Abel maps for reducible curves

Juliana Coelho

March 22, 2007

Agradecimentos

Agradeço à minha família pelo apoio durante o desenvolvimento deste trabalho. Ao professor Eduardo Esteves, pela paciência, boa vontade, entusiasmo e bom humor. Aos amigos da 327 e agregados por tudo mesmo. Ao Cayley, por estar comigo pelos longos meses de escrita desta tese. Aos membros da banca, professores Carolina Araújo, Karl Otto Stohr, Marco Paccini e Steven Kleiman. Por último ao Luís Carlos pela enorme paciência.

Contents

Chapter 0

Introduction

Let X be a smooth projective curve of genus g. Associated to X there is the Jacobian variety J^0_X parameterizing degree-zero divisors on X modulo rational equivalence, or equivalently, degree-zero line bundles on X modulo isomorphism. The Jacobian variety is smooth, projective and has dimension g. Moreover, J^0_X has a group structure given by addition of divisors (or tensor product of line bundles), so J^0_X is an algebraic group. To put it short, J^0_X is an Abelian variety.

For each fixed point $P \in X$, the Abel map

$$A_P : X \longrightarrow J^0_X$$

relates the curve X to its Jacobian J^0_X by taking a point Q of X to the class of the divisor $P - Q$ in J^0_X. If X is a genus 1 curve, this map is an isomorphism for every P, thus giving a group structure on X with identity element P. For higher genus, A_P is a closed embeding, so, in any case, we can view the curve X as a closed subvariety of the algebraic group J^0_X. This way, we can perform (extrinsic) group operations on X, by identifying X with its image in J^0_X (although in general the class of the divisor $(P - Q_1) + (P - Q_2) = 2P - Q_1 - Q_2$ is not in the image of A_P). This gives maps $X^d \to J^0_X$, where X^d is just the d-fold product of X, that is, the product of d copies of X. Actually, the map $X^d \to J^0_X$ factors through $X^{(d)}$, the d-fold symmetric product of X, that is, the quotient of X^d obtained identifying the d-tuples $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ if one is a permutation of the other. We obtain an induced map

$$A^{(d)}_P : X^{(d)} \longrightarrow J^0_X$$

called the d-th Abel map. The map A_P is then the first Abel map. If we denote by W_d the image of $A_P^{(d)}$ for each d, we have $W_1 \cong X$, because $A_P^{(1)} = A_P$ is a closed embedding. Furthermore

$$X \cong W_1 \subset W_2 \subset \ldots \subset W_d \subset \ldots \subset J_X^0.$$

It can be shown that each W_d has dimension d, and that $W_g = J_X^0$.

A lot of the properties of the curve X can be recovered from the varieties W_d and the morphisms $A_P^{(d)}$. For example, it can be shown that $A_P^{(2)}$ is a closed embedding if and only if X is not hyperelliptic. In fact, the whole curve can be reconstructed from its Jacobian J_X^0 together with its theta divisor (a translate of W_{g-1}). This is Torelli's theorem that, more precisely, states that if X and X' are smooth curves whose Jacobians are isomorphic, with an isomorphism identifying the theta divisors, then X and X' are isomorphic as well.

In addition, consider the pullback map A_P^* of line bundles on J_X^0 to line bundles on X. Since J_X^0 is a smooth projective variety, it has a Picard scheme parameterizing line bundles on J_X^0 modulo isomorphism. If we denote by $\mathrm{Pic}^0(J_X^0)$ the connected component of the Picard scheme containing the identity, the pullback map A_P^* takes an element of $\mathrm{Pic}^0(J_X^0)$ to an element of J_X^0. It is a classical result, known as the autoduality of the Jacobian, that the map

$$A_P^* : \mathrm{Pic}^0(J_X^0) \xrightarrow{\sim} J_X^0$$

is an isomorphism.

When we consider smooth curves in families, we feel the need to consider singular curves as well. So we might try to extend the above constructions and results for a larger class of curves. Thus, let X be a singular curve. Then the map A_P is not defined on the singular points of X, since these are not Cartier divisors. However, we may view the Abel map of a curve X as taking a point of X to its ideal sheaf (possibly twisted by some line bundle of X). So, in order to define an Abel map on the whole X, we must enlarge the target space of A_P to include the ideal sheaves of singular points of X. In other words, we need a natural compactification of J_X^0.

The problem of compactifying the relative Jacobian of a curve X, or even a family X/S of curves was first considered by Igusa in [I56] for pencils of integral curves with smooth general members and nodal special ones. Later on, Mayer and Mumford suggested in [MM64] an intrinsic characterization of Igusa's compactification, as a space parameterizing torsion-free rank-1 sheaves. (Roughly speaking, a torsion-free sheaf of rank 1 on a curve is the product of an ideal sheaf with an invertible sheaf.) Following that suggestion, D'Souza obtained in his 1974 thesis [D79] a compactification of the relative Jacobian $J_{X/S}$ of a family of irreducible curves X/S with nodes and cusps as singularities, under somewhat restrictive hypothesis.

In 1976 a good solution was found by Altman and Kleiman for a flat, locally projective, finitely presented family X/S of integral curves. They introduced in [AK76] a compactification $\overline{J}_{X/S}$ for the relative Jacobian $J_{X/S}$ as the moduli space parameterizing torsion-free rank-1 sheaves. They showed that the Abel map taking values in $\overline{J}_{X/S}$ is well defined, and is a closed embedding if the curves of the family have arithmetic genus greater than 0 (see [AK80, Theorem 8.8, p. 108]). Later, Esteves, Gagné and Kleiman showed in [EGK02, Theorem 2.1, p. 595] an autoduality theorem for the compactified Jacobian. More precisely, they showed that, if X is a projective curve with double points at worst, then the pullback map $A_P^* : \mathrm{Pic}^0(\overline{J}_X^0) \to J_X^0$ is an isomorphism, where $\mathrm{Pic}^0(\overline{J}_X^0)$ is the connected component of the Picard scheme of $\overline{J}_X^0$ containing the identity. (Actually, they worked with families as well.)

The problem of finding a good relative compactification for the relative Jacobian of a family of reducible curves is a more difficult one. This problem was considered by Oda and Seshadri [OS79] in 1979 for a single reduced (possibly reducible) nodal curve. Three years later, Seshadri treated the case of a general reduced curve in [S82]. As for families, in 1994 Caporaso constructed in [Cap94] a compactification of the relative Jacobian over the moduli space of Deligne–Mumford stable curves, considering invertible sheaves on curves derived from stable ones. And one year later, Pandharipande [P96] produced the same compactification as the space parameterizing torsion-free rank-1 sheaves. (He also worked with higher-rank bundles.) Both of these relative compactifications were constructed using Geometric Invariant Theory. Also, the compactifications in [Cap94] and [P96] are not fine moduli spaces, and thus do not carry a Poincaré sheaf. Furthermore, Caporaso and Pandharipande constructed their compactifications using the same setup and method that Gieseker [G] used construct the moduli space of stable curves, and hence, in principle, their method could not be extended to arbitrary families.

At last, Esteves considered in [E01] the algebraic space $\overline{J}_{X/S}$ parametrizing torsion-free rank-1 simple sheaves on the fibers of X/S. (A sheaf is simple if its automorphisms are homoteties.) This space was essentially introduced by Altman and Kleiman in [AK80]. Esteves showed that $\overline{J}_{X/S}$ is universally closed over S, and hence can be regarded as a compactification of the relative Jacobian of X/S. Esteves' compactification is a fine moduli space, and hence does admit a Poincaré sheaf after an étale base change. However, unlike Caporaso's and Pandharipande's compactifications, Esteves' is not a scheme, but only an algebraic space. (Although it does become a scheme after an étale base change.) In addition, $\overline{J}_{X/S}$ is too big, in the sense that it is not separated or of finite type over S. Thus Esteves considered certain small subspaces of $\overline{J}_{X/S}$ as well. He showed that, if the family admits a section, then we may consider open subspaces of $\overline{J}_{X/S}$ which are proper over the base. (Esteves' compactification will be better explained below.)

It is natural to consider for a family of reducible curves Abel maps taking values in the compactifications mentioned above. Esteves and Caporaso considered in [CE06] Abel maps for a pencil of stable curves taking values in Caporaso's compactification. They showed that, if the curves in the pencil are general enough (1-general), then the Abel map is well defined, and, if in addition the curves do not have disconnecting lines, the map is an injection [CE06, Theorem 5.5, p. 25 and Theorem 5.9, p. 27]. (A disconnecting line in a curve X is a rational component L of X such that the number of connected components of the complement $L^c := \overline{X - L}$ is equal to the number of points in $L \cap L^c$.)

In this work we consider Abel maps into Esteves' compactification [E01]. In particular, we do not need to restrict ourselves to stable curves. Before we present our results, we briefly explain this compactification. (For a more detailed account, see Section 1.3.) First, we need a polarization. A polarization on a curve X, in the sense of [E01], is simply a vector bundle E whose rank divides its degree. Esteves uses polarizations to bound the multidegrees or Euler characteristics of sheaves. More precisely, let I be a torsion-free rank-1 simple sheaf on X. Then we consider the Euler characteristic of the product $G \otimes E$, for each quotient G of I. As usual, I is stable with respect to E if $\chi(E \otimes G) > 0$, and is semi-stable if $\chi(E \otimes G) \geq 0$, for every proper quotient G of I. But Esteves introduced a new notion, that of quasi-stability, which is intermediate between the notions of stability and semi-stablity. This notion depends on the choice of a nonsingular point $P \in X$. A torsion-free rank-1 simple sheaf I is P-quasi-stable with respect to E if I is semi-stable, and $\chi(E \otimes G) > 0$ whenever P is contained in the support of a proper quotient G of I.

Now, fix a polarization E on a curve X. Consider a family of torsion-free rank-1 simple sheaves on X over a discrete valuation ring. If these sheaves are semi-stable with respect to E, then we can always find a limit that is also semi-stable, but this limit doesn't need to be unique. On the other hand, if the sheaves are stable, then the family does not necessarily have a stable limit, but if it does, then the limit is unique. Now, if the sheaves are P-quasi-stable, where P is a nonsingular point of X, then we can always find a P-quasi-stable limit, and this limit is unique. In other words, the locus $\overline{J}_E^P$ of sheaves that are P-quasi-stable with respect to the polarization E is complete, and the locus $\overline{J}_E^s$ of sheaves that are stable with respect to E is separated [E01, Theorem A, p. 3047]. These are the open subspaces of $\overline{J}_X$ we work with.

In Chapter 3 we work with a projective Gorenstein curve X of (arithmetic) genus g without disconnecting nodes, that is, nodes whose removal disconnects the curve. We then consider the

Abel map

$$\begin{aligned} A: X &\longrightarrow \overline{J}_X \\ Q &\mapsto \mathfrak{m}_Q \end{aligned}$$

taking each point Q of the curve to its ideal sheaf $\mathfrak{m}_Q$. The map A is well-defined, that is, $\mathfrak{m}_Q$ is simple for each $Q \in X$, because X has no disconnecting nodes. Now, since X is Gorenstein, its dualizing sheaf ω_X is a line bundle. We then give a polarization E that is a direct sum of copies of ω_X and of the structure sheaf $\mathcal{O}_X$, such that the map A has image in $\overline{J}_E^s$. Moreover, we show that A is a closed embedding in $\overline{J}_E^s$ if $g \geq 2$, and X satisfy an extra condition which holds, for instance, if the curve is stable; see Corollary 3.2.2.

Furthermore, for each nonsingular point $P \in X$, we consider also the Abel map

$$\begin{aligned} A_P: X &\longrightarrow \overline{J}_X \\ Q &\mapsto \mathfrak{m}_Q(P) \end{aligned}$$

taking each point Q of the curve to its ideal sheaf $\mathfrak{m}_Q$ twisted by P. Like A, the map A_P is well-defined because X has no disconnecting nodes. Then we show that the map A_P has image in $\overline{J}_E^P$, where the polarization E is simply the direct sum $\omega_X \oplus \mathcal{O}_X$. Also, if $g \geq 1$, we show that A_P is a closed embedding in $\overline{J}_E^P$; see Corollary 3.2.3.

To prove that A and A_P are closed embeddings we begin by noting that these maps are proper, because X is projective, and $\overline{J}_E^s$ and $\overline{J}_E^P$ are separated. We then show that the fibers of A and A_P are projective spaces, so we have only to see that these fibers are either empty or consist of a single point. This is not hard to show, though we use here again that X has no disconnecting nodes.

The results mentioned above hold more generally for Abel maps of families of curves without disconnecting nodes.

Now, suppose X has a disconnecting node N. Then the ideal sheaf of N is not simple (see Example 2.1.2), and hence the Abel maps A and A_P, having values in Esteves' compactification, are not well-defined. Nevertheless, for each smooth point $P \in X$, we manage to define in Chapter 4 an Abel map $\tilde{A}_P$ for X: for each $Q \in X$ that is not a disconnecting node $\tilde{A}_P(Q)$ is equal to $\mathfrak{m}_Q(P)$ twisted by a line bundle M. The line bundle M is constructed following [CE06], in a way that $\tilde{A}_P(Q)$ is P-quasi-stable with respect to the polarization $E = \omega_X \oplus \mathcal{O}_X$. If Q is a disconnecting node, we define $\tilde{A}_P(Q)$ to be a suitable P-quasi-stable invertible sheaf; see (4.2.1). This gives a (set-theoretical) map from X to $\overline{J}_E^P$. Then we show, by induction on the number of disconnecting nodes of X, that $\tilde{A}_P$ is indeed a morphism of schemes.

Furthermore, if the curve X has positive arithmetic genus, and no disconnecting lines, we show that the map $\tilde{A}_P$ is a closed embedding; see Theorem 4.2.3. We remark that, as in [CE06], the hypothesis that X does not have disconnecting lines is necessary for the injectivity of $\tilde{A}_P$. Indeed, in Subsection 4.2.6 we analyze the image of the disconnecting lines of X under $\tilde{A}_P$ and show that $\tilde{A}_P$ contracts every tree of disconnecting lines to a point. However, these are the only subcurves of X where $\tilde{A}_P$ is not injective. Roughly speaking, the fibers of the surjection $X \to \tilde{A}_P(X)$ are either points or maximal trees of disconnecting lines. The curve $\tilde{A}_P(X)$ seems to be a good substitute for X. In fact, for instance, $\tilde{A}_P(X)$ has the same arithmetic genus as X; see Theorem 4.2.10.

As we said, in Chapters 3 and 4 we give a complete description of the first Abel maps of a reducible Gorenstein curve. A natural continuation is to understand the d-th Abel maps $A_P^{(d)}$, induced from A_P by taking a collection of d points $\{Q_1, \dots, Q_d\}$ to the product of their ideal sheaves twisted by P, that is, $\mathfrak{m}_{Q_1}(P) \otimes \dots \otimes \mathfrak{m}_{Q_d}(P)$. Like the first Abel map, the maps $A_P^{(d)}$ are not well-defined if the curve X has disconnecting nodes. Actually, even when the curve has no disconnecting nodes, the d-th Abel maps are not necessarily well-defined. In fact, $A_P^{(d)}$ is not defined on a collection of d points $\{Q_1, \dots, Q_d\}$ which is the intersection of two complementary subcurves of X; see Subsection 2.2.1. Nevertheless, we might, in principle, expect to find a map from $X^{(d)}$ to the compactified Jacobian of X, by imitating the construction of $\tilde{A}_P$.

Now, the line bundles M we used to modify A_P can be shown to arise naturally from any family of smooth curves degenerating to X; see Remark 4.2.12. More precisely, let $\mathcal{X}/S$ be a smoothing of X, that is, a local pencil of curves whose generic member is smooth and whose special member is X. Assume that $\mathcal{X}$ is regular. Then the line bundles M used to define $\tilde{A}_P$ are the restrictions to X of line bundles on $\mathcal{X}$. (This is actually the approach used in [CE06].) So, to try to define the d-th Abel maps, we may first consider a smoothing $\mathcal{X}/S$ of X, and consider the d-th Abel maps of the family $\mathcal{X}/S$. These are not globally defined either. In addition, the strategy of first defining the map set-theoretically and then showing it is a morphism of schemes does not, in general, work for higher d. But on $\mathcal{X}^d$, or rather, on a resolution of it, we might expect to find the line bundles we need to modify the d-th Abel map.

Here we begin the study of the second Abel map $A_P^{(2)}$. Actually, we work with X^2 instead of $X^{(2)}$. As a first approach to the problem, we prefered to avoid the combinatorics of a general reducible curve, and focused rather on the geometry of the blowups needed to produce the exact line bunldes we wanted. So we consider in Chapter 5 a curve X having only two (smooth) irreducible components meeting at two nodes, the simplest interesting curve for which the second Abel map is not well-defined. (For any d, the d-th Abel map of a two-component one-node curve can be treated in a way similar to what is done in Chapter 4. Actually, we believe that the d-th Abel map of any tree-like curve can be thus defined.) Abel maps for two-component stable curves were studied by Caporaso and Esteves in [CE06], though they obtained only maps defined away from the singluarities.

Not only is the two-component two-node curve X the simplest case that is not too simple, but also this seems to be the key case. Indeed, for any nodal curve X', locally at a pair of points of X' where the second Abel map is not defined, the product $X' \times X'$ is isomorphic to $X \times X$. We thus feel that, once we have a thorough treatment of the second Abel map of X, we have only to tackle the combinatorics of the singular curves to solve the general case. Therefore, in Chapter 5, we completely describe the (six) blowups and the modifications needed to define a second Abel map for X. Not only we do manage to define a map $\tilde{A}_P^{(2)}$ from a blowup of the product $X \times X$ to the compactified Jacobian of X, but also we show that this map factors through $\overline{J}_E^P$, where $E = \omega_X \oplus \mathcal{O}_X$.

We point out that the polarization E used here is the same used in Chapters 3 and 4 for the first Abel map A_P. In other words, the image of $\tilde{A}_P^{(2)}$ lies in the same complete space as the image of A_P. Moreover, we expect the same to happen for any nodal curve X'. We could hope to, with a similar procedure, solve the problem of defining the d-th Abel map for a nodal curve, thus finding a map $\tilde{A}_P^{(d)}$ whose image lies in the same complete subspace of the compactified Jacobian of the curve, in our case $\overline{J}_E^P$. However, the technical difficulties seem huge.

This work is organized as follows. In Chapters 1 and 2 we recall some facts and definitions concerning the compactified Jacobian and the Abel maps. In Chapter 3 we study the first Abel map of a family of curves without disconnecting nodes, and in Chapter 4 we focus on curves having disconnecting nodes. At last, in Chapter 5, we treat the second Abel map of a two-component two-node curve.

Chapter 1

The compactified Jacobian

1.1 The compactified Jacobian functor

1.1.1 The Jacobian functor. A *curve* X is a geometrically reduced and connected projective scheme of pure dimension 1 over an algebraically closed field k. Let $\mathcal{O}_X$ be the structure sheaf of X. A sheaf I on X is said to be *invertible* if it is locally free of rank 1. If I is invertible, there exists another invertible sheaf J such that $I \otimes J \cong \mathcal{O}_X$. Indeed, J is the sheaf of homomorphisms $Hom(I, \mathcal{O}_X)$, since $I \otimes Hom(I, \mathcal{O}_X) \cong Hom(I, I) \cong \mathcal{O}_X$. The *Jacobian functor* of X is the contravariant functor

$$\mathbf{J}_X : (k\text{-schemes})^{\circ} \to (\text{sets}),$$

from the category of k-schemes to the category of sets, that associates to each k-scheme T the set of invertible sheaves on $X_T := X \times_k T$, modulo the equivalence relation that identifies two sheaves I_1 and I_2 if there is an invertible sheaf M on T such that $I_1 \cong I_2 \otimes p_2^* M$, where $p_2 : X_T \to T$ is the projection. The functor $\mathbf{J}_X$ is represented by a (formally) smooth k-scheme J_X, called the *Jacobian* of X. If X is smooth, then the connected components of J_X are proper over k.

If X is not smooth, then, in general, the connected components of J_X are not proper, that is, there are invertible sheaves degenerating to noninvertible ones. For example, let X be a nodal irreducible curve with a single node P. For any $Q \in X$, let $\mathfrak{m}_Q$ denote its ideal sheaf. If $Q \neq P$, then $\mathfrak{m}_Q$ is invertible, but $\mathfrak{m}_P$ isn't. Thus as Q "approaches" P, the invertible sheaves $\mathfrak{m}_Q$ "tend" to the noninvertible sheaf $\mathfrak{m}_P$. To overcome this problem, we consider a larger class of sheaves, namely *torsion-free simple sheaves of rank 1.*

1.1.2 Torsion-free rank-1 sheaves. A coherent sheaf I on X is said to be of *rank 1* if it is invertible on the generic points of X. If X has irreducible components $X_1, \dots, X_r$, let $\eta_1, \dots, \eta_r$ denote the corresponding generic points. We define the *torsion subsheaf* $\mathcal{T}(I)$ of I as the kernel of the map

$$I \to \prod_i I_{\eta_i},$$

where, for each $x \in X$, I_x denotes the sheaf that, on each open subscheme $U \subset X$, is equal to the stalk of I at the point x if $x \in U$, and is zero otherwise. If $\mathcal{T}(I) = 0$, we say that I is *torsion-free*. Note that an invertible sheaf is torsion-free and of rank 1.

Lemma 1.1.3 *Let I be a coherent sheaf on a curve X and $\mathcal{T}(I)$ its torsion subsheaf. Then the quotient sheaf $I/\mathcal{T}(I)$ is torsion-free.*

Proof. Denote by G the quotient $I/\mathcal{T}(I)$. So there is a short exact sequence

$$0 \to \mathcal{T}(I) \to I \to G \to 0.$$

We need to see that the natural map $G \to \prod_i G_{\eta_i}$ is an injection. Indeed, $\prod_i G_{\eta_i} = \prod_i I_{\eta_i}$ (since $\mathcal{T}(I)_{\eta_i} = 0$ for every generic point η_i of X) and, by definition of the quotient, there is an injection $G \hookrightarrow \prod_i I_{\eta_i}$. □

1.1.4 Subcurves and simple sheaves. A *subcurve* Y of X is a reduced subscheme of pure dimension 1, or, equivalently, a reduced union of irreducible components of X. (Note that a subcurve Y of X is not necessarily connected.) For convenience, the empty set is considered a subcurve of every curve. For a subcurve Y of X we let $Y^c = \overline{X - Y}$ be the minimum subcurve containing $X - Y$.

Let $Y \subset X$ be a subcurve of X and let I be a torsion-free rank-1 sheaf. We denote by I_Y the quotient of the restriction $I|_Y$ by its torsion subsheaf, that is,

$$I_Y = I|_Y / \mathcal{T}(I|_Y).$$

It follows from the lemma that I_Y is a torsion-free rank-1 sheaf on Y and it is easy to see that I_Y is the maximum torsion-free quotient of $I|_Y$. We say that I is *decomposable* if there are proper subcurves $Y, Z \subset X$ such that $I \cong I_Y \oplus I_Z$. (Note that the intersection $Y \cap Z$ is necessarily finite.) The sheaf I is said to be *simple* if $\mathrm{End}(I) = k$. An invertible sheaf I is clearly simple, since

$$\mathrm{End}(I) = \mathrm{Hom}(I, I) \cong \mathrm{Hom}(\mathcal{O}_X, I \otimes I^{-1}) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X),$$

and, since X is connected, $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) = k$.

Lemma 1.1.5 *A torsion-free rank-1 sheaf I is simple if and only if I is not decomposable.*

Proof. If I is decomposable, then it is obviously not simple. Now assume I is not simple. Then there exists an endomorphism $h : I \to I$ that is not a multiple of the identity. Let Y be the union of the irreducible components of X along whose generic points h is zero. Let $Z := Y^c$ be the complementary subcurve. Since h is not zero, $Z \neq \emptyset$. Furthermore, since by [AK80, Lemma 5.4, p. 83], I_W is simple for every irreducible component W of X, up to subtracting a multiple of the identity from h, we may assume $Z \neq X$. Note that h factors through and injection $h' : I_Z \hookrightarrow I$.

Let N be the kernel of the projection $I \twoheadrightarrow I_Y$ and consider the exact sequence

$$0 \to N \to I \to I_Y \to 0.$$

Since h is zero on Y, it factors through N, thus h' also factors through N. Let $g : I_Z \hookrightarrow N$ be the induced injection. Since the composition $f : N \hookrightarrow I \twoheadrightarrow I_Z$ is injective, composing g with f we have an isomorphism

$$f \circ g : I_Z \xrightarrow{\sim} I_Z.$$

Indeed, $f \circ g$ is clearly injective and, since I_Z is a torsion-free sheaf of rank 1, the cokernel of $f \circ g$ has finite support. On the other hand,

$$\chi(\mathrm{Coker}(f \circ g)) = \chi(I_Z) - \chi(I_Z) = 0,$$

showing that $\mathrm{Coker}(f \circ g)$ is zero, and hence, that $f \circ g$ is an isomorphism. In particular, this implies that $f : N \hookrightarrow I_Z$ is an isomorphism as well. Therefore we get a splitting for the sequence $0 \to N \to I \to I_Y \to 0$ by the projection $I \twoheadrightarrow I_Z \cong N$. So $I \cong I_Y \oplus N \cong I_Y \oplus I_Z$, and I is decomposable. □

1.1.6 Families of curves. The above definitions generalize to families of curves. A *family of curves* is a morphism of schemes $p : X \to S$ which is flat and projective and whose geometric fibers are curves. By a sheaf on the family X/S we mean a S-flat coherent sheaf on X. A sheaf $\mathcal{I}$ on X/S is said to be *torsion-free* (resp. *rank*-1, resp. *simple*) on X/S if $\mathcal{I}(s)$ is torsion-free (resp. rank-1, resp. simple) on $X(s)$ for every geometric point s of S. The *relative compactified Jacobian functor* of the family X/S is the contravariant functor

$$\overline{\mathbf{J}}_{X/S} : (S\text{-schemes})^{\circ} \to (\text{sets})$$

that associates to each S-scheme T the set of simple torsion-free rank-1 sheaves on X_T/T modulo equivalence, where we say that two sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ on X_T/T are equivalent if there exists an invertible sheaf $\mathcal{M}$ on T such that $\mathcal{I}_1 \cong \mathcal{I}_2 \otimes p^*\mathcal{M}$, with $p : X \times_S T \to T$ being the projection.

The *relative Jacobian functor* $\mathbf{J}_{X/S}$ of X/S, which associates to each S-scheme T the set of invertible sheaves on X_T/T modulo equivalence as above, is a subfunctor of $\overline{\mathbf{J}}_{X/S}$. Mumford showed that $\mathbf{J}_{X/S}$ is represented by an S-scheme $J_{X/S}$ if the irreducible components of each fiber of X/S are geometrically irreducible, see [BLR, Theorem 2, p. 210] or [FGIKNV, Theorem 9.4.8, p. 263]. But, as we saw in the case of a single curve, $J_{X/S}$ is still not proper if the curves of the family are not smooth.

The functor $\overline{\mathbf{J}}_{X/S}$ is in general not representable in the category of schemes, but Altman and Kleiman showed in [AK80] that its *associated sheaf in the étale topology* is representable in the larger category of *algebraic spaces*. In order to make sense of this sentence, we need to introduce these concepts.

1.2 Algebraic spaces and the étale topology

In this section we'll just set the definitions and state some facts about the étale topology and algebraic spaces. The main references are [BLR] and [FGIKNV, Chapters 1 and 2]. And for the facts about algebraic spaces, see [Kn].

1.2.1 Grothendieck topologies and representable functors. Let $\mathbf{P}$ be a property of morphisms of S-schemes, satisfied by isomorphisms, and stable under composition and base change. A *Grothendieck topology* on the category of S-schemes assigns to each S-scheme T a collection of *coverings* $\{V_i \to T\}$, that is, of morphisms of S-schemes with the property $\mathbf{P}$, such that

(i) if $V \to T$ is an isomorphism, then $\{V \to T\}$ is a covering;

(ii) if $\{V_i \to T\}$ is a covering and $T' \to T$ is a morphism of S-schemes, then $\{V_i \times_T T' \to T'\}$ is a covering;

(iii) if $\{V_i \to T\}$ is a covering and, for each i, $\{W_{i,j} \to V_i\}$ is a covering, then the composition $\{W_{i,j} \to V_i \to T\}$ is a covering.

The *Zariski topology* is the Grothendieck topology where the morphisms are open embeddings. The *étale topology* is the Grothendieck topology where the morphisms are *étale*, that is, smooth of relative dimension 0, and whose images cover T, that is, $\cup \mathrm{Im}(V_i \to T) = T$. Note that the class of étale morphisms is stable under base change and under composition. So, since an isomorphism is obviously étale, we have indeed a Grothendieck topology. We remark that, since an open embedding is étale, the étale topology is finer than the Zariski topology.

For an S-scheme X, we define the *functor of points* of X as the contravariant functor

$$h_X : (S\text{-schemes})^\circ \longrightarrow (\text{sets})$$

such that, for each S-scheme T, $h_X(T) = \mathrm{Hom}_S(T, X)$. A contravariant functor

$$F : (S\text{-schemes})^\circ \longrightarrow (\text{sets})$$

is *representable* if there exists a S-scheme X such that F is isomorphic to h_X. In this case we say that X *represents* F. Given an S-scheme W, and an isomorphism $h_X \to F$, a morphism $W \to X$ is uniquely determined by an object of $F(W)$. A necessary condition for the functor F to be representable is that F be a *sheaf* in the Zariski and the étale topologies, a notion we explain below.

A contravariant functor $F : (S\text{-schemes})^\circ \to (\text{sets})$ is a *sheaf* in a Grothendieck topology if it satisfies the following condition. Let $\{U_i \to T\}$ be a covering and, for each i, consider $a_i \in F(U_i)$. Let $p_1 : U_i \times_T U_j \to U_i$ and $p_2 : U_i \times_T U_j \to U_j$ be the projections on the first and second factors,

$$\begin{array}{ccc} U_i \times_T U_j & \xrightarrow{p_2} & U_j \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ U_i & \longrightarrow & T \end{array}$$

and assume that $p_1^*(a_i) = p_2^*(a_j) \in F(U_i \times_T U_j)$ for every i, j. Then F is a sheaf if, for each such data, there exists a unique $a \in F(T)$ whose pullback to $F(U_i)$ is a_i for every i.

As we said, if a functor is representable, then it is a sheaf in the Zariski and the étale topologies. Indeed, for any scheme X, its functor of point h_X is a sheaf in the Zariski topology (since morphisms to X can be defined locally), and also in the étale topology [Kn, Proposition 1, p. 200]. But the relative compactified Jacobian functor $\overline{\mathbf{J}}_{X/S}$ (in fact even the relative Jacobian functor) is not in general a sheaf in the Zariski topology (see [FGIKNV, p. 253] or [BLR]).

In order to expect representability for $\overline{\mathbf{J}}_{X/S}$, we need to *sheafify* it first. We will not describe here the proccess of sheafification; it is similar to the proccess of sheafification of a pre-sheaf of modules on a scheme. We say that the sheafification of $\overline{\mathbf{J}}_{X/S}$ in the étale topology is its *associated sheaf in the étale topology*, and we still denote it by $\overline{\mathbf{J}}_{X/S}$, as we'll only deal with it from now on.

1.2.2 Algebraic spaces. An *algebraic space* is a contravariant functor $A : (S\text{-schemes})^\circ \to (\text{sets})$ such that:

(i) A is a sheaf in the étale topology;

(ii) (Local representability) There exists an S-scheme U and a map of sheaves $h_U \to A$ such that for every S-scheme V and map of sheaves $h_V \to A$, the (sheaf) fiber product $h_U \times_A h_V$ is represented by some S-scheme W and the map $h_U \times_A h_V \to h_V$ is induced by an étale surjective map $W \to V$;

(iii) For U as in (ii), the map of schemes inducing the map of sheaves $h_U \times_A h_U \to h_U \times_{h_S} h_U$ is quasicompact.

A morphism of algebraic spaces is a morphism of functors and, with this, the algebraic spaces form a category.

An S-scheme X is clearly an algebraic space in the sense that its functor of points h_X satisfies the above definition with $U = X$. Therefore, the category of S-schemes is embedded in the category of algebraic spaces.

Although it seems, by the definition, that an algebraic space is mere formalism, Proposition 1.2.3 shows that algebraic spaces arise naturally when one considers *equivalence relations* on a scheme.

An *equivalence relation* on an S-scheme U can be defined as a subscheme $R \subset U \times_S U$ such that the following three conditions are satisfied.

(i) (Reflexivity) The diagonal Δ is a subscheme of R;

(ii) (Symmetry) If $\phi : U \times_S U \to U \times_S U$ is the morphism defined by taking a pair of T-points (u, v) to (v, u), then $\phi|_R$ is an isomorphism onto R;

(iii) (Transitivity) Consider the square diagram

$$\begin{array}{ccc} R \times_U R & \xrightarrow{q_2} & R \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ R & \xrightarrow{p_2} & U \end{array}$$

where $q_i : R \times_U R \to R$ and $p_i : R \to U$ are projections on the i-th factor, for $i = 1, 2$. Note that $R \times_U R$ is, set-theoretically, the set of pairs of pairs $((u, v), (v', w))$ such that $v = v'$, where $u, v, v', w \in U$. The compositions of projections $p_i \circ q_i : R \times_U R \to U$, $i = 1, 2$, give a morphism $\psi : R \times_U R \to U \times_S U$ (so that $\psi((u, v), (v', w)) = (u, w)$). The third condition is that the image of ψ lies in R.

We say that $R \subset U \times_S U$ is an *étale equivalence relation* if it is an equivalence relation and the projection maps $p_1, p_2 : R \to U$ are étale. Likewise, R is *flat* (resp. *proper*) if the projection maps are flat (resp. proper).

Proposition 1.2.3 (Kn, Proposition 1.3, p. 93) *Let U be an S-scheme and let $R \subset U \times_S U$ be an étale equivalence relation. Assume that the inclusion map $R \hookrightarrow U \times_S U$ is quasicompact. Then there exist an algebraic space A, unique up to isomorphism, and a map of sheaves $h_U \to A$ satisfying the local representability condition, with $h_U \times_A h_U = h_R$.*

Altman and Kleiman showed in [AK80, Corollary 2.10, p. 72] that if the equivalence relation R is also proper and U is a locally projective S-scheme, then the algebraic space A of the proposition is an S-scheme.

1.3 Representability of $\overline{\mathbf{J}}_{X/S}$

As we mentioned in the end of Section 1.1, $\overline{\mathbf{J}}_{X/S}$ is represented by an algebraic space:

Theorem 1.3.1 (AK80, Theorem 7.4, p. 99) *Let $f : X \to S$ be a family of curves. Then $\overline{\mathbf{J}}_{X/S}$ is represented by an algebraic space $\overline{J}_{X/S}$.*

We say that $\overline{J}_{X/S}$ is the *relative compactified Jacobian* or simply the *compactified Jacobian* of X/S. (In fact, Altman and Kleiman proved that the sheaf associated to the functor **Spl** in the étale topology is represented by an algebraic space, where **Spl** is the functor associating to each S-scheme T the set of T-simple $\mathcal{O}_{X_T}$-modules modulo an equivalence relation. The sheaf $\overline{\mathbf{J}}_{X/S}$ is an open subsheaf of **Spl**.)

Moreover, Altman and Kleiman also showed that if the geometric fibers of the family are integral, then each connected component of $\overline{J}_{X/S}$ is a proper scheme over S [AK76, Theorem 3, p. 948].

On the other hand, assume there are sections $\sigma_1, \dots, \sigma_n$ through the smooth locus of X/S such that, for every $s \in S$, every irreducible component of the fiber $X(s)$ contains $\sigma_i(s)$ for some i. Then every irreducible component of each fiber of X/S is geometrically integral. Indeed, assume by contradiction that a component Y of some fiber X_s of X/S is not geometrically integral. Let σ be the section of X/S passing through Y, and let $P = \sigma(s)$. Then the point P is not geometrically connected. But by [EGA, IV_2, Corollary 4.5.14, p. 65] a rational point is always geometrically connected. Thus Esteves showed [E01, Theorem B, p. 3048] that $\overline{J}_{X/S}$ is a scheme. In particular, since we can always make an étale base change to obtain enough sections [E01, Lemma 18, p. 3061], this says that after a suitable étale base change, $\overline{J}_{X/S}$ becomes a scheme.

Anyhow, the algebraic space (or scheme) $\overline{J}_{X/S}$ has a drawback: on one hand, $\overline{J}_{X/S}$ is big enough, in the sense that a "sequence" of torsion-free rank-1 simple sheaves does have a limit; but on the other, $\overline{J}_{X/S}$ is too big, in the sense that the limit doesn't need to be unique.

More precisely, let X/S be a local one-parameter family of curves, that is, assume S is the spectrum of a valuation ring with special point o and generic point η. Let X_η be the generic fiber of the family, and I_η a sheaf on X_η. We say that a sheaf $\mathcal{I}$ on X is an *extension* of I_η if $\mathcal{I}|_{X_\eta} = I_\eta$ and, in this case, $I_o := \mathcal{I}|_{X_o}$ is the *limit* sheaf on the special fiber X_o.

Now, assume that I_η is a rank-1 torsion-free sheaf on X_η. Then, there is an integer m such that $Hom(I_\eta, \mathcal{O}_{X_\eta})(m)$ is generated by global sections. So, since $Hom(I_\eta, \mathcal{O}_{X_\eta})$ is nonzero at the generic points of X_η, there is a morphism $u_\eta : I_\eta \to \mathcal{O}_{X_\eta}(m)$ that is injective at these points. Now, I_η is rank-1 and torsion-free, so u_η is injective, since otherwise the kernel of u_η would be a torsion subsheaf of I_η. Let C_η be the cokernel of u_η. By [EGA, IV_2, Proposition 2.8.1, p. 33], there exists a (unique) flat extension C of C_η, and a canonical map $q : \mathcal{O}_X \to C$. Let $\mathcal{I}$ be the kernel of q. Then the restriction of $\mathcal{I}$ to X_η is equal to I_η. In addition, since C is S-flat, also $\mathcal{I}$ is S-flat, and $I_o = \mathcal{I}|_{X_o}$ is contained in $\mathcal{O}_{X_o}(m)$. So I_o is torsion-free and on rank 1, and thus it is a torsion-free rank-1 limit of I_η.

In the next example we consider a local one-parameter family of curves with reducible special fiber and show that there are infinitely many limits of the trivial sheaf $\mathcal{O}_{X_\eta}$.

Example 1.3.2 Again, let X/S be a family of curves over the spectrum S of a discrete valuation ring, with special point o and generic point η. Assume that the generic fiber of the family is smooth and that the special fiber is the (reduced) reducible curve X_o with two irreducible components Y and Z meeting at a single simple node P. In addition, assume that X is regular. This situation is achieved by considering a deformation of X_o along a general direction.

Consider the structure sheaf $\mathcal{O}_{X_\eta}$ of X_η. Then $\mathcal{O}_{X_\eta}$ has the obvious extension $\mathcal{O}_X$. Now, since X is regular, any irreducible subscheme of codimension 1 of X is a Cartier divisor. In particular, Z is a Cartier divisor and, for any $n \in \mathbb{Z}$, the sheaf $\mathcal{O}_X(nZ)$ is invertible. Moreover, $\mathcal{O}_X(nZ)|_{X_\eta} = \mathcal{O}_{X_\eta}$, that is, $\mathcal{O}_X(nZ)$ is an extension of $\mathcal{O}_{X_\eta}$ for every n. Note that the sheaves $\mathcal{O}_X(nZ)|_{X_o}$ are all different.

Indeed, to see that these sheaves are diferent, recall that the *degree* of a torsion-free rank-1 sheaf I on a curve X is $\deg(I) = \chi(I) - \chi(\mathcal{O}_X)$, where $\chi(\cdot)$ is the Euler characteristic. So considering the coherent sheaf $I^{(n)} := \mathcal{O}_X(nZ)|_{X_o}$ on X_o and computing its degree $\deg_Y(I^{(n)}) := \deg(I^{(n)}|_Y)$ on the subcurve Y,

$$\deg_Y(I^{(n)}) = \deg(\mathcal{O}_X(nZ)|_Y) = \deg(\mathcal{O}_Y(nP)) = n.$$

This shows that all the limit sheaves are diferent, as they have diferent degrees on Y. □

1.3.3 Polarizations on a curve. To decompose the compactified Jacobian into smaller (and better behaved) pieces, we use *polarizations*. Polarizations were introduced by Seshadri in [S82] to construct the moduli space of vector bundles of given degree and rank on a fixed curve X. His polarizations were numerical; precisely, a polarization in the sense of [S82] is just a n-tuple of positive rational numbers adding up to 1, where n is the number of irreducible components of X. Since we are going to work with families of curves, we shall use polarizations in the sense of Esteves [E01]. Thus, a *polarization* on a curve X is a vector bundle E on X such that the slope of E is an integer; that is, $\mathrm{rk}(E)$ divides $\deg(E)$. (For the connection between the two types of polarizations see [E01, Observation 57, p. 3092].)

A torsion-free rank-1 sheaf I on a curve X is *stable* (resp. *semi-stable*) with respect to E if $\chi(E\otimes I)=0$ and $\chi(E\otimes I_Y)>0$ (resp. $\chi(E\otimes I_Y)\geq 0$) for all proper subcurves Y of X. Also, let W be an irreducible component of X and P a nonsingular point of X. We say that a torsion-free rank-1 sheaf I is *W-quasi-stable* (resp. *P-quasi-stable*) with respect to E if I is semi-stable and in addition $\chi(E\otimes I_Y)>0$ for every subcurve Y of X containing W (resp. containing P).

For a subcurve Y of X let

$$\beta_I(Y):=\chi(I_Y)+\frac{\deg_Y(E)}{\mathrm{rk}(E)}.$$

Then, as $\chi(E\otimes I_Y)=\mathrm{rk}(E)\chi(I_Y)+\deg_Y(E)$, the sheaf I is stable (resp. semi-stable) if and only if $\beta_I(Y)>0$ (resp. $\beta_I(Y)\geq 0$) for every proper subcurve Y of X. Moreover, if P is a nonsingular point of X and W is an irreducible component of X, then I is P-quasi-stable (resp. W-quasi-stable) if and only if it is semi-stable and $\beta_I(Y)>0$ for every subcurve Y such that $P\in Y$ (resp. $W\subseteq Y$).

The semi-stability condition gives bounds for the degree of the sheaf I on the subcurves of X. To see this we need to compare the Euler characterictic of a torsion-free rank-1 sheaf I on X with the ones of I_Y and I_{Y^c}, for each subcurve Y of X.

Lemma 1.3.4 *Let I be a torsion-free sheaf of rank 1 on a curve X. Let $\nu: X'\to X$ be the normalization of X, and $\delta:=\chi(\nu_*\mathcal{O}_{X'}/\mathcal{O}_X)$. Then for every subcurve Y of X we have*

$$\chi(I_Y)+\chi(I_{Y^c})\leq\chi(I)+\delta.$$

Proof. First note that the disjoint union $Y\coprod Y^c$ is a partial normalization of X along the points in the intersection $Y\cap Y^c$. Hence ν factors through the natural map $\rho: Y\coprod Y^c\twoheadrightarrow X$. Now, $\rho_*\rho^*I=I|_Y\oplus I|_{Y^c}$, so the maximum torsion-free quotient of $\rho_*\rho^*I$ is the sheaf $I_Y\oplus I_{Y^c}$. Let I' be the maximum torsion-free quotient of $\nu_*\nu^*I$ (that is, $I'=\nu_*\nu^*I/\mathcal{T}(\nu_*\nu^*I)$.) Then we have

$$I\hookrightarrow I_Y\oplus I_{Y^c}\hookrightarrow I'$$

and therefore

$$\chi(\mathrm{Coker}(I \hookrightarrow I')) \geq \chi(\mathrm{Coker}(I \hookrightarrow I_Y \oplus I_{Y^c})) = \chi(I_Y) + \chi(I_{Y^c}) - \chi(I).$$

So we need only show that $\chi(\mathrm{Coker}(I \hookrightarrow I')) \leq \delta$.

Now, since ν is a finite morphism, the sheaf $C := \mathrm{Coker}(I \hookrightarrow I')$ has finite support. In fact, it has support on the singular points of X. Thus let P be a singular point of X, and let $\mathcal{O}_P := \mathcal{O}_{X,P}$ denote the local ring of X at P. Let $\overline{\mathcal{O}}_P$ be the completion of $\mathcal{O}_P$. Since $\overline{\mathcal{O}}_P$ is a regular semilocal ring, it is a principal ideal domain.

Let I_P (resp. I'_P) denote the stalk of I (resp. I') at P. Then $I'_P = I_P\overline{\mathcal{O}}_P$. So I'_P is a fractional ideal of $\overline{\mathcal{O}}_P$. Hence there is a function $f \in I_P$ such that $I_P\overline{\mathcal{O}}_P = f\overline{\mathcal{O}}_P$. Thus we have

$$f\mathcal{O}_P \subset I_P \subset I_P\overline{\mathcal{O}}_P = f\overline{\mathcal{O}}_P$$

and hence

$$\ell(C_P) = \ell\left(\frac{I'_P}{I_P}\right) = \ell\left(\frac{I_P\overline{\mathcal{O}}_P}{I_P}\right) \leq \ell\left(\frac{f\overline{\mathcal{O}}_P}{f\mathcal{O}_P}\right),$$

where $\ell(\cdot)$ denotes length as $\mathcal{O}_P$-module. Now, since f is a nonzero-divisor, we have

$$\ell\left(\frac{f\overline{\mathcal{O}}_P}{f\mathcal{O}_P}\right) = \ell\left(\frac{\overline{\mathcal{O}}_P}{\mathcal{O}_P}\right).$$

Let $\delta_P := \ell(\overline{\mathcal{O}}_P/\mathcal{O}_P)$. Then

$$\delta = \chi\left(\frac{\nu_*\mathcal{O}_{X'}}{\mathcal{O}_X}\right) = \sum_{P\in X} \ell\left(\frac{\overline{\mathcal{O}}_P}{\mathcal{O}_P}\right) = \sum_{P\in X} \delta_P.$$

(Note that δ is clearly nonnegative and finite, since $\delta_P \geq 0$ for every point P of X, and $\delta_P = 0$ if P is nonsingular.) Then

$$\chi(C) = \sum_{P\in X} \ell(C_P) \leq \sum_{P\in X} \delta_P = \delta.$$

□

Now, let I be a sheaf on X, and assume that I is semi-stable with respect to a polarization E. First, note that the degree of I is determined by the polarization E. Indeed, since $\chi(E \otimes I) = 0$, we have

$$\deg(I) = -\frac{\deg(E)}{\mathrm{rk}(E)} + g - 1,$$

where g is the *arithmetic genus* of X, that is, g is the dimension of $H^1(X, \mathcal{O}_X)$. Now, let $Y \subset X$ be a proper subcurve and let

$$\deg_Y(I) := \deg(I_Y).$$

Then

$$0 \leq \beta_I(Y) = \deg_Y(I) + \chi(\mathcal{O}_Y) + \frac{\deg_Y(E)}{\mathrm{rk}(E)}$$

so that

$$\deg_Y(I) \geq -\chi(\mathcal{O}_Y) - \frac{\deg_Y(E)}{\mathrm{rk}(E)}.$$

On the other hand, by Lemma 1.3.4, $\chi(I_Y) + \chi(I_{Y^c}) \leq \chi(I) + \varepsilon$. Also, by the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_{Y^c} \to \mathcal{O}_{Y \cap Y^c} \to 0,$$

we have $\chi(\mathcal{O}_Y) + \chi(\mathcal{O}_{Y^c}) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_{Y \cap Y^c})$. Hence,

$$\deg_Y(I) + \deg_{Y^c}(I) \leq \deg(I) + \varepsilon - \chi(\mathcal{O}_{Y \cap Y^c}).$$

Thus, from $\beta_I(Y^c) \geq 0$ we get

$$\begin{aligned} \deg_Y(I) \quad &\leq \quad -\frac{\deg(E)}{\mathrm{rk}(E)} + g - 1 + \varepsilon - \chi(\mathcal{O}_{Y \cap Y^c}) \\ &\quad + \chi(\mathcal{O}_{Y^c}) + \frac{\deg_{Y^c}(E)}{\mathrm{rk}(E)}. \end{aligned}$$

So the degrees of a semi-stable sheaf on the subcurves of X are bounded.

Lemma 1.3.5 *Let I be a torsion-free rank-1 sheaf on a curve X. Let E be a polarization on X, and fix a smooth point $P \in X$. If I is P-quasi-stable with respect to E, then I is simple.*

Proof. Suppose that I is P-quasi-stable with respect to E, and assume that I is not simple. By Lemma 1.1.5, there are subcurves Y and Z of X such that $I \cong I_Y \oplus I_Z$. Now, since I has support on X, we have $Y \cup Z = X$. Hence, since P must be contained in either Y or Z, and since I is P-quasi-stable, we have

$$0 = \chi(E \otimes I) = \chi(E \otimes I_Y) + \chi(E \otimes I_Z) > 0,$$

an absurdity. □

1.3.6 Polarizations on families of curves. Let X/S be a family of curves and consider $\mathcal{E}$ a vector bundle on X. We say $\mathcal{E}$ is a *polarization* if $\mathrm{rk}(\mathcal{E})$ divides the *relative degree* $\deg(\mathcal{E}/S)$, which is defined to be simply $\deg_{X(s)}(\det(\mathcal{E}|_{X(s)}))$ for any $s \in S$, where $X(s)$ is the fiber of X/S over s. (Note that since the family X/S is flat, also $\mathcal{E}$ is S-flat, and thus the numbers $\chi(\mathcal{O}_{X(s)})$ and $\chi(\det(\mathcal{E}|_{X(s)}))$ do not depend on s.)

A torsion-free rank-1 sheaf $\mathcal{I}$ on X/S is said to be *stable* (resp. *semi-stable*) with respect to $\mathcal{E}$ if $\mathcal{I}(s)$ is stable (resp. semi-stable) with respect to $\mathcal{E}(s)$ for every geometric point s of S. Let $\sigma : S \to X$ be a section through the smooth locus of X; a torsion-free rank-1 sheaf $\mathcal{I}$ on X/S is *σ-quasi-stable* with respect to $\mathcal{E}$ if $\mathcal{I}(s)$ is $\sigma(s)$-quasi-stable with respect to $\mathcal{E}(s)$ for every geometric point s of S. It follows from the previous lemma that a σ-quasi-stable sheaf is simple on X/S.

Denote by $\overline{J}^{s}_{\mathcal{E}}$ (resp. $\overline{J}^{ss}_{\mathcal{E}}$) the subspaces of $\overline{J}_{X/S}$ parametrizing the torsion-free rank-1 simple sheaves $\mathcal{I}$ on X/S that are stable (resp. semi-stable) with respect to $\mathcal{E}$. Moreover, if $\sigma : S \to X$ is a section through the smooth locus of X, let $\overline{J}^{\sigma}_{\mathcal{E}}$ denote the subspace of $\overline{J}_{X/S}$ parametrizing the torsion-free rank-1 sheaves $\mathcal{I}$ on X/S that are σ-quasi-stable with respect to $\mathcal{E}$.

It is easy to see that

$$\overline{J}^{s}_{\mathcal{E}} \subset \overline{J}^{\sigma}_{\mathcal{E}} \subset \overline{J}^{ss}_{\mathcal{E}} \subset \overline{J}_{X/S}.$$

Moreover, it is shown in [E01] that all these subspaces are open and their formation commutes with base change. Furthermore:

Theorem 1.3.7 (E01, Theorem A, p. 3047) *The algebraic space $\overline{J}^{ss}_{\mathcal{E}}$ is of finite type over S. Also*

(1) $\overline{J}^{ss}_{\mathcal{E}}$ is universally closed over S;

(2) $\overline{J}^{s}_{\mathcal{E}}$ is a separated scheme over S;

(3) $\overline{J}^{\sigma}_{\mathcal{E}}$ is proper over S.

(That $\overline{J}^{s}_{\mathcal{E}}$ is a scheme is actually [E01, Corollary 50, p. 3086].)

Chapter 2

The Abel maps

2.1 The first Abel map

The degree of a coherent sheaf on a curve X is invariant under automorphisms, since the Euler characteristic is. Using the degree, we get stratifications of the Jacobian and the compactified Jacobian of X:

$$J_X = \coprod_d J_X^d \quad \text{and} \quad \overline{J}_X = \coprod_d \overline{J}_X^d,$$

where J_X^d (resp. $\overline{J}_X^d$) is the subset of J_X (resp. $\overline{J}_X$) parameterizing invertible sheaves (resp. simple torsion-free rank-1 sheaves) of degree d on X.

If X/S is a family of curves and $\mathcal{I}$ is a sheaf on X/S, define $\mathcal{I}(s)$ to be the restriction of $\mathcal{I}$ to the fiber $X(s)$ of X/S over $s \in S$. We say that $\mathcal{I}$ has *degree* d if for every s in S, $d = \chi(\mathcal{I}(s)) - \chi(\mathcal{O}_{X(s)})$. (Recall that $\mathcal{I}$ is coherent and S-flat, so the degree is well-defined.) Let $f : X \to S$ be the stucture morphism of the family X/S, and let M be an invertible sheaf on S. Then f^*M has degree zero, since $(f^*M)|_{X(s)} \cong M_s \otimes \mathcal{O}_{X(s)}$ is trivial. Thus we may write

$$J_{X/S} = \coprod_d J_{X/S}^d \quad \text{and} \quad \overline{J}_{X/S} = \coprod_d \overline{J}_{X/S}^d.$$

Set-theoretically, the *(first) Abel map* of a curve X associates to each closed point Q of X its ideal sheaf $\mathfrak{m}_Q$. As we see in the following lemma, if Q is a singular point of X, then $\mathfrak{m}_Q$ is not invertible. Nevertheless, $\mathfrak{m}_Q$ is a torsion-free sheaf of rank 1, and so we must consider the Abel map having image on the compactified Jacobian of X.

Lemma 2.1.1 *The ideal sheaf $\mathfrak{m}_Q$ of a closed point Q on a curve X is a torsion-free rank-1 sheaf of degree -1. Furthermore, $\mathfrak{m}_Q$ is invertible if and only if Q is a smooth point of X.*

Proof. Since $\mathfrak{m}_Q$ is a subsheaf of $\mathcal{O}_X$ which is isomorphic to $\mathcal{O}_X$ away from the point Q, it is torsion-free of rank 1. From the exact sequence

$$0 \to \mathfrak{m}_Q \to \mathcal{O}_X \to \mathcal{O}_Q \to 0,$$

we get that $\deg(\mathfrak{m}_Q) = \chi(\mathfrak{m}_Q) - \chi(\mathcal{O}_X) = -\chi(\mathcal{O}_Q)$, but $\chi(\mathcal{O}_Q) = h^0(\mathcal{O}_Q) = 1$.

For the last assertion, note that the stalk of $\mathfrak{m}_Q$ at Q, that is, $\mathfrak{m}_Q/\mathfrak{m}_Q^2$, is isomorphic to the cotangent space of X at Q, so it has dimension 1 if and only if Q is a smooth point of X. □

As we mentioned, the lemma shows that the Abel map has image in the Jacobian J_X^{-1} if and only if X is smooth. If X is not a smooth curve, we take the map to have image in its compactified Jacobian $\overline{J}_X^{-1}$. But this map may not be well defined, since the ideal sheaf of a singular point doesn't need to be simple.

Example 2.1.2 Let X be a curve. Assume first that there are subcurves Y and Z of X whose union $Y \cup Z$ is X and whose intersection $Y \cap Z$ is a reduced point P of X. Then $\mathfrak{m}_P$ is a torsion-free rank-1 sheaf, but is not simple. Indeed, consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}_P & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathfrak{m}_P)_Y \oplus (\mathfrak{m}_P)_Z & \longrightarrow & \mathcal{O}_Y \oplus \mathcal{O}_Z & \longrightarrow & \mathcal{O}_P \oplus \mathcal{O}_P & \longrightarrow & 0 \end{array}$$

whose horizontal sequences are exact. Since P is the intersection $Y \cap Z$, the cokernels of the second and third vertical maps are isomorphic to $\mathcal{O}_P$. Then, by the snake lemma, the cokernel of the first vertical map must be zero, and therefore, $\mathfrak{m}_P \cong (\mathfrak{m}_P)_Y \oplus (\mathfrak{m}_P)_Z$, showing that $\mathfrak{m}_P$ is not simple.

Now, on the other hand, let P be a point of X such that $\mathfrak{m}_P$ is not simple. By Lemma 1.1.5 there are subcurves Y and Z of X such that $Y \cup Z = X$ and $\mathfrak{m}_P \cong (\mathfrak{m}_P)_Y \oplus (\mathfrak{m}_P)_Z$. Now, we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}_P & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathfrak{m}_P)_Y \oplus (\mathfrak{m}_P)_Z & \longrightarrow & \mathcal{O}_Y \oplus \mathcal{O}_Z & \longrightarrow & \mathcal{O}_{P\cap Y} \oplus \mathcal{O}_{P\cap Z} & \longrightarrow & 0 \end{array}$$

where the first vertical arrow is an isomorphism, and $\mathcal{O}_{P\cap Z}$ and $\mathcal{O}_{P\cap Y}$ are either $\mathcal{O}_P$ or 0, depending on whether P is contained in the subcurve in question or not. We want to show the intersection

$Y \cap Z$ is the point P with multiplicity 1. By the diagram we have

$$\begin{aligned}\chi(\mathcal{O}_{Y\cap Z}) =&\chi(\mathrm{Coker}(\mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Z))\\ =&\chi(\mathrm{Coker}(\mathcal{O}_P \to \mathcal{O}_{P\cap Y} \oplus \mathcal{O}_{P\cap Z}))\\ =&\chi(\mathcal{O}_{P\cap Y} \oplus \mathcal{O}_{P\cap Z}) - \chi(\mathcal{O}_P)\\ =&\chi(\mathcal{O}_{P\cap Y}) + \chi(\mathcal{O}_{P\cap Z}) - \chi(\mathcal{O}_P).\end{aligned}$$

Now, $\chi(\mathcal{O}_P) = 1$, and $\chi(\mathcal{O}_{P\cap Y})$ and $\chi(\mathcal{O}_{P\cap Z})$ are either 0 or 1. But since the curve X is connected, we have $\chi(\mathcal{O}_{Y\cap Z}) \geq 1$, thus forcing $\chi(\mathcal{O}_{P\cap Y}) = \chi(\mathcal{O}_{P\cap Z}) = 1$. This means that $P \in Y \cap Z$ and also that $\chi(\mathcal{O}_{Y\cap Z}) = 1$, hence $Y \cap Z = \{P\}$ (scheme-theoretically). □

The above example shows that $\mathfrak{m}_P$ is not simple if and only if there is a subcurve Y of X such that $\{P\} = Y \cap Y^c$, where this intersection is to be understood scheme-theoretically, that is, with multiplicities. When this is the case we say that P is a *disconnecting node*. We remark that, if X is a Gorenstein curve, then the disconnecting nodes are indeed nodes of X, due to [Cat82, Proposition 1.10, p. 59].

Now we define the first Abel map scheme-theoretically as a morphism from a family of curves to its relative compactified Jacobian. Let X/S be a family of curves and assume that every geometric fiber of the family has no disconnecting nodes. Recall from Section 1.2 that a S-morphism $X \to \overline{J}_{X/S}$ can be determined by a simple torsion-free rank-1 sheaf on $X \times_S X/X$. For the Abel map, such a sheaf is the ideal sheaf I_Δ of the diagonal subscheme $\Delta \subset X \times_S X$. Considering $X \times_S X$ as a family of curves over X, let Q be a geometric point of X. We have $(I_\Delta)(Q) = \mathfrak{m}_Q$, so the map defined by I_Δ is indeed the Abel map. By Lemma 2.1.1, the image of the Abel map lies in $\overline{J}^{-1}_{X/S}$, that is, the *first Abel map* is the S-morphism

$$A : X \longrightarrow \overline{J}^{-1}_{X/S}.$$

(Sometimes the word first is ommited and we say simply that A is the Abel map of X/S.)

We remark that the map A is not well defined if some of the curves in the family have a disconnecting node. We'll see in Chapter 4 a way of overcoming this problem, that is, of defining an Abel map from a curve X to $\overline{J}_X$ even if the curve X has a disconnecting node, following the construction in [CE06]. Note that if the curves of the family are irreducible, then there is no disconnecting node and the map is well defined. Moreover, Altman and Kleiman showed in [AK80, Theorem 8.8, p. 108] that in this case, assuming that $g > 0$, where g is the arithmetic genus of the fibers of X/S, the map A is a closed embedding. We will recall the proof of this result in Theorem 2.1.6, but before we need two lemmas.

Lemma 2.1.3 *Let X be a curve, and I be a simple torsion-free rank-1 sheaf of degree -1 on X. Assume that every nonzero morphism $I \to \mathcal{O}_X$ is injective. Let $\mathrm{Spec}(k) \to \overline{J}_X^{-1}$ be the morphism induced by I. Then $A^{-1}(I) := X \times_{\overline{J}_X^{-1}} \mathrm{Spec}(k)$ is isomorphic to the projective space $\mathbb{P}(\mathrm{Hom}(I, \mathcal{O}_X)^\vee)$.*

Proof. We will construct morphisms

$$\Phi : \mathbb{P}(\mathrm{Hom}(I, \mathcal{O}_X)^\vee) \longrightarrow A^{-1}(I)$$

and

$$\Psi : A^{-1}(I) \longrightarrow \mathbb{P}(\mathrm{Hom}(I, \mathcal{O}_X)^\vee),$$

using the functorial property of $\mathbb{P}(\mathrm{Hom}(I, \mathcal{O}_X)^\vee)$ (see [H, Proposition 7.12, p. 162]) and the functor of points of $A^{-1}(I)$, such that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are identities. Let $g : T \to \mathrm{Spec}(k)$ be a morphism of schemes and consider the Cartesian diagram

$$\begin{array}{ccc} X \times T & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & \mathrm{Spec}(k). \end{array}$$

where p_1 and p_2 are the projections on the first and second factors, respectively.

A T-point of $\mathbb{P}(\mathrm{Hom}(I, \mathcal{O}_X)^\vee)$ corresponds to an isomorphism class of pairs (M, q) where M is an invertible sheaf on T and

$$q : \mathrm{Hom}(I, \mathcal{O}_X)^\vee \otimes \mathcal{O}_T \twoheadrightarrow M$$

is a surjective morphism. Dualizing q and pulling back to $X \times T$, we get the morphism

$$p_2^* M^\vee \longrightarrow \mathrm{Hom}(I, \mathcal{O}_X) \otimes \mathcal{O}_{X \times T}.$$

Now, tensoring with $p_1^* I$ we get

$$p_1^* I \otimes p_2^* M^\vee \longrightarrow p_1^* I \otimes \mathrm{Hom}(I, \mathcal{O}_X) \otimes \mathcal{O}_{X \times T} = p_1^*(I \otimes \mathrm{Hom}(I, \mathcal{O}_X)),$$

and thus composing with the evaluation map $I \otimes \mathrm{Hom}(I, \mathcal{O}_X) \xrightarrow{\mathrm{ev}} \mathcal{O}_X$, we get

$$u : p_1^* I \otimes p_2^* M^\vee \longrightarrow \mathcal{O}_{X \times T}.$$

Note that the restriction of u to each fiber of p_2 is nonzero, since q is surjective.

Moreover, u is injective with T-flat cokernel. Indeed, by hypothesis, for each $t \in T$ the morphism $u(t)$ is injective, because M is invertible and $u(t)$ is nonzero. Hence, u is injective and its cokernel is T-flat, by the local criterion of flatness [AK, Theorem 3.2, p. 91]. Therefore, $p_1^*I \otimes p_2^*M^\vee$ is isomorphic to the ideal sheaf of a T-flat subscheme Σ of $X \times T$. Moreover, since for each $t \in T$

$$\chi(p_1^*I \otimes p_2^*M^\vee)(t)) = \chi(I),$$

we have that $\Sigma(t) = \Sigma \cap (X \times \{t\})$ is a point of $X \times \{t\}$. Thus $\Sigma \cong T$, which gives a section of p_2,

$$\sigma : T \to X \times T.$$

Now, projecting to X, that is, composing σ with p_1, we get a T-point of X, that is, a morphism $\rho : T \to X$. In addition, for each $t \in T$, the image of $\rho(t)$ is a point $x \in X$ such that

$$\mathfrak{m}_x \cong I_\Sigma(t) \cong (p_1^*I \otimes p_2^*M^\vee)(t) = p_2^*(M^\vee(t)) \otimes I.$$

Hence, by the equivalence relation on $\overline{J}_X$, we have a morphism

$$\Phi : \mathbb{P}(\mathrm{Hom}(I, \mathcal{O}_X)^\vee) \longrightarrow A^{-1}(I).$$

On the other hand, a T-point of $A^{-1}(I)$ is morphism $T \to A^{-1}(I)$. Composing with the inclusion $A^{-1}(I) \hookrightarrow X$ we get a morphism $\rho : T \to X$, such that for each $t \in T$, the image of $\rho(t)$ is a point $x \in X$ having ideal sheaf

$$\mathfrak{m}_x \cong I.$$

So ρ induces a section $\sigma : T \to X \times T$ of p_2, whose image $\Sigma = \sigma(T)$ is a subscheme of $X \times T$, flat over T, and with ideal sheaf

$$I_\Sigma \cong p_1^*I \otimes p_2^*N,$$

for some invertible sheaf N on T. Thus consider the injective morphism

$$u : p_1^*I \otimes p_2^*N \xrightarrow{\sim} I_\Sigma \hookrightarrow \mathcal{O}_{X\times T}.$$

Then u induces a morphism

$$p_2^*N \longrightarrow Hom(p_1^*I, \mathcal{O}_{X\times T}).$$

Now, since p_1 is a flat morphism, we have

$$Hom(p_1^*I, \mathcal{O}_{X\times T}) = p_1^*Hom(I, \mathcal{O}_X).$$

And since

$$p_{2*}p_1^*Hom(I, \mathcal{O}_X) = \mathrm{Hom}(I, \mathcal{O}_X) \times \mathcal{O}_T,$$

by adjunction formula we have an induced morphism

$$N \longrightarrow \mathrm{Hom}(I, \mathcal{O}_X) \otimes \mathcal{O}_T.$$

Finally, dualizing we get a morphism

$$q : \mathrm{Hom}(I, \mathcal{O}_X)^\vee \otimes \mathcal{O}_T \longrightarrow N^\vee.$$

The map q is surjective by Nakayama's lemma, because for each $t \in T$, $q(t)$ is nonzero, since so is $u(t)$. This gives the morphism

$$\Psi : A^{-1}(I) \longrightarrow \mathbb{P}(\mathrm{Hom}(I, \mathcal{O}_X)^\vee).$$

By construction, Ψ is the inverse of Φ. $\square$

Remark 2.1.4 Actually, Altman and Kleiman have proved in [AK80, Theorem 4.2, p. 78] a stronger statement which, in particular, implies the following. Let X/S be a family of curves, and consider the Abel map of the family $A : X \longrightarrow \overline{J}^{-1}_{X/S}$. Then there is a sheaf $\mathcal{H}$ over $\overline{J}^{-1}_{X/S}$ such that $X = \mathbb{P}_{\overline{J}^{-1}_{X/S}}(\mathcal{H})$. Moreover, the Abel map A is the induced morphism $\mathbb{P}_{\overline{J}^{-1}_{X/S}}(\mathcal{H}) \longrightarrow \overline{J}^{-1}_{X/S}$.

In the case of a single curve X, the sheaf $\mathcal{H}$ is such that, for each torsion-free rank-1 sheaf I on X, the stalk of $\mathcal{H}$ at the point defined by I at $\overline{J}^{-1}_X$ is isomorphic to $\mathrm{Hom}(I, \mathcal{O}_X)^\vee$.

Let X be a curve, and let Y be a subcurve of X. Let

$$\delta_Y := \chi(\mathcal{O}_{Y \cap Y^c}).$$

Then X has a disconnecting node $P \in Y \cap Y^c$ if and only if $\delta_Y = 1$. Indeed, δ_Y is the number of points (with multiplicities) of the intersection $Y \cap Y^c$.

Lemma 2.1.5 *Let X be a curve and Q a point of X. Assume that X has no disconnecting nodes. Then every nonzero morphism $u : \mathfrak{m}_Q \to \mathcal{O}_X$ is injective.*

Proof. Since X has no disconnecting nodes, either X is irreducible or $\delta_Y \geq 2$ for every subcurve Y of X. If X is irreducible, the assertion is obvious, since $\mathfrak{m}_Q$ is a torsion-free rank-1 sheaf and the kernel of a noninjective nonzero morphism $\mathfrak{m}_Q \to \mathcal{O}_X$ would be a torsion subsheaf of $\mathfrak{m}_Q$.

Assume X is not irreducible, and let Y be the subcurve of X given by the union of all components of X along whose generic points u is zero. Assume, by contradiction, that Y is not empty.

Let $Z = Y^c$ be the complementary subcurve of Y. Since u is nonzero, Z is not empty. Then $u_Z : (\mathfrak{m}_Q)_Z \to \mathcal{O}_Z$ is injective.

Now, u_Z factors through the ideal sheaf $I_{Y\cap Z,Z}$ of $Y \cap Z$ in Z, because the composition $\mathfrak{m}_Q \to \mathcal{O}_X \to \mathcal{O}_Y$ is zero. Since $\delta_Y \geq 2$, the sheaf $I_{Y\cap Z,Z}$ has degree at most -2. On the other hand, if $Q \notin Z$ then $(\mathfrak{m}_Q)_Z$ equals $\mathcal{O}_Z$, and hence has degree 0; and if $Q \in Z$, then $(\mathfrak{m}_Q)_Z$ is the ideal sheaf of Q at Z, and hence has degree -1. In any case, it has degree at least -1.

Let $\mathcal{M}$ be the cokernel of u_Z. Then $\mathcal{M}$ has finite support, and thus, positive Euler characteristic. On the other hand, from the exact sequence $0 \to (\mathfrak{m}_Q)_Z \to I_{Z\cap Y,Z} \to \mathcal{M} \to 0$ we have

$$\chi(I_{Z\cap Y,Z}) = \chi((\mathfrak{m}_Q)_Z) + \chi(\mathcal{M}).$$

Subtracting $\chi(\mathcal{O}_Z)$ from both sides of the equality, we get

$$\deg(I_{Z\cap Y,Z}) = \deg((\mathfrak{m}_Q)_Z) + \chi(\mathcal{M})$$

and thus $\chi(\mathcal{M}) \leq -1$, an absurdity. □

Note that the proof of Lemma 2.1.5 in the case where X is irreducible is much simpler. Although we now apply this lemma only for an irreducible curve, later on it will be used in the proof of Theorem 3.2.1, for a (possibly) reducible curve without disconnecting nodes.

Theorem 2.1.6 (AK80, Theorem 8.8, p.108) *Let X/S be a family of irreducible curves with arithmetic genus $g > 0$. Then the Abel map $A : X \to \overline{J}^{-1}_{X/S}$ is a closed embedding.*

Proof. Since $\overline{J}_{X/S}$ is separated and X/S is projective, the map A is proper. Thus [EGA, IV_3, 8.11.5], A is a closed embedding if each of its geometric fibers is either empty or consists of a single reduced point. We may thus assume that S is the spectrum of an algebraically closed field. Now, by Lemmas 2.1.3 and 2.1.5, the fibers are projective spaces. Since X is a curve, this means that the fibers are empty, a single point or $\mathbb{P}^1$. But since X is irreducible and $g > 0$, the fibers must be empty or consist of a single point. □

2.2 Variations on the Abel map

2.2.1 Hilbert schemes and d-th Abel maps. The first Abel map of the previous section was constructed as the map taking a point of the curve to its ideal sheaf. One can try to generalize this map by considering a map taking a fixed number d of points of the curve to the tensor product of their ideal sheaves. In order to formalize this we introduce the *Hilbert schemes* $\mathrm{Hilb}^d(X)$.

For an integer $d \geq 1$, the *d-th Hilbert functor* of a family of curves X/S is the contravariant functor

$$\mathbf{Hilb}^d : (S\text{-schemes})^\circ \longrightarrow (\text{sets})$$

taking an S-scheme T to the set of relative length-d subschemes of X_T flat over T. This functor is represented by a projective S-scheme $\mathrm{Hilb}^d(X/S)$; see [FGIKNV, Chapter 5]. Let $\mathcal{U}$ be the universal subscheme of $X \times_S \mathrm{Hilb}^d(X/S)$ and let $\mathcal{M}$ be its ideal sheaf. Then $\mathcal{M}$ is a torsion-free rank-1 sheaf on the family $X \times_S \mathrm{Hilb}^d(X/S)/\mathrm{Hilb}^d(X/S)$ having degree $-d$. The *d-th Abel map* of X/S is the rational map

$$A^{(d)} : \mathrm{Hilb}^d(X/S) \longrightarrow \overline{J}_{X/S}^{-d}$$

associated to $\mathcal{M}$, where $A^{(d)}$ is defined only on the points over which $\mathcal{M}$ is simple. Note that $A^{(1)}$ is the first Abel map A defined in Section 1.1, because $\mathrm{Hilb}^1(X/S) = X$.

Even for a single curve X, the map $A^{(d)}$ is usually not defined everywhere if X is reducible. In fact, suppose there is a subcurve Y of X such that $\delta_Y \leq d$. Let $\Sigma = Y \cap Y^c$ and let I_Σ be its ideal sheaf. Since $\Sigma = Y \cap Y^c$, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_\Sigma & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_\Sigma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (I_\Sigma)_Y \oplus (\mathcal{I}_\Sigma)_{Y^c} & \longrightarrow & \mathcal{O}_Y \oplus \mathcal{O}_{Y^c} & \longrightarrow & \mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma & \longrightarrow & 0 \end{array}$$

has exact lines. Also, the cokernels of the second and third vertical maps are isomorphic to $\mathcal{O}_\Sigma$. Then, by the snake lemma, the cokernel of the first vertical map must be zero, and therefore $I_\Sigma \cong (I_\Sigma)_Y \oplus (I_\Sigma)_{Y^c}$, showing that I_Σ is not simple.

On the other hand, suppose there exists a length-d subscheme Σ of X whose ideal sheaf I_Σ is not simple. By Lemma 1.1.5 there is a subcurve Y of X such that $I_\Sigma = (I_\Sigma)_Y \oplus (I_\Sigma)_{Y^c}$. We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_\Sigma & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_\Sigma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (I_\Sigma)_Y \oplus (\mathcal{I}_\Sigma)_{Y^c} & \longrightarrow & \mathcal{O}_Y \oplus \mathcal{O}_{Y^c} & \longrightarrow & \mathcal{O}_{\Sigma\cap Y} \oplus \mathcal{O}_{\Sigma \cap Y^c} & \longrightarrow & 0 \end{array}$$

with exact lines, and whose first vertical map is an isomorphism. Thus, by the snake lemma, the cokernels of the second and third vertical maps are isomorphic. Then $\mathcal{O}_{Y\cap Y^c} = \mathcal{O}_{Y \cap Y^c \cap \Sigma}$, and so $\Sigma \supset Y \cap Y^c$. Thus $\delta_Y \leq d$.

To summarize, we showed that the d-th Abel map $A^{(d)}$ of a family X/S is well-defined if and only if the fibers of the family have no subcurve Y with $\delta_Y \leq d$.

2.2.2 Bigraded Abel maps. We can also consider a slight generalization of the d-th Abel maps defined above, the *bigraded Abel maps of bidegree* (d,n). Those are rational maps from $\mathrm{Hilb}^d(X/S)\times_S J^n_{X/S}$ to $\overline{J}^{n-d}_{X/S}$ defined by taking a pair of T-points $(\Sigma,\mathcal{M})$ to the sheaf $I_\Sigma\otimes\mathcal{M}$, where I_Σ is the ideal sheaf of a relative length-d subscheme Σ of X_T flat over T, and $\mathcal{M}$ is an invertible sheaf of degree n on X_T/T. To define these maps, we can also use the *multiplication map of bidegree* (r,s)

$$\mu_{r,s}:\overline{J}^r_{X/S}\times_S J^s_{X/S}\longrightarrow \overline{J}^{r+s}_{X/S}$$

defined naturally as a morphism of algebraic spaces, that is, a morphism of functors, as follows: If T is an S-scheme, $\mathcal{F}$ is a simple torsion-free sheaf of rank 1 and degree r on X_T/T and $\mathcal{L}$ is an invertible sheaf of degree s on X_T/T, set

$$\mu_{r,s}(T)(\mathcal{F},\mathcal{M})=\mathcal{F}\otimes\mathcal{M}$$

which is simple, torsion-free and of rank 1 on X_T/T.

The bigraded Abel map is the rational map given by the composition

$$A^{(d)}_n:\mathrm{Hilb}^d(X/S)\times_S J^n_{X/S}\xrightarrow{(A^{(d)},\mathrm{Id})}\overline{J}^{-d}_{X/S}\times_S J^n_{X/S}\xrightarrow{\mu_{-d,n}}\overline{J}^{n-d}_{X/S}$$

of the d-th Abel map of X/S with the multiplication map $\mu_{-d,n}$. Note that, like $A^{(d)}$, the map $A^{(d)}_n$ is usually just rational. More precisely, $A^{(d)}_n$ is well-defined if and only if the fibers of the family X/S have no subcurve Y with $\delta_Y\leq d$.

Now let $\mathcal{M}$ be an invertible sheaf of degree n on X/S. Then, for any S-scheme T, the sheaf $\mathcal{M}_T$ is invertible of degree n on X_T/T, where $\mathcal{M}_T$ is the pullback of $\mathcal{M}$ by $X_T\to X$. So, fixing $\mathcal{M}$ on X, we may define yet another Abel map

$$A^{(d)}_{\mathcal{M}}:\mathrm{Hilb}^d(X/S)\longrightarrow\overline{J}^{n-d}_{X/S},$$

given by the composition of the map $S\to J^n_{X/S}$ induced by $\mathcal{M}$ with $A^{(d)}_n$.

Example 2.2.3 (E01, Example 39, p. 3074) Let X be a curve of genus 1 without disconnecting nodes. For any integer n, Esteves found a proper subscheme of $\overline{J}^n_X$ which is isomorphic to the curve X. More precisely, for any smooth point P on X and any invertible sheaf M of degree $n+1$ on X, set $E:=M^{-1}\otimes\mathcal{O}_X(P)$. Then E is a polarization on X such that

(i) $\overline{J}^P_E$ is an open subscheme of $\overline{J}^n_X$;

(ii) $A^{(1)}_M:X\to\overline{J}^n_X$ is an isomorphism onto $\overline{J}^P_E$. □

2.2.4 Abel maps of families of pointed curves. At last, we define the first Abel map of a family of pointed curves. A *family of pointed curves* is a family of curves $f : X \to S$ with a section $\sigma : S \to X$ through the smooth locus of f. (The composition $f \circ \sigma$ is the identity on S, and for every $s \in S$, $\sigma(s)$ is a smooth point of $X(s)$.) Then the Abel map A_σ of $(X/S, \sigma)$ is the map that takes a point Q of $X(s)$ to the sheaf $\mathfrak{m}_Q(P) = \mathfrak{m}_Q \otimes \mathcal{O}_{X(s)}(P)$, where $P = \sigma(s)$. Since $\mathfrak{m}_Q(P)$ has degree 0, this map has image in $\overline{J}^0_{X/S}$. More precisely, if $\Sigma := \sigma(S)$ then $\mathcal{O}_X(\Sigma)$ is an invertible sheaf of degree 1 on X/S and $A_\sigma := A^{(1)}_{\mathcal{O}_X(\Sigma)}$. We say that

$$A_\sigma : X \longrightarrow \overline{J}^0_{X/S}$$

is the *first Abel map of* $(X/S, \sigma)$. In Chapter 3 we consider the first Abel maps A and A_σ for families of (pointed) curves without disconnecting nodes.

More generally, let $\mathcal{M}_d := \mathcal{O}_X(d\Sigma)$. The (rational) map

$$A^{(d)}_\sigma : \mathrm{Hilb}^d(X/S) \longrightarrow \overline{J}^0_{X/S}$$

defined as $A^{(d)}_\sigma := A^{(d)}_{\mathcal{M}_d}$ is the *d-th Abel map of the pointed curve* $(X/S, \sigma)$.

The second Abel map $A^{(2)}_P$ of a pointed curve (X, P) is well defined if there are no subcurves Y of X with $\delta_Y \leq 2$. In Chapter 5 we consider the map $A^{(2)}_P$ for a two-component nodal curve $X = X_1 \cup X_2$ such that $\delta_{X_1} = \delta_{X_2} = 2$. We'll deform the curve X to a family $\mathcal{X}/S$ and define a morphism from a blowup of $\mathcal{X} \times_S \mathcal{X}$ to $\overline{J}^0_{\mathcal{X}}$ whose image for a pair of smooth points of X is the image under $A^{(2)}_P$ twisted by a degree zero sheaf on X.

Chapter 3

On the first Abel map

Let X/S be a family of Gorenstein curves without disconnecting nodes, and consider the associated (first) Abel map A, or A_σ if X/S is pointed. (The case of curves with disconnecting nodes will be dealt with in the next chapter.) We already know that the first Abel maps are well-defined, but we still have the disadvantage of the nonseparatedness of $\overline{J}_{X/S}$. In this chapter we find separated subspaces $\overline{J}_0$ of $\overline{J}_{X/S}$ through which the Abel map A (or A_σ) factors. Furthermore, we show in Theorem 3.2.1 that the map A (or A_σ) is a closed embedding in $\overline{J}_0$.

Each of the subpaces $\overline{J}_0$ above will be given as $\overline{J}^s_{\mathcal{E}}$ or $\overline{J}^\sigma_{\mathcal{E}}$ for some polarization $\mathcal{E}$ on X/S. By Theorem 1.3.7, if $\overline{J}^s_{\mathcal{E}} = \overline{J}^{ss}_{\mathcal{E}}$ and $\overline{J}_0 = \overline{J}^s_{\mathcal{E}}$, then this space is in fact S-proper. Now, fix a geometric point s of S, let Y be a proper subcurve of $X(s)$, and set $E := \mathcal{E}(s)$. For a torsion-free rank-1 sheaf I on $X(s)$, we have $\beta_I(Y) = 0$ if and only if $\chi(I_Y) = -\deg_Y(E)/\mathrm{rk}(E)$. So stability will be the same as semi-stability with respect to $\mathcal{E}$ if and only if

$$\frac{\deg_Y(E)}{\mathrm{rk}(E)} \notin \mathbb{Z}$$

for every geometric s in S and every proper subcurve Y of $X(s)$. This will allow us to show that, in some cases, the subspace $\overline{J}_0$ is a proper S-scheme. (This will not be necessary for the Abel map A_σ of a pointed family $(X/S, \sigma)$.)

3.1 First Abel maps of a family of curves

In this section we define the subschemes $\overline{J}_0$ of $\overline{J}_{X/S}$ for the Abel maps A of X/S (Proposition 3.1.2) and A_σ of $(X/S, \sigma)$ (Proposition 3.1.5). For the Abel map A, we need an extra condition on the fibers of the family X/S. This condition, as we will see, is not a major drawback since it holds for families of stable curves. Alternatively, in Section 3.3 we consider the case of a single curve, without disconnecting nodes, and this condition is not needed.

Lemma 3.1.1 *Let X be a Gorenstein curve of arithmetic genus g, and let Y be a nonempty connected subcurve of X of arithmetic genus g_Y. Then* $\deg_Y(\omega_X) = 2g_Y - 2 + \delta_Y$.

Proof. (See [Cat82, Lemma 1.12, p. 61].) Let $\mathcal{K}$ be the kernel of the projection $\omega_X \twoheadrightarrow \omega_X|_{Y^c}$. We claim that $\mathcal{K}$ is the dualizing sheaf for Y. First, since ω_X is the dualizing sheaf on X, there is a trace morphism $t : H^1(X, \omega_X) \to k$. Now, consider the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \omega_X \longrightarrow \omega_X|_{Y^c} \longrightarrow 0. \tag{3.1}$$

Composing t with the induced morphism $H^1(Y, \mathcal{K}) \to H^1(X, \omega_X)$, we get a trace morphism for $\mathcal{K}$

$$t_Y : H^1(Y, \mathcal{K}) \longrightarrow k.$$

Let $\mathcal{F}$ be a coherent sheaf on Y. We must show that the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \mathcal{K}) \times H^1(Y, \mathcal{F}) \longrightarrow H^1(Y, \mathcal{K})$$

followed by t_Y gives an isomorphism $\mathrm{Hom}(\mathcal{F}, \mathcal{K}) \xrightarrow{\sim} H^1(Y, \mathcal{F})^\vee$. Let $i : Y \hookrightarrow X$ be the inclusion, and consider the coherent sheaf $i_*\mathcal{F}$ on X. Then, since ω_X is the dualizing sheaf of X, we have an isomorphism

$$\mathrm{Hom}(i_*\mathcal{F}, \omega_X) \xrightarrow{\sim} H^1(X, i_*\mathcal{F})^\vee = H^1(Y, \mathcal{F})^\vee$$

induced by t.

Also, from (3.1), we get an induced exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{K}) \longrightarrow \mathrm{Hom}(i_*\mathcal{F}, \omega_X) \longrightarrow \mathrm{Hom}(i_*\mathcal{F}, \omega_X|_{Y^c}).$$

We claim that $\mathrm{Hom}(i_*\mathcal{F}, \omega_X|_{Y^c}) = 0$. Indeed, $f : i_*\mathcal{F} \to \omega_X|_{Y^c}$ be a morphism. Note that the image of f has support on $Y \cap Y^c$, because $i_*\mathcal{F}$ has support on Y, and $\omega_X|_{Y^c}$ has support on Y^c. So the image of f is a torsion subsheaf of $\omega_X|_{Y^c}$. But $\omega_X|_{Y^c}$ is a torsion-free sheaf, since ω_X is invertible. Hence $\mathrm{Hom}(i_*\mathcal{F}, \omega_X|_{Y^c}) = 0$. Therefore, there is an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \mathcal{K}) \xrightarrow{\sim} H^1(Y, \mathcal{F})^\vee$$

induced by t_Y.

Since $\mathcal{K}$ is the dualizing sheaf of Y, we have $\deg(\mathcal{K}) = 2g_Y - 2$. Now, consider the following natural diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \omega_X & \longrightarrow & \omega_X|_{Y^c} & \longrightarrow & 0 \\ & & \phi\downarrow & & \psi\downarrow & & \| & & \\ 0 & \longrightarrow & \omega_X|_Y & \longrightarrow & \omega_X|_Y \oplus \omega_X|_{Y^c} & \longrightarrow & \omega_X|_{Y^c} & \longrightarrow & 0 \end{array}$$

where the horizontal sequences are exact. By the snake lemma, the cokernel of ϕ is equal to the cokernel of ψ. But the cokernel of ψ is $\omega_X|_{Y\cap Y^c}$, so

$$\chi(\mathrm{Coker}(\phi)) = \chi(\omega_X|_{Y\cap Y^c}).$$

Now, since ϕ is an injection, we have

$$\chi(\mathrm{Coker}(\phi)) = \chi(\omega_X|_Y) - \chi(\mathcal{K}) = \deg_Y(\omega_X) - \deg(\mathcal{K}).$$

On the other hand, since ω_X is invertible, and the intersection $Y \cap Y^c$ is finite, we have

$$\chi(\omega_X|_{Y\cap Y^c}) = \chi(\mathcal{O}_{Y\cap Y^c}) = \delta_Y$$

But $\deg(\mathcal{K}) = 2g_Y - 2$, so

$$\deg_Y(\omega_X) - (2g_Y - 2) = \chi(\mathrm{Coker}(\phi)) = \delta_Y,$$

and we are done. □

Proposition 3.1.2 *Let X/S be a family of Gorenstein curves of genus $g \geq 2$, and assume that the geometric fibers of the family have no disconnecting nodes. Let*

$$\mathcal{E} := \omega_{X/S}^{\oplus g} \oplus \mathcal{O}_X^{\oplus(g-2)}.$$

Then $\mathcal{E}$ is a polarization on X/S such that A factors through $\overline{J}_{\mathcal{E}}^{ss}$.

Moreover if, for every geometric point s of S, $X(s)$ contains no rational subcurve Y with $\delta_Y = 2$, then A actually factors through $\overline{J}_{\mathcal{E}}^{s}$.

Proof. To see $\mathcal{E}$ is a polarization, note that $\deg(\mathcal{E}) = g(2g-2)$ and $\mathrm{rk}(\mathcal{E}) = 2g-2$, so the slope of $\mathcal{E}$ is $g \in \mathbb{Z}$.

Now, to show A factors trough $\overline{J}_{\mathcal{E}}^{ss}$, we must show that for any geometric s in S, and any point $Q \in X(s)$, the sheaf $\mathfrak{m}_Q$ is semi-stable with respect to $E := \mathcal{E}(s)$. Since the degree of $\mathfrak{m}_Q$ is -1 we have $\chi(\mathfrak{m}_Q) = -g$, and hence

$$\chi(\mathfrak{m}_Q \otimes E) = \mathrm{rk}(E)\chi(\mathfrak{m}_Q) + \deg(E) = (2g-2)(-g) + g(2g-2) = 0.$$

Fix a connected subcurve Y of $X(s)$ and let g_Y be its arithmetic genus. Then the degree of $\mathfrak{m}_Q$ on Y is either -1 if $Q \in Y$ or 0 otherwise. In any case, $\deg_Y(\mathfrak{m}_Q) \geq -1$. Thus $\chi((\mathfrak{m}_Q)_Y) \geq -g_Y$. So $\chi(\mathfrak{m}_Q \otimes E|_Y) \geq (2g-2)(-g_Y) + g\deg_Y(\omega_X)$. By Lemma 3.1.1, $\deg_Y(\omega_X) = 2g_Y - 2 + \delta_Y$, so

$$\chi(\mathfrak{m}_Q \otimes E|_Y) \geq -g_Y(2g-2) + g(2g_Y - 2 + \delta_Y) = 2g_Y + g(\delta_Y - 2). \tag{3.2}$$

Now, since by hypothesis $X(s)$ has no disconnecting node, we have $\delta_Y \geq 2$, showing that (3.2) is positive. This already shows that A factors through $\overline{J}^{ss}_{\mathcal{E}}$.

Moreover, by (3.2), if $\chi(\mathfrak{m}_Q \otimes E|_Y) = 0$ then $g_Y = 0$ and $\delta_Y = 2$, thus showing the last statement. □

Remark 3.1.3 Let X/S be a family of Gorenstein curves of genus g. Note that stability is the same as semi-stability for any given polarization $\mathcal{E}$ on X/S if and only if, for every geometric point s of S and every proper subcurve Y of $X(s)$, the rank of $\mathcal{E}$ does not divide its degree on Y.

Assume that, for every geometric $s \in S$, the curve $X(s)$ contains no rational subcurve Y with $\delta_Y \leq 2$. Let s be a geometric point of S, and Y a proper subcurve of $X(s)$. We state that $\deg_Y(\omega_{X(s)}) > 0$. Indeed, assume first that Y is connected. Then by Lemma 3.1.1, we have

$$\deg_Y(\omega_{X(s)}) = 2g_Y - 2 + \delta_Y.$$

If $g_Y = 0$ then, by hypothesis, $\delta_Y > 2$, and thus $\deg_Y(\omega_{X(s)}) > 0$. If $g_Y \geq 1$ then

$$\deg_Y(\omega_{X(s)}) \geq \delta_Y > 0,$$

since $X(s)$ is connected. Now, assume that Y is not connected, and let $Y_1, \dots, Y_k$ be the connected components of Y. Then each Y_i is a nonempty connected subcurve of X, and thus $\deg_{Y_i}(\omega_{X(s)}) > 0$. But $\omega_{X(s)}|_Y = \omega_{X(s)}|_{Y_1} \oplus \ldots \oplus \omega_{X(s)}|_{Y_k}$ and hence,

$$\deg_Y(\omega_{X(s)}) = \deg_{Y_1}(\omega_{X(s)}) + \ldots + \deg_{Y_k}(\omega_{X(s)}) > 0.$$

On the other hand, since X is Gorenstein, ω_X is invertible, and thus

$$\deg(\omega_{X(s)}) = \deg_Y(\omega_{X(s)}) + \deg_{Y^c}(\omega_{X(s)}) > \deg_Y(\omega_{X(s)}).$$

So we showed that

$$0 < \deg_Y(\omega_{X(s)}) < \deg(\omega_{X(s)}) = 2g - 2 \tag{3.3}$$

for each proper nonempty subcurve $Y \subset X(s)$.

Now, assume $g \geq 2$. Consider the polarization $\mathcal{E} := \omega_{X/S}^{\oplus g} \oplus \mathcal{O}_X^{\oplus(g-2)}$ of the previous proposition. Then $\overline{J}_{\mathcal{E}}^s = \overline{J}_{\mathcal{E}}^{ss}$ if and only if for every geometric $s \in S$ and every proper subcurve Y of $X(s)$ we have

$$\frac{\deg_Y(E)}{\mathrm{rk}(E)} = \frac{g \deg_Y(\omega_{X(s)})}{(g-1)2} \notin \mathbb{Z},$$

where $E = \mathcal{E}(s)$. By (3.3), the only way this number can be an integer is when g is divisible by 2 and $\deg_Y(\omega_{X(s)}) = g - 1$.

We showed that if X/S is a family of curves of odd genus $g \geq 3$ such that, for every geometric point s of S, $X(s)$ contains no rational subcurve Y with $\delta_Y \leq 2$, then $\overline{J}_{\mathcal{E}}^s = \overline{J}_{\mathcal{E}}^{ss}$, and hence, not only is this space separated, but also a proper S-scheme, by Theorem 1.3.7.

Remark 3.1.4 Notice that if X/S is a family of stable curves, then the condition that $X(s)$ doesn't contain any rational subcurve Y with $\delta_Y \leq 2$ is satisfied.

Proposition 3.1.5 *Let $(X/S, \sigma)$ be a family of pointed curves, and assume the geometric fibers of the family have no disconnecting nodes. Let*

$$\mathcal{E} := \omega_{X/S} \oplus \mathcal{O}_X.$$

Then $\mathcal{E}$ is a polarization on X/S such that A_σ factors through $\overline{J}_{\mathcal{E}}^{\sigma}$.

Proof. The vector bundle $\mathcal{E}$ is indeed a polarization on X/S, since $\deg(\mathcal{E}) = 2g-2$ and $\mathrm{rk}(\mathcal{E}) = 2$, where g is the arithmetic genus of the curves of the family.

As in Proposition 3.1.2, we fix a geometric point s in S and a closed point Q in $X(s)$. Also let $E = \mathcal{E}(s)$ and let P be the image of $\sigma(s)$. Recall that P is a smooth point of $X(s)$. Since $\mathfrak{m}_Q(P)$ has degree 0, $\chi(\mathfrak{m}_Q(P) \otimes E) = 2(0 + 1 - g) + 2g - 2 = 0$.

Now, fix a connected proper subcurve Y of X of arithmetic genus g_Y. Then

$$\deg_Y(\mathfrak{m}_Q(P)) = \begin{cases} 1, & \text{if } P \in Y \text{ and } Q \notin Y; \\ 0, & \text{if } P,\, Q \in Y \text{ or } P,\, Q \notin Y; \\ -1, & \text{if } P \notin Y \text{ and } Q \in Y. \end{cases}$$

So $\deg_Y(\mathfrak{m}_Q(P)) \geq -1$, with $\deg_Y(\mathfrak{m}_Q(P)) \geq 0$ if $P \in Y$. Thus $\chi(\mathfrak{m}_Q(P)_Y) \geq -g_Y$, with $\chi(\mathfrak{m}_Q(P)_Y) > -g_Y$ if $P \in Y$, and hence

$$\chi(\mathfrak{m}_Q(P)_Y \otimes E|_Y) \geq 2(-g_Y) + (2g_Y - 2 + \delta_Y) = \delta_Y - 2. \tag{3.4}$$

Now, since $X(s)$ has no disconnecting nodes, we have $\delta_Y \geq 2$, so (3.4) is positive. Furthermore, the first inequality is strict if $P \in Y$, showing that $\mathfrak{m}_Q(P)$ is P-quasi-stable with respect to E. □

3.2 Closed embeddedness

Consider the Abel map A or A_σ, and assume it factors through a separated open subspace $\overline{J}_0$ of $\overline{J}$. Then we show that A or A_σ is a closed embedding in $\overline{J}_0$. To prove this, since the Abel map is proper and its fibers are projective spaces by Lemmas 2.1.3 and 2.1.5, we must only show that its fibers can't contain two distinct closed points.

Theorem 3.2.1 *Let X/S be a family of curves of arithmetic genus $g > 0$, and whose geometric fibers have no disconnecting nodes. Let A be either the Abel map of X/S or the Abel map of $(X/S, \sigma)$, where σ is a section of X/S. Assume there is a separated subspace $\overline{J}_0$ of $\overline{J}_{X/S}$ such that A factors through it. Then $A : X \to \overline{J}_0$ is a closed embedding.*

Proof. Since X is projective over S, and $\overline{J}_0$ is separated, the map A is proper by [H, Corollary 4.8 (e), p. 102]. Hence, by [EGA, IV_3, Proposition 8.11.5, p. 42], A is a closed embedding if and only if for each point $v \in \overline{J}_0$, the fiber $A^{-1}(v)$ is either empty or a point. Now, note that the fibers of A are projective spaces by Lemmas 2.1.3 and 2.1.5, so it is enough to show that $A^{-1}(v)$ has at most a single closed point. Since A commutes with base change, we may assume S is the spectrum of an algebraically closed field.

Assume that there are distinct points $Q_1, Q_2 \in X$ such that there is an isomorphism between their ideal sheaves $\mathfrak{m}_{Q_1} \xrightarrow{\sim} \mathfrak{m}_{Q_2}$. (Notice that if A is the Abel map of the pointed family $(X/S, \sigma)$, we should consider an isomorphism $\mathfrak{m}_{Q_1}(P) \xrightarrow{\sim} \mathfrak{m}_{Q_2}(P)$, where $P = \sigma(S)$. But such an isomorphism yields an isomorphism between $\mathfrak{m}_{Q_1}$ and $\mathfrak{m}_{Q_2}$, by tensoring with $\mathcal{O}_X(-P)$.)

If Q_1 is a singular point of X then we have $Q_1 = Q_2$, since otherwise $\mathfrak{m}_{Q_2}$ would be invertible at Q_1 and hence, by the isomorphism, $\mathfrak{m}_{Q_1}$ would be invertible at Q_1. Hence we may assume that Q_1 and Q_2 are smooth points of X, so $\mathfrak{m}_{Q_1}$ and $\mathfrak{m}_{Q_2}$ are both invertible. Let Y be the irreducible component of X to which Q_1 belongs. Then the degree of $\mathfrak{m}_{Q_1}$ on Y is -1, and thus the degree of $\mathfrak{m}_{Q_2}$ on Y must also be -1. Hence Q_2 is also on Y. By Theorem 2.1.6, $Q_1 = Q_2$ or Y is isomorphic to $\mathbb{P}^1$. Since by hypothesis $Q_1 \neq Q_2$, we have $Y \cong \mathbb{P}^1$. Note that Y is a proper subcurve of X, because $g \neq 0$.

Let f be the rational function that gives the isomorphism $\mathfrak{m}_{Q_1} \xrightarrow{\sim} \mathfrak{m}_{Q_2}$. Then f is constant on every irreducible component of X other than Y. Let Z be a connected component of Y^c. Then f is constant on Z, because f has no zeroes on Z. Now, $f|_Y$ is a function of degree 1 on Y, hence injective. Then the intersection $Y \cap Z$ consists of a single point, since f must have the same value on every point of $Y \cap Z$. Moreover, the intersection $Y \cap Z$ is transversal, since otherwise $f|_Y$ would be infinitesimally constant at $Y \cap Z$, because it is so on Z. But since f is of degree 1, this would mean that f is constant on Y. Hence the point of intersection between Y and Z is a node of X.

Therefore, every point of $Y \cap Y^c$ is a node and, since X has no disconnecting nodes, Y^c is connected. Also, as $\delta_Y \geq 2$ there are at least two points on $Y \cap Y^c$, and f has the same value on them. So f is constant also on Y, contradicting the fact that $f|_Y$ is a function of degree 1. Hence $Q_1 = Q_2$, proving that the nonempty fibers of A consist of a single point, and hence that A is indeed a closed embedding. $\square$

Now we apply the theorem to the subspaces of $\overline{J}_{X/S}$ defined in the previous section.

Corollary 3.2.2 *Let X/S be a family of curves of genus $g \geq 2$. Assume the geometric fibers of the family have no disconnecting nodes, and do not contain rational subcurves Y with $\delta_Y = 2$. Then there exists an open separated subspace $\overline{J}_0$ of $\overline{J}_{X/S}^{-1}$ such that $A : X \to \overline{J}_0$ is a closed embedding. Moreover, if g is odd then $\overline{J}_0$ is a proper S-scheme.*

Proof. Let $\mathcal{E}$ be as in Proposition 3.1.2 and take $\overline{J}_0$ to be $\overline{J}_{\mathcal{E}}^s$, so that A factors trough $\overline{J}_0$. By Theorem 1.3.7, $\overline{J}_0$ is separated and thus, by Theorem 3.2.1, $A : X \to \overline{J}_0$ is a closed embedding. The last claim follows from Remark 3.1.3. $\square$

Corollary 3.2.3 *Let $(X/S, \sigma)$ be a family of pointed curves of arithmetic genus $g > 0$, and whose geometric fibers have no disconnecting nodes. Then there is an S-proper open subspace $\overline{J}_0$ of $\overline{J}_{X/S}^0$ such that $A_\sigma : X \to \overline{J}_0$ is a closed embedding.*

Proof. Let $\mathcal{E}$ be as in Proposition 3.1.5, and let $\overline{J}_0 = \overline{J}_{\mathcal{E}}^\sigma$, so that A_σ factors trough $\overline{J}_0$. By Theorem 1.3.7, $\overline{J}_0$ is proper and thus, by Theorem 3.2.1, $A_\sigma : X \to \overline{J}_0$ is a closed embedding. $\square$

3.3 Other target spaces

For a curve X without disconnecting nodes, we will find complete open subschemes $\overline{J}_{\{p_i\}}$ of $\overline{J}_X^{-1}$ such that the Abel map A of X has image in $\overline{J}_{\{p_i\}}$ and thus, by Theorem 3.2.1, is a closed embedding in it. This improves Corollary 3.2.2, but the construction cannot usually be carried out for families.

Let $X_1, \dots, X_k$ be the irreducible components of X. For each collection of positive integers $p_1, \dots, p_k$, let $q := \sum_{i=1}^k p_i$, and define a subset $\overline{J}_{\{p_i\}}$ of $\overline{J}_X^{-1}$ as the set of those simple rank-1 torsion-free sheaves I of degree -1 such that

$$\deg_Y(I) > -\frac{\delta_Y}{2} - \sum_{X_i \subseteq Y} \frac{p_i}{q}$$

for every proper nonempty connected subcurve Y of X. The following proposition shows that $\overline{J}_{\{p_i\}}$ is a separated open subscheme of $\overline{J}_X^{-1}$, which is also complete if q is odd. To show this, we will find a polarization E on X such that $\overline{J}_{\{p_i\}} = \overline{J}_E^s$.

Proposition 3.3.1 *Let X be a curve of arithmetic genus $g > 0$, and assume X has no disconnecting nodes. Let $X_1, \dots, X_k$ be the irreducible components of X, and $p_1, \dots, p_k$ be a collection of positive integers. Then $\overline{J}_{\{p_i\}}$ is a separated open subscheme of $\overline{J}_X^{-1}$ that contains the image of the Abel map A, and $A : X \to \overline{J}_{\{p_i\}}$ is a closed embedding. Furthermore, if the sum $p_1 + \ldots + p_k$ is odd, then $\overline{J}_{\{p_i\}}$ is complete.*

Proof. Let $P_1, \dots, P_k$ be smooth points of X such that $P_i \in X_i$, and let $q := p_1 + \ldots + p_k$. We state that

$$E := \omega_X^{\otimes q}(2p_1P_1 + ... + 2p_kP_k) \oplus \mathcal{O}_X^{\oplus(2q-1)}$$

is a polarization for X such that $\overline{J}_E^s = \overline{J}_{\{p_i\}}$.

First we see that E is indeed a polarization. Let g be the arithmetic genus of X. Then

$$\deg(E) = q(2g-2) + 2p_1 + \ldots + 2p_k = 2qg$$

and $\mathrm{rk}(E) = 1 + 2q - 1 = 2q$, so the slope of E is g.

Now, let Y be a proper connected subcurve of X. Let $\epsilon(Y) := \sum_{X_i \subseteq Y} p_i$, and note that $\epsilon(Y) > 0$. Then we have

$$\deg_Y(E) = q \deg_Y(\omega_X) + 2\epsilon(Y).$$

By Lemma 3.1.1 we have $\deg_Y(\omega_X) = 2g_Y - 2 + \delta_Y$, and thus

$$\deg_Y(E) = q(2g_Y - 2 + \delta_Y) + 2\epsilon(Y), \tag{3.5}$$

where g_Y is the arithmetic genus of Y.

Let I be a torsion-free rank-1 sheaf on X of degree -1. First notice that $\chi(I \otimes E) = 0$. Now, by definition, $\beta_I(Y) > 0$ if and only if $\chi(I_Y) > -\deg_Y(E)/\mathrm{rk}(E)$. By (3.5), this happens if and only if

$$\deg_Y(I) + 1 - g_Y > 1 - g_Y - \frac{\delta_Y}{2} - \frac{\epsilon(Y)}{q},$$

thus showing that $\overline{J}_E^s = \overline{J}_{\{p_i\}}$. Therefore $\overline{J}_{\{p_i\}}$ is indeed a separated scheme.

Now, we must show that A factors through $\overline{J}_{\{p_i\}}$. (Since we showed that this scheme is separated, from Theorem 3.2.1 it will follow that $A : X \to \overline{J}_{\{p_i\}}$ is a closed embedding.) Let I be in the image of the Abel map A, that is, I is the ideal sheaf of a point on X. Then $\chi(I) = -g$ and

$$\beta_I(X) = -g + \frac{\deg(E)}{\mathrm{rk}(E)} = 0.$$

Also, for each proper connected subcurve Y of X, we have $\deg_Y(I) \geq -1$, and hence

$$\beta_I(Y) \geq (-1 + 1 - g_Y) + \frac{\deg_Y(E)}{2q} = -g_Y + \frac{\deg_Y(E)}{2q}.$$

Since X has no disconnecting nodes, $\delta_Y \geq 2$, and hence $\deg_Y(E) \geq 2qg_Y + 2\epsilon(Y)$, which implies that

$$\beta_I(Y) \geq \frac{\epsilon(Y)}{q} > 0, \tag{3.6}$$

showing that I is stable with respect to E.

In order for $\overline{J}_{\{p_i\}}$ to be complete it is enough that $\overline{J}_E^s = \overline{J}_E^{ss}$, that is, it is enough to show that the rank of E does not divide its degree on any proper subcurve Y of X. By (3.5), the latter is equivalent to saying

$$\frac{\delta_Y}{2} + \frac{\epsilon(Y)}{q} \notin \mathbb{Z} \tag{3.7}$$

for every proper subcurve Y. If δ_Y is even, then (3.7) is never an integer, because $\epsilon(Y) < q$. If δ_Y is odd, then (3.7) is an integer if and only if $1/2 + \epsilon(Y)/q$ is an integer, that is, if and only if $q + 2\epsilon(Y)$ is a multiple of $2q$. But, since $\epsilon(Y) < q$, this implies $2\epsilon(Y) = q$. So, if we choose the p_i's so that $q = p_1 + \ldots + p_k$ is odd, we get $\overline{J}_E^s = \overline{J}_E^{ss}$, and thus $\overline{J}_{\{p_i\}}$ is complete. (For instance, if the number k of irreducible components of X is odd, choose $p_i = 1$ for every i; and if k is even, choose $p_1 = 2$ and $p_i = 1$ for $i \neq 1$.) □

Remark 3.3.2 Let (X, P) be a pointed curve of arithmetic genus g without disconnecting nodes. Let $X_1, \ldots, X_k$ be the irreducible components of X, and assume $P \in X_1$. Fix $p_1 = 1$ and $p_i = 0$ for $i \neq 1$. The polarization of the previous proposition becomes $E = \omega_X(2P) \oplus \mathcal{O}_X$. Defining $\overline{J}_{\{p_i\}}$ as before, we still have $\overline{J}_{\{p_i\}} = \overline{J}_E^s$. Moreover, by (3.6), we see that A actually factors through $\overline{J}_E^P$ and, by Theorem 3.2.1, $A : X \to \overline{J}_E^P$ is a closed embedding if $g > 0$.

More generally, for a family of pointed curves $(X/S, \sigma)$ of arithmetic genus g, we may let $\mathcal{E} := \omega_{X/S}(2\Sigma) \oplus \mathcal{O}_X$, where Σ is the image of σ. Then $\mathcal{E}$ is a polarization. Also, the Abel map A of X/S factors through the proper S-space $\overline{J}_{\mathcal{E}}^\sigma$ and thus, by Theorem 3.2.1, it is a closed embedding in $\overline{J}_{\mathcal{E}}^\sigma$, if $g > 0$. Note that there is no hypothesis on the genus of the curves, so this improves Corollary 3.2.2 in the case of pointed families.

Chapter 4

On the first Abel map - II

In this chapter we focus on curves X having disconnecting nodes. As we saw in Example 2.1.2, in this case the Abel map is not well defined. Indeed, we showed that if Q is a disconnecting node of X, then its ideal sheaf $\mathfrak{m}_Q$ is not simple. For each fixed point P, we will define a map $\tilde{A}_P$ from the curve X to its compactified Jacobian, so that $\tilde{A}_P$ is a modification of A_P. More precisely, for each point Q of X that is not a disconnecting node, the image of Q under $\tilde{A}_P$ will be defined as $\mathfrak{m}_Q(P) \otimes M$, for certain invertible sheaf $M = M(Q)$ of degree zero on X. If Q is a disconnecting node, then the image of Q under $\tilde{A}_P$ will be a suitable invertible sheaf; see Proposition 4.2.2.

Furthermore, the map $\tilde{A}_P$ defined in this way factors through the locus of P-quasi-stable sheaves of X, and is in fact a closed embedding in it, if X has no rational component L such that every point on $L \cap L^c$ is a disconnecting node. We call such a component L a *disconnecting line.*

4.1 Tails

Let X be a curve, and assume X has a disconnecting node N. Then, by definition, there is a subcurve Z of X such that $Z \cap Z^c = \{N\}$ with multiplicity 1. In this case we say that Z and Z^c are the *tails* associated to N. Thus, a subcurve Z of X is a tail if and only if $\delta_Z = 1$. A tail is connected, because otherwise the curve X itself would be disconnected.

For Y and Z subcurves of X, we denote by $Y \wedge Z$ the maximum subcurve contained in $Y \cap Z$.

Lemma 4.1.1 *Let Z_1 and Z_2 be distinct tails of a curve X. Then exactly one of the following alternatives hold:*

(i) $Z_1 \cup Z_2 = X$;

(ii) $Z_1 \cap Z_2 = \emptyset$;

(iii) $Z_1 \subset Z_2$;

(iv) $Z_2 \subset Z_1$.

Proof. Let N_1 and N_2 be the disconnecting nodes of Z_1 and Z_2 respectively. If $Z_2 = Z_1^c$ then $Z_1 \cup Z_2 = X$ and we have (i). Assume $Z_1 \cup Z_2$ is not the whole curve X, so $Z_2 \neq Z_1^c$.

Let $\Sigma = Z_1 \cap Z_2$. If $\Sigma = \emptyset$ we have (ii). Note that (i) and (ii) cannot hold simultaneously, since otherwise the curve X would be disconnected. Now, assume that Σ is nonempty. If Σ is finite, then Z_2 contains no component of Z_1 and thus $Z_2 \subset Z_1^c$. So Σ consists of the separating node N_1 of Z_1. Hence $N_1 = N_2$, and so $Z_2 = Z_1^c$. Since we assumed the union $Z_1 \cup Z_2$ is not the whole X, we have that Σ is not finite. Therefore $Z_1 \wedge Z_2$ is not empty. We must show that in this case either (iii) or (iv) happen.

If $Z_1 \subset Z_2$ we have (iii). Now assume $Z_1 \not\subset Z_2$. Then $Z_1 \wedge Z_2 \neq Z_1$, and so $Z_1 \wedge Z_2^c$ is nonempty as well. Now, since Z_1 is connected, $Z_1 \wedge Z_2$ and $Z_1 \wedge Z_2^c$ must meet at the node N_2. So $N_2 \in Z_1$.

If $Z_2 \not\subset Z_1$, then $Z_1^c \not\subset Z_2^c$. So, as above, $Z_1^c \wedge Z_2^c$ and $Z_1^c \wedge Z_2$ meet at the node N_2, implying $N_2 \in Z_1^c$. Since Z_1 is a tail associated to N_1, and $N_2 \in Z_1 \cap Z_1^c$, we have that $N_2 = N_1$. Therefore $Z_1 = Z_2$ or $Z_1 = Z_2^c$, a contradiction. So $Z_2 \subset Z_1$ and we have (iv). □

Lemma 4.1.2 *Let Z be a tail of a curve X. There is a morphism*

$$\Gamma : \overline{J}_Z \times \overline{J}_{Z^c} \longrightarrow \overline{J}_X,$$

such that the image of a pair (L_1, L_2) of sheaves on Z and Z^c respectively is a sheaf L on X such that $L|_Z = L_1$ and $L|_{Z^c} = L_2$.

Proof. We'll construct the map Γ as a map of functors. Let T be a scheme. The T-points of $\overline{J}_Z$ (resp. $\overline{J}_{Z^c}$) correspond to torsion-free rank-1 sheaves on $Z \times T/T$ (resp. $Z^c \times T/T$). Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be torsion-free rank-1 sheaves on $Z \times T$ and $Z^c \times T$ respectively. We'll show that there is a unique (up to isomorphism) torsion-free rank-1 sheaf $\mathcal{L}$ on $X \times T$ that is invertible at $(Z \cap Z^c) \times T$, and such that $\mathcal{L}|_{Z \times T} = \mathcal{L}_1$ and $\mathcal{L}|_{Z^c \times T} = \mathcal{L}_2$.

Consider the following Cartesian diagram

$$\begin{array}{ccc} X \times T & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow \\ T & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

where p and q are the projections on the first and second factors, respectively. Let N be the separating node associated to the tail Z.

We may assume that $\mathcal{L}_1|_{\{N\}\times T}$ and $\mathcal{L}_2|_{\{N\}\times T}$ are trivial. Indeed, the sheaf $\mathcal{L}_1|_{\{N\}\times T}$ (resp. $\mathcal{L}_2|_{\{N\}\times T}$) is invertible, because $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) is torsion-free and rank-1 on $Z\times T/T$ (resp. $Z^c\times T/T$), and N is a simple point of Z (resp. Z^c). Now, by the equivalence relation on $\overline{J}_Z$ (resp. $\overline{J}_{Z^c}$), the sheaf $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) is equivalent to

$$\mathcal{L}_1' := \mathcal{L}_1\otimes q^*(\mathcal{L}_1|_{\{N\}\times T})^{-1} \text{ (resp. } \mathcal{L}_2' := \mathcal{L}_2\otimes q^*(\mathcal{L}_2|_{\{N\}\times T})^{-1}),$$

where $\{N\}\times T$ is identified with T by the projection q. Clearly, $\mathcal{L}_i'|_{\{N\}\times T}\cong \mathcal{O}_T$.

Since both $\mathcal{L}_1|_{\{N\}\times T}$ and $\mathcal{L}_2|_{\{N\}\times T}$ are isomorphic to $\mathcal{O}_T$, there is an isomorphism

$$\nu : \mathcal{L}_2|_{\{N\}\times T} \xrightarrow{\sim} \mathcal{L}_1|_{\{N\}\times T}.$$

Note that any two such isomorphisms differ by multiplication by a (unique) nonzero scalar. Let $\mathcal{L}$ be the kernel of the composition

$$\psi_\nu : \mathcal{L}_1\oplus\mathcal{L}_2 \longrightarrow \mathcal{L}_1|_{\{N\}\times T}\oplus\mathcal{L}_2|_{\{N\}\times T} \xrightarrow{(1,-\nu)} \mathcal{L}_1|_{\{N\}\times T}.$$

Then $\mathcal{L}$ is a torsion-free rank-1 sheaf on $X\times T$, invertible at $\{N\}\times T$, such that $\mathcal{L}|_{Z\times T}=\mathcal{L}_1$ and $\mathcal{L}|_{Z^c\times T}=\mathcal{L}_2$.

If $\nu' : \mathcal{L}_2|_{\{N\}\times T} \xrightarrow{\sim} \mathcal{L}_1|_{\{N\}\times T}$ is another isomorphism, then $\nu = a\nu'$ for some scalar a. Thus the map $(1,a):\mathcal{L}_1\oplus\mathcal{L}_2\to\mathcal{L}_1\oplus\mathcal{L}_2$ takes the kernel of ψ_ν isomorphically to the kernel of $\psi_{\nu'}$.

Conversely, let $\mathcal{I}$ be a torsion-free rank-1 sheaf on $X\times T$, invertible at $\{N\}\times T$. Assume there are isomorphisms $\lambda_1 : \mathcal{I}|_{Z\times T}\xrightarrow{\sim}\mathcal{L}_1$ and $\lambda_2 : \mathcal{I}|_{Z^c\times T}\xrightarrow{\sim}\mathcal{L}_2$. Then $\mathcal{I}$ is the kernel of ψ_ν, where $\nu = \lambda_1|_{\{N\}\times T}\circ\lambda_2^{-1}|_{\{N\}\times T}$.

Moreover, if the sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ are simple (over T), then the sheaf $\mathcal{L}$ is also simple (over T). Indeed, assume by contradiction that $\mathcal{L}$ is not simple, so for some $t\in T$, $\mathcal{L}(t)$ is not simple. We may thus assume that $X_T = X$ so $\mathcal{L}(t)=\mathcal{L}$. Then, by Lemma 1.1.5, there is a subcurve Y of X such that $\mathcal{L}\cong\mathcal{L}_Y\oplus\mathcal{L}_{Y^c}$. Thus, restricting to Z we get

$$\mathcal{L}_1=\mathcal{L}|_Z\cong\mathcal{L}_Y|_Z\oplus\mathcal{L}_{Y^c}|_Z$$

and, since $\mathcal{L}_1$ is simple and torsion-free, either $Y\wedge Z=\emptyset$, or $Y^c\wedge Z=\emptyset$. On the other hand, restricting $\mathcal{L}$ to Z^c we get either $Y\wedge Z^c=\emptyset$ or $Y^c\wedge Z^c=\emptyset$. Therefore, Z must be either Y or Y^c. Without loss of generality assume $Y=Z$ and $Y^c=Z^c$. Then

$$\mathcal{L}\cong\mathcal{L}_Z\oplus\mathcal{L}_{Z^c}=\mathcal{L}_1\oplus\mathcal{L}_2$$

and, restricting to the separating node N, we get $\mathcal{L}|_N \cong \mathcal{L}_1|_N \oplus \mathcal{L}_2|_N$, showing that $\mathcal{L}$ is not invertible at N, an absurd. $\square$

Remark 4.1.3 It can be shown that Γ is an isomorphism, with the inverse

$$\Gamma^{-1} : \overline{J}_X \longrightarrow \overline{J}_Z \times \overline{J}_{Z^c}$$

given by the restrictions to Z and Z^c. Since every simple torsion-free rank-1 sheaf on X must be invertible at the disconnecting nodes of X, the map Γ^{-1} is well-defined; see [E01, Example 37, p. 3073].

Let P be a simple point on a Gorenstein curve X, and take $Q \in X$. In Proposition 3.1.5 we showed that $\mathfrak{m}_Q(P)$ is P-quasi-stable with respect to the polarization $E := \omega_X \oplus \mathcal{O}_X$ if the curve has no disconnecting nodes. Now, assume the curve X has a disconnecting node. Then, by (3.4), if $P \in Y$, the condition $\chi(\mathfrak{m}_Q(P) \otimes E|_Y) > 0$ is still satisfied, and if $P \notin Y$ the condition $\chi(\mathfrak{m}_Q(P) \otimes E|_Y) \geq 0$ holds unless $Q \in Y$ and $\delta_Y = 1$. So we see that the P-quasi-stability condition fails exactly when $Q \in Y$, $\delta_Y = 1$ and $P \notin Y$, that is, when Y is a tail containing Q but not the fixed point P. Thus, to achieve P-quasi-stability we need only modify $\mathfrak{m}_Q(P)$ for Q belonging to tails not containing P.

Proposition 4.1.4 *Let P be a simple point of a curve X. Let $Q \in X$.*

(i) There is a (possibly empty) sequence of tails $Z_1 \subset \ldots \subset Z_r$ such that Z is a tail containing Q and not containing P if and only if $Z = Z_t$ for some t;

(ii) Let $N_1 \ldots, N_r$ be the disconnecting nodes associated to the tails $Z_1, \ldots, Z_r$ of (i). For each $t = 2, \ldots, r$ define $\tilde{Z}_t := \overline{Z_t - Z_{t-1}}$. Then there are invertible sheaves M and M' on X such that

$$\begin{aligned} M|_{Z_1} &= \mathcal{O}_{Z_1}(N_1), \qquad M'|_{Z_1} = \mathcal{O}_{Z_1}, \\ M|_{\tilde{Z}_t} &= M'|_{\tilde{Z}_t} = \mathcal{O}_{\tilde{Z}_t}(N_t - N_{t-1}) \ \textit{ for each } t = 2, \ldots, r, \\ M|_{Z_r^c} &= M'|_{Z_r^c} = \mathcal{O}_{Z_r^c}(-N_r). \end{aligned}$$

The sheaf M (resp. M') has degree 0 (resp. degree -1) on X, and degree 1 (resp. degree 0) on Z_t for each $t = 1, \ldots, r$.

Proof. Let Z_1 and Z_2 be two tails of X containg Q and not containing P. Since $Q \in Z_1 \cap Z_2$ and $P \notin Z_1 \cup Z_2$, by Lemma 4.1.1, we have either $Z_1 \subset Z_2$ or $Z_2 \subset Z_1$. Thus (i) is proved.

To construct M and M' in (ii) we apply Lemma 4.1.2 to Z_t and its tail Z_{t-1} for each $t = 2, \ldots, r$, and then to X and the tail Z_r. It follows from the definition of M (resp. M') that it has degree 0 (resp. degree -1) on the whole X and degree 1 (resp. degree 0) on each Z_t. $\square$

Note that if Q is a simple point of X, then the sheaves M and M' depend only on the irreducible component to which the point belongs.

Remark 4.1.5 Let P be a simple point of a Gorenstein curve X, and let $Q \in X$. Here we use the notation introduced in Proposition 4.1.4. Let $\tilde{Z}_{r+1} := Z_r^c$. For each t, denote by W_t (resp. W_t') the irreducible component of $\tilde{Z}_t$ containing N_t (resp. N_{t-1}). In addition, let $W_1' := W_{r+1} := \emptyset$. (Then both M and M' have degree 0 on W_1' and W_{r+1}.) It follows that, for each $t = 2, \ldots, r$, we have

$$\deg_{W_t}(M) = \deg_{W_t}(M') = 1 \quad \text{and} \quad \deg_{W_t'}(M) = \deg_{W_t'}(M') = -1.$$

In addition

$$\begin{aligned} \deg_{W_1}(M) &= 1, \quad \deg_{W_{r+1}'}(M) = -1 \\ \deg_{W_1}(M') &= 0, \quad \deg_{W_{r+1}'}(M') = -1. \end{aligned}$$

Note that, for each irreducible component W of X, if we have $\deg_W(M) \neq 0$ (resp. $\deg_W(M') \neq 0$) then either $W = W_t$ or $W = W_t'$ for some t.

Now, let Y be a connected subcurve of X. If Y does not contain any of the points $N_1, \ldots, N_r$, then clearly

$$\deg_Y(M) = \deg_Y(M') = 0.$$

Now assume Y contains some of the points $N_1, \ldots, N_r$. Let j (resp. k) be the minimum (resp. maximum) integer in $\{1, \ldots, r\}$ such that N_j (resp. N_k) belongs to Y. Then Y contains also N_t if $j \leq t \leq k$, because otherwise Y would not be a connected curve. It follows that Y contains all the irreducible components of $\tilde{Z}_{j+1}, \ldots, \tilde{Z}_k$ containing the nodes $N_j, \ldots, N_k$.

Furthermore, since M is invertible, and Y contains exactly the separating nodes $N_j, \ldots, N_k$, we have

$$\begin{aligned} \deg_Y(M) &= \sum_{t=1}^{r+1} (\deg_{W_t \wedge Y}(M) + \deg_{W_t' \wedge Y}(M)) \\ &= \deg_{W_j \wedge Y}(M) + \deg_{W_{j+1}' \wedge Y}(M) + \deg_{W_{j+1} \wedge Y}(M) + \cdots \\ &\quad + \deg_{W_k' \wedge Y}(M) + \deg_{W_k \wedge Y}(M) + \deg_{W_{k+1}' \wedge Y}(M) \\ &= \deg_{W_j \wedge Y}(M) - 1 + 1 - \cdots - 1 + 1 + \deg_{W_{k+1}' \wedge Y}(M) \\ &= \deg_{W_j \wedge Y}(M) + \deg_{W_{k+1}' \wedge Y}(M). \end{aligned} \tag{4.1}$$

Likewise, since M' is invertible,

$$\deg_Y(M') = \deg_{W_j \wedge Y}(M') + \deg_{W'_{k+1} \wedge Y}(M'). \tag{4.2}$$

In particular we have

$$-1 \leq \deg_Y(M),\ \deg_Y(M') \leq 1.$$

4.2 The Abel map

We use the line bundles M and M' of the previous section to define a map from the curve X to its compactified Jacobian.

4.2.1 The P-twist. Let P be a simple point on a Gorenstein curve X. For each $Q \in X$ we now define a sheaf $I(Q)$ which will be the image of Q under the map $\tilde{A}_P$. (In the following proposition we show that $\tilde{A}_P$ factors through $\overline{J}_E^P$, where $E := \omega_X \oplus \mathcal{O}_X$.) Let Q be a point of X, and let M and M' be the sheaves defined in Proposition 4.1.4 (ii).

- If Q is not a disconnecting node, set $I(Q) := \mathfrak{m}_Q(P) \otimes M$.
- If Q is a disconnecting node, set $I(Q) := M'(P)$.

We call the sheaf $I(Q)$ the *P-twist of Q in X*.

Note that, in any case, $I(Q)$ has degree 0 on X. In addition, if Q is not a disconnecting node of X, then $I(Q)$ is simple, torsion-free, and has rank 1, because $\mathfrak{m}_Q(P)$ is simple, torsion-free and of rank 1, and M is invertible. Also, if Q is a disconnecting node of X, then $I(Q)$ is invertible, whence simple.

Proposition 4.2.2 *Let P be a simple point on a Gorenstein curve X, and let $E := \omega_X \oplus \mathcal{O}_X$. Then there is a well-defined map of sets*

$$\tilde{A}_P : X \longrightarrow \overline{J}_E^P$$

sending a point Q of X to $I(Q)$, where $I(Q)$ is the P-twist of Q in X.

Proof. Let Q be a point of X, and let $I(Q)$ be the P-twist of Q in X. We have to verify that $I(Q)$ is P-quasi-stable with respect to E.

First, since $I(Q)$ has rank 1 and degree 0 on X we have

$$\chi(I(Q) \otimes E) = \chi(E) = \deg(E) + \mathrm{rk}(E)(1-g) = 0.$$

Now, let Y be a connected proper subcurve of X of arithmetic genus g_Y. Then

$$\begin{aligned} \beta_{I(Q)}(Y) &= \chi(I(Q)|_Y) + \deg_Y(E)/\mathrm{rk}(E) \\ &= (\deg_Y(I(Q)) + 1 - g_Y) + (g_Y - 1 + \delta_Y/2) \\ &= \deg_Y(I(Q)) + \delta_Y/2, \end{aligned}$$

where the second equality follows from Lemma 3.1.1. We have to show that $\beta_{I(Q)}(Y) \geq 0$ for all Y, with $\beta_{I(Q)}(Y) > 0$ if $P \in Y$.

Assume first that Q is not a disconnecting node of X. Then $I(Q) = \mathfrak{m}_Q(P) \otimes M$, where M is as in Proposition 4.1.4 (ii). Then

$$\deg_Y(I(Q)) = \deg_Y(\mathfrak{m}_Q(P)) + \deg_Y(M).$$

Let $Z_1 \subset \ldots \subset Z_r$ and $N_1, \ldots, N_r$ be as in Proposition 4.1.4 (i). Here we use the notation introduced in Remark 4.1.5. There are four cases to consider:

- If $P \in Y$ and $Q \notin Y$, then $\deg_Y(\mathfrak{m}_Q(P)) = 1$. Also, by Remark 4.1.5, we have $\deg_Y(M) \geq -1$, showing that $\beta_{I(Q)}(Y) \geq 1 - 1 + \delta_Y/2 > 0$.
- If $P, Q \in Y$, then $\deg_Y(\mathfrak{m}_Q(P)) = 0$. Also, since Y is connected, Y contains all disconnecting nodes $N_1, \ldots, N_r$. Moreover, since P and Q are not disconnecting nodes and both belong to Y, Y contains all the irreducible components of X containing $N_1, \ldots, N_r$. Hence, by (4.1), $\deg_Y(M) = 0$ and $\beta_{I(Q)}(Y) = 0 + 0 + \delta_Y/2 > 0$.
- If $P, Q \notin Y$ then $\deg_Y(\mathfrak{m}_Q(P)) = 0$. If $\delta_Y = 1$, then Y is a tail, and since $Q \notin Y$, we see that Y is not any of the Z_t. We state that $\deg_Y(M) \geq 0$. Indeed, assume by contradiction that $\deg_Y(M) = -1$. (Recall from Remark 4.1.5 that $-1 \leq \deg_Y(M) \leq 1$.) Then Y must contain some N_t. Let j (resp. k) be as in Remark 4.1.5. Then, by (4.1), $W'_{k+1} \subset Y$ and $W_j \not\subset Y$. Therefore the node N_j is contained in $Y \cap Y^c$. Since Y is a tail, N_j is the disconnecting node associated to Y. So $Y = Z_j$ and thus $Q \in Y$, or $Y = Z_j^c$ and thus $P \in Y$. As none of this is possible, we see that $\deg_Y(M) \geq 0$. We showed that, if $\delta_Y = 1$, then $\beta_{I(Q)}(Y) \geq 0 + 0 + 1/2 > 0$. In addition, if $\delta_Y \geq 2$, then $\beta_{I(Q)}(Y) \geq 0 - 1 + 1 = 0$, and we are done.
- If $P \notin Y$ and $Q \in Y$, then $\deg_Y(\mathfrak{m}_Q(P)) = -1$. First assume $Y = Z_t$ for some t. Then by Proposition 4.1.4, $\deg_Y(M) = 1$ so that $\beta(Y) = -1 + 1 + \delta_Y/2 > 0$. If Y is not any of the Z_t, then $\delta_Y \geq 2$, since otherwise Y would be a tail containing Q and not containing P. We claim that $\deg_Y(M) \geq 0$. Indeed, assume by contradiction that $\deg_Y(M) = -1$. Let $N_j, \ldots, N_k$ be as in Remark 4.1.5. Since $\deg_Y(M) = -1$, we have by (4.1) that $W'_{k+1} \subset Y$ and $W_j \not\subset Y$. In particular, Y does not contain W_1. Since W_1 is the irreducible component of Z_1 containing N_1, and Y

is connected, Y cannot contain Q, an absurd. So $\deg_Y(M) \geq 0$ and thus $\beta_{I(Q)}(Y) \geq -1+0+1 = 0$.

Now, assume that Q is a disconnecting node of X. Then $I(Q) = M'(P)$, where M' is as in Proposition 4.1.4 (ii). Let $Z_1 \subset \ldots \subset Z_r$ and $N_1, \ldots, N_r$ be as in Proposition 4.1.4 (i). Note that, since Q is a disconnecting node, and Z_1 is the smallest tail containing Q and not P, then $Q = N_1$. As before, we use the notation introduced in Remark 4.1.5. There are two cases to consider:

• If $P \in Y$, then $\deg_Y(I(Q)) = \deg_Y(M') + 1$. Now, since by Remark 4.1.5, $\deg_Y(M') \geq -1$, we have $\deg_Y(I(Q)) \geq 0$. Thus $\beta_{I(Q)}(Y) > 0$.

• If $P \notin Y$ then $\deg_Y(I(Q)) = \deg_Y(M')$. Since $\deg_Y(M') \geq -1$, if we have $\delta_Y \geq 2$, then $\beta_{I(Q)}(Y) \geq 0$. Now, assume $\delta_Y = 1$. Then Y is a tail of X. If $Y = Z_t$ for some t, then, by Proposition 4.1.4 (ii), we have $\deg_Y(M') = 0$, and thus $\beta_{I(Q)}(Y) > 0$. So, assume Y is not any of the Z_t. Note that in this case, $Q = N_1 \notin Y$. We state that $\deg_Y(M') \geq 0$. Indeed, assume by contradiction that $\deg_Y(M') = -1$. Let j and k be as in Remark 4.1.5. Then, since $N_1 \notin Y$, we have $j \geq 2$. Thus by (4.2), we have $W_j \not\subset Y$ and $W'_{k+1} \subset Y$. In particular, the node N_j is contained in $Y \cap Y^c$. Now, since Y is a tail, N_j is the node associated to Y. So $Y = Z_j$ and thus $N_1 \in Y$, or $Y = Z_j^c$ and thus $P \in Y$. As none of this is possible, we get that $\deg_Y(M') \geq 0$. Therefore $\beta_{I(Q)}(Y) > 0$ and we are done. □

Theorem 4.2.3 *Let P be a simple point on a Gorenstein curve X of arithmetic genus g. Then, the map $\tilde{A}_P$ defined in Proposition 4.2.2 is a morphism of schemes. In addition, if X contains no disconnecting lines and $g > 0$, then $\tilde{A}_P$ is a closed embedding.*

Proof. By Proposition 4.2.2 the map $\tilde{A}_P$ exists as a map of sets. Now we proceed by induction on the number of disconnecting nodes of X to prove this map is indeed a morphism of schemes.

If X has no disconnecting nodes, then X has no tails, and thus the map $\tilde{A}_P$ is the pointed Abel map A_P of (X, P) which, by Corollary 3.2.3, is a closed embedding.

Now assume X has a disconnecting node N, and let Z, Z^c be the tails associated to it. Then Z and Z^c have fewer disconnecting nodes than X. Indeed the disconnecting nodes of X are the disconnecting nodes of Z, the ones of Z^c, and N. Assume without loss of generality that $P \in Z^c$. Let

$$\tilde{A}_N^Z : Z \longrightarrow \overline{J}_Z$$

be the map taking a point Q of Z to the N-twist of Q in Z. And let

$$\tilde{A}_P^{Z^c} : Z^c \longrightarrow \overline{J}_{Z^c}$$

be the map taking a point Q of Z^c to the P-twist of Q in Z^c. By induction hypothesis, both $\tilde{A}_N^Z$ and $\tilde{A}_P^{Z^c}$ are morphims of schemes. (Note that N is a simple point of Z.)

We'll define, using $\tilde{A}_N^Z$ and $\tilde{A}_P^{Z^c}$, morphisms of schemes $B_1 : Z \to \overline{J}_X$ and $B_2 : Z^c \to \overline{J}_X$ such that $B_1(N) = B_2(N)$. Since Z and Z^c intersect transversally, B_1 and B_2 induce a morphism $B : X \to \overline{J}_X$. Then we show that B coincides with $\tilde{A}_P$ on every point of X.

Consider the morphism $\Gamma : \overline{J}_Z \times \overline{J}_{Z^c} \to \overline{J}_X$ of Remark 4.1.3. Define

$$B_1 : Z \xrightarrow{(\tilde{A}_N^Z, \tilde{A}_P^{Z^c}(N))} \overline{J}_Z \times \overline{J}_{Z^c} \xrightarrow{\Gamma} \overline{J}_X, \tag{4.3}$$

where the first morphism takes a point $Q \in Z$ to the pair $(\tilde{A}_N^Z(Q), \tilde{A}_P^{Z^c}(N))$. In addition, define

$$B_2 : Z^c \xrightarrow{(\mathcal{O}_Z, \tilde{A}_P^{Z^c})} \overline{J}_Z \times \overline{J}_{Z^c} \xrightarrow{\Gamma} \overline{J}_X, \tag{4.4}$$

where the first morphism takes a point $Q \in Z^c$ to the pair $(\mathcal{O}_Z, \tilde{A}_P^{Z^c}(Q))$. First we note that $B_1(N) = B_2(N)$, so $B : X \to \overline{J}_X$ is well-defined. Indeed, we clearly have $\tilde{A}_N^Z(N) = \mathcal{O}_Z$, because the N-twist of N in Z is the trivial sheaf $\mathcal{O}_Z$. (Since there are no tails in Z containing N and not N, the sequence of tails in Proposition 4.1.4 (i) is empty.)

Now we prove that $B = \tilde{A}_P$ set-theoretically. We must show that:

(i) $\tilde{A}_P(N)$ satisfies

$$\tilde{A}_P(N)|_Z = \mathcal{O}_Z \text{ and } \tilde{A}_P(N)|_{Z^c} = \tilde{A}_P^{Z^c}(N);$$

(ii) if $Q \in Z - \{N\}$, then $\tilde{A}_P(Q)$ satisfies

$$\tilde{A}_P(Q)|_Z = \tilde{A}_N^Z(Q) \text{ and } \tilde{A}_P(Q)|_{Z^c} = \tilde{A}_P^{Z^c}(N);$$

(iii) if $Q \in Z^c - \{N\}$, then $\tilde{A}_P(Q)$ satisfies

$$\tilde{A}_P(Q)|_Z = \mathcal{O}_Z \text{ and } \tilde{A}_P(Q)|_{Z^c} = \tilde{A}_P^{Z^c}(Q).$$

(Note that, in each case, there is a unique sheaf satisfying the conditions, by Lemma 4.1.2.) Now, let Q be a point of X and let $Z_1 \subset \ldots \subset Z_r$ be as in Proposition 4.1.4 (ii).

First we show (i), so assume that $Q = N$. In this case, since Z is a tail associated to the disconnecting node N and not containing P, and Z_1 is the smallest tail containing N but not P, we have $Z = Z_1$. Let $I(N)$ be the P-twist of N in X. By definition, see (4.2.1), we have $I(N) = M'(P)$, where M' is as in Proposition 4.1.4 (ii). Therefore, since $P \notin Z$, we have $I(N)|_{Z_1} = \mathcal{O}_{Z_1}$.

On the other hand, $\overline{Z_2 - Z_1}, \dots, \overline{Z_r - Z_1}$ are the tails of Z^c containing N but not P. Let M^{Z^c} be the sheaf on Z^c defined in Proposition 4.1.4 (ii) associated to the tails $\overline{Z_2 - Z_1}, \dots, \overline{Z_r - Z_1}$ of Z^c. Then $\tilde{A}_P^{Z^c}(N) = \mathfrak{m}_{N,Z^c}(P) \otimes M^{Z^c}$, where $\mathfrak{m}_{N,Z^c}$ is the ideal sheaf of N in Z^c. (Note that N is not a disconnecting node of Z^c.) Since N is a simple point of Z^c, we have

$$\tilde{A}_P^{Z^c}(N) = \mathcal{O}_{Z^c}(P - N) \otimes M^{Z^c} = M^{Z^c}(P - N).$$

Let $\tilde{Z}_t := \overline{Z_t - Z_{t-1}}$ for $t = 1, \dots, r$ and set $\tilde{Z}_{r+1} := Z_r^c$. Since $\tilde{A}_P(N) = M'(P)$, we need only see that M' and $M^{Z^c}(-N)$ coincide on $\tilde{Z}_t$ for each $t = 2, \dots, r+1$. But this follows from the definition of M' and M^{Z^c}. Therefore, $\tilde{A}_P(N)|_{Z^c} = \tilde{A}_P^{Z^c}(N)$, thus proving (i).

Now we show (ii), so assume Q is in Z and $Q \neq N$. Since $P \notin Z$, we have $Z = Z_t$ for some t. Thus $Z_1, \dots, Z_{t-1}$ are the tails of Z containing Q but not N, and $\overline{Z_{t+1} - Z_t}, \dots, \overline{Z_r - Z_t}$ are the tails of Z^c containing N but not P. By the definition of the maps $\tilde{A}_P$, $\tilde{A}_N^Z$ and $\tilde{A}_P^{Z^c}$ (see (4.2.1)), this shows that $\tilde{A}_P(Q)|_Z = \tilde{A}_N^Z(Q)$ and $\tilde{A}_P(Q)|_{Z^c} = \tilde{A}_P^{Z^c}(N)$.

At last we show (iii), so assume $Q \in Z^c$ and $Q \neq N$. Since P does not belong to Z or to Z_t, the union $Z \cup Z_t$ is not the whole curve X. By definition of Z_t, we have $Q \in Z_t$ for every t. Also, since $Q \neq N$, we have $Q \notin Z$. Hence, $Z_t \not\subseteq Z$ for all t. So, by Lemma 4.1.1, we have either $Z \cap Z_t = \emptyset$ or $Z \subsetneqq Z_t$. In any case, $Z_t' := Z^c \wedge Z_t$ is a tail of Z^c containing Q but not P. Moreover, $Z_1' \subset \dots \subset Z_r'$ are exactly the tails of Z^c containing Q and not P. Therefore $\tilde{A}_P(Q)|_{Z^c} = \tilde{A}_P^{Z^c}(Q)$.

On the other hand, Z does not contain any of the disconnecting nodes $N_1, \dots, N_r$ associated to the tails $Z_1, \dots, Z_r$. Indeed, we saw that either $Z \cap Z_t = \emptyset$ or $Z \subsetneqq Z_t$, for each t. So, if $N_t \in Z$, then $Z \cap Z_t \neq \emptyset$ and hence $Z \subsetneqq Z_t$. In particular, $N_t \in Z \cap Z^c$, showing that $Z = Z_t$, an absurd. Therefore $\tilde{A}_P(Q)|_Z = \mathcal{O}_Z$.

As for the last assertion of the theorem, assume that X contains no disconnecting lines. Then Z and Z^c contain no disconnecting lines. Indeed, first note that a disconnecting node of Z is a disconnecting node of X. Now, if $L \subset Z$ is a rational component, then

$$L \cap (\overline{X - L}) \subset L \cap (\overline{Z - L}) \cup \{N\}.$$

So if L were a disconnecting line of Z, then the points in the intersection $L \cap (\overline{Z - L})$ would be disconnecting nodes of Z, hence of X, and L would be a disconnecting line of X.

Thus, by induction hypothesis, $\tilde{A}_N^Z$ and $\tilde{A}_P^{Z^c}$ are closed embeddings. By the definition of B, this implies that $\tilde{A}|_Z = B_1$ and $\tilde{A}|_{Z^c} = B_2$ are closed embeddings. Note first that $\tilde{A}_P$ is injective. Indeed, recall that Γ is injective (see Remark 4.1.3). Therefore, it's enough to see that, if $Q_1 \in Z$ and $Q_2 \in Z^c$ are such that $\tilde{A}_P(Q_1) = \tilde{A}_P(Q_2)$, then $Q_1 = Q_2 = N$. Now, by the definition of B_1

and B_2, we have $\tilde{A}_N^Z(Q_1) = \mathcal{O}_Z$ and $\tilde{A}_P^{Z^c}(Q_2) = \tilde{A}_P^{Z^c}(N)$. The latter clearly implies that $Q_2 = N$, because $\tilde{A}_P^{Z^c}$ is injective. In addition, since $\tilde{A}_N^Z(N) = \mathcal{O}_Z$, we also have $Q_1 = N$, because $\tilde{A}_N^Z$ is injective. Therefore $\tilde{A}_P$ is injective.

Moreover, since $\tilde{A}_N^Z$ and $\tilde{A}_P^{Z^c}$ are embeddings, $\tilde{A}_P$ is an embedding except possibly at N. So we need only see that $\tilde{A}_P$ separates tangent vectors at N. Now, by Remark 4.1.3, $\overline{J}_Z \times \overline{J}_{Z^c} \cong \overline{J}_X$, and $\tilde{A}_P$ takes the tangent spaces of Z and Z^c at N to the subspaces $T_{\overline{J}_Z, \tilde{A}_N^Z(N)} \oplus 0$ and $0 \oplus T_{\overline{J}_{Z^c}, \tilde{A}_P^{Z^c}(N)}$ of $T_{\overline{J}_X, \tilde{A}_P(N)}$, respectively. Therefore $\tilde{A}_P$ separates tangent vectors also at N. At last, since X is complete and $\tilde{A}_P$ has image in the separated scheme $\overline{J}_E^P$, it follows that $\tilde{A}_P$ is a closed embedding. □

Remark 4.2.4 Let X be a Gorenstein curve, and consider the map $\tilde{A}_P$ defined in Proposition 4.2.2. Let Z be a tail of X with disconnecting node N. In the proof of the Theorem 4.2.3, we showed that the restriction of $\tilde{A}_P$ to Z is essentially the Abel map taking each point $Q \in Z$ to its P-twist in Z, if $P \in Z$; or to the N-twist of Q in Z, if $P \notin Z$.

Moreover, let $Y \subset X$ be a subcurve such that the points in the intersection $Y \cap Y^c$ are disconnecting nodes. Then each connected component of Y^c is a tail associated to its point of intersection with Y. If $P \notin Y$, let N be the node associated to the tail containing the point P. We state that $\tilde{A}_P|_Y$ is essentially the Abel map taking each point $Q \in Y$ to its P-twist in Y, if $P \in Y$; or to the N-twist of Q in Z, if $P \notin Y$.

We prove our statement by induction on δ_Y. Indeed, if $\delta_Y = 1$, then Y is a tail and the statement holds. Now, assume $\delta_Y > 1$. Let Z be a tail containing Y associated to a point in $Y \cap Y^c$. Note that if $P \notin Z$, then the point N defined in above paragraph is the disconnecting node associated to Z. Then $\tilde{A}_P|_Z$ is essentially the Abel map taking each point of Z to its P-twist in Z, if $P \in Z$; or to the N-twist of Q in Z, if $P \notin Z$. Now, Y is a subcurve of Z that meets $\overline{Z - Y}$ in $\delta_Y - 1$ points, and the points in $Y \cap (\overline{Z - Y})$ are disconnecting nodes of X, thus also of Z. By induction hypothesis, $(\tilde{A}_P|_Z)|_Y$ is essentially the Abel map taking each point of Y to its P-twist in Y, if $P \in Y$; or to the N-twist of Q in Y, if $P \notin Y$, as stated.

We now analyze the restriction of $\tilde{A}_P$ to an irreducible component W of X. We show that the restriction $\tilde{A}_P|_W$ is either the first Abel map of the curve W or that of the pointed curve (W, N), for some point N in W. In other words, for every $Q \in W$, the restriction $\tilde{A}_P(Q)|_W$ is either the sheaf $\mathfrak{m}_{Q,W}$ or the sheaf $\mathfrak{m}_{Q,W}(N)$, where $\mathfrak{m}_{Q,W}$ is the ideal sheaf of Q in W.

Assume first that $P \in W$. Then the restriction of $\tilde{A}_P$ to W is the first Abel map of the pointed curve (W, P). Indeed, let Q be a point in W. Note that, if Q is a disconnecting node of X, then W is contained in a tail Z of X such that Q is a simple point of Z. (In fact, Q is the disconnecting node associated to Z.) Thus, restricting first to Z, we may assume that Q is not a disconnecting node of X. So, since Q is not a disconnecting node of X, we have $\tilde{A}_P(Q) = \mathfrak{m}_Q(P) \otimes M$, where M is as in Proposition 4.1.4 (ii). But both P and Q are in the same irreducible component W, so there is no tail of X containing Q and not P. Hence $M = \mathcal{O}_X$ and $\tilde{A}_P(Q)|_W = \mathfrak{m}_{Q,W}(P)$.

Now, assume $P \notin W$. Let Q be a point of W, and let $Z_1, \ldots, Z_r$ be the ascending sequence of tails of X, defined in Proposition 4.1.4 (i), containing Q but not P. (Note that this sequence of tails depend only on the irreducible component W to which Q belongs.) Let N_1 be the disconnecting node associated to Z_1. If $N_1 \notin W$, then the restriction of $\tilde{A}_P$ to W is simply the first Abel map of the curve W. Indeed, we may again assume that Q is not a disconnecting node of X. So $\tilde{A}_P(Q) = \mathfrak{m}_Q(P) \otimes M$, where M is as in Proposition 4.1.4 (ii). Since $W \subset Z_1$ and $N_1 \notin W$, we have $M|_W \cong \mathcal{O}_W$. Therefore, since $P \notin W$, we have $\tilde{A}_P(Q)|_W = \mathfrak{m}_{Q,W}$. On the other hand, if $N_1 \in W$, we have $M|_W \cong \mathcal{O}_W(N_1)$, and hence $\tilde{A}_P(Q)|_W = \mathfrak{m}_{Q,W}(N_1)$. Thus if $N_1 \in W$, then the restriction of $\tilde{A}_P$ to W is the first Abel map of the pointed curve (W, N_1).

Remark 4.2.5 The hypothesis that X does not contain a disconnecting line is not just technical. Indeed, assume $L \subset X$ is a disconnecting line, and let $Z_1, \ldots, Z_k$ be the connected components of L^c. Note that $Z_1, \ldots, Z_k$ are tails of X. Then, by Remark 4.1.3, $\overline{J}_X \cong \overline{J}_{Z_1} \times \ldots \times \overline{J}_{Z_k} \times \overline{J}_L$. Also, as it can be derived from the proof of Theorem 4.2.3, $\tilde{A}_P$ sends essentially each Z_i to the factor $\overline{J}^0_{Z_i}$ of $\overline{J}^0_X$, and sends L to $\overline{J}^0_L$. But, since L is rational, $\overline{J}^0_L$ is a point, as all invertible sheaves of degree 0 on L are isomorphic. Therefore $\tilde{A}_P$ cannot be injective, as it contracts the disconnecting line L to a point in $\overline{J}_X$.

4.2.6 Trees of disconnecting lines. A *tree of disconnecting lines* in a curve X is a connected subcurve L of X having arithmetic genus 0 and such that every point of $L \cap L^c$ is a disconnecting node of X. As we will see, the removal of a tree of disconnecting lines from a Gorenstein curve doesn't change the genus of the curve. In addition, the map $\tilde{A}_P$ contracts every tree of disconnecting lines to a point. We show in Theorem 4.2.10 that the image $\tilde{X}$ of $\tilde{A}_P$ is a curve in $\overline{J}_X$ obtained from X by contracting every tree of disconnecting lines in X to a point and contracting nothing else. Furthermore, $\tilde{X}$ has the same arithmetic genus as X.

Lemma 4.2.7 *Let L be a tree of disconnecting lines in a Gorenstein curve X of arithmetic genus g. Let $Z_1, \ldots, Z_k$ be the connected components of L^c. Then $g = g_1 + \ldots + g_k$, where g_i is the arithmetic genus of Z_i.*

Proof. We proceed by induction on the number of connected components of $L^c = \overline{X - L}$.

First we note that, since X is Gorenstein, ω_X is invertible. Hence

$$\deg(\omega_X) = \deg_{Z_1}(\omega_X) + \deg_{Z_1^c}(\omega_X).$$

Assume first that $k = 1$. Then $X = Z_1 \cup L$. Therefore, since $\delta_L = \delta_{Z_1} = 1$, by Lemma 3.1.1, we have

$$2g - 2 = (2g_1 - 2 + 1) + (0 - 2 + 1) = 2g_1 - 2,$$

showing that $g = g_1$. Now, assume $k > 1$. Let g_1' be the arithmetic genus of Z_1^c. Again, by Lemma 3.1.1 we have,

$$2g - 2 = (2g_1 - 2 + 1) + (2g_1' - 2 + 1),$$

because $\delta_{Z_1} = \delta_{Z_1^c} = 1$. Then $g = g_1 + g_1'$. But $\overline{Z_1^c - L}$ has less connected components than $\overline{X - L}$, so by induction hypothesis, $g_1' = g_2 + \ldots + g_k$. (Note that the connected components of $\overline{Z_1^c - L}$ are $Z_2, \ldots, Z_k$.) Hence $g = g_1 + \ldots + g_k$. □

Lemma 4.2.8 *Let X be a Gorenstein curve, and L a tree of disconnecting lines in X. Then the irreducible components of L are indeed disconnecting lines. (In particular, for each simple point $P \in X$, the map $\tilde{A}_P$ of Proposition 4.2.2 contracts L to a point of $\overline{J}_X$.)*

Proof. We proceed by induction on the number of irreducible components of L. If L is irreducible, then L is clearly a disconnecting line. Assume L is not irreducible.

First we show that there is an irreducible component W of L, such that W meets $\overline{L - W}$ at a point with multiplicity 1. In other words, let $\delta_{W,L} := \chi(\mathcal{O}_{W \cap (\overline{L-W})})$. Then we state that $\delta_{W,L} = 1$ for some W. Assume by contradiction that $\delta_{W,L} \geq 2$ for every component $W \subset L$. Note that since X is Gorenstein, and L intersects L^c transversally, also L is Gorenstein. Since $\delta_{W,L} \geq 2$, we have

$$0 \leq \deg_W(\omega_L) \leq \deg(\omega_L) = -2,$$

an absurd. Thus $\delta_{W,L} = 1$ for some W.

Now we show that the irreducible components of L are disconnecting lines. Let W be an irreducible component of L such that $\delta_{W,L} = 1$. Let $Z = \overline{L - W}$. Since $\delta_{W,L} = 1$, the curve Z is connected. Now, since L is Gorenstein, ω_L is invertible, thus $\deg(\omega_L) = \deg_W(\omega_L) + \deg_Z(\omega_L)$. Then, by Lemma 3.1.1, we have

$$-2 = \deg(\omega_L) = (2g_W - 2 + 1) + (2g_Z - 2 + 1),$$

where g_W (resp. g_Z) is the arithmetic genus of W (resp. Z). Hence $g_W + g_Z = 0$. Thus, since g_W and g_Z are nonnegative integers, we have $g_W = g_Z = 0$. Therefore, W is a rational smooth curve.

Also, W is a disconnecting line. Indeed, if N is the point in the intersection $W \cap (\overline{L - W})$, then N is a disconnecting node of X. Moreover, we have $W \cap (\overline{X - W}) \subset L \cap (\overline{X - L}) \cup \{N\}$, and hence every point in $W \cap (\overline{X - W})$ is a disconnecting node of X. Now note that Z has less irreducible components than L, and Z is also a tree of disconnecting lines. Hence, by the induction hypothesis, the irreducible components of Z are disconnecting lines. □

Lemma 4.2.9 *Let P be a simple point on a Gorenstein curve X. Let Y be a subcurve of X such that $Y \cap Y^c$ consists of disconnecting nodes. Assume that Y contains no disconnecting lines. Let Q be a point of $Y - \{Y \cap Y^c\}$, and $R := \tilde{A}_P(Q)$. Then $\tilde{A}_P(Y^c)$ does not contain R.*

Proof. First recall that $\tilde{A}_P|_Y$ is essentially an Abel map. Indeed, by Remark 4.2.4, $\tilde{A}_P|_Y$ is essentially either the Abel map sending each point to its P-twist in Y, if $P \in Y$, or to its N-twist in Y, if $P \notin Y$. Thus, by Theorem 4.2.3, $\tilde{A}_P|_Y$ is a closed embedding, since Y does not contain a disconnecting line.

Let $Z_1, \dots, Z_k$ be the connected components of Y^c, and $N_1, \dots, N_k$ their intersections with Y. Note that $Z_1, \dots, Z_k$ are tails of X. Let $Q' \in Y^c$. Then $Q' \in Z_i$ for some i. We have to show that $\tilde{A}_P(Q') \neq R$. Recall the proof of Theorem 4.2.3, and in particular, the argument around (4.3) and (4.4).

Assume first that $P \notin Y$. Without loss of generality, assume $P \in Z_1$. Then $\tilde{A}_P(Q')|_Y = \tilde{A}{N_1}^Y(N_i)$. Since $Q \neq N_i$, the injectivity of $\tilde{A}^Y_{N_1} = \tilde{A}_P|_Y$ shows that $\tilde{A}_P(Q') \neq \tilde{A}_P(N_i)$. Finally, assume that $P \in Y$. Then $\tilde{A}_P(Q')|_Y = \tilde{A}^Y_P(N_i)$. Since $Q \neq N_i$, the injectivity of $\tilde{A}^Y_P = \tilde{A}_P|_Y$ shows that $\tilde{A}_P(Q') \neq \tilde{A}_P(Q)$ as well. □

We say that a tree of disconnecting lines L is *maximal* if, for every tree of disconnecting lines L' containing L, we have $L' = L$.

Theorem 4.2.10 *Let X be a Gorenstein curve of arithmetic genus $g > 0$, and consider the Abel map $\tilde{A}_P$ constructed in Proposition 4.2.2. Let L be a maximal tree of disconnecting lines, and set $R = \tilde{A}_P(L)$. Then $\tilde{X} = \tilde{A}_P(X)$ is a curve of arithmetic genus g, having an ordinary singularity of multiplicity δ_L at R, with linearly independent tangent lines.*

Furthermore, let Σ be the union of the disconnecting lines of X. Let $X_1, \dots, X_n$ be the connected components of $\overline{X - \Sigma}$, and $\tilde{X}_i := \tilde{A}_P(X_i)$ for $i = 1, \dots, n$. Then $\tilde{A}_P|_{X_i}$ is an isomorphism onto $\tilde{X}_i$. In addition, we have $\tilde{A}_P(Q_1) = \tilde{A}_P(Q_2)$ if and only if Q_1 and Q_2 belong to the same maximal tree of disconecting lines.

Proof. First note that L is a proper subcurve of X, because $g > 0$. Since X is connected, and L is neither empty nor equal to X, we have $\delta_L > 0$. Let $k = \delta_L$.

Let $N_1, \ldots, N_k \in X$ be the disconnecting nodes of L, that is, the points in the intersection $L \cap L^c$. Let $Z_1, \ldots, Z_k$ be the connected components of L^c, so that each Z_i is a tail associated to the disconnecting node N_i. Let g_i be the arithmetic genus of Z_i, for $i = 1, \ldots, k$. Note that if $g_i = 0$ then Z_i is a tree of disconnecting lines, because $Z_i \cap Z_i^c = Z_i \cap L$, and L is a tree of disconnecting lines. Therefore, since L is maximal, we have $g_i > 0$ for every i, since otherwise $Z_i \cup L$ would be a tree of disconnecting lines properly containing L.

Now, by Lemma 4.2.8, the irreducible components of L are disconnecting lines. Thus, since every point of $L \cap L^c$ is a disconnecting node, we have by Remark 4.1.3 that

$$\overline{J}_X \cong \overline{J}_{Z_1} \times \ldots \times \overline{J}_{Z_k}.$$

Note that, since $N_1, \ldots, N_k$ are points of L, their images by $\tilde{A}_P$ are equal to R. Moreover, by Remark 4.2.4, $\tilde{A}_P$ sends essentially each Z_i to the factor $\overline{J}_{Z_i}$. Thus the tangent space of $\tilde{X}$ at R is generated by the tangent spaces of $\tilde{A}_P(Z_i)$ at R, for $i = 1, \ldots, k$. But these tangent spaces are linearly independent. Thus the tangent space of $\tilde{X}$ at R is actually the direct sum of the tangent spaces of $\tilde{A}_P(Z_i)$ at R for $i = 1, \ldots, k$.

Let Y_i be the maximal connected subcurve of Z_i containing N_i, and not containing a disconnecting line. (It is easy to see that such a maximal subcurve exists. Indeed, let U be the union of the disconnecting lines of Z_i. By the maximality of L, we have that $N_i \notin U$, since otherwise the union of L and the connected component of U containing N_i would be a tree of disconnecting lines properly containing L. Then Y_i is the connected component of $\overline{Z_i - U}$ containing N_i.) Note that $Y_i \cap (\overline{Z_i - Y)i})$ consists of disconencting nodes, because

$$Y_i \cap (\overline{Z_i - Y_i}) \subset U \cap (\overline{Z_i - U})$$

and Z_i is a tail. Then, $\tilde{A}_P|_{Y_i}$ is a closed embedding. Indeed, by Remark 4.2.4, $\tilde{A}_P|_{Y_i}$ is an Abel map. Thus, by Theorem 4.2.3, $\tilde{A}_P|_{Y_i}$ is a closed embedding, since Y_i does not contain a disconnecting line.

Let $\tilde{Y}_i$ be the image of Y_i by $\tilde{A}_P$, so $\tilde{Y}_i$ is isomorphic to Y_i and contains the point R. Thus R is the intersection of $\tilde{Y}_1, \ldots, \tilde{Y}_k$. By Lemma 4.2.9, $\tilde{A}_P(\overline{Z_i - Y_i})$ does not contain R. Thus the tangent space of $\tilde{A}_P(Z_i)$ at R is the tangent space of $\tilde{Y}_i$ at R. Now, the tangent space of Y_i at N_i is isomorphic to the tangent space of $\tilde{Y}_i$ at R, because $\tilde{A}_P|_{Y_i}$ is a closed embedding, and $\tilde{Y}_i$ is the image of $\tilde{A}_P|_{Y_i}$. (Note that this tangent space has dimension 1, since N_i is a simple point of Y_i.) So the tangent space of $\tilde{X}$ at R is isomorphic to the direct sum of the tangent spaces of Y_i at

N_i. Thus the tangent space of $\tilde{X}$ at R has dimension k, and R is a point of $\tilde{X}$ with multiplicity k having k linearly independent tangent lines, each corresponding to a tail Z_i.

Now we show that the connected components $X_1, \dots, X_n$ of $\overline{X - \Sigma}$ are isomorphic to their images under $\tilde{A}_P$. First we note that X_i is a subcurve of X such that the intersection $X_i \cap X_i^c$ consists of disconnecting nodes. Indeed,

$$X_i \cap X_i^c = X_i \cap \Sigma \subset \Sigma^c \cap \Sigma,$$

and the points in $\Sigma^c \cap \Sigma$ are disconnecting nodes. In addition, by the definition of Σ, the curve X_i contains no disconnecting lines. So, since $\tilde{A}_P|_{X_i}$ is an Abel map (see Remark 4.2.4), it is a closed embedding by Theorem 4.2.3. Hence $\tilde{A}_P$ takes X_i isomorphically to $\tilde{X}_i$.

Moreover, let Q_1 and Q_2 be distinct points of X such that $\tilde{A}_P(Q_1) = \tilde{A}_P(Q_2)$. First assume that $Q_1 \notin \Sigma$. Then $Q_1 \in X_i - (X_i \cap X_i^c)$ for some i. Since $\tilde{A}_P|_{X_i}$ is a closed embedding, we have that $Q_2 \notin X_i$. Thus $Q_2 \in X_i^c$. But then, by Lemma 4.2.9, $\tilde{A}_P(Q_1) \neq \tilde{A}_P(Q_2)$, an absurd.

Now assume that $Q_1 \in \Sigma$. By the same token, we may assume $Q_2 \in \Sigma$ as well. Let L_1 and L_2 be the maximal trees of disconnecting lines to which Q_1 and Q_2 belong, respectively. We need to show that $L_1 = L_2$. Assume by contradiction that $L_1 \neq L_2$. Since X is connected, there is a connected component W of $\overline{X - (L_1 \cup L_2)}$ meeting L_1 and L_2. Let $N_1 \in W \cap L_1$ and $N_2 \in W \cap L_2$. Note that the points in $W \cap W^c$ are disconnecting nodes, because L_1 and L_2 are trees of disconnecting lines. So $\tilde{A}_P|_W$ is essentially an Abel map. Now, since $\tilde{A}_P$ contracts trees of disconnecting lines, we have

$$\tilde{A}_P(N_1) = \tilde{A}_P(Q_1) = \tilde{A}_P(Q_2) = \tilde{A}_P(N_2).$$

Moreover, by the maximality of L_1 and L_2, both N_1 and N_2 belong to components of W which are not disconnecting lines. But this is an absurd, by the first case applied to W. We thus showed that if Q_1 and Q_2 are distinct points such that $\tilde{A}_P(Q_1) = \tilde{A}_P(Q_2)$ then Q_1 and Q_2 must belong to the same maximal tree of disconnecting lines. On the other hand, by Lemma 4.2.8, if Q_1 and Q_2 are in the same tree of disconnecting lines, then $\tilde{A}_P(Q_1) = \tilde{A}_P(Q_2)$.

At last, we show that the arithmetic genus $\tilde{g}$ of $\tilde{X}$ is equal to g. Again we proceed by induction on the number of maximal trees of disconnecting lines in X. If X has no disconnecting line then, by Theorem 4.2.3, $\tilde{X}$ is isomorphic to X, so $\tilde{g} = g$. Now let L be a maximal tree of disconnecting lines of X, and let $Z_1, \dots, Z_k$ be the connected components of L^c. By Lemma 4.2.7, we have

$$g = g_1 + \ldots + g_k,$$

where g_i is the arithmetic genus of Z_i.

On the other hand, Z_i has less maximal trees of disconnecting lines than X. So, by induction hypothesis, the genus of $\tilde{Z}_i := \tilde{A}_P(Z_i)$ is g_i. Furthermore, $\tilde{g}$ is the sum of the genera of the $\tilde{Z}_i$. Indeed, since the subcurves $\tilde{Z}_1, \ldots, \tilde{Z}_k$ of $\tilde{X}$ intersect completely transversally at R, we have the exact sequences

$$0 \to \mathcal{O}_{\tilde{Z}_1}(-R) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{Z}_2 \cup \ldots \cup \tilde{Z}_k} \to 0$$

$$0 \to \mathcal{O}_{\tilde{Z}_2}(-R) \to \mathcal{O}_{\tilde{Z}_2 \cup \ldots \cup \tilde{Z}_k} \to \mathcal{O}_{\tilde{Z}_3 \cup \ldots \cup \tilde{Z}_k} \to 0$$

$$\vdots$$

$$0 \to \mathcal{O}_{\tilde{Z}_{k-1}}(-R) \to \mathcal{O}_{\tilde{Z}_{k-1} \cup \tilde{Z}_k} \to \mathcal{O}_{\tilde{Z}_k} \to 0.$$

Let $\tilde{g}_{i,\ldots,k}$ be the arithmetic genus of $\tilde{Z}_i \cup \ldots \cup \tilde{Z}_k$. Since the genus of $\tilde{Z}_i$ is g_i, taking Euler characteristics of the above sequences, we get

$$\begin{aligned} 1 - \tilde{g} &= 1 - g_1 - 1 + 1 - \tilde{g}_{2,\ldots,k} = 1 - g_1 - \tilde{g}_{2,\ldots,k} \\ 1 - \tilde{g}_{2,\ldots,k} &= 1 - g_2 - 1 + 1 - \tilde{g}_{3,\ldots,k} = 1 - g_2 - \tilde{g}_{3,\ldots,k} \\ &\vdots \\ 1 - \tilde{g}_{k-1,k} &= 1 - g_{k-1} - 1 + 1 - g_k = 1 - g_{k-1} - g_k. \end{aligned}$$

Hence $1 - \tilde{g} = 1 - g_1 - g_2 - \ldots - g_k = 1 - g$, showing that $\tilde{g} = g$. □

Remark 4.2.11 Let X be a *tree-like curve*, that is, a curve such that every point of intersection of two components of X is a disconnecting node. Assume X has double points at worst. Recall that, for every scheme Z, $\mathrm{Pic}(Z)$ parametrizes invertible sheaves on Z. Let $\mathrm{Pic}^0(Z)$ be the connected component of $\mathrm{Pic}(Z)$ containing the trivial sheaf $\mathcal{O}_Z$. So $\mathrm{Pic}(X) = J_X$, and $\mathrm{Pic}^0(X)$ parameterizes the invertible sheaves on X whose degree on every irreducible component of X is zero. Fix a simple point $P \in X$. Then the Abel map $\tilde{A}_P$ defined in Proposition 4.2.2 induces an isomorphism

$$\tilde{A}_P^* : \mathrm{Pic}^0(\overline{J}_E^P) \xrightarrow{\sim} \mathrm{Pic}^0(X).$$

This is a simple corollary of the autoduality theorem by Esteves, Gagné and Kleiman [EGK02]. Indeed, let $W_1, \ldots, W_k$ be the irreducible components of X. Since, for every i, the points in the intersection $W_i \cap W_i^c$ are disconnecting nodes we have, applying Remark 4.1.3 repeatedly, $\overline{J}_X \cong \prod \overline{J}_{W_i}$ and $J_X \cong \prod J_{W_i}$. In particular, $\mathrm{Pic}^0(X) \cong \prod \mathrm{Pic}^0(W_i)$. (Note that, since W_i is irreducible, $J^0_{W_i} = \mathrm{Pic}^0(W_i)$.) By [EGK02, Theorem 2.1, p. 595], for each degree-1 invertible sheaf

L_i on W_i we have an isomorphism $\mathrm{Pic}(\overline{J}^0_{W_i}) \xrightarrow{\sim} J^0_{W_i}$ induced by the Abel map $A^{(1)}_{L_i}$ of W_i. (Recall from Subsection 2.2.2 that $A^{(1)}_{L_i}(Q) = \mathfrak{m}_Q \otimes L_i$ for each $Q \in W_i$.)

Assume $P \in W_1$. Since X is a tree-like curve, the connected components of W_i^c are tails associated to the points in the intersection $W_i \cap W_i^c$. For $i = 2, \dots, k$, denote by N_i the point in $W_i \cap W_i^c$ associated to the connected component of W_i^c containing the point P. Set $N_1 := P$. By Remark 4.2.4, the restriction of $\tilde{A}_P$ to W_i is the Abel map of the pointed curve (W_i, N_i), that is, the map $W_i \to \overline{J}^0_{W_i}$ taking a point $Q \in W_i$ to the sheaf $\mathfrak{m}_Q(N_i)$. Taking $L_i := \mathcal{O}_{W_i}(N_i)$ for each i, we have an isomorphism induced by $\tilde{A}_P$

$$\prod \mathrm{Pic}^0(\overline{J}^0_{W_i}) \xrightarrow{\sim} \prod \mathrm{Pic}^0(W_i) = \mathrm{Pic}^0(X).$$

It remains only to show that $\prod \mathrm{Pic}^0(\overline{J}^0_{W_i}) = \mathrm{Pic}^0(\overline{J}^0_X)$.

But since each W_i is irreducible and has double points at worst, by [AIK76, Theorem 9, p. 8], $\overline{J}^0_{W_i}$ is integral and complete. So we can apply the theorem of the cube [M, Theorem, p. 91] to the product $\prod \mathrm{Pic}^0(\overline{J}^0_{W_i})$ and thus conclude that $\prod \mathrm{Pic}^0(\overline{J}^0_{W_i}) = \mathrm{Pic}^0(\prod \overline{J}^0_{W_i})$.

Remark 4.2.12 The P-twists used to define the Abel map $\tilde{A}_P$ can also be obtained in a way similar to what will be done in the next chapter. The procedure consists of considering a *smoothing of the pointed curve* (X, P), that is, a local 1-parameter family of curves $\mathcal{X}/S$ whose general member is smooth and special member is X, together with a section through the smooth locus of the family that restricts to P on the special fiber. The Abel map of the pointed family is still not defined on the whole $\mathcal{X}$. To correct this, we modify the sheaf giving the Abel map by certain Cartier divisors. The P-twists are the restrictions to X of the modification imposed by the invertible sheaves associated to these Cartier divisors.

More precisely, let Q be a nonsingular point of X, and let $Z_1 \subset \dots \subset Z_r$ be the sequence of tails described in Proposition 4.1.4 (i), and $N_1, \dots, N_r$ their respective disconnecting nodes. Assume that $\mathcal{X}$ is regular, so each tail Z_t is a Cartier divisor of $\mathcal{X}$. Then the P-twist of Q satisfies

$$I(Q) = \mathfrak{m}_Q(P) \otimes \mathcal{O}_{\mathcal{X}}(-Z_1 - \dots - Z_r)|_X.$$

Indeed, set $\tilde{Z}_t := \overline{Z_t - Z_{t-1}}$, for $t = 2, \dots, r$. Then

$$\begin{aligned}
\mathcal{O}_{\mathcal{X}}(-Z_1 - \dots - Z_r)|_{Z_1} &= \mathcal{O}_{\mathcal{X}}(-Z_1)|_{Z_1} = \mathcal{O}_{\mathcal{X}}(Z_1^c)|_{Z_1} = \mathcal{O}_{Z_1}(N_1), \\
\mathcal{O}_{\mathcal{X}}(-Z_1 - \dots - Z_r)|_{\tilde{Z}_t} &= \mathcal{O}_{\mathcal{X}}(-Z_t - Z_{t-1})|_{\tilde{Z}_t} = \mathcal{O}_{\mathcal{X}}(Z_t^c - Z_{t-1})|_{\tilde{Z}_t} \\
&= \mathcal{O}_{\tilde{Z}_t}(N_t - N_{t-1}), \text{ for each } t = 2, \dots, r, \\
\mathcal{O}_{\mathcal{X}}(-Z_1 - \dots - Z_r)|_{Z_r^c} &= \mathcal{O}_{\mathcal{X}}(-Z_r)|_{Z_r^c} = \mathcal{O}_{Z_r^c}(-N_r).
\end{aligned}$$

So $\mathcal{O}_{\mathcal{X}}(-Z_1 - \ldots - Z_r)|_X \cong M$, where M is the sheaf defined in Proposition 4.1.4.

Notice that $\mathcal{O}_{\mathcal{X}}(-Z_1 - \ldots - Z_r)$ restricts to the trivial sheaf on the generic curve X_η of the family $\mathcal{X}/S$. Therefore, the modification $\mathcal{O}_{\mathcal{X}}(-Z_1 - \ldots - Z_r)$ is simply a limit of the trivial sheaf $\mathcal{O}_{X_\eta}$ of the generic curve X_η.

The sheaves $I(Q)$ over X can be put together in a larger family, a sheaf on $\dot{\mathcal{X}} \times_S \mathcal{X}$, where $\dot{\mathcal{X}}$ is the smooth locus of $\mathcal{X}/S$. For a reasoning along these lines, and to see how to deal with the case where Q is singular, see [CE06].

Chapter 5

On the second Abel map of the two-component two-node curve

Let $A_P^{(2)} : \mathrm{Hilb}^2(X) \to \overline{J}_X^0$ be the second Abel map of a pointed curve (X, P) (see Section 2.2). As we observed in Chapter 2, $A_P^{(2)}$ is in general just a rational map. The usual approach to the problem of resolving this map would be to blow up the locus of points on $\mathrm{Hilb}^2(X)$ where the map is not defined. Here, before we perform a blowup, we consider the curve X as the special member of a one-parameter family of curves with smooth general member. We'll treat only the example of the "banana" curve, and thus focus more on the geometry of the blowups than on the combinatorics of a singular curve.

The *banana curve* is the two-component nodal curve X given by two smooth curves X_1 and X_2 joined together at two points R and S, as in Figure 5.1. Fix $P \in X_1$ a smooth point, that is, P is not R or S.

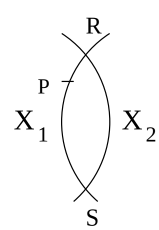


Figure 5.1: The banana curve

The image of a point of $\mathrm{Hilb}^2(X)$ corresponding to smooth points Q_1 and Q_2 of X by the map $A_P^{(2)}$ is the sheaf $\mathcal{O}_X(2P) \otimes \mathfrak{m}_{Q_1} \otimes \mathfrak{m}_{Q_2} = \mathcal{O}_X(2P - Q_1 - Q_2)$. Let $E_0 := \omega_X \oplus \mathcal{O}_X$. Then the P-quasi-stability condition for a sheaf I on X with respect to the polarization E_0 is that

$$\begin{aligned}\beta_I(X_i) &= \deg_{X_i}(I) + 1 - g_i + \deg_{X_i}(E_0)/\mathrm{rk}(E_0)\\ &= \deg_{X_i}(I) + 1 - g_i + g_i - 1 + \delta_{X_i}/2\\ &= \deg_{X_i}(I) + \delta_{X_i}/2,\end{aligned}$$

should be nonnegative for $i = 2$ and positive for $i = 1$, where g_i is the genus of X_i. Since $\delta_{X_i} = 2$ for $i = 1, 2$, and since $P \in X_1$, this condition for $I = \mathcal{O}_X(2P - Q_1 - Q_2)$ becomes

$$\deg_{X_1}(\mathcal{O}_X(-Q_1 - Q_2)) > -3 \quad \text{and} \quad \deg_{X_2}(\mathcal{O}_X(-Q_1 - Q_2)) \geq -1.$$

Since at worst, $\deg_{X_i}(\mathcal{O}_X(-Q_1 - Q_2)) = -2$, the P-quasi-stability condition fails if and only if $\deg_{X_2}(\mathcal{O}_X(-Q_1 - Q_2)) = -2$, that is, if and only if $Q_1, Q_2 \in X_2$. Thus, to achieve quasi-stability we have to modify the map $A_P^{(2)}$ on the points of $\mathrm{Hilb}^2(X)$ corresponding to a pair of points belonging to X_2.

Furthermore, if at most one of the points Q_i is a node of X then the sheaf $\mathcal{O}_X(2P) \otimes \mathfrak{m}_{Q_1} \otimes \mathfrak{m}_{Q_2}$ is simple, as seen in Section 2.2. Thus, the nonsimple sheaves appear when both Q_1 and Q_2 are (distinct) nodes and in any case we have to resolve the nodes by a blowup.

5.1 Our main result

Let $(\mathcal{X}/S, \sigma)$ be a *smoothing* of (X, P), that is, a local one-parameter family of curves with smooth generic fiber and special fiber X, carrying a section σ through the smooth locus of $\mathcal{X}/S$ which restricts to P on the special fiber. Denote by η the generic point of S and o the special point. We assume the total space $\mathcal{X}$ is regular. Let $\mathcal{X}^3 := \mathcal{X} \times_S \mathcal{X} \times_S \mathcal{X}$ and $\mathcal{X}^2 := \mathcal{X} \times_S \mathcal{X}$. The projection $p_{12} : \mathcal{X}^3 \to \mathcal{X}^2$ on the product of the first and second factors is a family of curves. Denote by $\psi : \mathcal{X} \to S$ the morphism of the smoothing. The fiber of p_{12} over a point (P_1, P_2) of $\mathcal{X}^2$ is just the fiber of $\mathcal{X}/S$ over $\psi(P_1) = \psi(P_2)$. Note that $p_{12} : \mathcal{X}^3 \to \mathcal{X}^2$ is obtained from ψ by the base change $\mathcal{X}^2 \to S$.

Let $\Delta_{13} := p_{13}^{-1}(\Delta)$ and $\Delta_{23} := p_{23}^{-1}(\Delta)$, where $\Delta \subset \mathcal{X}^2$ is the diagonal subscheme, and the map $p_{ij} : \mathcal{X}^3 \to \mathcal{X}^2$ is the projection on the product of the i-th and j-th factors. Consider the Abel map defined as the rational map from $\mathcal{X}^2$ to $\overline{J}^0_{\mathcal{X}/S}$ associated to the sheaf

$$\mathcal{I} := I_{\Delta_{13}} \otimes I_{\Delta_{23}} \otimes p_3^* I_\Sigma^{-2}$$

on $\mathcal{X}^3$, where $\Sigma = \sigma(S)$ and $p_3 : \mathcal{X}^3 \to \mathcal{X}$ is the projection on the third factor. There is a natural rational map from $\mathcal{X}^2$ to $\mathrm{Hilb}^2(\mathcal{X})$ taking a pair of points on $\mathcal{X}$ over the same point $s \in S$ to the corresponding length-2 subscheme of the fiber $\psi^{-1}(s)$, and the Abel map defined above factors through the second Abel map $A_\sigma^{(2)}$ of $(\mathcal{X}/S, \sigma)$.

Note that $\mathcal{X}^2$ is not regular exactly at points (N, N') with N and N' nodes of X; and $\mathcal{X}^3$ is not regular at points (Q_1, Q_2, Q_2) where at least two of the Q_i's are nodes of X. We resolve these singularities as follows:

(i) First, resolve the basis $\mathcal{X}^2$ of the family p_{12} by first blowing up along Δ, and then blowing up along the strict transform of $X_2 \times X_2$. Call the resulting scheme $\tilde{\mathcal{X}}^2$ and consider a base change of the family p_{12} by $\tilde{\mathcal{X}}^2$, that is, $\tilde{\mathcal{X}}^2 \times_{\mathcal{X}^2} \mathcal{X}^3 = \tilde{\mathcal{X}}^2 \times_S \mathcal{X} \to \tilde{\mathcal{X}}^2$. Note this base change does not affect the fibers of the family $\mathcal{X}^3/\mathcal{X}^2$ given by p_{12}.

(ii) Second, we modify the fibers by blowing up $\tilde{\mathcal{X}}^2 \times_S \mathcal{X}$ along the strict transform of the diagonals Δ_{23} and Δ_{13}, and then along the strict transforms of $X \times X_2 \times X_2$ and $X_2 \times X \times X_2$.

Notice that, at each step, we blow up along the strict transform of the subscheme in question with regard to the morphism to $\mathcal{X}^2$ or $\mathcal{X}^3$, that being done to assure that at each explosion we consider a pure codimension-1 subscheme.

We obtain a 4-fold $\tilde{\mathcal{X}}^3$ with a (birational) morphism

$$b : \tilde{\mathcal{X}}^3 \to \tilde{\mathcal{X}}^2 \times_S \mathcal{X},$$

which is the composition of the blowup maps of (ii). Composing with the projection, we get a family of curves:

$$\tilde{p}_{12} : \tilde{\mathcal{X}}^3 \to \tilde{\mathcal{X}}^2.$$

We show in Lemmas 5.2.1 and 5.2.4 (see Remark 5.2.5) that both $\mathcal{X}^2$ and $\mathcal{X}^3$ are regular. For each $i, j, k \in \{0, 1, 2\}$, denote by $\tilde{X}_{ijk}$ the strict transform of $X_i \times X_j \times X_k$ under the birational morphism $\tilde{\mathcal{X}}^3 \to \mathcal{X}^3$, where $X_0 := X$. Then $\tilde{X}_{ijk}$ is obviously a Cartier divisor of $\tilde{\mathcal{X}}^3$, since it is a codimension-1 subscheme of the regular scheme $\tilde{\mathcal{X}}^3$. Note that, as Cartier divisors,

$$\begin{aligned} \tilde{X}_{0jk} &= \tilde{X}_{1jk} + \tilde{X}_{2jk}, \\ \tilde{X}_{i0k} &= \tilde{X}_{i1k} + \tilde{X}_{i2k}, \\ \tilde{X}_{ij0} &= \tilde{X}_{ij1} + \tilde{X}_{ij2} \end{aligned}$$

Also, for each $i, j \in \{0, 1, 2\}$, denote by $\tilde{X}_{ij}$ the strict transform of $X_i \times X_j$ under the composition of blowups in (i),

$$f : \tilde{\mathcal{X}}^2 \to \mathcal{X}^2.$$

Note that the $\tilde{X}_{ij}$ are Cartier divisors of $\tilde{\mathcal{X}}^2$.

The section σ of $\mathcal{X}/S$ extends, by base change, to a section σ_0 of $p_{12} : \mathcal{X}^3 \to \mathcal{X}^2$. Thus, since σ is a section through the smooth locus of $\mathcal{X}/S$, σ_0 lifts to a section $\tilde{\sigma}$ of $\tilde{p}_{12} : \tilde{\mathcal{X}}^3 \to \tilde{\mathcal{X}}^2$ through the smooth locus of $\tilde{p}_{12}$. Let $\tilde{\Sigma} = \tilde{\sigma}(\tilde{\mathcal{X}}^2)$. Then

$$\tilde{\mathcal{I}} := I_{\tilde{\Delta}_{13}} \otimes I_{\tilde{\Delta}_{23}} \otimes I_{\tilde{\Sigma}}^{-2}$$

is an invertible sheaf on $\tilde{\mathcal{X}}^3$, where $\tilde{\Delta}_{13}$ and $\tilde{\Delta}_{23}$ are the strict transforms of the diagonals Δ_{13} and Δ_{23} by the birational morphism $\tilde{\mathcal{X}}^3 \to \mathcal{X}^3$.

Theorem 5.1.1 *The sheaf*

$$\mathcal{J} := \tilde{\mathcal{I}}(-\tilde{X}_{222}).$$

is simple and $\tilde{\sigma}$-quasi-stable with respect to $\mathcal{E} := \omega_{\tilde{\mathcal{X}}^3/\tilde{\mathcal{X}}^2} \oplus \mathcal{O}_{\tilde{\mathcal{X}}^3}$. In addition, $b_*\mathcal{J}$ induces a morphism

$$\tilde{A}_P^{(2)} : \tilde{\mathcal{X}}^2 \longrightarrow \overline{\mathcal{J}}_E^{\sigma},$$

where $E = \omega_{\mathcal{X}/S} \oplus \mathcal{O}_{\mathcal{X}}$.

5.2 The family $\tilde{\mathcal{X}}^3/\tilde{\mathcal{X}}^2$

In order to prove the theorem, we must understand the curves of the family $\tilde{p}_{12} : \tilde{\mathcal{X}}^3 \to \tilde{\mathcal{X}}^2$ in terms of the original family $\psi : \mathcal{X} \to S$. Consider the structure morphism $\tilde{\mathcal{X}}^3 \to S$. Then $\tilde{\mathcal{X}}^3$ is isomorphic to $\mathcal{X}^3$ off the inverse image of $o \in S$. So, to show the theorem, we need only examine the fibers of $\tilde{p}_{12}$ over points $s \in \tilde{\mathcal{X}}^2$ over the special point o of S. The first lemma gives us a local description of $\tilde{\mathcal{X}}^2$.

Lemma 5.2.1 (Description of the base) *Let $f : \tilde{\mathcal{X}}^2 \to \mathcal{X}^2$ be the composition of the blowups in (i). Then*

(a) f is an isomorphism away from (R,R), (S,S), (R,S) and (S,R);

(b) If $E = f^{-1}(R,R)$ or $E = f^{-1}(S,S)$, then E is a smooth rational curve which is a subscheme of $\tilde{X}_{12}$ and $\tilde{X}_{21}$. Moreover, E crosses $\tilde{X}_{11}$ and $\tilde{X}_{22}$ transversally at distinct points;

(c) If $E = f^{-1}(R,S)$ or $E = f^{-1}(S,R)$, then E is a smooth rational curve which is subscheme of $\tilde{X}_{11}$ and $\tilde{X}_{22}$. Moreover, E crosses $\tilde{X}_{12}$ and $\tilde{X}_{21}$ transversally at distinct points.

In addition, $\tilde{\mathcal{X}}^2$ is regular, and the composition $h_i : \tilde{\mathcal{X}}^2 \to \mathcal{X}$ of f and the i-th projection $\mathcal{X}^2 \to \mathcal{X}$ is flat, for each $i = 1, 2$.

Proof. Since both the diagonal Δ and $X_2 \times X_2$ are Cartier divisors on $\mathcal{X}^2$ everywhere but at (R,R), (S,S), (R,S) and (S,R), we have (a).

Denote by $f_1 : B \to \mathcal{X}^2$ and $f_2 : \tilde{\mathcal{X}}^2 \to B$ the first and second explosions described in (i), so that $f = f_1 \circ f_2$. So f_1 is the blowup along the diagonal Δ and f_2 is the blowup along the strict transform of $X_2 \times X_2$.

Now, let t be a local parameter of S at the special point o, so that $\hat{\mathcal{O}}_{S,o} \cong k[[t]]$. Let N and N' be nodes of X, not necessarily distinct. Since $\mathcal{X}$ is regular, we have

$$\hat{\mathcal{O}}_{\mathcal{X},N} \cong \frac{k[[x_1,x_2,t]]}{(x_1x_2 - t)} \quad \text{and} \quad \hat{\mathcal{O}}_{\mathcal{X},N'} \cong \frac{k[[y_1,y_2,t]]}{(y_1y_2 - t)},$$

where x_1 (resp. x_2) is a local equation of X_2 (resp. X_1) at N, and y_1 (resp. y_2) is a local equation of X_2 (resp. X_1) at N'. (If $N = N'$ we assume $x_i = y_i$ for $i = 1,2$.) Then, the restriction of x_i (resp. y_i) to X_i is a local parameter for X_i at N (resp. N'). We shall denote by x_i and y_i also the compositions of x_i and y_i with the first and second projections $\mathcal{X}^2 \to \mathcal{X}$ respectively, so that

$$\hat{\mathcal{O}}_{\mathcal{X}^2,(N,N')} \cong \frac{k[[x_1,x_2,y_1,y_2,t]]}{(x_1x_2 - t, y_1y_2 - t)} \cong \frac{k[[x_1,x_2,y_1,y_2]]}{(x_1x_2 - y_1y_2)}.$$

We prove (b) first. The diagonal Δ contains both points (R,R) and (S,S). Let N be either R or S. Locally at the point (N,N), the ideal of the diagonal is $(x_1 - y_1, x_2 - y_2)$. So B is the subscheme of $\mathcal{X}^2 \times \mathbb{P}^1$ given locally over (N,N) by

$$\alpha(x_1 - y_1) + \alpha'(x_2 - y_2) = 0,$$

where α, α' are homogeneous coordinates of $\mathbb{P}^1$. Assuming $\alpha' = 1$, the equations $x_1x_2 = y_1y_2$ and $\alpha(x_1 - y_1) + (x_2 - y_2) = 0$ give

$$x_2 = \alpha y_1 \ \text{ and } \ y_2 = \alpha x_1, \quad \text{and thus} \quad \hat{\mathcal{O}}_B \cong k[[x_1, y_1, \alpha - a]] \tag{5.1}$$

at $(a:1) \in B$. Likewise, assuming $\alpha = 1$, we get

$$y_1 = \alpha' x_2 \ \text{ and } \ x_1 = \alpha' y_2, \quad \text{and thus} \quad \hat{\mathcal{O}}_B \cong k[[x_2, y_2, \alpha' - a']] \tag{5.2}$$

at $(1:a') \in B$.

The ideal of $X_1 \times X_2$ at the point (N,N) is (x_2, y_1). If $\alpha' \neq 0$ we may assume $\alpha' = 1$ and, locally at $(N,N,(a:1)) \in B$, we have that $\tilde{X}_{12}$ is given by $y_1 = 0$. Also, if $\alpha \neq 0$ we may assume $\alpha = 1$ and, locally at $(N,N,(1:a')) \in B$, we have that $\tilde{X}_{12}$ is given by $x_2 = 0$. Since E is given by the equations $x_1 = x_2 = y_1 = y_2 = 0$, we get $E \subset \tilde{X}_{12}$. By an analogous reasoning we get $E \subset \tilde{X}_{21}$.

In addition, the ideal of $X_1 \times X_1$ at (N, N) is (x_2, y_2). Locally at $(N, N, (a:1)) \in B$, the strict transform $\tilde{X}_{11}$ is given by $\alpha = 0$. Thus, $\tilde{X}_{11}$ intersects E transversally at $(N, N, (0:1))$. (Note that $\tilde{X}_{11}$ does not intersect E in any other point since, locally at $(N, N, (1:a'))$, the equations $x_2 = 0$ and $y_2 = 0$ cut out a codimension-2 subscheme of B.) Again, by an analogous reasoning, we get that $\tilde{X}_{22}$ intersects E transversally at $(N, N, (1:0))$, and does not meet E is any other point. Note that the second explosion f_2 is an isomorphism on (a neighborhood of) $f_1^{-1}(N, N)$, since B is already regular on $f_1^{-1}(N, N)$.

For (c), we note that both points (R, S) and (S, R) are not in the diagonal Δ, and therefore f_1 is an isomorphism on a neighborhood of them. Let (N, N') be either (R, S) or (S, R). Since B is locally isomorphic to $\mathcal{X}^2$ at (N, N'), we have

$$\hat{\mathcal{O}}_{B,(N,N')} \cong \hat{\mathcal{O}}_{\mathcal{X}^2,(N,N')}.$$

Since the ideal of $X_2 \times X_2$ locally at (N, N') is (x_1, y_1), the blowup $\tilde{\mathcal{X}}^2$ of B along $X_2 \times X_2$ is the subscheme of $B \times \mathbb{P}^1$ given locally over (N, N') by

$$\alpha x_1 = \alpha' y_1,$$

where α, α' are homogeneous coordinates of $\mathbb{P}^1$. Assuming $\alpha' = 1$, the equations $x_1 x_2 = y_1 y_2$ and $y_1 = \alpha x_1$ give

$$y_1 = \alpha x_1 \quad \text{and} \quad x_2 = \alpha y_2, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{\mathcal{X}}^2} \cong k[[x_1, y_2, \alpha - a]] \tag{5.3}$$

at $(a:1) \in \tilde{\mathcal{X}}^2$. Likewise, assuming $\alpha = 1$ we get

$$x_1 = \alpha' y_1 \quad \text{and} \quad y_2 = \alpha' x_2, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{\mathcal{X}}^2} \cong k[[x_2, y_1, \alpha' - a']] \tag{5.4}$$

at $(1:a') \in \tilde{\mathcal{X}}^2$.

The ideal of $X_1 \times X_1$ at the point (N, N') is (x_2, y_2). Again, if $\alpha' \neq 0$ we may assume $\alpha' = 1$ and, locally at $(N, N', (a:1)) \in \tilde{\mathcal{X}}^2$, we have that $\tilde{X}_{11}$ is given by the equation $y_2 = 0$. Also, locally at $(N, N', (1:a')) \in \tilde{\mathcal{X}}^2$ we get that $\tilde{X}_{11}$ is given by the equation $x_2 = 0$. Since E is given by $x_1 = x_2 = y_1 = y_2 = 0$, we get $E \subset \tilde{X}_{11}$. By an analogous reasoning, we get $E \subset \tilde{X}_{22}$.

Now, the ideal of $X_1 \times X_2$ at (N, N') is (x_2, y_1). So, locally at $(N, N', (a:1)) \in \tilde{\mathcal{X}}^2$ we have that $\tilde{X}_{12}$ is given by the equation $\alpha = 0$, showing that $\tilde{X}_{12}$ intersects E transversally at the point $(N, N', (0:1))$. Moreover, $\tilde{X}_{12}$ does not meet E in any other point since, locally at $(N, N', (1:a'))$, the equations $x_2 = 0$ and $y_1 = 0$ cut out a codimension-2 subscheme of $\tilde{\mathcal{X}}^2$. By the same reasoning, $\tilde{X}_{21}$ intersects E transversally at $(N, N', (1:0))$, and does not meet E in any other point.

At last, we show that h_i is flat for $i = 1, 2$. Consider the induced map of rings

$$h_i^\# : \hat{\mathcal{O}}_{\mathcal{X}, h_i(s)} \to \hat{\mathcal{O}}_{\tilde{\mathcal{X}}^2, s}.$$

We need to show that, for each $s \in \tilde{\mathcal{X}}^2$, the map $h_i^\#$ is flat. Note that the projection $\mathcal{X}^2 \to \mathcal{X}$ on the i-th factor is flat. Thus, since f is an isomorphism away from (R, R), (S, S), (R, S) and (S, R), we need only verify that $h_i^\#$ is flat for points s belonging to the smooth rational curves described in (b) and (c).

First let $s \in E = f^{-1}(N, N)$, where N is a node of X. By (5.1), locally at $(a:1) \in B$, the map $h_1^\#$ is given by

$$\begin{array}{ccc} k[[x_1, x_2]] & \longrightarrow & k[[x_1, y_1, \alpha - a]] \\ x_1 & \mapsto & x_1 \\ x_2 & \mapsto & \alpha y_1. \end{array}$$

By the local criterion of flatness, this map is flat if the map $k[[x_2]] \to k[[y_1, \alpha - a]]$ taking x_2 to αy_1 is so. Now, $k[[x_2]]$ is a principal ideal domain, and αy_1 is a nonzero-divisor in $k[[y_1, \alpha - a]]$. Therefore $k[[y_1, \alpha - a]]$ is flat over $k[[x_2]]$. Hence h_1 is flat, locally at $(a : 1)$. By an analogous reasoning, h_1 is flat locally at $(1 : a')$.

Now, by (5.1), locally at $(a : 1) \in B$, the map $h_2^\#$ is given by

$$\begin{array}{ccc} k[[y_1, y_2]] & \longrightarrow & k[[x_1, y_1, \alpha - a]] \\ y_1 & \mapsto & y_1 \\ y_2 & \mapsto & \alpha x_1. \end{array}$$

Again, by the local criterion of flatness, this map is flat if the map $k[[y_2]] \to k[[x_1, \alpha - a]]$ taking y_2 to αx_1 is so. Now, since $k[[y_2]]$ is a principal ideal domain, and αx_1 is a nonzero-divisor in $k[[x_1, \alpha - a]]$, then $k[[x_1, \alpha - a]]$ is flat over $k[[y_2]]$. Thus h_2 is flat locally at $(a : 1)$. By an analogous reasoning, h_2 is flat locally at $(1 : a')$.

Finally, let $s \in E = f^{-1}(N, N')$, where N and N' are distinct nodes of X. By (5.3), locally at $(a : 1) \in \tilde{\mathcal{X}}^2$, the map $h_1^\#$ is given by

$$\begin{array}{ccc} k[[x_1, x_2]] & \longrightarrow & k[[x_1, y_2, \alpha - a]] \\ x_1 & \mapsto & x_1 \\ x_2 & \mapsto & \alpha y_2. \end{array}$$

As before, this map is flat if the map $k[[x_2]] \to k[[y_2, \alpha - a]]$ taking x_2 to αy_2 is so. But $k[[x_2]]$ is a principal ideal domain, and αy_2 is a nonzero-divisor in $k[[y_2, \alpha - a]]$, thus $k[[y_1, \alpha - a]]$ is flat over $k[[x_2]]$. Hence h_1 is flat locally at $(a : 1)$. By an analogous reasoning, h_1 is flat locally at $(1 : a')$.

In addition, by (5.3), locally at $(a:1) \in B$, the map $h_2^\#$ is given by

$$\begin{aligned} k[[y_1, y_2]] &\longrightarrow k[[x_1, y_2, \alpha - a]] \\ y_2 &\mapsto y_2 \\ y_1 &\mapsto \alpha x_1. \end{aligned}$$

Again, this map is flat if the map $k[[y_1]] \to k[[x_1, \alpha - a]]$ taking y_1 to αx_1 is so. But $k[[x_1, \alpha - a]]$ is flat over $k[[y_2]]$, since $k[[y_2]]$ is a principal ideal domain, and αx_1 is a nonzero-divisor in $k[[x_1, \alpha - a]]$. Thus h_2 is flat locally at $(a:1)$. By an analogous reasoning, h_2 is flat locally at $(1:a')$. □

Denote by $\tilde{\psi}_2 : \tilde{\mathcal{X}}^2 \to S$ the morphism induced by $\psi : \mathcal{X} \to S$. In Figure 5.2 we represent on the left a neighborhood of $E = f^{-1}(N, N)$, and on the right a neighborhood of $E = f^{-1}(N, N')$ in the surface $\tilde{\mathcal{X}}^2(o) := \tilde{\psi}_2^{-1}(o)$, for N and N' distinct nodes of X.

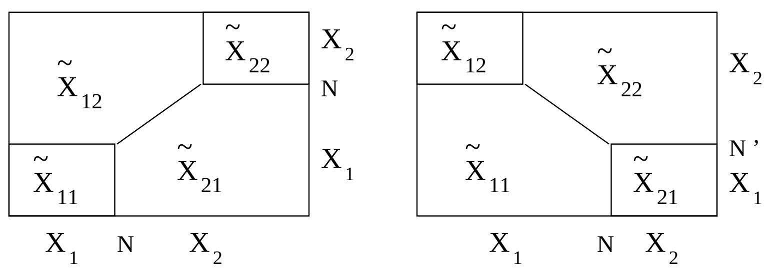


Figure 5.2: The basis types

For each $s \in \tilde{\mathcal{X}}^2$, denote by $\tilde{X}_s$ the fiber of $\tilde{p}_{12} : \tilde{\mathcal{X}}^3 \to \tilde{\mathcal{X}}^2$ over s, that is, $\tilde{X}_s = \tilde{p}_{12}^{-1}(s)$. The following three lemmas give a full description of the fibers $\tilde{X}_s$. But first, we set some notation. Recall that $f : \tilde{\mathcal{X}}^2 \to \mathcal{X}^2$ is the composition of the blowups in (i). Denote by Δ'_{23}, Δ'_{13}, X'_{022} and X'_{202} the strict transforms of Δ_{23}, Δ_{13}, $X \times X_2 \times X_2$ and $X_2 \times X \times X_2$ under $f \times \mathrm{Id} : \tilde{\mathcal{X}}^2 \times_S \mathcal{X} \to \mathcal{X}^3$.

Also, recall that $b : \tilde{\mathcal{X}}^3 \to \tilde{\mathcal{X}}^2 \times_S \mathcal{X}$ is the composition of the blowups in (ii). Let

$$b_n : \tilde{B}_n \longrightarrow \tilde{B}_{n-1}$$

be the n-th blowup map in (ii), for $n \in \{1, 2, 3, 4\}$, with $\tilde{B}_0 = \tilde{\mathcal{X}}^2 \times_S \mathcal{X}$ and $\tilde{B}_4 := \tilde{\mathcal{X}}^3$, so that

$$b : \tilde{B}_4 \xrightarrow{b_4} \tilde{B}_3 \xrightarrow{b_3} \tilde{B}_2 \xrightarrow{b_2} \tilde{B}_1 \xrightarrow{b_1} \tilde{B}_0.$$

Set

$$\tilde{q}^n : \tilde{B}_n \longrightarrow \tilde{\mathcal{X}}^2$$

to be the composition of $b_1 \circ \ldots \circ b_n$ with the projection $\tilde{\mathcal{X}}^2 \times_S \mathcal{X} \to \tilde{\mathcal{X}}^2$. Note that $\tilde{q}^n$ is a family of curves. Let $X_s := (\tilde{q}^0)^{-1}(s)$ for each $s \in \tilde{\mathcal{X}}^2$.

Let $q_{ij} : \tilde{\mathcal{X}}^2 \times_S \mathcal{X} \to \mathcal{X}^2$ be the composition of $f \times \mathrm{Id} : \tilde{\mathcal{X}}^2 \times_S \mathcal{X} \to \mathcal{X}^3$ with the projection $p_{ij} : \mathcal{X}^3 \to \mathcal{X}^2$ on the product of the i-th and j-th factors. More generaly, set

$$q_{ij}^n : \tilde{B}_n \longrightarrow \mathcal{X}^2$$

to be $q_{ij}^n := q_{ij} \circ b_1 \circ \ldots \circ b_n$ for $n \in \{1,2,3,4\}$, and $q_{ij}^0 = q_{ij}$. Note that, for $i = 1, 2$, the map q_{i3}^0 is flat. Indeed, we have a Cartesian diagram

$$\begin{array}{ccc} \tilde{\mathcal{X}}^2 \times_S \mathcal{X} & \longrightarrow & \tilde{\mathcal{X}}^2 \\ {\scriptstyle q_{i3}^0}\downarrow & & \downarrow{\scriptstyle h_i} \\ \mathcal{X} \times_S \mathcal{X} & \longrightarrow & \mathcal{X}, \end{array}$$

where $\tilde{\mathcal{X}}^2 \times_S \mathcal{X} \to \tilde{\mathcal{X}}^2$ is the projection on the first factor, and $\mathcal{X} \times_S \mathcal{X} \to \mathcal{X}$ is the projection on the i-th factor. Thus, q_{i3}^0 is obtained by base change from the map h_i defined in Lemma 5.2.1. Since h_i is flat, q_{i3}^0 is also flat.

Lemma 5.2.2 (Description of the fibers) *Let $s \in \tilde{\mathcal{X}}^2$. Then the fiber $\tilde{X}_s$ of $\tilde{p}_{12} : \tilde{\mathcal{X}}^3 \to \tilde{\mathcal{X}}^2$ over the point s is as follows:*

(a) If $\tilde{\psi}_2(s)$ is not the special point o of S, then $\tilde{X}_s \cong \psi^{-1}(\tilde{\psi}_2(s))$;

(b) If $\tilde{\psi}_2(s) = o$ and $f(s) = (Q_1, Q_2)$ with Q_1, Q_2 smooth points of X, then the fiber is isomorphic to X (see Figure 5.3 (1));

(c) If $\tilde{\psi}_2(s) = o$ and s is in the intersection of exactly two divisors $\tilde{X}_{ij}$ on $\tilde{\mathcal{X}}^2$, then $\tilde{X}_s$ is the four-component curve described in Figure 5.3 (2), with two components isomorphic to X_1 and X_2, not intersecting each other, and two rational components $E(R)$ and $E(S)$ intersecting both X_1 and X_2 transversally at R and S and not intersecting each other;

(d) If $\tilde{\psi}_2(s) = o$ and s is in the intersection of three divisors $\tilde{X}_{ij}$ on $\tilde{\mathcal{X}}^2$, then $\tilde{X}_s$ is the six-component curve described in Figure 5.3 (3), with two components isomorphic to X_1 and X_2, not crossing each other, and two trees of rational curves $E(R)$ and $E(S)$, of two components each, intersecting both X_1 and X_2 transversally at R and S and not crossing each other.

Proof. If $\tilde{\psi}_2(s) \neq o$ then the curve X_s is smooth, so b is an isomorphism over X_s, and thus $X_s \cong \tilde{X}_s$, showing (a). If $f(s) = (Q_1, Q_2)$ with Q_1, Q_2 smooth points of X, then all of Δ'_{23}, Δ'_{13},

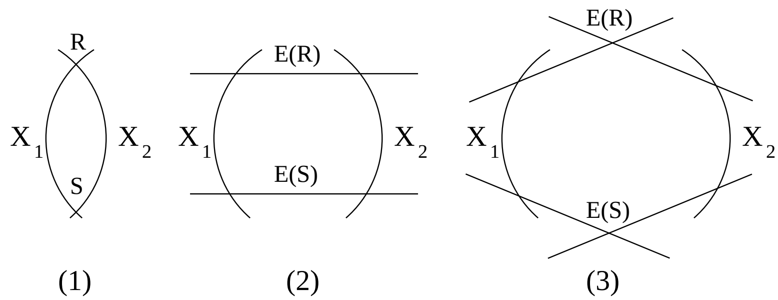


Figure 5.3: The fiber types

X'_{022} and X'_{202} intersect $X_s \cong X$ at smooth points, and hence b is an isomorphism over $\tilde{X}_s$ as well, showing (b).

Under the hypothesis of (c) we have actually two different cases: one if s belongs to $\tilde{X}_{ii} \cap \tilde{X}_{ij}$ or $\tilde{X}_{jj} \cap \tilde{X}_{ij}$, and the other if s belongs to $\tilde{X}_{ij} \cap \tilde{X}_{ji}$ or $\tilde{X}_{ii} \cap X_{jj}$, with $i \neq j$. Let $p, q : X \times X \to X$ be the first and second projections, respectively.

We first assume that $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij}$ (the case where $s \in \tilde{X}_{jj} \cap \tilde{X}_{ij}$ is analogous). Then $f(s) \in (X_i \times X_i) \cap (X_i \times X_j)$ and $p \circ f(s) \in X_i$. Since by hypothesis s belongs to exactly two components of $\tilde{\mathcal{X}}^2(o)$, we have $p \circ f(s) \notin X_j$. Thus the point $p \circ f(s)$ is a smooth point of X. Also, $q \circ f(s) \in X_i \cap X_j$ is a node (either R or S). Therefore $f(s) = (Q, N)$ where Q is a smooth point of X belonging to X_i and N is either R or S.

Let N' be a node of $X_s \cong X$, not necessarily distinct from N. We claim that the first of the blowups in (ii) that is not an isomorphism at N' can be realized as the base change of a blowup of $\mathcal{X}^2$ under the map q_{13}^{n-1} or the map q_{23}^{n-1},

$$\begin{array}{ccc} \tilde{B}_n & \xrightarrow{b_n} & \tilde{B}_{n-1} \\ \downarrow & & \downarrow{\scriptstyle q_{i3}^{n-1}} \\ \overline{\mathcal{X}}^2 & \longrightarrow & \mathcal{X}^2. \end{array}$$

Indeed, suppose the blowup b_n is along a subscheme Z of $\tilde{B}_{n-1}$ which is the strict transform under $b_1 \circ \ldots \circ b_{n-1}$ of a subscheme Z_0 of $\tilde{B}_0$. Since by hypothesis b_n is the first blowup to affect the node $N' \in X_s$, then $b_1 \circ \ldots \circ b_{n-1}$ is an isomorphism over N' and, in particular, an isomorphism over a neighborhood of Z_0 around N'. Now, Δ'_{23}, Δ'_{13}, X'_{022} and X'_{202} are inverse images under q_{i3}^0, for $i = 1$ or $i = 2$, of subschemes of $\mathcal{X}^2$. In addition, since $\tilde{B}_{n-1}$ is locally isomorphic to $\tilde{B}_0$ over (s, N'), and q_{13}^0 (resp. q_{23}^0) is flat, then q_{13}^{n-1} (resp. q_{23}^{n-1}) is locally flat at (Q, N') (resp. (N, N')). The flatness guarantees that the blowup of $\mathcal{X}^2$ along a subscheme base-changes under q_{i3}^{n-1} to the blowup of $\tilde{B}_{n-1}$ along the inverse image of this subscheme. Our claim follows.

This way, it is easy to see that the fiber over s of the blowup of $\tilde{B}_0 = \tilde{\mathcal{X}}^2 \times_S \mathcal{X}$ along $\Delta'_{23} \subset \tilde{B}_0$ is the inverse image by $q^0_{23} = q_{23}$ of the fiber over $q \circ f(s) = N$ of the blowup of $\mathcal{X}^2$ along the diagonal Δ, which produces the rational component $E(N)$ described in (c). Moreover, since $p \circ f(s) = Q$ is a smooth point, b_2 is an isomorphism over $b_1^{-1}(X_s)$.

Now, let N' be the other node of X. Away from $E(N)$, the fiber of $\tilde{B}_3$ over s is isomorphic to the inverse image by q^3_{23} of the fiber over N of the blowup of $\mathcal{X}^2$ along $X_2 \times X_2$, which produces the rational component $E(N')$ described in (c). (Note that the strict transform of $X \times X_2 \times X_2$ is a Cartier divisor on $\tilde{B}_2$ away from (s, N').) Also, since $p \circ f(s) = Q$ is a smooth point, the strict transform of $X_2 \times X \times X_2$ in $\tilde{B}_3$ crosses $(\tilde{q}^3)^{-1}(s)$ at a smooth point, so b_4 is an isomorphism on $(\tilde{q}^3)^{-1}(s)$.

Now, assume $s \in \tilde{X}_{ij} \cap \tilde{X}_{ji}$. Then s is on the "exceptional" component E of $\tilde{\mathcal{X}}^2(o)$ over (N, N), where N is either R or S. We will use the same notation used in the proof of Lemma 5.2.1, so that $\hat{\mathcal{O}}_{\tilde{\mathcal{X}}^2,s} \cong k[[x_1, y_1, \alpha - a]]$; see (5.1). (Note that, since s is not on X_{ii} or on X_{jj}, the homogeneous coordinates α and α' defined in the proof of Lemma 5.2.1 are both nonzero, and hence we may assume $\alpha' = 1$).

Recall that $\tilde{B}_0 = \tilde{\mathcal{X}}^2 \times_S \mathcal{X}$. Let z_1, z_2 be the composition of the functions defining X_2, X_1 locally at N, respectively, with the projection $\tilde{\mathcal{X}}^2 \times_S \mathcal{X} \to \mathcal{X}$. Then

$$\hat{\mathcal{O}}_{\tilde{B}_0,(s,N)} \cong \frac{k[[x_1, y_1, \alpha - a, z_1, z_2]]}{(z_1 z_2 - \alpha x_1 y_1)}.$$

(Note that the local parameter t of S at o chosen in Lemma 5.2.1, satisfying $t = x_1 x_2$, becomes $t = \alpha x_1 y_1$ by (5.1).) The first explosion in (ii), b_1, is along the diagonal $\Delta'_{23} \subset \tilde{B}_0$, whose ideal, locally at (s, N), is $(y_1 - z_1, \alpha x_1 - z_2)$, since $y_2 = \alpha x_1$ by (5.1). This way, $\tilde{B}_1$ is, locally over $(s, N) \in \tilde{B}_0$, the subscheme of $\tilde{B}_0 \times \mathbb{P}^1$ given by

$$\beta(y_1 - z_1) + \beta'(\alpha x_1 - z_2) = 0$$

where β, β' are homogeneous coordinates of $\mathbb{P}^1$. If $\beta' \neq 0$ we may assume $\beta' = 1$, and hence the equations $\beta(y_1 - z_1) + (\alpha x_1 - z_2) = 0$ and $z_1 z_2 = \alpha x_1 y_1$ give

$$\text{(5.5)} \qquad \alpha x_1 = \beta z_1 \ \text{ and } \ z_2 = \beta y_1, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_1} \cong \frac{k[[x_1, y_1, \alpha - a, z_1, \beta - b]]}{(\alpha x_1 - \beta z_1)}$$

at $(s, N, (b : 1))$. Since $\alpha \neq 0$, the above ring is regular (as we may write x_1 in terms of β, z_1 and α^{-1}). On the other hand, where $\beta \neq 0$ we may assume $\beta = 1$ and we get

$$\text{(5.6)} \qquad z_1 = \alpha \beta' x_1 \ \text{ and } \ y_1 = \beta' z_2, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_1} \cong k[[x_1, \alpha - a, z_2, \beta' - b']]$$

locally at $(s, N, (1 : b'))$. Once again we obtain a regular ring.

Now, note that the homogeneous coordinates β, β' indeed describe a rational component $E(N)$ of $\tilde{X}_s$, because $\tilde{p}_{12}^{-1}(s, N)$ is given by the equations $x_i = y_i = z_i = 0$, for $i = 1, 2$, and $\alpha = a$. Thus, modding out the rings in (5.5) and (5.6) by the equations defining $\tilde{p}_{12}^{-1}(s, N)$, we get on the one hand $k[[\beta - b]]$, and on the other hand $k[[\beta' - b']]$. Since $\tilde{B}_1$ is regular along $E(N)$, the composition $b_2 \circ b_3 \circ b_4$ is an isomorphism over $E(N)$, so there is only one rational component $E(N)$ over N in $\tilde{X}_s$.

The second explosion b_2, along the strict transform of Δ'_{13} in $\tilde{B}_1$, is an isomorphism over $(\tilde{q}^1)^{-1}(s)$, because Δ'_{13} does not contain (s, N') for $N' \neq N$ and is a Cartier divisor on $E(N)$, since $\tilde{B}_1$ is regular along $E(N)$.

The third explosion, b_3, along the strict transform of X'_{022} of $X \times X_2 \times X_2$ in $\tilde{B}_2$, produces the rational component $E(N')$ described in (c). Indeed, note that b_1 and b_2 are isomorphisms over (s, N'). Now, we let z_1, z_2 be the compositions of functions defining X_2, X_1 locally at N', respectively, with the projection $\tilde{\mathcal{X}}^2 \times_S \mathcal{X} \to \mathcal{X}$. Then

$$\hat{\mathcal{O}}_{\tilde{B}_0,(s,N')} \cong \frac{k[[x_1, y_1, \alpha - a, z_1, z_2]]}{(z_1 z_2 - \alpha x_1 y_1)}.$$

The ideal of the strict transform of $X \times X_2 \times X_2$ in $\tilde{B}_2$ is (y_1, z_1), locally at (s, N'), so the blowup $\tilde{B}_3$ is the subscheme of $\tilde{B}_2 \times \mathbb{P}^1$ given by

$$\beta y_1 = \beta' z_1,$$

where again β, β' are homogeneous coordinates of $\mathbb{P}^1$. If $\beta' \neq 0$ we may assume $\beta' = 1$, and hence the equations $z_1 z_2 = \alpha x_1 y_1$ and $\beta y_1 = z_1$ give

$$\text{(5.7)} \qquad \beta z_2 = \alpha x_1 \ \text{ and } \ z_1 = \beta y_1, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_3} \cong \frac{k[[x_1, y_1, \alpha - a, z_2, \beta - b]]}{(\beta z_2 - \alpha x_1)}$$

locally at $(s, N', (b : 1)) \in \tilde{B}_3$. Since $\alpha \neq 0$, the above ring is regular. On the other hand, if $\beta \neq 0$ we may assume $\beta = 1$, and we get

$$\text{(5.8)} \qquad z_2 = \beta' \alpha x_1 \ \text{ and } \ y_1 = \beta' z_1, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_3} \cong k[[x_1, \alpha - a, z_1, \beta' - b']]$$

locally at $(s, N', (1 : b'))$. Once again this ring is regular. Again, (s, N') is given by the equations $x_i = y_i = z_i = 0$, for $i = 1, 2$, and $\alpha = a$, so $\tilde{p}_{12}^{-1}(s, N')$ contains a rational component $E(N')$ described above, with homogeneous coordinates β, β', as we can see modding out the rings (5.7) and (5.8) by the equations defining $\tilde{p}_{12}^{-1}(s, N')$. We showed that there is only one rational component $E(N')$ over N' on $\tilde{X}_s$, since $\tilde{B}_3$ is regular at $E(N')$. Therefore the last explosion b_4 does not produce any rational component, that is, b_4 is an isomorphism over $(\tilde{q}^3)^{-1}(s)$.

The case where $s \in \tilde{X}_{ii} \cap X_{jj}$ is settled with a similar analysis, only noting that this time s lies on an "exceptional" component $E \subset \tilde{\mathcal{X}}^2(o)$ such that $f(E) = (N, N')$, where N and N' are either R or S but $N \neq N'$ (see Figure 5.2). Thus the first two explosions will produce the rational components $E(N)$ and $E(N')$ and $b_3 \circ b_4$ will be an isomorphism on $(\tilde{q}^2)^{-1}(s)$.

Indeed, if $s \in \tilde{X}_{ii} \cap \tilde{X}_{jj}$ and s is not in either $\tilde{X}_{ij}$ or $\tilde{X}_{ji}$ then $\hat{\mathcal{O}}_{\tilde{\mathcal{X}}^2,s} \cong k[[x_1, y_2, \alpha - a]]$ (see (5.3) in the proof of Lemma 5.2.1). Now, since $f(s) = (N, N')$, the diagonal Δ'_{23} does not contain $(s, N) \in \tilde{\mathcal{X}}^2 \times_S \mathcal{X} = \tilde{B}_0$, hence $\tilde{B}_1$ is isomorphic to $\tilde{B}_0$ over a neighborhood of $N \in X_s \subset \tilde{B}_0$. On the other hand, using the notation of the proof of Lemma 5.2.1, we have at N'

$$\hat{\mathcal{O}}_{\tilde{B}_0,(s,N')} \cong \frac{k[[x_1, y_2, \alpha - a, z_1, z_2]]}{(z_1 z_2 - \alpha x_1 y_2)},$$

where z_1, z_2 are functions defining X_2, X_1 locally at N', respectively, composed with the projection $\tilde{\mathcal{X}}^2 \times_S \mathcal{X} \to \mathcal{X}$.

The ideal of Δ'_{23} locally at (s, N') is $(z_1 - \alpha x_1, z_2 - y_2)$, since $y_1 = \alpha x_1$ (see (5.3) in the proof of Lemma 5.2.1). Therefore $\tilde{B}_1$ is the subscheme of $\tilde{B}_0 \times \mathbb{P}^1$ given locally over $(s, N') \in \tilde{B}_0$ by

$$\beta(z_1 - \alpha x_1) + \beta'(z_2 - y_2) = 0,$$

where β, β' are homogeneous coordinates of $\mathbb{P}^1$. If $\beta' \neq 0$ we may assume $\beta' = 1$, and hence the equations $\beta(z_1 - \alpha x_1) + (z_2 - y_2) = 0$ and $z_1 z_2 = \alpha x_1 y_2$ give

$$y_2 = \beta z_1 \quad \text{and} \quad z_2 = \beta \alpha x_1, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_1} \cong k[[x_1, \alpha - a, z_1, \beta - b]] \tag{5.9}$$

at $(s, N', (b : 1))$. This ring is clearly regular. On the other hand, if $\beta \neq 0$ we may assume $\beta = 1$, and we get

$$z_1 = \beta' y_2 \quad \text{and} \quad \alpha x_1 = \beta' z_2, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_1} \cong \frac{k[[x_1, y_2, \alpha - a, z_2, \beta' - b']]}{(\alpha x_1 - \beta' z_2)} \tag{5.10}$$

at $(s, N', (1 : b'))$. As $\alpha \neq 0$, this ring is regular (since we can write x_1 in terms of β', z_2 and α^{-1}). This shows there is at most one rational component over N' on $\tilde{X}_s$, since $\tilde{B}_1$ is regular on the rational component $E(N')$ defined above, which implies that the composition $b_2 \circ b_3 \circ b_4$ is an isomorphism over $E(N')$. Again, since $(s, N') \in \tilde{\mathcal{X}}^2 \times_S \mathcal{X}$ is given by $x_i = y_i = z_i = 0$, for $i = 1, 2$, and $\alpha = a$, the homogeneous coordinates β, β' defined above describe indeed a rational component $E(N')$ of $\tilde{X}_s$.

Now, since b_1 is an isomorphism over (s, N), we have

$$\hat{\mathcal{O}}_{\tilde{B}_1,(s,N)} \cong \frac{k[[x_1, y_2, \alpha - a, z_1, z_2]]}{(z_1 z_2 - \alpha x_1 y_2)},$$

where z_1, z_2 are functions defining X_2, X_1 locally at N, respectively, composed with the projection $\tilde{\mathcal{X}}^2 \times_S \mathcal{X} \to \mathcal{X}$. The ideal of the strict transform of Δ'_{13} in $\tilde{B}_1$ is $(x_1 - z_1, \alpha y_2 - z_2)$, locally at (s, N), since $x_2 = \alpha y_2$ (see (5.3) in the proof of Lemma 5.2.1). Then $\tilde{B}_2$ is the subscheme of $\tilde{B}_1 \times \mathbb{P}^1$ given locally over (s, N) by

$$\beta(x_1 - z_1) + \beta'(\alpha y_2 - z_2) = 0,$$

where β, β' are homogeneous coordinates of $\mathbb{P}^1$. If $\beta' \neq 0$ we may assume $\beta' = 1$, and so the equations $\beta(x_1 - z_1) + (\alpha y_2 - z_2) = 0$ and $z_1 z_2 = \alpha x_1 y_2$ give

$$(5.11) \qquad z_2 = \beta x_1 \text{ and } \beta z_1 = \alpha y_2, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_2} \cong \frac{k[[x_1, y_2, \alpha - a, z_1, \beta - b]]}{(\beta z_1 - \alpha y_2)}$$

at $(s, N, (b : 1))$. As $\alpha \neq 0$, this ring is regular. Also, if $\beta \neq 0$ we may assume $\beta = 1$, and we get

$$(5.12) \qquad x_1 = \beta' z_2 \text{ and } z_1 = \beta' \alpha y_2, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_2} \cong k[[y_2, \alpha - a, z_2, \beta' - b']]$$

at $(s, N, (1 : b'))$. This ring is clearly regular. Therefore $b_3 \circ b_4$ is an isomorphism over $(\tilde{q}^2)^{-1}(s)$ and $\tilde{X}_s$ is as stated. As before, the homogeneous coordinates β, β' describe indeed a rational component $E(N)$ of $\tilde{X}_s$, because $\tilde{p}_{12}^{-1}(s, N)$ is given by the equations $x_i = y_i = z_i = 0$, for $i = 1, 2$, and $\alpha = a$. Thus, modding out the rings in (5.11) and (5.12) by these equations, we get on the one hand $k[[\beta - b]]$, and on the other hand $k[[\beta' - b']]$.

As for (d), we note that these are the cases where either α or α' are zero, so the rings $\hat{\mathcal{O}}_{\tilde{B}_1}$ and $\hat{\mathcal{O}}_{\tilde{B}_3}$ of the first case of (c) (more precisely, (5.5) and (5.7)), and $\hat{\mathcal{O}}_{\tilde{B}_1}$ and $\hat{\mathcal{O}}_{\tilde{B}_2}$ of the second case (more precisely, (5.10) and (5.11)), are not regular and the remaining explosions produce the second rational component on each node N and N'. (These blowups will be analyzed in the proof of Lemma 5.2.4.) □

Let $s \in \tilde{\mathcal{X}}^2(o)$ be in the intersection of at least two divisors $\tilde{X}_{ij}$ of $\tilde{\mathcal{X}}^2$, see Figure 5.2. Let N be either R or S and let $E(N)$ be the rational chain on $\tilde{X}_s$ associated to N. By Lemma 5.2.2, $E(N)$ either consists of a single rational component, which we henceforth denote $E_1(N)$, or is the union of two rational components, one attached to X_1 and the other attached to X_2, and the composition $b = b_1 \circ b_2 \circ b_3 \circ b_4$ contracts both of them. In this case, we denote by $E_2(N)$ the first to be contracted, and by $E_1(N)$ the second. In any case, $E_1(N)$ is the exceptional component on the fiber over s that appears in the first blowup to affect $(s, N) \in \tilde{B}_0$ (see Figure 5.4).

Lemma 5.2.3 (The configuration of rational components, I) *Let $s \in \tilde{\mathcal{X}}^2$ and N be a node of X.*

(a) If $E_1(N)$ is not contracted by $b_2 \circ b_3 \circ b_4$ then $E_1(N)$ is a subscheme of both $\tilde{X}_{012}$ and $\tilde{X}_{021}$;

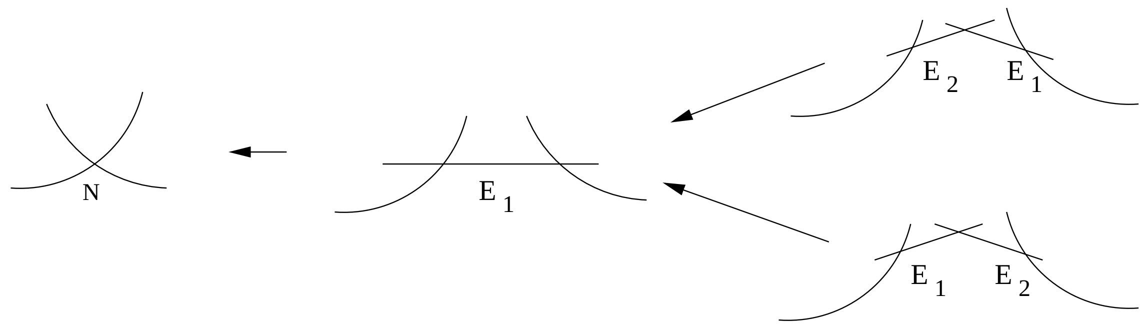


Figure 5.4: The definition of E_1 and E_2

(b) *If $E_1(N)$ is not contracted by $b_3 \circ b_4$ but is contracted by $b_2 \circ b_3 \circ b_4$ then $E_1(N)$ is a subscheme of both $\tilde{X}_{102}$ and $\tilde{X}_{201}$;*

(c) *If $E_1(N)$ is not contracted by b_4 but is contracted by $b_3 \circ b_4$ then $E_1(N)$ is a subscheme of both $\tilde{X}_{011}$ and $\tilde{X}_{022}$;*

(d) *If $E_1(N)$ is contracted by b_4, then $E_1(N)$ is a subscheme of both $\tilde{X}_{101}$ and $\tilde{X}_{202}$.*

Proof. Let $E = E_1(N)$. Since E is the smooth rational curve that appears in the first blowup to affect $(s, N) \in \tilde{B}_0$, we may apply the reasoning used in the proof of the previous lemma.

If E is not contracted by $b_2 \circ b_3 \circ b_4$ then it is contracted by b_1, since it is obviously contracted by b. (Here we identify E with its image $b_2 \circ b_3 \circ b_4(E)$.) The blowup b_1 can be realized as the base change of $f_1 : B \to \mathcal{X}^2$ (see proof of Lemma 5.2.1) under the map q^0_{23}. (Recall that b_1 is the blowup along $\Delta'_{23} \subset \tilde{B}_0$, and f_1 is the blowup along $\Delta \subset \mathcal{X}^2$.) So we have the folowing diagram

$$\begin{array}{cccc} E \subset & \tilde{B}_1 & \xrightarrow{b_1} & \tilde{B}_0 \\ & \downarrow & & \downarrow q^0_{23} \\ \overline{E} \subset & B & \xrightarrow{f_1} & \mathcal{X}^2, \end{array}$$

where $\overline{E}$ is the "exceptional" line over $q^1_{23}(E) = q^0_{23} \circ b_1(E) = (N, N)$. In addition, the diagram is Cartesian, because q^0_{23} is flat. By Lemma 5.2.1, $\overline{E}$ is a subscheme of both $\tilde{X}_{12}$ and $\tilde{X}_{21} \subset B$. Thus, E is a subscheme of both $\tilde{X}_{012}$ and $\tilde{X}_{021}$, showing (a).

Now assume E is not contracted by $b_3 \circ b_4$, but is by $b_2 \circ b_3 \circ b_4$. We identify E with its image $b_3 \circ b_4(E)$ and consider the diagram

$$\begin{array}{cccc} E \subset & \tilde{B}_2 & \xrightarrow{b_2} & \tilde{B}_1 \\ & \downarrow & & \downarrow q^1_{13} \\ \overline{E} \subset & B & \xrightarrow{f_1} & \mathcal{X}^2, \end{array}$$

where, once again, the map f_1 is the blowup along the diagonal and $\overline{E}$ is the "exceptional" line over $q^2_{13}(E) = q^1_{13} \circ b_2(E) = (N, N)$. (Recall that b_2 is the blowup of $\tilde{B}_1$ along the strict transform of Δ'_{13} under b_1, and, by hypothesis, b_1 is an isomorphism over (s, N).) Moreover, the diagram is Cartesian, because q^0_{13} is flat, and $\tilde{B}_1$ is locally isomorphic to $\tilde{B}_0$ over (s, N). Then, since $\overline{E} \subset \tilde{X}_{12} \cap \tilde{X}_{21}$, we have $E \subset \tilde{X}_{102} \cap \tilde{X}_{201}$, showing (b).

If E is not contracted by b_4, but is contracted by $b_3 \circ b_4$, we identify E with its image $b_4(E)$ and consider the diagram

$$\begin{array}{cccc} E \subset & \tilde{B}_3 & \xrightarrow{b_3} & \tilde{B}_2 \\ & \downarrow & & \downarrow q^2_{23} \\ \overline{E} \subset & \overline{\mathcal{X}}^2 & \longrightarrow & \mathcal{X}^2, \end{array}$$

where the map $\overline{\mathcal{X}}^2 \to \mathcal{X}^2$ is the blowup along $X_2 \times X_2$, and $\overline{E}$ is the "exceptional" line over $q^3_{23} = q^2_{23} \circ b_3(E) = (N', N)$, where N' is the other node of X. Again, the diagram is Cartesian because q^0_{23} is flat, and $\tilde{B}_2$ is locally isomorphic to $\tilde{B}_0$ over (s, N). By Lemma 5.2.1, $\overline{E}$ is a subscheme of both $\tilde{X}_{11}$ and $\tilde{X}_{22}$. Thus, E is a subscheme of both $\tilde{X}_{011}$ and $\tilde{X}_{022}$, proving (c).

At last, assume E is contracted by b_4 and consider the diagram

$$\begin{array}{cccc} E \subset & \tilde{B}_4 & \xrightarrow{b_4} & \tilde{B}_3 \\ & \downarrow & & \downarrow q^3_{13} \\ \overline{E} \subset & \overline{\mathcal{X}}^2 & \longrightarrow & \mathcal{X}^2, \end{array}$$

where again the map $\overline{\mathcal{X}}^2 \to \mathcal{X}^2$ is the blowup along $X_2 \times X_2$, and $\overline{E}$ is the exceptional component over $q^4_{13}(E) = q^3_{13} \circ b_4(E) = (N', N)$. Then $\overline{E} \subset \tilde{X}_{11} \cap \tilde{X}_{22}$. Now, since q^0_{13} is flat, and $\tilde{B}_3$ is locally isomorphic to $\tilde{B}_0$ over (s, N), the diagram is Cartesian. Therefore, we have $E \subset \tilde{X}_{101} \cap \tilde{X}_{202}$, proving (d). □

Lemma 5.2.4 (The configuration of rational components, II) *Let $s \in \tilde{\mathcal{X}}^2$ be in the intersection of three divisors $\tilde{X}_{ij}$ of $\tilde{\mathcal{X}}^2$. Let N be either R or S.*

(a) If $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij} \cap \tilde{X}_{ji}$ and $E_1(N) \subset \tilde{X}_{ii1}$ then $E_1(N)$ meets X_1 and $E_2(N)$ meets X_2;

(b) If $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij} \cap \tilde{X}_{ji}$ and $E_1(N) \subset \tilde{X}_{ii2}$ then $E_1(N)$ meets X_2 and $E_2(N)$ meets X_1;

(c) If $s \in \tilde{X}_{ij} \cap \tilde{X}_{ii} \cap \tilde{X}_{jj}$ and $E_1(N) \subset \tilde{X}_{ij1}$ then $E_1(N)$ meets X_1 and $E_2(N)$ meets X_2;

(d) If $s \in \tilde{X}_{ij} \cap \tilde{X}_{ii} \cap \tilde{X}_{jj}$ and $E_1(N) \subset \tilde{X}_{ij2}$ then $E_1(N)$ meets X_2 and $E_2(N)$ meets X_1.

Proof. Here we use the notation introduced in the proof of the previous lemmas.

First note that if $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij} \cap \tilde{X}_{ji}$ then $f(s) = (N, N)$ where N is a node of X, either R or S; see Figure 5.2. Moreover, $E_1(N)$ is a rational component not contracted by $b_2 \circ b_3 \circ b_4$, that is, $E_1(N)$ is produced by the first blowup in (ii). So, by Lemma 5.2.3, $E_1(N) \subset \tilde{X}_{0ij} \cap \tilde{X}_{0ji}$. Now, since $E_1(N) \subset \tilde{p}_{12}^{-1}(s)$ and $s \in \tilde{X}_{ii}$, we have $E_1(N) \subset \tilde{X}_{ii0}$. Thus $E_1(N) \subset \tilde{X}_{ii0} \cap \tilde{X}_{0ij} = \tilde{X}_{iij}$. So we must show that $E_1(N)$ is attached to X_j.

Now, s is in the rational line E of $\tilde{\mathcal{X}}^2(o)$ with homogeneous coordinates $(\alpha : \alpha')$ and, moreover, s is of the form $(0 : 1)$ or $(1 : 0)$. From (5.1) and (5.2) in the proof of Lemma 5.2.1 we see that, locally at $(0 : 1) \in E$ we have $\hat{\mathcal{O}}_{\tilde{\mathcal{X}}^2} \cong k[[x_1, y_1, \alpha]]$ with α a local parameter for E at $(0 : 1)$, and locally at $(1 : 0) \in E$ we have $\hat{\mathcal{O}}_{\tilde{\mathcal{X}}^2} \cong k[[x_2, y_2, \alpha']]$ with α' a local parameter for E at $(1 : 0)$. So, if $i = 2$ and $j = 1$ then $s = (1 : 0)$, and if $i = 1$ and $j = 2$ then $s = (0 : 1)$. Without loss of generality we may assume $i = 1$ and $j = 2$ so that, locally at $s = (0 : 1)$,

$$\hat{\mathcal{O}}_{\tilde{\mathcal{X}}^2, s} \cong k[[x_1, y_1, \alpha]].$$

Thus we have to show that $E_1(N)$ intersects X_2.

From (5.5) and (5.6) in the proof of Lemma 5.2.2 we have, locally at $(s, N, (b : 1))$,

$$\hat{\mathcal{O}}_{\tilde{B}_1} \cong \frac{k[[x_1, y_1, \alpha, z_1, \beta - b]]}{(\alpha x_1 - \beta z_1)}$$

and locally at $(s, N, (1 : b'))$,

$$\hat{\mathcal{O}}_{\tilde{B}_1} \cong k[[x_1, \alpha, z_2, \beta' - b']].$$

Recall that β, β' are the homogeneous coordinates of $b_1^{-1}(s, N) = E_1(N)$.

Consider first $\overline{\Delta}_{13} := (b_1)^{-1}(\Delta'_{13})$. Then the ideal of $\overline{\Delta}_{13}$ locally at $(s, N, (0 : 1)) \in \tilde{B}_1$ is $(x_1 - z_1, \alpha y_1 - \beta y_1)$, since $x_2 = \alpha y_1$ by (5.1) and $z_2 = \beta y_1$ by (5.5). Now,

$$(x_1 - z_1, \alpha y_1 - \beta y_1) = (x_1 - z_1, \alpha - \beta) \cap (x_1 - z_1, y_1)$$

and the ideal of the strict transform of Δ'_{13} in $\tilde{B}_1$ locally at $(s, N, (0 : 1))$ is $(x_1 - z_1, \alpha - \beta)$. (Note that $(x_1 - z_1, y_1)$ cuts out a codimension-2 subscheme of $\tilde{B}_1$.) Thus, locally at $(s, N, (0 : 1))$, the blowup $\tilde{B}_2$ is the subscheme of $\tilde{B}_1 \times \mathbb{P}^1$ given by

$$\gamma(x_1 - z_1) + \gamma'(\alpha - \beta) = 0$$

where γ, γ' are homogeneous coordinates of $\mathbb{P}^1 = E_2(N)$.

If $\gamma' \neq 0$, we may assume $\gamma' = 1$, and the equations $\gamma(x_1 - z_1) + (\alpha - \beta) = 0$ and $\alpha x_1 = \beta z_1$ give

$$\alpha = \gamma z_1 \text{ and } \beta = \gamma x_1, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_2} \cong k[[x_1, y_1, z_1, \gamma - c]] \tag{5.13}$$

at $(s, N, (0:1), (c:1)) \in \tilde{B}_2$. Notice that this ring is regular. On the other hand, if $\gamma \neq 0$ we may assume $\gamma = 1$, and we get

$$z_1 = \gamma' \alpha \text{ and } x_1 = \gamma' \beta, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_2} \cong k[[y_1, \alpha, \beta, \gamma' - c']] \tag{5.14}$$

at $(s, N, (0:1), (1:c'))$. Again we obtain a regular ring.

Therefore, the curve $(\tilde{q}^2)^{-1}(s)$ has two components isomorphic to X_1 and X_2 (and which we also denote by X_1 and X_2) plus the components $E_1(N)$ (described in Lemma 5.2.2) and $E_2(N)$. Indeed, the point $(s, N) \in \tilde{\mathcal{X}} \times_S \mathcal{X}$ is given by equations $x_i = y_i = z_i = 0$, for $i = 1, 2$, and $\alpha = 0$. Thus, modding out the rings in (5.13) and (5.14) by these equations, we get on the one hand $k[[\gamma - c]]$, and on the other hand $k[[\gamma' - c']]$, which shows the homogeneous coordinates γ, γ' define indeed a rational component $E_2(N)$.

Now, observe that γ is a local parameter for $E_2(N)$ at $(0:1)$ and z_1 is a local parameter for X_1 at N. Thus the equation $\alpha = \gamma z_1$ in (5.13) shows that $E_2(N)$ intersects X_1 at the points $(0:1) \in E_2(N)$ and $N \in X_1$. Also, since γ' is a local parameter for $E_2(N)$ at $(1:0)$, and β is a local parameter for $E_1(N)$ at $(0:1)$, the equation $x_1 = \gamma' \beta$ in (5.14) shows that $E_2(N)$ intersects $E_1(N)$ at the points $(1:0) \in E_2(N)$ and $(0:1) \in E_1(N)$. Moreover, at $(s, N, (1:0)) \in \tilde{B}_1$, we have $y_1 = \beta' z_2$ (see (5.6) in the proof of Lemma 5.2.2). Since z_2 is a local parameter for X_2 at N, and β' is a local parameter for $E_1(N)$ at $(1:0)$, we see that X_2 intersects $E_1(N)$ at the points $N \in X_2$ and $(1:0) \in E_1(N)$. So we showed that $E_1(N)$ meets X_2 as stated.

Now we examine $E_1(N')$ over $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij} \cap \tilde{X}_{ji}$. Recall that $f(s) = (N, N)$, and $N' \neq N$. The first two explosions, b_1 and b_2, are isomorphisms over (s, N'), so $E_1(N')$ is a rational component contracted by $b_3 \circ b_4$ but not by b_4. Then, by Lemma 5.2.3, $E_1(N') \subset \tilde{X}_{0ii} \cap \tilde{X}_{0jj}$. Now, since $E_1(N') \subset \tilde{p}_{12}^{-1}(s)$ and $s \in \tilde{X}_{ii}$, we have $E_1(N) \subset \tilde{X}_{ii0}$. Hence $E_1(N) \subset \tilde{X}_{ii0} \cap \tilde{X}_{0ii} = \tilde{X}_{iii}$. Therefore, we must show that $E_1(N')$ intersects X_i.

Again, we may assume $i = 1$ and $j = 2$. So, from (5.7) and (5.8) in the proof of Lemma 5.2.2, we have locally at $(s, N', (b:1)) \in \tilde{B}_3$:

$$\hat{\mathcal{O}}_{\tilde{B}_3} \cong \frac{k[[x_1, y_1, \alpha, z_2, \beta - b]]}{(\beta z_2 - \alpha x_1)}$$

and, locally at $(s, N', (1 : b'))$:

$$\hat{\mathcal{O}}_{\tilde{B}_3} \cong k[[x_1, \alpha, z_1, \beta' - b']],$$

where $(\beta : \beta')$ are the homogeneous coordinates of $E_1(N')$.

Consider $(b_1 \circ b_2 \circ b_3)^{-1}(X'_{202})$. This subscheme of $\tilde{B}_3$ is given locally at $(s, N', (0 : 1))$ by the equations $x_1 = 0$ and $\beta y_1 = 0$, since $z_1 = \beta y_1$ (see (5.7) in the proof of Lemma 5.2.2). Now,

$$(x_1, \beta y_1) = (x_1, \beta) \cap (x_1, y_1),$$

and the strict transform of X'_{202} in $\tilde{B}_3$ has ideal (x_1, β). (Note that (x_1, y_1) cuts out a codimension-2 subscheme of $\tilde{B}_3$.) So, the blowup $\tilde{B}_4$ is the subscheme of $\tilde{B}_3 \times \mathbb{P}^1$ given by

$$\gamma x_1 = \gamma' \beta$$

where γ, γ' are homogeneous coordinates of $\mathbb{P}^1 = E_2(N')$.

If $\gamma' \neq 0$ we may assume $\gamma' = 1$, and the equations $\gamma x_1 = \beta$ and $\beta z_2 = \alpha x_1$ give

$$\text{(5.15)} \qquad \alpha = \gamma z_2 \;\text{ and }\; \beta = \gamma x_1, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_4} \cong k[[x_1, y_2, z_2, \gamma - c]]$$

locally at $(s, N', (0 : 1), (c : 1)) \in \tilde{B}_4$. Notice that this ring is regular. On the other hand, if $\gamma \neq 0$ we may assume $\gamma = 1$, and we get

$$\text{(5.16)} \qquad z_2 = \alpha\gamma' \;\text{ and }\; x_1 = \beta\gamma', \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_4} \cong k[[y_1, \alpha, \beta, \gamma' - c']]$$

locally at $(s, N', (0 : 1), (1 : c'))$. Once again, this ring is regular.

Then the curve $\tilde{X}_s$ has two components isomorphic to X_1 and X_2 (and which we also denote by X_1 and X_2), plus the components $E_1(N')$ and $E_2(N')$. Indeed, the point $(s, N') \in \tilde{\mathcal{X}} \times_S \mathcal{X}$ is given by the equations $x_i = y_i = z_i = 0$, for $i = 1, 2$, and $\alpha = 0$. Thus, modding out the rings in (5.15) and (5.16) by these equations, we get on the one hand $k[[\gamma - c]]$, and on the other hand $k[[\gamma' - c']]$, which shows the homogeneous coordinates define indeed a rational component $E_2(N')$.

Now, since z_2 is a local parameter for X_2 at N', and γ is a local parameter for $E_2(N')$ at $(0 : 1)$, we see from the equation $\alpha = \gamma z_2$ in (5.15) that X_2 intersects $E_2(N')$ at the points $N' \in X_2$ and $(0 : 1) \in E_2(N')$. Also, since γ' is a local parameter for $E_2(N')$ at $(1 : 0)$, and β is a local parameter for $E_1(N')$ at $(0 : 1)$, we see from the equation $x_1 = \beta\gamma'$ in (5.16) that $E_1(N')$ intersects $E_2(N')$ at the points $(0 : 1) \in E_1(N')$ and $(1 : 0) \in E_2(N')$.

In addition, at $(s, N', (1 : 0)) \in \tilde{B}_3$ we have $y_1 = \beta' z_1$ (see (5.8) in the proof of Lemma 5.2.2). Since z_1 is a local parameter for X_1 at N' and β' is a local parameter for $E_1(N')$ at $(1 : 0)$ we see that X_1 intersects $E_1(N')$ transversally at the points $N' \in X_1$ and $(1 : 0) \in E_1(N')$. This proves both (a) and (b). (The curve $\tilde{X}_s$ is described in Figure 5.5 (I).)

If $s \in \tilde{X}_{ij} \cap \tilde{X}_{ii} \cap \tilde{X}_{jj}$ then $f(s) = (N, N')$ where N and N' are distinct nodes of X. Moreover, $E_1(N')$ is a rational component contracted by b_1 and not by $b_2 \circ b_3 \circ b_4$, that is, $E_1(N')$ is produced by the first blowup of (ii). Then, by Lemma 5.2.3, $E_1(N') \subset \tilde{X}_{0ij} \cap \tilde{X}_{0ji}$. On the other hand, since $E_1(N') \subset \tilde{p}_{12}^{-1}(s)$ and $s \in \tilde{X}_{ij}$, we have $E_1(N') \subset X_{ij0}$. In particular, $E_1(N') \subset \tilde{X}_{ij0} \cap \tilde{X}_{0ji} = \tilde{X}_{iji}$. So, we must show that $E_1(N')$ intersects X_i.

Recall that s is on the "exceptional" line $E \subset \tilde{\mathcal{X}}^2(o)$, which has homogeneous coordinates $(\alpha : \alpha')$. From (5.3) and (5.4) in the proof of Lemma 5.2.1 we get that, locally at $(0:1) \in E$, we have $\hat{\mathcal{O}}_{\tilde{\mathcal{X}}^2} \cong k[[x_1, y_2, \alpha]]$, and α is a local parameter for E at $(0:1)$, whereas locally at $(1:0) \in E$ we have $\hat{\mathcal{O}}_{\tilde{\mathcal{X}}^2} \cong k[[x_2, y_1, \alpha']]$, and α' is a local parameter for $(1:0) \in E$. So, if $i = 1$ and $j = 2$ then $s = (0:1)$, and if $i = 2$ and $j = 1$ then $s = (1:0)$. Without loss of generality we may assume $i = 1$ and $j = 2$.

From (5.9) and (5.10) in the proof of Lemma 5.2.2 we have that, locally at $(s, N', (b:1))$,

$$\hat{\mathcal{O}}_{\tilde{B}_1} \cong k[[x_1, \alpha, z_1, \beta - b]]$$

whereas locally at $(s, N', (1:b'))$

$$\hat{\mathcal{O}}_{\tilde{B}_1} \cong \frac{k[[x_1, \alpha, y_2, z_2, \beta' - b']]}{(\alpha x_1 - \beta' z_2)},$$

where $(\beta : \beta')$ are the homogeneous coordinates of $E_1(N')$.

The second blowup of (ii), along the strict transform of Δ'_{13} in $\tilde{B}_1$, is an isomorphism over $E_1(N')$ since $f(s) = (N, N')$. Thus $\tilde{B}_2$ is isomorphic to $\tilde{B}_1$ along $E_1(N')$.

Consider $(b_1 \circ b_2)^{-1}(X'_{022})$. This subscheme of $\tilde{B}_2$ is given locally at $(s, N', (1:0))$ by the equations $\beta' z_2 = 0$ and $\beta' y_2 = 0$, since $y_1 = \alpha x_1$ and $z_1 = \beta' y_2$ (see (5.3) and (5.10) in the proofs of Lemmas 5.2.1 and 5.2.2). Now, the strict transform is given by $\beta' = 0$, and hence is a Cartier divisor at $(s, N', (1:0))$. Thus $\tilde{B}_3$ is isomorphic to $\tilde{B}_2$ along $E_1(N')$.

Now, $(b_1 \circ b_2 \circ b_3)^{-1}(X'_{202})$ is a subscheme of $\tilde{B}_3$ given locally at $(s, N', (1:0))$ by the equations $x_1 = 0$ and $\beta' y_2 = 0$, since $z_1 = \beta' y_2$ (see (5.10) in the proof of Lemma 5.2.2). Note that

$$(\beta' y_2, x_1) = (\beta', x_1) \cap (y_2, x_1)$$

and that (y_2, x_1) defines a codimension-2 subscheme of $\tilde{B}_3$. The strict transform of X'_{202} in $\tilde{B}_3$ is then given by the equations $\beta' = 0$ and $x_1 = 0$. So $\tilde{B}_4$ is the subscheme of $\tilde{B}_3 \times \mathbb{P}^1$ given by

$$\gamma x_1 = \gamma' \beta'$$

where γ, γ' are homogeneous coordinates of $\mathbb{P}^1 = E_2(N')$.

If $\gamma' \neq 0$, we may assume $\gamma' = 1$ and the equations $\gamma x_1 = \beta'$ and $\beta' z_2 = \alpha x_1$ give

$$\alpha = \gamma z_2 \text{ and } \beta' = \gamma x_1, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_4} \cong k[[x_1, y_2, z_2, \gamma - c]] \tag{5.17}$$

locally at $(s, N', (1:0), (c:1)) \in \tilde{B}_4$. Notice that this ring is regular. Also, if $\gamma \neq 0$ we may assume $\gamma = 1$, and we get

$$z_2 = \alpha\gamma' \text{ and } x_1 = \gamma'\beta', \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_4} \cong k[[y_2, \alpha, \beta', \gamma' - c']] \tag{5.18}$$

locally at $(s, N', (1:0), (1:c'))$. Once again, this ring is regular. Modding out the rings above by the equations defining $\tilde{p}_{12}^{-1}(s, N')$, we see that the homogeneous coordinates γ, γ' define indeed a rational component $E_2(N')$.

The equation $\alpha = \gamma z_2$ in (5.17) shows that $E_2(N')$ intersects X_2 transversally at the point $(0:1) \in E_2(N')$ and $N' \in X_2$. Moreover, the equation $x_1 = \gamma'\beta'$ in (5.18) shows that $E_1(N')$ and $E_2(N')$ intersect transversally at $(1:0) \in E_1(N')$ and $(1:0) \in E_2(N')$. Also, from (5.9), we have $y_2 = \beta z_1$, which shows that $E_1(N')$ intersects X_1 transversally at $N' \in X_1$ and $(0:1) \in E_1(N')$. So we showed that $E_1(N')$ meets X_1, as stated.

Now, $E_1(N)$ is produced by the second blowup of (ii), that is, $E_1(N)$ is contracted by $b_2 \circ b_3 \circ b_4$ but not by $b_3 \circ b_4$. Thus, by Lemma 5.2.3, $E_1(N) \subset \tilde{X}_{i0j} \cap \tilde{X}_{j0i}$. Also, since $s \in \tilde{X}_{ij}$, we have $E_1(N) \subset \tilde{p}_{12}^{-1}(s) \subset \tilde{X}_{ij0}$. So, in particular $E_1(N) \subset \tilde{X}_{i0j} \cap \tilde{X}_{ij0} = \tilde{X}_{ijj}$. Hence, we must show that $E_1(N)$ intersects X_j. As before, we assume $i = 1$ and $j = 2$.

From (5.11) and (5.12) in the proof of Lemma 5.2.2 we have that, at $(s, N, (b:1)) \in \tilde{B}_2$,

$$\hat{\mathcal{O}}_{\tilde{B}_2} \cong \frac{k[[x_1, \alpha, y_2, z_1, \beta - b]]}{(\beta z_1 - \alpha y_2)},$$

and at $(s, N, (1:b')) \in \tilde{B}_2$,

$$\hat{\mathcal{O}}_{\tilde{B}_2} \cong k[[\alpha, y_2, z_2, \beta' - b']].$$

Recall that $(\beta : \beta')$ are homogeneous coordinates of $E_1(N)$. Consider $(b_1 \circ b_2)^{-1}(X'_{022})$. This is a subscheme of $\tilde{B}_2$ given locally at $(s, N, (0:1))$ by $\alpha x_1 = 0$ and $z_1 = 0$, because $y_1 = \alpha x_1$; see (5.3) in the proof of Lemma 5.2.1. Now,

$$(\alpha x_1, z_1) = (\alpha, z_1) \cap (x_1, z_1),$$

and (x_1, z_1) cuts out a codimension-2 subscheme of $\tilde{B}_2$. The strict transform of X'_{022} in $\tilde{B}_2$ is thus given by $\alpha = 0$ and $z_1 = 0$. Then $\tilde{B}_3$ is the subscheme of $\tilde{B}_2 \times \mathbb{P}^1$ given by

$$\gamma\alpha = \gamma' z_1$$

where γ, γ' are homogeneous coordinates of $\mathbb{P}^1 = E_2(N)$.

If $\gamma' \neq 0$ we may assume $\gamma' = 1$, and thus we get

$$z_1 = \gamma\alpha \;\text{ and }\; y_2 = \gamma\beta, \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_3} \cong k[[x_1, \alpha, \beta, \gamma - c]] \tag{5.19}$$

locally at $(s, N, (0:1), (c:1)) \in \tilde{B}_3$. Notice that this ring is regular. Also, if $\gamma \neq 0$ we may assume $\gamma = 1$, and we get

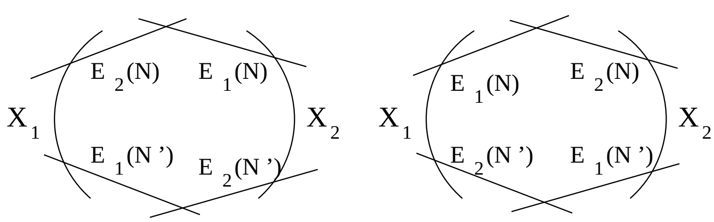
$$\alpha = \gamma' z_1 \;\text{ and }\; \beta = y_2\gamma', \quad \text{and thus} \quad \hat{\mathcal{O}}_{\tilde{B}_3} \cong k[[x_1, y_2, z_1, \gamma' - c']] \tag{5.20}$$

locally at $(s, N, (0:1), (1:c'))$. Once again, this ring is regular. Again, modding out these rings by the equations defining $p_{12}^{-1}(s, N)$, we see that the homogeneous coordinates γ, γ' define indeed a rational component $E_2(N)$.

In addition, the equation $y_2 = \gamma\beta$ in (5.19) shows that $E_1(N)$ intersects $E_2(N)$ transversally at the points $(0:1) \in E_1(N)$ and $(0:1) \in E_2(N)$. Also, the equation $\alpha = \gamma' z_1$ in (5.20) shows that $E_2(N)$ intersects X_1 transversally at $N \in X_1$ and $(1:0) \in E_2(N)$. This proves (c) and (d). (The curve $\tilde{X}_s$ is described in Figure 5.5 (I).)

Furthermore, note that, at each fiber $\tilde{X}_s$ of $\tilde{p}_{12}$, each blowup b_i contracts a single rational component of $\tilde{X}_s$ among $E_1(R), E_2(R), E_1(S)$ and $E_2(S)$. □

In Figure 5.5 we represent the curves $\tilde{X}_s$ for points $s \in \tilde{\mathcal{X}}^2$ in the intersection of three divisors $\tilde{X}_{ij}$. Let $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij} \cap \tilde{X}_{ji}$ or $s \in \tilde{X}_{ij} \cap \tilde{X}_{ii} \cap \tilde{X}_{jj}$. We observe that, although the second figure did not appear in the proof of Lemma 5.2.4 (where we considered the case $i = 1$ and $j = 2$), it is obtained when $i = 2$ and $j = 1$.



(I) (II)

Figure 5.5: $\tilde{X}_s$ for s in the intersection of three divisors $\tilde{X}_{ij}$

Remark 5.2.5 Note that in the proof of Lemma 5.2.4 we showed that $\tilde{\mathcal{X}}^3$ is regular.

If D is a divisor of $\tilde{\mathcal{X}}^3$ and $W \subset \tilde{\mathcal{X}}^3$ is a curve, we denote by $D \cdot W$ the *intersection number* between D and W, that is, $D \cdot W := \deg_W(\mathcal{O}_{\tilde{\mathcal{X}}^3}(D))$.

Lemma 5.2.6 (Intersection numbers) *Let $s \in \tilde{\mathcal{X}}^2(o)$, and denote by $X_{s,1}$ and $X_{s,2}$ the components of $\tilde{X}_s$ isomorphic to X_1 and X_2 respectively.*

(a) If $f(s) = (Q_1, Q_2)$ with Q_1 and Q_2 smooth points of X then, for $i, j \in \{1, 2\}$, we have

$$\tilde{\Delta}_{i3} \cdot X_{s,j} = \begin{cases} 1, & \text{if } Q_i \in X_j \\ 0, & \text{if } Q_i \notin X_j \end{cases} \quad \text{and} \quad \tilde{X}_{222} \cdot X_{s,j} = \begin{cases} 2, & \text{if } Q_1, Q_2 \in X_2 \text{ and } j = 1 \\ -2, & \text{if } Q_1, Q_2 \in X_2 \text{ and } j = 2 \\ 0, & \text{if } Q_1 \notin X_2 \text{ or } Q_2 \notin X_2. \end{cases}$$

(b) If $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij}$ but $s \notin \tilde{X}_{ji} \cup \tilde{X}_{jj}$, with $i \neq j$, then $f(s) = (Q, N)$ where Q is a smooth point of X_i, and N is a node of X. Let N' be the other node of X. Then the following intersection table holds.

	$X_{s,i}$	$X_{s,j}$	$E(N)$	$E(N')$
$\tilde{\Delta}_{13}$	1	0	0	0
$\tilde{\Delta}_{23}$	1	1	−1	0
$\tilde{X}_{iii}$	−1	1	1	−1
$\tilde{X}_{jjj}$	0	0	0	0

(c) If $s \in \tilde{X}_{ij} \cap \tilde{X}_{ji}$ but $s \notin \tilde{X}_{ii} \cup \tilde{X}_{jj}$, with $i \neq j$, then $f(s) = (N, N)$ where N is a node of X. Let N' be the other node of X. Then the following intersection table holds.

	$X_{s,1}$	$X_{s,2}$	$E(N)$	$E(N')$
$\tilde{\Delta}_{13}$	0	0	1	0
$\tilde{\Delta}_{23}$	1	1	−1	0
$\tilde{X}_{222}$	0	0	0	0

(d) If $s \in \tilde{X}_{ii} \cap \tilde{X}_{jj}$ but $s \notin \tilde{X}_{ij} \cup \tilde{X}_{ji}$, with $i \neq j$, then $f(s) = (N, N')$ where N and N' are distinct nodes of X. Then the following intersection table holds.

	$X_{s,1}$	$X_{s,2}$	$E(N)$	$E(N')$
$\tilde{\Delta}_{13}$	1	1	−1	0
$\tilde{\Delta}_{23}$	1	1	0	−1
$\tilde{X}_{222}$	0	−2	1	1

(e) If $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij} \cap \tilde{X}_{ji}$ then $f(s) = (N, N)$ with N a node of X. Let N' be the other node of X. Then the following intersection table holds.

	$X_{s,i}$	$X_{s,j}$	$E_1(N)$	$E_2(N)$	$E_1(N')$	$E_2(N')$
$\tilde{\Delta}_{13}$	1	0	1	-1	0	0
$\tilde{\Delta}_{23}$	1	1	-1	0	0	0
$\tilde{X}_{iii}$	-1	1	0	1	0	-1
$\tilde{X}_{jjj}$	0	0	0	0	0	0

(f) If $s \in \tilde{X}_{ij} \cap \tilde{X}_{ii} \cap \tilde{X}_{jj}$ then $f(s) = (N, N')$ with N and N' distinct nodes of X. Then the following intersection table holds.

	$X_{s,i}$	$X_{s,j}$	$E_1(N)$	$E_2(N)$	$E_1(N')$	$E_2(N')$
$\tilde{\Delta}_{13}$	1	1	-1	0	0	0
$\tilde{\Delta}_{23}$	1	1	0	0	-1	0
$\tilde{X}_{iii}$	-1	0	1	-1	1	0
$\tilde{X}_{jjj}$	0	-1	1	0	1	-1

We observe that the situation for a point $s \in \tilde{X}_{ii} \cap \tilde{X}_{ji}$, such that $s \notin \tilde{X}_{ij} \cup \tilde{X}_{jj}$, is analogous to the situation in (b), and hence doesn't need to be dealt with.

Proof. (a) Assume first that $f(s) = (Q_1, Q_2)$, with Q_1 and Q_2 smooth points of X. By Lemma 5.2.2, $\tilde{X}_s$ is as in Figure 5.3 (1), and b is an isomorphism around $\tilde{X}_s$. Recall that $X_s \subset \tilde{\mathcal{X}}^2 \times_S \mathcal{X}$ is the fiber over s of the projection to $\tilde{\mathcal{X}}^2$, thus $X_s = \{s\} \times X$. And, since $f(s)$ is a pair of smooth points, f is an isomorphism around s, that is, $f \times \mathrm{Id}$ is an isomorphism around X_s. Thus

$$\begin{aligned}
\tilde{\Delta}_{i3} \cap \tilde{X}_s &= b^{-1}(\Delta'_{i3} \cap X_s) \\
&= b^{-1}(f \times \mathrm{Id})^{-1}(\Delta_{i3} \cap (Q_1, Q_2) \times X) \\
&= b^{-1}(f \times \mathrm{Id})^{-1}(Q_1, Q_2, Q_i) = b^{-1}(s, Q_i),
\end{aligned}$$

for $i = 1, 2$, showing the first part of (a).

Let X'_{222} be the strict transform of $X_2 \times X_2 \times X_2$ by $f \times \mathrm{Id}$, then

$$\begin{aligned}
\tilde{X}_{222} \cap \tilde{X}_s &= b^{-1}(X'_{222} \cap X_s) \\
&= b^{-1}(f \times \mathrm{Id})^{-1}(X_2 \times X_2 \times X_2) \cap ((Q_1, Q_2) \times X).
\end{aligned}$$

Thus the intersection is clearly empty if Q_1 or Q_2 does not belong to X_2. Assume that Q_1 and Q_2 belong to X_2. Identifying $\tilde{X}_s$ with X, the intersection $\tilde{X}_{222} \cap \tilde{X}_s$ is simply $X_{s,2}$. This shows that $\tilde{X}_{222} \cap X_{s,1} = \{R, S\}$ and thus $\tilde{X}_{222} \cdot X_{s,1} = 2$. To see that $\tilde{X}_{222} \cdot X_{s,2} = -2$, we recall that, as divisors of $\tilde{\mathcal{X}}^3$, we have $\tilde{X}_{222} + \tilde{X}_{221} = \tilde{X}_{220}$. Now, $\tilde{X}_{220} = \tilde{p}_{12}^{-1}(\tilde{X}_{22})$, and hence this divisor is numerically equivalent to zero on $\tilde{X}_s$. Thus, since the intersection $\tilde{X}_{221} \cap \tilde{X}_s$ is $X_{s,1}$, we have $\tilde{X}_{222} \cdot X_{s,2} = -\tilde{X}_{221} \cdot X_{s,2} = -2$.

(b) Now assume $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij}$, but $s \in \tilde{X}_{ji} \cup \tilde{X}_{jj}$, for some $i \neq j$. Then $f(s) = (Q, N)$ where Q is a smooth point in X_i and N is a node of X. Let N' be the other node of X. Without loss of generality, assume $i = 1$ and $j = 2$. Thus the point s is given by the equations $x_2 = y_1 = y_2 = \alpha = 0$ and x_1 equals to a nonzero constant, where x_1, x_2, y_1, y_2 are as in Lemma 5.2.1. Modding out the rings in (5.5) or (5.9) by these equations, we get $k[[z_1, \beta - b]]/(z_1\beta)$. On the other hand, modding out the rings in (5.6) or (5.10) by the same equations, we get $k[[z_2, \beta' - b']]/(z_2\beta')$.

First we note that, since $(f \times \mathrm{Id}) \circ b(\tilde{\Delta}_{23}) = \Delta_{23}$, and $(f \times \mathrm{Id}) \circ b(E(N')) = (Q, N, N')$, which does not belong to Δ_{23}, we get

$$\tilde{\Delta}_{23} \cdot E(N') = 0.$$

Moreover, the ideal of Δ'_{23} is $(y_1 - z_1, y_2 - z_2)$. But, at $(s, N, (b : 1))$ we have $z_2 = \beta y_1$ and $y_2 = \beta z_1$, by (5.5) and (5.9). Thus, at this point, the Cartier divisor $\tilde{\Delta}_{23}$ is given by $y_1 - z_1 = 0$. Hence, the restriction of $\tilde{\Delta}_{23}$ to $\tilde{X}_s$ is given by $z_1 = 0$, locally at the point $(s, N, (b : 1))$. Analogously, at $(s, N, (1 : b'))$, we have $y_1 = \beta' z_2$ and $z_1 = \beta' y_2$, by (5.6) and (5.10). Thus, at this point, the Cartier divisor $\tilde{\Delta}_{23}$ is given by $y_2 - z_2 = 0$. Then the restriction of $\tilde{\Delta}_{23}$ to $\tilde{X}_s$ is given by $z_2 = 0$, locally at the point $(s, N, (1 : b'))$. So, since z_1 (resp. z_2) is a local parameter for $X_{s,1}$ (resp. $X_{s,2}$), we see that $\tilde{\Delta}_{23}$ intercects $X_{s,1}$ (resp. $X_{s,2}$) transversally at the points where $X_{s,1}$ (resp. $X_{s,2}$) meets $E(N)$, showing that

$$\tilde{\Delta}_{23} \cdot X_{s,1} = \tilde{\Delta}_{23} \cdot X_{s,2} = 1.$$

At last, by (a), $\tilde{\Delta}_{23}$ has degree 1 on the fibers of $\tilde{p}_{12}$ over the open set of $\tilde{\mathcal{X}}^2$ of points whose images by f are pairs of smooth points. Thus, since $\tilde{\Delta}_{23}$ is flat over $\tilde{\mathcal{X}}^2$, it is a divisor of degree 1 on $\tilde{\mathcal{X}}^3/\tilde{\mathcal{X}}^2$. So

$$\tilde{\Delta}_{23} \cdot E(N) = -1,$$

giving the second line of the table in (b). Also, $\tilde{\Delta}_{13}$ meets $\tilde{X}_s$ exactly at the point $b^{-1}(s, Q)$, showing the first line.

By Lemma 5.2.3, we have $E(N) \subset \tilde{X}_{112}$ and $E(N') \subset \tilde{X}_{111}$. In addition, we have $X_{s,1} \subset \tilde{X}_{111}$ and $X_{s,2} \subset \tilde{X}_{112}$. Now, since $\tilde{X}_{111} + \tilde{X}_{112} = \tilde{p}_{12}^{-1}(\tilde{X}_{11})$, and $s \in \tilde{X}_{11}$, we have that the union of $\tilde{X}_{111}$ and $\tilde{X}_{112}$ contains the curve $\tilde{X}_s$. Moreover, $\tilde{X}_{111}$ and $\tilde{X}_{112}$ intersect transversally (because so do the components X_1 and X_2 of X). Thus, since $\tilde{X}_{111}$ is numerically equivalent to $-\tilde{X}_{112}$, we have the third line of the table. At last, since $Q \notin X_2$, its clear that $\tilde{X}_{222}$ does not intersect $\tilde{X}_s$, and we have the last line, thus proving (b).

(c) Now assume that $s \in \tilde{X}_{ij} \cap \tilde{X}_{ji}$, but $s \notin \tilde{X}_{ii} \cup \tilde{X}_{jj}$, with $i \neq j$. Then $f(s) = (N, N)$, where N is a node of X. Let N' be the other node of X. The point s is on the line $E \subset \tilde{\mathcal{X}}^2(o)$ over (N, N). So s is given by the equations $x_1 = x_2 = y_1 = y_2 = 0$ and $\alpha = a$, where x_1, x_2, y_1, y_2 and α are as in (5.1). Thus, modding out the rings in (5.5) and (5.6) by these equations we have $k[[z_1, \beta - b]]/(\beta z_1)$ on the one hand, and $k[[z_2, \beta' - b']]/(\beta' z_2)$ on the other.

First we note that

$$\tilde{\Delta}_{23} \cdot E(N') = 0,$$

because $(f \times \mathrm{Id}) \circ b(E(N')) = (N, N, N')$ does not belong to $\Delta_{23} = (f \times \mathrm{Id}) \circ b(\tilde{\Delta}_{23})$. Now, the ideal of Δ'_{23} is $(y_1 - z_1, y_2 - z_2)$. Thus, as in (b), the restriction of $\tilde{\Delta}_{23}$ to $\tilde{X}_s$ is given by $z_1 = 0$ locally at the point $(s, N, (b : 1))$, and $z_2 = 0$ locally at the point $(s, N, (1 : b'))$. Then $\tilde{\Delta}_{23}$ crosses both $X_{s,1}$ and $X_{s,2}$ transversally at the points where these components meet $E(N)$, showing that

$$\tilde{\Delta}_{23} \cdot X_{s,1} = \tilde{\Delta}_{23} \cdot X_{s,2} = 1.$$

At last, $\tilde{\Delta}_{23}$ is a divisor of degree 1 on $\tilde{\mathcal{X}}^3/\tilde{\mathcal{X}}^2$, so

$$\tilde{\Delta}_{23} \cdot E(N) = -1,$$

giving the second line of the table in (c).

Now, since $(f \times \mathrm{Id}) \circ b(E(N')) = (N, N, N')$ does not belong to $\Delta_{13} = (f \times \mathrm{Id}) \circ b(\tilde{\Delta}_{13})$, we get that

$$\tilde{\Delta}_{13} \cdot E(N') = 0.$$

Also, the restriction of the diagonal $\tilde{\Delta}_{13}$ to $\tilde{X}_s$ is given, locally around $E(N)$, by the equation $\beta = \alpha$. Indeed, the ideal of Δ'_{13} is $(x_1 - z_1, x_2 - z_2)$, where $x_2 = \alpha y_1$ (see (5.1)) and $z_2 = \beta y_1$ (see (5.5)). Now,

$$(x_1 - z_1, \alpha y_1 - \beta y_1) = (x_1 - z_1, \alpha - \beta) \cap (x_1 - z_1, y_1),$$

and since $(x_1 - z_1, y_1)$ defines a codimension-2 subscheme of $\tilde{B}_1$, we have that the strict transform of Δ'_{13} in $\tilde{B}_1$ is given by $(x_1 - z_1, \alpha - \beta)$. Since at the point s we have $\alpha \neq 0$, from (5.5) we have $x_1 = \alpha^{-1}\beta z_1$, and thus the restriction of $\tilde{\Delta}_{13}$ to $\tilde{X}_s$ is given by $\beta = \alpha$. Therefore, $\tilde{\Delta}_{13}$ intersects $\tilde{X}_s$ transversally at the point of $E(N)$ where $\beta = \alpha$, and thus

$$\tilde{\Delta}_{13} \cdot E(N) = 1.$$

Moreover, the above reasoning shows that $\tilde{\Delta}_{13}$ does not intersect either $X_{s,1}$ or $X_{s,2}$, so

$$\tilde{\Delta}_{13} \cdot X_{s,1} = \tilde{\Delta}_{13} \cdot X_{s,2} = 0,$$

showing the first line of the table. As for the last line, it is enough to note that $b(\tilde{X}_{222}) = \tilde{X}_{22} \times X_2$ and $s \notin \tilde{X}_{22}$.

(d) Assume that $s \in \tilde{X}_{ii} \cap \tilde{X}_{jj}$, but $s \notin \tilde{X}_{ij} \cup \tilde{X}_{ji}$, with $i \neq j$. So $f(s) = (N, N')$, where N and N' are distinct nodes of X. Then s is given by the equations $x_1 = x_2 = y_1 = y_2 = 0$ and $\alpha = a$, where x_1, x_2, y_1, y_2 and α are as in (5.3). Modding out the rings in (5.11) and (5.12) by these equations, we get $k[[z_1, \beta - b]]/(z_1\beta)$ on the one hand, and $k[[z_2, \beta' - b']]/(z_2\beta')$ on the other.

Now, $\tilde{\Delta}_{13} \cdot E(N') = 0$, because

$$(f \times \mathrm{Id}) \circ b(E(N')) = (N, N', N') \notin \Delta_{13} = (f \times \mathrm{Id}) \circ b(\tilde{\Delta}_{13}).$$

In addition, the ideal of Δ'_{13} is $(x_1 - z_1, x_2 - z_2)$, so the restriction of $\tilde{\Delta}_{13}$ to $\tilde{X}_s$ is given by $z_1 = 0$ locally at $(s, N, (b : 1))$, and by $z_2 = 0$ locally at $(s, N, (1 : b'))$. Thus $\tilde{\Delta}_{13}$ intersects $X_{s,1}$ and $X_{s,2}$ transversally at the points where these components meet $E(N)$, showing that

$$\tilde{\Delta}_{13} \cdot X_{s,1} = \tilde{\Delta}_{13} \cdot X_{s,2} = 1.$$

Then, since $\tilde{\Delta}_{13}$ is a degree-1 divisor on $\tilde{\mathcal{X}}^3/\tilde{\mathcal{X}}^2$, we have $\tilde{\Delta}_{13} \cdot E(N) = -1$, proving the first line of the table in (d). The second line is analogous.

At last, by Lemma 5.2.3 we have $E(N), E(N') \subset \tilde{X}_{221}$. Clearly, we also have $X_{s,1} \subset \tilde{X}_{221}$ and $X_{s,2} \subset \tilde{X}_{222}$. Now, the union of $\tilde{X}_{222}$ and $\tilde{X}_{221}$ contains the curve $\tilde{X}_s$, and moreover $\tilde{X}_{222}$ and $\tilde{X}_{221}$ intersect transversally. Thus, since $\tilde{X}_{222}$ is numerically equivalent to $-\tilde{X}_{221}$, we have the last line of the table in (d).

(e) Let $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij} \cap \tilde{X}_{ji}$. Then $f(s) = (N, N)$ with N a node of X. Let N' be the other node of X. The point s belongs to the line $E \subset \tilde{\mathcal{X}}^2(o)$ over (N, N), and moreover s is either $(0 : 1)$ or $(1 : 0)$. Let's first describe $\tilde{X}_s$. By Lemma 5.2.2, the curve $\tilde{X}_s$ is as in Figure 5.3 (3). Now, $E_1(N)$ is not contracted by $b_2 \circ b_3 \circ b_4$, and $E_1(N')$ is contracted by $b_3 \circ b_4$ but not by b_4. Therefore, by Lemmas 5.2.3 and 5.2.4, the curve $\tilde{X}_s$ is as in Figure 6.5 (I) if $s \in \tilde{X}_{11} \cap \tilde{X}_{12} \cap \tilde{X}_{21}$, and (II) for $s \in \tilde{X}_{22} \cap \tilde{X}_{12} \cap \tilde{X}_{21}$.

First note that, since $(f \times \mathrm{Id}) \circ b(E_k(N')) = (N, N, N')$ does not belong to either Δ_{13} or Δ_{23}, for $k = 1, 2$, we get that

$$\tilde{\Delta}_{13} \cdot E_1(N') = \tilde{\Delta}_{23} \cdot E_1(N') = \tilde{\Delta}_{13} \cdot E_2(N') = \tilde{\Delta}_{23} \cdot E_2(N') = 0.$$

Now, let's examine $E_1(N)$ and $E_2(N)$. The point s is given by $x_1 = x_2 = y_1 = y_2 = 0$ and either $\alpha = 0$ or $\alpha' = 0$. From (5.1) and (5.2) we see that if $i = 1$ and $j = 2$ then $\alpha = 0$, and if $i = 2$ and $j = 1$ then $\alpha' = 0$. Without loss of generality, assume $\alpha = 0$, so $i = 1$ and $j = 2$.

Let $\overline{\Delta}_{23}$ be the strict transform of Δ'_{23} in $\tilde{B}_1$. Then $\overline{\Delta}_{23}$ is a Cartier divisor on $\tilde{B}_1$, and so $\tilde{\Delta}_{23}$ is the pullback of $\overline{\Delta}_{23}$ under the composition $b_2 \circ b_3 \circ b_4$. Therefore, by the projection formula,

$$\tilde{\Delta}_{23} \cdot W = \overline{\Delta}_{23} \cdot (b_2 \circ b_3 \circ b_4)(W)$$

for every irreducible component $W \subset \tilde{X}_s$. Now, modding out the rings in (5.5) and (5.6) by the equations defining $b_2 \circ b_3 \circ b_4(\tilde{X}_s)$ we get $k[[z_1, \beta - b]]/(z_1\beta)$ on the one hand, and $k[[z_2, \beta' - b']]/(z_2\beta')$ on the other.

The ideal of Δ'_{23} locally at (s, N) is $(y_1 - z_1, y_2 - z_2)$. Hence the restriction of the Cartier divisor $\overline{\Delta}_{23}$ to $b_2 \circ b_3 \circ b_4(\tilde{X}_s)$ is given by $z_1 = 0$, locally at the point $(s, N, (b : 1))$, and by $z_2 = 0$ locally at the point $(s, N, (1 : b'))$. Let $\overline{X}_{s,l} = b_2 \circ b_3 \circ b_4(X_{s,l})$ for $l = 1, 2$. So, since z_1 and z_2 are local parameters for $\overline{X}_{s,1}$ and $\overline{X}_{s,2}$, respectively, we have that $\overline{\Delta}_{23}$ intersects $\overline{X}_{s,1}$ and $\overline{X}_{s,2}$ transversally at the points where these components meet $b_2 \circ b_3 \circ b_4(E_1(N))$, showing that

$$\tilde{\Delta}_{23} \cdot X_{s,1} = \tilde{\Delta}_{23} \cdot X_{s,2} = 1.$$

Moreover, $E_2(N)$ is contracted by $b_2 \circ b_3 \circ b_4$, thus $(b_1 \circ b_2 \circ b_3)(E(N))$ is a point, and so

$$\tilde{\Delta}_{23} \cdot E_2(N) = 0.$$

At last, since $\tilde{\Delta}_{23}$ is a degree-1 divisor on $\tilde{\mathcal{X}}^3/\tilde{\mathcal{X}}^2$, we have $\tilde{\Delta}_{23} \cdot E_1(N) = -1$, giving the second line of the table on (e).

Now we check the first line. Modding out the rings in (5.13) and (5.14) by the equations defining $\tilde{X}_s$ we get $k[[z_1, \gamma - c]]/(z_1\gamma)$ on the one hand, and $k[[\beta, \gamma' - c']]/(\beta\gamma')$ on the other. The ideal of Δ'_{13} locally at (s, N) is $(x_1 - z_1, x_2 - z_2)$, where $x_2 = \alpha y_1$ and $z_2 = \beta y_1$ (see (5.1) and (5.5)). Now,

$$(x_1 - z_1, \alpha y_1 - \beta y_1) = (x_1 - z_1, \alpha - \beta) \cap (x_1 - z_1, y_1),$$

and since $(x_1 - z_1, y_1)$ defines a codimension-2 subscheme, we have that, around $E_2(N)$, the ideal of $\tilde{\Delta}_{13}$ is $(x_1 - z_1, \alpha - \beta)$. Then the restriction of $\tilde{\Delta}_{13}$ to $\tilde{X}_s$ is given by $z_1 = 0$ locally at $(s, N, (c : 1))$, and by $\beta = 0$ locally at $(s, N, (1 : c'))$. Therefore, since z_1 and β are local parameters for $X_{s.1}$ and $E_1(N)$, respectively, we have that $\tilde{\Delta}_{13}$ intersects $X_{s,1}$ and $E_1(N)$ transversally at the points where these components meet $E_2(N)$, showing that

$$\tilde{\Delta}_{13} \cdot X_{s,1} = \tilde{\Delta}_{13} \cdot E_1(N) = 1.$$

Moreover, the above reasoning shows that $\tilde{\Delta}_{13}$ does not intersect $X_{s,2}$, and thus

$$\tilde{\Delta}_{13} \cdot X_{s,2} = 0.$$

At last, since $\tilde{\Delta}_{13}$ is a degree-1 divisor on $\tilde{\mathcal{X}}^3/\tilde{\mathcal{X}}^2$, we have

$$\tilde{\Delta}_{13} \cdot E_2(N) = -1,$$

proving the first line of (e).

Since $s \notin \tilde{X}_{22}$, we have the last line of the table. (Recall that we assumed $i = 1$ and $j = 2$.) At last, by Lemma 5.2.3, we have $E_1(N) \subset \tilde{X}_{112}$ and $E_1(N') \subset \tilde{X}_{111}$. Also, $X_{s,1} \subset \tilde{X}_{111}$ and $X_{s,2} \subset \tilde{X}_{112}$. Let's show that $E_2(N) \subset \tilde{X}_{112}$ and $E_2(N') \subset \tilde{X}_{111}$.

Locally at (s, N), we have that X'_{112} is given by the equations $x_2 = y_2 = z_1 = 0$. Now, recall that $x_2 = \alpha y_1$ and $y_2 = \alpha x_1$ (see (5.1)), and that $\alpha = \gamma z_1$ (see (5.13)), so $\tilde{X}_{112}$ is given by $z_1 = 0$, locally around $E_2(N)$. Also, by (5.13), we see that $E_2(N)$ is given by $x_1 = y_1 = z_1 = 0$, showing that $E_2(N) \subset \tilde{X}_{112}$.

Similarly, locally at (s, N'), we have that X'_{111} is given by the equations $x_2 = y_2 = z_2 = 0$. Recall from (5.1) that $x_2 = \alpha y_1$ and $y_2 = \alpha x_1$, and from (5.15) that $\alpha = \gamma z_2$. Thus $\tilde{X}_{111}$ is given by $z_2 = 0$, around $E_2(N')$. On the other hand, from (5.15) we get that $E_2(N')$ is given by the equations $x_1 = y_2 = z_2 = 0$, thus showing that $E_2(N') \subset \tilde{X}_{111}$.

Now, the union of the divisors $\tilde{X}_{111}$ and $\tilde{X}_{112}$ contains the curve $\tilde{X}_s$, and moreover $\tilde{X}_{111}$ and $\tilde{X}_{112}$ intersect transversally. Thus, since $\tilde{X}_{111}$ is numerically equivalent to $-\tilde{X}_{112}$, we have the third line of the table in (e).

(f) Let $s \in \tilde{X}_{ij} \cap \tilde{X}_{ii} \cap \tilde{X}_{jj}$. Then $f(s) = (N, N')$ with N and N' distinct nodes of X. The point s belongs to the line $E \subset \tilde{\mathcal{X}}^2(o)$ over (N, N'), and moreover either $s = (0:1)$ or $s = (1:0)$. Let's describe the curve $\tilde{X}_s$. By Lemma 5.2.2, $\tilde{X}_s$ is as in Figure 6.3 (3). Now, $E_1(N')$ is not contracted by $b_2 \circ b_3 \circ b_4$, and $E_1(N)$ is not contracted by $b_3 \circ b_4$, but is by $b_2 \circ b_3 \circ b_4$. Therefore, by Lemmas 5.2.3 and 5.2.4, the curve $\tilde{X}_s$ is as in Figure 6.5 (I) if $s \in \tilde{X}_{12} \cap \tilde{X}_{11} \cap \tilde{X}_{22}$, and (II) if $s \in \tilde{X}_{21} \cap \tilde{X}_{11} \cap \tilde{X}_{22}$.

The point s is given by the equations $x_1 = y_1 = x_2 = y_2 = 0$ and either $\alpha = 0$ or $\alpha' = 0$, where $x_1, y_1, x_2, y_2, \alpha$ and α' are as in (5.3) and (5.4). Now, from (5.3) and (5.4) we see that if $s = (0:1)$, then $\alpha = 0$ and $i = 1$ and $j = 2$; and if $s = (1:0)$, then $\alpha' = 0$ and $i = 2$ and $j = 1$. Without loss of generality, we assume $s = (0:1)$, so $i = 1$ and $j = 2$.

First we note that, since $b \circ (f \times \mathrm{Id})(E(N')) = (N, N', N') \notin \Delta_{13}$, we have

$$\tilde{\Delta}_{13} \cdot E_1(N') = \tilde{\Delta}_{13} \cdot E_2(N') = 0.$$

Now let $\overline{\Delta}_{13}$ be the strict transform of Δ'_{13} in $\tilde{B}_2$. Then $\overline{\Delta}_{13}$ is a Cartier divisor on $\tilde{B}_2$, and so $\tilde{\Delta}_{13}$ is the pullback of $\overline{\Delta}_{13}$ under the composition $b_3 \circ b_4$. By the projection formula,

$$\tilde{\Delta}_{13} \cdot W = \overline{\Delta}_{13} \cdot b_3 \circ b_4(W)$$

for every irreducible component $W \subset \tilde{X}_s$.

Now, modding out the rings in (5.11) and (5.12) by the equations defining $b_3 \circ b_4(\tilde{X}_s)$ we get $k[[z_1, \beta - b]]/(z_1\beta)$ on the one hand, and $k[[z_2, \beta' - b']]/(z_2\beta')$ on the other. The restriction of $\overline{\Delta}_{13}$ to $b_3 \circ b_4(\tilde{X}_s)$ is given by $z_1 = 0$ locally at the point $(s, N, (b:1))$, and by $z_2 = 0$ locally at the point $(s, N, (1:b'))$. Let $\overline{X}_{s,l} = b_3 \circ b_4(X_{s,l})$, for $l = 1, 2$. So, since z_1 and z_2 are local parameters for $\overline{X}_{s,1}$ and $\overline{X}_{s,2}$, respectively, we have that $\overline{\Delta}_{13}$ intersects $\overline{X}_{s,1}$ and $\overline{X}_{s,2}$ transversally at the points where these components meet $b_3 \circ b_4(E_1(N))$, showing that

$$\tilde{\Delta}_{13} \cdot X_{s,1} = \tilde{\Delta}_{13} \cdot X_{s,2} = 1.$$

In addition, $E_2(N)$ is contracted by $b_3 \circ b_4$, so

$$\tilde{\Delta}_{13} \cdot E_2(N) = 0.$$

At last, since $\tilde{\Delta}_{13}$ is a degree-1 divisor on $\tilde{\mathcal{X}}^3/\tilde{\mathcal{X}}^2$, we have

$$\tilde{\Delta}_{13} \cdot E_1(N) = -1,$$

giving the first line of the table in (f). The verification of the second line is analogous.

We now check the two remaining lines. By Lemma 5.2.3, we have $E_1(N), E_1(N') \subset \tilde{X}_{112} \cap \tilde{X}_{221}$. In addition, clearly we have $X_{s,1} \subset \tilde{X}_{111} \cap \tilde{X}_{221}$ and $X_{s,2} \subset \tilde{X}_{112} \cap \tilde{X}_{222}$. Let's show that $E_2(N) \subset \tilde{X}_{111} \cap \tilde{X}_{221}$ and $E_2(N') \subset \tilde{X}_{112} \cap \tilde{X}_{222}$.

Locally at (s, N), we have that X'_{111} is given by $x_2 = y_2 = z_2 = 0$. Recall that, from (5.3) we have $x_2 = \alpha y_2$, from (5.11) we have $z_2 = \beta x_1$, and from (5.19) we have $y_2 = \gamma\beta$. Hence $\tilde{X}_{111}$ is given by the equation $\beta = 0$, locally around $E_2(N)$. Since by (5.19), $E_2(N)$ is given by $x_1 = \alpha = \beta = 0$, we get $E_2(N) \subset \tilde{X}_{111}$. In addition, locally at (s, N), X'_{221} is given by the equations $x_1 = y_1 = z_2 = 0$. From (5.3) we have $y_1 = \alpha x_1$, and from (5.11) we have $z_2 = \beta x_1$. Thus $\tilde{X}_{221}$ is given by the equation $x_1 = 0$. Since $E_2(N)$ is given by $x_1 = \alpha = \beta = 0$, we get $E_2(N) \subset \tilde{X}_{221}$.

Now, locally at (s, N'), we have that X'_{112} (resp. $\tilde{X}'_{222}$) is given by $x_2 = y_2 = z_1 = 0$ (resp. $x_1 = y_1 = z_1 = 0$). Recall that, from (5.3) we have $x_2 = \alpha y_2$ and $y_1 = \alpha x_1$, from (5.10) we have $z_1 = \beta' y_2$, and from (5.17) we have $\beta' = \gamma x_1$ and $\alpha = \gamma z_2$. So $\tilde{X}_{112}$ is given by the equation

$y_2 = 0$ (resp. $\tilde{X}_{222}$ is given by the equation $x_1 = 0$), locally around $E_2(N')$. Since by (5.17) the line $E_2(N')$ is given by $x_1 = y_2 = z_2 = 0$, we get $E_2(N') \subset \tilde{X}_{112} \cap \tilde{X}_{222}$.

At last, let l be either 1 or 2. The union of the divisors $\tilde{X}_{ll1}$ and $\tilde{X}_{ll2}$ contains the curve $\tilde{X}_s$, and moreover $\tilde{X}_{ll1}$ and $\tilde{X}_{ll2}$ intersect transversally. Hence, since $\tilde{X}_{ll1}$ is numerically equivalent to $-\tilde{X}_{ll2}$, we obtain the two remaining lines of the table in (f) from the above analysis. □

5.3 Proof of the theorem

In order to prove the theorem, we must first show that, for each $s \in \tilde{\mathcal{X}}^2$, the sheaf $b_*\mathcal{J}(s)$ is torsion-free and of rank 1. For this, we need yet two more lemmas.

Let Y be a curve, and $E \subset Y$ an irreducible component of Y. We say that E is an *exceptional component* of Y if E is a rational component such that the intersection $E \cap E^c$ is a pair of distinct nodes N_1, N_2. A *chain of exceptional components* is a connected subcurve of Y whose components are exceptional components of Y.

Let E be a chain of exceptional components $E_1, \dots, E_k$, in no particular order. Contracting the components $E_1, \dots, E_k$ to nodes, one at a time, we obtain a sequence of curves and birational maps

$$Y \xrightarrow{h_1} Y_1 \xrightarrow{h_2} \dots \xrightarrow{h_k} Y_k.$$

Then let $Y' := Y_k$, the curve obtained from Y by contracting all the exceptional components in E, and let $h : Y \to Y'$ be the composition $h_k \circ \dots \circ h_1$. Note that the intersection $E \cap E^c$ is a pair of points N_1, N_2, and that h is an isomorphism on $E^c - \{N_1, N_2\}$. Let $N \in Y'$ be the image of E under h. We say that the node N is obtained from Y by contracting the chain of exceptional components E.

Lemma 5.3.1 *Let Y be a curve, and $E \subset Y$ be a chain of exceptional components of Y. Let $h : Y \to Y'$ be the morphism contracting E to a node $N \in Y'$. Let L be an invertible sheaf on Y having degree 1 on at most one irreducible component of E and degree zero on the others. Then h_*L is a torsion-free rank-1 sheaf on Y'. Furthermore, h_*L is invertible at N if and only if $\deg_E(L) = 0$.*

Proof. We proceed by induction on the number of components of E. Assume first that E is irreducible.

Consider the complement $E^c \subset Y$. Denote by N_1, N_2 the points of Y in the intersection $E \cap E^c$.

We have the following exact sequences

$$0 \longrightarrow L|_E(-N_1 - N_2) \longrightarrow L \longrightarrow L|_{E^c} \longrightarrow 0,$$

$$0 \longrightarrow L|_{E^c}(-N_1 - N_2) \longrightarrow L \longrightarrow L|_E \longrightarrow 0.$$

Let $i = \deg_E(L)$. Then, since $E \cong \mathbb{P}^1$, we have $L|_E \cong \mathcal{O}_E(i)$, and so $L|_E(-N_1 - N_2) \cong \mathcal{O}_E(i-2)$.

Taking the direct image by h of the first sequence we get the exact sequence

$$(5.21) \qquad 0 \longrightarrow h_*(\mathcal{O}_E(i-2)) \longrightarrow h_*L \longrightarrow h_*(L|_{E^c}) \longrightarrow R^1h_*\mathcal{O}_E(i-2).$$

Now, as i is either 0 or 1, the direct image $h_*(\mathcal{O}_E(i-2))$ is zero, and thus h_*L is a subsheaf of $h_*(L|_{E^c})$, which is torsion-free since h is an isomorphism on $E^c - \{N_1, N_2\}$ and L is invertible. Therefore, h_*L is torsion-free. In addition, taking direct image of the second sequence yields the exact sequence

$$0 \longrightarrow h_*(L|_{E^c}(-N_1 - N_2)) \longrightarrow h_*L \longrightarrow h_*\mathcal{O}_E(i) \longrightarrow 0,$$

because $R^1h_*L|_{E^c} = 0$, since $h|_{E^c}$ is finite. If $i = 1$, then $h_*\mathcal{O}_E(1)$ is a rank-2 quotient of h_*L supported at the node N, so h_*L is not invertible at N.

Now, assume $i = 0$. We have that h_*L is a subsheaf of $h_*(L|_{E^c})$ of colength 1 containing the sheaf $h_*(L|_{E^c}(-N_1 - N_2))$. We claim that if h_*L is a subsheaf of $h_*(L|_{E^c}(-N_l))$ for some $l = 1, 2$, then it is equal to $h_*(L|_{E^c}(-N_1 - N_2))$. Indeed, if h_*L is a subsheaf of $h_*(L|_{E^c}(-N_l))$, then the local sections of h_*L are direct images of sections of L that are zero on N_l. Thus, restricting to E, we get a section of $L|_E$ that is zero on N_l. But since $i = 0$, we have $L|_E \cong \mathcal{O}_E$, and hence the section must be zero on E. Therefore, the section vanishes on both N_1 and N_2. So h_*L is actually a subsheaf of $h_*(L|_{E^c}(-N_1 - N_2))$. Now, since $h_*(L|_{E^c}(-N_1 - N_2))$ is a subsheaf of h_*L, we have $h_*L = h_*(L|_{E^c}(-N_1 - N_2))$. Thus, from the exact sequence

$$0 \longrightarrow h_*(L|_{E^c}(-N_1 - N_2)) \xrightarrow{\sim} h_*L \longrightarrow h_*\mathcal{O}_E \longrightarrow 0,$$

we get that $h_*\mathcal{O}_E$ is zero, an absurd since $h_*\mathcal{O}_E \cong \mathcal{O}_N$. We showed our claim, that h_*L cannot be a subsheaf of $h_*(L|_{E^c}(-N_l))$ for $l = 1, 2$. But these are the only noninvertible subsheaves of colength 1 of $h_*(L|_{E^c})$ containing h_*L. Therefore, $h_*(L|_{E^c}(-N_1 - N_2))$ is invertible.

Now assume that E has irreducible components $E_1, \dots, E_k$. Since by hypothesis the degree of L is 1 on at most one component of E, we may assume that the degree of L is 0 on $E_1, \dots, E_{k-1}$. Let $h' : Y \to Y_{k-1}$ be the birational morphism obtained by contracting $E_1, \dots, E_{k-1}$. By induction hypothesis, $L' := h'_*L$ is invertible on Y_{k-1}. Note that the degree of L' on E_k is equal to the degree of L on E_k. (Here we identify E_k with its image under h'.)

Finally, let $h_k : Y_{k-1} \to Y'$ be the birational morphism contracting E_k. Thus, by the first case, $h_{k*}(L') = h_*L$ is torsion-free of rank 1, and is invertible at $N = h(E) = h_k(E_k)$ if and only if $\deg_{E_k}(L) = 0$. □

Lemma 5.3.2 *Let $\mathcal{Y}/S$ and $\mathcal{Y}'/S$ be two families of curves over the same base. Let $h : \mathcal{Y} \to \mathcal{Y}'$ be a morphism of families. Assume that, for each $s \in S$, $f(s) : \mathcal{Y}(s) \to \mathcal{Y}'(s)$ is obtained by contracting exceptional components. Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{Y}$ such that $0 \le \deg_E(\mathcal{L}) \le 1$ for every chain of exceptional components of each fiber $\mathcal{Y}(s)$. Then $h_*\mathcal{L}$ is a torsion-free sheaf of rank 1 on $\mathcal{Y}'/S$, and its formation commutes with base change. Furthermore, $h_*\mathcal{L}$ is invertible at each node obtained by contracting a chain of exceptional components E such that $\deg_E(\mathcal{L}) = 0$.*

Proof. By Lemma 5.3.1, it is enough to show that $h_*\mathcal{L}$ is flat over S, and that its formation commutes with base change, that is, the induced morphism

$$(h_*\mathcal{L})(s) \longrightarrow h(s)_*\mathcal{L}(s)$$

is an isomorphism for every $s \in S$.

First, we show that $h_*\mathcal{L}$ is S-flat. Denote by $p : \mathcal{Y} \to S$ (resp. $p' : \mathcal{Y}' \to S$) the structure morphism of the family $\mathcal{Y}/S$ (resp. $\mathcal{Y}'/S$). Then $p = p' \circ h$. Let $\mathcal{O}_{\mathcal{Y}'}(1)$ be a relatively very ample sheaf on $\mathcal{Y}'/S$. The sheaf $h_*\mathcal{L}$ is flat over S if and only if $p'_*((h_*\mathcal{L})(m))$ is locally free for every sufficiently large integer m. By the projection formula,

$$p'_*((h_*\mathcal{L})(m)) = p_*(\mathcal{L} \otimes h^*\mathcal{O}_{\mathcal{Y}'}(m)).$$

Since $\mathcal{L} \otimes h^*\mathcal{O}_{\mathcal{Y}'}(m)$ is flat over S, we may use [H, Theorem 12.11, p. 290] to show that the sheaf $p_*(\mathcal{L} \otimes h^*\mathcal{O}_{\mathcal{Y}'}(m))$ is locally free. For that it is enough to show that

$$H^1((\mathcal{L} \otimes h^*\mathcal{O}_{\mathcal{Y}'}(m))(s)) = 0 \qquad (5.22)$$

for each $s \in S$, for a sufficiently large m.

So fix $s \in S$. Then, by hypothesis, $h(s)$ is a morphism contracting chains of exceptional components of $\mathcal{Y}(s)$. Suppose Σ is the union of all chains of exceptional components of $\mathcal{Y}(s)$ contracted by $h(s)$. Then $h(s)$ is an isomorphism away from Σ. Therefore $(h^*\mathcal{O}_{\mathcal{Y}'}(m))(s)$ is a sheaf having high multidegree on Σ^c, and degree 0 on each irreducible component of Σ. Let $F := (\mathcal{L} \otimes h^*\mathcal{O}_{\mathcal{Y}'}(m))(s)$. Thus, F has high muiltidegree on Σ^c, and degree 0 or 1 on each component E of Σ.

Now, consider the exact sequence

$$0 \to F|_{E^c}(-N_1 - N_2) \to F \to F|_E \to 0,$$

where N_1 and N_2 are the points of intersection of E with its complement in $\mathcal{Y}(s)$. Then, since E is rational, and F has degree 0 or 1 on E, we have $H^1(E, F|_E) = 0$. Thus, we have a surjection

$$H^1(E^c, F|_{E^c}(-N_1 - N_2)) \twoheadrightarrow H^1(\mathcal{Y}(s), F).$$

So it's enough to show that $H^1(E^c, F|_{E^c}(-N_1 - N_2)) = 0$. Repeating the proceedure for the sheaf $F|_{E^c}(-N_1 - N_2)$ in place of F, and another component of Σ in place of E, and thus successively, we get that it's enough to show that

$$H^1(\Sigma^c, F|_{\Sigma^c}(-N_1 - N_2 - \ldots - N_k)) = 0,$$

where $N_1, \ldots, N_k$ are the points in the intersection of Σ with its complement. Now, for $m >> 0$, the sheaf $F|_{\Sigma^c}(-N_1 - N_2 - \ldots - N_k)$ has high multidegree, and thus the first cohomology group of this sheaf vanishes. So we get (5.22). By [H, Theorem 12.11, p. 290], $F = p_*(\mathcal{L} \otimes h^*\mathcal{O}_{\mathcal{Y}'}(m))$ is locally free, and hence $h_*\mathcal{L}$ is flat over S.

Now, we show that the formation of $h_*\mathcal{L}$ commutes with base change. We must show that, for each point $s \in S$, we have a natural isomorphism $(h_*\mathcal{L})(s) \xrightarrow{\sim} h(s)_*\mathcal{L}(s)$. Since $\mathcal{Y}'(s)$ is projective, it is enough to show that, for each $s \in S$, and for every $m >> 0$, the direct images under $p'(s)$ of $(h_*\mathcal{L} \otimes \mathcal{O}_{\mathcal{Y}'}(m))(s)$ and $h(s)_*\mathcal{L}(s) \otimes \mathcal{O}_{\mathcal{Y}'(s)}(m)$ are equal. Now, by the projection formula,

$$p'(s)_*(h(s)_*\mathcal{L}(s) \otimes \mathcal{O}_{\mathcal{Y}'(s)}(m)) = p(s)_*(\mathcal{L}(s) \otimes h(s)^*\mathcal{O}_{\mathcal{Y}'(s)}(m)) = p(s)_*((\mathcal{L} \otimes h^*\mathcal{O}_{\mathcal{Y}'}(m))(s))$$

on the one hand, and, since $h_*\mathcal{L}$ is S-flat and $m >> 0$,

$$p'(s)_*((h_*\mathcal{L} \otimes \mathcal{O}_{\mathcal{Y}'}(m))(s)) = p'_*(h_*\mathcal{L} \otimes \mathcal{O}_{\mathcal{Y}'}(m))(s) = p_*(\mathcal{L} \otimes h^*\mathcal{O}_{\mathcal{Y}'}(m))(s)$$

on the other. Finally, since (5.22) holds, we have by [H, Theorem 12.11, p. 290]

$$p(s)_*((\mathcal{L} \otimes h^*\mathcal{O}_{\mathcal{Y}'}(m))(s)) = p_*(\mathcal{L} \otimes h^*\mathcal{O}_{\mathcal{Y}'}(m))(s),$$

and the result follows. □

Under the hypothesis of Lemma 5.3.2, we say that $\mathcal{L}$ *contracts to a torsion-free rank-1 sheaf* $h_*\mathcal{L}$ *on* $\mathcal{Y}'$. Moreover, for each $s \in S$, we say that $\mathcal{L}(s)$ *contracts to a torsion-free rank-1 sheaf* $h(s)_*\mathcal{L}(s)$ *on* $\mathcal{Y}'(s)$.

Proof of Theorem 5.1.1. Consider the families of curves $\tilde{p}_{12} : \tilde{\mathcal{X}}^3 \to \tilde{\mathcal{X}}^2$ and $\tilde{q}^0 : \tilde{\mathcal{X}}^2 \times_S \mathcal{X} \to \tilde{\mathcal{X}}^2$. (So, for each $s \in \tilde{\mathcal{X}}^2$, the curve $X_s = (\tilde{q}^0)^{-1}(s)$ is isomorphic to X.) The blowup $b : \tilde{\mathcal{X}}^3 \to \tilde{\mathcal{X}}^2$

is thus a morphism of families of curves such that $b(s)$ is obtained by contracting exceptional components, for each $s \in \tilde{\mathcal{X}}^2$. Now, the sheaf $b_*\mathcal{J}$ induces a morphism from $\tilde{\mathcal{X}}^2$ to $\overline{J}_{\mathcal{X}/S}$ if $b_*\mathcal{J}$ is a torsion-free rank-1 simple sheaf on $\tilde{\mathcal{X}}^2 \times_S \mathcal{X}/\tilde{\mathcal{X}}^2$. Thus, by Lemma 5.3.2, we have only to show that, for each $s \in \tilde{\mathcal{X}}^2(o)$, we have $0 \leq \deg_E(\mathcal{J}) \leq 1$ for every chain of exceptional components of each fiber $\tilde{p}_{12}^{-1}(s) = \tilde{X}_s$. Moreover, the morphism induced by $b_*\mathcal{J}$ factors through $\overline{J}_E^P$ if, for each $s \in \tilde{\mathcal{X}}^2(o)$, the restriction $\mathcal{J}(s)$ of $\mathcal{J}$ to $\tilde{X}_s$ is also $\tilde{\sigma}(s)$-quasi-stable with respect to $\mathcal{E}(s)$.

Thus, we must show that

$$0 \leq \deg_E(\mathcal{J}) \leq 1$$

for each exceptional component $E \subset \tilde{X}_s$ of the chains of rational components over the nodes of X, with equality $\deg_E(\mathcal{J}) = 1$ holding for at most one E, for each node. Furthermore, since $\tilde{\sigma}(s) \in X_{s,1}$, we must show that

$$\beta_{\mathcal{J}(s)}(X_{s,1}) > 0 \quad \text{and} \quad \beta_{\mathcal{J}(s)}(X_{s,2}) \geq 0,$$

where, as in Lemma 5.2.6, we denote by $X_{s,1}$ and $X_{s,2}$ the components of $\tilde{X}_s$ isomorphic to X_1 and X_2, respectively. This will imply that $\mathcal{J}(s)$ contracts to a torsion-free rank-1 sheaf $J := b_*\mathcal{J}(s)$ on X. Moreover, we get that J is P-quasi-stable with respect to $\omega_X \oplus \mathcal{O}_X$, because

$$\beta_J(X_i) = \beta_{\mathcal{J}(s)}(X_{s,i})$$

for $i = 1, 2$.

Let $s \in \tilde{\mathcal{X}}^2(o)$. As in Lemma 5.2.6, there are six cases to consider. First, if $f(s)$ is a pair of smooth points of X, then s belongs to only one divisor $\tilde{X}_{ij}$ of $\tilde{\mathcal{X}}^2$ and we have:

(a) $s \in \tilde{X}_{ij}$.

Second, if s is in the intersection of exactly two divisors, we have three cases:

(b) $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij}$ or $s \in \tilde{X}_{ii} \cap \tilde{X}_{ji}$, with $i \neq j$,

(c) $s \in \tilde{X}_{ij} \cap \tilde{X}_{ji}$, with $i \neq j$,

(d) $s \in \tilde{X}_{ii} \cap \tilde{X}_{jj}$, with $i \neq j$.

Finally, if s is in the intersection of three divisors, we have two more cases:

(e) $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij} \cap \tilde{X}_{ji}$, with $i \neq j$,

(f) $s \in \tilde{X}_{ij} \cap \tilde{X}_{ii} \cap \tilde{X}_{jj}$, with $i \neq j$.

Recall from the definition of $\mathcal{J}$ that, for each $s \in \tilde{\mathcal{X}}^2(o)$, and each irreducible component W of $\tilde{X}_s$, we have

$$\begin{aligned}\deg_W \mathcal{J}(s) &= \deg_W(\mathcal{O}_{\tilde{\mathcal{X}}^2}(-\tilde{\Delta}_{13})) + \deg_W(\mathcal{O}_{\tilde{\mathcal{X}}^2}(-\tilde{\Delta}_{23})) \\ &\quad + \deg_W(\mathcal{O}_{\tilde{\mathcal{X}}^2}(-\tilde{X}_{222})) + 2\deg_W(\mathcal{O}_{\tilde{\mathcal{X}}^2}(\tilde{\Sigma})) \\ &= -\tilde{\Delta}_{13}\cdot W - \tilde{\Delta}_{23}\cdot W - \tilde{X}_{222}\cdot W + 2\deg_W(\mathcal{O}_W(P))\end{aligned}$$

where $P = \tilde{\sigma}(s)$. Since we fixed $P \in X_{s,1}$, we have $\deg_{X_{s,1}}(\mathcal{O}_{X_{s,1}}(P)) = 1$, whereas $\deg_W(\mathcal{O}_{X_{s,2}}(P)) = 0$ if $W \neq X_{s,1}$. Also,

$$\beta_{\mathcal{J}(s)}(X_{s,i}) = \deg_{X_{s,i}}(\mathcal{J}(s)) + 1,$$

because $\delta_{X_{s,i}} = \delta_{X_i} = 2$ for $i = 1, 2$.

(a) Assume first that $f(s) = (Q_1, Q_2)$, with Q_1, Q_2 smooth points of X. By Lemma 5.2.2 the curve $\tilde{X}_s$ is isomorphic to X. By the intersection numbers calculated in Lemma 5.2.6, we have

$$\beta_{\mathcal{J}(s)}(X_{s,1}) = \begin{cases} -1-1-0+2+1=1, & \text{if } Q_1, Q_2 \in X_1 \\ -1-0-0+2+1=2, & \text{if } Q_1 \in X_1, \text{ and } Q_2 \in X_2 \\ -0-1-0+2+1=2, & \text{if } Q_1 \in X_2, \text{ and } Q_2 \in X_1 \\ -0-0-2+2+1=1, & \text{if } Q_1, Q_2 \in X_2 \end{cases}$$

and

$$\beta_{\mathcal{J}(s)}(X_{s,2}) = \begin{cases} -0-0-0+0+1=1, & \text{if } Q_1, Q_2 \in X_1 \\ -0-1-0+0+1=0, & \text{if } Q_1 \in X_1, \text{ and } Q_2 \in X_2 \\ -1-0-0+0+1=0, & \text{if } Q_1 \in X_2, \text{ and } Q_2 \in X_1 \\ -1-1+2+0+1=1, & \text{if } Q_1, Q_2 \in X_2 \end{cases}$$

and thus $\mathcal{J}(s)$ is the P-quasi-stable.

(b) If $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij}$, with $i \neq j$, then $f(s) = (Q, N)$, where Q is a smooth point of X in X_i, and N is a node of X. (The case where $s \in \tilde{X}_{ii} \cap \tilde{X}_{ji}$ is analogous.) By Lemma 5.2.2, $\tilde{X}_s$ has only one exceptional component over each node N and N' (where N' is the other node of X), denoted by $E(N)$ and $E(N')$, connecting the components $X_{s,1}$ and $X_{s,2}$ (see Figure 5.3 (2)).

First, we show that $\mathcal{J}(s)$ contracts to a torsion-free sheaf on X. By the intersection numbers calculated in Lemma 5.2.6, we have

$$\deg_{E(N)}(\mathcal{J}(s)) = \begin{cases} -0+1-0=1, & \text{if } i=1,\ j=2 \\ -0+1-1=0, & \text{if } i=2,\ j=1 \end{cases}$$

and

$$\deg_{E(N')}(\mathcal{J}(s)) = \begin{cases} -0-0-0=0, & \text{if } i=1,\ j=2 \\ -0-0+1=1, & \text{if } i=2,\ j=1 \end{cases}$$

Hence, by Lemma 5.3.1, the sheaf $\mathcal{J}(s)$ contracts to a torsion-free rank-1 sheaf on X, which we denote by J. Moreover, J is not invertible at N, but is invertible at N', if $i=1$ and $j=2$; and J is not invertible at N', but is invertible at N, if $i=2$ and $j=1$.

Now we have only to show that J is P-quasi-stable. By Lemma 5.2.6, we have

$$\beta_{\mathcal{J}(s)}(X_{s,1}) = \begin{cases} -1-1-0+2+1=1, & \text{if } i=1,\ j=2 \\ -0-1-1+2+1=1, & \text{if } i=2,\ j=1 \end{cases}$$

and

$$\beta_{\mathcal{J}(s)}(X_{s,2}) = \begin{cases} -0-1-0+0+1=0, & \text{if } i=1,\ j=2 \\ -1-1+1+0+1=0, & \text{if } i=2,\ j=1. \end{cases}$$

So J is P-quasi-stable on X.

(c) If $s \in \tilde{X}_{ij} \cap \tilde{X}_{ji}$, with $i \neq j$, then $f(s) = (N, N)$ with N a node of X. By Lemma 5.2.2, the curve $\tilde{X}_s$ has only one exceptional component over each node, denoted by $E(N)$ and $E(N')$, where N' is the other node of X (see Figure 5.3 (2)).

As in (b), we show first that $\mathcal{J}(s)$ contracts to a torsion-free rank-1 sheaf J on X. By Lemma 5.2.6 (c) we have

$$\deg_{E(N)}(\mathcal{J}(s)) = -1+1-0=0 \quad \text{and} \quad \deg_{E(N')}(\mathcal{J}(s)) = -0-0-0=0.$$

Therefore, $\mathcal{J}(s)$ contracts to an invertible sheaf J on X. Moreover, J is P-quasi-stable. Indeed, by Lemma 5.2.6, we have

$$\beta_{\mathcal{J}(s)}(X_{s,1}) = -0-1-0+2+1=2 \quad \text{and} \quad \beta_{\mathcal{J}(s)}(X_{s,2}) = -0-1-0+0+1=0,$$

finishing (c).

(d) If $s \in \tilde{X}_{ii} \cap \tilde{X}_{jj}$ with $i \neq j$, then $f(s) = (N, N')$, where N and N' are distinct nodes of X. By Lemma 5.2.2, the curve $\tilde{X}_s$ has only one exceptional component over each node, denoted by $E(N)$ and $E(N')$, where N' is the other node of X (see Figure 5.3 (2)).

As in (b) and (c), we first show that $\mathcal{J}(s)$ contracts to a torsion-free rank-1 sheaf J on X. By Lemma 5.2.6 (d) we have

$$\deg_{E(N)}(\mathcal{J}(s)) = +1-0-1=0 \quad \text{and} \quad \deg_{E(N')}(\mathcal{J}(s)) = -0+1-1=0.$$

Therefore, $\mathcal{J}(s)$ contracts to an invertible sheaf J on X. Moreover, by Lemma 5.2.6, we have

$$\beta_{\mathcal{J}(s)}(X_{s,1}) = -1-1-0+2+1 = 1 \quad \text{and} \quad \beta_{\mathcal{J}(s)}(X_{s,2}) = -1-1+2+0+1 = 1,$$

and so J is P-quasi-stable on X.

(e) If $s \in \tilde{X}_{ii} \cap \tilde{X}_{ij} \cap \tilde{X}_{ji}$ with $i \neq j$, then $f(s) = (N, N)$, where N is a node of X. Let N' be the other node. By Lemma 5.2.2, the curve $\tilde{X}_s$ is as in Figure 5.3 (3) and, to show that $\mathcal{J}(s)$ contracts to a torsion-free rank 1 sheaf on X, we must examine the degrees of $\mathcal{J}$ on $E_1(N)$, $E_2(N)$, $E_1(N')$ and $E_2(N')$.

By Lemma 5.2.6 (e) we have, for any $i \neq j$

$$\deg_{E_1(N)}(\mathcal{J}(s)) = -1+1-0 = 0 \quad \text{and} \quad \deg_{E_1(N')}(\mathcal{J}(s)) = -0-0-0 = 0.$$

Also,

$$\deg_{E_2(N)}(\mathcal{J}(s)) = \begin{cases} +1-0-0 = 1, & \text{if } i=1,\ j=2 \\ +1-0-1 = 0, & \text{if } i=2,\ j=1 \end{cases}$$

and

$$\deg_{E_2(N')}(\mathcal{J}(s)) = \begin{cases} -0-0-0 = 0, & \text{if } i=1,\ j=2 \\ -0-0+1 = 1, & \text{if } i=2,\ j=1 \end{cases}$$

showing that $\mathcal{J}(s)$ contracts to a torsion-free rank-1 sheaf J on X. Moreover, J is not invertible at N but is at N', if $i = 1$ and $j = 2$; and J is not invertible at N' but is at N, if $i = 2$ and $j = 1$. In addition, by Lemma 5.2.6 (e), we have

$$\beta_{\mathcal{J}(s)}(X_{s,1}) = \begin{cases} -1-1-0+2+1 = 1, & \text{if } i=1,\ j=2 \\ -0-1-1+2+1 = 1, & \text{if } i=2,\ j=1 \end{cases}$$

and

$$\beta_{\mathcal{J}(s)}(X_{s,2}) = \begin{cases} -0-1-0+0+1 = 0, & \text{if } i=1,\ j=2 \\ -1-1+1+0+1 = 0, & \text{if } i=2,\ j=1. \end{cases}$$

So J is P-quasi-stable.

(f) If $s \in \tilde{X}_{ij} \cap \tilde{X}_{ii} \cap \tilde{X}_{jj}$ with $i \neq j$, then $f(s) = (N, N')$ where N and N' are distinct nodes of X. By Lemma 5.2.2, the curve $\tilde{X}_s$ is as in Figure 5.3 (3).

By Lemma 5.2.6 (f), we have, for any $i \neq j$

$$\deg_{E_1(N)}(\mathcal{J}(s)) = +1-0-1 = 0, \quad \text{and} \quad \deg_{E_1(N')}(\mathcal{J}(s)) = -0+1-1 = 0.$$

Also

$$\deg_{E_2(N)}(\mathcal{J}(s)) = \begin{cases} -0-0-0=0, & \text{if } i=1,\ j=2 \\ -0-0+1=1, & \text{if } i=2,\ j=1 \end{cases}$$

and

$$\deg_{E_2(N')}(\mathcal{J}(s)) = \begin{cases} -0-0+1=1, & \text{if } i=1,\ j=2 \\ -0-0-0=0, & \text{if } i=2,\ j=1 \end{cases}$$

showing that $\mathcal{J}(s)$ contracts to a torsion-free rank-1 sheaf J on X. In addition, J is not invertible at N' but is at N, if $i=1$ and $j=2$; and J is not invertible at N but is at N', if $i=2$ and $j=1$.

Furthermore, by Lemma 5.2.6 (f), for any $i \neq j$ we have

$$\beta_{\mathcal{J}(s)}(X_{s,1}) = -1-1-0+2+1 = 1 \quad \text{and} \quad \beta_{\mathcal{J}(s)}(X_{s,2}) = -1-1+1+0+1 = 0,$$

and hence J is P-quasi-stable. □

Bibliography

[AIK76] A. ALTMAN, A. IARROBINO and S. KLEIMAN, *Irreducibility of the compactified Jacobian*, Real and complex singularities. Proceedings of Conference, Oslo, 1976, pp. 1–12.

[AK76] A. ALTMAN and S. KLEIMAN, *Compactifying the Jacobian*, Bull. of the Amer. Math. Soc. **6** (1976), 947–949.

[AK80] A. ALTMAN and S. KLEIMAN, *Compactifying the Picard scheme*, Adv. Math. **35** (1980), 50–112.

[AK] A. ALTMAN and S. KLEIMAN, *Introduction to Grothendieck duality theory*, Lecture Notes in Mathematics, **146**, Springer-Verlag, 1970.

[BLR] S. BOSCH, W. LÜTKEBOHMERT and M. RAYNAUD, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 21, Springer Verlag, 1990.

[Cap94] L. CAPORASO, *A compactification of the universal Picard variety over the moduli space of stable curves*, J. Amer. Math. Soc. **7** (1994), 589–660.

[CE06] L. CAPORASO and E. ESTEVES, *On Abel maps of stable curves*, Preprint, available at math.AG/0603476.

[Cat82] F. CATANESE, *Pluricanonical-Gorenstein-curves*, Progr. Math. **24** (1982), 51–95.

[D79] C. D'SOUZA, *Compactification of generalized Jacobians*, Proc. Indian Acad. Sci. Sect. A Math. Sci. **88** (1979), 418–457.

[E01] E. ESTEVES, *Compactifying the relative Jacobian over families of reduced curves*, Trans. of Amer. Math. Soc. **353** (2001), 3045–3095.

[EGK00] E. ESTEVES, M. GAGNÉ and S. KLEIMAN, *Abel maps and presentation schemes*, Comm. Algebra **28** (2000), 5961–5992.

[EGK02] E. ESTEVES, M. GAGNÉ and S. KLEIMAN, *Autoduality of the compactified Jacobian*, J. London Math. Soc. (2) **65** (2002), 591–610.

[EGA] A. GROTHENDIECK with J. DIEUDONNÉ, *Eléments de Geométrie Algébrique*, Publ. Math. IHES, **24** (1965) and **28** (1966).

[FGIKNV] B. FANTECHI, L. GÖTTSCHE, L. ILLUSIE, S. L. KLEIMAN, N. NITSURE, and A. VISTOLI, *Fundamental Algebraic Geometry: Grothendieck's FGA Explained*, Mathematical Surveys and Monographs, **123**, 2005.

[F] W. FULTON, *Algebraic Curves*, Benjamin, New York, 1969.

[G] D. GIESEKER, *Lectures on moduli of curves*, Tata Inst. Fund. Res. Lectures on Math. and Phys., vol 69, Springer-Verlag, Berlin and New York, 1982.

[H] R. HARTSHORNE, *Algebraic Geometry*, Springer-Verlag, New York - Heildelberg, 1977.

[I56] J. IGUSA, *Fibre systems of Jacobian varieties*, Amer. J. Math. **78** (1956), 171–199.

[Kn] D. KNUTSON, *Algebraic Spaces*, Lecture Notes in Mathematics 203, Springer-Verlag, Berlin - New York, 1971.

[MM64] A. MAYER and D. MUMFORD, *Further comments on boundary points*, Unpublished lecture notes distributed at the Amer. Math. Soc. Summer Institute, Woods Hole, 1964.

[M] D. MUMFORD, *Abelian varieties*, Oxford University Press, London, 1970.

[OS79] T. ODA and C. S. SESHADRI, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc. **253** (1979), 1–90.

[P96] R. PANDHARIPANDE, *A compactification over $\overline{M}_g$ of the universal moduli space of slope-semistable vector bundles*, J. Amer. Math. Soc. **9** (1996), 425–471.

[S82] C. S. SESHADRI, *Fibrés vectoriels sur les courbes algébriques*, Astérisque **96** (1982).