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Properties of Solutions to Some Nonlinear Dispersive Models

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To memory of my mother
To my father

Abstract

We study the local and global well-posedness issues of the initial value problem (IVP) associated to the coupled system of Korteweg-de Vries equations. Using the bilinear estimates established by Kenig, Ponce and Vega in the Fourier transform restriction space we prove a local result for given data in the Sobolev spaces of indices greater than -3/4. We prove that this local result is optimal by showing that the map data-solution is not twice differentiable at the origin. Further, under certain restrictions on the coefficients, we extend the local solution to a global one when the data is in the Sobolev spaces of indices greater than -3/10. We also consider the IVP associated to a coupled system of modified Korteweg-de Vries equations. We further refine the low-high frequency technique introduced by Bourgain and simplified by Fonseca, Linares and Ponce to develop an iteration process below the energy space and prove a global well-posedness result for data in the Sobolev spaces with indices greater than 4/9. Finally we consider a bi-dimensional generalization of the Korteweg-de Vries equation, called Zakharov-Kuznetsov equation, and prove that if a sufficiently smooth solution to the associated IVP is supported in a non-trivial time interval then it vanishes identically.

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Contents

Introduction

The first model we are interested to investigate in this work is the following system of nonlinear dispersive equations

$$\begin{cases} W_t + AW_{xxx} + B(W)W_x + CW_x = 0, & x,\, t \in \mathbb{R} \\ W(x,0) = W_0(x), \end{cases} \tag{0.0.1}$$

where $W = (u,v)^t$ with $u = u(x,t)$ and $v = v(x,t)$, real valued functions. The typical examples of the models we want to consider are the coupled system of the Korteweg-de Vries (KdV) equations. The above model arises in various physical contexts to describe several nonlinear phenomena.

Our main purpose here is to address the well-posedness issues to the initial value problem (IVP) (0.0.1). The notion of well-posedness we are going to use is the following:

Definition 1 *An IVP in a Banach space X is said to be locally well-posed if there exist a time $T > 0$ and a unique solution in the interval $[-T, T]$ such that the solution depends continuously upon the given data and satisfies the persistence property, it means, for given data $\phi \in X$ the solution $u(t) \in X$ for all $t \in [-T, T]$ describes a continuous curve in X. If the existence time interval is arbitrarily large we say that the IVP is globally well-posed and if any one of the above conditions fails to hold we say the IVP is ill-posed.*

A large amount of work has been devoted to study (0.0.1). For example, when

$$A = \begin{pmatrix} 1 & a_3 \\ \frac{b_2 a_3}{b_1} & \frac{1}{b_1} \end{pmatrix}, \quad B(W) = \begin{pmatrix} u + a_2 v & a_2 u + a_1 v \\ \frac{b_2 a_2}{b_1} u + \frac{b_2 a_1}{b_1} v & \frac{b_2 a_1}{b_1} u + \frac{1}{b_1} v \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & \frac{r}{b_1} \end{pmatrix}, \tag{0.0.2}$$

with $a_1, a_2, a_3, r \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}^+$, the system (0.0.1) is a well known model introduced by Gear and Grimshaw [26] to describe the strong interaction of two dimensional, long, internal gravity waves propagating on a neighboring pycnoclines in a stratified fluid.

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + uu_x + a_1 vv_x + a_2 (uv)_x = 0, & x,\, t \in \mathbb{R}, \\ b_1 v_t + v_{xxx} + b_2 a_3 u_{xxx} + vv_x + b_2 a_2 uu_x + b_2 a_1 (uv)_x + rv_x = 0, \\ u(x,0) = \phi(x), \\ v(x,0) = \psi(x). \end{cases} \tag{0.0.3}$$

The system (0.0.3) has the structure of the KdV equation

$$u_t + u_{xxx} + uu_x = 0, \qquad x,\, t \in \mathbb{R}, \tag{0.0.4}$$

coupled through dispersive as well as nonlinear effects. Several properties of the system (0.0.3) including existence theory for the associated IVP and the existence and stability of solitary wave solution can be found in the literature. For an extensive description of this model we refer to the work of Bona, Ponce, Saut and Tom [8]. They used Kato's theory for abstract evolution equations to obtain well-posedness results in classical Sobolev spaces. Further they utilized the theory developed by Kenig, Ponce and Vega [41] to get the local result in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Gear and Grimshaw [26] showed that the following quantities

$$\int u\, dx, \qquad \int v\, dx \qquad \text{and} \qquad \int (b_2 u^2 + b_1 v^2)\, dx$$

are conserved by the flow of (0.0.3). Bona et. al. [8] derived a new conserved quantity

$$\int \{ b_2(u_x^2 + 2a_3 u_x v_x - \frac{1}{3}u^3 - a_2 u^2 v - a_1 u v^2) + v_x^2 - \frac{1}{3}v^3 \}\, dx.$$

Using these four conserved quantities they were able to get an *a priori* estimate in the energy space $H^1(\mathbb{R})\times H^1(\mathbb{R})$ by imposing the condition $1-b_2a_3^2 > 0$ on the coefficients. This *a priori* estimate permits one to extend the local solution to a global one. This result is obtained by neglecting the dimensionless parameter r. Later, Ash, Cohen and Wang [4] studied this problem in the $X_{s,b}$ spaces introduced by Bourgain to deal with the nonlinear dispersive equations. Using bilinear estimates established by Kenig, Ponce and Vega [38] they proved local well-posedness for given data in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Also, they utilized the L^2-conserved quantity to extend the local solution to the global one in that space. Recently, Saut and Tzvetkov [55] considered the IVP (0.0.3) without neglecting the constant r and proved global well-posedness in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

Using the concentration compactness technique, Bona and Chen [7] proved the existence of solitary waves for the system (0.0.3) as global minimizers to the constrained variational problem. Later, Albert and Linares [3] proved that the solitary waves are stable in a weak sense by considering $a_3^2 b_2 < 1$. Recently, Menzala, Vasconcellos and Zuazua [48] showed that the solutions of the KdV equation in a bounded interval under the effect of a localized damping decay exponentially in time. The method of proof is a combination of multiplier techniques, compactness arguments and the unique continuation property of the KdV equation. A similar result for the IVP (0.0.3) was obtained by Bisognin, Bisognin and Menzala [6] whenever the conditions $b_1 = b_2$ and $0 < a_3 < 1$ hold.

Since (0.0.3) is a coupled system of KdV equations, it is natural to ask whether it shares similar results like KdV equations. In other words, whether we can lower the Sobolev index in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ as in the case of KdV equation to get well-posedness results? Using the scaling argument we can have an insight to this question. Observe that if (u, v) solves (0.0.3) (note that we are neglecting the parameter r, otherwise the scaling doesn't work) with initial data (ϕ, ψ) then for $\lambda > 0$ so does (u^λ, v^λ) with initial data $(\phi^\lambda, \psi^\lambda)$; where $u^\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^3 t)$, $v^\lambda(x,t) = \lambda^2 v(\lambda x, \lambda^3 t)$, $\phi^\lambda(x) = \lambda^2\phi(\lambda x)$ and $\psi^\lambda(x) = \lambda^2\psi(\lambda x)$. Note that,

$$\|(\phi^\lambda, \psi^\lambda)\|_{\dot{H}^s \times \dot{H}^s} = \lambda^{2s+3} \|(\phi, \psi)\|_{\dot{H}^s \times \dot{H}^s}, \tag{0.0.5}$$

where $\dot{H}^s(\mathbb{R})$ denotes the homogeneous Sobolev space of order s. Thus, we see from (0.0.5) that the well-posedness result for the IVP (0.0.3) could be achieved in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq -3/2$.

Bourgain [13] showed that the well-posedness result obtained by Kenig, Ponce and Vega [38] for the KdV equation in $H^s(\mathbb{R})$, $s > -3/4$, is essentially optimal if one strengthens the usual notion of well-posedness by requiring the flow-map

$$\phi \mapsto u_\phi(t), \qquad |t| < T$$

should act smoothly (for eg. C^3) on the space under consideration (instead of just continuous). This notion of well-posedness seems to be natural because, if one uses the contraction mapping principle to solve the integral equation associated with the Cauchy problem, the flow-map acts smoothly from H^s to itself. In fact, for $s > -3/4$ the flow-map is real analytic (see for eg., [38] [39] [15]). Takaoka [63] used this technique to show that the nonlinear Schrödinger equation with derivative in a nonlinear term is ill-posed in $H^s(\mathbb{R})$, $s < 1/2$. Further, Tzvetkov [67] showed that the KdV equation is locally ill-posed in $H^s(\mathbb{R})$ for $s < -3/4$ if one requires only C^2 regularity of the flow-map in the notion of well-posedness. Following the same scheme, we prove that the IVP (0.0.3) is ill-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s < -3/4$. This result is in agreement with the KdV results. So, one expects that the IVP (0.0.3) be locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > -3/4$. In fact, using the bilinear estimates established by Kenig, Ponce and Vega [38], we prove that the IVP (0.0.3) is locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$.

On the other hand, we should mention that, Nakanishi, Takaoka and Tsutsumi [52] constructed a counterexample to prove that the bilinear estimate established by Kenig, Ponce and Vega [38] fails when $s = -3/4$. Therefore the critical index $s = -3/4$ cannot be achieved using this method. However, Christ, Colliander and Tao [20] showed recently the existence of the solutions to the IVP associated to the KdV equation in $H^{-3/4}(\mathbb{R})$ using a generalized Miura transform to transfer the existing local theory for the modified KdV equation in $H^{1/4}(\mathbb{R})$.

Note that, in the Sobolev spaces of negative index, conservation laws are not available to extend the local solution to a global one. To overcome this difficulty, i.e. lack of conservation laws, quite recently, Colliander, Keel, Staffilani, Takaoka and Tao [22] introduced a variant of Bourgain's method [12] called *I-method* to obtain a global solution to KdV equation in Sobolev spaces of negative index. For this, they introduced a notion of an *almost conserved quantity* by utilizing an appropriate Fourier multiplier operator I. To obtain such almost conserved quantity they exploited some internal cancellation which the KdV equation satisfies. The cancellation plays a main role in this process. In our case, it is not possible to get such cancellation unless the coefficients $a_3 = 0$ and $b_1 = b_2$ (see Lemma 1.3 below). It seems that we cannot get more cancellation in the general case because the IVP under consideration is not completely integrable. Using this method, under above conditions on the coefficients, we prove that the IVP (0.0.3) is globally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/10$.

For $p \in \mathbb{Z}^+$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B(W) = \begin{pmatrix} \frac{1}{p}u^{p-1}v^{p+1} & \frac{1}{p+1}u^p v^p \\ \frac{1}{p+1}u^p v^p & \frac{1}{p}u^{p+1}v^{p-1} \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{0.0.6}$$

the model (0.0.1) turns out to be a coupled system of generalized KdV equations

$$\begin{cases} u_t + u_{xxx} + (u^p v^{p+1})_x = 0 \\ v_t + v_{xxx} + (u^{p+1} v^p)_x = 0, \qquad x,\, t \in \mathbb{R} \\ u(x,0) = \phi(x), \quad v(x,0) = \psi(x), \end{cases} \tag{0.0.7}$$

and arises in various physical situations. This system has the following conserved quantities

$$I_1(u,v) = \int_{\mathbb{R}} (u^2 + v^2)\, dx \tag{0.0.8}$$

and

$$I_2(u,v) = \int_{\mathbb{R}} \{u_x^2 + v_x^2 - \frac{2}{p+1} u^{p+1} v^{p+1}\}\, dx, \tag{0.0.9}$$

and admits sech solitary wave solutions. This system has been widely studied in the literature (see for example [2], [51] and references therein) and can also be solved by the inverse scattering method. In [2] Alarcon, Angulo and Montenegro established a general existence theory for the associated IVP along with the orbital stability of the solitary wave solutions. When $p = 1$, this model reduces to a system of modified KdV (mKdV) equations coupled through the nonlinear terms. Montenegro [51] used the theory developed by Kenig, Ponce and Vega [39] in the mKdV context to prove that the IVP associated to this particular model for given data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ has local solution when $s \geq 1/4$ and global solution when $s \geq 1$. So there is a gap in the Sobolev indices between the local and global existence results.

Bourgain [12] introduced a new technique to get the global solution below energy spaces. Let us explain in brief about how to implement the Bourgain's technique in the case of a general dispersive equation,

$$\begin{cases} w_t + P(D)w + f(w) = 0 \\ w(x,0) = \phi(x). \end{cases} \tag{0.0.10}$$

Suppose that the IVP (0.0.10) has local solution in H^{s_0} for some $s_0 \in (0,1)$ and its flow satisfies the H^1 conservation law. To extend the local solution to the global one below the energy space H^1, we proceed as follows.

We decompose the initial data $\phi \in H^s,\ s<1$, to $\phi = \phi_1 + \psi_1$, where ϕ_1 and ψ_1 are given by,

$$\hat{\phi}_1(\xi) = \chi_{\{|\xi|\leq N\}}\hat{\phi}(\xi), \qquad \hat{\psi}_1(\xi) = \chi_{\{|\xi|>N\}}\hat{\phi}(\xi).$$

In other words, we decompose ϕ into low and high frequency parts so that the low frequency part ϕ_1 is regular with H^1-norm large and the high frequency part ψ_1, although does not improve regularity, has H^s-norm small. In fact,

$$\|\phi_1\|_{H^1} \lesssim N^{1-s} \quad \text{and} \quad \|\psi_1\|_{\mathrm{H}^\rho} \lesssim \mathrm{N}^{\rho-\mathrm{s}}, \quad 0 \leq \rho \leq \mathrm{s} < 1.$$

We evolve the low frequency part ϕ_1 according to the original IVP (0.0.10) so that we have the existence result say in $[0,\delta]$. Let $\phi_1 \mapsto u_1(t)$ be the evolution of the low frequency part. Now we evolve the high frequency part ψ_1 according to the difference equation

$$\begin{cases} v_{1t} + P(D)v_1 + f(u_1+v_1) - f(u_1) = 0, \qquad x,\, t \in \mathbb{R} \\ v_1(x,0) = \psi_1(x), \end{cases} \tag{0.0.11}$$

with variable coefficients depending on the solution u_1. For simplicity, let us denote the evolution of the high frequency part $\psi_1 \mapsto v_1(t)$ by $v_1(t) = U(t)\psi_1 + z_1(t)$, where $U(t)$ is the unitary group associated to the linear problem. The main feature of this technique is that the existence interval $[0,\delta]$ is the same for both u_1 and v_1 and $w(t) = u_1(t) + v_1(t)$ solves the IVP (0.0.10). Note that the inhomogeneous part $z_1(t)$ of the evolution $v_1(t)$ of the high frequency part ψ_1 is smoother than the data itself. In fact, for some $\alpha = \alpha(s) > 0$

$$\|z_1(t)\|_{H^1} \lesssim N^{-\alpha}. \tag{0.0.12}$$

Thus at time $t=\delta$ we add $z_1(\delta)$ to $u_1(\delta)$ and repeat the above argument with new data

$$\phi_2 := u_1(\delta) + z_1(\delta) \qquad \text{and} \qquad \psi_2 := U(\delta)\psi_1$$

to obtain the solution in $[\delta, 2\delta]$. Then we iterate this process to cover the time interval $[0,T]$ for arbitrary $T>0$. In each step of iteration it is necessary to control the involved norms taking care of the contribution arising from (0.0.12) (also called as error term). In fact, we can proceed with this iteration process as long as the total error is at most comparable with the size of $\|\phi_1\|_{H^1}$ and at this point we obtain restriction on the Sobolev index s.

Soon after Bourgain [12] introduced this technique to get the global solution to the two-dimensional Schrödinger equation below energy space, several authors have applied it to obtain the global solution to various nonlinear dispersive models. Fonseca, Linares and Ponce [24] simplified this technique to get the global solution to the mKdV equation in $H^s(\mathbb{R})$, $s>3/5$. It is also applied to get the global solutions to the semi-linear wave equations (see Kenig, Ponce and Vega [37]) and critical generalized KdV equations (see Fonseca, Linares and Ponce [25]). Also, Takaoka used this technique to get the global solutions to KP-II equation in [64] and to the Schrödinger equation with derivative in [63]. Further, Pecher [54] followed the same technique to prove the global well-posedness for the 1D Zakharov system below the energy space. Recently, using the argument in [24], Carvajal and Linares [18] proved that the IVP associated to the higher order nonlinear Schrödinger equation is globally well-posed in $H^s(\mathbb{R})$, $s>5/9$.

Here we further refine this technique by exploiting the uniform bound of the solution (see (2.2.1) below) obtained by using iteration in the energy space. With proper choice of the Sobolev indices we develop an iteration process below the energy space and prove that the IVP (2.1.1) is globally well-posed for data in $H^s(\mathbb{R})\times H^s(\mathbb{R})$, $s>4/9$.

The next model we want to study is the two dimensional generalization of the KdV equation (0.0.4). The KdV model was obtained in [46] to describe the propagation of one dimensional surface gravity waves with small amplitude in a shallow channel of water. This is a widely studied model and arises in various physical contexts. It has very rich mathematical structure and can also be solved using inverse scattering technique. There are two dimensional generalizations of the KdV model that arise to govern the motion where transversal effects are also taken into consideration. The most known two dimensional generalizations of the KdV equation are the Kadomtsev-Petviashvelli (KP) equation

$$(u_t+u_{xxx}+uu_x)_x+\alpha u_{yy}=0, \qquad \alpha=\pm 1 \tag{0.0.13}$$

and Zakharov-Kuznetsov (ZK) equation

$$u_t+(u_{xx}+u_{yy})_x+uu_x=0. \tag{0.0.14}$$

The equation (0.0.13) derived by Kadomtsev and Petviashvelli [33] describes the evolution of weakly two dimensional long water waves of small amplitude while the equation (0.0.14) derived by Zakharov and Kuznetsov [66] models the propagation of nonlinear ion-acoustic waves in magnetized plasma. Much effort has been devoted to study several properties of these models, see for example [23], [5] and references therein. In particular, the well-posedness issue for the IVP associated to (0.0.14) has also been studied extensively in recent literature. Using the method developed by Kenig, Ponce and Vega [44] to show local well-posedness for the IVP associated with the KdV equation in $H^s(\mathbb{R})$, $s > 3/4$, Faminskii [23] proved the local well-posedness for the IVP associated to (0.0.14) when the given data is in $H^m(\mathbb{R}^2)$, $m \geq 1$, integer. He also proved the global well-posedness in the same space using the conserved quantities

$$\int_{\mathbb{R}^2} u^2(t)\, dxdy = \int_{\mathbb{R}^2} u_0^2\, dxdy \tag{0.0.15}$$

and

$$\int_{\mathbb{R}^2} (u_x^2 + u_y^2 - \frac{1}{3}u^3)(t)\, dxdy = \int_{\mathbb{R}^2} ({u_0}_x^2 + u_{0y}^2 - \frac{1}{3}u_0^3)\, dxdy, \tag{0.0.16}$$

satisfied by the flow of (0.0.14).

In this work, we are concerned about the following question: If a sufficiently smooth real valued solution $u = u(x, y, t)$ to the IVP associated to (0.0.14) is supported compactly on a certain time interval, is it true that $u \equiv 0$? In some sense, it is a weak version of the unique continuation property (UCP) which is defined as follows:

Definition 2 *If a solution u to certain evolution equation vanishes on some non-empty open set Ω_1 of Ω then it vanishes in the horizontal component of Ω_1 in Ω, where Ω is the domain of the evolution operator under consideration.*

A pioneer work in this direction is due to Carleman [17]. Carleman's method was based on the weighted estimates for the associated solutions. Later, Carleman's method was improved and extended to address the UCP for parabolic and hyperbolic operators (see [29] and [50]). As far as we know the first work dealing with the UCP for a general class of dispersive equations in one space dimension is due to Saut and Scheurer [56]. Carleman type estimates were the main tools used by them. In particular, the class considered in [56] includes the KdV equation. Also, D. Tataru [65] proved the UCP for Schrödinger equation by deriving some Carleman type estimates. Further, Isakov [32] considered a large class of evolution equations with nonhomogeneous principal part and proved the UCP. Later, Zhang [69] proved the UCP for the KdV and modified KdV (mKdV) equations using inverse scattering theory and Miura's transformation. This slightly stronger result implies the UCP for the KdV equation obtained in [56]. To prove this result, Zhang [69] introduced some decay condition to the solution and exploited the fact that the KdV and mKdV equations are integrable. Bourgain [14] introduced a new approach to address a wider class of evolution equations using complex variables techniques. The method introduced in [14] is more general and can also be applied to models in higher spatial dimensions. Recently, Kenig, Ponce and Vega

[36], using Carleman's type estimate and the result due to Saut and Scheurer [56] proved that; if a sufficiently smooth solution u of the generalized KdV equation is supported in $(-\infty, b)$ or in (a, ∞) at two different instants of time then $u \equiv 0$. The exponential decay property of the solution is essential in the argument employed in [36]. Quite recently, Carvajal and Panthee [19] extended the argument introduced in [14] and [36] to prove the UCP for Hasegawa-Kodama equation which is a mixed equation of type KdV and Schrödinger. Also there are recent works due to Iório in [30] and [31] dealing with the UCP for equations of Benjamin-Ono type and Kenig, Ponce Vega [35] for nonlinear Schrödinger equation.

Here we are going to generalize the scheme in [14] to address a bi-dimensional (spatial) model and provide an affirmative answer to the question posed above.

We organize this work as follows. First we list some notations that will be used throughout this work. Chapter 1 contains the results concerning the Cauchy problem for the IVP (0.0.3). Chapter 2 deals with the global solution to the system of mKdV equations. In Chapter 3 we establish the unique continuation property for the ZK equation. Finally, we present some discussion regarding the further scope and extension of the work conducted here.

Notation

Here we give some notations that we are going to use throughout this work.

- $\mathbb{N}$ - set of natural numbers
- $\mathbb{R}$ - set of real numbers
- $\mathbb{Z}$ - set of integers
- $\mathbb{C}$ - set of complex numbers
- $\partial_x^k u$ or $u_{x \cdots x}$ or $\frac{\partial^k u}{\partial x^k}$ - partial derivative of u w.r.t. variable x of order k
- $\hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$ - Fourier transform of f
- $f^\vee(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, d\xi$ - inverse Fourier transform of f
- $D_x^s f := (-\partial_x^2)^{s/2} f = [|\cdot|^s \hat{f}(\cdot)]^\vee$ - Riesz potential of order $-s$.
- $\mathcal{S}(\mathbb{R}^n)$ - Schwartz space on $\mathbb{R}^n$
- $C([0,T];X)$ - space of continuous functions from $[0,T]$ into X
- $\|f\|_s := (\int (1+|\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi)^{1/2}$
- $H^s(\mathbb{R}) := H^s$ - Sobolev space of order s with norm $\|f\|_s$
- $L_t^p(L_x^q)$, $(1 < p < \infty)$ - Banach spaces $L^p(\mathbb{R} : L^q(\mathbb{R}))$ for variables t and x respectively
- C, c - various constants whose exact values are immaterial
- $A \lesssim B$ - there exists a constant $C > 0$ such that $A < CB$
- $A \gtrsim B$ - there exists a constant $C > 0$ such that $A > CB$
- $A \sim B$ - $A \lesssim B$ and $A \gtrsim B$

- $\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p\, dx\right)^{1/p}, \quad 1 \leq p < \infty$, with usual modification for $p = \infty$
- $X^s := H^s(\mathbb{R}) \times H^s(\mathbb{R})$ - Cartesian product of Sobolev spaces
- $X := L^2(\mathbb{R}) \times L^2(\mathbb{R})$ - Cartesian product of L^2 spaces
- $\|\mathbf{f}\|_{X^s}^2 := \|f\|_{H^s}^2 + \|g\|_{H^s}^2$ for $\mathbf{f} = (f, g)$
- $\|\mathbf{f}\|_{L^p \times L^p} = \|f\|_{L^p} + \|g\|_{L^p}$
- $\|f\|_{L^p_x L^q_T} := \left(\int_{\mathbb{R}} \left(\int_0^T |f(x,t)|^q\, dt\right)^{p/q} dx\right)^{1/p}$ - mixed $L^p_x L^q_T$-norm with usual modification for $p = \infty$
- $\|\mathbf{f}\|_{L^p_x L^q_T} = \|f\|_{L^p_x L^q_T} + \|g\|_{L^p_x L^q_T}$
- $\operatorname{supp} \mathrm{f}$ - support of f
- $f * g$ - convolution product of f and g
- $a+ := a + \epsilon$ for $\epsilon > 0$

Chapter 1

Cauchy Problem for a Coupled System of KdV Equations

1.1 Introduction

This chapter is devoted to investigate the well-posedness issues associated to the IVP

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + u u_x + a_1 v v_x + a_2 (uv)_x = 0, \qquad x,\, t \in \mathbb{R}, \\ b_1 v_t + v_{xxx} + b_2 a_3 u_{xxx} + v v_x + b_2 a_2 u u_x + b_2 a_1 (uv)_x = 0, \\ u(x,0) = \phi(x), \\ v(x,0) = \psi(x), \end{cases} \tag{1.1.1}$$

where $u = u(x,t)$ and $v = v(x,t)$ are real valued functions and a_1, a_2, a_3, b_1, b_2 are real constants with b_1, b_2 positive.

Let us begin by introducing a function space where we are going to find solution to the IVP (1.1.1). For $s \in \mathbb{R}$ and $-1 < b < 1$, we define a Hilbert space $X_{s,b}$ as a completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\|f\|_{X_{s,b}} = \Big(\int_{\mathbb{R}^2} (1 + |\tau - \xi^3|)^{2b} (1 + |\xi|)^{2s} |\hat{f}(\xi,\tau)|^2 \, d\xi \, d\tau \Big)^{1/2},$$

where $\hat{f}$ given by

$$\hat{f}(\xi,\tau) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-i(x\xi + t\tau)} f(x,t)\, dx\, dt,$$

denotes the Fourier transform of f in both x and t variables.

Let us recall some properties of the space $X_{s,b}$ regarding the regularity. First, observe that for $f \in X_{s,b}$, one has,

$$\|f\|_{X_{s,b}} = \|(1+D_t)^b U(t) f\|_{L^2_t(H^s_x)},$$

where $U(t) = e^{-t\partial_x^3}$ is the unitary group associated with the linear problem.

If $b > 1/2$, the previous remark and the Sobolev lemma imply,

$$X_{s,b} \subset C(\mathbb{R}; H^s_x(\mathbb{R})).$$

Now we are in position to state the main results of this chapter.

1.2 Main Results

The first result is concerned about the local well-posedness for the IVP (1.1.1) and reads as follows.

Theorem 1.1 *For any $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$ and $b \in (1/2, 1)$, there exist $T = T(\|\phi\|_{H^s}, \|\psi\|_{H^s})$ and a unique solution of (1.1.1) in the time interval $[-T, T]$ satisfying*

$$u, v \in C([-T, T]; H^s(\mathbb{R})), \tag{1.2.1}$$

$$u, v \in X_{s,b} \subseteq L^p_{x,loc}(\mathbb{R}; L^2_t(\mathbb{R})), \quad for\ 1 \leq p \leq \infty, \tag{1.2.2}$$

$$(u^2)_x, (v^2)_x \in X_{s,b-1}, \tag{1.2.3}$$

and

$$u_t, v_t \in X_{s-3,b-1}. \tag{1.2.4}$$

Moreover, given $T' \in (0, T)$, the map $(\phi, \psi) \mapsto (u(t), v(t))$ is smooth from $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ to $C([-T', T']; H^s(\mathbb{R})) \times C([-T', T']; H^s(\mathbb{R}))$.

Our next theorem deals with the global well-posedness for the IVP (1.1.1). More precisely, we prove the following result.

Theorem 1.2 *The initial value problem (1.1.1) is globally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/10$ in the case when $a_3 = 0$ and $b_1 = b_2$.*

The final result of this chapter is concerned about the ill-posedness for the IVP (1.1.1). In fact, we prove the following theorem showing that the local result given by Theorem 1.1 is sharp.

Theorem 1.3 *Let $s < -3/4$, then there is no $T > 0$ such that the flow-map*

$$(\phi, \psi) \mapsto (u(t), v(t)), \qquad t \in (0, T]$$

be C^2 Frechet-differentiable at the origin from $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R}) \times H^s(\mathbb{R}))$.

1.3 Reduction of the Problem and Preliminary Estimates

In this section we decouple the dispersive terms in the system (1.1.1). Also we recall some estimates that will be useful in the proof of Theorem 1.1.

If $a_3 = 0$ there is no coupling in the dispersive terms. So we suppose $a_3 \neq 0$. As mentioned above, we are interested to decouple the dispersive terms in the system (1.1.1). For this, let $a_3^2 b_2 \neq 1$ and define,

$$\lambda = \left\{\left(1 - \frac{1}{b_1}\right)^2 + \frac{4b_2 a_3^2}{b_1}\right\}^{1/2} > 0 \quad \text{and} \quad \alpha_\pm = \frac{1}{2}\left(1 + \frac{1}{b_1} \pm \lambda\right).$$

Our assumption $a_3^2 b_2 \neq 1$ guarantees that $\alpha_\pm \neq 0$. Thus we can write the system (1.1.1) in a matrix form and then diagonalize the matrix of coefficients corresponding to the dispersive terms as in [55]. After that we make the change of scale

$$u(x,t) = \tilde{u}(\alpha_+^{-1/3}x, t) \qquad \text{and} \qquad v(x,t) = \tilde{v}(\alpha_-^{-1/3}x, t).$$

Then we obtain the system of equations

$$\begin{cases} \tilde{u}_t + \tilde{u}_{xxx} + a\tilde{u}\tilde{u}_x + b\tilde{v}\tilde{v}_x + c(\tilde{u}\tilde{v})_x = 0, \\ \tilde{v}_t + \tilde{v}_{xxx} + \tilde{a}\tilde{u}\tilde{u}_x + \tilde{b}\tilde{v}\tilde{v}_x + \tilde{c}(\tilde{u}\tilde{v})_x = 0, \\ \tilde{u}(x,0) = \tilde{\phi}(x), \\ \tilde{v}(x,0) = \tilde{\psi}(x), \end{cases} \tag{1.3.1}$$

where a, b, c and $\tilde{a}, \tilde{b}, \tilde{c}$ are constants.

Remark 1.1 *Notice that the nonlinear terms involving the functions $\tilde{u}$ and $\tilde{v}$ are not evaluated at the same point. Therefore those terms are not local anymore. Hence we should be careful in making the estimates. In the existing literature, see for instance [55] and [4], this feature of the nonlinear terms was not pointed out which may lead to wrong conclusions.*

Remark 1.2 *Due to the previous remark, in Proposition 1.1 below, we need to estimate terms of the form $\partial_x(u(Ax,t)v(Bx,t))$ or more generally $\partial_x(u(Ax+C,t)v(Bx+D,t))$. It is not difficult to prove the same inequality since the only changes coming out are from the contributions given by the constants A, B, C and D.*

The system (1.3.1) has a pair of KdV equations coupled only in the nonlinear terms. It is enough to prove local well-posedness for system (1.3.1) because the results for the IVP (1.1.1) can be obtained in the obvious way.

Hence, our interest is to solve the system (1.3.1) for initial data $(\tilde{\phi}, \tilde{\psi})$ in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$. For this we use the Fourier transform restriction space $X_{s,b}$ discussed above. For the sake of simplicity, from now onwards we will drop '~' and use the notation u, v, ϕ, ψ in the system (1.3.1).

Using Duhamel's principle, we study the following system of integral equations equivalent to the system (1.3.1),

$$\begin{cases} u(t) = U(t)\phi - \int_0^t U(t-t')F(u,v,u_x,v_x)(t')\,dt', \\ v(t) = U(t)\psi - \int_0^t U(t-t')G(u,v,u_x,v_x)(t')\,dt', \end{cases} \tag{1.3.2}$$

where $U(t) = e^{-t\partial_x^3}$ is the unitary group associated with the linear problem and F and G are respective nonlinearities.

To find a local solution to (1.3.1) we can replace (1.3.2) with the following system of integral equations,

$$\begin{cases} u(t) = \psi_1(t)U(t)\phi - \psi_1(t)\int_0^t U(t-t')\psi_\delta(t')F(u,v,u_x,v_x)(t')\,dt', \\ v(t) = \psi_1(t)U(t)\psi - \psi_1(t)\int_0^t U(t-t')\psi_\delta(t')G(u,v,u_x,v_x)(t')\,dt', \end{cases} \tag{1.3.3}$$

where $\psi_1 \in C_0^\infty(\mathbb{R})$, $0 \le \psi_1 \le 1$ is a cut-off function given by,

$$\psi_1 = \begin{cases} 1, & |t| < 1 \\ 0, & |t| \ge 2 \end{cases}$$

and $\psi_\delta = \psi_1(t/\delta)$, $0 < \delta \le 1$.

Now, let us recall some estimates which will be used to prove the local well-posedness result.

Lemma 1.1 *Let $s \in \mathbb{R}$, $b', b \in (1/2, 1)$ with $b' < b$ and $\delta \in (0,1)$; then we have,*

$$\|\psi_\delta(t)U(t)\phi\|_{X_{s,b}} \le C\,\delta^{\frac{(1-2b)}{2}}\|\phi\|_{H^s}, \tag{1.3.4}$$

$$\|\psi_\delta F\|_{X_{s,b-1}} \le C\delta^{\frac{b-b'}{8(1-b')}}\|F\|_{X_{s,b'-1}}, \tag{1.3.5}$$

$$\left\|\psi_\delta(t)\int_0^t U(t-t')F(t')\,dt'\right\|_{X_{s,b}} \le C\,\delta^{\frac{(1-2b)}{2}}\|F\|_{X_{s,b-1}} \tag{1.3.6}$$

and

$$\left\|\psi_\delta(t)\int_0^t U(t-t')F(t')\,dt'\right\|_{H^s} \le C\,\delta^{\frac{(1-2b)}{2}}\|F\|_{X_{s,b-1}}. \tag{1.3.7}$$

Proposition 1.1 *Let $s > -3/4$, then there exists $1/2 < b < 1$ such that the following bilinear estimate holds,*

$$\|(uv)_x\|_{X_{s,b-1}} \leq C\|u\|_{X_{s,b}}\|v\|_{X_{s,b}}. \tag{1.3.8}$$

The proof of Lemma 1.1 can be found in ([40], [38]) and that of Proposition 1.1 in [38], so we skip the details.

1.4 Local Well-posedness Result

In this section we supply the proof of Theorem 1.1, the local well-posedness result for the IVP (1.1.1).

Proof.[Proof of Theorem 1.1:] We consider the following function space where we seek a solution to the IVP (1.3.1). For given $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ and $b > 1/2$, let us define,

$$\mathcal{H}_{MN} := \{(u, v) \in X_{s,b} \times X_{s,b} : \|u\|_{X_{s,b}} \leq M,\ \|v\|_{X_{s,b}} \leq N\},$$

where $M = 2C_0\|\phi\|_{H^s}$ and $N = 2C_0\|\psi\|_{H^s}$. Then $\mathcal{H}_{MN}$ is a complete metric space with norm,

$$|||(u, v)|||_{\mathcal{H}_{MN}} := \|u\|_{X_{s,b}} + \|v\|_{X_{s,b}}.$$

Without loss of generality, we may assume that $M > 1$ and $N > 1$. For $(u, v) \in \mathcal{H}_{MN}$, let us define the maps,

$$\begin{cases} \Phi_\phi[u, v] = \psi_1(t)U(t)\phi - \psi_1(t)\int_0^t U(t-t')\psi_\delta(t')F(u, v, u_x, v_x)(t')\,dt' \\ \Psi_\psi[u, v] = \psi_1(t)U(t)\psi - \psi_1(t)\int_0^t U(t-t')\psi_\delta(t')G(u, v, u_x, v_x)(t')\,dt'. \end{cases} \tag{1.4.1}$$

We prove that $\Phi \times \Psi$ maps $\mathcal{H}_{MN}$ into $\mathcal{H}_{MN}$ and is a contraction.
Using (1.3.4) and (1.3.6) we get from (1.4.1),

$$\begin{cases} \|\Phi[u, v]\|_{X_{s,b}} \leq C_0\|\phi\|_{H^s} + C\|\psi_\delta F(u, v, u_x, v_x)\|_{X_{s,b-1}} \\ \|\Psi[u, v]\|_{X_{s,b}} \leq C_0\|\psi\|_{H^s} + C\|\psi_\delta G(u, v, u_x, v_x)\|_{X_{s,b-1}}. \end{cases} \tag{1.4.2}$$

Now, using (1.3.5) we get from (1.4.2) for $b' < b$ and $\theta = \frac{b-b'}{8(1-b')}$,

$$\begin{cases} \|\Phi[u, v]\|_{X_{s,b}} \leq C_0\|\phi\|_{H^s} + C\delta^\theta\|F(u, v, u_x, v_x)\|_{X_{s,b'-1}} \\ \|\Psi[u, v]\|_{X_{s,b}} \leq C_0\|\psi\|_{H^s} + C\delta^\theta\|G(u, v, u_x, v_x)\|_{X_{s,b'-1}}. \end{cases} \tag{1.4.3}$$

Using the bilinear estimate (1.3.8), the estimate (1.4.3) yields,

$$\begin{cases}\|\Phi[u,v]\|_{X_{s,b}} \leq C_0\|\phi\|_{H^s} + C_1\delta^\theta\{\|u\|^2_{X_{s,b}} + \|v\|^2_{X_{s,b}} + \|u\|_{X_{s,b}}\|v\|_{X_{s,b}}\}\\ \|\Psi[u,v]\|_{X_{s,b}} \leq C_0\|\psi\|_{H^s} + C_2\delta^\theta\{\|u\|^2_{X_{s,b}} + \|v\|^2_{X_{s,b}} + \|u\|_{X_{s,b}}\|v\|_{X_{s,b}}\}.\end{cases} \tag{1.4.4}$$

As $(u,v)\in\mathcal{H}_{MN}$, with our choice of M and N we get from (1.4.4),

$$\begin{cases}\|\Phi[u,v]\|_{X_{s,b}} \leq \frac{M}{2} + C_1\delta^\theta\{M^2+N^2+MN\}\\ \|\Psi[u,v]\|_{X_{s,b}} \leq \frac{N}{2} + C_2\delta^\theta\{M^2+N^2+MN\}.\end{cases} \tag{1.4.5}$$

If we choose δ such that

$$\delta^\theta \leq \big(2\max\{C_1,C_2\}(M+N)^2\big)^{-1},$$

then we get from (1.4.5),

$$\|\Phi[u,v]\|_{X_{s,b}} \leq M \quad \text{and} \quad \|\Psi[u,v]\|_{X_{s,b}} \leq N.$$

Therefore,

$$(\Phi[u,v],\Psi[u,v])\in\mathcal{H}_{MN}.$$

Now, we need to show that $\Phi\times\Psi : (u,v)\mapsto(\Phi[u,v],\Psi[u,v])$ is a contraction. For this, let $(u,v),(u_1,v_1)\in\mathcal{H}_{MN}$, then as above using Lemma 1.1 and Proposition 1.1 we get,

$$\begin{cases}\|\Phi[u,v]-\Phi[u_1,v_1]\|_{X_{s,b}} \leq C_1\delta^\theta(M+N)\big[\|u-u_1\|_{X_{s,b}} + \|v-v_1\|_{X_{s,b}}\big]\\ \|\Psi[u,v]-\Psi[u_1,v_1]\|_{X_{s,b}} \leq C_2\delta^\theta(M+N)\big[\|u-u_1\|_{X_{s,b}} + \|v-v_1\|_{X_{s,b}}\big].\end{cases} \tag{1.4.6}$$

If we choose δ such that

$$\delta^\theta \leq \big(4\max\{C_1,C_2\}(M+N)^2\big)^{-1},$$

then (1.4.6) yields,

$$\begin{cases}\|\Phi[u,v]-\Phi[u_1,v_1]\|_{X_{s,b}} \leq \frac{1}{4}[\|u-u_1\|_{X_{s,b}} + \|v-v_1\|_{X_{s,b}}]\\ \|\Psi[u,v]-\Psi[u_1,v_1]\|_{X_{s,b}} \leq \frac{1}{4}[\|u-u_1\|_{X_{s,b}} + \|v-v_1\|_{X_{s,b}}].\end{cases} \tag{1.4.7}$$

Therefore the map $\Phi\times\Psi$ is a contraction and we obtain a unique fixed point (u,v) which solves the IVP (1.3.1) for $t\in[-T,T]$ with $T\leq\delta$. The remainder of the proof follows a standard argument so we skip it. Just to be precise, the smoothness of the flow-map follows by using Implicit Function Theorem. $\square$

1.5 Global Well-posedness Result

This section is devoted to extend the local solution obtained in the previous section to the global one. Using usual conservation laws we have global solution in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 0$. So, we suppose $s < 0$ throughout this section. Our aim here is to derive an *almost conserved quantity* and use it to prove Theorem 1.2. For this, let us define the Fourier multiplier operator I by,

$$\widehat{Iu}(\xi) = m(\xi)\hat{u}(\xi),$$

where $m(\xi)$ is a smooth and monotone function given by

$$m(\xi) = \begin{cases} 1, & |\xi| < N, \\ N^{-s}|\xi|^s, & |\xi| \geq 2N, \end{cases}$$

with $N \gg 1$ to be fixed later.

Note that, I is the identity operator in low frequencies, $\{\xi : |\xi| < N\}$, and simply an integral operator in high frequencies. In general, it commutes with differential operators and satisfies the following property.

Lemma 1.2 *Let $-3/4 < s < 0$ and $N \gg 1$. Then the operator I maps $H^s(\mathbb{R})$ to $L^2(\mathbb{R})$ and*

$$\|If\|_{L^2(\mathbb{R})} \lesssim N^{-s}\|f\|_{H^s(\mathbb{R})}. \tag{1.5.1}$$

Proof.

$$\begin{aligned}
\|If\|_{L^2}^2 &= \|\widehat{If}\|_{L^2}^2 = \|m(\cdot)\hat{f}\|_{L^2}^2 \\
&= \int_{|\xi|<N} |\hat{f}(\xi)|^2\, d\xi + \int_{N\leq|\xi|\leq 2N} |m(\xi)|^2|\hat{f}(\xi)|^2\, d\xi + \int_{|\xi|>2N} N^{-2s}|\xi|^{2s}|\hat{f}(\xi)|^2\, d\xi \\
&\leq CN^{-2s}\|f\|_{H^s}^2 + N^{-2s}\int_{|\xi|>2N} (1+\xi^2)^s|\hat{f}(\xi)|^2(1+\frac{1}{\xi^2})^{-s}\, d\xi \\
&\leq CN^{-2s}\|f\|_{H^s}^2.
\end{aligned}$$

□

As discussed in the introduction let us consider the IVP (1.1.1) with $a_3 = 0$ and $b_1 = b_2$, that is,

$$\begin{cases} u_t + u_{xxx} + uu_x + a_1vv_x + a_2(uv)_x = 0, \\ b_1v_t + v_{xxx} + vv_x + b_2a_2uu_x + b_2a_1(uv)_x = 0, \\ u(x,0) = \phi(x), \\ v(x,0) = \psi(x). \end{cases} \tag{1.5.2}$$

After introducing the multiplier operator I, we have the following variant of the local well-posedness for the IVP (1.5.2).

Theorem 1.4 *For any $(\phi,\psi)\in H^s(\mathbb{R})\times H^s(\mathbb{R})$, $s>-3/4$, the IVP (1.5.2) is locally well-posed in the Banach space $I^{-1}L^2\times I^{-1}L^2=\{(\phi,\psi)\in H^s\times H^s,\ \text{with norm } \|I\phi\|_{L^2}+\|I\psi\|_{L^2}\}$ with the existence lifetime satisfying,*

$$\delta\gtrsim(\|I\phi\|_{L^2}^2+\|I\psi\|_{L^2}^2)^{-\theta},\qquad \theta>0. \tag{1.5.3}$$

Moreover,

$$\begin{cases}\|\psi_\delta Iu\|_{X_{0,b}}\leq C\|I\phi\|_{L^2}\\ \|\psi_\delta Iv\|_{X_{0,b}}\leq C\|I\psi\|_{L^2}.\end{cases} \tag{1.5.4}$$

The proof of this theorem is not difficult and follows by using the same procedure used to prove the local well-posedness for the IVP (1.3.1) (see the Proof of Theorem 1.1) once we have the bilinear estimate

$$\|\partial_x I(uv)\|_{X_{0,-\frac12+}}\leq C\|Iu\|_{X_{0,\frac12+}}\|Iv\|_{X_{0,\frac12+}}. \tag{1.5.5}$$

The proof of the bilinear estimate (1.5.5) is easy and follows by using the usual bilinear estimate (1.3.8) due to Kenig, Ponce and Vega [38] combined with the following extra smoothing bilinear estimate whose proof is given in Colliander, Keel, Staffilani, Takaoka and Tao [22].

Proposition 1.2 *The bilinear estimate*

$$\|\partial_x\{(IuIv-I(uv)\}\|_{X^\delta_{0,-\frac12-}}\leq CN^{-\frac34+}\|Iu\|_{X^\delta_{0,-\frac12+}}\|Iv\|_{X^\delta_{0,-\frac12+}}, \tag{1.5.6}$$

holds.

Now we proceed to introduce the almost conserved quantity. Using the Fundamental Theorem of Calculus, the equation and integration by parts we get,

$$\begin{aligned}\|Iu(\delta)\|_{L^2}^2=&\|Iu(0)\|_{L^2}^2+\int_0^\delta\frac{d}{dt}(Iu(t),Iu(t))\,dt\\ =&\|Iu(0)\|_{L^2}^2+2\int_0^\delta\Big(\frac{d}{dt}Iu(t),Iu(t)\Big)\,dt\\ =&\|Iu(0)\|_{L^2}^2+2\int_0^\delta(I(-u_{xxx}-uu_x-a_1vv_x-a_2(uv)_x),Iu(t))\ dt\\ =&\|Iu(0)\|_{L^2}^2+2\int_0^\delta(I(-uu_x-a_1vv_x-a_2(uv)_x),Iu(t))\ dt\\ =&\|Iu(0)\|_{L^2}^2+R_1(\delta),\end{aligned} \tag{1.5.7}$$

where

$$R_1(\delta) = \int_0^\delta \int_{\mathbb{R}} \partial_x(-Iu^2 - a_1 Iv^2 - 2a_2 I(uv)) Iu\, dxdt. \tag{1.5.8}$$

Similarly,

$$\|Iv(\delta)\|_{L^2}^2 = \|Iv(0)\|_{L^2}^2 + R_2(\delta), \tag{1.5.9}$$

where

$$R_2(\delta) = \int_0^\delta \int_{\mathbb{R}} \partial_x(-\frac{1}{b_1} Iv^2 - \frac{b_2 a_2}{b_1} Iu^2 - \frac{2b_2 a_1}{b_1} I(uv)) Iv\, dxdt. \tag{1.5.10}$$

Let us define $R(\delta) := R_1(\delta) + R_2(\delta)$, so that we have from (1.5.7) and (1.5.9),

$$\|Iu(\delta)\|_{L^2}^2 + \|Iv(\delta)\|_{L^2}^2 = \|Iu(0)\|_{L^2}^2 + \|Iv(0)\|_{L^2}^2 + R(\delta). \tag{1.5.11}$$

We will obtain the so called almost conserved quantity from (1.5.11) by treating $R(\delta)$ as an error term. In what follows we prove a cancellation property which plays a vital role in our analysis.

Lemma 1.3 *The following cancellations hold.*

$$\int_0^\delta \int_{\mathbb{R}} \partial_x(Iu)^2 Iu\, dxdt = 0 = \int_0^\delta \int_{\mathbb{R}} \partial_x(Iv)^2 Iv\, dxdt \tag{1.5.12}$$

and

$$a_1 b_1 I_1 + 2a_2 b_1 I_2 + b_2 a_2 I_3 + 2b_2 a_1 I_4 = 0, \quad \text{if} \quad b_1 = b_2, \tag{1.5.13}$$

where

$$I_1 = \int_0^\delta \int_{\mathbb{R}} \partial_x(Iv)^2 Iu\, dxdt, \qquad I_2 = \int_0^\delta \int_{\mathbb{R}} \partial_x(IuIv) Iu\, dxdt,$$
$$I_3 = \int_0^\delta \int_{\mathbb{R}} \partial_x(Iu)^2 Iv\, dxdt \quad \text{and} \quad I_4 = \int_0^\delta \int_{\mathbb{R}} \partial_x(IuIv) Iv\, dxdt.$$

Proof. The proof of (1.5.12) is trivial and (1.5.13) follows by using integration by parts. In fact, for $b_1 = b_2$,

$$\begin{aligned}
a_1I_1+&2a_2I_2 + a_2I_3 + 2a_1I_4 = \\
&= a_1\int_0^\delta\int_{\mathbb{R}}\partial_x(Iv)^2Iu\,dxdt + 2a_2\int_0^\delta\int_{\mathbb{R}}\partial_x(IuIv)Iu\,dxdt \\
&\qquad + a_2\int_0^\delta\int_{\mathbb{R}}\partial_x(Iu)^2Iv\,dxdt + 2a_1\int_0^\delta\int_{\mathbb{R}}\partial_x(IuIv)Iv\,dxdt \\
&= - a_1\int_0^\delta\int_{\mathbb{R}}(Iv)^2\partial_xIu\,dxdt - a_2\int_0^\delta\int_{\mathbb{R}}Iv\partial_x(Iu)^2\,dxdt \\
&\qquad - a_2\int_0^\delta\int_{\mathbb{R}}(Iu)^2\partial_xIv\,dxdt - a_1\int_0^\delta\int_{\mathbb{R}}Iu\partial_x(Iv)^2\,dxdt \\
&= - a_1\int_0^\delta\int_{\mathbb{R}}\partial_x[Iu(Iv)^2]\,dxdt - a_2\int_0^\delta\int_{\mathbb{R}}\partial_x[Iv(Iu)^2]\,dxdt \\
&= 0.
\end{aligned}$$

□

Remark 1.3 *Note that, it is here in Lemma 1.3, where the restriction $b_1 = b_2$ on the coefficients appears. From here onwards we will use this restriction on the coefficients.*

Using Lemma 1.3, $R(\delta)$ can be written as,

$$\begin{aligned}
&R(\delta) = \\
&\int_0^\delta\int_{\mathbb{R}}\partial_x\{(Iu)^2 - I(u^2)\}Iu\,dxdt + \frac{1}{b_1}\int_0^\delta\int_{\mathbb{R}}\partial_x\{(Iv)^2 - (Iv^2)\}Iv\,dxdt \\
&+ a_1\int_0^\delta\int_{\mathbb{R}}\partial_x\{(Iv)^2 - I(v^2)\}Iu\,dxdt + a_2\int_0^\delta\int_{\mathbb{R}}\partial_x\{(Iu)^2 - (Iu^2)\}Iv\,dxdt \\
&+ 2a_2\int_0^\delta\int_{\mathbb{R}}\partial_x\{IuIv - I(uv)\}Iu\,dxdt + 2a_1\int_0^\delta\int_{\mathbb{R}}\partial_x\{IuIv - I(uv)\}Iv\,dxdt.
\end{aligned} \tag{1.5.14}$$

Using Plancherel identity and Cauchy-Schwarz inequality as in [60] we get,

$$\begin{aligned}
|R(\delta)| \leq C\Big\{&\|\partial_x\{(IuIv - I(uv)\}\|_{X^\delta_{0,-\frac12-}}\big(\|Iu\|_{X^\delta_{0,-\frac12+}} + \|Iv\|_{X^\delta_{0,-\frac12+}}\big) \\
&+ \big(\|\partial_x\{(Iu)^2 - I(u^2)\}\|_{X^\delta_{0,-\frac12-}} + \|\partial_x\{(Iv)^2 - I(v^2)\}\|_{X^\delta_{0,-\frac12-}}\big)\|Iu\|_{X^\delta_{0,-\frac12+}} \\
&+ \big(\|\partial_x\{(Iv)^2 - I(v^2)\}\|_{X^\delta_{0,-\frac12-}} + \|\partial_x\{(Iu)^2 - I(u^2)\}\|_{X^\delta_{0,-\frac12-}}\big)\|Iv\|_{X^\delta_{0,-\frac12+}}\Big\}.
\end{aligned} \tag{1.5.15}$$

Now, using (1.5.15) and Proposition 1.2 the identity (1.5.11) yields the following almost conservation law,

$$\begin{aligned}\|Iu(\delta)\|_{L^2}^2 + \|Iv(\delta)\|_{L^2}^2 \leq \|Iu(0)\|_{L^2}^2 + \|Iv(0)\|_{L^2}^2 \\ + CN^{-\frac{3}{4}+}\Big\{\|Iu\|^3_{X^\delta_{0,-\frac{1}{2}+}} + \|Iu\|^2_{X^\delta_{0,-\frac{1}{2}+}}\|Iv\|_{X^\delta_{0,-\frac{1}{2}+}} \\ + \|Iv\|^2_{X^\delta_{0,-\frac{1}{2}+}}\|Iu\|_{X^\delta_{0,-\frac{1}{2}+}} + \|Iv\|^3_{X^\delta_{0,-\frac{1}{2}+}}\Big\}.\end{aligned} \tag{1.5.16}$$

Now we are in position to prove the global well-posedness result.

Proof.[Proof of Theorem 1.2:] To prove the theorem it is enough to show that the local solution to the IVP (1.5.2) can be extended to $[0,T]$ for arbitrary $T > 0$. To make the analysis easy we use the scaling introduced in the introduction. That is, if (u,v) solves the IVP (1.5.2) with initial data (ϕ,ψ) then for $1 > \lambda > 0$ so does (u^λ, v^λ); where $u^\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^3 t)$, $v^\lambda(x,t) = \lambda^2 v(\lambda x, \lambda^3 t)$; with initial data $(\phi^\lambda, \psi^\lambda)$ given by $\phi^\lambda(x) = \lambda^2\phi(\lambda x)$, $\psi^\lambda(x) = \lambda^2\psi(\lambda x)$. Observe that, (u,v) exists in $[0,T]$ if and only if (u^λ, v^λ) exists in $[0, T/\lambda^3]$. So we are interested in extending (u^λ, v^λ) to $[0, T/\lambda^3]$.

Using Lemma 1.2 we have,

$$\begin{cases}\|I\phi^\lambda\|_{L^2} \leq C\lambda^{\frac{3}{2}+s}N^{-s}\,\|\phi\|_{H^s}, \\ \|I\psi^\lambda\|_{L^2} \leq C\lambda^{\frac{3}{2}+s}N^{-s}\,\|\psi\|_{H^s}.\end{cases} \tag{1.5.17}$$

$N = N(T)$ will be selected later, but let us choose $\lambda = \lambda(N)$ right now by requiring that,

$$\begin{cases}C\,\lambda^{\frac{3}{2}+s}N^{-s}\,\|\phi\|_{H^s} = \sqrt{\frac{\epsilon_0}{2}} \ll 1, \\ C\,\lambda^{\frac{3}{2}+s}N^{-s}\,\|\psi\|_{H^s} = \sqrt{\frac{\epsilon_0}{2}} \ll 1.\end{cases} \tag{1.5.18}$$

From (1.5.18) we get, $\lambda \sim N^{\frac{2s}{3+2s}}$ and using (1.5.18) in (1.5.17) we get,

$$\begin{cases}\|I\phi^\lambda\|_{L^2}^2 \leq \frac{\epsilon_0}{2} \ll 1 \\ \|I\psi^\lambda\|_{L^2}^2 \leq \frac{\epsilon_0}{2} \ll 1.\end{cases} \tag{1.5.19}$$

Therefore, if we choose ϵ_0 arbitrarily small then from Theorem 1.4 we see that the IVP (1.5.2) is well-posed for all $t \in [0,1]$.

Now, using the almost conserved quantity (1.5.16), the identity (1.5.19) and Theorem 1.4, we get,

$$\begin{aligned}\|Iu^\lambda(1)\|_{L^2}^2 + \|Iv^\lambda(1)\|_{L^2}^2 &\leq \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} + CN^{-\frac{3}{4}+}\{3\frac{\epsilon_0}{2}(\frac{\epsilon_0}{2})^{1/2}\} \\ &\leq \epsilon_0 + CN^{-\frac{3}{4}+}\epsilon_0.\end{aligned} \tag{1.5.20}$$

So, we can iterate this process $C^{-1}N^{\frac{3}{4}-}$ times before doubling $\|Iu^\lambda(t)\|_{L^2}^2+\|Iv^\lambda(t)\|_{L^2}^2$. By this process we can extend the solution to the time interval $[0,\, C^{-1}N^{\frac{3}{4}-}]$ by taking $C^{-1}N^{\frac{3}{4}-}$ time steps of size $O(1)$. As we are interested in extending the solution to the time interval $[0,\, T/\lambda^3]$, let us select $N = N(T)$ such that, $C^{-1}N^{\frac{3}{4}-} \geq T/\lambda^3$. That is,

$$N^{\frac{3}{4}-} \geq C\frac{T}{\lambda^3} \sim TN^{\frac{-6s}{3+2s}}.$$

Therefore for large N, the existence interval will be arbitrarily large if we choose s such that $s > -3/10$. This completes the proof of the theorem. □

1.6 Ill-posedness Result

As in the local well-posedness result, we consider the system (1.3.1). Note that we have dropped '~' and retained the notation u, v, ϕ and ψ. Let $W = (u, v)^T$, then the IVP (1.3.1) can be written as,

$$\begin{cases} W_t + W_{xxx} + B(W)W_x = 0, \\ W(x,0) = W_0(x), \end{cases} \tag{1.6.1}$$

where,

$$B(W) = \begin{pmatrix} au + dv & cu + bv \\ \tilde{a}u + \tilde{d}v & \tilde{c}u + \tilde{b}v \end{pmatrix}.$$

For fixed $\Phi \in \dot{H}^s(\mathbb{R}) \times \dot{H}^s(\mathbb{R})$, consider the solution $W = W^\delta$ of the IVP

$$\begin{cases} W_t + W_{xxx} + B(W)W_x = 0, \\ W(x,0) = \delta\Phi(x), \qquad \delta \in \mathbb{R}. \end{cases} \tag{1.6.2}$$

We will show that the flow-map $\delta\Phi \mapsto W^\delta(x,t)$ fails to be C^2 at the origin when $s < -3/4$. More precisely, we prove the following theorem which in turn implies Theorem 1.3.

Theorem 1.5 *Let $s < -3/4$, then there is no $T > 0$ such that the flow-map*

$$\delta\Phi \mapsto W^\delta(t), \qquad t \in (0, T]$$

be C^2 Frechet-differentiable at the origin from $\dot{H}^s(\mathbb{R}) \times \dot{H}^s(\mathbb{R})$ to $C([0,T]; \dot{H}^s(\mathbb{R}) \times \dot{H}^s(\mathbb{R}))$.

Proof. We prove it by contradiction. Suppose that the flow-map be C^2 differentiable at the origin. Using Duhamel's formula we have from (1.6.2),

$$W^\delta(x,t) = \delta U(t)\Phi(x) - \int_0^t U(t-t')B(W^\delta(x,t'))W_x^\delta(x,t')\,dt', \tag{1.6.3}$$

where $U(t)$ is the unitary group associated to the linear problem. Differentiating (1.6.3) with respect to δ we get,

$$\frac{\partial W^\delta(x,t)}{\partial \delta}\Big|_{\delta=0} = U(t)\Phi(x) := W_1(x,t), \tag{1.6.4}$$

$$\frac{\partial^2 W^\delta(x,t)}{\partial \delta^2}\Big|_{\delta=0} = -2\int_0^t U(t-t')B(W_1(x,t'))W_{1x}(x,t')\,dt' := W_2(x,t). \tag{1.6.5}$$

Our assumption of the C^2 regularity of the flow-map at the origin implies that,

$$\|W_2(\cdot,t)\|_{\dot H^s\times\dot H^s} \le C\,\|\Phi\|^2_{\dot H^s\times\dot H^s}. \tag{1.6.6}$$

Now, we look for $\Phi \in \dot H^s(\mathbb{R}) \times \dot H^s(\mathbb{R})$ so that (1.6.6) fails to hold whenever $s < -3/4$. For this, let $I_1 := [-N, -N+\alpha]$, $I_2 := [N+\alpha, N+2\alpha]$ with $N \gg 1$ and $\alpha \ll 1$, define ϕ by the formula

$$\hat\phi(\xi) = \alpha^{-\frac12}N^{-s}\big\{\chi_{I_1}(\xi) + \chi_{I_2}(\xi)\big\}, \tag{1.6.7}$$

and take $\Phi = (\phi,\phi)^T$.

It is easy to see that

$$\|\Phi\|_{\dot H^s\times\dot H^s} \sim 1. \tag{1.6.8}$$

Now we proceed to calculate $\|W_2(\cdot,t)\|_{\dot H^s\times\dot H^s}$. For this, let us first calculate $W_1(x,t)$ and $W_2(x,t)$.

From (1.6.4) we have ,

$$\widehat{W_1}^{(x)}(\xi,t) = e^{it\xi^3}\hat\Phi(\xi).$$

Therefore,

$$W_1(x,t) \sim \alpha^{-\frac12}N^{-s}\begin{pmatrix} \int_{\xi\in I_1\cup I_2} e^{ix\xi+it\xi^3}\,d\xi \\ \int_{\xi\in I_1\cup I_2} e^{ix\xi+it\xi^3}\,d\xi \end{pmatrix}.$$

From (1.6.5) we get,

$$\begin{aligned} W_2(x,t) &= -2\int_0^t U(t-t')B(W_1(x,t'))W_{1x}(x,t')\,dt' \\ &= \int_{\mathbb{R}^2} \xi e^{ix\xi+it\xi^3}\hat{\phi}(\xi-\xi_1)\hat{\phi}(\xi_1)\begin{pmatrix} a'\,\frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{3\xi\xi_1(\xi-\xi_1)} \\ b'\,\frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{3\xi\xi_1(\xi-\xi_1)} \end{pmatrix} d\xi_1\,d\xi, \end{aligned}$$

where $a' = a+b+c+d$ and $b' = \tilde{a}+\tilde{b}+\tilde{c}+\tilde{d}$. Therefore, using (1.6.7) we get,

$$W_2(x,t) \sim \alpha^{-1}N^{-2s}\int_{\substack{\xi_1\in I_1\cup I_2 \\ \xi-\xi_1\in I_1\cup I_2}} \xi e^{ix\xi+it\xi^3}\begin{pmatrix} a'\,h(\xi,\xi_1,t) \\ b'\,h(\xi,\xi_1,t)\end{pmatrix} d\xi\,d\xi_1,$$

where $h(\xi,\xi_1,t) = \frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{\xi\xi_1(\xi-\xi_1)}$. Hence, formally we have,

$$\begin{aligned} \widehat{W_2}^{(x)}(\xi,t) &\sim \alpha^{-1}N^{-2s}\xi e^{it\xi^3}\begin{pmatrix} a'\,\sum_{j=1}^3\int_{A_j(\xi)} h(\xi,\xi_1,t)\,d\xi_1 \\ b'\,\sum_{j=1}^3\int_{A_j(\xi)} h(\xi,\xi_1,t)\,d\xi_1 \end{pmatrix} \\ &:= \begin{pmatrix} p_1(\xi,t)+p_2(\xi,t)+p_3(\xi,t) \\ q_1(\xi,t)+q_2(\xi,t)+q_3(\xi,t)\end{pmatrix}, \end{aligned}$$

where,

$$\begin{cases} A_1(\xi) = \{\xi_1 : \xi_1\in I_1\,,\ \xi-\xi_1\in I_1\} \\ A_2(\xi) = \{\xi_1 : \xi_1\in I_2\,,\ \xi-\xi_1\in I_2\} \\ A_3(\xi) = \{\xi_1 : \xi_1\in I_1\,,\ \xi-\xi_1\in I_2 \quad \text{or} \quad \xi_1\in I_2\,,\ \xi-\xi_1\in I_1\}. \end{cases}$$

Let $\widehat{f_j}^{(x)} = p_j$ and $\widehat{g_j}^{(x)} = q_j$, $j=1,2,3$, then,

$$\begin{aligned} W_2(x,t) &= \begin{pmatrix} f_1(x,t) \\ g_1(x,t)\end{pmatrix} + \begin{pmatrix} f_2(x,t) \\ g_2(x,t)\end{pmatrix} + \begin{pmatrix} f_3(x,t) \\ g_3(x,t)\end{pmatrix} \\ &:= F_1(x,t)+F_2(x,t)+F_3(x,t). \end{aligned} \tag{1.6.9}$$

Let us find an upper bound for $\dot{H}^s\times\dot{H}^s$ norm of F_1. If $\xi_1\in I_1$ and $\xi-\xi_1\in I_1$ then $|\xi_1|\sim|\xi-\xi_1|\sim|\xi|\sim N$ and we get,

$$\begin{aligned} \|F_1\|^2_{\dot{H}^s\times\dot{H}^s} &= \int_{\mathbb{R}} |\xi|^{2s}(|p_1(\xi,t)|^2+|q_1(\xi,t)|^2)\,d\xi \\ &\sim \int_{\mathbb{R}} |\xi|^{2s}\alpha^{-2}N^{-4s}|\xi|^2(a'^2+b'^2)\Big|\int_{A_1(\xi)} \frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{\xi\xi_1(\xi-\xi_1)}\,d\xi_1\Big|^2 d\xi \\ &\le C\alpha^{-2}N^{-2s-4}(a'^2+b'^2)\alpha^2\alpha \\ &= C\alpha N^{-2s-4}. \end{aligned}$$

Therefore,

$$\|F_1\|^2_{\dot{H}^s\times\dot{H}^s} \le C\alpha^{\frac{1}{2}}N^{-s-2}. \tag{1.6.10}$$

Similarly,

$$\|F_2\|^2_{\dot{H}^s\times\dot{H}^s} \le C\alpha^{\frac{1}{2}}N^{-s-2}. \tag{1.6.11}$$

Now we find, with proper choice of α and N, the lower bound for the $\dot{H}^s \times \dot{H}^s$ norm of F_3.

If $\xi_1 \in I_1$ and $\xi-\xi_1 \in I_2$ or $\xi_1 \in I_2$ and $\xi - \xi_1 \in I_1$ then $|\xi_1| \sim N$, $|\xi-\xi_1| \sim N$ and $|\xi| \sim \alpha$. Therefore,

$$|\xi\xi_1(\xi-\xi_1)| \sim N^2\alpha.$$

For $0 < \epsilon \ll 1$, choose N and α such that $N^2\alpha = N^{-\epsilon}$. Hence for $\xi_1 \in A_3(\xi)$ we have, for fixed t and large N,

$$\Big|\frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{\xi\xi_1(\xi-\xi_1)}\Big| \ge C > 0. \tag{1.6.12}$$

Now,

$$\|F_3\|^2_{\dot{H}^s\times\dot{H}^s} \sim \int_\alpha^{3\alpha} |\xi|^{2s}\alpha^{-2}N^{-4s}|\xi|^2(a'^2+b'^2)\Big|\int_{A_3(\xi)} \frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{\xi\xi_1(\xi-\xi_1)}\,d\xi_1\Big|^2\,d\xi.$$

Using the Mean Value Theorem for integrals and (1.6.12) it is easy to see that,

$$\Big|\int_{A_3(\xi)} \frac{e^{-3it\xi\xi_1(\xi-\xi_1)}-1}{\xi\xi_1(\xi-\xi_1)}\,d\xi_1\Big| \ge C|A_3(\xi)|. \tag{1.6.13}$$

Also, it is easy to see that $|A_3(\xi)| \sim \alpha$. In fact, $|A_3(\xi)| \le |I_1| + |I_2| \le 2\alpha$. On the other hand, if $\xi \in (7\alpha/4,\, 9\alpha/4) \subset (\alpha, 3\alpha)$ then $[-N+\alpha/4,\, -N+3\alpha/4] \subset A_3(\xi)$ and we get $|A_3(\xi)| \ge \alpha/2$.

Now using (1.6.13), we get,

$$\|F_3\|^2_{\dot{H}^s\times\dot{H}^s} \ge CN^{-4s}\alpha^{-2}(a'^2+b'^2)\int_{\frac{7}{4}\alpha}^{\frac{9}{4}\alpha} |\xi|^{2s+2}\alpha^2\,d\xi \sim CN^{-4s}\alpha^{2s+3}.$$

Therefore,

$$\|F_3\|^2_{\dot{H}^s\times\dot{H}^s} \ge CN^{-2s}\alpha^{s+\frac{3}{2}}. \tag{1.6.14}$$

Observe that supp $\widehat{\mathrm{F}_1} \subseteq [-2\mathrm{N}, -2\mathrm{N}+2\alpha]$, supp $\widehat{\mathrm{F}_2} \subseteq [2\mathrm{N}+2\alpha, 2\mathrm{N}+4\alpha]$ and supp $\widehat{\mathrm{F}_3} \subseteq [\alpha, 3\alpha]$, which are clearly disjoint, therefore using (1.6.8), (1.6.9), (1.6.10), (1.6.11) and (1.6.14) in (1.6.6) we obtain,

$$\begin{aligned} 1 \sim \|\Phi\|^2_{\dot{H}^s\times\dot{H}^s} &\geq \|W_2(\cdot,t)\|^2_{\dot{H}^s\times\dot{H}^s} \\ &\geq \|F_3(\cdot,t)\|^2_{\dot{H}^s\times\dot{H}^s} \\ &\geq CN^{-2s}\alpha^{s+\frac{3}{2}} \\ &= CN^{-4s-3}N^{-\epsilon(s+\frac{3}{2})}. \end{aligned}$$

Hence

$$N^{-4s-3} \leq CN^{\epsilon(s+3/2)}. \tag{1.6.15}$$

Case I: If $-3/2 < s < -3/4$ then $s+3/2 > 0$ and (1.6.15) gives $N^{(-4s-3)-\epsilon(s+3/2)} \leq C$ which is a contradiction for $N \gg 1$ if we choose $0 < \epsilon < (-4s-3)/(s+3/2)$.

Case II: If $s \leq -3/2$ then $s+3/2 \leq 0$ and (1.6.15) gives $N^{-4s-3} \leq C$, which is again a contradiction for $N \gg 1$.

Hence for $s < -3/4$, (1.6.6) fails to hold for our choice of Φ, which completes the proof of the theorem. $\square$

Chapter 2

Global Well-posedness for a Coupled System of mKdV Equations

2.1 Introduction

This chapter is concerned about the global solution to the following IVP

$$\begin{cases} u_t + u_{xxx} + (uv^2)_x = 0 \\ v_t + v_{xxx} + (u^2v)_x = 0, \qquad x,\, t \in \mathbb{R} \\ u(x,0) = \phi(x), \quad v(x,0) = \psi(x), \end{cases} \tag{2.1.1}$$

where $u = u(x,t)$ and $v = v(x,t)$ are real valued functions. This is the particular case of the model (0.0.7) when $p = 1$.

The system (2.1.1) has a structure of the modified Korteweg-de Vries (mKdV) equation coupled through nonlinear effects and is a special case of a broad class of nonlinear evolution equations of physical interest (see for eg [1]). Many complex physical phenomena can be modeled as mKdV equation. In recent years much effort has been made to study the mKdV model (see for example [9], [11], [24], [39], [61] and references therein). This model has also been studied using inverse scattering theory (see [49], [57] and references therein).

The Cauchy problem as well as the existence and stability of solitary wave solutions to (2.1.1) is widely studied in the literature (see for example [51] and [2]). Recently using the argument developed by Kenig, Ponce and Vega [39] in context of the mKdV equation, Montenegro [51] proved that the IVP (2.1.1) is locally well-posed for given data $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1/4$. More precisely, the following theorem has been proved in [51].

Theorem 2.1 *Let $s \geq 1/4$. Then for all $(\phi, \psi) \in X^s$, there exist $T = T(\|\phi\|_{H^{1/4}}, \|\psi\|_{H^{1/4}})$ [in fact $T \sim c\|(\phi,\psi)\|_{X^{1/4}}^{-4} > 0$] and a unique solution $(u(t), v(t))$ to the IVP (2.1.1) such that*

$$(u, v) \in C([-T, T] : X^s)$$

$$\|D_x^s \partial_x u\|_{L_x^\infty L_T^2} < \infty, \qquad \|D_x^s \partial_x v\|_{L_x^\infty L_T^2} < \infty, \tag{2.1.2}$$

$$\|\partial_x u\|_{L_x^{20} L_T^{5/2}} < \infty, \qquad \|\partial_x v\|_{L_x^{20} L_T^{5/2}} < \infty, \tag{2.1.3}$$

$$\|D_x^s u\|_{L_x^5 L_T^{10}} < \infty, \qquad \|D_x^s v\|_{L_x^5 L_T^{10}} < \infty, \tag{2.1.4}$$

$$\|u\|_{L_x^4 L_T^\infty} < \infty, \qquad \|v\|_{L_x^4 L_T^\infty} < \infty. \tag{2.1.5}$$

Moreover, for any $T' \in (0, T)$, there exists a neighborhood $\mathcal{V}$ of (ϕ, ψ) in X^s such that the map $(\tilde{\phi}, \tilde{\psi}) \mapsto (\tilde{u}, \tilde{v})$ from $\mathcal{V}$ into the class defined by (2.1.2) to (2.1.5) with T' in place of T is Lipschitz.

Using the conservation laws (0.0.8) and (0.0.9) satisfied by the flow of (2.1.1), the local solution can be extended to the global one for the initial data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1$. Hence, there is a gap in Sobolev indices between the existence of the local and global solution to the IVP (2.1.1). In this work, we will fill this gap to some extent.

2.2 Main Result and the Scheme of the Proof

Here we further refine the high-low frequency technique introduced by Bourgain [12] and more simplified by Fonseca, Linares and Ponce [24], in the context of the mKdV equation. For this we exploit the uniform bound of the solution (see (2.2.1) below) obtained by using iteration in the energy space. With proper choice of the Sobolev indices we develop an iteration process below the energy space and prove that the IVP (2.1.1) is globally well-posed for data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 4/9$. More precisely, we prove the following result.

Theorem 2.2 *For any $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 4/9$, the unique solution to the IVP (2.1.1) provided by Theorem 2.1 extends to any time interval $[0, T]$.*

To prove this theorem we use the sharp smoothing effects present in the solution of the linear problem associated to the IVP (2.1.1) combined with the iteration process introduced by Bourgain [12]. The proof will be carried out in two steps. In the first step we closely follow the modified techniques developed by Fonseca, Linares and Ponce [24] to perform iteration in the energy space and prove that the local solution to the IVP (2.1.1) can be extended to a global one for given data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 3/5$. Moreover, we obtain that the solution grows according as

$$\sup_{[0,T]} \|(u(t), v(t))\|_{H^s \times H^s} \leq cT^{2s(1-s)/(5s-3)}, \qquad 3/5 < s < 1, \tag{2.2.1}$$

with $N = N(T) \sim T^{2/(5s-3)}$, sufficiently large.

In the second step, we take $s_0 \in (3/5, 1)$ and the data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $1/4 \leq s < s_0$. Utilizing the uniform bound (2.2.1) of the solution in $H^{s_0}(\mathbb{R}) \times H^{s_0}(\mathbb{R})$ obtained in the first step, we develop an iteration process in this space by controlling the involved norms and complete the proof (for details, see proof of the Theorem 2.2 below).

2.3 Linear Estimates

In this section we give some linear estimates associated to the IVP (2.1.1). These estimates are not new and can be found in literature. We will not give the details of the proofs rather we just sketch the idea of the proof and mention the references where these can be found. Let $U(t)$ be the group generated by the operator ∂_x^3. First let us state the smoothing effects.

Theorem 2.3 *If $\phi \in L^2(\mathbb{R})$, then*

$$\|\partial_x U(t)\phi\|_{L_x^\infty L_t^2} \leq \|\phi\|_{L^2}. \tag{2.3.1}$$

If $g \in L_x^1 L_t^2$ then for any $T > 0$

$$\|\partial_x \int_0^t U(t-t')g(\cdot, t')\,dt'\|_{L_T^\infty L_x^2} \leq \|g\|_{L_x^1 L_T^2}. \tag{2.3.2}$$

If $g \in L_x^{2/(1+\theta)} L_t^2$, $0 \leq \theta \leq 1$, then for any $T > 0$

$$\|D_x^\theta \int_0^t U(t-t')g(\cdot, t')\,dt'\|_{L_T^\infty L_x^2} \leq cT^{\frac{1}{2}(1-\theta)} \|g\|_{L_x^{2/(1+\theta)} L_T^2}. \tag{2.3.3}$$

Proof. For the proof of the homogeneous smoothing effect (2.3.1) see section 4 in [42] (see also [39]). Inequality (2.3.2) is the dual version of the smoothing effect (2.3.1). The estimate (2.3.3) can be found in [18]. In fact, the Minkowski integral inequality and the Cauchy-Schwarz inequality yield,

$$\|\partial_x \int_0^t U(t-t')g(\cdot, t')\,dt'\|_{L_x^2} \leq \int_0^T \|D_x g(\cdot, t')\|_{L_x^2}\,dt' \leq T^{1/2} \|D_x g\|_{L_x^2 L_T^2}. \tag{2.3.4}$$

Now, application of Stein's interpolation (see [62]) between the dual version of the smoothing effect (2.3.2) and the estimate (2.3.4) by considering the analytic family of operators $T_z f = D_x^{-z}(\int_0^t \partial_x U(t-t')f(\cdot,t')\,dt')$, $z \in \mathbb{C}$, $0 \leq \Re z \leq 1$ gives,

$$\|\partial_x \int_0^t U(t-t')g(\cdot,t')\,dt'\|_{L_T^\infty L_x^2} \leq cT^{\theta/2}\|D_x^\theta g\|_{L_x^{2/(2-\theta)}L_T^2}, \tag{2.3.5}$$

which implies the required estimate. □

Observe that, if $D_x^\theta \phi \in L^2$ for $0 < \theta \leq 1$, then using Stein's interpolation between the homogeneous smoothing effect (2.3.1) and $\|U(t)f\|_{L_x^2 L_T^2} \leq T^{1/2}\|f\|_{L_x^2}$ we get,

$$\|\partial_x U(t)\phi\|_{L_x^{2/\theta}L_T^2} \leq cT^{\theta/2}\|D_x^\theta \phi\|_{L^2}. \tag{2.3.6}$$

Now we give the maximal function estimates.

Theorem 2.4 *If $\phi \in H^{1/4}$, then*

$$\|U(t)\phi\|_{L_x^4 L_T^\infty} \leq c\|D_x^{1/4}\phi\|_{L^2}. \tag{2.3.7}$$

If $\phi \in H^s$, $s > 3/4$ and $0 < T < 1$ then

$$\|U(t)\phi\|_{L_x^2 L_T^\infty} \leq c\|\phi\|_{H^s}. \tag{2.3.8}$$

If $\phi \in H^{(1+2\theta)/4}$, $0 \leq \theta < 1$ and $0 < T < 1$ then

$$\|U(t)\phi\|_{L_x^{4/(1+\theta)}L_T^\infty} \leq c\|\phi\|_{H^{(1+2\theta)/4}}. \tag{2.3.9}$$

Proof. The proof of the estimates (2.3.7) and (2.3.8) can be found in [41] and [45]. The estimate (2.3.9) can be obtained by interpolating (2.3.7) and (2.3.8). □

Theorem 2.5 *If $\phi \in L^2(\mathbb{R})$, then*

$$\|U(t)\phi\|_{L_x^5 L_t^{10}} \leq c\|\phi\|_{L^2} \tag{2.3.10}$$

and

$$\|\partial_x U(t)\phi\|_{L_x^{20}L_t^{5/2}} \leq c\|D_x^{1/4}\phi\|_{L^2}. \tag{2.3.11}$$

Proof. The estimates of this theorem can be found in [39]. The estimate (2.3.10) follows by interpolating (2.3.1) and (2.3.8). The estimate (2.3.11) follows by using Stein's interpolation between (2.3.1) and (2.3.7). □

Theorem 2.6 *Let $1/4 \leq \theta \leq 1$. If $D_x^\theta \phi \in L^2$, then*

$$\|D_x U(t)\phi\|_{L_x^{40/(20\theta-3)} L_t^{5/2}} \leq cT^{\theta/2-1/8}\|D_x^\theta \phi\|_{L_x^2}. \tag{2.3.12}$$

Proof. The proof of this estimate can be found in [18] which follows by interpolating (2.3.6) and

$$\|D_x U(t)\phi\|_{L_x^{5/\theta} L_t^{10/(5-4\theta)}} \leq \|D_x^\theta \phi\|_{L_x^2},\ 0 \leq \theta \leq 1. \tag{2.3.13}$$

The estimate (2.3.13) can be obtained by interpolating the homogeneous smoothing effect (2.3.1) and the maximal function estimate (2.3.7). □

Finally we have the Leibniz's rule for fractional derivatives whose proof is given in [39].

Theorem 2.7 *Let $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in [0,\alpha]$, $\alpha_1 + \alpha_2 = \alpha$. Let $p, p_1, p_2, q, q_1, q_2 \in (1,\infty)$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p L_T^q} \leq c\|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}} \tag{2.3.14}$$

Moreover, for $\alpha_1 = 0$ the value $q_1 = \infty$ is allowed.

2.4 Preliminary Results

We decompose the given data $(\phi,\psi) \in X^s$, $s < 1$ to low and high frequency parts as,

$$\begin{cases} \phi(x) = (\chi_{\{|\xi|\leq N\}}\hat{\phi}(\xi))^\vee(x) + (\chi_{\{|\xi|>N\}}\hat{\phi}(\xi))^\vee(x) := \phi_1(x) + \phi_2(x) \\ \psi(x) = (\chi_{\{|\xi|\leq N\}}\hat{\psi}(\xi))^\vee(x) + (\chi_{\{|\xi|>N\}}\hat{\psi}(\xi))^\vee(x) := \psi_1(x) + \psi_2(x), \end{cases} \tag{2.4.1}$$

where $N \gg 1$ arbitrary but fixed.

Then we have, $(\phi_1, \psi_1) \in X^\beta$, $0 \leq \beta \leq 1$ and $(\phi_2, \psi_2) \in X^\rho$, $\quad 0 \leq \rho \leq s < 1$.

As discussed in the introduction, we evolve (ϕ_1, ψ_1) according to the IVP*

$$\begin{cases} u_{1t} + u_{1xxx} + (u_1 v_1^2)_x = 0 \\ v_{1t} + v_{2xxx} + (u_1^2 v_1)_x = 0 \\ u_1(x,0) = \phi_1(x), \quad v_1(x,0) = \psi_1(x), \end{cases} \tag{2.4.2}$$

*We use the notations $u_{1t} := (u_1)_t$, $u_{1x} := (u_1)_x$ and similar for other terms.

which is the same as the IVP (2.1.1). We evolve (ϕ_2, ψ_2) according to the difference equation

$$\begin{cases} u_{2t} + u_{2xxx} + ((u_1+u_2)(v_1+v_2)^2)_x - (u_1v_1^2)_x = 0 \\ v_{2t} + v_{2xxx} + ((u_1+u_2)^2(v_1+v_2))_x - (u_1^2v_1)_x = 0 \\ u_2(x,0) = \phi_2(x), \quad v_2(x,0) = \psi_2(x), \end{cases} \tag{2.4.3}$$

with coefficients depending on the solution (u_1, v_1) to the IVP (2.4.2). It is clear that $u = u_1 + u_2$ and $v = v_1 + v_2$ solve the IVP (2.1.1). For simplicity, let us write (2.4.3) as

$$\begin{cases} u_{2t} + u_{2xxx} + F = 0 \\ v_{2t} + v_{2xxx} + G = 0 \\ u_2(x,0) = \phi_2(x), \quad v_2(x,0) = \psi_2(x), \end{cases} \tag{2.4.4}$$

where

$$\begin{aligned} F =& 2u_1v_1v_{2x} + 2u_1v_2v_{1x} + 2v_1v_2u_{1x} + 2u_1v_2v_{2x} + 2u_2v_1v_{1x} + 2u_2v_1v_{2x} \\ &+ 2u_2v_2v_{1x} + 2v_1v_2u_{2x} + 2u_2v_2v_{2x} + v_2^2u_{1x} + v_1^2u_{2x} + v_2^2u_{2x} \end{aligned} \tag{2.4.5}$$

and

$$\begin{aligned} G =& 2v_1u_1u_{2x} + 2v_1u_2u_{1x} + 2u_1u_2v_{1x} + 2v_1u_2u_{2x} + 2v_2u_1u_{1x} + 2v_2u_1u_{2x} \\ &+ 2v_2u_2u_{1x} + 2u_1u_2v_{2x} + 2v_2u_2u_{2x} + u_2^2v_{1x} + u_1^2v_{2x} + u_2^2v_{2x}. \end{aligned} \tag{2.4.6}$$

Note that from Theorem 2.1 we have the existence result for the IVP (2.4.2). To get the existence result for the IVP (2.4.4) we need the following theorem.

Theorem 2.8 *Suppose the initial data (ϕ_1, ψ_1) of the IVP (2.4.2) satisfy*

$$\begin{cases} \|(\phi_1, \psi_1)\|_X \le c \\ \|(\phi_1, \psi_1)\|_{X^1} \le cN^{1-s}. \end{cases} \tag{2.4.7}$$

Then for the existence time $T \sim c\|(\phi_1, \psi_1)\|_{X^{1/4}}^{-4} \sim cN^{-(1-s)}$ obtained in Theorem 2.1
(i) The solution (u_1, v_1) to the IVP (2.4.2) satisfies,

$$\sup_t \|(u_1(t), v_1(t))\|_{X^1} = \sup_t [\|u_1(t)\|_{H^1} + \|v_1(t)\|_{H^1}] \le cN^{1-s}. \tag{2.4.8}$$

(ii) Moreover, for any $\beta \in [1/4, 1)$, the solution (u_1, v_1) to the IVP (2.4.2) satisfies,

$$|||(u_1, v_1)|||_\beta \sim N^{(1-s)\beta}, \tag{2.4.9}$$

where $|||(u_1, v_1)|||_\beta = \max\{|||u_1|||_\beta, |||v_1|||_\beta\}$ and,

$$\begin{aligned} |||f|||_\beta =& \|f\|_{L_T^\infty H^\beta} + \|D_x^\beta \partial_x f\|_{L_x^\infty L_T^2} + \|\partial_x f\|_{L_x^{20} L_T^{5/2}} \\ &+ \|D_x^\beta f\|_{L_x^5 L_T^{10}} + \|f\|_{L_x^4 L_T^\infty} + \|\partial_x f\|_{L_x^\infty L_T^2}. \end{aligned} \tag{2.4.10}$$

Proof. The proof of (2.4.8) follows by using the conservation laws (0.0.8) and (0.0.9) combined with the Gagliardo-Nirenberg inequality. The estimate (2.4.9) can be obtained by using the hypothesis (2.4.7) and the local well-posedness result. □

The following theorem provides the existence result for the IVP (2.4.4).

Theorem 2.9 *Let* $(\phi_2, \psi_2) \in X^s$, $s \geq 1/4$ *and* (u_1, v_1) *be the unique solution given by Theorem 2.8. Then there exists a unique solution* (u_2, v_2) *to the IVP (2.4.4) in the same interval of existence of* (u_1, v_1), $[0,T]$ *such that,*

$$(u_2, v_2) \in C([0,T] : X^s)$$

$$\|D_x^s \partial_x u_2\|_{L_x^\infty L_T^2} < \infty, \qquad \|D_x^s \partial_x v_2\|_{L_x^\infty L_T^2} < \infty, \tag{2.4.11}$$

$$\|\partial_x u_2\|_{L_x^{20} L_T^{5/2}} < \infty, \qquad \|\partial_x v_2\|_{L_x^{20} L_T^{5/2}} < \infty, \tag{2.4.12}$$

$$\|D_x^s u_2\|_{L_x^5 L_T^{10}} < \infty, \qquad \|D_x^s v_2\|_{L_x^5 L_T^{10}} < \infty, \tag{2.4.13}$$

$$\|u_2\|_{L_x^4 L_T^\infty} < \infty, \qquad \|v_2\|_{L_x^4 L_T^\infty} < \infty. \tag{2.4.14}$$

Proof. We will prove this theorem following the argument in [39]. As in [39] we will give details only for the case $s = 1/4$, for this we consider the equivalent integral equation associated to the IVP (2.4.4), i.e,

$$\begin{cases} u_2(t) = U(t)\phi_2 - \int_0^t U(t-t')F(t')\,dt' \\ v_2(t) = U(t)\psi_2 - \int_0^t U(t-t')G(t')\,dt', \end{cases} \tag{2.4.15}$$

where F and G are defined in (2.4.5) and (2.4.6) respectively.

Let us define,

$$\mathcal{X}_{a,T} = \{(u_2, v_2) \in C([0,T] : X^{1/4}(\mathbb{R})) : \|\!|(u_2, v_2)\|\!|_{1/4} < a\},$$

where

$$\|\!|(u_2, v_2)\|\!|_{1/4} = \max\{\|\!|u_2\|\!|_{1/4}, \|\!|v_2\|\!|_{1/4}\},$$

with

$$\begin{aligned} \|\!|f\|\!|_{1/4} = \|f\|_{L_T^\infty H^{1/4}} + \|D_x^{1/4}\partial_x f\|_{L_x^\infty L_T^2} + \|\partial_x f\|_{L_x^{20} L_T^{5/2}} \\ + \|D_x^{1/4} f\|_{L_x^5 L_T^{10}} + \|f\|_{L_x^4 L_T^\infty} + \|\partial_x f\|_{L_x^\infty L_T^2}. \end{aligned} \tag{2.4.16}$$

Finally, we define,

$$\begin{cases} \Phi_{\phi_2}[u_2, v_2] = U(t)\phi_2 - \int_0^t U(t-t')F(t')\,dt' \\ \Psi_{\psi_2}[u_2, v_2] = U(t)\psi_2 - \int_0^t U(t-t')Gt')\,dt'. \end{cases} \tag{2.4.17}$$

and show that $\Phi \times \Psi$ maps $\mathfrak{X}_{a,T}$ into $\mathfrak{X}_{a,T}$ and is a contraction.

Using the linear estimates established in section 2.3 we obtain,

$$\begin{aligned} |\!|\!|(\Phi, \Psi)|\!|\!|_{1/4} \leq c\|(\phi_2, \psi_2)\|_{X^{1/4}} &+ cT^{1/2}\{\|D_x^{1/4}F\|_{L_x^2L_T^2} \\ &+ \|F\|_{L_x^2L_T^2} + \|D_x^{1/4}G\|_{L_x^2L_T^2} + \|G\|_{L_x^2L_T^2}\} \\ = c\|(\phi_2, \psi_2)\|_{X^{1/4}} &+ cT^{1/2}\{A_1 + A_2 + A_3 + A_4\}. \end{aligned} \tag{2.4.18}$$

Now using the definition of F we get,

$$\begin{aligned} A_1 \leq\ & c\big[\|D_x^{1/4}(u_1v_1v_{2x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(u_1v_2v_{1x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(v_1v_2u_{1x})\|_{L_x^2L_T^2} \\ &+ \|D_x^{1/4}(u_1v_2v_{2x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(u_2v_1v_{1x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(u_2v_1v_{2x})\|_{L_x^2L_T^2} \\ &+ \|D_x^{1/4}(u_2v_2v_{1x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(v_1v_2u_{2x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(u_2v_2v_{2x})\|_{L_x^2L_T^2} \\ &+ \|D_x^{1/4}(v_2^2u_{1x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(v_1^2u_{2x})\|_{L_x^2L_T^2} + \|D_x^{1/4}(v_2^2u_{2x})\|_{L_x^2L_T^2}\big] \\ :=\ & A_{1,1} + A_{1,2} + \cdots + A_{1,12}. \end{aligned} \tag{2.4.19}$$

Also, we can have the similar expressions for A_2, A_3 and A_4.

Using the Leibniz's rule for fractional derivative and Hölder's inequality we get,

$$\begin{aligned} A_{1,1} &= c\|D_x^{1/4}(u_1v_1v_{2x})\|_{L_x^2L_T^2} \\ &\leq c\big[\|D_x^{1/4}(u_1v_1)\|_{L_x^{20/9}L_T^{10}}\|v_{2x}\|_{L_x^{20}L_T^{5/2}} + \|u_1v_1D_x^{1/4}v_{2x}\|_{L_x^2L_T^2} \\ &\quad + \|v_{2x}D_x^{1/4}(u_1v_1)\|_{L_x^2L_T^2}\big] \\ &\leq c\big[\|D_x^{1/4}(u_1v_1)\|_{L_x^{20/9}L_T^{10}}\|v_{2x}\|_{L_x^{20}L_T^{5/2}} + \|u_1\|_{L_x^4L_T^\infty}\|v_1\|_{L_x^4L_T^\infty}\|D_x^{1/4}v_{2x}\|_{L_x^\infty L_T^2}\big] \\ &\leq c\big[\|D_x^{1/4}(u_1v_1)\|_{L_x^{20/9}L_T^{10}}|\!|\!|v_2|\!|\!|_{1/4} + |\!|\!|u_1|\!|\!|_{1/4}|\!|\!|v_1|\!|\!|_{1/4}|\!|\!|v_2|\!|\!|_{1/4}\big]. \end{aligned} \tag{2.4.20}$$

Again, using the Leibniz's rule for fractional derivative and Hölder's inequality we have,

$$\begin{aligned} \|D_x^{1/4}(u_1v_1)\|_{L_x^{20/9}L_T^{10}} \leq\ & c\big[\|u_1\|_{L_x^4L_T^\infty}\|D_x^{1/4}v_1\|_{L_x^5L_T^{10}} + \|u_1D_x^{1/4}v_1\|_{L_x^{20/9}L_T^{10}} \\ &+ \|v_1D_x^{1/4}u_1\|_{L_x^{20/9}L_T^{10}}\big] \\ \leq\ & c\big[\|u_1\|_{L_x^4L_T^\infty}\|D_x^{1/4}v_1\|_{L_x^5L_T^{10}} + \|v_1\|_{L_x^4L_T^\infty}\|D_x^{1/4}u_1\|_{L_x^5L_T^{10}}\big] \\ \leq\ & c|\!|\!|u_1|\!|\!|_{1/4}|\!|\!|v_1|\!|\!|_{1/4}. \end{aligned} \tag{2.4.21}$$

Inserting (2.4.21) in (2.4.20) we obtain,

$$A_{1,1} \leq c|||(u_1, v_1)|||_{1/4}^2 |||(u_2, v_2)|||_{1/4}.$$

Also, we can get the analogous estimates for $A_{1,j},\ j = 2, 3, \ldots, 12$.

Therefore, from (2.4.19) we obtain,

$$A_1 \leq c\big[|||(u_1, v_1)|||_{1/4}^2 |||(u_2, v_2)|||_{1/4} + |||(u_1, v_1)|||_{1/4} |||(u_2, v_2)|||_{1/4}^2 + |||(u_2, v_2)|||_{1/4}^3\big].$$

Using the similar argument we can get for $j = 2, 3, 4$,

$$A_j \leq c\big[|||(u_1, v_1)|||_{1/4}^2 |||(u_2, v_2)|||_{1/4} + |||(u_1, v_1)|||_{1/4} |||(u_2, v_2)|||_{1/4}^2 + |||(u_2, v_2)|||_{1/4}^3\big].$$

Hence,

$$\begin{aligned} |||(\Phi, \Psi)|||_{1/4} \leq c\|(\phi_2, \psi_2)\|_{H^{1/4}} + cT^{1/2}\big\{ |||(u_1, v_1)|||_{1/4}^2 \\ + |||(u_1, v_1)|||_{1/4} |||(u_2, v_2)|||_{1/4} + |||(u_2, v_2)|||_{1/4}^2 \big\} |||(u_2, v_2)|||_{1/4}. \end{aligned} \tag{2.4.22}$$

Let us set $a = 2c\max\{\|(\phi_1, \psi_1)\|_{X^{1/4}}, \|(\phi_2, \psi_2)\|_{X^{1/4}}\}$. With this choice, if we take T such that $ca^2T^{1/2} < 1/10$, then (2.4.22) yields,

$$|||(\Phi, \Psi)|||_{1/4} \leq \frac{a}{2} + \frac{3}{10}\, a < a.$$

Therefore, $\Phi \times \Psi$ maps $\mathcal{X}_{a,T}$ into $\mathcal{X}_{a,T}$. Using the same argument we can show that $\Phi \times \Psi$ is a contraction. The rest of the proof follows a standard argument. □

In what follows we need the following results.

Corollary 2.1 *Let (u_1, v_1) and (u_2, v_2) be solutions to the IVPs (2.4.2) and (2.4.4) with initial data $(\phi_1, \psi_1), (\phi_2, \psi_2) \in X^s$, $s \geq 1/4$ respectively. For $1/4 < \rho \leq s < 1$, let (ϕ_2, ψ_2) satisfies,*

$$\|(\phi_2, \psi_2)\|_{X^\rho} \sim N^{\rho - s} \tag{2.4.23}$$

and (ϕ_1, ψ_1) satisfies the conditions of Theorem 2.8. If

$$|||(u_2, v_2)|||_\rho = \max\{|||u_2|||_\rho, |||v_2|||_\rho\},$$

where,

$$|||f|||_\rho = \|f\|_{L_T^\infty H^\rho} + \|D_x^\rho \partial_x f\|_{L_x^\infty L_T^2} + \|\partial_x f\|_{L_x^{20} L_T^{5/2}} + \|D_x^\rho f\|_{L_x^5 L_T^{10}}.$$

Then,

$$|||(u_2, v_2)|||_\rho \sim N^{\rho - s}.$$

Proof. Using the definition of $\|\!|\cdot|\!\|_\rho$ and the linear estimates established in section 2.3 we obtain,

$$\begin{aligned}\|\!|(u_2,v_2)|\!\|_\rho &\le c\|(\phi_2,\psi_2)\|_{X^\rho} + cT^{1/2}\big\{\|D_x^\rho F\|_{L_x^2L_T^2} + \|F\|_{L_x^2L_T^2} + \|D_x^\rho G\|_{L_x^2L_T^2} + \|G\|_{L_x^2L_T^2}\big\}\\ &= c\|(\phi_2,\psi_2)\|_{X^\rho} + B_1 + B_2 + B_3 + B_4.\end{aligned} \tag{2.4.24}$$

From the definition of F we get,

$$\begin{aligned}B_1 \le cT^{1/2}\big[&\|D_x^\rho(u_1v_1v_{2x})\|_{L_x^2L_T^2} + \|D_x^\rho(u_1v_2v_{1x})\|_{L_x^2L_T^2} + \|D_x^\rho(v_1v_2u_{1x})\|_{L_x^2L_T^2}\\ &+ \|D_x^\rho(u_1v_2v_{2x})\|_{L_x^2L_T^2} + \|D_x^\rho(u_2v_1v_{1x})\|_{L_x^2L_T^2} + \|D_x^\rho(u_2v_1v_{2x})\|_{L_x^2L_T^2}\\ &+ \|D_x^\rho(u_2v_2v_{1x})\|_{L_x^2L_T^2} + \|D_x^\rho(v_1v_2u_{2x})\|_{L_x^2L_T^2} + \|D_x^\rho(u_2v_2v_{2x})\|_{L_x^2L_T^2}\\ &+ \|D_x^\rho(v_2^2u_{1x})\|_{L_x^2L_T^2} + \|D_x^\rho(v_1^2u_{2x})\|_{L_x^2L_T^2} + \|D_x^\rho(v_2^2u_{2x})\|_{L_x^2L_T^2}\big]\\ := B_{1,1} &+ B_{1,2} + \cdots + B_{1,12}.\end{aligned} \tag{2.4.25}$$

Using the Leibniz's rule for fractional derivative and Hölder's inequality one gets,

$$\begin{aligned}B_{1,1} &= T^{1/2}\|D_x^\rho(u_1v_1v_{2x})\|_{L_x^2L_T^2}\\ &\le cT^{1/2}\big[\|D_x^\rho(u_1v_1)\|_{L_x^{20/9}L_T^{10}}\|v_{2x}\|_{L_x^{20}L_T^{5/2}} + \|v_{2x}D_x^\rho(u_1v_1)\|_{L_x^2L_T^2}\\ &\quad + \|u_1v_1D_x^\rho v_{2x}\|_{L_x^2L_T^2}\big]\\ &\le cT^{1/2}\big[\|D_x^\rho(u_1v_1)\|_{L_x^{20/9}L_T^{10}}\|v_{2x}\|_{L_x^{20}L_T^{5/2}} + \|u_1\|_{L_x^4L_T^\infty}\|v_1\|_{L_x^4L_T^\infty}\|D_x^\rho v_{2x}\|_{L_x^\infty L_T^2}\big]\\ &\le cT^{1/2}\big[\|\!|v_2|\!\|_{1/4}\|D_x^\rho(u_1v_1)\|_{L_x^{20/9}L_T^{10}} + \|\!|u_1|\!\|_{1/4}\|\!|v_1|\!\|_{1/4}\|\!|v_2|\!\|_\rho\big].\end{aligned} \tag{2.4.26}$$

Again using the Leibniz's rule for fractional derivative we get,

$$\begin{aligned}\|D_x^\rho(u_1v_1)\|_{L_x^{20/9}L_T^{10}} &\le c\big[\|u_1\|_{L_x^4L_T^\infty}\|D_x^\rho v_1\|_{L_x^5L_T^{10}} + \|u_1D_x^\rho v_1\|_{L_x^{20/9}L_T^{10}} + \|v_1D_x^\rho u_1\|_{L_x^{20/9}L_T^{10}}\big]\\ &\le c\big[\|\!|u_1|\!\|_{1/4}\|\!|v_1|\!\|_\rho + \|v_1\|_{L_x^4L_T^\infty}\|D_x^\rho u_1\|_{L_x^5L_T^{10}}\big]\\ &\le c\big[\|\!|u_1|\!\|_{1/4}\|\!|v_1|\!\|_\rho + c\|\!|v_1|\!\|_{1/4}\|\!|u_1|\!\|_\rho\big].\end{aligned} \tag{2.4.27}$$

Inserting (2.4.27) in (2.4.26) we obtain,

$$B_{1,1} \le cT^{1/2}[\|\!|(u_1,v_1)|\!\|_{1/4}\|\!|(u_1,v_1)|\!\|_\rho\|\!|(u_2,v_2)|\!\|_{1/4} + \|\!|(u_1,v_1)|\!\|_{1/4}^2\|\!|(u_2,v_2)|\!\|_\rho]. \tag{2.4.28}$$

Using the similar argument it is easy to get, for $j = 2, 3, 5, 11$,

$$B_{1,j} \le cT^{1/2}\big[\|\!|(u_1,v_1)|\!\|_{1/4}\|\!|(u_1,v_1)|\!\|_\rho\|\!|(u_2,v_2)|\!\|_{1/4} + \|\!|(u_1,v_1)|\!\|_{1/4}^2\|\!|(u_2,v_2)|\!\|_\rho\big], \tag{2.4.29}$$

for $j = 4, 6, 7, 8, 10$,

$$B_{1,j} \le cT^{1/2}\big[\|(u_1,v_1)\|_\rho \|(u_2,v_2)\|_{1/4}^2 + \|(u_1,v_1)\|_{1/4}\|(u_2,v_2)\|_{1/4}\|(u_2,v_2)\|_\rho\big], \tag{2.4.30}$$

and for $j = 9, 12$,

$$B_{1,j} \le cT^{1/2}\big[\|(u_2,v_2)\|_{1/4}^2 \|(u_2,v_2)\|_\rho\big]. \tag{2.4.31}$$

Now, using Theorem 2.8, Theorem 2.9 and $T \sim N^{-(1-s)}$ we get from (2.4.28) - (2.4.31),

$$B_{1,j} \le cN^{\rho-s} + \frac{1}{20}\|(u_2,v_2)\|_\rho, \qquad j = 1, 2, 3, 5, 11, \tag{2.4.32}$$

$$B_{1,j} \le cN^{\rho-s} + cN^{-\frac{3}{4}s}\|(u_2,v_2)\|_\rho, \quad j = 4, 6, 7, 8, 10 \tag{2.4.33}$$

and

$$B_{1,j} \le cN^{-\frac{3}{2}s}\|(u_2,v_2)\|_\rho, \qquad j = 9, 12. \tag{2.4.34}$$

The use of (2.4.32) - (2.4.34) in (2.4.25) yields,

$$B_1 \le c\frac{1}{4}\|(u_2,v_2)\|_\rho + cN^{-\frac{3}{4}s}\|(u_2,v_2)\|_\rho + cN^{\rho-s}. \tag{2.4.35}$$

Now we estimate the term B_2. From the definition of F we get,

$$\begin{aligned} B_2 \le cT^{1/2}\big[&\|u_1v_1v_{2x}\|_{L_x^2L_T^2} + \|u_1v_2v_{1x}\|_{L_x^2L_T^2} + \|v_1v_2u_{1x}\|_{L_x^2L_T^2} + \|u_1v_2v_{2x}\|_{L_x^2L_T^2} \\ &+ \|u_2v_1v_{1x}\|_{L_x^2L_T^2} + \|u_2v_1v_{2x}\|_{L_x^2L_T^2} + \|u_2v_2v_{1x}\|_{L_x^2L_T^2} + \|v_1v_2u_{2x}\|_{L_x^2L_T^2} \\ &+ \|u_2v_2v_{2x}\|_{L_x^2L_T^2} + \|v_2^2u_{1x}\|_{L_x^2L_T^2} + \|v_1^2u_{2x}\|_{L_x^2L_T^2} + \|v_2^2u_{2x}\|_{L_x^2L_T^2}\big] \\ := B_{2,1} &+ B_{2,2} + \cdots + B_{2,12}. \end{aligned} \tag{2.4.36}$$

Using Hölder's inequality, Theorem 2.8, Theorem 2.9 and $T \sim N^{-(1-s)}$ we obtain,

$$\begin{aligned} B_{2,1} &\le cT^{1/2}\|u_1\|_{L_x^4L_T^\infty}\|v_1\|_{L_x^4L_T^\infty}\|v_{2x}\|_{L_x^\infty L_T^2} \\ &\le cT^{1/2}\|u_1\|_{1/4}\|v_1\|_{1/4}\|v_2\|_{1/4} \\ &\le cT^{1/2}\|(u_1,v_1)\|_{1/4}^2\|(u_2,v_2)\|_{1/4} \\ &\le cN^{-\frac{1}{2}(1-s)}N^{\frac{1}{2}(1-s)}N^{\frac{1}{4}-s} \le cN^{\rho-s}. \end{aligned}$$

We can have the similar estimates for $B_{2,j},\ j = 2, 3, \ldots, 12,$ so that

$$B_2 \leq cN^{\rho - s}. \tag{2.4.37}$$

Also, with the argument applied in B_1 and B_2 we get the similar estimates for B_3 and B_4 respectively.

Therefore, using (2.4.35), (2.4.37) and the analogous estimates for B_3 and B_4 we obtain from (2.4.24),

$$|||(u_2, v_2)|||_\rho \leq c\|(\phi_2, \psi_2)\|_{X^\rho} + \{\frac{1}{2} + cN^{-\frac{3}{4}s}\}|||(u_2, v_2)|||_\rho + cN^{\rho - s}.$$

Choosing $N \gg 1$ such that $cN^{-\frac{3}{4}s} < 1/3$ we get the required result. □

Proposition 2.1 *Define* $|||(u_2, v_2)|||_0 = \max\{|||u_2|||_0, |||v_2|||_0\}$ *where,*

$$|||f|||_0 = \|f\|_{L^\infty_T L^2_x} + \|\partial_x f\|_{L^\infty_x L^2_T} + \|f\|_{L^5_x L^{10}_T}.$$

Let (u_1, v_1) *and* (u_2, v_2) *be solutions to the IVPs (2.4.2) and (2.4.4) with* $(\phi_1, \psi_1) \in X^1$ *and* $(\phi_2, \psi_2) \in X^s$ *respectively satisfying* $\|(\phi_1, \psi_1)\|_{X^1} \sim N^{1-s}$ *and* $\|(\phi_2, \psi_2)\|_X \sim N^{-s}$, $1/4 \leq s < 1$. *Then*

$$|||(u_2, v_2)|||_0 \sim N^{-s}. \tag{2.4.38}$$

Proof. By the definition of $|||\cdot|||_0$ and linear estimates established in section 2.3 we obtain,

$$|||(u_2, v_2)|||_0 \leq c\|(\phi_2, \psi_2)\|_X + cT^{1/2}\{\|F\|_{L^2_x L^2_T} + \|G\|_{L^2_x L^2_T}\}. \tag{2.4.39}$$

From the definition of F we get,

$$\begin{aligned}
\|F\|_{L^2_x L^2_T} \leq c[&\|u_1 v_1 v_{2x}\|_{L^2_x L^2_T} + \|u_1 v_2 v_{1x}\|_{L^2_x L^2_T} + \|v_1 v_2 u_{1x}\|_{L^2_x L^2_T} + \|u_1 v_2 v_{2x}\|_{L^2_x L^2_T} \\
&+ \|u_2 v_1 v_{1_x}\|_{L^2_x L^2_T} + \|u_2 v_1 v_{2x}\|_{L^2_x L^2_T} + \|u_2 v_2 v_{1x}\|_{L^2_x L^2_T} + \|v_1 v_2 u_{2x}\|_{L^2_x L^2_T} \\
&+ \|u_2 v_2 v_{2x}\|_{L^2_x L^2_T} + \|v_2^2 u_{1x}\|_{L^2_x L^2_T} + \|v_1^2 u_{2x}\|_{L^2_x L^2_T} + \|v_2^2 u_{2x}\|_{L^2_x L^2_T}] \\
:= &F_1 + F_2 + \cdots + F_{12}.
\end{aligned} \tag{2.4.40}$$

Now, using Hölder's inequality and the definition of $|||\cdot|||_0$ and $|||\cdot|||_{1/4}$ we obtain,

$$F_1 \leq c\|u_1\|_{L^4_x L^\infty_T}\|v_1\|_{L^4_x L^\infty_T}\|v_{2x}\|_{L^\infty_x L^2_T} \leq c|||u_1|||_{1/4}|||v_1|||_{1/4}|||v_2|||_0 \leq c|||(u_1, v_1)|||^2_{1/4}|||(u_2, v_2)|||_0.$$

Similarly,

$$F_2 \le c\|u_1\|_{L_x^4 L_T^\infty}\|u_1 v_2\|_{L_x^5 L_T^{10}}\|v_{1x}\|_{L_x^{20} L_T^{5/2}} \le c\|u_1\|_{1/4}\|v_2\|_0\|v_1\|_{1/4} \le c\|(u_1,v_1)\|_{1/4}^2\|(u_2,v_2)\|_0.$$

We can apply the similar argument to get,

$$\begin{aligned} &F_j \le c\|(u_1,v_1)\|_{1/4}^2\|(u_2,v_2)\|_0, \quad j=3,5,11.\\ &F_j \le c\|(u_1,v_1)\|_{1/4}\|(u_2,v_2)\|_{1/4}\|(u_2,v_2)\|_0, \quad j=4,6,7,8,10.\\ &F_j \le c\|(u_2,v_2)\|_{1/4}^2\|(u_2,v_2)\|_0, \quad j=9,12.\end{aligned}$$

Also, we have the similar estimates for $\|G\|_{L_x^2 L_T^2}$. Collecting all these estimates we get from (2.4.39),

$$\begin{aligned}\|(u_2,v_2)\|_0 \le c\|(\phi_2,\psi_2)\|_X &+ cT^{1/2}\big\{\|(u_1,v_1)\|_{1/4}^2\\ &+\|(u_1,v_1)\|_{1/4}\|(u_2,v_2)\|_{1/4}+\|(u_2,v_2)\|_{1/4}^2\big\}\|(u_2,v_2)\|_0.\end{aligned} \tag{2.4.41}$$

Finally, considering $cT^{1/2}\big\{\|(u_1,v_1)\|_{1/4}^2+\|(u_1,v_1)\|_{1/4}\|(u_2,v_2)\|_{1/4}+\|(u_2,v_2)\|_{1/4}^2\big\} < 3/10$ for the choice of T in Theorem 2.9 we get from (2.4.41),

$$\|(u_2,v_2)\|_0 \le c\|(\phi_2,\psi_2)\|_X,$$

which gives the required result. □

Proposition 2.2 *If (u_1,v_1) is a solution to the IVP (2.4.2) in $[0,T]$, $T<1$, then*

$$\|u_1\|_{L_x^{4/(1+\theta)} L_T^\infty} + \|v_1\|_{L_x^{4/(1+\theta)} L_T^\infty} \le c\|(u_1,v_1)\|_{(1+2\theta)/4}, \qquad 0\le\theta<1 \tag{2.4.42}$$

and

$$\|u_{1x}\|_{L_x^{40/(20\theta-3)} L_T^{5/2}} + \|v_{1x}\|_{L_x^{40/(20\theta-3)} L_T^{5/2}} \le cT^{\theta/2-1/8}\|(u_1,v_1)\|_\theta, \;\; 1/4\le\theta\le 1. \tag{2.4.43}$$

Moreover, the solution (u_2,v_2) to the IVP (2.4.4) satisfies,

$$\|u_2\|_{L_x^{4/(1+\theta)} L_T^\infty} + \|v_2\|_{L_x^{4/(1+\theta)} L_T^\infty} \le c|\!|\!|(u_2,v_2)|\!|\!|_{(1+2\theta)/4}, \qquad 0\le\theta<1 \tag{2.4.44}$$

and

$$\|u_{2x}\|_{L_x^{40/(20\theta-3)} L_T^{5/2}} + \|v_{2x}\|_{L_x^{40/(20\theta-3)} L_T^{5/2}} \le cT^{\theta/2-1/8}|\!|\!|(u_2,v_2)|\!|\!|_\theta, \quad 1/4\le\theta\le 1. \tag{2.4.45}$$

Proof. The estimate (2.4.42) follows by using the equivalent integral formula for (u_1, v_1), the estimate (2.3.9) and the choice of T in the local well-posedness result. The estimate (2.4.43) follows by using the similar argument along with the estimate (2.3.12). The other estimates follow analogously. □

The following Proposition gives the estimates for the X^1 and X norms of the inhomogeneous part of the evolution of the high frequency part.

Proposition 2.3 *Let F and G be given by (2.4.5) and (2.4.6) with (u_1, v_1) and (u_2, v_2) solutions to the IVPs (2.4.2) and (2.4.4) respectively. Define,*

$$(z_1(t), z_2(t)) = \left(-\int_0^t U(t-t')F(t')\,dt',\ -\int_0^t U(t-t')G(t')\,dt'\right). \tag{2.4.46}$$

Let (ϕ_1, ψ_1) and (ϕ_2, ψ_2) satisfy the hypothesis of Corollary 2.1 and Proposition 2.1. If $3/5 < s < 1$, then

$$\sup_{t\in[0,T]} \|(z_1(t), z_2(t))\|_{X^1} \leq cN^{\frac{3-5s}{2}} \tag{2.4.47}$$

and

$$\sup_{t\in[0,T]} \|(z_1(t), z_2(t))\|_{X} \leq cN^{-s}. \tag{2.4.48}$$

Proof. Applying (2.3.3) and the definition of F we get,

$$\begin{aligned}
\|D_x z_1\|_{L^2} &= \|D_x \int_0^t U(t-t')F(t')\,dt'\|_{L^2} \leq c\|F\|_{L_x^1 L_T^2}\\
&\leq c\big\{\|u_1v_1v_{2x}\|_{L_x^1L_T^2} + \|u_1v_2v_{1x}\|_{L_x^1L_T^2} + \|v_1v_2u_{1x}\|_{L_x^1L_T^2} + \|u_1v_2v_{2x}\|_{L_x^1L_T^2}\\
&\quad + \|u_2v_1v_{1x}\|_{L_x^1L_T^2} + \|u_2v_1v_{2x}\|_{L_x^1L_T^2} + \|u_2v_2v_{1x}\|_{L_x^1L_T^2} + \|v_1v_2u_{2x}\|_{L_x^1L_T^2}\\
&\quad + \|u_2v_2v_{2x}\|_{L_x^1L_T^2} + \|v_2^2u_{1x}\|_{L_x^1L_T^2} + \|v_1^2u_{2x}\|_{L_x^1L_T^2} + \|v_2^2u_{2x}\|_{L_x^1L_T^2}\big\}\\
&:= z_{1,1} + z_{1,2} + \cdots + z_{1,12}.
\end{aligned} \tag{2.4.49}$$

Now we estimate $z_{1,j}, j = 1, 2, \ldots, 12$. To get estimates for all these terms we use similar argument utilizing Theorem 2.8, Corollary 2.1, Proposition 2.1, Proposition 2.2 and the choice of T. For the sake of clarity let us consider the most difficult terms $u_1v_1v_{2x}$ and $u_1v_2v_{1x}$ in F and obtain,

$$\begin{aligned}
z_{1,1} &= c\|u_1v_1v_{2x}\|_{L_x^1L_T^2}\\
&\leq c\|u_1\|_{L_x^2L_T^\infty}\|v_1\|_{L_x^2L_T^\infty}\|v_{2x}\|_{L_x^\infty L_T^2}\\
&\leq c|\!|\!|(u_1, v_1)|\!|\!|_{3/4}^2|\!|\!|(u_2, v_2)|\!|\!|_0\\
&\leq cN^{\frac{3-5s}{2}}
\end{aligned}$$

and

$$\begin{aligned} z_{1,2} &= c\|u_1 v_2 v_{1x}\|_{L^1_x L^2_T} \\ &\leq c\|u_1\|_{L^{8/3}_x L^\infty_T} \|v_2\|_{L^5_x L^{10}_T} \|v_{1x}\|_{L^{40/17}_x L^{5/2}_T} \\ &\leq cT^{3/8} \|\!|(u_1, v_1)|\!\|_{1/2} \|\!|(u_2, v_2)|\!\|_0 \|\!|(u_1, v_1)|\!\|_1 \\ &\leq cN^{\frac{9-17s}{8}} \leq cN^{\frac{3-5s}{2}}. \end{aligned}$$

We can obtain similar estimates for the other terms in (2.4.49) too.

Using an analogous argument we can get,

$$\|z_1\|_{L^2_x} \leq cN^{-s}.$$

Finally, we can also obtain similar estimates for z_2 and that concludes the proof. □

Next, we derive some estimates that will be useful in the second step of the proof of the main result. Now, we consider the given data in X^s, $1/4 \leq s \leq s_1 < 1$, and split into low and high frequency parts according to the formula (2.4.1). For $s_0 := s_1 + \epsilon < 1$, the low frequency part $(\phi_1, \psi_1) \in X^{s_0}$ with $\|(\phi_1, \psi_1)\|_{X^{s_0}} \leq cN^{s_0 - s}$ and the high frequency part $(\phi_2, \psi_2) \in X^\rho$, $0 \leq \rho \leq s < s_0$, with $\|(\phi_2, \psi_2)\|_{X^\rho} \leq cN^{\rho - s}$. Moreover, $\|(\phi_1, \psi_1)\|_X \leq c$ and by interpolation we obtain $\|(\phi_1, \psi_1)\|_{X^\beta} \leq cN^{\frac{\beta}{s_0}(s_0 - s)}$, $0 \leq \beta \leq s_0$. As earlier we evolve (ϕ_1, ψ_1) and (ϕ_2, ψ_2) according to the IVPs (2.4.2) and (2.4.4) respectively. In this case also, we can have the results analogous to Theorem 2.8 and Theorem 2.9 with the local existence time replaced by

$$T \sim \|(\phi_1, \psi_1)\|^{-4}_{X^{1/4}} \sim N^{-\frac{1}{s_0}(s_0 - s)}$$

and the estimate (2.4.9) replaced by

$$\|\!|(u_1, v_1)|\!\|_{X^\beta} \leq cN^{\frac{\beta}{s_0}(s_0 - s)}, \qquad 1/4 \leq \beta \leq s_0. \tag{2.4.50}$$

Also, it is easy to obtain the results analogous to Corollary 2.1 and Proposition 2.1, i.e.,

$$\|\!|(u_2, v_2)|\!\|_\rho \leq cN^{\rho - s}, \qquad 1/4 < \rho \leq s < s_0, \tag{2.4.51}$$

$$\|\!|(u_2, v_2)|\!\|_0 \leq cN^{-s}. \tag{2.4.52}$$

Our next result is similar to Proposition 2.3. Before establishing it let us explain in brief the argument we are going to employ. We want to develop an iteration process in X^{s_0} by incorporating the inhomogeneous part of the evolution of the high frequency part with the evolution of the low frequency part. For this, we need to know the growth of the X^{s_0} and

X norms of the inhomogeneous part (z_1, z_2), given by (2.4.46), of the solution to the IVP (2.4.4).

For simplicity we analyze $z_1(t)$ considering one of the worst terms $u_1v_1v_{2x}$ in F and get estimate for

$$\|D_x^{s_0} z_1\|_{L^2} = \|D_x^{s_0} \int_0^t U(t-t')(u_1v_1v_{2x})(t')\,dt'\|_{L^2}.$$

Using (2.3.3) we get,

$$\|D_x^{s_0} z_1\|_{L^2} \le cT^{\frac{1}{2}(1-s_0)}\|u_1v_1v_{2x}\|_{L_x^{2/(1+s_0)}L_T^2}. \tag{2.4.53}$$

Applying Hölder's inequality, Proposition 2.1 and Proposition 2.2 we get from (2.4.53),

$$\begin{aligned}\|D^{s_0} z_1\|_{L^2} &\le cT^{\frac{1}{2}(1-s_0)}\|u_1\|_{L_x^{4/(1+s_0)}L_T^\infty}\|v_1\|_{L_x^{4/(1+s_0)}L_T^\infty}\|v_{2x}\|_{L_x^\infty L_T^2}\\ &\le cT^{\frac{1}{2}(1-s_0)}|||(u_1,v_1)|||^2_{(1+2s_0)/4}|||(u_2,v_2)|||_0.\end{aligned} \tag{2.4.54}$$

We will get the same estimate if we consider z_2 too.

Note that we have control on $\|D^{s_0}(u_1,v_1)\|_{L^2}$ and are interested to control $\|D^{s_0}(z_1,z_2)\|_{L^2}$ by it. From (2.4.54) it is clear that we will have such control on $\|D^{s_0}(z_1,z_2)\|_{L^2}$ only if

$$s_0 \ge \frac{1+2s_0}{4}, \qquad \text{i.e.} \quad s_0 \ge 1/2, \tag{2.4.55}$$

which is true, since in the second step of the proof of the main result we take $s_0 \in (3/5, 1)$ (see proof of Theorem 2.2). This condition also implies that the number of derivatives of (z_1, z_2) is not less than those of (u_1, v_1) justifying the incorporation of (z_1, z_2) with (u_1, v_1) in the iteration process.

The following Proposition provides the estimates for the X^{s_0} and X norms of (z_1, z_2).

Proposition 2.4 *Let z_1 and z_2 be defined in* (2.4.46) *with F and G given by (2.4.5) and (2.4.6) respectively. Let (ϕ_1, ψ_1) and (ϕ_2, ψ_2) satisfy the respective hypotheses of Corollary 2.1 and Proposition 2.1, then for $s_0 \ge 1/2$,*

$$\sup_{t\in[0,T]} \|(z_1(t), z_2(t))\|_{X^{s_0}} \le cN^{\frac{3s_0-5s}{2}} \tag{2.4.56}$$

and

$$\sup_{t\in[0,T]} \|(z_1(t), z_2(t))\|_{X} \le cN^{-s}. \tag{2.4.57}$$

Proof. Using (2.3.3) and the definition of F we get,

$$
\begin{aligned}
\|D_x^{s_0} z_1\|_{L^2} = \ & \|D_x^{s_0} \int_0^t U(t-t')F(t')\,dt'\|_{L^2} \leq cT^{\frac{1}{2}(1-s_0)}\|F\|_{L_x^{2/(1+s_0)}L_T^2} \\
\leq \ & cT^{\frac{1}{2}(1-s_0)}\Big\{ \|u_1 v_1 v_{2x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|u_1 v_2 v_{1x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|v_1 v_2 u_{1x}\|_{L_x^{2/(1+s_0)}L_T^2} \\
& + \|u_1 v_2 v_{2x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|u_2 v_1 v_{1x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|u_2 v_1 v_{2x}\|_{L_x^{2/(1+s_0)}L_T^2} \\
& + \|u_2 v_2 v_{1x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|v_1 v_2 u_{2x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|u_2 v_2 v_{2x}\|_{L_x^{2/(1+s_0)}L_T^2} \\
& + \|v_2^2 u_{1x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|v_1^2 u_{2x}\|_{L_x^{2/(1+s_0)}L_T^2} + \|v_2^2 u_{2x}\|_{L_x^{2/(1+s_0)}L_T^2} \Big\} \\
:= \ & z_{1,1} + z_{1,2} + \cdots + z_{1,12}.
\end{aligned}
\tag{2.4.58}
$$

Now we estimate $z_{1,j}, j = 1, 2, \ldots, 12$. To get estimates for all these terms we use similar argument utilizing (2.4.50), (2.4.51), (2.4.52), Proposition 2.2 and the choice of T. For the sake of clarity let us consider one of the most difficult terms $u_1 v_1 v_{2x}$ in F and obtain,

$$
\begin{aligned}
z_{1,1} &= cT^{\frac{1}{2}(1-s_0)}\|u_1 v_1 v_{2x}\|_{L_x^{2/(1+s_0)}L_T^2} \\
&\leq cT^{\frac{1}{2}(1-s_0)}\|u_1\|_{L_x^{4/(1+s_0)}L_T^\infty}\|v_1\|_{L_x^{4/(1+s_0)}L_T^\infty}\|v_{2x}\|_{L_x^\infty L_T^2} \\
&\leq cT^{\frac{1}{2}(1-s_0)}|||(u_1, v_1)|||_{(1+2s_0)/4}^2 |||(u_2, v_2)|||_0 \\
&\leq cN^{-\frac{1}{2}(1-s_0)\frac{1}{s_0}(s_0-s)} N^{\frac{1+2s_0}{2}\frac{1}{s_0}(s_0-s)} N^{-s} \\
&\leq cN^{\frac{3s_0-5s}{2}}.
\end{aligned}
$$

Similar estimates can also be obtained for the other terms in (2.4.58).

An analogous argument leads to,

$$
\|z_1\|_{L_x^2} \leq cN^{-s}.
$$

Finally, we can also obtain similar estimates for z_2 and that concludes the proof. □

2.5 Proof of the Global Well-posedness Result

In this section we give the proof of Theorem 2.2, the main result of this chapter.

Proof.[Proof of Theorem 2.2:] As mentioned in the introduction we carry-out the proof in two steps.

First step: Let $(\phi, \psi) \in X^s(\mathbb{R})$, $3/5 < s < 1$ and $N \gg 1$ be arbitrary but fixed. Let us decompose the initial data as in (2.4.1) to

$$\begin{cases} \phi(x) = \phi_1(x) + \phi_2(x), \\ \psi(x) = \psi_1(x) + \psi_2(x). \end{cases} \tag{2.5.1}$$

Then we have,

$$\begin{cases} \|(\phi_1, \psi_1)\|_X \leq c \\ \|(\phi_1, \psi_1)\|_{X^\beta} \leq cN^{\beta(1-s)}, \qquad 0 \leq \beta \leq 1. \end{cases} \tag{2.5.2}$$

$$\|(\phi_2, \psi_2)\|_{X^\rho} \leq cN^{\rho-s}, \quad 0 \leq \rho \leq s < 1. \tag{2.5.3}$$

Consider the IVP (2.4.2) with initial data $(\phi_1, \psi_1) \in X^\beta$, $1/4 \leq \beta \leq 1$. From Theorem 2.1 there exists T_0 satisfying

$$T_0 \leq c\|(\phi_1, \psi_1)\|_{X^{1/4}}^{-4} \sim cN^{-(1-s)}, \tag{2.5.4}$$

such that the IVP (2.4.2) has a unique solution (u_1, v_1) in the interval $[0, T_0]$. Moreover,

$$\sup_{t\in[0,T_0]} \|(u_1(t), v_1(t))\|_{X^1} \leq cN^{(1-s)}. \tag{2.5.5}$$

Now, we consider the IVP (2.4.4) with initial data (ϕ_2, ψ_2). In Theorem 2.9 we found that the IVP (2.4.4) has a unique solution (u_2, v_2) defined in the same interval of existence of the solution (u_1, v_1), $[0, T_0]$ and is given by (2.4.15), i.e.

$$\begin{cases} u_2(t) = U(t)\phi_2 + z_1(t) \\ v_2(t) = U(t)\psi_2 + z_2(t). \end{cases} \tag{2.5.6}$$

where $z_1(t)$ and $z_2(t)$ are given by (2.4.46).

As mentioned in the introduction, $u = u_1 + u_2$ and $v = v_1 + v_2$ solve the IVP (2.1.1) in the time interval $[0, T_0]$.

Given $T > 0$ arbitrary, we are interested in extending the solution (u, v) of the IVP (2.1.1) to the interval $[0, T]$. For this, we iterate the above process in each interval of size T_0 unless covering the whole interval. Now, at the time $t = T_0$ we have,

$$\begin{cases} u(T_0) = u_1(T_0) + U(T_0)\phi_2 + z_1(T_0) \\ v(T_0) = v_1(T_0) + U(T_0)\psi_2 + z_2(T_0). \end{cases} \tag{2.5.7}$$

Now we decompose $(u(T_0), v(T_0))$ as,

$$\begin{cases} u(T_0) = \tilde{u}_1(T_0) + \tilde{u}_2(T_0) \\ v(T_0) = \tilde{v}_1(T_0) + \tilde{v}_2(T_0), \end{cases} \tag{2.5.8}$$

where,

$$\begin{cases} \tilde{u}_1(T_0) = u_1(T_0) + z_1(T_0), \qquad \tilde{u}_2(T_0) = U(T_0)\phi_2 \\ \tilde{v}_1(T_0) = v_1(T_0) + z_2(T_0), \qquad \tilde{v}_2(T_0) = U(T_0)\psi_2, \end{cases} \tag{2.5.9}$$

and evolve $(\tilde{u}_1(T_0), \tilde{v}_1(T_0))$ and $(\tilde{u}_2(T_0), \tilde{v}_2(T_0))$ according to the IVPs (2.4.2) and (2.4.4) respectively. Using previous procedure, to get solution to the IVP (2.1.1) in $[T_0, 2T_0]$ we must guarantee that $(\tilde{u}_1(T_0), \tilde{v}_1(T_0))$ and $(\tilde{u}_2(T_0), \tilde{v}_2(T_0))$ satisfy the respective conditions (2.5.2) and (2.5.3).

Since $U(t)$ is unitary in H^ρ, $(\tilde{u}_2(T_0), \tilde{v}_2(T_0))$ satisfies the same growth condition as that of (ϕ_2, ψ_2), i.e, $(\tilde{u}_2(T_0), \tilde{v}_2(T_0)) \in X^\rho$ and

$$\|(\tilde{u}_2(T_0), \tilde{v}_2(T_0))\|_{X^\rho} = \|(\phi_2, \psi_2)\|_{X^\rho} \le cN^{\rho - s}, \quad \rho \le s.$$

Now, let us check how is the growth of the X^1-norm and the X-norm of $(\tilde{u}_1(T_0), \tilde{v}_1(T_0))$. Using Proposition 2.3 and estimate (2.5.5) we get

$$\begin{aligned} \|(\tilde{u}_1(T_0), \tilde{v}_1(T_0))\|_{X^1} &\le \|(u_1(T_0), v_1(T_0))\|_{X^1} + \|(z_1(T_0), z_2(T_0))\|_{X^1} \\ &\le cN^{(1-s)} + cN^{\frac{3-5s}{2}}. \end{aligned} \tag{2.5.10}$$

On the other hand, using conservation law (0.0.8) and Proposition 2.3 we obtain,

$$\begin{aligned} \|(\tilde{u}_1(T_0), \tilde{v}_1(T_0))\|_{X} &\le \|(u_1(T_0), v_1(T_0))\|_{X} + \|(z_1(T_0), z_2(T_0))\|_{X} \\ &\le \|(\phi_1, \psi_1)\|_X + cN^{-s} \\ &\le c, \qquad \text{for sufficiently large N.} \end{aligned} \tag{2.5.11}$$

So, from (2.5.10) it is clear that the solution to the IVP (2.1.1) can be extended to the interval $[T_0, 2T_0]$ if we can guarantee that $N^{\frac{3-5s}{2}} \le cN^{(1-s)}$ for large N and some appropriate values of s. In what follows we select these values not only to guarantee this condition for a single iteration but to cover the whole interval $[0, T]$.

To cover the interval $[0, T]$ we must iterate the above process T/T_0 times. As seen earlier, in each iteration, there will be a contribution of $\|(z_1, z_2)\|_{X^1}$ and $\|(z_1, z_2)\|_X$. From (2.5.10) we see that the total contribution of $\|(z_1, z_2)\|_{X^1}$ to cover $[0, T]$ is, $(T/T_0)N^{\frac{3-5s}{2}}$.

Thus the X^1-norm of (z_1, z_2) will grow uniformly as $N^{(1-s)}$ on the interval $[0,T]$ if we have,

$$\frac{T}{T_0}N^{\frac{3-5s}{2}} < cN^{(1-s)}. \tag{2.5.12}$$

Now, using $T_0 \sim N^{-(1-s)}$ from (2.5.4) we see that (2.5.12) is equivalent to,

$$TN^{\frac{3-5s}{2}} < c. \tag{2.5.13}$$

Therefore, to guarantee (2.5.12) we must choose $N = N(T)$ satisfying

$$N(T) = T^{\frac{2}{5s-3}},$$

with $\frac{5s-3}{2} > 0$, i.e. $s > 3/5$.

Let us show, with this choice the X-norm is also of $O(1)$. We know from (2.5.11) that the total contribution of $\|(z_1, z_2)\|_X$ to cover the interval $[0,T]$ is $(T/T_0)N^{-s}$. Now, with the choice of N we get for $3/5 < s \leq 1$,

$$\frac{T}{T_0}N^{-s} \leq cTN^{1-s}N^{-s} \leq c,$$

as required.

Hence, we conclude that the IVP (2.1.1) has global solution whenever $s > 3/5$.

Also, it is easy to see that the solution can be written in the form

$$u(t) = U(t)\phi + w_1(t) \qquad \text{and} \qquad v(t) = U(t)\psi + w_2(t),$$

with

$$\sup_{[0,T]} \|w_j(t)\|_{H^1} \leq cT^{2(1-s)/(5s-3)}. \tag{2.5.14}$$

From (2.5.14) and the choice of N we can obtain the following upper bound for the solution to the IVP (2.1.1) in the X^s norm

$$\sup_{[0,T]} \|(u(t), v(t))\|_{X^s} \leq cN^{s(1-s)}, \qquad 3/5 < s < 1. \tag{2.5.15}$$

Second step: Let $(\phi_0, \psi_0) \in X^s(\mathbb{R})$, $1/4 \leq s < s_0$, $s_0 \in (3/5, 1)$ and $N \gg 1$ be arbitrary but fixed. Let us decompose the initial data as in (2.4.1) to low and high frequency parts, then we have,

$$\begin{cases} \|(\phi_1, \psi_1)\|_X \leq c \\ \|(\phi_1, \psi_1)\|_{X^{s_0}} \leq cN^{s_0-s} \end{cases} \tag{2.5.16}$$

and

$$\|(\phi_2,\psi_2)\|_{X^\rho} \le cN^{\rho-s}, \quad 0\le \rho \le s. \tag{2.5.17}$$

Also, by interpolating the estimates in (2.5.16) we get,

$$\|(\phi_1,\psi_1)\|_{X^\beta} \le cN^{\frac{\beta}{s_0}(s_0-s)}, \qquad 0\le\beta\le s_0. \tag{2.5.18}$$

Consider the IVP (2.4.2) with initial data $(\phi_1,\psi_1)\in X^{s_0}$. In the first step we saw that the solution to the IVP (2.4.2) for given data in X^{s_0} exists in any time interval $[0,T]$. Moreover, from (2.5.15) we have the following uniform bound for the X^{s_0}-norm of the solution,

$$\sup_{[0,T]} \|(u_1(t),v_1(t))\|_{X^{s_0}} \le cN^{s_0(1-s_0)}. \tag{2.5.19}$$

As mentioned earlier, in this step we develop an iteration process in the space X^{s_0}, $s_0<1$ to extend the local solution to any time interval $[0,T]$. From (2.5.16) we have $\|(\phi_1,\psi_1)\|_{X^{s_0}} \le cN^{s_0-s}$, so we expect that the evolution of (ϕ_1,ψ_1) i.e., $(u_1(t),v_1(t))$ in $[0,T]$ also satisfy the same growth condition. In other words, we want the following uniform bound

$$\sup_{[0,T]} \|(u_1(t),v_1(t))\|_{X^{s_0}} \le cN^{s_0(1-s_0)} \le cN^{s_0-s}. \tag{2.5.20}$$

But for the validity of (2.5.20) we must have

$$s_0(1-s_0)\le s_0-s, \quad \text{i.e.,} \quad s_0^2\ge s. \tag{2.5.21}$$

Therefore, from here onwards we take s_0 satisfying (2.5.21) such that (2.5.20) is valid.

Now, we consider the IVP (2.4.4) with initial data (ϕ_2,ψ_2). From an analysis analogous to Theorem 2.9 we see that there exist T_0 satisfying

$$T_0 \le c\|(\phi_1,\psi_1)\|_{X^{1/4}}^{-4} \le cN^{-\frac{1}{s_0}(s_0-s)}, \tag{2.5.22}$$

and a unique solution (u_2,v_2) to the IVP (2.4.4) in the interval $[0,T_0]$ given by,

$$u_2(t)=U(t)\phi_2+z_1(t), \quad v_2(t)=U(t)\psi_2+z_2(t), \tag{2.5.23}$$

where $z_1(t)$ and $z_2(t)$ are as in (2.4.46).

In this case also $u=u_1+v_1$ and $v=u_2+v_2$ solve the IVP (2.1.1) in the time interval $[0,T_0]$.

Given $T>0$ arbitrary, we are interested to extend the solution (u,v) of the IVP (2.1.1) to the interval $[0,T]$. For this, we will iterate the above process in each interval of size T_0 unless covering the whole interval. At the time $t=T_0$ we have,

$$\begin{cases} u(T_0)=u_1(T_0)+U(T_0)\phi_2+z_1(T_0)\\ v(T_0)=v_1(T_0)+U(T_0)\psi_2+z_2(T_0). \end{cases} \tag{2.5.24}$$

Now we decompose $(u(T_0), v(T_0))$ by the formula,

$$\begin{cases} u(T_0) = \tilde{u}_1(T_0) + \tilde{u}_2(T_0) \\ v(T_0) = \tilde{v}_1(T_0) + \tilde{v}_2(T_0) \end{cases} \tag{2.5.25}$$

where,

$$\begin{cases} \tilde{u}_1(T_0) = u_1(T_0) + z_1(T_0), \qquad \tilde{u}_2(T_0) = U(T_0)\phi_2 \\ \tilde{v}_1(T_0) = v_1(T_0) + z_2(T_0), \qquad \tilde{v}_2(T_0) = U(T_0)\psi_2. \end{cases} \tag{2.5.26}$$

To extend the solution (u, v) to $[T_0, 2T_0]$ we proceed as in the first step by evolving $(\tilde{u}_1, \tilde{v}_1)$ and $(\tilde{u}_2, \tilde{v}_2)$ according to the IVPs (2.4.2) and (2.4.4) respectively. For this we need to guarantee that $(\tilde{u}_1, \tilde{v}_1)$ and $(\tilde{u}_2, \tilde{v}_2)$ satisfy the conditions (2.5.16) and (2.5.17) respectively. Because of the group property, $(\tilde{u}_2, \tilde{v}_2)$ satisfies the desired condition, i.e.,

$$\|(\tilde{u}_2, \tilde{v}_2)\|_{X^\rho} \sim \|(\phi_2, \psi_2)\|_{X^\rho} \sim N^{\rho - s}, \qquad 1/4 \leq \rho \leq s < s_0. \tag{2.5.27}$$

As in (2.5.11), it is easy to show that the first condition in (2.5.16) holds. Now, let us move to check the X^{s_0}-norm of $(\tilde{u}_1, \tilde{v}_1)$.

Note that, from (2.5.20) and Proposition 2.4 we have,

$$\begin{cases} \|(u_1(T_0), v_1(T_0))\|_{X^{s_0}} \leq cN^{s_0 - s} \\ \|(z_1(T_0), z_2(T_0))\|_{X^{s_0}} \leq cN^{\frac{3s_0 - 5s}{2}}. \end{cases} \tag{2.5.28}$$

Therefore, at $t = T_0$, from (2.5.26) and (2.5.28) we see that $(\tilde{u}_1, \tilde{v}_1)$ increases as

$$\|(\tilde{u}_1, \tilde{v}_1)\|_{X^{s_0}} \leq cN^{s_0 - s} + cN^{\frac{3s_0 - 5s}{2}}. \tag{2.5.29}$$

So, from (2.5.29) it is clear that the solution to the IVP (2.1.1) can be extended to the interval $[T_0, 2T_0]$ if we can guarantee that $N^{\frac{3s_0 - 5s}{2}} \leq cN^{s_0 - s}$ for large N and some appropriate values of s and s_0. In what follows we select these values not only to guarantee this condition for a single iteration but to cover the whole interval $[0, T]$.

To cover the interval $[0, T]$ we must iterate the above process T/T_0 times. As seen above, in each iteration, there will be a contribution of $\|(z_1, z_2)\|_{X^{s_0}}$. From (2.5.29) we see that the total contribution of $\|(z_1, z_2)\|_{X^{s_0}}$ to cover $[0, T]$ is $(T/T_0)N^{\frac{3s_0 - 5s}{2}}$.

Thus the X^{s_0}-norm of (z_1, z_2) will grow uniformly as $N^{s_0 - s}$ on the interval $[0, T]$ if we have,

$$\frac{T}{T_0} N^{\frac{3s_0 - 5s}{2}} < cN^{s_0 - s}. \tag{2.5.30}$$

Now, using $T_0 \sim N^{-\frac{1}{s_0}(s_0-s)}$ from (2.5.22) we see that (2.5.30) is equivalent to,

$$TN^{\frac{1}{2s_0}(s_0-s)(2+s_0)-s} < c. \tag{2.5.31}$$

Therefore, to guarantee (2.5.30) we must choose $N = N(T)$ satisfying

$$N(T) = T^{2s_0/\{2ss_0-(s_0-s)(2+s_0)\}},$$

with $2ss_0 - (s_0 - s)(2 + s_0) > 0$, which in turn gives,

$$s > \frac{2+s_0}{2+3s_0}s_0. \tag{2.5.32}$$

Thus we need to choose s_0 in such a way that the RHS of (2.5.32) is a minimum positive number. Taking into account the identity (2.5.21), we must have $s_0 > \max\{3/5, \sqrt{s}\}$. But, as $1/4 \leq s \leq 3/5$, we need to select $s_0 > \sqrt{s}$. Now, using this selection in (2.5.32) we obtain $1 > s > 4/9$. Therefore, the IVP (2.1.1) has global solution if $s > 4/9$ and this completes the proof of the theorem. □

Chapter 3

Unique Continuation Property for Zakharov-Kuznetsov Equation

3.1 Introduction

Let us consider the following initial value problem (IVP),

$$\begin{cases} u_t + (u_{xx} + u_{yy})_x + uu_x = 0, & (x, y) \in \mathbb{R}^2, \ t \in \mathbb{R} \\ u(x, y, 0) = u_0(x, y), \end{cases} \tag{3.1.1}$$

where $u = u(x, y, t)$ is a real valued function.

This two dimensional generalization of the KdV equation was obtained by Zakharov and Kuznetsov [66] to describe the propagation of nonlinear ion-acoustic waves in magnetized plasma. Several properties of this equation including existence and stability of solitary wave solutions have extensively been studied in the literature (see for eg. [5], [23], [58]).

As mentioned earlier, our concern is about the following question: If a sufficiently smooth real valued solution $u = u(x, y, t)$ to the IVP (3.1.1) is supported compactly on a certain time interval, is it true that $u \equiv 0$? In some sense, it is a weak version of the unique continuation property (see Definition 2).

Our result in this work is in the same spirit to that of Bourgain in [14]. We derive some new estimates to address a bi-dimensional (spatial) model and provide an affirmative answer to the question posed above. More precisely, we prove the following result.

Theorem 3.1 *Let $u = u(x, y, t)$ be a smooth solution to the IVP (3.1.1) and $I = [-T, T]$ be a non trivial time interval. If for some $B > 0$*

$$\operatorname{supp} u(t) \subseteq [-B, B] \times [-B, B], \quad \forall\, t \in I,$$

then $u \equiv 0$.

As mentioned in the introduction, there are much stronger results of UCP for the KdV and mKdV equation. For example, the UCP result due to Zhang in [69] implies the results in [56] and [14] for the KdV equation. Zhang [69] used inverse scattering theory and Miura's transformation to get these results. In fact, he introduced some decay condition to the solution and exploited the fact that the KdV and mKdV equations are completely integrable. Recently, Kenig, Ponce and Vega [36] proved that if a sufficiently smooth solution u of the generalized KdV equation is supported in $(-\infty, b)$ or in (a, ∞) at two different instants of time then $u \equiv 0$. To get this result they used Carleman's type estimate and the result due to Saut and Scheurer [56]. The exponential decay property of the solution is essential in the argument employed in [36].

Remark 3.1 *The equation* (3.1.1) *is not integrable (see [58] and [59]) and also we do not know whether its solution has exponential decay property. So the methods in [69] and [36] cannot be applied to get much stronger results as mentioned above.*

3.2 Preliminary Estimates

This section is devoted to establish some preliminary estimates that will play fundamental role in our analysis. Let us begin with the following result.

Lemma 3.1 *Let $u = u(x, y, t)$ be a smooth solution to the IVP (3.1.1). If for some $B > 0$,*

$$\operatorname{supp} u(t) \subseteq \mathcal{B} := [-B, B] \times [-B, B],$$

then for all $\lambda = (\xi, \eta), \sigma = (\theta, \delta) \in \mathbb{R}^2$, we have,

$$|\widehat{u(t)}(\lambda + i\sigma)| \lesssim e^{c|\sigma|B}. \tag{3.2.1}$$

Where we have used $|(x, y)| = \max\{|x|, |y|\}$.

Proof. Using the Cauchy-Schwarz inequality and the conservation law (0.0.15) we have,

$$\begin{aligned}
|\widehat{u(t)}(\lambda + i\sigma)| &\leq \int_{\mathbb{R}^2} |e^{-i\mathbf{x}\cdot(\lambda+i\sigma)} u(t)(\mathbf{x})|\, d\mathbf{x} \\
&\leq \int_{\mathcal{B}} e^{\mathbf{x}\cdot\sigma} |u(t)(\mathbf{x})|\, d\mathbf{x} \\
&\leq \max_{\mathbf{x}\in\mathcal{B}} e^{\mathbf{x}\cdot\sigma} \int_{\mathcal{B}} |u(t)(\mathbf{x})|\, d\mathbf{x} \\
&\leq c \max_{-B\leq x,y\leq B} e^{x\theta+y\delta} \Big(\int_{\mathbb{R}^2} |u(t)(\mathbf{x})|^2\, d\mathbf{x} \Big)^{1/2} \\
&\leq c e^{B(|\theta|+|\delta|)} \lesssim e^{c|\sigma|B}.
\end{aligned}$$

□

For $\lambda = (\xi, \eta)$ and $\lambda' = (\xi', \eta')$ define

$$u^*(\lambda) = \sup_{t \in I} |\widehat{u(t)}(\lambda)| \tag{3.2.2}$$

and

$$m(\lambda) = \sup_{\substack{|\xi'| \geq |\xi| \\ |\eta'| \geq |\eta|}} |u^*(\lambda')|. \tag{3.2.3}$$

Considering $u(0)$ sufficiently smooth and taking into account the well-posedness theory for the IVP (3.1.1) (see for example, Biagioni and Linares [5]), we have the following result.

Lemma 3.2 *Let $u \in C([-T, T]; H^s)$ be a sufficiently smooth solution to the IVP (3.1.1) with* $\operatorname{supp} u(t) \subseteq \mathcal{B}$, $t \in I$, *then for some constant B_1, we have,*

$$m(\lambda) \lesssim \frac{B_1}{1 + |\lambda|^4}. \tag{3.2.4}$$

Proof. The Cauchy-Schwarz inequality and the conservation law (0.0.15) yield,

$$\int_{\mathbb{R}^2} |u(t)(\lambda)|\, d\lambda \leq |\mathcal{B}|^{1/2} \|u(t)\|_{L^2} \lesssim 1. \tag{3.2.5}$$

Now, using properties of the Fourier transform and (3.2.5) we get,

$$\|\widehat{u(t)}\|_{L^\infty} \leq c\|u(t)\|_{L^1} \lesssim 1. \tag{3.2.6}$$

Therefore,

$$|\widehat{u(t)}(\lambda)| \leq \|\widehat{u(t)}\|_{L^\infty} \lesssim 1, \tag{3.2.7}$$

and consequently,

$$\sup_t |\widehat{u(t)}(\lambda)| \leq c. \tag{3.2.8}$$

From the local well-posedness result (see [5]), we have,

$$\|D^s u(t)\|_{L^\infty_T L^2_{xy}} \leq c. \tag{3.2.9}$$

Next, using the Cauchy-Schwarz inequality and (3.2.9) one gets,

$$\int_{\mathbb{R}^2} |D^s u(t)(x,y)|\, dxdy \leq c\Big(\int_{\mathbb{R}^2} |D^s u(t)(x,y)|^2\, dxdy\Big)^{1/2} \leq c. \tag{3.2.10}$$

Since,

$$|\lambda|^s \widehat{u(t)}(\lambda) = \widehat{D^s u(t)}(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}^2} D^s u(t)(x,y) e^{-i(x\xi + y\eta)}\, dxdy,$$

the estimate (3.2.10) implies,

$$|\lambda|^s |\widehat{u(t)}(\lambda)| \leq c \int_{\mathbb{R}^2} |D^s u(t)(x,y)|\, dxdy \leq c_1. \tag{3.2.11}$$

Therefore,

$$|\widehat{u(t)}(\lambda)| \leq \frac{c_1}{|\lambda|^s}. \tag{3.2.12}$$

If we consider $s = 4$ (which is possible, because we have local well-posedness for the IVP (3.1.1) in H^1) and combine (3.2.8) and (3.2.12) we get,

$$\sup_t |\widehat{u(t)}(\lambda)| \leq \frac{B_1}{1 + |\lambda|^4}. \tag{3.2.13}$$

If λ' is such that $|\xi'| \geq |\xi|$ and $|\eta'| \geq |\eta|$, then $\frac{1}{1+|\lambda|^4} \geq \frac{1}{1+|\lambda'|^4}$. Hence,

$$m(\lambda) = \sup_{\substack{|\xi'| \geq |\xi| \\ |\eta'| \geq |\eta|}} \sup_{t \in I} |\widehat{u(t)}(\lambda')| \leq \frac{B_1}{1 + |\lambda'|^4} \leq \frac{B_1}{1 + |\lambda|^4},$$

as required. □

Now, using Lemma 3.2, we have the following result.

Proposition 3.1 *Let $u(t)$ be compactly supported and suppose that there exists $t \in I$ with $u(t) \neq 0$. Then there exists a number $c > 0$ such that for any large number $Q > 0$ there are arbitrary large $|\lambda|$-values such that,*

$$m(\lambda) > c(m * m)(\lambda) \tag{3.2.14}$$

and

$$m(\lambda) > e^{-\frac{|\lambda|}{Q}}. \tag{3.2.15}$$

Proof. The argument is similar to the one given in the proof of lemma in page 440 in [14], so we omit it. □

Using the definition of $m(\lambda)$ and Proposition 3.1 we choose $|\lambda|$ large (with $|\xi|$, $|\eta|$ large) and $t_1 \in I$ such that,

$$|\widehat{u(t_1)}(\lambda)| = u^*(\lambda) = m(\lambda) > c(m * m)(\lambda) + e^{-\frac{|\lambda|}{Q}}. \tag{3.2.16}$$

In what follows we prove some estimates regarding derivative of an entire function. First, let us recall a lemma whose proof is given in [14].

Lemma 3.3 *Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function which is bounded and integrable on the real axis and satisfies,*

$$|\phi(\xi + i\theta)| \lesssim e^{|\theta| B}, \qquad \xi, \theta \in \mathbb{R}.$$

Then, for $\lambda_1 \in \mathbb{R}^+$ we have,

$$|\phi'(\lambda_1)| \lesssim B\Big(\sup_{\xi' \geq \lambda_1} |\phi(\xi')|\Big)\Big[1 + \Big|\log\Big(\sup_{\xi' \geq \lambda_1} |\phi(\xi')|\Big)\Big|\Big]. \tag{3.2.17}$$

Using this lemma, we have the following result.

Lemma 3.4 *Let $\Phi : \mathbb{C}^2 \to \mathbb{C}$ be an entire function satisfying*

$$|\Phi(\lambda + i\sigma)| \lesssim e^{c|\sigma| B} \qquad \lambda, \sigma \in \mathbb{R}^2,$$

such that for z_2 fixed, $\Phi_1(z_1) := \Phi(z_1, z_2)$ and for z_1 fixed, $\Phi_2(z_1) := \Phi(z_1, z_2)$ are bounded and integrable on the real axis. Then for λ_1, $\lambda_2 \in \mathbb{R}^+$ we have,

$$|\nabla\Phi(\lambda_1, \lambda_2)| \lesssim B\Big(\sup_{\substack{\xi' \geq \lambda_1 \\ \eta' \geq \lambda_2}} |\Phi(\xi', \eta')|\Big)\Big[1 + \Big|\log\Big(\sup_{\substack{\xi' \geq \lambda_1 \\ \eta' \geq \lambda_2}} |\Phi(\xi', \eta')|\Big)\Big|\Big]. \tag{3.2.18}$$

Proof. Let $\lambda' = (\xi', \eta')$ and fix z_2 such that $\eta' \geq \lambda_2$. Applying the Lemma 3.3 for Φ_1 we obtain,

$$|\Phi_1'(\lambda_1)| \lesssim B\Big(\sup_{\xi' \geq \lambda_1} |\Phi_1(\xi')|\Big)\Big[1 + \Big|\log\Big(\sup_{\xi' \geq \lambda_1} |\Phi_1(\xi')|\Big)\Big|\Big]. \tag{3.2.19}$$

Now, let us fix z_1 such that $\xi' \geq \lambda_1$. Again applying the Lemma 3.3 for Φ_2 we get,

$$|\Phi_2'(\lambda_2)| \lesssim B\Big(\sup_{\eta' \geq \lambda_2} |\Phi_1(\eta')|\Big)\Big[1 + \Big|\log\Big(\sup_{\eta' \geq \lambda_2} |\Phi_2(\eta')|\Big)\Big|\Big]. \tag{3.2.20}$$

Since

$$\nabla\Phi(\lambda_1, \lambda_2) := (\Phi_1'(\lambda_1), \Phi_2'(\lambda_2)),$$

we obtain,

$$\begin{aligned}|\nabla\Phi(\lambda_1,\lambda_2)| &\lesssim B\ \max\Big\{\Big(\sup_{\xi'\geq\lambda_1}|\Phi(\xi',\eta')|\Big)\Big[1+\Big|\log\Big(\sup_{\xi'\geq\lambda_1}|\Phi(\xi',\eta')|\Big)\Big|\Big],\\ &\qquad\Big(\sup_{\eta'\geq\lambda_2}|\Phi(\xi',\eta')|\Big)\Big[1+\Big|\log\Big(\sup_{\eta'\geq\lambda_2}|\Phi(\xi',\eta')|\Big)\Big|\Big]\Big\}\\ &\leq B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi',\eta')|\Big)\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi',\eta')|\Big)\Big|\Big],\end{aligned}$$

as needed. □

Corollary 3.1 *Let $\sigma\in\mathbb{R}^2$ be such that,*

$$|\sigma|\leq B^{-1}\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1>0\\ \eta'\geq\lambda_2>0}}|\Phi(\xi',\eta')|\Big)\Big|\Big]^{-1}. \tag{3.2.21}$$

Then,

$$\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda'+i\sigma)|\leq 4\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')| \tag{3.2.22}$$

and

$$\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\nabla\Phi(\lambda'+i\sigma)|\lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big|\Big]. \tag{3.2.23}$$

Proof. The proof of (3.2.22) is immediate by using Corollary 2.9 in [14]. In fact, first fixing $\eta'\geq\lambda_2$ and then fixing $\xi'\geq\lambda_1$ we obtain,

$$\begin{aligned}\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi'+i\theta,\eta'+i\delta)| &\leq \sup_{\eta'\geq\lambda_2}\Big(2\sup_{\xi'\geq\lambda_1}|\Phi(\xi',\eta'+i\delta)|\Big)\\ &\leq 4\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|.\end{aligned} \tag{3.2.24}$$

To prove (3.2.23), we use the estimate (3.2.22) and Lemma 3.4. For this, let us define $\tilde{\Phi}(z)=\Phi(z+i\sigma)$, then $\tilde{\Phi}$ is an entire function and moreover we have,

$$|\tilde{\Phi}(z+i\sigma')|=|\Phi(\lambda+i(\sigma+\sigma'))|\lesssim e^{c_1|\sigma+\sigma'|B}\lesssim e^{c_1|\sigma'|B}.$$

Therefore, $\tilde{\Phi}$ satisfies the conditions of Lemma 3.4 and we get,

$$|\nabla\tilde{\Phi}(\lambda_1,\lambda_2)| \lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\tilde{\Phi}(\xi',\eta')|\Big)\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\tilde{\Phi}(\xi',\eta')|\Big)\Big|\Big].$$

for any $\lambda_1, \lambda_2 \in \mathbb{R}^+$.

Hence, using the definition of $\tilde{\Phi}$ and (3.2.22) we obtain,

$$\begin{aligned}
&|\nabla\Phi(\lambda_1+i\theta,\lambda_2+i\delta)|\\
&\quad\lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi'+i\theta,\eta'+i\delta)|\Big)\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi'+i\theta,\eta'+i\delta)|\Big)\Big|\Big]\\
&\quad\lesssim 4B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi',\eta')|\Big)\Big[1+\Big|\log\Big(4\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi',\eta')|\Big)\Big|\Big]\\
&\quad\lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big[(1+\log 4)+(1+\log 4)\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big|\Big]\\
&\quad\lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big|\Big].
\end{aligned} \tag{3.2.25}$$

Therefore,

$$\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\nabla\Phi(\lambda'+i\sigma')| \lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi',\eta')|\Big)\Big|\Big],$$

which concludes the proof. □

Corollary 3.2 *Let $t \in I$, $\Phi(z) = \widehat{u(t)}(z)$, σ be as in Corollary 3.1 and $m(\lambda)$ be as in (3.2.3). Then, for $|\sigma'| \leq |\sigma|$ fixed, we have*

$$|\nabla\Phi(\lambda-\lambda'+i\sigma')| \lesssim B\big[m(\lambda)+m(\lambda-\lambda')\big]\big[1+|\log m(\lambda)|\big]. \tag{3.2.26}$$

Proof. Let $\tilde{\Phi}(z) := \Phi(z+i\sigma')$, $z=(z_1,z_2)=(\xi+i\theta,\eta+i\delta)$. First, let us use (3.2.23) with $\sigma = 0$ and then use (3.2.22) to get, for $|\bar{\lambda_1}| = \min\{|\xi_1|, |\xi-\xi_1|\}$, $|\bar{\lambda_2}| = \min\{|\eta|, |\eta-\eta'|\}$ and

$\bar{\lambda} = (\bar{\lambda_1}, \bar{\lambda_2})$,

$$\begin{aligned}
|\nabla\Phi(\lambda - \lambda' + i\sigma')| &\leq \sup_{\substack{\xi' \geq |\bar{\lambda_1}| \\ \eta' \geq |\bar{\lambda_2}|}} |\nabla\Phi(\lambda' + i\sigma')| = \sup_{\substack{\xi' \geq |\bar{\lambda_1}| \\ \eta' \geq |\bar{\lambda_2}|}} |\nabla\tilde{\Phi}(\lambda')| \\
&\lesssim B\Big(\sup_{\substack{\xi' \geq |\bar{\lambda_1}| \\ \eta' \geq |\bar{\lambda_2}|}} |\tilde{\Phi}(\lambda')|\Big)\Big[1 + \Big|\log\Big(\sup_{\substack{\xi' \geq |\bar{\lambda_1}| \\ \eta' \geq |\bar{\lambda_2}|}} |\tilde{\Phi}(\lambda')|\Big)\Big|\Big] \\
&\lesssim B\Big(\sup_{\substack{\xi' \geq |\bar{\lambda_1}| \\ \eta' \geq |\bar{\lambda_2}|}} |\Phi(\lambda')|\Big)\Big[1 + \Big|\log\Big(\sup_{\substack{\xi' \geq |\bar{\lambda_1}| \\ \eta' \geq |\bar{\lambda_2}|}} |\Phi(\lambda')|\Big)\Big|\Big] \\
&\lesssim B\, m(\bar{\lambda})[1 + |\log m(\bar{\lambda})|] \\
&\leq B\,[m(\lambda) + m(\lambda - \lambda')][1 + |\log m(\lambda)|],
\end{aligned} \tag{3.2.27}$$

which is the desired estimate. □

3.3 Proof of the Main Result

In this section we present the proof of the main result of this chapter.

Proof.[Proof of Theorem 3.1:] Suppose that there exists $t \in I$ such that $u(t) \neq 0$. We will use the estimates derived in the previous section to arrive at a contradiction. For this we proceed as follows. Form Duhamel's principle, we have for $t_1, t_2 \in I$

$$u(t_2) = U(t_2 - t_1)u(t_1) - \frac{1}{2}\int_{t_1}^{t_2} U(t_2 - t')(u^2)_x(t')\, dt', \tag{3.3.1}$$

where $U(t)$ given by,

$$U(t)f(x, y) = \int_{\mathbb{R}^2} e^{i(t(\xi^3 + \xi\eta^2) + x\xi + y\eta)} \hat{f}(\xi, \eta)\, d\xi d\eta,$$

is the unitary group associated to the linear problem. Taking Fourier transform in the space variables we get from (3.3.1),

$$\widehat{u(t_2)}(\lambda) = e^{i(t_2 - t_1)(\xi^3 + \xi\eta^2)}\widehat{u(t_1)}(\lambda) - \frac{i\xi}{2}\int_{t_1}^{t_2} e^{i(t_2 - t')(\xi^3 + \xi\eta^2)}\widehat{u^2(t')}(\lambda)\, dt'. \tag{3.3.2}$$

Let $t_2 - t_1 = \Delta t$, then from (3.3.2) we obtain,

$$\widehat{u(t_2)}(\lambda) = e^{i\Delta t(\xi^3 + \xi\eta^2)}\Big[\widehat{u(t_1)}(\lambda) - \frac{i\xi}{2}\int_{t_1}^{t_2} e^{i(t_1 - t')(\xi^3 + \xi\eta^2)}\widehat{u^2(t')}(\lambda)\, dt'\Big]. \tag{3.3.3}$$

Let us change variable in the integral in (3.3.3) by defining $s = t' - t_1$, to get,

$$\widehat{u(t_2)}(\lambda) = e^{i\Delta t(\xi^3+\xi\eta^2)}\Big[\widehat{u(t_1)}(\lambda) - \frac{i\xi}{2}\int_0^{\Delta t} e^{-is(\xi^3+\xi\eta^2)}\widehat{u^2(s+t_1)}(\lambda)\,ds\Big]. \tag{3.3.4}$$

Since $u(t), t \in I$ has compact support, by Paley-Wiener theorem, $\widehat{u(t)}(\lambda)$ has analytic continuation in $\mathbb{C}^2$, and we have,

$$\begin{aligned}\widehat{u(t_2)}(\lambda + i\sigma) = e^{i\Delta t\{(\xi+i\theta)^3+(\xi+i\theta)(\eta+i\delta)^2\}}\Big[&\widehat{u(t_1)}(\lambda+i\sigma)\\ &- \frac{i(\xi+i\theta)}{2}\int_0^{\Delta t} e^{-is\{(\xi+i\theta)^3+(\xi+i\theta)(\eta+i\delta)^2\}}\widehat{u^2(s+t_1)}(\lambda+i\sigma)\,ds\Big].\end{aligned} \tag{3.3.5}$$

Since,

$$\begin{aligned}(\xi+i\theta)^3 + (\xi+i\theta)(\eta+i\delta)^2 = \xi^3 &- 3\xi\theta^2 - \xi\eta^2 - \xi\delta^2 - 2\eta\theta\delta\\ &+ i(3\xi^2\theta - \theta^3 + 2\xi\eta\delta + \theta\eta^2 - \theta\delta^2),\end{aligned}$$

the use of Lemma 3.1 in (3.3.5) yields,

$$\begin{aligned}&ce^{\Delta t(3\xi^2\theta-\theta^3+2\xi\eta\delta+\theta(\eta^2-\delta^2))} \geq\\ &\geq \big|\widehat{u(t_1)}(\lambda+i\sigma)\big| - \frac{|\xi+i\theta|}{2}\int_0^{\Delta t} e^{s(3\xi^2\theta-\theta^3+2\xi\eta\delta+\theta(\eta^2-\delta^2))}\big|\widehat{u^2(s+t_1)}(\lambda+i\sigma)\big|\,ds.\end{aligned} \tag{3.3.6}$$

Let us take $|\lambda| = \max\{|\xi|, |\eta|\}$ very large with both $|\xi|$ and $|\eta|$ large such that

$$\xi\eta > 0. \tag{3.3.7}$$

Choose $\sigma = \sigma(\lambda)$ with $|\sigma| = \max\{|\theta|, |\delta|\} \approx 0$ such that,

$$\theta\Delta t < 0 \qquad \text{and} \qquad \delta\Delta t < 0. \tag{3.3.8}$$

Moreover, let us suppose the following conditions are satisfied

$$\frac{1}{|\xi|} \ll \begin{cases}|\theta|\\ |\delta|\end{cases} \qquad \text{and} \qquad \frac{1}{|\eta|} \ll \begin{cases}|\theta|\\ |\delta|.\end{cases} \tag{3.3.9}$$

With these choices, (3.3.6) can be written as

$$\begin{aligned}&ce^{\Delta t(3\xi^2\theta+2\xi\eta\delta+\theta\eta^2)} \gtrsim\\ &\gtrsim \big|\widehat{u(t_1)}(\lambda+i\sigma)\big| - |\xi|\int_0^{\Delta t} e^{s(3\xi^2\theta+2\xi\eta\delta+\theta\eta^2)}\big|\widehat{u^2(s+t_1)}(\lambda+i\sigma)\big|\,ds.\end{aligned} \tag{3.3.10}$$

Now, using (3.3.7) and (3.3.8) in (3.3.10) we obtain for $\Delta t > 0$,

$$\Big|\int_0^{\Delta t} e^{s(3\xi^2\theta+2\xi\eta\delta+\theta\eta^2)}\widehat{u^2(s+t_1)}(\lambda+i\sigma)\,ds\Big| = \Big|\int_0^{\Delta t} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)}\widehat{u^2(s+t_1)}(\lambda+i\sigma)\,ds\Big|$$

and for $\Delta t < 0$, making change of variables, $s \leftrightarrow -s$,

$$\begin{aligned}\Big|\int_0^{\Delta t} & e^{s(3\xi^2\theta+2\xi\eta\delta+\theta\eta^2)}\widehat{u^2(s+t_1)}(\lambda+i\sigma)\,ds\Big| \\ &= \Big|\int_0^{-\Delta t} e^{-s(3\xi^2\theta+2\xi\eta\delta+\theta\eta^2)}\widehat{u^2(t_1-s)}(\lambda+i\sigma)\,ds\Big| \\ &= \Big|\int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)}\widehat{u^2(t_1-s)}(\lambda+i\sigma)\,ds\Big|.\end{aligned}$$

Therefore, in any case we have,

$$\begin{aligned}e^{-|\Delta t|(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)} \gtrsim & \big|\widehat{u(t_1)}(\lambda+i\sigma)\big| \\ & - |\xi|\int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)}\big|\widehat{u^2(t_1\pm s)}(\lambda+i\sigma)\big|\,ds.\end{aligned} \tag{3.3.11}$$

In what follows we consider the case $\Delta t > 0$ (the analysis for $\Delta t < 0$ is similar). Since $e^{-x} < 1$ for $x > 0$, the estimate (3.3.11) can be written as,

$$e^{-(3\xi^2+\eta^2)|\theta\Delta t|} \gtrsim \big|\widehat{u(t_1)}(\lambda+i\sigma)\big| - |\xi|\int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)}\big|\widehat{u^2(t_1+s)}(\lambda+i\sigma)\big|\,ds.$$

Finally, we write this last estimate in the following manner,

$$\begin{aligned}e^{-(3\xi^2+\eta^2)|\theta\Delta t|} \gtrsim & \big|\widehat{u(t_1)}(\lambda)\big| - |\xi|\int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)}\big|\widehat{u^2(t_1+s)}(\lambda)\big|\,ds \\ & - \big|\widehat{u(t_1)}(\lambda+i\sigma) - \widehat{u(t_1)}(\lambda)\big| \\ & - |\xi|\int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)}\big|\widehat{u^2(t_1+s)}(\lambda+i\sigma) - \widehat{u^2(t_1+s)}(\lambda)\big|\,ds \\ := & I_1 - I_2 - I_3.\end{aligned} \tag{3.3.12}$$

Now, our aim is to find appropriate estimates for I_1, I_2 and I_3 to get a contradiction in (3.3.12).

Let us estimate I_1: Use of (3.2.2) and (3.2.14) yields,

$$\begin{aligned}
|\xi| \int_0^{|\Delta t|} & e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)} \big|\widehat{u(t_1+s)}\big| * \big|\widehat{u(t_1+s)}\big|(\lambda)\, ds \\
&\leq |\xi|(u^* * u^*)(\lambda) \int_0^{|\Delta t|} e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)} ds \\
&\leq |\xi|(m*m)(\lambda) \frac{1-e^{-|\Delta t|(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)}}{3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2} \\
&\leq \frac{|\xi|(m*m)(\lambda)}{2|\xi\eta\delta|} \\
&\lesssim \frac{m(\lambda)}{2|\eta\delta|}.
\end{aligned}$$

Therefore,

$$I_1 \gtrsim m(\lambda) - \frac{m(\lambda)}{2|\eta\delta|} \geq \frac{m(\lambda)}{2}. \tag{3.3.13}$$

Now, we estimate I_2: For this let us define $\Phi(z) = \widehat{u(t_1)}(z)$, $z = (z_1, z_2) \in \mathbb{C}^2$. Using (3.2.16) we get,

$$|\Phi(\lambda)| = |\widehat{u(t_1)}(\lambda)| = \sup_{\substack{|\xi'|\geq|\xi| \\ |\eta'|\geq|\eta|}} |\Phi(\lambda')| = m(\lambda). \tag{3.3.14}$$

Let us choose $|\sigma|$ satisfying

$$|\sigma| \lesssim B^{-1}\big[1 + |\log m(\lambda)|\big]^{-1}, \tag{3.3.15}$$

and use Corollary 3.1 to obtain,

$$\begin{aligned}
I_2 &= \big|\widehat{u(t_1)}(\lambda + i\sigma) - \widehat{u(t_1)}(\lambda)\big| \\
&\lesssim |\sigma| \sup_{\substack{|\xi'|\geq|\xi| \\ |\eta'|\geq|\eta|}} |\nabla\widehat{u(t_1)}(\lambda' + i\sigma)| \\
&\lesssim |\sigma| B\, m(\lambda)\big[1 + |\log m(\lambda)|\big] \\
&\lesssim m(\lambda) \\
&\lesssim \frac{1}{5} m(\lambda).
\end{aligned} \tag{3.3.16}$$

Next, we estimate I_3: Using Proposition 3.1, Corollary 3.2 and taking $|\sigma|$ as in (3.3.15) we get,

$$\begin{aligned}
&\left|\widehat{u^2(t_1+s)}(\lambda+i\sigma)-\widehat{u^2(t_1+s)}(\lambda)\right| \\
&= \left|\int_{\mathbb{R}^2}\widehat{u(t_1+s)}(\lambda+i\sigma-\lambda')\widehat{u(t_1+s)}(\lambda')\,d\lambda' - \int_{\mathbb{R}^2}\widehat{u(t_1+s)}(\lambda-\lambda')\widehat{u(t_1+s)}(\lambda')\,d\lambda'\right| \\
&\le \int_{\mathbb{R}^2}\left|\widehat{u(t_1+s)}(\lambda-\lambda'+i\sigma)-\widehat{u(t_1+s)}(\lambda-\lambda')\right|\left|\widehat{u(t_1+s)}(\lambda')\right|d\lambda' \\
&\le |\sigma|\int_{\mathbb{R}^2}\sup_{|\sigma'|\le|\sigma|}\left|\nabla\widehat{u(t_1+s)}(\lambda-\lambda'+i\sigma')\right| m(\lambda')\,d\lambda' \\
&\le \int_{\mathbb{R}^2}[m(\lambda)+m(\lambda-\lambda')]\,m(\lambda')\,d\lambda' \\
&\le m(\lambda)c_2+(m*m)(\lambda) \\
&\le m(\lambda)(c_2+c^{-1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_3 &\le |\xi|m(\lambda)(c_2+c^{-1})\int_0^{|\Delta t|}e^{-s(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)}ds \\
&= |\xi|m(\lambda)(c_2+c^{-1})\frac{1-e^{-|\Delta t|(3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2)}}{3\xi^2|\theta|+2|\xi\eta\delta|+|\theta|\eta^2} \\
&\le \frac{|\xi|m(\lambda)(c_2+c^{-1})}{2|\xi\eta\delta|} \\
&\lesssim \frac{m(\lambda)}{|\eta\delta|} \\
&< \frac{m(\lambda)}{5}.
\end{aligned} \tag{3.3.17}$$

Now using (3.3.13), (3.3.16) and (3.3.17) in (3.3.12) we get,

$$e^{-(3\xi^2+\eta^2)|\theta\Delta t|} \gtrsim \frac{m(\lambda)}{2}-\frac{m(\lambda)}{5}-\frac{m(\lambda)}{5}=\frac{1}{10}m(\lambda)\gtrsim e^{-\frac{|\lambda|}{Q}}. \tag{3.3.18}$$

Our choice in (3.3.9) gives, $|\xi\theta|\gg 1$ and $|\eta\theta|\gg 1$. Therefore,

$$\begin{aligned}
e^{-(3\xi^2+\eta^2)|\theta||\Delta t|} &= e^{-(3|\xi||\xi\theta|+|\eta||\eta\theta|)|\Delta t|} \\
&\le e^{-(|\xi|+|\eta|)|\Delta t|} \\
&\le e^{-c|\lambda||\Delta t|}.
\end{aligned} \tag{3.3.19}$$

Finally, using (3.3.19) in (3.3.18), we obtain,

$$e^{-c|\lambda||\Delta t|} \gtrsim e^{-\frac{|\lambda|}{Q}},$$

which is a contradiction for $|\lambda|$ large, if we choose Q large such that $\frac{1}{Q} < c|\Delta t|$. This completes the proof of the theorem. $\square$

Conclusions and Remarks

Here we give a brief description of the main results obtained in this work and also mention some future works.

In the first chapter we considered the initial value value problem (IVP) associated to the coupled system of KdV equations (1.1.1) that describes the strong interaction of long internal gravity waves in stratified fluids. This model consists of a pair of KdV equations coupled through nonlinear as well as dispersive parts.

We studied the associated IVP (1.1.1) in the Fourier transform truncation space $X_{s,b}$ introduced by Bourgain [15] in the context of the general class of evolution equations. Using the bilinear estimates established by Kenig, Ponce and Vega [38] we proved the local well-posedness result for given data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$. Also, using the idea introduced by Bourgain [13], we proved that the local result is optimal by showing that the map data-solution cannot be C^2 at the origin for given data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s < -3/4$. Further, under certain conditions on the coefficients, we exploited the symmetry of the model to derive an *almost conserved quantity* and used it to implement the *I-method*, a variant of the method of Bourgain [12] recently introduced by Colliander et. al. [22], and proved that the local solution can be extended to a global one for given data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/10$. The results in Chapter 1 improve those obtained by Ash, Cohen and Wang in [4].

Observe that, the method used in Chapter 1 can also be applied to obtain analogous results to the IVP associated to the following coupled KdV system of Nutku and Oğuz [53]

$$\begin{cases} u_t = u_{xxx} + 2\alpha u u_x + v v_x + (uv)_x \\ v_t = v_{xxx} + 2\beta v v_x + u u_x + (uv)_x, \end{cases} \tag{3.3.20}$$

where α, β are constants.

In the second chapter we addressed the global well-posedness problem to the IVP associated to the system (2.1.1). The model in question has a pair of mKdV equations coupled through nonlinear terms and arises in several physical situations. We studied this problem using the low-high frequency technique introduced by Bourgain [12], and further simplified by Fonseca, Linares and Ponce [24] in the mKdV context. Using the uniform bound of the solution in certain Sobolev spaces below the energy space, we developed an iteration process in those spaces and proved that the local solution to the IVP associated to (2.1.1) can be extended to any time interval $[0, T]$ for given data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 4/9$. This result improves the result obtained in Montenegro [51] where he showed that the IVP associated to (2.1.1) for given data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, is locally well-posed when $s \geq 1/4$ and globally well-posed when $s \geq 1$.

The third chapter was concerned about the bi-dimensional generalization (3.1.1) of the KdV equation. This model was proposed by Zakharov and Kuznetsov [66] and governs the propagation of nonlinear ion-acoustics waves in magnetized plasma. Various properties related to this model like Cauchy problem, existence and stability of solitary waves are studied by many authors (see [5], [23]). We extended the recent work of Bourgain [14] and derived some new estimates to address a bi-dimensional model and proved that, if sufficiently smooth solution to the Zakharov-Kuznetsov equation (3.1.1) is supported in a nontrivial time interval then it vanishes identically.

There is still ample room to improve and extend the results obtained in this work. Also there is strong possibility to use these techniques to new models as well. In what follows we mention some open problem that may be of interest in the future.

It would be interesting to obtain the best possible global well-posedness result for the IVP associated to (1.1.1), i.e., for data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$. For this, one can expect to extend the variant of the method of Bourgain [12] introduced by Colliander et. al. [21] in the KdV context. Also, it would be interesting to get global solution without imposing restriction on the coefficients. The presence of the arbitrary constants suggests one to proceed in some other way to implement the theory used in this work. Another nice problem is to utilize the recent work of Lopes [47] to study the orbital stability of the solitary wave solutions to the system (1.1.1). The next problem is to obtain the best possible global well-posedness result for the IVP associated to (2.1.1).

Other interesting problem is to extend the argument used to prove the unique continuation property for the Zakharov-Kuznetsov equation (3.1.1) to apply for the KP-II equation.

The study of the Cauchy problem associated to the following new coupled KdV and mKdV systems [16], [28], [68],

$$\begin{cases} u_t = \frac{1}{2}(u_{xxx} - 6uu_x) + 6vv_x \\ v_t = -v_{xxx} + 3uv_x \end{cases} \tag{3.3.21}$$

and

$$\begin{cases} p_t = \frac{1}{2}p_{xxx} - 3p^2p_x + 3(qq_x)_x + 3(pq^2)_x \\ q_t = -q_{xxx} - 3(qp_x)_x + 6pqp_x + 3(p^2 - q^2)_x. \end{cases} \tag{3.3.22}$$

will also be of interest.

We believe that the methods employed to the models studied during the doctoral project can be utilized (with necessary modification and generalization) to obtain similar results to these new models but it has to be done.

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