On the Dynamics of Torus Homeomorphisms

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Contents

Notation

As usual, we denote the two-torus $\mathbb{R}^2/\mathbb{Z}^2$ by $\mathbb{T}^2$, with quotient projection $\pi\colon (x,y) \mapsto (x,y)+\mathbb{Z}^2$. The integer translations are

$$T_1\colon (x,y) \mapsto (x+1,y) \text{ and } T_2\colon (x,y) \mapsto (x,y+1),$$

and $\mathrm{pr}_i\colon \mathbb{R}^2 \to \mathbb{R}$, for $i = 1,2$ are the projections onto the first and second coordinate, respectively.

By $\mathrm{Homeo}(X)$ we mean the set of homeomorphisms of X to itself, and by $\mathrm{Homeo}_*(X)$ the set of elements of $\mathrm{Homeo}(X)$ which are isotopic to the identity. We remark that a torus homeomorphism is isotopic to the identity if and only if it is homotopic to the identity [Eps66].

Given $F \in \mathrm{Homeo}(\mathbb{T}^2)$, a lift of F to $\mathbb{R}^2$ is a map $f \in \mathrm{Homeo}(\mathbb{R}^2)$ such that $\pi f = F\pi$. An homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of an element of $\mathrm{Homeo}_*(\mathbb{T}^2)$ if and only if f commutes with T_1 and T_2. Any two lifts of a given homeomorphism of $\mathbb{T}^2$ always differ by an integer translation. We will usually denote maps of $\mathbb{T}^2$ to itself by uppercase letters, and lifts of such maps to $\mathbb{R}^2$ by their corresponding lowercase letters.

By $\mathbb{Z}^2_{\mathrm{cp}}$ we denote the set of pairs of integers (m,n) such that m and n are coprime. We will say that $(x,y) \in \mathbb{R}^2$ is an integer point if both x and y are integer, and a rational point if both x and y are rational. When we write a rational number as p/q, we assume that it is in reduced form, i.e. that p and q are coprime, except when we are talking about a rational point $(p_1/q, p_2/q)$, in which case we assume that p_1, p_2 and q are mutually coprime (i.e. $\mathrm{lcd}\{p_1,p_2,q\} = 1$).

The set $\mathbb{A}$ will denote the open annulus $\mathbb{R} \times S^1 \simeq \mathbb{R}^2/\langle T_2 \rangle$. The map T_1 induces a map $\tau\colon \mathbb{A} \mapsto \mathbb{A}$, defined by $(x,v) \mapsto (x+1,v)$. We denote by $\pi_1\colon \mathbb{A} \to \mathbb{T}^2$ the covering map $(x,v) \mapsto (x+\mathbb{Z},v)$, which is the same as the quotient projection $\mathbb{A} \mapsto \mathbb{A}/\langle\tau\rangle$. Similarly, $\overline{\mathbb{A}}$ denotes the closed annulus $[0,1] \times S^1$.

Introduction

The rotation set is one of the most important tools for the study of homeomorphisms of the torus in the homotopy class of the identity[1]. Although not as easily as in the case of circle homeomorphisms, one can obtain useful *a priori* information about the dynamics by studying the rotation set. A central question in this direction is under what conditions a point of rational coordinates in the rotation set is realized as the rotation vector of a periodic orbit. It is clear that periodic orbits have rotation vectors with rational coordinates. Franks proved in [Fra88] that extremal points of the rotation set are always realized by periodic orbits; but it is generally not true that *every* rational point in the rotation set is realized (c.f. Example 0.1). In this aspect, the case that is best understood is when the rotation set has non-empty interior. For such homeomorphisms, a theorem by Llibre and Mackay [LM91] guarantees positive topological entropy. Moreover, Franks [Fra89] proved that every rational point in the interior of the rotation set is realized as the rotation vector of a periodic orbit, and this is optimal in the sense that there are examples (even area-preserving ones) where the rotation set has nonempty interior and many rational points on the boundary, but the only ones that are realized by periodic orbits are extremal or interior rational points (see [MZ91, §3]).

On the other hand, when the rotation set has empty interior the situation is more delicate. Since the rotation set is compact and convex, if it has no interior it must be a line segment or a single point. The following simple

[1] In this work, unless stated otherwise, all homeomorphisms are assumed to be in the homotopy class of the identity. Note that by [Eps66], this is equivalent to being in the isotopy class of the identity.

example shows that one can have homeomorphisms with the rotation set being a segment with many rational points, but which have no periodic orbits at all.

Example 0.1. Let $P\colon S^1 \to S^1$ be the north pole-south pole map on S^1, e.g. the map lifted by $x \mapsto x + 0.1\sin(2\pi x)$. Define $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (P(x), y + r\sin(2\pi x))$ for some small irrational r. Then f is the lift of a map $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ isotopic to the identity, and $\rho(f) = \{0\} \times [-r, r]$, which contains many rational points; but F has no periodic points.

This example shows that some additional hypothesis is required to guarantee that rational points are realized by periodic orbits. The following theorem gives a sufficient condition.

Theorem 0.2 (Franks, [Fra95])**.** *If an area-preserving homeomorphism of the torus has a rotation set with empty interior, then every rational point in its rotation set is realized as the rotation vector of a periodic orbit.*

In the same article, Franks asks whether the area-preserving hypothesis is really necessary for the conclusion of the theorem. It is natural to expect that a weaker, more topological hypothesis should suffice to obtain the same result. This topological substitute for the area-preserving hypothesis turns out to be, to some extent, the *curve intersection property*. An essential simple closed curve is *free* for F if $F(\gamma) \cap \gamma = \emptyset$. We say that F has the curve intersection property if F has no free curves. In Chapter 1 we prove the following:

Theorem A. *If a homeomorphism of the torus satisfying the curve intersection property has a rotation set with empty interior, then every rational point in its rotation set is realized by a periodic orbit.*

Our proof is essentially different of that of Theorem 0.2, since the latter relies strongly on chain-recurrence properties that are guaranteed by the area preserving hypothesis but not by the curve intersection property.

Variations of the curve intersection property are already present in some fixed point theorems. An interesting case is a generalization of the following classic theorem:

Poincaré-Birkhoff Theorem (Birkhoff, [Bir25]). *Let $F\colon \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ be an area-preserving homeomorphism of the closed annulus, verifying the boundary twist condition. Then F has at least two fixed points.*

The *boundary twist condition* means that F preserves boundary components and there exists a lift $f\colon [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$ of F such that the rotation numbers of $f|_{\{0\}\times\mathbb{R}}$ and $f|_{\{1\}\times\mathbb{R}}$ have opposite signs.

Birkhoff and Kerékjártó already noted that the area preserving hypothesis was not really necessary, and that it could be replaced by a more topological one, and they obtained the following

Theorem ([Bir25], [Ker29]). *If $F\colon \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ is a homeomorphism satisfying the boundary twist condition and such that $F(\gamma) \cap \gamma \neq \emptyset$ for each essential simple closed curve γ, then F has at least one fixed point in the interior of the annulus.*

An even more topological version of this theorem was proven by Guillou, who substituted the twist condition by the property that every simple arc joining the boundary components intersects its image by F:

Theorem (Guillou, [Gui94]). *If $F\colon \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ is an orientation-preserving homeomorphism such that every essential simple closed curve or simple arc joining boundary components intersects its image by F, then F has a fixed point.*

The hypotheses of the above theorem can be regarded as the curve intersection property in the setting of the closed annulus: they say that every "interesting" curve intersects its image by F. This led Guillou to ask if a similar result holds for the torus:

> *Let $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism isotopic to the identity. Does the curve intersection property imply the existence of a fixed point for F?*

The answer is no, as an example by Bestvina and Handel shows [BH92]. Their example relies in the fact that the existence of a free curve imposes a restriction on the "size" of the rotation set (see Lemma 1.4). They construct a homeomorphism such that any strip bounded by straight lines and containing the rotation set, also contains a point of integer coordinates. By the previously mentioned lemma, and some simple considerations, this homeomorphism cannot have any free curves; but by their construction, there are no points of integer coordinates inside the rotation set, which implies that the homeomorphism has no fixed points.

Nevertheless, the existing examples have a rotation set with nonempty interior, which implies that they have infinitely many periodic points of arbitrarily high periods, and positive topological entropy ([LM91], [Fra89]). The question that arises is whether the presence of this kind of "rich" dynamics is the only new obstruction to the existence of free curves in $\mathbb{T}^2$. In other words, *is the answer to Guillou's question affirmative if the rotation set has empty interior?* This leads to our next result, which is proved in Chapter 2:

Theorem B. *Let $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism satisfying the curve intersection property. Then either F has a fixed point, or its rotation set has nonempty interior.*

Thus, if F has the curve intersection property, then either F has periodic orbits of arbitrarily high periods or it has a fixed point. In the latter case, one might expect the existence of a second fixed point (as in the case of the annulus) but no more than that. Figure 1 shows that one cannot expect more than two fixed points in the annulus; the time-one map of the flow sketched there is a homeomorphism with the curve intersection property, which has two fixed points and no other periodic points. Gluing a symmetric copy of this homeomorphism through the boundaries of the annulus, one obtains an example with the same properties in $\mathbb{T}^2$.

In Chapter 3 we consider a situation opposite to the curve intersection property: what can be said if there is a simple closed curve that is "always free" for F (i.e. such that all iterates of the curve by F are pairwise disjoint)? In general we cannot say much, since the existence of an attractor "wrapping around the torus" (e.g. an attracting homotopically non-trivial

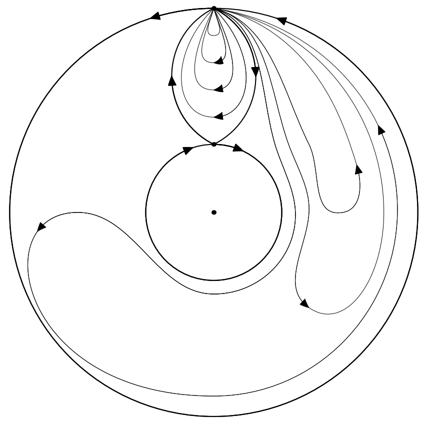

Figure 1: A curve-intersecting flow with no periodic points of period > 1

closed curve) usually implies the existence of such a curve; and this is a local property, meaning that we cannot say anything about the dynamics outside a neighborhood of the attractor. On the other hand, if we assume that this curve has a dense orbit (which is true, for example, if F is transitive), we obtain the following:

Theorem C. *Let F be a homeomorphism of $\mathbb{T}^2$, and suppose there exists a simple closed curve that is disjoint from all its iterates by F, whose orbit is dense in $\mathbb{T}^2$. Then F is semi-conjugate to an irrational rotation of the circle.*

Moreover, the proof of Theorem C can easily be generalized to homeomorphisms possessing an always free continuum with a dense orbit, provided that the continuum "wraps around the torus" and is disjoint from some essential simple closed curve (see the precise statement in Chapter 4, Theorem C')

The above result is related to a question arising from an article of Fathi and Herman [FH77]. In that classical paper, they prove the existence of minimal and uniquely ergodic diffeomorphisms in manifolds admitting a C^∞ free action of S^1. Their techniques are based on the *fast approximation by conjugations* method, introduced by Anosov and Katok in [AK70], combining it with Baire's theorem in an appropriate space (for a comprehensive overview of these methods and their applications, see [FK04]). In the setting of $\mathbb{T}^2$, this space corresponds to $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$, which is the C^∞-closure of diffeomorphisms which are C^∞-conjugate to rigid translations of $\mathbb{T}^2$. The following is a quotation from [FH77]:

> *Le second auteur montrera ailleurs que la propriété suivante:*
> *"f n'admet pas de feuilletage $\mathcal{F}$ (de $\mathbb{T}^2$) C^0 de codimension 1 invariant (i.e. f envoide chaque feuille de $\mathcal{F}$ sur une feuille de $\mathcal{F}$ pas nécessairement la même)"*
> *est vraie sur un ensemble qui contient un G_δ dense de $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$.*

The claim says that generic diffeomorphisms in $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$ have no invariant C^0 foliation of codimension 1 (i.e. this property holds in a residual set). However, the proof of this fact never appeared in the subsequent publications of Herman.

In Chapter 4, we prove some partial results regarding the above claim. Combining Theorem C and some results of Herman, we obtain a quick proof that generic diffeomorphisms in $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$ have no invariant C^0 foliation (of codimension 1) with a compact leaf. In fact, we have something stronger: generic diffeomorphisms in $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$ have no always free curves (§4.4).

In addition, we prove that generically in $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$ there are no invariant C^0 foliations induced by a C^0 line field (§4.3). This is obtained as a corollary of the following more general result (§4.2):

Theorem D. *For a generic diffeomorphism $f \in \overline{\mathcal{O}}^\infty(\mathbb{T}^2)$, its associated dynamic cocycle*

$$(x, v) \mapsto \left(f(x), \frac{Df_x v}{\|Df_x v\|} \right)$$

is minimal.

A similar result holds if we restrict our attention to $\overline{\mathcal{O}}_m^\infty(\mathbb{T}^2)$, the closure of the set of area-preserving diffeomorphisms which are C^∞-conjugated to a rigid translation. It is worth mentioning that, while one can easily construct $\mathrm{SL}(2,\mathbb{R})$-cocycles over minimal diffeomorphisms of $\mathbb{T}^2$ such that their normal action on $\mathbb{T}^2 \times S^1$ is minimal, it is not obvious that one can obtain *dynamic* cocycles with that property. However, Theorem D provides a wide family of such examples.

Finally, in §4.5 we also show, using the fast approximation method, that there is a dense subset of $\overline{\mathcal{O}}^{\infty}(\mathbb{T}^2)$ consisting of minimal diffeomorphisms which have no invariant C^0 foliation of codimension 1 with C^1 leaves.

To give a complete proof of Herman's claim, it remains to show that the subset of $\overline{\mathcal{O}}^{\infty}(\mathbb{T}^2)$ consisting of diffeomorphisms which have a C^0 foliation which is conjugated to the foliation induced by a minimal translation of $\mathbb{T}^2$, is a meager set in $\overline{\mathcal{O}}^{\infty}(\mathbb{T}^2)$. We are not able to prove this at this time. In fact, even the following question remains open (to our knowledge):

Question 0.3. *Is there a minimal homeomorphism of $\mathbb{T}^2$ possessing no invariant C^0 foliation?*

The results presented in this work, with the exception of Theorem C, were obtained jointly with Alejandro Kocsard.

Chapter 0

Background

0.1 The rotation set

Throughout this section we assume that $F \in \mathrm{Homeo}_*(\mathbb{T}^2)$, and $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of F.

Definition 0.4 (Misiurewicz & Ziemian, [MZ89])**.** The *rotation set of* f is defined as

$$\rho(f) = \bigcap_{m=1}^{\infty} \mathrm{cl}\left(\bigcup_{n=m}^{\infty} \left\{ \frac{f^n(x) - x}{n} : x \in \mathbb{R}^2 \right\} \right) \subset \mathbb{R}^2$$

The *rotation set of a point* $x \in \mathbb{R}^2$ is defined by

$$\rho(f, x) = \bigcap_{m=1}^{\infty} \mathrm{cl} \left\{ \frac{f^n(x) - x}{n} : n > m \right\}.$$

If the above set consists of a single point v, we say that v is the rotation vector of x.

Remark 0.5. $\rho(f)$ is just the set of all limits of convergent sequences of the form

$$\frac{f^{n_k}(x_k) - x_k}{n_k}$$

where $x_k \in \mathbb{R}^2$ and $n_k \to \infty$.

Proposition 0.6. *For all integers* n, m_1, m_2*,*

$$\rho(T_1^{m_1} T_2^{m_2} f^n) = n\,\rho(f) + (m_1, m_2).$$

Remark 0.7. In particular, the rotation set of any other lift of F is an integer translate of $\rho(f)$, and we can talk about the "rotation set of F" if we keep in mind that it is defined modulo integer translations.

Theorem 0.8 ([MZ89])**.** *The rotation set is compact and convex, and every extremal point of* $\rho(f)$ *is the rotation vector of some point.*

By *extremal* point in the theorem above, we mean extremal in the usual sense for convex sets, i.e. a point that is not in the interior of a line segment contained in $\rho(f)$.

0.1.1 The rotation set and periodic orbits

For a homeomorphism of the circle, if the rotation number is a rational p/q, then there exists a periodic orbit of period q. This motivates the following question in the two-dimensional case.

Question 0.9. *If* $(p_1/q, p_2/q) \in \rho(f)$*, when can we find a point* $x \in \mathbb{R}^2$ *such that* $f^q(x) = x + (p_1, p_2)$*?*

Whenever $f^q(x) = x + (p_1, p_2)$, with p_1, p_2 and q mutually coprime, we will say that $(p_1/q, p_2/q)$ is *realized* as the rotation vector of a periodic orbit of F. This is because $\pi(x)$ is a periodic orbit of F of period q, and its rotation vector is $(p_1/q, p_2/q)$ (modulo $\mathbb{Z}^2$). So the question is *when can we realize rational points of the rotation set by periodic orbits of* F.

There are several results in this direction (including the already mentioned Theorem 0.2)

Theorem 0.10 (Franks, [Fra88])**.** *If a rational point of* $\rho(f)$ *is extremal, then it is realized by a periodic orbit.*

Theorem 0.11 (Franks, [Fra89])**.** *If a rational point is in the interior of* $\rho(f)$*, then it is realized by a periodic orbit.*

Theorem 0.12 (Jonker & Zhang, [JZ98]). *If $\rho(f)$ is a segment with irrational slope, and it contains a point of rational coordinates, then this point is realized by a periodic orbit.*

0.2 Curves, lines

We denote by I the interval $[0,1]$. A *curve* on a manifold M is a continuous map $\gamma\colon I \to M$. As usual, we represent by γ both the map and its image, as it should be clear from the context which is the case.

We say that the curve γ is *closed* if $\gamma(0) = \gamma(1)$, and *simple* if the restriction of the map γ to the interior of I is injective. If γ is a closed curve, we say it is *essential* if it is homotopically non-trivial.

Definition 0.13. A curve $\gamma \subset M$ is *free* for F if $F(\gamma) \cap \gamma = \emptyset$. We say that F has the *curve intersection property* (CIP) if there are no free essential simple closed curves for F.

Remark 0.14. For convenience, from now on by a *free curve* for F we will usually mean an essential simple closed curve that is free for F, unless stated otherwise.

By a *line* we mean a proper topological embedding $\ell\colon \mathbb{R} \to \mathbb{R}^2$. Again, we use ℓ to represent both the function and its image.

Definition 0.15. Given $(p,q) \in \mathbb{Z}^2_{\mathrm{cp}}$, a (p,q)-line in $\mathbb{R}^2$ is a line ℓ that is invariant by $T_1^p T_2^q$, such that $\ell(s) = T_1^p T_2^q \ell(t)$ implies $s > t$, and such that its projection to $\mathbb{T}^2$ by π is a simple closed curve. A (p,q)-curve in $\mathbb{T}^2$ is the projection by π of a (p,q)-line. We will say that a simple closed curve is *vertical* if it is either a $(0,1)$-curve or a $(0,-1)$-curve. Similarly, a line will be called vertical if it is a $(0,1)$-line or a $(0,-1)$-line.

Remark 0.16. If ℓ is a line in $\mathbb{R}^2$ that is invariant by $T_1^p T_2^q$, then $\pi(\ell)$ is a always a closed curve in $\mathbb{T}^2$. We are requiring that this curve be simple to call ℓ a (p,q)-line. Conversely, if γ is an essential simple closed curve in $\mathbb{T}^2$, taking a lift $\tilde{\gamma}\colon I \to \mathbb{R}^2$, we have that $(p,q) = \tilde{\gamma}(1) - \tilde{\gamma}(0)$ is an integer point independent of the choice of the lift. The curve $\tilde{\gamma}$ can be extended naturally to $\mathbb{R}$ by $\tilde{\gamma}(t+n) = \tilde{\gamma}(t) + n(p,q)$, if $n \in \mathbb{Z}$ and $t \in [0,1]$; in this way we obtain a (p,q)-line that projects to γ. It is not hard to see that p and q must be coprime, from the fact that γ is simple and essential. We will say that a (p,q)-line ℓ is a lift of $\pi(\ell)$.

Remark 0.17. Two disjoint essential simple closed curves must be either both (p,q)-curves for some $(p,q) \in \mathbb{Z}^2_{\text{cp}}$, or one a $(p,q)-curve$ and the other a $(-p,-q)$-curve. Note that the difference between these two is just orientation; if we only regard the curves as sets, then (p,q)-curves and $(-p,-q)$-curves are the same thing.

Remark 0.18. It is easy to see that any (p,q)-line is contained in a strip bounded by two straight lines of slope q/p. In particular, if ℓ is a vertical line, there is $M > 0$ such that $\text{pr}_1(\ell) \subset [-M, M]$.

0.2.1 Ordering lines

A line ℓ in $\mathbb{R}^2$ can be seen as a Jordan curve through ∞ in $S^2 = \mathbb{R}^2 \cup \{\infty\}$. Thus, $\mathbb{R}^2 \setminus \ell$ has exactly two connected components, both unbounded. Using the orientation of ℓ, we may define the *left* and the *right* components, which we denote by $L\ell$ and $R\ell$. We also denote by $\overline{L}\ell$ and $\overline{R}\ell$ their respective closures, which correspond to $L\ell \cup \ell$ and $R\ell \cup \ell$.

There is a natural partial ordering between lines in $\mathbb{R}^2$, defined by $\ell_1 < \ell_2$ if $\ell_1 \subset L\ell_2$ (and $\ell_2 \subset R\ell_1$). With an abuse of notation we will write $\ell_1 \leq \ell_2$ when $\ell_1 \subset \overline{L}\ell_2$, which means that the lines may intersect but only "from one side". Naturally, if two lines ℓ_1 and ℓ_2 do not intersect, then either $\ell_1 < \ell_2$ or $\ell_2 < \ell_1$. If $\ell_1 < \ell_2$, we denote by $S(\ell_1, \ell_2)$ the strip $L\ell_2 \cap R\ell_1$, and by $\overline{S}(\ell_1, \ell_2)$ its closure.

Remark 0.19. If $f \in \text{Homeo}(\mathbb{R}^2)$ preserves orientation, then f preserves order: if $\ell_1 < \ell_2$, then $f(\ell_1) < f(\ell_2)$.

0.2.2 Brouwer lines

Definition 0.20. A Brouwer line for $h \in \text{Homeo}(\mathbb{R}^2)$ is a line ℓ in $\mathbb{R}^2$ such that $h(\ell) > \ell$.

The classic Brouwer Translation Theorem guarantees the existence of a Brouwer line through any point of $\mathbb{R}^2$ for any fixed-point free, orientation-preserving homeomorphism (see [Bro12, Ker29, Fat87, Fra92, Gui94]; also see [LC05] for a very useful equivariant version). The following will be much more useful for our purposes:

Theorem 0.21 (Guillou, [Gui06])**.** *Let f be a lift of $F \in \mathrm{Homeo}_*(\mathbb{T}^2)$, and suppose f has no fixed points. Then f has a Brouwer (p, q)-line, for some $(p, q) \in \mathbb{Z}^2_{\mathrm{cp}}$.*

0.3 More about the rotation set

0.3.1 The rotation set and Brouwer lines

The following lemma is particularly useful when there is a Brouwer (p, q)-line:

Lemma 0.22. *Let S be a closed semiplane determined by a straight line containing the origin, and for $y \in \mathbb{R}^2$ denote by $S_y = \{w + y : w \in S\}$ its translate by y. Suppose that $x \in \mathbb{R}^2$ is such that for some y,*

$$f^n(x) \in S_y \text{ for all } n > 0.$$

Then $\rho(f, x) \subset S$. Moreover, if for all $x \in \mathbb{R}^2$ there is y such that the above holds, then $\rho(f) \subset S$.

Proof. Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ be a linear functional such that

$$S_y = \{w \in \mathbb{R}^2 : \phi(w) \geq \phi(y)\}.$$

Given x such that $f^n(x) \in S_y$ for all $n > 0$, we have then

$$\phi\left(\frac{f^n(x) - x}{n}\right) = \frac{\phi(f^n(x)) - \phi(x)}{n} \geq \frac{\phi(y) - \phi(x)}{n} \to 0.$$

Thus, if z is the limit of the sequence $(f^{n_i}(x) - x)/n_i$, then $\phi(z) \geq 0$. This implies that $\rho(f, x) \subset S$. The other claim follows from Theorem 0.8. □

Remark 0.23. If f has a Brouwer (p,q)-line, the above lemma and Remark 0.18 imply that $\rho(f)$ is contained in one of the closed semiplanes determined by the straight line of slope q/p through the origin.

0.3.2 The rotation set and free curves

Besides the existence of periodic orbits, other practical dynamical information that can be obtained from the rotation set is the existence of free curves. Recall that an interval with rational endpoints $[p/q, p'/q']$ is a Farey interval if $qp' - pq' = 1$. The following result was proved by Kwapisz for diffeomorphisms, and by Beguin, Crovisier, LeRoux and Patou in the general case.

Theorem 0.24 ([Kwa02], [BCLP04])**.** *Suppose there exists a Farey interval* $[p/q, p'/q']$ *such that*

$$\operatorname{pr}_1 \rho(f) \subset \left(\frac{p}{q}, \frac{p'}{q'}\right).$$

Then there exists a simple closed $(0,1)$*-curve* γ *in* $\mathbb{T}^2$ *such that all the curves* γ, $F(\gamma)$, $F^2(\gamma)$, $\dots$, $F^{q+q'-1}(\gamma)$ *are mutually disjoint. In particular, if* $\operatorname{pr}_1 \rho(f) \cap \mathbb{Z} = \emptyset$*, then* F *has a free* $(0,1)$*-curve.*

0.3.3 The rotation set and conjugations

Given $A \in \mathrm{GL}(2,\mathbb{Z})$, we denote by $\tilde{A}$ the homeomorphism of $\mathbb{T}^2$ lifted by it. If $H \in \mathrm{Homeo}(\mathbb{T}^2)$, there is a unique $A \in \mathrm{GL}(2,\mathbb{Z})$ such that for every lift h of H, the map $h - A$ is bounded (in fact, $\mathbb{Z}^2$-periodic). From this it follows that H is isotopic to $\tilde{A}$, and $h^{-1} - A^{-1}$ is bounded. In fact, $H \mapsto A$ induces an isomorphism of the isotopy group of $\mathbb{T}^2$ to $\mathrm{GL}(2,\mathbb{Z})$.

Lemma 0.25. *If* $H, F \in \mathrm{Homeo}(\mathbb{T}^2)$ *with* H *isotopic to* $A \in \mathrm{GL}(2,\mathbb{Z})$ *and* F *isotopic to the identity, and* $h, f \in \mathrm{Homeo}(\mathbb{R}^2)$ *are their respective lifts, then* $\rho(hfh^{-1}) = A\,\rho(f)$*. In particular,* $\rho(AfA^{-1}) = A\,\rho(f)$*.*

Proof. We can write $((hfh^{-1})^n(x) - x)/n$ as

$$\frac{(h-A)(f^n h^{-1}(x))}{n} + A\left(\frac{f^n(h^{-1}(x)) - h^{-1}(x)}{n}\right) + A\left(\frac{(h^{-1} - A^{-1})(x)}{n}\right)$$

and using the fact that $h - A$ and $h^{-1} - A^{-1}$ are bounded, we see that the leftmost and rightmost terms of the above expression vanish when $n \to \infty$. Thus if $n_k \to \infty$ and $x_k \in \mathbb{R}^2$, we have

$$\lim_{k\to\infty} \frac{(hfh^{-1})^{n_k}(x_k) - x_k}{n_k} = A\left(\lim_{k\to\infty} \frac{f^{n_k}(h^{-1}(x_k)) - h^{-1}(x_k)}{n_k}\right)$$

whenever the limits exist. Since h is a homeomorphism, it follows from the definition that $\rho(hfh^{-1}) = A\,\rho(f)$. □

We will use the above lemma extensively: when trying to prove some property that is invariant by topological conjugation (like the existence of a free curve or a periodic point for F), it allows us to consider just the case where the rotation set is the image of $\rho(f)$ by some convenient element of $\mathrm{GL}(2,\mathbb{Z})$.

Remark 0.26. A particular case that will often appear is when $\rho(f)$ is a segment of rational slope. In that case, there exists a map $A \in \mathrm{GL}(2,\mathbb{Z})$ such that $A\,\rho(f)$ is a vertical segment. Indeed, if $\rho(f)$ is a segment of slope p/q, then given $x, y \in \mathbb{Z}$ such that $px + qy = 1$, and letting

$$A = \begin{pmatrix} p & -q \\ y & x \end{pmatrix}$$

we have that $\det(A) = 1$, and $A(q,p) = (0,1)$, so that $A\,\rho(f)$ is vertical. Note that the above A also maps (q,p)-curves to $(0,1)$-curves.

Chapter 1

Realizing periodic orbits

In this chapter we present some results regarding the curve intersection property and its relation with the rotation set, which will allow us to prove Theorem A. Throughout this chapter, $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ will be a homeomorphism isotopic to the identity and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ a lift of F.

1.1 Preliminary results

The following result will be essential in the proof of Theorem A.

Theorem 1.1. *Suppose F^n has a free (p, q)-curve for some $n \geq 1$. Then F has a free (p, q)-curve.*

We will also use the following lemmas, the proofs of which are postponed to the end of this chapter.

Lemma 1.2. *Suppose f^n has a Brouwer $(0, 1)$-line, for some $n \in \mathbb{N}$. Then f has a Brouwer $(0, 1)$-line.*

Lemma 1.3. *Suppose that some lift f of F has a Brouwer $(0, 1)$-line. Then, either F has a free $(0, 1)$-curve, or* $\max(\mathrm{pr}_1\, \rho(f)) \geq 1$.

The next lemma is essentially Lemma 3 of [BH92]; we include it here for the sake of completeness.

Lemma 1.4. *Suppose F has a free $(0,1)$-curve. Then for any lift f of F there is $k \in \mathbb{Z}$ such that* $\operatorname{pr}_1 \rho(f) \subset [k, k+1]$.

1.2 Proof of Theorem 1.1

Conjugating the involved maps by an element of $\mathrm{GL}(2, \mathbb{Z})$, we may assume that $(p,q) = (0,1)$ (see Lemma 0.25 and the remark below it).

Suppose F^n has a free $(0,1)$-curve for some $n \geq 2$. Any lift to $\mathbb{R}^2$ of this curve is a vertical Brouwer line for f^n, so Lemma 1.2 implies that there is a vertical Brouwer line for f. This holds for any lift f of F.

If $\operatorname{pr}_1 \rho(f) \cap \mathbb{Z} = \emptyset$, then by Theorem 0.24 there is a free $(0,1)$-curve for F, and we are done.

Otherwise, let $k \in \operatorname{pr}_1 \rho(f) \cap \mathbb{Z}$, and consider the lift $f_0 = T_1^{-k} f$ of F. By Proposition 0.6, it is clear that $0 \in \operatorname{pr}_1 \rho(f_0)$. On the other hand, as we already saw, f_0 has a vertical Brouwer line ℓ.

Assume ℓ is a Brouwer $(0,1)$-line. Then by Lemma 1.3, either F has a free $(0,1)$-curve, or $\max(\operatorname{pr}_1 \rho(f_0)) \geq 1$. In the latter case, since we know that $0 \in \rho(f_0)$, it follows from connectedness that $\operatorname{pr}_1 \rho(f_0) \supset [0,1]$. But this implies that

$$\operatorname{pr}_1 \rho(f_0^n) \supset [0,n] \supset [0,2],$$

which contradicts Lemma 1.4 (since F^n has a free $(0,1)$-curve). Thus the only possibility is that F has a free $(0,1)$-curve.

If ℓ is a Brouwer $(0,-1)$-line, then using the previous argument with f_0^{-1} instead of f_0, we see that F^{-1} (and thus F) has a free $(0,-1)$-curve γ; and inverting the orientation of γ we get a free $(0,1)$-curve for F. This completes the proof. □

1.3 Proof of Theorem A

Suppose $F \in \mathrm{Homeo}_*(\mathbb{T}^2)$ has the curve intersection property and $\rho(f)$ has empty interior, where f is a lift of F. We have three cases.

$\rho(f)$ is a single point

In this case, the unique point of $\rho(f)$ is extremal; and if it is rational, Theorem 0.10 implies that it is realized by a periodic orbit of F.

$\rho(f)$ is a segment of irrational slope

In this case $\rho(f)$ contains at most one rational point and, by Theorem 0.12, this point is realized by a periodic orbit.

Remark 1.5. Theorem 0.21 provides a simple way of proving this as well. In fact, it suffices to consider the case where the unique rational point in $\rho(f)$ is the origin, and to show that in this case f has a fixed point. If the origin is an extremal point, this follows from Theorem 0.10. If the origin is strictly inside the rotation set, then there is only one straight line through the origin such that $\rho(f)$ is contained in one of the closed semiplanes determined by the line. This unique line is the one with the same slope as $\rho(f)$, which is irrational. If f has no fixed points, then Theorem 0.21 implies that f has a Brouwer (p,q)-line for some $(p,q) \in \mathbb{Z}^2_{\mathrm{cp}}$; but then our previous claim contradicts Lemma 0.22.

$\rho(f)$ is a segment of rational slope

Fix a rational point $(p_1/q, p_2/q) \in \rho(f)$. Recall that this point is realized as the rotation vector of a periodic orbit of F if and only if $g = T_1^{-p_1}T_2^{-p_2}f^q$ has a fixed point. Note that $(0,0) \in \rho(g)$, and g is a lift of F^q. Moreover,

$$\rho(g) = T_1^{-p_1}T_2^{-p_2}(q \cdot \rho(f)),$$

which is a segment of rational slope containing the origin. Conjugating all the involved maps by an element of $\mathrm{GL}(2,\mathbb{Z})$, we may assume that $\rho(g)$ is a vertical segment containing the origin.

We will show by contradiction that g has a fixed point. Suppose this is not the case. Then, by Theorem 0.21, g has a Brouwer (p,q)-line ℓ, for some $(p,q) \in \mathbb{Z}^2_{\mathrm{cp}}$. Moreover, $(0,0)$ must be strictly inside $\rho(g)$ (i.e. it cannot be extremal, since otherwise g would have a fixed point by Theorem 0.10), and by Remark 0.23 this implies that ℓ is a vertical Brouwer line.

Assume ℓ is a $(0,1)$-line (if it is a $(0,-1)$-line, we may consider g^{-1} instead of g and use a similar argument). Since $\operatorname{diam}(\operatorname{pr}_1 \rho(g)) = 0$, Lemma 1.3 implies that F^q, the map lifted by g, has a free $(0,1)$-curve; but then by Theorem 1.1, F has a free curve, contradicting the curve intersection property. This concludes the proof. □

1.4 The wedge

We devote the rest of this chapter to the proof of Lemmas 1.2-1.4. But first we need to introduce an operation between lines, which will be a fundamental tool in what follows. Recall the definitions of L and R from §0.2.1.

Definition 1.6. Given two (p,q)-lines ℓ_1 and ℓ_2 in $\mathbb{R}^2$, their *wedge* $\ell_1 \wedge \ell_2$ is the line defined as the boundary of the unique unbounded connected component of $L\ell_1 \cap L\ell_2$, oriented so that this component corresponds to $L(\ell_1 \wedge \ell_2)$.

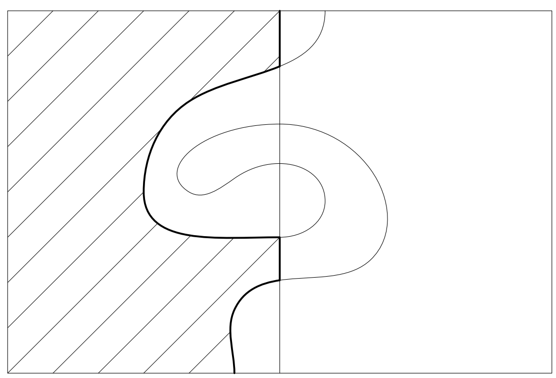

Figure 1.1: The wedge of two lines

This operation is called 'join' in [BCLP04] and denoted by $\vee$.

Recall that a Jordan domain is a topological disk bounded by a simple closed curve. The following theorem guarantees that the wedge is well defined.

Theorem 1.7 (Kerékjártó, [Ker23]). *If U_1 and U_2 are two Jordan domains in the two-sphere, then each connected component of $U_1 \cap U_2$ is a Jordan domain.*

In fact, identifying $\mathbb{R}^2 \cup \{\infty\}$ with the two-sphere S^2, we may regard lines in $\mathbb{R}^2$ as simple closed curves in S^2 containing ∞. From the fact that ℓ_1 and ℓ_2 are (p, q)-lines, it is easy to see that the intersection of $L\ell_1$ and $L\ell_2$ has a unique connected component containing ∞ in its boundary. By the above theorem, this boundary is a simple closed curve, so that it corresponds to a line in $\mathbb{R}^2$. One can easily see that this new line is also a (p, q)-line.

We denote the wedge of multiple lines $\ell_1, \ldots, \ell_n$ by

$$\ell_1 \wedge \ell_2 \wedge \cdots \wedge \ell_n = \bigwedge_{i=1}^{n} \ell_i.$$

This is well defined because the wedge is commutative.

The following proposition resumes the interesting properties of the wedge.

Proposition 1.8. *The wedge is commutative, associative and idempotent. Furthermore,*

1. *The wedge of (p, q)-lines is a (p, q)-line;*
2. *If $h \in \mathrm{Homeo}(\mathbb{R}^2)$ is a lift of a torus homeomorphism, then $h(\ell_1 \wedge \ell_2) = h(\ell_1) \wedge h(\ell_2)$;*
3. *$\ell_1 \wedge \ell_2 \leq \ell_1$ and $\ell_1 \wedge \ell_2 \leq \ell_2$;*
4. *If $\ell_1 < \ell_2$ and $\xi_1 < \xi_2$, then $\ell_1 \wedge \xi_1 < \ell_2 \wedge \xi_2$;*
5. *The wedge of Brouwer lines is a Brouwer line.*

1.5 Proof of Lemma 1.2

Let ℓ be a Brouwer $(0, 1)$-line for f^n, for some $n > 1$. We will show that there is a Brouwer $(0, 1)$-line for f^{n-1}; by induction, it follows that there is a Brouwer $(0, 1)$-line for f.

We know that $f^n(\ell) > \ell$. Let ξ be a $(0,1)$-line such that $f^n(\ell) > \xi > \ell$, and define

$$\ell' = \xi \wedge \bigwedge_{i=1}^{n-1} f^i(\ell).$$

By Proposition 1.8, ℓ' is still a $(0,1)$-line. We claim that it is a Brouwer line for f^{n-1}. In fact,

$$\begin{aligned}
f^{n-1}(\ell') &= f^{n-1}(\xi) \wedge \bigwedge_{i=1}^{n-1} f^{n-1}(f^i(\ell)) \\
&= f^{n-1}(\xi) \wedge \bigwedge_{i=1}^{n-1} f^{i-1}(f^n(\ell)) \\
&= f^{n-1}(\xi) \wedge f^n(\ell) \wedge \bigwedge_{i=2}^{n-1} f^{i-1}(f^n(\ell)) \\
&= f^{n-1}(\xi) \wedge f^n(\ell) \wedge \bigwedge_{i=1}^{n-2} f^i(f^n(\ell)).
\end{aligned}$$

Using the facts that

$$f^{n-1}(\xi) > f^{n-1}(\ell), \qquad f^n(\ell) > \xi, \quad \text{and} \quad f^i(f^n(\ell)) > f^i(\ell),$$

and Proposition 1.8, we see that

$$f^{n-1}(\ell') > f^{n-1}(\ell) \wedge \xi \wedge \bigwedge_{i=1}^{n-2} f^i(\ell) = \xi \wedge \bigwedge_{i=1}^{n-1} f^i(\ell) = \ell',$$

so that ℓ' is a Brouwer $(0,1)$-line for f^{n-1}. This concludes the proof.

1.6 Proof of Lemma 1.3

Let ℓ be a Brouwer $(0,1)$-line for f. We consider two cases.

Case 1. For all $n > 0$, $f^n(\ell) \not< T_1^n(\ell)$

In this case, for each $n > 0$ we can choose $x_n \in \ell$ such that $f^n(x_n) \in \overline{R}(T_1^n \ell)$. From the fact that ℓ is a $(0,1)$-line we also know that $\mathrm{pr}_1(\ell) \subset [-M, M]$ for some $M > 0$, and therefore

$$\mathrm{pr}_1(T_1^n(\ell)) \subset [-M+n, M+n].$$

Hence,

$$\mathrm{pr}_1(f^n(x_n)) \in \mathrm{pr}_1(R(T_1^n(\ell))) \subset [-M+n, \infty).$$

It then follows that

$$\mathrm{pr}_1\left(\frac{f^n(x_n) - x_n}{n}\right) \geq \frac{(-M+n) - M}{n} = -2\frac{M}{n} + 1 \xrightarrow{n\to\infty} 1,$$

and by the definition of rotation set this implies that some point $(x, y) \in \rho(f)$ satisfies $x \geq 1$; i.e. $\max \mathrm{pr}_1(\rho(f)) \geq 1$.

Case 2. $f^n(\ell) < T_1^n \ell$ for some $n > 0$

We will show that in this case F has a free $(0,1)$-curve. The idea is similar to the proof of Lemma 1.2. Let n be the smallest positive integer such that $f^n(\ell) < T_1^n \ell$. If $n = 1$, we are done, since $\ell < f(\ell) < T_1 \ell$ so that ℓ projects to a free $(0,1)$-curve for F.

Now assume $n > 1$. We will show how to construct a new Brouwer $(0,1)$-line β for f such that $f^{n-1}(\beta) < T_1^{n-1}\beta$. Repeating this argument $n-1$ times, we end up with a $(0,1)$-line ℓ' such that $\ell' < f(\ell') < T_1(\ell')$, so that ℓ' projects to a free curve, completing the proof.

Let ξ be a $(0,1)$-curve such that

$$f^n(\ell) < \xi < T_1^n \ell \tag{1.1}$$

We may choose ξ such that $f(\xi) > \xi$, by taking it close enough to $f^n(\ell)$. This is possible because $f(f^n(\ell)) > f^n(\ell)$, and these two curves are separated by a positive distance, since they lift $(0,1)$-curves in $\mathbb{T}^2$. Thus ξ is also a Brouwer

$(0,1)$-line for f.

Define

$$\beta = \xi \wedge \bigwedge_{i=1}^{n-1} T_1^{n-i} f^i(\ell).$$

Let us see that $f^{n-1}(\beta) < T^{n-1}(\beta)$. Since T_1 commutes with f, we have

$$\begin{aligned} f^{n-1}(\beta) &= f^{n-1}(\xi) \wedge \bigwedge_{i=1}^{n-1} f^{n-1} T_1^{n-i} f^i(\ell) \\ &= f^{n-1}(\xi) \wedge \bigwedge_{i=1}^{n-1} f^{i-1} T_1^{n-i} f^n(\ell) \\ &= f^{n-1}(\xi) \wedge T_1^{n-1} f^n(\ell) \wedge \bigwedge_{i=2}^{n-1} f^{i-1} T_1^{n-i} f^n(\ell) \end{aligned}$$

By (1.1), we also have

- $f^{n-1}(\xi) < f^{n-1}(T_1^n \ell)$,
- $T_1^{n-1} f^n(\ell) < T_1^{n-1}\xi$, and
- $f^{i-1} T_1^{n-i} f^n(\ell) < f^{i-1} T_1^{n-i} T_1^n \ell$.

Using these facts and Proposition 1.8 we see that

$$\begin{aligned} f^{n-1}(\beta) &< f^{n-1}(T_1^n \ell) \wedge T_1^{n-1}\xi \wedge \bigwedge_{i=2}^{n-1} f^{i-1} T_1^{n-i} T_1^n(\ell) \\ &= T_1^{n-1}(T_1 f^{n-1}(\ell)) \wedge T_1^{n-1}(\xi) \bigwedge_{i=2}^{n-1} T_1^{n-1}(T_1^{n-(i-1)} f^{i-1}(\ell)) \\ &= T_1^{n-1}\left(\xi \wedge T_1 f^{n-1}(\ell) \wedge \bigwedge_{i=1}^{n-2} T_1^{n-i} f^i(\ell)\right) \\ &= T_1^{n-1}\left(\xi \wedge \bigwedge_{i=1}^{n-1} T_1^{n-i} f^i(\ell)\right) \\ &= T_1^{n-1}(\beta). \end{aligned}$$

Since β is a Brouwer $(0,1)$-line, this completes the proof. □

1.7 Proof of Lemma 1.4

Suppose that $N \in \mathrm{int}(\mathrm{pr}_1\, \rho(f)) \cap \mathbb{Z}$. We will show that F cannot have any free $(0,1)$-curves. Note that this also implies that F cannot have a free $(0,-1)$-curve.

Considering the lift $T_1^{-N} f$ instead of f, we may assume that $N = 0$. If F has a free $(0,1)$-curve, then any lift of this curve is a $(0,1)$-line ℓ such that $f(\ell) \cap \ell = \emptyset$. This means that either $f(\ell) > \ell$ or $f(\ell) < \ell$, and by Lemma 0.22 this implies that $\rho(f)$ is contained in one of the semiplanes $\{(x,y) : x \geq 0\}$ or $\{(x,y) : x \leq 0\}$. Hence 0 is an extremal point of $\mathrm{pr}_1(\rho(f))$, contradicting our assumption that it was an interior point.

Thus we know that $\mathrm{pr}_1(\rho(f))$ has no integer points in its interior. But $\mathrm{pr}_1(\rho(f))$ is an interval, so that it must be contained in $[k, k+1]$ for some integer k. □

Chapter 2

Free curves and fixed points

As before, we assume throughout this chapter that F is a homeomorphism of $\mathbb{T}^2$ isotopic to the identity and f is a lift of F.

To prove Theorem B, we have two main cases. The first one is when the rotation set is either a segment of rational slope or a single point; this is dealt with Theorem A and the results stated in Chapter 0. The second case is when the rotation set is a segment of irrational slope. In that case, the main idea is to find $A \in \mathrm{GL}(2, \mathbb{R})$ such that $A\,\rho(f)$ has no integers in the first or second coordinate, so that we may apply directly Theorem 0.24 to AfA^{-1} (c.f. §0.3.3). In fact, using this argument, we obtain the following more general result:

Theorem 2.1. *Suppose $\rho(f)$ is a segment of irrational slope with no rational points. Then for each $n > 0$ there is an essential simple closed curve γ such that γ, $F(\gamma), \dots, F^n(\gamma)$ are pairwise disjoint.*

The problem of finding the map A previously mentioned is mainly an arithmetic one, and we consider it first. In the next section, we briefly discuss a few facts about continued fractions that will be needed in the proof; in §2.2 we prove the two arithmetic lemmas that allow us to find the map A; in §2.3 we prove Theorem 2.1; finally, in §2.4 we complete the proof of Theorem B.

2.1 Continued fractions

Given an integer a_0 and positive integers $a_1, \ldots, a_n$, we define

$$[a_0; a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}.$$

Given $\alpha \in \mathbb{R}$, define $\{\alpha_n\}$ and $\{a_n\}$ recursively by $a_0 = \lfloor \alpha \rfloor$, $\alpha_0 = \alpha - a_0$, and

$$a_{n+1} = \left\lfloor \alpha_n^{-1} \right\rfloor, \ \alpha_{n+1} = \alpha_n^{-1} - a_{n+1},$$

whenever $\alpha_n \neq 0$. This gives the continued fractions representation of α: If α is rational, we get a finite sequence $a_0, \ldots, a_n$, and

$$\alpha = [a_0; a_1, \ldots, a_n].$$

If α is irrational, then the sequence is infinite and

$$\alpha = [a_0; a_1, a_2, \ldots] \doteq \lim_{n\to\infty} [a_0; a_1, \ldots, a_n].$$

The rational number $p_n/q_n = [a_0; a_1 \ldots, a_n]$ is called the n-th *convergent* to α. Convergents may be regarded as the "best rational approximations" to α, in view of the following properties (see, for instance, [HW90])

Proposition 2.2. *If p_n/q_n are the convergents to α, then*

1. *$\{q_n\}$ is an increasing sequence of positive integers, and*

$$\frac{1}{q_n + q_{n+1}} < (-1)^n(\alpha q_n - p_n) < \frac{1}{q_{n+1}}.$$

2. $\frac{p_{2n}}{q_{2n}} < \frac{p_{2n+2}}{q_{2n+2}} < \alpha < \frac{p_{2n+3}}{q_{2n+3}} < \frac{p_{2n+1}}{q_{2n+1}}$.

3. $p_{n+1}q_n - p_n q_{n+1} = (-1)^n$

A Farey interval is a closed interval with rational endpoints $[p/q, p'/q']$ such that $p'q - q'p = 1$. Note that two consecutive convergents give Farey intervals.

Proposition 2.3. *Let $[p/q, p'/q']$ be a Farey interval. Then*

$$\left[\frac{p+p'}{q+q'}, \frac{p'}{q'}\right] \quad \textit{and} \quad \left[\frac{p}{q}, \frac{p+p'}{q+q'}\right]$$

are Farey intervals, and if $p''/q'' \in (p/q, p'/q')$, then $q'' > q + q'$.

2.2 Arithmetic lemmas

Define the vertical and horizontal inverse *Dehn twists* by

$$D_1 : (x, y) \mapsto (x - y, y), \; D_2 : (x, y) \mapsto (x, y - x).$$

Let Q be the set of vectors of $\mathbb{R}^2$ with positive coordinates; for $u = (x, y) \in Q$, we denote by $\operatorname{slo}(u) = y/x$ the *slope* of u. Let Q_1 and Q_2 be the sets of elements of Q having slope smaller than one, and greater than one, respectively.

Remark 2.4. Note the following simple properties

1. for $i = 1, 2$, $D_i Q_i = Q$; and if $u \in Q$, then $D_i^{-k} u \in Q_i$ for all $k > 0$;
2. for $i = 1, 2$, $\|D_i u\| < \|u\|$ if $u \in Q_i$;
3. $\operatorname{slo}(D_2^k u) = \operatorname{slo}(u) - k$ and $\operatorname{slo}(D_1^k u)^{-1} = \operatorname{slo}(u)^{-1} - k$.

Lemma 2.5. *Let u and v be elements of Q with different slopes. Then there is $A \in \mathrm{GL}(2, \mathbb{Z})$ such that*

1. *$\|Au\| \leq \|u\|$;*
2. *$\|Av\| \leq \|v\|$;*
3. *Both Au and Av are in Q, and either one of these points is on the diagonal and the other in Q_1, or one is in Q_1 and the other in Q_2.*

Proof. We first note that it suffices to consider the case where both u and v are in Q_1. Indeed, if one of the vectors is in Q_i and the other is not (for $i = 1$ or 2), there is nothing to do; and if both u and v are in Q_2 then we may use Su and Sv instead, where S is the isometry $(x, y) \mapsto (y, x)$.

Given $u \in Q_1$, we define a sequence of matrices $A_n \in \mathrm{SL}(2, \mathbb{Z})$ and integers a_n by $A_0 = I$, $a_0 = 0$, and recursively (see Figure 2.1)

- If $\mathrm{slo}(A_n u) = 1$ stop the construction.
- a_{n+1} is the smallest integer such that $D_i^{a_{n+1}} A_n u \notin Q_i$, where $i = 2$ if n is odd, 1 if n is even;
- $A_{n+1} = D_i^{a_{n+1}} A_n$.

In this way we get either an infinite sequence, or a finite sequence $A_1, \dots A_N$ such that $A_N u$ lies on the diagonal and has positive coordinates. Furthermore, given $0 \leq n < N$ if the sequence is finite, or $n \geq 0$ if it is infinite, we have

- $A_n u \in Q_i$ where $i = 2$ if n is odd, 1 if n is even;
- If $\alpha_n = \mathrm{slo}(A_n u)^{(-1)^n}$, then $\alpha_n = \alpha_{n-1}^{-1} - a_n$
- $\|A_0 u\|, \|A_1 u\|, \dots$ is a decreasing sequence;

The first property is a consequence of the definition. The second follows from Remark 2.4, since for odd n, we have

$$\alpha_n = \mathrm{slo}(D_1^{a_n} A_{n-1} u)^{-1} = \mathrm{slo}(A_{n-1} u)^{-1} - a_n = \alpha_{n-1}^{-1} - a_n$$

while for even n,

$$\alpha_n = \mathrm{slo}(D_2^{a_n} A_{n-1} u) = \mathrm{slo}(A_{n-1} u) - a_n = \alpha_{n-1}^{-1} - a_n.$$

The last property also follows from the construction, since if $A_n u = D_i^{a_n} A_{n-1} u$ then $D_i^k A_{n-1} u \in Q_i$ for all $0 \leq k < a_n$, so that Remark 2.4 implies that $\|A_n u\| < \|A_{n-1} u\|$.

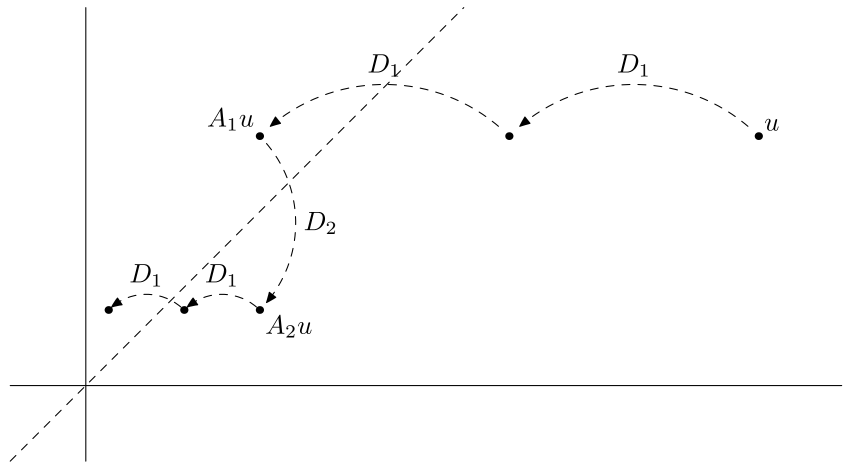

Figure 2.1: The sequence A_iu

If n is odd, a_n is the smallest integer such that $D_1^{a_n}A_{n-1}u \notin Q_1$, or equivalently (assuming that $n < N$ if the sequence a_n is finite), the smallest integer such that

$$\mathrm{slo}(D_1^{a_n}A_{n-1}u) = (\mathrm{slo}(A_{n-1}u)^{-1} - a_n)^{-1} = (\alpha_{n-1}^{-1} - a_n)^{-1} \geq 1.$$

Since $A_{n-1}u \in Q_1$, $\mathrm{slo}(A_{n-1}u) < 1$ so that $\alpha_{n-1}^{-1} > 1$, and $a_n > 0$. Note that α_n^{-1} cannot be an integer, since otherwise $\mathrm{slo}(A_nu) = 1$, which contradicts the fact that $A_nu \in Q_2$; thus

$$a_n = \left\lfloor \alpha_{n-1}^{-1} \right\rfloor.$$

If n is even, the above equation holds by a similar argument. One easily sees from these facts that α_n coincides with the sequence obtained in the definition of the continued fractions expression of $\alpha_0 = \mathrm{slo}(u)$, and thus a_n coincides with the continued fractions coefficients of $\mathrm{slo}(u)$.

Now given $v \in Q_1$ with $\mathrm{slo}(v) \neq \mathrm{slo}(u)$, define in the same way as above sequences of positive integers b_n and of matrices $B_n \in \mathrm{SL}(2, \mathbb{Z})$ such that $B_0 = I$, b_n is given by the continued fractions expression of $\mathrm{slo}(v)$, and $B_{n+1} = D_i^{b_{n+1}}B_n$ where $i = 2$ if n is odd, 1 if n is even. As before, we have that $\|B_n v\|$ is a (finite or infinite) decreasing sequence, and if it is finite of length N then $\mathrm{slo}(B_N v) = 1$. Also $B_n v \in Q_i$ where $i = 2$ if n is odd, 1 if n is even (given that $n < N$ if the sequence is finite).

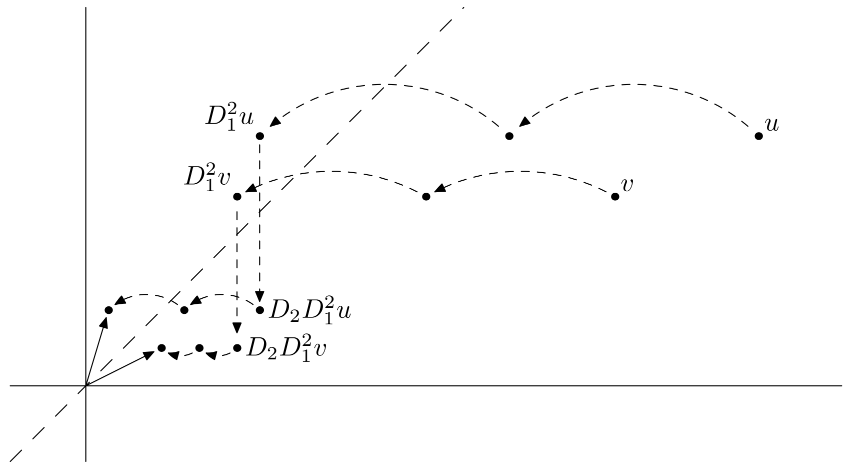

Figure 2.2: Example

Since u and v have different slopes, their continued fractions expression cannot coincide. Thus there exists $m \geq 0$ such that $a_0 = b_0, \dots, a_m = b_m$ but $b_{m+1} \neq a_{m+1}$. We may further assume that $a_{m+1} < b_{m+1}$, by swapping u and v if necessary. This means that $A_k = B_k$ for $0 \leq k \leq m$; so that if m is even, $A_m u \in Q_1$ and $A_m v \in Q_1$, but since $a_{m+1} < b_{m+1}$, it holds that

$$A_{m+1} u = D_1^{a_{m+1}} A_m u \notin Q_1 \ \text{ but } \ A_{m+1} v = D_1^{a_{m+1}} B_m v \in Q_1;$$

that is, $D_1^{a_{m+1}}$ "pushes" $A_m u$ out of Q_1, while leaving $A_m v$ in Q_1 (see Figure 2.2).

Moreover, since a_{m+1} is minimal with that property, either $\mathrm{slo}(A_{m+1} u) = 1$ or $A_{m+1} u \in Q_2$; and in either case $A_{m+1} u$ has positive coordinates. By construction, it also holds that $\|A_{m+1} v\| \leq \|v\|$ and $\|A_{m+1} u\| \leq \|u\|$.

If m is odd, a similar argument holds, and we see that $A_{m+1} u \in Q_2$ while $A_{m+1} v$ is either on the diagonal or in Q_1.

Starting from u and v in Q_1 we obtained $A = A_{m+1} \in \mathrm{GL}(2, \mathbb{Z})$ such that Au and Av have positive coordinates and either one of them is on the diagonal or one is in Q_1 and the other in Q_2; furthermore, $\|Au\| \leq \|u\|$ and $\|Av\| \leq \|v\|$. This completes the proof. $\square$

Lemma 2.6. *Let $w = (x, y)$ be a vector with irrational slope. Then, for any $\epsilon > 0$ there exists $A \in \mathrm{SL}(2, \mathbb{Z})$ such that $\|Aw\| < \epsilon$.*

Proof. Let p_i/q_i be the convergents to y/x, and define

$$A_k = \begin{pmatrix} (-1)^k p_k & (-1)^{k+1} q_k \\ p_{k+1} & -q_{k+1} \end{pmatrix}.$$

By Proposition 2.2 we have that $\det A = 1$, the sequence $q_1, q_2, \ldots$ is increasing, and

$$\left| p_i - \frac{y}{x} q_i \right| < \frac{1}{q_{i+1}}$$

for all $i \geq 1$. Hence,

$$|\mathrm{pr}_1 A_k w| = \left| x \left(p_k - \frac{y}{x} q_k \right) \right| < \frac{|x|}{q_{k+1}},$$

and similarly

$$|\mathrm{pr}_2 A_k w| = \left| x \left(p_{k+1} - \frac{y}{x} q_{k+1} \right) \right| < \frac{|x|}{q_{k+2}};$$

Choosing k large enough so that $q_{k+1} > \sqrt{2}\,|x|\,\epsilon^{-1}$, we have $\|A_k w\| < \epsilon$. $\square$

2.3 Proof of Theorem 2.1

2.3.1 The case $n = 1$

We first assume $n = 1$, i.e. we prove that F has a free curve, assuming that $\rho(f)$ is a segment of irrational slope containing no rational points. The problem is reduced, by Lemma 0.25 and Theorem 0.24, to finding $A \in \mathrm{GL}(2, \mathbb{Z})$ such that the projection of $A\,\rho(f)$ to the first or the second coordinate contains no integers. Note that Lemma 2.6 allows us to assume that

$$\mathrm{diam}(\rho(f)) < \epsilon < \frac{1}{2\sqrt{5}}.$$

We may also assume that there are $m_1 \in \mathrm{pr}_1(\rho(f)) \cap \mathbb{Z}$ and $m_2 \in \mathrm{pr}_2(\rho(f)) \cap \mathbb{Z}$, for otherwise there is nothing to do.

Then using $T_1^{-m_1} T_2^{-m_2} f$ (which also lifts F) instead of f, we have that the extremal points of $\rho(f)$ are in opposite quadrants. By conjugating f with a rotation by $\pi/2$, we may assume that $\rho(f)$ is the segment joining $u = (-u_1, -u_2)$ and $v = (v_1, v_2)$ where $v_i \geq 0$ and $u_i \geq 0$, $i = 1, 2$. From this, and the fact that $\mathrm{diam}(\rho(f)) < \epsilon$, it follows that

$$\|u\| < \epsilon, \text{ and } \|v\| < \epsilon.$$

Case 1. One of the points has a zero coordinate

It is clear that neither u nor v can have both coordinates equal to 0. Conjugating by an appropriate isometry in $\mathrm{GL}(2, \mathbb{Z})$, we may assume the generic case that $u = (-u_1, 0)$, with $u_1 > 0$. Then $v_2 > 0$: in fact if $v_2 = 0$ then $\rho(f)$ contains the origin, which is not possible. Let $k > 0$ be the greatest integer such that $\mathrm{pr}_1 D_1^k v > -u_1$, i.e.

$$k = \left\lfloor \frac{u_1 + v_1}{v_2} \right\rfloor .$$

Note that $D_1^k u = u$, so that $D_1^k \rho(f)$ is the segment joining u to $D_1^k v$ (see Figure 2.3).

Moreover, $D_1^{k+1} v = (v_1', v_2)$ where $v_1' = v_1 - (k+1)v_2 < -u_1$. Thus

$$\max \mathrm{pr}_1(D_1^{k+1} \rho(f)) = -u_1 < 0.$$

On the other hand,

$$\min \mathrm{pr}_1(D_1^{k+1} \rho(f)) \geq -u_1 - v_1 > -2\epsilon > -1,$$

so that taking $A = D_1^{k+1}$ we have $\mathrm{pr}_1(\rho(A f A^{-1})) \subset (-1, 0)$. By Theorem 0.24, it follows that $\tilde{A} F \tilde{A}^{-1}$ has a free $(0,1)$ curve. Thus F has a free curve.

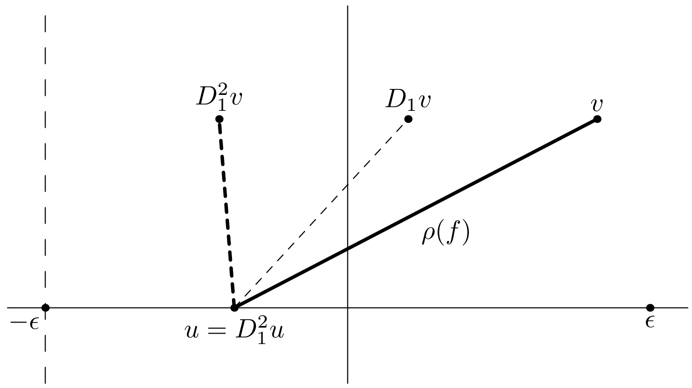

Figure 2.3: Avoiding integers in the first coordinate

Case 2. None of the points has a zero coordinate

In this case, $u \in -Q$ and $v \in Q$. Since the segment joining u to v cannot contain the origin, $-u$ and v are elements of Q with different slopes; thus Lemma 2.5 implies that there is $A \in \mathrm{GL}(2, \mathbb{Z})$ such that $\|Au\| \le \|u\|$, $\|Av\| \le \|v\|$, both $-Au$ and Av are in Q, and either one of them lies on the diagonal, or they are in opposite sides of the diagonal. This means that Au and Av are both contained in one of the closed semiplanes determined by the diagonal. By using $-A$ instead of A if necessary, we may assume that both are in the closed semiplane above the diagonal, which is mapped by D_2 to the upper semiplane $H = \{(x, y) : y \ge 0\}$. Note that

$$\|D_2 Au\| \le \|D_2\| \, \|Au\| \le \sqrt{5} \, \|u\| \, ,$$

and similarly $\|D_2 Av\| \le \sqrt{5} \, \|v\|$. If $\mathrm{pr}_2(D_2 Av) > 0$ and $\mathrm{pr}_2(D_2 Au) > 0$, then (see Figure 2.4) we have that

$$\mathrm{pr}_2 \, \rho(f) \subset (0, \sqrt{5}\epsilon) \subset (0, 1),$$

and by Theorem 0.24 (as in Case 1) it follows that F has a free curve. On the other hand, if either of $D_2 Av$ or $D_2 Au$ has zero second coordinate, the argument in Case 1 implies that F has a free curve.

This completes the proof when $\rho(f)$ has irrational slope and $n = 1$.

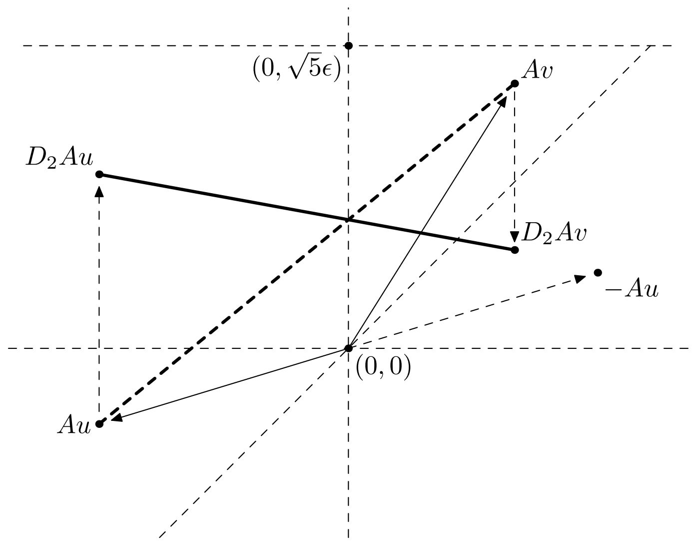

Figure 2.4: Avoiding integers in the second coordinate

2.3.2 The case $n > 1$

Note that when $\rho(f)$ has irrational slope, $\rho(f^n) = n\,\rho(f)$ has irrational slope for all n.

Let $N = n!$. As we saw in the previous case, conjugating our maps by some $A \in \mathrm{GL}(2, \mathbb{Z})$, we may assume $\mathrm{pr}_1\,\rho(f^N) \cap \mathbb{Z} = \emptyset$; thus

$$\mathrm{pr}_1\,\rho(f^N) \subset (K, K+1) \text{ for some } K \in \mathbb{Z},$$

and by Theorem 0.8,

$$\mathrm{pr}_1\,\rho(f) \subset \left(\frac{K}{N}, \frac{K+1}{N}\right).$$

Let $[p/q, p'/q']$ be the smallest Farey interval containing $\mathrm{pr}_1(\rho(f))$. We claim that $q + q' > n$. In fact, if $q + q' \leq n$, then $[K/N, (K+1)/N]$ must be contained in one of the smaller Farey intervals (see Proposition 2.3)

$$\left[\frac{p}{q}, \frac{p+p'}{q+q'}\right] \text{ or } \left[\frac{p+p'}{q+q'}, \frac{p'}{q'}\right].$$

This is because $N(p+p')/(q+q')$ is an integer, so that it cannot be in the interior of $[K, K+1]$. But we chose our Farey interval to be the smallest, so $[p/q, p'/q']$ must as well be contained in one of these two intervals, which is a contradiction. Thus $q+q' > n$, and Theorem 0.24 guarantees that there is an essential simple closed curve γ such that its first n iterates by F are pairwise disjoint. This completes the proof.

2.4 Proof of Theorem B

Assume that $\rho(f)$ has empty interior. We will show that either F has a fixed point, or it has a free curve. There are several cases:

- $\rho(f)$ is a segment of irrational slope which contains no rational points. Then there is a free curve, by Theorem 2.1.

- $\rho(f)$ is a segment of irrational slope containing a rational non-integer point $(p_1/q, p_2/q)$. By Lemma 2.6 there exists $A \in \mathrm{GL}(2, \mathbb{Z})$ such that $\mathrm{diam}(A\,\rho(f)) < 1/q$. One of the two coordinates of $A(p_1/q, p_2/q)$ must be non-integer. We assume $p'/q' = \mathrm{pr}_1 A(p_1/q, p_2/q) \notin \mathbb{Z}$ (otherwise, we can conjugate f with a rotation by $\pi/2$, as usual). Since $A(p_1, p_2)$ is an integer point, it follows that $q' \leq q$ (if we assume p'/q' is irreducible). Thus,

$$\mathrm{pr}_1\left(\rho(AfA^{-1})\right) = \mathrm{pr}_1(A\,\rho(f)) \subset \mathrm{pr}_1\left(\frac{p'}{q'} - \frac{1}{q}, \frac{p'}{q'} + \frac{1}{q}\right).$$

  It is clear that the interval above contains no integers, so that $\tilde{A}F\tilde{A}^{-1}$ (and, consequently, F) has a free curve by Theorem 0.24.

- $\rho(f)$ is a segment of irrational slope with an integer point. Then F has a fixed point by Theorem 0.12 (see also Remark 1.5).

- $\rho(f)$ is a single point. Then either this point is integer, and F has a fixed point by Theorem 0.10 or it is not integer, and F has a free curve by Theorem 0.24.

- $\rho(f)$ is a segment of rational slope. Conjugating all the maps by an element of $\mathrm{GL}(2,\mathbb{Z})$ we may assume it is a vertical segment; and with this assumption, if both $\mathrm{pr}_1(\rho(f))$ and $\mathrm{pr}_2(\rho(f))$ contain an integer, it follows that $\rho(f)$ contains an integer point, and by Theorem A, F has either a fixed point or a free curve. On the other hand, if either of the two projections contains no integer, Theorem 0.24 implies the existence of a free curve for F as before.

This concludes the proof.

Chapter 3

Always free curves

Given a homeomorphism $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ (not necessarily in the homotopy class of the identity), we say that a simple closed curve γ is *always free* if

$$F^n(\gamma) \cap \gamma = \emptyset \text{ for all } n \in \mathbb{Z},\, n \neq 0.$$

In this chapter we prove Theorem C, which says that if F has an always free curve with a dense orbit, then F is semi-conjugate to an irrational rotation of the circle; i.e. there exists a continuous surjection $h\colon \mathbb{T}^2 \to S^1$ such that $hF = Rh$, where R is some irrational rotation of S^1.

The natural idea for proving the theorem is defining the map h on the orbit of γ by F, mapping it to the orbit of some point x_0 in S^1 by some appropriately chosen map (which eventually turns out to be conjugated to an irrational rotation), and then extending h to $\mathbb{T}^2$. We follow this idea in a slightly indirect way: we explicitly construct a partition of $\mathbb{T}^2$ which will correspond to $\{h^{-1}(x) : x \in S^1\}$ and then we show that by collapsing elements of this partition, the map induced by F on the quotient space is conjugated to an irrational rotation of S^1.

With minor modifications, the proof presented here also works if γ is an essential continuum (i.e. a continuum such that $\pi^{-1}(\gamma)$ has an unbounded connected component), with the additional hypothesis that there is some essential simple closed curve disjoint from γ. The latter condition is necessary to ensure that F has a lift to the annulus. Thus we have

Theorem C'. *Let $F \in \mathrm{Homeo}(\mathbb{T}^2)$, and let $\gamma \subset \mathbb{T}^2$ be an essential continuum with a dense orbit, such that all its iterates by F are pairwise disjoint. Assume further that γ is disjoint from some essential simple closed curve. Then f is semi-conjugated to an irrational rotation of the circle.*

Homeomorphisms in the homotopy class of the identity satisfying the hypothesis of Theorem C are "almost" fibered, in the sense that they are similar to skew-products of the form $(x, y) \mapsto (f(x), g_x(y))$ where $f\colon S^1 \to S^1$ has an irrational rotation number. The partition found in Theorem C resembles the foliation by vertical circles in skew-products of this kind, except that it could consist of general continua instead of curves.

In [Her83], Herman proved that skew-products always have a rotation set consisting of a single point (i.e. they are pseudo-rotations). Thus, the previous discussion motivates the following

Question 3.1. *If $F \in \mathrm{Homeo}_*(\mathbb{T}^2)$ has an always free curve with a dense orbit, must its rotation set consist of a single point?*

3.1 Proof of theorem C

Let γ_0 be an always free curve for F with a dense orbit.

Proposition 3.2. *γ_0 is homotopically non-trivial.*

Proof. Suppose γ_0 is homotopic to a point, and let D be the topological disk bounded by it in $\mathbb{T}^2$. We know that γ_0 has a dense orbit, so that it must hold $F^n(\gamma_0) \cap D \neq \emptyset$ for some n which we may assume positive (otherwise consider F^{-1}). Moreover, we assume n to be the smallest positive integer with that property. Since γ_0 is always free, we must have $F^n(D) \subset D$ and $F^k(D) \cap D = \emptyset$ when $0 < k < n$. If $A = D - \mathrm{cl}(F^n(D))$, then it follows easily that $F^k(\gamma_0) \cap A = \emptyset$ for all $k \in \mathbb{Z}$. But since A is open and nonempty, this contradicts the fact that the orbit of γ_0 is dense. □

By the above proposition, γ_0 is not homotopic to a point. This means that γ_0 is a (p,q)-curve for some $(p,q) \in \mathbb{Z}^2$. Moreover, since γ_0 is simple, p and q are coprime; hence we can find $A \in \mathrm{GL}(2,\mathbb{Z})$ such that $A(p,q) = (0,1)$ (see Remark 0.26). Such A induces a homeomorphism $\tilde{A}$ of $\mathbb{T}^2$, such that $\tilde{A}\gamma_0$ is an always free $(0,1)$-curve for $\tilde{A}F\tilde{A}^{-1}$; hence, it suffices to prove the theorem when γ_0 is a $(0,1)$-curve, which we will assume from now on.

Since $F(\gamma_0)$ is a simple closed curve which is disjoint from γ_0, it must be either a $(0,1)$-curve or a $(0,-1)$ curve (see Remark 0.17). This easily implies (by the lifting theorem) that F can be lifted to $f\colon \mathbb{A} \mapsto \mathbb{A}$ by the covering π_1, and γ_0 (and any iterate of it by F) is lifted to a simple closed curve γ on $\mathbb{A}$.

Recall that τ is the translation $(x,v) \mapsto (x+1,v)$ in $\mathbb{A}$. Let

$$\Gamma = \{\tau^m f^n(\gamma) : (n,m) \in \mathbb{Z} \times \mathbb{Z}\}.$$

Note that the elements of Γ are pairwise disjoint simple closed curves, and $\cup\Gamma$ is dense in $\mathbb{A}$.

For any $\omega \in \Gamma$, there are exactly two connected components in $\mathbb{A} \setminus \omega$, both unbounded. We write $L\omega$ for the one "on the left", and $R\omega$ for the remaining one. We also write $\overline{L}\omega$ and $\overline{R}\omega$ for their respective closures. Since all elements of Γ are disjoint, the relation defined by $\omega < \omega'$ if $\omega \subset L\omega'$ (which is the same as $L\omega \subset L\omega'$) is a linear ordering of Γ.

Proposition 3.3. *For any ω_0 and ω_1 in Γ,*

1. *$f(L\omega_0) = Lf(\omega_0)$, $f(R\omega_0) = Rf(\omega_0)$, and similarly for τ;*
2. *If $\omega_0 < \omega_1$ then $f(\omega_0) < f(\omega_1)$ and $\tau(\omega_0) < \tau(\omega_1)$;*
3. *$\partial L\omega_0 = \omega_0 = \partial R\omega_0$;*
4. *$\tau(\omega_0) > \omega_0$. In particular, there is no maximum or minimum element of Γ;*
5. *If $\omega_0 < \omega_1$, then there exists $\omega \in \Gamma$ such that $\omega_0 < \omega < \omega_1$.*

Proof. (1) The assertion for τ is obvious since $L\omega_0$ and $R\omega_0$ are left and right unbounded, respectively, and τ is just a translation.

As it is easy to see, either $f(L\omega) = Lf(\omega)$ and $f(R\omega) = Rf(\omega)$ for all $\omega \in \Gamma$, or $f(L\omega) = Rf(\omega)$ and $f(R\omega) = Lf(\omega)$ for all $\omega \in \Gamma$; thus it suffices to prove the property for the original curve $\omega_0 = \gamma$. We may also assume that f is such that $\gamma < f(\gamma) < \tau(\gamma)$, by choosing an appropriate m and using $\tau^m f$, which is also a lift of F, instead of f. Suppose that $f(R\gamma) = Lf(\gamma)$. Then, letting $U = R\gamma \cap Lf(\gamma)$, we have $f(U) \subset U$, and $\pi_1(U) \subset \pi_1(R\gamma \cap L\tau(\gamma))$ which is a proper open subset of $\mathbb{T}^2$, whose complement has nonempty interior. Since $F(\pi_1(U)) \subset \pi_1(U)$, this contradicts the fact that γ has a dense orbit.

(2) By item 1, $Lf(\omega_0) = f(L\omega_0) \subset f(L\omega_1) = Lf(\omega_1)$, and a similar argument for τ.

(3) Follows from the fact that ω_0 is simple.

(4) Suppose $\tau(\omega_0) < \omega_0$; then, by item 2, $\tau^n(\omega_0) < \omega_0$ for all $n > 0$. But it is clear that if $x \in \omega_0$, then for n large enough, $\tau^n(x) \in R\omega_0$, which is a contradiction. Thus $\tau(\omega_0) > \omega_0$.

(5) Since $U = L\omega_1 \cap R\omega_0$ is open and $\cup\Gamma$ is dense, there exists $\omega \in \Gamma$ such that $\omega \cap U \neq \emptyset$. But then ω must be entirely contained in U. $\square$

For $\omega_1, \omega_2 \in \Gamma$, we define the open strip $S(\omega_1, \omega_2) = R\omega_1 \cap L\omega_2$, and the closed strip $\overline{S}(\omega_1, \omega_2) = \overline{R}\omega_1 \cap \overline{L}\omega_2 = S(\omega_1, \omega_2) \cup \omega_1 \cup \omega_2$.

Remark 3.4. Note that if $\omega_1 < \omega_2$, then $S(\omega_1, \omega_2)$ is homeomorphic to $\mathbb{A}$. Also note that finite intersections of open (resp. closed) strips are open (resp. closed) strips.

3.1.1 Construction of an invariant partition

Given $x \in \mathbb{A}$, we will say that a strip $S_0 = S(\omega_1, \omega_2)$ is *good* for x if x belongs to some closed strip which is contained in S_0. Define

$$P_x = \cap\{\overline{S}_0 : S_0 \text{ is good for x }\}$$

and let $\mathcal{P} = \{P_x : x \in \mathbb{A}\}$. This will be our partition.

Remark 3.5. By proposition 3.3, for any x we can find ω_1', ω_2' such that $x \in \overline{R}\omega_1' \subset R\omega_2'$. Thus, if we have $\omega_1 < \omega_2$ such that $x \in \overline{L}\omega_1$, then necessarily $P_x \subset \overline{L}\omega_2$; moreover, since there exists some ω such that $\omega_1 < \omega < \omega_2$, we have $P_x \subset L\omega_2$. An analogous property holds in the other direction. As a consequence, we may take the intersection of open strips instead of closed ones in the definition of P_x.

Proposition 3.6. *The following properties hold:*

1. *$\mathcal{P}$ is a partition;*
2. *Each element of the partition is compact, connected, has empty interior, and splits $\mathbb{A}$ into exactly two connected components.*
3. *$\mathcal{P}$ is f-invariant, i.e. $f(P_x) = P_{f(x)}$ for all $x \in \mathbb{A}$;*
4. *$\mathcal{P}$ is τ-invariant, and consequently $\{\pi_1(P) : P \in \mathcal{P}\}$ is an F-invariant partition of $\mathbb{T}^2$;*
5. *For any $x, y \in \mathbb{A}$, if $P_x \neq P_y$ then there exists $\omega \in \Gamma$ that separates P_x and P_y, so that $P_x \subset L\omega$ and $P_y \subset R\omega$ or vice-versa;*
6. *For each $x \in \mathbb{A}$, at most one element of Γ intersects P_x, and when it does it is contained in P_x.*

Proof. (1): It is clear that $x \in \mathcal{P}$ for all $x \in \mathbb{A}$; thus it suffices to show that whenever $P_x \neq P_y$, it holds that $P_x \cap P_y = \emptyset$. Fix $x, y \in \mathbb{A}$, and suppose there are $\omega_1 < \omega_2$ such that $x \in \overline{L}\omega_1$ and $y \in \overline{R}\omega_2$. Then by Proposition 3.3 we may find ω such that $\omega_1 < \omega < \omega_2$, so that $x \in \overline{L}\omega_1 \subset L\omega$ and $y \in \overline{R}\omega_2 \subset R\omega$. By Remark 3.5, $P_x \subset L\omega$ and $P_y \subset R\omega$; thus, $P_x \cap P_y = \emptyset$ and ω separates P_x from P_y.

Now suppose that no element of Γ separates P_x from P_y. Then whenever $\omega_1 < \omega_2$ and $x \in \overline{L}\omega_1$, we must have $y \in L\omega_2$; by a symmetric argument, also if $\omega_1 < \omega_2$ and $x \in \overline{R}\omega_2$, we have $y \in R\omega_1$. This implies that y belongs to every good strip of x, so that $x \in P_y$. Analogously, $y \in P_x$. We will show that every good strip for x is a good strip for y; that will imply that $P_x \subset P_y$, and by symmetry that $P_y \subset P_x$ finishing the proof of this item.

Let $S_0 = S(\omega_1', \omega_2')$ be a good strip for x. By definition, there is a closed strip $\overline{S}_1 = \overline{S}(\omega_1, \omega_2)$ such that $x \in \overline{S}_1 \subset S_0$. We know that $y \subset P_x$, so that $y \in S_0$. If $y \in \overline{S}_1$, then S_0 is a good strip for y and we are done. Assume that $y \notin \overline{S}_1$. Then either $y \in L\omega_1$ or $y \in R\omega_2$. Suppose without loss of generality that $y \in L\omega_1$. By Proposition 3.3 there is $\omega \in \Gamma$ such that $\omega_1' < \omega < \omega_1$, and we must have $y \in \overline{R}\omega$. In fact if this is not the case, $y \in \overline{L}\omega$ and $x \in \overline{R}\omega_1$, which by the argument in the beginning of the proof implies that P_x and P_y are disjoint, contradicting our assumption. Hence, $y \in \overline{S}(\omega, \omega_1) \subset S_0$ which means that S_0 is a good strip for y, as we wanted.

(3 and 4): This is clear since Proposition 3.3 implies that the image by f (resp. f^{-1},τ,τ^{-1}) of a good strip for x is a good strip for $f(x)$ (resp. $f^{-1}(x)$, $\tau(x)$,$\tau^{-1}(x)$).

(5) Follows from the proof of item 1.

(6) Let $\omega \in \Gamma$, $x \in \omega$. Then it is clear that any good strip for x contains ω, so $\omega \subset P_x$. Given $\omega' \neq \omega$, suppose without loss of generality that $\omega' > \omega$. Then $x \in \overline{L}\omega \subset L\omega'$, so $P_x \subset L\omega'$ by remark 3.5. Hence P_x does not intersect ω'.

(2) Since $\cup\Gamma$ is dense and is a union of simple closed curves, any open set intersects infinitely many elements of Γ; by part 6, P_x intersects at most one such element, so it has empty interior.

It is clear that $x \in P_x$ for all $x \in \mathbb{A}$, and it is easy to see that finite intersections of good strips for x are good strips for x. Any family of compact connected nonempty sets which is closed under finite intersections has a compact, connected intersection; thus P_x is compact and connected.

It remains to show that $\mathbb{A} - P_x$ has exactly two connected components. Suppose first that $\mathbb{A} - P_x$ is connected. Since it is open, it is arc-connected. Take $\omega_1 < \omega_2$ such that $P_x \subset \overline{S}(\omega_1, \omega_2)$, let N be large enough so that both ω_1 and ω_2 are contained in $[-N, N] \times S^1$, and let σ be a compact arc joining $(-N-1, 0)$ to $(N+1, 0)$, contained in $\mathbb{A} - P_x$. Then any ω such that $\omega_1 < \omega < \omega_2$ intersects σ, so that any good strip for x intersects σ. Hence, the family $\{\text{good strips for } x\} \cup \{\sigma\}$ has the finite intersection property, and therefore its intersection is nonempty. This means that P_x intersects σ, which contradicts our choice of σ. Thus $\mathbb{A} - P_x$ has more than one connected component.

Now suppose $\mathbb{A} - P_x$ has more than two components. Then, since P_x is bounded, there is some connected component U of $\mathbb{A} - P_x$ that is bounded. Since U is open, infinitely many elements of Γ intersect it; each one of these elements must then be contained in U, or intersect $\partial U \subset P_x$. Since we already saw that at most one element of Γ intersects P_x, there must exist $\omega \in \Gamma$ such that $\omega \subset U$. Since P_x is connected, either $P_x \subset L\omega$ or $P_x \subset R\omega$. But U intersects both $L\omega$ and $R\omega$, so that it has boundary points on both sets, which is a contradiction. This shows that $\mathbb{A} - P_x$ has exactly two connected components. $\square$

3.1.2 Defining order in $\mathcal{P}$

Define an order in $\mathcal{P}$ by $P < Q$ if there is $\omega \in \Gamma$ such that $P \subset L\omega$ and $Q \subset R\omega$. In view of the previous propositions, it is clear that this defines a total order. For $P \in \mathcal{P}$, we will denote by $L(P)$ and $R(P)$ the two connected components of $\mathbb{A} - P$, being $L(P)$ the one "on the left"; and we will endow $\mathcal{P}$ with the topology induced by this order. Note that for $P, Q \in \mathcal{P}$, $P < Q$ means that $P \subset L(Q)$ and $Q \subset R(P)$.

Proposition 3.7. $(\mathcal{P}, <)$ *is separable and has the property of the supremum.*

Proof. From Propositions 3.3 and 3.6 it easily follows that

$$\mathcal{D} = \{P \in \mathcal{P} : P \text{ contains some element of } \Gamma\}$$

is dense; and it is countable because Γ is countable.

Now let $P_0 \in \mathcal{P}$ and let $\mathcal{S} \subset \mathcal{P}$ be such that $P < P_0$ for all $P \in \mathcal{S}$. Let

$$L = \bigcup_{P \in \mathcal{S}} L(P).$$

Then L is open and $L \subset L(P_0)$. Since $\mathbb{A} - L \neq \emptyset$, we may choose $x \in \partial L$. We claim that $L \cap P_x$ is empty. In fact, if $y \in L \cap P_x$ then $y \in L(P)$ for some $P \in \mathcal{S}$, so that $P_x = P_y < P$, which implies that $x \in L(P) \subset L$ and this contradicts the choice of x. Thus we have $P < P_x$ for all $P \in \mathcal{S}$.

Furthermore, if $Q \in \mathcal{P}$ is such that $P < Q$ for all $P \in \mathcal{S}$, then $L \subset L(Q)$, so that $\partial L \subset \mathrm{cl}(L(Q))$. Since P_x contains a boundary point of L, we have $P_x \cap \mathrm{cl}(L(Q)) \neq \emptyset$, which implies that $P_x \leq Q$. Hence P_x is the lowest upper bound of $\mathcal{S}$. □

Let $p\colon \mathbb{A} \to \mathcal{P}$ be the quotient projection that maps $x \in \mathbb{A}$ to the unique element of $\mathcal{P}$ that contains x. The quotient topology on $\mathcal{P}$ is given by

$$\left\{\mathcal{A} \subset \mathcal{P} : p^{-1}(\mathcal{A}) \text{ is open in } \mathbb{A}\right\}.$$

Note that $p^{-1}(\mathcal{A}) = \cup\mathcal{A}$.

Proposition 3.8. *The order topology of $\mathcal{P}$ coincides with the quotient topology induced by p.*

Proof. We use the interval notation $(P_1, P_2) = \{P \in \mathcal{P} : P_1 < P < P_2\}$. Such intervals form a basis of open sets for the order topology. Given $I = (P_1, P_2)$, we see that $p^{-1}(I) = L(P_2) \cap R(P_1)$ which is open in $\mathbb{A}$, so that I is open in the quotient topology.

Now let $\mathcal{A} \subset \mathcal{P}$ be such that $U = p^{-1}(\mathcal{A})$ is open in $\mathbb{A}$. We will find, for every $P \in \mathcal{A}$, an interval $I = (P_1, P_2)$ such that $P \in I \subset \mathcal{A}$, thus showing that $\mathcal{A}$ is open in the order topology. Suppose that this is not the case for some $P \in \mathcal{A}$. Since $\partial L(P) \subset P$ and $\partial R(P) \subset P$, we can choose $x, y \in P$ such that any neighborhood of x intersects $L(P)$ and any neighborhood of y intersects $R(P)$. Since U is open and contains P, we may find ϵ small enough so that $B_\epsilon(x) \subset U$ and $B_\epsilon(y) \subset U$. Let $x' \in B_\epsilon(x) \cap L(P)$, $y' \in B_\epsilon(y) \cap R(P)$, and let σ be an arc contained in U and joining x to x'. Then any $Q \in \mathcal{P}$ such that $P_{x'} < Q < P_x$ must intersect σ, because σ connects $L(Q)$ and $R(Q)$, and Q separates those sets. Since $\sigma \subset U$ and $U = \cup\mathcal{A}$, every element of $\mathcal{P}$ that intersects U must be contained in U; thus $Q \subset U$. This shows that $(P_{x'}, P_x) \subset \mathcal{A}$. An analogous argument with y and y' shows that $(P_y, P_{y'}) \subset \mathcal{A}$. Since $P_y = P_x = P$, it follows that $(P_{x'}, P_{y'}) \subset \mathcal{A}$ as we wanted. □

3.1.3 Construction of the semi-conjugation

We now know that the order topology in $(\mathcal{P},<)$ coincides with the quotient topology induced by p. Moreover, $(\mathcal{P},<)$ is a separable totally ordered space which has no maximum or minimum element (i.e. it is unbounded), and has the property of the supremum (i.e. it is complete). It is well known that any such space is homeomorphic to $\mathbb{R}$ by an order-isomorphism (see, for example, [Ros82, §4.2]).

By Proposition 3.6, the maps f and τ induce homeomorphisms $\tilde{f}$ and $\tilde{\tau}$ of $\mathcal{P}$ which commute. Moreover, $\{\tilde{\tau}^n : n \in \mathbb{Z}\}$ is a properly discontinuous action of $\mathbb{Z}$ on $\mathcal{P}$. Thus, $\mathcal{S} \doteq \mathcal{P}/\langle\tilde{\tau}\rangle$ is homeomorphic to $\mathbb{R}/\mathbb{Z} \simeq S^1$. The quotient projection p also induces a projection $\tilde{p}\colon \mathbb{T}^2 \to \mathcal{S}$, and since $\tilde{f}$ commutes with $\tilde{\tau}$, it induces a map $\tilde{F}\colon \mathcal{S} \to \mathcal{S}$ such that the following diagrams commute:

$$
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{f} & \mathbb{A} \\
{\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\
\mathcal{P} & \xrightarrow{\tilde{f}} & \mathcal{P}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{T}^2 & \xrightarrow{F} & \mathbb{T}^2 \\
{\scriptstyle \tilde{p}}\downarrow & & {\scriptstyle \tilde{p}}\downarrow \\
\mathcal{S} & \xrightarrow{\tilde{F}} & \mathcal{S}
\end{array}
$$

It is clear that F is semi-conjugated to $\tilde{F}$ (by means of $\tilde{p}$). We show that $\tilde{F}$ is transitive. If I, J are nonempty intervals in $\mathcal{P}$, then $U = p^{-1}(I)$ and $V = p^{-1}(J)$ are open subsets of $\mathbb{A}$, and both U and V must contain an element of Γ. If $\tau^{m_1} f^{n_1}\gamma \in U$ and $\tau^{m_2} f^{n_2}\gamma \in V$, then

$$\tau^{m_2-m_1} f^{n_2-n_1}(U) \cap V \neq \emptyset.$$

This implies that $\tilde{\tau}^{m_2-m_1}\tilde{f}^{n_2-n_1}(I) \cap J \neq \emptyset$. Since this holds for any pair of intervals I, J in $\mathcal{P}$, it follows easily that $\tilde{F}$ is transitive.

By the theory of Poincaré, any transitive homeomorphism of S^1 is conjugated to an irrational rotation. Since F is semi-conjugated to $\tilde{F}$, we conclude that F is semi-conjugated to an irrational rotation of S^1. This completes the proof of Theorem C.

Chapter 4

Invariant foliations and minimal diffeomorphisms

In this chapter, we study the diffeomorphisms in the closure of the conjugacy class of the set of rigid translations of $\mathbb{T}^2$. This is motivated by the work of Fathi and Herman [FH77], where they prove the existence of strictly ergodic (i.e. uniquely ergodic and minimal) diffeomorphisms on any manifold M admitting a free C^∞ action Γ of S^1. In their article, they prove that the set of such diffeomorphisms is residual in the space $\overline{\mathcal{O}}^\infty(\Gamma)$, the C^∞-closure of the set of diffeomorphisms C^∞-conjugated to elements of the action.

We denote by $R_{(\lambda_1,\lambda_2)}$ either the translation $(x, y) \mapsto (x + \lambda_1, y + \lambda_2)$, or the diffeomorphism of $\mathbb{T}^2$ lifted by it (with a slight abuse of notation). Let

$$\mathcal{O}(\mathbb{T}^2) = \left\{ hR_\alpha h^{-1} : h \in \mathrm{Diff}^\infty(\mathbb{T}^2),\, \alpha \in \mathbb{T}^2 \right\}.$$

If Γ is the action $t \mapsto R_{(0,t)}$ of S^1 on $\mathbb{T}^2$, it is easy to see that $\overline{\mathcal{O}}^\infty(\Gamma) = \overline{\mathcal{O}}^\infty(\mathbb{T}^2)$, where the closure is taken in the C^∞-topology.

Recall that $\mathrm{Diff}^\infty(\mathbb{T}^2)$ with the C^∞-topology is a Polish space, so that any closed subset of it (in particular, $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$) is a Baire space. A property is said to be generic in a Baire space if it holds in a residual subset (i.e. a dense G_δ) of the space.

A homeomorphism $f\colon X \to X$ of the compact metric space X is *minimal* if the f-orbit of every point is dense in X or, equivalently, if the only compact f-invariant sets are the whole space X and the empty set.

By a C^r *foliation* of codimension 1 of $\mathbb{T}^2$, we mean a partition $\mathcal{F}$ of the space into 1-dimensional, connected topological sub-manifolds (the *leaves*) such that there is a C^r atlas of $\mathbb{T}^2$ formed by charts (ϕ_i, U_i) such that $\phi_i \phi_j^{-1}$ has the form $(u(x), v(x,y))$ on its domain, and if S is a vertical segment in the domain of ϕ_i, then $\phi_i^{-1}(S) \subset \mathcal{F}_{\phi^{-1}(p)}$ for any $p \in S$ (where $\mathcal{F}_x$ denotes the unique leaf of the partition that contains x).

The foliation $\mathcal{F}$ is said to be invariant for a homeomorphism $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ if $f(\mathcal{F}_x) = \mathcal{F}_{f(x)}$ for each $x \in \mathbb{T}^2$. A foliation of $\mathbb{T}^2$ always has a lift to $\mathbb{R}^2$, i.e. a foliation $\mathcal{F}'$ such that $\pi(\mathcal{F}'_z) = \mathcal{F}_{\pi(z)}$ for any $z \in \mathbb{R}^2$. This foliation is invariant for any lift of f to $\mathbb{R}^2$ if $\mathcal{F}$ is invariant for f.

We say that a C^0 foliation is induced by a C^0 line field if its leaves are C^1, and if the tangent spaces to the leaves form a continuous sub-bundle of the tangent space of $\mathbb{T}^2$. Note that this is weaker than saying that the foliation is of class C^1, but stronger than just saying that the leaves are of class C^1.

4.1 Dynamic cocycles

Since the tangent space of $\mathbb{T}^2$ is parallelizable, we identify it with $\mathbb{T}^2 \times \mathbb{R}^2$.

Definition 4.1. Given $f \in \mathrm{Diff}(\mathbb{T}^2)$, we define its associated *dynamic cocycle* by

$$\hat{f}\colon \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1, \qquad (x, v) \mapsto \left(f(x), \frac{Df_x v}{\|Df_x v\|} \right),$$

which is of class C^∞ if f is C^∞.

It is useful to note that $\widehat{fg} = \hat{f}\hat{g}$. This is true because

$$\frac{D(fg)_x v}{\|D(fg)_x v\|} = \frac{Df_{g(x)} Dg_x v}{\left\|Df_{g(x)} Dg_x v\right\|} = \frac{Df_{g(x)} \left(\frac{Dg_x v}{\|Dg_x v\|}\right)}{\left\|Df_{g(x)} \left(\frac{Dg_x v}{\|Dg_x v\|}\right)\right\|}$$

Hence, if $g = hfh^{-1}$, where $h \in \mathrm{Diff}^\infty(\mathbb{T}^2)$, then $\hat{g} = \hat{h}\hat{f}\hat{h}^{-1}$.

We will denote by d_2 the flat metric on $\mathbb{T}^2$, by d_1 the one on S^1, and we endow $\mathbb{T}^2 \times S^1$ with the metric

$$d\left((x,v),(y,w)\right) = \max\left\{d_2(x,y), d_1(v,w)\right\}.$$

4.2 Generic minimality of the dynamic cocycle

To prove Theorem D, we define an appropriate family $\mathcal{U}_n$ of open subsets of $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$ such that every element of $\cap_n \mathcal{U}_n$ satisfies the required property. Then, following the ideas of [FH77], we prove that $R_{(0,p/q)}$ is in the closure of $\mathcal{U}_n$, for each $p/q \in \mathbb{Q}$ and each n. Using this fact, we show that $hR_{(0,p/q)}h^{-1}$ is also in the closure of $\mathcal{U}_n$ for all $h \in \mathrm{Diff}^\infty(\mathbb{T}^2)$. Since such diffeomorphisms are dense in $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$, we conclude that $\mathcal{U}_n$ is dense in $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$ for each n. It then follows that $\cap_n \mathcal{U}_n$ is a residual set with the required properties.

Remark 4.2. With almost no modifications, the proof we present below is valid if we restrict it to the space $\overline{\mathcal{O}}^\infty_m(\mathbb{T}^2)$ of elements of $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$ preserving Lebesgue measure. This is because the conjugations used in the proof are measure preserving.

If (X,ρ) is a compact metric space, we say that a map $f\colon (X,\rho) \to (X,\rho)$ is *ϵ-minimal* if every point has an ϵ-dense orbit, *i.e.* for each $x, y \in X$ there is $n \in \mathbb{Z}$ such that $\rho(f^n(x), y) < \epsilon$.

Remark 4.3. By compactness, if f is ϵ-minimal then there is $N \in \mathbb{N}$, depending only on f, such that $\{f^k(x) : |k| < N\}$ is ϵ-dense.

Let

$$\mathcal{U}_n = \left\{ f \in \overline{\mathcal{O}}^\infty(\mathbb{T}^2) : \hat{f} \text{ is } \frac{1}{n}\text{-minimal} \right\}.$$

Note that by the previous remark, and by continuity of $f \mapsto f^k$ in the C^0-topology, $\mathcal{U}_n$ is C^0-open for each $n \in \mathbb{N}$.

Claim 1. *Let $\alpha = (0, p/q)$. Then for each $N > 0$, there exists $h \in \mathrm{Diff}^\infty(\mathbb{T}^2)$ such that $hR_\alpha = R_\alpha h$, and for each integer k with $0 \le k \le 2\pi N$ there is an open $1/N$-dense set $U_k \subset \mathbb{T}^2$ such that Df_z is a rigid rotation with angle*

k/N for each $z \in U_k$. Moreover, h is $1/N$-close to the identity in the C^0 topology.

Proof. Let us briefly sketch the idea of the construction. We will define $\tilde{h}\colon \mathbb{R}^2 \to \mathbb{R}^2$ with the desired properties, which commutes with R_α (i.e. with $(x, y) \mapsto (x, y + p/q)$) and with the translations T_1, T_2 (so that it lifts a map of $\mathbb{T}^2$); then the map h lifted by $\tilde{h}$ is the map we are looking for. We construct $\tilde{h}$ as follows: first we choose a disjoint family of $1/N$-dense sets S_k, $0 \leq k < 2\pi N$, each consisting of equally spaced points and such that each S_k is invariant by R_α (and by integer translations). Then we define $\tilde{h}$ in a way that it coincides with a rigid rotation by angle k/N in a small disk around each point $z \in S_k$, and with the identity outside a neighborhood of $\cup_k S_k$. Essentially, the action of h near any point of S_k is a "copy" of its action near any other such point, which guarantees that $\tilde{h}$ commutes with R_α. We now proceed to the explicit construction.

Let $\psi\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be defined by

$$\psi(x, y, t) = (x\cos(t) + y\sin(t), x\sin(t) - y\cos(t)),$$

i.e. $\psi(\cdot, \cdot, t)$ is a rotation with angle t around the origin. Let $b\colon \mathbb{R} \to \mathbb{R}$ be a C^∞ bump function such that:

- $b(t) = 1$ if $|t| \leq 1/6$;
- $b(t) = 0$ if $|t| \geq 1/2$;
- $|b(t)| \leq 1$ for all $t \in \mathbb{R}$.

Define $\Phi_t\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\Phi_t(x, y) = \psi(x, y, tb(\sqrt{x^2 + y^2}))$. It is clear that Φ_t is a C^∞ map. Moreover, it is a diffeomorphism: in fact it has an inverse given by

$$(x, y) \mapsto \psi\left(x, y, -tb\left(\sqrt{x^2 + y^2}\right)\right),$$

which is also C^∞. Also note that $D\Phi_t(z)$ is a rigid rotation with angle t when $\|z\| < 1/6$, and the support of Φ_t is contained in the disk of radius $1/2$ centered at the origin.

Now let $M \in \mathbb{Z}$ be large enough so that $1/(qM) < 1/N$, let $K = 7qMN$, and for $0 \le k < 2\pi N$, define

$$S_k = \left\{ \left(\frac{iN+k}{K}, \frac{jN+k}{K} \right) : (i,j) \in \mathbb{Z}^2 \right\}.$$

Note that each S_k is an $1/N$-dense subset of $\mathbb{R}^2$, and $R_\alpha S_k = S_k$ because K is a multiple of qN. Moreover, the sets S_k, $0 \le k < 2\pi 8N$ are pairwise disjoint, and any pair of different points in $\cup_k S_k$ are at least a distance of $1/K$ apart. Define

$$\tilde{h}_{k,z}(x,y) = R_z \left(\frac{1}{2K} \Phi_{k/N} \Big(2KR_{-z}(x,y) \Big) \right),$$

i.e. $\tilde{h}_{k,z}(x,y)$ is a diffeomorphism with support in the disk of radius $1/(2K)$ around z, and which is a rotation with angle k/N in a small disk around z. Note that if $z \in S_k$ and $z' \in S_{k'}$, with $z \neq z'$, then the supports of $\tilde{h}_{k,z}$ and $\tilde{h}_{k',z'}$ are disjoint; thus they commute. Moreover, if $z \in S_k$, then

$$\tilde{h}_{k,z} R_\alpha = R_\alpha \tilde{h}_{k,R_{-\alpha}z}. \tag{4.1}$$

Let

$$\tilde{h}_k = \bigcirc_{z \in S_k} \tilde{h}_{k,z},$$

i.e. the composition of all $\tilde{h}_{k,z}$ with $z \in S_k$. The order does not matter because any pair of such diffeomorphisms commute. From the fact that $R_{-\alpha} S_k = S_k$ and from (4.1), it easily follows that $\tilde{h}_k R_\alpha = R_\alpha \tilde{h}_k$. Also, the supports of $\tilde{h}_i$ and $\tilde{h}_j$ are disjoint if $i \neq j$, so that $\tilde{h}_i \tilde{h}_j = \tilde{h}_j \tilde{h}_i$.

$$\tilde{h} = \bigcirc_{0 \le k < 2\pi N} \tilde{h}_k.$$

Since each $\tilde{h}_k$ commutes with R_α, so does $\tilde{h}$. By a similar argument, $\tilde{h}$ commutes with any integer translation, so that it is a lift of a torus diffeomorphism. If $z \in S_k$, then $\tilde{h}$ coincides with a rotation by k/N around z in a small disk (of radius $1/(12K)$) around z. Thus if $\tilde{U}_k$ is an $1/(12K)$-

neighborhood of S_k we have that $D\tilde{h}_z$ is a rotation with angle k/N around the origin for each $z \in \tilde{U}_k$. Since $\tilde{U}_k$ is $1/N$-dense (because S_k is $1/N$-dense), $U_k = \pi(\tilde{U}_k)$ gives an $1/N$-dense family of open sets with the required properties for the map $h \in \mathrm{Diff}^\infty(\mathbb{T}^2)$ lifted by $\tilde{h}$. The fact that h is $1/N$-close to the identity in the C^0 topology follows from the fact that this is true for each $\tilde{h}_{k,z}$ and their supports are disjoint. This completes the proof of the claim. □

Claim 2. *$R_\alpha \in \overline{\mathcal{U}_n^\infty}$ for every $n \in \mathbb{N}$ and $\alpha = (0, p/q)$.*

Proof. Fix $n \in \mathbb{N}$ and $\alpha = (0, p/q)$ with $p/q \in \mathbb{Q}$ in reduced form, and let h be as in the previous claim with $N = 4n$. If α_k is a sequence of rationally independent pairs (so that R_{α_k} is minimal) such that $\alpha_k \to \alpha$, then

$$hR_{\alpha_k}h^{-1} \xrightarrow[k\to\infty]{C^\infty} hR_\alpha h^{-1} = R_\alpha.$$

Thus, it suffices to show that $hR_{\alpha_k}h^{-1} \in \mathcal{U}_n$ for each k. This is possible due to the minimality of R_{α_k}.

Let $\{U_i : 0 < i < 2\pi N\}$ be the open sets given by the previous claim for h, and fix $(z, v) \in \mathbb{T}^2 \times S^1$. If $(z', v') \in \mathbb{T}^2 \times S^1$, then $\hat{h}^{-1}(z', v') = (h^{-1}(z'), w)$ for some $w \in S^1$. If the angle (measured in $[0, 2\pi)$) between w and v is θ, we can find m such that

$$|\theta - m/N| < 1/N < 1/n,$$

and there is $z_0 \in U_m$ such that $d_2(z_0, h^{-1}(z)) < 1/N$. By the properties of U_m, there is a small neighborhood U of z_0 such that Dh_{z_1} is a rotation with angle m/N for each $z_1 \in U$, and we may assume that $d_2(z_1, z_0) < 1/N$ for each $z_1 \in U$ by taking a smaller neighborhood, if necessary. Since R_{α_k} is minimal, there exists j such that $R^j_{\alpha_k}(h^{-1}(z')) \in U$. Since $DR_{\alpha_k} = Id$, it follows that

$$\hat{R}^j_{\alpha_k}\hat{h}^{-1}(z', v') = (R^j_{\alpha_k}h^{-1}(z'), w).$$

Hence, letting $z_1 = R^j_{\alpha_k} h^{-1}(z')$, we have $z_1 \in U$ and

$$\hat{h}\hat{R}^j_{\alpha_k}\hat{h}^{-1}(z',v') = (h(z_1), Dh_{z_1}w) = \hat{h}(z_1,w).$$

But since $d_2(z_1,z_0) < 1/N$ and $d_2(z_0,h^{-1}(z)) < 1/N$, and since h is $1/N$-close to the identity, it follows that $d(h(z_1),z) < 4/N = 1/n$. We also know that $d_1(Dh_{z_1}w,v) < 1/n$ (because Dh_{z_1} is a rotation by m/N and $|\theta - m/N| < 1/N < 1/n$); thus

$$d\left((\widehat{hR_{\alpha_k}h}^{-1})^j(z',v'),\,(z,v)\right) = d(\hat{h}(z_1,w),\,(z,v)) < 1/n.$$

Since (z,v) and k were arbitrary, this shows that $hR_{\alpha_k}h^{-1} \in \mathcal{U}_n$ for each k. This completes the proof of the claim. ☐

Claim 3. $\overline{\mathcal{O}}^\infty(\mathbb{T}^2) = \overline{\mathcal{U}}^\infty_n$.

Proof. First note that $\{hR_{(0,p/q)}h^{-1} : p/q \in \mathbb{Q},\, h \in \mathrm{Diff}^\infty(\mathbb{T}^2)\}$ is C^∞-dense in $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$. In fact, it suffices to show that every element of $\mathcal{O}(\mathbb{T}^2)$ is the limit of a sequence in that set. If $hR_\lambda h^{-1} \in \mathcal{O}(\mathbb{T}^2)$, we can choose a sequence $\lambda_n \in \mathbb{Q}^2$ such that $\lambda_n \to \lambda$; by Remark 0.26, there is $A_n \in \mathrm{GL}(2,\mathbb{Z})$ and $p_n/q_n \in \mathbb{Q}$ such that $\tilde{A}_n R_{\lambda_n}\tilde{A}_n^{-1} = R_{(0,p_n/q_n)}$, so that

$$(h\tilde{A}_n)R_{(0,p_n/q_n)}(h\tilde{A}_n)^{-1} = hR_{\lambda_n}h^{-1} \xrightarrow[n\to\infty]{C^\infty} hR_\lambda h^{-1}.$$

From this fact, to prove the claim it suffices to show that $hR_{(0,p/q)}h^{-1} \in \overline{\mathcal{U}}^\infty_n$ for each n and $p/q \in \mathbb{Q}$.

Fix $n \in \mathbb{N}$, $p/q \in \mathbb{Q}$, and let $\delta > 0$ be such that $d(\hat{x},\hat{y}) < \delta$ implies $d(\hat{h}^{-1}(\hat{x}),\hat{h}^{-1}(\hat{y})) < 1/n$. Let $k > \delta^{-1}$. Given $f \in \mathcal{U}_k$, and $\hat{x},\hat{y} \in \mathbb{T}^2 \times S^1$, we know from the $1/k$-minimality that there is $i \in \mathbb{Z}$ such that

$$d\left(\hat{f}^i(\hat{h}^{-1}(\hat{x})),\hat{h}^{-1}(\hat{y})\right) < 1/k < \delta.$$

Hence,

$$d\left((\hat{h}\hat{f}\hat{h}^{-1})^i(\hat{x}),\hat{y}\right) = d\left(\hat{h}\hat{f}^i\hat{h}^{-1}(\hat{x}),\hat{h}(\hat{h}^{-1}(\hat{y}))\right) < \frac{1}{n}.$$

Since $\hat{x}$ and $\hat{y}$ are arbitrary, it follows that $\hat{h}\hat{f}\hat{h}^{-1}$ is $1/n$-minimal, i.e.

$hfh^{-1} \in \mathcal{U}_n$. Since this is true for any $f \in \mathcal{U}_k$, and by the previous claim we can choose a sequence $f_i \in \mathcal{U}_k$ such that $f_i \to R_{(0,p/q)}$, we have a sequence hf_ih^{-1} of elements of $\mathcal{U}_n$ which converges to $hR_{(0,p/q)}h^{-1}$. Thus $hR_{(0,p/q)}h^{-1} \in \overline{\mathcal{U}}_n^{\infty}$.

This completes the proof of the claim. □

We proved that each $\mathcal{U}_n$ is C^0-open (thus C^∞-open) and C^∞-dense in $\overline{\mathcal{O}}^{\infty}(\mathbb{T}^2)$. Thus,

$$\mathcal{R} = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$$

is a residual subset of $\overline{\mathcal{O}}^{\infty}(\mathbb{T}^2)$. The fact $\hat{f}$ is minimal for each $f \in \mathcal{R}$ is obvious from our definition of the sets $\mathcal{U}_n$. This proves Theorem D.

4.3 Generic non-existence of invariant foliations

The following is an easy consequence of Theorem D:

Corollary 4.4. *If $\mathcal{R}$ is the residual set from Theorem D, and $f \in \mathcal{R}$, there is no Df-invariant continuous sub-bundle of $T(\mathbb{T}^2)$ of codimension 1. In particular, f has no invariant foliation induced by a C^0 line field.*

Proof. If $z \mapsto E_z$ is a continuous map assigning to each $z \in \mathbb{T}^2$ a 1-dimensional subspace of $\mathbb{R}^2$, then whenever $f^n(z)$ is close to z, it must be the case that $Df^n(z)E_z = E_{f^n(z)}$ is close to E_z. But if $v \in E_z$ is a unit vector, and $w \notin E_z$ is another, by minimality of $\hat{f}$ we can find a sequence $n_k \to \infty$ such that $\hat{f}^{n_k}(z,v) \to (z,w)$. This clearly implies that $E_{f^{n_k}(z)} \to \mathbb{R}w \neq E_z$, which is a contradiction. □

4.4 Generic non-existence of always-free curves

Combining Theorem C and a result of Herman, we have:

Corollary 4.5. *There is a residual set in $\overline{\mathcal{O}}^{\infty}(\mathbb{T}^2)$ such that given any f in the set and any simple closed curve γ in $\mathbb{T}^2$, there is $n \neq 0$ such that*

$f^n(\gamma) \cap \gamma \neq \emptyset$. In particular, f cannot have an invariant C^0 foliation with a compact leaf.

Proof. By [Her92, §9.8], there exists a residual set $G_{P_8} \subset \overline{\mathcal{O}}^\infty(\mathbb{T}^2)$ such that if $f \in G_{P_8}$, then f is not semi-conjugated (not even metrically semi-conjugated) to a translation on a compact Abelian group. This follows from the fact that all elements of G_{P_8} are minimal and uniquely ergodic, and they are weak-mixing with respect to its unique invariant measure.

Let $f \in G_{P_8}$ and let γ be a simple closed curve. Suppose γ is always free. Then, by the minimality of f, the orbit of γ is dense. Thus, Theorem C implies that f is semi-conjugated to an irrational rotation of S^1. But this contradicts the aforementioned property of G_{P_8}. Hence there exists $n \neq 0$ such that $f^n(\gamma) \cap \gamma \neq \emptyset$.

If f has an invariant foliation with a compact leaf γ, then γ is a simple closed curve, so that by the previous argument there exists $n \neq 0$ such that $f^n(\gamma) \cap \gamma \neq \emptyset$. But since γ is the leaf of an invariant foliation, this implies that $f^n(\gamma) = \gamma$, so that γ is a periodic curve for f. In particular, the orbit of γ is a proper compact f-invariant subset of $\mathbb{T}^2$. This contradicts the minimality of f; hence there is no such foliation. □

4.5 C^0 foliations with non-differentiable leaves

We will now show how to construct diffeomorphisms in $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$ such that any invariant foliation has a leaf which is not C^1.

For simplicity, we will work in the spaces $\mathcal{H}^0$ of all homeomorphisms $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ which are lifts of torus homeomorphisms in the isotopy class of the identity, and $\mathcal{H}^\infty$ of all C^∞ diffeomorphisms which are lifts of torus diffeomorphisms in the isotopy class of the identity. We endow $\mathcal{H}^0$ with the C^0-metric

$$d_0(f, g) = \max\left\{\sup_{z\in\mathbb{R}^2} \|f(z) - g(z)\|, \sup_{z\in\mathbb{R}^2} \left\|f^{-1}(z) - g^{-1}(z)\right\|\right\}$$

and $\mathcal{H}^\infty$ with the C^∞-metric

$$d_\infty(f,g) = \min\left\{1, \sup_{z\in\mathbb{R}^2,\, k\geq 0} \left\| D_z^k(f-g) \right\| \right\}.$$

These metrics make both spaces complete, and if a sequence $f_n \in \mathcal{H}^0$ (resp. $\mathcal{H}^\infty$) converges to F, then the corresponding sequence F_n of maps lifted by f_n converges in the C^0 (resp. C^∞) topology to F (the map lifted by F).

Theorem 4.6. *There is a dense subset $\mathcal{S}$ of $\overline{\mathcal{O}}^\infty(\mathbb{T}^2)$ such that every $f \in \mathcal{S}$ is conjugated to a minimal translation but has no invariant foliation with C^1 leaves.*

To prove this theorem, it suffices to show that any translation R_γ is C^∞-approximated by elements of $\mathcal{S}$. In fact, since the property of having no invariant foliation with C^1 leaves is invariant by C^∞ conjugation, this automatically implies that $\overline{\mathcal{S}}^\infty = \overline{\mathcal{O}}^\infty(\mathbb{T}^2)$.

Let $\Phi_{\pi/2}$ be as in the proof of Theorem D (Claim 1). That is, $\Phi_{\pi/2}$ is a rotation by $\pi/2$ in the disk of radius 1/6 centered at the origin, and the identity outside the disk of radius 1/2. Note that $d_0(\Phi, Id) \leq 1$.

Now let $\phi_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by requiring that $\phi_0 = \phi$ on $[-1/2, 1/2]^2$ and that $\phi_0(x+m, y+n) = \phi_0(x,y) + (m,n)$ for all $(m,n) \in \mathbb{Z}^2$, $(x,y) \in \mathbb{R}^2$. That is the same as

$$\phi_0 = \bigcirc_{(m,n)\in\mathbb{Z}^2} \Phi_{\pi/2}(x-m, y-n) + (m,n).$$

Finally, define ϕ_n by $\phi_n(x,y) = 3^{-n}\phi_0(3^n x, 3^n y)$. The following properties are easily verified:

- $\phi_n \in \mathcal{H}^\infty$ and it preserves Lebesgue measure;
- if $p = 3^{-n}(i,j)$ for some $(i,j) \in \mathbb{Z}^2$, ϕ_n is a rotation by $\pi/2$ around p on $B(p, 3^{-(n+1)}/2)$ and the identity on $\partial B(p, 3^{-n}/2)$;
- ϕ_n commutes with R_λ whenever $\lambda = 3^{-n}(i,j)$ for some $(i,j) \in \mathbb{Z}^2$;
- $d_0(\phi_n, id) \leq 3^{-n}$.

Fix $\gamma \in \mathbb{R}^2$ and $\delta > 0$. We now show how to find $f \in \mathcal{H}^\infty$ which is the lift of a map with the required properties, and such that $d_\infty(f, R_\gamma) < \delta$.

Let $\alpha_0 = (p_0/3^{n_0}, r_0/3^{m_0})$ be such that $|\gamma - \alpha_0| < \delta/2$, where m_0, n_0, p_0, r_0 are nonzero integers with $m_0 > n_0 > 0$, and let $h_0 = H_0 = id$. We define recursively sequences $\{p_i/3^{n_i}\}_{n\in\mathbb{N}}$ and $\{r_i/3^{m_i}\}_{i\in\mathbb{N}}$ of rational numbers ($p_i \neq 0 \pmod 3$, $r_i \neq 0 \pmod 3$) and $\{u_i\}_{i\in\mathbb{N}}$ of natural numbers such that for $i \geq 1$, if $h_i = \phi_{u_i}$, $\alpha_i = (p_i/3^{n_i}, r_i/3^{m_i})$, $H_i = h_0 \cdots h_i$, and $f_i = H_i R_{\alpha_i} H_i^{-1}$, then

1. $R_{\alpha_{i-1}} h_i = h_i R_{\alpha_{i-1}}$;
2. $\alpha_i = \alpha_{i-1} + (3^{-n_i}, 3^{-m_i})$;
3. $d_0(H_{i-1}, H_i) < 1/2^i$;
4. $d_\infty(f_{i-1}, f_i) < \delta/2^{i+1}$;
5. $m_i - 2i > n_i > m_{i-1} + i$;
6. $u_i \geq u_{i-1} + 3$.

Assume that we have constructed such sequences up to step $i = k$. Since $d_0(\phi_n, id) \to 0$ as $n \to \infty$, there is N such that $d_0(H_k\phi_n, H_k) < 2^{-(k+1)}$ for all $n \geq N$. Letting $u_{k+1} = \max\{m_k, u_k + 3, N\}$, properties (1), (3) and (6) hold.

Now note that for any $\beta \in \mathbb{R}^2$,

$$H_{k+1} R_{\alpha_k+\beta} H_{k+1}^{-1} = H_k h_{k+1} R_\beta R_{\alpha_k} h_{k+1}^{-1} H_k^{-1} = H_k (h_{k+1} R_\beta h_{k+1}^{-1}) R_{\alpha_k} H_k^{-1}$$

so that there is $\epsilon > 0$ such that if $|\beta| < \epsilon$, then

$$d_\infty(H_{k+1} R_{\alpha_k+\beta} H_{k+1}^{-1}, f_k) < \delta/2^{k+1}.$$

Let M be such that $3^{-M} < \epsilon$, and define

$$n_{k+1} = \max\{m_k + (k+1), M, u_k\} + 1,$$

$m_{k+1} = n_{k+1} + i$, and α_{k+1} as in (2). It is easy to see that the remaining properties hold, and the recursion step is complete.

Claim 1. *There exist $f \in \mathcal{H}^\infty$, $h \in \mathcal{H}^0$ and $\alpha \in \mathbb{R}^2$ such that $f_i \to f$ in the C^∞ topology, $H_i \to h$ in the C^0 topology, and $\alpha_i \to \alpha$. Moreover, R_α is minimal, $f = hR_\alpha h^{-1}$, and $d_\infty(f, R_\gamma) < \delta$.*

Proof. The existence of f, h and α follows by completeness of the respective spaces, since the corresponding sequences are Cauchy by construction. By continuity of composition in the C^0 topology, $f = hR_\alpha h^{-1}$. Since $d_\infty(f_0, f_i) \leq \delta \sum_{i=1}^{\infty} 2^{-(i+1)} = \delta/2$ and $d_\infty(f_0, R_\gamma) < \delta/2$, we have $d_\infty(R_\gamma, f) < \delta$. Finally, R_α is minimal because, by the conditions we required on the sequences m_i and n_i, Lemma 4.7 implies that the two coordinates of α are rationally independent. □

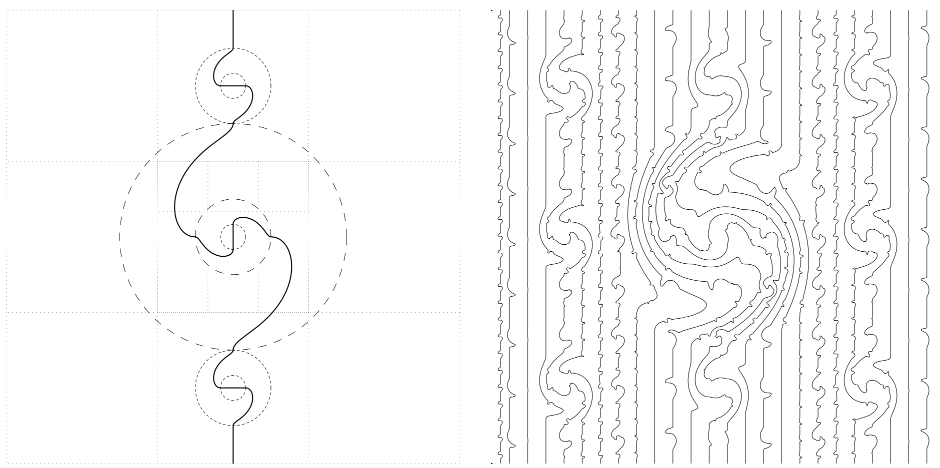

Image of a vertical line by H_1 Image of the vertical foliation by h

Figure 4.1: The conjugation h

Claim 2. *Let $F \in \overline{\mathcal{O}}^\infty(\mathbb{T}^2)$ be the map lifted by f. Then, any F-invariant codimension 1 foliation of $\mathbb{T}^2$ has leaves which are not C^1.*

Proof. We first note the simple fact that any R_α-invariant codimension 1 foliation of $\mathbb{T}^2$ lifts to a foliation of the plane by straight lines. Thus, in order to prove the claim, it suffices to show that there is a point p such that any straight line containing p is mapped by h to a curve which is not differentiable. We will see that this is the case for $p = (0, 0)$.

Since the origin is fixed by ϕ_n for all n, it is fixed by h as well. Let $r_k = 3^{-(u_i+1)}/2$ and $B_i = B(0, r_k)$, and let $L \subset \mathbb{R}^2$ be a straight line containing the origin. We will find two sequences of points in L, $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$, both converging to the origin, such that $\angle(h(x_k), h(y_k))$ is uniformly away from 0 or π. This will imply that $h(L)$ does not have a well-defined tangent direction at the origin. Let L' be one of the components of $L - \{(0,0)\}$.

Note that $H_k^{-1}h = \lim_{i\to\infty} H_k^{-1}H_i = \lim_{i\to\infty} h_{k+1}h_{k+2}\cdots h_i$, so that

$$d_0(H_k^{-1}h, id) = \lim_{i\to\infty} d_0(H_k^{-1}H_i, id) \le \sum_{j=k+1}^{\infty} d_0(h_j, id) \le \sum_{j=k+1}^{\infty} 3^{-u_j}$$
$$\le \frac{3^{-(u_{k+1}-1)}}{2} \le \frac{3^{-(u_k+2)}}{2} = r_k/3.$$

Thus, if $z_k = L' \cap \partial B_k$, we have that $\left\|H_k^{-1}h(z_k) - z_k\right\| \le r_k/3$. This implies that $\angle(H_k^{-1}h(z_k), z_k) \le 1/3$. For $1 \le i \le k$, h_i is a rotation by $\pi/2$ on $B_i \supset B_k$, so that H_k is a rotation by $k\pi/2$ on B_k; thus when $k = 1$ (mod 4), we have

$$|\angle(h(z_k), z_k) - \pi/2| < 1/3.$$

By the same argument, if $k = 0$ (mod 4),

$$|\angle(h(z_k), z_k)| \le 1/3.$$

Letting $x_i = z_{4i+1}$ and $y_i = z_{4i}$, observing that $|\angle(x_i, y_i)| = 0$, we have two sequences of points of L converging to the origin, such that

$$|\angle(h(x_i), h(y_i)) - \pi/2| \le 2/3 < \pi/2.$$

This means that the above angle is uniformly away from 0 or π. If $h(L)$ were differentiable, we would have $|\angle(h(x_i), h(x_i))| \to 0$ (or π). Thus, we just showed that $h(L)$ is not differentiable. This completes the proof of the theorem. □

Lemma 4.7. *Let $\{m_i\}_{i\in\mathbb{N}}$ and $\{n_i\}_{i\in\mathbb{N}}$ be sequences of natural numbers such that $n_i < m_i < n_{i+1}$ for all i and*

$$\lim_{i\to\infty} m_i - n_i = \lim_{i\to\infty} n_{i+1} - m_i = \infty.$$

Then, for any integer $q > 1$, the numbers $\alpha = \sum_{i=0}^{\infty} q^{-n_i}$ and $\beta = \sum_{i=0}^{\infty} q^{-m_i}$ are rationally independent.

Proof. Suppose $x\alpha + y\beta = z$ for integers x, y, z. Let $\alpha_k = \sum_{i=0}^{k} q^{-n_i}$ and $\beta_k = \sum_{i=0}^{k} q^{-m_i}$. Note that $\alpha_k = p_k/q^{n_k}$ for some p_k relatively prime to q and similarly $\beta_k = r_k/q^{m_k}$. We may write $y = q^b y'/a$ for some natural numbers b, a and some integer y' coprime with q. Then,

$$aq^{m_k-b}(x\alpha_k + y\beta_k - z) = q^{m_k-n_k-b}(axp_k - aq^{n_k}z) + y'r_k.$$

If k is large enough we have $m_k - n_k > b$, and since $y'r_k$ is coprime with q, it follows from the above that $aq^{m_k-b}(x\alpha_k + y\beta_k - z)$ is an integer coprime with q, so that

$$a|x\alpha_k + y\beta_k - z| \geq q^{-m_k+b}.$$

On the other hand,

$$\begin{aligned}|x\alpha_k + y\beta_k - z| &= |x(\alpha_k - \alpha) + y(\beta_k - \beta)| \\ &\leq |x| \sum_{i=k+1}^{\infty} q^{-n_i} + |y| \sum_{i=k+1}^{\infty} q^{-m_i} \\ &\leq |x| \frac{q^{-n_{k+1}+1}}{q-1} + |y| \frac{q^{-m_{k+1}+1}}{q-1} \leq 2(|x| + |y|)q^{-n_{k+1}}.\end{aligned}$$

Thus if k is large enough we have

$$2a(|x| + |y|)q^{-n_{k+1}} \geq a|x\alpha_k + y\beta_k - z| \geq q^{-m_k+b},$$

so that

$$q^{n_{k+1}-m_k} \leq 2a(|x| + |y|)q^{-b}$$

which is a contradiction. □

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