Toward the classification of cohomology-free vector fields

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Abstract

Given a smooth vector field X on a closed orientable d-manifold M, many questions about the dynamics of its induced flow can be studied analyzing the following *cohomological equation*:

$$\mathcal{L}_X u = \xi,$$

where ξ is a given real function on M, $u\colon M \to \mathbb{R}$ is the solution that we look for (in a certain regularity class) and $\mathcal{L}_X$ is the Lie derivative in the X direction.

In 1984, Anatole Katok [Hur85, KR01, Kat03] proposed to characterize those vector fields which are cohomologicaly trivial. More precisely, he conjectured that if X is so that for all smooth function $\xi\colon M \to \mathbb{R}$, there exist a constant $c = c(\xi) \in \mathbb{R}$ and $u \in C^\infty(M, \mathbb{R})$ verifying

$$\mathcal{L}_X u = \xi - c,$$

then X should be smoothly conjugated to a Diophantine (constant) vector field on $\mathbb{T}^d$. In particular, M should be diffeomorphic to $\mathbb{T}^d$.

The main goal of this work is to prove the validity of Katok Conjecture for 3-manifolds.

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Contents

Chapter 1

Introduction

The main goal in Differentiable Dynamics consists in understanding the global behavior of "most" of the orbits of systems, where the *phase space* is represented by a compact differential manifold M, and the evolution by a diffeomorphism $f \in \mathrm{Diff}^r(M)$ (discrete time case), or by a C^r flow $\Phi\colon M \times \mathbb{R} \to M$ (continuous time case).

Looking for the unification of the notation, we can assume that the dynamics of our system is given by a C^r Lie group action on M. More precisely, if $(\mathbb{G}, +)$ denotes any analytic abelian Lie group, we shall suppose that $\mathbb{G}$ represents the time and the evolution of the system is given by a C^r $\mathbb{G}$-action $\Gamma\colon M \times \mathbb{G} \to M$.

1.1 Cocycles and Coboundaries

When we analyze different questions about the dynamics of Γ, there is a family of objects that appears repeatedly and in a very natural way (see Section 1.2 for some examples). These are the *real cocycles over* Γ:

Definition 1.1. Given a C^r $\mathbb{G}$-action $\Gamma\colon M \times \mathbb{G} \to M$, a *real cocycle over* Γ (or simply a *cocycle*) is a C^k map (usually $k \leq r$) $\Xi\colon M \times \mathbb{G} \to \mathbb{R}$ such that

$$\Xi(p, g_0 + g_1) = \Xi(\Gamma(p, g_0), g_1) + \Xi(p, g_0), \quad \forall p \in M,\ \forall g_0, g_1 \in \mathbb{G}. \tag{1.1}$$

Within this framework it is natural to consider the following equivalence relation between cocycles:

Definition 1.2. We shall say that two C^k cocycles $\Xi, \Theta\colon M \times \mathbb{G} \to M$ over Γ are C^s-*cohomologous* (with $s \leq k \leq r$) if there exists a C^s map $\alpha\colon M \to \mathbb{R}$ verifying

$$\Xi(p,g) = \alpha(\Gamma(p,g)) + \Theta(p,g) - \alpha(p), \quad \forall p \in M,\ \forall g \in \mathbb{G}.$$

On the other hand, notice that given any real C^k function $\xi\colon M \to \mathbb{R}$, we can easily construct a cocycle Ξ over Γ defining

$$\Xi(p,g) \doteq \xi(\Gamma(p,g)) - \xi(p), \quad \forall p \in M,\ \forall g \in \mathbb{G}. \tag{1.2}$$

Cocycles constructed as above are very important and deserve a special name: they are called *coboundaries*. In other words, we may say that a cocycle is a coboundary if and only if it is cohomologous to the null cocycle.

These names come from (abstract) Group Cohomology Theory. In fact, if we suppose that Γ is C^∞, then it induces in a natural way a $\mathbb{G}$-action on $C^\infty(M,\mathbb{R})$, turning $C^\infty(M,\mathbb{R})$ into a $\mathbb{G}$-module. So, in a purely algebraic way we can define the cohomology complex $H^*(\mathbb{G},\Gamma)$ (see [AW67] for example). In this way, $H^1(\mathbb{G},\Gamma)$ happens to be canonically isomorphic to the quotient vector space of all smooth cocycles over Γ by the subspace of all smooth coboundaries. However, since in the future we shall not make any other reference to higher cohomology groups, the reader can simply consider this algebraic construction as a justification for the chosen names.

1.2 Cohomological Equations

Since we are mainly interested in the "classical group actions", from now on we shall assume that Γ is a differentiable $\mathbb{G}$-action, being $\mathbb{G} = \mathbb{Z}$ or $\mathbb{R}$.

As it was already mentioned, cocycles appear naturally in different contexts when we want to study some dynamical properties of Γ. Among the problems in Differentiable Dynamics that can be reduced to cohomological considerations we can mention:

1. *Existence of invariant volume forms* (see Section 5.1 in the book of Katok and Hasselblat [KH95]).

2. *Stability of hyperbolic torus automorphisms* (see Section 2.6 in [KH95]).

3. *Livšic Theory* (see Section 19.2 in [KH95], Section 3.4 in the survey of Katok and Robinson [KR01], or the original work Livšic [Liv71]).

4. *KAM Theory* (see the survey of R. de la Llave [dlL99]).

5. *Constructions of minimal conservative but non uniquely ergodic diffeomorphisms* (see the classical work of H. Furstenberg [Fur61]).

In all the cases listed above the main problem consists in proving that a given cocycle is or is not a coboundary, or more generally, that it is or it is not C^s-cohomologous to another given cocycle.

This is the reason why it is so important to analyze the existence of solutions $u\colon M \to \mathbb{R}$ (in a certain regularity class) for the following difference equation:

$$u(\Gamma(p,g)) - u(p) = \Xi(p,g), \quad \forall p \in M,\ \forall g \in \mathbb{G}, \tag{1.3}$$

where Ξ is a given real cocycle over the $\mathbb{G}$-action Γ. These equations deserve a special name:

Definition 1.3. A difference equation like (1.3) will be called a *cohomological equation.*

In the particular case that $\mathbb{G} = \mathbb{Z}$, the cocycle Ξ is "generated" by the function $\xi(p) \doteq \Xi(p,1)$. Indeed, it holds

$$\Xi(p,n) = \begin{cases} 0, \text{ if } n = 0, \\ \sum_{i=0}^{n-1} \xi(f^i(p)), \text{ if } n > 0, \\ -\sum_{i=n}^{-1} \xi(f^i(p)), \text{ if } n < 0, \end{cases}$$

where $f \doteq \Gamma(\cdot, 1) \in \mathrm{Diff}^r(M)$. And so, in this case the cohomological equation (1.3) can be written as

$$u \circ f - u = \xi. \tag{1.4}$$

On the other hand, when $\mathbb{G} = \mathbb{R}$ the cocycle Ξ has an "infinitesimal generator" defined by

$$\xi(p) \doteq \partial_t \Xi(p,t)\Big|_{t=0}, \quad \forall p \in M.$$

In this case, $\Xi(p,t) = \int_0^t \xi(\Gamma(p,s))\, \mathrm{d}s$, and hence, derivating equation (1.3) with respect to the time variable, we get the following differential equation:

$$\mathcal{L}_X u = \xi, \tag{1.5}$$

where $X \in \mathfrak{X}(M)$ is the vector field generating Γ, i.e. $X(p) \doteq \partial_t \Gamma(p,t)\big|_{t=0}$, and $\mathcal{L}_X$ denotes the Lie derivative along X.

1.3 Obstructions

In general it is not an easy task to determine if a particular cohomological equation admits some solution in a particular regularity class. So, it appears as an important problem to characterize the *"set of obstructions"* for the existence of (L^p, C^r, *etc.*) solutions for equations like (1.3).

For example, if Γ is a given $\mathbb{R}$-action, then the very first obstructions that we can find for the existence of continuous solutions for equation (1.5) are the elements of $\mathfrak{M}(\Gamma)$, the set of Borel finite measures on M which are left invariant by the flow Γ. More precisely, if u is a continuous solution of equation (1.5), since

$$\frac{1}{T}\int_0^T \xi(\Gamma(p,t))\,\mathrm{d}t = \frac{1}{T}\big(u(\Gamma(p,t)) - u(p)\big) \xrightarrow{T\to\infty} 0,$$

as a straight-forward consequence of Birkhoff ergodic theorem we have that

$$\int_M \xi\,\mathrm{d}\mu = 0, \quad \text{for every } \mu\in\mathfrak{M}(\Gamma).$$

In Section 2.1 we shall see that the set of Γ-invariant distributions, in the sense of Schwartz, is the most natural space for looking for obstructions for the existence of smooth solutions for equation (1.5) (or (1.4)).

Two very classical results which completely characterize this *set of obstructions* in two particular, and in some sense, extremal opposite situations, are due to Gottschalk and Hedlund [GH55] and to Livšic [Liv71].

In the first one, if Γ is a continuous minimal $\mathbb{Z}$-action (i.e. every point in M has a dense Γ-orbit) generated by a homeomorphism f on M and if $\xi\in C^0(M,\mathbb{R})$, Gottschalk and Hedlund proved that equation (1.4) admits a continuous solution u if and only if the family of functions

$$\left\{\sum_{i=0}^{n-1}\xi\circ f^i\right\}_{n\geq 1}$$

is uniformly bounded in $C^0(M,\mathbb{R})$.

In the second one, Livšic studied the case where Γ is a C^2 hyperbolic $\mathbb{R}$-action (i.e. an Anosov flow) induced by a vector field $X\in\mathfrak{X}^3(M)$. Assuming that ξ is Hölder continuous, he proved that the only obstruction for the existence of a Hölder continuous solution u for equation (1.5) is given by the set of probabilities concentrated on the periodic orbits, i.e. there is a Hölder continuous solution u as long as

$$\int_0^{\tau(z)} \psi(\Gamma(z,s))\,\mathrm{d}s = 0, \quad \forall z \in \mathrm{Per}(\Gamma),$$

and where $\tau(z) \doteq \inf\{t > 0 : \Gamma(z,t) = z\}$.

It is interesting to remark that both results [GH55] and [Liv71] hold for $\mathbb{R}$-actions as well as for $\mathbb{Z}$-actions.

There are more recent results that completely characterize the sets of obstructions in some other cases. For example, cohomological equations associated to area-preserving flows on higher genus surfaces have been studied by Giovanni Forni [For97, For01] (the torus case is rather classical); and associated to their "very close relatives", the interval exchanged maps, by Stefano Marmi, Pierre Moussa and Jean-Christophe Yoccoz [MMY03, MMY05]. Other very important flows that nowadays are very well understood from the cohomological point of view are homogeneous ones on nilmanifolds: this study is due to Livio Flaminio and Giovanni Forni. They started studying some particular cases (horocycle flows, nilflows on Heisenberg manifolds) in [FF03, FF06], and the general case was settled in [FF07].

1.4 Cohomology-free Dynamical Systems

As it was already mentioned in Section 1.3, in general it is very difficult to characterize the set of obstructions for the existence of solutions for a cohomological equation like (1.5). With the aim of understanding the nature (topological, analytical, *etc.*) of this set of obstructions, Anatole Katok, in the early '80, proposed the following

Definition 1.4. Given a closed manifold M, we say that a smooth $\mathbb{G}$-action $\Gamma\colon M \times \mathbb{G} \to \mathbb{M}$ is *cohomology-free* if any smooth real cocycle over Γ is C^∞-cohomologous to a constant one.

Notice that two different constant cocycles are never smoothly cohomologous, and so, the first cohomology group of any smooth action always contains a subgroup isomorphic to $\mathbb{R}$. Therefore, we can say that a smooth action is cohomology-free if and only if its first cohomology group is as small as possible.

For the sake of clarity of the exposition, from now on we shall mainly concentrate on smooth $\mathbb{R}$-actions, i.e. flows induced by C^∞ vector fields. In this particular case, Definition 1.4 can be restated in the following way:

We say that $X \in \mathfrak{X}(M)$ is *cohomology-free* if given any $\xi \in C^\infty(M, \mathbb{R})$, there exist a constant $c(\xi) \in \mathbb{R}$ and $u \in C^\infty(M, \mathbb{R})$ verifying

$$\mathcal{L}_X u = \xi - c(\xi). \tag{1.6}$$

Remark 1.5. It is clear that the set of cohomology-free vector fields is closed under C^∞-conjugacy.

To introduce the prototypical example of cohomology-free vector fields, first we need to state the following

Definition 1.6. We say that $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is a *Diophantine vector* if there exist real constants $C, \tau > 0$ satisfying

$$\left| \sum_{i=1}^{d} \alpha_i p_i \right| > C \Big(\max_{1 \leq i \leq d} |p_i| \Big)^{-\tau}, \tag{1.7}$$

for every $p = (p_1, \ldots, p_d) \in \mathbb{Z}^d \setminus \{0\}$.

A vector field X_α on the d-dimensional torus $\mathbb{T}^d$ verifying $X_\alpha \equiv \alpha$ will be called a *Diophantine vector field*.

Example 1.7. Diophantine vector fields on tori are cohomology-free.

In fact, let $\alpha \in \mathbb{R}^d$ be a Diophantine vector. The Haar measure on $\mathbb{T}^d$ is the only X_α-invariant probability measure and if $\xi \colon \mathbb{T}^d \to \mathbb{R}$ is an arbitrary C^∞ function, considering its Fourier expansion

$$\xi(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{\xi}_k e^{2\pi i k \cdot \theta},$$

we can define u, at first just formally, writing

$$u(\theta) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{\hat{\xi}_k}{k \cdot \alpha} e^{2\pi i k \cdot \theta}.$$

Taking into account estimate (1.7), we easily see that $u \in C^\infty(\mathbb{T}^n, \mathbb{R})$ and, by construction, it holds

$$\mathcal{L}_{X_\alpha} u = \xi - \hat{\xi}_0.$$

As we will see in Theorem 2.5, these are the only cohomology-free vector fields on tori, of course, modulo C^∞-conjugacy.

Considering this example and the previous work of Stephen Greenfield and Nolan Wallach [GW73] on globally hypoelliptic vector fields (see Section 5.2 for precise definitions), Anatole Katok proposed in [Hur85] the following conjecture characterizing the cohomology-free vector fields:

Conjecture 1.8 (Katok Conjecture [Hur85]). *If M is a compact, connected, orientable d-manifold, and X is a cohomology-free vector field on M, then M is diffeomorphic to the torus $\mathbb{T}^d$, and therefore, X is C^∞ conjugated to a Diophantine constant vector field on $\mathbb{T}^d$.*

Some results supporting Katok Conjecture have recently appeared. First, Federico and Jana Rodriguez-Hertz [RHRH06] have proved that a manifold supporting a cohomology-free vector field must fiber over the torus of dimension equal to the first Betti number of the manifold (see Theorem 2.7 for the precise statement); and secondly, Livio Flaminio and Giovanni Forni [FF07] have proved that tori are the only nilmanifolds supporting cohomology-free homogeneous flows.

1.5 Main Results and Outline of this Work

The main goal of this work is to present a complete proof of Katok Conjecture in dimension 3. For this, the rest of the work will be organized as follows:

In Chapter 2 we shall present general properties about cohomology-free vector fields, some of which are very classical, like strict ergodicity, and the rather new result due to Federico and Jana Rodríguez-Hertz, Theorem 2.7.

In Chapter 3 we shall expose the proof of our first result toward the classification of cohomology-free vector fields on 3-manifolds:

Theorem A. *Let M be a closed and orientable 3-manifold verifying*

$$\beta_1(M) = \dim H_1(M, \mathbb{Q}) \geq 1,$$

and suppose that there exists a smooth cohomology-free vector field $X \in \mathfrak{X}(M)$. Then M is diffeomorphic to $\mathbb{T}^3$ and X is C^∞-conjugated to a Diophantine constant vector field.

While this work was in progress, Giovanni Forni [For06] communicated to us that he had proved the following result[1]:

Theorem B. *If M is a closed orientable 3-manifold with $H_1(M, \mathbb{Q}) = 0$ and $X \in \mathfrak{X}(M)$ is a cohomology-free vector field, then there exists a 1-form α on M verifying*

$$\alpha \wedge d\alpha \neq 0, \quad i_X\alpha \equiv 1, \quad i_X d\alpha \equiv 0.$$

In other words, α is a contact form and X is its induced Reeb vector field.

On the other hand, Clifford Taubes [Tau] has recently proved Weinstein Conjecture which asserts that every Reeb vector field on a 3-manifold must exhibit a periodic

[1] Forni independently got a proof of Theorem A, too.

orbit. This clearly contradicts the minimality (see Corollary 2.4) of the flow induced by a cohomology-free vector field. Therefore, Theorem B lets us affirm that there is no cohomology-free vector field on 3-manifolds with vanishing first Betti number and thus we get

Corollary 1.9 (Katok Conjecture in dimension 3)**.** *If M is closed and orientable 3-manifold and $X \in \mathfrak{X}(M)$ is a smooth cohomology-free vector field on M, then M is diffeomorphic to $\mathbb{T}^3$ and X is C^∞-conjugated to a constant Diophantine vector field.*

In Chapter 4 we will sketch very briefly Forni's proof of Theorem B (that he kindly communicated to us) and we will present another proof, using completely different techniques. We hope this can help to get a better comprehension of the whole problem.

For the sake of completeness, in that chapter we will also recall some fundamental facts about Contact Geometry and we shall precisely state Weinstein Conjecture.

Finally, in Chapter 5 we propose some open problems and consider some final remarks about the results presented in this work.

1.6 Notation and Conventions

For simplicity, we will mainly work in the C^∞ category and the word *smooth* will be used as a synonymous of C^∞.

We shall say that a manifold is *closed* if it is compact, connected and its boundary is empty.

Along this work, M will denote a smooth closed orientable d-dimensional manifold, and most of the time $d = 3$.

The linear space of all C^r vector fields on M will be denoted by $\mathfrak{X}^r(M)$, and to simplify the notation, we shall just write $\mathfrak{X}(M)$ for the space of smooth vector fields.

Analogously, $\mathrm{Diff}^r(M)$ will stand for the set of C^r diffeomorphism of M and we will simply write $\mathrm{Diff}(M)$ for the set of smooth diffeomorphisms.

The expression $\Lambda^k(M)$ will be used for the space of smooth k-forms on M, and given any $X \in \mathfrak{X}(M)$, $i_X \colon \Lambda^k(M) \to \Lambda^{k-1}(M)$ shall denote the contraction by X (also called interior product).

As usual, we shall identify $\Lambda^0(M)$ with $C^\infty(M, \mathbb{R})$.

Given any $X \in \mathfrak{X}(M)$, $\{\Phi_X^t\}_{t\in\mathbb{R}}$ will denote the flow induced by X.

If T denotes any smooth tensor field on M, the Lie derivative of T along X will be denoted by $\mathcal{L}_X T$ and defined by

$$\mathcal{L}_X T(x) \doteq \lim_{t\to 0} \frac{(\Phi_X^t)^* T(x) - T(x)}{t}, \quad \forall x \in M.$$

The set of all finite signed Borel measures on M (i.e. real continuous linear functionals on $C^0(M, \mathbb{R})$) shall be denoted by $\mathfrak{M}(M)$, and we will write $\mathcal{D}'(M)$ for the space of all real continuous linear functionals on $C^\infty(M, \mathbb{R})$.

Given any smooth fibration $p \colon N \to M$, the fiber over any $x \in M$ shall be denoted by N_x, and we will write $\Gamma(N)$ for the space of smooth sections (i.e. maps $s \colon M \to N$ verifying $p \circ s = id_M$). The only exception for this notational convention is the tangent bundle over M: in this case $\pi \colon TM \to M$ will denote the canonical projection and we will write T_xM for $\pi^{-1}(x)$ and $\mathfrak{X}(M)$ for $\Gamma(TM)$.

Similarly, given any foliation $\mathscr{F}$ on M, $\mathscr{F}_x$ or $\mathscr{F}(x)$ will stand for the leaf of $\mathscr{F}$ through $x \in M$.

It is very important to remark that, in order to avoid confusions along this work we shall use the term *distribution* in the "sense of Schwartz," i.e. for us a *distribution* will be any element of $\mathcal{D}'(M)$. This word has a completely different meaning in Differential Geometry. Indeed, we shall use the expression *k-plane field*, or *line field* when $k = 1$, to denote the objects that are commonly named distributions in Differential Geometry.

There are some relationships between the linear spaces $C^\infty(M, \mathbb{R})$, $\Lambda^d(M)$, $\mathfrak{M}(M)$ and $\mathcal{D}'(M)$. First, since the elements of $\mathfrak{M}(M)$ can be considered as linear continuous functionals on $C^0(M,\mathbb{R})$, it can be canonically embedded in $\mathcal{D}'(M)$. On the other hand, it is very easy to see that each element of $\Lambda^d(M)$ (where $d = \dim M$) naturally induces a signed measure, i.e. we can assume that $\Lambda^d(M) \subset \mathfrak{M}(M)$. And finally, since M is supposed to be orientable, we can choose a volume form on M and use it to get a bijection between $C^\infty(M,\mathbb{R})$ and $\Lambda^d(M)$. However, it is important to remark that in this case this identification is not canonical at all.

Since we have defined the Lie derivative $\mathcal{L}_X$ on $C^\infty(M,\mathbb{R})$, we can easily extend it by duality to $\mathcal{D}'(M)$. In fact, we can define $\mathcal{L}_X\colon \mathcal{D}'(M) \to \mathcal{D}'(M)$ writing

$$\langle \mathcal{L}_X T, \psi\rangle \doteq -\langle T, \mathcal{L}_X \psi\rangle, \quad \forall T \in \mathcal{D}'(M),\ \forall \psi \in C^\infty(M,\mathbb{R}).$$

In this way, it is reasonable to define the set of X-invariant distributions and measures by

$$\begin{aligned}
\mathcal{D}'(X) &\doteq \{T \in \mathcal{D}'(M) : \mathcal{L}_X T = 0\},\\
\mathfrak{M}(X) &\doteq \{\mu \in \mathfrak{M}(M) : (\Phi_X^t)_*\mu = \mu,\ \forall t \in \mathbb{R}\}\\
&= \{\mu \in \mathfrak{M}(M) : \mu \in \mathcal{D}'(X)\}
\end{aligned}$$

Finally, the d-dimensional torus will be denoted by $\mathbb{T}^d$ and the quotient Lie group $\mathbb{R}^d/\mathbb{Z}^d$ will be our favorite model for it. $\mathrm{pr}_{\mathbb{Z}^d}\colon \mathbb{R}^d \to \mathbb{T}^d$ will denote the canonical quotient projection. The Haar probability measure on $\mathbb{T}^d$, also called the Lebesgue measure, will be denoted by Leb^d.

In general, an arbitrary point of $\mathbb{T}^d$ shall be denoted by $\theta = (\theta^0, \theta^1, \dots, \theta^{d-1})$.

It is a very well-known fact that there exists a canonical group isomorphism between the group of automorphisms of $\mathbb{T}^d$ and $\mathrm{GL}(d,\mathbb{Z})$. Taking this into account, if $A \in \mathrm{Diff}(\mathbb{T}^d)$ is any Lie group automorphism of $\mathbb{T}^d$, the corresponding element of $\mathrm{GL}(d,\mathbb{Z})$ will be denoted by $\hat{A}$. Notice that A and $\hat{A}$ are related by $A \circ \mathrm{pr}_{\mathbb{Z}^d} = \mathrm{pr}_{\mathbb{Z}^d} \circ \hat{A}$.

Chapter 2

General Properties of Cohomology-free Vector Fields

2.1 Strict Ergodicity

This section is devoted to proving that every cohomology-free vector field is strictly ergodic, i.e. it is uniquely ergodic and every orbit of its induced flow is dense on the whole manifold. These results are very classical and, as the reader will see, the proofs are rather simple. Nevertheless, we decided to include them here for the sake of completeness and with the purpose of making easier the reading of this work.

As it was already mentioned in Section 1.6, we shall assume that M is a closed orientable d-manifold.

Proposition 2.1. *If $X \in \mathfrak{X}(M)$ is a cohomology-free vector field, then its induced flow $\{\Phi_X^t\}$ is uniquely ergodic.*

Proof. Let $\psi\colon M \to \mathbb{R}$ be any smooth function and let $c(\psi) \in \mathbb{R}$ and $u \in C^\infty(M,\mathbb{R})$ be as in equation (1.6). Then we have

$$\frac{1}{T}\bigg(\int_0^T \psi(\Phi_X^s(p))\,\mathrm{d}s\bigg) = \frac{1}{T}\big(u(\Phi_X^T(p)) - u(p)\big) + c(\psi), \tag{2.1}$$

for every $p \in M$ and every $T > 0$.

Then, if μ is an arbitrary X-invariant ergodic probability measure, by the Birkhoff ergodic theorem we know that the left side of equation (2.1) must converge to $\int \psi \, \mathrm{d}\mu$, for μ-almost every $p \in M$, when $T \to \infty$. On the other hand, since u is bounded, the right side of (2.1) converges to c. Therefore, $\int \psi \, \mathrm{d}\mu = c(\psi)$, for every $\mu \in \mathfrak{M}(X)$, and since $C^\infty(M, \mathbb{R})$ is dense in $C^0(M, \mathbb{R})$, we conclude that $\mathfrak{M}(X)$ contains only one element. □

In fact, we can prove a stronger result:

Proposition 2.2. *If $X \in \mathfrak{X}(M)$ is a cohomology-free vector field, then*

$$\dim \mathcal{D}'(X) = 1.$$

Proof. Given an arbitrary $\psi \in C^\infty(M, \mathbb{R})$, let $u \in C^\infty(M, \mathbb{R})$ and $c(\psi) \in \mathbb{R}$ be such that

$$\mathcal{L}_X u = \psi - c(\psi).$$

Then, for any $T \in \mathcal{D}'(X)$ we have

$$\langle T, \psi \rangle = \langle T, \mathcal{L}_X u + c(\psi) \rangle = -\langle \mathcal{L}_X T, u \rangle + \langle T, c(\psi) \rangle = \langle T, c(\psi) \rangle.$$

From this we can easily conclude that $\dim \mathcal{D}'(X) = 1$. □

We can also get the following regularity result for the elements of $\mathcal{D}'(X)$:

Proposition 2.3. *Let $X \in \mathfrak{X}(M)$ be a cohomology-free vector field. Then there exists a smooth volume form $\Omega \in \Lambda^d(M)$ such that $\mathcal{L}_X \Omega \equiv 0$.*

Proof. Since we are assuming that M is orientable, let $\tilde{\Omega} \in \Lambda^d(M)$ be an arbitrary smooth volume form. Let us define $\operatorname{div}_{\tilde{\Omega}} X \in C^\infty(M, \mathbb{R})$ as the only smooth function verifying

$$\mathcal{L}_X\tilde{\Omega} = (\operatorname{div}_{\tilde{\Omega}} X)\tilde{\Omega}.$$

Hence there exist a smooth function $u\colon M \to \mathbb{R}$ and a real constant $c = c(\operatorname{div}_{\tilde{\Omega}} X)$ satisfying

$$\mathcal{L}_X u = -(\operatorname{div}_{\tilde{\Omega}} X) + c.$$

Therefore, if we define $\Omega \doteq \exp(u)\tilde{\Omega}$, we obtain

$$\begin{aligned}
\mathcal{L}_X\Omega &= e^u\mathcal{L}_X\tilde{\Omega} + (e^u\mathcal{L}_X u)\tilde{\Omega} \\
&= e^u(\operatorname{div}_{\tilde{\Omega}} X)\tilde{\Omega} + e^u(-(\operatorname{div}_{\tilde{\Omega}} X) + c)\tilde{\Omega} \\
&= ce^u\tilde{\Omega} = c\Omega.
\end{aligned}$$

Finally, this clearly implies that $(\Phi_X^t)^*\Omega = (1+tc)\Omega$, and since the total Ω-volume of M is invariant, we have $c = 0$. □

As a direct consequence of Propositions 2.1 and 2.3 we get the following

Corollary 2.4. *If $X \in \mathfrak{X}(M)$ is a cohomology-free vector field, then the induced flow $\{\Phi_X^t\}_{t\in\mathbb{R}}$ is minimal, i.e. it holds*

$$\operatorname{cl}\left\{\Phi_X^t(p) : t \in \mathbb{R}\right\} = M, \quad \forall p \in M.$$

2.2 Cohomology-free Vector Fields on Tori

The aim of this section consists in proving that the Diophantine vector fields are the only cohomology-free ones on tori, modulo C^∞-conjugacy. More precisely, we shall prove the following

Theorem 2.5. *If $X \in \mathfrak{X}(\mathbb{T}^d)$ is a cohomology-free vector field on $\mathbb{T}^d$, then there exist a Diophantine vector $\alpha \in \mathbb{R}$ (see Definition 1.6) and $f \in \operatorname{Diff}(\mathbb{T}^d)$ homotopic to the* *identity such that*

$$Df(X(\theta)) \equiv \alpha.$$

This result is essentially due to Richard Luz and Nathan dos Santos. In fact, in [LdS98] they proved that the only cohomology-free diffeomorphisms on $\mathbb{T}^d$ homotopic to the identity are those C^∞-conjugated to Diophantine translations. The proof of Theorem 2.5 is just a slight modification of their proof.

Proof of Theorem 2.5. Let $X(\theta) = (X_1(\theta), X_2(\theta), \ldots, X_d(\theta))$ be the coordinates of X in the canonical trivialization of $T\mathbb{T}^d$ and let $\Omega \in \Lambda^d(\mathbb{T}^d)$ be the only normalized X-invariant volume form given by Proposition 2.3. Let us define

$$\alpha_i \doteq \int_{\mathbb{T}^d} X_i \Omega \in \mathbb{R}, \quad \text{for } i = 1, \ldots, d.$$

So there exist smooth functions u_i such that $\mathcal{L}_X u_i = -X_i + \alpha_i$. Then we can define a smooth map $f\colon \mathbb{T}^d \to \mathbb{T}^d$ writing

$$f(\theta) \doteq \theta + (u_1(\theta), u_2(\theta), \ldots, u_d(\theta)) \mod 1, \quad \forall \theta \in \mathbb{T}^d.$$

And then we have

$$Df(X) = (X_i + \mathcal{L}_X u_i)_{i=1}^d = (\alpha_1, \alpha_2, \ldots, \alpha_d). \tag{2.2}$$

From equation (2.2) we can easily see that $f(\mathbb{T}^d)$ must be a coset of a closed connected subgroup of $\mathbb{T}^d$. By construction, f is isotopic to the identity and so f must be surjective. On the other hand, the set of critical points for f is Φ_X-invariant, and by Sard's theorem, it is not the whole torus. Therefore, every point of $\mathbb{T}^d$ is regular and f is a diffeomorphism, since tori do not admit any non-injective self-covering maps homotopic to the identity.

As we already observed in Remark 1.5, the set of cohomology-free vector fields is invariant by C^∞-conjugacy. Hence, $X_\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ must be cohomology-free too. Finally, it is rather easy to verify that then, X_α must be a Diophantine vector field (see §3.2.2 in [KR01]). □

2.3 Topological Restrictions

As it was already proved in Section 2.1, the flow $\{\Phi_X^t\}$ induced by a cohomology-free vector field $X \in \mathfrak{X}(M)$ is minimal and uniquely ergodic. In particular, X cannot exhibit any singularity, and so, the Euler characteristic of M must vanish.

For a very long time this was the only known topological restriction for manifolds supporting cohomology-free vector fields, until Federico and Jana Rodríguez-Hertz produced a breakthrough in [RHRH06], finding additional restrictions on the first Betti number of the manifold.

For simplifying the exposition, let us first present a definition that will be used all along this work:

Definition 2.6. Given a closed d-manifold M and a smooth vector field $Y \in \mathfrak{X}(M)$, we say that a $p\colon M \to \mathbb{T}^n$ (where $n \leq d$) is a *good fibration for* Y if it is a smooth submersion and there exists a Diophantine vector $\alpha \in \mathbb{R}^n$ verifying $Dp(Y) \equiv \alpha$.

Now we can state the main result of this section:

Theorem 2.7 (F. & J. Rodríguez-Hertz [RHRH06])**.** *Let $X \in \mathfrak{X}(M)$ be a cohomology-free vector field on the closed manifold M and let us write $\beta_1 \doteq \dim H_1(M, \mathbb{Q})$. Then there exists a good fibration $p\colon M \to \mathbb{T}^{\beta_1}$ for X, where $Dp(X) \equiv X_\alpha \in \mathfrak{X}(\mathbb{T}^{\beta_1})$ and α is a Diophantine vector. In particular, it holds $\beta_1(M) \leq \dim M$.*

This fundamental result gives non-trivial information on the topology of M in all but one case: when M has trivial first rational homology group.

This is the main reason why it is necessary to attack Katok Conjecture with different techniques, depending on the vanishing or not of the first Betti number of the manifold.

Chapter 3

The case $\beta_1(M) \geq 1$

In this chapter we present the proof of Theorem A.

We continue assuming that M is a closed orientable manifold and from now on, we shall assume that $\dim M = 3$ and

$$\beta_1(M) \doteq \dim H_1(M, \mathbb{Q}) \geq 1.$$

For a better organization, we shall base the proof of Theorem A on the following two propositions:

Proposition 3.1. *Let us suppose that there exists $X \in \mathfrak{X}(M)$ verifying:*

1. *The flow $\{\Phi_X^t\}_{t\in\mathbb{R}}$ induced by X does not have any periodic orbit;*
2. *and there is a good fibration $q\colon M \to \mathbb{T}^1$ for X.*

Then $\beta_1(M) \geq 2$.

Proposition 3.2. *Let X be a smooth vector field on M and suppose that the induced flow $\{\Phi_X^t\}$ preserves a smooth volume form Ω, i.e. $\mathcal{L}_X\Omega \equiv 0$. Besides, assume that there exists a good fibration $p\colon M \to \mathbb{T}^2$ for X verifying $Dp(X) = X_\alpha$.*

Then, if M is not diffeomorphic to $\mathbb{T}^3$, $\mathcal{D}'(X)$ has infinite dimension.

3.1 Proof of Theorem A

This short section is devoted to prove Theorem A, assuming Propositions 3.1 and 3.2.

We are supposing that M is a closed orientable 3-manifold, with $\beta_1(M) \geq 1$ and $X \in \mathfrak{X}(M)$ is a cohomology-free vector field. By Proposition 2.4 we know that the induced flow $\{\Phi_X^t\}$ is minimal, so in particular, it does not exhibit any periodic orbit. On the other hand, by Theorem 2.7, we know that there exists a good fibration $q\colon M \to \mathbb{T}^1$ for X, with $Dq(X)$ verifying a Diophantine condition. Notice that in the one-dimensional case, being Diophantine is equivalent to be different from zero. Hence, we can apply Proposition 3.1 for concluding that $\beta_1(M) \geq 2$.

Therefore, we can apply Theorem 2.7 once again for getting a good fibration $p\colon M \to \mathbb{T}^2$ for X such that $Dp(X)$ is a Diophantine vector in $\mathbb{R}^2$. On the other hand, by Proposition 2.3 we know that there exists a smooth X-invariant volume form Ω. And by Proposition 2.2, we can assure that $\dim \mathcal{D}'(X) = 1$. So, if we apply Proposition 3.2, we conclude that M is diffeomorphic to $\mathbb{T}^3$.

Finally, by Theorem 2.5, X is C^∞-conjugated to a constant vector field on $\mathbb{T}^3$, which satisfies a Diophantine condition like estimate (1.7), and we finish the proof of Theorem A.

3.2 Proof of Proposition 3.1

Let $X_\alpha \in \mathfrak{X}(\mathbb{T}^1)$ be the Diophantine vector field given by $X_\alpha \equiv Dq(X)$. We know that $\alpha \neq 0$ and there is no loss of generality supposing that $\alpha > 0$.

Notice that for any $\theta \in \mathbb{T}^1$, the fiber $q^{-1}(\theta)$ is a global transverse section for the flow $\{\Phi_X^t\}_{t\in\mathbb{R}}$. So, it makes sense to define the Poincaré return map to $q^{-1}(\theta)$ and this will be denoted by $\mathscr{P}_\theta$. Observe that $\mathscr{P}_\theta = {\Phi_X}^{\alpha^{-1}}\Big|_{q^{-1}(\theta)}$.

Since the flow $\{\Phi_X^t\}$ does not have any periodic orbit, the Poincaré return map $\mathscr{P}_\theta$ does not have any periodic point. Hence, the Euler characteristic of the fiber

$q^{-1}(\theta)$ must vanish. Taking into account that the fiber is an orientable (maybe non-connected) surface, we can affirm that it is diffeomorphic to a disjoint union of k 2-torus. Our next step consists in proving that we can modify our good fibration q for getting another one with connected fiber. This is the contents of our next

Lemma 3.3. *If the fibration $q\colon M \to \mathbb{T}^1$ is such that $q^{-1}(\theta)$ has exactly k connected components for some (and hence for any) $\theta \in \mathbb{T}^1$, then there exists another smooth good fibration $\tilde{q}\colon M \to \mathbb{T}^1$ satisfying:*

1. *$\tilde{q}^{-1}(\theta)$ is diffeomorphic to the 2-torus;*
2. *$D\tilde{q}(X) \equiv X_{k^{-1}\alpha}$;*
3. *and the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\quad q \quad} & \mathbb{T}^1 \\ & {\scriptstyle \tilde{q}}\searrow \quad \nearrow{\scriptstyle E_k} & \\ & \mathbb{T}^1 & \end{array},$$

   is commutative, where $E_k : \theta \mapsto k\theta$ is the canonical k-fold covering of the circle.

Proof. Let θ_0 be an arbitrary point of $\mathbb{T}^1$ and let us write M_0 for denoting a connected component of $q^{-1}(\theta_0)$. Since our manifold M is connected, the Poincaré return map $\mathscr{P}_{\theta_0}$ must cyclically interchange all the connected components of $q^{-1}(\theta_0)$. Then, if we define

$$M_t \doteq {\Phi_X}^{t\alpha^{-1}}(M_0), \quad \text{for every } t \in \mathbb{R},$$

it holds

$$M_t = M_{t+k}, \quad \text{for every } t \in \mathbb{R};$$

$$M = \bigcup_{t\in\mathbb{R}} M_t.$$

Therefore, if we define $\tilde{q} \colon M \to \mathbb{T}^1$ by

$$\tilde{q}(x) \doteq k^{-1}t + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}, \quad \text{if } x \in M_t,$$

we easily see that $\tilde{q}$ is a good fibration for X, and it clearly satisfies properties (1), (2) and (3). □

With the purpose of simplifying our notation, we shall make the assumption that our original good fibration $q \colon M \to \mathbb{T}^1$ was such that its fibers $q^{-1}(\theta)$ were connected, and hence, diffeomorphic to $\mathbb{T}^2$.

Now, let us fix a point $\theta_0 \in \mathbb{T}^1$ and let $f \colon q^{-1}(\theta_0) \to \mathbb{T}^2$ denote any diffeomorphism. Hence, we can use the diffeomorphism f to write the Poincaré return map $\mathscr{P}_{\theta_0}$ as a diffeomorphism of $\mathbb{T}^2$, i.e. we have $f \circ \mathscr{P}_{\theta_0} \circ f^{-1} \in \mathrm{Diff}(\mathbb{T}^2)$. Then we can choose an appropriate matrix $\hat{A} \in \mathrm{SL}(2,\mathbb{Z})$ such that its induced linear automorphism $A \in \mathrm{Diff}(\mathbb{T}^2)$ is isotopic to $f \circ \mathscr{P}_{\theta_0} \circ f^{-1}$.

By Lefschetz fixed point theorem, and since $\mathscr{P}_{\theta_0}$ is fixed-point free, we know that

$$0 = L(\mathscr{P}_{\theta_0}) = \det(\hat{A} - id_{\mathbb{R}^2}). \tag{3.1}$$

In this way, since $\hat{A} \in \mathrm{SL}(2,\mathbb{Z})$, equation (3.1) implies that 1 is the only element in the spectrum of $\hat{A}$. Therefore, $\hat{A}$ must be $\mathrm{SL}(2,\mathbb{Z})$-conjugated to a matrix of the following form:

$$\begin{pmatrix} 1 & 0 \\ n_0 & 1 \end{pmatrix}, \tag{3.2}$$

commonly named the Jordan form of $\hat{A}$.

Then, post-composing f with an appropriate element of $\mathrm{SL}(2,\mathbb{Z})$ if necessary, we can assume that $\hat{A}$ equals to matrix (3.2).

On the other hand, notice that since $\mathscr{P}_{\theta_0}$ is the time-α^{-1} map of the flow $\{\Phi_X^t\}$, matrix $\hat{A}$ (in fact, the conjugacy class of $\hat{A}$ in $\mathrm{SL}(2,\mathbb{Z})$) determines the topology of M. More precisely, we know that M is a $\mathbb{T}^2$-bundle over $\mathbb{T}^1$, and so there exists a matrix $\hat{B} \in \mathrm{SL}(2,\mathbb{Z})$ such that M is smoothly diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}/(B,1)$, where $(B,1) \in \mathrm{Diff}(\mathbb{T}^2 \times \mathbb{R})$ is defined by $(B,1)\colon (x,t) \mapsto (Bx, t-1)$. Furthermore, it is well-known that, given $\hat{B}_1, \hat{B}_2 \in \mathrm{SL}(2,\mathbb{Z})$, $\mathbb{T}^2 \times \mathbb{R}/(B_1,1)$ is homeomorphic to $\mathbb{T}^2 \times \mathbb{R}/(B_2,1)$ if and only if $\hat{B}_1$ and $\hat{B}_2$ are $\mathrm{SL}(2,\mathbb{Z})$-conjugated (see for instance [Hat], Theorem 2.6, p. 36). Taking this into account, it is not difficult to verify that $\hat{A}$, the only matrix which induces an automorphism in the isotopy class of $f \circ \mathscr{P}_{\theta_0} \circ f^{-1}$, and $\hat{B}$ must be conjugated in $\mathrm{SL}(2,\mathbb{Z})$.

Therefore, there exists a smooth diffeomorphism $\Gamma\colon \mathbb{T}^2 \times \mathbb{R}/(A,1) \to M$.

Having gotten this nice topological characterization of M, our next aim consists in studying the algebraic properties of the fundamental group $\pi_1(M)$. For this, we define diffeomorphisms $\tau_0, \tau_1, \tau_2 \colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\tau_0 : (x^0, x^1, x^2) \mapsto (x^0 - 1, x^1, x^2); \tag{3.3}$$

$$\tau_1 : (x^0, x^1, x^2) \mapsto (x^0, x^1 - 1, x^2); \tag{3.4}$$

$$\tau_2 : (x^0, x^1, x^2) \mapsto (x^0, x^1 + n_0 x^0, x^2 - 1). \tag{3.5}$$

If $G(\tau_i)$ denotes the subgroup of $\mathrm{Diff}(\mathbb{R}^3)$ generated by $\{\tau_0, \tau_1, \tau_2\}$, we easily see that

$$\mathbb{R}^3 \big/ G(\tau_i) = \mathbb{T}^2 \times \mathbb{R} \big/ (\hat{A}, 1),$$

an therefore, we have that $G(\tau_i)$ is (algebraically) isomorphic to $\pi_1(M)$. As a consequence of this, we have that $H_1(M, \mathbb{Q})$ is isomorphic to

$$\Big(G(\tau_i) \big/ [G(\tau_i), G(\tau_i)] \Big) \otimes \mathbb{Q},$$

where $[G(\tau_i), G(\tau_i)]$ denotes the commutator subgroup of $G(\tau_i)$.

Hence, we finish the proof of Proposition 3.1 with the following

Lemma 3.4. *It holds*

$$\operatorname{rank}\left(G(\tau_i)\Big/[G(\tau_i), G(\tau_i)]\right) \geq 2.$$

Proof. Let $H \doteq \operatorname{span}\{\tau_0, \tau_2\}$ be the subgroup of $G(\tau_i)$ generated by τ_0 and τ_2. Let us write $\operatorname{pr}_i \colon \mathbb{R}^3 \to \mathbb{R}$ for the canonical projection on the i-th coordinate, where $i = 0, 1, 2$.

First, notice that for any $g \in [G(\tau_i), G(\tau_i)]$, we have

$$\operatorname{pr}_i \circ g - \operatorname{pr}_i \equiv 0, \quad \text{for } i = 0, 2. \tag{3.6}$$

Secondly, observe that H is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, being a possible group isomorphism defined by

$$h \mapsto (\operatorname{pr}_0 \circ h - \operatorname{pr}_0, \operatorname{pr}_2 \circ h - \operatorname{pr}_2). \tag{3.7}$$

Finally, taking into account (3.6) and (3.7), we easily conclude that the restriction to H of the canonical projection of $G(\tau_i)$ on its abelianization is injective. In other words, $G(\tau_i)/[G(\tau_i), G(\tau_i)]$ contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. □

3.3 Proof of Proposition 3.2

This is the last section of the current chapter and it is devoted to proving Proposition 3.2.

By hypothesis, there exists a good fibration $p \colon M \to \mathbb{T}^2$ for X. Since M is a closed 3-manifold fibering over $\mathbb{T}^2$, the fibers of p must be diffeomorphic to the union of k copies of $\mathbb{T}^1$. If $k > 1$, then the idea is that we can apply Lemma 3.3 "twice" to get a new good fibration with connected fibers. This is what we are going to do first:

Lemma 3.5. *There exists another good fibration* $\tilde{p}\colon M \to \mathbb{T}^2$ *for* X *verifying the following conditions:*

1. $\tilde{p}^{-1}(\theta)$ *is connected (and then, diffeomorphic to* $\mathbb{T}^1$*), for every* $\theta \in \mathbb{T}^2$*.*

2. *There exists* $k_0, k_1 \in \mathbb{N}$*, such that* $D\tilde{p}(X) \equiv X_{\tilde{\alpha}}$*, where* $\tilde{\alpha} \doteq (k_0^{-1}\alpha_0, k_1^{-1}\alpha_1)$*.*

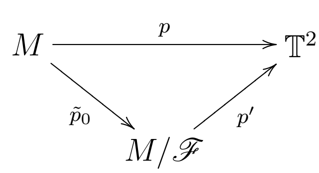
Remark 3.6. A very simple but fundamental observation for the future is that the new vector $\tilde{\alpha}$ continues to be Diophantine. This can be simply proved observing that

$$\left| r(k_0^{-1}\alpha_0) + s(k_1^{-1}\alpha_1) \right| \geq \frac{C}{(\max\{|rk_0^{-1}|, |sk_1^{-1}|\})^\tau} \geq \frac{C(\min\{k_0, k_1\})^\tau}{(\max\{|r|, |s|\})^\tau}$$

Proof of Lemma 3.5. Heuristically, we could apply twice the method used in the proof of Lemma 3.3 to "unfold" the good fibration p along each direction of $\mathbb{T}^2$. Nevertheless, here we shall develop a different technique that makes the proof a little clearer.

Let us start noticing that the fibration $p\colon M \to \mathbb{T}^2$ induces a smooth foliation $\mathscr{F}$ on M which leaves are the connected components of the fibers of p. Since $\mathscr{F}$ is a foliation with all its leaves compact, the space of leaves of $\mathscr{F}$, which will be denoted by $M/\mathscr{F}$, is a Hausdorff surface.

Moreover, $p\colon M \to \mathbb{T}^2$ clearly factors through $M/\mathscr{F}$, i.e. if $\tilde{p}_0\colon M \to M/\mathscr{F}$ denotes the canonical quotient map, then there exists a continuous map $p'\colon M/\mathscr{F} \to \mathbb{T}^2$ making the following diagram commutative:



Then, we can easily see that $p'\colon M/\mathscr{F} \to \mathbb{T}^2$ is a k-fold covering map (k is the number of connected components of any fiber of p) and therefore, $M/\mathscr{F}$ must be homeomorphic to $\mathbb{T}^2$. So, we can find two integers $k_0, k_1 \in \mathbb{N}$, with $k = k_0 k_1$, and a homeomorphism $h\colon M/\mathscr{F} \to \mathbb{T}^2$ expanding the previous diagram and getting

$$\begin{array}{ccc} M & \xrightarrow{\quad p \quad} & \mathbb{T}^2 \\ {\scriptstyle \tilde{p}_0}\big\downarrow & \overset{p'}{\nearrow} & \big\uparrow {\scriptstyle E_{k_0,k_1}} \\ M/\mathscr{F} & \xrightarrow{\quad h \quad} & \mathbb{T}^2 \end{array} \tag{3.8}$$

where $E_{k_0,k_1} : (\theta^0, \theta^1) \mapsto (k_0\theta^0, k_1\theta^1)$ is a k-fold covering. In this way, the map $\tilde{p} \doteq h \circ \tilde{p}_0$ is a smooth fibration satisfying $D\tilde{p}(X) \equiv (k_0^{-1}\alpha_0, k_1^{-1}\alpha_1)$ as desired. □

Having proved that we can find a good fibration with connected fibers satisfying all the hypotheses of Proposition 3.2, to simplify the notation, we shall assume that our original good fibration $p\colon M \to \mathbb{T}^2$ has connected fibers.

So, writing $q_0 \doteq \mathrm{pr}_0 \circ p\colon M \to \mathbb{T}^1$, we get a good fibration for X over $\mathbb{T}^1$, having connected fibers diffeomorphic to $\mathbb{T}^2$. On the other hand, since $Dp(X) \equiv X_\alpha$, being α a Diophantine vector of $\mathbb{R}^2$, we know that $\{\Phi_X^t\}$ cannot exhibit any periodic orbit. Hence, we are within the same context of Proposition 3.1. Repeating the same arguments exposed there, we may ensure that our manifold M is smoothly diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}/(A, 1)$, where

$$\hat{A} \doteq \begin{pmatrix} 1 & 0 \\ n_0 & 1 \end{pmatrix}, \tag{3.9}$$

and $(A, 1) \in \mathrm{Diff}(\mathbb{T}^2 \times \mathbb{R})$ is defined by $(A, 1) : (x, t) \mapsto (Ax, t - 1)$.

In this way we can reformulate the conclusion of Proposition 3.2 saying that $\mathcal{D}'(X)$ has infinite dimension, provided that $n_0 \neq 0$.

Continuing with the notation introduced in Section 3.2, the Poincaré return map to the fiber $q_0^{-1}(\theta)$ shall be denoted by $\mathscr{P}_\theta$, i.e. $\mathscr{P}_\theta = \Phi_X^{\alpha_0^{-1}}\big|_{q_0^{-1}(\theta)}$.

Observe that all the things that we have done so far had the purpose of returning to the setting of Proposition 3.1. Nevertheless, in this context we have additional geometric information about the Poincaré return map $\mathscr{P}_\theta$. First, it preserves a smooth foliation where all the leaves are circles (see (3.11) for the definition of the invariant foliation). As we will see in paragraph 3.3.1, this will let us get a fine system of coordinates for $\mathscr{P}_\theta$. Secondly, since $\{\Phi_x^t\}$ preserves a smooth volume form, we easily see that $\mathscr{P}_\theta$ also preserves a smooth volume form. We shall use this in paragraph 3.3.2 to improve our system of coordinates proving that, in fact, $\mathscr{P}_\theta$ is linearizable, i.e. it is smoothly conjugated to an affine map in $\mathbb{T}^2$.

Finally, using the fact that $\mathscr{P}_\theta$ is C^∞-conjugated to an affine map and applying a classical construction, attributed to Anatole Katok [KR01], we shall prove in paragraph 3.3.3 that there exist infinitely many linear independent X-invariant distributions, provided $\mathscr{P}_\theta$ is not isotopic to the identity.

3.3.1 The Invariant Foliation

In this paragraph we shall use the fact that $\mathscr{P}_\theta \in \mathrm{Diff}(q_0^{-1}(\theta))$ preserves a smooth foliation for proving that $\mathscr{P}_\theta$ is smoothly conjugated to a skew-product over a rigid rotation of $\mathbb{T}^1$.

For this, first notice that the flow $\{\Phi_X^t\}$ preserves the codimension-two foliation in M induced by the fibers of p. In fact it holds

$$\Phi_X^t(p^{-1}(\theta^0,\theta^1)) = p^{-1}(\theta^0 + t\alpha_0, \theta^1 + t\alpha_1), \quad \forall(\theta^0,\theta^1) \in \mathbb{T}^2,\ \forall t \in \mathbb{R}. \tag{3.10}$$

Moreover, by definition, each fiber of p is contained in a fiber of q_0. In other words, the fibration p is inducing a codimension-one foliation on each fiber $q_0^{-1}(\theta)$, and this foliation happens to be $\mathscr{P}_\theta$-invariant.

Then, let us fix some point $\theta \in \mathbb{T}^1$ and consider any smooth diffeomorphism $f_0\colon \mathbb{T}^2 \to q_0^{-1}(\theta)$. To simplify forthcoming notation, let us define

$$\mathscr{P}_1 \doteq f_0^{-1} \circ \mathscr{P}_\theta \circ f_0 \in \mathrm{Diff}(\mathbb{T}^2).$$

Let $\mathcal{F}$ be the codimension-one foliation on $\mathbb{T}^2$ defined by

$$\mathcal{F}(x) \doteq f_0^{-1}(p^{-1}(p(f_0(x)))), \quad \forall x \in \mathbb{T}^2, \tag{3.11}$$

where $\mathcal{F}(x)$ denotes the leaf of $\mathcal{F}$ passing through x.

On the other hand, if we define the *vertical foliation* $\mathcal{V}$ in $\mathbb{T}^2$ by

$$\mathcal{V}(\theta^0, \theta^1) \doteq \{\theta^0\} \times \mathbb{T}^1, \tag{3.12}$$

and since all the leaves of $\mathcal{F}$ are diffeomorphic to $\mathbb{T}^1$, it is a very well-known fact that there exists $f_1 \in \mathrm{Diff}(\mathbb{T}^2)$ verifying

$$f_1(\mathcal{V}(x)) = \mathcal{F}(f_1(x)), \quad \forall x \in \mathbb{T}^2. \tag{3.13}$$

Once again, for the sake of simplicity, let us define

$$\mathscr{P}_2 \doteq f_1^{-1} \circ \mathscr{P}_1 \circ f_1.$$

From (3.11) and (3.13) we easily see that there exists $g_1 \in \mathrm{Diff}(\mathbb{T}^1)$ satisfying

$$\mathrm{pr}_0(\mathscr{P}_2(\theta^0, \theta^1)) = g_1(\theta^0), \quad \forall (\theta^0, \theta^1) \in \mathbb{T}^2. \tag{3.14}$$

Then we have the following

Lemma 3.7. *g_1 is smoothly linearizable, i.e. there exists an orientation-preserving smooth diffeomorphism $h_1\colon \mathbb{T}^1 \to \mathbb{T}^1$ such that $h_1^{-1} \circ g_1 \circ h_1$ is an irrational rigid rotation on $\mathbb{T}^1$.*

Proof. First observe g_1 preserves orientation on $\mathbb{T}^1$, and hence it makes sense to consider its rotation number $\rho(g_1) \in \mathbb{T}^1$. Since $\mathscr{P}_2$ is a minimal diffeomorphism on

$\mathbb{T}^2$, we have that g_1 does not exhibit any periodic point, and therefore, the rotation number $\rho(g_1)$ is irrational and hence, it is completely determined by the order of the points of any orbit.

Then, notice that the order of the points of $\{g_1^n(x)\}_{n\in\mathbb{Z}}$ in $\mathbb{T}^1$, for any x in $\mathbb{T}^1$, is the same that the order of the leaves $\{\mathcal{F}(\mathscr{P}_\theta^n(z))\}_{n\in\mathbb{Z}}$ in $\mathbb{T}^2$, for any $z \in \mathbb{T}^2$. On the other hand, we know the order of the leaves $\{\mathcal{F}(\mathscr{P}_\theta^n(z))\}_{n\in\mathbb{Z}}$ is given by the Poincaré return map to the global section $\{\theta_0\} \times \mathbb{T}^1 \subset \mathbb{T}^2$ of the flow on $\mathbb{T}^2$ induced by the constant vector field (α_0, α_1). We can easily see that the dynamics of this return map is given by the rigid rotation $x \mapsto x + \alpha_1/\alpha_0$. Therefore, we can affirm that $\rho(g_1) = \alpha_1/\alpha_0 \mod \mathbb{Z}$.

Besides, we know that, by hypothesis, there exist real positive constants C and τ verifying

$$|m\alpha_0 + n\alpha_1| \geq \frac{C}{(\max\{|m|, |n|\})^\tau}, \quad \forall (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\},$$

and thus, elementary computations show that, indeed, it holds

$$\left| m + n\frac{\alpha_1}{\alpha_0} \right| \geq \frac{C'}{|n|^\tau} \quad \forall n \in \mathbb{Z} \setminus \{0\}, \tag{3.15}$$

for some other real constant $C' > 0$.

Finally, taking into account (3.15), we can apply Yoccoz linearization theorem [Yoc84] to guarantee that g_1 is smoothly conjugated to the rigid rotation R_{α_1/α_0}. □

This diffeomorphism h_1 can be used for defining $f_2 \in \mathrm{Diff}(\mathbb{T}^2)$ by $f_2 : (\theta^0, \theta^1) \mapsto (h_1(\theta^0), \theta^1)$, getting as result

$$f_2^{-1}(\mathscr{P}_2(f_2((\theta^0, \theta^1)) = \left(\theta^0 + \frac{\alpha_1}{\alpha_0}, \theta^1 + n_0\theta^0 + \eta(\theta^0, \theta^1) \right), \tag{3.16}$$

for some $\eta \in C^\infty(\mathbb{T}^2, \mathbb{R})$ and for every $(\theta^0, \theta^1) \in \mathbb{T}^2$.

Once again let us write

$$\mathscr{P}_3 \doteq f_2^{-1} \circ \mathscr{P}_2 \circ f_2. \tag{3.17}$$

3.3.2 The invariant Volume Form

In this paragraph we show that there exists a smooth $\mathscr{P}_\theta$-invariant volume form and analyze the consequences of this.

By hypothesis we know that there exists a smooth X-invariant volume form $\Omega \in \Lambda^3(M)$. So, if we write

$$\omega \doteq i_X\Omega, \tag{3.18}$$

we get an X-invariant 2-form. And since X is transverse to $\ker Dq_0$, we easily see that $\omega\big|_{q_0^{-1}(\theta)}$ is a $\mathscr{P}_\theta$-invariant area form on $q_0^{-1}(\theta)$.

Therefore, defining

$$\omega_3 \doteq (f_0 \circ f_1 \circ f_2)^*\omega \in \Lambda^2(\mathbb{T}),$$

we get a $\mathscr{P}_3$-invariant area form. Making some abuse of notation we can consider ω_3 as an element of $\mathfrak{M}(\mathscr{P}_3) \subset \mathfrak{M}(\mathbb{T}^2)$, identifying the area form with the Borel finite measure that it induces on $\mathbb{T}^2$. Then, by (3.16), we know that it holds

$$(\mathrm{pr}_0)_*\omega_3 = K\mathrm{Leb}^1, \tag{3.19}$$

where $K \doteq \int_{\mathbb{T}^2} \omega_3$ is a positive real constant and Leb^1 denotes the Haar measure on $\mathbb{T}^1$.

At this point it would be desirable to know that the invariant measure ω_3 is a constant multiple of Leb^2, the Haar measure of $\mathbb{T}^2$. We could easily achieve our goal applying the classical Moser's isotopy theorem [Mos65], but *a priori* we could not continue to have the skew-product structure of our diffeomorphism. This is the reason why it is necessary to get a "foliated version" of Moser's isotopy theorem. The following can be considered a two-dimensional reformulation of a more general result due to Richard Luz and Nathan dos Santos [LdS98]:

Theorem 3.8. *Let* $\Omega_1, \Omega_2 \in \Lambda^2(\mathbb{T}^2)$ *be two volume forms and suppose they satisfy:*

$$\int_{\mathbb{T}^2} \Omega_1 = \int_{\mathbb{T}^2} \Omega_2, \quad \textit{and} \quad \Omega_1(\mathrm{pr}_0^{-1}(C)) = \Omega_2(\mathrm{pr}_0^{-1}(C)),$$

for every Borel measurable set $C \subset \mathbb{T}^1$*, where we are considering* Ω_1 *and* Ω_2 *as elements of* $\mathfrak{M}(\mathbb{T}^2)$*. Then there exists* $H \in \mathrm{Diff}(\mathbb{T}^2)$ *isotopic to the identity verifying*

$$H^*\Omega_1 = \Omega_2, \quad \textit{and} \quad H(\mathcal{V}(x)) = \mathcal{V}(H(x)), \quad \forall x \in \mathbb{T}^2,$$

where $\mathcal{V}$ *is the vertical foliation in* $\mathbb{T}^2$ *defined in (3.12).*

Proof. See the proof of Theorem 6.1 in [LdS98]. □

Therefore, if we take into account (3.19), Theorem 3.8 lets us affirm that there exists a skew product map $f_3 \in \mathrm{Diff}(\mathbb{T}^2)$ verifying

$$f_3{}^*\big(K(d\theta^0 \wedge d\theta^1)\big) = \omega_3,$$

From this we see that the diffeomorphism $\mathscr{P}_4 \doteq f_3^{-1} \circ \mathscr{P}_3 \circ f_3 \in \mathrm{Diff}(\mathbb{T}^2)$ preserves the Haar measure and therefore, we can conclude that

$$\mathscr{P}_4(\theta^0, \theta^1) = \left(\theta^0 + \frac{\alpha_1}{\alpha_0}, \theta^1 + n_0\theta^0 + \chi(\theta^0)\right),$$

for some real function $\chi \in C^\infty(\mathbb{T}^1, \mathbb{R})$.

Since $\frac{\alpha_1}{\alpha_0}$ satisfies Diophantine condition (3.15), arguments analogous to those used in Example 1.7 let us prove that the rigid rotation $x \mapsto x + \frac{\alpha_1}{\alpha_0}$ on $\mathbb{T}^1$ is cohomology-free, and hence, we can find a function $\zeta \in C^\infty(\mathbb{T}^1, \mathbb{R})$ verifying

$$\zeta(x + \alpha_1\alpha_0^{-1}) - \zeta(x) = \chi(x) - \int_{\mathbb{T}^1} \chi \, \mathrm{d}(\mathrm{Leb}^1), \quad \forall x \in \mathbb{T}^1.$$

This function ζ can be used for linearizing $\mathscr{P}_4$. More precisely, if we define $f_4 : (\theta^0, \theta^1) \mapsto (\theta^0, \theta^1 + \zeta(\theta^0))$, we get

$$f_4^{-1}\Big(\mathscr{P}_4\big(f_4(\theta^0, \theta^1)\big)\Big) = \left(\theta^0 + \frac{\alpha_1}{\alpha_0}, \theta^1 + n_0\theta^0 + \int_{\mathbb{T}^1} \chi \, \mathrm{d}(\mathrm{Leb}^1)\right). \tag{3.20}$$

3.3.3 Invariant Distributions

Summarizing what we have done in previous paragraphs, we can simply say that there exists a diffeomorphism $F\colon \mathbb{T}^2 \to q_0^{-1}(\theta)$ verifying

$$F^{-1} \circ \mathscr{P}_\theta \circ F = A + (\alpha_1\alpha_0^{-1}, \beta), \tag{3.21}$$

where A is the automorphism of $\mathbb{T}^2$ induced by matrix $\hat{A}$ defined in (3.9) and $\beta = \int_{\mathbb{T}^1} \chi \, \mathrm{d}(\mathrm{Leb}^1)$ is obtained in (3.20).

By (3.9), we know that if $n_0 = 0$, then M is diffeomorphic to $\mathbb{T}^3$. Hence, we shall assume that $n_0 \neq 0$ and applying a construction due to Katok [KR01], we will get infinitely many linearly independent $\mathscr{P}_\theta$-invariant distributions on $\mathbb{T}^2$.

For this, let us start defining $T_m \in \mathcal{D}'(\mathbb{T}^2)$, for each $m \in \mathbb{Z} \setminus \{0\}$, writing

$$\langle T_m, \psi \rangle \doteq \sum_{k\in\mathbb{Z}} \hat{\psi}(kn_0m, m) e^{-2\pi ikm\left(\beta + \frac{k-1}{2} n_0\alpha_1\alpha_0^{-1}\right)}, \tag{3.22}$$

for each $\psi \in C^\infty(\mathbb{T}^2, \mathbb{R})$ and where $\hat{\psi}\colon \mathbb{Z}^2 \to \mathbb{C}$ denotes, as usual, the Fourier transform of ψ. Clearly, the set $\{T_m : m \in \mathbb{Z} \setminus \{0\}\}$ is linearly independent. Furthermore, we can make the following

Claim 1. If we define $B \doteq A + (\alpha_1\alpha_0^{-1}, \beta) \in \mathrm{Diff}(\mathbb{T}^2)$, it holds

$$\langle T_m, \psi \circ B\rangle = \langle T_m, \psi\rangle, \quad \forall m \in \mathbb{Z} \setminus \{0\},\ \forall \psi \in C^\infty(\mathbb{T}^2, \mathbb{R}), \tag{3.23}$$

In fact, we have

$$\begin{aligned}
\widehat{\psi \circ B}(k,\ell) &= \widehat{\psi \circ A}(k,\ell) \exp(2\pi i(k\alpha_1\alpha_0^{-1} + \ell\beta)) \\
&= \hat{\psi}((A^*)^{-1}(k,\ell)) \exp(2\pi i(k\alpha_1\alpha_0^{-1} + \ell\beta)) \\
&= \hat{\psi}(k - n_0\ell, \ell) \exp(2\pi i(k\alpha_1\alpha_0^{-1} + \ell\beta)).
\end{aligned}$$

And hence, it holds

$$\begin{aligned}
\langle T_m, \psi \circ B\rangle &= \sum_{k\in\mathbb{Z}} \widehat{\psi \circ B}(kn_0m, m) e^{-2\pi ikm\left(\beta + \frac{k-1}{2} n_0\alpha_1\alpha_0^{-1}\right)} \\
&= \sum_{k\in\mathbb{Z}} \hat{\psi}((k-1)n_0m, m) e^{2\pi i(kn_0\alpha_1\alpha_0^{-1}+\beta)m} e^{-2\pi ikm\left(\beta + \frac{k-1}{2} n_0\alpha_1\alpha_0^{-1}\right)} \\
&= \sum_{k\in\mathbb{Z}} \hat{\psi}((k-1)n_0m, m) e^{-2\pi i(k-1)m\left(\beta + \frac{k-2}{2} n_0\alpha_1\alpha_0^{-1}\right)} \\
&= \langle T_m, \psi\rangle,
\end{aligned}$$

for every $\psi \in C^\infty(\mathbb{T}^2, \mathbb{R})$.

Then, taking into account equation (3.21), we see that we may push-forward each T_m by F for getting infinitely many linearly independent $\mathscr{P}_\theta$-invariant distributions on $q_0^{-1}(\theta)$.

Finally, if we define $\tilde{T}_m \in \mathcal{D}'(M)$, for $m \in \mathbb{Z} \setminus \{0\}$, writing

$$\langle \tilde{T}_m, \psi\rangle \doteq \int_{\mathbb{T}^1} \left\langle F_* T_m, (\psi \circ \Phi_X^{-t})\big|_{q_0^{-1}(\theta+t)} \right\rangle \mathrm{d}t, \tag{3.24}$$

for every $\psi \in C^\infty(M, \mathbb{R})$, we easily see that each $\tilde{T}_m \in \mathcal{D}'(X) \setminus \mathfrak{M}(X)$ and they clearly form a linearly independent set.

Chapter 4

The case $\beta_1(M) = 0$

This chapter aims to prove that there is no cohomology-free vector field on closed orientable 3-manifolds with vanishing first Betti number.

First of all, notice that by Poincaré duality, a closed 3-manifold with trivial first rational cohomology group must also have trivial second cohomology group. So, from now on and until the end of the current chapter, M will denote a rational homological 3-sphere and we shall suppose that there exists a cohomology-free $X \in \mathfrak{X}(M)$.

The general strategy for getting a contradiction from our assumptions consists in proving first that there exists an X-invariant one-form with no singularity. Then, we shall analyze the integrability of its kernel, getting two possible cases: either the kernel of the invariant form is everywhere integrable, or it is a contact structure, being X collinear with the induced Reeb vector field (see Section 4.2 for more details). The rest of the proof consists in proving that both cases lead to a contradiction.

4.1 The Invariant 1-form

The main purpose of this section is to prove that the derivative of the flow $\{\Phi_X^t\}$ preserves a smooth two-dimensional plane field.

At this point the author would like to thank Giovanni Forni who kindly communicated the following result to us:

Theorem 4.1. *Let M be a closed 3-manifold such that $H^1(M,\mathbb{Q}) = H^2(M,\mathbb{Q}) = 0$ and let $X \in \mathfrak{X}(M)$ be a cohomology-free vector field. Then there exists $\lambda \in \Lambda^1(M)$ verifying*

$$\mathcal{L}_X\lambda \equiv 0 \quad \textit{and} \quad \lambda(p) \neq 0,$$

for every $p \in M$.

Proof. By Proposition 2.3, we know that there exists an X-invariant volume form $\Omega \in \Lambda^3(M)$. Hence, if we write $\omega \doteq i_X\Omega$, Cartan's formula lets us affirm

$$0 = \mathcal{L}_X\Omega = d(i_X\Omega) + i_X(d\Omega) = d\omega,$$

i.e. $\omega \in \Lambda^2(M)$ is a closed form. On the other hand, by the Universal Coefficient Theorem we know that $H^2(M,\mathbb{R}) = 0$, and thus, there exists a 1-form $\tilde{\lambda}$ such that $\omega = d\tilde{\lambda}$. Applying Cartan's formula once again we obtain

$$\mathcal{L}_X\tilde{\lambda} = d(i_X\tilde{\lambda}) + i_X(d\tilde{\lambda}) = d(i_X\tilde{\lambda}) + i_X(i_X\Omega) = d(i_X\tilde{\lambda}).$$

Notice that $i_X\tilde{\lambda}$ is an element of $C^\infty(M,\mathbb{R})$, so there exists a smooth function $u\colon M \to \mathbb{R}$ verifying

$$\mathcal{L}_X u = -i_X\tilde{\lambda} + \int_M (i_X\tilde{\lambda})\Omega. \tag{4.1}$$

Therefore, if we define $\lambda \doteq \tilde{\lambda} + du$, it still holds $d\lambda = \omega$ and besides,

$$\begin{aligned}\mathcal{L}_X\lambda &= \mathcal{L}_X\tilde{\lambda} + \mathcal{L}_X du = d(i_X\tilde{\lambda}) + d(i_X du) \\ &= d\big(i_X\tilde{\lambda} + \mathcal{L}_X u\big) = d\bigg(\int_M (i_X\tilde{\lambda})\Omega\bigg) = 0,\end{aligned}$$

i.e. λ is an X-invariant 1-form.

Then, taking into account the minimality of $\{\Phi_X^t\}$, we easily see that λ exhibits a singularity if and only if $\lambda \equiv 0$. On the other hand, since $d\lambda = i_X\Omega \neq 0$, we know that $\lambda \not\equiv 0$, and therefore, λ does not have any singularity. $\square$

So, we have proved the existence of a singularity-free 1-form λ on M which is invariant under the flow $\{\Phi_X^t\}$. This lets us define an invariant two-dimensional plane field

$$\Sigma \doteq \ker \lambda \subset TM.$$

Now, it seems to be natural to ask ourselves about the integrability of the plane field Σ. For this, it is interesting to notice that the minimality of $\{\Phi_X^t\}$ implies that Σ is either a contact structure or it is everywhere integrable.

These cases will be analyzed in the following two sections.

4.2 The Contact Structure Case

Let us start this section recalling some fundamental facts about Contact Geometry.

Given a $(2n+1)$-manifold N, we say that $\alpha \in \Lambda^1(N)$ is a *contact form* if $\alpha \wedge (d\alpha)^{\wedge 2n}$ is a volume form on N. This clearly implies that $\ker\alpha \oplus \ker d\alpha = TM$, and consequently, there exists a unique vector field $Y \in \mathfrak{X}(N)$, called the *Reeb vector field induced by* α, verifying

$$i_Y\alpha \equiv 1, \quad \text{and} \quad i_Y d\alpha \equiv 0.$$

As the completely opposite case we know by Froebenius theorem that the kernel of a singularity-free 1-form β is completely integrable (i.e. there exists a smooth foliation $\mathscr{F}$ verifying $T\mathscr{F} = \ker\beta$) if and only if $\beta \wedge d\beta \equiv 0$.

As it was already mentioned at the end of Section 4.1, we have a strict dichotomy: either λ is a contact form, or Σ is completely integrable. In this section we shall analyze the first case, i.e. we shall assume that λ is a contact form.

We know that $d\lambda = \omega = i_X\Omega$, and therefore, $\ker d\lambda = \mathbb{R}X$. This implies that

$X \notin \Sigma$ and consequently, by equation (4.1), we have

$$\lambda(X) \equiv \int_M (i_X \tilde{\lambda}) \Omega \neq 0.$$

All this implies that X is a constant multiple of the Reeb vector field of λ, and in particular, they have the same orbits.

A very important problem in Contact Geometry that has received a lot of attention and has led much of the research in this area during the last decades is the following conjecture proposed by Alan Weinstein in [Wei79]:

Conjecture 4.2 (Weinstein's Conjecture)**.** *Let N be a closed 3-manifold, $\alpha \in \Lambda^1(N)$ be a smooth contact form and $Y \in \mathfrak{X}(N)$ be its Reeb vector field. Then Y exhibits a periodic orbit.*

Clifford Taubes has recently proved the validity of this conjecture in [Tau], and in our setting, this leads us to a contradiction: since $\{\Phi_X^t\}$ is minimal, it cannot have any periodic orbit.

4.3 The Completely Integrable Case

In this section we shall analyze the situation where Σ is a completely integrable plane field. As it was already mentioned in Section 1.5, while this work was in progress, Giovanni Forni communicated to the author that he had been able to exclude this case using the foliation tangent to Σ to prove that M should be diffeomorphic to a nilmanifold and $\{\Phi_X^t\}$ smoothly conjugated to a homogeneous flow. On the other hand, Stephen Greenfield and Nolan Wallach had already proved in [GW73] that $\mathbb{T}^3$ was the only 3-dimensional nilmanifold that supported cohomology-free homogeneous vector fields[1] (see [FF07] for higher dimensional nilmanifolds).

[1] In fact, in [GW73] they proved this for globally hypoelliptic vector fields.

Nevertheless, in this work we propose a completely different proof that does not use the integrability condition in a direct form. In fact, the only information we need for our proof is that our vector field is contained in the plane field Σ. This approach has the advantage that seems to be more "usable" for solving the contact structure case independently of Taubes' proof of Weinstein's conjecture, which would be very desirable (see Section 5.1 for a more detailed discussion about this point).

Our general strategy consists in proving that, under our assumptions about the topology of M, $\{\Phi_X^t\}$ must be a positively expansive flow (see Definition 4.13).

For getting this, we will have to carefully study the dynamics of the derivative of the flow $\{\Phi_X^t\}$ on TM. This analysis starts in paragraph 4.3.3, where we get our first result about the angular behavior of $D\Phi_X : TM \times \mathbb{R} \to TM$, studying the dynamics of the *projective flow* (see paragraph 4.3.1 for definitions). Then, in paragraph 4.3.2, we shall get some results about the radial behavior of flow $\{D\Phi_X^t\}$ analyzing the dynamics of the *normal flow* and proving that it exhibits a parabolic behavior. And in paragraph 4.3.6, we will prove that our flow $\{\Phi_X^t\}$ is indeed positively expansive.

On the other hand, using a nice result due to Miguel Paternain [Pat93] about expansive flows on 3-manifolds, we shall prove in paragraph 4.3.6 that there is no closed 3-manifold supporting positively expansive flows, getting our desired contradiction.

4.3.1 The Normal and Projective Flows

This short paragraph is devoted to introduce some terminology that we shall repeatedly use in subsequent paragraphs.

Let us start defining the relation $\sim$ on TM by

$$v \sim w \iff \pi(v) = \pi(w) \text{ and } v - w \in \operatorname{span}(X),$$

where $\pi : TM \to M$ stands for the canonical vector bundle projection. This is clearly an equivalence relation and thus, we can define NX to be the quotient of TM by this relation. This set NX can be naturally endowed with a unique C^∞ vector bundle structure $\pi_N\colon NX \to M$ such that the quotient map $\mathrm{pr}_X\colon TM \to NX$ given by $\mathrm{pr}_X : v \mapsto \hat{v} \doteq \{w \in TM : v \sim w\}$ is a smooth vector bundle map. This shall be called the *normal vector bundle induced by* X.

Observe that since $D\Phi_X^t(X(p)) = X(\Phi_X^t(p))$, we easily see the derivative of $\{\Phi_X^t\}$ induces a *vector bundle flow* $N\Phi_X\colon NX \times \mathbb{R} \to NX$ over $\{\Phi_X^t\}$, i.e. it makes sense to define

$$N\Phi_X^t(\hat{v}) \doteq \mathrm{pr}_X(D\Phi_X^t(v)), \quad \text{for any } v \in \mathrm{pr}_X^{-1}(\hat{v}),$$

and any $t \in \mathbb{R}$. This flow $\{N\Phi_X^t\}$, which will be called the *normal flow* induced by $\{\Phi_X^t\}$, clearly verifies $\pi_N \circ N\Phi_X^t = \Phi_X^t \circ \pi_N$, being $N\Phi_X^t\colon NX_p \to NX_{\Phi_X^t(p)}$ a linear isomorphism.

Then, since $\{N\Phi_X^t\}$ is a vector bundle flow, it induces a new flow on $\pi_{\mathbb{P}}\colon \mathbb{P}(NX) \to M$, the projectivization of the normal bundle $\pi_N\colon NX \to M$. This will be called the *projective flow* induced by $\{\Phi_X^t\}$ and it shall be denoted by $P\Phi_X\colon \mathbb{P}(NX) \times \mathbb{R} \to \mathbb{P}(NX)$. We will write $\mathrm{pr}_{\mathbb{P}}\colon NX\backslash\{0\} \to \mathbb{P}(NX)$[2] for the canonical quotient projection given by $\mathrm{pr}_{\mathbb{P}} : \hat{v} \mapsto (\mathbb{R} \setminus \{0\})\hat{v}$.

4.3.2 Dynamics of the Normal Flow I

In this paragraph we start the analysis of the dynamics of the normal flow $\{N\Phi_X^t\}$.

Let us start introducing any smooth Riemannian structure $\langle \cdot, \cdot \rangle$ on TM. This naturally induces another Riemannian structure $\langle \cdot, \cdot \rangle_{NX}$ on NX defining, for each $p \in M$,

$$\langle v_1, v_2 \rangle_{NX} \doteq \langle v_1', v_2' \rangle, \quad \forall v_1, v_2 \in NX_p,$$

where v_i' is defined as the only element of T_pM verifying simultaneously $\langle X(p), v_i'(p) \rangle =$

[2] In this context $\{0\}$ means the zero section of NX.

0 and $\mathrm{pr}_X(v_i') = v_i$. The Finsler structures induced by $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{NX}$ will be denoted by $\|\cdot\|$ and $\|\cdot\|_{NX}$, respectively. As usual, we shall also use the Riemannian structures $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{NX}$ for measuring angles between non-null vectors of the same fiber. Making some abuse of notation, we shall use the symbol $\sphericalangle(\cdot,\cdot)$ for both.

Before we state our first result about the radial behavior of vectors in NX, we need to recall some notions of hyperbolic dynamics:

Given a closed d-manifold B, a vector bundle $\pi\colon E \to B$ and a non-singular vector field $Y \in \mathfrak{X}^r(B)$ ($r \geq 2$), we say that $A\colon E \times \mathbb{R} \to E$ is a *linear cocycle over* $\{\Phi_Y^t\}$ if it holds $\Phi_X^t \circ \pi = \pi \circ A(\cdot,t)$, for any t, being the maps $A(\cdot,t)\colon \pi^{-1}(p) \to \pi^{-1}(\Phi_X^t(p))$ linear isomorphisms that verify

$$A(p, t_0 + t_1) = A(\Phi_X^{t_0}(p), t_1)A(p, t_0), \quad \forall p \in B,\ \forall t_0, t_1 \in \mathbb{R}.$$

We shall say that cocycle A is *Anosov* if there exist two sub-bundles $E^s, E^u \subset E$ and real constants $C > 0$ and $\rho \in (0,1)$ verifying

- $E^s \oplus E^u = E$,
- $A(E^\sigma_p, t) = E^\sigma_{\Phi_Y^t(p)}$, for every $p \in M$, every $t \in \mathbb{R}$ and $\sigma = s, u$.
- $\left\|A(\cdot,t)\big|_{E^s}\right\| \leq C\rho^t$, and $\left\|A(\cdot,-t)\big|_{E^u}\right\| \leq C\rho^t$, for any $t > 0$,

where $\|\cdot\|$ is any Finsler structure on $\pi\colon E \to B$.

We shall say that cocycle A is *quasi Anosov* if, given any $v \in E$, it holds

$$\sup_{t\in\mathbb{R}} \|A(v,t)\| < \infty \Rightarrow v = 0. \tag{4.2}$$

The following result appears in different forms, and in fact with different hypothesis, in the works of Ricardo Mañé [Mn77], Robert Sacker and George Sell [SS74], and James Selgrade [Sel75, Sel76]:

Proposition 4.3. *If the flow $\{\Phi_Y^t\}$ does not have wandering points, then a cocycle A is quasi Anosov if and only if it is Anosov.*

On the other hand, we shall say that a vector field $Y \in \mathfrak{X}^r(B)$ is *Anosov* if there exists a codimension-one $D\Phi_Y$-invariant sub-bundle $F \subset TB$ verifying $F \oplus \mathbb{R}X = TB$ and such that $D\Phi_Y|_F \colon F \times \mathbb{R} \to F$ is an Anosov linear cocycle.

Then, we can state the following result due to Claus Doering:

Proposition 4.4 (Doering [Doe87])**.** *Let suppose that $\{\Phi_Y^t\}$ does not have any wandering point. Then, Y is an Anosov vector field if and only if its normal flow $\{N\Phi_Y^t\}$ is an Anosov linear cocycle (over $\{\Phi_Y^t\}$).*

Now, we can present our first result about the dynamics of our normal flow $\{N\Phi_X^t\}$:

Lemma 4.5. *There exists $\hat{v}_0 \in NX$ such that $\hat{v}_0 \neq 0$ and*

$$\sup_{t\in\mathbb{R}} \|N\Phi_X^t(\hat{v}_0)\|_{NX} < \infty. \tag{4.3}$$

Proof. Let us suppose that estimate (4.3) is not satisfied by any non-vanishing vector in NX. In other words, let suppose that $N\Phi_X \colon NX \to \mathbb{R} \to NX$ is quasi Anosov. By Proposition 4.3, $\{N\Phi_X^t\}$ is an Anosov cocycle. Then, Proposition 4.4 lets us affirm that X is indeed Anosov.

Finally, it is a very well-known fact that any Anosov flow exhibits (infinitely many) periodic orbits, which clearly contradicts the minimality of $\{\Phi_X^t\}$. $\square$

Our second result about the dynamics of the normal flow is the following

Lemma 4.6. *The normal flow $\{N\Phi_X^t\}$ is conservative. More precisely, there exists a symplectic form κ on the vector bundle $\pi_N \colon NX \to M$ which is invariant under the action of $\{N\Phi_X^t\}$.*

Proof. Notice that $\omega = i_X\Omega = d\lambda$ is a 2-form on TM verifying $i_X\omega \equiv 0$. This implies that we may push-forward this form by pr_X on NX, i.e. we can find a smooth 2-form κ on NX such that

$$\kappa(\mathrm{pr}_X(v), \mathrm{pr}_X(w)) = \omega(v, w), \quad \forall v, w \in T_pM,\ \forall p \in M.$$

It is very easy to verify that κ is symplectic on NX and that it is $N\Phi_X$-invariant. □

4.3.3 Dynamics of the Projective Flow

This paragraph aims to prove that the dynamics of the projective flow is very simple. In fact, we shall get that the limit set of $\{P\Phi_X^t\}$ is a smooth submanifold of $\mathbb{P}(NX)$ which happens to be a graph over M, being the dynamics on this set smoothly conjugated to $\{\Phi_X^t\}$.

For this, first we will need the following result due to Hiromichi Nakayama and Takeo Noda about the geometry and amount of minimal sets for the projective flow:

Theorem 4.7 (Nakayama & Noda [NN05])**.** *Let V be a closed 3-manifold and let $Y \in \mathfrak{X}(V)$ be such that its induced flow $\Phi_Y \colon V \times \mathbb{R} \to V$ is minimal.*

Let $P\Phi_Y \colon \mathbb{P}(NY) \times \mathbb{R} \to \mathbb{P}(NY)$ be the projective flow induced by $\{\Phi_Y^t\}$. Hence, we have:

1. *If $\{P\Phi_Y^t\}$ exhibits more than two minimal sets, then V is diffeomorphic to $\mathbb{T}^3$ and $\{\Phi_X^t\}$ is continuously conjugate to an irrational translation.*

2. *If $\{P\Phi_Y^t\}$ exhibits exactly two minimal sets $M_1, M_2 \subset \mathbb{P}(NY)$ and $\{\Phi_X^t\}$ is not C^0-conjugate to an irrational translation on $\mathbb{T}^3$, then for any $z \in V$ it holds: $M_1 \cap \pi_{\mathbb{P}}^{-1}(z)$ or $M_2 \cap \pi_{\mathbb{P}}^{-1}(z)$ consists of a single point. Moreover, there exists a residual subset $B \subset V$ such that both sets $M_1 \cap \pi_{\mathbb{P}}^{-1}(z)$ and $M_2 \cap \pi_{\mathbb{P}}^{-1}(z)$ contain just a point, for every $z \in B$.*

Since we are assuming that $H_1(M, \mathbb{Q}) = 0$, Theorem 4.7 lets us affirm that the flow $\{P\Phi_X^t\}$ exhibits at most two minimal sets.

One is given by the plane field Σ. In fact, we have $X(p) \in \Sigma_p$ for each $p \in M$, and hence,

$$E_\Sigma \doteq \mathrm{pr}_X(\Sigma) \subset NX, \tag{4.4}$$

is a smooth one-dimensional vector sub-bundle of NX. In this way, E_Σ determines exactly one point on each fiber of $\pi_{\mathbb{P}} \colon \mathbb{P}(NX) \to M$. More precisely, we may define the point $\theta_p \in \pi_{\mathbb{P}}^{-1}(p)$ by $\theta_p \doteq \mathrm{pr}_{\mathbb{P}}(E_{\Sigma p} \setminus \{0\})$.

Notice that since the plane field Σ is invariant under the action of $\{D\Phi_X^t\}$, we have the flow $\{N\Phi_X^t\}$ leaves invariant the line field E_Σ, and therefore, it holds $P\Phi_X^t(\theta_p) = \theta_{\Phi_X^t(p)}$, for any $p \in M$ and any $t \in \mathbb{R}$. So, summarizing we have that

$$K_\Sigma \doteq \{\theta_p : p \in M\} \subset \mathbb{P}(NX) \tag{4.5}$$

is a minimal set for $\{P\Phi_X^t\}$.

Finally, as it was mentioned above, we shall prove that K_Σ is indeed the only minimal set, and consequently, it is the α- and ω-limit of any point in $\mathbb{P}(NX)$:

Theorem 4.8. *$K_\Sigma \subset \mathbb{P}(NX)$ defined in (4.5) is the only minimal set for $\{P\Phi_X^t\}$.*

For proving Theorem 4.8, we shall suppose that there exists another $P\Phi_X$-invariant minimal set $K_0 \subset \mathbb{P}(NX)$ (i.e. different from K_Σ), and for the sake of clarity of the exposition, we will separate the proof in several lemmas:

Lemma 4.9. *Sub-bundle $E_\Sigma \subset NX$ defined in (4.4) is orientable, and therefore, it admits a non-vanishing section $\hat{Y}_0 \in \Gamma(E_\Sigma)$.*

Proof. Since Σ was defined as the kernel of a non-singular 1-form and by hypothesis, M is orientable, we have that $\Sigma \to M$ is orientable. On the other hand, our vector field X can be considered as a non-singular element of $X \in \Gamma(\Sigma)$.

This lets us affirm that $\Sigma \to M$ is a globally trivial vector bundle, and therefore, we can find a smooth section $Y_0 \in \Gamma(\Sigma)$ verifying $\Sigma_p = \operatorname{span}\{X(p), Y_0(p)\}$, for every $p \in M$.

Finally, defining $\hat{Y}_0 \doteq \operatorname{pr}_X(Y_0)$ we get our desired section of $E_\Sigma \to M$. $\square$

Lemma 4.10. *Assuming that there exists another minimal set $K_0 \subset \mathbb{P}(NX)$, we can find a non-vanishing $\hat{Y} \in \Gamma(E_\Sigma)$ verifying*

$$N\Phi_X^t\big(\hat{Y}(p)\big) = \hat{Y}(\Phi_X^t(p)), \quad \forall p \in M, \ \forall t \in \mathbb{R}. \tag{4.6}$$

Proof. Let $L_\Sigma \in C^\infty(M, \mathbb{R})$ be defined by

$$L_\Sigma(p)\hat{Y}_0(p) = \lim_{t\to 0} \frac{N\Phi_X^{-t}(\hat{Y}_0(\Phi_X^t(p))) - \hat{Y}_0(p)}{t}, \quad \forall p \in M.$$

Using the fact that X is cohomology-free, we get a function $u \in C^\infty(M, \mathbb{R})$ verifying

$$\mathcal{L}_X u = -L_\Sigma + \int_M L_\Sigma \Omega. \tag{4.7}$$

Then, if we define $\hat{Y} \doteq e^u \hat{Y}_0$, applying equation (4.7) we clearly get

$$\lim_{t\to 0} \frac{N\Phi_X^{-t}(\hat{Y}(\Phi_X^t(p))) - \hat{Y}(p)}{t} = \left(\int_M L_\Sigma \Omega\right) \hat{Y}(p), \quad \forall p \in M,$$

and therefore, it holds

$$N\Phi_X^t(\hat{Y}(p)) = \exp\left(t \int_M L_\Sigma \Omega\right) \hat{Y}(\Phi_X^t(p)), \tag{4.8}$$

for every $p \in M$ and every $t \in \mathbb{R}$.

Notice that by equation (4.8), $\int_M L_\Sigma \Omega$ is a Lyapunov exponent of the linear cocycle $\{N\Phi_X^t\}$. So, let us suppose that $\int_M L_\Sigma \Omega \neq 0$. In this case, the one-dimensional subbundle $E_\Sigma \subset NX$ is uniformly hyperbolic.

On the other hand, by Theorem 4.7, we know that K_0 and K_Σ are the only minimal sets on $\mathbb{P}(NX)$, and moreover, we can find a point $p_0 \in M$ such that $\theta' \in \mathbb{P}(NX)$ is the only point in $K_0 \cap \pi_{\mathbb{P}}^{-1}(p_0)$.

Observe that, since K_0 and K_Σ are disjoint closed sets, we have that there exists a real constant $C > 0$ such that

$$\operatorname{dist}_{\mathbb{P}}\left(P\Phi_X^t(\theta_{p_0}), P\Phi_X^t(\theta')\right) > C, \quad \forall t \in \mathbb{R}, \tag{4.9}$$

where $\operatorname{dist}_{\mathbb{P}}$ denotes the distance function on $\mathbb{P}(NX)$ induced by the Riemannian structure $\langle\cdot,\cdot\rangle_{NX}$.

Then, taking into account conservativeness proved in Lemma 4.6, estimate (4.9) and equation (4.8), we have that any vector $\hat{v} \in NX_{p_0}$ whose pr_X-projection is equal to $\theta' \in K_0 \cap \pi_{\mathbb{P}}^{-1}(p_0)$ will satisfies the following estimate:

$$\left\|N\Phi_x^t(\hat{v})\right\|_{NX} \leq C' \exp\left(-t\int_M L_\Sigma\Omega\right)\|\hat{v}\|_{NX}, \quad \forall t \in \mathbb{R}, \tag{4.10}$$

and for some real constant $C' > 0$, that just depends on constant C of estimate (4.9).

From equation (4.8) and estimate (4.10) (and supposing that $\int_M L_\Sigma\Omega \neq 0$), we clearly conclude that Oseldets splitting (see [Ose68]) of the linear cocycle $\{N\Phi_X^t\}$ is not just measurable, but continuous and uniformly hyperbolic. This implies that $\{N\Phi_X^t\}$ is an Anosov cocycle, and by Proposition 4.4, we know that X must be Anosov, which is clearly impossible, since $\{\Phi_X^t\}$ does not have any periodic orbit.

Therefore, the absurd comes from our supposition that $\int_M L_\Sigma\Omega$ could be non-null. Finally, equation (4.8) let us assure that $\hat{Y}$ is a $N\Phi_X$-invariant section, as desired. □

Now, we are ready for proving the theorem:

Proof of Theorem 4.8. Let $K_0 \subset \mathbb{P}(NX)$, $p_0 \in M$ and $\theta' \in K_0 \cap \pi_{\mathbb{P}}^{-1}(p_0) \in \mathbb{P}(NX)$ as above. Let $\hat{v} \in NX_{p_0}$ verifying $\operatorname{pr}_{\mathbb{P}}(\hat{v}) = \theta'$.

We can rewrite estimate (4.9) as

$$\inf_{t\in\mathbb{R}} \sphericalangle\left(Y(\Phi_X^t(p)), N\Phi_X^t(\hat{v})\right) > 0. \tag{4.11}$$

Putting together equation (4.6), estimate (4.11) and Lemma 4.6, we get that there exists a real constant $C'' > 1$ verifying

$$\frac{1}{C''} < \left\|N\Phi_X^t(\hat{v})\right\|_{NX} < C'', \quad \forall t \in \mathbb{R}. \tag{4.12}$$

Now, consider another vector $\hat{w} \in NX_{p_0} \setminus \{0\}$ such that $\mathrm{pr}_{\mathbb{P}}(\hat{w}) \notin K_\Sigma \cup K_0$. Since K_Σ and K_0 are the only minimal sets for $\{P\Phi_X^t\}$, we know that the ω-limit of $\mathrm{pr}_{\mathbb{P}}(\hat{w})$ must be either K_0 or K_Σ. Let us suppose that the positive semi-orbit of $\mathrm{pr}_{\mathbb{P}}(\hat{w})$ accumulates on K_Σ. This implies that

$$\lim_{t\to+\infty} \sphericalangle\left(\hat{Y}\left(\Phi_X^t(p_0)\right), N\Phi_X^t(\hat{w})\right) = 0. \tag{4.13}$$

Once again, taking into account that $\{N\Phi_X^t\}$ preserves the symplectic form κ and section $\hat{Y} \in \Gamma(NX)$, we see that equation (4.13) implies that

$$\left\|N\Phi_x^t(\hat{w})\right\| \longrightarrow \infty, \quad \text{as } t \to +\infty. \tag{4.14}$$

Finally, we clearly see that estimates (4.11), (4.12) and (4.14) violate conservativeness.

We can analogously get a contradiction supposing that the ω-limit of $\mathrm{pr}_{\mathbb{P}}(\hat{w})$ is K_0, and the we conclude that K_Σ is the only minimal set for $\{P\Phi_X^t\}$. □

4.3.4 Dynamics of the Normal Flow II

In paragraph 4.3.2 we begun the analysis of the dynamics of the normal flow $\{N\Phi_x^t\}$. After what we have just done in paragraph 4.3.3, here we shall see that some of the results previously gotten can be considerably improved. In fact, we will completely characterize the dynamics of $\{N\Phi_X^t\}$, showing that it exhibits a parabolic behavior.

In Lemma 4.5 we showed that there was some non-null vector in NX such that its whole $N\Phi_X$-orbit was bounded. On the other hand, in Lemma 4.10, under the assumption that there were two different minimal sets for $\{P\Phi_X^t\}$, we proved that there existed $\hat{Y} \in \Gamma(E_\Sigma)$ which was invariant under the action of $\{N\Phi_X^t\}$. Our first result of this paragraph consists in proving that we can get the same invariant section assuming in this case that K_Σ is the only minimal set:

Lemma 4.11. *There exists a non-vanishing section $\hat{Y} \in \Gamma(E_\Sigma)$ verifying*

$$N\Phi_X^t\big(\hat{Y}(p)\big) = \hat{Y}(\Phi_X^t(p)), \quad \forall p \in M,\ \forall t \in \mathbb{R}. \tag{4.15}$$

Proof. Let $\hat{Y}_0, \hat{Y} \in \Gamma(E_\Sigma)$ and $L_\Sigma \in C^\infty(M, \mathbb{R})$ be as in Lemma 4.10.

Recalling equation (4.8), we have

$$N\Phi_X^t(\hat{Y}(p)) = \exp\left(t \int_M L_\Sigma \Omega\right) \hat{Y}(\Phi_X^t(p)), \quad \forall t \in \mathbb{R}.$$

On the other hand, by Lemma 4.5, we know that there exists $\hat{v}_0 \in NX \setminus \{0\}$ which $N\Phi_X$-orbit is bounded, and applying Theorem 4.8 we get

$$\lim_{t\to\pm\infty} \mathrm{dist}_{\mathbb{P}}\left(\mathrm{pr}_{\mathbb{P}}\left(\hat{Y}\left(\Phi_X^t\big(\pi_N(\hat{v}_0)\big)\right)\right), \mathrm{pr}_{\mathbb{P}}\left(N\Phi_x^t(\hat{v}_0)\right)\right) = 0. \tag{4.16}$$

This clearly implies that $\|N\Phi_X^t(\hat{Y})\|_{NX}$ cannot exhibit exponential growth, and therefore, $\int_M L_\Sigma \Omega = 0$, getting the desired invariance of $\hat{Y}$. $\square$

Next, notice that $\pi_N\colon NX \to M$ is an orientable vector bundle with 2-dimensional fibers and $\hat{Y}$ is non-singular section of this bundle. This clearly implies that $\pi_N\colon NX \to M$ is globally trivial, in particular, we can find a smooth section $\hat{Z}_0 \in \Gamma(NX)$ verifying

$$\kappa\left(\hat{Y}(p), \hat{Z}_0(p)\right) = 1, \quad \forall p \in M, \tag{4.17}$$

and in particular, it holds $\operatorname{span}\{\hat{Y}, \hat{Z}_0\} = NX$.

Theorem 4.8 let us affirm that there exists $\sigma \in \{-1, 1\}$ satisfying

$$\begin{aligned} \lim_{t\to+\infty} \sphericalangle\left(N\Phi_X^t\big(\hat{Z}_0(p)\big), \sigma\hat{Y}\big(\Phi_X^t(p)\big)\right) &= 0, \\ \lim_{t\to-\infty} \sphericalangle\left(N\Phi_X^t\big(\hat{Z}_0(p)\big), -\sigma\hat{Y}\big(\Phi_X^t(p)\big)\right) &= 0. \end{aligned} \tag{4.18}$$

There is no lost of generality if we suppose that $\sigma = 1$ in (4.18).

Using $\{\hat{Y}, \hat{Z}_0\}$ as an ordered basis for NX, $N\Phi_X^t\colon NX_p \to NX_{\Phi_X^t(p)}$ can be represented as an element of $\mathrm{SL}(2, \mathbb{R})$, and indeed, it will have the following form:

$$N\Phi_X^t(p) = \begin{pmatrix} 1 & \hat{a}(p,t) \\ 0 & 1 \end{pmatrix}, \tag{4.19}$$

where $\hat{a}\colon M \times \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\hat{a}(\cdot, 0) = 0$.

Then, if we define $\hat{A} \in C^\infty(M, \mathbb{R})$ by $\hat{A}(p) \doteq \partial_t \hat{a}(p,t)\big|_{t=0}$, we can find a smooth real function $\hat{B}$ verifying

$$\mathcal{L}_X \hat{B} = -\hat{A} + \int_M \hat{A}\Omega. \tag{4.20}$$

Function $\hat{B}$ can be used for defining a new section

$$\hat{Z} \doteq \hat{Z}_0 + \hat{B}\hat{Y} \in \Gamma(NX),$$

and in this way we clearly have

$$N\Phi_X^t(\hat{Z}(p)) = \hat{Z}\big(\Phi_X^t(p)\big) + t\left(\int_M \hat{A}\Omega\right)\hat{Y}\big(\Phi_X^t(p)\big). \tag{4.21}$$

for any $t \in \mathbb{R}$ and $p \in M$.

From (4.18) and (4.21) we easily see that $\int_M \hat{A}\Omega > 0$, proving that in fact, $\{N\Phi_X^t\}$ exhibits a parabolic behavior as desired.

4.3.5 Dynamics on Σ

In this short paragraph we shall analyze the dynamics of the flow $D\Phi_X \colon TM \times \mathbb{R} \to TM$ restricted to the invariant sub-bundle $\Sigma \to M$.

Our main result consists in proving that $\{D\Phi_X^t\}$ on $\Sigma \subset TM$, as $\{N\Phi_X^t\}$ on NX, has a parabolic behavior. In fact, the techniques used in here are very similar to those used in paragraph 4.3.2. The only novelty is that *a priori* we do not have any information about the projective flow induced by $D\Phi_X \colon \Sigma \times \mathbb{R} \to \Sigma$.

In this case we know that, for each p and t, $D\Phi_X^t(X(p)) = X(\Phi_X^t(p))$ and therefore, we should prove that all the vectors non-collinear with X have polynomial growth and their directions converge to the direction of X.

Let us start considering any smooth vector field $Y_0 \in \Gamma(\Sigma) \subset \mathfrak{X}(M)$ verifying

$$\mathrm{pr}_X(Y_0(p)) = \hat{Y}(p), \quad \forall p \in M. \tag{4.22}$$

Then, notice that putting together equations (4.6) and (4.22) we can affirm that there exists a smooth function $A \in C^\infty(M, \mathbb{R})$ verifying

$$\mathcal{L}_X Y_0 = AX. \tag{4.23}$$

Once again, since X is cohomology-free, there exists $B \in C^\infty(M, \mathbb{R})$ satisfying

$$\mathcal{L}_X B = -A + \int_M A\Omega. \tag{4.24}$$

We use this function B for defining a new vector field

$$Y \doteq Y_0 + BX \in \Gamma(\Sigma) \subset \mathfrak{X}(M). \tag{4.25}$$

Notice that it continues to hold $\operatorname{span}\{X, Y\} = \Sigma \subset TM$ and, additionally, we get

$$\mathcal{L}_X Y \equiv \left(\int_M A\Omega \right) X. \tag{4.26}$$

Thus, we have the following

Lemma 4.12. *Function $A \in C^\infty(M, \mathbb{R})$ given by equation (4.23) satisfies*

$$\int_M A\Omega \neq 0.$$

Proof. Contrarily, let us suppose that $\int_M A\Omega = 0$.

Then, equation (4.26) is equivalent to say that $[X, Y] \equiv 0$, i.e. X and Y commute. Since X and Y generate Σ, in particular we have that they are everywhere linearly independent, and so, these vector fields induce a locally free $\mathbb{R}^2$-action on M.

Finally, a classical result due to Harold Rosenberg, Robert Roussarie and David Weil [RRW70] affirms that the only orientable closed 3-manifolds admitting locally free $\mathbb{R}^2$-actions are 2-torus bundles over a circle, and our manifold M clearly does not satisfies this property since we are assuming that $H_1(M, \mathbb{Q}) = 0$. □

As a corollary of this lemma we easily see that, given any $p \in M$, it holds

$\|D\Phi_X^t(Y(p))\| \to \infty$, uniformly as $t \to \pm\infty$, and

$$\begin{aligned} \lim_{t\to+\infty} \sphericalangle \left(D\Phi_X^t(Y(p)), \sigma_0 X(\Phi_X^t(p))\right) = 0, \\ \lim_{t\to-\infty} \sphericalangle \left(D\Phi_X^t(Y(p)), -\sigma_0 X(\Phi_X^t(p))\right) = 0, \end{aligned} \tag{4.27}$$

where $\sigma_0 \doteq \operatorname{sign}\left(\int_M A\Omega\right) \in \{1, -1\}$ and $\sphericalangle(\cdot,\cdot)$ stands for the angle (measured with respect to the Riemannian structure $\langle\cdot,\cdot\rangle$) between two non-null tangent vectors.

For the sake of simplicity, and since we do not loose any generality, we shall assume that $\int A\Omega > 0$, and thus, $\sigma_0 = 1$.

Summarizing what we have just proved, $D\Phi_X^t \colon \Sigma_p \to \Sigma_{\Phi_X^t(p)}$ is a parabolic linear map, and taking the ordered set $\{X, Y\}$ as basis of $\Sigma \subset TM$, we can represent it by

$$D\Phi_X^t\Big|_\Sigma = \begin{pmatrix} 1 & t\left(\int_M A\Omega\right) \\ 0 & 1 \end{pmatrix}. \tag{4.28}$$

4.3.6 Expansiveness

Let us start this paragraph recalling the definition of *expansive flow* due to Rufus Bowen and Peter Walters [BW72]:

Definition 4.13. Given a compact metric space (K, d), a continuous flow $\Psi \colon K\times\mathbb{R} \to K$ is called *expansive* if it satisfies the following property:

For every $\epsilon > 0$, there is a $\delta > 0$ such that if there exists a pair of points $x, y \in K$ and a homeomorphism $h \colon \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ verifying

$$d(\Psi^t(x), \Psi^{h(t)}(y)) < \delta, \quad \forall t \in \mathbb{R}, \tag{4.29}$$

then $y = \Psi^\tau(x)$, for some $\tau \in (-\epsilon, \epsilon)$.

Moreover, we shall say that Ψ is *positively expansive* (respec. *negatively expansive*) if above condition is satisfied replacing $\mathbb{R}$ by $(0, +\infty)$ (respec. $(-\infty, 0)$) in equation (4.29). More precisely, if it holds $y = \Psi^\tau(x)$, for some $\tau \in (-\epsilon, \epsilon)$, whenever

$$d(\Psi^t(x), \Psi^{h(t)}(y)) < \delta, \quad \forall t \in (0, +\infty) \ (\forall t \in (-\infty, 0)).$$

Our main goal now consists in proving that our flow $\{\Phi^t_X\}$ is positively (and in fact also negatively) expansive.

For this, let us start observing that in paragraph 4.3.4 we have constructed a smooth section $\hat{Z} \in \Gamma(NX)$ that verifies equation (4.21), where $\int_M \hat{A}\Omega \neq 0$ (in fact, we have supposed that this constant is positive). Then, if $Z \in \mathfrak{X}(M)$ is any smooth vector field verifying $\mathrm{pr}_X(Z) = \hat{Z}$, we will clearly have that for every $p \in M$,

$$\left\| D\Phi^t_X(Z(p)) \right\| \to \infty, \quad \text{when } t \to \pm\infty, \tag{4.30}$$

being the convergence uniform.

On the other hand, equations (4.18) and (4.27) let us affirm that (modulo our sign assumptions made there) for every p it holds

$$\sphericalangle \left(D\Phi^t_X\big(Z(p)\big), X\big(\Phi^t_X(p)\big)\right) \to 0, \quad \text{when } t \to +\infty, \tag{4.31}$$

being this convergence uniform, too.

Then, taking into account that $\{X, Y, Z\}$ is a global basis for TM, jointly with equations (4.12), (4.27), (4.30) and (4.31), we easily get

Proposition 4.14. *The flow $\{\Phi^t_X\}$ is positively expansive.*

And then we are very close to the end of our proof. In fact, as we will shortly see, there is no closed 3-manifold supporting positively expansive flows. The essential tool for getting this is the work due to Miguel Paternain [Pat93] about the existence of stable and unstable foliations for expansive flows on 3-manifolds.

Let us briefly recall Paternain's results. For this we need to introduce some additional notation. Let K be any closed manifold, $\operatorname{dist}\colon K \times K \to \mathbb{R}$ be any distance compatible with the topology of K and $\Psi\colon K \times \mathbb{R} \to \mathbb{K}$ be a continuous expansive flow.

As usual, given any $x \in K$, we can define its *stable* and *unstable sets* writing

$$W^s(x, \Psi) \doteq \left\{y \in K : d\big(\Psi^t(x), \Psi^t(y)\big) \to 0, \text{ as } t \to +\infty\right\},$$
$$W^u(x, \Psi) \doteq \left\{y \in K : d\big(\Psi^{-t}(x), \Psi^{-t}(y)\big) \to 0, \text{ as } t \to +\infty\right\},$$

respectively.

Thus, we can precisely state

Theorem 4.15 (Paternain [Pat93])**.** *If K is a closed 3-manifold and Ψ is an expansive flow on K, then there exists a finite set (maybe empty) of periodic orbits $\gamma_1, \gamma_2, \dots, \gamma_n$ of Ψ such that the partitions*

$$\mathscr{F}^\sigma = \left\{W^\sigma(x, \Psi) : x \in M \setminus \bigcup_{i=1}^{n} \gamma_i\right\}, \quad \textit{for } \sigma = s, u,$$

are C^0 codimension-two foliations on $M \setminus \bigcup \gamma_i$.

In our particular case the flow $\{\Phi_X^t\}$ has no periodic orbit, and hence, since we have proved that it is positively expansive, in particular, it is expansive and then, this theorem lets us affirm that, given any point $p \in M$, the set $W^s(p, \Phi_X)$ does not just reduce to $\{p\}$. This clearly contradicts the fact that $\{\Phi_X^t\}$ is positively expansive, and we finish our proof.

Chapter 5

Final Remarks and Problems

5.1 On Manifolds with $\beta_1(M) = 0$

5.1.1 3-manifolds and Weinstein Conjecture

As it was already explained in the introduction of this work, the main goal behind Katok Conjecture is to understand all possible (topological and analytical) obstructions than can appear when we look for smooth solutions of cohomological equations.

In Chapter 3 we analyzed the existence of cohomology-free vector fields on 3-manifolds with non-zero first Betti number. In all the stages of the proof of Theorem A it was rather clear how the topology of the manifold imposed different obstructions for the existence of cohomology-free vector fields, and all those obstructions let us completely characterize the supporting manifold.

Unfortunately, the situation is not that clear when we have to prove that there is no rational homological 3-sphere supporting cohomology-free vector fields. First, in Section 4.1, we proved that a hypothetical cohomology-free vector field on such a manifold had to preserve a non-singular 1-form and the analysis of the existence of obstructions was very satisfactory in the case that the kernel of the invariant 1-form was integrable (Section 4.3): solving some cohomological equations we completely characterized the dynamics of the derivative of the flow and we saw that there was no flow with that behavior on the tangent bundle.

Nevertheless, when we had to analyze the case where the kernel of the invariant 1-form determined a contact structure, we just proved that our hypothetical vector field was collinear with the Reeb vector field induced by the invariant 1-form, and then we finished our proof invoking Taubes' work on Weinstein Conjecture. From a purely formal point of view, this is a correct and complete proof, but if we take into account the real goal behind Katok Conjecture, we cannot affirm that this is a satisfactory one, because we are not understanding the nature of the obstructions that appear in this case. This is mainly due to the fact that Taubes' techniques used in [Tau] are extremely different to those used in the rest of this work.

Hence, it would be very desirable to complete the analysis that we started in Section 4.2 not invoking Taubes' proof of Weinstein Conjecture, getting a more "cohomological" proof.

5.1.2 Higher Dimensional Manifolds

As the reader could see in Chapter 3, Theorem 2.7 due to Federico and Jana Rodríguez-Hertz had a very important role in the proof of Theorem A.

Nevertheless, if the first Betti number of our supporting manifold is zero, then this result does not supply any non-trivial information.

Therefore, it seems reasonable to propose the following

Problem 5.1. *Let M be a closed d-manifold, with $d \geq 5$. Let us assume that there exists $X \in \mathfrak{X}(M)$ cohomology-free. Then, does there exist a good fibration for X $p\colon M \to \mathbb{T}^1$? In particular, must it hold $\beta_1(M) \geq 1$?*

Another problem that seems to be very helpful (but difficult) for understanding the dynamics of cohomology-free diffeomorphisms on higher dimensional manifolds, is the following one proposed by Richard Luz and Nathan dos Santos [LdS98]:

Problem 5.2. *If M is a closed manifold, $f \in \mathrm{Diff}(M)$ is cohomology-free and $n \in \mathbb{Z} \setminus \{0\}$, is it true that f^n is cohomology-free?*

Motivated by this problem, we propose the following one for vector fields:

Problem 5.3. *If M is a closed manifold, $p \colon \tilde{M} \to M$ a k-fold covering with $k \geq 2$ and $X \in \mathfrak{X}(M)$ cohomology-free, is the p-lift vector field $\tilde{X} \doteq p^*(X) \in \mathfrak{X}(M)$ cohomology-free?*

5.2 Globally Hypoelliptic Vector Fields

In the theory of Partial Differential Equations there is a family of smooth vector fields that has been extensively studied and that, a priori, strictly contains the family of cohomology-free vector fields. These are the *globally hypoelliptic vector fields*:

Definition 5.4. Let M be a closed orientable manifold and $X \in \mathfrak{X}(M)$. We say that X is *globally hypoelliptic* if given any $T \in \mathcal{D}'(M)$, it holds

$$\mathcal{L}_X T \in C^\infty(M, \mathbb{R}) \subset \mathcal{D}'(M) \Rightarrow T \in C^\infty(M, \mathbb{R}).$$

It is very easy to see that every cohomology-free vector field is indeed globally hypoelliptic, but a priori these two concepts are not equivalent.

The first result concerning the classification of globally hypoelliptic vector fields is due to Stephen Greenfield and Nolan Wallach who proved in [GW73] that, modulo C^∞ conjugacy, the constant vector fields on $\mathbb{T}^2$ verifying a Diophantine condition like (1.7) are the only examples on closed surfaces. This led them to propose the following

Conjecture 5.5 (Greenfield-Wallach Conjecture [GW73])**.** *Tori are the only closed manifolds that support globally hypoelliptic vector fields.*

It is interesting to remark that this implies Katok Conjecture. In fact, Chen Wenyi and M. Y. Chi have proved in [CC00] that the only globally hypoelliptic vector fields on tori are those smoothly conjugated to constant vector fields satisfying a Diophantine condition like (1.7).

However, one of the main results in [CC00] is Theorem 2.2 which asserts that any globally hypoelliptic vector field on $\mathbb{T}^d$ is cohomology-free. As Federico Rodríguez-Hertz has recently observed, the proof of this result, presented by Chen and Chi in [CC00], continues to hold on any closed manifold, and consequently, both families of vector fields coincide. Therefore, we have that Greenfield-Wallach Conjecture and Katok Conjecture are indeed equivalent.

5.3 Positively Expansive Flows

Given a compact metric space (K, d) and a homeomorphism $h\colon K \to K$, we say that h is *expansive* if there exists $\varepsilon > 0$ such that, for any pair of distinct points $x, y \in K$, it holds

$$\sup_{n\in\mathbb{Z}} d\left(f^n(x), f^n(y)\right) > \varepsilon,$$

and we say that h is *positively expansive* if it holds

$$\sup_{n\in\mathbb{N}_0} d\left(f^n(x), f^n(y)\right) > \varepsilon,$$

It is a very well known fact that $h\colon K \to K$ is positively expansive if and only if K is a finite set.

On the other hand, in Section 4.3.6, invoking a result due to Miguel Paternain [Pat93], we easily proved that there does not exist any positively expansive flow on closed 3-manifolds. However, we do not have any knowledge about the existence of positively expansive flows on higher dimensional manifolds. In fact, taking into account the simple classification of positively expansive homeomorphisms, it seems natural to ask:

Problem 5.6. *If $\Psi\colon K \times \mathbb{R} \to K$ is a fixed-point free positively expansive flow, is it true that K is homeomorphic to a finite disjoint union of copies of $\mathbb{T}^1$?*

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