Contribuições à teoria ergódica de sistemas não hiperbólicos

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Resumo

O estudo dos sistemas dinâmicos data dos trabalhos de Poincaré sobre a mecânica celeste, nos quais propriedades qualitativas de certas EDO's são estudadas (ou seja, o comportamento assintótico das órbitas), sendo estes trabalhos retomados por Birkhoff nos anos 30. Porém, só na decáda de 60 houve uma revolução na teoria dos sistemas dinâmicos com os trabalhos de Smale e Anosov. Nestes trabalhos uma classe de sistemas (chamada *sistemas hiperbólicos*) as quais são, localmente, bastante simples (a derivada apresenta contração numa direção e expansão numa direção complementar) apesar de serem ricos a ponto de admitirem certas propriedades *caóticas*. A partir daí se desenvolveu uma teoria muito detalhada de sistemas hiperbólicos, tanto do ponto de vista geométrico quanto ergódico.

Este último aspecto remonta à criação da Mecânica Estatística, com os trabalhos de Boltzmann, Maxwell e Gibbs. Dada a complexidade dos sistemas com que lida a teoria cinética dos gases, esses pioneiros propouseram uma abordagem estatística de tais sistemas. Os anos 70 do século 20 trouxeram uma aplicação muito frutífera dessas idéias em dinâmica diferenciável, que também foi muito influenciada pela visão de Kolmogorov.

Então, com o surgimento dos trabalhos de Sinai, Ruelle, Bowen, obteve-se uma teoria completa sobre as propriedades ergódicas de sistemas hiperbólicos: eles admitem um número finito de medidas (chamadas medidas físicas ou SRB) que descrevem o comportamento estatístico de órbitas típicas.

Entretanto, foi constatado que os sistemas hiperbólicos não representam, de fato, a "maioria" dos sistemas dinâmicos: *existem sistemas não hiperbólicos que são robustos (ou seja, formam um conjunto aberto no espaço dos sistemas dinâmicos).*

Por outro lado, atualmente não se tem uma teoria completa sobre as características ergódicas dos sistemas não hiperbólicos, apesar de hoje em dia já se ter bastante resultados nesta direção. Neste trabalho, como o título indica, iremos estudar alguns aspectos ergódicos destes sistemas. Mas antes temos algo a explicar sobre a organização deste artigo.

Na verdade, este trabalho é fruto da junção de resultados obtidos em outros artigos (alguns já publicados e outros aceitos para publicação). Sendo estes artigos de certa forma independentes, no primeiro capítulo faremos uma introdução a cada um dos resultados que aparecerão nos outros capítulos. Em particular, após a leitura do primeiro capítulo, o leitor pode ler os restantes em qualquer ordem.

Capítulo 1

Introdução

1.1 Estados de Equílibrio

Considere um cristal 1-dimensional sujeito a um campo magnético de intensidade ϕ. É constatado fisicamente que após um certo tempo vibrando, os *spins* das moléculas tendem a um *estado de equilíbrio*. Matematicamente, o estado de equilíbrio pode ser representado por uma medida invariante associada ao sistema acima.

Mais geralmente, se $f : M \to M$ é um sistema dinâmico e $\phi : M \to \mathbb{R}$ é um potencial, uma medida f-invariante μ é um estado de equilíbrio para f (com respeito ao potencial ϕ) caso valha

$$h_\mu(f) + \int \phi d\mu = \max_{\nu \ f-\text{invariante}} h_\nu(f) + \int \phi d\nu,$$

ou seja, se μ realiza o máximo de um certo princípio variacional. Aqui $h_\mu(f)$ denota a *entropia* da função f com respeito a μ (i.e., a quantidade de desordem de f sobre conjuntos que são relevantes para μ).

No caso de sistemas hiperbólicos, a existência e unicidade (além de propriedades probabilísticas como *decaimento de correlações, teorema central do limite, etc.*) de estados de equilíbrio foram estabelecidos por vários autores como Sinai, Ruelle, Bowen, Walters e Parry.

Entretanto, só recentemente com os trabalhos de Oliveira consegui-se um resultado de existência de estados de equilíbrio num contexto não-hiperbólico. Porém, tudo isto se refere a sistemas determinísticos, mas, como ocorre em diversas situações práticas, é interessante supor que ao longo das iterações da dinâmica cometemos pequenos erros. Para isto foi desenvolvida a teoria ergódica das transformações *aleatórias*.

Os teoremas de existência e unicidade de estados de equilíbrio no contexto hiperbólico foram provados por Liu, e num certo contexto não-hiperbólico, por Khanin-Kifer.

No capítulo 2 provaremos que para perturbações aleatórias de certas transformações não-uniformemente expansoras existem estados de equilíbrio para potenciais próximos de serem constantes.

1.2 Estabilidade Ergódica

Uma das propriedades mais básicas da teoria ergódica é o conceito de ergodicidade, o qual diz que, do ponto de vista métrico, o sistema é indecomponível. A motivação desta definição reside no fato de que se temos um recipiente com gás (por exemplo, uma sala cheia de ar) e queremos medir a temperatura desta sala, então, se soubermos a priori que este sistema é ergódico, basta medir a temperatura ao longo da órbita de uma partíciula típica.

Para sistemas que preservam o volume (ditos *conservativos*), a ergodicidade de sistemas hiperbólicos foi provada por Anosov. Porém, como os sistemas hiperbólicos não são "maioria", Pugh e Shub fizeram a seguinte conjectura:

Um pouco de hiperbolicidade é suficiente para garantir ergodicidade.

No capítulo 3 provaremos que a conjectura de Pugh-Shub é verdadeira, onde um pouco de hiperbolicidade significará hiperbolicidade parcial, ou seja, podem existir direções sem expansão ou contração, mas isto ocorre de modo controlado (i.e., dominado).

1.3 A propriedade de Bernoulli

Como já observamos, o conceito de ergodicidade aparece em dinâmica caótica por simplificar o estudo deste sistemas, dado que ele nos diz que todo sistema pode ser aproximado por órbitas de pontos típicos. Entretanto, voltando ao exemplo da medição da temperatura da sala (veja seção 1.1), por razões práticas, não é apenas suficiente saber que a temperatura da sala é aproximada pela órbita genérica, mas também saber *com que velocidade isto ocorre.*

Quanto a isso, existem vários conceitos ligados a este problema, sendo um dos mais famosos a propriedade de ser *Bernoulli*.

Em poucas palavras, um sistema é dito Bernoulli se for equivalente a um shift de Bernoulli, i.e., a um processo aleatório independente identicamente distribuído, por exemplo, um lançamento de uma moeda (cara ou coroa).

Como analisar um lançamento de moeda é relativamente fácil do ponto de vista probabilístico, saber que um certo sistema é Bernoulli é uma informação útil.

Atualmente, é conhecido que sistemas hiperbólicos são Bernoulli. Entretanto, os exemplos físicos de bilhares (que modelam iterações de partículas de um gás), que são não hiperbólicos (mas são parcialmente hierbólicos, i.e., estão próximos da hiperbolicidade), não são cobertos pelos teoremas clássicos.

Então, o objetivo do capítulo 4 será mostrar que alguma hiperbolicidade garante a propriedade de Bernoulli.

1.4 A dimensão de Hausdorff do conjunto de pontos de curvatura zero

No estudo de hipersuperfícies do espaço euclidiano, o conceito de curvatura certamente é importante. Em particular, saber se o conjunto de pontos de curvatura zero C de uma dada imersão é "pequeno" ou "grande" é uma pergunta pertinente. Aqui o conceito de ser pequeno ou grande pode variar. por exemplo, podemos dizer que C é pequeno se está contido numa união de subvariedades de codimensão alta.

Este conceito foi utilizado por Barbosa-Fukuoka-Mercuri e do Carmo-Elbert para obter caracterizações topólogicas de imersões com conjuntos pequenos de pontos de curvatura zero. Entretanto, como foi conjecturado por do Carmo-Santos (e provado posteriormente por Arbieto-Matheus), o conjunto de pontos de curvatura zero nem sempre é bem-comportado (ou seja, pode ser *fractal*).

Neste caso, para definir o que significa ser pequeno, precisamos utilizar um conceito familiar e recorrente na teoria dos sistemas dinâmicos, a chamada dimensão de Hausdorff.

A partir deste novo conceito de pequenez, o que provaremos no capítulo 5 é que se M^n é uma variedade imersa em $\mathbb{R}^{n+1}$ tal que o conjunto de pontos de curvatura zero (ou seja, os pontos singulares da aplicação normal de Gauss) é pequeno (tem dimensão de Hausdorff baixa), então M^n é topologicamente a esfera S^n. Mais ainda, se a hipersuperfície tem tipo geométrico finito, ela é a esfera menos um número finito de pontos. Em particular, provaremos uma teorema de caracterização do $2m$-catenóide como a única hipersuperfície *mínima* com tipo geométrico finito e com conjunto de pontos de curvatura zero pequeno.

Capítulo 2

Equilibrium States for Random Non-uniformly Expanding Maps

Os resultados abaixo foram obtidos em conjunto com Alexander Arbieto e Krerley Oliveira no artigo *Equilibrium States for Random Non-uniformly Expanding Maps* o qual foi publicado em *Nonlinearity, vol.17, n.2, 581–593 (2004).* Abaixo segue o conteúdo (em inglês) deste artigo.

2.1 Introduction

Particles systems, as they appear in kinetic theory of gases, have been an important model motivating much development in the field of Dynamical Sytems and Ergodic Theory. While these are deterministic systems, ruled by Hamiltonian dynamics, the evolution law is too complicated, given the huge number of particles involved. Instead, one uses a stochastic approach to such systems.

More generally, ideais from statistical mechanics have been brought to the setting of dynamical systems, both discrete-time and continuous-time, by Sinai, Ruelle, Bowen, leading to a beautiful and very complete theory of equilibrium states for uniformly hyperbolic diffeomorphisms and flows. In a few words, equilibrium states are invariant probabilities in the phase space which maximaze a certain variational principle (corresponding to the Gibbs free energy in the statistical mechanics context). The theory of Sinai-Ruelle-

Bowen gives that for uniformly hyperbolic systems equilibrium states exist, and they are unique if the system is topologically transitive and the potential is Hölder continuous.

Several authors have worked on extending this theory beyond the uniformly hyperbolic case. See e.g. [5], [13], among other important authors. Our present work is more directly motivated by the results of Oliveira [12] where he constructed equilibrium states associated to potentials with not-too-large variation, for a robust (C^1-open) class of non-uniformly expanding maps introduced by Alves-Bonatti-Viana [2].

On the other hand, corresponding problems have been studied also in the context of the theory of random maps, which was much developed by Kifer [6] and Arnold [3], among other mathematicians. Indeed, Kifer [6] proved the existence of equilibrium states for random uniformly exapnding systems, and Liu [8] extended this to uniformly hyperbolic systems.

In the present work, we combine these two approaches to give a construction of equilibrium states for non-uniformly hyperbolic maps. In fact, some attempts to show the existence of equilibrium states beyond uniform hyperbolicity were made by Khanin-Kifer [7]. However, our point of view is quite different. Before stating the main result, we recall that a random map is a continuous map $f : \Omega \to C^r(M, M)$ where M is a compact manifold Ω is a Polish space (i.e., a separable complete metric space), and $T : \Omega \to \Omega$ a measurably invertible continuous map with an invariant ergodic measure $\mathbb{P}$. The main result is the following :

"For a C^2-open set $\mathcal{F}$ of non-uniformly expanding local diffeomorphisms, potentials ϕ with low variation *and $f : \Omega \to \mathcal{F}$, there are equilibrium states for the random system associated to f and T. In particular, f admits measures with maximal entropy."*

A potential has low variation if it is not far from being constant. See the precise definition in section 3. In particular, constant functions have low variation; their equilibrium states are measures of maximal entropy.

The proof, which we present in the next sections extends ideias from Alves-Araújo [1], Alves-Bonatti-Viana [2] and Oliveira [12].

It is very natural to ask whether these equilibrium states we construct are unique and whether they are (weak) Gibbs states. Another very interesting question is whether existence (and uniqueness) of equilibrium states extends to (random or deterministic) non-uniformly hyperbolic maps with singularities, such as the Viana maps [1]. Although our present methods do not solve these questions, we believe the answers are affirmative.

2.2 Definitions

Random Transformations and Invariant Measures

Let M^l be a compact l-dimensional Riemannian manifold and $\mathcal{D}$ the space of C^2 local diffeomorphisms of M. Let $(\Omega, T, \mathbb{P})$ a measure preserving system, where $T : \Omega \to \Omega$ is $\mathbb{P}$-invariant ($\mathbb{P}$ is a Borel measure) and Ω is a Polish space, i.e., Ω is a complete separable metric space. By a random transformation we understand a continuous map $f : \Omega \to \mathcal{D}$. Then we define:

$$f^n(w) = f(T^{n-1}(w)) \circ \cdots \circ f(w), \ f^{-n}(w) = (f^n(w))^{-1}. \tag{2.1}$$

We also define the skew-product generated by f:

$$F : \Omega \times M \to \Omega \times M, \ F(w, x) = (Tw, f(w)x).$$

We denote $\mathcal{P}(\Omega \times M)$ the space of probability measures μ on $\Omega \times M$ such that the marginal of μ on Ω is $\mathbb{P}$. Let $\mathcal{M}(\Omega \times M) \subset \mathcal{P}(\Omega \times M)$ be the measures μ which are F-invariant.

Because M is compact, invariant measures always exists and the property of $\mathbb{P}$ be the marginal on Ω of a invariant measures can be characterized by its disintegration:

$$d\mu(w, x) = d\mu_w(x) d\mathbb{P}(w).$$

μ_w are called *samples measures* of μ (see [9], [10]).

An invariant measure is called *ergodic* if (F, μ) is ergodic, the set of all ergodic measures is denoted by $\mathcal{M}_e(\Omega \times M)$. Furthermore, each invariant measure can be decomposed into its ergodic components by integration when the σ-algebra on Ω is countably generated and $\mathbb{P}$ is ergodic.

In what follows, as usual, **we always assume** $(\Omega, \mathcal{A}, \mathbb{P})$ **is a Lebesgue space,** $(T, \mathbb{P})$ **is ergodic and** T **is measurably invertible and continuous**. Observe that these assumptions are satified in the canonical case of left-shift operators τ, Ω being $C^r(M, M)^{\mathbb{N}}$ or $C^r(M, M)^{\mathbb{Z}}$.

Entropy

We follow Liu [9] on the definition of the Kolmogorov-Sinai entropy for random transformations:

Let μ be an F-invariant measure as above. Let ξ be a finite Borel partition of M. We set:

$$h_\mu(f, \xi) = \lim_{n \to +\infty} \frac{1}{n} \int H_{\mu_w}(\vee_{k=0}^{n-1} f^{-k}(w)\xi) d\mathbb{P}(w), \tag{2.2}$$

where $H_\nu(\eta) := -\sum_{C\in\eta} \nu(C)\log\nu(C)$ (and $0\log 0 = 0$), for a finite partition η and ν a probability on M (and μ_w are the sample measures of μ).

Definition 2.2.1. The entropy of (f,μ) is:

$$h_\mu(f) := \sup_\xi h_\mu(f,\xi)$$

with the supremum taken over all finite Borel partitions of M.

Definition 2.2.2. The *topological entropy* of f is $h_{top}(f) = \sup_\mu h_\mu(f)$

Theorem 2.2.3 ("Random" Kolmogorov-Sinai theorem)**.** *If $\mathcal{B}$ is the Borel σ-algebra of M and ξ is a generating partition of M, i.e.,*

$$\bigvee_{k=0}^{+\infty} f^{-k}(w)\ \xi = \mathcal{B} \quad \textit{for } \mathbb{P} - a.e.\ w,$$

then

$$h_\mu(f) = h_\mu(f,\xi).$$

For a proof of this theorem see [10] or [4].

Lyapunov Exponents

Let μ an F-invariant measure as before. The Oseledet's Theorem says that under an integrability condition:

$$\int_{\Omega\times M} \log^+ |D_x f(w)| < \infty.$$

The following quantities called the *Lyapunov exponents* are well defined for μ almost every point. More precisely, for almost every point there exists a decomposition $T_xM = E^1(x,w)\oplus\cdots\oplus E^{\tau(x,w)}(x,w)$ and numbers $\lambda^i(x,w)$ such that for every $v \in E^i(x,w)\backslash\{0\}$ we have:

$$\lim_{n\to+\infty} \frac{1}{n}\log|Df^n(w)(x).v| = \lambda^i(x,w).$$

Also all of these functions are measurable with respect to μ and invariant by F. So if the measure is ergodic these quantities are constant almost everywhere. For more details on this (and some others) properties of the Lyapounov exponents for random dynamical systems, see [9], for instance.

Equilibrium States

Let $L^1(\Omega, C(M))$ the set of all families $\{\phi = \{\phi_w \in C^0(M)\}\}$ such that the map $(w,x) \to \phi_w(x)$ is a measurable map and $\|\phi\|_1 := \int_\Omega |\phi_w|_\infty d\mathbb{P}(w) < +\infty$.

For a $\phi \in L^1(\Omega, C(M))$, $\varepsilon > 0$ and $n \geq 1$, we define:

$$\pi_f(\phi)(w,n,\varepsilon) = \sup\{\sum_{x\in K} e^{S_f(\phi)(w,n,x)}; K \text{ is a } (n,\varepsilon) - \text{separated set}\},$$

where $S_f(\phi)(w,n,x) := \sum_{k=0}^{n-1} \phi_{T^k(w)}(f^k(w)x)$ and a set K is called (n,ε)-separeted if for any $x, y \in K$, the distance $d(f^k(w)(x), f^k(w)(y))$ of the points $f^k(w)(x)$ and $f^k(w)(y)$ is at least ε, for all $0 \leq k \leq n$.

Definition 2.2.4. The map $\pi_f : L^1(\Omega, C(M)) \to R \cup \{\infty\}$ given by:

$$\pi_f(\phi) = \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \int_\Omega \log \pi_f(\phi)(w,n,\varepsilon) d\mathbb{P}(w).$$

is called *the pressure map.*

It is well know that the variational principle occurs (see [9]):

Theorem 2.2.5. *If Ω is a Lebesgue space, then for any $\phi \in L^1(\Omega, C(M))$ we have:*

$$\pi_f(\phi) = \sup_{\mu\in\mathcal{M}(\Omega\times M)} \{h_\mu(f) + \int \phi d\mu\} \tag{2.3}$$

Remark 2.2.6. If $\mathbb{P}$ is ergodic then we can take the supremum over the set of ergodic measures (see [9]).

Definition 2.2.7. A measure $\mu \in \mathcal{M}(\Omega \times M)$ is an equilibrium state for f, if μ attains the supremum of (2.3).

Physical Measures

As in the deterministic case, we follow [1] on the definition of *physical measure* in the context of random transformations :

Definition 2.2.8. A measure μ is a physical measure if for positive Lebesgue measure set of points $x \in M$ (called the *basin* $B(\mu)$ of μ),

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n-1} \phi(f^j(w)(x)) = \int \phi d\mu \text{ for all continuous } \phi : M \to \mathbb{R}. \tag{2.4}$$

for $\mathbb{P}$-ae w.

2.3 Statement of the results

Before starting abstract definitions, we comment that in next section, it is showed that there are examples of random transformations satisfying our hypothesis below.

We say that a local diffeomorphism f of M is in $\widetilde{\mathcal{F}}$ if f is in $\mathcal{D}$ and satisfies, for positive constants δ_0, β, δ_1, σ_1, and p, $q \in N$, the following properties :

(H1) There exists a covering $B_1, \dots, B_p, \dots, B_{p+q}$ of M such that every $f|B_i$ is injective and

- f is uniformly expanding at every $x \in B_1 \cup \dots \cup B_p$:
$$\|Df(x)^{-1}\| \le (1+\delta_1)^{-1}.$$
- f is never too contracting: $\|Df(x)^{-1}\| \le (1+\delta_0)$ for every $x \in M$.

(H2) f is everywhere volume-expanding: $|\det Df(x)| \ge \sigma_1$ with $\sigma_1 > q$.

(H3) There is an open set V of M such that $V \supset \{x \in M; \|Df(x)^{-1}\| > (1+\delta_1)^{-1}\}$ and an open set $W \subset B_{p+1} \cup \dots \cup B_{p+q}$ containing V such that
$$M_1 > m_2 \quad \text{and} \quad m_2 - m_1 < \beta$$
where m_1 and m_2 are the infimum and the supremum of $|\det Df|$ on V, respectively, and M_1 and M_2 are the infimum and the supremum of $|\det Df|$ on W^c, respectively. In particular, this condition means that the volume expansion in the "bad" region V is not too different from the volume expansion in the "good" region W^c.

This kind of transformations was considered by [2], [12], [1], where they construct C^1-open sets of such maps.

We will consider a subset $\mathcal{F} \subset \widetilde{\mathcal{F}}$ such that :

(C1) There is a uniform constant A_0 s.t. $|\log \|f\|_{C^2}| \leq A_0$ for any $f \in \mathcal{F}$ and the constants m_1, m_2, M_1, M_2 are uniform on $\mathcal{F}$;

From now on, our random transformations will be given by a map $F : \Omega \to \mathcal{F}$, and f satisfies the following condition :

(C2) f admits an ergodic absolutely continuous *physical measure* $\mu_{\mathbb{P}}$ (see section 2).

Remark 2.3.1. We will show in the appendix that $(H1), (H2)$ implies the following property:

(F1) There exists some $\gamma_0 = \gamma_0(\delta_1, \sigma_1, p, q) < 1$ such that the random orbits of Lebesgue almost every point spends at most a fraction of time $\gamma_0 < 1$ inside $B_{p+1} \cup \cdots \cup B_{p+q}$, depending only on σ_1, p, q. I.e., for $\mathbb{P}$-a.e. w and Lebesgue almost every x

$$\lim_{n\to\infty} \frac{\#\{0 \leq j \leq n-1 : f^j(w)(x) \in B_{p+1} \cup \cdots \cup B_{p+q}\}}{n} \leq \gamma_0.$$

Then we analyse the existence of an equilibrium state for low-variation potentials:

Definition 2.3.2. A potential $\phi \in L^1(\Omega, C(M))$ has ρ_0-low variation if

$$\|\phi\|_1 < \pi_f(\phi) - \rho_0 h_{top}(f). \tag{2.5}$$

Remark 2.3.3. We call ϕ above a ρ_0-low variation potential because in the deterministic case (i.e., $\phi(w,x) = \phi(x)$), if $\max \phi - \min \phi < (1 - \rho_0) h_{top}(f)$ then ϕ satisfies (2.5).

The main result is :

Theorem A. *Assume hypotheses (H1), (H2), (H3) hold, with δ_0 and β sufficiently small and assume also conditions (C1), (C2). Then, there exists ρ_0 such that if ϕ is a continuous potential with ρ_0-low variation then ϕ has* *some equilibrium state. Moreover, these equilibrium states are hyperbolic measures, with all Lyapunov exponents bigger than some* $c = c(\delta_1, \sigma_1, p, q) > 0$.

As pointed out in the introduction, an interesting question related to the theorem A is the uniqueness of equilibrium states, and if they are (weak) Gibbs measures. In the deterministic context, Oliveira [12] obtained this results using the Perron-Frobenius operator, a semi-conjugacy with a shift in a symbolic space and using a weak Gibbs property. Although a work in progress by the authors says that assuming that the partition B_i is transitive (in some sense), then the equilibrium states are unique, the main difficult is that Oliveira [12] uses the Brin-Katok formula in the proof of uniqueness of equilibrium states for deterministic systems, but this formula still unknown for the random case (see [9, page 1289] for more details).

2.4 Examples

In this section we exhibit a C^1-open class of C^2-diffeomorphism which are contained in $\widetilde{\mathcal{F}}$. To start the construction, we now follow [12] *ipsis-literis* and construct examples of 'deterministic' non-uniformly expanding maps. After this, we construct the desired random non-uniformly expanding maps in $\mathcal{F}$ a C^2-neighborhood of a fixed diffeomorphism of $\widetilde{\mathcal{F}}$.

We observe that the class $\mathcal{F}$ contains an open set of non-uniformly expanding which *are not uniformly expanding*.

We start by considering any Riemannian manifold that supports an expanding map $g : M \to M$. For simplicity, choose $M = \mathbb{T}^n$ the n-dimensional torus, and g an endomorphism induced from a linear map with eigenvalues $\lambda_n > \cdots > \lambda_1 > 1$. Denote by $E_i(x)$ the eigenspace associated to the eigenvalue λ_i in T_xM.

Since g is an expanding map, g admits a transitive Markov partition $R_1, \ldots, R_d$ with arbitrary small diameter. We may suppose that $g|R_i$ is injective for every $i = 1, \ldots, d$. Replacing by an iterate if necessary, we may suppose that there exists a fixed point p_0 of g and, renumbering if necessary, this point is contained in the interior of the rectangle R_d of the Markov partition.

Considering a small neighborhood $W \subset R_d$ of p_0 we deform g inside W along the direction E_1. This deformation consists essentially in rescaling the expansion along the invariant manifold associated to E_1 by a real function

α. Let us be more precise:

Considering W small, we may identify W with a neighborhood of 0 in $\mathbb{R}^n$ and p_0 with 0. Without loss of generality, suppose that $W = (-2\epsilon, 2\epsilon) \times B_{3r}(0)$, where $B_{3r}(0)$ is the ball or radius $3r$ and center 0 in $\mathbb{R}^{n-1}$. Consider a function $\alpha : (-2\epsilon, 2\epsilon) \to \mathbb{R}$ such $\alpha(x) = \lambda_1 x$ for every $|x| \geq \epsilon$ and for small constants γ_1, γ_2:

1. $(1+\gamma_1)^{-1} < \alpha'(x) < \lambda_1 + \gamma_2$
2. $\alpha'(x) < 1$ for every $x \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$;
3. α is C^0-close to λ_1: $\sup\limits_{x\in(-\epsilon,\epsilon)} |\alpha(x) - \lambda_1 x| < \gamma_2$,

Also, we consider a bump function $\theta : B_{3r}(0) \to \mathbb{R}$ such $\theta(x) = 0$ for every $2r \leq |x| \leq 3r$ and $\theta(x) = 1$ for every $0 \leq |x| \leq r$. Suppose that $\|\theta'(x)\| \leq C$ for every $x \in B_{3r}(0)$. Considering coordinates $(x_1, \ldots, x_n)$ such that $\partial_{x_i} \in E_i$, define f_0 by:

$$f_0(x_1, \ldots, x_n) = (\lambda_1 x_1 + \theta(x_2, \ldots, x_n)(\alpha(x_1) - \lambda_1 x_1), \lambda_2 x_2, \ldots, \lambda_n x_n)$$

Observe that by the definition of θ and α we can extend f_0 smoothly to $\mathbb{T}^n$ as $f_0 = g$ outside W. Now, is not difficult to prove that f_0 satisfies the conditions (H1), (H2), (H3) above.

First, we have that $\|Df_0(x)^{-1}\|^{-1} \geq \min\limits_{i=1,\ldots,n} \|\partial_{x_i} f_0\|$. Observe that:

$$\partial_{x_1} f_0(x_1, \ldots, x_n) = (\alpha'(x_1)\theta(x_2, \ldots, x_n) + (1 - \theta(x_2, \ldots, x_n))\lambda_1, 0, \ldots, 0)$$

$$\partial_{x_i} f_0(x_1, \ldots, x_n) = ((\alpha(x_1) - \lambda_1)\partial_{x_i}\theta(x_2, \ldots, x_n), 0, \ldots, \lambda_i, 0, \ldots, 0), \text{ for } i \geq 2.$$

Then, since $\|\partial_{x_i}\theta(x)\| \leq C$ for every $x \in B_{3r}(0)$, and $\alpha(x_1) - \lambda_1 x_1 \leq \gamma_2$ we have that $\|\partial_{x_i} f_0\| > (\lambda_i - \gamma_2 C)$ for every $i = 2, \ldots, n$. Moreover, by condition 1, $\|\partial_{x_1} f_0\| \leq \max\{\alpha'(x_1), \lambda_1\} \leq \lambda_1 + \gamma_2$, if we choose γ_2 small in such way that $\lambda_2 - \gamma_2 C > \lambda_1 + \gamma_2$ then:

$$\|\partial_{x_i} f_0\| > \|\partial_{x_1} f_0\|, \text{ for every } i \geq 2.$$

Notice also that $\|\partial_{x_1} f_0\| \geq \min\{\alpha'(x_1), \lambda_1\} \geq (1+\gamma_1)^{-1}$. This prove that:

$$\|Df_0(x)^{-1}\|^{-1} \geq \min_{i=1,\dots,n} \|\partial_{x_i} f_0\| \ (1+\gamma_1)^{-1}.$$

Since f coincides with g outside W, we have $\|Df_0(x)^{-1}\| \leq \lambda_1^{-1}$ for every $x \in W^c$. Together with the above inequality, this proves condition (H1), with $\delta_0 = \gamma_1$.

Choosing γ_1 small and $p = d-1$, $q = 1$, $B_i = R_i$ for every $i = 1, \dots, d$, condition (H2) is immediate. Indeed, observe that the Jacobian of f_0 is given by the formula:

$$\det Df_0(x) = (\alpha'(x_1)\theta(x_2,\dots,x_n) + (1-\theta(x_2,\dots,x_n))\lambda_1) \prod_{i=2}^{n} \lambda_i.$$

Then, if we choose $\gamma_1 < \prod_{i=2}^{n} \lambda_i - 1$:

$$\det Df_0(x) > (1+\gamma_1)^{-1} \prod_{i=2}^{n} \lambda_i > 1.$$

Therefore, we may take $\sigma_1 = (1+\gamma_1)^{-1} \prod_{i=2}^{n} \lambda_i > 1$.

To verify property (H3) for f_0, observe that if we denote by

$$V = \{x \in M; \|Df_0(x)^{-1}\| > (1+\delta_1)^{-1}\},$$

with $\delta_1 < \lambda_1 - 1$, then $V \subset W$. Indeed, since $\alpha(x_1)$ is constant equal to $\lambda_1 x_1$ outside W we have that $\|Df_0(x)^{-1}\| \leq \lambda_1^{-1} < (1+\delta_1)^{-1}$, for every $x \in W^c$. Given γ_3 close to 0, we may choose δ_1 close to 0 and α satisfying the conditions above in such way that,

$$\sup_{x,y \in V} \alpha'(x_1) - \alpha'(y_1) < \gamma_3.$$

If m_1 and m_2 are the infimum and the supremum of $|\det Df_0|$ on V, respectively,

$$m_2 - m_1 \leq C(\sup_{x,y \in V} \alpha'(x_1) - \alpha'(y_1)) < \gamma_3 C,$$

where $C = \prod_{i=2}^{n} \lambda_i$. Then, we may take $\beta = \gamma_3 C$ in (H3). If M_1 is the infimum of $|\det Df_0|$ on W^c, $M_1 > m_2$, since $\lambda_1 > (1+\delta_1) \geq \sup_{x \in V} \alpha'(x)$.

The arguments above show that the hypotheses $(H1), (H2), (H3)$ are satisfied by f_0. Moreover, if we one takes $\alpha(0) = 0$, then p_0 is fixed point for f_0, which is not a repeller, since $\alpha'(0) < 1$. Therefore, f_0 is not a uniformly expanding map.

It is not difficult to see that this construction may be carried out in such way that f_0 does not satisfy the expansiveness property: there is a fixed hyperbolic saddle point p_0 such that the stable manifold of p_0 is contained in the unstable manifold of two other fixed points.

Now, if $\mathcal{F}$ denotes a small C^2-neighborhood of f_0 in $\widetilde{\mathcal{F}}$, and $H : \Omega \to \widetilde{\mathcal{F}}$ is a continuous map, Alves-Araújo [1] shows that if $w^* \in \Omega$ is such that $F(w^*) = f_0$ and θ_ε is a sequence of measures, $\text{supp}(\theta_\varepsilon) \to \{w_0\}$ then for small $\varepsilon > 0$ there are physical measures for the RDS $F : \Omega^{\mathbb{Z}} \to \mathcal{F}$, $F(\ldots, w_{-k}, \ldots, w_0, \ldots, w_k, \ldots) = H(w_0)$. This concludes the construction of examples satisfying $(H1), (H2), (H3), (C1), (C2)$.

2.5 Non-uniformly expanding measures and hyperbolic times

We now precise the conditions on δ_0 and β. We consider γ_0 given in condition $(F1)$. By condition $(C1)$, there exists $\epsilon_0 > 0$ s.t. for any $\eta \in B_{\epsilon_0}(\xi)$ and $\mathbb{P}$-a.e. w holds :

$$\frac{||Df(w)^{-1}(\xi)||}{||Df(w)^{-1}(\eta)||} \leq e^{\frac{c}{2}},$$

where c is such that for some $\alpha > \gamma_0$, we have $(1+\delta_0)^\alpha(1+\delta_1)^{-(1-\alpha)} < e^{-2c} < 1$ and $\alpha m_2 + (1-\alpha)M_2 < \gamma_0 m_1 + (1-\gamma_0)M_1 - l\log(1+\delta_0)$ $(l := \dim(M))$, if δ_0 and β are sufficiently small. Now, the constants fixed above allows us to prove good properties for the objects defined below, which are of fundamental interest in the proof of theorem A.

Definition 2.5.1. We say that a measure $\nu \in \mathcal{M}(\Omega \times M)$ is *non-uniformly expanding with exponent* c if for ν-almost every $(w, x) \in \Omega \times M$ we have:

$$\lambda(w, x) = \limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(T^j(w))(f^j(w)(x))^{-1}\| \leq -2c < 0.$$

Definition 2.5.2. We say that n is a *c-hyperbolic time* for (w, x), if for every $1 \leq k \leq n$:

$$\prod_{j=n-k}^{n-1} \|Df(T^{j+1}(w))(f^j(w)(x))^{-1}\| \leq e^{-ck}.$$

As in lemma 3.1 of [2], lemma 4.8 of [12] and lemma 2.2 of [1], we have infinity many hyperbolic times for expanding measures. For this we need a lemma due to Pliss (see [2]).

Lemma 2.5.3. *Let $A \geq c_2 > c_1 > 0$ and $\zeta = \frac{c_2 - c_1}{A - c_1}$. Given real numbers $a_1, \cdots, a_N$ satisfying:*

$$\sum_{j=1}^{N} a_j \geq c_2 N \text{ and } a_j \leq H \text{ for all } 1 \leq j \leq,$$

there are $l > \zeta N$ and $1 < n_1 < \cdots < n_l \leq N$ such that:

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_1 - n) \text{ for each } 0 \leq n < n_i, \; i = 1, \cdots, l.$$

Lemma 2.5.4. *For every invariant measure ν with exponent c, there exists a full ν-measure set $H \subset \Omega \times M$ such that every $(w, x) \in H$ has infinitely many c-hyperbolic times $n_i = n_i(w, x)$ and, in fact, the density of hyperbolic times at infinity is larger than some $d_0 = d_0(c) > 0$:*

1. $\displaystyle\prod_{j=n-k}^{n-1} \|Df(T^{j+1}(w))(f^j(w)(x))^{-1}\| \leq e^{-cj}$ *for every* $1 \leq k \leq n_i$

2. $\displaystyle\liminf_{n \to \infty} \frac{\sharp\{0 \leq n_i \leq n\}}{n} \geq d_0 > 0.$

Demonstração. Let $H \subset \Omega \times M$ with full ν-measure. For any $(w, x) \in H$ and n large enough, we have:

$$\sum_{j=0}^{n-1} \log \|Df(T^j(w))(f^j(w)(x))^{-1}\| \leq -\frac{3c}{2} n$$

Now, by $(C1)$ we can apply lemma 2.5.3 with $A = \sup_{(w,x)} (-\log ||Df(w)^{-1}(x)||)$, $c_1 = c$, $c_2 = \frac{3c}{2}$ and $a_i = -\log \|Df(T^j(w))(f^j(w)(x))^{-1}\|$ and the statement follows. □

Lemma 2.5.5. *$\exists\ \varepsilon_0 > 0$ such that for $\mathbb{P}$-a.e. w, if n_i is a hyperbolic time of (w,x) then there exists a neighborhood V_w around x satisfying that $f^{n_i}(w)$ maps V_w diffeomorphically onto the ball $B_{\varepsilon_0}(f^{n_i}(w)(x))$ such that if $f^{n_i}(w)(z) \in B_{\epsilon_0}(f^{n_i}(w)(x))$ and $z \in V$ then*

$$d(f^{n_i-j}(w)(z), f^{n_i-j}(w)(x)) \le e^{\frac{-cj}{2}} \cdot d(f^{n_i}(w)(z), f^{n_i}(w)(x)),$$

$\forall 1 \le j \le n_i$.

Demonstração. By $(C1)$ we know that there exists $\varepsilon_0 > 0$ such that for any $\eta \in B_{\varepsilon_0}(\xi)$ we have:

$$\frac{||Df(w)^{-1}(\xi)||}{||Df(w)^{-1}(\eta)||} \le e^{\frac{c}{2}} \text{ for } \mathbb{P}\text{-ae } w.$$

In fact, this hold in the T-orbit of w $\mathbb{P}$-ae. Indeed, let $C = \{w; \frac{||Df(w)^{-1}(\xi)||}{||Df(w)^{-1}(\eta)||} \le e^{\frac{c}{2}}\}$ for any ξ and $\eta \in B_{\varepsilon_0}(\xi)$, then $\bigcap T^j(C)$ has full measure and the estimate follows. Because $f^{n_i}(w)(z) \in B_{\epsilon_0}(f^{n_i}(w)(x))$, by the estimative above, we have that w $\mathbb{P}$-ae if we take the inverse branch of $f^{n_i}(w)$ which sends $f^{n_i}(w)(x)$ to $f^{n_i-1}(w)(x)$ (restricted to $B_{\varepsilon_0}(f^{n_i}(w)(x))$) and has derivative with norm less than $e^{-\frac{c}{2}}$, then we have $d(f^{n_i-1}(w)(z), f^{n_i-1}(w)(x)) \le \varepsilon_0$. Using the estimate along the orbit (and induction), we construct the neighborhood V as the successive images of the ball $B_{\varepsilon_0}(f^{n_i}(w,x))$ by the inverses branchs consider before and we have:

$$\prod_{j=n-k}^{n-1} \|Df(T^{j+1}(w))(f^j(w)(z))^{-1}\| \le e^{-\frac{ck}{2}} \text{ for all } 0 \le k \le n_i.$$

The statement follows. □

2.6 Proof of theorem A

Now we define a set of measures where the "bad set" V has small measure.

Definition 2.6.1. We define the convex set K_α by

$$K_\alpha = \{\mu : \mu(\Omega \times V) \le \alpha\}$$

Lemma 2.6.2. *$K_\alpha \neq \emptyset$ is a compact set.*

Demonstração. Let $\{\mu_n\} \subset K_\alpha$. By compacity, we can assume that $\mu_n \to \mu$. Since V is open then $\mu(\Omega \times V) \leq \liminf(\mu_n)(\Omega \times V) \leq \alpha$. This implies compacity. The physical measure given by condition (C2) (see equation (4)) is in K_α, because *Leb*-a.e. random orbit stay at most $\gamma_0 < \alpha$ inside V (by $(F1)$). By definition of physical measure (limit of average of Dirac measures supported on random orbits) and the absolute continuity with respect to the Lebesgue measure, $\mu_w(V) \leq \alpha$ for w $\mathbb{P}-$ a.e. holds. In particular, $\mu(\Omega \times V) \leq \alpha$. □

We recall that the ergodic decomposition theorem holds for RDS. With this in mind, we distinguish a set $\mathcal{K} \subset K_\alpha$:

Definition 2.6.3. $\mathcal{K} = \{\mu : \mu_{(w,x)} \in K_\alpha$ for $\mu - a.e.(w,x)\}$ ($\mu_{(w,x)}$ is the ergodic decomposition of μ).

Lemma 2.6.4. *Every measure $\mu \in \mathcal{K}$ is f-expanding with exponent c :*

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(T^j(w))(f^j(w)(x))^{-1}\| \leq -2c$$

for μ-a.e. $(w,x) \in M$.

Demonstração. We assume first that μ is ergodic. By definition of K_α, we have $\mu(\Omega \times V) \leq \alpha$. But Birkhoff's Ergodic Theorem applied to (F,μ) says that in the random orbit of (w,x) $\mu-$a.e. we have:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_V(f^i(w)(x)) \leq \alpha.$$

Now, we use hypothesis (H1): $\|Df(w,y)^{-1}\| \leq (1+\delta_0)$ for any $y \in V$ and $\|Df(w,y)^{-1}\| \leq (1+\delta_1)^{-1}$ for any $y \in V^c$, obtaining:

$$\frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(T^j(w))(f^j(w)(x))^{-1}\| \leq \log[(1+\delta_0)^\alpha (1+\delta_1)^{1-\alpha}] \leq -2c < 0$$

$(w,x) - \mu-$a.e.

In the general case we use the ergodic decomposition theorem (see [12] and [10]). □

Entropy lemmas

Definition 2.6.5. Given $\varepsilon > 0$, we define :

$$A_\varepsilon(w,x) = \{y : d(f^n(w)(x), f^n(w)(y)) \leq \varepsilon \text{ for every } n \geq 0\}.$$

Lemma 2.6.6. *Suppose that $\mu \in \mathcal{K}$ is ergodic and let ε_0 given by lemma 2.5.5. Then, for $\mathbb{P}$-almost every w and any $\varepsilon < \varepsilon_0$,*

$$A_\varepsilon(w,x) = x.$$

Demonstração. By lemma 2.5.4 we have infinity hyperbolic times $n_i = n_i(w,x)$ for $(w,x) \in H$ (where $\mu(H) = 1$). For each w set $H_w = \{x; (w,x) \in H\}$, then $\mathbb{P}$-a.e. w we have $\mu_w(H_w) = 1$ and infinity hyperbolic times for μ_w-a.e. x. Now, by lemma 2.5.5, if $z \in A_\varepsilon(w,x)$ with $\varepsilon < \varepsilon_0$ we have:

$$d(x,z) \leq e^{-\frac{cn_i}{2}} d(f^{n_i}(w)(x), f^{n_i}(w)(z)) \leq e^{-\frac{cn_i}{2}} \varepsilon.$$

The lemma follows. □

Let $\mathcal{P}$ be a partition of M in measurable sets with diameter less than ε_0. From the above lemma, we get :

Lemma 2.6.7. *Let $\mathcal{P}$ be a partition of M in measurable sets with diameter less that ε_0. Then, $\mathcal{P}$ is a generating partition for every $\mu \in \mathcal{K}$.*

Demonstração. As usual we will write:

$$\mathcal{P}_w^n = \{\mathcal{C}_w^n = (\mathcal{P}_w)_{i_0} \cap \cdots \cap f^{-(n-1)}(w)(\mathcal{P}_w)_{i_{n-1}})\} \text{ for each } n \geq 1,$$

where $(\mathcal{P}_w)_{i_k}$ is an element of the partition $\mathcal{P}$. By the previous lemma, we know that for $\mathbb{P}$-a.e. w, we have $A_\varepsilon(w,x) = x$ for x μ_w-a.e. Let A a measurable set of M and $\delta > 0$. Take $K_1 \subset A$ and $K_2 \subset A^c$ two compact sets such that $\mu_w(K_1 \triangle A) \leq \delta$ and $\mu_w(K_2 \triangle A^c) \leq \delta$. Now if $r = d(K_1, K_2)$, the previous lemma says that if n is big enough then $diam \mathcal{P}_w^n(x) \leq \frac{r}{2}$ for x in a set of μ_w-measure bigger than $1 - \delta$. The sets $(\mathcal{C}_w^n)_1, \cdots, (\mathcal{C}_w^n)_k$ that intersects K_1 satisfy:

$$\begin{aligned} \mu(\bigcup (C_w^n)_i \Delta A) &= \mu(\bigcup (C_w^n)_i - A) + \mu(A - \bigcup (C_w^n)_i) \\ &\leq \mu(A - K_1) + \mu(A^c - K_2) + \delta \leq 3\delta. \end{aligned}$$

This end the proof.

□

Corollary 2.6.8. *For every $\mu \in \mathcal{K}$, $h_\mu(f) = h_\mu(f, \mathcal{P})$*

Demonstração. The result follows from lemma 2.6.7 and the theorem 2.2.3. □

Lemma 2.6.9. *The map $\mu \to h_\mu(f, \mathcal{P})$ is upper semi-continuous at μ_0 measure s.t. $(\mu_0)_w(\partial P) = 0$ for $\mathbb{P}$-a.e. w, $P \in \mathcal{P}$.*

Demonstração. In fact, we have :

$$h_\mu(f, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} \int H_{\mu_w}(\mathcal{P}_w^n) d\mathbb{P} = \inf_n \frac{1}{n} \int H_{\mu_w}(\mathcal{P}_w^n) d\mathbb{P}(w).$$

But, if $(\mu_0)_w(\partial P) = 0$ for any $P \in \mathcal{P}$ and $\mathbb{P}$-a.e. w, then the function $H(\mu, n)$ given by $\mu \to \int H_{\mu_w}(\mathcal{P}_w^n) d\mathbb{P}$ is upper semi-continuous at μ_0. Indeed, since we are assuming that T is continuous, the same argument in the proof of theorem 1.1 of [11] shows this result. In particular, because the infimum of a sequence of upper semi-continuous functions is itself upper semi-continuous, this proves the claim. □

Lemma 2.6.10. *All ergodic measures η outside $\mathcal{K}$ have small entropy : there exists $\rho_0 < 1$ such that*

$$h_\eta(f) \leq \rho_0 h_{top}(f).$$

Demonstração. By the random versions of Oseledet's theorem and Ruelle's inequality (see [9]), we have:

$$h_\eta(f) \leq \int \sum_{i=1}^{s} \lambda^{(i)}(w, x) m^{(i)}(w, x) d\eta.$$

where $\lambda^{(1)}(w, x), \cdots, \lambda^{(s)}(w, x)$ are the positive Lyapunov exponents of f at (w, x) and $m^{(1)}(w, x), \ldots, m^{(s)}(w, x)$ their multiplicity respectively. Furthermore, by hypothesis the measure is ergodic, then these objects are constant a.e. then $h_\eta(f) \leq \sum_{i=1}^{s} \lambda^{(i)}$ and $\int \log \| \det Df(w)(x) \| d\eta = \sum_i \lambda^{(i)}$. Since $\|Df(w)(x)^{-1}\| \leq (1 + \delta_0)$ we have $\lambda_l > -\log(1 + \delta_0)$. By the definitions of m_2, M_2 and the above estimates, we have by (C1):

$$\begin{aligned} h_\eta(f) &\leq \int \log \|Df(w)(x)\| d\eta - \sum_{i=s+1}^{l} \lambda^i \\ &\leq \eta(\Omega \times V) m_2 + (1 - \eta(\Omega \times V)) M_2 + (l - s)(1 + \delta_0) \\ &\leq \alpha m_2 + (1 - \alpha) M_2 + l \log(1 + \delta_0) \end{aligned}$$

Now the physical measure $\mu_{\mathbb{P}}$ given by condition (C2) satisfy $\mu_{\mathbb{P}}(W) < \gamma_0$ (by $(F1)$). The Random Pesin's formulae gives:

$$h_{\mu_{\mathbb{P}}}(f) = \int \log \|detDf\| d\mu_{\mathbb{P}} \geq \mu_{\mathbb{P}}(W)m_1 + (1-\mu_{\mathbb{P}}(W))M_1.$$

But $m_1 < M_1$ then $\gamma_0 m_1 + (1-\gamma_0)M_1 \leq h_{\mu_{\mathbb{P}}}(f)$. Using that $\eta \notin K$, $m_2 < M_2$ and (C1) we have:

$$\alpha m_2 + (1-\alpha)M_2 < \gamma_0 m_1 + (1-\gamma_0)M_1 - l\log(1+\delta_0).$$

Then, we can choose $\rho_0 < 1$ such that

$$\alpha m_2 + (1-\alpha)M_2 + l\log(1+\delta_0) < \rho_0(\gamma_0 m_1 + (1-\gamma_0)M_1) < \rho_0 h_{\mu_{\mathbb{P}}}(f)$$

This gives: $h_\eta(f) \leq \rho_0 h_{top}(f)$. □

Corollary 2.6.11. $\pi_f(\phi) = \sup\limits_{\mu\in\mathcal{K}}\{h_\mu(f) + \int \phi d\mu\}$.

Demonstração. By remark 2.2.6, we need to show that:

$$\sup_{\mu\in\mathcal{K}}\{h_\mu(f) + \int \phi d\mu\} = \sup_{\mu\in\mathcal{M}_e(\Omega\times M)}\{h_\mu(f) + \int \phi d\mu\}$$

By the previous lemma, if $\eta \notin \mathcal{K}$ then:

$$h_\eta(f) + \int \phi d\eta \leq \rho_0 h_{top}(f) + \|\phi\|_1 < \pi_f(\phi)$$

□

Proof of theorem A. We will use the following notation: $\Psi(\mu) = h_\mu(f) + \int \phi d\mu$. Let $\{\mu_k\} \subset \mathcal{K}$ such that $\Psi(\mu_k) \to \pi_f(\phi)$, by compacity we can suppose that μ_k converge to μ weakly.

Fix $\mathcal{P}$ a partition with diameter less than ε_0, and for w-a.e., $\mu_w(\partial P) = 0$, for any $P \in \mathcal{P}$. By corollary 2.6.8 we have $h_{\mu_k}(f) = h_{\mu_k}(f,\mathcal{P})$. Then $\pi_f(\phi) = \sup\limits_{\eta\in\mathcal{K}} \Psi(\eta) = \limsup \Psi(\mu_k)$. By the comments after corollary 2.6.8 we know that $\eta \to h_\eta(f,\mathcal{P})$ is upper semicontinuous in η *over* $\mathcal{K}$, then:

$$\limsup \Psi(\mu_k) \leq h_\mu(f,\mathcal{P}) + \int \phi d\mu \leq \Psi(\mu).$$

But, $\Psi(\mu) \leq \pi_f(\phi)$. This implies that μ is an equilibrium state.

In the other hand, if η is a measure which attain the supremum in (2.3) then let $\eta_{(w,x)}$ the ergodic decomposition of η. Then the entropy of η is equal to the integral of entropies of its ergodic components (see [9], page 1289 and references there in), of course the same occurs with the $\Psi(\eta)$ (*). If $(x, w) \notin \{(x, w); \eta_{(x,w)} \in \mathcal{K}_\alpha\}$ then by lemma 2.6.10:

$$\Psi(\eta_x) = h_{\eta_{(x,w)}}(f) + \int \phi d\eta_{(x,w)} \leq \rho_0 h_{top}(f) + \|\phi\|_1 < \pi_f(\phi).$$

Then if $\eta(\{(x, w); \eta_{(x,w)} \in \mathcal{K}_\alpha\}^c) > 0$, (*) says that $\Psi(\eta) < \pi_f(\eta)$ a contradiction, so every equilibrium state is in $\mathcal{K}$. The proof of the theorem is now complete. □

Remark 2.6.12. Liu-Zhao [11] show the semi-continuity of the entropy under the hypothesis that $T : \Omega \to \Omega$ is *continuous* and f is expansive *at every point of* M. From this result, a natural question is : "What about the semi-continuity without topological assumptions (e.g., continuity) ? And the case of weak expansiveness assumptions ?". We point out that the proof of theorem A shows the semicontinuity of the entropy map *in the set* $\mathcal{K}$. This partially answers the question since, although we need to assume continuity, only a weak expansion at Lebesgue a.e. point of M is required (*this assumption is the sole reason of the restriction to the set of measures* $\mathcal{K}$). Indeed, non-uniform expansion on Lebesgue a.e. point obligates us to restrict the proof of our lemmas on semicontinuity to the set $\mathcal{K}$.

Remark 2.6.13. Our theorem A holds in the context of *RDS bundles* (see [9] or [11]) with the extra assumption that T *and* the skew-product F are continuous.

2.7 Appendix

We now prove that $(F1)$ follows from (H1) and (H2), in fact, this is a well known argument (see for example [1]), but for sake of completeness we give the proof.

Fix (w, x), if $i = (i_0, \cdots i_{n-1}) \in \{1, \cdots, p+q\}^n$ let $[i] = B_{i_0} \cap f^{-1}(w)(B_{i_1}) \cap \cdots \cap f^{-n+1}(w)(B_{i_{n-1}})$ and $g(i) = \#\{0 \leq j < n; I_j \leq p\}$.

If $\gamma > 0$ then $\#\{i; g(i) < \gamma n\} \leq \sum_{k \leq \gamma n} \binom{n}{k} p^{\gamma n} q^n$. By Stirling's formula this is bounded by $(e^\xi p^\gamma q)^n$ (here ξ depends of γ) and $\xi(\gamma) \to 0$ if $\gamma \to 0$.

Now (H1) and (H2) says that $m([i]) \leq \sigma_1^{-n}\sigma_1^{\gamma n}$. If we set $I(n,w) = \bigcup\{[i]; g([i]) < \gamma n\}$ then $m(I(n,w)) \leq \sigma_1^{-(1-\gamma)n}(e^{\xi}p^{\gamma}q)^n$ and since $\sigma_1 > q$ there is a γ_0 (small) such that $(e^{\xi}p^{\gamma}q)^n < \sigma_1^{(1-\gamma_0)}$. Then there is a $\tau = \tau(\gamma_0) < 1$ and $N = N(\gamma_0)$ such that if $n \geq N$ then $m(I(n,w)) \leq \tau^n$

Let $I_n = \bigcup_w(\{w\} \times I(n,w))$ and by Fubini's theorem $\mathbb{P} \times Leb(I_n) \leq \tau^n$ if $n \geq N$. But $\sum_n \mathbb{P} \times Leb(I_n) < \infty$ then Borel-Cantelli's lemma implies:

$$\mathbb{P} \times Leb(\bigcap_{n\geq 1}\bigcup_{n\geq k} I_k) = 0$$

Using Birkhoff's theorem we have that the set:

$$\{(w,x); \exists\, n \geq 1, \forall k \geq n, \lim \frac{\#\{0 \leq j < n; f^j(w)(x) \in B_1 \cup \cdots \cup B_p\}}{n}\}.$$

has $\mathbb{P} \times Leb$-measure at least γ_0. Now by Fubini's theorem again, we have (F1).

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Capítulo 3

Abundance of stable ergodicity

Os resultados abaixo foram obtidos em conjunto com Christian Bonatti, Marcelo Viana e Amie Wilkinson no artigo *Abundance of stable ergodicity* o qual foi aceito para publicação em *Commentarii Math. Helvetici.* Abaixo segue o conteúdo (em inglês) deste artigo.

3.1 History

A fundamental problem, going back to Boltzmann and the foundation of the kinetic theory of gases, is to decide how frequently conservative dynamical systems are ergodic.

A first striking answer was provided by KAM (Kolmogorov, Arnold, Moser) theory: ergodicity is not a generic property, in fact there are open sets of conservative systems exhibiting positive volume sets consisting of invariant tori supporting minimal translations.

In sharp contrast with this elliptic type of behavior, ergodicity prevails at the other end of the spectrum, namely, among strongly hyperbolic systems. Indeed, after partial results of Hopf and Hedlund, Anosov proved that the geodesic flow of any compact manifold with negative curvature is ergodic. In fact, the same is true for any sufficiently smooth conservative uniformly hyperbolic flow or diffeomorphism.

By the mid-nineties, Pugh and Shub proposed to address the ergodicity problem in the context of partially hyperbolic systems, where the tangent space splits into uniformly contracting (stable), uniformly expanding (unstable), and "neutral" (central) directions. To summarize their main theme:

A little hyperbolicity goes a long way in guaranteeing ergodicity.

In more precise terms, in [11] they proposed the following

Conjecture. Stable ergodicity is a dense property among C^2 volume preserving partially hyperbolic diffeomorphisms.

At about the same time, there was a renewed interest in the geometric and ergodic properties of partially hyperbolic systems in the broader context of possibly non-conservative dynamical systems. A main goal here was to establish existence and finiteness of SRB (Sinai, Ruelle, Bowen) measures, and to characterize their basins of attraction.

Thus the general theme of partially hyperbolic dynamics evolved into a very active research field, with contributions from a large number of mathematicians. See, for instance, [2, 6] for detailed accounts of much progress attained in the last few years.

3.2 Result

The purpose of this note is to point out that, putting together recent results by Shub, Wilkinson [12] followed by Baraviera, Bonatti [1], by Bonatti, Viana [3] followed by Burns, Dolgopyat, Pesin [5], and by Dolgopyat, Wilkinson [7], one obtains a proof of the conjecture stated above, when the central direction is 1-dimensional.

Theorem. Let M be a compact manifold endowed with a smooth volume form ω, and $\mathcal{PH}_\omega(M)$ be the set of all partially hyperbolic diffeomorphisms having 1-dimensional center bundle and preserving the volume form.

Then the volume measure defined by ω is ergodic, and even Bernoulli, for any C^2 diffeomorphisms in a C^1 open and dense subset of $\mathcal{PH}_\omega(M)$.

The proof of the theorem follows. In fact, we prove a bit more: every C^2 diffeomorphism in $\mathcal{PH}_\omega(M)$ is C^1 approximated by another C^2 diffeomorphism in $\mathcal{PH}_\omega(M)$ which is stably Bernoulli. Note that it is not known whether C^2 maps are dense in $\mathcal{PH}_\omega(M)$.

Throughout, all maps are assumed to be volume preserving. First, [1] extends the technique of [12], to prove that every partially hyperbolic diffeomorphism may be C^1 approximated by another for which the integrated sum of all Lyapunov exponents along the central direction is non-zero. Under our dimension assumption, this just means that the integrated central Lyapunov exponent is non-zero, for a C^1 open and dense subset $\mathcal{O}_1$ of partially hyperbolic diffeomorphisms. Let us decompose $\mathcal{O}_1$ as $\mathcal{O}_- \cup \mathcal{O}_+$, according to whether the integrated central exponent is negative or positive.

Up to replacing f by its inverse, we may suppose that $f \in \mathcal{O}_-$. For such f, there is a positive volume set of points with negative central Lyapunov exponent. Assuming f is C^2, the arguments in [3] show that there exists an invariant ergodic Gibbs u-state μ with negative central Lyapunov exponent. See the Lemma below for a proof. Then μ is an SRB measure and, as observed in [5], its basin contains a full volume measure subset of some open set $O(\mu)$, which is saturated by both strong foliations.

Also for f in a C^1 open and dense subset $\mathcal{O}_2$, [7] proves that the diffeomorphism has the accessibility property: any two points may be joined by a path formed by finitely many segments contained in leaves of the strong-stable foliation or the strong-unstable foliation. Taking f of class C^2 in $\mathcal{O}_- \cap \mathcal{O}_2$ we obtain that $O(\mu)$ is the whole manifold so that the basin of μ has total volume in M. This implies ergodicity.

Finally, the same arguments extend directly to any iterate f^n, $n \geq 1$. Indeed, $f^n \in \mathcal{O}_\pm$ if and only if $f \in \mathcal{O}_\pm$, and μ is an SRB-measure also for f^n. Moreover, f^n is accessible if and only if f is, since the two maps have the same strong foliations. This shows that f^n is ergodic, for every $n \geq 1$, whenever $f \in \mathcal{O}_\pm \cap \mathcal{O}_2$. Using Theorem 8.1 of Pesin [10], we conclude that f is Bernoulli.

3.3 Conclusion

To conclude, we give the technical definitions of the notions involved, and we state and prove the Lemma.

Let M be a compact manifold endowed with a volume form ω. A volume preserving diffeomorphism $f : M \to M$ is *stably ergodic* if the volume measure defined by ω is ergodic for any C^2 diffeomorphism in a C^1-neighborhood of f.

A diffeomorphism $f : M \to M$ is *partially hyperbolic* if there is a splitting $TM = E^s \oplus E^c \oplus E^u$ of the tangent bundle into three invariant bundles (with positive dimension) and there exists $m \geq 1$ such that

$$\|Df^m \mid E^s\| \leq \frac{1}{2} \quad \text{and} \quad \|Df^{-m} \mid E^u\| \leq \frac{1}{2}$$

and

$$\|Df^m \mid E^s\| \, \|(Df^m \mid E^c)^{-1}\| \le \frac{1}{2} \text{ and } \|(Df^m \mid E^u)^{-1}\| \, \|Df^m \mid E^c\| \le \frac{1}{2}.$$

The first condition means that E^s is uniformly contracting and E^u is uniformly expanding. The last one means that the splitting is *dominated.*

We denote $\mathcal{PH}(M)$ the space of partially hyperbolic C^1 diffeomorphisms on M with $\dim E^c = 1$, and $\mathcal{PH}_\omega(M)$ the subset of volume preserving diffeomorphisms.

Let $f \in \mathcal{PH}(M)$. Then the stable bundle E^s and the unstable bundle E^u are uniquely integrable. The corresponding integral foliations, respectively strong-stable $\mathcal{F}^s$ and strong-unstable $\mathcal{F}^u$ are invariant, and their leaves are uniformly contracted by all forward and backward iterates of f, respectively.

We say that $f \in \mathcal{PH}(M)$ has the accessibility property if any two points of M may be joined by a path formed by finitely many segments contained in leaves of the strong-stable foliation or the strong-unstable foliation.

A *Gibbs u-state* is an invariant probability with absolutely continuous conditional measures along the leaves of the strong-unstable foliation. Gibbs s-states are defined in the same fashion. In the partially hyperbolic context, such measures were first constructed by Pesin, Sinai [9].

An invariant probability measure μ is an *SRB measure* if the set of points $x \in M$ whose time averages

$$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x))} \to \mu \qquad \text{(weakly)}$$

has positive volume. This set is called the *basin* of μ.

Recall that $\mathcal{O}_-$ denotes the subset of diffeomorphisms in $\mathcal{O}$ with negative integrated central Lyapunov exponent. The following lemma is essentially contained in [3]:

Lemma. Any C^2 diffeomorphism $f \in \mathcal{O}_-$ has some Gibbs state with negative central Lyapunov exponent.

Demonstração. Let $f \in \mathcal{O}_-$ be a C^2 diffeomorphism. Denote by $\mathcal{R}$ the set of points $x \in M$ for which the Lyapunov exponent

$$\lambda^c(x) = \lim_{n \to \pm\infty} \frac{1}{n} \log |Df^n \mid E^c_x|$$

is well defined; that is, both limits exist and they coincide. As f preserves volume, the ergodic theorem ensures that $\mathcal{R}$ has full volume in M. Since the strong-unstable foliation $\mathcal{F}^u$ is absolutely continuous [4], there is a full measure subset $\mathcal{R}_0$ of points $x \in \mathcal{R}$ for which the intersection $\mathcal{R} \cap \mathcal{F}^u(x)$ has full Lebesgue measure inside $\mathcal{F}^u(x)$, where $\mathcal{F}^u(x)$ denotes the strong-unstable leaf through x.

On the other hand, the hypothesis

$$\int_M \lambda^c(x)\, d\omega(x) = \int_M \log |Df \mid E^c_x|\, d\omega(x) < 0$$

implies that there exists a positive volume set $\mathcal{R}^-$ of points x such that $\lambda^c(x)$ is negative. Observe that $\lambda^c(x) = \lambda^c(y)$ if x, $y \in \mathcal{R}$ belong to the same strong-unstable leaf. Choose $x_0 \in \mathcal{R}^- \cap \mathcal{R}_0$ and let $D \subset \mathcal{F}^u(x)$ be a disk centered at x. Let m_0 be the normalized Lebesgue measure induced on D by some Riemannian metric of M. Then m_0 is a probability measure, and $m_0(D \cap \mathcal{R}^-) = 1$. Let

$$m_n = \frac{1}{n} \sum_0^{n-1} f^i_*(m_0).$$

By [9], every accumulation point μ of the sequence m_n is a Gibbs u-state for f. Moreover, since λ^c is well defined and equal to $\lambda^c(x_0)$ at m_0-almost every point,

$$\int_M \log |Df \mid E^c_x|\, d\mu(x) = \lim_{n \to +\infty} \int_M \log |Df \mid E^c_x|\, dm_n(x) = \lambda^c(x_0) < 0.$$

Then at least one ergodic component μ_0 of μ must have

$$\int_M \log |Df \mid E^c_x|\, d\mu_0(x) \leq \lambda^c(x_0) < 0.$$

Finally, [3] asserts that each ergodic component of a Gibbs u-state is again a Gibbs u-state. Hence, μ_0 is the announced ergodic Gibbs u-state with negative central Lyapunov exponent. □

3.4 Questions

One would like to remove the assumption on the central dimension.

Another important open problem is the C^r version of the conjecture, any $r > 1$. In this direction, Niţică, Török [8] prove C^r density of accessibility assuming a r-normally hyperbolic 1-dimensional, integrable central bundle with at least two compact leaves.

Here we prove ergodicity assuming C^2 regularity. While ergodic systems always form a G_δ, it is not known whether C^2 maps are dense in the space C^1 volume preserving diffeomorphisms; see Zehnder [13]. So it remains open whether ergodicity is generic (dense G_δ) among C^1 partially hyperbolic with 1-dimensional central bundle.

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Capítulo 4

The Bernoulli property for weakly hyperbolic systems

Os resultados abaixo foram obtidos em conjunto com Alexander Arbieto e Maria José Pacífico no artigo *The Bernoulli property for weakly hyperbolic systems* o qual foi aceito para publicação em *Journal of Statistical Physics*. Abaixo segue o conteúdo (em inglês) deste artigo.

4.1 Introduction

Chaotic dynamics is associated to loss of memory and creation of information (two aspects of the same phenomenon) as the system envolves time. Indeed, orbits starting at nearby points *forget* this fact rather rapidly; the evolution of each orbit yields *new* information, which can not be deduced from the initial data nor from the evolution of another orbit.

This idea can be formalized in several (non-equivalent) ways. One is encapsulated in the notion of *entropy*, the exponential rate of creation of information by the system. Another, which concerns us more directly here, is through the mixing property: a system is called *mixing* if measurements of any observable quantity at same latter time correlate poorly with the initial measurements of the same, or any other, observable quantity. There are several stronger (and a few weaker) versions of this notion. The strongest is the Bernoulli property: a system is *Bernoulli* if it is ergodically equivalent to an i. i. d. random process. In simple terms, iterations of the system are as chaotic (unpredictable) as sucessive throws of an havest coin.

It is now well stablished that mixing is closely related to hyperbolicity properties of the dynamical system. On the other hand, there was the fundamental work of Anosov [3] proving that the geodesic flow on any negatively curved manifold is ergodic. The strategy was to prove that these flows are *uniformly hyperbolic* (meaning that the tangent space transverse to the flow splits into two invariant directions which are expanding and contracted, respectively, at uniform rates, under time evolution) and to deduce ergodicity from it. A powerful machinery developed for hyperbolic systems in the sixties and the seventies shows, in particular, that Anosov flows are Bernoulli.

On the other hand, there was the equally remarkable theory of Kolmogorov, Arnold, Moser showing that most *elliptic* systems are not ergodic, let alone mixing or Bernoulli. For instance, close to an elliptic point most of phase space is occupied by invariant tori restricted to which the dynamics is given by a rigid rotation, up to a smooth change of coordinates.

Roughly speaking, this connection between mixing properties and hyperbolicity goes as follows. Expansion along certain directions of the tangent space means that most nearby points tend to move away from each other, so that their orbits decorrelate rapidly. The same is true for contraction, considering backward iterates. For smooth systems, as we are considering here, this local behavior is reflected at the global level.

Over the last decade, there has been a great deal of attention devoted to investigating the mixing properties of systems lying somewhere in between the two extreme situations, hyperbolic and elliptic, that we discussed before. Most successful attempts dealt with *partially hyperbolic* systems, where one still asks for expanding and contracting invariant directions, but one allows for additional so-called central directions, where the behavior is rather arbitrary. Concrete examples of partially hyperbolic systems arise in several applications, for instance hard ball systems with many balls [16] that model the motion of ideal gases (however hard ball systems correspond to piecewise smooth maps). Our goal in this paper is to prove that, in fact, quite *weak hyperbolicity features suffice for the system to be mixing and even Bernoulli.*

Before we give precise statemants of our results, let us mention a few previous related results. On one hand, there are the works of Pugh, Shub [15] and their collaborators, investigating stable ergodicity for conservative (volume-preserving) diffeomorphisms. A dynamical property is *stable* (or *robust*) if it is shared by all systems in a C^1 neighborhood. A key ingredient in this approuch is the notion of accessibility: a system is *accessible* if any two points may be joined by a smooth path whose velocity is everywhere contained in the union of the stable and unstable direction. Dolgopyat, Wilkinson [12] recently proved that accessibility holds for generic (residual subset of) C^1 diffeomorphisms, conservative or not. Moreover, Bonatti, Matheus, Viana, Wilkinson [9] proved that generic partially hyperbolic diffeomorphisms with 1-dimensional central direction are stably ergodic.

On the other hand, there is the work of Alves, Bonatti, Viana [2, 10] on the ergodic properties of partially hyperbolic diffeomorphisms, not necessarily conservative. They exploit the combination of partial hyperbolicity and non-uniform hyperbolicity (non-zero Lyapunov exponents) to prove existence and finiteness of physical measures for those systems. We prove here that the examples that appear in both papers above have the Bernoulli property. Let us point out that Bochi, Fayad, Pujals [5] prove that generic stably ergodic conservative systems are non-uniformly hyperbolic.

The two approaches have been put together by Burns, Dolgopyat, Pesin [11] in a work which may be considered a predecessor to the present paper. In a few words we push their analysis further to obtain the Bernoulli property rather than just ergodicity.

We point out that ergodicity implies chaotic properties like, for instance, topological transitivity (existence of dense orbits). But the sole assumption of transitiveness does not guaranty that the system is ergodic: Furstenberg exhibited in [13] a minimal but non-ergodic diffeomorphism. So, insteady of a topological property of a single system we ask for robustness (it holds in a neighborhood of the system) of such topological behavior in the attempt to derive any statistical/ergodic property. Related to this Bonatti, Diaz and Pujals [8] proved that robustly transitive dissipative diffeomorphisms have a dominated splitting (the tangent bundle splits into two invariant directions, one contracting direction and one central direction). Arbieto and Matheus [4] proved that the same result holds for C^2 robustly transitive conservative diffeomorphisms. Tahzibi [17] proved that robustly transitive partially hyperbolic diffeomorphisms with central direction *mostly contracting* are stably ergodic. We extend this last result of Tahzibi replacing the robust transitiveness hypothesis by robust topologically mixing hypothesis, and obtain, insteady of ergodicity, the Bernoulli property for these systems. We also prove that the systems in $\mathcal{T}^4$ studied by Tahzibi in [18] have the Bernoulli property.

The paper is organized as follows. In section 4.2 we give the definitions of the objects treated here, state the results and give some sketchs of the proofs. In section 4.3 we give the proofs of the theorems. In section 4.4 we study a construction due to Bonatti and Viana [10] and as an application of our methods prove that it yelds Bernoulli systems. Finally, in section 4.5, we point out how obtain some extentions and discuss some open problems.

4.2 Definitions and Statement of the Results

Throughout we will use the notation $\mathrm{Diff}^{1+}_m(M) := \bigcup_{\alpha>0} \mathrm{Diff}^{1+\alpha}_m$ and the diffeomorphisms considered here will be always in $\mathrm{Diff}^{1+}_m(M)$. We will deal with *robust* properties, but since they arise from different nature we need to specify the topologies involved.

Definition 4.2.1. A diffeomorphism f is robustly transitive (resp. robustly topologically mixing) if there exists a neighborhood $\mathcal{U} \subset \mathrm{Diff}^{1+}_m(M)$ in the C^1-topology such that any $g \in \mathcal{U}$ is transitive (resp. topologically mixing).

Obviously any topological mixing diffeomorphism is transitive.

Definition 4.2.2. A diffeomorphism f is robustly ergodic (resp. robustly Bernoulli, robustly mixing) if there exists a neighborhood $\mathcal{U} \subset \mathrm{Diff}^{1+}_m(M)$ in the C^1-topology such that any $g \in \mathcal{U}$ is ergodic (resp. Bernoulli, mixing).

We recall that by definition, a Bernoulli system is equivalent to a Bernoulli shift. It is easy to see that if f is Bernoulli then it is mixing.

We say that a property is generic robustly if it holds in a neighborhood intersected by a residual set. For example, f is generic transitive if there exists a neighborhood $\mathcal{U} \subset \mathrm{Diff}^{1+}_m(M)$ in the C^1-topology and a residual set $\mathcal{R}$ such that any $g \in \mathcal{U} \cap \mathcal{R}$ is transitive.

Next we state our results, in the different settings of partial hyperbolicity.

Partially Hyperbolic Systems

First let us recall some definitions.

Definition 4.2.3. A Df-invariant splitting $TM = E \oplus F$ is a dominated splitting if there is $\lambda < 1$ such that:

$$\|Df|_{E_x}\|.\|Df^{-1}|_{F_{f(x)}}\| \leq \lambda \text{ for all } x \in M.$$

We will use also the notion of a $k-$dominated splitting of $E \oplus F$ along the orbit of a point x. We require that for all $n \in \mathbb{Z}$:

$$\frac{\|Df^k_{f^n(x)}|_F\|}{\mathbf{m}(Df^k_{f^n(x)}|_E)} \leq \frac{1}{2},$$

where $\mathbf{m}(A) = \|A^{-1}\|^{-1}$. By a $k-$dominated splitting over an invariant set D we mean a $k-$dominated splitting for all orbits in D

A diffeomorphism f is *partially hyperbolic* if it has a dominated splitting $E \oplus F$ such that at least one of the subbundles is hyperbolic (either uniformly contracting or expanding). The complement of the hyperbolic subbundle is called the *central bundle* or equivalentely *central direction*. We denote by $\mathcal{PH}^r(M)$ (respectively $\mathcal{PH}^r_m(M)$) the set of partially hyperbolic (respectively conservative partially hyperbolic) diffeomorphisms.

We can define the Lyapunov exponents of the system with respect to an invariant measure as the following:

Definition 4.2.4. Let $f : M \to M$ be a C^1 diffeomorphism of a compact manifold that preserves a volume m. Oseledets theorem states that, for $m-$almost every point $x \in M$, there exist real numbers $\lambda_1(x) > \cdots > \lambda_{k(x)}(x)$ and

$$T_x M = E^1_x \oplus \cdots \oplus E^{k(x)}_x$$

such that:

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)(v_j)\| = \lambda_j(x) \text{ for all } \quad v_j \in E^j_x \backslash \{0\}.$$

For each j, λ_j is the Lyapunov exponent along the sub-bundle E^j and it depends measurably on x.

Definition 4.2.5. We say that the central direction of $f \in \mathcal{PH}^r_m(M)$ is non-uniformly hyperbolic if m-almost every point x has non-zero Lyapunov exponents along the central direction.

Now we can state our first result.

Theorem A. *Let $f \in \mathrm{Diff}^{1+}_m(M)$ be a topologically mixing partially hyperbolic diffeomorphism with decomposition $TM = E^u \oplus E^{cs}$ such that E^{cs} has only negative Lyapounov exponents. Then (f, m) is Bernoulli and, in particular mixing.*

From this theorem we obtain immediately an extension of a result by Thazibi [17]:

Corollary 4.2.6. *If f is robustly topologically mixing and $TM = E^u \oplus E^{cs}$ with only negative exponents in the E^{cs} direction then f is robustly Bernoulli.*

We note that in [10] the authors constructed an example in $\mathcal{T}^3$ which is robustly transitive and stably ergodic. In section 4.4 we show that this example is in fact robustly topologically mixing.

We can relax the hypotheses in Theorem A requiring only non-zero Lyapunov exponents (non-uniformly hyperbolicity). Unfortunately we do not obtain the Bernoulli property in the whole manifold, but it holds for an arbitrarily large region in the sense of Lebesgue measure. To announce our next result let us introduce the following definition.

Definition 4.2.7. A diffeomorphism f is ε-Bernoulli if there exist an ergodic component C of the Lebesgue measure m such that $m(C) > 1 - \varepsilon$ and if m_C is the normalization of the Lebesgue measure to C, $(f|_C, m_C)$ is Bernoulli. A diffeomorphism is quasi-robustly Bernoulli if for any $\varepsilon > 0$ there exist an open set $\mathcal{U}_\varepsilon \subset \mathrm{Diff}^{1+}_m(M)$ ε-close to f such that any $g \in \mathcal{U}_\varepsilon$ is ε-Bernoulli.

With this we obtain an extension of a theorem by Tahzibi:

Theorem B. *If* $\dim(M) = 3$ *and* $\mathcal{U} \subset \mathcal{PH}^{1+}_m(M)$ *is an open set such that generically in* U *any diffeomorphism is non-uniformly hyperbolic and topologically mixing then generically in* $\mathcal{U}$ *any diffeomorphism is quasi-robustly Bernoulli.*

Strongly Partially Hyperbolic Systems

Now we focus in systems that have two genuine hyperbolic directions (contracting and expanding) and a center direction. That is, the tangent bundle admits a dominated splitting $TM = E^u \oplus E^c \oplus E^s$ where E^u (respectively E^s) is uniformly expanding (respectively contracting). This allow us to use accessibility, instead of topological mixing, to spread out the negative Lyapunov exponents as in Theorems 2, 3 and 4 of [11] and obtain the Bernoulli property rather than just ergodicity.

Definition 4.2.8. We say that f is accessible if any two points $p, q \in M$ can be joined by piecewise smooth paths such that each piece is a path entirely contained on a stable leaf or a unstable leaf. We call these paths a us-path. We say that f has the essentially accessible if any measurable union of accessible sets (i.e. any two points in each of these sets can be joined by a us-path) must have zero or full measure. Each piece of the us-path is called a leg.

Next we state the generalizations of the results by Burns, Dolgopyat and Pesin [11].

Theorem C. *Let f be a partially hyperbolic diffeomorphism with negative Lyapunov exponents along the central direction for a positive measure set A and suppose that f is essentially accessible. Then A has full measure. In particular f is non-uniformly hyperbolic and Bernoulli.*

Theorem D. *Let f be an accessible partially hyperbolic diffeomorphism satisfying*

$$\int_M \log \|Df|_{E^c_f(x)}\| dm(x) < 0 \,.$$

Then f is robustly Bernoulli.

Theorem E. *Let f be an accessible partially hyperbolic diffeomorphism with only negative Lyapounov exponents in the central direction. Then f is robustly Bernoulli.*

Remark 4.2.9. We observe that the hypotheses in the previous theorem as well the hypotheses of corollary 4.2.6 imply that f is C^1-robustly mostly contracting (see [2]), with the difference that in our case we require strong partial hyperbolicity.

We can strength these theorems using a denseness result by Dolgopyat and Wilkinson [12]. This will be done in section 4.5.

4.2.1 Comments on the proofs

The key result that we use here is a corollary of some results by Pesin [14], which in brief terms says that if the system is non-uniformly hyperbolic and every iterate is ergodic then the system is Bernoulli.

For the proof of theorems A and B we use the mostly contracting condition to obtain local ergodicity and then we use the topological mixing property to spread the ergodcity to the whole manifold and obtain the same result for all the iterates of the diffeomorphism. To conclude we use Pesin's result described above. We stress that in the theorem B we only obtain the Bernoulli property on a set with large measure.

Now, in theorems C, D and E, the accessibility is enough to study the iterates of the diffeomorphism. But in the proof of theorem D, we use the existence of hyperbolic times. So we need to investigate the key element on the Pesin's result: the Pinsker partition. In brief words, the methods used before give that an infinite set of iterates are ergodic. But we need all of them to be ergodic. s Hence we use the fact that the atoms of the Pinsker partition are permutated and the estimates given by the hyperbolic times to reduce the problem to prove ergodicity for only a finite number N of iterates. So we can intersect only a finite number of neighborhoods of f obtaining in this way a neighborhood where every diffeomorphism is Bernoulli.

4.3 Proof of the Theorems

4.3.1 Proof of Theorem A

We will follow Hopf's argument as used in [10],[11] and [17]. For the sake of completeness we give such argument.

By non-uniform hyperbolicity we have a countable number of ergodic components. Now, take an ergodic component C and $R \subset C$ (with full Lebesgue measure in C) the set of regular points in the sense of Birkhoff's, i.e., if $x \in R$ then:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) = \lim_{n\to-\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) \text{ for any } \varphi \in C^0(M).$$

Then by Pesin's theory, any $x \in R$ has a local stable manifold $W^{cs}_\varepsilon(x)$ and if m_{cs} is the induced measure in $W^{cs}_\varepsilon(x)$ then m_{cs} is absolutely continous. This implies that there is a $x \in C$ and $C_x \subset W^{cs}_\varepsilon(x) \cap C \cap R$ such that $m_{cs}(W^{cs}_\varepsilon(x)\backslash C_x) = 0$. By partial hyperbolicity, any point has unstable manifolds with size uniformly away from zero. Now take $U_x = \bigcup_{y\in W^{cs}_\varepsilon(x)} W^u(y)$. Then, by continuity of the unstable foliation, U_x contains an open set. And for every $y \in C_x$ and $z \in W^u(y)$ we have:

$$\begin{aligned} \lim_{n\to-\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(z)) &= \lim_{n\to-\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(y)) = \\ = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(y)) &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(z)). \end{aligned}$$

Then, using absolute continuity of W^u, $\bigcup_{y\in C_x} W^u(y)$ has full measure in U_x and hence C contains a total Lebesgue measure subset of the open set U_x.

Finally, transitiveness shows that there exists a unique ergodic component with full Lebesgue measure and thus f is ergodic.

Now we use a theorem by Pesin (which is in fact a corollary of Theorem 8.1 in [14]):

Theorem 4.3.1. *If $f \in \mathrm{Diff}^{1+}_m(M)$ is non-uniformly hyperbolic such that (f^n, m) is ergodic for any $n \geq 0$ then f is Bernoulli.*

Because being topologically mixing and have negative Lyapunov exponents in the E^{cs} direction is an invariant property for all iterates f^k. Then, by the previous argument, all of (f^k, m) are ergodic and non-uniformly hyperbolic then the above theorem shows that (f, m) is Bernoulli. □

Proof of Corollary 4.2.6. Let U be the open set given by the hypothesis. Then by Theorem A any $g \in U$ is Bernoulli. This implies that f is C^1-robustly Bernoulli. □

The proof above shows the following:

Corollary 4.3.2. *If $f \in \mathrm{Diff}^{1+}_m(M)$ is generic robustly topologically mixing and $TM = E^u \oplus E^{cs}$ with central direction non-uniformly hyperbolic then f is generic robustly Bernoulli.*

Also [11] uses the same arguments to prove the following:

Theorem 4.3.3 (Burns, Dolgopyat and Pesin)**.** *If $f \in \mathrm{Diff}^{1+}_m(M)$ has an invariant subset $A \subset M$ with $m(A) > 0$, such that $f|_A$ is strongly partially hyperbolic with negative exponents along the central direction then every ergodic component of $f|_A$ (and A) is open (mod 0). If f is topologically transitive then A is dense and $f|_A$ is ergodic.*

This theorem will be used later for the proofs of theorems C, E and D.

4.3.2 Proof of Theorem B

We will follow the arguments in [17]. We can assume that $TM = E^u \oplus E^{cs}$, since the other case is analogous.

By the Bochi-Viana's theorem [6], there exists a C^1-residual subset $\mathcal{R}$ of $\mathrm{Diff}^1_m(M)$ such that for every $f \in \mathcal{R}$, the Oseledets splitting is dominated or else trivial, at almost every point. Let $g \in \mathcal{R} \cap \mathcal{U}$ where $\mathcal{U}$ is an open set such that every $g \in \mathcal{U}$ is topologically mixing.

We recall that the residual set given by Bochi-Viana's theorem is characterized as the continuity points of the maps: $\Lambda_i(f) = \lambda_1(f) + \cdots + \lambda_i(f)$, where $\lambda_j(f) = \int_M \lambda_j(x) dm$. Now, let $\mathcal{V}$ be an open set containing g such that for any $f, h \in \mathcal{V}$ we have $|\Lambda_i(f) - \Lambda_i(h)| \leq \delta_0$.

We know that there exists a countable number of ergodic components, and for any ergodic component C we consider the normalized Lebesgue measure m_C on $supp(C)$ and we can use $supp(C)$ instead of C. Recall that the basin of m_C is the set of points z such that $\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(z)}$ converges to m_C, this set has full measure in C and every orbit in the basin is transitive. Now we use the basin of m_C instead of C and we continue denoting it by C.

An ergodic component C is "good" if $\lambda_2(x) < 0$ (the central exponent) for every $x \in C$ or $\lambda_2(x) > 0$ and $E^2 \oplus E^3$ is dominated. Other components are called "bad" components.

Any "good" component contains open sets (mod 0). Indeed, the case of $\lambda_2 < 0$ is in the proof of Theorem A and the other case follows from the fact that for a conservative system, the dominated splitting $E^{cu} \oplus E^{cs}$ is in fact volume hyperbolic (see [8]). Recall that

Definition 4.3.4. A dominated splitting $TM = E^1 \oplus E^2 \cdots \oplus E^k$ is volume hyperbolic if there exist some $K > 0$:

$$|\det(Df^{-n}|E^k(x)| \leq K\lambda^n\,, \quad \text{and} \quad |\det(Df^n|E^1(x)| \leq K\lambda^n\,.$$

Because the dimension is 3 and that $\dim E^{cs} = 1$ this subbundle is actually uniformly contracting, and so, a stable bundle. Hence, we have in fact a strong stable foliation, and the argument is analogous of the one in the proof of TheoremA (using strong unstable/stable leaves).

We recall that if C is a "good" component for f then it is a "good" component for f^k for any $k \geq 1$ and the same holds for "bad" components. And by topologically mixing there exists only one "good" ergodic component for f^k, $k \geq 1$.

Now we prove that the measure of the union of "bad" components can be made arbitrarily small. Let $\Gamma(f,k)$ be the subset of points such that $E^2 \oplus E^3$ does not admit a $k-$dominated splitting and let $\Gamma(f,\infty) = \cap_{k\in\mathbb{N}}\Gamma(f,k)$. For the "bad" ergodic components C, we have that $\lambda_2 > 0$ and $E^2 \oplus E^3$ does not admit a $k-$dominated splitting over C for any $k \in \mathbb{N}$ and by transitivity every point $x \in C$ doesn't have a k-dominated splitting. This shows that $C \subset \Gamma(f,\infty)$ (mod 0).

Denote $J(f) = \int_{\Gamma(f,\infty)} \frac{\lambda_2 - \lambda_3}{2} dm(x)$. Then we can use the following:

Proposition 4.3.5 (Proposition 4.17 [6])**.** *Given any $\delta > 0$ and $\varepsilon > 0$, there exists a diffeomorphism f_1, ε near to f such that*

$$\int_M \Lambda_2(f_1, x)dm < \int_M \Lambda_2(f, x)dm(x) - J(f) + \delta.$$

¿From the above proposition we will deduce that if the measure of bad components is not small enough then after perturbing f a little, the average of $\lambda_1 + \lambda_2$ drastically drops. Indeed, as $C \subset \Gamma(f, \infty)$ and on $C, \lambda_2(x) > 0$ by the above proposition we get

$$\Lambda_2(f) - \Lambda_2(f_1) \geq \frac{1}{2}\int_C (\lambda_2 - \lambda_3)(f)dm - \delta \geq \frac{1}{2}\int_C -\lambda_3(f)dm - \delta$$

$$\geq m(C) \inf_{x \in C} \frac{-\lambda_3(f, x)}{2} - \delta.$$

Now f is volume hyperbolic and partially hyperbolic $TM = E^u \oplus E^{cs}$, so $\det(Df|E^{cs}(x)) < \alpha < 1$ for all $x \in M$ and we can take α uniform in a C^1 neighborhood of g by continuity on the C^1 topology of $f|_{E^{cs}}(x, f)$. If we take $x \in C$ then:

$$\lambda_2(x) + \lambda_3(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log\, \det(Df|E^{cs}(f^i(x))) \leq \log(\alpha)$$

and as $\lambda_2(x) > 0$, we have $\lambda_3(x) < \log(\alpha)$ for every $x \in C$. So

$$\inf_{x \in C} \frac{-\lambda_3(f, x)}{2} \geq \frac{-\log(\alpha)}{2}$$

and this estimate is uniform in $\mathcal{V}$. Hence

$$\delta_0 \geq \Lambda_2(f) - \Lambda_2(f_1) \geq m(C)\frac{-\log(\alpha)}{2} - \delta.$$

Thus $m(C) \leq \frac{\delta + \delta_0}{-\log(\alpha)}$. Taking δ_0 and δ small, for $f \in \mathcal{V} \cap \mathrm{Diff}^{1+}_m(M)$, $m(C)$ is small enough. We observe that we can do this for the set C_{bad}, the union of all of the "bad" ergodic components. Then the measure of the "good" components is large enough (which are the same for all the iterates of f).

Finally we conclude the proof as follows. Taking $\mathcal{E}_n := \frac{1}{n}$-ergodic diffeomorphisms in $\mathrm{Diff}^{1+}_m \cap \mathcal{U}$, then $\mathcal{E}_n$ is open and dense in the C^1 induced topology , so $\mathcal{E} = \bigcap \mathcal{E}_n$ is a residual subset and $f \in \mathcal{E}$ is Bernoulli.

4.3.3 Proof of theorems C, D, and E

We follow the proof of [11], so we deal with the notion of ε-accessibility. That is, given $\varepsilon > 0$, g is ε-accessible if for any open Ball B of radius ε, the union of points that can be accessible from a point in B is the whole of M. We will also use the following lemmas that can be found in [11]:

Lemma 4.3.6. *If f is accessible, then for any $\varepsilon > 0$ there exists $l > 0$ and $R > 0$ such that for any $p, q \in M$ there exists a us-path that starts at p, ends within distance $\varepsilon/2$ of q an has at most l legs, each of them with length at most R. And there exist a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^2(M)$ such that any $g \in \mathcal{U}$ is ε-accessible.*

Lemma 4.3.7. *For any f ε-accessible every orbit is ε-dense (i.e. the set $\{f^n(x)\}_{n\in\mathcal{Z}}$ is an ε-net set). Also, if f is essentially accessible then almost every point has a dense orbit.*

Proof of theorem C. We observe that any stable/unstable leaf of f is also a stable/unstable leaf of any iterate of f, then f is (essentially) accessible if and only if f^k is (essentially) accessible for any $k \geq 0$. Then, the conclusions of Lemma 4.3.6 and 4.3.7 hold for any iterate of f (of course the neighborhood $\mathcal{U}$ and the constants can be smaller when the iterate growth).

So for any iterate f^k, $k \geq 0$, almost every point has a dense orbit. Then f^k is ergodic and non-uniformly hyperbolic by Theorem 4.3.3 for all $k \geq 0$. Thus, Theorem 4.3.1 implies that f is Bernoulli. □

Proof of theorem D. We will need the following result by Burns-Dolgopyat-Pesin :

Theorem 4.3.8 (Theorem 4 of [11]). *Let f be a $C^{1+\alpha}$ partially hyperbolic, volume preserving diffeomorphism. Assume that f is accessible and*

$$\int \log \|df|_{E^c_f(x)}\| d\mu < 0.$$

Then f is stably ergodic.

We recall that the *Pinsker partition* is the maximal partition with zero entropy. This partition was used by Pesin in the proof of Theorem 4.3.1, which we now consider. First of all, Pesin shows that any ergodic component Λ of a non-uniformly hyperbolic conservative $C^{1+\alpha}$ diffeomorphism can be decomposed $\Lambda = \bigcup_{i=1}^{N} \Lambda_i$ into disjoint sets such that $f(\Lambda_i) = \Lambda_{i+1} (i = 1, \dots, N-1)$, $f(\Lambda_N) = \Lambda_1$ and $f^N|\Lambda_1$ is a K-automorphism (in fact, it is Bernoulli). In Pesin's proof, the number N is the number of elements of the Pinsker partition of $f|\Lambda$ and this number is bounded from above by $1/\theta$, where θ is the measure of an open set defined by the union of unstable local manifolds along center-stable manifolds. Also, the elements of the Pinsker partition are permuted cyclically and all of them have the same measure.

We recall now lemmas 1 and 2 of [11]. The first states that there exists an $\alpha > 0$ such that for any $g \in \mathcal{U}$ there exists a subset A_g with positive measure such that any $x \in A_g$ has *hyperbolic times*, that is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j-0}^{n-1} \|Dg|_{E^c_g(g^j(x))}\| \leq -\alpha.$$

This is used to prove:

Lemma 4.3.9 (Lemma 2 of [11]). *There exists a neighborhood $\mathcal{V}$ of f and $s_0 > 0$ such that if $g \in \mathcal{V}$ is C^2 and $x \in A_g$ then there exists an integer $n \geq 0$ such that the size of $W^{cs}(g^{-n}(x))$ is at least s_0.*

So, if we take an atom of the Pinsker partition of $g \in \mathcal{V}$ which intersects A_g we have, by permutation, a "rectangle" contained in this atom with size at least s_0. So the measure of each atom is at least r_0 (the measure of the rectangle) for some $r_0 > 0$.

In particular, N is bounded by $1/r_0$ for any $g \in \mathcal{V}$. Take T the intersection of the neighborhoods such that f^j is stably ergodic for $j = 1, ..., N$ and $\mathcal{V}$. For any $g \in T$ we have that g^j is ergodic for any $j = 1, ..., N$ and hence all the iterates of g are ergodic. This shows that g is Bernoulli. $\square$

Proof of theorem E. By theorem C we know that f is Bernoulli and has negative exponents in the central direction almost everywhere. So, by ergodicity, there exists $\beta > 0$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \log \|df^n|_{E^c_f(x)}\| < -\beta \text{ a.e. } x \in M.$$

Hence there exists $n_0 > 0$ such that

$$\int_M \log \|df^{n_0}|_{E^c_f(x)}\| dm \leq -\beta.$$

The same estimate holds for any $n \geq n_0$ and then, by Theorem D, f^n is robustly Bernoulli for $n \geq n_0$. Also, as in the previous proof, if we take $\mathcal{V}$ the neighborhood of f such that if $g \in \mathcal{V}$ then g^{n_0} is Bernoulli and g^{n_0} satisfies the hypothesis of lemma 4.3.9, we obtain that g satisfies the same conclusion of that lemma. Reasoning as in the previous proof (analysing the Pinsker partition) we obtain a (uniform) bound for the number of atoms of the Pinsker partition for any $g \in \mathcal{W}$, where $\mathcal{W}$ is a smaller neighborhood of f. Thus there exists $m_0 \geq n_0$ such that if $g, ..., g^{m_0}$ are ergodic then g is Bernoulli. Now we take $\mathcal{V}_i$ the neighborhoods of f such that if $g \in \mathcal{V}_i$ then g^i is Bernoulli and consider $\mathcal{T} = (\bigcap_{i=1}^{m_0} \mathcal{V}_i) \cap \mathcal{W}$. This is a neighborhood of f such that any $g \in ST$ is Bernoulli, completing the proof. □

4.4 Examples

In this section we prove that an open set of robustly transitive and stably ergodic partially hyperbolic in $\mathcal{T}^3$ studied by Bonatti-Viana [10] are in fact stably Bernoulli.

We recall the construction of Bonatti and Viana [10]. Start with a linear Anosov diffeomorphism in $\mathcal{T}^3$ and fix a fixed point. Then perform a pitchfork bifurcation and obtain 2 hyperbolic fixed points with different indices of stability and make the two contracting eigenvalues of one of these fixed points to be complex. The resulting system is robustly transitive, stably ergodic, partially hyperbolic with central direction mostly contracting. That is, for m-almost every point x the Lyapunov exponents along the central direction are negative. The main property of this construction is that "in a neighborhood of this system every strong-stable leaf is dense in $\mathcal{T}^3$" and this implies robust transitiveness and, since there is uniform expansion in a neighborhood of the starting diffeomorphism, also gives topologically mixing.

We can prove that these systems have the Bernoulli property by our methods because they are partially hyperbolic with mostly contracting central direction and the denseness of the unstable manifolds holds for every iterate of any system in that neighborhood. So we obtain that every iterate is ergodic. This implies that the system is stably Bernoulli, in particular, robustly topologically mixing, because it preserves Lebesgue measure.

In the same way we can analyze the open set of ergodic diffeomorphisms (non-partially hyperbolic) studied by Tahzibi [18] and get that they are in fact Bernoulli. Indeed, let us define the open set $\mathcal{V}$ considered by Tahzibi. Let f_0 be an Anosov diffeomorphism on $\mathcal{T}^n$ whose foliations lifted to the universal covering are global graphs of C^1 functions. Let $V = \cup V_i$ be a finite union of small balls, such that f_0 has a periodic orbit q outside V. Then $f \in \mathcal{V}$ if:

- TM has small invariant continuous cone fields C^{cu} and C^{cs} containing E^u and E^s (the hyperbolic directions of f_0.
- f is C^1-close to f_0 on V^c. So there exist a $\sigma > 1$ such that

$$\|(DF|_{TxD^{cu}})-1\| < \sigma \text{ and } \|Df|_{TxD^{cs}}\| < \sigma.$$

- There exists some small δ_0 such that for $x \in V$:

$$\|(DF|_{TxD^{cu}})-1\| < 1+\delta_0 \text{ and } \|Df|_{TxD^{cs}}\| < 1+\delta_0.$$

Where D^{cu} and D^{cs} are disks tangent to C^{cu} and C^{cs}.

Theorem 4.4.1. *Every $f \in \mathcal{V} \cap \mathrm{Diff}^2_m(\mathcal{T}^n)$ is stably Bernoulli. Also, any $f \in \mathcal{V}$ having volume hyperbolic property for $E^{cu} \oplus E^{cs}$ has an unique SRB measure which is a Bernoulli measure with full Lebesgue measure basin.*

Demonstração. We proceed as in the proof in [18]. Given any $f \in \mathcal{V}$ we will prove that f is Bernoulli. By the arguments in [18] (using dominated splitting and volume hyperbolicity), there exist a $c_0 > 0$ and a full Lebesgue measure set H such that for $x \in H$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|(Df|_{E^{cu}_{f^i(x)}})^{-1}\| \leq -c_0,$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df|_{E^{cs}_{f^i(x)}}\| \leq -c_0.$$

Then, get a disk tangent to C^{cu} everywhere and intersecting H in a positive Lebesgue measure (of the disk). And construct an invariant measure ν that is an accumulation point of the sequence of averages of forward iterates of Lebesgue measure restricted to the disk. By [2, Proposition 4.1], there exist a cylinder $\mathcal{C}$ (diffeomorphic to $B^u \times B^s$, balls with dimension $\dim(E^{cs})$

and $\dim(E^{cu})$ respectively), and a family $\mathcal{K}_\infty$ of pairwise disjoint disks D_i contained in $\mathcal{C}$ which are graphics over B^u, and such that the union has positive ν measure and ν (restricted to that union) has absolutely continuous conditional measure along the disks in $\mathcal{K}_\infty$. This measure is used to construct a *cu*-Gibbs state such that an ergodic component has positive measure (with respect to this measure). Hence, can write $M = \cup B(\mu_i)$ where μ_i are *cu*-Gibbs states (in fact ergodic SRB measures). We observe that Tahzibi constructed stable and unstable manifolds and proved absolute continuity for these systems.

Now we use the fact that if V is small enough then the stable manifold of q intersects any disk tangent to C^{cu} with radius bigger than ε_0 (for some small ε_0), the same holds for the unstable manifold. With this we have

Proposition 4.4.2 (Proposition 5.1 of [18])**.** *The stable manifold of q is dense and intersects transversally each D_i.*

Following Tahzibi, with help of this proposition, we can prove that the intersection of the basins $B(\mu_i)$ of each μ_i is non-empty. Because the $\mu_i's$ are ergodic, the $\mu_i's$ are all the same and hence f is ergodic. Now we stress that the construction of the $\mu_i's$ is getting ergodic components of the original measure. So, in the end, all of the $\mu_i's$ are equal to this measure.

Now fix $k > 0$. Since, by construction, ν is invariant for all f^k we can repeat the argument above. For each i we can write $B(\mu_i) = \cup B(\mu_i^k)$ where μ_i^k are the f^k-ergodic components of μ_i, and there exist $D_i^{k,\infty}$ a disk on $\mathcal{K}_\infty$ contained in $B(\mu_i^k)$ with the same property as D_i. The number of ergodic components can grow up, but to prove that their basins itersects depend only from proposition 4.4.2, which also holds for any iterate f^k. So again, all the μ_i^k are equal to the original measure μ_i, which coincides with ν. Thus, f^k is also ergodic all k.

Now applying Theorem 4.3.1 we conclude that f is Bernoulli. The dissipative case is analogous. The proof of Theorem 4.4.1 is complete.

□

Remark 4.4.3. We observe that this open set has robustly topologically mixing non partially hyperbolic diffeomorphisms.

4.5 Final Remarks

We point out that as we can obtain "generic" statements of our theorems using some "generic" tools as for example, the following theorem:

Theorem 4.5.1 (Dolgopyat, Wilkinson)**.** *Generically a strongly partially hyperbolic diffeomorphism f is stably accessible.*

So we can drop the accessibility hypothesis over a residual set.

Finally, in the volume-preserving case we can use a strong corollary by Bonatti and Crovisier [7]:

Theorem 4.5.2. *Generically a volume preserving diffeomorphism is transitive.*

So using theorem 4.5.2 and, if one could prove that generic robustly transitive systems are topologically mixing (as *indicate* the results in [1]), we can drop the robust topologically mixing condition over a generic set of partially hyperbolic diffeomorphisms.

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Capítulo 5

Geometrical *versus* Topological Properties of Manifolds

Os resultados abaixo foram obtidos em conjunto com Krerley Oliveira no artigo *Geometrical* versus *Topological Properties of Manifolds*. Abaixo segue o conteúdo (em inglês) deste artigo.

5.1 Introduction

Let $f : M^n \to N^m$ be a C^1 map. We denote by

$$rank(f) := \min_{p \in M} rank(D_p f).$$

If $n = \dim M = \dim N = m$, let $C := \{p \in M : \det D_p f = 0\}$ the set of *critical points* of f and $S := f(C)$ the set of *critical values* of f.

Now, let M^n a compact, connected, boundaryless, n-dimensional manifold. Denote by H_s the s-dimensional Hausdorff measure and $\dim_H(A)$ the Hausdorff dimension of $A \subset M^n$. For definitions see section 5.2 below. Let x an immersion $x : M^n \to \mathbb{R}^{n+1}$. In this case, let $G : M^n \to S^n$ the Gauss map associated to x, C the critical points of G and S the critical values of G. We denote by $dim_H(x) := dim_H(S)$. By Moreira's improvement of Morse-Sard theorem (see [Mo]), since G is a smooth map, we have that $\dim_H(S) \leq n-1$.

In other words, if $\mathcal{I}mm = \{x : M \to \mathbb{R}^{n+1} : x \text{ is an immersion}\}$, then $\sup_{x \in \mathcal{I}mm} dim_H(x) \leq n-1$. Clearly, this supremum could be equal to $n-1$, as some immersions of S^n in $\mathbb{R}^{n+1}$ show (e.g., immersions with "cylindrical pieces"). Our interest here is the number $\inf dim_H(x)$. Before discuss this, we introduce some definitions.

Definition 5.1.1. Given an immersion $x : M^n \to \mathbb{R}^{n+1}$ we define $rank(x) := rank(G)$, where G is the Gauss map for x.

Definition 5.1.2. We denote by $\mathcal{R}(k)$ the set $\mathcal{R}(k) = \{x \in \mathcal{I}mm : rank(x) \geq k\}$. Define by $\alpha_k(M)$ the numbers:

$$\alpha_k(M) = \inf_{x\in\mathcal{R}(k)} dim_H(x), k = 0, \ldots, n$$

If $\mathcal{R}(k) = \emptyset$ we define $\alpha_k(M) = n - 1$.

Now, we are in position to state our first result:

Theorem F. *If M^n is a compact manifold such that $\alpha_k(M^n) < k - [\frac{n}{2}]$, for some integer k, then $M^n \simeq S^n$ ($[r]$ is the integer part of r).*

The proof of this theorem in the cases $n = 3$ and $n \geq 4$ are quite different. For higher dimensions, we can use the generalized Poincaré Conjecture (Smale and Freedman) to obtain that the given manifold is a sphere. Since the Poincaré Conjecture is not available in three dimensions, the proof, in this case, is a little bit different. We use a characterization theorem due to Bing to compensate the loss of Poincaré Conjecture, as commented before.

To prove this theorem in the case $n = 3$, we proceed as follows:

- By a theorem of Bing (see [B]), we just need to prove that every piecewise smooth simple curve γ in M^3 lies in a topological cube $\mathcal{R}$ of M^3;
- In order to prove it, we shall show that it is enough to prove for $\gamma \subset M - G^{-1}(S)$ and that $G : M - G^{-1}(S) \to S^3 - S$ is a diffeomorphism;
- Finally, we produce a cube $\tilde{\mathcal{R}} \supset G(\gamma)$ in $S^3 - S$ and we obtain $\mathcal{R}$ pulling back this cube by G

Observe that by [C], in three dimensions always there are Euclidean codimension 1 immersions. In particular, it is reasonable to consider the following consequence of the Theorem A:

Corollary 5.1.3. *The following statement is equivalent to Poincaré Conjecture : "Simply connected 3-manifolds admits Euclidean codimension one immersions with rank at least 2 and Hausdorff dimension of the singular set for his Gauss map less than 1".*

For a motivation of this conjecture and some comments about three dimensional manifolds see the section 5.7.

Our motivation in this theorem are results by do Carmo, Elbert [dCE] and Barbosa, Fukuoka, Mercuri [BFM]. Roughly speaking, they obtain topological results about certain manifolds provides that there are special codimension 1 immersions of them. These results motivates the question : how the space of immersions (extrinsic information) influenciates the topology of M (intrinsic information)? The theorems A and B are a partial answer to this question. The theorems needs the concept of Hausdorff dimension. Essentially, Hausdorff dimension is a fractal dimension that measures how "small" is a given set with respect to usual "regular" sets (e.g., smooth submanifolds, that always has integer Hausdorff dimension).

In section 6 of this paper we obtain the following generalizations of theorems A, B, the results of do Carmo, Elbert [dCE] and Barbosa, Fukuoka, Mercuri [BFM].

Definition 5.1.4. Let $\overline{M}^n$ a compact (oriented) manifold and $p_1, \ldots, p_k \in \overline{M}^n$. Let $M = \overline{M}^n - \{p_1, \ldots, p_k\}$. An immersion $x : M^n \to \mathbb{R}^{n+1}$ is of *finite geometrical type* (in a weaker sense than that of [BFM]) if M^n is complete in the induced metric, the Gauss map $G : M^n \to S^n$ extends continuously to a function $\overline{G} : \overline{M}^n \to S^n$ and the set $G^{-1}(S)$ has $H_{n-1}(G^{-1}(S)) = 0$ (this last condition occurs if $rank(x) \geq m$ and $H_{m-1}(S) = 0$).

The conditions in the previous definition are satisfied by complete hypersurfaces with finite total curvature whose Gauss-Kronecker curvature $H_n = k_1 \ldots k_n$ does not change of sign and vanish in a small set, as showed by [dCE]. Recall that a hypersurface $x : M^n \to \mathbb{R}^{n+1}$ has total finite curvature if $\int_M |A|^n dM < \infty$, $|A| = (\sum_i k_i^2)^{1/2}$, k_i are the principal curvatures. With this observations, one has :

Theorem G. *If $x : M^n \to \mathbb{R}^{n+1}$ is a hypersurface with finite geometrical type and $H_{k-[\frac{n}{2}]}(S) = 0$, $rank(x) \geq k$. Then M^n is topologically a sphere minus a finite number of points, i.e., $\overline{M}^n \simeq S^n$. In particular, this holds for complete hypersurfaces with finite total curvature and $H_{k-[\frac{n}{2}]}(S) = 0$, $rank(x) \geq k$.*

For even dimensions, we follow [BFM] and improve theorem G. In particular, we obtain the following characterization of $2n$-catenoids, as the unique minimal hypersurfaces of finite geometrical type.

Theorem H. *Let $x : M^{2n} \to \mathbb{R}^{2n+1}$, $n \geq 2$ an immersion of finite geometrical type with $H_{k-n}(S) = 0$, $rank(x) \geq k$. Then M^{2n} is topologically a sphere minus two points. If M^{2n} is minimal, M^{2n} is a $2n$-catenoid.*

5.2 Notations and Statements

Let M^n be a smooth manifold. Before starting the proof of the statements we fix some notations and collect some (useful) standard propositions about Hausdorff dimension (and limit capacity, another fractal dimension). For the proofs of these propositions we refer [Fa].

Let X a compact metric space and $A \subset X$. We define the s-dimensional Hausdorff measure of A by

$$H_s(A) := \lim_{\varepsilon \to 0} \inf\{\sum_i (diam\, U_i)^s : A \subset \bigcup U_i, U_i \text{ is open and } diam(U_i) \leq \varepsilon, \forall i \in \mathbb{N}\}.$$

The *Hausdorff dimension* of A is $\dim_H(A) := \sup\{d \geq 0 : H_d(A) = \infty\} = \inf\{d \geq 0 : H_d(A) = 0\}$. A remarkable fact is that H_n coincides with Lebesgue measure in smooth manifolds M^n.

A related notion are the lower and upper *limit capacity* (sometimes called box counting dimension) defined by

$$\underline{\dim_B}(A) := \liminf_{\varepsilon \to 0} \log n(A, \varepsilon)/(-\log \varepsilon),\ \overline{\dim_B}(A) := \limsup_{\varepsilon \to 0} \log n(A, \varepsilon)/(-\log \varepsilon),$$

where $n(A, \varepsilon)$ is the minimum number of ε-balls that cover A. If $d(A) = \underline{\dim_B}(A) = \overline{\dim_B}(A)$, we say that the limit capacity of A is $\dim_B(A) = d(A)$.

These fractal dimensions satisfy the properties expected for "natural" notions of dimensions. For instance, $\dim_H(A) = m$ if A is a smooth m-submanifold.

Proposition 5.2.1. *The properties listed below hold :*

1. $\dim_H(E) \leq \dim_H(F)$ *if* $E \subset F$;
2. $\dim_H(E \cup F) = \max\{\dim_H(E), \dim_H(F)\}$;
3. *If f is a Lipschitz map with Lipschitz constant C, then $H_s(f(E)) \leq C \cdot H_s(E)$. As a consequence,* $\dim_H(f(E)) \leq \dim_H E$;

4. If f is a bi-Lipschitz map (e.g., diffeomorphisms), $\dim_H(f(E)) = \dim_H(E)$*;*

5. $dim_H(A) \leq \underline{dim_B}(A)$*.*

Analogous properties holds for lower and upper limit capacity. If E is countable, $\dim_H(E) = 0$ *(although we may have* $\dim_B(E) > 0$*).*

When we are dealing with product spaces, the relationship between Hausdorff dimension and limit capacity are the product formulae :

Proposition 5.2.2. $\dim_H(E) + \dim_H(F) \leq \dim_H(E \times F) \leq \dim_H(E) + \overline{\dim_B}(F)$*. Moreover,* $c \cdot H_s(E) \cdot H_t(F) \leq H_{s+t}(E \times F) \leq C \cdot H_s(E)$*, where c depends only on s and t, C depends only on s and* $\overline{\dim_B}(F)$*.*

Before stating the necessary lemmas to prove the central results, we observe that follows from lemma above that if M and N are diffeomorphic n-manifolds then $\alpha_k(M) = \alpha_k(N)$. This proves :

Lemma 5.2.3. *The numbers*

$$\alpha_k(M) = \inf_{x \in \mathcal{R}(k)} dim_H(x), \text{ for } k = 0, \ldots, n$$

are smooth invariants of M.

In particular, if $n = 3$ we also have that α_k are topological invariants. It is a consequence of a theorem due to Moise [M], which state that if M and N are homeomorphic 3-manifolds then they are diffeomorphic. Then, the following conjecture arises from the Theorem F

Conjecture 1. *If M^3 is simply connected, then*

$$\alpha_2(M^3) = \inf_{x \in \mathcal{R}(2)} dim_H(x) < 1$$

R. Cohen's theorem [C] says that there are immersions of compact n-manifolds M^n in $\mathbb{R}^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of 1's in the binary expansion of n. This implies, for the case $n = 3$, we always have that $\mathcal{I}mm \neq \emptyset$. In particular, the implicit hypothesis of existence of codimension 1 immersions in theorem F is not too restrictive and our conjecture is reasonable. We point out that conjecture 1 is true if Poincaré conjecture holds and, in this case, $\sup_{x \in \mathcal{I}mm} rank(x) = 3$ and $\inf_{x \in \mathcal{R}(k)} dim_H(x) = 0$, for all $0 \leq k \leq 3$. A corollary of the theorem F and this observation is:

Corollary 5.2.4. *The Poincaré Conjecture is equivalent to the conjecture 1.*

From this, a natural approach to conjecture 1 is a deformation and desingularization argument for metrics given by pull-back of immersions in $\mathcal{I}mm$. We observe that Moreira's theorem give us $\alpha_2(M^3) \leq 2$. This motivates the following question, which is a kind of step toward Poincaré Conjecture. However, this question is of independent interest, since it can be true even if Poincaré Conjecture is false :

Question 1. *For simply connected* 3*-manifolds, is true that* $\alpha_2(M^3) < 2$ *?*

5.3 Some lemmas

In this section, we prove some useful facts in the way to establish the theorems F, B. The first one relates the Hausdorff dimension of subsets of smooth manifolds and rank of smooth maps :

Proposition 5.3.1. *Let* $f : M^m \to N^n$ *a* C^1*-map and* $A \subset N$. *Then* $\dim_H f^{-1}(A) \leq \dim_H(A) + n - rank(f)$.

Demonstração. The computation of Hausdorff dimension is a local problem. So, we can consider $p \in f^{-1}(S)$, coordinate neighborhoods $p \in U$, $f(p) \in V$ fixed and $f = (f_1, \ldots, f_n) : U \to V$. Making a change of coordinates (which does not change Hausdorff dimensions), we can suppose that $\widetilde{f} = (f_1, \ldots, f_r)$ is a submersion, where $r = rank(f)$. By the local form of submersions, there is φ a diffeomorphism s.t. $\widetilde{f} \circ \varphi(y_1, \ldots, y_m) = (y_1, \ldots, y_r)$. This implies that $f \circ \varphi(y_1, \ldots, y_m) = (y_1, \ldots, y_r, g(\varphi(y_1, \ldots, y_m))$. Then, if π denotes the projection in the r first variables, $x \in f^{-1}(S) \Rightarrow \pi\varphi^{-1}(x) \in \pi(S)$, i.e., $f^{-1}(S) \subset \varphi(\pi(S) \times \mathbb{R}^{n-r})$. By properties of Hausdorff dimension (see section 2), we have $dim_H f^{-1}(S) \leq dim_H(\pi(S) \times \mathbb{R}^{n-r}) \leq dim_H \pi(S) + \overline{dim_B}(\mathbb{R}^{n-r}) \leq dim_H(S) + n - r$. This concludes the proof. □

The second proposition relates Hausdorff dimension with topological results.

Proposition 5.3.2. *Let* $n \geq 3$ *and* F *is a closed subset of a* n*-dimensional connected (not necessarily compact) manifold* M^n. *If the Hausdorff dimension of* F *is strictly less than* $n-1$ *then* $M^n - F$ *is connected. If* $M^n = \mathbb{R}^n$ *or* $M^n = S^n$, F *is compact and the Hausdorff dimension of* F *is strictly less* *than $n-k-1$ then $M^n - F$ is k-connected (i.e., its homotopy groups π_i vanishes for $i \leq k$).*

Demonstração. First, if F is a closed subset of M^n with Hausdorff dimension strictly less than $n-1$, $x, y \in M^n - F$, take γ a path from x to y in M^n. Since $n \geq 3$, we can suppose γ a smooth simple curve (by transversality). In this case, γ admits some compact tubular neighborhood $\mathcal{L}$. For each $p \in \gamma$, denote $\mathcal{L}_p$ the $\mathcal{L}$-fiber passing throught p. By hypothesis, $dim_H(F \cap \mathcal{L}_p) < n-1 \; \forall p$. In this case, the tubular neighborhood $\mathcal{L}$ is diffeomorphic to $\gamma \times D^{n-1}$, the fibers $\mathcal{L}_p$ are $p \times D^{n-1}$ (D^{n-1} is the $(n-1)$-dimensional unit disk centered at 0) and γ is $\gamma \times 0$. Then, since F is closed, it is easy that every $x \in \gamma$ admits a neighborhood $V(x)$ s.t. for some sequence $v_n = v_n(x) \to 0$ holds $(V(x) \times v_n) \cap F = \emptyset$. Moreover, again by the fact that F is closed, any vector v sufficiently close to some v_n satisfies $(V(x) \times v) \cap F = \emptyset$. With this in mind, by compactness of γ, we get some finite cover of γ by neighborhoods as described before. This guarantees the existence of v_0 arbitrarily small s.t. $(\gamma \times v_0) \cap F = \emptyset$. This implies that $M - F$ is connected.

Second, if F is a compact subset of $M^n = \mathbb{R}^n$, $dim_H F < n - k - 1$, let $[\Gamma] \in \pi_i(\mathbb{R}^n - F)$ a homotopy class for $i \leq k$. Choose a smooth representant $\Gamma \in [\Gamma]$. Define $f : \Gamma \times F \to S^{n-1}$, $f(x,y) := (y-x)/||y-x||$. We will consider in $\Gamma \times F$ the sum norm, i.e., if $p, q \in \Gamma \times F$, $p = (x,y), q = (z,w)$ then $||p-q|| := ||x-z|| + ||y-w||$. For this choice of norm we have

$$||f(p)-f(q)|| = \frac{1}{||y-x|| \cdot ||z-w||} \cdot \Big\| \big\{ (y-x) \cdot ||z-w|| + ||y-x|| \cdot (z-w) \big\} \Big\| \Rightarrow$$

$$||f(p)-f(q)|| \leq \frac{\Big\| (y-x) \cdot ||z-w|| - ||z-w|| \cdot (w-z) \Big\|}{||y-x|| \cdot ||z-w||} + \frac{\Big\| ||z-w|| \cdot (w-z) - ||y-x|| \cdot (w-z) \Big\|}{||y-x|| \cdot ||z-w||} \cdot \Rightarrow$$

$$||f(p)-f(q)|| \leq \frac{1}{||y-x||} \cdot \Big\{ ||(z-x)+(y-w)|| \Big\} + \frac{1}{||y-x||} \cdot \big| \{ ||(z-w)|| - ||(y-x)|| \} \big| \Rightarrow$$

$$||f(p)-f(q)|| \leq 2 \cdot C \cdot ||p-q||$$

where $C = 1/d(\Gamma, F)$. We have $d(\Gamma, F) > 0$ since these are compact disjoint sets. This computation shows that f is Lipschitz.

Then, we have (prop. 5.2.1, 5.2.2) $dim_H f(\Gamma \times F) \leq dim_H(\Gamma \times F) \leq \overline{dim_B}(\Gamma) + dim_H(F) < i + n - k - 1 \leq n - 1 \Rightarrow \exists \; v \notin f(\Gamma \times F)$. Now, F is compact implies that there is a real N s.t. $F \subset B_N(0)$. Then, making a translation of Γ at v direction, we can put, using this translation as homotopy, Γ outside B_N. Since $\mathbb{R}^n - B_N$ is n-connected (for $n \geq 3$), $\pi_i(\mathbb{R}^n - F) = 0$. This concludes the proof. □

Remark 5.3.3. We remark that the hypothesis F is closed in the previous proposition is necessary. For example, take $F = \mathbb{Q}^n$, $M^n = \mathbb{R}^n$. We have $dim_H(F) = 0$ (F is a countable set) but $M^n - F$ is not connected.

We can think proposition 5.3.2 as a weak type of transversality. In fact, if F is a compact $(n-2)$-submanifold of M^n then $M - F$ is connected and if F is a compact $(n-3)$-submanifold of $\mathbb{R}^n$ (or S^n) then $\mathbb{R}^n - F$ is simply connected. This follows from basic transversality. However, our previous proposition does not assume regularity of F, but allows us to conclude the same results. It is natural these results are true because Hausdorff dimension translates the fact that F is, in some sense, "smaller" than a $(n-1)$-submanifold N which has optimal dimension in order to disconnect M^n.

For later use, we generalize the first part of proposition 5.3.2 as follows :

Lemma 5.3.4. *Suppose that $\Gamma \in \pi_i(M^n)$ is Lipschitz (e.g., if $i = 1$ and Γ is a piecewise smooth curve) and let $K \subset M^n$ compact, $dim_H K < n - i$. Then there are diffeomorphisms h of M, arbitrarily close to identity map, s.t. $h(\Gamma) \cap K = \emptyset$. In particular, if $[\Gamma] \in \pi_i(M^n)$ a homotopy class, $K \subset M^n$ a compact set, $dim_H(K) < n - i$, there is a smooth representant $\Gamma \in [\Gamma]$ s.t. $\Gamma \cap K = \emptyset$, i.e., $\Gamma \in \pi_i(M^n - K)$.*

Demonstração. First, consider a parametrized neighbourhood $\phi : U \to B_3(0) \subset \mathbb{R}^n$ and suppose that Γ lies in $\overline{V_1}$, where $V_1 = \phi^{-1}(B_1(0))$. Let $K_1 = \phi(K) \subset \mathbb{R}^n$ and $\Gamma_1 = \phi(\Gamma) \subset \mathbb{R}^n$. Consider the map:

$$F : \Gamma_1 \times K_1 \to \mathbb{R}^n \text{ defined by } F(x, y) = x - y$$

Observe that, since Γ is Lipschitz and ϕ is a diffeomorphism, $\overline{dim_B}\Gamma = \overline{dim_B}\Gamma_1 \leq i$. This implies that $dim_H(F(\Gamma_1 \times K_1)) < n$, since $dim_H(K) < n - i$. This implies, in particular, that $\mathbb{R}^n - F(\Gamma_1 \times K_1)$ is an open and dense subset, since K is compact. Then, we may choose a vector $v \in \mathbb{R}^n - F(\Gamma_1 \times K_1)$ arbitrarily close to 0 such $(\Gamma_1 + v) \subset B_2(0)$. Since, $v \in \mathbb{R}^n - F(\Gamma_1 \times K_1)$ we have that $(\Gamma_1 + v) \cap K_1 = \emptyset$.

To construct h we consider a bump function $\beta : \mathbb{R}^n \to [0, 1]$, such that $\beta(x) = 1$ if $x \in B_1(0)$ and $\beta(x) = 0$ for every $x \in \mathbb{R}^n - B_2(0)$. It is easy to see that h defined by:

$$h(y) = y \text{ if } x \in M - U \text{ and } h(y) = \phi^{-1}(\beta(\phi(y))v + \phi(y)),$$

is a diffeomorphism that satisfies $h(\Gamma) \cap K = \emptyset$, since $(\Gamma_1 + v) \cap K_1 = \emptyset$.

In the general case, we proceed as follows : first, considering a finite number of parametrized neighbourhoods $\phi_i : U_i \to B_3(0), i \in \{1, \dots, n\}$ and $V_i = \phi_i^{-1}(B_1(0))$ covering Γ, by the previous case, there exists h_1 arbitrarily close to the identity such $h_1(\Gamma) \subset \bigcup_{i=1}^{n} V_i$ and such that $h_1(\Gamma \cap \overline{V_1}) \cap K = \emptyset$. Observe that, $d(h_1(\Gamma \cap \overline{V_1}), K) > \epsilon_1 > 0$, since $h_1(\Gamma \cap \overline{V_1})$ is a compact set.

The next step is to repeat the previous argument considering h_2 arbitrarily close to the identity, in such way that $h_2(h_1(\Gamma) \cap V_2) \cap K = \emptyset$ and $h_2(h_1(\Gamma)) \subset \bigcup_{i=1}^{n} V_i$. If $d(h_2, id) < \frac{\epsilon_1}{2}$ then $h_2(h_1(\Gamma) \cap V_1) \cap K = \emptyset$. Repeating this argument by induction, we obtain that $h = h_n \circ \cdots \circ h_1$ is a diffeomorphism such that $h(\Gamma) \cap K = \emptyset$. This concludes the proof. $\square$

5.4 Proof of Theorem A in the case $n = 3$

Before giving a proof for theorem F, we mention a lemma due to Bing [B] :

Lemma 5.4.1 (Bing)**.** *A compact, connected, 3-manifold M is topologically S^3 if and only if each piecewise smooth simple closed curve in M lies in a topological cube in M.*

A modern proof of this lemma can be found in [R]. In modern language, Bing's proof shows that the hypothesis above imply that *Heegaard* splitting of M is in two balls. This is sufficient to conclude the result.

In fact, Bing's theorem is not stated in [B], [R] as above. But the lemma holds. Actually, to prove that M is homeomorphic to S^3, Bing uses only that, if a triangulation of M is fixed, every simple *polyhedral* closed curve lies in a topological cube. Observe that *polyhedral* curves are piecewise smooth curves, if we choose a smooth triangulation (smooth manifolds always can be smooth triangulated, see [T], page 194; see also [W], page 124).

Proof of theorem A in the case $n = 3$. If $\alpha_2(M) < 1$, there is an immersion $x : M^3 \to \mathbb{R}^4$ s.t. $rank(x) \geq 2, dim_H(x) < 1$. Let G the Gauss map associated to x. By propositions 5.3.2, 5.3.1, since $dim_H(S) < 1$, $M - G^{-1}(S), S^3 - S$ are connected manifolds. Consider $G : M - G^{-1}(S) \to S^3 - S$. This is a proper map between connected manifolds whose jacobian never vanishes. So it is a surjective and covering map (see [WG]). Since, moreover, $S^3 - S$ is simply connected (by proposition 5.3.2), $G : M - G^{-1}(S) \to S^3 - S$ is a diffeomorphism. To prove that M^3 is homeomorphic to S^3, it is necessary and sufficient that every piecewise smooth simple closed curve $\gamma \subset M^3$ is contained in a topological cube $Q \subset M^3$ (by lemma 5.4.1).

In order to prove that every piecewise smooth curve γ lies in a topological cube, observe that we may suppose that $\gamma \cap K = \emptyset$ (here $K = G^{-1}(S)$). Indeed, by lemma 5.3.4 there exists a diffeomorphism h of M such $h(\gamma) \cap K = \emptyset$. Then, if $h(\gamma)$ lies in a topological cube R, the γ itself lies in the topological cube $h^{-1}(R)$ too, thus we can, in fact, make this assumption.

Now, since $\gamma \subset M - K$ and $M - K$ is diffeomorphic to $S^3 - S$, we may consider $\gamma \subset \mathbb{R}^3 - S$, S a compact subset of $\mathbb{R}^3$ with Hausdorff dimension less than 1 via identification by the diffeomorphism G and stereographic projection. In this case, we can follow the proof of proposition 5.3.2 to conclude that $f : \gamma \times S \to S^2$, $f(x, y) = (x-y)/||x-y||$ is Lipschitz. Because $\overline{\dim}_B \gamma \leq 1, \dim_H S < 1$ (here we are using that γ is piecewise smooth), we obtain a direction $v \in S^2$ s.t. $F := \bigcup_{t \in \mathbb{R}} (L_t(\gamma))$ is disjoint from S, where $L_t(p) := p + t \cdot v$. By compactness of γ it is easy that F is a closed subset of $\mathbb{R}^n$. This implies that $3\ \epsilon = d(F, S) > 0$. Consider $F_\epsilon = \{x : d(x, F) \leq \epsilon\}$ and $S_\epsilon = \{x : d(x, S) \leq \epsilon\}$. By definition of $\epsilon > 0$, $F_\epsilon \cap S_\epsilon = \emptyset$, then we can choose $\varphi : \mathbb{R}^3 \to \mathbb{R}$ a smooth function s.t. $\varphi|_{F_\epsilon} = 1$, $\varphi|_{S_\epsilon} = 0$. Consider the vector field $X(p) = \varphi(p) \cdot v$ and let X_t, $t \in \mathbb{R}$ the X-flow. We have $X_t(p) = p + tv\ \forall p \in \gamma$ and $X_t(p) = p\ \forall p \in S$, for any $t \in \mathbb{R}$. Choosing N real s.t. $S \subset B_N(0)$ and T s.t. $t \geq T \Rightarrow L_t(\gamma) \cap B_N(0) = \emptyset$, we obtain a global homeomorphism X_t which sends γ outside $B_N(0)$ and keep fixed S, $\forall\ t \geq T$.

Observe that $X_t(\gamma)$ is contained in the interior of a topological cube $Q \subset \mathbb{R}^3 - B_N(0)$. Then, observing that X_t is a diffeomorphism and that $X_t(x) = x$ for every $x \in S$ and $t \in \mathbb{R}$, we have that $\gamma \subset X_{-t}(Q) \subset \mathbb{R}^3 - S$, $\forall\ t \geq T$. This concludes the proof. □

5.5 Proof of Theorem A in the case $n \geq 4$

We start this section with the statement of generalized Poincaré Conjecture :

Theorem 5.5.1. *A compact simply connected homological sphere M^n is homeomorphic to S^n, if $n \geq 4$ (diffeomorphic for $n = 5, 6$).*

The proof of generalized Poincaré Conjecture is due to Smale [S] for $n \geq 5$

and to Freedman [F] for $n = 4$. This lemma makes the proof of the theorem B a little bit easier than the proof of theorem A.

Proof of Theorem A in the case $n \geq 4$. If $k = n$, there is nothing to prove. Indeed, in this case, $G : M^n \to S^n$ is a diffeomorphism, by definition. I.e., without loss of generality we can suppose $k \leq n-1$; $\alpha_k(M) < k - [\frac{n}{2}] \Rightarrow \exists\ x : M^n \to \mathbb{R}^{n+1}$ immersion, $rank(x) \geq k$, $dim_H(x) < k - [\frac{n}{2}]$. The hypothesis implies that $M - G^{-1}(S)$ is connected, $S^n - S$ is simply connected and G is a proper map whose jacobian never vanishes. By [WG], G is a surjective, covering map. So, we conclude that $G : M - G^{-1}(S) \to S^n - S$ is diffeomorphism. But $S^n - S$ is $(n-1-k+[\frac{n}{2}])$-connected, by proposition 5.3.2. In particular, because $k \leq n-1$, $S^n - S$ is $[\frac{n}{2}]$-connected and so, using the diffeomorphism G, $M - K$ is $[\frac{n}{2}]$-connected, where $K = G^{-1}(S)$. It is sufficient to prove that M^n is a simply connected homological sphere, by theorem 5.5.1. By lemma 5.3.4, $M - K$ is $[\frac{n}{2}]$-connected and $dim_H(K) < n - [\frac{n}{2}]$ (by prop.5.3.2) implies M itself is $[\frac{n}{2}]$-connected. It is know that $H^i(M) = L(H_i(M)) \oplus T(H_{i-1}(M))$, L and T denotes the free part and the torsion part of the group. By Poincaré duality, $H_{n-i}(M) \simeq H^i(M)$. The fact that M is $[\frac{n}{2}]$-connected and these informations give us $H_i(M) = 0$, for $0 < i < n$. This concludes the proof. □

5.6 Proof of theorems B and C

In this section we make some comments on extensions of theorem F. Although these extensions are quite easy, they were omitted so far to make the presentation of the paper more clear. Now, we are going to improve our previous results. First, all preceding arguments works with assumption that $H_{k-[\frac{n}{2}]}(S) = 0$ and $rank(x) \geq k$ in theorems A, B (where H_s is the s-dimensional Hausdorff measure). We prefer consider the hypothesis as its stands in these theorems because it is more interesting define the invariants $\alpha_k(M)$. The reason to this "new" hypothesis works is that our proofs, essentially, depends on the existence of special directions $v \in S^{n-1}$. But this directions exists if the singular sets have Hausdorff measure 0. Second, M need not to be compact. It is sufficient that M is of *finite geometric type* (here finite geometrical type is a little bit different from [BFM]). We will make more precise these comments in proof of theorem 6.2 below, after recall the definition :

Definition 5.6.1. Let $\overline{M}^n$ a compact (oriented) manifold and $q_1, \ldots, q_k \in \overline{M}^n$. Let $M^n = \overline{M}^n - \{q_1, \ldots, q_k\}$. An immersion $x : M^n \to \mathbb{R}^{n+1}$ is of finite geometrical type if M^n is complete in the induced metric, the Gauss map $G : M^n \to S^n$ extends continuously to a function $\overline{G} : \overline{M}^n \to S^n$ and the set $G^{-1}(S)$ has $H_{n-1}(G^{-1}(S)) = 0$ (this last condition occurs if $rank(x) \geq k$ and $\dim_H(x) < k - 1$, or more generally, if $rank(x) \geq m$ and $H_{m-1}(S) = 0$).

As pointed out in the introduction, the conditions in the previous definition are satisfied, for example, by complete hypersurfaces with finite total curvature whose Gauss-Kronecker curvature $H_n = k_1 \ldots k_n$ does not change of sign and vanish in a small set, as showed by [dCE]. Recall that a hypersurface $x : M^n \to \mathbb{R}^{n+1}$ has total finite curvature if $\int_M |A|^n dM < \infty$, $|A| = (\sum_i k_i^2)^{1/2}$, k_i are the principal curvatures. Then, there are examples satisfying the definition. With this observations, it is interesting to show our theorem C. Recall that the statement of this theorem is :

Theorem 5.6.2 (Theorem G)**.** *If $x : M^n \to \mathbb{R}^{n+1}$ is a hypersurface with finite geometrical type and $H_{k-[\frac{n}{2}]}(S) = 0$, $rank(x) \geq k$. Then M^n is topologically a sphere minus a finite number of points, i.e., $\overline{M}^n \simeq S^n$. In particular, this holds for complete hypersurfaces with finite total curvature and $H_{k-[\frac{n}{2}]}(S) = 0$, $rank(x) \geq k$.*

Proof of theorem G. To avoid unnecessary repetitions, we will only indicate the principal modifications needed in proof of theorems A, B by stating "new" propositions, which are analogous to the previous ones, and making few comments in their proofs. The details are left to reader.

Proposition 5.6.3 (Prop. 5.3.1')**.** *Let $f : M^m \to N^n$ a C^1-map and $A \subset N$. If $H_s(A) = 0$, then $H_{s+n-rank(f)}(f^{-1}(A)) = 0$.*

Demonstração. It suffices to show that for any $p \in f^{-1}(A)$, there is an open set $U = U(p) \ni p$ s.t. $H_{s+n-r}(f^{-1}(A) \cap U) = 0$. However, if U is chosen as in proof of proposition 5.3.1, we have $f^{-1}(A) \cap U \subset \varphi(\pi(A) \times \mathbb{R}^{n-r})$, where φ is a diffeomorphism, $r = rank(f)$ and π is the projection in first r variables. By propositions 5.2.1, 5.2.2, $H_{s+n-r}(f^{-1}(A) \cap U) \leq C_1 \cdot H_{s+n-r}(\pi(A) \times \mathbb{R}^{n-r}) \leq C_1 \cdot C_2 \cdot H_s(A) = 0$, where C_1 depends only on φ and C_2 depends only on $(n - r)$. This finishes the proof. □

Proposition 5.6.4 (Prop. 5.3.2'). *Let $n \geq 3$ and F a closed subset of M^n s.t. $H_{n-1}(F) = 0$ then $M - F$ is connected. If $M^n = \mathbb{R}^n$ or $M^n = S^n$, F is compact and $H_{n-k}(F) = 0$ then $M^n - F$ is k-connected.*

Demonstração. First, if γ is a path in M^n from x to y, $x, y \notin F$, we can suppose γ a smooth simple curve. So, there is a compact tubular neighborhood $\mathcal{L} = \gamma \times D^{n-1}$ of γ. Since $dim(\mathcal{L}_p) = n-1$, $F \cap \mathcal{L}_p$ has Lebesgue measure 0 for any p. Thus, using that F is closed and γ is compact, we obtain some arbitrarily small vector v s.t. $(\gamma \times v) \cap F = \emptyset$. Then, $M^n - F$ is connected.

Second, if $[\Gamma] \in \pi_i(\mathbb{R}^n - F)$, $i \leq k$ is a homotopy class and Γ is a smooth representant, define $f : \Gamma \times F \to S^{n-1}$, $f(x, y) = (x-y)/||x-y||$. Following the proof of proposition 5.3.2, f is Lipschitz. Now, since $H_{n-k-1}(F) = 0$, we have, by proposition 5.2.2, $H_{n-1}(\Gamma \times F) = 0$. Thus, prop. 5.2.1 imply $H_{n-1}(f(\Gamma \times F)) = 0$. This concludes the proof. □

Lemma 5.6.5 (Lemma 5.3.4'). *Suppose that $\Gamma \in \pi_i(M^n)$ is Lipschitz and $K \subset M^n$ is compact, $H_{n-i}(K) = 0$. Then there are diffeomorphisms h of M, arbitrarily close to identity map, s.t. $h(\Gamma) \cap K = \emptyset$.*

Demonstração. If Γ is Lipschitz and Γ lies in a parametrized neighborhood, we can take $F : \Gamma \times K \to \mathbb{R}^n$, $F(x, y) = x - y$ a Lipschitz function. Because $H_n(\Gamma \times K) = 0$, this imply $H_n(F(\Gamma \times K)) = 0$. In general case we proceed as in proof of Lemma 5.3.4. Take, by compactness, a finite number of parametrized neighborhoods and apply the previous case. By finiteness of number of parametrized neighborhoods and using that K is compact, an induction argument achieve the desired diffeomorphisms h. This concludes the proof. □

Returning to proof of theorem G, observe that in theorem A, we need $\overline{G} : \overline{M}^n - \overline{G}^{-1}(\widetilde{S}) \to S^n - \widetilde{S}$ is diffeomorphism, where $\widetilde{S} = S \cup \{\overline{G}(q_i) : i = 1, \ldots, k\}$. This remains true because $(*)$ $H_{k-[\frac{n}{2}]}(S) = 0$ implies $S^n - \widetilde{S}$ is $(n-1-k+[\frac{n}{2}])$-connected. In fact, this is a consequence of $(*)$, proposition 5.6.4 and $\{p_i : i = 1, \ldots, k\}$ is finite ($p_i := \overline{G}(q_i)$). Moreover, $rank(x) \geq k$ imply, by prop. 5.6.3, 5.6.4, $\overline{M} - G^{-1}(\widetilde{S})$ is connected. Indeed, these propositions says that $rank(x) \geq k \Rightarrow H_{n-[\frac{n}{2}]}(G^{-1}(S)) = 0$ and $H_{n-1}(G^{-1}(S)) = 0 \Rightarrow M - G^{-1}(S)$ is connected. However, if $\overline{G}^{-1}(\widetilde{S}) - (G^{-1}(S) \cup \{q_i : i = 1, \ldots, k\}) := A$, then, for all $x \in A$, $(**)$ $\det D_x G \neq 0$. In particular, since $G(A) \subset \{p_i : i = 1, \ldots, k\}$, $(**)$ imply $\dim_H(A) = 0$. Then,

$H_{n-[\frac{n}{2}]}(\overline{G}^{-1}(\widetilde{S})) = H_{n-[\frac{n}{2}]}(G^{-1}(S)) = 0$. Thus, by [WG], G is surjective and covering map (because it is proper and its jacobian never vanishes). In particular, by simple connectivity, G is a diffeomorphism. At this point, using the previous lemma and propositions, it is sufficient follow proof of theorem A, if $n = 3$, and proof of theorem B, if $n \geq 4$, to obtain $\overline{M}^n \simeq S^n$. This concludes the proof. □

For even dimensions, we can follow [BFM] and improve theorem B :

Theorem 5.6.6 (Theorem H)**.** *Let $x : M^{2n} \to \mathbb{R}^{2n+1}$, $n \geq 2$ an immersion of finite geometrical type with $H_{k-n}(S) = 0$, $rank(x) \geq k$. Then M^{2n} is topologically a sphere minus two points. If M^{2n} is minimal, M^{2n} is a $2n$-catenoid.*

For sake of completeness we present an outline of proof of theorem C.

Outline of proof of theorem D. Barbosa, Fukuoka, Mercuri define to each *end* p of M a geometric index $I(p)$ that is related with the topology of M by the formula (see theorem 2.3 of [BFM]):

$$\chi(\overline{M}^{2n}) = \sum_{i=1}^{k}(1 + I(p_i)) + 2\sigma m \tag{5.1}$$

where σ is the sign of Gauss-Kronecker curvature and m is the degree of $G : M^n \to S^n$. Now, the hypothesis $2n > 2$ implies (see [BFM]) $I(p_i) = 1, \forall\, i$. Since we know, by theorem 6.2, $\overline{M}^{2n}$ is a sphere, we have $2 = 2k + 2\sigma m$. But, it is easy that $m = deg(G) = 1$ because G is a diffeomorphism outside the singular set. Then, $2 = 2k + 2\sigma \Rightarrow k = 2,\ \sigma = -1$. In particular, M is a sphere minus two points.

If M is minimal, we will use the following theorem of Schoen : *The only minimal immersions, which are regular at infinity and have two ends, are the catenoid and a pair of planes.* The regularity at infinity in our case holds if the ends are embedded. However, $I(p) = 1$ means exactly this. So, we can use this theorem in the case of minimal hypersurfaces of finite geometric type. This concludes the outline of proof. □

Remark 5.6.7. We can extend theorem A in a different direction (without mention $rank(x)$). In fact, using only that G is Lipschitz, it suffices assume that $H_{n-[\frac{n}{2}]}(C) = 0$ (C is the set of points where Gauss-Kronecker curvature vanishes). This is essentially the hypothesis of Barbosa, Fukuoka and

Mercuri [BFM]. We prefer state theorems C and D as before since the classical theorems concerning estimatives for Hausdorff dimension (Morse-Sard, Moreira) deal only with the critical values S and, in particular, our corollary 2.4 will be more difficult if the hypothesis is changed to $H_1(C) = 0$ for some immersion $x : M^3 \to \mathbb{R}^4$ (although, in this assumption, we have no problems with $rank(x)$, i.e., this assumption has some advantages).

Remark 5.6.8. It is interesting to know if there are examples of codimension 1 immersion with singular set which is not in the situation of Barbosa-Fukuoka-Mercuri and do Carmo-Elbert but it satisfies our hypothesis. This question was posed to the second author by Walcy Santos during the Differential Geometry seminar at IMPA. In fact, these immersions can be constructed with some extra work. Some examples will be presented in another work to appear elsewhere.

5.7 Final Remarks

The corollary 5.1.3 is motivated by Anderson's program for Poincaré Conjecture. In order to coherently describe this program, we briefly recall some facts about topology of 3-manifolds.

An attempt to better understand the topology of 3-manifolds (in particular, give an answer to Poincaré Conjecture) is the so called "Thurston Geometrization Conjecture". Thurston's Conjecture goes beyond Poincaré Conjecture (which is a very simple corollary of this conjecture). In fact, its goal is the understanding of 3-manifolds by decomposing them into pieces which could be "geometrizated", i.e., one could put complete locally homogeneous metric in each of this pieces. Thurston showed that, in three dimensions, there are exactly *eight* geometries, all of which are realizable. Namely, they are : the constant curvature spaces $\mathbb{H}^3$, $\mathbb{R}^3$, S^3, the products $\mathbb{H}^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$ and the twisted products $\widetilde{SL}(2, \mathbb{R})$, Nil, Sol (for details see [T]). Thurston proved his conjecture in some particular cases (e.g., for *Haken* manifolds). These particular cases are not easy. To prove the result Thurston developed a wealth of new geometrical ideas and machinery to carry this out. In few words, Thurston's proof is made by induction. He decomposes the manifold M in an appropriate hierarchy of submanifolds $M_k = M \supset \cdots \supset$ union of balls $= M_0$ (this is possible if M is *Haken*). Then, if M_{i-1} has a metric with some properties, it is possible glue certain ends of M_{i-1} to obtain M_i. Moreover, by a deformation and isometric gluing of ends argument, M_i has a metric with the same properties of that from M_{i-1}. This is the most difficult part of the proof. So, the induction holds and M itself satisfies the Geometrization Conjecture.

Recently, M. Anderson [A] formulate three conjectures that imply Thurston's Conjecture. Morally, these three conjectures says that information about *sigma* constant give us information about geometry and topology of 3-manifolds. We recall the definition of sigma constant. If $S(g) := \int_M s_g \, dV_g$ is the total scalar curvature functional (g is a metric with unit volume, i.e., $g \in \mathcal{M}_1$, dV_g is volume form determined by g and s_g is the scalar curvature) and $[g] := \{\widetilde{g} \in \mathcal{M}_1 : \widetilde{g} = \psi^2 g$, for some smooth positive function $\psi\}$ is the conformal class of g, then S is a bounded below functional in $[g]$. Thus, we can define $\mu[g] = \inf_{g \in [g]} S(g)$ called *Yamabe* constant of $[g]$. An elementary comparison argument shows $\mu[g] \leq \mu(S^n, g_{can})$, where g_{can} is the canonical metric of S^n with unit 1 and positive constant curvature. Then makes sense define the sigma constant :

$$\sigma(M) = \sup_{[g] \in \mathcal{C}} \mu[g] \tag{5.2}$$

where $\mathcal{C}$ is the space of all conformal classes. The sigma constant is a smooth invariant defined by a minimax principle (see equation 5.2). The first part of this minimax procedure was solved by Yamabe [Y]. More precisely, for any conformal class $[g] \in \mathcal{C}$, $\mu[g]$ is realized by a (smooth) metric $g_\mu \in [g]$ s.t. $s_{g_\mu} \equiv \mu[g]$ (a such g_μ is called *Yamabe* metric). The second part of this procedure is more difficult since it depends on the underlying topology. The sigma constant is important since it is know that critical points of the scalar curvature functional S are Einstein metrics. But it is not know if $\sigma(M)$ is a critical value of S (partially by non-uniqueness of Yamabe metrics). Then, if one show that is possible to realize the second part of minimax procedure and that $\sigma(M)$ is a critical value of S, we obtain the Geometrization Conjecture.

This approach is very difficult. To see this, we remark that all of three Anderson's Conjectures are necessary to obtain the "Elliptization Conjecture" (the particular case of Thurston's Conjecture which implies Poincaré Conjecture). In others words, we have to deal with all cases of Thurston Conjecture to obtain Poincaré Conjecture. This inspirates our definition of another minimax smooth invariants. The advantage in these invariants is it does not requires construction of metrics with positive constant curvature. But the disadvantage is we always work extrinsically.

To finish the paper, we comment that there are many others attacks and approachs to Poincaré Conjecture. For example, see [G] for an accessible exposition of V. Poénaru's program and [P] for recent proof of one step of this program. In the other hand, some authors (e.g., Bing [B]) believes that only simple connectivity is not sufficient that a manifold be S^3.

Added in proof. The first version of this paper was written in October 22, 2002, when the works of Perelman was not available. Nowdays, it is well-known that Perelman's works *seems* to give a complete answer to the geometrization conjecture (and so, Poincaré conjecture). In particular, although our proof of theorem A only uses results which are simpler than Perelman's ones, this result follows (as in proof of theorem A in the case $n \geq 4$) from the Poincaré conjecture.

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