On the Central Limit Theorem for a tagged particle in the simple exclusion process

Aluno: Milton David Jara Valenzuela
Orientador: Cláudio Landim
IMPA

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À minha mãe, por me ensinar o bom caminho, e à minha mulher, por me manter nele...

Agradecimientos

Gostaria de aproveitar esta oportunidade para agradecer às pessoas que, de uma ou outra forma ajudaram a dar esta forma final à minha tese.

Para começar, gostaria de agradecer ao meu orientador, Prof. Cláudio Landim, pelo apoio que me deu durante todos estes anos, pela disposição e paciência que teve para ouvir minhas perguntas e discutir minhas idêias, e por ter escolhido problemas desafiantes para trabalhar durante a minha tese.

Gostaria de agradecer também aos professores Marcelo Fragoso, Antônio Galves, Welington de Melo, Sergei Popov e Vladas Sidoravicius por terem aceito participar da banca, e pelas sugestões feitas para melhorar a redação desta tese.

Gostaria de agradecer aos meus colegas do IMPA, com os quais compartilhei momentos de alegria durante estes anos, e especialmente a Johel Beltrán, Jesus Zapata, Rudy Rozas e Fidel Jiménez, por compartilhar comigo o caminho da formação acadêmica no IMPA.

Não menos importante, é a contribuição do IMPA em geral, e da cidade de Rio de Janeiro, pela beleza das suas paissagens que inspiram a qualquer um.

Finalmente quero agradecer à minha mulher, à minha família e a todos os que passaram por minha vida nestes anos, colaborando para que pudesse chegar neste momento.

Resumo

O estudo do comportamento assintótico de um apartícula marcada começa com o trabalho de Einstein de 1905 sobre o movimento Browniano. No trabalho de Einstein, a partícula marcada é diferente das outras, e é assumido que ela não influi na evolução das outras partículas. Esta hipótese corresponde a supor que o sistema está em equilíbrio. No caso em que a partícula marcada é igual às outras partículas, também é possivel provar que o comportamento assintótico da partícula marcada é dado por um movimento Browniano. A variância do movimento Browniano limite depende da densidade de partículas, e da natureza da interação. No processo de exclusão simples na rede infinita, a interação das partículas é dada pelo princípio de exclusão: a evolução das partículas é dada por um passeio aleatório simples, condicionado a não ter mais de uma partícula por sítio.

No caso especial em dimensão um, no qual os saltos são simétricos e a vizinhos próximos, a posição relativa das partículas não é alterada pela dinâmica. Em conseqüência, a variância do movimento Browniano limite para uma partícula marcada neste caso se anula, independentemente da densidade de partículas. Para obter um comportamento assintótico não trivial, uma escala sub-difusiva é introduzida. O limite de escala neste caso é dado por um moviemnto Browniano fracionário.

Este trabalho é composto de duas partes. Na primeira parte, é considerado o modelo de exclusão simples no caso em que o limite de escala para a partícula marcada é dado por um movimento Browniano não degenerado. Provaremos que o coeficiente de difusão do processo em volume infinito pode ser aproximado pelo coeficiente de difusão do processo de exclusão simples em volume finito, quando o volume é arbitráriamente grande.

Na segunda parte deste trabalho, consideraremos o caso especial em que o limite de escal sub-difusivo. Obteremos o limite de escala da partícula marcada para o processo fora de equilíbrio. Neste caso, a posição assintótica da partícula marcada é dada pela equação diferencial de transporte associada á equação do calor, e as flutuações da posição da partícula marcada são dadas por um movimento Browniano fracionário não-homogêneo de parâmetros calculados em termos das flutuações da densidade empírica de partículas com respeito ao limite hidrodinâmico do modelo.

Chapter 1

Finite approximations of the diffusion coefficient

In [KV], Kipnis and Varadhan proved an invariance principle for the position of a marked particle in a symmetric simple exclusion process in equilibrium. Their proof relies on a central limit theorem for additive functionals of a Markov process. Later, this result was generalized to mean zero simple exclusion process (see [V]), and asymmetric simple exclusion process in dimension $d \geq 3$ in [SVY].

The diffusion matrix of the limiting Brownian process is a function $D(\alpha)$ of the density of particles, and is given by a variational formula.

The method of proof used by Kipnis and Varadhan works directly in infinite systems, and it raises naturally the question about the stability of the diffusion coefficient under finite-dimensional approximations. More precisely, consider a finite-dimensional version of the simple exclusion process on the torus $\{-N, \dots, 0, \dots, N\}^d$. In order to obtain an ergodic process, fix the total number K of particles. When N is large enough, the motion of a tagged particle on this finite system has a unique canonical lifting to $\mathbb{Z}^d$. We obtain in this manner a process $X_N(t)$ with values in $\mathbb{Z}^d$. Let $D_{N,K}$ the variance of the limiting Brownian motion of the scaled process $\varepsilon X_N(t/\varepsilon^2)$ when $\varepsilon \to 0$. We prove that

$$\lim_{\substack{N\to\infty \\ K/(2N)^d \to \alpha}} D_{N,K} = D(\alpha)$$

for mean zero simple exclusion process, and for asymmetric simple exclusion process in dimension $d \geq 3$.

This limit was first considered in [LOV2] for symmetric simple exclusion process, and the proof presented there follows from a variational formula for the diffusion coefficient that depends on the Sobolev dual norm associated to the generator of the process, and from a convergence result for the Sobolev dual norms of the finite-dimensional approximations. Let h, g be local functions with mean zero with respect to all the Bernoulli product measures μ_α, that assign density α to each coordinate. Denote by $\langle\ ,\ \rangle_\alpha$ the inner product in $\mathcal{L}^2(\mu_\alpha)$. Let $\mu_{N,K}$ be the uniform measure over the configurations with K particles on the torus $\{-N, \dots, 0, \dots, N\}^d$, and $\langle\ ,\ \rangle_{\mu_{N,K}}$ the inner product in $\mathcal{L}^2(\mu_{N,K})$. Let L (resp. L_N) be the generator of the process in $\mathbb{Z}^d$ (resp. the torus). Suppose for a moment that $(-L)^{-1}g$ exists and is local. Then,

$$\lim_{\substack{N\to\infty \\ K/(2N)^d \to \alpha}} \langle h, (-L_N)^{-1} g\rangle_{N,K} = \langle h, (-L)^{-1} g\rangle_\alpha,$$

because $(-L)^{-1}g$ is local and the equivalence of ensembles. The desired result will be consequence of a generalization of this result for a larger class of functions h, g.

1.1 Notation and Results

Consider a probability measure $p(\cdot)$ of finite range on $\mathbb{Z}^d$: $p(z) = 0$ if $|z|$ is large enough. Suppose that $p(0) = 0$ and that the random walk with transition rate $p(\cdot)$ is irreducible, that is, the (finite) set $\{z; p(z) > 0\}$ generates the group $\mathbb{Z}^d$. The simple exclusion process associated to $p(\cdot)$ corresponds to the Markov process defined on $\mathcal{X} = \{0,1\}^{\mathbb{Z}^d}$, whose generator $\mathcal{L}_0$ acting on local functions f is given by

$$\mathcal{L}_0 f(\eta) = \sum_{x,y\in\mathbb{Z}^d} p(y-x)\eta(x)(1-\eta(y))[f(\sigma^{xy}\eta) - f(\eta)],$$

Here $\eta \in \mathcal{X}$ denotes a configuration of particles in $\mathbb{Z}^d$. In particular, $\eta(x) = 1$ if there is a particle at the site x, and $\eta(x) = 0$ otherwise, and $\sigma^{xy}\eta$ is the configuration obtained from η exchanging the occupation numbers at x and y:

$$\sigma^{xy}\eta(z) = \begin{cases} \eta(y), & \text{if } z = x \\ \eta(x), & \text{if } z = y \\ \eta(z), & \text{otherwise.} \end{cases}$$

If $p(z) = p(-z)$ for all z, the process will be called symmetric; if $\sum zp(z) = 0$, it will be called of mean zero, and if $\sum_{z\in\mathbb{Z}^d} zp(z) = m \neq 0$, the process will be called asymmetric.

For each $\alpha \in [0,1]$, let ν_α be the Bernoulli product measure in $\mathcal{X}$, that is, the product measure such that $\nu_\alpha[\eta(x) = 1] = \alpha$ for each $x \in \mathbb{Z}^d$. It is not hard to prove that ν_α is an invariant measure for the process generated by $\mathcal{L}_0$.

In this model, particles are indistinguishable. In order to study the time evolution of a single particle, we proceed in the next way: let $\eta \in \mathcal{X}$ be an initial state with a particle at the origin (that is, $\eta(0) = 1$). Tag this particle, and let η_t, resp. X_t, be the time evolution of the exclusion process starting from η and the tagged particle starting from $x = 0$. Let $\xi_t(x) = \eta_t(x + X_t)$ be the process as seen by the tagged particle. We call ξ_t the environment process.

It is clear that X_t is not a Markov process, due to the interaction between the tagged particle and the environment, but (η_t, X_t) and ξ_t are Markov processes, the last one defined in the state space $\mathcal{X}_* = \{0,1\}^{\mathbb{Z}^d_*}$, where $\mathbb{Z}^d_* = \mathbb{Z}^d \backslash \{0\}$. The generator of the process ξ_t, acting on local functions f, is given by $L = L_0 + L_\tau$, where

$$L_0 f(\xi) = \sum_{x,y\in\mathbb{Z}^d_*} p(y-x)\xi(x)(1-\xi(y))[f(\sigma^{xy}\xi) - f(\xi)],$$
$$L_\tau f(\xi) = \sum_{z\in\mathbb{Z}^d_*} p(z)(1-\xi(z))[f(\tau_z\xi) - f(\xi)].$$

The first part of the generator, L_0, takes into account the jumps of the environment (that is, all the particles but the tagged one), while the second part takes into account the jumps of the tagged particle.

In this formula, $\tau_z\xi$ is the configuration obtained making the tagged particle (at the origin) jump to site z, and then bringing it back to the origin with a translation:

$$\tau_z\xi(x) = \begin{cases} 0, & \text{if } x = -z \\ \xi(x+z), & \text{if } x \neq -z. \end{cases}$$

For the process ξ_t, we have a one-parameter family of invariant ergodic measures $\{\mu_\alpha\}_{\alpha\in[0,1]}$, where μ_α is the Bernoulli product measure defined in $\mathcal{X}_*$ of density α: $\mu_\alpha[\xi(x) = 1] = \alpha$ for all $x \in \mathbb{Z}^d_*$, independently for each site (see [S]).

Note that the position of the tagged particle can be calculated in terms of jump processes associated to ξ_t. Define N_t^z as the number of translations by z of ξ_t, that is, $N_t^z = N_{t-}^z + 1 \iff \xi_t = \tau_z \xi_{t-}$. Then, $X_t = \sum_z z N_t^z$.

In this context, Kipnis and Varadhan proved a central limit theorem for the position of the tagged particle when the environment process is in equilibrium, with distribution μ_α. They proved that $\varepsilon X_{t/\varepsilon^2}$ converges, when ε goes to zero, to a Brownian motion with diffusion coefficient $D(\alpha)$, which can be described in terms of the Sobolev norms associated to the operator L in $\mathcal{L}(\mu_\alpha)$.

This result has been generalized by Varadhan to the mean zero case (in any dimension), and by Sethuraman, Varadhan and Yau for the asymmetric case in dimension $d \geq 3$, in which case it is proved that $\varepsilon[X_{t/\varepsilon^2} - mt(1-\alpha)/\varepsilon^2]$ converges to a Brownian motion with diffusion coefficient $D(\alpha)$, given by

$$a^t D(\alpha) a = (1-\alpha) \sum_{z \in \mathbb{Z}_*^d} (z \cdot a)^2 p(z) - 2\langle w_a, (-L)^{-1} v_a \rangle_\alpha, \tag{1.1}$$

where $a \in \mathbb{R}^d$, $\langle \ , \ \rangle_\alpha$ is the inner product in $\mathcal{L}^2(\mu_\alpha)$, and the functions v_a, w_a are local functions defined by

$$v_a = \sum_{z \in \mathbb{Z}^d} (z \cdot a) p(z)[\alpha - \eta(z)]$$
$$w_a = \sum_{z \in \mathbb{Z}^d} (z \cdot a) p(z)[\alpha - \eta(-z)].$$

In general, L is not an invertible operator, and the meaning of this expression must be clarified. This will be done in sections 1.2 and 1.3.

Let N be a positive integer and define $T_N^d = \{-N, ..., 0, ..., N\}^d$, the d-dimensional discrete torus of $(2N)^d$ points, with $-N$ and N identified. Using the same probability measure $p(\cdot)$, we can define a simple exclusion process evolving in T_N^d. The space state now will be $\mathcal{X}_N = \{0,1\}^{T_N^d}$, and the generator $\mathcal{L}_N$ acting on any function f will be given by

$$\mathcal{L}_N f(\xi) = \sum_{x,y \in T_N^d} p(y-x)\eta(x)(1-\eta(y))[f(\sigma^{xy}\eta) - f(\eta)].$$

In the same way, it is possible to define the environment process in the torus $T_{N,*}^d = T_N^d \backslash \{0\}$. In this case, the environment as seen by the tagged particle is a Markov process evolving in the space $\mathcal{X}_{N,*} = \{0,1\}^{T_{N,*}^d}$ and generated by the operator $L_N = L_{0,N} + L_{\tau,N}$, where

$$L_{0,N} f(\xi) = \sum_{x,y \in T_{N,*}^d} p(y-x)\xi(x)(1-\xi(y))[f(\sigma^{xy}\xi) - f(\xi)],$$
$$L_{\tau,N} f(\xi) = \sum_{z \in T_{N,*}^d} p(z)(1-\xi(z))[f(\tau_z \xi) - f(\xi)].$$

It is clear, by conservation of the number of particles, that for $0 < K \leq (2N)^d$, the probability measure $\mu_{N,K}$, uniform over the set $\mathcal{X}_{N,K} = \{\xi \in \mathcal{X}_{N,*}; \sum_{x \in T_{N,*}^d} \xi(x) = K-1\}$ of configurations with K particles, is an invariant ergodic measure for the jprocess generated by L_N.

For N large enough it is possible to lift the motion of the tagged particle to $\mathbb{Z}^d$. Let X_t^N the position of the tagged particle in $\mathbb{Z}^d$. It is not hard to prove an invariance principle for X_t^N: $\varepsilon[x^N_{t/\varepsilon^2} - mt(1-\alpha_{N,K})/\varepsilon^2]$ converges to a Brownian motion of variance $D_{N,K}$ given by

$$\begin{aligned} a^t D_{N,K} a &= (1-\alpha_{N,K}) \sum_{z \in \mathbb{Z}^d} (a \cdot z)^2 p(z) \\ &\quad - 2\big\langle w_a - \langle w_a \rangle_{N,K}, L_N^{-1}(v_a - \langle v_a \rangle_{N,K})\big\rangle_{N,K}. \end{aligned} \tag{1.2}$$

In this formula, $\langle\ ,\ \rangle_{N,K}$ (resp. $\langle\ \ \rangle_{N,K}$) stands for the inner product in $\mathcal{L}(\mu_{N,K})$ (resp. the mean with respect to $\mu_{N,K}$), and

$$\alpha_{N,K} = \frac{K-1}{(2N)^d - 1}.$$

Note that, for $f : \mathcal{X}_{N,K} \to \mathbb{R}$ with $\langle f \rangle_{N,K} = 0$, $L_N^{-1} f$ is well defined. In fact, for f we have that

$$\begin{aligned} \mathcal{D}_{N,K}(f) &= \langle f, -L_N f \rangle_{N,K} \\ &= \frac{1}{4} \sum_{x,y \in T^d_{N,*}} (p(y-x) + p(x-y)) \int [f(\sigma_{x,y}\eta) - f(\eta)]^2 d\mu_{N,K}. \end{aligned}$$

In particular, $L_N f = 0$ if and only if f is constant, and L_N is an invertible operator in $\mathcal{C}_{0,N,K} = \{f; \langle f \rangle_{N,K} = 0\}$.

For the symmetric simple exclusion process, Landim, Olla and Varadhan [LOV2] proved that $D_{N,K} \to D(\alpha)$ if $\alpha_{N,K} \to \alpha$. We extend this result to the asymmetric case:

Theorem 1.1.1. *$D_{N,K} \to D(\alpha)$ if $\alpha_{N,K} \to \alpha$, for mean zero simple exclusion process (in any dimension), and for asymmetric simple exclusion process in dimension $d \geq 3$.*

1.2 The Sobolev Spaces H_1, H_{-1}

In this section we prove the stability of the H_{-1} norm under finite approximations. We discuss it in the more general context of functional analysis, since our results can in principle be applied to a broad range of models of interacting particle systems, and we will used repeatedly in the sequel.

Let H be a real Hilbert space with inner product $\langle\ ,\ \rangle$. An operator (not necessarily bounded) $L : D(L) \subseteq H \to H$ is called positive if $\langle g, Lg \rangle > 0$ for all $g \in D(L) \backslash \{0\}$.

Given a positive closed operator L, we define, for $f \in D(L)$,

$$||f||_1^2 =: \langle f, Lf \rangle.$$

It is easy to see that $||\cdot||_1$ defines a norm in $D(L)$ that satisfies the parallelogram rule. Therefore, $||\cdot||_1$ can be extended to an inner product in $D(L)$. Define $\mathcal{H}_1 = \mathcal{H}_1(L)$, the Sobolev space associated to the operator L, as the completion of $D(L)$ under $||\cdot||_1$.

In the same way, we see that

$$||g||_{-1}^2 =: \sup_{f \in D(L)} \{2\langle g, f \rangle - \langle f, Lf \rangle\}$$

defines a norm in the set $\{g \in H; ||g||_{-1} < \infty\}$, that can be extended to a inner product. Define $\mathcal{H}_{-1}$ as the completion of this set under $||\cdot||_{-1}$.

In the next proposition, some well known properties of the spaces $\mathcal{H}_1, \mathcal{H}_{-1}$ are listed:

Proposition 1.2.1. *For $f \in H \cap \mathcal{H}_1$, $g \in H \cap \mathcal{H}_{-1}$, we have*

$$\begin{aligned} i) \quad & ||g||_{-1} = \sup_{h \in D(L)\setminus\{0\}} \frac{\langle h, g\rangle}{||h||_1} \\ ii) \quad & |\langle f, g\rangle| \leq ||f||_1 ||g||_{-1} \\ iii) \quad & ||f||_1 \leq ||Lf||_{-1} \end{aligned}$$

Proof. For $i)$:

$$\begin{aligned} \sup_{h \in D(L)} \{2\langle g, h\rangle - \langle h, Lh\rangle\} &= \sup_{||h||_1=1} \sup_{\alpha \in \mathbb{R}} \{2\alpha\langle g, h\rangle - \alpha^2\} \\ &= \sup_{||h||_1=1} \langle g, h\rangle^2. \end{aligned}$$

For $ii)$:

$$||g||_{-1} = \sup_{h \in D(L)\setminus\{0\}} \frac{|\langle g, h\rangle|}{||h||_1} \geq \frac{|\langle g, f\rangle|}{||f||_1}.$$

For $iii)$:

$$||Lf||_{-1}^2 = \sup_{h \in D(L)} \{2\langle Lf, h\rangle - \langle h, Lh\rangle\} \geq \langle f, Lf\rangle.$$

□

From property $i)$ can be concluded that $\mathcal{H}_{-1}$ is the dual of $\mathcal{H}_1$ with respect to H. Thanks to property $ii)$, the inner product $\langle\ ,\ \rangle$ can be extended to continuous a bilinear form $\langle\ ,\ \rangle : \mathcal{H}_{-1} \times \mathcal{H}_1 \to \mathbb{R}$. Property $iii)$ assures that the operator $L^{-1}:\ \mathrm{Im}(L) \cap \mathcal{H}_{-1} \to \mathcal{H}_1$ is bounded, from which it can be continuously extended to an operator defined in the closure of $\mathrm{Im}(L) \cap \mathcal{H}_{-1}$ under $||\cdot||_{-1}$.

If the operator L is symmetric, that is, if $\langle f, Lg\rangle = \langle Lf, g\rangle$ for $f, g \in D(L)$, then the inequality in $iii)$ becomes equality, and L can be extended to an isometry from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$ (not necessarily surjective).

Let $\{H_n\}_n$ be an increasing sequence of finite-dimensional subspaces of H and define $\mathcal{L}oc = \mathcal{L}oc(H) =: \cup_n H_n$. Suppose that $\mathcal{L}oc$ is a kernel for L, that is, the closure of the operator L restricted to $\mathcal{L}oc$ is the operator L itself. Suppose also that $\mathcal{L}oc$ is a kernel for the adjoint L^* of L. Consider on each subspace H_n an inner product $\langle\ ,\ \rangle_n$ such that for all $f, g \in \mathcal{L}oc$,

$$\lim_{n\to\infty} \langle f, g\rangle_n = \langle f, g\rangle,$$

where $\langle f, g\rangle_n$ is well defined for n large enough.

A sequence $\{L_n\}_n$ of operators is called a finite approximation of L if:

$i)$ $L_n : H_n \to H_n$

$ii)$ $\langle f, L_n f\rangle_n > 0$ for $f \in H_n\setminus\{0\}$

$iii)$ For all $f \in \mathcal{L}oc$ there exist $n_0 \in \mathbb{N}$ such that $L_n f = Lf$ for $n \geq n_0$.

$iv)$ If L is a symmetric operator, then L_n is also a symmetric operator.

In H_n, define the $||\cdot||_{1,n}$, $||\cdot||_{-1,n}$ norms associated to L_n, as before:

$$||f||^2_{1,n} = \langle f, L_n f\rangle_n$$
$$||f||^2_{-1,n} = \sup_{g\in H_n} \{2\langle f, g\rangle_n - \langle g, L_n g\rangle_n\}.$$

Observe that $\mathrm{Ker}(L_n) = \{0\}$, from which L_n is invertible. The purpose of this section is to establish sufficient conditions to ensure that

$$\lim_{n\to\infty} \langle h', L_n^{-1} h\rangle_n = \langle h', L^{-1} h\rangle \qquad (\star)$$

for $h, h' \in \mathcal{L}oc \cap \mathcal{H}_{-1}$ with h in the closure of $\mathrm{Im}(L) \cap \mathcal{H}_{-1}$

While $L_n^{-1}h$ is always well defined, h might not be in the image of L, and the left side of this equality would not be well defined. However, when h is in the closure of $\mathrm{Im}(L) \cap \mathcal{H}_{-1}$ under $||\cdot||_{-1}$, the product $\langle h', L^{-1}h\rangle$ can be defined by continuity. Remind that the product $\langle h', L_n^{-1}h\rangle_n$ is well defined for n large enough, because $h, h' \in \mathcal{L}oc$, and each time a limit like the one appearing in $(\star)$ is considered, this comment must be taken into account.

The next theorem is a perturbative result that asserts that if $(\star)$ is satisfied for an operator S_0 (and a suitable finite approximation $\{S_{0,n}\}_n$ of S_0), then it is also satisfied for a class of perturbations of S_0:

Theorem 1.2.2. *Let L be a positive closed operator. Let $S_0 : D(S_0) \subseteq H \to H$ be a symmetric positive operator such that $\mathcal{L}oc$ is a kernel for S_0 and $\langle g, S_0 g\rangle \leq \langle g, Lg\rangle$. Let $\{S_{0,n}\}_n$ be a finite approximation of S_0 such that $\langle f, S_{0,n} f\rangle_n \leq \langle f, L_n f\rangle_n$ for all $f \in H_n$. Define the norms $||\cdot||_{0,1}$, $||\cdot||_{0,-1}$ ($||\cdot||_{0,1,n}$, $||\cdot||_{0,-1,n}$ resp.) associated to S_0 ($S_{0,n}$ resp.) as before. Consider $h, h' \in \mathcal{L}oc \cap \mathcal{H}_{-1}$, with h in the closure of $Im(L) \cap \mathcal{H}_{-1}$.*

Assume that

A) For each $\varepsilon > 0$ exists $g_\varepsilon \in \mathcal{L}oc$ such that

$$||h - Lg_\varepsilon||_{0,-1} < \varepsilon.$$

B)

$$\lim_{n\to\infty} ||h'||_{0,-1,n} = ||h'||_{0,-1},$$

and for $u_\varepsilon = h - Lg_\varepsilon$,

$$\lim_{n\to\infty} ||u_\varepsilon||_{0,-1,n} = ||u_\varepsilon||_{0,-1}.$$

Then,

$$\lim_{n\to\infty} \langle h', L_n^{-1} h\rangle_n = \langle h', L^{-1} h\rangle.$$

Proof. First, we observe that the operator S_0 ($S_{0,n}$ resp.) is dominated by L_0 ($L_{0,n}$ resp.), from which we have, for all f, the inequalities

$$||f||_{-1} \leq ||f||_{0,-1}$$
$$||f||_{0,1} \leq ||f||_1$$
$$||f||_{-1,n} \leq ||f||_{0,-1,n}$$
$$||f||_{0,1,n} \leq ||f||_{1,n}.$$

Fix $\varepsilon > 0$, and let $u_\varepsilon = h - Lg_\varepsilon$ be chosen according to Assumption A. Then,

$Lg_\varepsilon = L_n g_\varepsilon$ for n large enough, from which Lg_ε belongs to H_n and

$$\begin{aligned}\langle h', L_n^{-1} h\rangle_n &= \langle h', L_n^{-1}(u_\varepsilon + Lg_\varepsilon)\rangle_n \\ &= \langle h', g_\varepsilon\rangle_n + \langle h', L_n^{-1} u_\varepsilon\rangle_n.\end{aligned}$$

Since h' and g_ε are in $\mathcal{L}oc$,

$$\langle h', g_\varepsilon\rangle_n \xrightarrow[n\to\infty]{} \langle h', g_\varepsilon\rangle.$$

We also have that

$$\begin{aligned}|\langle h', L_n^{-1} u_\varepsilon\rangle_n| &\le ||h'||_{0,-1,n} \cdot ||L_n^{-1} u_\varepsilon||_{0,1,n} \\ &\le ||h'||_{0,-1,n} \cdot ||L_n^{-1} u_\varepsilon||_{1,n} \\ &\le ||h'||_{0,-1,n} \cdot ||u_\varepsilon||_{-1,n} \\ &\le ||h'||_{0,-1,n} \cdot ||u_\varepsilon||_{0,-1,n}.\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} |\langle h', L_n^{-1} u_\varepsilon\rangle_n| \le ||h'||_{0,-1} \cdot ||u_\varepsilon||_{0,-1} \le \varepsilon \cdot ||h'||_{0,-1}.$$

In the other hand, $\langle h', L^{-1} h\rangle = \langle h', L^{-1} u_\varepsilon\rangle + \langle h', g_\varepsilon\rangle$ and

$$|\langle h', L^{-1} u_\varepsilon\rangle| \le ||h'||_{0,-1} \cdot ||L^{-1} u_\varepsilon||_{0,1} \le \varepsilon \cdot ||h'||_{0,-1}.$$

In consequence,

$$\limsup_{n\to\infty} |\langle h', L^{-1} h\rangle - \langle h', L_n^{-1} h\rangle_n| \le 2\varepsilon ||h'||_{0,-1}.$$

□

1.3 Proof of Theorem 1.1.1

This section is organized as follows. First, we show in which sense the sequence $\{L_N\}_N$ is a finite approximation of the operator L. Once this has been done, the proof of Theorem 1.1.1 is reduced to the verification of the hypothesis of Theorem 1.2.2, as we will see. Then, we verify these hypothesis separately for symmetric, mean zero and asymmetric simple exclusion process.

1.3.1 Finite Approximations for the generator L

Let $\alpha \in [0,1]$ be fixed. Let $\{K_N\}_N$ a sequence such that, as N goes to infinity, $\alpha_{N,K_N} \to \alpha$, $K_N \to \infty$ and $(2N)^d - K_N \to \infty$. From now on, we drop out the index K_N if there is no risk of confusion. Let f, g be in $\mathcal{L}^2(\mu_\alpha)$. First, we take care of irrelevant constants. We say that $f \sim g$ if $\int (f-g) d\mu_\alpha = 0$. Define $H = \mathcal{L}^2(\mu_\alpha)/\sim$. It is easy to see that H is isomorphic to the set of functions with mean zero in $\mathcal{L}^2(\mu_\alpha)$. Let $\mathcal{L}oc = \mathcal{L}oc(H)$ be the set of local functions in H. We define $H_N \cong \mathcal{C}_{0,N,K_N}$ as follows: consider the canonical projection $\pi_N : \mathcal{X}_* \to \mathcal{X}_{N,*}$. For $f \in \mathcal{C}_{0,N,K_N}$, define $\pi_N^{-1} f \in \mathcal{L}oc$ by

$$\pi_N^{-1} f(\eta) = \begin{cases} f(\pi_N \eta), & \text{if } \pi_N \eta \in \mathcal{X}_{N,K_N} \\ 0, & \text{if } \pi_N \eta \notin \mathcal{X}_{N,K_N} \end{cases}$$

Then, $H_N = \pi_N^{-1}(\mathcal{C}_{0,N,K_N})/\sim$. It is not hard to see that $\mathcal{L}oc = \cup_N H_N$. In fact, for a local function f, denote by $\text{supp}(f)$ the support of f. Then, if $\text{supp}(f) \subseteq T^d_{N,*}$, $\#\,\text{supp}(f) < \min\{K_N, (2N)^d - K_N\}$, then $f \in H_N$, and clearly $H_N \subseteq \mathcal{L}oc$. In H_N we define the inner product $\langle\ ,\ \rangle_N$ induced by the measure μ_{N,K_N}.

It is clear that for $f, g \in \mathcal{L}oc$ and N large enough (note that f, g are not in $\mathcal{C}_{0,N,K_N}$ necessarily),

$$\begin{aligned}\langle f, g\rangle_N &= \int \Big(f - \int f d\mu_{N,K_N}\Big)\Big(g - \int g d\mu_{N,K_N}\Big) d\mu_{N,K_N}\\ &= \int fg d\mu_{N,K_N} - \int f d\mu_{N,K_N} \int g d\mu_{N,K_N},\end{aligned}$$

We have already seen that the operator $-L_N$ is positive, and it is clear that $-L_N f = Lf$ for $f \in \mathcal{L}oc$ and N large enough. From the ergodicity of μ_α with respect the process generated by L and the fact that L is a generator of a Markov process, we deduce that $\mathcal{D}_\alpha(f) = \langle f, -Lf\rangle_\alpha > 0$ if $f \neq 0$, from which we see that $-L$ is a positive operator. In consequence, $\{L_N\}_N$ is a finite approximation of $-L$, if it were not for the fact that $H_N \subsetneq H_{N+1}$, because K_N, $(2N)^d - K_N$ are not necessarily increasing sequences. However, what is true is that $H_N \subseteq H_M$ for M large enough, depending both in the range of the transition probability $p(\cdot)$ and in the sequence K_N (here we use that $K_N \to \infty$ and $(2N)^d - K_N \to \infty$). Of course, Theorem 1.1.1 applies in this situation, by taking subsequences or slightly modifying it to fit this case. Anyway, we will say that $\{L_N\}_N$ is a finite approximation of L.

Observe that the inner product $\langle\ ,\ \rangle_N$ is exactly the product appearing in the equation 1.2. Comparing equations 1.1 and 1.2, it is clear that Theorem 1.1.1 follows from Theorem 1.2.2 applied to the operators $-L$ and $-L_N$. So, it only rest to find suitable operators S_0, $\{S_{0,N}\}$ to compare with L and $\{L_N\}$ and to check the hypothesis of Theorem 1.2.2 for them.

1.3.2 Symmetric case

Suppose that the transition probability $p(\cdot)$ is symmetric, that is, $p(x) = p(-x)$ for all $x \in \mathbb{Z}^d$. This case has been considered in [LOV2], but in order to make the exposition clear, we outline here the proof in our setting.

Choose $S_0 = -L_0$ and $S_{0,N} = -L_{0,N}$, the part of the generator corresponding to jumps of the environment. It is clear that $\{S_{0,N}\}$ is a finite approximation of S_0, and that $\langle f, S_0 f\rangle_\alpha \leq \langle f, -Lf\rangle_\alpha$, $\langle g, S_{0,N} g\rangle_N \leq \langle g, -L_N g\rangle_N$. Conditions A and B of Theorem 1.2.2 are consequence, on this case, of the next results, that we state as lemmas:

Lemma 1.3.1. *$w_a, v_a \in H_{0,-1}$, and for all $g \in \mathcal{L}oc$, $Lg \in H_{0,-1}$.*

Proof. For a criteria of Sethuraman and Xu ([SX]), a sufficient condition for a local function v to be in $H_{0,-1}$ is that $\langle v\rangle_\alpha = 0$ for all $\alpha \in [0,1]$. Therefore, it is enough to observe that for all $\alpha \in [0,1]$, $\langle w_a\rangle_\alpha = \langle v_a\rangle_\alpha = \langle Lg\rangle_\alpha = 0$. □

Lemma 1.3.2. *If $g \in \mathcal{L}oc$ and $\langle g\rangle_\alpha = 0$ for all $\alpha \in [0,1]$, then*

$$\lim_{N\to\infty} ||g||_{0,-1,N} = ||g||_{0,-1}.$$

Proof. This is just a consequence of Corollaries 2.2 and 2.4 of [LOV2], that are based in the so called Liouville D property of the lattice $\mathbb{Z}^d_*$ □

The following lemma is just Theorem 4.2 of [LOV2]:

Lemma 1.3.3. *If $v \in \mathcal{L}oc$ and $\langle v \rangle_\alpha = 0$ for all $\alpha \in [0,1]$, then for all $\varepsilon > 0$ there exists $g_\varepsilon \in \mathcal{L}oc$ such that*

$$||v - Lg_\varepsilon||_{0,-1} < \varepsilon.$$

Once these three lemmas are stated, by Theorem 1.2.2 we have the following result:

Theorem 1.3.4. *For all $v \in \mathcal{L}oc$ such that $\langle v \rangle_\alpha = 0$ for all $\alpha \in [0,1]$,*

$$\lim_{N\to\infty} ||v||_{-1,N} = ||v||_{-1}.$$

1.3.3 Mean zero case

Now suppose that the transition probability has mean zero, that is, $\sum_z zp(z) = 0$. Define $S = -(L+L^*)/2$, $S_N = -(L_N + L_N^*)/2$, the symmetric part of the generator. A simple computation shows that

$$\begin{aligned} Sf(\xi) &= \sum_{x,y\in\mathbb{Z}^d_*} s(y-x)\xi(x)(1-\xi(y))[f(\sigma^{xy}\xi) - f(\xi)] \\ &+ \sum_{z\in\mathbb{Z}^d_*} s(z)(1-\xi(z))[f(\tau_z\xi) - f(\xi)], \\ S_N f(\xi) &= \sum_{x,y\in T^d_{N,*}} s(y-x)\xi(x)(1-\xi(y))[f(\sigma^{xy}\xi) - f(\xi)] \\ &+ \sum_{z\in T^d_{N,*}} s(z)(1-\xi(z))[f(\tau_z\xi) - f(\xi)], \end{aligned}$$

where $s(x) = (p(x) + p(-x))/2$, the symmetrization of $p(\cdot)$. It is clear that $s(\cdot)$ is a symmetric, finite range, irreducible transition probability, from which S (resp. S_N) is the generator of a symmetric exclusion process in $\mathbb{Z}^d_*$ (resp. $T^d_{N,*}$). We choose $S_0 = S$ and $S_{0,N} = S_N$. Like in the symmetric case, S_{0,N_N} is a finite approximation of S_0, and by definition, $\langle f, S_0 f\rangle = \langle f, -Lf\rangle$ and $\langle f, S_{0,N} f\rangle_N = \langle f, -L_N f\rangle_N$. Observe that in this case, S_0 and $-L$ generates the same Sobolev norms.

Like in the symmetric case, we need to verify Assumptions A and B of Theorem 1.2.2. First, we need to prove that $w_a, v_a \in H_{-1}$ and for $g \in \mathcal{L}oc$, $Lg \in H_{-1}$. But this is true because $\langle v_a\rangle_\alpha = \langle w_a\rangle_\alpha = \langle Lg\rangle_\alpha = 0$ for all $\alpha \in [0,1]$, $H_{-1} \subseteq H_{0,-1}$ (in the notation of the previous subsection) and by the criteria of [SX], $v_a, w_a, Lg \in H_{0,-1}$.

After this, Assumption B of Theorem 1.2.2 follows from Theorem 1.3.4. Therefore, in order to apply Theorem 1.2.2 to prove Theorem 1.1.1, it only remains to prove Assumption A. We state it as a lemma:

Lemma 1.3.5. *For all $v \in \mathcal{L}oc$ such that $\langle v \rangle_\alpha = 0$ for all $\alpha \in [0,1]$ and for all $\varepsilon > 0$, there exists $g_\varepsilon \in \mathcal{L}oc$ such that*

$$||v - Lg_\varepsilon||_{-1} < \varepsilon.$$

Proof. In [V], Varadhan proved a sector condition for the mean zero exclusion process, which roughly states that the asymmetric part of the operator can be bounded by the symmetric part. More precisely, there exists a constant $C = C(p(\cdot))$ such that for all $f, g \in \mathcal{L}oc$,

$$\langle f, Lg\rangle_\alpha^2 \le C\langle f, -Lf\rangle_\alpha \langle g, -Lg\rangle_\alpha.$$

In particular, $||Lg||^2_{-1} \le C||g||^2_1$, from which L is a bounded and densely defined operator from H_1 to H_{-1}. So, it is enough to prove that $v \in L(H_1)$. To this end, we use the resolvent method. Let h be in $H_{-1} \cap \mathcal{L}oc$. For each $\lambda > 0$, let u_λ be the solution of the resolvent equation

$$\lambda u_\lambda - Lu_\lambda = h.$$

This is always possible because L is a negative operator in $\mathcal{L}^2(\mu_\alpha)$, and $u_\lambda \in D(L)$, from which $u_\lambda \in H_1$. The idea is to prove that u_λ (or at least a subsequence) converges in some sense to a certain u, that satisfies $Lu = -h$. In fact, in [LOV1] it is proven that there exists such $u \in H_1$ such that $u_\lambda \to u$ strongly in H_1 and $Lu_\lambda \to -h$ weakly in H_{-1}. Since L is a continuous operator, by unicity of the limit, $-Lu = h$. Approximating u by local functions, the lemma follows. □

1.3.4 Asymmetric case for $d \geq 3$

In dimension $d \geq 3$, a necessary and sufficient condition for a local function v to be in $H_{0,-1}$, is $\langle v\rangle_\alpha = 0$ [SX]. In particular, $w_a, v_a \in H_{0,-1}$ and for $g \in \mathcal{L}oc$, $Lg \in H_{0,-1}$. As for the mean zero case, we choose $S_0 = -(L + L^*)/2$, $S_{0,N} = -(L_N + L_N^*)/2$, and we apply Theorem 1.2.2. The difference here is that for $\alpha' \neq \alpha$, $\langle v_a\rangle_{\alpha'} \neq 0$, and we can not invoke Theorem 1.3.4 in order to prove Assumption B. The next lemma says that condition B is true for this case. The proof of this lemma will be presented in the next section.

Lemma 1.3.6. *In dimension $d \geq 3$, for a local function h with $\langle h\rangle_\alpha = 0$,*

$$\lim_{N\to\infty} ||h - \int h d\mu_{N,K}||_{-1,N} = ||h||_{-1}.$$

A proof of Assumption A for this case can be found in [SVY]. Once Assumptions A and B are verified, Theorem 1.1.1 follows from Theorem 1.2.2.

1.4 Proof of Lemma 1.3.6

First note that Lemma 1.3.6 is just the generalization, in dimension $d \geq 3$, of Theorem 1.3.4 to the case in which $\langle v\rangle_\alpha = 0$ just for the fixed $\alpha \in [0, 1]$. In consequence, in order to prove Lemma 1.3.6 it is enough to prove the corresponding generalizations of Lemmas 1.3.1, 1.3.2 and 1.3.3 to this case. Note that the $|| \cdot ||_{-1}$ norm depends only on the symmetric part S of the generator L. Define the operators $S_0 = (L_0 + L_0^*)/2$ and $S_{0,N} = (L_{0,N} + L_{0,N}^*)/2$, the symmetric part of the jumps of the environment:

$$S_0 f(\xi) = \sum_{x,y\in\mathbb{Z}_*^d} s(y - x)\xi(x)(1 - \xi(y))[f(\sigma^{xy}\xi) - f(\xi)]$$
$$S_{0,N} f(\xi) = \sum_{x,y\in T_{N,*}^d} s(y - x)\xi(x)(1 - \xi(y))[f(\sigma^{xy}\xi) - f(\xi)].$$

The generalizations of Lemmas 1.3.1 and 1.3.3 are proven in [SX] and [LOV2]:

Lemma 1.4.1. *In dimension $d \geq 3$, if $v \in \mathcal{L}oc$ satisfies $\langle v\rangle_\alpha = 0$, then $v \in H_{0,-1}$.*

Lemma 1.4.2. *In dimension $d \geq 3$, if $v \in \mathcal{L}oc$ and $\langle v\rangle_\alpha = 0$, then for all $\varepsilon > 0$ there exists $g_\varepsilon \in \mathcal{L}oc$ such that*

$$||v - Sg_\varepsilon||_{0,-1} < \varepsilon.$$

So, it only rests to prove the generalization of Lemma 1.3.2 to this case:

Lemma 1.4.3. *Let v be a local function such that $\langle v\rangle_\alpha = 0$. Define $\langle v\rangle_N = \int v d\mu_{N,K_N}$. In dimension $d \geq 3$,*

$$\lim_{N\to\infty} ||v - \langle v\rangle_N||_{0,-1,N} = ||v||_{0,-1}.$$

Proof. Using the variational formula for $||v||_{0,-1}$, it is not hard to prove that

$$\liminf_{N\to\infty} ||v - \langle v\rangle_N||_{0,-1,N} \geq ||v||_{0,-1}.$$

In fact, by definition, for all $\varepsilon > 0$ there exists a local function f_ε such that

$$\begin{aligned} ||v||_{0,-1}^2 &\leq 2\langle v, f_\varepsilon\rangle_\alpha - \langle f_\varepsilon, -S_0 f_\varepsilon\rangle_\alpha + \varepsilon \\ &= \lim_{N\to\infty} \{2\langle v - \langle v\rangle_N, f_\varepsilon\rangle_N - \langle f_\varepsilon, -S_{0,N} f_\varepsilon\rangle_N\} + \varepsilon \\ &\leq \liminf_{N\to\infty} \sup_f \{2\langle v - \langle v\rangle_N, f\rangle_N - \langle f, -S_{0,N} f\rangle_N\} + \varepsilon \\ &= \liminf_{N\to\infty} ||v - \langle v\rangle_N||_{0,-1,N}^2 + \varepsilon. \end{aligned}$$

The converse inequality is harder to prove. The idea is to approximate v in $H_{0,-1}$ by local functions with mean zero for all densities $\alpha \in [0,1]$. The proof requires two auxiliary lemmas. The first one is just a version of Lemma 3.6 of [LOV2]:

Lemma 1.4.4. *Let w be a local function with $\langle w\rangle_\alpha = 0$ for all $\alpha \in [0,1]$. Let $\{f_N\}_N$ be a sequence of functions defined in $H_{0,1,N}$ such that*

$$\langle f_N, -S_{0,N} f_N\rangle_N \leq 1,$$

$$\lim_{N\to\infty} \langle w, f_N\rangle_N = A.$$

Then, there exist $f \in H_{0,1}$ and subsequence N' such that $\langle w, f\rangle_\alpha = A$, $\langle f, -S_0 f\rangle_\alpha \leq 1$, and for all local functions h with $\langle h\rangle_\alpha = 0$ for each $\alpha \in [0,1]$,

$$\lim_{N'\to\infty} \langle f_{N'}, h\rangle_N = \langle f, h\rangle_\alpha.$$

Before state the second auxiliary lemma, we need to introduce some notation. Let $\Lambda_N = \{-N+1, ..., N\}^d \setminus \{0\}$ be the cube of radius N. Note that $\Lambda_N \neq T^d_{N,*}$, because Λ_N has no periodic conditions. For each $x \in \mathbb{Z}^d_*$, define $\theta_x(\xi) =: \xi(x)$, and for each $l > 0$, define $\varphi_l(\xi) = \sum_{x\in\Lambda_l} \xi(x)$. Let $\mathcal{F}_{\Lambda_N}$ be the σ-algebra generated by φ_l and $\{\theta_x; x \in \Lambda_N^c\}$. For $l > 0$ such that $\mathrm{supp}(v) \subseteq \Lambda_l$, define $v_l = \mathbb{E}[v|\mathcal{F}_{\Lambda_l}]$. Note that there is a natural way to define v_l that does not depend on the particular value of α. The next lemma is an easy consequence of the equivalence of ensembles:

Lemma 1.4.5. *Fix positive integers l, q such that* $\mathrm{supp}(v) \subseteq \Lambda_l$ *and $q > 2$. Define $g_n = v_{lq^n}$. There is a finite constant κ such that*

i) $\langle (g_n - g_{n-1})^2\rangle_\alpha \leq \kappa(lq^n)^{-d}$

ii) $\langle (g_n - g_{n-1})^2\rangle_N \leq \kappa(lq^n)^{-d}$.

The proof follows in the next way: for each N there exist a function $f_N \in H_{1,N}$ such that $\langle f_N, -S_{0,N} f_N\rangle_N \leq 1$ and $||v - \langle v\rangle_N||_{0,-1,N} = \langle f_N, v - \langle v\rangle_N\rangle_N$. Consider a subsequence $\tilde{N}$ such that

$$\lim_{\tilde{N}\to\infty} ||v - \langle v\rangle_{\tilde{N}}||_{0,-1,\tilde{N}} = \limsup_{N\to\infty} ||v - \langle v\rangle_N||_{0,-1,N} =: A.$$

By Lemma 1.4.4, there are function $f \in H_1$ and sub-subsequence N' such that $\langle f_{N'}, h\rangle_{N'} \to \langle f, h\rangle_\alpha$ for all local functions h with mean zero for each μ_α. In particular,

$$\lim_{N'\to\infty} \langle f_{N'}, v - v_l\rangle_{N'} = \langle f, v - v_l\rangle_\alpha.$$

Let $l, q > 2$ be fixed. Define, as in Lemma 1.4.4, $g_n = v_{lq^n}$. Just to make notation simpler, suppose that $N' = lq^n$, and denote N' simply by N. The changes needed if it is not the case are straightforward. Then, we have that

$$\begin{aligned}\langle f_N, v - \langle v\rangle_N\rangle_N &= \langle f_N, v - v_l\rangle_N + \langle f_N, v_l - \langle v\rangle_N\rangle_N \\ &= \sum_{k=1}^{n}\langle f_N, g_{k-1} - g_k\rangle_N + \langle f_N, v - v_l\rangle_N\end{aligned}$$

Define $\mathcal{L}_k$ as the generator of a exclusion process in Λ_{lq^k}. Notice that, due to the boundary effects, $\mathcal{L}_{lq^k} \neq S_{0,lq^k}$. We see that $\langle v - v_l\rangle_\alpha = 0$, $\langle g_{k-1} - g_k\rangle_\alpha = 0$ for all $\alpha \in [0,1]$. By linear algebra, there exists a local function G_k defined in $\{0,1\}^{\Lambda_{lq^k}}$ such that $g_{k-1} - g_k = \mathcal{L}_k G_k$. Therefore,

$$\sum_{k=1}^{n}\langle f_N, g_{k-1} - g_k\rangle_N = \sum_{k=1}^{n}\langle f_N, \mathcal{L}_k G_k\rangle_N = \sum_{k=1}^{n}\sum_{b\in\Gamma_k}\langle \nabla_b f_N, \nabla_b G_k\rangle_N,$$

where $\sum_{b\in\Gamma_k}$ means sum over all bonds $b = \langle xy\rangle$ such that $x, y \in \Lambda_{lq^k}$ and $\nabla_b g = s(y-x)^{1/2}[g(\sigma^{xy}\eta) - g(\eta)]$.

Choose $a_k = \varepsilon 2^k$. By Cauchy's inequality with weights a_k, we have

$$\begin{aligned}|\sum_{k=1}^{n}\langle f_N, g_{k-1} - g_k\rangle_N| &\le \sum_{k=1}^{n}\sum_{b\in\Gamma_k}\frac{1}{a_k}\langle(\nabla_b f_N)^2\rangle_N + a_k\langle(\nabla_b G_k)^2\rangle_N \\ &\le \sum_{b\in\Gamma_n}\sum_{k:b\in\Gamma_k}\frac{1}{a_k}\langle(\nabla_b f_N)^2\rangle_N + \sum_{k=1}^{n}\sum_{b\in\Gamma_k} a_k\langle(\nabla_b G_k)^2\rangle_N \\ &\le \frac{1}{\varepsilon}\sum_{b\in\Gamma_n}\langle(\nabla_b f_N)^2\rangle_N + \sum_{k=1}^{n} a_k\langle g_k - g_{k-1}, -\mathcal{L}_k^{-1}(g_k - g_{k-1})\rangle_N \\ &\le \frac{1}{\varepsilon}\langle f_N, -\mathcal{L}_n f_N\rangle_N + \varepsilon\sum_{k=1}^{n} 2^k\langle g_k - g_{k-1}, -\mathcal{L}_k^{-1}(g_k - g_{k-1})\rangle_N \\ &\le \frac{1}{\varepsilon}\langle f_N, -S_{0,lq^n} f_N\rangle_N + \varepsilon\sum_{k=1}^{n} 2^k C\cdot 2^k (lq^k)^2\langle(g_{k-1} - g_k)^2\rangle_N,\end{aligned}$$

where in the last line we have used the spectral gap inequality for the exclusion process [Q].

Using Lemma 1.4.5 and minimizing in ε,

$$\begin{aligned}|\sum_{k=1}^{n}\langle f_N, g_{k-1} - g_k\rangle_N| &\le \frac{1}{\varepsilon} + \varepsilon\sum_{k=1}^{n} C\kappa\cdot 2^k (lq^k)^{2-d} \\ &\le \frac{1}{\varepsilon} + \varepsilon\Big[\frac{C\kappa l^{2-d}}{1 - 2q^{2-d}}\Big] \\ &\le 2\sqrt{\frac{C\kappa l^{2-d}}{1 - 2q^{2-d}}} \le C_1 l^{\frac{2-d}{2}}.\end{aligned}$$

By the law of large numbers, as $l \to \infty$, $v_l \to 0$ $\mu_\alpha - a.s.$ and in $\mathcal{L}^2(\mu_\alpha)$. We also have that

$$\begin{aligned}||g_k - g_{k-1}||^2_{0,-1} &= \langle g_k - g_{k-1}, (-S_0)^{-1}(g_k - g_{k-1})\rangle_\alpha \\ &\leq \langle g_k - g_{k-1}, (-\mathcal{L}_{k+1})^{-1}(g_k - g_{k-1})\rangle_\alpha \\ &\leq C(lq^{k+1})^2\langle (g_k - g_{k-1})^2\rangle_\alpha \\ &\leq C\kappa q^2(lq^k)^{2-d}.\end{aligned}$$

Therefore, the sequence $\{g_k - g_{k-1}\}_k$ is absolutely summable, and there exists $g \in H_{0,-1}$ such that

$$\lim_{n\to\infty}(v_l - v_{lq^n}) = \sum_{k=1}^{\infty} g_k - g_{k-1} = g.$$

In the other hand, we know that $v_{lq^n} \to 0$ in $\mathcal{L}^2(\mu_\alpha)$, from which $\langle F, v_{lq^n}\rangle_\alpha$ goes to zero for all $F \in \mathcal{L}^2(\mu_\alpha)$, and $v_{lq^n} \to v_l - g$ in $H_{0,-1}$, from which $\langle F, v_{lq^n}\rangle_\alpha \to \langle F, v_l - g\rangle_\alpha$ for all $F \in H_{0,1}$. Since $D(S_0) \subseteq \mathcal{L}^2(\mu_\alpha) \cap H_{0,-1}$ and $D(S_0)$ is dense in $H_{0,-1}$, we have $g = v_l$.

As before, by using part $i)$ of Lemma 1.4.5, we can prove that there exists a constant C_2 such that

$$|\langle f, v_l\rangle_\alpha| \leq C_2 \cdot l^{\frac{2-d}{2}}.$$

Combining both inequalities, we see that

$$\begin{aligned}\limsup_{N\to\infty} ||v - \langle v\rangle_N||_{0,-1,N} &= \limsup_{N\to\infty}\langle f_N, v - \langle v\rangle_N\rangle_N \\ &= \limsup_{N\to\infty}\Big\{\langle f_N, v - v_l\rangle_N + \langle f_N, v_l - \langle v\rangle_N\rangle_N\Big\} \\ &\leq \langle f, v - v_l\rangle_\alpha + C_1 \cdot l^{\frac{2-d}{2}} \\ &\leq (C_1 + C_2)l^{\frac{2-d}{2}} + \langle f, v\rangle_\alpha.\end{aligned}$$

Since $d \geq 3$ and l is arbitrary,

$$\limsup_{N\to\infty} ||v - \langle v\rangle_N||_{0,-1,N} \leq \langle f, v\rangle_\alpha \leq ||f||_{0,1} \cdot ||v||_{0,-1} \leq ||v||_{0,-1}.$$

$\square$

Chapter 2

The sub-diffusive CLT for the tagged particle

Consider the one-dimensional nearest neighbor symmetric situation. In this context, as already observed by Arratia [A] and differently from the previous sections, the scaling changes dramatically since to displace the tagged particle from the origin to a site $N > 0$, all particles between the origin and N need to move to the right of N. This observation relates the asymptotic behavior of the tagged particle to the hydrodynamic behavior of the system. The correct scaling for the law of large numbers should therefore be X_{tN^2}/N and we expect $(X_{tN^2} - E[X_{tN^2}])/\sqrt{N}$ to converge to a Gaussian variable.

The central limit theorem in equilibrium was obtained by Rost and Vares [RV] for a slightly different model. They proved that for each fixed $t > 0$, $X_{tN^2}/\sqrt{N}$ converges to a fractional Brownian motion W_t with variance given by $E[W_t^2] = \alpha t^{1/2}$. We extend their result to the nonequilibrium case.

The idea of the proof is to relate the position of the tagged particle to the well known hydrodynamic behavior of the symmetric exclusion process. Since particles cannot jump over other particles, the position of the tagged particle is determined by the current over one bond and the density profile of particles. Therefore, a nonequilibrium central limit theorem for the position of the tagged particle follows from a joint central limit theorem for the current and the density profile. Since the current over a bond can itself, at least formally, be written as the difference between the mass at the right of the bond at time t and the mass at time 0, a central limit theorem for the position of the tagged particle should follow from a nonequilibrium central limit theorem for the density field. This is the content of the article.

There are three main ingredients in the proof. In Section 2.2 we present a nonequilibrium central limit theorem for the current over a bond and show how it relates to the fluctuations of the density field. In section 2.4 we obtain a formula which relates the position of the tagged particle to the current over one bond and the density field. Finally, in Section 2.5 we present a sharp estimate on the difference of the solution of the hydrodynamic equation and the solution of a discretized version of the hydrodynamic equation.

2.1 Notation and Results

The nearest neighbor one-dimensional symmetric exclusion process is the simple exclusion process defined on $\{0, 1\}^{\mathbb{Z}}$ for which $p(1) = p(-1) = 1/2$ and $p(z) = 0$ if $z \neq -1, 1$.

In order to make the exposition clear, we slightly modify the notation of the previous section. Denote byL_N the generator of the process speeded up by N^2:

$$(L_N f)(\eta) \;=\; N^2 \sum_{x\in\mathbb{Z}} [f(\sigma^{x,x+1}\eta) - f(\eta)].$$

For each configuration η, denote by $\pi(\eta)$ the positive measure on $\mathbb{R}$ obtained by assigning mass N^{-1} to each particle:

$$\pi(\eta) \;=\; N^{-1} \sum_{x\in\mathbb{Z}} \eta(x)\delta_{x/N}$$

and let $\pi_t = \pi(\eta_t)$.

Fix a profile $\rho_0 : \mathbb{R} \to [0,1]$ with the first four derivatives bounded. Denote by $\nu^N_{\rho_0(\cdot)}$ the product measure on $\mathcal{X}$ associated to ρ_0:

$$\nu^N_{\rho_0(\cdot)}\{\eta, \eta(x) = 1\} \;=\; \rho_0(x/N)$$

for x in $\mathbb{Z}$. For each $Ng_\varepsilon 1$ and each measure μ on $\mathcal{X}$, denote by $\mathbb{P}_\mu$ the probability on the path space $D(\mathbb{R}_+, \mathcal{X})$ induced by the measure μ and the exclusion process with generator L_N. Expectation with respect to $\mathbb{P}_\mu$ is denoted by $\mathbb{E}_\mu$. Note that we have omitted the dependence of the probability $\mathbb{P}_\mu$ on N to keep notation simple. This convention is adopted below for several other quantities which also depend on N. The hydrodynamic behavior of the symmetric simple exclusion process is trivial and described by the heat equation.

Theorem 2.1.1. *Fix a profile $\rho_0 : \mathbb{R} \to [0,1]$. Then, for all time $t \geq 0$, under $\mathbb{P}_{\nu^N_{\rho_0(\cdot)}}$ the sequence of random measures π_t converges in probability to the absolutely continuous measure $\rho(t,u)du$ whose density ρ is the solution of the heat equation with initial condition ρ_0:*

$$\begin{cases} \partial_t \rho = \Delta\rho \\ \rho(0,\cdot) = \rho_0(\cdot). \end{cases} \tag{2.1}$$

Here and below, Δ stands for the Laplacian.

This theorem establishes a law of large numbers for the empirical measure. To state the central limit theorem some notation is required. For $kg_\varepsilon 0$, denote by $\mathcal{H}_k$ the Hilbert space induced by smooth rapidly decreasing functions and the scalar product $< \cdot, \cdot >_k$ defined by

$$< f, g >_k = < f, (x^2 - \Delta)^k g > ,$$

where $< \cdot, \cdot >$ stands for the usual scalar product in $\mathbb{R}^d$. Notice that $\mathcal{H}_0 = L^2(\mathbb{R}^d)$ and denote by $\mathcal{H}_{-k}$ the dual of $\mathcal{H}_k$.

Let $\rho^N_t(x) = \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}[\eta_t(x)]$. A trivial computation shows that $\rho^N_t(x)$ is the solution of the discrete heat equation:

$$\begin{cases} \partial_t \rho^N_t(x) \;=\; \Delta_N \rho^N_t(x) \;, \\ \rho^N_0(x) = \rho_0(x/N) \;, \end{cases} \tag{2.2}$$

where $(\Delta_N h)(x) = N^2 \sum_{y,|y-x|=1} [h(y) - h(x)]$.

Fix $kg_\varepsilon 4$ and denote by $\{Y^N_t, tg_\varepsilon 0\}$ the so called density field, a $\mathcal{H}_{-k}$-valued process given by

$$Y^N_t(G) \;=\; \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} G(x/N)\{\eta_t(x) - \rho^N_t(x)\}$$

for G in $\mathcal{H}_k$. Denote by Q_N the probability measure on the path space $D(\mathbb{R}_+, \mathcal{H}_{-k})$ induced by the process Y^N_t and the measure $\nu^N_{\rho_0(\cdot)}$. Next result is due to Galves, Kipnis and Spohn in dimension $d = 1$ and to Ravishankar [R1] in dimension $dg_\varepsilon 2$.

Theorem 2.1.2. *The sequence Q_N converges to Q, the probability measure concentrated on $C(\mathbb{R}_+, \mathcal{H}_{-k})$ corresponding to the Orsntein-Uhlenbeck process Y_t with mean zero and covariance given by*

$$\mathbb{E}[Y_t(H)Y_s(G)] \;=\; \int_{\mathbb{R}} (T_{t-s}H)\, G\, \chi_s \;-\; \int_0^s dr \int_{\mathbb{R}} (T_{t-r}H)\,(T_{s-r}G)\,\{\partial_r \chi_r - \Delta \chi_r\}$$

for $0 \le s < t$ and $G, H \in \mathcal{H}_k$. In this formula, $\{T_t : t g_\varepsilon 0\}$ stands for the semigroup associated to the Laplacian and χ_s for the function $\chi(s,u) = \rho(s,u)[1-\rho(s,u)]$.

Note that in the case of the heat equation, $\partial_r \chi_r - \Delta \chi_r = 2(\partial_x \rho)^2$. Also, in the equilibrium case, χ is constant in space and time so that the second term vanishes and we recover the equilibrium covariances. Finally, integrating by parts twice the expression with $\Delta\chi_r$, we rewrite the limiting covariances as

$$\mathbb{E}[Y_t(H)Y_s(G)] \;=\; \int_{\mathbb{R}} (T_tH)\,(T_sG)\,\chi_0 \;+\; 2\int_0^s dr \int_{\mathbb{R}} (\nabla T_{t-r}H)\,(\nabla T_{s-r}G)\,\chi_r\;, \qquad (2.3)$$

where ∇f is the space derivative of f.

We will examine nonequilibrium central limit theorems for the current through a bond and the position of a tagged particle. For a bond $(x, x+1)$, denote by $J_{x,x+1}(t)$ the current over this bond. This is the total number of jumps from site x to site $x+1$ in the time interval $[0,t]$ minus the total number of jumps from site $x+1$ to site x in the same time interval.

Theorem 2.1.3. *Fix u in $\mathbb{R}$ and let*

$$Z_t^N \;=\; \frac{1}{\sqrt{N}}\Big\{ J_{x_N,x_N+1}(t) - E_{\nu^N_{\rho_0(\cdot)}}[J_{x_N,x_N+1}(t)]\Big\}\;,$$

where $x_N = [uN]$. Then, for every $k g_\varepsilon 1$ and every $0 \le t_1 < \cdots < t_k$, $(Z^N_{t_1}, \dots, Z^N_{t_k})$ converges in law to a Gaussian vector $(Z_{t_1}, \dots, Z_{t_k})$ with covariance given by

$$\begin{aligned} E[Z_sZ_t] \;&=\; \int_{-\infty}^0 dx\, P[B_s \le x]\, P[B_t \le x]\, \chi_0(x) \\ &+\; \int_0^\infty dx\, P[B_s g_\varepsilon x]\, P[B_t g_\varepsilon x]\, \chi_0(x) \\ &+\; 2\int_0^s dr \int_{-\infty}^\infty dx\, p_{t-r}(0,x)\, p_{s-r}(0,x)\, \chi_r(x) \end{aligned}$$

provided $s \le t$ and $u = 0$. In this formula, B_t is a standard Brownian motion starting from the origin and $p_t(x,y)$ is the Gaussian kernel.

By translation invariance, in the case $u \ne 0$, we just need to translate χ by $-u$ in the covariance.

Let $H_0 = \mathbf{1}\{[0,\infty)\}$. The covariance appearing in the previous theorem is easy to understand. Formally the current $N^{-1/2}J_{-1,0}(t)$ centered by its mean corresponds to $Y^N_t(H_0)\ -Y^N_0(H_0)$ since both processes increase (resp. decrease) by $N^{-1/2}$ whenever a particle jumps from -1 to 0 (resp. 0 to -1). The limiting covariance $E[Z_sZ_t]$ corresponds to the formal covariance

$$E\Big[\big\{Y_t(H_0) - Y_0(H_0)\big\}\big\{Y_s(H_0) - Y_0(H_0)\big\}\Big]\;.$$

Denote by $\nu^{N,*}_{\rho_0(\cdot)}$ the measure $\nu^N_{\rho_0(\cdot)}$ conditioned to have a particle at the origin.

Remark 2.1.4. *The law of large numbers and the central limit theorem for the empirical measure and for the current starting from $\nu^{N,*}_{\rho_0(\cdot)}$ follow from the law of large numbers* *and the central limit theorem for the empirical measure and the current starting from the measure* $\nu^N_{\rho_0(\cdot)}$ *since we may couple both processes in such a way that they differ at most at one site at any given time.*

Fix a profile ρ_0 with the first four derivatives limited, and consider the product measure $\nu^{N,*}_{\rho_0(\cdot)}$. Denote by X_t the position at time $tg_\varepsilon 0$ of the particle initially at the origin. A law of large numbers for X_t follows from the hydrodynamic behavior of the process:

Theorem 2.1.5. *Fix* $tg_\varepsilon 0$. X_t/N *converges in* $\mathbb{P}_{\nu^{N,*}_{\rho_0(\cdot)}}$*-probability to* u_t*, the solution of*

$$\dot{u}_t \;=\; -\frac{(\partial_u \rho)(t,u_t)}{\rho(t,u_t)} \;.$$

Note that the solution of the previous equation is given by

$$\int_0^{u_t} du\, \rho(t,u) \;=\; -\int_0^t ds\, (\partial_u \rho)(s,0) \;.$$

Theorem 2.1.6. *Assume that* ρ_0 *has a bounded fourth derivative. Let* $W_t = N^{-1/2}$ $(X_t - N u_t)$*. Under* $\mathbb{P}_{\nu^{N,*}_{\rho_0(\cdot)}}$*, For every* $kg_\varepsilon 1$ *and every* $0 \le t_1 < \cdots < t_k$*,* $(W^N_{t_1}, \dots, W^N_{t_k})$ *converges in law to a Gaussian vector* $(W_{t_1}, \dots, W_{t_k})$ *with covariance given by*

$$\begin{aligned}
\rho(s,u_s)\rho(t,u_t)E[W_s W_t] \;&=\; \int_{-\infty}^0 dx\, P_{u_s}[B_s \le x]\, P_{u_t}[B_t \le x]\, \chi_0(x) \\
&+\; \int_0^\infty dx\, P_{u_s}[B_s g_\varepsilon x]\, P_{u_t}[B_t g_\varepsilon x]\, \chi_0(x) \\
&+\; 2\int_0^s dr \int_{-\infty}^\infty dx\, p_{t-r}(u_t,x)\, p_{s-r}(u_s,x)\, \chi_r(x) \;.
\end{aligned}$$

In this formula, P_u *stands for the probability corresponding to a standard Brownian motion starting from* u*.*

The assumption made on the smoothness of ρ_0 appears because in the proof of Theorem 2.1.6 we need a sharp estimate on the difference of the discrete approximation of the heat equation (2.2) and the heat equation (2.1). In section 2.5 we show that there exists a finite constant C_0 for which $|\rho^N_t(x) - \rho(t,x/N)| \le C_0 t N^{-2}$ for all $Ng_\varepsilon 1$, x in $\mathbb{Z}$ and $tg_\varepsilon 0$ under the assumption that ρ_0 has a bounded fourth derivative.

2.2 Nonequilibrium fluctuations of the current

Suppose for a moment that the profile ρ_0 has a compact support. Then, η_0 is almost surely a configuration with a finite number of particles, and it is easy to see that we have a simple formula for the current $J_{-1,0}(t)$:

$$J_{-1,0}(t) = \sum_{x \ge 0} \eta_t(x) - \eta_0(x) \;. \tag{2.4}$$

In particular, we can write $J_{-1,0}(t)$ in terms of the fluctuation field:

$$\frac{1}{\sqrt{N}} \Big\{ J_{-1,0}(t) - E_{\nu^N_{\rho_0(\cdot)}}[J_{-1,0}(t)] \Big\} \;=\; Y^N_t(H_0) - Y^N_0(H_0\,),$$

where H_a is the indicator function of the interval $[a,\infty)$:

$$H_a(u) \;=\; \mathbf{1}\{[a,\infty)\}(u) \;.$$

Since the profile has compact support, it is possible to define $Y_t(H_0)$ as the limit $Y_t(G_n)$ for some sequence G_n of compact supported function converging to H_0 on compact subsets of $\mathbb{R}$ and to prove that $Y_t^N(H_0)$, defined in a similar way, converges to $Y_t(H_0)$.

In the general case, however, when ρ_0 is an arbitrary profile, neither formula (2.4) makes sense, nor the fluctuation field $Y_t^N(H_0)$ is well defined. Nevertheless, there is a way to calculate the fluctuations of the current by appropriated approximations of the function G, as made by Rost and Vares [RV] in the equilibrium case.

Define the sequence $\{G_n : n g_\varepsilon 1\}$ of approximating functions of H_0 by

$$G_n(u) = \{1 - (u/n)\}^+ \mathbf{1}\{u g_\varepsilon 0\} \ .$$

From here we use the next convention: if X is a random variable, we denote by $\overline{X}$ the centered variable $X - E_{\nu^N_{\rho_0(\cdot)}}[X]$.

Proposition 2.2.1. *For every $t g_\varepsilon 0$,*

$$\lim_{n\to\infty} \mathbb{E}_{\nu^N_{\rho_0(\cdot)}} \Big[N^{-1/2} \overline{J}_{-1,0}(t) - Y_t^N(G_n) + Y_0^N(G_n) \Big]^2 \ = \ 0$$

uniformly in N.

Proof. Clearly,

$$M_{x,x+1}(t) \ := \ J_{x,x+1}(t) \ - \ N^2 \int_0^t ds \, \{\eta_s(x) - \eta_s(x+1)\}$$

is a martingale with quadratic variation given by

$$< M_{x,x+1} >_t \, = \ N^2 \int_0^t ds \, \{\eta_s(x) - \eta_s(x+1)\}^2 \ .$$

The goal is to express the difference $Y_t^N(G_n) - Y_0^N(G_n)$ in terms of the martingales $M_{x,x+1}(t)$ and to notice that these martingales are orthogonal, since they have no common jumps.

Since

$$J_{x-1,x}(t) - J_{x,x+1}(t) \ = \ \eta_t(x) - \eta_0(x)$$

for all x in $\mathbb{Z}^d$, $t g_\varepsilon 0$,

$$Y_t^N(G_n) - Y_0^N(G_n) \ = \ N^{-1/2} \sum_{x\in\mathbb{Z}} G_n(x/N) \{ \overline{J}_{x-1,x}(t) - \overline{J}_{x,x+1}(t) \} \ .$$

A summation by parts and the explicit form of G_n permits to rewrite this expression as

$$N^{-1/2} \overline{J}_{-1,0}(t) - N^{-1/2} \sum_{x=1}^{nN} \frac{1}{nN} \overline{J}_{x-1,x}(t) \ .$$

Representing the currents $J_{x,x+1}(t)$ in terms of the martingales $M_{x,x+1}(t)$, we obtain that

$$\begin{aligned} & N^{-1/2} \overline{J}_0(t) - \big[Y_t^N(G_n) - Y_0^N(G_n) \big] \\ & = \ \frac{1}{\sqrt{N}} \sum_{x=1}^{nN} \frac{1}{nN} M_{x-1,x}(t) \ + \ \frac{1}{\sqrt{N}} \int_0^t ds \, \frac{N}{n} \, [\overline{\eta}_s(0) - \overline{\eta}_s(nN)] \ . \end{aligned}$$

We claim that the martingale and the integral term converge to 0 in $L^2(\mathbb{P}_{\nu^N_{\rho_0(\cdot)}})$. In fact, since the martingales are orthogonal, estimating their quadratic variations by tN^2, an elementary computation shows that

$$\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\Big[\frac{1}{\sqrt{N}}\sum_{x=1}^{nN}\frac{1}{nN}M_{x-1,x}(t)\Big]^2 \;\le\; \frac{t}{n}\;.$$

The integral term is more demanding, because in non-equilibrium the two-point correlations are not easy to estimate. Expanding the square we have that

$$\begin{aligned}
&\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\Big[\frac{1}{\sqrt{N}}\int_0^t ds\,\frac{N}{n}[\overline{\eta}_s(0)-\overline{\eta}_s(nN)]ds\Big]^2\\
&\quad=\;\frac{2N}{n^2}\int_0^t ds\int_0^s dr\,\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\Big[\big(\overline{\eta}_s(0)-\overline{\eta}_s(nN)\big)\big(\overline{\eta}_r(0)-\overline{\eta}_r(nN)\big)\Big]\;.
\end{aligned}$$

By Lemma 2.2.2 the previous expression is less than or equal to $C_0 t^{5/2} n^{-2}$ for some finite constant C_0 depending only on ρ_0. This concludes the proof of the proposition. □

A central limit theorem for the current $\overline{J}_{-1,0}(t)$ is a consequence of this proposition.
Proof of Theorem 2.1.3. Fix $tg_\varepsilon 0$ and $ng_\varepsilon 1$. By approximating G_n in $L^2(\mathbb{R})\cap L^1(\mathbb{R})$ by a sequence $\{H_{n,k} : kg_\varepsilon 1\}$ of smooth functions with compact support, recalling Theorem 2.1.2, we show that $Y^N_t(G_n)$ converges in law to a Gaussian variable denoted by $Y_t(G_n)$.

By Proposition 2.2.1, $\{Y^N_t(G_n) - Y^N_0(G_n) : ng_\varepsilon 1\}$ is a Cauchy sequence uniformly in N. In particular, $Y_t(G_n)-Y_0(G_n)$ is a Cauchy sequence and converges to a Gaussian limit denoted by $Y_t(H_0) - Y_0(H_0)$. Therefore, by Proposition 2.2.1, $\overline{J}_{-1,0}(t)$ converges in law to $Y_t(H_0) - Y_0(H_0)$.

The same argument show that any vector $(\overline{J}_{-1,0}(t_1),\dots,\overline{J}_{-1,0}(t_k))$ converges in law to $(Y_{t_1}(H_0) - Y_0(H_0),\dots,Y_{t_k}(H_0) - Y_0(H_0))$. The covariances can be computed since by (2.3)

$$\begin{aligned}
&E\Big[\big\{Y_t(H_0)-Y_0(H_0)\big\}\big\{Y_s(H_0)-Y_0(H_0)\big\}\Big]\\
&\quad=\;\lim_{n\to\infty}E\Big[\big\{Y_t(G_n)-Y_0(G_n)\big\}\big\{Y_s(G_n)-Y_0(G_n)\big\}\Big]\\
&\quad=\;\lim_{n\to\infty}\Big\{\int_{\mathbb{R}}\big\{(T_tG_n)(T_sG_n)+G_n^2-(T_tG_n)G_n-(T_sG_n)G_n\big\}\chi_0\\
&\qquad\qquad+2\int_0^s dr\int_{\mathbb{R}}(\nabla T_{t-r}G_n)\,(\nabla T_{s-r}G_n)\chi_r\Big\}\;.
\end{aligned}$$

A long but elementary computation permits to recover the expression presented in the statement of the theorem. This concludes the proof. □

We conclude this section with some elementary estimates on two points correlation functions. For $0\le s\le t$ and $x\neq y$ in $\mathbb{Z}$, let

$$\varphi(t;x,y)=E_{\nu^N_{\rho_0(\cdot)}}[\eta_t(x);\eta_t(y)]\;,\quad \varphi(s,t;x,y)=E_{\nu^N_{\rho_0(\cdot)}}[\eta_s(x);\eta_t(y)]\;.$$

In this formula and below, $E_\mu[f;g]$ stands for the covariance of f and g with respect to μ.

Lemma 2.2.2. *There exists a finite constant $C_0 = C_0(\rho_0)$ depending only on the initial profile ρ_0 such that*

$$\sup_{x,y\in\mathbb{Z}}|\varphi(t;x,y)|\;\le\;\frac{C_0\sqrt{t}}{N}\;,\quad \sup_{x,y\in\mathbb{Z}}|\varphi(s,t;x,y)|\;\le\;\frac{C_0}{N}\Big\{\sqrt{s}+\frac{1}{\sqrt{t-s}}\Big\}\;.$$

The first statement is a particular case of an estimate proved in [FPSV]. In sake of completeness, we present an elementary proof of this lemma.

Proof. Let L_2 be the generator of 2 nearest-neighbor symmetric simple exclusion processes on $\mathbb{Z}$. An elementary computation shows that $\varphi(t;x,y)$ satisfies the difference equation

$$\begin{cases} (\partial_t\varphi)(t;x,y) = N^2(L_2\varphi)(t;x,y) - \mathbf{1}\{|x-y|=1\}N^2[\rho^N(t,x)-\rho^N(t,y)]^2 \;, \\ \varphi(0;x,y) = 0 \;. \end{cases}$$

This equation has an explicit solution which is (negative and) absolutely bounded by

$$C_0(\rho_0)\int_0^t ds\,\mathbb{P}_{x,y}\big[|X_s - Y_s| = 1\big]$$

for $C_0 = \|\partial\rho_0\|_\infty^2$. In this formula, (X_s, Y_s) represent the position of the symmetric exclusion process speeded up by N^2 and starting from $\{x, y\}$. A coupling argument shows that $\mathbb{P}_{x,y}[|X_s - Y_s| = 1] \le \mathbb{P}^0_{x,y}[|X_s - Y_s| = 1]$ where in the second probability particles are evolving independently. Since $\mathbb{P}^0_{x,y}[|X_s - Y_s| = 1] \le C(sN^2)^{-1/2}$, the first part of the lemma is proved.

To prove the second statement, recall that we denote by Δ_N the discrete Laplacian in $\mathbb{Z}$. $\varphi(t;y) = \varphi(s,t;x,y)$ satisfies the difference equation

$$\begin{cases} (\partial_t\varphi)(t;y) = (\Delta_N\varphi)(t;y) \\ \varphi(s;y) = \varphi(s;x,y) \text{ if } y \neq x, \\ \varphi(s;y) = \rho^N(s,x)[1-\rho^N(s,x)] \text{ for } y = x. \end{cases}$$

This equation has an explicit solution

$$\varphi(s;y) \;=\; \sum_{z\neq x} p_{t-s}(y,z)\varphi(s;x,z) \;+\; p_{t-s}(y,x)\rho^N(s,x)[1-\rho^N(s,x)] \;,$$

where $p_s(x,y)$ stands for the transition probability of a nearest neighbor symmetric random walk speeded up by N^2. The first part of the lemma together with well known estimates on p_s permit to conclude. □

2.3 Law of Large Numbers for the Tagged Particle

In this section we assume the initial measure to be $\nu^{N,*}_{\rho_0(\cdot)}$, the product measure $\nu^N_{\rho_0(\cdot)}$ conditioned to have a particle at the origin. Keep in mind Remark 2.1.4.

Fix a positive integer n. The tagged particle is at the right of n at time t if and only if the total number of particles in the interval $\{0,\dots,n-1\}$ is less than or equal to the current $J_{-1,0}(t)$:

$$\{X_t \ge n\} \;=\; \{J_{-1,0}(t) \ge \sum_{x=0}^{n-1}\eta_t(x)\} \;. \tag{2.5}$$

This equation indicates that a law of large numbers and a central limit theorem for the position of the tagged particle are intimately connected to the joint asymptotic behavior of the current and the empirical measure. We prove in this section the law of large numbers.

Denote by $\lceil a\rceil$ the smallest integer larger than or equal to a. Fix $u > 0$ and set $n = \lceil uN\rceil$ in (2.5) to obtain that

$$\{X_t \ge uN\} \;=\; \Big\{N^{-1}J_{-1,0}(t) \;\ge\; <\pi^N_t, \mathbf{1}\{[0,u]\}> \;+ O(N^{-1})\Big\} \,. \tag{2.6}$$

By Theorem 2.1.1, $<\pi^N_t, \mathbf{1}\{[0,u]\}>$ converges in probability to $\int_0^u \rho(t,w)dw$, where ρ is the solution of the heat equation (2.1).

On the other hand, the law of large numbers for $J_{-1,0}(t)$ under $\mathbb{P}_{\nu^N_{\rho_0(\cdot)}}$ is an elementary consequence of the central limit theorem proved in the last section and the convergence of the expectation of $N^{-1}J_{-1,0}(t)$. By the martingale decomposition of the current and by Theorem 2.5.1,

$$\begin{aligned}\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\big[N^{-1}J_{-1,0}(t)\big] &= \int_0^t ds\, N\big[\rho^N_s(-1)-\rho^N_s(0)\big] \\ &= -\int_0^t \partial_u\rho(s,0)ds \;+\; O(N^{-1})\,.\end{aligned}$$

Hence, $N^{-1}J_{-1,0}(t)$ converges in probability to $-\int_0^t \partial_u\rho(s,0)ds$.

In view of (2.6) and the law of large numbers for the current and the empirical measure,

$$\lim_{N\to\infty}\mathbb{P}_{\nu^{N,*}_{\rho_0(\cdot)}}\big[N^{-1}X_t g_\varepsilon u\big] \;=\; \begin{cases}0 & \text{if } -\int_0^t\partial_u\rho(s,0)ds < \int_0^u\rho(t,w)dw\,,\\ 1 & \text{if } -\int_0^t\partial_u\rho(s,0)ds > \int_0^u\rho(t,w)dw\,.\end{cases}$$

By symmetry around the origin, a similar statement holds for $u<0$. Thus, X^N_t/N converges to u_t in probability, where u_t is the solution of the implicit equation

$$\int_0^{u_t}\rho(t,w)dw \;=\; -\int_0^t\partial_u\rho_s(0)ds\,.$$

2.4 Central Limit Theorem for the Tagged Particle

In this section we prove Theorem 2.1.6 developing the ideas of the previous section. Assume first that $u_t>0$ and fix a in $\mathbb{R}$. By equation (2.5), the set $\{X_t g_\varepsilon N u_t + a\sqrt{N}\}$ is equal to the set in which

$$\overline{J}_{-1,0}(t) \ge \sum_{x=0}^{Nu_t}\overline{\eta}_t(x) \;+\; \sum_{x=1}^{a\sqrt{N}-1}\eta_t(x+Nu_t) \;-\; \Big\{\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}[J_{-1,0}(t)] - \sum_{x=0}^{Nu_t}\rho^N_t(x)]\Big\}\,, \quad (2.7)$$

where $\rho^N_t(x)$ is the solution of the discrete heat equation (2.2).

We claim that second term on the right hand side of (2.7) divided by $\sqrt{N}$ converges to its mean in L^2. Indeed, by Lemma 2.2.2, its variance is bounded by $C_0a^2N^{-1}$ for some finite constant C_0. Since by Theorem 2.5.1,

$$\frac{1}{\sqrt{N}}\sum_{x=1}^{a\sqrt{N}-1}\rho^N_t(x+Nu_t)$$

converges to $a\rho(t,u_t)$, the second term on the right hand side of (2.7) converges in probability to $a\rho(t,u_t)$.

An elementary computation based on the definition of u_t and on Theorem 2.5.1 shows that the third term on the right hand side of (2.7) divided by $\sqrt{N}$ vanishes as $N\uparrow\infty$.

Finally, by Proposition 2.2.1, for fixed t, $N^{-1/2}\{\overline{J}_{-1,0}(t) - \sum_{x=0}^{Nu_t}\overline{\eta}_t(x)\}$ behaves as $Y^N_t(G_n) - Y^N_0(G_n) - Y^N_t(\mathbf{1}\{[0,u_t]\})$, as $N\uparrow\infty$, $n\uparrow\infty$. Repeating the arguments presented at the beginning of the proof of Theorem 2.1.3, we show that this latter variable converges in law to a centered Gaussian variable, denoted by W_t, and which is formally equal to $Y_t(H_{u_t}) - Y_0(H_0)$.

Up to this point we proved that

$$\lim_{N\to\infty}\mathbb{P}_{\nu^N_{\rho_0(\cdot)}}\Big[\frac{X_t-u_tN}{\sqrt{N}}g_\varepsilon a\Big] \;=\; P[W_t g_\varepsilon a\rho(t,u_t)]$$

provided $u_t>0$. The same arguments permit to prove the same statement in the case $u_t=0$, $a>0$. By symmetry around the origin, we can recover the other cases: $u_t<0$

and a in $\mathbb{R}$, $u_t = 0$ and $a < 0$.

Putting all these facts together, we conclude that for each fixed t, $(X_t - Nu_t)/\sqrt{N}$ converges in distribution to the Gaussian $W_t/\rho(t, u_t) = [Y_t(H_{u_t}) - Y_0(H_0)]/\rho(t, u_t)$. The same arguments show that any vector $(N^{-1/2}[X_{t_1} - Nu_{t_1}], \dots, N^{-1/2}[X_{t_k} - Nu_{t_k}])$ converges to the corresponding centered Gaussian vector.

It remains to compute the covariances, which is long but elementary.

2.5 Finite approximations for the heat equation

In sake of completeness, we present in this section a result on the approximation of the heat equation by solutions of discrete heat equations.

Fix a profile $\rho_0 : \mathbb{R} \to \mathbb{R}$ with a bounded fourth derivative. Let $\rho : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be the solution of the heat equation with initial profile ρ_0:

$$\begin{cases} \partial_t \rho(t, x) = \partial_x^2 \rho(t, x) \\ \rho(0, x) = \rho_0(x) \ . \end{cases}$$

Recall that we denote by Δ_N the discrete Laplacian. For each $N \in \mathbb{N}$, define $\rho_t^N(x)$ as the solution of the system of ordinary differential equations

$$\begin{cases} (d/dt)\rho_t^N(x) = (\Delta_N \rho_t^N)(x) \\ \rho_0^N(x) = \rho_0(x/N) \ . \end{cases} \tag{2.8}$$

The main result of this section asserts that ρ^N approximates ρ up to order N^{-2}:

Theorem 2.5.1. *Assume that $\rho_0 : \mathbb{R} \to [0, 1]$ is a function with a bounded fourth derivative. There exists a finite constant C_0 such that*

$$\left| \rho_t^N(x) - \rho(t, \frac{x}{N}) \right| \ \le \ \frac{C_0 t}{N^2}$$

for all $N g_\varepsilon 1$, $t \ge 0$, $x \in \mathbb{Z}$.

An easy way to prove this statement is to introduce a time discrete approximation of the heat equation. For each N in $\mathbb{N}$ and each $\delta > 0$, we define $\rho_l^{\delta,N}(k)$, k in $\mathbb{Z}$, $l \ge 0$ by the recurrence formula

$$\begin{cases} \rho_{l+1}^{\delta,N}(k) = \rho_l^{\delta,N}(k) + \delta N^2 [\rho_l^{\delta,N}(k+1) + \rho_l^{\delta,N}(k-1) - 2\rho_l^{\delta,N}(k)] \\ \rho_0^{\delta,N}(k) = \rho_0(k/N) \ . \end{cases} \tag{2.9}$$

We now recall two well known propositions whose combination leads to the proof of Theorem 2.5.1. The first one states that the solution of (2.9) converges as $\delta \downarrow 0$ to the solution of (2.8) uniformly on compact sets. The second one furnishes a bound on the distance between the solution of the discrete equation (2.9) and the solution of the heat equation.

For a in $\mathbb{R}$, denote by $\lfloor a \rfloor$ the largest integer smaller or equal to a.

Proposition 2.5.2. *For each $N g_\varepsilon 1$,*

$$\lim_{\delta \to 0} \rho_{\lfloor t/\delta \rfloor}^{N,\delta}(k) \ = \ \rho_t^N(k)$$

uniformly on compacts of $\mathbb{R}_+ \times \mathbb{Z}$.

Proposition 2.5.3. *Suppose that $\delta N^2 < 1/2$. Then, there exist a finite constant $C_0 = C_0(\rho_0)$ such that*

$$\left| \rho_l^{\delta,N}(k) - \rho\Big(\delta l, k/N\Big) \right| \ \le \ C_0 \Big\{ \delta^2 l + \frac{\delta l}{N^2} \Big\}$$

for all $l \geq 0, k \in \mathbb{Z}$.

Clearly, Theorem 2.5.1 is an immediate consequence of Propositions 2.5.2 and 2.5.3. Proposition 2.5.2 is a consequence of Proposition 2.5.3 and the Cauchy-Peano existance theorem for ordinary differential equations. Proposition 2.5.3 is a standard result on numerical analysis (see [T] for instance).

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