general. It can be shown, for instance, that for cases where $\tilde{b}$ is decreasing (as in example 1 of JSSZ) and for some specifications of v, there is a continuum of tie-breaking rules (like that defined by JSSZ for their example 1), which ensures the existence of equilibrium. All these tie-breaking rules nevertheless imply different revenues. In light of this observation, the existence of equilibrium with an *endogenous* tie-breaking rule seems even more problematic as a solution concept, since it can sustain many different behaviors at equilibrium.

The reader may observe that the expression of the payment in Theorem 3 depends only on the conjugation, which is fixed for all kind of auctions. Also, the payment is exactly the same under the all-pay auction tie-breaking rule. So, we have the following:

Theorem 5 (Revenue Equivalence Theorem). Consider auctions that satisfy (H0) -(H3) and with the all-pay auction tie-breaking rule. Then, any format of the auction

[6] Observe that in the second auction (the tie-breaking auction), the bids and payments can be less than in the first auction.

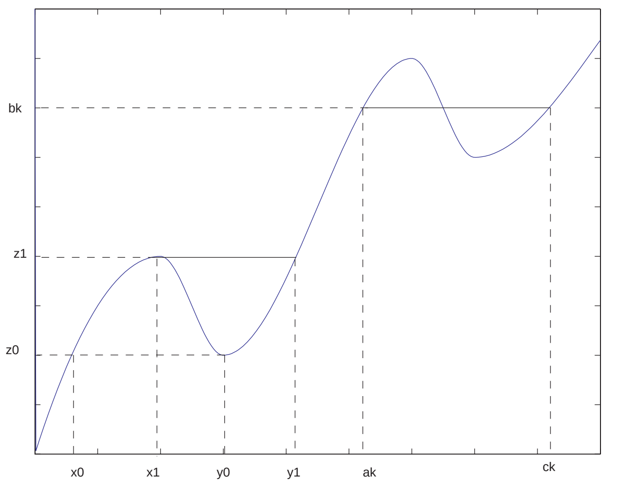

FIGURE 4. Possible specifications for the level of the tie.

gives the same revenue, provided the bidders follow the symmetric equilibrium specified by Theorem 4.

Proof. See Appendix C.■

6. CONCLUSION

Now we will briefly highlight what are the most important contributions of this paper and to discuss possible extensions.

6.1. **The Contributions.** Our contributions can be summarized as follows:

- Equilibrium Existence in the Multidimensional Setting - McAdams (2003a) generalizes Athey (2001) for multidimensional types and actions. He works with discrete bids and continuous types. Our approach gives the existence with continuum types and bids. Our result provides the expressions of the bidding functions, while his is an existence result only. His assumptions requires monotonicity and are not on the fundamentals of the model. On the other hand, our results do not cover multi-unit auctions nor asymmetries as his. JSSZ give the existence for multidimensional games, including cases with dependence, while we require independence. However, they need an endogenous tie breaking rule and the existence is in mixed strategies, while our results are in pure strategies. Jackson and Swinkels (2004) show the existence of equilibria for a large class of multidimensional private value auctions. Their setting is private values, while ours is interdependent values. They allow asymmetries, dependence of signals and multi-units, but the existence is given in mixed strategies.
- Equilibrium Existence in Non-Monotonic Settings - We are not aware of any general non-monotonic equilibrium existence results in pure strategies. Zheng (2001), Athey and Levin (2001) and Ewerhart and Fieseler (2003) present cases

where non-monotonicity arises. So, our results develop a theory to deal with the situations where the usual monotonicity is not fulfilled.[7]

- Uniqueness of Equilibrium - We are able to ensure the uniqueness of equilibrium in the symmetric interdependent values auctions that satisfy assumption H3, extending the well known uniqueness of unidimensional and monotonic auctions.
- Necessary and Sufficient Conditions for the Existence of Equilibrium without Ties - The results of JSSZ do not allow one to distinguish when special tie-breaking is needed or not. Our approach clarifies, under assumption H3, whether ties occur with positive probability (and there is a potential need for special tie-breaking rules).
- All-Pay Auction Tie-Breaking Rule - When there is a need for a tie with positive probability, we are able to offer an exogenous tie-breaking rule, which is implemented through an all-pay auction. Moreover, the equilibrium that the rule implements is in pure strategies. For private value auctions, Jackson and Swinkels (2004) show that the equilibrium is invariant for any trade-maximizing tie-breaking rule. Nevertheless, this does not need to hold for the interdependent values auctions that we treat.
- Revenue Equivalence Theorem - We have also generalized the Revenue Equivalence Theorem (Theorem 5). Furthermore, Theorem 2 and Appendix B show that there is a deep connection between the revenue equivalence and the existence of equilibrium. Riley and Samuelson (1981) and Myerson (1981) establish that revenue equivalence is a consequence of the equilibrium behavior. Proposition 5 and Corollary 1 in Appendix B show that the revenue equivalence is also sufficient for the existence of equilibrium (if an extra condition is satisfied).

Thus, our results have clarified some aspects of the equilibrium existence problem in auctions. The theory shows that, under assumption H3, there is no additional difficult in working with the more general setting of multidimensional types and non-monotonic utilities besides those difficulties already possible in the unidimensional setting.[8] Moreover, this approach allows the equilibrium bidding functions to be expressed in a simple manner. This is so because the equilibrium bidding function of a general auction can be expressed by the equilibrium bidding function of an auction with two bidders and types uniformly distributed on $[0, 1]$.

6.2. **The Limits of the Method.** Our theory makes two important assumptions: independence of the types and symmetry.

The generalization of this approach for dependent types involves some difficulties, because the conjugation would depend in a complicated way on types. Nevertheless, we believe that some extension can be done if we assume conditional independence.[9] It is worth remembering that the problem with dependence is not specific to our approach. Jackson (1999) gives a counter-example for the equilibrium existence of an auction with bidimensional affiliated types. Fang and Morris (2003) also obtain negative results, not only for the existence of equilibrium but also for the revenue equivalence.

[7] Of course, papers that provide existence in distributional (mixed) strategies can treat non-monotonic settings as well.

[8] Theorem 3 shows that the non-existence of the equilibrium comes from the non-monotonicity of the indirect bidding function. This can occur also in unidimensional setting, although it can be more usual in multidimensional models.

[9] de Castro (2004b) proposes the use of conditional independence as an alternative for affiliation.

On the other hand, asymmetry does not seem to impose severe restriction on the existence of equilibrium. We believe that the approach of the indirect auction can be adapted to this case, although not in a straightforward way. If this can be done, it is unlikely that we will obtain simple expressions as in this paper.

Another limitation of our theory is that it is applied to single-unit auctions.

Finally, the relaxation of assumption H3 is an obvious direction to pursue, although H3 seems to encompass many important economical examples.

7. Appendix A - Proof of the Basic Results

We will need the following result, which was proved, in a more general setting, by de Castro (2004a).

Lemma 1 (Payoff Characterization) — Assume (H0)-(H2). Fix $b(\cdot) \in \mathcal{S}$. The bidder i's payoff can be expressed by

$$\Pi(t_i, b_i, b(\cdot)) = \Pi_i(t_i, b_{\min}) + \int_{(b_{\min}, b_i)} \partial_{b_i} \Pi(t_i, \beta, b(\cdot)) d\beta,$$

where $\partial_{b_i} \Pi(t_i, \beta, b(\cdot))$ exists for almost all β with

$$\begin{aligned}\partial_{b_i} \Pi(t_i, \beta, b(\cdot)) = E\Big[&-\partial_1 p^W\left(\beta, b_{(-i)}(t_{-i})\right) 1_{\left[\beta > b_{(-i)}(t_{-i})\right]} - \partial_1 p^L\left(\beta, b_{(-i)}(t_{-i})\right) 1_{\left[\beta < b_{(-i)}(t_{-i})\right]}\Big] \\ &+ E[v(t_i, t_{-i}) - p^W(\beta, \beta) + p^L(\beta, \beta) \,|b_{(-i)}(t_{-i}) = \beta] f_{b_{(-i)}}(\beta).\end{aligned}$$

a.e., where $b_{(-i)}(t_{-i}) \equiv \max_{j \neq i} b(t_j)$.

Proof. Let us define $b_{(-i)}(t_{-i}) \equiv \max_{j \neq i} b(t_j)$. Since $b_i = b_{(-i)}(t_{-i})$ with probability zero, we have

$$\begin{aligned}\Pi(t_i, b_i, b(\cdot)) &= \int_{S^{N-1}} \left[v(t_i, t_{-i}) - p^W\left(b_i, b_{(-i)}(t_{-i})\right)\right] 1_{\left[b_i > b_{(-i)}(t_{-i})\right]} \prod_{j \neq i} \mu(dt_j) \\ &\quad + \int_{S^{N-1}} \left[-p^L\left(b_i, b_{(-i)}(t_{-i})\right)\right] 1_{\left[b_i < b_{(-i)}(t_{-i})\right]} \prod_{j \neq i} \mu(dt_j).\end{aligned}$$

Take a sequence $a_n \to b_i^-$, i.e., $a^n < b_i$ (the other case is analogous). We want to prove that there exists $\lim_{n\to\infty} D_n(b_i) / (b_i - a_n)$ for almost all b_i, where

$$D_n(b_i) = \Pi(t_i, b_i, b(\cdot)) - \Pi(t_i, a_n, b(\cdot)).$$

In the sequel, we will omit the measure $\prod_{j \neq i} \mu(dt_j)$ and the terms t_{-i}. We have:

$$\begin{aligned}D_n(b_i) &= \int \left[v(t_i, \cdot) - p^W\left(b_i, b_{(-i)}(\cdot)\right)\right] 1_{\left[b_i > b_{(-i)}(\cdot)\right]} 1_{\left[a_n > b_{(-i)}(\cdot)\right]} \\ &\quad + \int \left[-p^L\left(b_i, b_{(-i)}(\cdot)\right)\right] 1_{\left[b_i < b_{(-i)}(\cdot)\right]} \\ &\quad - \int \left[v(t_i, \cdot) - p^W\left(a_n, b_{(-i)}(\cdot)\right)\right] \\ &\quad - \int \left[-p^L\left(a_n, b_{(-i)}(\cdot)\right)\right] 1_{\left[a_n < b_{(-i)}(\cdot)\right]} \\ &= \int \left[v(t_i, \cdot) - p^W\left(b_i, b_{(-i)}(\cdot)\right) + p^L\left(a_n, b_{(-i)}(\cdot)\right)\right] 1_{\left[b_i > b_{(-i)}(\cdot) > a_n\right]} \\ &\quad + \int \left[-p^W\left(b_i, b_{(-i)}(\cdot)\right) + p^W\left(a_n, b_{(-i)}(\cdot)\right)\right] 1_{\left[a_n > b_{(-i)}(\cdot)\right]} \\ &\quad + \int \left[-p^L\left(b_i, b_{(-i)}(\cdot)\right) + p^L\left(a_n, b_{(-i)}(\cdot)\right)\right] 1_{\left[b_i < b_{(-i)}(\cdot)\right]}\end{aligned}$$

Let us call the three last integrals as $D_n^1(b_i)$, $D_n^2(b_i)$ and $D_n^3(b_i)$, respectively. Since p^W and p^L are differentiable, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{D_n^2(b_i)}{b_i-a_n} &= -\lim_{n\to\infty}\int\frac{p^W\left(a_n,b_{(-i)}(\cdot)\right)-p^W\left(b_i,b_{(-i)}(\cdot)\right)}{b_i-a_n}1_{\left[b_i>b_{(-i)}(\cdot)\right]}\\ &= -\int\partial_1 p^W\left(b_i,b_{(-i)}(\cdot)\right)1_{\left[b_i>b_{(-i)}(\cdot)\right]}\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\frac{D_n^3(b_i)}{b_i-a_n} &= -\lim_{n\to\infty}\int\frac{p^L\left(a_n,b_{(-i)}(\cdot)\right)-p^L\left(b_i,b_{(-i)}(\cdot)\right)}{b_i-a_n}1_{\left[a_n<b_{(-i)}(\cdot)\right]}\\ &= -\int\partial_1 p^L\left(b_i,b_{(-i)}(\cdot)\right)1_{\left[b_i<b_{(-i)}(\cdot)\right]}.\end{aligned}$$

everywhere. So, if we put $a_0 = b_{\min}$ and $b_i \geqslant b_{\min}$, the Fundamental Theorem of Calculus gives

$$D_0^2(b_i) = \int_{(b_{\min},b_i)}\int -\partial_1 p^W\left(\alpha,b_{(-i)}(\cdot)\right)1_{\left[b_i>b_{(-i)}(\cdot)\right]}d\alpha \tag{23}$$

and

$$D_0^3(b_i) = \int_{(b_{\min},b_i)}\int -\partial_1 p^L\left(\alpha,b_{(-i)}(\cdot)\right)1_{\left[b_i<b_{(-i)}(\cdot)\right]}d\alpha. \tag{24}$$

Now, define the measure ρ over $\mathbb{R}_+$ by

$$\rho(V)\equiv\int_{S^{N-1}}\left[v(t_i,t_{-i})-p^W\left(b_i,b_{(-i)}(t_{-i})\right)+p^L\left(b_i,b_{(-i)}(t_{-i})\right)\right]1_{\left[b_{(-i)}(t_{-i})\in V\right]}\prod_{j\neq i}\mu(dt_j).$$

Observe that, since $b\in\mathcal{S}$, ρ is absolutely continuous with respect the Lebesgue measure λ. We have

$$\lim_{n\to\infty}\frac{D_n^1(b_i)}{b_i-a_n}=\lim_{n\to\infty}\frac{\rho([a_n,b_i))}{b_i-a_n}=\lim_{a^n\to b_i}\left\{\frac{\rho([a_n,b_i))}{\lambda([a_n,b_i))}\right\}=\frac{d\rho}{d\lambda}(b_i),$$

where $\frac{d\rho}{d\lambda}(.)$ is the Radon-Nikodym derivative of ρ with respect to λ. Indeed, the existence of such limit is ensured by Theorem 8.6 of Rudin (1966) for almost all b_i, that is,

$$\lambda\left(\left\{b_i : \nexists\lim_{a^n\to b_i}\left\{\frac{\rho([a_n,b_i))}{\lambda([a_n,b_i))}\right\}\right\}\right)=0.$$

It is easy to see that the Radon-Nikodym derivative $\frac{d\rho}{d\lambda}(b_i)$ is simply

$$E[v(t_i,t_{-i})-p^W(\beta,\beta)+p^L(\beta,\beta)\,|b_{(-i)}(t_{-i})=\beta]f_{b_{(-i)}}(\beta),$$

where $f_{b_{(-i)}}(\beta)$ is the Radon-Nikodym derivative of the distribution of maximum bids, $\int 1_{\left[b_{(-i)}(t_{-i})\in V\right]}$. Moreover, Theorem 8.6 of Rudin says that

$$\rho((b_{\min},b_i))=\int_{(b_{\min},b_i)}\frac{d\rho}{d\lambda}(\alpha)\,d\alpha. \tag{25}$$

Thus, (23), (24) and (25) imply that

$$\Pi(t_i,b_i,b(\cdot))=\Pi(t_i,b_{\min})+\int_{(b_{\min},b_i)}\partial_{b_i}\Pi(t_i,\beta,b(\cdot))d\beta,$$

where

$$\partial_{b_i}\Pi(t_i,\beta,b(\cdot)) = E\left[-\partial_1 p^W\left(\beta, b_{(-i)}(t_{-i})\right)1_{\left[\beta>b_{(-i)}(t_{-i})\right]} - \partial_1 p^L\left(\beta, b_{(-i)}(t_{-i})\right)1_{\left[\beta<b_{(-i)}(t_{-i})\right]}\right]$$
$$+ E[v(t_i,t_{-i}) - p^W(\beta,\beta) + p^L(\beta,\beta)\,|\max_{j\neq i} b(t_j) = \beta] f_{b_{(-i)}}(\beta).$$

This concludes the proof.■

Proof of Proposition 1. Let us introduce the following notation:

$$\Pi_i^+(t_i,c) = \int \left[v(t_i,\cdot) - p^W\left(c, b_{(-i)}(\cdot)\right)\right]1_{\left[c>b_{(-i)}(\cdot)\right]}\Pi_{j\neq i}\mu(dt_j)$$
$$\Pi_i^-(t_i,c) = \int p^L\left(c, b_{(-i)}(\cdot)\right)1_{\left[c<b_{(-i)}(\cdot)\right]}\Pi_{j\neq i}\mu(dt_j),$$
$$\tilde{\Pi}_i^{+,-}(\phi_i,c) \equiv E\left[\Pi_i^{+,-}(t_i,c)\,|P(t_i) = \phi_i\right].$$

Let us begin with the proof for $\tilde{\Pi}_i^+$ and Π_i^+. Let us denote the conditional expectation by

(26) $$g^{t_i,c}(\alpha) \equiv E\left[v(t_i,t_{-i}) - p^W\left(c, b_{(-i)}(t_{-i})\right)|P_{(-i)}^b(t_{-i}) = \alpha\right].$$

The event $\left[c > b_{(-i)}(t_{-i})\right]$ occurs if and only if $\left[\tilde{P}^b(c) > P_{(-i)}^b(t_{-i})\right]$ occurs. Then, we have

$$\Pi_i^+(t_i,c) = \int g^{t_i,c}\left(P_{(-i)}^b(t_{-i})\right)1_{\left[\tilde{P}^b(c)>P_{(-i)}^b(t_{-i})\right]}\Pi_{j\neq i}\mu(dt_j).$$

Now we appeal to the Lemma 2.2, p. 43, of Lehmann (1959). This lemma says the following: if R is a transformation and if $\mu^*(B) = \mu\left(R^{-1}(B)\right)$, then

$$\int_{R^{-1}(B)} g[R(t)]\,\mu(dt) = \int_B g(\alpha)\,\mu^*(d\alpha).$$

In our case, $R = P_{(-i)}^b$ and $\mu^*([0,c]) = \mu^*([0,c)) = \tau_{-i}\left(\left(P^b\right)_{(-i)}^{-1}([0,c))\right) = \Pr\{t_{-i} \in S^{N-1} : P^b(t_j) < c\} = c$, by (4). So, μ^* is exactly the Lebesgue measure, and we have

(27) $$\Pi_i^+(t_i,c) = \int_0^{\tilde{P}^b(c)} g^{t_i,c}(\alpha)\,d\alpha.$$

From this and the definition of $\tilde{\Pi}_i^+$, we have

$$\tilde{\Pi}_i^+(\phi_i,c) = E\left[\int_0^{\tilde{P}^b(c)} g^{t_i,c}(\alpha)\,d\alpha|P^b(t_i) = \phi_i\right]$$
$$= \int_0^{\tilde{P}^b(c)} E\left[g^{t_i,c}(\alpha)\,|P^b(t_i) = \phi_i\right]d\alpha$$
$$= \int_0^{\tilde{P}^b(c)} \left[\tilde{v}(\phi_i,\alpha) - p^W\left(c,\tilde{b}(\alpha)\right)\right]d\alpha,$$

where the second line comes from a interchange of integrals (Fubbini's Theorem) and the last line comes from independency and the definition of $\tilde{v}(\phi_i,\alpha)$ and $g^{t_i,c}(\alpha)$ (see

(5) and (26)). Also from the fact that $\tilde{b} = \left(\tilde{P}^b\right)^{-1}$, we can substitute $\tilde{P}^b$, to obtain

$$\tilde{\Pi}_i^+ (\phi_i, c) = \int_0^{\tilde{b}^{-1}(c)} \left[\tilde{v}(\phi_i, \alpha) - p^W \left(c, \tilde{b}(\alpha)\right)\right] d\alpha. \tag{28}$$

Now, we can repeat the above procedures with $\Pi_i^- (\phi_i, c)$ and obtain:

$$\tilde{\Pi}_i^- (\phi_i, c) = \int_{\tilde{b}^{-1}(c)}^1 p^L \left(c, \tilde{b}(\alpha)\right) d\alpha. \tag{29}$$

Adding up, that is, putting $\tilde{\Pi}_i (\phi_i, c) = \tilde{\Pi}_i^+ (\phi_i, c) - \tilde{\Pi}_i^- (\phi_i, c)$, we obtain the interim payoff of the indirect auction. This concludes the proof of the first part.

For the second part, observe that the equality (8) implies that for all t_i such that $P^b(t_i) = P^b(s) = x$,

$$\begin{aligned} E\left[g^{t_i,c}(\alpha) | P^b(t_i) = x\right] &= E\left[\left(v(t_i, t_{-i}) - p^W\left(c, b_{(-i)}(t_{-i})\right)\right) | P^b(t_i) = x, P^b_{(-i)}(t_{-i}) = \alpha\right] \\ &= E\left[\left(v(t_i, t_{-i}) - p^W\left(c, b_{(-i)}(t_{-i})\right)\right) | t_i = s, P^b_{(-i)}(t_{-i}) = \alpha\right] \\ &= E\left[\left(v(s, t_{-i}) - p^W\left(c, b_{(-i)}(t_{-i})\right)\right) | P^b_{(-i)}(t_{-i}) = \alpha\right] \\ &= g^{s,c}(\alpha). \end{aligned}$$

Then,

$$\begin{aligned} \tilde{\Pi}_i^+ (x, c) &= E\left[\int_0^{\tilde{P}(c)} g^{t_i,c}(\alpha) \, d\alpha | P^b(t_i) = x\right] \\ &= \int_0^{\tilde{P}(c)} E\left[g^{t_i,c}(\alpha) | P^b(t_i) = x\right] d\alpha \\ &= \int_0^{\tilde{P}(c)} g^{s,c}(\alpha) \, d\alpha \\ &= \Pi_i^+ (s, c), \end{aligned}$$

where the last line comes from (27). Obviously, the same can be shown for Π_i^- and $\tilde{\Pi}_i^-$. So, the proof is complete.■

8. Appendix B - Indirect Auction Equilibria

In this appendix, we will analyze auctions between two players, with independent types uniformly distributed on $[0, 1]$. Since this is the setting of the indirect auction, we will use notation consistent with that, although the results of this appendix are independent from the results of section 4. For $(i, -i) = (1, 2)$ or $(2, 1)$ let

$$\tilde{u}_i(x, b) = \begin{cases} \tilde{v}(x_i, x_{-i}) - p^W(b_i, b_{-i}), & \text{if } b_i > b_{-i} \\ -p^L(b_i, b_{-i}), & \text{if } b_i < b_{-i} \\ \frac{\tilde{v}(x_i, x_{-i}) - b_i}{2}, & \text{if } b_i = b_{-i} \end{cases}$$

be the ex-post payoff. We will assume

(**H0**)' The types are independent and uniformly distributed on $[0, 1]$. $\tilde{v}$ is positive, measurable and bounded above.

By the definition of the indirect auction, we are interested only in nondecreasing equilibria $\tilde{b}$, strictly increasing in the range of winning types. That is, for a nondecreasing strategy $\tilde{b}$, define x_0 to be the minimum type that bids at least $b_{\min}$. So we require $\tilde{b}$ to be increasing in $[x_0, 1]$ and equal to $\tilde{b}(x) = -1$ for $x < x_0$. In order to be an equilibrium, $\tilde{b}$ must satisfy the following:

$$\int_0^{x_0} \left[\tilde{v}(x_0, \alpha) - p^W\left(\tilde{b}(x_0), \tilde{b}(\alpha)\right)\right] d\alpha - \int_{x_0}^1 p^L\left(\tilde{b}(x_0), \tilde{b}(\alpha)\right) d\alpha = 0. \tag{30}$$

Indeed, the above integral is the payoff of x_0. If it is negative, then x_0 can do better by bidding -1. If the above integral is positive, then $x_0 > 0$, otherwise the payoff of $x_0 = 0$ would reduce to $-\int_{x_0}^1 p^L\left(\tilde{b}(x_0), \tilde{b}(\alpha)\right) d\alpha$ which is nonpositive because p^L is positive. If $x_0 > 0$, for a $x < x_0$ sufficiently close of x_0, the payoff

$$\int_0^{x_0} \left[\tilde{v}(x, \alpha) - p^W\left(\tilde{b}(x_0), \tilde{b}(\alpha)\right)\right] d\alpha - \int_{x_0}^1 p^L\left(\tilde{b}(x_0), \tilde{b}(\alpha)\right) d\alpha$$

is still positive (because of the continuity of $\tilde{v}$). So, x is receiving zero but could obtain a strictly positive payoff by bidding $\tilde{b}(x_0) = b_{\min}$.

So, for a fixed $\tilde{b}$, we assume the following:

(**H2**)' There exists x_0, the mininum indirect type that satisfies (30) and $x_0 \in [0, 1)$.

We have the following:

Proposition 2 *(Case 1).* Assume (H0)', (H1)-1, that is, $\partial_1 p^W(\cdot) > 0$ or $\partial_1 p^L(\cdot) > 0$, (H2)' and that $\tilde{v}$ is continuous. Let $\tilde{b}$ be an increasing equilibrium of the indirect auction. Then, $\tilde{b}$ is differentiable and satisfies

$$\tilde{b}'(x) = \frac{\tilde{v}(x, x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) > \tilde{b}(\alpha)\right]} + \partial_1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) < \tilde{b}(\alpha)\right]}\right]}. \tag{31}$$

Proof. Suppose that player $-i$ follows $\tilde{b}$. The interim payoff of player i with (indirect) type x_i is

$$\begin{aligned}\tilde{\Pi}(x_i, b_i, \tilde{b}(\cdot)) &= \int \left[\tilde{v}(x_i, x_{-i}) - p^W(b_i, b(x_{-i}))\right] 1_{\left[b_i > \tilde{b}(x_{-i})\right]} dx_{-i} \\ &\quad - \int p^L(b_i, b(x_{-i})) 1_{\left[b_i < \tilde{b}(t_{-i})\right]} dx_{-i}.\end{aligned}$$

If $\tilde{b}$ is discontinuous, there exists $x^* \geqslant x_0$, with

$$\limsup_{x < x^*} \tilde{b}(x) < \liminf_{x > x^*} \tilde{b}(x).$$

Consider bidders $x_i^{+\varepsilon}$ that bids $\beta^{+\varepsilon} = \tilde{b}\left(x_i^{+\varepsilon}\right) = \liminf_{x > x^*} \tilde{b}(x) + \varepsilon$ and $x_i^{-\varepsilon}$ that bids $\beta^{-\varepsilon} = \tilde{b}\left(x_i^{-\varepsilon}\right) = \limsup_{x < x^*} \tilde{b}(x) - \varepsilon$. For $\varepsilon > 0$ sufficiently small, the event $\left[\beta^{+\varepsilon} > \tilde{b}(x_{-i})\right]$ is arbitrarily close of $\left[\beta^{-\varepsilon} > \tilde{b}(x_{-i})\right]$. So,

$$\begin{aligned}&\int \tilde{v}(x_i, x_{-i}) 1_{\left[\beta^{+\varepsilon} > \tilde{b}(x_{-i})\right]} dx_{-i} - \int \tilde{v}(x_i, x_{-i}) 1_{\left[\beta^{-\varepsilon} > \tilde{b}(x_{-i})\right]} dx_{-i} \\ = &\int \tilde{v}(x_i, x_{-i}) 1_{\left[\beta^{+\varepsilon} > \tilde{b}(x_{-i}) \square\, \beta^{-\varepsilon}\right]} dx_{-i}\end{aligned}$$

is arbitrarily small. On the other hand,

$$
\begin{aligned}
& -\int \left[p^W\left(\beta^{+\varepsilon}, b\left(x_{-i}\right)\right) 1_{\left[\beta^{+\varepsilon}>\tilde{b}\left(x_{-i}\right)\right]}+p^L\left(\beta^{+\varepsilon}, b\left(x_{-i}\right)\right) 1_{\left[\beta^{+\varepsilon}<\tilde{b}\left(t_{-i}\right)\right]}\right] d x_{-i} \\
& +\int\left[p^W\left(\beta^{-\varepsilon}, b\left(x_{-i}\right)\right) 1_{\left[\beta^{-\varepsilon}>\tilde{b}\left(x_{-i}\right)\right]}+p^L\left(\beta^{-\varepsilon}, b\left(x_{-i}\right)\right) 1_{\left[\beta^{-\varepsilon}<\tilde{b}\left(t_{-i}\right)\right]}\right] d x_{-i} \\
= & \int\left[\left(p^W\left(\beta^{-\varepsilon}, b\left(x_{-i}\right)\right)-p^W\left(\beta^{+\varepsilon}, b\left(x_{-i}\right)\right)\right)\right] 1_{\left[\beta^{-\varepsilon}>\tilde{b}\left(x_{-i}\right)\right]} d x_{-i} \\
& +\int\left[p^L\left(\beta^{-\varepsilon}, b\left(x_{-i}\right)\right)-p^L\left(\beta^{+\varepsilon}, b\left(x_{-i}\right)\right)\right] 1_{\left[\beta^{-\varepsilon}<\tilde{b}\left(t_{-i}\right)\right]} d x_{-i} \\
& +r \\
= & \int \int_{\beta^{-\varepsilon}}^{\beta^{+\varepsilon}}-\partial_1 p^W\left(z, b\left(x_{-i}\right)\right) 1_{\left[\beta^{-\varepsilon}>\tilde{b}\left(x_{-i}\right)\right]} d x_{-i} \\
& +\int \int_{\beta^{-\varepsilon}}^{\beta^{+\varepsilon}}-\partial_1 p^L\left(z, b\left(x_{-i}\right)\right) 1_{\left[\beta^{-\varepsilon}<\tilde{b}\left(x_{-i}\right)\right]} d x_{-i} \\
& +r
\end{aligned}
$$

where r denotes integrals over the event $\left[\beta^{+\varepsilon}>\tilde{b}\left(x_{-i}\right) \geqslant \beta^{-\varepsilon}\right]$. Observe that the sum of the two integrals is negative and bounded away from zero, because of $\partial_1 p^W(\cdot)>0$ or $\partial_1 p^L(\cdot)>0$. So, it is not optimal for bidder $x_i^{+\varepsilon}$ to bid $\beta^{+\varepsilon}$ and this contradicts $\tilde{b}$ to be equilibrium. So, $\tilde{b}$ is continuous. Let us prove that it is differentiable. We have

$$
\begin{aligned}
& \tilde{\Pi}(x, \tilde{b}(x), \tilde{b}(\cdot))-\tilde{\Pi}(x, \tilde{b}(x+h), \tilde{b}(\cdot)) \\
= & \int_0^x\left[\tilde{v}(x, \alpha)-p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right)\right] d \alpha-\int_x^1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) d \alpha \\
& -\int_0^{x+h}\left[\tilde{v}(x, \alpha)-p^W\left(\tilde{b}(x+h), \tilde{b}(\alpha)\right)\right] d \alpha+\int_{x+h}^1 p^L\left(\tilde{b}(x+h), \tilde{b}(\alpha)\right) d \alpha \\
= & \int_x^{x+h}\left[-\tilde{v}(x, \alpha)+p^W\left(\tilde{b}(x+h), \tilde{b}(\alpha)\right)-p^L\left(\tilde{b}(x+h), \tilde{b}(\alpha)\right)\right] d \alpha \\
& +\int_0^x\left[p^W\left(\tilde{b}(x+h), \tilde{b}(\alpha)\right)-p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right)\right] d \alpha \\
& +\int_x^1\left[p^L\left(\tilde{b}(x+h), \tilde{b}(\alpha)\right)-p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right)\right] d \alpha .
\end{aligned}
$$

Since the first integrand is continuous, by the Mean Value Theorem, there exists x^* between x and $x+h$ such that

$$
\begin{aligned}
& \int_{x+h}^x\left[\tilde{v}(x, \alpha)-p^W\left(\tilde{b}(x+h), \tilde{b}(\alpha)\right)+p^L\left(\tilde{b}(x+h), \tilde{b}(\alpha)\right)\right] d \alpha \\
= & h\left[-\tilde{v}\left(x, x^*\right)+p^W\left(\tilde{b}(x+h), \tilde{b}\left(x^*\right)\right)-p^L\left(\tilde{b}(x+h), \tilde{b}\left(x^*\right)\right)\right]
\end{aligned}
$$

Because p^W and p^L are differentiable and $\tilde{b}$ continuously increasing, there exists x^{**} and x^{***} between x and $x+h$ such that

$$\begin{aligned}\int_0^x \left[p^W\left(\tilde{b}(x+h), \tilde{b}(\alpha)\right) - p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right)\right] d\alpha &= \left[\tilde{b}(x+h) - \tilde{b}(x)\right] \int_0^x \partial_1 p^W\left(\tilde{b}(x^{**}), \tilde{b}(\alpha)\right) d\alpha \\ \int_x^1 \left[p^L\left(\tilde{b}(x+h), \tilde{b}(\alpha)\right) - p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right)\right] d\alpha &= \left[\tilde{b}(x+h) - \tilde{b}(x)\right] \int_x^1 \partial_1 p^L\left(\tilde{b}(x^{***}), \tilde{b}(\alpha)\right) d\alpha.\end{aligned}$$

So, since $\tilde{\Pi}(x, \tilde{b}(x), \tilde{b}(\cdot)) - \tilde{\Pi}(x, \tilde{b}(x+h), \tilde{b}(\cdot)) \geqslant 0$, we have

$$\text{(32)} \quad \frac{\tilde{b}(x+h) - \tilde{b}(x)}{h} \geqslant \frac{\left[\tilde{v}(x, x^*) - p^W\left(\tilde{b}(x+h), \tilde{b}(x^*)\right) + p^L\left(\tilde{b}(x+h), \tilde{b}(x^*)\right)\right]}{\int_0^x \partial_1 p^W\left(\tilde{b}(x^{**}), \tilde{b}(\alpha)\right) d\alpha + \int_x^1 \partial_1 p^L\left(\tilde{b}(x^{***}), \tilde{b}(\alpha)\right) d\alpha.}$$

Now, consider the difference:

$$\begin{aligned} & 0 \leqslant \tilde{\Pi}(x+h, \tilde{b}(x+h), \tilde{b}(\cdot)) - \tilde{\Pi}(x+h, \tilde{b}(x), \tilde{b}(\cdot)) \\ = & \int_x^{x+h} \left[\tilde{v}(x+h, \alpha) - p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) + p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right)\right] d\alpha \\ & - \int_0^{x+h} \left[p^W\left(\tilde{b}(x+h), \tilde{b}(\alpha)\right) - p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right)\right] d\alpha \\ & - \int_{x+h}^1 \left[p^L\left(\tilde{b}(x+h), \tilde{b}(\alpha)\right) - p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right)\right] d\alpha \\ = & \; h\left[\tilde{v}\left(x+h, x'\right) - p^W\left(\tilde{b}(x), \tilde{b}(x')\right) + p^L\left(\tilde{b}(x), \tilde{b}(x')\right)\right] \\ & - \left[\tilde{b}(x+h) - \tilde{b}(x)\right] \int_0^{x+h} \partial_1 p^W\left(x'', \tilde{b}(\alpha)\right) d\alpha \\ & - \left[\tilde{b}(x+h) - \tilde{b}(x)\right] \int_{x+h}^1 \partial_1 p^L\left(x''', \tilde{b}(\alpha)\right) d\alpha, \end{aligned}$$

where the existence of x', $x''$and x'''' between x and $x+h$ is ensured by the Mean Value Theorem. Thus, we obtain

$$\text{(33)} \quad \frac{\tilde{b}(x+h) - \tilde{b}(x)}{h} \leqslant \frac{\left[\tilde{v}(x+h, x') - p^W\left(\tilde{b}(x), \tilde{b}(x')\right) + p^L\left(\tilde{b}(x), \tilde{b}(x')\right)\right]}{\int_0^{x+h} \partial_1 p^W\left(x'', \tilde{b}(\alpha)\right) d\alpha + \int_{x+h}^1 \partial_1 p^L\left(x''', \tilde{b}(\alpha)\right) d\alpha}$$

When we make $h \to 0$, the right hand side in (32) and (33) both converge to

$$\frac{\tilde{v}(x, x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) > \tilde{b}(\alpha)\right]} + \partial_1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) < \tilde{b}(\alpha)\right]}\right]}.$$

So, $\tilde{b}$ is differentiable at $x \in (x_0, 1)$ and $\tilde{b}'(x)$ is equal to the expression above.■

Proposition 3 *(Case 2)***.** Assume (H0)', (H1)-2, that is, $\partial_1 p^W = \partial_1 p^L = 0$ and $\partial_2\left(p^W - p^L\right) > 0$ and that $\tilde{v}$ is continuous. Let $\tilde{b}$ be an increasing equilibrium of the indirect auction. Then

$$\text{(34)} \quad x \in (x_0, 1) \Rightarrow \tilde{v}(x, x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right) = 0.$$

Proof. Given b, b', let us define the function h as

$$h(z) \equiv p^W(b,z) - p^L(b',z).$$

Since $\partial_1 p^W = \partial_1 p^L \equiv 0$ and $\partial_2 \left(p^W - p^L\right) > 0$, h does not depend on b or b' and is differentiable and increasing.

By contradiction, assume that (34) is false, that is, there exists $x^* \in (x_0, 1)$ such that

$$\tilde{v}(x^*, x^*) > h\left(\tilde{b}(x^*)\right). \tag{35}$$

Because $\tilde{v}$ is continuous and $\tilde{b}$ is increasing, for sufficiently small $\delta > 0$, we have

$$\tilde{v}(x^* - \delta, x^* - \delta) > h\left(\tilde{b}(x^*)\right) > h\left(\tilde{b}(x^* - \delta)\right).$$

Since the set of the points of discontinuity of $\tilde{b}$ is enumerable, we may assume that (35) holds for a point x^* where $\tilde{b}$ is continuous. Thus, for sufficiently small ε and $\delta > 0$, $\forall \alpha \in [x^*, x^* + \varepsilon]$,

$$\tilde{v}(x^*, \alpha) > h\left(\tilde{b}(x^*) + \delta\right) > h\left(\tilde{b}(\alpha)\right).$$

Consider the following difference:

$$\begin{aligned}
&\tilde{\Pi}(x^*, \tilde{b}(x^* + \varepsilon), \tilde{b}(\cdot)) - \tilde{\Pi}(x^*, \tilde{b}(x^*), \tilde{b}(\cdot)) \\
= &\int_0^{x^*+\varepsilon} \left[\tilde{v}(x^*, \alpha) - p^W\left(\tilde{b}(x^* + \varepsilon), \tilde{b}(\alpha)\right)\right] d\alpha - \int_{x^*+\varepsilon}^1 p^L\left(\tilde{b}(x^* + \varepsilon), \tilde{b}(\alpha)\right) d\alpha \\
&- \int_0^{x^*} \left[\tilde{v}(x^*, \alpha) - p^W\left(\tilde{b}(x^*), \tilde{b}(\alpha)\right)\right] d\alpha + \int_{x^*}^1 p^L\left(\tilde{b}(x^*), \tilde{b}(\alpha)\right) d\alpha \\
= &\int_{x^*}^{x^*+\varepsilon} \left[\tilde{v}(x^*, \alpha) - h\left(\tilde{b}(\alpha)\right)\right] d\alpha > 0
\end{aligned}$$

where we used the property $\partial_1 p^W = \partial_1 p^L \equiv 0$ in order to obtain the last equality. This contradicts the optimality of $\tilde{b}(x^*)$ for x^*.

Now, assume that there is a $x^* \in (x_0, 1)$ such that

$$\tilde{v}(x^*, x^*) < h\left(\tilde{b}(x^*)\right).$$

Again, we may assume that x^* is a point of continuity of $\tilde{b}$. Hence, for $\varepsilon, \delta > 0$ sufficiently small, $\forall \alpha \in [x^* - \varepsilon, x^*]$,

$$\tilde{v}(x^*, \alpha) < h\left(\tilde{b}(x^*) - \delta\right) < h\left(\tilde{b}(\alpha)\right).$$

Similarly,

$$\begin{aligned}
&\tilde{\Pi}(x^*, \tilde{b}(x^*), \tilde{b}(\cdot)) - \tilde{\Pi}(x^*, \tilde{b}(x^* - \varepsilon), \tilde{b}(\cdot)) \\
= &\int_0^{x^*} \left[\tilde{v}(x^*, \alpha) - p^W\left(\tilde{b}(x^*), \tilde{b}(\alpha)\right)\right] d\alpha - \int_{x^*}^1 p^L\left(\tilde{b}(x^*), \tilde{b}(\alpha)\right) d\alpha \\
&- \int_0^{x^*-\varepsilon} \left[\tilde{v}(x^*, \alpha) - p^W\left(\tilde{b}(x^* - \varepsilon), \tilde{b}(\alpha)\right)\right] d\alpha + \int_{x^*-\varepsilon}^1 p^L\left(\tilde{b}(x^* - \varepsilon), \tilde{b}(\alpha)\right) d\alpha \\
= &\int_{x^*-\varepsilon}^{x^*} \left[\tilde{v}(x^*, \alpha) - h\left(\tilde{b}(\alpha)\right)\right] d\alpha < 0.
\end{aligned}$$

This completes the proof of (34). So, we have

$$\tilde{b}(x) = h^{-1}(\tilde{v}(x,x)), \tag{36}$$

which shows that $\tilde{b}$ is continuous. Moreover, $\tilde{b}$ is increasing if and only if $x \longmapsto \tilde{v}(x,x)$ is also increasing.■

Now, we will analyze the equilibrium existence in both cases 1 and 2. Instead of assuming that $\tilde{v}$ is continuous, as we did in the last two propositions, we will assume directly its consequence, that is, we suppose that there exists a function $\tilde{b}$ that satisfies the following:

Case 1: $\partial_1 p^W(\cdot) > 0$ or $\partial_1 p^L(\cdot) > 0$, $\tilde{b}$ is differentiable and

$$\tilde{b}'(x) = \frac{\tilde{v}(x,x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) > \tilde{b}(\alpha)\right]} + \partial_1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) < \tilde{b}(\alpha)\right]}\right]}.$$

Case 2: $\partial_1 p^W = \partial_1 p^L = 0$, $\partial_2\left(p^W - p^L\right) > 0$, $\tilde{b}$ is continuous, and

$$x \in (x_0, 1) \Rightarrow \tilde{v}(x,x) = h\left(\tilde{b}(x)\right) = p^W\left(\tilde{b}(x), \tilde{b}(x)\right) - p^L\left(\tilde{b}(x), \tilde{b}(x)\right).$$

Observe that we do not assume that $\tilde{b}$ is increasing. This is so because this is exactly the setting of Theorem 4. To treat non-increasing $\tilde{b}$, we define the following:

Modified Auction - The bidder submits a type $y \in [0,1]$. In any event, the payment is determined as if the bidder has submitted the bid $\tilde{b}(y)$. The bidder wins against opponents who announce types below y and loses to opponents who announce types above y. If there is a tie, the object is given with probability $1/2$ for each bidder.

Observe that if $\tilde{b}$ is increasing, the modified auction is simply the indirect auction. If $\tilde{b}$ is not increasing, the difference is that the events of winning are not determined by $\tilde{b}$ but by the announced type y. The rule of the modified auction implies the following interim payoff:

$$\hat{\Pi}(x,y) = \begin{cases} \int_0^y \left[\tilde{v}(x,\alpha) - p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right)\right] d\alpha - \int_y^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha, & \text{if } y \geqslant x_0 \\ 0 & \text{if } y < x_0 \end{cases}$$

We can simplify the above expression to

$$\hat{\Pi}(x,y) = \begin{cases} \int_0^y \tilde{v}(x,\alpha)\, d\alpha - \hat{p}(y), & \text{if } y \geqslant x_0 \\ 0 & \text{if } y < x_0 \end{cases} \tag{37}$$

where

$$\hat{p}(y) \equiv \begin{cases} \int_0^y p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha, & \text{if } y \geqslant x_0 \\ 0, & \text{if } y < x_0 \end{cases}$$

In case 1, $\tilde{b}$, p^W and p^L are differentiable on $(x_0, 1)$, $\hat{p}$ and $\hat{\Pi}$ are also differentiable. So, for every $y \in (x_0, 1)$, we have

$$\hat{p}'(y) = \partial_y\left\{\int_0^y \tilde{v}(x,\alpha)\, d\alpha - \hat{\Pi}(x,y)\right\} = \tilde{v}(x,y) - \partial_y \hat{\Pi}(x,y).$$

Truth-telling is always optimal if

$$\hat{\Pi}(x,x) - \hat{\Pi}(x,y) \geqslant 0. \tag{38}$$

In case 1, this is equivalent to

$$\int_y^x \partial_y \hat{\Pi}(x,\alpha)\, d\alpha \geqslant 0$$

if $x, y \geqslant x_0$. Also, if $x, y \geqslant x_0$, $\partial_y \hat{\Pi}(x,y)\mid_{y=x}$ must be zero, so that

$$\partial_y \hat{\Pi}(x,y)\mid_{y=x} = 0 \Rightarrow \hat{p}'(x) = \tilde{v}(x,x). \tag{39}$$

Indeed, these are simply the second- and the first-order conditions, respectively. So, for $y \geqslant x_0$,

$$\hat{p}(y) = \int_{x_0}^y \tilde{v}(\alpha,\alpha)\, d\alpha + \hat{p}(x_0).$$

Now, let us turn to case 2. Since $\tilde{b}$ is only continuous, $\hat{p}$ is not necessarily differentiable. Nevertheless, if $y \geqslant x_0$,

$$\begin{aligned}
\hat{p}(y) &= \int_0^y p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha \\
&= \int_{x_0}^y \left[p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) - p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right)\right] d\alpha + \hat{p}(x_0) \\
&= \int_{x_0}^y h\left(\tilde{b}(\alpha)\right) d\alpha + \hat{p}(x_0) \\
&= \int_{x_0}^y \tilde{v}(\alpha,\alpha)\, d\alpha + \hat{p}(x_0).
\end{aligned}$$

Observe that the expression above is exactly the same of case 1. For $y < x_0$, the payment is zero. For $y = x_0$, $\hat{p}(y)$ is obtained from (30):

$$\int_0^{x_0} \tilde{v}(x_0,\alpha)\, d\alpha - \int_0^{x_0} p^W\left(\tilde{b}(x_0), \tilde{b}(\alpha)\right) d\alpha + \int_{x_0}^1 p^L\left(\tilde{b}(x_0), \tilde{b}(\alpha)\right) d\alpha = 0.$$

So, we have proved the following:

Proposition 4 (Payment Rule). Assume (H0)', (H1) and (H2)'. Then, both for case 1 or case 2, we have

$$\hat{p}(y) = \begin{cases} \hat{p}(x_0) + \int_{x_0}^y \tilde{v}(\alpha,\alpha)\, d\alpha, & \text{if } y > x_0 \\ \int_0^{x_0} \tilde{v}(x_0,\alpha)\, d\alpha, & \text{if } y = x_0 \\ 0, & \text{if } y < x_0 \end{cases} \tag{40}$$

Now, we turn to the equilibrium existence.

Proposition 5 (Equilibrium). Assume (H0)', (H1), (H2)' and (40). Then, truth-telling is equilibrium of the modified auction if and only if, for all x, $y \in [0,1]$,

$$\begin{cases} \int_y^x \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha \geqslant 0, & \text{if } x, y \geqslant x_0 \\ \int_{x_0}^x \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha + \int_0^{x_0} \left[\tilde{v}(x,\alpha) - \tilde{v}(x_0,\alpha)\right] d\alpha \geqslant 0, & \text{if } x \geqslant x_0 > y \\ 0 \geqslant \int_{x_0}^y \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha + \int_0^{x_0} \left[\tilde{v}(x,\alpha) - \tilde{v}(x_0,\alpha)\right] d\alpha & \text{if } y \geqslant x_0 > x \end{cases} \tag{41}$$

Proof. Given (40), the optimality condition for truth-telling, namely, $\hat{\Pi}(x,x) - \hat{\Pi}(x,y) \geqslant 0$, is equivalent to

$$\begin{aligned} & \int_0^x \tilde{v}(x,\alpha)\,d\alpha - \int_{x_0}^x \tilde{v}(\alpha,\alpha)\,d\alpha - \hat{p}(x_0) \\ & - \int_0^y \tilde{v}(x,\alpha)\,d\alpha + \int_{x_0}^y \tilde{v}(\alpha,\alpha)\,d\alpha + \hat{p}(x_0) \\ = & \int_y^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\,d\alpha \geqslant 0 \end{aligned} \tag{42}$$

if $x, y \geqslant x_0$. The other cases are immediate.■

As we have said before, if $\tilde{b}$ is increasing, the modified auction is just the original (unmodified) auction. Then, we have

Corollary 1. Assume (H0)', (H1), (H2)' and that $\tilde{b}$ is increasing and implies (40). Then, if (41) holds, $\tilde{b}$ is equilibrium of the indirect auction.

Observe that Corollary 1 does not require $\tilde{v}$ to be continuous.

9. Appendix C - Proofs of the Theorems

Proof of Theorem 1.

(i) If $b \in \mathcal{S}$, it defines a conjugation P^b by (2). The bid $b(t_i) = \beta$ is optimal for bidder t_i against the strategy $b(\cdot)$ of the opponents. This and the fact that $\partial_b \Pi(s, b(s)) = 0$ imply that

$$\begin{aligned} & E\left[v(t_i,\cdot)\,|t_i = s, b_{(-i)}(t_{-i}) = \beta\right] \\ & \qquad = p^W(\beta,\beta) - p^L(\beta,\beta) - \frac{E_{t_{-i}}\left[\partial_{b_i} p^W 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} + \partial_{b_i} p^L 1_{\left[b_i < \mathbf{b}_{(-i)}\right]}\right]}{f_{b_{(-i)}}(\beta)}. \end{aligned}$$

Observe that the right-hand side does not depend on s (it depends on it only by the fact that $\beta = b(s)$ is the optimum bid for such bidder). Thus, the left-hand side has to be the same for all s that are bidding the same bid in equilibrium, which implies that (10) holds.

(ii) If $b(t_i)$ maximizes $\Pi(t_i, c)$ for t_i, and $P(t_i') = P(t_i)$, then $b(t_i') = b(t_i)$. Then, $b(t_i)$ maximizes $\tilde{\Pi}(P(t_i'), c)$ for all t_i' such that $P(t_i') = P(t_i)$, from the definition of $\tilde{\Pi}(P(t_i), c)$ given by (6). In other words, $\tilde{b}(x) = \left(\tilde{P}^b\right)^{-1}(x) = b\left(P^{-1}(x)\right)$ is the equilibrium of the indirect auction.

If $\tilde{v}$ is continuous, we appeal to the results of the Appendix B. Propositions 2 and 3 proves (iii), Proposition 4 proves (iv) and Proposition 5 gives (v).■

Proof of Theorem 2. Corollary 1 of Appendix B proves that conditions (ii) and (iii) are sufficient for $\tilde{b}$ to be the equilibrium of the indirect auction. Now, Proposition

1 proves that condition (i)' implies that for all s such that $P(s) = x$, $\tilde{\Pi}(x,c) = \Pi(s,c)$ (see (9)). Now, if we put $b(s) = \tilde{b}(P(s))$, then

$$\Pi(s, b(s)) = \tilde{\Pi}\left(P(s), \tilde{b}(P(s))\right) \text{ and}$$
$$\Pi(s,c) = \tilde{\Pi}(P(s), c).$$

But this is sufficient to show the equilibrium existence in the direct auction, since $\tilde{b}$ is the equilibrium in the indirect auction, which implies that

$$\tilde{\Pi}\left(P(s), \tilde{b}(P(s))\right) \geqslant \tilde{\Pi}(P(s), c),$$

for all $c \in \mathbb{R}$. If $\tilde{v}$ is continuous, $\tilde{\Pi}(x,c)$ is differentiable at all $c \in \mathbb{R}$. This concludes the proof.■

Through the proof of Theorem 3, we will make successive use of the following fact:

Lemma 2. Assume (H1), (H2) and (H3). For any $\sigma-$field Σ on S^{N-1}, we have

$$\begin{aligned}
&\exists t_{-i} : v\left(s', t_{-i}\right) > v(s, t_{-i}) \\
&\quad \Leftrightarrow \forall t_{-i} : v\left(s', t_{-i}\right) > v(s, t_{-i}) \\
&\quad \Leftrightarrow E\left[v(t_i, t_{-i})\,|t_i = s', \Sigma\right] > E\left[v(t_i, t_{-i})\,|t_i = s, \Sigma\right], \text{ a.s.}
\end{aligned}$$

Proof. (H3) gives the first equivalence. By (H2), v is continuous over a compact. So, if $\forall t_{-i} : v(s', t_{-i}) > v(s, t_{-i})$, there is $\delta > 0$ so that $d(t_{-i}) \equiv v(s', t_{-i}) - v(s, t_{-i}) - \delta \geqslant 0$ for all t_{-i}. Then, for any Σ, $E[d(t_{-i})\,|\Sigma] \geqslant 0$ almost surely.[10] This implies that $E[v(t_i, t_{-i}) \mid t_i = s', \Sigma] > E[v(t_i, t_{-i}) \mid t_i = s, \Sigma]$, a.s. On the other hand, $E[v(t_i, t_{-i}) \mid t_i = s', \Sigma] > E[v(t_i, t_{-i}) \mid t_i = s, \Sigma]$ a.s. implies that $\exists t_{-i} : v(s', t_{-i}) > v(s, t_{-i})$.■

Proof of Theorem 3. *Equilibrium Existence.* If we define P by (19), it is a conjugation. Let us prove that it satisfies condition (i)' of Theorem 2. If for some x, y and s, such that $P(s) = x$, we have

$$\tilde{v}(x,y) = E[v(t_i,t_{-i})|P(t_i) = x, P_{(-i)}(t_{-i}) = y] < E[v(t_i,t_{-i})|t_i = s, P_{(-1)}(t_{-i}) = y],$$

then, for at least one t_{-i} and s', $P(s') = x$, $v(s, t_{-i}) > v(s', t_{-i})$. But then, by (H3), $v(s, t_{-i}) > v(s', t_{-i})$ for all t_{-i} which implies $v^1(s) > v^1(s')$ and $P(s) > P(s')$, a contradiction with the assumption that $P(s) = P(s') = x$. So, condition (i)$'$ is satisfied.

Let us prove condition (ii) of Theorem 2. If $x > y$, for all t_i and t_i' such that $P(t_i') = x$ and $P(t_i) = y$, we have $v(t_i', t_{-i}) > v(t_i, t_{-i})$ for all t_{-i}, by (H3). Then, for all $z \in [0,1]$,

$$\begin{aligned}
\tilde{v}(x,z) &\equiv E\left[v(t_i,t_{-i})\,|P(t_i) = x, P_{(-i)}(t_{-i}) = z\right] \\
&> E\left[v(t_i,t_{-i})\,|P(t_i) = y, P_{(-i)}(t_{-i}) = z\right] = \tilde{v}(y,z).
\end{aligned}$$

Then, if $y < \alpha < x$, $\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha) > 0$ and we have:

$$\int_y^x \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha \geqslant 0.$$

[10]See, for instance, Kallenberg (2002), Theorem 6.1, p. 104.

Now if $x < \alpha < y$, we have $\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha) < 0$ so that condition (ii) is satisfied. Since our assumption is the condition (iii) of Theorem 2, this implies the existence of equilibrium, with the equilibrium bidding function given by $b = \tilde{b} \circ P$.

Sufficiency. Conditions (i)' and (ii) of the Theorem 2 was shown in the first part, above. Proposition 4 in appendix B proves condition (iii) of Theorem 2. Then, there exists a equilibrium $b = \tilde{b} \circ P$. Since $\tilde{v}$ is continuous, Theorem 2 shows the existence of $\partial_b \Pi(s, b(s))$ for all s.

Necessity. According to Theorem 1, given a $b \in \mathcal{S}$, the associated conjugation P^b (given by (2)) is such that for all $s \in \left(P^b\right)^{-1}(x)$,

$$E[v(t_i,t_{-i})|P^b(t_i) = x, P^b_{(-i)}(t_{-i}) = x] = E[v(t_i,t_{-i})|t_i = s, P^b_{(-i)}(t_{-i}) = x].$$

If $P^b(s) = P^b(s')$ and there is some t_{-i} such that $v(s,t_{-i}) < v(s',t_{-i})$, Lemma 2 implies that

$$E[v(t_i,t_{-i})|t_i = s, P^b_{(-i)}(t_{-i}) = x] < E[v(t_i,t_{-i})|t_i = s', P^b_{(-i)}(t_{-i}) = x],$$

which contradicts the previous equality between the conditional expectations. We conclude that

$$P^b(s) = P^b\left(s'\right) \Rightarrow v(s,t_{-i}) = v\left(s',t_{-i}\right) \text{ for all } t_{-i}. \tag{43}$$

Let us define $\tilde{v}^1(x)$ as $E\left[v(t_i,t_{-i})|P^b(t_i) = x\right]$ and prove that it is non-decreasing. Suppose by absurd that there exist x and y, $x > y$, such that $\tilde{v}^1(x) < \tilde{v}^1(y)$.

First, we claim that for all t_i and t_i' such that $P^b(t_i) = x$ and $P^b(t_i') = y$, we have $v(t_i,t_{-i}) < v(t_i',t_{-i})$ for all t_{-i}. Otherwise, $v(t_i,t_{-i}) \geqslant v(t_i',t_{-i})$ for some t_{-i} and, by (H3), $v\left(t_i,t_{-i}'\right) \geqslant v\left(t_i',t_{-i}'\right)$ for all t_{-i}'. Then, Lemma 2 and (43) would imply that $\tilde{v}^1(x) = E\left[v(t_i,t_{-i})|P^b(t_i) = x\right] \geqslant E\left[v(t_i,t_{-i})|P^b(t_i) = y\right] = \tilde{v}^1(y)$, a contradiction with our (absurd) assumption. Thus, the claim is proved.

This claim and Lemma 2 imply that

$$\begin{aligned}\tilde{v}(x,z) &\equiv E\left[v(t_i,t_{-i})|P^b(t_i) = x, P^b_{(-i)}(t_{-i}) = z\right] \\ &< E\left[v(t_i,t_{-i})|P^b(t_i) = y, P^b_{(-i)}(t_{-i}) = z\right] = \tilde{v}(y,z),\end{aligned}$$

for all $z \in [0,1]$, a.s. Thus,

$$\int_y^x \left[\tilde{v}(x,\alpha) - \tilde{v}(y,\alpha)\right] d\alpha < 0.$$

By condition (v) of Theorem 1, we also have that

$$\int_y^x \left[\tilde{v}(y,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha \leqslant 0.$$

Summing up these two integrals, we obtain

$$\int_y^x \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha < 0,$$

which contradicts condition (v) of Theorem 1. This contradiction establishes that $x > y$ $\Rightarrow \tilde{v}^1(x) \geqslant \tilde{v}^1(y)$.

Suppose now that there exists $x > y$ such that $\tilde{v}^1(x) = \tilde{v}^1(y)$. Then, the monotonicity of $\tilde{v}^1$ (just proved) gives

$$\forall \phi \in [y, x], \tilde{v}^1(\phi) = \tilde{v}^1(x) = \tilde{v}^1(y). \tag{44}$$

Let $S' = \left\{ s \in S : \tilde{b}(y) \leqslant b(s) < \tilde{b}(x) \right\}$. From (2), for all $s \in S'$, $P^b(s) \in [y, x]$. Then, (43) and (44) imply that $s \in S' \Rightarrow v^1(s) = \tilde{v}^1(x)$. Assumption (H3) requires that $\mu(S') = 0$. Observe that $S' = A \backslash B$, where $A \equiv \left\{ s \in S : b(s) < \tilde{b}(x) \right\}$ and $B = \left\{ s \in S : b(s) < \tilde{b}(y) \right\}$. But then, $\mu(A) = \mu(B)$. However, from the definition of $\tilde{b}$ as the inverse of $\tilde{P}^b$, we have the following:

$$0 < x - y = \tilde{P}^b\left(\tilde{b}(x)\right) - \tilde{P}^b\left(\tilde{b}(y)\right) = (\mu(A))^{N-1} - (\mu(B))^{N-1},$$

which is a contradiction. So, we have proved that $x = P^b(s') > P^b(s) = y$ implies $v^1(s') = \tilde{v}^1(x) > \tilde{v}^1(y) = v^1(s)$ and $P^b(s') = P^b(s)$ implies $v^1(s') = v^1(s)$. In other words, $P^b(s') \lesseqgtr P^b(s)$ if and only if $v^1(s') \lesseqgtr v^1(s)$ which allows us to conclude that

$$P^b(t_i) = \Pr\left\{ t_{-i} \in T_{-i} = S^{N-1} : v^1(t_j) < v^1(t_i), j \neq i \right\},$$

as we have defined in (19). In other words, the conjugation is unique.

Now, $\tilde{v}$ and $\tilde{b}$ in Theorem 1 are exactly those defined in the statement of Theorem 3. So, Theorem 1 implies the claims about $\tilde{b}$.

Uniqueness. Since $\tilde{v}$ is continuous, Propositions 2 and 3 in Appendix B says that any equilibrium $\tilde{b}$ satisfy the conditions given. If there is just one $\tilde{b}$ that satisfy such conditions, then the equilibrium of the indirect auction is unique. Since the previous step (necessity) shows that the conjugation is unique, the equilibrium of the direct auction is unique.■

Proof of Theorem 4. If $\tilde{b}$ is strictly increasing, then $b = \tilde{b} \circ P$ is an equilibrium of direct auction, by Theorem 3.

So, we have to show that an equilibrium exists if $\tilde{b}$ is not increasing. For future use, remember that in the first part of the proof of Theorem 3, we have established conditions (i)' and (ii) of Theorem 2 and that

$$x > y \Rightarrow \tilde{v}(x, z) > \tilde{v}(y, z), \forall z \in [0, 1]. \tag{45}$$

Let us define $\overline{b}(x) = \sup_{\alpha \in [0,x]} \tilde{b}(\alpha)$. As we discussed after the statement of Theorem 4, this is just one of the possible specification for the equilibrium bidding function. The only exception is when the tie is to occur including the highest bidder. In such a case, it is mandatory to have the bid of the tieing bidders following the above definition. The reason will become clear in the sequel.

Remember that $\tilde{b}$ is absolutely continuous. Then, there is an enumerable set of intervals $[a_k, c_k]$ where $\overline{b}(x)$ is constant. Let $b_k \equiv \overline{b}(x)$ for $x \in [a_k, c_k]$. (See Figure 5.)

Therefore, there is a tie among the indirect types in $[a_k, c_k]$ for the bidding function $\overline{b}$. Let b_k be the specified bid for indirect types in $[a_k, c_k]$, that is, $\overline{b}([a_k, c_k]) = \{b_k\}$. The tie is solved by an all-pay auction among the tying bidders.

The unique information that bidders have for the second auction is that there is a tie in b_k, that is, $P_{(-i)}(t_{-i}) \in [a_k, c_k]$.

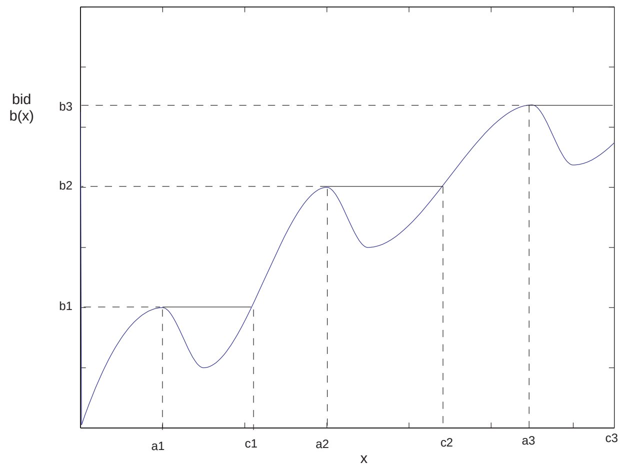

FIGURE 5. Indirect Equilibrium Bidding Function

By the definition of P (19), $P_{(-i)}$ satisfies the following:

$$\Pr\left(\left\{t_{-i} \in S^{N-1} : P_{(-i)}\left(t_{-i}\right) < x\right\} | P_{(-i)}\left(t_{-i}\right) \in \left[a_k, c_k\right]\right) = \frac{x - a_k}{c_k - a_k}.$$

So, in the tie-breaking auction, the (direct) type t_i of bidder i is competing against players t_j in the set $\{s \in S : P(s) \in [a_k, c_k]\}$ and the equilibrium is to bid the increasing function[11]

$$\tilde{b}^2(x) = \frac{1}{c_k - a_k} \int_{a_k}^{x} \tilde{v}(\alpha, \alpha)\, d\alpha.$$

Indeed, from condition (ii) of Theorem 2, we have that

$$\begin{aligned} & \frac{1}{c_k - a_k}\left[\int_{a_k}^{x} \tilde{v}(x, \alpha)\, d\alpha - \int_{a_k}^{x} \tilde{v}(\alpha, \alpha)\, d\alpha\right] \\ \geqslant & \frac{1}{c_k - a_k}\left[\int_{a_k}^{y} \tilde{v}(x, \alpha)\, d\alpha - \int_{a_k}^{y} \tilde{v}(\alpha, \alpha)\, d\alpha\right] \end{aligned}$$

for any $x, y \in [a_k, c_k]$.

Thus, in the whole auction, the bidder of indirect type $x \in [a_k, c_k]$ who follows the strategy $\overline{b}(x)$ and, in case of a tie, the above strategy, will receive the expected payoff

$$\begin{aligned} & \int_0^{a_k} \left[\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)\right] d\alpha + (c_k - a_k)\left\{\frac{1}{c_k - a_k} \int_{a_k}^{x} \left[\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)\right] d\alpha\right\} \\ = & \int_0^{x} \left[\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)\right] d\alpha \end{aligned}$$

Deviation in the second auction is suboptimal. By deviating from $\overline{b}$, but bidding in the range of $\overline{b}$, that is, bidding $\overline{b}(y) \neq \overline{b}(x)$, he will get

$$\tilde{\Pi}_i\left(x, \overline{b}(y)\right) = \int_0^{y} \left[\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)\right] d\alpha,$$

[11] It is increasing because $\tilde{v}$ is positive.

if $\overline{b}(y)$ is not a bid with positive probability. This cannot be profitable by condition (ii). If it is a bid with positive probability, the second stage will be again an all-pay auction, where the bidder cannot improve its payoff, again by condition (ii).

Now, if x bids $\beta < \inf_{x\in[0,1]} \overline{b}(x)$, then his payoff will be

$$\int_0^1 p^L \left(\beta, \overline{b}(\alpha)\right) d\alpha \leqslant 0,$$

because $p^L \leqslant 0$. Therefore, this deviation cannot be profitable.

If x bids $\beta > \sup_{x\in[0,1]} \overline{b}(x) = \tilde{b}(\overline{x}) \geqslant \tilde{b}(1)$, for some $\overline{x}$. Since $\partial_1 p^W(\cdot) \geqslant 0$, $p^W\left(\beta, \overline{b}(z)\right) \geqslant p^W\left(\tilde{b}(1), \overline{b}(z)\right)$. Then,

$$\int_0^1 p^W \left(\beta, \overline{b}(\alpha)\right) \geqslant \int_0^1 p^W \left(\tilde{b}(1), \overline{b}(\alpha)\right) d\alpha = \int_0^1 \tilde{v}(\alpha,\alpha)\, d\alpha.$$

Then, the payoff of the bidder with indirect type x that bids β will be

$$\begin{aligned} & \int_0^1 \left[\tilde{v}(x,\alpha) - p^W\left(\beta, \overline{b}(\alpha)\right)\right] d\alpha \\ \leqslant & \int_0^1 \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha \\ = & \int_0^x \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha + \int_x^1 \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha \\ < & \int_0^x \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha, \end{aligned}$$

where the last inequality comes from (45). Thus, the deviation to β is unprofitable.

In Theorem 3, we also proved condition (i)'. Then, the equilibrium in the indirect auction gives the equilibrium for the direct one.■

Proof of Theorem 5. If y is an indirect type that is not involved in ties, the payment is given by

$$\int_0^y p^W \left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L \left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha = \int_0^y \tilde{v}(\alpha,\alpha)\, d\alpha.$$

If $x \in [a_k, c_k]$, in the notation of the previous proof, the expected payment of x will be

$$\int_0^{a_k} \tilde{v}(\alpha,\alpha)\, d\alpha + (c_k - a_k)\left\{\frac{1}{c_k - a_k}\int_{a_k}^x \tilde{v}(\alpha,\alpha)\, d\alpha\right\} = \int_0^x \tilde{v}(\alpha,\alpha)\, d\alpha.$$

So, if the equilibrium specified in the proof of Theorem 4 is followed, the expected payment does not of the auction format.■

10. Appendix D - Proofs for the Examples

Proof of example 2.

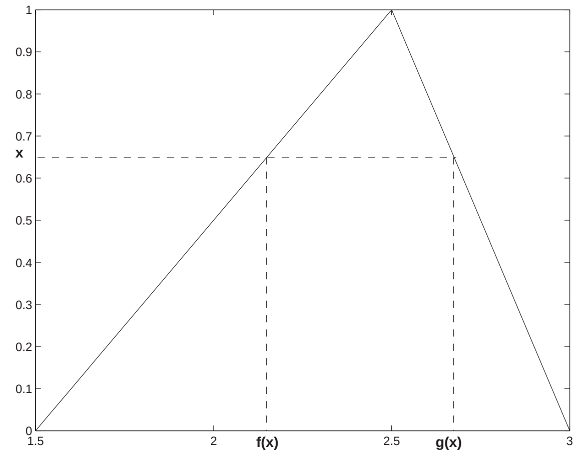

FIGURE 6. Equilibrium bidding function in Example 1.

First, let us show that there is no monotonic equilibria for this auction. By contradiction, assume that there is an increasing equilibrium bidding function. Then, $P(t_i) = \frac{t_i - 1.5}{1.5}$ and condition (i)' is trivial. We have

$$\begin{aligned}\tilde{v}(x,y) &= (1.5x+1.5)\left[1.5y+1.5-\frac{1.5x+1.5}{2}\right] \\ &= \frac{9(x+1)(2y-x+1)}{8}.\end{aligned}$$

Thus, the necessary condition (ii) is not satisfied, because $x > y$ implies

$$\int_y^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha = -\frac{3(x-y)^3}{8} < 0.$$

Thus, there is no monotonic equilibrium.

Now, we will show that there are multiple equilibria non-monotonic for this auction. Assume that there exists a bell-shaped equilibrium and that, for each x, there are two types, $f(x)$ and $g(x)$, such that $P(t_i) = x = \frac{3-g(x)+f(x)-1.5}{1.5}$, which implies that $g(x) = f(x) + 1.5(1-x)$. (See Figure 6).

Condition (i)' requires

$$\begin{aligned}& f(x)\left(\frac{f(y)+g(y)}{2} - \frac{f(x)}{2}\right) = \frac{f(x)+g(x)}{2}\left(\frac{f(y)+g(y)}{2}\right) - \frac{f^2(x)+g^2(x)}{4} \\ \Leftrightarrow\ & \frac{f(y)+g(y)}{2}\left[f(x) - \frac{f(x)+g(x)}{2}\right] = \frac{f(x)^2 - g(x)^2}{4} \\ \Leftrightarrow\ & \frac{f(y)+g(y)}{2} = \frac{f(x)+g(x)}{2}\end{aligned}$$

Then, $f(y)+g(y)$ is a constant, and we have $f(x)=k+3/4x$. Since $f(0)=1.5$, $k=1.5$. We obtain:

$$\begin{aligned}\tilde{v}(x,y) &= \frac{f(x)+g(x)}{2}\left(\frac{f(y)+g(y)}{2}\right)-\frac{f^2(x)+g^2(x)}{4}\\ &= \left(\frac{9}{4}\right)^2-\frac{(3/2+3/4x)^2+(3-3/4x)^2}{4}\\ &= \left(\frac{9}{4}\right)\left[1+\frac{x}{4}-\frac{x^2}{8}\right],\end{aligned}$$

which satisfies condition (ii) because it is increasing in x on $[0,1]$. Condition (iii) and (iv) are also satisfied, since

$$\tilde{b}(x)=\frac{1}{x}\int_0^x \tilde{v}(\alpha,\alpha)\,d\alpha=\frac{3\left(24+3x-x^2\right)}{32}$$

is increasing on $[0,1]$.■

Proof for Example 3 - Spectrum Auction

Let us assume that $t_i=\left(t_i^1,t_i^2,t_i^3\right)$ are independent and uniformly distributed on $\left[\underline{s}^1,\overline{s}^1\right]\times\left[\underline{s}^2,\overline{s}^2\right]\times\left[\underline{s}^3,\overline{s}^3\right]$, with $\underline{s}^1$, $\underline{s}^2$, $\underline{s}^3\geqslant 0$. We have

$$v^1\left(t_i^1,t_i^2,t_i^3\right)=\frac{t_i^1}{N}-t_i^2-\frac{N-1}{N}t_i^3+\frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2-\left(\underline{s}^1\right)^2+\left(\overline{s}^3\right)^2-\left(\underline{s}^3\right)^2\right]$$

Let us denote by $\overline{v}^1$ the expression in the first line above, that is,

$$\overline{v}^1\left(t_i^1,t_i^2,t_i^3\right)=\frac{t_i^1}{N}-t_i^2-\frac{N-1}{N}t_i^3.$$

The conjugation P and the c.d.f. $\tilde{P}$ are given by:

$$P\left(t_i^1,t_i^2,t_i^3\right)=\left[\Pr\left\{\left(s^1,s^2,s^3\right):\overline{v}^1\left(s^1,s^2,s^3\right)<\overline{v}^1\left(t_i^1,t_i^2,t_i^3\right)\right\}\right]^{N-1}.$$

and

$$\tilde{P}(k)=\left[\Pr\left\{\left(s^1,s^2,s^3\right):\overline{v}^1\left(s^1,s^2,s^3\right)+\frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2-\left(\underline{s}^1\right)^2+\left(\overline{s}^3\right)^2-\left(\underline{s}^3\right)^2\right]<k\right\}\right]^{N-1}.$$

We can reparameterize the problem so that

$$\tilde{P}(k)=\left[\Pr\left\{(x,y,z)\in[0,1]^3:ax+by+cz<l(k)\right\}\right]^{N-1},$$

where $a=\left(\overline{s}^1-\underline{s}^1\right)/N>0$, $b=-\left(\overline{s}^2-\underline{s}^2\right)<0$, $c=-(N-1)\left(\overline{s}^3-\underline{s}^3\right)/N<0$ and

$$l(k)=k-\frac{\underline{s}^1}{N}+\overline{s}^2+\frac{N-1}{N}\underline{s}^3-\frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2-\left(\underline{s}^1\right)^2+\left(\overline{s}^3\right)^2-\left(\underline{s}^3\right)^2\right].$$

It is elementary to obtain that, for a uniform distribution on $[0,1]^3$ and $a>0$, $b<0$, $c<0$ and $k>b+c$,

$$\Pr\left\{(x,y,z)\in[0,1]^3:ax+by+cz<l\right\}=\frac{(l-b-c)^3}{6abc}.$$

So,

$$\tilde{P}(k)=\frac{[l(k)-b-c]^{3(N-1)}}{(6abc)^{N-1}}$$

and

$$\tilde{v}(x,y) = \left\{ \tilde{P}^{-1}(x) - \frac{N-1}{2N} \left[\left(\overline{s}^1\right)^2 - \left(\underline{s}^1\right)^2 + \left(\overline{s}^3\right)^2 - \left(\underline{s}^3\right)^2 \right] \right\} y$$
$$+ E \left[\frac{\sum_{j \neq i} \left(t_j^1 + t_j^3\right)}{N} | \max_{j \neq i} P(t_j) = y \right].$$

The candidate for the equilibrium on the first-price indirect auction is

$$\tilde{b}(x) = \frac{1}{x} \int_0^x \tilde{v}(\alpha, \alpha)\, d\alpha,$$

which is differentiable, with $\tilde{b}'(x) = [\tilde{v}(x,x) - x]/x$. Then, Theorem 3 tells us that there exists an equilibrium in regular pure strategies for this auction if and only if

$$\tilde{v}(x,x) - x = \left\{ \tilde{P}^{-1}(x) - \frac{N-1}{2N} \left[\left(\overline{s}^1\right)^2 - \left(\underline{s}^1\right)^2 + \left(\overline{s}^3\right)^2 - \left(\underline{s}^3\right)^2 \right] - 1 \right\} x$$
$$+ E \left[\frac{\sum_{j \neq i} \left(t_j^1 + t_j^3\right)}{N} | \max_{j \neq i} v^1(t_j) = \tilde{P}^{-1}(x) \right] \geqslant 0$$

Depending on the values of $\underline{s}^n, \overline{s}^n$, for $n = 1, 2, 3$, the above expression can be positive or negative. If it is always positive, $\tilde{b}$ is increasing and it is the equilibrium of the indirect auction. In the other case, there is no equilibrium without ties. For instance, a sufficient condition for the existence of equilibrium in pure strategy is $\frac{\underline{s}^1}{N} - \overline{s}^2 - \overline{s}^3 \frac{N-1}{N} - 1 \geqslant 0$.■

Proof for Example 4 - Job Market

We assume that there are two players with unidimensional signals uniformly distributed on $[0,1]$ and that $m \in [0,1]$, $b \geqslant 0$. Following the method given by Theorem 3, we first obtain

$$v^1(t_i) = am + \frac{c}{2} - b(t_i - m)^2.$$

We will consider two cases.

First case: $m \leqslant 1/2$. We have

$$P(t_i) = \begin{cases} 1 - 2m + 2t_i, & \text{if } 0 \leqslant t_i < m \\ 1 - 2t_i + 2m, & \text{if } m \leqslant t_i < 2m \\ 1 - t_i, & \text{if } 2m \leqslant t_i \leqslant 1 \end{cases}$$

So,

$$\tilde{v}(x,y) = \begin{cases} am + c(1-y) - b(1-x-m)^2, & \text{if } 0 \leqslant x, y < 1 - 2m \\ am + c(1-y) - \frac{b}{4}(1-x)^2, & \text{if } 0 \leqslant y < 1 - 2m \leqslant x \leqslant 1 \\ (a+c)m - b(1-x-m)^2, & \text{if } 0 \leqslant x < 1 - 2m \leqslant y \leqslant 1 \\ (a+c)m - \frac{b}{4}(1-x)^2, & \text{if } 1 - 2m \leqslant x, y \leqslant 1 \end{cases}$$

Now, it is easy to obtain, for $x < 1 - 2m$,

$$\begin{aligned}\tilde{b}(x) &= \frac{1}{x}\int_0^x \left[am + c(1-y) - b(1-y-m)^2\right] dy, \\ &= am + c - b(1-m)^2 - x\left[\frac{c}{2} + b(m-1)\right] - \frac{b}{3}x^2,\end{aligned}$$

which is increasing if $c \leqslant \frac{2b(m+1)}{3}$. For $x > 1 - 2m$,

$$\begin{aligned}\tilde{b}(x) &= \frac{1}{x}\left\{\frac{1}{6}(1-2m)\left[6am + 3c(1+2m) - 2b\left(1-m+m^2\right)\right]\right. \\ &\quad \left. + \int_{1-2m}^x \left[(a+c)m - \frac{b}{4}(1-y)^2\right] dy\right\} \\ &\Rightarrow \tilde{b}(x) = \frac{(1-2m)\left[2c - b(1-2m)\right]}{4x} + m(a+c) - \frac{b}{4} + \frac{b\left(3x - x^2\right)}{12}\end{aligned}$$

whose derivative can be simplified to

$$\tilde{b}'(x) = -\frac{(1-2m)\left[2c - b(1-2m)\right]}{4x^2} + \frac{b(3-2x)}{12}.$$

Since the term $x^2(3-2x)$ is increasing, the bidding function will be increasing if and only if $\tilde{b}'(1-2m) \geqslant 0$, that is,

$$c \leqslant \frac{2b(1-2m)(1+m)}{3}.$$

We conclude that in the case of $m < 1/2$, there exists a pure strategy equilibrium in regular strategies if and only if

$$c \leqslant \min\left\{\frac{2b(m+1)}{3}, \frac{2b(1-2m)(1+m)}{3}\right\}.$$

Second case: $m > 1/2$. We have

$$P(t_i) = \begin{cases} t_i, & \text{if } 0 \leqslant t_i < 2m-1, \\ 1 - 2m + 2t_i, & \text{if } 2m-1 \leqslant t_i < m \\ 1 - 2t_i + 2m, & \text{if } m \leqslant t_i \leqslant 1 \end{cases}$$

and

$$\tilde{v}(x,y) = \begin{cases} am + cy - b(x-m)^2, & \text{if } 0 \leqslant x, y < 2m-1 \\ am + cy - \frac{b}{4}(1-x)^2, & \text{if } 0 \leqslant y < 2m-1 \leqslant x \leqslant 1 \\ (a+c)m - b(x-m)^2, & \text{if } 0 \leqslant x < 2m-1 \leqslant y \leqslant 1 \\ (a+c)m - \frac{b}{4}(1-x)^2, & \text{if } 2m-1 \leqslant x, y \leqslant 1 \end{cases}$$

For $x < 2m - 1$,

$$\begin{aligned}\tilde{b}(x) &= \frac{1}{x}\int_0^x \left[am + cy - b(y-m)^2\right] dy, \\ &= am - bm^2 + x\left(\frac{c}{2} + bm\right) - \frac{b}{3}x^2,\end{aligned}$$

which is increasing in the considered interval if and only if $c \geqslant \frac{2}{3}b(m-2)$.

For $x > 2m - 1$,

$$\tilde{b}\left(x\right) = \frac{-2c\left(2m-1\right) - b\left(2m-1\right)^2}{4x} + \frac{12\left(a+c\right)m - b\left(3 - 3x + x^2\right)}{12}$$

which gives

$$\tilde{b}'\left(x\right) = \frac{2c\left(2m-1\right) + b\left(2m-1\right)^2}{4x^2} + \frac{b\left(3-2x\right)}{12}.$$

Following the same procedure of the first case, $\tilde{b}'\left(x\right) \geqslant 0, \forall x \in \left[2m-1, 1\right]$ if and only if

$$c \geqslant -\frac{2b\left(2m-1\right)\left(1+m\right)}{3}.$$

We conclude that, if $m > 1/2$, there exists a pure strategy equilibrium in regular strategies if and only if

$$c \geqslant \max\left\{\frac{2}{3}b\left(m-2\right), \frac{2b\left(1-2m\right)\left(1+m\right)}{3}\right\}.$$

■

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CHAPTER 4
SINGLE OBJECT AUCTIONS WITH MULTIDIMENSIONAL BIDS

Abstract. We study some formats of single object auctions with multidimensional bids. We show that it is possible to extend the results of Athey and Levin (2001) and Ewerhart, C. and K. Fieseler (2003). We observe that such kind of auctions can have interesting properties, as that of revealing information more efficiently.

JEL Classification Numbers: C62, C72, D44, D82.

Keywords: auctions, pure strategy equilibria, non-monotonic bidding functions, tie-breaking rules

1. Introduction

In the previous chapter, we have extended equilibrium existence results in auctions with unitary demands, from unidimensional to multidimensional types. Nevertheless, we considered only unidimensional bids. It is worth wondering what can be said about multidimensional bids.

Maybe the most obvious example of auctions with multidimensional bids are multi-unit auctions. Indeed, in these auctions each bidder submits prices for each unit to be received. The models used in the previous chapters need important modifications to approach this case. This comes from the fact that our assumption of unitary demand allows us to consider only two situations for each bidder: to receive the object or not (ignoring the ties). Then, it is sufficient to consider just two utility functions, $\overline{u}_i$ and $\underline{u}_i$, one for each of these situations. When there are K objects in the auction, and the bidders have multiunit demand, we need to consider $K+1$ outcomes for each bidder: to end with $k = 0, 1, ..., K$ objects and, for the outcome of receiving k objects, to consider the utility function u_i^k. This will require the lemma of characterization and the basic principle of bidding to be rephrased in order to take into account all these new possibilities. It seems reasonable to hope that the approach will be fruitful in this case, but, of course, careful work is needed to obtain valuable results.

Nevertheless, multi-unit auctions are not the only interesting case of auctions with multidimensional bids. Indeed, many single-object auctions have multidimensional bids. For instance, in the timber auctions conducted by the U.S. Forest Service, the bidders generally are required to submit individual prices for each kind of trees to be harvested in the tract. Also, in a procurement auction for an engineering service, a buyer may request prices of the materials and of the working hours to be spent on the service. Yet another example is a procurement auction of non-homogenous products. In this case the bidders have to submit not only the price of the object but also its characteristics (quality, durability, warranty, reliability, capacity, time to delivery, etc.), that affect the utility of that product to the buyer. So, it is reasonable for the buyer to take into account such characteristics (part of the multidimensional bid) when deciding which proposal to accept.

Since the result of the auction for each bidder is only winning or losing, the seller has to specify a complete order to the multidimensional bids. We can assume that this order is given by a scoring function. For a real example, if b_i^1 and b_i^2 are the prices (bids) submitted by bidder i for the two species of trees in a tract, the U.S. Forest Service declares the winner to be the bidder with the highest expected payment $b_i^1 t_0^1 + b_i^2 t_0^2$, where t_0^1 and t_0^2 are the estimates for the quantity of each species made previously by the U.S. Forest Service. Doing so, the Forest Service is trying to maximize the expected payment that it will receive from the bidders.

In the example of procurement of non-homogenous products, the bid is (p_i, q_i), where q_i stands for the quality of the product offered and p_i, for its price. The scoring rule can be given by $U(q_i) - p_i$, where U tries to capture the value that the auctioneer attributes to quality. That is, the bid (p_i, q_i) that leads to the higher surplus $U(q_i) - p_i$ is the winning bid.

Another example is the auction for the B Band of mobile phones in Brazil. The government asked for bids that include not only the price for the license (p_i), but also the price to the consumers (c_i). The winner was the company with highest $B(p_i, c_i) = 0,6p_i - 0,4c_i$.

In this chapter, we present two models that analyze the previous examples. In section 2, we analyze a procurement auction of unit-price contracts. In section 3, we treat the case of non-homogenous products. In section 4, we conclude.

2. Unit-Price Contracts

In this subsection, we present a model of procurement auction with multidimensional bids that generalizes the model of Ewerhart and Fieseler (2003). Although our model is phrased for procurement auctions, easy adaptations can be made in order to deal with the situation analyzed by Athey and Levin (2001): the timber auctions conducted by the U.S. Forest Service.

A firm (or a government) procures a service to be executed. Its engineers estimate the amount of each input to be used to execute it: materials, working hours, etc. If there are m input factors to the service, the engineers estimate the amounts t_0^1, ..., t_0^m that will be used. We denote the vector of estimates by $t_0 = (t_0^1, ..., t_0^m)$.

The potential suppliers of the service (who will be called sellers) have private information about their technologies. That is, seller i knows the quantity of inputs t_i^1, ..., t_i^m that he will need to complete the service. Let $t_i = (t_i^1, ..., t_i^m)$.

The buyer then conducts a procurement auction, and request the potential suppliers to submit multidimensional bids $b_i = \left(b_i^1, ..., b_i^m\right) \in \mathbb{R}_+^m$. The non-negative number b_i^k is the price that seller i asks for each unit of the $k-th$ input. Based on the vector of bids, the buyer decides to buy the service from the bidder with the least cost, that is, bidder i such that $b_i \cdot t_0 = \min_j b_j \cdot t_0$, where $b_j \cdot t_0$ denotes the inner product $\sum_{k=1}^m b_j^k t_0^k$. In other words, there is a scoring function that is used by the buyer to evaluate the bids. It is just a function $B : \mathbb{R}^m \to \mathbb{R}$, given by $B(b_i) = b_i \cdot t_0$. The bid with the lowest score (expected payment) is the winner.

Once the winner is chosen, say bidder i, the buyer signs a contract with him, specifying the unit price that will be charged, $p = \left(p^1, ..., p^m\right)$. The signed contract can be a lowest-score contract (corresponding to a first-price auction), where $p \cdot t_0 = \min_j B(b_j)$, or a second-score contract, in which case $p \cdot t_0 = B_{(-i)} \equiv \min_{j \neq i} B(b_j)$.[1] In the first

[1]All pay auctions and war of attrition seem inadequate in this setting: the buyer pays something even to those who do not win. We will not consider these formats.

case, $p = b_i$ is, then, the contract signed. In the second case the bidder is free to choose p in order to met the requirement $p \cdot t_0 = B_{(-i)}$.

After the contract is signed, the service is executed, the true amount of inputs used, $t_i^1, ..., t_i^m$, is revealed and the transfer (payment) $p \cdot t_i$ is made by the buyer to the seller. We assume that the buyer can observe the efforts made by the contractor so that there is no moral hazard. It would be possible to include in the model the possibility of moral hazard, but this will turn the problem much more complex. However, the reader should note that our assumption is not so restrictive. We can understand $t_i^1, ..., t_i^m$ as the optimal level of the observable variables that are chosen by the contractor i, given the technology of observation of the buyer and the technology available to the contractor. So, the unique true restriction of the model is that, at the moment of bidding at the auction, the seller has solved all uncertainties regarding its technology, so that his choices deterministically imply the outcome of the observable.

This kind of contract is called unit-price contract and it is widely used in the real world. A natural question is "why?" Indeed, one could guess that it would be better (or at least equivalent) for the buyer to ask for an unidimensional bid: the price of the whole project. Then the buyer could contract the seller with the cheapest proposal. The intuition for the use of unit-price contracts is that this enables the contractor and the buyer to share risks. With the unidimensional bid, the risk becomes entirely on the part of the contractor.

We assume that seller i faces a cost $c(t_i)$ of providing the service. The profit of seller i is, then,

$$p \cdot t_i - c(t_i).$$

So, the problem of the seller is

$$\begin{aligned}&\max_{b_i \in \mathbb{R}_+^m} E\left\{[p \cdot t_i - c(t_i)]\, 1_{\left[t_0 \cdot b_i < B_{(-i)}\right]}\right\}\\ &= \max_{b_i \in \mathbb{R}_+^m} [p \cdot t_i - c(t_i)] \Pr\left[t_0 \cdot b_i < B_{(-i)}\right]\end{aligned}$$

Observe that this problem can be broken into two parts. The level $\beta = t_0 \cdot b_i$ determines the probability of winning the auction. Under an optimum level β, the seller is free to choose b_i (and hence, p), which maximizes $p \cdot t_i - c(t_i)$.

So, in a first-scoring auction, where $p = b_i$, this problem is

$$\max_{b_i : b_i \cdot t_0 = \beta} b_i \cdot t_i,$$

since $-c(t_i)$ is a constant. In a second-scoring auction, the problem is

$$\max_{p \in \mathbb{R}_+^m, p \cdot t_0 = B_{(-i)}} p \cdot t_i.$$

Both problems are linear with linear restrictions and they are formally equivalent. So, the maximum is obtained by a corner solution, which is very easy to obtain. For a fixed level β or for a $B_{(-i)} = \beta$, the problem is

$$\max_{p \in \mathbb{R}_+^m, p \cdot t_0 = \beta} [p \cdot (t_i - t_0) + p \cdot t_0]$$

Let $k(t_i)$ be defined as $\arg\max_k \left(t_i^k - t_0^k\right)$. Then, the solution is, clearly,

$$b(t_i, \beta) = \left(0, ..., 0, \frac{\beta}{t_0^{k(t_i)}}, 0, ..., 0\right),$$

where all entries are zero, but that in position $k(t_i)$. With this bid, the profit is

$$\frac{\beta}{t_0^{k(t_i)}} t_i^{k(t_i)} - c(t_i) = \frac{t_i^{k(t_i)}}{t_0^{k(t_i)}} \left[\beta - c(t_i) \frac{t_0^{k(t_i)}}{e_i^{k(t_i)}}\right].$$

The problem of the bidder now becomes to choose the score level β in

$$\begin{aligned}(1) \qquad & \arg\max_{\beta > 0} \frac{t_i^{k(t_i)}}{t_0^{k(t_i)}} \left[\beta - c(t_i) \frac{t_0^{k(t_i)}}{e_i^{k(t_i)}}\right] \Pr\left[\beta < B_{(-i)}\right] \\ & = \arg\max_{\beta > 0} \left[\beta - c(t_i) \frac{t_0^{k(t_i)}}{e_i^{k(t_i)}}\right] \Pr\left[\beta < B_{(-i)}\right]\end{aligned}$$

So, all types t_i that have the same $c(t_i)\, t_0^{k(t_i)} / t_i^{k(t_i)}$ must choose the same optimum bid. Then, we define the conjugation:[2]

$$P(t_i) = \Pr\left\{t_{-i} \in T_{-i} : c(t_j) \frac{t_0^{k(t_j)}}{t_j^{k(t_j)}} > c(t_i) \frac{t_0^{k(t_i)}}{t_i^{k(t_i)}}, \forall j \neq i\right\}.$$

Also, define, for all $x = P(t_i)$,

$$\tilde{c}(x) \equiv c(t_i) \frac{t_0^{k(t_i)}}{t_i^{k(t_i)}},$$

which is well defined from the definition of the conjugation. Observe that types t_i with higher $P(t_i)$ are eager to win, because they have a lesser adjusted cost $c(t_i)\, t_0^{k(t_i)} / t_i^{k(t_i)}$. Interestingly, in the unit-price auction, it is not the seller with the lower costs that wins. Indeed, the auction favors those players who have types with high difference $t_i^{k(t_i)} - t_0^{k(t_i)}$, because the term $t_0^{k(t_i)} / t_i^{k(t_i)}$ lowers the true costs to the "virtual cost" $c(t_i)\, t_0^{k(t_i)} / t_i^{k(t_i)}$. Another important observation is that, as we have said before, the players that conjugated do not need to have the same payoff. This comes from the fact that factor $t_i^{k(t_i)} / t_0^{k(t_i)}$ adjusts the "virtual payoff", $\left[\beta - c(t_i)\, t_0^{k(t_i)} / e_i^{k(t_i)}\right]$. See (1).

Turning back to the solution of the auction, in the first-score auction the problem of the seller now simplifies to

$$\max_{\beta} \left[\beta - \tilde{c}(x)\right] \Pr\left[\beta < B_{(-i)}\right].$$

By the definition of conjugation, $F_{B_{(-i)}}(\beta(x)) = 1 - x$, so that $\Pr\left[\beta(x) < B_{(-i)}\right] = x$. The first-order condition becomes

$$\beta'(x) = \frac{\beta(x) - \tilde{c}(x)}{x},$$

which, together with the initial condition $\beta(0) = \tilde{c}(0)$, gives the symmetric equilibrium:

$$\beta^1(x) = x\left[\tilde{c}(0) - \int_0^x \frac{\tilde{c}(\alpha)}{\alpha^2} d\alpha\right].$$

[2] Of course, we again work under the assumption of non-atoms in the distribution of $c(t_i)\, t_0^{k(t_i)} / t_i^{k(t_i)}$.

For a second-score auction, the problem is

$$\max_{\beta} \left[p\left(B_{(-i)}\right) \cdot t_0 - \tilde{c}\,(x)\right] \Pr\left[\beta > B_{(-i)}\right].$$

Observe that the first term does not depend on β (besides the dependence on $B_{(-i)}$) and the other is increasing in β. Since the strategy is increasing in the conjugation, then the solution is given simply by that β such that $p\left(B_{(-i)}\right) \cdot t_0 - \tilde{c}\,(x) > 0$ if and only if $\beta > B_{(-i)}$. That is,

$$\beta^2\,(x) = \tilde{c}\,(x)\,,$$

and, after the result of the auction, the contract $p\left(B_{(-i)}\right)$ is signed.

Ewerhart and Fieseler (2003) solve just the first-score auction for the particular case where there are two players, the types are unidimensional and the costs linear. The interpretation is that all sellers are assumed to have the same type (equal to one) for one of the inputs (materials). The cost is given by $c\,(t_i) = c_M + c_L t_i^L$, where c_M is the cost of materials and c_L is the cost of labor. Under these simplifications, they obtain unimodal behavior (with increasing and decreasing regions fixed). They can thus show the existence of equilibrium with the standard monotonic methods.

3. Non-Homogeneous Products

In this section we consider a procurement auction where the product to be delivered may have different characteristics. In other words, the products are non-homogenous. So, the buyer requires each seller to submit, together with a price b_i^0, a vector of characteristics, $b_i^c = \left(b_i^1, ..., b_i^m\right)$, of the product the seller plans to deliver. So, the whole bid is the vector $b_i = (b_i^0,\, b_i^1,\, ...,\, b_i^m)$.

The bids are ranked through a scoring function that we will assume to be of the form: $B\,(b_i) = V\left(b_i^1, ..., b_i^m\right) - b_i^0$, where V can be (or not) the utility that the buyer attributes to the good with characteristics $\left(b_i^1, ..., b_i^m\right)$. We assume this form of the scoring rule for the sake of simplicity.

Each seller has multidimensional private information t_i. The private information is related to the cost of producing the good, that is, the cost of delivering a good with characteristics $b_i^c = \left(b_i^1, ..., b_i^m\right)$ by a seller with type t_i is $c\,(t_i, b_i^c)$.

The payment to the seller is p_i in a first-score auction. In a second-score auction, the second highest score, $B_{(-i)} \equiv \max_{j \neq i} B\,(b_j)$, has to be matched, but the firm is free to choose the price and the characteristics to do so. That is, the firm chooses $\overline{b}_i$ such that $B\left(\overline{b}_i\right) = B_{(-i)}$.[3] If the contract $p = \left(p^0, p^c\right) = (p^0,\, p^1,\, ...,\, p^m) = \overline{b}_i$ is signed, the seller ends up with a profit of $p^0 - c\,(t_i, p^c)$ and the buyer a utility $U\,(p^c) - p^0$, where U can (or not) be equal to V. The problem of the bidder is to choose b_i in order to

$$\begin{aligned} &\max_{b_i \in \mathbb{R}_+^m} E\left\{\left[p^0 - c\,(t_i, p^c)\right] 1_{\left[B(b_i) > B_{(-i)}\right]}\right\} \\ &= \max_{b_i \in \mathbb{R}_+^m} \left[p^0 - c\,(t_i, p^c)\right] \Pr\left[B\,(b_i) > B_{(-i)}\right] \end{aligned}$$

[3] Other variations are possible. For instance, the seller may be required to meet the exact bid b_j of an opponent j such that $B\,(b_j) = B_{(-i)}$. Another possibility is to require that the price b_j^0 of this bidder is matched and to choose a vector of characteristics $\overline{b}_i^c$ that is at least as good as that of j, that is, $V\left(\overline{b}_i^c\right) > V\,(b_j)$. For the sake of simplicity, we will restrict our attention to the two rules described.

Again, the problem can be broken into two parts. For each score level β, the bidder finds the contract $p = p(t_i, \beta)$ to solve

$$h(t_i, \beta) \equiv \max_{p:B(p)=\beta} p^0 - c(t_i, p^c).$$

The second problem is to choose the β in order to maximize

$$\max_{\beta > 0} h(t_i, \beta) \Pr\left[\beta > B_{(-i)}\right].$$

Let us analyze the first problem. The condition is that $B(p) = V(p^c) - p^0 = \beta$. So, the problem can be simplified to obtain p^c that solves

$$\max_{p^c \in \mathbb{R}^m} V(p^c) - c(t_i, p^c),$$

since the choice $p^0 = V(p^c) - \beta$ ensures the restriction of the original problem. Suppose that there is a unique $p^c = p^c(t_i)$ that solves the above problem.

We obtain $h(t_i, \beta) = V(p^c(t_i)) - c(t_i, p^c(t_i)) - \beta$. The second problem is now

$$\max_{\beta > 0} \left[V(p^c(t_i)) - c(t_i, p^c(t_i)) - \beta\right] \Pr\left[\beta > B_{(-i)}\right].$$

It becomes clear that the types with the same level $V(p^c(t_i)) - c(t_i, p^c(t_i))$ will bid the same β. Let $v^1(t_i)$ be defined as $V(p^c(t_i)) - c(t_i, p^c(t_i))$. Then, it is natural to define the conjugation:

$$P(t_i) = \Pr\left\{t_{-i} \in T_{-i} : v^1(t_j) < v^1(t_i), \forall j \neq i\right\}.$$

Define $\tilde{v}^1(x)$ as $E\left[v^1(t_i) | P(t_i) = x\right]$. Observe that $\tilde{v}^1(x) = v^1(t_i)$ if $P(t_i) = x$ and that $v^1(t_i) = \tilde{v}^1 \circ P(t_i)$.

Then, the solution of the first-score auction is given by

$$\beta^1(x) = \frac{1}{x} \int_0^x \tilde{v}^1(\alpha)\, d\alpha.$$

For the second-score auction, the strategy is simply $\beta^2(x) = \tilde{v}(x)$.

4. Conclusion

Although some papers are dedicated to the analysis of auctions with multidimensional bids (Athey and Levin (2001) and Ewerhart, C. and K. Fieseler (2003)), this seems still insufficient to cope with the reasons to use such auctions.

We conjecture that the main reason for the use auctions with multidimensional bids is to obtain a better revelation of the (multidimensional) information of the bidders.

In this paper, we give some steps in the direction of a unifying approach that allows the study of auctions with multidimensional bids.

CHAPTER 5
MONOTONIC EQUILIBRIA OF AUCTIONS

Abstract. We prove that the correspondence of best-reply is non-decreasing for various auction formats, with independent types. Also, we offer an alternative method to prove the existence of asymmetric equilibrium for N bidders with independent types. This method gives new results for the existence of equilibrium in double auctions.

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Keywords: equilibrium existence in auctions, pure strategy Nash equilibrium, monotonic equilibrium.

1. Introduction

This paper provides an alternative method of proving the existence of equilibrium for monotonic asymmetrical auctions. One of its contribution is to offer a single approach that works for all kind of single object auctions normally considered in the literature. Doing that, we are able to offer an existence result that covers new cases, as that of the double auction with asymmetric utilities and independent types.

In section 2, we describe the model and present the preliminary results. In section 3, we prove our monotonic best reply result, which is the base for the equilibrium existence result given in section 4. We briefly conclude in section 5. The appendix contains the details of the proofs.

2. The Model and Preliminary Results

There are n players: $\{1, ..., n\}$. Player $i \in \{1, ..., n\}$ receives a private information, t_i, and choose an action that is a real number (i.e., she submits a bid b_i). The auctioneer compares the bids and determines who "wins" and who "looses". If player i wins, she receives $\overline{u}_i(t, b)$ and if she looses, she receives $\underline{u}_i(t, b)$, where $t = (t_i, t_{-i})$ is the profile of all signals and $b = (b_i, b_{-i})$ is the profile of bids submitted.[1]

Information

Types are independent. Because $\overline{u}_i(t, b)$ and $\underline{u}_i(t, b)$ can have any form, we may assume without loss of generality that the private signal of each player, t_i, is a real number uniformly distributed on $[0, 1]$.[2]

Bidding

[1]We consider the dependence on b instead of b_i because we want to include in our results auctions where the payoff depends on bids of the opponents, as the second price auction, for instance. This also allows us to study "exotic" auctions, i.e., auctions where the payment is an arbitrary function of all bids.

[2]Our assumption rules out just the case of atoms in the distribution of types.

After receiving the private information, each player submits a sealed proposal, that is, a bid (or offer) that is a real number. There is a reserve price $b_{\min} \geqslant 0$ and a maximum allowed bid ($b_{\max}$), which are commonly know.[3] In addition, the bidders can take a non-participation decision (-1). Then, the space of bids is $B = \{-1\} \cup [b_{\min}, b_{\max}]$.[4]

Allocation and Payoff

We suppose that each bidder sees a number that depends only on the submitted bids by the opponents and that determines the threshold of the winning and losing events. We denote such number as $b_{(-i)}$. For instance, if the auction is an one-object auction where all players are buyers, $b_{(-i)}$ is the maximal bid of the opponents, that is, $b_{(-i)} \equiv \max_{j \neq i} b_j$, provided $b_j \geqslant b_{\min}$ for at least one player $j \neq i$. If there are K objects for selling and a reserve price $b_{\min}$, then $b_{(-i)} \equiv \max \left\{ b_{\min}, b^{-i}_{(K)} \right\}$, where $b^{-i}_{(m)}$ is the m-th order statistic of $(b_1, ..., b_{i-1}, b_{i+1}, ..., b_n)$, that is, $b^{-i}_{(1)} \geqslant b^{-i}_{(2)} \geqslant ... \geqslant b^{-i}_{(N-1)}$.

In double auctions between m sellers and $k = n - m$ buyers, there are m objects for selling and the m highest bids are "winners" in the sense that they end the auction with one object, being the player a buyer or a seller. Then, for a player i (buyer or seller) $b_{(-i)} \equiv \max \left\{ b_{\min}, b^{-i}_{(m)} \right\}$.

If $b_i < b_{\min}$ (that is, player i does not participate), the payoff is 0. If $b_i > b_{(-i)}$, player i is "holder of an object" (and she has a ex-post payoff $\overline{u}_i(t, b)$ in this situation). If $b_{\min} \leqslant b_i < b_{(-i)}$, player i receives $\underline{u}_i(t, b)$.[5]

Observe that the model permits to treat buyers and sellers in the same manner. Only, if player i is a seller, she begins with a object and if $b_i < b_{(-i)}$, she sells her object. If she is a buyer, the situation $b_i < b_{(-i)}$ corresponds to maintain her previous situation: without the object. Also, the model allows for any specification of the price to be paid by the bidders.

If $b_i = b_{(-i)}$, there is a tie and a specific rule (that may include a random device and/or the requirement of a further action a_i) may determine if the player is a winner or a looser.[6] We model this by saying that the player receives $u_i^T(t, b, a)$, a value between $\overline{u}_i(t, b)$ and $\underline{u}_i(t, b)$.[7] We do not need to specify $u_i^T(t, b, a)$ for the two first results.

This setting is very general and applies to a broad class of discontinuous games. For example, $\overline{u}_i(t, b) = v_i(t) - b_i$ and $\underline{u}_i(t, b) = 0$ correspond to a first price auction with risk neutrality. If $\overline{u}_i(t, b) = v_i(t) - b_i$ and $\underline{u}_i(t, b) = -b_i$ we have the all-pay auction. If $\overline{u}_i(t, b) = v_i(t) - b_{(-i)}$ and $\underline{u}_i(t, b) = -b_i$, this is the war of attrition. As pointed out by Lizzeri and Persico (2000), we can have also combinations of these games. For

[3]If there is no reserve price (in the usual sense), let $b_{\min} = 0$.

[4]We assume a maximum permitted bid to rule out behaviors (equilibria) in which one bidder bids arbitrarily high and the others bid zero. This could happen in third price auctions, for instance.

[5]In most auctions, $\underline{u}_i$ is normalized as 0. However, in double and all-pay auctions or if there is an entry fee, this is not the case.

[6]The required action can be the submission of another bid for a Vickrey auction that will decide who will receive the object (as in Maskin and Riley (2000)) or the announcement of the type (as in Jackson et. al. (2002)). Since the only revealed information in the case of a tie is its occurrence, the action can be required together with the submission of the bid.

[7]The specification of a tie-breaking rule is important for the existence of equilibria, as shown by Jackson *et al.* (2002). With this terminology, the proposal of an "endogenous tie-breaking rule" of Simon and Zame (1990) corresponds to specify endogenously u_i^T in order to ensure the equilibrium existence.

example, $\overline{u}_i(t,b) = v_i(t) - \alpha b_i - (1-\alpha) b_{(-i)}$ and $\underline{u}_i(t,b) = 0$, with $\alpha \in (0,1)$, gives a combination of the first and second price auctions. Another possibility is the "third price auction" or an auction where the payment is a general function of the others' bids.

Notation and Definitions

Let I and N be, respectively, the sets of (strictly) increasing and nondecreasing functions from $[0,1]$ to B. We endow I and N with the norm topology of $\mathbb{L}^1([0,1], B)$. Thus, the elements of I and N are, indeed, equivalence classes of the functions that differ only in a set of zero measure and where at least one representative of each class is non-decreasing or strictly increasing. It is not difficult to see that the set N is compact.[8]

In order to avoid confusion, we will use bold letters to denote bidding functions, i.e., $\mathbf{b} = (\mathbf{b}_1, ..., \mathbf{b}_n) \in N^n$. If we fix the other's strategies, $\mathbf{b}_{-i}$, let $F_{b_{(-i)}}(b_i) \equiv \lambda^{n-1}(\{t_{-i} : \mathbf{b}_{-i}(t_{-i}) \leqslant b_i\})$ and $f_{b_{(-i)}}(\cdot)$ be its Radon-Nykodim derivative with respect to the Lebesgue measure λ^{n-1}, i.e., the density function.

A profile of functions $\mathbf{b}_{-i} \in N^{n-1}$ is regular if $F_{b_{(-i)}}(\cdot)$ is strictly increasing, that is, $\mathbf{b}_{-i}$ do not have gaps in its range. If the profile $\mathbf{b}_{-i}$ is fixed, the expected payoff of bidder i of type t_i, when bidding b_i, is:[9]

$$\begin{aligned}\Pi_i(t_i, b_i, \mathbf{b}_{-i}) \equiv \int \Big[& \overline{u}_i(t, b_i, \mathbf{b}_{-i}(t_{-i}))\, 1_{[b_i > \mathbf{b}_{(-i)}(t_{-i})]} \\ & + u_i^T(t, b_i, \mathbf{b}_{-i}(t_{-i}), a)\, 1_{[b_i = \mathbf{b}_{(-i)}(t_{-i})]} \\ & + \underline{u}_i(t, b_i, \mathbf{b}_{-i}(t_{-i}))\, 1_{[b_i < \mathbf{b}_{(-i)}(t_{-i})]} \Big] dt_{-i}\end{aligned}$$

if $b_i \in [b_{\min}, b_{\max}]$ and $\Pi_i(t_i, b_i, \mathbf{b}_{-i}) = 0$ otherwise.

We will adopt the following notation for events:

$$\begin{aligned} W_i(\beta) &= \left\{ t_{-i} \in [0,1]^{n-1} : \beta > \mathbf{b}_{(-i)}(t_{-i}) \right\}; \\ T_i(\beta) &= \left\{ t_{-i} \in [0,1]^{n-1} : \beta = \mathbf{b}_{(-i)}(t_{-i}) \right\}; \\ L_i(\beta) &= \left\{ t_{-i} \in [0,1]^{n-1} : \beta < \mathbf{b}_{(-i)}(t_{-i}) \right\}. \end{aligned}$$

When there is no possibility of confusion, we will write $\Pi_i(t_i, b_i)$ for $\Pi_i(t_i, b_i, \mathbf{b}_{-i})$ and omit the arguments, the set of integration, $[0,1]^{n-1}$ and the measure (dt_{-i}). So, we have

[8] One way to see this is to remember Helly's Theorem, that says that a sequence of nondecreasing functions has a subsequence that converges pointwise to a nondecreasing function for all the continuity points of the limit function. The pointwise convergence implies the convergence in L^1. Thus, the representative function in each equivalence class $\mathbf{b}_i^m \in N$ has a convergent subsequence that converges to $\mathbf{b}_i \in N$. Another way to see this is to prove that N is totally bounded, construting, for each $\varepsilon > 0$, a finite covering of N with sets of diameter less than ε. This can be done with step functions for a sufficiently fine grid.

[9] If the probability of bid b_i of being equal to $\mathbf{b}_{(-i)}$, conditional on t_i, is zero, the tie-breaking rule is not important and the second term in the integral may be omitted.

$$
\begin{aligned}
&\Pi_i(t_i, b_i) \\
&= \int \left\{ \overline{u}_i 1_{W_i(b_i)} + u_i^T 1_{T_i(b_i)} + \underline{u}_i \left(1 - 1_{W_i(b_i)} - 1_{T_i(b_i)}\right) \right\} \\
&= \int \left\{ u_i 1_{W_i(b_i)} + \left(u_i^T - \underline{u}_i\right) 1_{T_i(b_i)} + \underline{u}_i \right\} \\
&= \int_{W_i(b_i)} u_i + \int_{T_i(b_i)} \left(u_i^T - \underline{u}_i\right) + \int \underline{u}_i,
\end{aligned}
$$

where $u_i \equiv \overline{u}_i - \underline{u}_i$ is the net payoff.

Finally, we define the interim and the ex-ante best-reply correspondence, respectively, by

$$\Theta_i\left(t_i, \mathbf{b}_{-i}\right) \equiv \arg\max_{\beta \in B} \Pi_i\left(t_i, \beta, \mathbf{b}_{-i}\right),$$

and

$$\Gamma_i\left(\mathbf{b}_{-i}\right) \equiv \arg\max_{\mathbf{b}_i \in L^1([0,1],B)} V_i\left(\mathbf{b}_i, \mathbf{b}_{-i}\right),$$

where $V_i\left(\mathbf{b}_i, \mathbf{b}_{-i}\right) = \int \Pi_i\left(t_i, \mathbf{b}_i\left(t_i\right), \mathbf{b}_{-i}\right) dt_i$ is the ex-ante payoff.

Preliminary Result

Our results are based in the Basic Principle of Bidding, stated by Araujo, de Castro and Moreira (2004) under a more general setting. For our purposes is sufficient to known an implication of their result, namely that against regular bidding functions $\mathbf{b}_{-i}$, such that $b_* \equiv \inf\{\beta \geqslant b_{\min}\colon\ F_{b_{(-i)}}(\beta) \geqslant 0\}$, the payoff can be written in the following format:

$$\Pi_i(t_i, b_i, \mathbf{b}_{-i}) = \Pi_i(t_i, b_*) + \int_{[b_*, b_i)} \partial_{b_i} \Pi_i(t_i, \beta) d\beta.$$

where $\partial_{b_i}\Pi_i(t_i,\beta)$ exists for almost all $\beta \in [b_{\min}, b_{\max}]$ and is given by

$$
\begin{aligned}
\partial_{b_i}\Pi_i(t_i,\beta) = E\left[\partial_{b_i}\overline{u}_i\left(t_i,\beta,\cdot\right) 1_{\left[\beta > \mathbf{b}_{(-i)}\right]} + \partial_{b_i}\underline{u}_i\left(t_i,\beta,\cdot\right) 1_{\left[\beta < \mathbf{b}_{(-i)}\right]}\right] \\
+ E[u_i\left(t_i,\beta,\cdot\right) | \mathbf{b}_{(-i)} = \beta] f_{b_{(-i)}}(\beta).
\end{aligned}
$$

3. Monotonic Best Reply

The literature has focused on auctions whose properties are such that the equilibrium bidding functions are non-decreasing or, more precisely, strictly increasing. In this section, we deal with hypotheses that lead to such conclusions. Let us begin with the some definitions.

From now on, we will assume that the following assumptions hold for all $i \in I$:

(A0) *$\overline{u}_i$ and $\underline{u}_i$ are continuous on t and b.*[10]

(A1) *Ttypes are independent and uniformly distributed on* $[0,1]$.

(A2) *$u_i \equiv \overline{u}_i - \underline{u}_i$ is strictly increasing in t_i.*

[10]Since the domains are compact sets, this implies that the functions are absolutely continuous and bounded.

(A3) *For almost all* $t \leqslant t'$ *and* b, $\partial_{b_i}\overline{u}_i(b,t) \leqslant \partial_{b_i}\overline{u}_i(b,t')$.

(A4) *For almost all* $t \leqslant t'$ *and* b, $\partial_{b_i}\underline{u}_i(b,t) \leqslant \partial_{b_i}\underline{u}_i(b,t')$.

It is worth to discuss the hypotheses. (A1) and (A2) are standard in auction theory. Assumption (A3) and (A4) are weaker versions of supermodularity ($\partial^2_{b_it_i}\overline{u}_i \geqslant 0$ and $\partial^2_{b_it_i}\underline{u}_i \geqslant 0$). Roughly speaking, it means that a bidder with higher valuation is less sensible to changes in his bid. This assumption is always satisfied in the second price auction. For the first price auction, $\overline{u}_i(t_i', t_{-i}, b) = U(v(t) - b_i)$, then $\partial^2_{b_it_i}u_i = U''\cdot(-1)\cdot v'$. If $v' \geqslant 0$, as usual, then $\partial^2_{b_it_i}u_i \geqslant 0 \Leftrightarrow U'' \leqslant 0$, i.e., in this setting, supermodularity is equivalent to weak risk aversion.

Our firs result is related to Proposition 1 of Maskin and Riley (2000). Such proposition says that if there is a best reply, it is monotonic, but they proved it for first price auctions only. Theorem 1 says that there exists a monotonic best reply to regular functions and it is unique, in the sense made clear in the Remark 1, below.[11]

Theorem 1. Assume (A0)-(A4). Fix a profile $\mathbf{b}_{-i}$ of regular functions. Then, for each t_i, $\Theta_i(t_i, \mathbf{b}_{-i})$ is no empty. Moreover, if $t_i^1 < t_i^2$, $b_i^1 \in \Theta_i(t_i^1, \mathbf{b}_{-i})$, $F_{b_{(-i)}}(b_i^1) > 0$ and $b_i^2 \in \Theta_i(t_i^2, \mathbf{b}_{-i})$, then $b_i^1 \leqslant b_i^2$.[12]
Proof. See the appendix.■

Remark1. Theorem 1 has an important consequence. It implies that the best-reply to purely increasing strategies is unique. To see why, let $\mathbf{b}_{-i}$ be a profile of regular functions. The set of types t_i where $\Theta_i(t_i, \mathbf{b}_{-i})$ has diameter greater than $\varepsilon > 0$ is finite. Then, $\Theta_i(t_i, \mathbf{b}_{-i})$ is uni-valued except for a countable set of types t_i. Thus, $t_i \longmapsto \Theta_i(t_i, \mathbf{b}_{-i})$ is a non-decreasing function, well defined except in its points of discontinuity.

4. Equilibrium Existence

In this section we present a method to obtain the existence of equilibrium in monotonic auctions from Theorem 1. Remark 1 made clear that the function $t_i \longmapsto \Theta_i(t_i, \mathbf{b}_{-i})$ is in N if $\mathbf{b}_{-i} \in I^{n-1}$. Thus, for each $i = 1, ..., n$, the correspondence of best replies to $\mathbf{b}_{-i}$ is, in fact a function $\Gamma_i : I^{n-1} \to N$. Let us prove that Γ_i is continuous.

Consider a sequence $\{\mathbf{b}_{-i}^m\}_{m\in\mathsf{N}} \subset I^{n-1}$, $\mathbf{b}_{-i}^m \to \overline{\mathbf{b}}_{-i}$, $\overline{\mathbf{b}}_{-i} \in I^{n-1}$, and let $\overline{\mathbf{b}}_i \equiv \Gamma_i(\overline{\mathbf{b}}_{-i})$. Consider also $\{\mathbf{b}_i^m\}_m \subset I$, $\mathbf{b}_i^m = \Gamma_i(\mathbf{b}_{-i}^m)$. Then, $V_i(\mathbf{b}_i^m, \mathbf{b}_{-i}^m) \geqslant V_i(\mathbf{b}_i, \mathbf{b}_{-i}^m)$, $\forall \mathbf{b}_i \in N$. Since N is compact, there is a subsequence of $\mathbf{b}_i^m$ converging to a function $\mathbf{b}_i$. Since $\mathbf{b}_{-i}$ is strictly increasing, $V_i(\mathbf{b}_i, \mathbf{b}_{-i})$ is continuous at $\mathbf{b}_i$ and $\mathbf{b}_{-i}$. Then, we have $V_i(\mathbf{b}_i, \overline{\mathbf{b}}_{-i}) \geqslant V_i(\overline{\mathbf{b}}_i, \overline{\mathbf{b}}_{-i})$ by the continuity and $V_i(\overline{\mathbf{b}}_i, \overline{\mathbf{b}}_{-i}) \geqslant V_i(\mathbf{b}_i, \overline{\mathbf{b}}_{-i})$ because $\overline{\mathbf{b}}_i \equiv \Gamma_i(\overline{\mathbf{b}}_{-i})$. But then, $\mathbf{b}_i$ is also a best-reply, what we have seen to be unique. Hence, $\overline{\mathbf{b}}_i \equiv \mathbf{b}_i$ and Γ_i is continuous.

Now, for each $\overline{\mathbf{b}}_i \in I$, let $U^m(\overline{\mathbf{b}}_i)$ be the open set

$$U^m(\overline{\mathbf{b}}_i) = \{\mathbf{b}_i \in I : \|\mathbf{b}_i - \overline{\mathbf{b}}_i\|_1 < \frac{1}{m}\},$$

[11]For the definition of regular functions, see section 2, notation.

[12]Remember that, $\Theta_i(t_i, \mathbf{b}_{-i})$ is the best-reply interim correspondence, that is,

$$\Theta_i(t_i, \mathbf{b}_{-i}) \equiv \arg\max_{\beta\in[-1,M]} \Pi_i(t_i, \beta, \mathbf{b}_{-i}).$$

where $\|\cdot\|_1$ is the norm of $\mathbb{L}^1$. It is easy to see that $\cup_{\overline{\mathbf{b}}_i \in S} U^m\left(\overline{\mathbf{b}}_i\right)$ is a open cover of N. Since N is compact, it has a finite subcover. Let K^m be the finite set of indices λ such that $\cup_{\lambda\in K^m} U^m\left(\mathbf{b}_i^\lambda\right) \supset N$. Let $\left\{\psi^\lambda\right\}_{\lambda\in K^m}$ be a partition of the unity subordinate to this finite open cover. That is, $\psi^\lambda : N \to [0,1]$, $\sum_{\lambda\in K^m} \psi^\lambda\left(\mathbf{b}_i\right) = 1$ for all $\mathbf{b}_i \in N$ and $\psi^\lambda\left(\mathbf{b}_i\right) = 0$, unless $\mathbf{b}_i \in U^m\left(\mathbf{b}_i^\lambda\right)$. Define the continuous transformation:

$$\Lambda_i^m\left(\mathbf{b}_i\right) = \sum_{\lambda\in K^m} \psi^\lambda\left(\mathbf{b}_i\right)\mathbf{b}_i^\lambda.$$

Since each $\mathbf{b}_i^\lambda$ is purely increasing, so it is $\Lambda_i^m\left(\mathbf{b}_i\right)$.[13] For later use, observe that $\Lambda_i^m\left(\mathbf{b}_i\right)$ is purely increasing and it is strictly above $\mathbf{b}_i$.

It is clear that Λ_i^m:$N \to I$ is continuous. Now we define $\Lambda_{-i}^m : N^{n-1} \to I^{n-1}$ as $\Lambda_{-i}^m \equiv \times_{j\neq i}\Lambda_j^m$ and $\Lambda^m : N^n \to I^n$ as $\Lambda^m \equiv \left(\Lambda_i^m, \Lambda_{-i}^m\right)$. We can conclude that for all $m \in \mathbb{N}$, the transformation $\Gamma \circ \Lambda^m : N^n \to N^n$, defined by

$$\Gamma \circ \Lambda^m\left(\mathbf{b}\right) \equiv \left(\Gamma_i\left(\Lambda_{-i}^m\left(\mathbf{b}_{-i}\right)\right)\right)_{i=1}^n,$$

is continuous. By the Schauder-Tychonoff Theorem, $\Gamma \circ \Lambda^m$ has a fixed point, which we denote by $\mathbf{b}^m$.[14]

To understand the meaning of $\mathbf{b}_i^m$, suppose that for all $j \neq i$, player j follows $\mathbf{b}_j^m$, but player $i \neq j$ mistakenly considers that every player $j \neq i$ is using strategy $\Lambda_j^m\left(\mathbf{b}_j^m\left(\cdot\right)\right)$. Then, the best strategy for bidder i is to follow $\mathbf{b}_i^m$.

Now, since N is compact in the strong topology of $\mathbb{L}^1$, there is a convergent subsequence that converges to a bidding function $\mathbf{b}^*$. Now, we have just to prove that $\mathbf{b}^*$ is equilibrium, that is,

$$V_i\left(\mathbf{b}_i^*, \mathbf{b}_{-i}^*\right) \geqslant V_i\left(\mathbf{b}_i, \mathbf{b}_{-i}^*\right), \forall \mathbf{b}_i \in \mathbb{L}^1\left(\left[0,1\right], B\right), \forall i.$$

Equivalently, we need to show that for all i and almost all $t_i \in [0,1]$,

$$\Pi_i\left(t_i, \mathbf{b}_i^*\left(t_i\right), \mathbf{b}_{-i}^*\right) \geqslant \Pi_i\left(t_i, \beta, \mathbf{b}_{-i}^*\right), \forall \beta \in B.$$

If $\mathbf{b}^*$ does not have ties with positive probability, the event $\{t \in T : \mathbf{b}_i^*\left(t_i\right) = \mathbf{b}_{(-i)}^*\left(t_{-i}\right)$ for at least one player $i\}$ has zero measure. Then, the continuity of $\overline{u}_i$ in the event $\{t \in T : \mathbf{b}_i^*\left(t_i\right) > \mathbf{b}_{(-i)}^*\left(t_{-i}\right)\}$ and the continuity of $\underline{u}_i$ in the event $\{t \in T : \mathbf{b}_i^*\left(t_i\right) < \mathbf{b}_{(-i)}^*\left(t_{-i}\right)\}$ implies that V_i is continuous. The result now follows from the definition of $\mathbf{b}_i^m$, that says that $V_i\left(\mathbf{b}_i^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \geqslant V_i\left(\mathbf{b}_i, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right), \forall \mathbf{b}_i, \forall i$. Then, we need only to deal with the possibility of ties.

For solve this, let us define the allocation function $a^m : [0,1]^n \to \{0,1\}^n$ as

$$a^m\left(t\right) = \left(a_1^m\left(t\right), ..., a_n^m\left(t\right)\right),$$

where

$$a_i^m\left(t_i, t_{-i}\right) = \begin{cases} 1, & \text{if } \mathbf{b}_i^m\left(t_i\right) > \Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)\left(t_{-i}\right) \\ 0, & \text{if } \mathbf{b}_i^m\left(t_i\right) < \Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)\left(t_{-i}\right) \end{cases}$$

---

[13] If $t_i < t_i'$, $\mathbf{b}_i^\lambda\left(t_i\right) < \mathbf{b}_i^\lambda\left(t_i'\right)$ and $\psi^\lambda\left(\mathbf{b}_i\right)\mathbf{b}_i^\lambda\left(t_i\right) < \psi^\lambda\left(\mathbf{b}_i\right)\mathbf{b}_i^\lambda\left(t_i'\right)$ for each $\lambda \in K_i^m$ such that $\psi^\lambda\left(\mathbf{b}_i\right) > 0$. Then, $\Lambda_i^m\left(\mathbf{b}_i\right)\left(t_i\right) < \Lambda_i^m\left(\mathbf{b}_i\right)\left(t_i'\right)$, since they are finite sums of $\psi^\lambda\left(\mathbf{b}_i\right)\mathbf{b}_i^\lambda\left(t_i\right)$ and $\psi^\lambda\left(\mathbf{b}_i\right)\mathbf{b}_i^\lambda\left(t_i'\right)$, respectively.

[14] A reference for Schauder-Tychonoff Theorem is Theorem V.10.5, p. 456, of Dunford and Schwartz (1958). Observe that N is convex and compact.

Observe that if $a_i^m(t)$ is well defined for almost all t because $\Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)$ is increasing. So, $a^m(t)$ is well defined in $\mathbb{L}^1\left([0,1]^n, \{0,1\}^n\right)$ for all $m \in \mathbb{N}$. The set $\{a^m\}_{m\in\mathsf{N}}$ is compact in $\mathbb{L}^1\left([0,1]^n, \{0,1\}^n\right)$. To see this, observe that for each i, $a_i^m(t_i, t_{-i})$ is nondecreasing in t_i and nonincreasing in t_{-i}. Thus, for each i, $\{a_i^m(\cdot)\}_{m\in\mathsf{N}}$ is compact and the claim follows. So, there is a convergent subsequence (that we will denote by the same superscript), $a^m(t) \to a(t)$, where the convergence is in the $\mathbb{L}^1$ sense.

Tie-Breaking Rule: If there is a tie, all bidders are requested to reveal their types and the final allocation is given by $a(t)$, that is, bidder i receives an object if and only if $a_i(t) = 1$.

With the Tie-Breaking Rule just defined (which is an endogenous tie-breaking rule), we have equilibrium. This will follow from two lemmas. The first shows that there is no profitable deviation from bidding differently of the bid specified by $\mathbf{b}^*$. The second says that it is optimum to state the true type in case of bidding.

Lemma 1. If $\mathbf{b}_i^m(t_i) \to \mathbf{b}_i^*(t_i) = \beta$, there is no $\beta' \in B$ such that

$$\Pi_i\left(t_i, \mathbf{b}_i^*(t_i), \mathbf{b}_{-i}^*\right) < \Pi_i\left(t_i, \beta', \mathbf{b}_{-i}^*\right).$$

Proof. The idea is very simple. If there is such β', then β' would be a profitable deviation along the sequence, which is impossible because $\mathbf{b}_i^m(t_i)$ is the best reply, by definition. The details are given in the appendix.■

Lemma 2. In case of a tie, it is optimum for all bidders to reveal their true types.
Proof. The idea is to prove that they cannot make a profit by misreporting. See the appendix. ■

These two lemmas prove the following:

Theorem 2. Assume (A0)-(A4) and the tie-breaking rule just specified. Then, there exists a pure strategy non-decreasing equilibrium.

5. Possible Extensions and Conclusion

The aim of this paper is to show a method of prove that can ensure the existence of equilibrium of asymmetrical single object auctions. Possible extensions can be the analysis of multidimensional settings, as McAdams (2003) did with Athey (2001)'s method.

In any case, our work can be extended, at least with some adaptations, to multidimensional settings. For instance, we can consider an individual demanding m units as m individuals with a very good knowledge of the signals of the others. Such knowledge can be parametrized by an epsilon and the behavior as epsilon goes to zero is analyzed. When the actions are multidimensional, some function can summarize the bid, changing it to the unidimensional case again. For example, in a procurement auction, a firm bids offering both price and quality, and the buyer may have a function to combine the two dimensions in order to rank the proposals. So, the action can be taken as unidimensional again.

Another possible extension is to relax assumptions like (A3) and (A4), that are related to Single Crossing Properties, and, consequently, investigate non-monotonic equilibria. A possible method to be used is that of Araujo and Moreira (2001).

Appendix

Proof of Theorem 1.

Fix types $t_i^1 < t_i^2$, $b_i^1 \in \Theta_i\left(t_i^1, \mathbf{b}_{-i}\right)$ and $b_i^2 \in \Theta_i\left(t_i^2, \mathbf{b}_{-i}\right)$, where $\mathbf{b}_{-i}$ is a fixed regular strategy. For a contradiction, suppose that $b_i^2 < b_i^1$. Since $[0,1]^{n-1}$ and B^n are compact and u_i is (absolutely) continuous, there exists $\delta > 0$ such that $u_i\left(t_i^1, t_{-i}, b\right) + 2\delta < u_i\left(t_i^2, t_{-i}, b\right)$ for all $t_{-i} \in [0,1]^{n-1}$ and all $b \in B^n$. For a bid $\beta \in B$, define the functions

$$g^1\left(t_{-i}\right) = u_i\left(t_i^1, t_{-i}, \beta, \mathbf{b}_{-i}\left(t_{-i}\right)\right), \text{ and}$$
$$g^2\left(t_{-i}\right) = u_i\left(t_i^2, t_{-i}, \beta, \mathbf{b}_{-i}\left(t_{-i}\right)\right).$$

Then, $g^1\left(t_{-i}\right) + 2\delta < g^2\left(t_{-i}\right)$. By the positivity of conditional expectations,

$$E\left[g^2 - g^1 - 2\delta | \mathbf{b}_{(-i)} = \beta\right] \geqslant 0.$$

So, by the independence (A1), we conclude that

$$E[u_i\left(t_i^1, \cdot\right) | \mathbf{b}_{(-i)} = \beta] + \delta < E[u_i\left(t_i^1, \cdot\right) | \mathbf{b}_{(-i)} = \beta]. \tag{1}$$

By assumptions A3 and A4,

$$E\left[\partial_{b_i}\overline{u}_i\left(t_i^1, \cdot\right) 1_{\left[\beta > \mathbf{b}_{(-i)}\right]}\right] \leqslant E\left[\partial_{b_i}\overline{u}_i\left(t_i^2, \cdot\right) 1_{\left[\beta > \mathbf{b}_{(-i)}\right]}\right]. \tag{2}$$

and

$$E\left[\partial_{b_i}\underline{u}_i\left(t_i^1, \cdot\right) 1_{\left[\beta < \mathbf{b}_{(-i)}\right]}\right] \leqslant E\left[\partial_{b_i}\underline{u}_i\left(t_i^2, \cdot\right) 1_{\left[\beta < \mathbf{b}_{(-i)}\right]}\right]. \tag{3}$$

Then, (1), (2), (3) and the expression of $\partial_{b_i}\Pi_i(t_i, \beta, \mathbf{b}_{-i})$ given by (**??**) imply that for almost all β,

$$\partial_{b_i}\Pi_i(t_i^2, \beta, \mathbf{b}_{-i}) > \partial_{b_i}\Pi_i(t_i^1, \beta, \mathbf{b}_{-i}) + \delta f_{b_{(-i)}}(\beta). \tag{4}$$

Since $\mathbf{b}_{-i}$ is regular, the difference $\Pi_i(t_i^2, b_i^1, \mathbf{b}_{-i}) - \Pi_i(t_i^2, b_i^2, \mathbf{b}_{-i})$ can be write as the integral:

$$\begin{aligned}
&\int_{\left[b_i^2, b_i^1\right)} \partial_{b_i}\Pi_i(t_i^2, \beta, \mathbf{b}_{-i}) d\beta \\
&> \int_{\left[b_i^2, b_i^1\right)} \partial_{b_i}\Pi_i(t_i^1, \beta, \mathbf{b}_{-i}) d\beta + \delta \int_{\left[b_i^2, b_i^1\right)} f(\beta)\, d\beta \\
&\geqslant \delta\left[F_{b_{(-i)}}\left(b_i^1\right) - F_{b_{(-i)}}\left(b_i^2\right)\right],
\end{aligned}$$

where the first inequality comes from (4) and the second comes from the fact that $b_i^1 \in \Theta_i\left(t_i^1, \mathbf{b}_{-i}\right)$, that is,

$$\int_{\left[b_i^2, b_i^1\right)} \partial_{b_i}\Pi_i(t_i^1, \beta, \mathbf{b}_{-i}) d\beta \geqslant 0.$$

Now, since $\mathbf{b}_{-i}$ is regular and $F_{b_{(-i)}}\left(b_i^1\right) > 0$, then $\varepsilon = \delta[F_{b_{(-i)}}\left(b_i^1\right) - F_{b_{(-i)}}\left(b_i^2\right)] > 0$. So, $\Pi_i\left(t_i^2, b_i^1, \mathbf{b}_{-i}\right) > \Pi_i\left(t_i^2, b_i^2, \mathbf{b}_{-i}\right) + \varepsilon$. This contradicts the fact that $b_i^2 \in \Theta_i\left(t_i^2, \mathbf{b}_{-i}\right)$.■

Proof of Lemma 1.

From now on, the type t_i is fixed and let us denote $\mathbf{b}_i^*\left(t_i\right)$ by β^* and $\mathbf{b}_i^m\left(t_i\right)$ by β^m. By contradiction, suppose that there is β' and $\eta > 0$ such that

$$\Pi_i(t_i,\beta',\mathbf{b}_{-i})-\Pi_i\left(t_i,\beta^*,\mathbf{b}^*_{-i}\right)>\eta.$$

To fix ideas, suppose that $\beta'>\beta^*$ (the other case is completely analogous). Then, $W_i\left(\beta^*\right)\subset W_i\left(\beta'\right)$ and $L_i\left(\beta^*\right)\supset L_i\left(\beta'\right)$, where

$$W_i\left(\beta\right)=\{t_{-i}\in[0,1]^n:\beta>\mathbf{b}^*_{(-i)}\left(t_{-i}\right)\text{ or }\beta=\mathbf{b}^*_{(-i)}\left(t_{-i}\right)\text{ and }a_i\left(t\right)=1\},$$

and

$$L_i\left(\beta\right)=\{t_{-i}\in[0,1]^n:\beta<\mathbf{b}^*_{(-i)}\left(t_{-i}\right)\text{ or }\beta=\mathbf{b}^*_{(-i)}\left(t_{-i}\right)\text{ and }a_i\left(t\right)=0\}.$$

Just to see the continuity, remember that:

$$\begin{aligned}
&\Pi_i(t_i,\beta',\mathbf{b}^*_{-i})-\Pi_i\left(t_i,\beta^*,\mathbf{b}^*_{-i}\right)\\
&=\left(\int_{W_i(\beta')}\overline{u}_i+\int_{L_i(\beta')}\underline{u}_i\right)-\left(\int_{W_i(\beta^*)}\overline{u}_i+\int_{L_i(\beta^*)}\underline{u}_i\right)\\
&=\int_{W_i(\beta^*)}\left[\overline{u}_i\left(\beta',\cdot\right)-\overline{u}_i\left(\beta^*,\cdot\right)\right]+\int_{L_i(\beta')}\left[\underline{u}_i\left(\beta',\cdot\right)-\underline{u}_i\left(\beta^*,\cdot\right)\right]\\
&+\int_{W_i(\beta')\backslash W_i(\beta^*)}\overline{u}_i\left(\beta',\cdot\right)-\int_{L_i(\beta^*)\backslash L_i(\beta')}\underline{u}_i\left(\beta^*,\cdot\right)
\end{aligned}$$

Let us define the following events:

$$\begin{aligned}
W_i^m\left(\beta\right)&=\left\{t_{-i}\in[0,1]^{n-1}:\beta>\Lambda^m_{(-i)}\left(\mathbf{b}^m_{-i}\right)\left(t_{-i}\right)\right\};\\
T_i^m\left(\beta\right)&=\left\{t_{-i}\in[0,1]^{n-1}:\beta=\Lambda^m_{(-i)}\left(\mathbf{b}^m_{-i}\right)\left(t_{-i}\right)\right\};\\
L_i^m\left(\beta\right)&=\left\{t_{-i}\in[0,1]^{n-1}:\beta<\Lambda^m_{(-i)}\left(\mathbf{b}^m_{-i}\right)\left(t_{-i}\right)\right\},
\end{aligned}$$

Then, we have that (passing to a subsequence, if needed) when $m\to\infty$, $\beta^m\to\beta^*$, $1_{W_i^m(\beta^m)}\to 1_{W_i(\beta^*)}$, $1_{L_i^m(\beta^m)}\to 1_{L_i(\beta^*)}$, $\Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\left(t_{-i}\right)\to\mathbf{b}^*_{-i}\left(t_{-i}\right)$ for almost all t_{-i}. From the continuity of the $\overline{u}_i$ and $\underline{u}_i$, we have that for sufficiently high m,

$$\left|\Pi_i(t_i,\beta^m,\Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right))-\Pi_i\left(t_i,\beta^*,\mathbf{b}^*_{-i}\right)\right|<\frac{\eta}{3}$$

and

$$\left|\Pi_i(t_i,\beta',\Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right))-\Pi_i\left(t_i,\beta',\mathbf{b}^*_{-i}\right)\right|<\frac{\eta}{3}.$$

So,

$$\Pi_i(t_i,\beta',\Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right))-\Pi_i(t_i,\beta^m,\Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right))>\frac{\eta}{3}>0,$$

which is an absurd, since

$$\beta^m\in\Theta_i\left(t_i,\Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right)=\arg\max_{\beta\in B}\Pi_i\left(t_i,\beta,\Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right).$$

This conclues the proof.■

Proof of Lemma 2.

Proof. As in lemma 1, let the type t_i be fixed and denote $\mathbf{b}_i^*\left(t_i\right)$ by β^* and $\mathbf{b}_i^m\left(t_i\right)$ by β^m. Now, we have to distinguish the winning events for the announced types. So, let

$$W_i\left(\beta,\tilde{t}_i\right)=\{t_{-i}\in[0,1]^n:\beta>\mathbf{b}^*_{(-i)}\left(t_{-i}\right)\text{ or }\beta=\mathbf{b}^*_{(-i)}\left(t_{-i}\right)\text{ and }a_i\left(\tilde{t}_i,t_{-i}\right)=1\},$$

and

$$L_i\left(\beta, \tilde{t}_i\right) = \{t_{-i} \in [0,1]^n : \beta < \mathbf{b}^*_{(-i)}\left(t_{-i}\right) \text{ or } \beta = \mathbf{b}^*_{(-i)}\left(t_{-i}\right) \text{ and } a_i\left(\tilde{t}_i, t_{-i}\right) = 0\}.$$

By contradiction assume that there is a type $t_i' \neq t_i$ such that for $\eta > 0$, we have

$$= \left(\int_{W_i\left(\beta^*, t_i'\right)} \overline{u}_i + \int_{L_i\left(\beta^*, t_i'\right)} \underline{u}_i\right) - \left(\int_{W_i\left(\beta^*, t_i\right)} \overline{u}_i + \int_{L_i\left(\beta^*, t_i\right)} \underline{u}_i\right) > 10\eta. \tag{5}$$

To fix ideas, assume that $t_i' > t_i$, so that $W_i\left(\beta^*, t_i\right) \subset W_i\left(\beta^*, t_i'\right)$ and $L_i\left(\beta^*, t_i\right) \supset L_i\left(\beta^*, t_i'\right)$, because $\mathbf{b}^m_{-i}$, $\mathbf{b}^*_{-i} \in N^{n-1}$. Simplifying the expression above, we obtain:

$$\begin{aligned}
&\int_{W_i\left(\beta^*, t_i\right)} \left[\overline{u}_i\left(t_i, t_{-i}, \beta^*, \mathbf{b}^*_{-i}\left(t_{-i}\right)\right) - \overline{u}_i\left(t_i, t_{-i}, \beta^*, \mathbf{b}^*_{-i}\left(t_{-i}\right)\right)\right] \\
&+ \int_{L_i\left(\beta^*, t_i'\right)} \left[\underline{u}_i\left(t_i, t_{-i}, \beta^*, \mathbf{b}^*_{-i}\left(t_{-i}\right)\right) - \underline{u}_i\left(t_i, t_{-i}, \beta^*, \mathbf{b}^*_{-i}\left(t_{-i}\right)\right)\right] \\
&+ \int_{W_i\left(\beta^*, t_i'\right) \backslash W_i\left(\beta^*, t_i\right)} \overline{u}_i\left(\beta^*, \cdot\right) - \int_{L_i\left(\beta^*, t_i\right) \backslash L_i\left(\beta^*, t_i'\right)} \underline{u}_i\left(\beta^*, \cdot\right)
\end{aligned}$$

The first two integrals are zero. Observe that the set $W_i\left(\beta^*, t_i'\right) \backslash W_i\left(\beta^*, t_i\right)$ is exactly $L_i\left(\beta^*, t_i\right) \backslash L_i\left(\beta^*, t_i'\right)$. Let us call it A. It is easy and useful to see that

$$A \subset \left\{t_{-i} : \mathbf{b}^*_{(-i)}\left(t_{-i}\right) = \beta^*\right\}. \tag{6}$$

If we remember that $u_i \equiv \overline{u}_i - \underline{u}_i$, we can rewrite (5) as

$$\int_A u_i(t_i, t_{-i}, \beta^*, \mathbf{b}^*_{-i}\left(t_{-i}\right)) dt_{-i} > 10\eta. \tag{7}$$

Let M be an upper bound for $\max\left\{\left|u_i\right|, \left|\underline{u}_i\right|, \left|\overline{u}_i\right|\right\}$. Because $\mathbf{b}^*_{-i}\left(\cdot\right)$ is nondecreasing, there exists $\delta_1 > 0$ such that

$$\Pr\{t_{-i} : \beta^* - 2\delta_1 < \mathbf{b}^*_{(-i)}\left(t_{-i}\right) < \beta^*\} < \eta/M. \tag{8}$$

and

$$\Pr\{t_{-i} : \beta^* < \mathbf{b}^*_{(-i)}\left(t_{-i}\right) < \beta^* + 2\delta_1\} < \eta/M. \tag{9}$$

Indeed, this comes from the continuity of the probability:

$$\begin{aligned}
&\lim_{\delta \downarrow 0} \Pr\left(\{t_{-i} : \beta^* - 2\delta < \mathbf{b}^*_{(-i)}\left(t_{-i}\right) < \beta^*\}\right) \\
&= \Pr\left(\bigcap_{\delta > 0} \{t_{-i} : \beta^* - 2\delta < \mathbf{b}^*_{(-i)}\left(t_{-i}\right) < \beta^*\}\right) \\
&= 0,
\end{aligned}$$

and analogously for $\Pr\{t_{-i} : \beta^* < \mathbf{b}^*_{(-i)}\left(t_{-i}\right) < \beta^* + 2\delta_1\}$.

Since $\overline{u}_i$, $\underline{u}_i$ and u_i are absolutely continuous, there exists $\delta_2 > 0$, such that for all t_i, t_{-i}, b_i, b_{-i}, b'_{-i}, β'' and β',

$$\left|\beta'' - \beta'\right| < 4\delta_2 \Rightarrow \left|\overline{u}_i(t_i, t_{-i}, \beta'', b_{-i}) - \overline{u}_i(t_i, t_{-i}, \beta', b_{-i})\right| < \eta, \tag{10}$$

$$\left|\beta'' - \beta'\right| < 4\delta_2 \Rightarrow \left|\underline{u}_i(t_i, t_{-i}, \beta'', b_{-i}) - \underline{u}_i(t_i, t_{-i}, \beta', b_{-i})\right| < \eta. \tag{11}$$

There exists δ_3 such that

$$\max_{k\neq i}\left|b_k - b'_k\right| < 4\delta_3 \Rightarrow \left|u_i(t_i,t_{-i},b_i,b_{-i}) - u_i(t_i,t_{-i},b_i,b'_{-i})\right| < \eta. \tag{12}$$

Fix $0 < \delta < \min\{\delta_1,\delta_2,\delta_3\}$.

The functions $\Lambda_j^m\left(\mathbf{b}_j^m\right)$ are nondecreasing and converge to $\mathbf{b}_j^*$. Moreover, there exists a set $U \subset [0,1]^{n-1}$ such that $\Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right) \to \mathbf{b}_{-i}^*$ uniformly on U and such that $\Pr\left([0,1]^{n-1}\backslash U\right) < \eta/M$. So, there exists m_1 such that $m \geqslant m_1$ implies that

$$\sup_{t_{-i}\in U}\max_{j\neq i}\left|\Lambda_j^m\left(\mathbf{b}_j^m\right)(t_j) - \mathbf{b}_j^*(t_j)\right| < \delta. \tag{13}$$

Also, there is m_2 such that $m \geqslant m_2$ implies $|\beta^m - \beta^*| < \delta$. We will define

$$\begin{aligned}A^m &\equiv W_i^m\left(\mathbf{b}_i^m\left(t'_i\right)\right)\backslash W_i^m\left(\beta^m\right)\\&= \left\{t_{-i} : a^m\left(t'_i,t_{-i}\right) = 1 \text{ and } a^m\left(t_i,t_{-i}\right) = 0\right\}.\end{aligned}$$

Remember that, since $\mathbf{b}_i^*\left(t'_i\right) = \mathbf{b}_i^*\left(t_i\right) = \beta^*$,

$$\begin{aligned}A &\equiv W_i\left(\beta^*,t'_i\right)\backslash W_i\left(\beta^*,t_i\right)\\&= \left\{t_{-i} : a\left(t'_i,t_{-i}\right) = 1 \text{ and } a\left(t_i,t_{-i}\right) = 0\right\}.\end{aligned}$$

We know that $a^m \to a$ in $\mathbb{L}^1$. Finally, there is m_3 such that $m \geqslant m_3$ implies

$$\Pr\left(A^m \Delta A\right) < \frac{\eta}{M}. \tag{14}$$

From now on, fix some $m > \max\{m_1,m_2,m_3\}$.

Because $\Lambda^m\left(\mathbf{b}_{-i}^m\right)$ is increasing, we can omit the terms with u_i^T in the following difference:

$$\begin{aligned}&\Pi_i\left(t_i,\mathbf{b}_i^m\left(t'_i\right),\Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i,\beta^m,\Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right)\\&= \int_{W_i^m\left(\mathbf{b}_i^m\left(t'_i\right)\right)} \overline{u}_i(t_i,t_{-i},\mathbf{b}_i^m\left(t'_i\right),\Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}))\\&+ \int_{L_i^m\left(\mathbf{b}_i^m\left(t'_i\right)\right)} \underline{u}_i(t_i,t_{-i},\mathbf{b}_i^m\left(t'_i\right),\Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}))\\&- \int_{W_i^m(\beta^m)} \overline{u}_i(t_i,t_{-i},\beta^m,\Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}))\\&- \int_{L_i^m(\beta^m)} \underline{u}_i(t_i,t_{-i},\beta^m,\Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}))\end{aligned}$$

From now on, we will substitute the arguments $(t_i,t_{-i},\mathbf{b}_i^m\left(t'_i\right),\Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}))$ and $(t_i,t_{-i},\beta^m,\Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}))$ by $\left(\mathbf{b}_i^m\left(t'_i\right),\cdot\right)$ and $(\beta^m,\cdot)$, respectively. Since $u_i > -M$,

$\underline{u}_i > -M$ and $\overline{u}_i > -M$, we have:

$$\begin{aligned}
&\Pi_i\left(t_i, \mathbf{b}_i^m\left(t_i'\right), \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \\
&> \int_{U\cap W_i^m\left(\mathbf{b}_i^m\left(t_i'\right)\right)} \overline{u}_i(\mathbf{b}_i^m\left(t_i'\right), \cdot) + \int_{U\cap L_i^m\left(\mathbf{b}_i^m\left(t_i'\right)\right)} \underline{u}_i(\mathbf{b}_i^m\left(t_i'\right), \cdot) \\
&- \int_{U\cap W_i^m(\beta^m)} \overline{u}_i(\beta^m, \cdot) - \int_{U\cap L_i^m(\beta^m)} \underline{u}_i(\beta^m, \cdot) \\
&+ \int_{[0,1]^{n-1}\setminus U} (-M)
\end{aligned}$$

Since $\Pr\left([0,1]^{n-1}\setminus U\right) < \eta/M$, the last integral is greater than $-\eta$. Rearranging the terms,

$$\begin{aligned}
&\Pi_i\left(t_i, \mathbf{b}_i^m\left(t_i'\right), \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \\
&> -\eta + \int_{U\cap W_i^m\left(\mathbf{b}_i^m\left(t_i'\right)\right)\setminus W_i^m(\beta^m)} \left[\overline{u}_i(\mathbf{b}_i^m\left(t_i'\right), \cdot) - \underline{u}_i(\beta^m, \cdot)\right] \\
&+ \int_{U\cap W_i^m(\beta^m)} \left[\overline{u}_i(\mathbf{b}_i^m\left(t_i'\right), \cdot) - \overline{u}_i(\beta^m, \cdot)\right] \\
&+ \int_{U\cap L_i^m\left(\mathbf{b}_i^m\left(t_i'\right)\right)} \left[\underline{u}_i(\mathbf{b}_i^m\left(t_i'\right), \cdot) - \underline{u}_i(\beta^m, \cdot)\right]
\end{aligned}$$

From (13) and (10),

$$\int_{U\cap W_i^m(\beta^m)} \left[\overline{u}_i(\mathbf{b}_i^m\left(t_i'\right), \cdot) - \overline{u}_i(\beta^m, \cdot)\right] > \int_{U\cap W_i^m(\beta^m)} (-\eta) \geqslant -\eta.$$

Analogously, from (13) and (11),

$$\int_{U\cap L_i^m\left(\mathbf{b}_i^m\left(t_i'\right)\right)} \left[\underline{u}_i(\mathbf{b}_i^m\left(t_i'\right), \cdot) - \underline{u}_i(\beta^m, \cdot)\right] > \int_{U\cap L_i^m\left(\mathbf{b}_i^m\left(t_i'\right)\right)} (-\eta) \geqslant -\eta$$

Remember that $A^m \equiv W_i^m\left(\mathbf{b}_i^m\left(t_i'\right)\right)\setminus W_i^m\left(\beta^m\right)$. Thus,

$$\begin{aligned}
&\Pi_i\left(t_i, \mathbf{b}_i^m\left(t_i'\right), \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \\
&\geqslant \int_{U\cap A^m} \left[\overline{u}_i(\mathbf{b}_i^m\left(t_i'\right), \cdot) - \underline{u}_i(\beta^m, \cdot)\right] - 3\eta
\end{aligned}$$

From (10) and (11), for $t_{-i} \in U \cap A^m$,

$$\begin{aligned}
&\overline{u}_i\left(t_i, t_{-i}, \mathbf{b}_i^m\left(t_i'\right), \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i})\right) - \underline{u}_i\left(t_i, t_{-i}, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i})\right) \\
&\geqslant u_i\left(t_i, t_{-i}, \beta^*, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i})\right) - 2\eta.
\end{aligned}$$

We obtain:

$$\begin{aligned}
&\Pi_i\left(t_i, \mathbf{b}_i^m\left(t_i'\right), \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \\
&\geqslant \left(\int_{U\cap A^m} u_i(\beta^*, \cdot)\right) - 5\eta.
\end{aligned}$$

For $t_{-i} \in U \cap A^m$, we have $\max_{k\neq i}\left|\Lambda_k^m\left(\mathbf{b}_k^m\right)(t_k) - \mathbf{b}_k^*\left(t_k\right)\right| < \delta$ and $\Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}) \in [\beta^m, \beta^* + \delta) \subset (\beta^* - \delta, \beta^* + \delta)$, that is, $\left|\Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}) - \beta^*\right| < \delta$.

So, for such t_{-i}, $\left|\mathbf{b}^*_{(-i)}\left(t_{-i}\right) - \beta^*\right| < 2\delta$. The event $U \cap A^m$ is contained in the union of the following events:

$$U_- = U \cap A^m \cap \left[\beta^* - 2\delta_1 < \mathbf{b}^*_{(-i)}\left(t_{-i}\right) < \beta^*\right];$$
$$U_0 = U \cap A^m \cap \left[\mathbf{b}^*_{(-i)}\left(t_{-i}\right) = \beta^*\right];$$
$$U_+ = U \cap A^m \cap \left[\beta^* < \mathbf{b}^*_{(-i)}\left(t_{-i}\right) < \beta^* + 2\delta_1\right].$$

By (8) and (9), $\Pr U_- < \eta/M$ and $\Pr U_+ < \eta/M$. Thus, we have

$$\Pi_i\left(t_i, \mathbf{b}^m_i\left(t'_i\right), \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right) \geqslant \int_{U_0} u_i(\beta^*, \cdot) - 7\eta.$$

The argument in the function above is $\left(t_i, t_{-i}, \beta^*, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\left(t_{-i}\right)\right)$. Observe that in U_0, $\max_{k \neq i}\left|\Lambda^m_k\left(\mathbf{b}^m_k\right)\left(t_k\right) - \mathbf{b}^*_k\left(t_k\right)\right| < \delta$. So, (12) implies that

$$\begin{aligned}&\Pi_i\left(t_i, \mathbf{b}^m_i\left(t'_i\right), \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right)\\ &\geqslant \int_{U_0} u_i(t_i, t_{-i}, \beta^*, \mathbf{b}^*_{-i}\left(t_{-i}\right)) - 8\eta.\end{aligned}$$

From (6), we know that $A \subset \left[\mathbf{b}^*_{(-i)}\left(t_{-i}\right) = \beta^*\right]$. So, we have

$$\begin{aligned}\int_{U_0} u_i &= \int_{U \cap A^m \cap \left[\mathbf{b}^*_{(-i)}(t_{-i}) = \beta^*\right]} u_i\\ &= \int_{U \cap A^m \cap A} u_i + \int_{U \cap A^m \cap \left[\mathbf{b}^*_{(-i)}(t_{-i}) = \beta^*\right] \setminus A} u_i\\ &= \int_A u_i - \int_{A \setminus (U \cap A^m)} u_i + \int_{(A^m \setminus A) \cap U \cap \left[\mathbf{b}^*_{(-i)}(t_{-i}) = \beta^*\right]} u_i\\ &\geqslant \int_A u_i - \int_{A \setminus A^m} M - \int_{A^m \setminus A} M\\ &= \int_A u_i - M \Pr\left(A \Delta A^m\right)\\ &> \int_A u_i - \eta,\end{aligned}$$

where the last line comes from (14). Now we can use (7) to conclude that

$$\begin{aligned}&\Pi_i\left(t_i, \mathbf{b}^m_i\left(t'_i\right), \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right)\\ &> \int_A u_i(t_i, t_{-i}, \beta^*, \mathbf{b}^*_{-i}\left(t_{-i}\right)) dt_{-i} - 9\eta\\ &> 10\eta - 9\eta\\ &= \eta > 0.\end{aligned}$$

But the fact that $\beta^m \in \Theta_i\left(t_i, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right)$, implies

$$\Pi_i\left(t_i, \beta', \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right) \leqslant 0$$

for all β'. This contradiction concludes the proof.■

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CHAPTER 6
IS AFFILIATION A GOOD ASSUMPTION?

Abstract. We offer an alternative approach to the problem of dependence of signals (private information), especially in auctions. Since the traditional solution to this problem is given through the Milgrom and Weber's concept of affiliation, we begin by arguing that affiliation is excessively strong and does not appropriately cover the notion of positive dependence. We provide an alternative explanation for the predominant usage of English auctions in the real world, which has the additional advantage of discriminating whether open or sealed-bid first-price should be used. This is in accordance with the observation that one or other format is almost always used in specific contexts. Empirical and experimental literatures are briefly reviewed, but we are unable to find support for affiliation. Finally, we describe our method, that also applies to multidimensional and asymmetrical auctions, to the contrary of affiliation.

JEL Classification Numbers: C62, C72, D44, D82.

Keywords: affiliation, independence, auctions, monotonic equilibria, revenue rank.

1. Introduction

One of the most important contributions of the remarkable and influential paper of Paul R. Milgrom and Robert J. Weber (1982a) is the introduction of the concept of affiliation in Auction Theory.[1] Affiliation is a generalization of independence that seems very appealing, at least at first glance, in the models of auctions.[2] Under this assumption, Milgrom and Weber (1982a) obtain two main results:[3]

- the equilibrium bidding functions are monotonic (this is only a generalization of the independent types case);[4]
- the second price auction gives greater revenue than the first price auction (a truly new result that breaks the important Revenue Equivalence Theorem).

In face of these results, it is possible to cite at least three reasons for the deep influence of that paper in Auction Theory: (i) its mathematical generality and elegance; (ii) the plausibility of the hypothesis of affiliation, as explained by a clear economic intuition; (iii) the fact that it implies that English auctions yield higher revenues than first price auctions (which, in turn, can explain the dominance of English auctions in practice).

The beauty and deepness of the paper is uncontestable. However, we dispute the plausibility of affiliation, by arguing that it is hardly satisfied, despite the economic

[1] In two previous papers, Milgrom presented results that use the same concept, under the traditional statistical name "monotone likelihood ratio property" (MLRP): Milgrom (1981a, 1981b). Nevertheless, the concept is fully developed and the term affiliation first appears in Milgrom and Weber (1982a). See also Milgrom and Weber (1982b).

[2] We give a definition of affiliation in section 2.

[3] The paper has many other conclusions. We restrict ourselves to these because they seem the most important and relevant for our discussion.

[4] This is proved for the symmetric English, first price and second price auctions.

intuition. This is the discussion made in section 2. There, we present a series of conditions that correspond to the idea that the evaluations of the bidders have positive dependence, i.e., if one's value is high, then it is likely that the others' values are also high. Affiliation is the most restrictive of all such properties. Thus, an auction theorist may be aware that the intuition provided is not compelling to accept the hypothesis of affiliation.

In section 3, we argue that in the context of auctions, the assumptions of Milgrom and Weber's model are very unlikely to be met. In this discussion, we departure from comments of Milgrom and Weber (1982) and present an example which illustrates the difficulties with the standard model of unidimensional affiliated types. After the example, we prove a theorem that indicates that the difficulties are robust.

In section 4, we treat the question of the monotonicity of the equilibrium bidding function. We generalize the existence result for a condition strictly weaker than affiliation, in private value auctions. Then, we give a counterexample where a minor modification of an affiliated distribution leads to a non-monotonic equilibrium. Section 5 parallels section 4 in the analysis of the revenue predominance of open auctions.

We argue in section 6 that affiliation is hardly acceptable as a reason for the predominance of English auction in practice. We also offer an alternative explanation that has the advantage of having a clear and easy economic intuition. Moreover, the explanation can justify the specific and consistent uses of first price auctions and English auctions in certain applications. That is, it has the predictive power to distinguish where one or other auction has to be used. Affiliation, in turn, does not have such virtue.

In face of these arguments, we conclude that affiliation is not supported by theoretical means. Thus, in section 7 we turn our attention to the empirical and experimental literature and we find that there is no support for affiliation either. Section 8 examines some further arguments in defense of affiliation. Then, in section 9 we present the Conditional Independence Program, which we offer as a substitute for affiliation to approach the question of dependence between signals.[5] Section 10 is the conclusion.

2. Affiliation Is Not Synonymous of Positive Dependence

The introduction of the affiliation concept was made through a very appealing (at least at first sight) economic intuition:

> "Roughly, this [affiliation] means that a high value of one bidder's estimate makes high values of the others' estimates more likely."[6]

If there exists a $f : \mathbb{R}^N \to \mathbb{R}_+$ for the random variables X_1, ..., X_N, affiliation can be defined as the requirement that $f(x) f(y) \leqslant f(x \wedge y) f(x \vee y)$, where x and y are realizations of $(X_1, ..., X_N)$ and $x \wedge y = (\min\{x_i, y_i\})_{i=1}^N$ and $x \vee y = (\max\{x_i, y_i\})_{i=1}^N$.[7]

We illustrate the definition in Figure 1, where $N = 2$.

Affiliation requires that the product of the weights at the points (x', y') and (x, y) (where both values are high or both are low) is greater than (x, y') and (x', y) (where they are high and low, alternatively).

Thus, affiliation *seems* to be a good concept to express positive dependence.

[5]Conditional Independence Program does not consist in assuming that the signals are independent given the true value of the object, as do Wilson (1977).

[6]Milgrom and Weber (1982a), p. 1096.

[7]The definition in the general case, i.e., without density function, is given in the appendix. For the most parts of the paper, we will work with this definition.

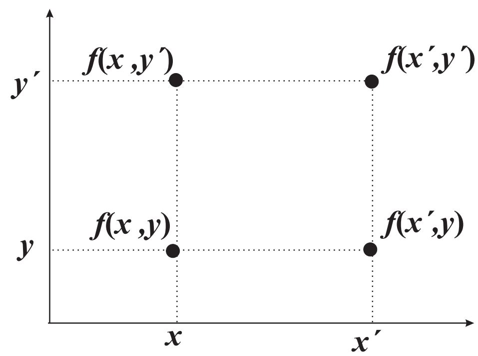

FIGURE 1. Affiliated Variables.

Definition (informal) — Positive dependence is the statistical notion that describes those situations where a high value of one of the variables makes more likely that the other variables also take on high values.

Indeed, there is a predominant view in Auction Theory that understands affiliation as a suitable synonymous of positive dependence. To see this, it is sufficient to consider some standard comments about affiliation:

> "(...) with affiliation, a higher value of the item for one bidder makes higher values for other bidders more likely."[8]
>
> "This is similar to assuming positive correlation." And, in a footnote: "It is actually a stronger assumption, but it is probably typically approximately satisfied."[9]
>
> "Intuitively, this means that a high value of one of the variables, S_j or X_i , makes it more likely that the other variables also take on high values."[10]
>
> "(...) affiliated random variables, which means roughly that high σ_i are good news in the sense of signalling high values of v."[11]
>
> "Intuitively, affiliation implies that large values for some of the components make the other components more likely to be large than small. In particular, it implies that a bidder who evaluates the object highly will expect others to evaluate the object highly too."[12]
>
> "If reservation prices are 'affiliated' (technically, pair-wise positively correlated), (...)"[13]
>
> "The notion that bidder's valuations may to some extent be correlated is captured by the concept of *affiliation*: The vector of random variables (s, x) is affiliated if, roughly, some variables' being large makes it likely

[8]Kagel, Harstad and Levin (1987), p. 1275.
[9]Paul Klemperer (2003), p. 5.
[10]Vijay Krishna and Morgan (1997).
[11]Laffont (1997), p. 9.
[12]Li, Perrigne and Vuong (2002), p. 173.
[13]Maskin and Riley (2000), p. 413.

that the other variables are large: If variables are affiliated, then they are positively correlated."[14]

From the above quotations, we can agree that the literature seems to mix two different ideas that we would like to state separately:
(1) Positive dependence is a sensible assumption; and
(2) affiliation is the suitable mathematical description of positive dependence.
We agree with the first notion, but the second is misleading, in our opinion. Hereafter we call it "Rough Identification", after the use of the word "rough" to identify affiliation and positive dependence. The problem with the Rough Identification is that it hides how strong is the affiliation hypothesis, as we will see.

A disclaim is needed, however. We are not saying that the quoted authors were wrong in their attempt to provide an intuition for affiliation. An intuition is often very helpful. Our point is only against the notion that neglects the particularity of affiliation and the difference between it and the more general concept of positive dependence.

To clarify our arguments, let us begin by remembering that, in the statistical literature, various concepts were proposed to correspond to the informal notion of positive dependence.[15] Let us restrict to the case of two real random variables, X and Y, and define some of these concepts:[16]

Property I - X and Y are positively correlated (PC) if $cov(X, Y) \geqslant 0$.

Property II - X and Y are said to be positively quadrant dependent (PQD) if $cov(f(X), g(Y)) \geqslant 0$, for all f and g non-decreasing.

Property III - The real random variables X and Y are said to be associated (As) if $cov(f(X, Y), g(X, Y)) \geqslant 0$, for all f and g non-decreasing.

Property IV - Y is said to be left-tail decreasing in X (LTD($Y|X$)) if $\Pr[Y \leqslant y|X \leqslant x]$ is non-increasing in x for all y. X and Y satisfy property IV if LTD($Y|X$) and LTD($X|Y$).

Property V - Y is said to be positively regression dependent on X (PRD($Y|X$)) if $\Pr[Y \leqslant y|X = x] = F_{Y|X}(y|x)$ is non-increasing in x for all y. X and Y satisfy property V if PRD($Y|X$) and PRD($X|Y$).

Property VI - Y is said to be Inverse Hazard Rate Decreasing in X (IHRD($Y|X$)) if $\frac{F_{Y|X}(y|x)}{f_{Y|X}(y|x)}$ is non-increasing in x for all y, when both are in their respective support and where $f_{Y|X}(y|x)$ is the p.d.f. of Y conditional to X. X and Y satisfy property VI if IHRD($Y|X$) and IHRD($X|Y$).

[14] McAfee and McMillan (1987), p. 706.
[15] See, e.g., Lehmann (1966) and Esary, Proschan and Walkup (1967).
[16] Most of the concepts can be properly generalized to multivariate distributions. All of them, but (VI), were previously defined and used.

Property VII - Y and X are said to be affiliated (or that they satisfy property (VII)) if $\frac{f_{Y|X}(y'|x)}{f_{Y|X}(y|x)}$ is non-increasing in x for all y and y' with $y > y'$, both in their respective support.[17]

Although Property VII seems asymmetric, it is indeed symmetric. To see this and that Property (VII) is equivalent to the previous definition of affiliation, observe that Property (VII) holds if $x \geqslant x'$, $y \geqslant y'$ (in the support of the distribution) imply

$$\frac{f_{Y|X}(y'|x)}{f_{Y|X}(y|x)} \leqslant \frac{f_{Y|X}(y'|x')}{f_{Y|X}(y|x')} \Leftrightarrow f_{Y|X}\left(y'|x\right) f_{Y|X}\left(y|x'\right) \leqslant f_{Y|X}\left(y'|x'\right) f_{Y|X}(y|x) .$$

Multiplying both sides on the right by $f_X(x) f_X(x')$ we obtain the affiliation inequality and now dividing by $f_Y(y) f_Y(y')$, we obtain the symmetrical condition for property (VII). Due to the fact that Property VII is equivalent to the monotonicity of $\frac{f_{Y|X}(y|x')}{f_{Y|X}(y|x)}$, it is also known as Monotone Likelihood Ratio Property (MLRP).

We have the following:

Theorem 1. The above properties are successively strong and are all different. In other words,

$$(VII) \Rightarrow (VI) \Rightarrow (V) \Rightarrow (IV) \Rightarrow (III) \Rightarrow (II) \Rightarrow (I)$$

and all implications are strict.

This theorem shows how strong is the affiliation assumption.[18] What is striking about Theorem 1 is not its novelty, but rather the fact that it seems ignored by auction theorists.[19] Indeed, it is possible that the following reaction is raised:

Reaction 1 - We already know that affiliation is a strong assumption.

Our first answer to such position is that we do not know any paper that states that affiliation is a restrictive assumption. Perry and Reny (1999), for instance, have a very interesting example concerning the failure of the linkage principle (that is a consequence of affiliation) for multi-unit auctions. They do not question the plausibility of affiliation nor that it is not likely to be valid even for single object auctions. Klemperer (2003) also criticizes affiliation, but his point is different from ours. Indeed, as one of the previous quotation shows, he seems sympathetic to the possibility that affiliation is (approximately) satisfied. His criticism of affiliation is that there are other more important issues, like asymmetry and collusion. We agree with him, although for a stronger reason: we believe that affiliation is excessively strong, as suggested by Theorem 1.

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[17] In statistic literature, affiliation is known as positively likelihood ratio dependent (PLRD). The reason for this name becomes clear from this form of the definition. In the Appendix, we give a more general definition of affiliation, that do not need the use of density functions.

[18] The reader should not be impressed for affiliation being the most particular of *seven* concepts that describes positive dependence. Yanagimoto (1972) defines more than *thirty* concepts of positive dependence and, again, affiliation is the most particular of every one.

[19] Some implications of Theorem 1 are trivial and most of them were previously established. Our main contribution is around Property VI, that we use later to prove convenient generalizations of equilibrium existence and revenue rank results, that were previously established for affiliation. We prove that Property VI is strictly weaker than affiliation and that it is sufficient for but not equivalent to Property V.

A clue that affiliation is considered important and useful comes from its practical uses. For instance, the auction theorists that were consultants to the Federal Communications Commission (FCC) in the spectral auction of 1993/4 have preferred an open format from the faith on the linkage principle, despite other relevant problems.[20] Theorem 1 shows that such use is questionable and the faith in positive dependence does not imply affiliation. In other words, the Rough Identification is hardly acceptable. At least, we should be cautious in applying it to derive conclusions for real applications.

Finally, if it is known the particularity of the affiliation assumption, this has to be reflected in clear statements in the literature and in a correspondent research effort to relax it. We were unable to find another paper with any of these two characteristics.[21]

But, another reaction can be raised as follows.

Reaction 2 - We admit that there are counterexamples for each implications above, but such counterexamples can be atypical and affiliation can be true in the majority of cases where positive correlation (property I) holds.

In this regard, we would like to mention a result of Kotz, Wang and Hung (1988). They made the following computational experiment:

(1) Generate random numbers g_{ij}, for $i = 1, ..., m$ and $j = 1, ..., n$.
(2) Define the distribution of two random variables X and Y by the probabilities $p_{ij} = \Pr(X = i, Y = j) = g_{ij} / \left[\sum_{i=1}^{m} \sum_{j=1}^{n} g_{ij}\right]$, for $i = 1, ..., m$ and $j = 1, ..., n$.[22]
(3) Test if X and Y have any positive dependence property.[23]

They repeated the above procedures 3000 times, with $m = n = 3$. As expected, they obtained property I (positive correlation) in 49.8% of the simulations. Affiliation, in turn, appears in just 1.1% of the trials. We have reproduced their experiment with $m = n = 5$ and 5000 trials and we do not find a single trial satisfying affiliation. The reason for that becomes apparent if the reader consults the general definition of affiliation given in the appendix. Affiliation requires a condition that has to be tested for *every* sublattice. This is a rather strong property to require and it is the source of its rarity.

This should suggest that affiliation is, indeed, a narrow condition and probably not a good description of the world. Of course, our statements are intentionally provocative of the following reaction:

Reaction 3 - There is no problem if an assumption is very rare or has zero measure. The set where it holds, although of zero measure, can include exactly the important cases.

[20]This is a point borrowed from Perry and Reny (1999). Of course, such preference can be also justified by other means. See McMillan (1994).

[21]Jackson and Swinkels (2003) provide an existence result under distributional strategies for all kind of dependence. Nevertheless, they are driven for the sake of obtaining a general theory, that is always desirable. They do not present a criticism of affiliation.

[22]It is clear that $p_{ij} \geq 0$ and $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1$.

[23]They tested among others, all the properties above, but VI.

We agree with the position expressed in Reaction 3. One can remember, for instance, that independence is also a very rare assumption and, in theoretical application, independence is constantly used. Nevertheless, affiliation was offered as a generalization of independence and the above remarks suggest that, out of independence, affiliation embraces just a few of the possible cases. So, why bother with affiliation if independence gives almost the same cases?

We can make the argument from another point: the intuition says that positive dependence is the set of important cases. Affiliation covers just a very tiny part of these cases. So, the assumption is unsatisfactory because most of the "important cases" are out of its scope.

Reaction 4 - It is not true that affiliation does not include important cases out of independence. One has, for instance, the conditional independence models.

Conditional independence models assumes that the signals of the bidders are independent conditional to the value of the object (see Wilson 1969, 1977).[24] Assume that $f(t_1, ..., t_N|v)$ is the p.d.f. of the signals conditional to the value and that it is C^2. It can be proved that the signals is affiliated if and only if

$$\frac{\partial^2 \log f(t_1, ..., t_N|v)}{\partial t_i \partial t_j} \geqslant 0,$$

and

$$\frac{\partial^2 \log f(t_1, ..., t_N|v)}{\partial t_i \partial v} \geqslant 0, \tag{1}$$

for all i, j.[25] Conditional independence implies only that

$$\frac{\partial^2 \log f(t_1, ..., t_N|v)}{\partial t_i \partial t_j} = 0.$$

So, conditional independence is not sufficient for affiliation. To obtain the later, one needs to assume (1) or that t_i and v are affiliated. In other words, to obtain affiliation from conditional independence, one has to assume affiliation itself. So, the justification of affiliation through conditional independence is meaningless.

This can bring to the reader's mind some usual method of obtaining affiliated signals: to assume that the signals t_i are a common value plus an individual error, that is, $t_i = z + \varepsilon_i$, where the ε_i are independent and identically distributed. This is yet not sufficient for the affiliation of t_1, ..., t_N. Indeed, let g be the p.d.f. of the ε_i, $i = 1$, ..., N. Then, t_1, ..., t_N are affiliated if and only if g is a strongly unimodal function.[26,27]

Reaction 5 - But affiliation is equivalent to the first order stochastic dominance, as shown by Milgrom, and this is a property with a reasonable economic meaning.

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[24]See also Fundenberg et. al. (2003). See section 9, where we propose a new approach, named Conditional Independence Program.

[25]This follows from a theorem of Topkis (), also cited by Milgrom and Weber (1982).

[26]The term is borrowed from Lehmann (1959). A function is strongly unimodal if $\log g$ is concave. A proof of the affirmation can be found in Lehmann (1959), p. 509, or be obtained directly from the previous discussion.

[27]It is a consequence of Theorem 2 in the next section that even if g is strongly unimodal, so that $t_1, ..., t_N$ are affiliated, we have that t_1, ..., t_N, ε_1, ..., ε_N, z are not affiliated.

Milgrom (1981a) states the following:

> "Proposition" 2. The family of densities $\{f(\cdot|x)\}$ has the strict MLRP if and only if for every no degenerate prior distribution G on Y and every y and y' in the range of Y with $y' > y$, the posterior distribution $G(\cdot|Y = y')$ dominates the posterior distribution $G(\cdot|Y = y)$ in the sense of strict first-order stochastic dominance.[28]

In other words, the proposition says that affiliation is equivalent to Property V (it is easy to see that first order stochastic dominance is, indeed, property V). This "proposition" is misstated, as Theorem 1 shows: affiliation implies first order stochastic dominance, but the converse is false. In the proof of Theorem 1, in the appendix, we give two counterexamples for this, with symmetric and strictly positive density functions on the support $[0,1]^2$.

Reaction 6 - This criticism is challenging a property that is equivalent to the Monotone Likelihood Ratio Property, which is widely used in Statistics, Reliability Theory and in many areas of economics. Then, it cannot be so important, otherwise the property would be previously rejected in those areas.

In Statistics, affiliation is known as Positive Likelihood Ratio Dependence (PLRD), the name given by Lehmann (1966) when he introduced the concept. PLRD is widely known by statisticians to be a strong property and many papers in the field do use weaker concepts (such as given by properties V, IV and III).

In Reliability Theory, affiliation is generally referred as Total Positivity of order two (TP_2), after Karlin (1968). Historical notes in Barlow and Proschan (1965) suggest why TP_2 is convenient for the theory. It is generally assumed that the failure rates of components or systems follow specific probabilistic distributions and such special distributions usually have the TP_2 property. So, it is natural to study its consequences. In contrast, in Auction Theory, the signals represent information gathered by the bidders and usually there is no reason for assuming that they have a specific distribution. Indeed, this is rarely assumed. Thus, the reason for the use of TP_2 in Reliability Theory does not apply to Auction Theory.

Finally, we stress that our criticism is of the use of affiliation in Auction Theory, as the subsequent sections emphasize. It is a work for the specialists in the other fields to analyze whether it is appropriate for their applications. It can happen that affiliation is not particularly restrictive given the setting where it is assumed. This leads to the following:

Reaction 7 - Even if affiliation is a strong assumption, it works so well with the rest of the model that we cannot abandon it.

In other words: even if we recognize the particularity of affiliation, this could be not sufficient to preclude its use if it combines in a perfect way with the other assumed conditions. However, this is not the case of Auction Theory, as we show in the next section.

[28] See also Theorem 2.1 of Milgrom (1981b).

3. And If Affiliation Were Positive Dependence?

In the previous section we show that the Rough Identification is misleading. In other words, we argued that the agreement about positive dependence of signals is very far from implying acceptance of affiliation.

Despite those arguments, in this section we make a concession: we will assume that the Rough Identification is valid. In other words, we will assume that positive dependence is synonymous of affiliation and that the estimates of the bidders have positive dependence (hence, are affiliated). We will show, nevertheless, that even such extreme assumption is not sufficient for the received theory.

The departure of our argument is a quote of Milgrom and Weber (1982), when they explain why it is interesting to analyze common value auctions:

> "(...) consider the situation in an auction for mineral rights on a tract of land where the value of the rights depends on the unknown amount of recoverable ore, its quality, its ease of recovery, and the prices that will prevail for the processed mineral. To a first approximation, the values of these mineral rights to the various bidders can be regard as equal, but bidders may have differing estimates of the common value."[29]

In the above context, it is likely that the expected value of an object is a function of various variables (quality, price, etc.), that is, private information is a multidimensional signal.[30] In other words, if we want to work just with the "estimate" that a bidder makes, then we have to accept that such estimate is built on a number of variables.

Now, the Rough Identification can be applied only to each variable taking in account by the bidders. For instance, in an auction of offshore oil and gas lease auction, we accept that the bidders' estimates of the price of the oil are affiliated, as well the bidders' estimates for the amount of oil in it.

In order to concrete the discussion, consider an auction of one object where each buyer observes private signals $X_i^1, ..., X_i^m$ that are his signals for the relevant variables.[31] With such observations, buyer i computes the value of the object as $\tau_i \equiv v_i\left(X_i^1, ..., X_i^m\right)$. The Rough Identification leads to accept that for each fixed characteristic - market share, X_i^k, for example - the variables $X_1^k, ..., X_N^k$ are affiliated. Would this imply that the τ_i are affiliated? If so, then the Rough Identification would pass from the X_i^k to the τ_i. Unfortunately, the answer is no, as the following example shows.

Example 1 - Auction of an Oil Lease

Consider the auction of a tract between two bidders. Buyer i has a private estimation of the oil quality in the field, (q_i), and the amount of recoverable ore (s_i). Estimates of these two variables are drawn from independent distributions, but q_1 and q_2 are affiliated, as well s_1 and s_2. The value of the field is calculated as $\tau_i = p\,(q_i)\,s_i - c\,(s_i)$, where $p\,(.)$ denotes the price of the oil according to its quality and $c\,(.)$ is the cost of oil extraction, depending, obviously, on the size.

[29] Milgrom and Weber (1982), p. 1093-4.

[30] Of course, this cannot be taken from a naive point of view. In many cases, multidimensional random variables can be reduced to unidimensional ones. Nevertheless, we will illustrate that such reduction is not free of consequences.

[31] In yet another example, if the object is a firm, the signals X_i^k are the evaluation of buyer i for the assets of the firm, its market share, its technology, the locked-in consumers and so on.

For simplicity, we will give numerical examples with discrete values: the size can be S (small), M (medium) or B (big). The quality can be L (low) or H (high). There are two bidders and their signals obey the distributions below that are easily checked to be affiliated.

$2\uparrow$ / $1\rightarrow$	small	medium	big
big	1/18	1/12	1/6
medium	1/12	1/6	1/12
small	1/6	1/9	1/18

Table 1 - Joint Distribution of Bidder's Estimates for the Amount of Oil

$2\uparrow$ / $1\rightarrow$	Low	High
High	1/3	1/3
Low	1/6	1/6

Table 2 - Joint Distribution for the Estimates of the Quality of the Oil

We represent the small (S) size as 1, the medium (M) size as 2 and the big (B) as 3, take a non-decreasing function of costs, $c(1) = c(2) = 1$ and $c(3) = 3$. Let $p(L) = 1$ and $p(H) = 3$. With this, the possible values of the hole are $\tau_i = 0, 1$ or 3. Such specification leads us to the following distribution of types τ_i :

$\tau_2 \uparrow / \tau_1 \rightarrow$	0	1	3
3	5/36	5/36	1/6
1	23/216	5/36	7/72
0	2/27	5/72	5/72

Table 3 - Distribution of Values

It is easy to see that this distribution is not affiliated: for example, using the four probabilities in the right down corner, we have $\frac{5}{36} \cdot \frac{5}{72} > \frac{7}{72} \cdot \frac{5}{72}$.

We can go further. Suppose that the small size (S) is $\frac{1}{42}$, the medium size (M) is $\frac{5}{6}$ and the big size (B) is $\frac{6}{7}$. We take an increasing function of costs, $c\left(\frac{1}{42}\right) = 0$, $c\left(\frac{5}{6}\right) = \frac{29}{6}$ and $c\left(\frac{6}{7}\right) = 5$. Let $p(L) = 6$ and $p(H) = 7$. With these values, the possible values of the petroleum field are $\tau_i = \frac{1}{7}, \frac{1}{6}$ or 1 and the distribution showed in Table 3 remains the same, just substituting 0, 1 and 3 by $\frac{1}{7}, \frac{1}{6}$ and 1. Then, if bidder 1 has common value utility $u_1 = \frac{\tau_1+\tau_2}{2}$, as usual, the expected utility turns out to be non-monotonic. Indeed, $E\left[\frac{\tau_1+\tau_2}{2}|\tau_1 = \frac{1}{7}\right] = 0.3332 > 0.3310 = E\left[\frac{\tau_1+\tau_2}{2}|\tau_1 = \frac{1}{6}\right]$. ■

The above example suggests that there are serious problems when affiliation has to be applied to multidimensional settings.[32] We go further, by proving that one of the most important property of affiliation *has to be* restricted to unidimensional settings.

The result is the following:

Theorem 2. Suppose that the real random variables X, Y and $Z = v(X,Y)$ are affiliated, where $v : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a non-decreasing function. Then, there are not real numbers $x < x'$ and $y < y'$ in the convex support of X and Y such that v is strictly increasing in both arguments in $[x, x'] \times [y, y']$.[33]

[32] Previous examples of these problems were provided by Reny and Perry (1999) and Reny and Zamir (2002).

[33] In other words, v is a function of just one of its two variables.

To understand the consequences of Theorem 2, let us return to the setting previously analyzed. Suppose that variables X_i^k, for $k = 1, ..., m$ and $i = 1, ..., n$, are affiliated and have convex support. Then, in order to the signals $\tau_i = v_i\left(X_i^1, ..., X_i^m\right)$ be affiliated for $i = 1, ..., n$, it is necessary that v_i depends on just one of the signals X_i^k!

We conclude that even under the assumption that the Rough Identification is valid, this is not sufficient to accept the use of affiliation.[34]

Thus, the justification for affiliation (or positive correlation) is based on a case where it hardly holds. Consider the following reasoning, which closely relates the use of many variables with the need of positive correlation (and, hence, affiliation, under the Rough Identification):

> "What are we to use in place of the independence assumption? When the bidder's costs or valuation depend on some common random factors, so that all the bidders are estimating the same variables, their estimates will be positively correlated even if their estimation *errors* are independent. Positive correlation has been especially prominent in models of auctions for oil and gas drilling rights, where the rights being acquired are, to a first approximation, of equal value to each of the bidders, and the main uncertainties concern such common factors as the quantities of recoverable hydrocarbons, the cost of recovery, the costs of transporting the product to market (perhaps through as yet undeveloped pipelines over the Arctic Slope), future world energy prices, and so on. The common uncertainties found in these auctions also play a large role in the sale of items like wine or art which are purchased at least partly for their savings or investment value, as the parties estimate what it would cost to purchase the same vintage in the future or what the eventual resale price for the painting will be. So there is good reason to believe that positive correlations among value estimates will often be present.
>
> The actual equilibrium analysis of auctions relies on a stronger notion than positive correlation. The appropriate concept, known as affiliation, was introduced by Milgrom and Weber (1982)."[35]

We want to emphasize that this problem is related only to affiliation and not with the notion of positive dependence. Indeed, our proof is heavily based in the use of sublattice conditioning, as required by the general definition of affiliation (see the Appendix). Under association (property III), we have a result more favorable. Consider the following:

Definition. A set of N functions $g_1, ..., g_N$, each defined on $\mathbb{R}^{Nm} \to \mathbb{R}$ is said to be m-concordant if the functions are monotone in each of the Nm arguments and the direction of monotonicity is the same for each block of m arguments, $jm+1, ..., jm+m$, where $0 \leqslant j \leqslant m-1$.

Jogdeo (1977) proves the following:

[34]A qualification is need, however. The theorem does not impede that the τ_1, ..., τ_N are affiliated. Only, the standard method to prove affiliation does not work if v depends on more than one argument. On the absence of other results, assuming that τ_1, ..., τ_N were affiliated is to impose an unjustified and heroic assumption. Even if we do that, however, we do not solve all the problems, as we discuss below.

[35]Milgrom (1989), p. 13-4.

Proposition 1. Assume that $g_1, ..., g_N$ are m-concordant functions defined for Nm-tuples, and that $X^1, ..., X^m$ are independent N-dimensional random variables such that the components of $X^k = \left(X_1^k, ..., X_N^k\right)$ are associated for each k. Let $X = (X_1^1, ..., X_1^m, ..., X_N^1, ..., X_N^m)$. Then, the random variables $V_i = g_i\left(X\right), i = 1, ..., N$ are associated (As).

Then, if each g_i gives the value of an object as a function of all signals received by all players, we just require that each signal contributes monotonically for the value, and in the same manner for all players. In Proposition 1, each X^k, for $k = 1, ..., m$, accounts for one kind of information (for example, the price of the oil, in the previous example). The vector $X^k = \left(X_1^k, ..., X_N^k\right)$ represents the valuations of the players of the information k. Proposition 1 requires that X_i^k are associated. Under its assumptions, it implies that the values of the object for the bidders are associated.

It is also worth to note that if the concept is independence then the difficult also disappears, since although τ_i is not independent of the X_i^k $(k = 1, ..., m)$ it is independent of the τ_j, for $j \neq i$.

Let us summarize our findings. We began this section by assuming that the Rough Identification is true. Theorem 2 teaches us that affiliation would pass to the value of the object if we take care of just one real-valued estimate. A situation where it occurs is in auctions of non-durable goods, as fish. But even in fish auctions, the bidders shall not take into account altogether, the color, the size and the smell of the fish. They are permitted to consider just one of these variables.

Nevertheless, our faith in the theory could lead us to expect that, under the assumptions of 1) Rough Identification and 2) bidders' estimates are just one real variable, the theory could finally be applied. Nevertheless, this is yet not possible.

To describe the remaining difficulties, let us quote again Milgrom and Weber (1982):

> "To represent a bidder's information by a single real-valued signal is to make two substantive assumptions. Not only must his signal be a sufficient statistic for all of the information he possesses concerning the value of the object to him, it must also adequately summarize his information concerning the signals received by the other bidders. The derivation of such a statistic from several separate pieces of information is in general a difficult task. It is in the light of these difficulties that we choose to view each X_i as a "value estimate," which may be correlated with the "estimates" of others but is the only piece of information available to bidder i."[36]

Now, the reasonable positive dependence assumption and the Rough Identification, that were previously justified for the estimates of the bidders, have to apply, indeed, for sufficient statistics. But it is not clear why their appeal would pass for the sufficient statistics.

The problems do not stop here, however. Even if the sufficient statistics are affiliated, it is not clear that the other assumptions of the model can be sustained, as that the utility has to be monotonic with the sufficient statistics. This is reasonable if the signal is a value, but why the value is monotonic with the signal if this is just a sufficient statistic?

[36] Milgrom and Weber (1982), footnote 14, p. 1097.

This discussion shows that we need a theory to derive and characterize the sufficient statistics from the pieces of information possessed by the bidders. In the absence of such a theory, we cannot be sure of what we are really assuming when we think in sufficient statistics as values. Of course, it is possible that in many settings the assumption is suitable, that is, the sufficient statistic is the value itself (for instance, in private values). But we need to classify in which situations this is true.

4. Monotonic Equilibrium

We hope that the discussion of the previous section has made clear that affiliation is hardly satisfied (out of the independency). Nevertheless, another reaction can be raised as follows:

Reaction 8 - We assume affiliation because it entails the mathematical conditions that enable us to prove interesting facts. Even if it is not satisfied, it is likely that its consequences remain true in a great proportion of cases.

This reaction calls for a investigation of the robustness of the main consequences of affiliation: the existence of monotonic equilibrium and the greater revenue of English/second price auctions. In this section, we address the first question, while the second is treated in section 5.

Let us consider the private value, first price auction, where the bidders are risk neutrals and receive a unidimensional signal, but the distribution of the signals is not affiliated. We have the following:

Theorem 3. Consider a symmetric first price, private value auction between 2 bidders. Suppose also that bidders are risk-neutrals and there is a joint p.d.f., $f : [0,1]^2 \to \mathbb{R}_+$.[37] Then, if the distribution satisfies Property VI, there is a symmetric monotonic equilibrium.

Theorem 3 strictly generalizes the equilibrium existence, but we are not saying that this generalization is important, since Property VI is too close of affiliation.[38] Nevertheless, it highlights the fact that the existence result is bounded to be valid under Property VI. We conjecture that are counterexamples for the existence of equilibrium under Property V.

The next result shows that minor perturbations in affiliation break the conclusion that the bidding function is monotonic, even in unidimensional settings.

Proposition 2. Given $\eta > 0$ and a affiliated density function, that is, a function $f : [0,1]^n \to [0,1]$ that is the joint density function of n affiliated variables, there is a density function f^η such that $\|f^\eta - f\| < \eta$ and there is a non-monotonic equilibrium under f^η, where $\|\cdot\|$ represents the L^1 norm.

Proof. See the appendix.

If we want a norm more restrictive than L^1 norm, such as a C^k norm, $\|\cdot\|_k$, it is still possible to show that there is a f^η such that $\|f^\eta - f\| < \eta$ and there is no monotonic

[37] It is not necessary that the domain of f be $[0,1]^2$. We assume this for convenience.

[38] In some sense, its proof is already contained in Milgrom and Weber (1982a).

equilibrium under f^{η}. This shows that a small perturbation in affiliation leads to the failure of the property of existence of a monotonic equilibrium bidding function.

5. Revenue Rank

In this section we derive an expression for the difference in revenue from second and first price symmetric auctions. Indeed, we are interested in answering whether the result on the rank of the auctions also holds for a concept weaker than affiliation.

Consider the auction of an indivisible object with 2 risk neutrals bidders and with private values. Let $f(x)$ be a symmetric probability density function. We have the following:

Theorem 4. If f satisfies Property VI, then the second price auction gives greater revenue than the first price auction. More than that, the revenue difference is given by

$$\int_{[\underline{t}_i,\bar{t}_i]} \int_{[\underline{t}_i,x]} b'(y) \left[\frac{F_Y(y|y)}{f_Y(y|y)} - \frac{F_Y(y|x)}{f_Y(y|x)} \right] f_Y(y|x)\, dy \cdot f(x)\, dx$$

where $b(\cdot)$ is the first price equilibrium bidding function, or by

$$\int_{[\underline{t}_i,\bar{t}_i]} \int_{[\underline{t}_i,x]} \left[\int_{[\underline{t}_i,y]} L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F_Y(y|x)}{f_Y(y|x)} \cdot \frac{f_Y(y|y)}{F_Y(y|y)} \right] \cdot f_Y(y|x)\, dy \cdot f(x)\, dx,$$

where $L(\alpha|t) = \exp\left[-\int_{\alpha}^{t} \frac{f_Y(s|s)}{F_Y(s|s)} ds \right]$.

From Theorem 4 we learn that the revenue predominance of the English (second price) auction over the first price auction seems to be strongly dependent of the condition required for Property VI. In other words, one can conjecture that the revenue rank can be broken for the other properties that express positive dependence. We conjecture that this is possible even for Property V, but we do not have an example.[39]

6. Alternative Explanation for the Dominance of English Auctions

In the last two sections we saw that some provisions of affiliation are not robust. It is still possible, however, to raise the following argument:

Reaction 9 - Affiliation is the best explanation for the huge dominance of English auctions in the real world. So, this is an uncontestable fact that supports affiliation.

Indeed, another quote illustrates the common understanding in the theory:

> "(...) the English (open, ascending) auctions are much more common than any other form. Here we relax the independence assumption of the standard model in order to explain the prevalence of English auctions and of certain other auctions practices."[40]

Let us discuss such explanation. First we need to clarify all the reasoning, with its implicit assumptions and logical conclusions.

[39] Of course, the example should exhibit an equilibrium. Then, this has to be verified directly, since Theorem 3 does ensure the equilibrium existence under Property V.

[40] Milgrom (1989), p. 13.

(1) Under symmetry, affiliation implies that open auctions raise higher revenue;
(2) Types are affiliated;
(3) The players are symmetric;
(4) Bidders follow their (symmetric) equilibrium strategies;
(5) Open auctions leads to a higher expected revenue;
(6) The sellers learn that the open auctions produces higher expected revenue;
(7) Sellers seek expected revenue maximization;
(8) The choice of the auction's format is a rational decision of the sellers;
(9) The open (or English) format is used.[41]

The claim 1 is the theoretical result of Milgrom and Weber. Claim 5 is the logical consequence of 1, 2, 3 and 4. Now, with 5, 6, 7 and 8, one can conclude 9, the real world fact that we want to explain through affiliation. Note that 2, 3, 4, 6, 7 and 8 are all extra assumptions required to the conclusion.

It is easy to see how weak this reasoning is. We will analyze the argument backward.

It is worth to say that the whole attempt in explaining of the predominance of English format through an economic reason can be misleading. The origin of the word "auction" is "augere", that means to rise (the price), as in an ascending auction. It is clear that this form of auction is the oldest, for a good reason: in the antiquity, to write and submit a sealed bid was not a trivial task. Since this is the first used form of auctions, the predominance can be justified appealing just for inertia or cultural preferences. Of course, an economist would prefer an economic explanation. But there is no problem in translate the cultural inertia in terms of costs of change. In other words, point 8 above can be no valid.

Another circumstance that challenges point 8 is the influence of bidders in the choice of the auction's format. The bidders may prefer English auction by a number of reasons as because it may facilitate collusion. Also, open auctions are simpler and less computationally demanding than a sealed-bid first-price auction.

Concerning point 7, it is far from obvious that sellers pursue (only) the revenue maximization. Concerns as efficiency can be important in some cases. However, we do not insist in a criticism for this point.

Point 6 is hard to believe. The empiricists and experimentalists could not established it yet, even in laboratories and with the help of the theory.[42] Thus, it is difficult to accept that lay people, with no scientific expertise could grasp such conclusion, with no help of an intuition. Of course, one can appeal to an evolutionary argument to sustain that such conclusion is reached. The argument goes as follows: those that do not reach to the correct format are ruled out of the market. This is problematic. Since the open format is the first to be used in the history, then the "mutation" has to be from the open to the first-price sealed bid auction. According to the theory, this "mutation" is non-profitable. So, not only we are unable to justify the predominance of open auctions through an evolutionary argument, but also we reach the absurd conclusion that the sealed-bid format will never be used.

Point 5 is the heart of the contribution of affiliation to the argument. Our criticism of this point is that we can reach it with a better condition that we describe below.

[41]There is an alternative reasoning based in ideas of evolutionary selection that leads to the same conclusion. We comment it below.

[42]We discuss their results in section 7.

In some experimental studies, the players do not follow equilibrium strategies. This puts in doubt the plausibility of assumption 4.[43] Asymmetries are very common in practice, which puts in doubt assumption 3. We think that the previous sections were sufficient to comment assumption 2.

In order to provide an alternative explanation, we will try to reach the conclusion in 5, without the use of 1 and 2, where affiliation plays a role. In other words, we will offer an alternative reason for the consequence in 5, which is simpler and more natural than affiliation. More than that, it has the advantage of being easily grasped by intuition, which facilitates point 6.

This is an important advantage, because the hope that the sellers would learn the predominance of a format is very problematic. As we will discuss in the next section, the experimentalists and empiricists are still trying to establish such ranks. Why the learning process of sellers would be better? Another problem is that if the learning cannot be summarized in a clear intuition, then it is likely to take much more time to the conclusion be accepted. Unfortunately, affiliation does not provide such intuition. The revenue rank, as we have shown, depends on very technical statistical concepts, developed only on the second half of the twenty century. Thus, it is hard to accept that the learning of the affiliation consequence has spread out the world and explained the preference for open auctions. This is especially difficult to believe once one realizes that before Milgrom and Weber's paper, nobody has provide any intuition of this consequence, to the best of our knowledge.[44] Also, it is hard to accept that a lay person could grasp the point. In sum, affiliation does not perform well in the task of providing the explanation.

6.1. **Alternative explanation of the revenue rank.** Our alternative theory for the revenue priority of ascending auction is based on the easy concept of regret.

Suppose that someone wins a first price auction with the bid b_i, and where the second highest bid is $b_{(-i)}$. The winner becomes happy with the profit $t_i - b_i$, but he/she can think that she might win even more, if his/her bid were $b_{(-i)} + \varepsilon$, for a very small $\varepsilon > 0$. So, he/she becomes sad with the money left on the table, $b_i - b_{(-i)}$, that is, the money that could be saved in the auction. In other words, the money left on the table can be wrongly perceived as a loss.

The regret is very common and spread out, not only among individuals, but also among firms. If the difference between the two highest bids is high, the winner can be perceived as a fool. If he is a director of a firm, he has to give explanations to the board.[45]

We emphasize that the "regret" is very simple and easy to understand for everyone that works with auction. The interesting thing is the consequences of it: the open auction performs better even in the circumstances where it is expected by the receive theory

[43]See Kagel (1995).

[44]As Klemperer (2002) argues, the explanation through "winner's curse" seems of little help. It is valid only for common value auctions, while the revenue predominance implied by affiliation remains valid under private values. Moreover, the "winner's curse" effect occurs also with independence.

[45]In privatization of Banespa, a bank in Sao Paulo, Brazil, the highest bid was about US$ 7 billion, made by Santander (a Spanish bank), and the second highest bid was US$ 1 billion, made by Bradesco (another Brazilian bank). Through some days, the newspapers and magazines published declarations of the executives of Santander justifying why they bid so high. Of course, we do not have access to the explanations given to the council.

that they lead to the same revenue (that is, under the hypotheses of the equivalence theorem).

Consider a symmetric first price auction, with risk neutral bidders that give an weight $a > 0$ to the loss, so if a bidder wins, he receives $v(t) - b_i - a\left(b_i - b_{(-i)}\right) = v(t) + ab_{(-i)} - b_i(1+a)$ and receives 0 otherwise. The Payoff Characterization Lemma of Aloisio Araujo, Luciano de Castro and Humberto Moreira (2004) implies that in equilibrium,

$$E[v(t) + ab_{(-i)} - b_i(1+a)|t_i, \mathbf{b}_{(-i)} = b_i]f_{b_{(-i)}}(b_i|t_i) = (1+a)F_{b_{(-i)}}(b_i|t_i),$$

which simplifies to

$$\mathbf{b}'(t_i) = E[v(t) - b_i|t_i, t_{(-i)} = t_i]\frac{f_{t_{(-i)}}(t_i|t_i)}{(1+a)F_{t_{(-i)}}(t_i|t_i)}.$$

As in Milgrom and Weber, define $v^E(x,y) = E[v(t)|t_i = x, t_{(-i)} = y]$, $\widetilde{v}(t) = v^E(t,t)$, $L(\alpha|t) = \exp\left[-\int_\alpha^t \frac{f_{t_{(-i)}}(s|s)}{F_{t_{(-i)}}(s|s)}ds\right]$ and

$$\begin{aligned} L^{rg}(\alpha|t) &= \exp\left[-\frac{1}{(1+a)}\int_\alpha^t \frac{f_{t_{(-i)}}(s|s)}{F_{t_{(-i)}}(s|s)}ds\right] \\ &= \left\{\exp\left[-\int_\alpha^t \frac{f_{t_{(-i)}}(s|s)}{F_{t_{(-i)}}(s|s)}ds\right]\right\}^{\frac{1}{(1+a)}} \\ &= L(\alpha|t)^{\frac{1}{(1+a)}} > L(\alpha|t). \end{aligned}$$

Thus, the bidding function with regret is

$$\begin{aligned} \mathbf{b}^{rg}(t) &= \int_{[\underline{t}_i,t]} v^E(\alpha,\alpha)\,dL^{pl}(\alpha|t) = v^E(t,t) - \int_{[\underline{t}_i,t]} L^{pl}(\alpha|t)\,d\widetilde{v}(\alpha) \\ &< \mathbf{b}(t), \end{aligned}$$

where $\mathbf{b}(t)$ is the equilibrium bidding function of the standard auction. Thus, from the revenue equivalence theorem, the revenue in the open auction is greater than that of the first price auction.

Thus, we proved the following:

Theorem 5. Under regret, symmetric common value setting with risk neutrality and independent types, the second price auction leads to higher revenue than the first price auction.[46]

The reader may note that we do not need the assumption of independence for the result. If affiliation is added, the rank is only accentuated.

We can list some advantages of regret over affiliation as an explanation for English revenue dominance. First, regret is simple to understand and provides a clear intuition for the revenue superiority in open formats: in these auctions there is no regret. So, in any auction where the regret is possible, the bidders will try to bidder less to reduce

[46]As the proof makes clear, the result remains true if we assume affiliation instead of independence. This would not make clear our point: regret is another kind of relaxation of the assumptions of the Revenue Equivalence Theorem.

their posterior regret. With regret, the auctioneer may receive ready complaints for the money left on the table and this can be a further reason for avoiding sealed-bid formats.[47]

Another advantage of this explanation is that it allows to understand why some auctions are always conducted via the same form (a feature that affiliation cannot capture). Indeed, if the auctions are conducted in such way that at the end, the bidders do not know the bids of the opponents, then there is no regret effect. In these cases, other effects, like asymmetries and risk aversion, may lead to the situation where first price auction performs better. For an example, consider the job markets auctions. The employers do not know the bid of the others. So, there is no regret effect and we can verify that these markets are almost always conducted via first price auctions.[48]

A final observation is worth. Many departures from the assumptions of the Revenue Equivalence Theorem (RET) have been studied. Its main assumptions are: (1) independence; (2) risk neutrality; (3) lack of collusion; (4) symmetry. Although the standard relaxation of (1) — affiliation — implies the predominance of the open format, the relaxation of the other assumptions implies, in general, the contrary. It is well known that risk aversion favors first price auctions (see Matthews (1983) and Maskin and Riley (1984)). Since during an open auction, it is possible to identify and punish an agreement's defector, the possibility of collusion is more likely in the English format than in the first price sealed bid format. So, again the latter performs better than the former.

The consequences of asymmetries are less clear. Maskin and Riley (2000a) classify three cases of asymmetries that lead to clear predominance of one of the formats. Two of them favor first price auctions. Open auctions are superior only in the case of shifts of probability to a low end point. This happens in this case because of the behavior of "low balling" in the first price auction, from the part of the strongest bidder. This seems to be a less representative case of asymmetry. So, it is likely that most of the departures from the assumptions of the RET are likely to favor first price auction (taking in account our argument that affiliation is very rare).[49] Thus, in general terms, the regret explanation is a relaxation of the assumptions of the RET that leads to a more robust support to point 5. If the reasoning expressed by points 5-9 is intended as explanation of the predominance of English auction, it is better to appeal to regret than to affiliation.

In other words, the (considered strong) argument in favor of affiliation (expressed in Reaction 9) is mild, in fact.

7. The Tests of Affiliation

Despite our (theoretical) arguments, another reaction could require more:

Reaction 10: A true test for an assumption must to be empirical and/or experimental. So, we cannot dismiss affiliation without such tests.

[47]Boyes and Happel (1989), p. 40, relate such complaints in a first price auction conducted in Arizona State University.

[48]Maskin and Riley (2000) explain this fact via an asymmetric reasoning. Nevertheless, we consider that regret explains better the phenomenon.

[49]Of course, this is, also, a "rough" affirmative. We do not have a measure to indicate which departure of the assumptions of the RET is the most important. We hope empiricists and experimentalists can give a contribution in this matter.

This reaction points out the need of checking the validity of affiliation.

Fortunately, Milgrom and Weber's results include many testable predictions, most of them strongly based on the affiliation assumption. So, we can briefly survey the empirical and experimental literature to look for support for affiliation. This is the purpose of this section, although this cannot be taken as a complete or exhaustive survey of the field. Indeed, for a survey of the experimental literature, see Kagel (1995). Laffont (1997) is a comprehensive account of the empirical studies in auctions.

Anticipating the findings, it seems that the literature lacks a true test of affiliation. Such absence can be understood only if the affiliation assumption is considered so good that it is unquestionable. Our hope is that our arguments can stimuli the necessary studies.

7.1. **Experimental Literature.** In the experimental literature, an important work is Kagel, Harstad and Levin (1987).[50] They produced affiliated signals that were distributed to individuals. The bidding behavior was collected. The main finding is that individuals follow the Nash equilibrium strategies in first price auctions (the equilibrium describe better the behavior than ad-hoc models proposed). Doses of public information raised revenue, but lesser than predicted. Also, the second price auctions raise more revenue than the English auction, mainly because there is overbidding. These facts could be a falsification of the affiliation assumption, but they are rather weak and the authors do not interpret the results in this way. It is interesting to look at their table X, where they reproduce their results on revenue rank. From three sets of experiments, two have produced more revenue on the first price than on the English auction. In comparison with second price auction, the first price performed better in one set of experiments. However, what is more striking in their table X is the predicted, i.e., the theoretical differences in revenue. The signals range from \$25 to \$125 and they are truly affiliated, but the theory says to expected revenue differences just from \$0.08 to \$0.69.

A more recent experiment was reported by Lucking-Reiley (1999). He conducted real auctions of Magic cards through the internet, with experienced bidders. He found that the Dutch auction leads to higher revenue than the first price auction (to the contrary of the theory) and that the English and the second price are roughly equivalent. Both facts are against the implications of Milgrom and Weber (1982) model, but, again, the finding is not conclusive. Unfortunately, he does not compare the first and the second price auctions.

We find that these experiments do not check properly affiliation. In fact, this is not the authors' purpose. An experimental test for affiliation has to include the investigation of the values of the individuals in their experiments. It is needed to discover the estimates that each bidder makes and, then, check if such estimates are affiliated but not independent (in which case affiliation matters). Of course, a number of issues have to be considered when performing such tests. Some of them are whether the individuals have well defined estimates and their behavior is consistent with such estimates. Also, for the check of the affiliation hypothesis, a large set of data is probably needed. Nevertheless, the process of collecting this data cannot interfere in the outcome, as would happen if the repetition of the game could produce some kind of collusion, for instance.

[50]See also Cox, Roberson and Smith (1982), Cox, Smith and Walker (1985a, 1985b) and Kagel and Levin (1986).

In sum, there is a large avenue to be covered by experimentalists. In the benefit of the progress of Auction Theory, it is highly worth to test the plausibility of the affiliation assumption.

7.2. **Empirical Literature.** We could split the vast empirical literature in at least two distinct set of works. In the first, the theoretical one, the main attention is paid to the structural econometric approach and the development of methods for nonparametric (and parametric) estimation of auctions, mainly first price auctions. Another point of interest is the question of identification, that is, the property of being possible to recover the distribution of the unobservable from the distribution of the observable. See Paarsch (1992), Laffont and Vuong (1993), Laffont, Ossard and Vuong (1995), Donald and Paarsch (1996), Guerre, Perrigne and Vuong (2000), Haile and Tamer (2002), Li, Perrigne and Vuong (2002), Athey and Haile (2002).

Most of these papers is devoted to independent models. More recently, attempts have been made to extend the models to affiliation.[51] As in the purely theoretical literature, affiliation is taken as a good generalization of independence.

The other branch of the literature embraces the truly empirical works. In a series of papers, Ken Hendricks and Robert H. Porter have extensively studied the U.S. offshore oil and gas lease auctions conducted by the Department of the Interior, i.e., the sales of the Outer Continental Shelf. (See Hendricks, Porter and Boudreau (1987), Hendricks, Porter and Spady (1989) and Porter (1995).) The main concern of them is the bidding behavior with respect to the information. They distinguished the wildcat leases auctions, where any bidder has a special informational advantage because the tracts in the neighborhood are unexplored, and the drainage auctions, where some bidder can be better informed because of the experience in a close tract. A length discussion is made over questions as whether the common value paradigm is suitable for wildcat leases auctions and about the proper model for drainage auctions. We find little or none relation with affiliation.

The tests for revenue equivalence seem more relevant. For a set of data on English and first price auctions run by the U.S. Forest Service for timber haversting rights, Mead (1966) and Johnson (1979) found a tendency for the first price auctions to raise more revenue than the English auctions. However, Hansen (1985, 1986) argued that there was a selection bias caused by the way the Forest Service chose which auction to use for each timber lot. After correcting for this bias, he found that the conclusion does not sustain. Another related, but still inconclusive study was made by Tenorio (1993) for the multi-unit currency auctions in Zambia.

Other references to empirical works in auctions are found in the survey of Laffont (1997). We do not find a direct test of the assumption: it seems to exist an amazing lack of experimental support for affiliation.

8. Discussion

Affiliation seems to have a very good reputation as an assumption within Auction Theory. In this paper we show how far this notion is from being reasonable. Also, we provide an alternative explanation to the predominance of English auctions. This explanation has a very important feature: it allows to distinguish the situations where the English auctions are preferred and where the first price auctions are better. Indeed, if the auctions are conducted in such way that at the end, the bidders do not know the

[51]See Li, Perrigne and Vuong (2002) and Athey and Haile (2002).

bids of the opponents, then there is no regret effect and the first price can dominate. This is precisely the case of job markets.

Some final reactions need further comments.

Reaction 11 - Affiliation is a generalization of independence. Then, no matter what we prove, affiliation is strictly preferred to independence.

We disagree. First, the theoretical manipulation under affiliation is much more cumbersome than under independence. That is, there is a considerable cost of passing from independence to affiliation. The benefits, on the other hand, are poor: the monotonicity of equilibrium is maintained (at hard work, in some cases) and the revenue differences are likely to be of small magnitude.[52] Moreover, the conclusions obtained under affiliation can be misleading if we try to apply them to the reality, where affiliation is hardly satisfied. Then, in this sense, affiliation is worse than independence.

Reaction 12 - Affiliation enables us to prove monotonic equilibrium and this is of help in comparative statics.

Affiliation is problematic even for the task of proving the equilibrium existence. In asymmetrical and/or multidimensional settings, affiliation fails to ensure the necessary monotonicity.[53] Even the monotone comparative statics are not always possible, as McAdams (2003) observes. Then, for comparative statics purposes, it is better to restrict ourselves to the independent case.

9. Alternative Method for Treating Dependence: The Conditional Independence Program

An entirely negative paper can be useful, but it would be better to present some alternative. Otherwise, the following reaction could emerge:

Reaction 13: Independence is a very strong assumption, not likely to be satisfied in the real world. So, we need to treat the case where the signals of the bidders are, in some extent, correlated. We need to stick on affiliation if we do not have an alternative to treat dependent signals.

The purpose of this section is to discuss an alternative method to affiliation, that we call the Conditional Independence Program (CIP). Conditional Independence is, indeed, a concept widely and intensively used in Statistics, Psychology, Artificial Intelligence, Scientific Methodology and Science Philosophy. Dawid (1979) seems to be a precursor of many of these applications. The concept is also not new in Auction Theory. Wilson (1969, 1977) analyzes some models with conditional independence. Nevertheless, differently of the CIP, he conditions in the true value of the object (and do not show the existence of equilibrium). Unfortunately, after the introduction of affiliation, Conditional Independence seems to be banned from Auction Theory, although some recent papers use conditional independence in a close form of that of Wilson (see Fundenberg et. al. (2003) and also Reny and Perry (2003)). To describe the contribution of the CIP, let us begin with an informal description.

[52] Although we have not proved this last point, there are indications in this direction.

[53] A counterexample is given by Reny and Zamir (2002).

Consider, for instance, the behavior of two experienced bidders in wine auctions. If we analyze only their bids, obviously they will be highly correlated: for a poor quality lot they will bid a few; for a lot of good wine, they will bid high. Nevertheless, this high correlation is not a reason to abandon the independent case model. Simply, one has to condition the bids (and the signals) on the commonly known information: the lot of the wine being sold. Similar reasoning applies to all auctions, as those of paintings, drilling rights, timber, wool, etc.

When conditioning in these common knowledge characteristics, C, the dependence may disappear. If it does, we are done, as we show in the subsection 9.2 below.

In the other case, even conditioning in C, the signals t_1, ..., t_N are dependent. Nevertheless, there exists a statistics P such that, conditioning in P (and in C), t_1, ..., t_N are independent.[54] In other words, P represents an information that is not commonly known, but that is sufficient to make the bidders' signals independent. For grasping what P can be, consider an auction where the bidders consult the same report or the same consultant in order to make their estimates, but this behavior is not commonly known. The content of the report is P and, conditioning in it, the estimates are independent. One could say that P embraces the characteristics of the observational technology used by all the bidders. From a scientific point of view, it is better to explicitly include in our models such variable P, otherwise we preclude ourselves to learn something about the structure of dependence between the signals. A central idea of the program is to take into account P. Now, we will describe more detailed the CIP.[55]

9.1. **Preliminaries.** Let t_1, ..., t_N be multidimensional and (possibly) dependent variables. Let C_i be a commonly known factor that influences bidder i's information, t_i, for $i = 1$, ..., N. Then, $C = (C_1, ..., C_N)$ is common knowledge. Observe that C may take many different values. We assume that there is a conditional cumulative distribution function (c.d.f.) $F(t_1, ..., t_N|C) = F^C(t_1, ..., t_N)$.[56] Let P be another statistic such that the signals t_i are independent, conditioning in P, $F^C(t_1, ..., t_N|P) = F^C(t_1|P) \cdot \ldots \cdot F^C(t_N|P)$. Let $F_i^{C,P}$ be the marginal for t_i, when conditioning in both C and P. Let us define $I_i \equiv F_i^{C,P}(t_i)$, so that I_i and I_j are i.i.d. variables, with uniform distribution in $[0,1]$. Let

$$Q_i(u; C, P) = \sup\left\{t_i : F_i^{C,P}(t_i) \leqslant u\right\}.$$

Then, $t_i = Q_i(I_i; C, P)$ for almost all t_i. In other words, we have split each bidder's signal in three parts: a commonly know piece of information, C; an individually known information, independent of the other bidders' information, I_i and a statistics that summarizes the dependence between the bidders' opinion, but it is not commonly known, P. The CIP consists in approaching the problem of the dependence of the signals by, first, splitting the bidders' information as above and then taking advantage of the conditional independence that it describes, as the following methods show.

[54]We discuss below the existence of such P.

[55]This will not be a formal exposition. Especially in the subsequent subsections, some claims are not rigorously stated and proved. Nevertheless, they are more or less well known. It is a matter of future work to put CIP in mathematical statements and methods.

[56]It is not necessary to adopt the Bayesian doctrine that there is also a distribution for C or that there is a joint distribution for $(t_1, ..., t_N)$ and C, although this is obviously possible.

9.2. **Conditional Independence Approach (CIA).** If we assume that P =constant, that is, the t_i are independent conditioning in C, then the existence and characterization of equilibrium follow immediately from the correspondent results for independent types. Indeed, let us consider the auction where the signals are I_i, $i = 1, ..., N$, independent and let $b^C = (b_1^C(\cdot), ..., b_N^C(\cdot))$ be an equilibrium bidding functions profile, for a fixed C. Then, since $t_i = Q_i(I_i; C)$, we can define the bidding functions

$$b_i(t_i) = b_i(Q_i(I_i; C)) = b_i^C(I_i),$$

for $i = 1, ..., N$. We have the following

Lemma 1. If $b^C(I) = (b_1^C(I_1), ..., b_N^C(I_N))$ specifies an equilibrium of an auction with independent types $I_1, ..., I_N$ for each C, and if $t_i = Q_i(I_i; C)$, where C is commonly known, then the profile $b = (b_1(\cdot), ..., b_N(\cdot))$, where $b_i(t_i) = b_i^C(I_i)$, is equilibrium of the auction with types t_i.

Proof. It is a trivial consequence of the fact that $b_i^C(I_i)$ is the optimal bid for each C.■

Observe that the strategies so obtained will depend on C, but since C is common knowledge, there is no problem. Observe also that we do not take C to be the true value of the object, which is unknown in general.

Of course, an empiricist has to be careful in order to apply the CIA. What is common knowledge for the bidders participating in the auction is not necessarily easy to known by the empiricist. For instance, it can be common knowledge for the bidders that a tract is in a region of great productivity. This can be difficult to known from outside, but if the empiricists does not condition in this information, their results will be correlated, although the bidders' signals and, hence, bids, are independent (conditioning in that common knowledge information).

It is important to observe that, since we do not need to specify the relationship between C and the signals, t_i, affiliation does not hold, in general. We also emphasize that the assumption that P is constant is not too strong. If we are considering, for instance, a mineral rights auction, it is commonly known the specific region of the tract and the public reports about it. If we make the conditioning in all these public information, it may remain little that can be a source of dependence. Of course, this is still possible and for treat these cases, we propose the following:

9.3. **Conditional Independence Classification (CIC).** If CIA cannot be applied, there is a piece of information, P, which can be (partially) known to some bidders, but is not common knowledge. We can classify some cases for P:

(1) If it is common knowledge that P is know to one (or some) bidder(s) – but not for the others, we are in a situation in which some bidders have superior information, a setting similar to the case analyzed by Engelbrecht-Wiggans, Milgrom and Weber (1983). Their results suggest that, in this case, there is equilibrium in mixed strategies and the less informed bidders have zero expected rent.
(2) Another situation occurs when P is known (or can be obtained) by all bidders, but it is not common knowledge. For instance, bidder i does not know that bidder j knows P or bidder j does not know whether bidder i knows that bidder k knows P, and so on. In this case, it is natural to expect that P becomes common knowledge after some rounds of the auction game and we

return to CIA. A good example of this case is given by Ashenfelter (1989). He reports that some wine auctions in United States used to end with low prices for the lot, in comparison with similar lots sold in Europe. He explains that the reason for this was the fact that the auctions' participant in USA consult the same report for wines, and are deeply influenced by it. Since the report have misleading indications for some wines — for instance, good wines considered bad —, the price of those wine paid in USA were in general lower than the price in Europe. Ashenfelter says that the aware bidders can take advantage from this and buy good wines for small prices. Nevertheless, he concludes by saying that the opportunities were disappearing. We interpret this by the fact that the information of the report P was becoming common knowledge.

(3) If P can be discovered (even at some cost), then a) if it is not too costly, just one or some bidders can make efforts to learn it (and we turn to the case 1); or b) all the bidders try to learn P, and the situation turns to the case 2. In both situations, this case is not stable.

(4) If it is too costly (or if it is impossible) to discover P, then it may be the case of nonexistence of pure strategy equilibrium. If there is a mixed strategy equilibrium, it is possible to interpret that the bidders are trying to obtain an extra source of randomness, since they are not able to access the relevant random variable P. We propose some alternative solutions for this in the next subsections.

Of course, many questions arise for a convenient use of Conditional Independence Classification. How can we distinguish a case from the other? Can we develop some test? And what is, in reality, P, in specific cases? Do the bidders really try to learn it? The different auction formats imply different dynamics for the learning of P? How this affects the Revenue Equivalence Theorem?

9.4. **Conditional Independence Approach with Correlated Strategies (CIA-CS).** Suppose that P cannot be commonly known by the players (because it is impossible or because it is too cost, as in case 4 of CIC above). For instance, P can be formed by the experience, education, culture, propaganda, concepts, values or even by the common consequences of the evolutionary process of the human beings. To be more concrete, consider the following: if one bidder likes a painting (because it is colorful or because she finds it beautiful), the Darwinian evolution can make the other bidders' appreciation of colors or the concept of beauty similar. In any case, there is a non assessable variable P that influences the bidders' valuation.

The bidders cannot explicitly derive their strategies upon the unknown P, but the equilibrium strategies are not a matter of calculation. In a real auction, the bidders do not "calculate" their equilibrium strategies. At best, the equilibrium behavior is reached after some trials or rounds of the game. Then, if there is a random device that can be used by all players — even not explicitly knowing it — this can sustain an equilibrium behavior. Formally, we propose the following:

(1) Let I_i be obtained as above, and let $b^{C,P} = (b_1^{C,P}(\cdot), ..., b_N^{C,P}(\cdot))$ be a profile of equilibrium bidding functions (if it there exists) obtained for the auction with independent types I_i.

(2) Remembering that $t_i = Q_i(I_i; C, P)$, define the bidding functions

$$b_i(t_i) = b_i(Q_i(I_i; C, P)) = b_i^{C,P}(I_i),$$

for $i = 1, ..., N$.

(3) Observe that, exactly as in Lemma 1, $b = (b_1(\cdot), ..., b_N(\cdot))$ constitutes a (correlated) equilibrium of the game.

The unusual part of the above solution is that we allow the description of the bidding strategies to depend on a not commonly known random variable, P. So, we are using the correlated equilibrium concept as solution, but in a special form.[57] It is a special form because the players do not receive a direct indication of the action to adopt (as in Aumann (1987)), but rather they play a function $P \mapsto b_i^{C,P}(I_i)$. The real bid is calculated from the realization of P. This special form is also consistent with Aumann's definitions.

Two difficulties can arise with this solution. The first is the impression that the bidders do not know their bids. Some interpretations of mixed strategies assume exactly this. Nevertheless, this is needed only for the interim stage, when we calculate the payoff to determine the optimality of the strategy. The interim stage, however, is only a theoretical moment — it does not correspond to a "real time" instant. In other words, when the players meet at the auction, they can know their real bids $b_i^{C,P}(I_i)$.

The second difficult concerns the possibility of randomization based in the same (not commonly known) device, P. But if the source of correlation is the education[58], the cultural inheritance or the propaganda[59], why this cannot influence not only the opinions but also the actions, even if it is not commonly known the values that it takes?

Of course, CIA-CS is likely to be controversial. We believe, however, that the approach is valuable at least because it raises many questions: what are the real sources of the dependence of opinions and actions? How they can be accessed and learned? Is it possible to play without the knowledge of the true mechanism producing the action? In sum, CIA-CS puts again in perspective the need for a deeper understanding of how the bidders form their strategies.

It is natural to expect that some auction theoreticians will be unsatisfied with CIA-CS and we do not insist in its applicability for them. Rather, we point out two more options. First, as in the case 4 of the CIC, an equilibrium may exists only in mixed strategy (or do not exist at all). This may have interesting interpretations, as we discuss in subsection 9.6 below. Another possibility is to appeal to the following:

9.5. **Conditional Independence Approach with Monotonicity (CIA-M).** It is possible to assume, without loss of generality, that P is unidimensional. So, the general problem is that of conditional independence with P an unknown unidimensional parameter. If one assumes monotonicity of the function $F(t|P) = F_1(t_1|P) \cdot ... \cdot F_N(t_N|P)$ with respect to P, we are impose a strong restriction, but that is still weaker than affiliation.

Probably, the best examples of CIA-M are the works of Wilson (1977, 1979), Fundenberg et. al. (2003) and Reny and Perry (2003). Observe that the first papers do not have existence results and the latter require large number of bidders.

[57] For the concept of correlated equilibrium, see Aumann (1974, 1987).

[58] Aumann (1987, p. 16) gives an example where the correlation of the strategies is intuitively justified from the fact that the players went "to the same business school".

[59] For instance, the propaganda viewed by the participants in a business event or by some clubs of investments, parties, etc.

It is likely that the above procedure will not give the existence of pure strategy equilibria in general cases.[60] This does not preclude the need for an analysis of what are the cases where the equilibrium in pure strategy there exists. Before we finish this discussion with the presentation of the last part of CIP, we make some comments.

9.6. **Digression.** The previous discussion made clear that the Conditional Independence Program (CIP) puts in other perspective the importance of the results based in the assumption of independent signals. Indeed, the equilibrium existence in the case of dependent signals is derived trivally from such results under the cases treated by CIA or CIA-CS. So, the following reaction can emerge:

Reaction 14. CIP brings nothing new. We already used conditional independence assumptions and there is no new equilibrium existence results.

Conditional independence *is not an assumption* for CIP. Rather, the program specifies all the possible situations that can happen and offer an solution or interpretation for all of them. The classification (CIC) is one of its important functions.

To the contrary of previous works, CIP does not assume the existence of a variable upon which the bidder's signals are independent. The central feature of CIP is exactly to decompose the bidder's signals in the commonly known part (C) and in a uncommonly known part (P). This (very simple) idea opens the door to:

(1) a reassessment of the value and utilization of the independent signals' results;
(2) the investigation of how evolves the dependence of the bidder's signals and how it is related to their learning efforts;
(3) the investigation of the situations not covered by existence results.

Although the idea is very simple, these contributions seem worth and are certainly original. Of course, some mathematical developments are still needed. We let then to future works. Nevertheless, it is worth to note that CIP decouples the theoretical investigation in two: the study of the dynamics of the information and the equilibrium existence result in its own.

In any case, CIP can ensure the existence of equilibrium in cases where affiliation does not work, like that of multidimensional signals and asymmetrical auctions. Also, it puts in another perspective the importance of understanding the independent types case. Indeed, it has the advantage of directly using the theory constructed for independence.

9.7. **Conditional Independence Classification of Equilibrium Strategies (CIC-ES).** When the signals are independent, in general there exists equilibrium in pure strategy. See, e.g., Athey 2001, Maskin and Riley (2000b), McAdams (2003), Araujo, de Castro and Moreira (2004), etc.[61] As we have already observed, out of the independence, general results only exists in mixed strategy (see Matthew O. Jackson and Jeroen M. Swinkels (2004)). So, our theoretical interests can point exactly to the cases where P is not constant and cannot be used as a basis for randomizing actions (so that CIA and CIA-CS do not apply). The alternative just presented, CIA-M, is already known. Thus, it is possible the following:

[60] Indeed, we think that CIA-M is not the best research strategy to Auction Theory, as we explain in the next subsection.

[61] Of course, this is "in general terms". A completely general result seems impossible, as the counterexample of M. Ali Khan, Kali P. Rath and Yeneng Sun (1999) shows.

Reaction 15. CIP solves any problem. The truly problem of equilibrium existence under dependence remains untouched under CIP.

This reaction gives the opportunity to explain a different research strategy that CIP allows to pursue (which we name Conditional Independence Classification of Equilibrium Strategies — CIC-ES). We base it in the following excerpt of the iluminating paper of Ariel Rubinstein (1991, p. 922):

> "*The meaning of nonexistence.* If what we are trying to model in game theory are situations in which we expect regular behavior, then it is not true that all descriptions of the world should have an equilibrium. The mere fact that a game theoretician constructs a game does not mean that the game corresponds to a regular mode of behavior. The modeller should check the adequacy of the model as a decription of conveived regularity.
>
> "This brings me back to the mixed strategies issue. One of the reasons that mixed strategies are popular in both game and economic theory, in spite of being so unintuitive, is that many models do not have an equilibrium in pure strategy. However, the nonexistence of a solution concept in pure strategy does not necessarily mean that we should look for stochastic explanations. It means that the description of the game and the assumptions embedded in the solution concept are not consistent with regularity. Expanding the model or changing the basic assumptions are alternatives which the modeller should consider at least as favorably as mixed strategies."

A reexamination of the cases 3 and 4 in CIC shows that the problem with such cases is exactly the lack of regularity in the play of the game. In case 3, this is excessively clear. In case 4, the absence of regularity is more subtle. It comes from the description of this case as one where the learning of the important variable P is impossible (or too costly) for all the players. This means that the bidders are always lacking the necessary information and cannot reach regular (stable) behavior in the game. In turn, this brings the possibility of using mixed strategies: to appeal to stochastic behavior when simple (pure strategy) behavior is not stable. This shows the deep connection between the use of mixed strategy and the irregularity of the modeled situation.

In simple terms, CIC-ES consist in the classification of the models that have equilibria in pure strategies and the models where equilibria exist only in mixed strategy. A deeper understanding of these conditions will facilitate the description of the important determinants of the strategic behavior of the players. Of course, these considerations illustrate that CIP is, indeed, valid for a more general context than auctions. It fits well for all incomplete information game, where the players' signals can be correlated.

In sum, CIC-ES consists in solving the following problems:

(1) Decide whether there exists or not pure strategy equilibria in games of incomplete information (auctions);
(2) Study alternative modifications of the model to describe the behavior of the bidders, rather than just accept the mixed strategies solution concept.

The justificative for the second point above come from the quotation of Rubinstein, but also for the well known fact that the concept of mixed strategies has many problems.

The discussion make clear that we view mixing as an expression of unstable behavior, but this does not contradict the traditional views about mixed strategies.

10. Conclusion

We argued that affiliation is not a suitable assumption and have offered an alternative program for the analysis of equilibrium existence in auction. The Conditional Independence Program suggests ways for understanding the complex problem of dependence of signals, while opens the door for many questions. Also, it builds a bridge between Auction Theory and the fields where conditional independence is actively used and studied, as in Psychology, Statistics, Artificial Intelligence, Scientific Methodology and Philosophy of Sciences. This bridge can be valuable in bringing new ideas and insights. In any case, we consider it a more convenient and robust tool for approaching the cases of dependence, because it allows to approach asymmetries and multidimensionality, to the contrary of affiliation.

This paper also has the purpose to remember to the experimental, empirical and theoretical auction community that there is important and open works to be undertaken:

- Experimental: to develop methods to learn the values that people attribute to objects in an auction and whether they are correlated; to make relations between their values and their bids;
- Empirical: to describe what are the kind of correlation (if any) of the bids in real auctions;
- Theoretical: to develop models and to explore implications of CIP (or another alternative for affiliation) for the description of strategic and rational behavior of the bidders;

We hope that these efforts might allow Auction Theory to move forward and to supersede affiliation as work assumption.

Appendix

We begin by giving more general definitions of Association (As) and Affiliation without the use of density functions.

Definition. A subset A of R^N is called increasing if its indicator function 1_A is non-decreasing. A set $S \subset R^N$ is a sublattice if it contains $x \wedge x' \equiv (\min\{x_i, x_i'\})_{i=1}^N$ and $x \vee x' \equiv (\max\{x_i, x_i'\})_{i=1}^N$ as long as x and x' are in S.

Definition. Random variables $X_1, ..., X_N$ are associated if for all increasing sets A and B,

$$\Pr[(X_1, ..., X_N) \in A \cap B] \geq \Pr[(X_1, ..., X_N) \in A] \cdot \Pr[(X_1, ..., X_N) \in B].$$

Definition. Random variables $X_1, ..., X_N$ are affiliated if they are associated conditional on any sublattice. In other words, if for all increasing sets A and B, and every sublattice S,

$$\Pr[(X_1, ..., X_N) \in A \cap B|S] \geq \Pr[(X_1, ..., X_N) \in A|S] \cdot \Pr[(X_1, ..., X_N) \in B|S].$$

Proof of Theorem 1. The implication $(VII) \Rightarrow (VI)$ is Lemma 1 of Milgrom and Weber (1982). The implication $(V) \Rightarrow (IV)$ is proved by Tong (1980), chap. 5. The implication $(IV) \Rightarrow (III)$ is Theorem 4.3. of Esary, Proschan and Walkup (1967). It is obvious that $(III) \Rightarrow (II) \Rightarrow (I)$. Thus, we have just to prove that $(VI) \Rightarrow (V)$. Assume that $H(y|x) \equiv \frac{f_{Y|X}(y|x)}{F_{Y|X}(y|x)}$ is non-decreasing in x for all y. Then, $H(y|x) = \partial_y \left[\ln F_{Y|X}(y|x)\right]$ and we have

$$1 - \ln\left[F_{Y|X}(y|x)\right] = \int_y^\infty H(s|x)\, ds \geqslant \int_y^\infty H\left(s|x'\right) ds = 1 - \ln\left[F_{Y|X}\left(y|x'\right)\right],$$

if $x \geqslant x'$. Then, $\ln\left[F_{Y|X}(y|x)\right] \leqslant \ln\left[F_{Y|X}(y|x')\right]$, which implies that $F_{Y|X}(y|x)$ is non-increasing in x for all y, as required by the property (V).

The counterexamples for each passage are given by Tong (1980), chap. 5, except those involving property (VI): $(V) \nRightarrow (VI)$, $(VI) \nRightarrow (VII)$. For the first counterexample, consider the following symmetric p.d.f. defined on $[0,1]^2$:

$$f(x,y) = \frac{k}{1 + 4(y-x)^2}$$

where $k = [\arctan(2) - \ln(5)/4]^{-1}$ is the suitable constant for f to be a p.d.f. We have the marginal given by

$$f(y) = \frac{k}{2}[\arctan 2(1-y) + \arctan 2(y)]$$

so that we have, for $(x,y) \in [0,1]^2$:

$$f(x|y) = 2\left[1 + 4(y-x)^2\right]^{-1}[\arctan 2(1-y) + \arctan 2(y)]^{-1},$$

$$F(x|y) = \frac{[\arctan 2(x-y) + \arctan 2(y)]}{\arctan 2(1-y) + \arctan 2(y)}$$

and

$$\frac{F(x|y)}{f(x|y)} = 2\left[1 + 4(y-x)^2\right][\arctan 2(x-y) + \arctan 2(y)].$$

Observe that for $y' = 0.91$, $y = 0.9$, $x = 0.1$,

$$\frac{F(x|y')}{f(x|y')} = 0.366863 > 0.366686 = \frac{F(x|y)}{f(x|y)},$$

which violates property (VI). On the other hand,

$$\partial_y [F(x|y)] = \frac{\frac{2}{1+4y^2} - \frac{2}{1+4(x-y)^2}}{\arctan(2-2y) + \arctan(2y)} - \frac{[\arctan(2x-2y) + \arctan(2y)]\left[\frac{2}{1+4y^2} - \frac{2}{1+4(1-y)^2}\right]}{[\arctan(2-2y) + \arctan(2y)]^2}$$

In the considered range, the above expression is non-positive, so that property (V) is satisfied. Then, $(V) \nRightarrow (VI)$.

Now, fix an $\varepsilon < 1/2$ and consider the continuous and symmetric density function over $[0,1]^2$:

$$f(x,y) = \begin{cases} k_1, & \text{if } x + y \leqslant 2 - \varepsilon \\ k_2, & \text{otherwise} \end{cases}$$

where $k_1 > 1 > k_2 = \left[2 - k_1\left(2 - \varepsilon^2\right)\right]/\varepsilon^2$. The conditional density function is given by

$$f(y|x) = \begin{cases} 1, & \text{if } x \leqslant 1 - \varepsilon \\ \frac{k_1}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{if } x > 1 - \varepsilon \text{ and if } y \leqslant 2 - \varepsilon - x \\ \frac{k_2}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{otherwise} \end{cases}$$

and the conditional c.d.f. is given by:

$$F(y|x) = \begin{cases} 1, & \text{if } x \leqslant 1 - \varepsilon \\ \frac{k_1 y}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{if } x > 1 - \varepsilon \text{ and if } y \leqslant 2 - \varepsilon - x \\ \frac{k_2(y+x+\varepsilon-2)+k_1(2-\varepsilon-x)}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{otherwise} \end{cases}$$

and

$$\frac{F(y|x)}{f(y|x)} = \begin{cases} 1, & \text{if } x \leqslant 1 - \varepsilon \\ y, & \text{if } x > 1 - \varepsilon \text{ and if } y \leqslant 2 - \varepsilon - x \\ y + x + \varepsilon - 2 + k_1/k_2(2 - \varepsilon - x), & \text{otherwise} \end{cases}$$

Since $1 - k_1/k_2 < 0$, the above expression is non-increasing in x for all y, so that property (VI) is satisfied. On the other hand, it is obvious that property (VII) does not hold:

$$f(1,1) f(1-\varepsilon, 1-\varepsilon) = k_2 k_1 < k_1^2 = f(1-\varepsilon, 1) f(1, 1-\varepsilon).$$

This shows that $(VI) \nRightarrow (VII)$.■

Proof of Theorem 2. For a contradiction, assume the existence of $x < x'$ and $y < y'$ in the support of X and Y such that v is strictly increasing in both arguments in $[x,x'] \times [y,y']$. Let $z_4 = v(x',y') > z_3 = v(x',y)$, $z_2 = v(x,y') > z_1 = v(x,y)$. (See Figure 2.) Without loss of generality, we assume that $z_3 > z_2$. (If they were equal, we can reduce y' and obtain the desired inequality.)

Let $S = \{(X,Y,Z) : Z = z_2 \text{ or } z_3 \text{ and } X = x \text{ or } x'\}$ be a sublattice and let $A = \{(X,Y,Z) : Y \geqslant y' \text{ and } Z \geqslant z_2\}$ and $B = \{(X,Y,Z) : X \geqslant x' \text{ and } Z \geqslant z_3\}$ be increasing sets. Figures 3 and 4 illustrate such sets. (One of the axes are omitted in each figure.)

It is clear that $AB = A \cap B = \{(X,Y,Z) : X \geqslant x', Y \geqslant y',\ Z \geqslant z_3\}$ and $A \cap B \cap S = \{(x',y',z_3)\}$. Such point is out of the support of X, Y, Z because $v(x',y') = z_4 > z_3$.

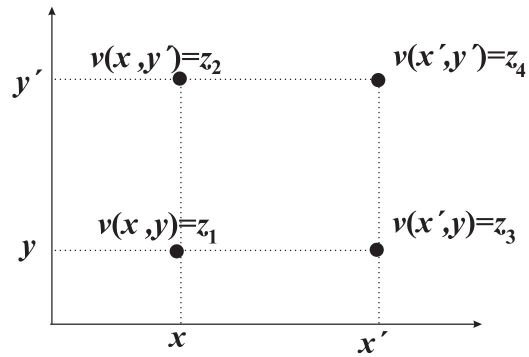

FIGURE 2

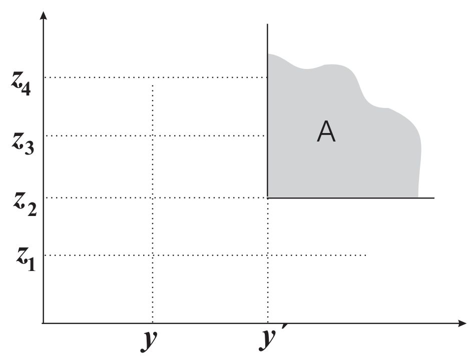

FIGURE 3

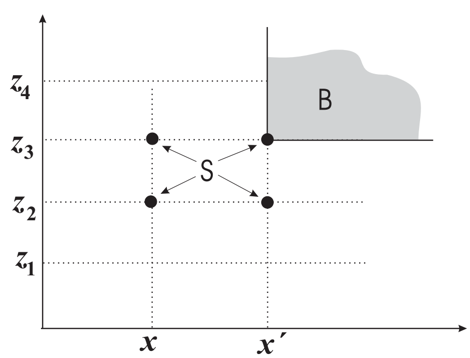

FIGURE 4

This implies that $\Pr[AB|S] = 0$.[62] Nevertheless, $P[A|S] > 0$ and $P[B|S] > 0$. Then, the necessary condition for affiliation, namely, $\Pr[AB|S] \geqslant P[A|S]\, P[B|S]$ does not hold, that is, X, Y and Z are not affiliated.■

[62]Although S has zero probability, the conditional probability $\Pr[AB|S]$ can be well defined as the limit of $\Pr[AB|S + \Delta\varepsilon]$ when $\varepsilon \to 0$, where $S + \Delta\varepsilon \equiv S = \{(X, Y, Z) : Z \in [z_2 - \varepsilon,\, z_2 + \varepsilon] \cup [z_3 - \varepsilon,\, z_3 + \varepsilon]$ and $X \in [x - \varepsilon,\, x + \varepsilon] \cup [x' - \varepsilon,\, x' + \varepsilon]\}$. It is easy to see that $\Pr[AB|S + \Delta\varepsilon] = 0$ for sufficiently small ε.

Proof of Theorem 3. The Lemma of Characterization of Araujo, de Castro and Moreira (2004) implies that $\partial_b \Pi(t,b)$ can be writing as

$$\partial_b \Pi(t,b) = (t-b)\, f_{b_{(-i)}}(b_i|t_i) - F_{b_{(-i)}}(b_i|t_i),$$

where $f_{b_{(-i)}}(b_i|t_i)$ and $F_{b_{(-i)}}(b_i|t_i)$ are the p.d.f. and the c.d.f., respectively, of the maximum bid of the opponents, conditioned to the signal t_i. Let us consider the existence of a symmetric monotonic equilibrium, $b^*(t_i)$. If it is differentiable,

$$(b^*)'(t) = (t - b^*(t)) \frac{f_Y(t|t)}{F_Y(t|t)}.$$

The solution to this equation is given by $b(t) = \int_{[\underline{t},t]} \alpha dL(\alpha|t) = t - \int_{[\underline{t},t]} L(\alpha|t)\, d\alpha$, where $L(\alpha|t) = \exp\left[-\int_\alpha^t \frac{f_Y(s|s)}{F_Y(s|s)} ds\right]$. So, we have

$$\begin{aligned} b'(y) &= 1 - L(y|y) - \int_{[\underline{t},y]} \partial_y L(\alpha|y)\, d\alpha \\ &= \frac{f_Y(y|y)}{F_Y(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\, d\alpha. \end{aligned}$$

It is clear, then, that $b'(y) > 0$, as long as $y > \underline{t}$. Now, suppose that a bidder of type t is considering to bid $b(y)$. Then,

$$\begin{aligned} \partial_b \Pi(t, b(y)) &= f_Y(t|t) \left[\frac{t - b(y)}{b'(y)} - \frac{F_Y(y|t)}{f_Y(y|t)}\right] \\ &= f_Y(t|t) \left[\frac{t-y}{b'(y)} + \frac{F_Y(y|y)}{f_Y(y|y)} - \frac{F_Y(y|t)}{f_Y(y|t)}\right]. \end{aligned}$$

So, by IHRD$(Y|X)$, the term in brackets has the signal of $t-y$, which shows that it is optimal for bidder of type t to bid $b(t)$.■

Proof. of Proposition 2. Consider a symmetric first price auction with two bidders and private values. Let $f(t_1,t_2)$ be the density function of joint distribution of the affiliated signals t_1 and t_2 with support $T = [\underline{t},\overline{t}] \times [\underline{t},\overline{t}]$. Let E be the triangle with vertices in $(\overline{t}, \overline{t}-\varepsilon)$, $(\overline{t}-\varepsilon, \overline{t})$ and $(\overline{t},\overline{t})$ and let $1 - \frac{1}{k}$ be the probability of E, such that

$$f^m(t_1,t_2) = k f(t_1,t_2)\, 1_{T \setminus E}$$

is a density function with support $T\setminus E$. Let F and F^m denote the c.d.f. of f and f^m, respectively. It is easy to see that $f^m(t_2|t_1) = f(t_2|t_1) \cdot g(t_1)$, where

$$g(t_1) = \begin{cases} 1, & \text{if } \underline{t} \leqslant t_1 \leqslant \overline{t} - \varepsilon \\ f(t_1) / \left[\int_{\underline{t}}^{2\overline{t}-\varepsilon-t_1} f(t_1,t_2)\, dt_2\right] & \text{if } \overline{t} - \varepsilon \leqslant t_1 \leqslant \overline{t} \end{cases}$$

Observe that, for a sufficiently small $\varepsilon > 0$, f^m is $\eta-$close to f. We have the following:

Claim. The first price auction with the modified p.d.f. f^m possess a symmetric equilibrium that can be expressed by

$$\mathbf{b}^m(t) = \begin{cases} \mathbf{b}(t), & \text{if } \underline{t} \leqslant t \leqslant \overline{t} - \varepsilon/2 \\ \mathbf{b}\left(2\overline{t} - \varepsilon - t\right), & \text{if } \overline{t} - \varepsilon/2 \leqslant t \leqslant \overline{t} \end{cases}$$

where $b(t)$ is the symmetric equilibrium under the p.d.f. f. Consequently, this auction has a non-monotonic equilibrium bidding function.

$$\mathbf{b}^m(t) = \begin{cases} \mathbf{b}(t), & \text{if } \underline{t} \leqslant t \leqslant \overline{t} - \varepsilon/2 \\ \mathbf{b}\left(2\overline{t} - \varepsilon - t\right), & \text{if } \overline{t} - \varepsilon/2 \leqslant t \leqslant \overline{t} \end{cases}$$

where $b(t)$ is the symmetric equilibrium under the p.d.f. f.

Consider that bidder 2 uses the strategy $b^m(t)$ above. The payoff of bidder 1, of type t_1, bidding b_1, is $\Pi^m(t_1, b_1) = (t_1 - b_1) F^m_{b_2}(b_1|t_1)$, where $F^m_{b_2}(b_1|t_1) \equiv \Pr\left[\mathbf{b}^m(t_2) < b_1|t_1\right]$.

The Characterization Lemma implies that, if $f^m_{b_2}(b_1|t_1) > 0$,

$$\begin{aligned} \partial_b \Pi^m_1(t_1, b_1) &\lesseqqgtr 0 \Longleftrightarrow (t_1 - b_1) f^m_{b_2}(b_1|t_1) - F^m_{b_2}(b_1|t_1) \lesseqqgtr 0 \\ &\Leftrightarrow \quad (t_1 - b_1) \lesseqqgtr \frac{F^m_{b_2}(b_1|t_1)}{f^m_{b_2}(b_1|t_1)} \end{aligned}$$

We have three cases to analyze:

a) $\underline{t} \leqslant t < \overline{t} - \varepsilon$:

$$F^m_{b_2}(b_1|t_1) = \begin{cases} 0, & \text{if } b_1 < \mathbf{b}(\underline{t}) \\ F_{b_2}(b_1|t_1) & \text{if } \mathbf{b}(\underline{t}) \leqslant b_1 \leqslant \mathbf{b}\left(\overline{t} - \varepsilon\right) \\ F_{b_2}(b_1|t_1) & \\ \quad + 1 - F\left(2\overline{t} - \varepsilon - \mathbf{b}^{-1}(b_1)|t_1\right), & \text{if } \mathbf{b}\left(\overline{t} - \varepsilon\right) \leqslant b_1 \leqslant \mathbf{b}\left(\overline{t} - \varepsilon/2\right) \\ 1, & \text{if } b_1 > \mathbf{b}\left(\overline{t} - \varepsilon/2\right) \end{cases}$$

where $F_{b_2}(b_1|t_1) \equiv F\left(\mathbf{b}^{-1}(b_1)|t_1\right)$.

This permits to obtain

$$\frac{F^m_{b_2}(b_1|t_1)}{f^m_{b_2}(b_1|t_1)} = \begin{cases} 0, & \text{if } b_1 < \mathbf{b}(\underline{t}) \\ \frac{F_{b_2}(b_1|t_1)}{f_{b_2}(b_1|t_1)}, & \text{if } \mathbf{b}(\underline{t}) \leqslant b_1 \leqslant \mathbf{b}\left(\overline{t} - \varepsilon\right) \\ \frac{F_{b_2}(b_1|t_1)}{f_{b_2}(b_1|t_1)} & \\ \quad + \frac{1 - F\left(2\overline{t} - \varepsilon - \mathbf{b}^{-1}(b_1)|t_1\right)}{f_{b_2}(b_1|t_1)}, & \text{if } \mathbf{b}\left(\overline{t} - \varepsilon\right) \leqslant b_1 \leqslant \mathbf{b}\left(\overline{t} - \varepsilon/2\right) \\ +\infty, & \text{if } b_1 > \mathbf{b}\left(\overline{t} - \varepsilon/2\right) \end{cases}$$

Since $b(t_1)$ is the best reply for the original auction, and since

$$\frac{1 - F\left(2\overline{t} - \varepsilon - \mathbf{b}^{-1}(b_1)|t_1\right)}{f_{b_2}(b_1|t_1)} > 0,$$

it is easy to see that the best reply is $b^*_1 = b^m(t_1) = b(t_1) < b\left(\overline{t} - \varepsilon\right)$.

b) $\overline{t} - \varepsilon \leqslant t \leqslant \overline{t} - \varepsilon/2$:

$$F^m_{b_2}(b_1|t_1) = \begin{cases} 0, & \text{if } b_1 < \mathbf{b}(\underline{t}) \\ F_{b_2}(b_1|t_1) \cdot g(t_1) & \text{if } \mathbf{b}(\underline{t}) \leqslant b_1 \leqslant \mathbf{b}\left(\overline{t} - \varepsilon\right) \\ F_{b_2}(b_1|t_1) \cdot g(t_1) + 1 & \text{if } \mathbf{b}\left(\overline{t} - \varepsilon\right) \leqslant \\ -F(2\overline{t} - \varepsilon - \mathbf{b}^{-1}(b_1)|t_1) \cdot g(t_1), & \quad b_1 \leqslant \mathbf{b}\left(\overline{t} - \varepsilon/2\right) \\ 1, & \text{if } b_1 > \mathbf{b}\left(\overline{t} - \varepsilon/2\right) \end{cases}$$

Fortunately, the term $g(t_1)$ also appears in $f^m_{b_2}(b_1|t_1)$. So, we have

$$\frac{F_{b_2}^m(b_1|t_1)}{f_{b_2}^m(b_1|t_1)} = \begin{cases} 0, & \text{if } b_1 < \mathbf{b}(\underline{t}) \\ \frac{F_{b_2}(b_1|t_1)}{f_{b_2}(b_1|t_1)}, & \text{if } \mathbf{b}(\underline{t}) \leqslant b_1 \leqslant \mathbf{b}(\overline{t}-\varepsilon) \\ \frac{F_{b_2}(b_1|t_1)}{f_{b_2}(b_1|t_1)} & \\ \quad + \frac{1/g(t_1) - F\left(2\overline{t}-\varepsilon-\mathbf{b}^{-1}(b_1)|t_1\right)}{f_{b_2}(b_1|t_1)}, & \text{if } \mathbf{b}\left(\overline{t}-\varepsilon\right) \leqslant b_1 \\ & \leqslant \mathbf{b}\left(\overline{t}-\varepsilon/2\right) \\ +\infty, & \text{if } b_1 > \mathbf{b}\left(\overline{t}-\varepsilon/2\right) \end{cases}$$

Observe that

$$\frac{1}{g(t_1)} = \frac{\int_{\underline{t}}^{2\overline{t}-\varepsilon-t_1} f(t_1,t_2)\,dt_2}{f(t_1)} = F\left(2\overline{t}-\varepsilon-t_1|t_1\right).$$

Then, $\partial_b \Pi_1^m(t_1,b_1) \lesseqgtr 0 \Leftrightarrow b^m(t_1) = b(t_1) \lesseqgtr b_1$, as is required for the best reply to be $b^m(t_1)$.

c) $\overline{t}-\varepsilon/2 \leqslant t \leqslant \overline{t}-\varepsilon$:

$$F_{b_2}^m(b_1|t_1) = \begin{cases} 0, & \text{if } b_1 < \mathbf{b}(\underline{t}) \\ F_{b_2}(b_1|t_1)\cdot g(t_1) & \text{if } \mathbf{b}(\underline{t}) \leqslant b_1 \leqslant \mathbf{b}\left(\overline{t}-\varepsilon/2\right) \\ 1, & \text{if } b_1 > \mathbf{b}\left(\overline{t}-\varepsilon/2\right) \end{cases}$$

from the fact that there is no $t_2 \geqslant 2\overline{t}-\varepsilon-t_1$ in the support of the conditional distribution, given t_1. It is easy to see that

$$\partial_b \Pi_1^m(t_1,b_1) = (t_1-b_1)\,f_{b_2}^m(b_1|t_1) - F_{b_2}^m(b_1|t_1) > 0$$

for all $b_1 \leqslant b\left(\overline{t}-\varepsilon/2\right)$ because

$$\begin{aligned} \partial_b \Pi_1(t_1,b_1) &= (t_1-b_1)\,f_{b_2}(b_1|t_1) - F_{b_2}(b_1|t_1) \lesseqgtr 0 \\ &\Leftrightarrow \mathbf{b}\left(\overline{t}-\varepsilon/2\right) \leqslant \mathbf{b}(t) \lesseqgtr b_1 \end{aligned}$$

Since it does not improve the probability of winning but it is costly to bid above $b\left(2\overline{t}-\varepsilon-t_1\right)$, this is the best reply bid, as we wanted to show.■

Proof of Theorem 4. The dominant strategy for each bidder in the second price auction is to bid his value: $b^2(t) = t$. Then, the expected payment by a bidder in the second price auction, P^2, is given by:

$$\begin{aligned} P^2 &= \int_{[\underline{t},\overline{t}]}\int_{[\underline{t},x]} y f_Y(y|x)\,dy\cdot f(x)\,dx = \\ &= \int_{[\underline{t},\overline{t}]}\int_{[\underline{t},x]} [y-b(y)]\,f_Y(y|x)\,dy\cdot f(x)\,dx + \int_{[\underline{t},\overline{t}]}\int_{[\underline{t},x]} b(y)\,f_Y(y|x)\,dy\cdot f(x)\,dx, \end{aligned}$$

where $b(\cdot)$ gives the equilibrium strategy for symmetric first price auctions. Thus, the first integral can be substituted by $\int_{[\underline{t},\overline{t}]}\int_{[\underline{t},x]} b'(y)\,\frac{F_Y(y|y)}{f_Y(y|y)} f_Y(y|x)\,dy\cdot f(x)\,dx$, from the equality: $b'(y)\,\frac{F_Y(y|y)}{f_Y(y|y)} = y - b(y)$. On the other hand, the last integral can be integrated by parts, to:

$$
\begin{aligned}
& \int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]} b(y)\, f_Y(y|x)\, dy\cdot f(x)\, dx \\
= & \int_{[\underline{t},\bar{t}]}\left[b(x)\, F(x|x) - \int_{[\underline{t},x]} b^{'}(y)\, F_Y(y|x)\, dy\right]\cdot f(x)\, dx \\
= & \int_{[\underline{t},\bar{t}]} b(x)\, F(x|x)\cdot f(x)\, dx - \int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]} b^{'}(y)\, F_Y(y|x)\, dy\cdot f(x)\, dx
\end{aligned}
$$

In the last line, the first integral is just the expected payment for the first price auction, P_i^1. Thus, we have

$$
\begin{aligned}
D &= P^2 - P^1 \\
&= \int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]} b^{'}(y)\,\frac{F_Y(y|y)}{f_Y(y|y)} f_Y(y|x)\, dy\cdot f(x)\, dx \\
&\quad -\int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]} b^{'}(y)\, F_Y(y|x)\, dy\cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]} b^{'}(y)\left[\frac{F_Y(y|y)}{f_Y(y|y)} f_Y(y|x) - F_Y(y|x)\, dy\right]\cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]} b^{'}(y)\left[\frac{F_Y(y|y)}{f_Y(y|y)} - \frac{F_Y(y|x)}{f_Y(y|x)}\right] f_Y(y|x)\, dy\cdot f(x)\, dx
\end{aligned}
$$

Remember that $b(t) = \int_{[\underline{t},t]} \alpha dL(\alpha|t) = t - \int_{[\underline{t},t]} L(\alpha|t)\, d\alpha$, where $L(\alpha|t) = \exp\left[-\int_\alpha^t \frac{f_Y(s|s)}{F_Y(s|s)} ds\right]$. So, we have

$$
\begin{aligned}
b^{'}(y) &= 1 - L(y|y) - \int_{[\underline{t},y]} \partial_y L(\alpha|y)\, d\alpha \\
&= \frac{f_Y(y|y)}{F_Y(y|y)}\int_{[\underline{t},y]} L(\alpha|y)\, d\alpha.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
D &= \int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]} \frac{f_Y(y|y)}{F_Y(y|y)}\int_{[\underline{t},y]} L(\alpha|y)\, d\alpha\left[\frac{F_Y(y|y)}{f_Y(y|y)} - \frac{F_Y(y|x)}{f_Y(y|x)}\right] f_Y(y|x)\, dy\cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]}\left[\int_{[\underline{t},y]} L(\alpha|y)\, d\alpha\right]\cdot\left[1 - \frac{F_Y(y|x)}{f_Y(y|x)}\cdot\frac{f_Y(y|y)}{F_Y(y|y)}\right]\cdot f_Y(y|x)\, dy\cdot f(x)\, dx
\end{aligned}
$$

This is the wanted expression.■

CHAPTER 7
A FIXED POINT THEOREM FOR NON-CONVEX SET-VALUED MAPS

Abstract. We prove a theorem analogous to the Kakutani-Fan-Glicksberg's theorem in the Bochner-Lebesgue spaces, whose main feature is that the values of the set-valued map can be nonconvex. In the place of convexity, we require decomposability. In contrast with previous papers, we give our results in the weak topology and show why this is important.

1. Introduction

Let $(T, \mathcal{T}, \tau)$ be a non-atomic probability space and let E be a separable Banach space. Let $L = L^1(T, \mathcal{T}, \tau, E)$ be the space of Bochner integrable functions $s : T \to E$. A set $K \subset L$ is said to be decomposable if for any $r, s \in K$, $r1_P + s1_{T \setminus P} \in K$, $\forall P \in \mathcal{T}$, where $1_P(t) = 1$ if $t \in P$ and 0 otherwise.

Decomposability is a property that is a good substitute for convexity in Bochner spaces (see Olech (1984)). This paper treats such substitutability with respect to the Kakutani-Fan-Glicksberg's fixed point theorem:

Theorem (Kakutani-Fan-Glicksberg). Let K be a nonempty compact convex subset of a locally convex Hausdorf space. Assume that the set-valued map $\Gamma : K \rightrightarrows K$ is upper semicontinuous and, for all $x \in K$, $\Gamma(x)$ is nonempty, closed and convex. Then, there exists $x \in \Gamma(x)$.

Cellina, Colombo and Fonda (1986) provides an analogue to the above theorem where K is a subset of $L = L^1(T, E)$ and where the convexity is replaced by decomposability:

Theorem (Cellina, Colombo and Fonda). Let K be a nonempty closed subset of $L^1(T, E)$, with the norm topology, and let $\Gamma : K \rightrightarrows K$ be upper semicontinuous, with closed graph such that for all $x \in K$, $\Gamma(x)$ is nonempty, closed and decomposable. Moreover, assume that $\Gamma(K)$ is totally bounded and decomposable. Then, there exists $x \in \Gamma(x)$.

Observe that they specify the strong (norm) topology for L. This is undesirable, since compact sets in the norm topology are rare. Indeed, in section 3 we prove that if E has the Radon-Nikodým property, their assumptions imply that $\Gamma(K)$ is unitary.

In section 4, we provide an alternative theorem to theirs, that do not require the use of the strong topology, but of the weak topology of L^p, for $1< p < \infty$.

As an aplication, we obtain a Nash equilibrium existence result, in section 5.

The next section is dedicated to the definitions and to the statement of well known results used in the sequel.

2. Preliminaries

If X and Y are topological spaces, we say that the set-valued map $\Gamma : X \rightrightarrows Y$ is upper semicontinuous, abbreviated usc if, whenever $\Gamma(x) \subset V$ for some open set $V \subset Y$, there exists a neighborhood U of x, such that $\forall x' \in U \Rightarrow \Gamma(x') \subset V$.

As in the introduction, let $(T, \mathcal{T}, \tau)$ be a non-atomic probability space, E be a separable Banach space and $L = L^1(T, \mathcal{T}, \tau, E)$ be the space of Bochner integrable functions $s : T \to E$. Let Π be the set of finite partitions π of T. For each partition $\pi \in \Pi$, define the linear map $U_\pi : L \to L$ by

$$U_\pi f = \sum_{P \in \pi} \frac{\int_P f d\tau}{\tau(P)} 1_P$$

with the convention that $0/0 = 0$. Introduce a partial ordering on Π by saying that $\pi \geqslant \pi_0$ if for each $P \in \pi$, there is a $P_0 \in \pi_0$, such that $P \subset P_0$. Thus, with a fixed topology in L, $\lim_\pi U_\pi f$ exists if the net $\{U_\pi f\}_{\pi \in \Pi}$ is convergent for such topology. Lemma III.2.1 of Diestel and Uhl (1977), p. 67, shows that U_π is norm continuous and $\|U_\pi\| \leqslant 1$. It is easy to see that U_π is a compact operator.

Theorem 2 of Brooks and Dinculeanu (1979) states that a set $K \subset L$ is relatively norm compact if and only if: (1) K is uniformly integrable; (2) The set $\left\{\int_P f d\tau : f \in K\right\}$ is relatively norm compact in E; (3) For a net of partitions, $U_\pi f \to f$ uniformly in K in the norm topology.

Theorem IV.2.1. of Diestel and Uhl (1977), p. 101, states that, if E and $E*$ have the Radon-Nikodým property, a set $K \subset L$ is relatively weakly compact if and only if: (1) K is bounded; (2)K is uniformly integrable; and (3) for each $P \in \mathcal{T}$, the set $\left\{\int_P f d\tau : f \in K\right\}$ is relatively weakly compact in E.

3. Strong Compactness and Decomposability

Cellina, Colombo and Fonda (1986)'s Theorem 2 assume that $\Gamma(K)$ is totally bounded (and decomposable) so that $\overline{\Gamma(K)}$ is norm compact (see the introduction). Observe that $\overline{\Gamma(K)}$ is also decomposable.[1] Then, the assumptions of the cited theorem imply that, if E has the Radon-Nikodým property (e.g., if E is reflexive), then $\Gamma(K)$ is unitary. This follows from the following:

Proposition 1. Assume that E has the Radon-Nykodým Property. Let $M \subset L = L^1(T, \mathcal{T}, \tau, E)$ be norm compact and decomposable. Then $M = \{f\}$, for some $f \in L$.

Proof. For an absurd, suppose that there exists f, $g \in M$, and

$$D = \{t \in T : |f(t) - g(t)| > \eta\}$$

such that $\tau(D) > \delta$, with η, $\delta > 0$. Let $\varepsilon = \eta\delta/6$. Since M is norm compact, U_π converges uniformly on M, that is, we can find $\pi_\varepsilon \in \Pi$ such that $\pi \geqslant \pi_\varepsilon$ and $h \in M$ implies $\|U_\pi h - h\| = \int |h(t) - U_\pi h(t)| \tau(dt) < \varepsilon$. We will obtain a contradiction if M is also decomposable. Without loss of generality, we can fix a $\pi \geqslant \pi_\varepsilon$ that contains sets $P_1, ..., P_n \in \pi$ such that $P_i \subset D$ and $\cup_{i=1}^n P_i = D$. Let f_i be the value of $U_\pi f(t)$ in P_i and analogously for g_i. We have

[1] Indeed, let $M \subset L$ be decomposable and take $P \in \mathcal{T}$, $r, s \in \overline{M}$. Since we are using the norm topology, there exist sequences $\{r^n\}$, $\{s^n\} \subset M$, so that $r^n \to r$, $s^n \to s$. Since M is decomposable, $u^n = r^n 1_P + s^n 1_{T \setminus P} \in M$, $\forall n$ and $u^n \to u$. Thus, $u \in \overline{M}$.

$$\begin{aligned}\|U_\pi f - f\| &= \sum_{P\in\pi}\int_P |f(t) - U_\pi f(t)|\,\tau(dt) < \varepsilon \\ &\Rightarrow \sum_{i=1}^n \int_{P_i} |f(t) - f_i|\,\tau(dt) < \varepsilon,\end{aligned}$$

and an analogous property is valid for g. This permits to obtain that

$$\begin{aligned}\sum_{i=1}^n \int_{P_i} |g_i - f_i|\,\tau(dt) &\geqslant \sum_{i=1}^n \Bigg[\int_{P_i} |f(t) - g(t)|\,\tau(dt) \\ &\quad - \int_{P_i} |f(t) - f_i|\,\tau(dt) - \int_{P_i} |g(t) - g_i|\,\tau(dt)\Bigg] \\ &> \eta\delta - 2\varepsilon = \frac{2\eta\delta}{3}.\end{aligned}$$

For each i, consider the measure μ over P_i defined by $\mu(C) = \left(\int_C f d\tau, -\int_C g d\tau\right)$, which has bounded variation.[2] Theorem IX.1.10, p. 266 of Diestel and Uhl (1977) implies that the s-closure of $\{\mu(C) : C \subset P_i, C \in \mathcal{T}\}$ is s-compact and convex. So, there exists $B_i \subset P_i, B_i \in \mathcal{T}$ such that $x_i \equiv \int_{B_i} f d\tau - \frac{1}{2}\int_{P_i} f d\tau$ and $y_i \equiv -\int_{B_i} g d\tau + \frac{1}{2}\int_{P_i} g d\tau$ $= \int_{P_i\setminus B_i} g d\tau - \frac{1}{2}\int_{P_i} g d\tau$ satisfy $|x_i| < \varepsilon/2n$ and $|y_i| < \varepsilon/2n$.

Put $B = \cup_i B_i$ and define $h = f1_B + g1_{T\setminus B}$. Clearly, $h \in M$, because M is decomposable and $B \in \mathcal{T}$. We have

$$\begin{aligned}U_\pi h|_{P_i} &= \frac{1}{\tau(P_i)}\int_{P_i}\left[f1_{B_i} + g1_{P_i\setminus B_i}\right] d\tau \\ &= \frac{1}{\tau(P_i)}\int_{B_i} f d\tau + \frac{1}{\tau(P_i)}\int_{P_i\setminus B_i} g d\tau \\ &= \frac{1}{\tau(P_i)}\left[\frac{1}{2}\int_{P_i} f d\tau + \frac{1}{2}\int_{P_i} g d\tau + x_i + y_i\right],\end{aligned}$$

For each $i = 1,\ldots,n$, define $m_i = (f_i + g_i)/2$. Thus,

$$U_\pi h|_{P_i} = m_i + \frac{x_i + y_i}{\tau(P_i)},$$

and we compute $\|U_\pi h - h\|$ as

$$\begin{aligned}&\sum_{P\in\pi}\int_P |h(t) - U_\pi h(t)|\,\tau(dt) \\ &= \sum_{i=1}^n \int_{P_i} |h(t) - U_\pi h(t)|\,\tau(dt) + \sum_{P\subset D^C}\int_P |g(t) - U_\pi g(t)|\,\tau(dt).\end{aligned}$$

This is not lesser than

[2]This can be seen from the fact that M is bounded, which implies that μ is bounded. Then, Lemma III.1.5, p. 97, of Dunford and Schwartz (1958) proves that μ is of bounded variation.

$$\sum_{i=1}^{n}\int_{P_i}|h(t)-m_i|\,\tau(dt)-\sum_{i=1}^{n}\int_{P_i}\frac{|x_i|}{\tau(P_i)}\tau(dt)-\sum_{i=1}^{n}\int_{P_i}\frac{|y_i|}{\tau(P_i)}\tau(dt)$$
$$=\sum_{i=1}^{n}\int_{P_i}\left[\left|f_i+\frac{g_i-f_i}{2}-f(t)\right|1_{B_i}+\left|g_i+\frac{f_i-g_i}{2}-g(t)\right|1_{P_i\setminus B_i}\right]\tau(dt)$$
$$-\sum_{i=1}^{n}|x_i|-\sum_{i=1}^{n}|y_i|\,.$$

Again, this is not lesser than

$$\sum_{i=1}^{n}\left\{\int_{P_i}\frac{|g_i-f_i|}{2}\tau(dt)-\int_{B_i}|f(t)-f_i|\,\tau(dt)\right.$$
$$\left.-\int_{P_i\setminus B_i}|g(t)-g_i|\,\tau(dt)\right\}-\varepsilon$$
$$>\frac{2\eta\delta}{3}-\frac{\eta\delta}{6}-\frac{\eta\delta}{6}-\frac{\eta\delta}{6}$$
$$=\frac{\eta\delta}{6}=\varepsilon.$$

This contradicts the uniform integrability of M and, hence, its compacity.■

4. Main Results

Once we learn that the norm topology is not suitable for working with compact and decomposable sets, we turn to the weak topology. We need to particularize the sets involved. Let E be a reflexive Banach space and let $L = L^p(T,\mathcal{T},\tau,E)$ be separable, with $1\leqslant p<\infty$, with τ a nonatomic probability measure. Let W be a weakly compact and convex subset of L. By Theorem V.6.3. of Dunford and Schwartz (1958), p. 434, the weak topology is metrizable in W. Let $\{x_n^*\}_{n\in\mathsf{N}}$ be an enumerable dense set in $L^q(T,\mathcal{T},\tau,E^*)$, where $p^{-1}+q^{-1}=1$. Then, the topology in W is generated by the metric:[3]

$$d(f,g)=\sum_{n}\frac{1}{2^n}z\left(\left|\int_T\langle x_n^*(t),f(t)-g(t)\rangle\,d\tau\right|\right),$$

where $\langle\cdot,\cdot\rangle : E^*\times E\to\mathbb{R}$ is the bilinear form between the duals E^* and E and $z:\mathbb{R}_+\to[0,1]$ is the increasing function defined by

$$z(x)=\frac{x}{1+x}.$$

We have the following:

Theorem 1. Let $\Gamma : W\rightrightarrows W$ be an upper semicontinuous map such that $\Gamma(m)$ is nonempty, closed and decomposable for all $m\in W$. Then, for every $\varepsilon>0$, there exists a continuous $\varepsilon-$ approximate selection of Γ.

Proof. It is sufficient an adaptation of the proof of Theorem 1 of Cellina, Colombo and Fonda (1986). We give all the steps here for reader's convenience. Fix $\varepsilon>0$. By

[3]See, for instance, Dunford and Schwartz (1958), V.6.3., p. 434.

the upper semicontinuity of Γ, for each $m \in M$, there is $\delta(m) \in (0, \varepsilon/3)$ such that $\rho(m, m') < \delta(m)$ imply $\Gamma(m') \subset B(\Gamma(m), \varepsilon/3) = \{f \in L : d(f, \Gamma(m)) < \varepsilon/3\}$, where we are abusing the notation by defining $d(f, \Gamma(m)) = \inf\{d(f, g) : g \in \Gamma(m)\}$.

The compacity of W ensures the existence of a finite set $\{m_1, ..., m_n\} \subset W$, such that the balls $B(m_i, \delta_i) = \{m \in W : \rho(m, m_i) < \delta_i\}$ form a subcovering of W, with $\delta_i = \delta(m_i)/2$. Let $\{p_i : i = 1, ..., n\}$ be a continuous partition of the unity subordinate to it. For each $i = 1, ..., n$, fix a $u_i \in \Gamma(m_i)$. For each i and j=1,...,n, take $v_{ij} \in \Gamma(m_j)$ such that

$$d(u_i, v_{ij}) < d(u_i, \Gamma(m_j)) + \frac{\varepsilon}{3}.$$

Now, we define

$$\mu_{ij}(A) \equiv \sum_n \frac{1}{2^n} z\left(\left|\int_A \langle x_n^*(t), u_i(t) - v_{ij}(t)\rangle\, d\tau\right|\right).$$

Thus $\mu_{ij}(T) = d(u_i, v_{ij}) \leqslant d(u_i, \Gamma(m_j)) + \varepsilon/3$.

Observe that μ_{ij} is non-atomic since τ is. We know, by a standard application of Lyapunov's theorem, that there exists a family $(T_\alpha)_{\alpha \in [0,1]}$ such that

$(P1)$ $T_\alpha \subset T_\beta$, if $\alpha \leqslant \beta$.

$(P2)$ $\mu_{ij}(T_\alpha) = \alpha \mu_{ij}(T)$.

$(P3)$ $\tau(T_\alpha) = \alpha \tau(T)$.

Set $\alpha_0 \equiv 0$, $\alpha_i(m) = p_1(m) + ... + p_i(m)$ and define the approximate selection as

$$s_\varepsilon(m) = \sum_{i=1}^n u_i 1_{T_{\alpha_i(m)} \setminus T_{\alpha_{i-1}(m)}}$$

For each $m \in W$, define $I(m) = \{i \in \{1, ..., n\} : p_i(s) > 0\}$. Then, the above sum is given by

$$s_\varepsilon(m) = \sum_{i \in I(m)} u_i 1_{T_{\alpha_i(m)} \setminus T_{\alpha_{i-1}(m)}}.$$

We claim that s_ε has the required properties. Let us check that it is an $\varepsilon-$ approximate selection of Γ. Fix m and let k be such that $\delta_k = \max\{\delta_i : i \in I(m)\}$.

Then, for every $i \in I(m)$, we have $m_i \in B(m_k, 2\delta_k)$ so that $\Gamma(m_i) \subset B(\Gamma(m_k), \varepsilon/3)$. Thus, $d(u_i, \Gamma(m_k)) < \varepsilon/3$. Define $v_k = \sum_{i \in I(m)} v_{ij} 1_{T_{\alpha_i(m)} \setminus T_{\alpha_{i-1}(m)}}$.We have

$$d_{W \times L}((m, s_\varepsilon(m)), (m_k, v_k)) = \rho(m, m_k) + d(s_\varepsilon(m), v_k) \leqslant \varepsilon/3 + d(s_\varepsilon(m), v_k).$$

It is sufficient to prove that $d(s_\varepsilon(m), v_k) \leqslant 2\varepsilon/3$. This term is given by

$$\begin{aligned} & d\left(\sum_{i \in I(m)} u_i 1_{T_{\alpha_i(m)} \setminus T_{\alpha_{i-1}(m)}}, \sum_{i \in I(m)} v_{ik} 1_{T_{\alpha_i(m)} \setminus T_{\alpha_{i-1}(m)}}\right) \\ = & \sum_n \frac{1}{2^n} z\left(\left|\int \left\langle x_n^*(t), \sum_{i \in I(m)} (u_i(t) - v_{ik}(t)) 1_{T_{\alpha_i(m)} \setminus T_{\alpha_{i-1}(m)}}(t)\right\rangle d\tau\right|\right) \\ = & \sum_n \frac{1}{2^n} z\left(\left|\sum_{i \in I(m)} \int \left\langle x_n^*(t), (u_i(t) - v_{ik}(t)) 1_{T_{\alpha_i(m)} \setminus T_{\alpha_{i-1}(m)}}(t)\right\rangle d\tau\right|\right) \\ \leqslant & \sum_n \frac{1}{2^n} z\left(\sum_{i \in I(m)} \left|\int_{T_{\alpha_i(m)} \setminus T_{\alpha_{i-1}(m)}} \langle x_n^*(t), u_i(t) - v_{ik}(t)\rangle\, d\tau\right|\right). \end{aligned}$$

If $a, b \geqslant 0$, it is easy to verify that $z(a+b) \leqslant z(a) + z(b)$. Since the limits are finite, we can interchange the sums. Then, by the definition of μ_{ik}, this is not greater than

$$\sum_{i \in I(m)} \mu_{ik} \left(T_{\alpha_i(m)} \backslash T_{\alpha_{i-1}(m)}\right)$$

Now, by (P2), this is equal to

$$\begin{aligned}
& \sum_{i \in I(m)} \left[\alpha_i(m) - \alpha_{i-1}(m)\right] \mu_{ik}(T) \\
\leqslant & \sum_{i \in I(m)} p_i(m) \left[d\left(u_i, \Gamma(m_k)\right) + \frac{\varepsilon}{3}\right] \\
\leqslant & \frac{2\varepsilon}{3} \sum_{i \in I(m)} p_i(m) = \frac{2\varepsilon}{3},
\end{aligned}$$

as we wanted to show. So, s_ε is a ε–approximate selection of Γ. Let us prove that it is continuous. For m and $m' \in W$, $d\left(s_\varepsilon(m), s_\varepsilon(m')\right)$ is given by

$$d\left(\sum_{i \in I(m)} u_i 1_{T_{\alpha_i(m)} \backslash T_{\alpha_{i-1}(m)}}, \sum_{i \in I(m)} u_i 1_{T_{\alpha_i(m')} \backslash T_{\alpha_{i-1}(m')}}\right)$$

In turn, this is equal to

$$\begin{aligned}
& \sum_n \frac{1}{2^n} z\left(\left|\int \left\langle x_n^*(t), \sum_{i \in I(m)} u_i(t) \left[1_{T_{\alpha_i(m)} \backslash T_{\alpha_{i-1}(m)}} - 1_{T_{\alpha_i(m')} \backslash T_{\alpha_{i-1}(m')}}\right](t)\right\rangle d\tau\right|\right) \\
& = \sum_n \frac{1}{2^n} \sum_{i \in I(m)} z\left(\int \left|\left\langle x_n^*(t), u_i(t) \left[1_{T_{\alpha_i(m)} \backslash T_{\alpha_{i-1}(m)}} - 1_{T_{\alpha_i(m')} \backslash T_{\alpha_{i-1}(m')}}\right]\right\rangle\right| d\tau\right) \\
& \leqslant \sum_n \frac{1}{2^n} \sum_{i \in I(m)} z\left(\int |u_i(t)| \left|1_{T_{\alpha_i(m)} \backslash T_{\alpha_{i-1}(m)}} - 1_{T_{\alpha_i(m')} \backslash T_{\alpha_{i-1}(m')}}\right| d\tau\right) \\
& \leqslant \sum_{i \in I(m)} \sum_n \frac{1}{2^n} z\left(\int |u_i(t)| \left(\left|1_{T_{\alpha_i(m)}} - 1_{T_{\alpha_i(m')}}\right| + \left|1_{T_{\alpha_{i-1}(m)}} - 1_{T_{\alpha_{i-1}(m')}}\right|\right) d\tau\right) \\
& \leqslant \sum_{i \in I(m)} \sum_n \frac{1}{2^n} z\left(\int_{T_{\alpha_i(m)} \triangle T_{\alpha_i(m')}} |u_i(t)| \, d\tau + \int_{T_{\alpha_{i-1}(m)} \triangle T_{\alpha_{i-1}(m')}} |u_i(t)| \, d\tau\right).
\end{aligned}$$

Let M be an upper bound for W. Given $\eta > 0$, there exists $\delta > 0$ such that $\rho(m, m') < \delta$ implies $\tau\left(T_{\alpha_i(m)} \triangle T_{\alpha_i(m')}\right), \tau\left(T_{\alpha_{i-1}(m)} \triangle T_{\alpha_{i-1}(m')}\right) < \eta/2M$, which proves the continuity.■

Remark 1. It seems possible to follow the proofs of Bressan and Colombo (1988) to obtain the extension of Theorem 1 to the case of a general metric space M. It also seems possible to obtain the generalization of the continuous selection theorem of Fryszkowski (1983). We do not undertake these lenght works here because we are mainly interested in the next result, which does not need such generalizations.

Theorem 2. Let W be as above. Assume that $\Gamma : W \rightrightarrows W$ is upper semicontinuous and $\Gamma(x)$ is nonempty, closed and decomposable for all $x \in W$. Then, there exists $\bar{x} \in \Gamma(\bar{x})$.

Proof. By Theorem 1, there is a continuous $1/n-$ approximate selection f^n of Γ, that is, $Graph\{f^n\} \subset B\left(Graph\ \Gamma, \frac{1}{n}\right)$. By the Schauder-Tychonoff Theorem, there exists a fixed point $s^n = f(s^n)$, such that $s^n = f(s^n)$. From the definition of $f^{,n}$, there exist x^n and r^n in W, with $r^n \in \Gamma(x^n)$, such that

$$d(s^n, x^n) + d(s^n, r^n) < \frac{1}{n}. \tag{4.1}$$

Since W is compact, s^n has a subsequence that converges weakly to a point $\overline{s}$. Thus, $r^n \rightharpoonup \overline{s}$ and $x^n \rightharpoonup \overline{s}$. Since Γ is upper semicontinuous, $\overline{s} \in \Gamma(\overline{s})$. Since $\Gamma(W) \subset K$, $\overline{s} \in \Gamma(\overline{s}) \cap K$, and it is a fixed point of the original set-valued map.■

5. Application

We will now show how Theorem 2 can be used to prove the existence of pure strategy equilibria in continuous games. There are I players. For $i = 1, ..., I$, let E_i be a separable Banach space and A_i be a compact subset of E_i, representing the set of actions of player i. Let T_i be the set of Harsanyi types of the players. Let W_i be the set of pure strategies $s_i : T_i \to A_i$. Let us denote $W_{-i} \equiv \times_{j\neq i} W_j$. Each player has a payoff function $u_i : T_i\times\ A_i\times\ W_{-i} \to \mathbb{R}$. Let us assume that u_i satisfies the following continuity assumption:

(C) Let $\{s_i^n\}_{n\in\mathsf{N}} \subset W_i$ and $\left\{s_{-i}^n\right\}_{n\in\mathsf{N}} \subset W_{-i}$ be converging sequences. If $s_i^n \rightharpoonup s_i$, $s_{-i}^n \rightharpoonup s_{-i}$, then

$$u_i\left(t_i, s_i^n(t_i), s_{-i}^n\right) \to u_i\left(t_i, s_i(t_i), s_{-i}\right).$$

Let us define

$$F_i(t_i, s_{-i}) \equiv \arg\max_{a_i\in A_i} u_i(t_i, a_i, s_{-i})$$

and the selection set-valued map

$$S_{F_i}(s_{-i}) = \{f \in L^p(T_i, \mathcal{T}_i, \tau_i, E_i) : f(\omega) \in F_i(\omega)\ \ a.e.\}.$$

Obviously, F_i has nonempty, closed values. So, the set-valued map $S_{F_i} : W_{-i} \to W_i$ has nonempty and weakly compact values. Indeed, this comes from the weak compactness characterization given in section 2. Theorem 3.1 of Hiai and Umegaki (1977) shows that S_{F_i} has decomposable values.

We say that $s = (s_1, ..., s_n)$ is an equilibrium for the game if for all $i = 1, ..., n$, we have $s_i \in S_{F_i}(s_{-i})$. Then, we have the following:

Theorem 3. Assume (C). Then, the game has a pure strategy equilibrium.

Proof. Let us verify that the S_{F_i} is upper semicontinuous. Since its values are weakly compact, it is sufficient to verify that its graph is closed. Let $\{s_i^n\}_{n\in\mathsf{N}} \subset W_i$ and $\left\{s_{-i}^n\right\}_{n\in\mathsf{N}} \subset W_{-i}$, with $s_i^n \rightharpoonup s_i$, $s_{-i}^n \rightharpoonup s_{-i}$ and $s_i^n \in S_{F_i}\left(s_{-i}^n\right)$. By definition,

$$u_i\left(t_i, s_i^n(t_i), s_{-i}^n\right) \geqslant u_i\left(t_i, a_i, s_{-i}^n\right), \forall a_i \in A_i, \text{ a.e.}$$

Now, by (C),

$$u_i\left(t_i, s_i(t_i), s_{-i}\right) \geqslant u_i\left(t_i, a_i, s_{-i}\right), \forall a_i \in A_i, \text{ a.e.},$$

which implies that $s_i \in S_{F_i}(s_{-i})$, as we want to show. Now, let us define the set-valued map $S : W \to W$, where $W = \times_{i=1}^I W_i$, by $S(s) \equiv \times_{i=1}^I S_{F_i}(s_{-i})$.Then, S satisfy the assumptions of Theorem 2. The fixed point $s \in S(s)$ is easily seen to be an equilibrium.■

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