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Dynamics of sets of zero density

Yuri Gomes Lima

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Orientador: Prof. Carlos Gustavo Moreira.
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To my beloved mom, Glaucia.

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“Ora (direis) ouvir estrelas! Certo
Perdeste o senso!” E eu vos direi, no entanto,
Que, para ouvi-las, muita vez desperto
E abro as janelas, pálido de espanto…

E conversamos toda a noite, enquanto
A via láctea, como um pálio aberto,
Cintila. E, ao vir do sol, saudoso e em pranto,
Inda as procuro pelo céu deserto.

Direis agora: “Tresloucado amigo!
Que conversas com elas? Que sentido
Tem o que dizem, quando estão contigo?”

E eu vos direi: “Amai para entendê-las!
Pois só quem ama pode ter ouvido
Capaz de ouvir e de entender estrelas”.

Via Láctea, de Olavo Bilac.

Abstract

This thesis is devoted to the study of sets of zero density, both continuous - inside $\mathbb{R}$ - and discrete - inside $\mathbb{Z}$. Most of the arguments are combinatorial in nature.

The interests are twofold. The first investigates arithmetic sums: given two subsets E, F of $\mathbb{R}$ or $\mathbb{Z}$, what can be said about $E + \lambda F$ for most of the parameters $\lambda \in \mathbb{R}$? The continuous case dates back to Marstrand. We give an alternative combinatorial proof of his theorem for the particular case of products of regular Cantor sets and, in the sequel, extend these techniques to give the proof of the general case.

We also discuss Marstrand's theorem in discrete spaces. More specifically, we propose a fractal dimension for subsets of the integers and establish a Marstrand type theorem in this context.

The second interest is ergodic theoretical. It contains constructions of $\mathbb{Z}^d$-actions with prescribed topological and ergodic properties, such as total minimality, total ergodicity and total strict ergodicity. These examples prove that Bourgain's polynomial pointwise ergodic theorem has not a topological version.

Resumo

O presente trabalho estuda conjuntos de densidade zero, tanto contínuos - subconjuntos de $\mathbb{R}$ - quanto discretos - subconjuntos de $\mathbb{Z}$. Os argumentos são, em sua maioria, de natureza combinatória.

Os interesses são divididos em duas partes. O primeira investiga somas aritméticas: dados subconjuntos E, F de $\mathbb{R}$ ou $\mathbb{Z}$, o que se pode inferir a respeito do conjunto $E + \lambda F$ para a maioria dos parâmetros $\lambda \in \mathbb{R}$? O caso contínuo tem origem nos trabalhos de Marstrand. Aqui, é dada uma prova alternativa desse teorema no caso em que o conjunto é produto de dois conjuntos de Cantor regulares e, posteriormente, as técnicas aplicadas são estendidas para a obtenção de uma prova do caso geral.

Ainda na primeira parte, discutimos o Teorema de Marstrand em espaços discretos. Mais especificamente, propomos uma dimensão fractal para subconjuntos de inteiros e provamos um resultado do tipo Marstrand nesse contexto. Uma grande quantidade de exemplos é discutida, além de contraexemplos para possíveis extensões do resultado.

O segundo interesse é ergódico. Construímos $\mathbb{Z}^d$-ações com propriedades ergódicas e topológicas prescritas, como totalmente minimal, totalmente ergódico e totalmente estritamente ergódico. Tais exemplos provam que não existe uma versão topológica para o teorema de Bourgain sobre a convergência pontual de médias ergódicas polinomiais.

List of Publications

- $\mathbb{Z}^d$*-actions with prescribed topological and ergodic properties.* To appear in Ergodic Theory & Dynamical Systems.
- *A combinatorial proof of Marstrand's Theorem for products of regular Cantor sets*, with Carlos Gustavo Moreira. To appear in Expositiones Mathematicae.
- *Yet another proof of Marstrand's theorem*, with Carlos Gustavo Moreira. To appear in Bulletin of the Brazilian Mathematical Society.
- *A Marstrand theorem for subsets of integers*, with Carlos Gustavo Moreira. Available at www.impa.br/∼yurilima.

Contents

Chapter 1

Brief description

This thesis is devoted to the study of sets of zero density, both continuous - inside $\mathbb{R}$ - and discrete - inside $\mathbb{Z}$. Most of the arguments are combinatorial in nature.

The interests are twofold. The first investigates arithmetic sums: given two subsets E, F of $\mathbb{R}$ or $\mathbb{Z}$, what can be said about $E + \lambda F$ for most of the parameters $\lambda \in \mathbb{R}$? The continuous case dates back to Marstrand, as we will describe in the next section. The discrete case is new [24]. The second interest is ergodic theoretical. Specifically, it is shown that Bourgain's polynomial pointwise ergodic theorem has not a topological version.

Each chapter originated an article and for this reason is self-contained. Chapters 2, 3, 4 were written together with one of my Ph.D. advisors, Carlos Gustavo Moreira (Gugu), and Chapter 5 by myself as I was a visiting scholar at The Ohio State University under the supervision of Vitaly Bergelson.

The present chapter describes the main goals of this work and gives a brief description of each of the next chapters.

1.1 Goals

Consider the following table.

	Continuous	Discrete
Positive density	✓	✓
Zero density	✓	?

For each box with the checkmark ✓, we represent a situation for which there is an active research and a solid theory has been developed, as we will describe in the sequel. The only unchecked box concerns discrete sets of zero density. These objects are of great importance, not only for intrisic reasons, but also because many of them are natural arithmetic sets that appear in diverse areas of Mathematics, from which we mention

- the prime numbers: even having zero density, B. Green and T. Tao [16] proved that the prime numbers are combinatorially rich in the sense that they contain infinitely long arithmetic progressions;
- J. Bourgain [6] has proved almost sure pointwise convergence of ergodic averages along polynomials.

The above results strongly rely on the particular combinatorial and analytical structure of each set. In this thesis, we obtain results in a more general point of view. Yet concerning the table above, let us contrast the continuous and discrete situations.

1.1.1 Largeness in $\mathbb{R}$ and $\mathbb{Z}$

Let m denote the Lebesgue measure of $\mathbb{R}$. We ask the following question: if $K \subset \mathbb{R}$ is large in the sense of m, does it inherit some structure from $\mathbb{R}$? A satisfactory answer is given by

Theorem 1.1.1 (Lebesgue differentiation)**.** *If $K \subset \mathbb{R}$ is a Borel set with $m(K) > 0$, then*

$$\lim_{\varepsilon \to 0^+} \frac{m(K \cap (x-\varepsilon, x+\varepsilon))}{2\varepsilon} = 1$$

for m-almost every $x \in K$.

In other words, copies of intervals of the real line "almost" appear in large sets of $\mathbb{R}$. Surprisingly, the same phenomenon holds for subsets of $\mathbb{Z}$. Given a subset $E \subset \mathbb{Z}$, let $d^*(E)$ denote the *upper-Banach density* of E, defined as

$$d^*(E) = \limsup_{N-M \to \infty} \frac{|E \cap \{M+1, \ldots, N\}|}{N-M} .$$

We thus have a discrete version of Theorem 1.1.1.

Theorem 1.1.2 (Szemerédi [45])**.** *If $E \subset \mathbb{Z}$ has positive upper Banach density, then E contains arbitrarily long arithmetic progressions.*

One can interpret this result by saying that density represents the correct notion of largeness needed to preserve finite configurations of $\mathbb{Z}$. These two situations just described are examples of *Ramsey's principle* which, roughly speaking, asserts that

If a substructure occupies a positive portion of a structure, then it must contain "copies" of the structure.

1.1.2 Zero density in $\mathbb{R}$ and $\mathbb{Z}$

However, theorems 1.1.1 and 1.1.2 can not infer anything if the density is zero, but these sets might posses rich combinatorial properties as well. For example, the ternary Cantor set

$$K = \left\{ \sum_{n \geq 1} a_i \cdot 3^{-i}\,;\, a_i = 0 \text{ or } 2 \right\}$$

satisfies $K + K = [0, 2]$ and naturally appears in the theory of homoclonic bifurcations on surfaces. Regarding the continuous case, one can investigate largeness of a set of zero density via its Hausdorff dimension. It is a classical result in Geometric Measure Theory the following

Theorem 1.1.3 (Marstrand [27])**.** *If $K \subset \mathbb{R}^2$ is a Borel set with Hausdorff dimension greater than one, then the projection of K into $\mathbb{R}$ in the direction of λ has positive Lebesgue measure for m-almost every $\lambda \in \mathbb{R}$.*

Informally speaking, this means that sets of Hausdorff dimension greater than one are "fat" in almost every direction. In particular, if $K = K_1 \times K_2$ is a cartesian product of two one-dimensional subsets of $\mathbb{R}$, Marstrand's theorem translates to "$m(K_1 + \lambda K_2) > 0$ for m-almost every $\lambda \in \mathbb{R}$".

Up to now, a theory of fractal sets in $\mathbb{Z}$ has not been developed. This constitutes the main goal of this thesis. We will investigate these sets from geometrical and ergodic points of view, according to the diagram below.

PART 1: GEOMETRY

$\boxed{\text{Marstrand's theorem for subsets of } \mathbb{Z}}$

$\uparrow$

$\boxed{\text{Marstrand's theorem in } \mathbb{R}\text{: new proof}}$

PART 2: ERGODIC

$\boxed{\text{Ergodic theorems along sets of zero density}}$

$\uparrow$

$\boxed{\mathbb{Z}^d\text{-actions with prescribed topological and ergodic properties}}$

Part 1 is the content of chapters 2, 3 and 4 and Part 2 of chapter 5. In order to obtain a Marstrand's theorem for subsets of $\mathbb{Z}$, we first establish a new proof of the classical Marstrand's theorem that can be adapted to the discrete case. Regarding Part 2, we prove that Bourgain's polynomial pointwise ergodic theorem has not a topological version by constructing $\mathbb{Z}^d$-actions with prescribed tolopogical and ergodic properties. In the next sections of this chapter, we will give a brief description of the ideas involved.

1.2 Chapter 2

Let m denote the Lebesgue measure of $\mathbb{R}$. It is a classical result in Geometric Measure Theory the following

Theorem 1.2.1 (Marstrand [27]). *If $K \subset \mathbb{R}^2$ is a Borel set with Hausdorff dimension greater than one, then the projection of K into $\mathbb{R}$ in the direction of λ has positive Lebesgue measure for m-almost every $\lambda \in \mathbb{R}$.*

In particular, if $K = K_1 \times K_2$ is a cartesian product of two one-dimensional subsets of $\mathbb{R}$, Marstrand's theorem translates to "$m(K_1 + \lambda K_2) > 0$ for m-almost every $\lambda \in \mathbb{R}$". The investigation of such *arithmetic sums* $K_1 + \lambda K_2$ has been an active area of Mathematics, in special when K_1 and K_2 are dynamically defined Cantor sets. Remarkably, M. Hall [18] proved, in 1947, that the Lagrange spectrum[1] contains a whole half line, by showing that the arithmetic sum $K(4) + K(4)$ of a certain Cantor set $K(4) \subset \mathbb{R}$ with itself contains $[6, \infty)$.

Marstrand's Theorem for product of Cantor sets is also fundamental in certain results of dynamical bifurcations, namely homoclinic bifurcations in surfaces. For instance, in [38] it is used to prove that hyperbolicity holds for most small positive values of parameter in homoclinic bifurcations associated to horseshoes with Hausdorff dimension smaller than one; in [39] it is used to show that hyperbolicity is not prevalent in homoclinic bifurcations associated to horseshoes with Hausdorff dimension larger than one.

Looking for some kind of converse to Marstrand's theorem, J. Palis conjectured in [36] that for generic pairs of regular Cantor sets (K_1, K_2) of the real line either $K_1 + K_2$ has zero measure or else it contains an interval. Observe that the first part is a consequence of Marstrand's theorem. In [33], it is proved that stable intersections of regular Cantor sets are dense in the region where the sum of their Hausdorff dimensions is larger than one, thus establishing the remaining of Palis conjecture. It is remarkable to point out that Marstrand's theorem is used in the proof.

[1]The Lagrange spectrum is the set of best constants of rational approximations of irrational numbers. See [9] for the specific description.

In the connection of these two applications, I would like to point out that a formula for the Hausdorff dimension of $K_1 + K_2$, under mild assumptions of non-linear Cantor sets K_1 and K_2, has been obtained by Gugu [31] and applied in [32] to prove that the Hausdorff dimension of the Lagrange spectrum increases continuously. In parallel to this non-linear setup, Y. Peres and P. Shmerkin proved the same phenomena happen to self-similar Cantor sets without algebraic resonance [41]. Finally, M. Hochman and P. Shmerkin extended and unified many results concerning projections of products of self-similar measures on regular Cantor sets [19].

The techniques used in [31] were based on estimates obtained in [33]. With these tools, Gugu and myself were able to develop in [25] a purely combinatorial proof of Marstrand's theorem for products of regular dynamically defined Cantor sets. Denoting by HD the Hausdorff dimension, the main theorem is

Theorem 1.2.2 (Lima-Moreira [25]). *If K_1, K_2 are regular dynamically defined Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, such that $d = \mathrm{HD}(K_1) + \mathrm{HD}(K_2) > 1$, then $m\,(K_1 + \lambda K_2) > 0$ for m-almost every $\lambda \in \mathbb{R}$.*

Let m_d denote the Hausdorff d-measure on $\mathbb{R}$. The idea of the proof relies on the fact that a regular Cantor set of Hausdorff dimension d is regular in the sense that the m_d-measure of arbitrarily small portions of it can be uniformly controlled. Once this is true, a double counting argument proves that the arithmetic sum has Lebesgue measure bounded away from zero at advanced stages of the construction of the Cantor sets, except for a small set of parameters. The good feature of the proof is that this discretization idea may be applied to the discrete context, as we shall see in Chapter 4.

The same idea of homogeneity is applied in [26] to prove the general case of Marstrand's theorem. Namely, once some upper regularity on the m_d-measure of K is assumed, it is possible to apply a weighted version of [25] (the covers of K may be composed of subsets with different diameter scales) to conclude the result.

1.3 Chapter 3

Chapter 3 extends the techniques developed in Chapter 2 and encloses a combinatorial proof of Marstrand's Theorem without any restrictions on K. The proof makes a study on the one-dimensional fibers of K in every direction and relies on two facts:
(I) Transversality condition: given two squares on the plane, the Lebesgue measure of the set of angles $\theta \in \mathbb{R}$ for which their projections in the direction of θ have nonempty intersection has an upper bound.
(II) After a regularization of K, (I) enables us to conclude that, except for a small set of angles $\theta \in \mathbb{R}$, the fibers in the direction of θ are not concentrated in a thin region. As a consequence,

K projects into a set of positive Lebesgue measure.

The idea of (II) is based on [31] and was employed in [26] to develop a combinatorial proof of Marstrand's Theorem when K is the product of two regular Cantor sets (which is the content of Chapter 2).

Compared to other proofs of Marstrand's Theorem, the new ideas here are the discretization of the argument and the use of dyadic covers, which allow the simplification of the method employed.

1.4 Chapter 4

Given a subset $E \subset \mathbb{Z}$, let $d^*(E)$ denote the *upper-Banach density* of E, defined as

$$d^*(E) = \limsup_{N-M\to\infty} \frac{|E \cap \{M+1,\dots,N\}|}{N-M} \cdot$$

Szemerédi's theorem asserts that if $d^*(E) > 0$, then E contains arbitrarily long arithmetic progressions. One can interpret this result by saying that density represents the correct notion of largeness needed to preserve finite configurations of $\mathbb{Z}$. On the other hand, Szemerédi's theorem cannot infer any property about subsets of zero density, and many of these are important. A class of examples are the integer polynomial sequences. The prime numbers are another example. These sets may, as well, contain combinatorially rich patterns.

A set $E \subset \mathbb{Z}$ of zero upper-Banach density is characterized as occupying portions in intervals of $\mathbb{Z}$ that grow in a sublinear fashion as the length of the intervals grow. On the other hand, there still may exist some kind of growth speed. For example, the number of perfect squares on an interval of length n is about $n^{0.5}$. This exponent means, in some sense, a dimension of $\{n^2\,;\, n \in \mathbb{Z}\}$ inside $\mathbb{Z}$. In general, define the *counting dimension* of $E \subset \mathbb{Z}$ as

$$D(E) = \limsup_{|I|\to\infty} \frac{\log|E \cap I|}{\log|I|}\ ,$$

where I runs over the intervals of $\mathbb{Z}$.

In [24], Gugu and myself investigated this dimension when one considers arithmetic sums of the form $E + \lfloor \lambda F \rfloor$, $\lambda \in \mathbb{R}$, for a class of subsets $E, F \subset \mathbb{Z}$. At first sight, by an ingenuous application of the multiplicative principle, one should expect that $E + \lfloor \lambda F \rfloor$ has dimension at least $D(E) + D(F)$. This is not true at all, due to two reasons:

(1) E and F may be very irregular as subsets of $\mathbb{Z}$.

(2) E and F may exhibit growth speed inside intervals of very different lengths.

This led to the definition of *regular compatible sets*: E is *regular* if

$$\frac{|E \cap I|}{|I|^{D(E)}} \lesssim 1,$$

where I runs over all intervals of $\mathbb{Z}$, and there exists a sequence of intervals $(I_n)_{n\geq 1}$ such that $|I_n| \to \infty$ and

$$\frac{|E \cap I_n|}{|I_n|^{D(E)}} \gtrsim 1. \tag{1.4.1}$$

Two regular subsets $E, F \subset \mathbb{Z}$ are *compatible* if there exist two sequences $(I_n)_{n\geq 1}$, $(J_n)_{n\geq 1}$ of intervals with increasing lengths satisfying (1.4.1) and such that $I_n \sim J_n$ (above, we used asymptotic Vinogradov notations $\lesssim$ and $\sim$). The main theorem of [24] is

Theorem 1.4.1 (Lima-Moreira [24])**.** *Let $E, F \subset \mathbb{Z}$ be two regular compatible sets. Then*

$$D(E + \lfloor \lambda F \rfloor) \geq \min\{1, D(E) + D(F)\}$$

for Lebesgue almost every $\lambda \in \mathbb{R}$. If in addition $D(E) + D(F) > 1$, then $E + \lfloor \lambda F \rfloor$ has positive upper-Banach density for Lebesgue almost every $\lambda \in \mathbb{R}$.

The above theorem has direct consequences when applied to integer polynomial sequences. For any $p(x) \in \mathbb{Z}[x]$ of degree d, the set $E = \{p(n)\,;\, n \in \mathbb{Z}\}$ is regular and has dimension $1/d$. Also, it is compatible with any other regular set. These observations imply the second result of the chapter.

Theorem 1.4.2 (Lima-Moreira [24])**.** *Let $p_i \in \mathbb{Z}[x]$ with degree d_i and let $E_i = \{p_i(n)\,;\, n \in \mathbb{Z}\}$, $i = 0, \ldots, k$. If*

$$\frac{1}{d_0} + \frac{1}{d_1} + \cdots + \frac{1}{d_k} > 1,$$

then the arithmetic sum $E_0 + \lfloor \lambda_1 E_1 \rfloor + \cdots + \lfloor \lambda_k E_k \rfloor$ has positive upper-Banach density for Lebesgue almost every $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$.

1.5 Chapter 5

Let T be a mesure-preserving transformation on the probability space $(X, \mathcal{B}, \mu)$ and $f : X \to \mathbb{R}$ a measurable function. A successful area in ergodic theory deals with the convergence of averages $n^{-1} \cdot \sum_{k=1}^{n} f\left(T^k x\right)$, $x \in X$, when n converges to infinity. The well known *Birkhoff's Theorem* states that such limit exists for almost every $x \in X$ whenever f is an L^1-function. Several results have been (and still are being) proved when, instead of $\{1, 2, \ldots, n\}$, average is made along other sequences of natural numbers. A remarkable result on this direction was given by J. Bourgain [6], where he proved that if $p(x)$ is a polynomial with integer coefficients and f is an

L^p-function, for some $p > 1$, then the averages $n^{-1} \cdot \sum_{k=1}^{n} f\left(T^{p(k)}x\right)$ converge for almost every $x \in X$. In other words, convergence fails to hold for a negligible set with respect to the measure μ. We mention this result is not true for L^1-functions, as recently proved by Z. Buczolich and D. Mauldin [7].

In [3], V. Bergelson asked if this set is also negligible from the topological point of view. It turned out, by a result of R. Pavlov [40], that this is not true. He proved that, for every sequence $(p_n)_{n\geq 1} \subset \mathbb{Z}$ of zero upper-Banach density, there exist a totally minimal, totally uniquely ergodic and topologically mixing transformation (X, T) and a continuous function $f : X \to \mathbb{R}$ such that $n^{-1} \cdot \sum_{k=1}^{n} f\left(T^{p_k}x\right)$ fails to converge for a residual set of $x \in X$. The conditions of minimality, unique ergodicity, although they seem artificial, are natural for the question posed by Bergelson, in order to avoid pathological counterexamples. The construction of Pavlov was inspired in results of Hahn and Katznelson [17].

In [23], I extended the results of Pavlov [40] to the setting of $\mathbb{Z}^d$-actions. The method consisted of constructing (totally) minimal and (totally) uniquely ergodic $\mathbb{Z}^d$-actions. The program was carried out by constructing closed shift invariant subsets of a sequence space. More specifically, a sequence of finite configurations $(\mathcal{C}_k)_{k\geq 1}$ of $\{0,1\}^{\mathbb{Z}^d}$ was built, where $\mathcal{C}_{k+1}$ is essentially formed by the concatenation of elements in $\mathcal{C}_k$ such that each of them occurs statistically well-behaved in each element of $\mathcal{C}_{k+1}$. Finally, the set of limits of shifted $\mathcal{C}_k$-configurations as $k \to \infty$ constitutes the shift invariant subset. The main results on [23] is

Theorem 1.5.1 (Lima [23]). *Given a set $P \subset \mathbb{Z}^d$ of zero upper-Banach density, there exist a totally strictly ergodic $\mathbb{Z}^d$-action $(X, \mathcal{A}, \mu, T)$ and a continuous function $f : X \to \mathbb{R}$ such that the ergodic averages*

$$\frac{1}{|P \cap (-n,n)^d|} \sum_{g \in P \cap (-n,n)^d} f\left(T^g x\right)$$

fail to converge for a residual set of $x \in X$.

Yet in the arithmetic setup, let $p_1, \ldots, p_d \in \mathbb{Z}[x]$. Bergelson and Leibman proved in [4] that if T is a minimal $\mathbb{Z}^d$-action, then there is a residual set $Y \subset X$ for which $x \in \overline{\{T^{(p_1(n),\ldots,p_d(n))}x\,;\, n \in \mathbb{Z}\}}$, for every $x \in Y$. The same method employed in Theorem 1.5.1 also gives the best topological result one can expect in this context.

Theorem 1.5.2 (Lima [23]). *Given a set $P \subset \mathbb{Z}^d$ of zero upper-Banach density, there exists a totally strictly ergodic $\mathbb{Z}^d$-action $(X, \mathcal{A}, \mu, T)$ and an uncountable set $Y \subset X$ for which $x \notin \overline{\{T^p x\,;\, p \in P\}}$, for every $x \in Y$.*

Chapter 2

A combinatorial proof of Marstrand's Theorem for products of regular Cantor sets

In a paper from 1954 Marstrand proved that if $K \subset \mathbb{R}^2$ has Hausdorff dimension greater than 1, then its one-dimensional projection has positive Lebesgue measure for almost-all directions. In this chapter, we give a combinatorial proof of this theorem when K is the product of regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, for which the sum of their Hausdorff dimension is greater than 1.

2.1 Introduction

If U is a subset of $\mathbb{R}^n$, the diameter of U is $|U| = \sup\{|x-y|; x, y \in U\}$ and, if $\mathcal{U}$ is a family of subsets of $\mathbb{R}^n$, the *diameter* of $\mathcal{U}$ is defined as

$$\|\mathcal{U}\| = \sup_{U \in \mathcal{U}} |U|.$$

Given $d > 0$, the *Hausdorff d-measure* of a set $K \subseteq \mathbb{R}^n$ is

$$m_d(K) = \lim_{\varepsilon \to 0} \left(\inf_{\substack{\mathcal{U} \text{ covers } K \\ \|\mathcal{U}\| < \varepsilon}} \sum_{U \in \mathcal{U}} |U|^d \right).$$

In particular, when $n = 1$, $m = m_1$ is the Lebesgue measure of Lebesgue measurable sets on $\mathbb{R}$. It is not difficult to show that there exists a unique $d_0 \geq 0$ for which $m_d(K) = \infty$ if $d < d_0$ and $m_d(K) = 0$ if $d > d_0$. We define the Hausdorff dimension of K as $\mathrm{HD}(K) = d_0$. Also, for each $\theta \in \mathbb{R}$, let $v_\theta = (\cos\theta, \sin\theta)$, L_θ the line in $\mathbb{R}^2$ through the origin containing v_θ

and $\text{proj}_\theta : \mathbb{R}^2 \to L_\theta$ the orthogonal projection. From now on, we'll restrict θ to the interval $[-\pi/2, \pi/2]$, because $L_\theta = L_{\theta+\pi}$.

In 1954, J. M. Marstrand [27] proved the following result on the fractal dimension of plane sets.

Theorem. *If $K \subseteq \mathbb{R}^2$ is a Borel set such that $\text{HD}(K) > 1$, then $m(\text{proj}_\theta(K)) > 0$ for m-almost every $\theta \in \mathbb{R}$.*

The proof is based on a qualitative characterization of the "bad" angles θ for which the result is not true. Specifically, Marstrand exhibits a Borel measurable function $f(x,\theta)$, $(x,\theta) \in \mathbb{R}^2 \times [-\pi/2, \pi/2]$, such that $f(x,\theta) = \infty$ for m_d-almost every $x \in K$, for every "bad" angle. In particular,

$$\int_K f(x,\theta) dm_d(x) = \infty. \tag{2.1.1}$$

On the other hand, using a version of Fubini's Theorem, he proves that

$$\int_{-\pi/2}^{\pi/2} d\theta \int_K f(x,\theta) dm_d(x) = 0$$

which, in view of (2.1.1), implies that

$$m\left(\{\theta \in [-\pi/2, \pi/2]\,;\, m(\text{proj}_\theta(K)) = 0\}\right) = 0.$$

These results are based on the analysis of rectangular densities of points.

Many generalizations and simpler proofs have appeared since. One of them appeared in 1968 by R. Kaufman who gave a very short proof of Marstrand's theorem using methods of potential theory. See [21] for his original proof and [29], [37] for further discussion.

In this chapter, we prove a particular case of Marstrand's Theorem.

Theorem 2.1.1. *If K_1, K_2 are regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, such that $d = \text{HD}(K_1) + \text{HD}(K_2) > 1$, then $m\left(\text{proj}_\theta(K_1 \times K_2)\right) > 0$ for m-almost every $\theta \in \mathbb{R}$.*

The argument also works to show that the push-forward measure of the restriction of m_d to $K_1 \times K_2$, defined as $\mu_\theta = (\text{proj}_\theta)_*(m_d|_{K_1 \times K_2})$, is absolutely continuous with respect to m, for m-almost every $\theta \in \mathbb{R}$. Denoting its Radon-Nykodim derivative by $\chi_\theta = d\mu_\theta / dm$, we also give yet another proof of the following result.

Theorem 2.1.2. *χ_θ is an L^2 function for m-almost every $\theta \in \mathbb{R}$.*

Our proof makes a study on the fibers $\text{proj}_\theta{}^{-1}(v) \cap (K_1 \times K_2)$, $(\theta, v) \in \mathbb{R} \times L_\theta$, and relies on two facts:

(I) A regular Cantor set of Hausdorff dimension d is regular in the sense that the m_d-measure of small portions of it has the same exponential behavior.

(II) This enables us to conclude that, except for a small set of angles $\theta \in \mathbb{R}$, the fibers ${\mathrm{proj}_\theta}^{-1}(v) \cap (K_1 \times K_2)$ are not concentrated in a thin region. As a consequence, $K_1 \times K_2$ projects into a set of positive Lebesgue measure.

The idea of (II) is based on [31]. It proves that, if K_1 and K_2 are regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, and at least one of them is non-essentially affine (a technical condition), then the arithmetic sum $K_1 + K_2 = \{x_1 + x_2; x_1 \in K_1, x_2 \in K_2\}$ has the expected Hausdorff dimension:

$$\mathrm{HD}(K_1 + K_2) = \min\{1, \mathrm{HD}(K_1) + \mathrm{HD}(K_2)\}.$$

Marstrand's Theorem for products of Cantor sets is a source of very relevant applications in dynamical systems, as pointed out in Chapter 1. It is of fundamental importance in certain results of dynamical bifurcations, namely homoclinic bifurcations. For instance, in [38] it is used to prove that hyperbolicity holds for most small positive values of parameter in homoclinic bifurcations associated to horseshoes with Hausdorff dimension smaller than one; in [39] it is used to show that hyperbolicity is not prevalent in homoclinic bifurcations associated to horseshoes with Hausdorff dimension larger than one; in [33] it is used to prove that stable intersections of regular Cantor sets are dense in the region where the sum of their Hausdorff dimensions is larger than one; in [34] to show that, for homoclinic bifurcations associated to horseshoes with Hausdorff dimension larger than one, typically there are open sets of parameters with positive Lebesgue density at the initial bifurcation parameter corresponding to persistent homoclinic tangencies.

2.2 Regular Cantor sets of class $C^{1+\alpha}$

We say that $K \subset \mathbb{R}$ is a *regular Cantor set of class* $C^{1+\alpha}$, $\alpha > 0$, if:

(i) there are disjoint compact intervals $I_1, I_2, \ldots, I_r \subseteq [0,1]$ such that $K \subset I_1 \cup \cdots \cup I_r$ and the boundary of each I_i is contained in K;

(ii) there is a $C^{1+\alpha}$ expanding map ψ defined in a neighbourhood of $I_1 \cup I_2 \cup \cdots \cup I_r$ such that $\psi(I_i)$ is the convex hull of a finite union of some intervals I_j, satisfying:

(ii.1) for each $i \in \{1, 2, \ldots, r\}$ and n sufficiently big, $\psi^n(K \cap I_i) = K$;

(ii.2) $K = \bigcap\limits_{n \in \mathbb{N}} \psi^{-n}(I_1 \cup I_2 \cup \cdots \cup I_r)$.

The set $\{I_1, \ldots, I_r\}$ is called a Markov partition of K. It defines an $r \times r$ matrix $B = (b_{ij})$ by

$$\begin{aligned} b_{ij} &= 1, \quad \text{if } \psi(I_i) \supseteq I_j \\ &= 0, \quad \text{if } \psi(I_i) \cap I_j = \emptyset, \end{aligned}$$

which encodes the combinatorial properties of K. Given such matrix, consider the set $\Sigma_B = \left\{\underline{\theta} = (\theta_1, \theta_2, \ldots) \in \{1, \ldots, r\}^{\mathbb{N}}\,;\, b_{\theta_i\theta_{i+1}} = 1, \forall\, i \geq 1\right\}$ and the shift transformation $\sigma : \Sigma_B \to \Sigma_B$ given by $\sigma(\theta_1, \theta_2, \ldots) = (\theta_2, \theta_3, \ldots)$.

There is a natural homeomorphism between the pairs (K, ψ) and (Σ_B, σ). For each finite word $\underline{a} = (a_1, \ldots, a_n)$ such that $b_{a_i a_{i+1}} = 1$, $i = 1, \ldots, n-1$, the intersection

$$I_{\underline{a}} = I_{a_1} \cap \psi^{-1}(I_{a_2}) \cap \cdots \cap \psi^{-(n-1)}(I_{a_n})$$

is a non-empty interval with diameter $|I_{\underline{a}}| = |I_{a_n}|/|(\psi^{n-1})'(x)|$ for some $x \in I_{\underline{a}}$, which is exponentially small if n is large. Then, $\{h(\underline{\theta})\} = \bigcap_{n \geq 1} I_{(\theta_1, \ldots, \theta_n)}$ defines a homeomorphism $h : \Sigma_B \to K$ that commutes the diagram

$$\begin{tikzcd} \Sigma_B \arrow[r, "\sigma"] \arrow[d, "h"'] & \Sigma_B \arrow[d, "h"] \\ K \arrow[r, "\psi"'] & K \end{tikzcd}$$

If $\lambda = \sup\{|\psi'(x)|; x \in I_1 \cup \cdots \cup I_r\} \in (1, \infty)$, then $\left|I_{(\theta_1, \ldots, \theta_{n+1})}\right| \geq \lambda^{-1} \cdot \left|I_{(\theta_1, \ldots, \theta_n)}\right|$ and so, for $\rho > 0$ small and $\underline{\theta} \in \Sigma_B$, there is a positive integer $n = n(\rho, \underline{\theta})$ such that

$$\rho \leq \left|I_{(\theta_1, \ldots, \theta_n)}\right| \leq \lambda\rho.$$

Definition 2.2.1. A ρ-decomposition of K is any finite set $(K)_\rho = \{I_1, I_2, \ldots, I_r\}$ of disjoint closed intervals of $\mathbb{R}$, each one of them intersecting K, whose union covers K and such that

$$\rho \leq |I_i| \leq \lambda\rho\,,\ i = 1, 2, \ldots, r.$$

Remark 2.2.2. Although ρ-decompositions are not unique, we use, for simplicity, the notation $(K)_\rho$ to denote any of them. We also use the same notation $(K)_\rho$ to denote the set $\bigcup_{I \in (K)_\rho} I \subset \mathbb{R}$ and the distinction between these two situations will be clear throughout the text.

Every regular Cantor set of class $C^{1+\alpha}$ has a ρ-decomposition for $\rho > 0$ small: by the compactness of K, the family $\left\{I_{\left(\theta_1, \ldots, \theta_{n(\rho, \underline{\theta})}\right)}\right\}_{\underline{\theta} \in \Sigma_B}$ has a finite cover (in fact, it is only necessary for ψ to be of class C^1). Also, one can define ρ-decomposition for the product of two Cantor sets K_1 and K_2, denoted by $(K_1 \times K_2)_\rho$. Given $\rho \neq \rho'$ and two decompositions $(K_1 \times K_2)_{\rho'}$ and $(K_1 \times K_2)_\rho$, consider the partial order

$$(K_1 \times K_2)_{\rho'} \prec (K_1 \times K_2)_\rho \iff \rho' < \rho \text{ and } \bigcup_{Q' \in (K_1 \times K_2)_{\rho'}} Q' \subseteq \bigcup_{Q \in (K_1 \times K_2)_\rho} Q.$$

In this case, $\text{proj}_\theta((K_1 \times K_2)_{\rho'}) \subseteq \text{proj}_\theta((K_1 \times K_2)_\rho)$ for any θ.

A remarkable property of regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, is bounded distortion.

Lemma 2.2.3. *Let (K, ψ) be a regular Cantor set of class $C^{1+\alpha}$, $\alpha > 0$, and $\{I_1, \dots, I_r\}$ a Markov partition. Given $\delta > 0$, there exists a constant $C(\delta) > 0$, decreasing on δ, with the following property: if $x, y \in K$ satisfy*

(i) $|\psi^n(x) - \psi^n(y)| < \delta$;

(ii) The interval $[\psi^i(x), \psi^i(y)]$ *is contained in* $I_1 \cup \dots \cup I_r$, *for* $i = 0, \dots, n-1$,

then

$$e^{-C(\delta)} \le \frac{|(\psi^n)'(x)|}{|(\psi^n)'(y)|} \le e^{C(\delta)}.$$

In addition, $C(\delta) \to 0$ *as* $\delta \to 0$.

A direct consequence of bounded distortion is the required regularity of K, contained in the next result.

Lemma 2.2.4. *Let K be a regular Cantor set of class $C^{1+\alpha}$, $\alpha > 0$, and let $d = \mathrm{HD}(K)$. Then $0 < m_d(K) < \infty$. Moreover, there is $c > 0$ such that, for any $x \in K$ and $0 \le r \le 1$,*

$$c^{-1} \cdot r^d \le m_d(K \cap B_r(x)) \le c \cdot r^d.$$

The same happens for products $K_1 \times K_2$ of Cantor sets (without loss of generality, considered with the box norm).

Lemma 2.2.5. *Let K_1, K_2 be regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, and let $d = \mathrm{HD}(K_1) + \mathrm{HD}(K_2)$. Then $0 < m_d(K_1 \times K_2) < \infty$. Moreover, there is $c_1 > 0$ such that, for any $x \in K_1 \times K_2$ and $0 \le r \le 1$,*

$${c_1}^{-1} \cdot r^d \le m_d\left((K_1 \times K_2) \cap B_r(x)\right) \le c_1 \cdot r^d.$$

See chapter 4 of [37] for the proofs of these lemmas. In particular, if $Q \in (K_1 \times K_2)_\rho$, there is $x \in (K_1 \cup K_2) \cap Q$ such that $B_{\lambda^{-1}\rho}(x) \subseteq Q \subseteq B_{\lambda\rho}(x)$ and so

$$\left(c_1 \lambda^d\right)^{-1} \cdot \rho^d \le m_d((K_1 \times K_2) \cap Q) \le c_1 \lambda^d \cdot \rho^d.$$

Changing c_1 by $c_1 \lambda^d$, we may also assume that

$${c_1}^{-1} \cdot \rho^d \le m_d\left((K_1 \times K_2) \cap Q\right) \le c_1 \cdot \rho^d,$$

which allows us to obtain estimates on the cardinality of ρ-decompositions.

Lemma 2.2.6. *Let K_1, K_2 be regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, and let $d = \mathrm{HD}(K_1) + \mathrm{HD}(K_2)$. Then there is $c_2 > 0$ such that, for any ρ-decomposition $(K_1 \times K_2)_\rho$, $x \in K_1 \times K_2$ and $0 \le r \le 1$,*

$$\#\left\{Q \in (K_1 \times K_2)_\rho; Q \subseteq B_r(x)\right\} \le c_2 \cdot \left(\frac{r}{\rho}\right)^d.$$

In addition, ${c_2}^{-1} \cdot \rho^{-d} \le \#(K_1 \times K_2)_\rho \le c_2 \cdot \rho^{-d}$.

Proof. We have

$$\begin{aligned} c_1 \cdot r^d &\geq m_d\left((K_1 \times K_2) \cap B_r(x)\right) \\ &\geq \sum_{Q \subseteq B_r(x)} m_d\left((K_1 \times K_2) \cap Q\right) \\ &\geq \sum_{Q \subseteq B_r(x)} {c_1}^{-1} \cdot \rho^d \\ &= \#\left\{Q \in (K_1 \times K_2)_\rho; Q \subseteq B_r(x)\right\} \cdot {c_1}^{-1} \cdot \rho^d \end{aligned}$$

and then

$$\#\left\{Q \in (K_1 \times K_2)_\rho; Q \subseteq B_r(x)\right\} \leq {c_1}^2 \cdot \left(\frac{r}{\rho}\right)^d.$$

On the other hand,

$$m_d(K_1 \times K_2) = \sum_{Q \in (K_1 \times K_2)_\rho} m_d\left((K_1 \times K_2) \cap Q\right) \leq \sum_{Q \in (K_1 \times K_2)_\rho} c_1 \cdot \rho^d,$$

implying that

$$\#(K_1 \times K_2)_\rho \geq {c_1}^{-1} \cdot m_d(K_1 \times K_2) \cdot \rho^{-d}.$$

Taking $c_2 = \max\{{c_1}^2\,, \, c_1/m_d(K_1 \times K_2)\}$, we conclude the proof. $\square$

2.3 Proof of Theorem 2.1.1

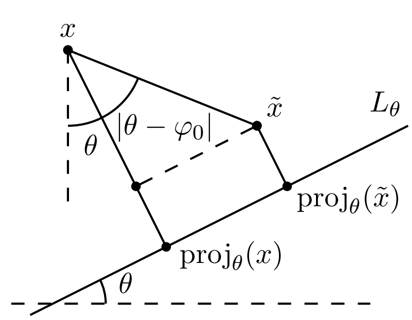
Given rectangles Q and $\tilde{Q}$, let

$$\Theta_{Q,\tilde{Q}} = \left\{\theta \in [-\pi/2, \pi/2]; \mathrm{proj}_\theta(Q) \cap \mathrm{proj}_\theta(\tilde{Q}) \neq \emptyset\right\}.$$

Lemma 2.3.1. *If $Q, \tilde{Q} \in (K_1 \times K_2)_\rho$ and $x \in (K_1 \times K_2) \cap Q, \tilde{x} \in (K_1 \times K_2) \cap \tilde{Q}$, then*

$$m\left(\Theta_{Q,\tilde{Q}}\right) \leq 2\pi\lambda \cdot \frac{\rho}{d(x,\tilde{x})}\cdot$$

Proof. Consider the figure.



Since $\mathrm{proj}_\theta(Q)$ has diameter at most $\lambda\rho$, $d(\mathrm{proj}_\theta(x),\mathrm{proj}_\theta(\tilde{x})) \le 2\lambda\rho$ and then, by elementary geometry,

$$\begin{aligned} \sin(|\theta-\varphi_0|) &= \frac{d(\mathrm{proj}_\theta(x),\mathrm{proj}_\theta(\tilde{x}))}{d(x,\tilde{x})} \\ &\le 2\lambda\cdot\frac{\rho}{d(x,\tilde{x})} \\ \Longrightarrow \qquad |\theta-\varphi_0| &\le \pi\lambda\cdot\frac{\rho}{d(x,\tilde{x})}\,, \end{aligned}$$

because $\sin^{-1} y \le \pi y/2$. As φ_0 is fixed, the lemma is proved. $\square$

We point out that, although simple, Lemma 2.3.1 expresses the crucial property of transversality that makes the proof work, and all results related to Marstrand's theorem use a similar idea in one way or another. See [43] where this tranversality condition is also exploited.

Fixed a ρ-decomposition $(K_1\times K_2)_\rho$, let

$$N_{(K_1\times K_2)_\rho}(\theta) = \#\left\{(Q,\tilde{Q})\in (K_1\times K_2)_\rho\times(K_1\times K_2)_\rho; \mathrm{proj}_\theta(Q)\cap\mathrm{proj}_\theta(\tilde{Q})\neq\emptyset\right\}$$

for each $\theta\in[-\pi/2,\pi/2]$ and

$$E((K_1\times K_2)_\rho) = \int_{-\pi/2}^{\pi/2} N_{(K_1\times K_2)_\rho}(\theta)d\theta.$$

Proposition 2.3.2. *Let K_1, K_2 be regular Cantor sets of class $C^{1+\alpha}$, $\alpha>0$, and let $d = \mathrm{HD}(K_1)+\mathrm{HD}(K_2)$. Then there is $c_3>0$ such that, for any ρ-decomposition $(K_1\times K_2)_\rho$,*

$$E((K_1\times K_2)_\rho)\le c_3\cdot\rho^{1-2d}.$$

Proof. Let $s_0 = \lceil \log_2\rho^{-1}\rceil$ and choose, for each $Q\in(K_1\times K_2)_\rho$, a point $x\in(K_1\times K_2)\cap Q$. By a double counting and using Lemmas 2.2.6 and 2.3.1, we have

$$\begin{aligned} E((K_1\times K_2)_\rho) &= \sum_{Q,\tilde{Q}\in(K_1\times K_2)_\rho} m\left(\Theta_{Q,\tilde{Q}}\right) \\ &= \sum_{s=1}^{s_0}\sum_{\substack{Q,\tilde{Q}\in(K_1\times K_2)_\rho \\ 2^{-s}<d(x,\tilde{x})\le 2^{-s+1}}} m\left(\Theta_{Q,\tilde{Q}}\right) \\ &\le \sum_{s=1}^{s_0} c_2\cdot\rho^{-d}\left[c_2\cdot\left(\frac{2^{-s+1}}{\rho}\right)^d\right]\cdot\left(2\pi\lambda\cdot\frac{\rho}{2^{-s}}\right) \\ &= 2^{d+1}\pi\lambda {c_2}^2\cdot\left(\sum_{s=1}^{s_0} 2^{s(1-d)}\right)\cdot\rho^{1-2d}. \end{aligned}$$

Because $d>1$, $c_3 = 2^{d+1}\pi\lambda {c_2}^2\cdot\sum_{s\ge1}2^{s(1-d)} < +\infty$ satisfies the required inequality. $\square$

This implies that, for each $\varepsilon > 0$, the upper bound

$$N_{(K_1\times K_2)_\rho}(\theta) \le \frac{c_3\cdot\rho^{1-2d}}{\varepsilon} \tag{2.3.1}$$

holds for every θ except for a set of measure at most ε. Letting $c_4 = {c_2}^{-2}\cdot{c_3}^{-1}$, we will show that

$$m\left(\mathrm{proj}_\theta\left((K_1\times K_2)_\rho\right)\right) \ge c_4\cdot\varepsilon \tag{2.3.2}$$

for every θ satisfying (2.3.1). For this, divide $[-2,2]\subseteq L_\theta$ in $\lfloor 4/\rho\rfloor$ intervals $J_1^\rho,\ldots,\ J_{\lfloor 4/\rho\rfloor}^\rho$ of equal lenght (at least ρ) and define

$$s_{\rho,i} = \#\left\{Q\in(K_1\times K_2)_\rho\,;\,\mathrm{proj}_\theta(x)\in J_i^\rho\right\},\quad i=1,\ldots,\lfloor 4/\rho\rfloor\,.$$

Then $\sum_{i=1}^{\lfloor 4/\rho\rfloor} s_{\rho,i} = \#(K_1\times K_2)_\rho$ and

$$\sum_{i=1}^{\lfloor 4/\rho\rfloor}{s_{\rho,i}}^2 \le N_{(K_1\times K_2)_\rho}(\theta) \le c_3\cdot\rho^{1-2d}\cdot\varepsilon^{-1}.$$

Let $S_\rho = \{1\le i\le\lfloor 4/\rho\rfloor\,;\,s_{\rho,i}>0\}$. By Cauchy-Schwarz inequality,

$$\#S_\rho \ge \frac{\left(\displaystyle\sum_{i\in S_\rho} s_{\rho,i}\right)^2}{\displaystyle\sum_{i\in S_\rho}{s_{\rho,i}}^2} \ge \frac{{c_2}^{-2}\cdot\rho^{-2d}}{c_3\cdot\rho^{1-2d}\cdot\varepsilon^{-1}} = \frac{c_4\cdot\varepsilon}{\rho}\,\cdot$$

For each $i\in S_\rho$, the interval J_i^ρ is contained in $\mathrm{proj}_\theta((K_1\times K_2)_\rho)$ and then

$$m\left(\mathrm{proj}_\theta((K_1\times K_2)_\rho)\right) \ge c_4\cdot\varepsilon,$$

which proves (2.3.2).

Proof of Theorem 2.1.1. Fix a decreasing sequence

$$(K_1\times K_2)_{\rho_1} \succ (K_1\times K_2)_{\rho_2} \succ \cdots \tag{2.3.3}$$

of decompositions such that $\rho_n\to 0$ and, for each $\varepsilon>0$, consider the sets

$$G_\varepsilon^n = \left\{\theta\in[-\pi/2,\pi/2]\,;\,N_{(K_1\times K_2)_{\rho_n}}(\theta)\le c_3\cdot{\rho_n}^{1-2d}\cdot\varepsilon^{-1}\right\},\quad n\ge 1.$$

Then $m\left([-\pi/2,\pi/2]\backslash G_\varepsilon^n\right)\le\varepsilon$, and the same holds for the set

$$G_\varepsilon = \bigcap_{n\ge 1}\bigcup_{l=n}^{\infty} G_\varepsilon^l\,.$$

If $\theta\in G_\varepsilon$, then

$$m\left(\mathrm{proj}_\theta((K_1\times K_2)_{\rho_n})\right)\ge c_4\cdot\varepsilon\,,\quad\text{for infinitely many } n,$$

which implies that $m\left(\mathrm{proj}_\theta(K_1\times K_2)\right)\ge c_4\cdot\varepsilon$. Finally, the set $G=\bigcup_{n\ge 1}G_{1/n}$ satisfies $m([-\pi/2,\pi/2]\backslash G)=0$ and $m\left(\mathrm{proj}_\theta(K_1\times K_2)\right)>0$, for any $\theta\in G$. □

2.4 Proof of Theorem 2.1.2

Given any $X \subset K_1 \times K_2$, let $(X)_\rho$ be the restriction of the ρ-decomposition $(K_1 \times K_2)_\rho$ to those rectangles which intersect X. As done in Section 2.3, we'll obtain estimates on the cardinality of $(X)_\rho$. Being a subset of $K_1 \times K_2$, the upper estimates from Lemma 2.2.6 also hold for X. The lower estimate is given by

Lemma 2.4.1. *Let X be a subset of $K_1 \times K_2$ such that $m_d(X) > 0$. Then there is $c_6 = c_6(X) > 0$ such that, for any ρ-decomposition $(K_1 \times K_2)_\rho$ and $0 \leq r \leq 1$,*

$$c_6 \cdot \rho^{-d} \leq \#(X)_\rho \leq c_2 \cdot \rho^{-d}.$$

Proof. As $m_d(X) < \infty$, there exists $c_5 = c_5(X) > 0$ (see Theorem 5.6 of [11]) such that

$$m_d\left(X \cap B_r(x)\right) \leq c_5 \cdot r^d\,, \quad \text{for all } x \in X \text{ and } 0 \leq r \leq 1,$$

and then

$$m_d(X) = \sum_{Q \in (X)_\rho} m_d\left(X \cap Q\right) \leq \sum_{Q \in (X)_\rho} c_5 \cdot (\lambda\rho)^d = \left(c_5 \cdot \lambda^d\right) \cdot \rho^d \cdot \#(X)_\rho\,.$$

Just take $c_6 = {c_5}^{-1} \cdot \lambda^{-d} \cdot m_d(X)$. $\square$

Proposition 2.4.2. *The measure $\mu_\theta = (\mathrm{proj}_\theta)_*(m_d|_{K_1 \times K_2})$ is absolutely continuous with respect to m, for m-almost every $\theta \in \mathbb{R}$.*

Proof. Note that the implication

$$X \subset K_1 \times K_2\,,\ m_d(X) > 0 \quad \Longrightarrow \quad m(\mathrm{proj}_\theta(X)) > 0 \tag{2.4.1}$$

is sufficient for the required absolute continuity. In fact, if $Y \subset L_\theta$ satisfies $m(Y) = 0$, then

$$\mu_\theta(Y) = m_d(X) = 0\,,$$

where $X = {\mathrm{proj}_\theta}^{-1}(Y)$. Otherwise, by (2.4.1) we would have $m(Y) = m(\mathrm{proj}_\theta(X)) > 0$, contradicting the assumption.

We prove that (2.4.1) holds for every $\theta \in G$, where G is the set defined in the proof of Theorem 2.1.1. The argument is the same made after Proposition 2.3.2: as, by the previous lemma, $\#(X)_\rho$ has lower and upper estimates depending only on X and ρ, we obtain that

$$m\left(\mathrm{proj}_\theta((X)_{\rho_n})\right) \geq {c_3}^{-1} \cdot {c_6}^2 \cdot \varepsilon\,, \quad \text{for infinitely many } n,$$

and then $m(\mathrm{proj}_\theta(X)) > 0$. $\square$

Let $\chi_\theta = d\mu_\theta/dm$. In principle, this is a L^1 function. We prove that it is a L^2 function, for every $\theta \in G$.

Proof of Theorem 2. Let $\theta \in G_{1/m}$, for some $m \in \mathbb{N}$. Then

$$N_{(K_1\times K_2)_{\rho_n}}(\theta) \le c_3 \cdot {\rho_n}^{1-2d} \cdot m\,, \text{ for infinitely many } n. \tag{2.4.2}$$

For each of these n, consider the partition $\mathcal{P}_n = \{J_1^{\rho_n}, \ldots, J_{\lfloor 4/\rho_n \rfloor}^{\rho_n}\}$ of $[-2,2] \subset L_\theta$ into intervals of equal length and let $\chi_{\theta,n}$ be the expectation of χ_θ with respect to $\mathcal{P}_n$. As $\rho_n \to 0$, the sequence of functions $(\chi_{\theta,n})_{n\in\mathbb{N}}$ converges pointwise to χ_θ. By Fatou's Lemma, we're done if we prove that each $\chi_{\theta,n}$ is L^2 and its L^2-norm $\|\chi_{\theta,n}\|_2$ is bounded above by a constant independent of n.

By definition,

$$\mu_\theta(J_i^{\rho_n}) = m_d\left((\mathrm{proj}_\theta)^{-1}\left(J_i^{\rho_n}\right)\right) \le s_{\rho_n,i} \cdot c_1 \cdot {\rho_n}^d, \quad i = 1, 2, \ldots, \lfloor 4/\rho_n \rfloor,$$

and then

$$\chi_{\theta,n}(x) = \frac{\mu_\theta(J_i^{\rho_n})}{|J_i^{\rho_n}|} \le \frac{c_1 \cdot s_{\rho_n,i} \cdot {\rho_n}^d}{|J_i^{\rho_n}|}\ , \quad \forall\, x \in J_i^{\rho_n},$$

implying that

$$\begin{aligned}
\|\chi_{\theta,n}\|_2^2 &= \int_{L_\theta} |\chi_{\theta,n}|^2\, dm \\
&= \sum_{i=1}^{\lfloor 4/\rho_n \rfloor} \int_{J_i^{\rho_n}} |\chi_{\theta,n}|^2\, dm \\
&\le \sum_{i=1}^{\lfloor 4/\rho_n \rfloor} |J_i^{\rho_n}| \cdot \left(\frac{c_1 \cdot s_{\rho_n,i} \cdot {\rho_n}^d}{|J_i^{\rho_n}|}\right)^2 \\
&\le {c_1}^2 \cdot {\rho_n}^{2d-1} \cdot \sum_{i=1}^{\lfloor 4/\rho_n \rfloor} {s_{\rho_n,i}}^2 \\
&\le {c_1}^2 \cdot {\rho_n}^{2d-1} \cdot N_{(K_1\times K_2)_{\rho_n}}(\theta).
\end{aligned}$$

In view of (2.4.2), this last expression is bounded above by

$$\left({c_1}^2 \cdot {\rho_n}^{2d-1}\right) \cdot \left(c_3 \cdot {\rho_n}^{1-2d} \cdot m\right) = {c_1}^2 \cdot c_3 \cdot m\,,$$

which is a constant independent of n. □

2.5 Concluding remarks

The proofs of Theorems 2.1.1 and 2.1.2 work not just for the case of products of regular Cantor sets, but in greater generality, whenever $K \subset \mathbb{R}^2$ is a Borel set for which there is a constant $c > 0$ such that, for any $x \in K$ and $0 \leq r \leq 1$,

$$c^{-1} \cdot r^d \leq m_d(K \cap B_r(x)) \leq c \cdot r^d,$$

since this alone implies the existence of ρ-decompositions for K.

Chapter 3

Yet another proof of Marstrand's Theorem

In a paper from 1954 Marstrand proved that if $K \subset \mathbb{R}^2$ is a Borel set with Hausdorff dimension greater than 1, then its one-dimensional projection has positive Lebesgue measure for almost-all directions. In the present chapter, we give a combinatorial proof of this theorem, extending the techniques developed in Chapter 2. It corresponds to a joint paper with C.G. Moreira [26] to appear in Bulletin of the Brazilian Mathematical Society.

3.1 Introduction

If U is a subset of $\mathbb{R}^n$, the diameter of U is $|U| = \sup\{|x-y|; x, y \in U\}$ and, if $\mathcal{U}$ is a family of subsets of $\mathbb{R}^n$, the *diameter* of $\mathcal{U}$ is defined as

$$\|\mathcal{U}\| = \sup_{U \in \mathcal{U}} |U|.$$

Given $s > 0$, the *Hausdorff s-measure* of a set $K \subset \mathbb{R}^n$ is

$$m_s(K) = \lim_{\varepsilon \to 0} \left(\inf_{\substack{\mathcal{U} \text{ covers } K \\ \|\mathcal{U}\| < \varepsilon}} \sum_{U \in \mathcal{U}} |U|^s \right).$$

In particular, when $n = 1$, $m = m_1$ is the Lebesgue measure of Lebesgue measurable sets on $\mathbb{R}$. It is not difficult to show that there exists a unique $s_0 \geq 0$ for which $m_s(K) = \infty$ if $s < s_0$ and $m_s(K) = 0$ if $s > s_0$. We define the Hausdorff dimension of K as $\mathrm{HD}(K) = s_0$. Also, for each $\theta \in \mathbb{R}$, let $v_\theta = (\cos\theta, \sin\theta)$, L_θ the line in $\mathbb{R}^2$ through the origin containing v_θ and $\mathrm{proj}_\theta : \mathbb{R}^2 \to L_\theta$ the orthogonal projection. From now on, we'll restrict θ to the interval $[-\pi/2, \pi/2]$, because $L_\theta = L_{\theta+\pi}$.

In 1954, J. M. Marstrand [27] proved the following result on the fractal dimension of plane sets.

Theorem 3.1.1. *If $K \subset \mathbb{R}^2$ is a Borel set such that* $\mathrm{HD}(K) > 1$*, then* $m(\mathrm{proj}_\theta(K)) > 0$ *for m-almost every $\theta \in \mathbb{R}$.*

The proof is based on a qualitative characterization of the "bad" angles θ for which the result is not true. Specifically, Marstrand exhibits a Borel measurable function $f(x, \theta)$, $(x, \theta) \in \mathbb{R}^2 \times [-\pi/2, \pi/2]$, such that $f(x, \theta) = \infty$ for m_s-almost every $x \in K$, for every "bad" angle. In particular,

$$\int_K f(x, \theta) dm_s(x) = \infty. \tag{3.1.1}$$

On the other hand, using a version of Fubini's Theorem, he proves that

$$\int_{-\pi/2}^{\pi/2} d\theta \int_K f(x, \theta) dm_s(x) = 0$$

which, in view of (3.1.1), implies that

$$m\left(\{\theta \in [-\pi/2, \pi/2]\,;\, m(\mathrm{proj}_\theta(K)) = 0\}\right) = 0.$$

These results are based on the analysis of rectangular densities of points.

Many generalizations and simpler proofs have appeared since. One of them came in 1968 by R. Kaufman who gave a very short proof of Marstrand's Theorem using methods of potential theory. See [21] for his original proof and [29], [37] for further discussion.

In this chapter, we give a new proof of Theorem 3.1.1. Our proof makes a study on the fibers $K \cap {\mathrm{proj}_\theta}^{-1}(v)$, $(\theta, v) \in \mathbb{R} \times L_\theta$, and relies on two facts:

(I) Transversality condition: given two squares on the plane, the Lebesgue measure of the set of angles for which their projections have nonempty intersection has an upper bound. See Subsection 3.3.2.

(II) After a regularization of K, (I) enables us to conclude that, except for a small set of angles $\theta \in \mathbb{R}$, the fibers $K \cap {\mathrm{proj}_\theta}^{-1}(v)$ are not concentrated in a thin region. As a consequence, K projects into a set of positive Lebesgue measure.

The idea of (II) is based on [31] used in [25] to develop a combinatorial proof of Theorem 3.1.1 when K is the product of two regular Cantor sets. In the present chapter, we give a combinatorial proof of Theorem 3.1.1 without any restrictions on K. Compared to other proofs of Marstrand's Theorem, the new ideas here are the discretization of the argument and the use of dyadic covers, which allow the simplification of the method employed. These covers may be composed of sets with rather different scales and so a weighted sum is necessary to capture the Hausdorff s-measure of K.

The theory developed in [26] works whenever K is an *Ahlfors-David regular set*, namely when there are constants $a, b > 0$ such that

$$a \cdot r^d \leq m_s(K \cap B_r(x)) \leq b \cdot r^d\,, \quad \text{for any } x \in K \text{ and } 0 < r \leq 1.$$

Unfortunately, the general situation can not be reduced to this one, as proved by P. Mattila and P. Saaranen: in [30], they constructed a compact set of $\mathbb{R}$ with positive Lebesgue measure such that it contains no nonempty Ahlfors-David subset.

We also give yet another proof that the push-forward measure of the restriction of m_s to K, defined as $\mu_\theta = (\mathrm{proj}_\theta)_*(m_s|_K)$, is absolutely continuous with respect to m, for m-almost every $\theta \in \mathbb{R}$, and its Radon-Nykodim derivative is square-integrable.

Theorem 3.1.2. *The measure μ_θ is absolutely continuous with respect to m and its Radon-Nykodim derivative is an L^2 function, for m-almost every $\theta \in \mathbb{R}$.*

Marstrand's Theorem is a classical result in Geometric Measure Theory. In particular, if $K = K_1 \times K_2$ is a cartesian product of two one-dimensional subsets of $\mathbb{R}$, Marstrand's theorem translates to "$m(K_1 + \lambda K_2) > 0$ for m-almost every $\lambda \in \mathbb{R}$". The investigation of such *arithmetic sums* $K_1 + \lambda K_2$ has been an active area of Mathematics, in special when K_1 and K_2 are dynamically defined Cantor sets. Remarkably, M. Hall [18] proved, in 1947, that the Lagrange spectrum[1] contains a whole half line, by showing that the arithmetic sum $K(4)+K(4)$ of a certain Cantor set $K(4) \subset \mathbb{R}$ with itself contains $[6, \infty)$.

In the connection of these two applications, we point out that a formula for the Hausdorff dimension of $K_1 + K_2$, under mild assumptions of non-linear Cantor sets K_1 and K_2, has been obtained by the second author in [31] and applied in [32] to prove that the Hausdorff dimension of the Lagrange spectrum increases continuously. In parallel to this non-linear setup, Y. Peres and P. Shmerkin proved the same phenomena happen to self-similar Cantor sets without algebraic resonance [41]. Finally, M. Hochman and P. Shmerkin extended and unified many results concerning projections of products of self-similar measures on regular Cantor sets [19].

The content of this chapter is organized as follows. In Section 3.2 we introduce the basic notations and definitions. Section 3.3 is devoted to the main calculations, including the transversality condition in Subsection 3.3.2 and the proof of existence of good dyadic covers in Subsection 3.3.3. Finally, in Section 3.4 we prove Theorems 3.1.1 and 3.1.2. We also collect final remarks in Section 3.5.

[1]The Lagrange spectrum is the set of best constants of rational approximations of irrational numbers. See [9] for the specific description.

3.2 Preliminaries

3.2.1 Notation

The distance in $\mathbb{R}^2$ will be denoted by $|\cdot|$. Let $B_r(x)$ denote the open ball of $\mathbb{R}^2$ centered in x with radius r. As in Section 3.1, the diameter of $U \subset \mathbb{R}^2$ is $|U| = \sup\{|x-y|; x, y \in U\}$ and, if $\mathcal{U}$ is a family of subsets of $\mathbb{R}^2$, the diameter of $\mathcal{U}$ is defined as

$$\|\mathcal{U}\| = \sup_{U \in \mathcal{U}} |U|.$$

Given $s > 0$, the Hausdorff s-measure of a set $K \subset \mathbb{R}^2$ is $m_s(K)$ and its Hausdorff dimension is $\mathrm{HD}(K)$. In this work, we assume K is contained in $[0,1)^2$.

Definition 3.2.1. A Borel set $K \subset \mathbb{R}^2$ is an *s-set* if $\mathrm{HD}(K) = s$ and $0 < m_s(K) < \infty$.

Let m be the Lebesgue measure of Lebesgue measurable sets on $\mathbb{R}$. For each $\theta \in \mathbb{R}$, let $v_\theta = (\cos\theta, \sin\theta)$, L_θ the line in $\mathbb{R}^2$ through the origin containing v_θ and $\mathrm{proj}_\theta : \mathbb{R}^2 \to L_\theta$ the orthogonal projection onto L_θ.

A square $[a, a+l) \times [b, b+l) \subset \mathbb{R}^2$ will be denoted by Q and its center, the point $(a+l/2, b+l/2)$, by x.

Let X be a set. We use the following notation to compare the asymptotic of functions.

Definition 3.2.2. Let $f, g : X \to \mathbb{R}$ be two real-valued functions. We say $f \lesssim g$ if there is a constant $C > 0$ such that

$$|f(x)| \leq C \cdot |g(x)|, \quad \forall\, x \in X.$$

If $f \lesssim g$ and $g \lesssim f$, we write $f \sim g$.

3.2.2 Dyadic squares

Let $\mathcal{D}_0$ be the family of unity squares of $\mathbb{R}^2$ congruent to $[0,1)^2$ and with vertices in the lattice $\mathbb{Z}^2$. Dilating this family by a factor of 2^{-i}, we obtain the family $\mathcal{D}_i$, $i \in \mathbb{Z}$.

Definition 3.2.3. Let $\mathcal{D}$ denote the union of $\mathcal{D}_i$, $i \in \mathbb{Z}$. A *dyadic square* is any element $Q \in \mathcal{D}$.

The dyadic squares possess the following properties:

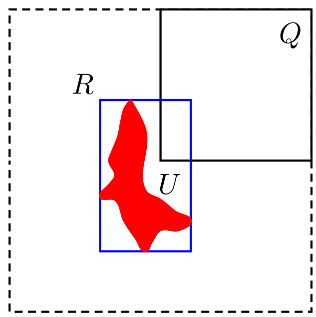
(1) Every $x \in \mathbb{R}^2$ belongs to exactly one element of each family $\mathcal{D}_i$.

(2) Two dyadic squares are either disjoint or one is contained in the other.

(3) A dyadic square of $\mathcal{D}_i$ is contained in exactly one dyadic square of $\mathcal{D}_{i-1}$ and contains exactly four dyadic squares of $\mathcal{D}_{i+1}$.

(4) Given any subset $U \subset \mathbb{R}^2$, there are four dyadic squares, each with side length at most $2 \cdot |U|$, whose union contains U.

(1) to (3) are direct. To prove (4), let R be smallest rectangle of $\mathbb{R}^2$ with sides parallel to the axis that contains $\overline{U}$. The sides of R have length at most $|U|$. Let $i \in \mathbb{Z}$ such that $2^{-i-1} < |U| \leq 2^{-i}$ and choose a dyadic square $Q \in \mathcal{D}_i$ that intersects R. If Q contains U, we're done. If not, Q and three of its neighbors cover U.



Definition 3.2.4. A *dyadic cover* of K is a finite subset $\mathcal{C} \subset \mathcal{D}$ of disjoint dyadic squares such that

$$K \subset \bigcup_{Q \in \mathcal{C}} Q.$$

First used by A.S. Besicovitch in his demonstration that closed sets of infinite m_s-measure contain subsets of positive but finite measure [5], dyadic covers were later employed by Marstrand to investigate the Hausdorff measure of cartesian products of sets [28].

Due to (4), for any family $\mathcal{U}$ of subsets of $\mathbb{R}^2$, there is a dyadic family $\mathcal{C}$ such that

$$\bigcup_{Q \in \mathcal{C}} Q \supset \bigcup_{U \in \mathcal{U}} U \quad \text{and} \quad \sum_{Q \in \mathcal{C}} |Q|^s < 64 \cdot \sum_{U \in \mathcal{U}} |U|^s$$

and so, if K is an s-set, there exists a sequence $(\mathcal{C}_i)_{i \geq 1}$ of dyadic covers of K with diameters converging to zero such that

$$\sum_{Q \in \mathcal{C}_i} |Q|^s \sim 1\,. \tag{3.2.1}$$

3.3 Calculations

Let $K \subset \mathbb{R}^2$ be a Borel set with Hausdorff dimension greater than one. From now on, we assume every cover of K is composed of dyadic squares of sides at most one. Before going into the calculations, we make the following reduction.

Lemma 3.3.1. *Let K be a Borel subset of $\mathbb{R}^2$. Given $s < \mathrm{HD}(K)$, there exists a compact s-set $K' \subset K$ such that*

$$m_s(K' \cap B_r(x)) \lesssim r^d\,, \quad x \in \mathbb{R}^2 \text{ and } 0 \le r \le 1.$$

In other words, there exists a constant $b > 0$ such that

$$m_s(K' \cap B_r(x)) \le b \cdot r^d\,, \quad \text{for any } x \in \mathbb{R}^2 \text{ and } 0 \le r \le 1. \tag{3.3.1}$$

See Theorem 5.4 of [11] for a proof of the above lemma when K is closed and [10] for the general case. From now on, we assume K is a compact s-set, with $s > 1$, that satisfies (3.3.1).

Given a dyadic cover $\mathcal{C}$ of K, let, for each $\theta \in [-\pi/2, \pi/2]$, $f_\theta^{\mathcal{C}} : L_\theta \to \mathbb{R}$ be the function defined by

$$f_\theta^{\mathcal{C}}(x) = \sum_{Q \in \mathcal{C}} \chi_{\mathrm{proj}_\theta(Q)}(x) \cdot |Q|^{s-1}\,,$$

where $\chi_{\mathrm{proj}_\theta(Q)}$ denotes the characteristic function of the set $\mathrm{proj}_\theta(Q)$. The reason we consider this function is that it captures the Hausdorff s-measure of K in the sense that

$$\begin{aligned} \int_{L_\theta} f_\theta^{\mathcal{C}}(x)dm(x) &= \sum_{Q \in \mathcal{C}} |Q|^{s-1} \cdot \int_{L_\theta} \chi_{\mathrm{proj}_\theta(Q)}(x)dm(x) \\ &= \sum_{Q \in \mathcal{C}} |Q|^{s-1} \cdot m(\mathrm{proj}_\theta(Q)) \end{aligned}$$

which, as $|Q|/2 \le m(\mathrm{proj}_\theta(Q)) \le |Q|$, satisfies

$$\int_{L_\theta} f_\theta^{\mathcal{C}}(x)dm(x) \sim \sum_{Q \in \mathcal{C}} |Q|^s.$$

If in addition $\mathcal{C}$ satisfies (3.2.1), then

$$\int_{L_\theta} f_\theta^{\mathcal{C}}(x)dm(x) \sim 1\ , \quad \forall\, \theta \in [-\pi/2, \pi/2]. \tag{3.3.2}$$

Denoting the union $\bigcup_{Q \in \mathcal{C}} Q$ by $\mathcal{C}$ as well, an application of the Cauchy-Schwarz inequality gives that

$$m(\mathrm{proj}_\theta(\mathcal{C})) \cdot \left(\int_{\mathrm{proj}_\theta(\mathcal{C})} \left(f_\theta^{\mathcal{C}}\right)^2 dm \right) \ge \left(\int_{\mathrm{proj}_\theta(\mathcal{C})} f_\theta^{\mathcal{C}} dm \right)^2 \sim 1.$$

The above inequality implies that if $(\mathcal{C}_i)_{i \ge 1}$ is a sequence of dyadic covers of K satisfying (3.2.1) with diameters converging to zero and the L^2-norm of $f_\theta^{\mathcal{C}_i}$ is uniformly bounded, that is

$$\int_{\mathrm{proj}_\theta(\mathcal{C}_i)} \left(f_\theta^{\mathcal{C}_i}\right)^2 dm \lesssim 1, \tag{3.3.3}$$

then

$$m(\mathrm{proj}_\theta(K)) = \lim_{i\to\infty} m(\mathrm{proj}_\theta(\mathcal{C}_i)) \gtrsim 1$$

and so $\mathrm{proj}_\theta(K)$ has positive Lebesgue measure, as wished. This conclusion will be obtained for m-almost every $\theta \in [-\pi/2, \pi/2]$ by showing that

$$I_i \doteq \int_{-\pi/2}^{\pi/2} d\theta \int_{L_\theta} \left(f_\theta^{\mathcal{C}_i}\right)^2 dm \lesssim 1. \tag{3.3.4}$$

3.3.1 Rewriting the integral I_i

For simplicity, let f denote $f_\theta^{\mathcal{C}_i}$. Then the interior integral of (3.3.4) becomes

$$\begin{aligned}
\int_{L_\theta} f^2 dm &= \int_{L_\theta} \left(\sum_{Q\in\mathcal{C}_i} \chi_{\mathrm{proj}_\theta(Q)} \cdot |Q|^{s-1}\right) \cdot \left(\sum_{\tilde{Q}\in\mathcal{C}_i} \chi_{\mathrm{proj}_\theta(\tilde{Q})} \cdot |\tilde{Q}|^{s-1}\right) dm \\
&= \sum_{Q,\tilde{Q}\in\mathcal{C}_i} |Q|^{s-1} \cdot |\tilde{Q}|^{s-1} \cdot \int_{L_\theta} \chi_{\mathrm{proj}_\theta(Q)\cap\mathrm{proj}_\theta(\tilde{Q})} dm \\
&= \sum_{Q,\tilde{Q}\in\mathcal{C}_i} |Q|^{s-1} \cdot |\tilde{Q}|^{s-1} \cdot m(\mathrm{proj}_\theta(Q) \cap \mathrm{proj}_\theta(\tilde{Q}))
\end{aligned}$$

and, using the inequalities

$$m(\mathrm{proj}_\theta(Q) \cap \mathrm{proj}_\theta(\tilde{Q})) \le \min\{m(\mathrm{proj}_\theta(Q)), m(\mathrm{proj}_\theta(\tilde{Q})\} \le \min\{|Q|, |\tilde{Q}|\},$$

it follows that

$$\int_{L_\theta} f^2 dm \lesssim \sum_{Q,\tilde{Q}\in\mathcal{C}_i} |Q|^{s-1} \cdot |\tilde{Q}|^{s-1} \cdot \min\{|Q|, |\tilde{Q}|\}. \tag{3.3.5}$$

We now proceed to prove (3.3.4) by a double-counting argument. To this matter, consider, for each pair of squares $(Q, \tilde{Q}) \in \mathcal{C}_i \times \mathcal{C}_i$, the set

$$\Theta_{Q,\tilde{Q}} = \left\{\theta \in [-\pi/2, \pi/2]; \mathrm{proj}_\theta(Q) \cap \mathrm{proj}_\theta(\tilde{Q}) \ne \emptyset\right\}.$$

Then

$$\begin{aligned}
I_i &\lesssim \sum_{Q,\tilde{Q}\in\mathcal{C}_i} |Q|^{s-1} \cdot |\tilde{Q}|^{s-1} \cdot \min\{|Q|, |\tilde{Q}|\} \cdot \int_{-\pi/2}^{\pi/2} \chi_{\Theta_{Q,\tilde{Q}}}(\theta) d\theta \\
&= \sum_{Q,\tilde{Q}\in\mathcal{C}_i} |Q|^{s-1} \cdot |\tilde{Q}|^{s-1} \cdot \min\{|Q|, |\tilde{Q}|\} \cdot m(\Theta_{Q,\tilde{Q}}). \qquad (3.3.6)
\end{aligned}$$

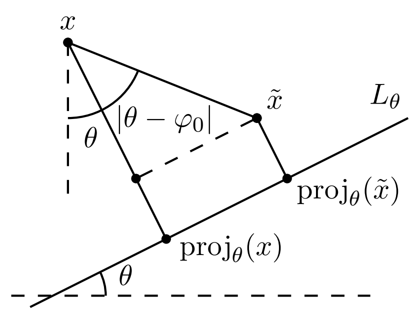
3.3.2 Transversality condition

This subsection estimates the Lebesgue measure of $\Theta_{Q,\tilde{Q}}$.

Lemma 3.3.2. *If $Q, \tilde{Q}$ are squares of $\mathbb{R}^2$ and $x, \tilde{x} \in \mathbb{R}^2$ are its centers, respectively, then*

$$m\left(\Theta_{Q,\tilde{Q}}\right) \leq 2\pi \cdot \frac{\max\{|Q|, |\tilde{Q}|\}}{|x - \tilde{x}|}.$$

Proof. Let $\theta \in \Theta_{Q,\tilde{Q}}$ and consider the figure.



Since $\text{proj}_\theta(Q)$ has diameter at most $|Q|$ (and the same happens to $\tilde{Q}$), we have $|\text{proj}_\theta(x) - \text{proj}_\theta(\tilde{x})| \leq 2 \cdot \max\{|Q|, |\tilde{Q}|\}$ and then, by elementary geometry,

$$\begin{aligned} \sin(|\theta - \varphi_0|) &= \frac{|\text{proj}_\theta(x) - \text{proj}_\theta(\tilde{x})|}{|x - \tilde{x}|} \\ &\leq 2 \cdot \frac{\max\{|Q|, |\tilde{Q}|\}}{|x - \tilde{x}|} \\ \implies \qquad |\theta - \varphi_0| &\leq \pi \cdot \frac{\max\{|Q|, |\tilde{Q}|\}}{|x - \tilde{x}|}, \end{aligned}$$

because $\sin^{-1} y \leq \pi y/2$. As φ_0 is fixed, the lemma is proved. □

We point out that, although ingenuous, Lemma 3.3.2 expresses the crucial property of transversality that makes the proof work, and all results related to Marstrand's Theorem use a similar idea in one way or another. See [43] where this tranversality condition is also exploited.

By Lemma 3.3.2 and (3.3.6), we obtain

$$I_i \lesssim \sum_{Q,\tilde{Q} \in \mathcal{C}_i} |x - \tilde{x}|^{-1} \cdot |Q|^s \cdot |\tilde{Q}|^s. \tag{3.3.7}$$

3.3.3 Good covers

The last summand will be estimated by choosing appropriate dyadic covers $\mathcal{C}_i$. Let $\mathcal{C}$ be an arbitrary dyadic cover of K. Remember K is an s-set satisfying (3.3.1).

Definition 3.3.3. The dyadic cover $\mathcal{C}$ is *good* if

$$\sum_{\substack{\tilde{Q}\in\mathcal{C}\\ \tilde{Q}\subset Q}} |\tilde{Q}|^s < \max\{128b, 1\} \cdot |Q|^s \ , \quad \forall\, Q \in \mathcal{D}. \tag{3.3.8}$$

Any other constant depending only on K would work for the definition. The reason we chose this specific constant will become clear below, where we provide the existence of good dyadic covers.

Proposition 3.3.4. *Let $K \subset \mathbb{R}^2$ be a compact s-set satisfying (3.3.1). Then, for any $\delta > 0$, there exists a good dyadic cover of K with diameter less than δ.*

Proof. Let $i_0 \geq 1$ such that $2^{-i_0-1} < \delta \leq 2^{-i_0}$. Begin with a finite cover $\mathcal{U}$ of K with diameter less than $\delta/4$ such that

$$\sum_{\substack{U\in\mathcal{U}\\ U\subset Q}} |U|^s < 2 \cdot m_s(K \cap Q) \ , \quad \forall\, Q \in \mathcal{D}_{i_0}.$$

Now, change $\mathcal{U}$ by a dyadic cover $\mathcal{C}$ according to property (4) of Subsection 3.2.1. $\mathcal{C}$ has diameter at most δ and satisfies

$$\sum_{\substack{\tilde{Q}\in\mathcal{C}\\ \tilde{Q}\subset Q}} |\tilde{Q}|^s < 64 \cdot \sum_{\substack{U\in\mathcal{U}\\ U\subset Q}} |U|^s < 128 \cdot m_s(K \cap Q) \ , \quad \forall\, Q \in \mathcal{D}_{i_0}.$$

By additivity, the same inequality happens for any $Q \in \bigcup_{0\leq i\leq i_0} \mathcal{D}_i$ and so, as $m_s(K \cap Q) \leq b \cdot |Q|^s$, it follows that

$$\sum_{\substack{\tilde{Q}\in\mathcal{C}\\ \tilde{Q}\subset Q}} |\tilde{Q}|^s < 128b \cdot |Q|^s \ , \quad \forall\, Q \in \bigcup_{0\leq i\leq i_0} \mathcal{D}_i \,, \tag{3.3.9}$$

that is, (3.3.8) holds for large scales. To control the small ones, apply the following operation: whenever $Q \in \bigcup_{i>i_0} \mathcal{D}_i$ is such that

$$\sum_{\substack{\tilde{Q}\in\mathcal{C}\\ \tilde{Q}\subset Q}} |\tilde{Q}|^s > |Q|^s,$$

we change $\mathcal{C}$ by $\mathcal{C}\cup\{Q\}\backslash\{\tilde{Q} \in \mathcal{C}\,;\, \tilde{Q} \subset Q\}$. It is clear that such operation preserves the inequality (3.3.9) and so, after a finite number of steps, we end up with a good dyadic cover. $\square$

As the constant in (3.3.8) does not depend on δ, there is a sequence $(\mathcal{C}_i)_{i\geq 1}$ of good dyadic covers of K with diameters converging to zero such that

$$\sum_{\substack{\tilde{Q}\in\mathcal{C}_i \\ \tilde{Q}\subset Q}} |\tilde{Q}|^s \lesssim |Q|^s\,, \quad Q\in\mathcal{D} \text{ and } i\geq 1. \tag{3.3.10}$$

3.4 Proof of Theorems 3.1.1 and 3.1.2

Proof of Theorem 3.1.1. Let $(\mathcal{C}_i)_{i\geq 1}$ be a sequence of good dyadic covers satisfying (3.3.10) such that $\|\mathcal{C}_i\|\to 0$. By (3.3.7),

$$\begin{aligned}
I_i &\lesssim \sum_{Q,\tilde{Q}\in\mathcal{C}_i} |x-\tilde{x}|^{-1}\cdot|Q|^s\cdot|\tilde{Q}|^s \\
&= \sum_{Q\in\mathcal{C}_i}\sum_{j=0}^{\infty}\sum_{\substack{\tilde{Q}\in\mathcal{C}_i \\ 2^{-j-1}<|x-\tilde{x}|\leq 2^{-j}}} |x-\tilde{x}|^{-1}\cdot|Q|^s\cdot|\tilde{Q}|^s \\
&\leq \sum_{Q\in\mathcal{C}_i}\sum_{j=0}^{\infty}\sum_{\substack{\tilde{Q}\in\mathcal{C}_i \\ \tilde{Q}\subset B_{3\cdot 2^{-j}}(x)}} |x-\tilde{x}|^{-1}\cdot|Q|^s\cdot|\tilde{Q}|^s \\
&\lesssim \sum_{Q\in\mathcal{C}_i}|Q|^s\sum_{j=0}^{\infty}2^j\cdot\left(2^{-j}\right)^s \\
&= \sum_{Q\in\mathcal{C}_i}|Q|^s\sum_{j=0}^{\infty}\left(2^j\right)^{1-s} \\
&\lesssim \sum_{Q\in\mathcal{C}_i}|Q|^s \\
&\lesssim 1,
\end{aligned}$$

establishing (3.3.4). Define, for each $\varepsilon>0$, the sets

$$G_\varepsilon^i=\left\{\theta\in[-\pi/2,\pi/2]\,;\int_{L_\theta}\left(f_\theta^{\mathcal{C}_i}\right)^2 dm<\varepsilon^{-1}\right\},\quad i\geq 1.$$

Then $m\left([-\pi/2,\pi/2]\backslash G_\varepsilon^i\right)\lesssim\varepsilon$, and the same holds for the set

$$G_\varepsilon=\bigcap_{i\geq 1}\bigcup_{j=i}^{\infty}G_\varepsilon^j\,.$$

If $\theta\in G_\varepsilon$, then

$$m\left(\mathrm{proj}_\theta(\mathcal{C}_i)\right)\gtrsim\varepsilon\,,\quad\text{for infinitely many } n,$$

which implies that $m\left(\mathrm{proj}_\theta(K)\right)>0$. Finally, the set $G=\bigcup_{i\geq 1}G_{1/i}$ satisfies $m([-\pi/2,\pi/2]\backslash G)=0$ and $m\left(\mathrm{proj}_\theta(K)\right)>0$, for any $\theta\in G$. □

A direct consequence is the

Corollary 3.4.1. *The measure* $\mu_\theta = (\mathrm{proj}_\theta)_*(m_s|_K)$ *is absolutely continuous with respect to* m*, for* m*-almost every* $\theta \in \mathbb{R}$*.*

Proof. By Theorem 3.1.1, we have the implication

$$X \subset K\,,\ m_s(X) > 0 \implies m(\mathrm{proj}_\theta(X)) > 0, \quad m\text{-almost every } \theta \in \mathbb{R}, \tag{3.4.1}$$

which is sufficient for the required absolute continuity. Indeed, if $Y \subset L_\theta$ satisfies $m(Y) = 0$, then

$$\mu_\theta(Y) = m_s(X) = 0\,,$$

where $X = {\mathrm{proj}_\theta}^{-1}(Y)$. Otherwise, by (3.4.1) we would have $m(Y) = m(\mathrm{proj}_\theta(X)) > 0$, contradicting the assumption. ☐

Let $f_\theta = d\mu_\theta/dm$. By the proof of Theorem 3.1.1, we have

$$\left\| f_\theta^{\mathcal{C}_i} \right\|_{L^2} \lesssim 1, \quad m\text{-a.e. } \theta \in \mathbb{R}. \tag{3.4.2}$$

Proof of Theorem 3.1.2. Define, for each $\varepsilon > 0$, the function $f_{\theta,\varepsilon} : L_\theta \to \mathbb{R}$ by

$$f_{\theta,\varepsilon}(x) = \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f_\theta(y)dm(y), \quad x \in L_\theta.$$

As f_θ is an L^1-function, the Lebesgue differentiation theorem gives that $f_\theta(x) = \lim_{\varepsilon\to 0} f_{\theta,\varepsilon}(x)$ for m-almost every $x \in L_\theta$. If we manage to show that[2]

$$\|f_{\theta,\varepsilon}\|_{L^2} \lesssim 1, \quad m\text{-a.e. } \theta \in \mathbb{R}, \tag{3.4.3}$$

then Fatou's lemma establishes the theorem. To this matter, first observe that

$$\begin{aligned} f_{\theta,\varepsilon}(x) &= \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f_\theta(y)dm(y) \\ &= \frac{1}{2\varepsilon}\cdot \mu_\theta([x-\varepsilon, x+\varepsilon]) \\ &= \frac{1}{2\varepsilon}\cdot m_s\left((\mathrm{proj}_\theta)^{-1}([x-\varepsilon, x+\varepsilon]) \cap K\right). \end{aligned}$$

[2] We consider $\|f_{\theta,\varepsilon}\|_{L^2}$ as a function of $\varepsilon > 0$.

In order to estimate this last term, fix $\varepsilon > 0$ and let $i > 0$ such that $\mathcal{C} = \mathcal{C}_i$ has diameter less than ε. Then

$$\begin{aligned} m_s\left((\mathrm{proj}_\theta)^{-1}([x-\varepsilon, x+\varepsilon]) \cap K\right) &\le \sum_{\substack{Q\in\mathcal{C} \\ \mathrm{proj}_\theta(Q)\subset[x-2\varepsilon,x+2\varepsilon]}} m_s(Q\cap K) \\ &\lesssim \sum_{\substack{Q\in\mathcal{C} \\ \mathrm{proj}_\theta(Q)\subset[x-2\varepsilon,x+2\varepsilon]}} |Q|^s \\ &\sim \sum_{\substack{Q\in\mathcal{C} \\ \mathrm{proj}_\theta(Q)\subset[x-2\varepsilon,x+2\varepsilon]}} |Q|^{s-1}\cdot m(\mathrm{proj}_\theta(Q)) \\ &\le \int_{x-2\varepsilon}^{x+2\varepsilon} f_\theta^{\mathcal{C}}(y)dm(y), \end{aligned}$$

where in the second inequality we used (3.3.1). By the Cauchy-Schwarz inequality,

$$|f_{\theta,\varepsilon}(x)|^2 \lesssim \frac{1}{2\varepsilon}\int_{x-2\varepsilon}^{x+2\varepsilon} \left|f_\theta^{\mathcal{C}}(y)\right|^2 dm(y)$$

and so

$$\begin{aligned} \|f_{\theta,\varepsilon}\|_{L^2}^2 &\lesssim \int_{L_\theta} \frac{1}{2\varepsilon}\int_{x-2\varepsilon}^{x+2\varepsilon} \left|f_\theta^{\mathcal{C}}(y)\right|^2 dm(y)dm(x) \\ &\sim \int_{L_\theta} \left|f_\theta^{\mathcal{C}}\right|^2 dm \\ &= \left\|f_\theta^{\mathcal{C}}\right\|_{L^2}^2 \end{aligned}$$

which, by (3.4.2), establishes (3.4.3). □

3.5 Concluding remarks

The good feature of the proof is that the discretization idea may be applied to other contexts. For example, we prove in [24] a Marstrand type theorem in an arithmetical context.

Chapter 4

A Marstrand theorem for subsets of integers

We prove a Marstrand type theorem for a class of subsets of the integers. More specifically, after defining the counting dimension $D(E)$ of $E \subset \mathbb{Z}$ and the concepts of regularity and compatibility, we show that if $E, F \subset \mathbb{Z}$ are two regular compatible sets, then $D(E + \lfloor \lambda F \rfloor) \geq \min\{1, D(E) + D(F)\}$ for Lebesgue almost every $\lambda \in \mathbb{R}$. If in addition $D(E) + D(F) > 1$, it is also shown that $E + \lfloor \lambda F \rfloor$ has positive upper Banach density for Lebesgue almost every $\lambda \in \mathbb{R}$. The result has direct consequences when applied to arithmetic sets, such as the integer values of a polynomial with integer coefficients.

4.1 Introduction

The purpose of this chapter is to prove a Marstrand type theorem for a class of subsets of the integers.

The well-known theorem of Marstrand [27] states the following: if $K \subset \mathbb{R}^2$ is a Borel set such that its Hausdorff dimension is greater than one then, for almost every direction, its projection to $\mathbb{R}$ in the respective direction has positive Lebesgue measure. In other words, this means K is "fat" in almost every direction. When K is the product of two real subsets K_1, K_2, Marstrand's theorem can be stated in a more analytical form as: for Lebesgue almost every $\lambda \in \mathbb{R}$, the arithmetic sum $K_1 + \lambda K_2$ has positive Lebesgue measure. Much research has been done around this topic, specially because the analysis of such arithmetic sums has applications in the theory of homoclinic birfurcations and also in diophantine approximations.

Given a subset $E \subset \mathbb{Z}$, let $d^*(E)$ denote its *upper Banach density*[1]. A remarkable result in

[1]See Subsection 4.2.2 for the proper definitions.

additive combinatorics is Szemerédi's theorem [45]. It asserts that if $d^*(E) > 0$, then E contains arbitrarily long arithmetic progressions. One can interpret this result by saying that density represents the correct notion of largeness needed to preserve finite configurations of $\mathbb{Z}$. On the other hand, Szemerédi's theorem cannot infer any property about subsets of zero upper Banach density, and many of these sets are of interest. A class of examples are the integer values of a polynomial with integer coefficients and the prime numbers. These sets may, as well, contain combinatorially rich patterns. See, for example, [4] and [16].

A set $E \subset \mathbb{Z}$ of zero upper Banach density is characterized as occupying portions in intervals of $\mathbb{Z}$ that grow in a sublinear fashion as the length of the intervals grow. On the other hand, there still may exist some kind of growth speed. For example, the number of perfect squares on an initial interval of length n is about $n^{0.5}$. This exponent means, in some sense, a dimension of $\{n^2\,;\, n \in \mathbb{Z}\}$ inside $\mathbb{Z}$. In the present chapter, we suggest a *counting dimension* $D(E)$ for E and establish the following Marstrand type result on the counting dimension of most arithmetic sums $E + \lfloor \lambda F \rfloor$ for a class of subsets $E, F \subset \mathbb{Z}$.

Theorem 4.1.1. *Let $E, F \subset \mathbb{Z}$ be two regular compatible sets. Then*

$$D(E + \lfloor \lambda F \rfloor) \geq \min\{1, D(E) + D(F)\}$$

for Lebesgue almost every $\lambda \in \mathbb{R}$. If in addition $D(E) + D(F) > 1$, then $E + \lfloor \lambda F \rfloor$ has positive upper Banach density for Lebesgue almost every $\lambda \in \mathbb{R}$.

An immediate consequence of Theorem 4.1.1 is that the same result holds for subsets of the natural numbers.

Corollary 4.1.2. *Let $E, F \subset \mathbb{N}$ be two regular compatible sets. Then*

$$D(E + \lfloor \lambda F \rfloor) \geq \min\{1, D(E) + D(F)\}$$

for Lebesgue almost every $\lambda > 0$. If in addition $D(E) + D(F) > 1$, then $E + \lfloor \lambda F \rfloor$ has positive upper Banach density for Lebesgue almost every $\lambda > 0$.

The reader should make a parallel between the quantities $d^*(E)$ and $D(E)$ of subsets $E \subset \mathbb{Z}$ and Lebesgue measure and Hausdorff dimension of subsets of $\mathbb{R}$. It is exactly this association that allows us to call Theorem 4.1.1 a Marstrand theorem for subsets of integers.

The notions of *regularity* and *compatibility* are defined in Subsections 4.4.1 and 4.4.2, respectively. Both are fulfilled for many arithmetic subsets of $\mathbb{Z}$, such as the integer values of a polynomial with integer coefficients and, more generally, for the universal sets (see Definition 4.4.5). These are the sets that exhibit the expected growth rate along intervals of arbitrary length. The second result of this work is

Theorem 4.1.3. *Let $E_0, \ldots, E_k$ be universal subsets of $\mathbb{Z}$. Then*

$$D(E_0 + \lfloor \lambda_1 E_1 \rfloor + \cdots + \lfloor \lambda_k E_k \rfloor) \geq \min\{1, D(E_0) + D(E_1) + \cdots + D(E_k)\}$$

for Lebesgue almost every $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$. If in addition $\sum_{i=0}^{k} D(E_i) > 1$, then the arithmetic sum $E_0 + \lfloor \lambda_1 E_1 \rfloor + \cdots + \lfloor \lambda_k E_k \rfloor$ has positive upper Banach density for Lebesgue almost every $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$.

The integer values of a polynomial with integer coefficients have special interest in ergodic theory and its connections with combinatorics, due to ergodic theorems along these subsets [6], as well as its combinatorial implications. See [4] for the remarkable work of V. Bergelson and A. Leibman on the polynomial extension of Szemerédi's theorem. Theorem 4.1.3 has a direct consequence when applied to this setting.

Corollary 4.1.4. *Let $p_i \in \mathbb{Z}[x]$ with degree $d_i > 0$ and let $E_i = (p_i(n))_{n \in \mathbb{Z}}$, $i = 0, \ldots, k$. Then*

$$D(E_0 + \lfloor \lambda_1 E_1 \rfloor + \cdots + \lfloor \lambda_k E_k \rfloor) \geq \min\left\{1, \frac{1}{d_0} + \frac{1}{d_1} + \cdots + \frac{1}{d_k}\right\}$$

for Lebesgue almost every $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$. If in addition $\sum_{i=0}^{k} d_i{}^{-1} > 1$, then the arithmetic sum $E_0 + \lfloor \lambda_1 E_1 \rfloor + \cdots + \lfloor \lambda_k E_k \rfloor$ has positive upper Banach density for Lebesgue almost every $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$.

The proof of Theorem 4.1.1 is based on the ideas developed in [24] and [26]. It relies on the fact that the cardinality of a regular subset of $\mathbb{Z}$ along an increasing sequence of intervals exhibits an exponential behavior ruled out by its counting dimension. As this holds for two regular subsets $E, F \subset \mathbb{Z}$, the compatibility assumption allows to estimate the cardinality of the arithmetic sum $E + \lfloor \lambda F \rfloor$ along the respective arithmetic sums of intervals and, finally, a double-counting argument estimates the size of the "bad" parameters for which such cardinality is small. Theorem 4.1.3 follows from Theorem 4.1.1 by a fairly simple induction.

This chapter is organized as follows. In Section 4.2 we introduce the basic notations and definitions. Section 4.3 is devoted to the discussion of examples. In particular, the sets given by integer values of a polynomial with integer coefficients are investigated in Subsection 4.3.1. In Section 4.4 we introduce the notions of regularity and compatibility. Subsection 4.4.3 provides a counterexample to Theorem 4.1.1 when the sets are no longer compatible and Subsection 4.4.4 a counterexample to the same theorem when the space of parameters is $\mathbb{Z}$. Finally, in Section 4.5 we prove Theorems 4.1.1 and 4.1.3. We also collect final remarks and further questions in Section 4.6.

4.2 Preliminaries

4.2.1 General notation

Given a set X, $|X|$ denotes the cardinality of X. $\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ the set of positive integers. We use the following notation to compare the asymptotic of functions.

Definition 4.2.1. Let $f, g : \mathbb{Z}$ or $\mathbb{N} \to \mathbb{R}$ be two real-valued functions. We say $f \lesssim g$ if there is a constant $C > 0$ such that

$$|f(n)| \leq C \cdot |g(n)|, \quad \forall n \in \mathbb{Z} \text{ or } \mathbb{N}.$$

If $f \lesssim g$ and $g \lesssim f$, we write $f \sim g$. We say $f \approx g$ if

$$\lim_{|n| \to \infty} \frac{f(n)}{g(n)} = 1.$$

For each $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the integer part of x. For each $k \geq 1$, m_k denotes the Lebesgue measure of $\mathbb{R}^k$. For $k = 1$, let $m = m_1$. The letter I will always denote an interval of $\mathbb{Z}$:

$$I = (M, N] = \{M + 1, \ldots, N\}.$$

The *length* of I is equal to its cardinality, $|I| = N - M$.

For $E \subset \mathbb{Z}$ and $\lambda \in \mathbb{R}$, λE denotes the set $\{\lambda n\,;\, n \in E\} \subset \mathbb{R}$ and $\lfloor \lambda E \rfloor$ the set $\{\lfloor \lambda n \rfloor\,;\, n \in E\} \subset \mathbb{Z}$.

4.2.2 Counting dimension

Definition 4.2.2. The *upper Banach density* of $E \subset \mathbb{Z}$ is equal to

$$d^*(E) = \limsup_{|I| \to \infty} \frac{|E \cap I|}{|I|},$$

where I runs over all intervals of $\mathbb{Z}$.

Definition 4.2.3. The *counting dimension* or simply *dimension* of a set $E \subset \mathbb{Z}$ is equal to

$$D(E) = \limsup_{|I| \to \infty} \frac{\log |E \cap I|}{\log |I|},$$

where I runs over all intervals of $\mathbb{Z}$.

Obviously, $D(E) \in [0, 1]$ and $D(E) = 1$ whenever $d^*(E) > 0$. The counting dimension allows us to distinguish largeness between two sets of zero upper Banach density. Similar definitions to $D(E)$ have appeared in [2] and [13]. We now give another characterization to it that is similar in spirit to the Hausdorff dimension of subsets of $\mathbb{R}$. Let α be a nonnegative real number.

Definition 4.2.4. The *counting α-measure* of $E \subset \mathbb{Z}$ is equal to

$$H_\alpha(E) = \limsup_{|I|\to\infty} \frac{|E \cap I|}{|I|^\alpha},$$

where I runs over all intervals of $\mathbb{Z}$.

Clearly, $H_\alpha(E) \in [0, \infty]$. For a fixed $E \subset \mathbb{Z}$, the numbers $H_\alpha(E)$ are decreasing in α and one easily checks that

$$\alpha < D(E) \implies H_\alpha(E) = \infty \quad \text{and} \quad \alpha > D(E) \implies H_\alpha(E) = 0,$$

which in turn implies the existence and uniqueness of $\alpha \geq 0$ such that

$$\begin{aligned} H_\beta(E) &= \infty \quad , \quad \text{if } 0 \leq \beta < \alpha, \\ &= 0 \quad , \quad \text{if } \beta > \alpha. \end{aligned}$$

The above equalities imply that $D(E) = \alpha$, that is, the counting dimension is exactly the parameter α where $H_\alpha(E)$ decreases from infinity to zero. Also, if $\beta > D(E)$, then

$$|E \cap I| \lesssim |I|^\beta, \tag{4.2.1}$$

where I runs over all intervals of $\mathbb{Z}$ and, conversely, if (4.2.1) holds, then $D(E) \leq \beta$.

Below, we collect basic properties of the counting dimension and α-measure.

(i) $E \subset F \Longrightarrow D(E) \leq D(F)$.

(ii) $D(E \cup F) = \max\{D(E), D(F)\}$.

(iii) For any $\lambda > 0$,
$$H_\alpha(\lfloor \lambda E \rfloor) = \lambda^{-\alpha} \cdot H_\alpha(E). \tag{4.2.2}$$

The first two are direct. Let's prove (iii). For any interval $I \subset \mathbb{Z}$, we have $|E \cap I| \approx |\lfloor \lambda E \rfloor \cap \lfloor \lambda I \rfloor|$ and so

$$\begin{aligned} \frac{|E \cap I|}{|I|^\alpha} &\approx \lambda^\alpha \cdot \frac{|\lfloor \lambda E \rfloor \cap \lfloor \lambda I \rfloor|}{|\lfloor \lambda I \rfloor|^\alpha} \\ \Longrightarrow \quad H_\alpha(E) &= \lambda^\alpha \cdot H_\alpha(\lfloor \lambda E \rfloor). \end{aligned}$$

Remark 4.2.5. As $\lfloor -x \rfloor = -\lfloor x \rfloor$ or $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$, the sets $\lfloor -\lambda E \rfloor$ and $\lfloor \lambda E \rfloor$ have the same counting dimension. Also, $0 < H_\alpha(\lfloor -\lambda E \rfloor) < \infty$ if and only if $0 < H_\alpha(\lfloor \lambda E \rfloor) < \infty$. For these reasons, we assume from now on that $\lambda > 0$.

4.3 Examples

Example 1. Let $\alpha \in (0,1]$ and

$$E_\alpha = \left\{ \left\lfloor n^{1/\alpha} \right\rfloor \, ; \, n \in \mathbb{N} \right\}. \tag{4.3.1}$$

We infer that $H_\alpha(E_\alpha) = 1$. To prove this, we make use of the inequality[2]

$$(x+y)^\alpha \leq x^\alpha + y^\alpha \,, \;\; x, y \geq 0 \,,$$

to conclude that

$$\frac{|E_\alpha \cap (M,N]|}{(N-M)^\alpha} \leq \frac{(N+1)^\alpha - (M+1)^\alpha}{(N-M)^\alpha} \leq 1.$$

This proves that $H_\alpha(E_\alpha) \leq 1$. On the other hand,

$$\frac{|E_\alpha \cap (0,N]|}{N^\alpha} \geq \frac{N^\alpha - 1}{N^\alpha}$$

and so $H_\alpha(E_\alpha) \geq 1$. Then $H_\alpha(E_\alpha) = 1$ and, in particular, $D(E_\alpha) = \alpha$.

Example 2. The set

$$E = \bigcup_{n \in \mathbb{N}} (n^n, (n+1)^n] \cap E_{1-1/n}$$

has zero upper Banach density and $D(E) = 1$.

Example 3. The prime numbers have dimension one. This follows from the prime number theorem:

$$\lim_{n \to \infty} \frac{\log |\{1 \leq p \leq n \, ; \, p \text{ is prime}\}|}{\log n} = \lim_{n \to \infty} \frac{\log n - \log \log n}{\log n} = 1 \,.$$

Example 4. Sets of zero upper Banach density appear naturally in infinite ergodic theory. Let $(X, \mathcal{A}, \mu, T)$ be a sigma-finite measure-preserving system, with $\mu(X) = \infty$, and let $A \in \mathcal{A}$ have finite measure. Fixed $x \in A$, let $E = \{n \geq 1 \, ; \, T^n x \in A\}$. By Hopf's Ratio Ergodic Theorem, E has zero upper Banach density almost surely. In many specific cases, its dimension can be calculated or at least estimated. See [1] for further details.

4.3.1 Polynomial subsets of $\mathbb{Z}$

Definition 4.3.1. A *polynomial subset of* $\mathbb{Z}$ is a set $E = \{p(n) \, ; \, n \in \mathbb{Z}\}$, where $p(x)$ is a non-constant polynomial with integer coefficients.

[2]For each $t \geq 0$, the function $x \mapsto x^\alpha + (t-x)^\alpha$, $0 \leq x \leq t$, is concave and so attains its minimum in $x = 0$ and $x = t$. Then $x^\alpha + (t-x)^\alpha \geq t^\alpha$ for any $x \in [0,t]$.

These are the sets we consider in Theorem 4.1.3. Their counting dimension is easily calculated as follows. Given $E, F \subset \mathbb{Z}$, let E and F be *asymptotic* if $E = \{\cdots < a_{-1} < a_0 < a_1 < \cdots\}$, $F = \{\cdots < b_{-1} < b_0 < b_1 < \cdots\}$ and there is $i \geq 0$ such that

$$a_{n-i} \leq b_n \leq a_{n+i} \ , \quad \text{for every } n \in \mathbb{Z}. \tag{4.3.2}$$

Denote this by $E \approx F$.

Lemma 4.3.2. *If $E, F \subset \mathbb{Z}$ are asymptotic then $H_\alpha(E) = H_\alpha(F)$, for any $\alpha \geq 0$. In particular, $D(E) = D(F)$.*

Proof. Let $I = (M, N]$ be an interval and assume $E \cap I = \{a_{m+1}, a_{m+2}, \ldots, a_n\}$. By relation (4.3.2),

$$b_{m-i} \leq a_m \leq M < a_{m+1} \leq b_{m+i+1} \quad \text{and} \quad b_{n-i} \leq a_n \leq N < a_{n+1} \leq b_{n+i+1},$$

which imply the inclusions

$$\{b_{m+i+1}, \ldots, b_{n-i}\} \subset F \cap I \subset \{b_{m-i}, \ldots, b_{n+i+1}\}.$$

Then $|E \cap I| \approx |F \cap I|$ and so

$$H_\alpha(E) = \limsup_{|I| \to \infty} \frac{|E \cap I|}{|I|^\alpha} = \limsup_{|I| \to \infty} \frac{|F \cap I|}{|I|^\alpha} = H_\alpha(F)\,.$$

□

Let $E = \{p(n)\,;\, n \in \mathbb{Z}\}$, where $p(x) \in \mathbb{Z}[x]$ has degree d. Assuming p has leading coefficient $a > 0$, there is $i \geq 0$ such that $a \cdot (n-i)^d < p(n) < a \cdot (n+i)^d$ for every $n \in \mathbb{Z}$ and then $E \approx aE_{1/d}$, where $E_{1/d}$ is defined as in (4.3.1). By Lemma 4.3.2, we get

$$D(E) = \frac{1}{d} \ \text{ and } \ H_{1/d}(E) = 1.$$

4.3.2 Cantor sets in $\mathbb{Z}$

The famous ternary Cantor set of $\mathbb{R}$ is formed by the real numbers of $[0,1]$ with only 0's and 2's on their base 3 expansion. In analogy to this, let $E \subset \mathbb{Z}$ be defined as

$$E = \left\{ \sum_{i=0}^{n} a_i \cdot 3^i \,;\, n \in \mathbb{N} \text{ and } a_i = 0 \text{ or } 2 \right\}. \tag{4.3.3}$$

The set E has been slightly investigated in [11]. There, A. Fisher proved that

$$H_{\log 2/\log 3}(E) > 0\,.$$

We prove below that $H_{\log 2/\log 3}(E) \leq 2$, which in particular gives that $D(E) = \log 2/\log 3$, as expected. Let $I = (M, N]$ be an interval of $\mathbb{Z}$. We can assume $M+1, N \in E$. Indeed, if $\tilde{I} = (\tilde{M}, \tilde{N}]$, where $\tilde{M}+1$ and $\tilde{N}$ are the smallest and largest elements of $E \cap I$, respectively, then

$$\frac{|E \cap I|}{|I|^{\alpha}} \leq \frac{|E \cap \tilde{I}|}{|\tilde{I}|^{\alpha}} \cdot$$

Let $M+1, N \in E$, say

$$\begin{aligned} M+1 &= a_0 \cdot 3^0 + a_1 \cdot 3^1 + \cdots + a_{m-1} \cdot 3^m + 2 \cdot 3^m \\ N &= b_0 \cdot 3^0 + b_1 \cdot 3^1 + \cdots + b_{n-1} \cdot 3^n + 2 \cdot 3^n. \end{aligned}$$

We can also assume that $m < n$. If this is not the case, the quotient $|E \cap I|/|I|^{\alpha}$ does not decrease if we change I by $(M - 2 \cdot 3^n, N - 2 \cdot 3^n]$. With these assumptions,

$$N - M \geq 2 \cdot 3^n - (3^{m+1} - 2) \geq 3^n$$

and then

$$\frac{|E \cap I|}{|I|^{\log 2/\log 3}} \leq \frac{|E \cap (0, N]|}{|I|^{\log 2/\log 3}} \leq \frac{2^{n+1}}{(3^n)^{\log 2/\log 3}} = 2.$$

Because I is arbitrary, this gives that $H_{\log 2/\log 3}(E) \leq 2$.

Observe that the renormalization of $E \cap (0, 3^n)$ via the linear map $x \mapsto x/3^n$ generates a subset of the unit interval $(0,1)$ that is equal to the set of left endpoints of the remaining intervals of the n-th step of the construction of the ternary Cantor set of $\mathbb{R}$. In other words, if $K = \bigcup_{n \in E} [n, n+1]$, then $K/3^n$ is exactly the n-th step of the construction of the ternary Cantor set of $\mathbb{R}$.

More generally, let us define a class of Cantor sets in $\mathbb{Z}$. Fix a basis $a \in \mathbb{N}$ and a binary matrix $A = (a_{ij})_{0 \leq i,j \leq a-1}$. Let

$$\Sigma_n(A) = \left\{ (d_0 d_1 \cdots d_{n-1} d_n) \, ; \, a_{d_{i-1} d_i} = 1, \ 1 \leq i \leq n \right\}, \ n \geq 0,$$

denote the set of admissible words of length $n+1$ and $\Sigma^*(A) = \bigcup_{n \geq 0} \Sigma_n(A)$ the set of all finite admissible words.

Definition 4.3.3. The *integer Cantor set* $E_A \subset \mathbb{Z}$ associated to the matrix A is the set

$$E_A = \{ d_0 \cdot a^0 + \cdots + d_n \cdot a^n \, ; \, (d_0 d_1 \cdots d_n) \in \Sigma^*(A) \}.$$

Our definition was inspired on the fact that dynamically defined topologically mixing Cantor sets of the real line are homeomorphic to subshifts of finite type, which is exactly what we did above, after truncating the numbers. See [24] for more details. The dimension of E_A, as in the inspiring case, depends on the Perron-Frobenius eigenvalue of A. Remember that the *Perron-Frobenius eigenvalue* is the largest eigenvalue $\lambda_+(A)$ of A. It has multiplicity one and maximizes the absolute value of the eigenvalues of A. Also, there is a constant $c = c(A) > 0$ such that

$$c^{-1} \cdot \lambda_+(A)^n \leq |\Sigma_n(A)| \leq c \cdot \lambda_+(A)^n\,, \quad \text{for every } n \geq 0, \tag{4.3.4}$$

whose proof may be found in [22].

Lemma 4.3.4. *If A is a binary $a \times a$ matrix, then*

$$D(E_A) = \frac{\log \lambda_+(A)}{\log a} \quad \textit{and} \quad 0 < H_{\frac{\log \lambda_+(A)}{\log a}}(E_A) < \infty\,.$$

Proof. Let $I = (M, N]$. Again, we may assume $M+1, N \in E_A$, say

$$\begin{aligned} M+1 &= x_0 \cdot a^0 + \cdots + x_n \cdot a^n \\ N &= y_0 \cdot a^0 + \cdots + y_n \cdot a^n, \end{aligned}$$

with $y_n > x_n$. If $y_n \geq x_n + 2$, then

$$\left\{ \begin{aligned} M+1 &\leq (x_n+1) \cdot a^n \\ N &\geq (x_n+2) \cdot a^n \end{aligned} \right. \implies |I| \geq a^n$$

and, as $I \subset (0, a^{n+1})$, we have

$$\frac{|E_A \cap I|}{|I|^{\frac{\log \lambda_+(A)}{\log a}}} \leq \frac{|\Sigma_n(A)|}{a^{\frac{n \log \lambda_+(A)}{\log a}}} \leq \frac{c \cdot \lambda_+(A)^n}{\lambda_+(A)^n} = c\,. \tag{4.3.5}$$

If $y_n = x_n + 1$, let $i, j \in \{0, 1, \ldots, n-1\}$ be the indices for which

(i) $x_i < a-1$ and $x_{i+1} = \cdots = x_{n-1} = a-1$,

(ii) $y_j > 0$ and $y_{j+1} = \cdots = y_{n-1} = 0$.

Then

$$\left\{ \begin{aligned} M+1 &\leq (x_n+1) \cdot a^n - a^i \\ N &\geq (x_n+1) \cdot a^n + a^j \end{aligned} \right. \implies |I| \geq a^i + a^j \geq a^{\max\{i,j\}}.$$

In order to $\sum_{k=0}^n z_k \cdot a^k$ belong to I, one must have $z_n = x_n$ or $z_n = x_n + 1$. In the first case $z_{i+1} = \cdots = z_{n-1} = a-1$ and in the second $z_{j+1} = \cdots = z_{n-1} = 0$. Then

$$|E_A \cap I| \leq |\Sigma_i(A)| + |\Sigma_j(A)| \leq 2c \cdot \lambda_+(A)^{\max\{i,j\}}$$

and so

$$\frac{|E_A \cap I|}{|I|^{\frac{\log \lambda_+(A)}{\log a}}} \leq \frac{2c \cdot \lambda_+(A)^{\max\{i,j\}}}{a^{\frac{\max\{i,j\} \log \lambda_+(A)}{\log a}}} = 2c\,. \tag{4.3.6}$$

Estimates (4.3.5) and (4.3.6) give $H_{\log\lambda_+(A)/\log a} < \infty$. On the other hand,

$$\frac{|E_A\cap(0,a^n]|}{a^{\frac{n\log\lambda_+(A)}{\log a}}} \geq \frac{c^{-1}\cdot\lambda_+(A)^{n-1}}{\lambda_+(A)^n} = c^{-1}\cdot\lambda_+(A)^{-1} \quad\Longrightarrow\quad H_{\log\lambda_+(A)/\log a} > 0\,,$$

which concludes the proof. □

By the above lemma, if $X\subset\{0,\ldots,a-1\}$ and $A=(a_{ij})$ is defined by $a_{ij}=1$ iff $i,j\in X$, then $D(E_A)=\log|X|/\log a$, which extends the results about the ternary Cantor set defined in (4.3.3).

In general, if E,F are subsets of $\mathbb{Z}$ such that $D(E)+D(F)>1$, it is not true that $d^*(E+F)>0$, because the elements of $E+F$ may have many representations as the sum of one element of E and other of F. This resonance phenomenon decreases the dimension of $E+F$. Lemma 4.3.4 provides a simple example to this situation: if $E=E_A$ and $F=E_B$, where $A=(a_{ij})_{0\leq i,j\leq 11}$, $B=(b_{ij})_{0\leq i,j\leq 11}$ are defined by

$$a_{ij}=1 \iff 0\leq i,j\leq 3 \quad\text{and}\quad b_{ij}=1 \iff 4\leq i,j\leq 7,$$

then $D(E)+D(F)=2\log 4/\log 12$, while $E+F=E_C$ for $C=(c_{ij})_{0\leq i,j\leq 11}$ given by

$$c_{ij}=1 \iff 4\leq i,j\leq 10.$$

$E+F$ has counting dimension equal to $\log 7/\log 12$ and so $d^*(E+F)=0$. What Theorem 4.1.1 proves is that resonance is avoided if one is allowed to change the scales of the sets, multiplying one of them by a factor $\lambda\in\mathbb{R}$.

4.3.3 Generalized IP-sets

This class of sets was suggested to us by Simon Griffiths and Rob Morris.

Definition 4.3.5. The *generalized IP-set* associated to the sequences of positive integers $(k_n)_{n\geq 1}$, $(d_n)_{n\geq 1}$ is the set

$$E=\left\{\sum_{i=1}^n x_i\cdot d_i\,;\, n\in\mathbb{N} \text{ and } 0\leq x_i<k_i\right\}.$$

We assume $d_n>\sum_{i=1}^{n-1}k_i\cdot d_i$, in which case the map $\sum_{i=1}^n x_i\cdot d_i\mapsto(x_1,\ldots,x_n)$ is a bijection between E and the set of finite sequences

$$\Sigma^*=\{(x_1,\ldots,x_n)\,;\, n\in\mathbb{N}, x_n>0 \text{ and } 0\leq x_i<k_i\}\,.$$

Also, if Σ^* is lexicographically ordered[3], then the map is order-preserving.

[3]The sequence $(x_1,\ldots,x_n)$ is smaller than $(y_1,\ldots,y_m)$ if $n<m$ or if there is $i\in\{1,\ldots,n\}$ such that $x_i<y_i$ and $x_j=y_j$ for $j=i+1,\ldots,n$.

Lemma 4.3.6. *Let E be the generalized IP-set associated to $(k_n)_{n\geq 1}$, $(d_n)_{n\geq 1}$ and let $p_n = k_1 \cdots k_n$. Then*

$$\limsup_{n\to\infty} \frac{\log p_n}{\log k_n d_n} \leq D(E) \leq \limsup_{n\to\infty} \frac{\log p_n}{\log d_n} \cdot \tag{4.3.7}$$

In particular, if $\log k_n / \log p_n \to 0$, then $D(E) = \limsup_{n\to\infty} \log p_n / \log d_n$.

Proof. The calculations are similar to those of Lemma 4.3.4. Let $I = (M, N]$, say

$$\begin{aligned} M+1 &= x_1 \cdot d_1 + \cdots + x_n \cdot d_n \\ N &= y_1 \cdot d_1 + \cdots + y_n \cdot d_n\,, \end{aligned}$$

with $y_n > x_n$. If $y_n \geq x_n + 2$, then

$$\left\{ \begin{array}{rcl} M+1 & \leq & (x_n+1)\cdot d_n \\ N & \geq & (x_n+2)\cdot d_n \end{array} \right. \implies |I| \geq d_n$$

and $|E \cap I| \leq p_n$, thus giving

$$\frac{\log |E\cap I|}{\log |I|} \leq \frac{\log p_n}{\log d_n} \cdot \tag{4.3.8}$$

If $y_n = x_n + 1$, let $i, j \in \{1, \ldots, n-1\}$ be the indices for which

(i) $x_i < k_i - 1$ and $x_j = k_j - 1$, $j = i+1, \ldots, n-1$,

(ii) $y_j > 0$ and $y_{j+1} = \cdots = y_{n-1} = 0$.

Then

$$M+1 \leq x_n \cdot d_n + x_i \cdot d_i + \sum_{\substack{l=1\\ l\neq i}}^{n-1} k_l \cdot d_l < x_n \cdot d_n - d_i + \sum_{l=1}^{n-1} k_l \cdot d_l < (x_n+1)\cdot d_n - d_i$$

and

$$N \geq y_j \cdot d_j + y_n \cdot d_n \geq (x_n+1)\cdot d_n + d_j,$$

implying that $|I| \geq d_i + d_j \geq d_{\max\{i,j\}}$. Now, in order to $\sum_{l=1}^n z_l \cdot d_l$ belong to I, one must have $z_n = x_n$ or $z_n = x_n + 1$. In the first case $z_{i+1} = k_{i+1} - 1, \ldots, z_{n-1} = k_{n-1} - 1$ and in the second $z_{j+1} = \cdots = z_{n-1} = 0$. Then $|E \cap I| \leq p_i + p_j \leq 2p_{\max\{i,j\}}$ and so

$$\frac{\log |E\cap I|}{\log |I|} \leq \frac{\log 2p_{\max\{i,j\}}}{\log d_{\max\{i,j\}}} \cdot \tag{4.3.9}$$

Relations (4.3.8) and (4.3.9) prove the right hand inequality of (4.3.7). The other follows by considering the intervals $(0, \sum_{i=1}^n k_i \cdot d_i]$. □

4.4 Regularity and compatibility

4.4.1 Regular sets

Definition 4.4.1. A subset $E \subset \mathbb{Z}$ is called *regular* or $D(E)$-*set* if $0 < H_{D(E)}(E) < \infty$.

By Lemmas 4.3.2 and 4.3.4, polynomial sets and Cantor sets are regular.

Definition 4.4.2. Given two subsets $E = \{\cdots < x_{-1} < x_0 < x_1 < \cdots\}$ and F of $\mathbb{Z}$, let $E * F$ denote the set

$$E * F = \{x_n\,;\, n \in F\}.$$

This is a subset of E whose counting dimension is at most $D(E)D(F)$. To see this, consider an arbitrary interval $I \subset \mathbb{Z}$. If $E \cap I = \{x_{i+1}, \ldots, x_j\}$, then

$$(E * F) \cap I = \{x_n\,; n \in F \cap (i,j]\}\,.$$

Given $\alpha > D(E)$ and $\beta > D(F)$, relation (4.2.1) guarantees that

$$|(E * F) \cap I| = |F \cap (i,j]| \lesssim (j-i)^{\beta} = |E \cap I|^{\beta} \lesssim |I|^{\alpha\beta}$$

and so $D(E * F) \le \alpha\beta$. Choosing α, β arbitrarily close to $D(E)$, $D(F)$, respectively, it follows that $D(E * F) \le D(E)D(F)$.

If E is regular, it is possible to choose F with arbitrary dimension in such a way that $E * F$ is also regular and has dimension equal to $D(E)D(F)$. To this matter, choose disjoint intervals $I_n = (a_n, b_n]$, $n \ge 1$, with lengths going to infinity such that

$$\frac{|E \cap I_n|}{|I_n|^{D(E)}} \gtrsim 1$$

and let $E \cap I_n = \{x_{i_n+1} < x_{i_n+2} < \cdots < x_{j_n}\}$, where $i_n < j_n$. Given $\alpha \in [0,1]$, let

$$F = \bigcup_{n \ge 1} (E_\alpha + i_n) \cap (i_n, j_n]\,,$$

where E_α is defined as in (4.3.1). Then $D(F) = D(E_\alpha) = \alpha$ and

$$\begin{aligned}
|(E * F) \cap I_n| &\ge |F \cap (i_n, j_n]| \\
&= |E_\alpha \cap (0, j_n - i_n]| \\
&\gtrsim (j_n - i_n)^{D(F)} \\
&= |E \cap I_n|^{D(F)} \\
&\gtrsim |I_n|^{D(E)D(F)},
\end{aligned}$$

implying that

$$|(E * F) \cap I_n| \gtrsim |I_n|^{D(E)D(F)}. \tag{4.4.1}$$

This proves the reverse inequality $D(E*F) \geq D(E)D(F)$. We thus obtain that, given a regular subset $E \subset \mathbb{Z}$ and $0 \leq \alpha \leq D(E)$, there exists a regular subset $E' \subset E$ such that $D(E') = \alpha$. It is a harder task to prove that this holds even when E is not regular.

Proposition 4.4.3. *Let $E \subset \mathbb{Z}$ and $0 \leq \alpha \leq 1$. If $H_\alpha(E) > 0$, then there exists a regular subset $E' \subset E$ such that $D(E') = \alpha$. In particular, for any $0 \leq \alpha < D(E)$, there is $E' \subset E$ regular such that $D(E') = \alpha$.*

Proof. The idea is to apply a dyadic argument in E to decrease $H_\alpha(E)$ in a controlled way. Given an interval $I \subset \mathbb{Z}$ and a subset $F \subset \mathbb{Z}$, define

$$s_F(I) \doteq \sup_{\substack{J \subset I \\ J \text{ interval}}} \frac{|F \cap J|}{|J|^\alpha}\cdot$$

If $F = \{a_1, a_2, \ldots, a_k\} \subset I$, the dyadic operation of alternately discard the elements $a_2, a_4, \ldots, a_{2\lfloor (k-1)/2 \rfloor}$ of the set $\{a_2, a_3, \ldots, a_{k-1}\}$,

$$F = \{a_1, a_2, \ldots, a_k\} \rightsquigarrow F' = \{a_1, a_3, a_5, \ldots, a_{2\lceil (k-1)/2 \rceil - 1}, a_k\},$$

decreases $s_F(I)$ to approximately $s_F(I)/2$. More specifically, if $s_F(I) > 2$, then $s_{F'}(I) > 1/2$. Indeed, for every interval $J \subset I$

$$\frac{|F' \cap J|}{|J|^\alpha} \leq \frac{1}{2} \cdot \frac{|F \cap J| + 1}{|J|^\alpha} \leq \frac{s_F(I)}{2} + \frac{1}{2} < s_F(I) - \frac{1}{2}$$

and, for J maximizing $s_F(I)$,

$$\frac{|F' \cap J|}{|J|^\alpha} \geq \frac{1}{2} \cdot \frac{|F \cap J| - 1}{|J|^\alpha} > 1 - \frac{1}{2 \cdot |J|^\alpha} \geq \frac{1}{2}\cdot$$

After a finite number of these dyadic operations, one obtains a subset $F' \subset F$ such that

$$\frac{1}{2} < s_{F'}(I) \leq 2.$$

If $H_\alpha(E) < \infty$, there is nothing to do. Assume that $H_\alpha(E) = \infty$. We proceed inductively by constructing a sequence $F_1 \subset F_2 \subset \cdots$ of finite subsets of E contained in a sequence of intervals $I_n = (a_n, b_n]$ with increasing lengths such that

(i) $1/2 < s_{F_n}(I_n) \leq 3$;

(ii) there is an interval $J_n \subset I_n$ such that $|J_n| \geq n$ and

$$\frac{|F_n \cap J_n|}{|J_n|^\alpha} > \frac{1}{2}\cdot$$

Once these properties are fulfilled, the set $E' = \bigcup_{n\geq 1} F_n$ will satisfy the required conditions.

Take any $a \in E$ and $I_1 = \{a\}$. Assume I_n, F_n and J_n have been defined satisfying (i) and (ii). As $H_\alpha(E) = \infty$, there exists an interval J_{n+1} disjoint from $(a_n - |I_n|^{1/\alpha}, b_n + |I_n|^{1/\alpha}]$ for which

$$\frac{|E \cap J_{n+1}|}{|J_{n+1}|^\alpha} \geq (n+1)^{1-\alpha}\,.$$

This inequality allows to restrict J_{n+1} to a smaller interval of size at least $n+1$, also denoted J_{n+1}, such that

$$s_E(J_{n+1}) = \frac{|E \cap J_{n+1}|}{|J_{n+1}|^\alpha}\,. \tag{4.4.2}$$

Consider $F'_{n+1} = E \cap J_{n+1}$ and apply the dyadic operation to F'_{n+1} until

$$\frac{1}{2} < s_{F'_{n+1}}(J_{n+1}) = \frac{|F'_{n+1} \cap J_{n+1}|}{|J_{n+1}|^\alpha} \leq 2. \tag{4.4.3}$$

Let $I_{n+1} = I_n \cup K_n \cup J_{n+1}$ be the convex hull[4] of I_n and J_{n+1} and $F_{n+1} = F_n \cup F'_{n+1}$. Condition (ii) is satisfied because of (4.4.3). To prove (i), let I be a subinterval of I_{n+1}. We have three cases to consider.

- $I \subset I_n \cup K_n$: by condition (i) of the inductive hypothesis,

$$\frac{|F_{n+1} \cap I|}{|I|^\alpha} \leq \frac{|F_n \cap (I \cap I_n)|}{|I \cap I_n|^\alpha} \leq 3.$$

- $I \subset K_n \cup J_{n+1}$: by (4.4.3),

$$\frac{|F_{n+1} \cap I|}{|I|^\alpha} \leq \frac{|F'_{n+1} \cap (I \cap J_{n+1})|}{|I \cap J_{n+1}|^\alpha} \leq 2.$$

- $I \supset K_n$: as $|K_n| \geq |I_n|^{1/\alpha}$,

$$\begin{aligned}
\frac{|F_{n+1} \cap I|}{|I|^\alpha} &= \frac{|F_n \cap (I \cap I_n)| + |F'_{n+1} \cap (I \cap J_{n+1})|}{(|I \cap I_n| + |K_n| + |I \cap J_{n+1}|)^\alpha} \\
&\leq \frac{|F_n \cap (I \cap I_n)|}{(|I \cap I_n| + |K_n|)^\alpha} + \frac{|F'_{n+1} \cap (I \cap J_{n+1})|}{(|I \cap J_{n+1}|)^\alpha} \\
&\leq \frac{|I_n|}{|K_n|^\alpha} + s_{F'_{n+1}}(J_{n+1}) \\
&\leq 3.
\end{aligned}$$

This proves condition (i) and completes the inductive step. □

By the above proposition and equation (4.2.2), for any $\alpha \in [0,1]$ and $h \in (0,\infty)$, there exists an α-set $E \subset \mathbb{Z}$ such that $H_\alpha(E) = h$. We'll use this fact in Subsection 4.4.3.

[4]Observe that $|K_n| \geq |I_n|^{1/\alpha}$.

4.4.2 Compatible sets

Definition 4.4.4. Two regular subsets $E, F \subset \mathbb{Z}$ are *compatible* if there exist two sequences $(I_n)_{n\geq 1}$, $(J_n)_{n\geq 1}$ of intervals with increasing lengths such that

1. $|I_n| \sim |J_n|$,
2. $|E \cap I_n| \gtrsim |I_n|^{D(E)}$ and $|F \cap J_n| \gtrsim |J_n|^{D(F)}$.

The notion of compatibility means that E and F have comparable intervals on which the respective intersections obey the correct growth speed of cardinality. Some sets have these intervals in all scales.

Definition 4.4.5. A regular subset $E \subset \mathbb{Z}$ is *universal* if there exists a sequence $(I_n)_{n\geq 1}$ of intervals such that $|I_n| \sim n$ and $|E \cap I_n| \gtrsim |I_n|^{D(E)}$.

It is clear that E and F are compatible whenever one of them is universal and the other is regular. Every E_α is universal and the same happens to polynomial subsets E, due to the asymptotic relation $E \approx aE_{1/d}$, where d is the degree of the polynomial that defines E (see Subsection 4.3.1). In particular, any two polynomial subsets are compatible.

4.4.3 A counterexample of Theorem 4.1.1 for regular non-compatible sets

In this subsection, we construct regular sets $E, F \subset \mathbb{Z}$ such that $D(E)+D(F) > 1$ and $E + \lfloor \lambda F \rfloor$ has zero upper Banach density for every $\lambda \in \mathbb{R}$. The idea is, in the contrast to compatibility, construct E and F such that the intervals $I, J \subset \mathbb{Z}$ for which $\dfrac{|E \cap I|}{|I|^{D(E)}}$ and $\dfrac{|F \cap J|}{|J|^{D(F)}}$ are bounded away from zero have totally different sizes.

Let $\alpha \in (1/2, 1)$. We will define

$$E = \bigcup_{i=1,3,\ldots} (E_i \cap I_i) \quad \text{and} \quad F = \bigcup_{i=2,4,\ldots} (E_i \cap I_i)$$

such that the following conditions hold:

(i) $E_i = \lfloor \mu_i \lfloor {\mu_i}^{-1} E_0 \rfloor \rfloor$, $i \geq 1$, where $E_0 \subset \mathbb{N}$ is an α-set with $H_\alpha(E_0) = 1/2$.

(ii) $I_i = (a_i, b_i]$, $i \geq 1$, is a disjoint sequence of intervals of increasing length such that

$$\lim_{i\to\infty} \frac{|E_i \cap I_i|}{|I_i|^\alpha} = \frac{1}{2} \cdot$$

(iii) $\mu_i > {b_{i-1}}^{\frac{2}{1-\alpha}}$ for every $i \geq 1$.

It is clear that we can inductively construct the sequences $(\mu_i)_{i\geq 1}, (E_i)_{i\geq 1}$ and $(I_i)_{i\geq 1}$ in such a way that $0 < b_i < a_{i+1}$ for every $i \geq 1$. Before going into the calculations, let us explain the above conditions. The dilations in (i) guarantee that consecutive elements of E_i differ at least by (the order of) μ_i; (ii) ensures that E, F are α-sets; (iii) ensures that the referred incompatibility between E and F holds and, also, that the left endpoint of I_i is much bigger than the right endpoint of I_{i-1}.

Lemma 4.4.6. *Let $E, F \subset \mathbb{Z}$ with $D(E), D(F) < 1$, $A, B \subset \mathbb{Z}$ finite and $E' = E \cup A$, $F' = F \cup B$. Then*

$$d^*(E' + \lfloor \lambda F' \rfloor) = d^*(E + \lfloor \lambda F \rfloor)\,,\ \forall\, \lambda \in \mathbb{R}.$$

Proof. We have

$$E' + \lfloor \lambda F' \rfloor = (E + \lfloor \lambda F \rfloor) \cup (A + \lfloor \lambda F \rfloor) \cup (E + \lfloor \lambda B \rfloor) \cup (A + \lfloor \lambda B \rfloor)$$

and each of the sets $A + \lfloor \lambda F \rfloor$, $E + \lfloor \lambda B \rfloor$, $A + \lfloor \lambda B \rfloor$ has dimension smaller than one. $\square$

Fix $\lambda > 0$. By removing, if necessary, finitely many intervals I_i, we may also assume that

(iv) $\lfloor \lambda I_i \rfloor \cap \lfloor \lambda I_j \rfloor = \emptyset$ for all $i \neq j$.

(v) $b_i > \max\{4\lambda^{-1}, 4\lambda\}^{\frac{1-\alpha}{1+\alpha}}$ for all $i \geq 1$.

By the previous lemma, this does not affect the value of $d^*(E + \lfloor \lambda F \rfloor)$. With these assumptions,

$$(I_i + \lfloor \lambda I_j \rfloor) \cap (I_k + \lfloor \lambda I_l \rfloor) \neq \emptyset \iff i = k \text{ and } j, l < i \ \text{ or } \ j = l \text{ and } i, k < j$$

and then

$$\begin{aligned}(E + \lfloor \lambda F \rfloor) \cap (a_i, a_{i+1}] \ &= \bigsqcup_{\substack{a \in I_i \\ j = 2,4,\ldots,i-1}} (a + \lfloor \lambda I_j \rfloor) \quad , \text{ if } i \text{ is odd} \\ &= \bigsqcup_{\substack{b \in I_i \\ j = 1,3,\ldots,i-1}} (I_j + \lfloor \lambda b \rfloor) \quad , \text{ if } i \text{ is even}\end{aligned}$$

In addition, (v) implies that, if i is odd, $a, a' \in I_i$ are distinct and $j, k < i$ are even, then the gap between $a + \lfloor \lambda I_j \rfloor$ and $a' + \lfloor \lambda I_k \rfloor$ is at least $|a - a'|/2$; if i is even, $b, b' \in I_i$ are distinct and $j, k < i$ are odd, then the gap between $I_j + \lfloor \lambda b \rfloor$ and $I_k + \lfloor \lambda b' \rfloor$ is at least $|b - b'|/4\lambda$.

By the above fractal-like structure, it is expected that $E + \lfloor \lambda F \rfloor$ has zero upper Banach density. Let us formally prove this. Consider an interval $I = (M, N] \subset \mathbb{Z}$ and, as usual, assume that $M + 1, N \in E + \lfloor \lambda F \rfloor$, say

$$\begin{aligned} M + 1 \ &= \ a' + \lfloor \lambda b' \rfloor \ \in \ I_k + \lfloor \lambda I_l \rfloor \\ N \ &= \ a + \lfloor \lambda b \rfloor \ \in \ I_i + \lfloor \lambda I_j \rfloor\,. \end{aligned}$$

The largest index among i, j, k, l is either i or j. We may assume that $i > j$. In fact, the reverse case is symmetric, because $E + \lfloor \lambda F \rfloor$ is basically $\lfloor \lambda(F + \lfloor \lambda^{-1} E \rfloor) \rfloor$ and, with this interpretation, the roles of I_i and I_j are interchanged. In this situation, we have two cases to consider:

- $a = a'$: an element $r + \lfloor \lambda s \rfloor$ belongs to $(E + \lfloor \lambda F \rfloor) \cap I$ iff $r = a$ and $s \in F \cap [b', b]$ and then

$$\frac{|(E + \lfloor \lambda F \rfloor) \cap I|}{|I|} \leq \frac{|b - b' + 1|^\alpha}{|\lfloor \lambda b \rfloor - \lfloor \lambda b' \rfloor|} \sim \frac{|b - b'|^\alpha}{\lambda \cdot |b - b'|} = \frac{1}{\lambda \cdot |b - b'|^{1-\alpha}} \cdot \tag{4.4.4}$$

- $a > a'$: for $r + \lfloor \lambda s \rfloor$ belong to $(E + \lfloor \lambda F \rfloor) \cap I$, we necessarily have $r \in E \cap [a', a]$ and $s \in I_2 \cup I_4 \cup \cdots \cup I_{i-1}$ and so

$$\frac{|(E + \lfloor \lambda F \rfloor) \cap I|}{|I|} \leq \frac{|E \cap [a', a]| \cdot b_{i-1}}{|a - a'|/2} \sim \frac{|a - a'|^\alpha \cdot b_{i-1}}{|a - a'|} \leq \frac{b_{i-1}}{{\mu_i}^{1-\alpha}} \leq \frac{1}{b_{i-1}} \ , \tag{4.4.5}$$

where in the last inequality we used (ii).

Estimates (4.4.4) and (4.4.5) establish that $E + \lfloor \lambda F \rfloor$ has zero upper Banach density.

4.4.4 Another counterexample of Theorem 4.1.1

We now prove that one can not expect that the set of parameters given by Theorem 4.1.1 contains integers. More specifically, we construct a regular set $E \subset \mathbb{Z}$ such that $D(E) = D(E + \lambda E)$ for every $\lambda \in \mathbb{Z}$. In particular, if $D(E) \in (1/2, 1)$, $E + \lambda E$ has zero upper Banach density and so every parameter $\lambda \in \mathbb{R}$ for which $d^*(E + \lfloor \lambda E \rfloor) > 0$ is not an integer.

Fixed $\alpha \in (1/2, 1)$ and $c \in \mathbb{Z}$, consider the generalized IP-set associated to the sequences

$$k_n = c \cdot 2^n \quad \text{and} \quad d_n = \left\lfloor 2^{n^2/2\alpha} \right\rfloor \ , \quad n \geq 1.$$

Observe that

$$\sum_{i=1}^{n-1} k_i \cdot d_i \leq c \cdot \sum_{i=1}^{n-1} 2^{\frac{i^2}{2\alpha} + i} \leq c \cdot \sum_{j=1}^{\frac{(n-1)^2}{2\alpha} + n - 1} 2^j < c \cdot 2^{\frac{(n-1)^2}{2\alpha} + n} < d_n$$

for large n. Also, by Lemma 4.3.6, this set has counting dimension α. Then, if E is the generalized IP-set associated to

$$k_n = 2^n \quad \text{and} \quad d_n = \left\lfloor 2^{n^2/2\alpha} \right\rfloor ,$$

the arithmetic sum $E + \lambda E$ is also a generalized IP-set, associated to the sequences $\tilde{k}_n = (\lambda + 1) \cdot k_n$ and d_n, which also has counting dimension equal to α. By Proposition 4.4.3, E contains a regular subset with dimension greater than 1/2. This subset gives the required counterexample.

4.5 Proofs of the Theorems

Let E, F be two regular compatible subsets of $\mathbb{Z}$. Throughout the rest of the proof, fix a compact interval Λ of positive real numbers. Given distinct points $z = (a, b)$ and $z' = (a', b')$ of $E \times F$, let

$$\Lambda_{z,z'} = \left\{\lambda \in \Lambda\,;\, a + \lfloor \lambda b \rfloor = a' + \lfloor \lambda b' \rfloor\right\}.$$

Clearly, $\Lambda_{z,z'}$ is empty if $b = b'$. For $b \neq b'$, it is possible to estimate its Lebesgue measure.

Lemma 4.5.1. *Let $z = (a, b)$ and $z' = (a', b')$ be distinct points of $\mathbb{Z}^2$. If $\Lambda_{z,z'} \neq \emptyset$, then*

(a) $m(\Lambda_{z,z'}) \lesssim |b - b'|^{-1}$.

(b) $\min \Lambda \cdot |b - b'| - 1 \leq |a - a'| \leq \max \Lambda \cdot |b - b'| + 1$.

Proof. Assume $b > b'$ and let $\lambda \in \Lambda_{z,z'}$, say $a + \lfloor \lambda b \rfloor = n = a' + \lfloor \lambda b' \rfloor$. Then

$$\left\{\begin{array}{ccccc} n - a & \leq & \lambda b & < & n - a + 1 \\ n - a' & \leq & \lambda b' & < & n - a' + 1 \end{array}\right. \implies a' - a - 1 < \lambda(b - b') < a' - a + 1$$

and so

$$\Lambda_{z,z'} \subset \left(\frac{a' - a - 1}{b - b'}\,, \frac{a' - a + 1}{b - b'}\right),$$

which proves (a). The second part also follows from the above inclusion, as

$$\frac{a' - a - 1}{b - b'} \leq \min \Lambda_{z,z'} \leq \max \Lambda \quad \implies \quad a' - a \leq \max \Lambda \cdot (b - b') + 1$$

and

$$\frac{a' - a + 1}{b - b'} \geq \max \Lambda_{z,z'} \geq \min \Lambda \quad \implies \quad a' - a \geq \min \Lambda \cdot (b - b') - 1.$$

$\square$

By item (b) of the above lemma, $|a - a'| \sim |b - b'|$ whenever $\Lambda_{z,z'} \neq \emptyset$. We point out that, although naive, Lemma 4.5.1 expresses the crucial property of transversality that makes the proof work, and all results related to Marstrand's theorem use a similar idea in one way or another.

Let $(I_n)_{n\geq 1}$ and $(J_n)_{n\geq 1}$ be sequences of intervals satisfying the compatibility conditions of Definition 4.4.4. Associated to these intervals, consider, for each pair $(n, \lambda) \in \mathbb{N} \times \Lambda$, the set

$$N_n(\lambda) = \left\{((a, b), (a', b')) \in ((E \cap I_n) \times (F \cap J_n))^2\,;\, a + \lfloor \lambda b \rfloor = a' + \lfloor \lambda b' \rfloor\right\}$$

and, for each $n \geq 1$, the integral

$$\Delta_n = \int_\Lambda |N_n(\lambda)| dm(\lambda)\,.$$

By a double counting, one has the equality

$$\Delta_n = \sum_{z,z'\in(E\cap I_n)\times(F\times J_n)} m(\Lambda_{z,z'})\,. \tag{4.5.1}$$

Lemma 4.5.2. *Let $D(E)=\alpha$ and $D(F)=\beta$ as above.*

(a) If $\alpha+\beta<1$, then $\Delta_n \lesssim |I_n|^{\alpha+\beta}$.

(b) If $\alpha+\beta>1$, then $\Delta_n \lesssim |I_n|^{2\alpha+2\beta-1}$.

Proof. Using (4.5.1),

$$\begin{aligned}
\Delta_n &= \sum_{z,z'\in(E\cap I_n)\times(F\times J_n)} m(\Lambda_{z,z'}) \\
&= \sum_{\substack{a\in E\cap I_n\\ b\in F\cap J_n}} \sum_{s=1}^{\log|I_n|} \sum_{\substack{a'\in E\cap I_n\\ |a-a'|\sim e^s}} \sum_{\substack{b'\in F\cap J_n\\ |b-b'|\sim e^s}} m(\Lambda_{z,z'}) \\
&\lesssim \sum_{\substack{a\in E\cap I_n\\ b\in F\cap J_n}} \sum_{s=1}^{\log|I_n|} e^{-s}\cdot(e^s)^{\alpha}\cdot(e^s)^{\beta} \\
&= \sum_{\substack{a\in E\cap I_n\\ b\in F\cap J_n}} \sum_{s=1}^{\log|I_n|} (e^s)^{\alpha+\beta-1} \\
&\lesssim |I_n|^{\alpha+\beta}\cdot \sum_{s=1}^{\log|I_n|} \left(e^{\alpha+\beta-1}\right)^s
\end{aligned}$$

and then

$$\begin{aligned}
\Delta_n &\lesssim |I_n|^{\alpha+\beta}\cdot|I_n|^{\alpha+\beta-1} = |I_n|^{2\alpha+2\beta-1} \quad &&\text{, if } \alpha+\beta>1, \\
&\lesssim |I_n|^{\alpha+\beta}\cdot 1 = |I_n|^{\alpha+\beta} &&\text{, if } \alpha+\beta<1.
\end{aligned}$$

□

Proof of Theorem 4.1.1. The proof is divided in three parts.

Part 1. $\alpha+\beta<1$: fix $\varepsilon>0$. By Lemma 4.5.2, the set of parameters $\lambda\in\Lambda$ for which

$$|N_n(\lambda)| \lesssim \frac{|I_n|^{\alpha+\beta}}{\varepsilon} \tag{4.5.2}$$

has Lebesgue measure at least $m(\Lambda)-\varepsilon$. We will prove that

$$\frac{|(E+\lfloor\lambda F\rfloor)\cap(I_n+\lfloor\lambda J_n\rfloor)|}{|I_n+\lfloor\lambda J_n\rfloor|^{\alpha+\beta}} \gtrsim \varepsilon \tag{4.5.3}$$

whenever $\lambda \in \Lambda$ satisfies (4.5.2). For each $(m,n,\lambda) \in \mathbb{Z} \times \mathbb{Z} \times \Lambda$, let

$$s(m,n,\lambda) = |\{(a,b) \in (E \cap I_n) \times (F \cap J_n)\,;\, a + \lfloor \lambda b \rfloor = m\}|\,.$$

Then

$$\sum_{m \in \mathbb{Z}} s(m,n,\lambda) = |E \cap I_n| \cdot |F \cap J_n| \sim |I_n|^{\alpha+\beta} \tag{4.5.4}$$

and

$$\sum_{m \in \mathbb{Z}} s(m,n,\lambda)^2 = |N_n(\lambda)| \lesssim \frac{|I_n|^{\alpha+\beta}}{\varepsilon}\,\cdot \tag{4.5.5}$$

The numerator in (4.5.3) is at least the cardinality of the set $S(n,\lambda) = \{m \in \mathbb{Z}\,;\, s(m,n,\lambda) > 0\}$, because $(E + \lfloor \lambda F \rfloor) \cap (I_n + \lfloor \lambda J_n \rfloor)$ contains $S(n,\lambda)$. By the Cauchy-Schwarz inequality and relations (4.5.4) and (4.5.5),

$$\begin{aligned} |S(n,\lambda)| &\geq \frac{\left(\displaystyle\sum_{m \in \mathbb{Z}} s(m,n,\lambda)\right)^2}{\displaystyle\sum_{m \in \mathbb{Z}} s(m,n,\lambda)^2} \\ &\gtrsim \frac{\left(|I_n|^{\alpha+\beta}\right)^2}{\dfrac{|I_n|^{\alpha+\beta}}{\varepsilon}} \\ &= \varepsilon \cdot |I_n|^{\alpha+\beta} \end{aligned}$$

and so, as $|I_n + \lfloor \lambda J_n \rfloor| \sim |I_n|$,

$$\frac{|(E + \lfloor \lambda F \rfloor) \cap (I_n + \lfloor \lambda J_n \rfloor)|}{|I_n + \lfloor \lambda J_n \rfloor|^{\alpha+\beta}} \gtrsim \frac{|S(n,\lambda)|}{|I_n|^{\alpha+\beta}} \gtrsim \varepsilon\,,$$

establishing (4.5.3).

For each $n \geq 1$, let $G_\varepsilon^n = \{\lambda \in \Lambda\,;\, (4.5.3) \text{ holds}\}$. Then $m\left(\Lambda \backslash G_\varepsilon^n\right) \leq \varepsilon$, and the same holds for the set

$$G_\varepsilon = \bigcap_{n \geq 1} \bigcup_{l=n}^{\infty} G_\varepsilon^l\,.$$

For each $\lambda \in G_\varepsilon$,

$$D_{\alpha+\beta}(E + \lfloor \lambda F \rfloor) \geq \varepsilon \quad \Longrightarrow \quad D(E + \lfloor \lambda F \rfloor) \geq \alpha + \beta$$

and then, as the set $G = \bigcup_{n \geq 1} G_{1/n} \subset \Lambda$ has Lebesgue measure $m(\Lambda)$, Part 1 is completed.

Part 2. $\alpha + \beta > 1$: for a fixed $\varepsilon > 0$, Lemma 4.5.2 implies that the set of parameters $\lambda \in \Lambda$ for which

$$|N_n(\lambda)| \lesssim \frac{|I_n|^{2\alpha+2\beta-1}}{\varepsilon} \tag{4.5.6}$$

has Lebesgue measure at least $m(\Lambda) - \varepsilon$. In this case,

$$\begin{aligned} |S(n,\lambda)| &\geq \frac{\left(\displaystyle\sum_{m\in\mathbb{Z}} s(m,n,\lambda)\right)^2}{\displaystyle\sum_{m\in\mathbb{Z}} s(m,n,\lambda)^2} \\ &\gtrsim \frac{\left(|I_n|^{\alpha+\beta}\right)^2}{\dfrac{|I_n|^{2\alpha+2\beta-1}}{\varepsilon}} \\ &= \varepsilon \cdot |I_n| \end{aligned}$$

and then

$$\frac{|(E + \lfloor \lambda F \rfloor) \cap (I_n + \lfloor \lambda J_n \rfloor)|}{|I_n + \lfloor \lambda J_n \rfloor|} \gtrsim \frac{|S(n,\lambda)|}{|I_n|} \gtrsim \varepsilon.$$

The Borel-Cantelli argument is analogous to Part 1.

Part 3. $\alpha + \beta = 1$: let $n \geq 1$. Being regular, E has a regular subset $E_n \subset E$, also compatible[5] with F, such that $D(E) - 1/n < D(E_n) < D(E)$. Then

$$1 - \frac{1}{n} < D(E_n) + D(F) < 1$$

and so, by Part 1, there is a full Lebesgue measure set Λ_n such that

$$D(E_n + \lfloor \lambda F \rfloor) \geq 1 - \frac{1}{n}\ , \quad \forall\, \lambda \in \Lambda_n.$$

The set $\Lambda = \bigcap_{n\geq 1} \Lambda_n$ has full Lebesgue measure as well and

$$D(E + \lfloor \lambda F \rfloor) \geq 1, \quad \forall\, \lambda \in \Lambda.$$

□

Proof of Theorem 4.1.3. We also divide it in parts.

Part 1. $\sum_{i=0}^{k} D(E_i) \leq 1$: by Theorem 4.1.1,

$$D(E_0 + \lfloor \lambda_1 E_1 \rfloor) \geq D(E_0) + D(E_1)\ , \quad m - \text{a.e. } \lambda_1 \in \mathbb{R}.$$

To each of these parameters, apply Proposition 4.4.3 to obtain a regular subset $F_{\lambda_1} \subset E_0 + \lfloor \lambda_1 E_1 \rfloor$ such that

$$D(F_{\lambda_1}) = D(E_0) + D(E_1)\,.$$

[5] This may be assumed because of relation (4.4.1).

As E_2 is universal, another application of Theorem 4.1.1 guarantees that

$$D(F_{\lambda_1} + \lfloor \lambda_2 E_2 \rfloor) \geq D(E_0) + D(E_1) + D(E_2)\,, \quad m - \text{a.e. } \lambda_2 \in \mathbb{R}$$

and them, by Fubini's theorem,

$$D(E_0 + \lfloor \lambda_1 E_1 \rfloor + \lfloor \lambda_2 E_2 \rfloor) \geq D(E_0) + D(E_1) + D(E_2)\,, \quad m_2 - \text{a.e. } (\lambda_1, \lambda_2) \in \mathbb{R}^2.$$

Iterating the above arguments, it follows that

$$D(E_0 + \lfloor \lambda_1 E_1 \rfloor + \cdots + \lfloor \lambda_k E_k \rfloor) \geq D(E_0) + \cdots + D(E_k)\,, \quad m_k - \text{a.e. } (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k.$$

Part 2. $\sum_{i=0}^{k} D(E_i) > 1$: without loss of generality, we can assume

$$D(E_0) + \cdots + D(E_{k-1}) \leq 1 < D(E_0) + \cdots + D(E_{k-1}) + D(E_k)\,.$$

By Part 1,

$$D(E_0 + \lfloor \lambda_1 E_1 \rfloor + \cdots + \lfloor \lambda_{k-1} E_{k-1} \rfloor) \geq D(E_0) + \cdots + D(E_{k-1})$$

for m_{k-1} – a.e. $(\lambda_1, \ldots, \lambda_{k-1}) \in \mathbb{R}^{k-1}$. To each of these $(k-1)$-tuples, consider a regular subset $F_{(\lambda_1,\ldots,\lambda_{k-1})}$ of $E_0 + \cdots + \lfloor \lambda_{k-1} E_{k-1} \rfloor$ such that

$$D\left(F_{(\lambda_1,\ldots,\lambda_{k-1})}\right) = D(E_0) + \cdots + D(E_{k-1})\,.$$

Finally, because $D(F_{(\lambda_1,\ldots,\lambda_{k-1})}) + D(E_k) > 1$, Theorem 4.1.1 guarantees that

$$d^*\left(F_{(\lambda_1,\ldots,\lambda_{k-1})} + \lfloor \lambda_k E_k \rfloor\right) > 0\ , \quad m - \text{a.e. } \lambda_k \in \mathbb{R}\,,$$

which, after another application of Fubini's theorem, concludes the proof. □

4.6 Concluding remarks

We think there is a more specific way of defining the counting dimension that encodes the conditions of regularity and compatibility. A natural candidate would be a prototype of a Hausdorff dimension, where one looks to all covers, properly renormalized in the unit interval, and takes a $\liminf$. An alternative definition has appeared in [35]. It would be a natural program to prove Marstrand type results in this context.

Another interesting question is to consider arithmetic sums $E + \lambda F$, where $\lambda \in \mathbb{Z}$. These are genuine arithmetic sums and, as we saw in Subsection 4.4.4, their dimension may not increase. We think very strong conditions on the sets E, F are needed to prove analogous results about $E + \lambda F$ for $\lambda \in \mathbb{Z}$.

We also think the results obtained in this chapter work to subsets of $\mathbb{Z}^k$. Given $E \subset \mathbb{Z}^k$, the *upper Banach density* of E is equal to

$$d^*(E) = \limsup_{|I_1|,\ldots,|I_k| \to \infty} \frac{|E \cap (I_1 \times \cdots \times I_k)|}{|I_1 \times \cdots \times I_k|},$$

where $I_1, \ldots, I_k$ run over all intervals of $\mathbb{Z}$, the *counting dimension* of E is

$$D(E) = \limsup_{|I_1|,\ldots,|I_k| \to \infty} \frac{\log |E \cap (I_1 \times \cdots \times I_k)|}{\log |I_1 \times \cdots \times I_k|},$$

where $I_1, \ldots, I_k$ run over all intervals of $\mathbb{Z}$ and, for $\alpha \geq 0$, the *counting α-measure* of E is

$$H_\alpha(E) = \limsup_{|I_1|,\ldots,|I_k| \to \infty} \frac{|E \cap (I_1 \times \cdots \times I_k)|}{|I_1 \times \cdots \times I_k|^\alpha},$$

where $I_1, \ldots, I_k$ run over all intervals of $\mathbb{Z}$. These quantities satisfy similar properties to those in Subsection 4.2.2. The notion of regularity is defined in an analogous manner. For compatibility, we have to take into account the geometry of $\mathbb{Z}^k$. Two regular subsets $E, F \subset \mathbb{Z}^k$ are compatible if there exist sequences of rectangles $R_n = I_1^n \times \cdots \times I_k^n$ and $S_n = J_1^n \times \cdots \times J_k^n$, $n \geq 1$, such that

(i) $|I_i^n| \sim |J_i^n|$ for every $i = 1, 2, \ldots, k$, and

(ii) $|E \cap R_n| \gtrsim |R_n|^{D(E)}$ and $|F \cap S_n| \gtrsim |S_n|^{D(F)}$.

We think the theory of this chapter may be extended to prove that if $E, F \subset \mathbb{Z}^k$ are two regular compatible subsets, then

$$D(E + \lfloor \lambda F \rfloor) \geq \min\{1, D(E) + D(F)\}$$

for Lebesgue almost every $\lambda \in \mathbb{R}$ and, if in addition $D(E) + D(F) > 1$, then $E + \lfloor \lambda F \rfloor$ has positive upper Banach density for Lebesgue almost every $\lambda \in \mathbb{R}$.

Chapter 5

$\mathbb{Z}^d$-actions with prescribed topological and ergodic properties

We extend constructions of Hahn-Katznelson [17] and Pavlov [40] to $\mathbb{Z}^d$-actions on symbolic dynamical spaces with prescribed topological and ergodic properties. More specifically, we describe a method to build $\mathbb{Z}^d$-actions which are (totally) minimal, (totally) strictly ergodic and have positive topological entropy.

5.1 Introduction

Ergodic theory studies statistical and recurrence properties of measurable transformations T acting in a probability space $(X, \mathcal{B}, \mu)$, where μ is a measure invariant by T, that is, $\mu(T^{-1}A) = \mu(A)$, for all $A \in \mathcal{B}$. It investigates a wide class of notions, such as ergodicity, mixing and entropy. These properties, in some way, give qualitative and quantitative aspects of the randomness of T. For example, ergodicity means that T is indecomposable in the metric sense with respect to μ and entropy is a concept that counts the exponential growth rate for the number of statistically significant distinguishable orbit segments.

In most cases, the object of study has topological structures: X is a compact metric space, $\mathcal{B}$ is the Borel σ-algebra of X, μ is a Borel measure probability and T is a homeomorphism of X. In this case, concepts such as minimality and topological mixing give topological aspects of the randomness of T. For example, minimality means that T is indecomposable in the topological sense, that is, the orbit of every point is dense in X.

A natural question arises: how do ergodic and topological concepts relate to each other? How do ergodic properties forbid topological phenomena and vice-versa? Are metric and topological indecomposability equivalent? This last question was answered negatively in [15] via the construction of a minimal diffeomorphism of the torus $\mathbb{T}^2$ which preserves area but is not ergodic.

Another question was raised by W. Parry: suppose T has a unique Borel probability invariant measure and that (X, T) is a minimal transformation. Can (X, T) have positive entropy? The difficulty in answering this at the time was the scarcity of a wide class of minimal and uniquely ergodic transformations. This was solved affirmatively in [17], where F. Hahn and Y. Katznelson developed an inductive method of constructing symbolic dynamical systems with the required properties. The principal idea of the paper was the weak law of large numbers.

Later, works of Jewett and Krieger (see [42]) proved that every ergodic measure-preserving system $(X, \mathcal{B}, \mu, T)$ is metrically isomorphic to a minimal and uniquely ergodic homeomorphism on a Cantor set and this gives many examples to Parry's question: if an ergodic system $(X, \mathcal{B}, \mu, T)$ has positive metric entropy and $\Phi : (X, \mathcal{B}, \mu, T) \to (Y, \mathcal{C}, \nu, S)$ is the metric isomorphism obtained by Jewett-Krieger's theorem, then (Y, S) has positive topological entropy, by the variational principle.

It is worth mentioning that the situation is quite different in smooth ergodic theory, once some regularity on the transformation is assumed. A. Katok showed in [21] that every $C^{1+\alpha}$ diffeomorphism of a compact surface can not be minimal and simultaneously have positive topological entropy. More specifically, he proved that the topological entropy can be written in terms of the exponential growth of periodic points of a fixed order.

Suppose that T is a mesure-preserving transformation on the probability space $(X, \mathcal{B}, \mu)$ and $f : X \to \mathbb{R}$ is a measurable function. A successful area in ergodic theory deals with the convergence of averages $n^{-1} \cdot \sum_{k=1}^{n} f\left(T^k x\right)$, $x \in X$, when n converges to infinity. The well known *Birkhoff's Theorem* states that such limit exists for almost every $x \in X$ whenever f is an L^1-function. Several results have been (and still are being) proved when, instead of $\{1, 2, \ldots, n\}$, average is made along other sequences of natural numbers. A remarkable result on this direction was given by J. Bourgain [6], where he proved that if $p(x)$ is a polynomial with integer coefficients and f is an L^p-function, for some $p > 1$, then the averages $n^{-1} \cdot \sum_{k=1}^{n} f\left(T^{p(k)} x\right)$ converge for almost every $x \in X$. In other words, convergence fails to hold for a negligible set with respect to the measure μ. We mention this result is not true for L^1-functions, as recently proved by Z. Buczolich and D. Mauldin [7].

In [3], V. Bergelson asked if this set is also negligible from the topological point of view. It turned out, by a result of R. Pavlov [40], that this is not true. He proved that, for every sequence $(p_n)_{n \geq 1} \subset \mathbb{Z}$ of zero upper-Banach density, there exist a totally minimal, totally uniquely ergodic and topologically mixing transformation (X, T) and a continuous function $f : X \to \mathbb{R}$ such that $n^{-1} \cdot \sum_{k=1}^{n} f\left(T^{p_k} x\right)$ fails to converge for a residual set of $x \in X$.

Suppose now that (X, T) is totally minimal, that is, (X, T^n) is minimal for every positive integer n. Pavlov also proved that, for every sequence $(p_n)_{n\geq 1} \subset \mathbb{Z}$ of zero upper-Banach density, there exists a totally minimal, totally uniquely ergodic and topologically mixing continuous transformation (X,T) such that $x \notin \overline{\{T^{p_n}x\,;\, n \geq 1\}}$ for an uncountable number of $x \in X$.

In this work, we extend the results of Hahn-Katznelson and Pavlov, giving a method of constructing (totally) minimal and (totally) uniquely ergodic $\mathbb{Z}^d$-actions with positive topological entropy. We carry out our program by constructing closed shift invariant subsets of a sequence space. More specifically, we build a sequence of finite configurations $(\mathcal{C}_k)_{k\geq 1}$ of $\{0,1\}^{\mathbb{Z}^d}$, $\mathcal{C}_{k+1}$ being essentially formed by the concatenation of elements in $\mathcal{C}_k$ such that each of them occurs statistically well-behaved in each element of $\mathcal{C}_{k+1}$, and consider the set of limits of shifted $\mathcal{C}_k$-configurations as $k \to \infty$. The main results are

Theorem 5.1.1. *There exist totally strictly ergodic $\mathbb{Z}^d$-actions $(X,\mathcal{B},\mu,T)$ with arbitrarily large positive topological entropy.*

We should mention that this result is not new, because Jewett-Krieger's Theorem is true for $\mathbb{Z}^d$-actions [46]. This formulation emphasizes to the reader that the constructions, which may be used in other settings, have the additional advantage of controlling the topological entropy.

Theorem 5.1.2. *Given a set $P \subset \mathbb{Z}^d$ of zero upper-Banach density, there exist a totally strictly ergodic $\mathbb{Z}^d$-action $(X,\mathcal{B},\mu,T)$ and a continuous function $f : X \to \mathbb{R}$ such that the ergodic averages*

$$\frac{1}{|P \cap (-n,n)^d|} \sum_{g \in P \cap (-n,n)^d} f\left(T^g x\right)$$

fail to converge for a residual set of $x \in X$. In addition, $(X,\mathcal{B},\mu,T)$ can have arbitrarily large topological entropy.

The above theorem has a special interest when P is an *arithmetic set* for which classical ergodic theory and Fourier analysis have established almost-sure convergence. This is the case (also proved in [6]) when

$$P = \{(p_1(n),\ldots,p_d(n))\,;\, n \in \mathbb{Z}\},$$

where $p_1,\ldots,p_d$ are polynomials with integer coefficients: for any $f \in L^p$, $p > 1$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(T^{(p_1(k),\ldots,p_d(k))}x\right)$$

exists almost-surely. Note that P has zero upper-Banach density whenever one of the polynomials has degree greater than 1.

Theorem 5.1.3. *Given a set $P \subset \mathbb{Z}^d$ of zero upper-Banach density, there exists a totally strictly ergodic $\mathbb{Z}^d$-action $(X,\mathcal{B},\mu,T)$ and an uncountable set $X_0 \subset X$ for which $x \notin \overline{\{T^p x\,;\, p \in P\}}$, for every $x \in X_0$. In addition, $(X,\mathcal{B},\mu,T)$ can have arbitrarily large topological entropy.*

Yet in the arithmetic setup, Theorem 5.1.3 is the best topological result one can expect. Indeed, Bergelson and Leibman proved in [4] that if T is a minimal $\mathbb{Z}^d$-action, then there is a residual set $Y \subset X$ for which $x \in \overline{\{T^{(p_1(n),\dots,p_d(n))}x\,;\,n \in \mathbb{Z}\}}$, for every $x \in Y$.

5.2 Preliminaries

We begin with some notation. Consider a metric space X, $\mathcal{B}$ its Borel σ-algebra and a G group with identity e. Throughout this work, G will denote $\mathbb{Z}^d$, $d > 1$, or one of its subgroups.

5.2.1 Group actions

Definition 5.2.1. A G-action on X is a measurable transformation $T : G \times X \to X$, denoted by (X, T), such that

(i) $T(g_1, T(g_2, x)) = T(g_1 g_2, x)$, for every $g_1, g_2 \in G$ and $x \in X$.

(ii) $T(e, x) = x$, for every $x \in X$.

In other words, for each $g \in G$, the restriction

$$\begin{array}{cccc} T^g & : & X & \longrightarrow & X \\ & & x & \longmapsto & T(g,x) \end{array}$$

is a bimeasurable transformation on X such that $T^{g_1 g_2} = T^{g_1} T^{g_2}$, for every $g_1, g_2 \in G$. When G is abelian, $(T^g)_{g \in G}$ forms a commutative group of bimeasurable transformations on X. For each $x \in X$, the *orbit* of X with respect to T is the set

$$\mathcal{O}_T(x) \doteq \{T^g x\,;\, g \in G\}.$$

If F is a subgroup of G, the restriction $T|_F : F \times X \to X$ is clearly a F-action on X.

Definition 5.2.2. We say that (X, T) is *minimal* if $\mathcal{O}_T(x)$ is dense in X, for every $x \in X$, and *totally minimal* if $\mathcal{O}_{T|_F}(x)$ is dense in X, for every $x \in X$ and every subgroup $F < G$ of finite index.

Remind that the *index* of a subgroup F, denoted by $(G : F)$, is the number of cosets of F in G. The above definition extends the notion of total minimality of $\mathbb{Z}$-actions. In fact, a $\mathbb{Z}$-action (X, T) is totally minimal if and only if $T^n : X \to X$ is a minimal transformation, for every $n \in \mathbb{Z}$.

Consider the set $\mathcal{M}(X)$ of all Borel probability measures in X. A probability $\mu \in \mathcal{M}(X)$ is *invariant under* T or simply T-*invariant* if

$$\mu\,(T^g A) = \mu(A),\ \forall\, g \in G,\ \forall\, A \in \mathcal{B}.$$

Let $\mathcal{M}_T(X) \subset \mathcal{M}(X)$ denote the set of all T-invariant probability measures. Such set is non-empty whenever G is amenable, by a Krylov-Bogolubov argument applied to any Fϕlner sequence[1] of G.

Definition 5.2.3. A G *measure-preserving system* or simply G-*mps* is a quadruple $(X, \mathcal{B}, \mu, T)$, where T is a G-action on X and $\mu \in \mathcal{M}_T(X)$.

We say that $A \in \mathcal{B}$ is T-invariant if $T^g A = A$, for all $g \in G$.

Definition 5.2.4. The G-mps $(X, \mathcal{B}, \mu, T)$ is *ergodic* if it has only trivial invariant sets, that is, if $\mu(A) = 0$ or 1 whenever A is a measurable set invariant under T.

Definition 5.2.5. The G-action (X, T) is *uniquely ergodic* if $\mathcal{M}_T(X)$ is unitary, and *totally uniquely ergodic* if, for every subgroup $F < G$ of finite index, the restricted F-action $(X, T|_F)$ is uniquely ergodic.

Definition 5.2.6. We say that (X, T) is *strictly ergodic* if it is minimal and uniquely ergodic, and *totally strictly ergodic* if, for every subgroup $F < G$ of finite index, the restricted F-action $(X, T|_F)$ is strictly ergodic.

The result below was proved in [47] and states the pointwise ergodic theorem for $\mathbb{Z}^d$-actions.

Theorem 5.2.7. *Let $(X, \mathcal{B}, \mu, T)$ be a $\mathbb{Z}^d$-mps. Then, for every $f \in L^1(\mu)$, there is a T-invariant function $\tilde{f} \in L^1(\mu)$ such that*

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{g \in [0,n)^d} f\left(T^g x\right) = \tilde{f}(x)$$

for μ-almost every $x \in X$. In particular, if the action is ergodic, $\tilde{f}$ is constant and equal to $\int f d\mu$.

Above, $[0, n)$ denotes the set $\{0, 1, \ldots, n-1\}$, $[0, n)^d$ the d-dimensional cube $[0, n) \times \cdots \times [0, n)$ of $\mathbb{Z}^d$ and by a T-invariant function we mean that $f \circ T^g = f$, for every $g \in G$. These averages allow the characterization of unique ergodicity. Let $C(X)$ denote the space of continuous functions from X to $\mathbb{R}$.

Proposition 5.2.8. *Let (X, T) be a $\mathbb{Z}^d$-action on the compact metric space X. The following items are equivalent.*

(a) (X, T) is uniquely ergodic.

[1] $(A_n)_{n\geq 1}$ is a Fϕlner sequence of G if $\lim_{n\to\infty} |A_n \Delta g A_n|/|A_n| = 0$ for every $g \in G$.

(b) For every $f \in C(X)$ and $x \in X$, the limit

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{g\in[0,n)^d} f\left(T^g x\right)$$

exists and is independent of x.

(c) For every $f \in C(X)$, the sequence of functions

$$f_n = \frac{1}{n^d} \sum_{g\in[0,n)^d} f \circ T^g$$

converges uniformly in X to a constant function.

Proof. The implications (c)⇒(b)⇒(a) are obvious. It remains to prove (a)⇒(c). Let $\mathcal{M}_T(X) = \{\mu\}$. We'll show that f_n converges uniformly to $\tilde{f} = \int f d\mu$. By contradiction, suppose this is not the case for some $f \in C(X)$. This means that there exist $\varepsilon > 0$, $n_i \to \infty$ and $x_i \in X$ such that

$$\left| f_{n_i}(x_i) - \int f d\mu \right| \geq \varepsilon.$$

For each i, let $\nu_i \in \mathcal{M}(X)$ be the probability measure associated to the linear functional $\Theta_i : C(X) \to \mathbb{R}$ defined by

$$\Theta_i(\varphi) = \frac{1}{{n_i}^d} \sum_{g\in[0,n_i)^d} \varphi(T^g x_i)\,,\ \varphi \in C(X).$$

Restricting to a subsequence, if necessary, we assume that $\nu_i \to \nu$ in the weak-star topology. Because the cubes $A_i = [0, n_i)^d$ form a Fϕlner sequence in $\mathbb{Z}^d$, $\nu \in \mathcal{M}_T(X)$. In fact, for each $h \in \mathbb{Z}^d$,

$$\begin{aligned}
\left| \int \left(\varphi \circ T^h\right) d\nu - \int \varphi d\nu \right| &= \lim_{i\to\infty} \frac{1}{{n_i}^d} \left| \sum_{g\in A_i+h} \varphi(T^g x_i) - \sum_{g\in A_i} \varphi(T^g x_i) \right| \\
&\leq \max_{x\in X} |\varphi(x)| \cdot \lim_{i\to\infty} \frac{\#A_i \Delta (A_i + h)}{\#A_i} \\
&= 0.
\end{aligned}$$

But

$$\left| \int f d\nu - \int f d\mu \right| = \lim_{i\to\infty} \left| \int f d\nu_i - \int f d\mu \right| = \lim_{i\to\infty} \left| f_{n_i}(x_i) - \int f d\mu \right| \geq \varepsilon$$

and so $\nu \neq \mu$, contradicting the unique ergodicity of (X, T). □

5.2.2 Subgroups of $\mathbb{Z}^d$

Let $\mathcal{F}$ be the set of all subgroups of $\mathbb{Z}^d$ of finite index. This set is countable, because each element of $\mathcal{F}$ is generated by d linearly independent vectors of $\mathbb{Z}^d$. Consider, then, a subset $(F_k)_{k\in\mathbb{N}}$ of $\mathcal{F}$ such that, for each $F \in \mathcal{F}$, there exists $k_0 > 0$ such that $F_k < F$, for every $k \geq k_0$. For this, just consider an enumeration of $\mathcal{F}$ and define F_k as the intersection of the first k elements. Such intersections belong to $\mathcal{F}$ because

$$(\mathbb{Z}^d : F \cap F') \leq (\mathbb{Z}^d : F) \cdot (\mathbb{Z}^d : F'),\ \forall\, F, F' < \mathbb{Z}^d.$$

Restricting them, if necessary, we assume that $F_k = m_k \cdot \mathbb{Z}^d$, where $(m_k)_{k\geq 1}$ is an increasing sequence of positive integers. Observe that $(\mathbb{Z}^d : F_k) = {m_k}^d$. Such sequence will be fixed throughout the rest of the chapter.

Definition 5.2.9. Given a subgroup $F < \mathbb{Z}^d$, we say that two elements $g_1, g_2 \in \mathbb{Z}^d$ are *congruent modulo* F if $g_1 - g_2 \in F$ and denote it by $g_1 \equiv_F g_2$. The set $\bar{F} \subset \mathbb{Z}^d$ is a *complete residue set modulo* F if, for every $g \in \mathbb{Z}^d$, there exists a unique $h \in \bar{F}$ such that $g \equiv_F h$.

Every complete residue set modulo F is canonically identified to the quocient $\mathbb{Z}^d/F$ and has exactly $(\mathbb{Z}^d : F)$ elements.

5.2.3 Symbolic spaces

Let $\mathcal{C}$ be a finite alphabet and consider the set $\Omega(\mathcal{C}) = \mathcal{C}^{\mathbb{Z}^d}$ of all functions $x : \mathbb{Z}^d \to \mathcal{C}$. We endow $\mathcal{C}$ with the discrete topology and $\Omega(\mathcal{C})$ with the product topology. By Tychonoff's theorem, $\Omega(\mathcal{C})$ is a compact metric space. We are not interested in a particular metric in $\Omega(\mathcal{C})$. Instead, we consider a basis of topology $\mathcal{B}_0$ to be defined below.

Consider the family $\mathcal{R}$ of all finite d-dimensional cubes $A = [r_1, r_1 + n) \times \cdots \times [r_d, r_d + n)$ of $\mathbb{Z}^d$, $n \geq 0$. We say that A has length n and is centered at $g = (r_1, \ldots, r_d) \in \mathbb{Z}^d$.

Definition 5.2.10. A *configuration* or *pattern* is a pair $b_A = (A, b)$, where $A \in \mathcal{R}$ and b is a function from A to $\mathcal{C}$. We say that b_A is *supported in A with encoding function b.*

Let $\Omega_A(\mathcal{C})$ denote the space of configurations supported in A and $\Omega^*(\mathcal{C})$ the space of all configurations in $\mathbb{Z}^d$:

$$\Omega^*(\mathcal{C}) \doteq \{b_A\,;\, b_A \text{ is a configuration}\}.$$

Given $A \in \mathcal{R}$, consider the map $\Pi_A : \Omega(\mathcal{C}) \to \Omega_A(\mathcal{C})$ defined by the restriction

$$\begin{array}{rcccl} \Pi_A(x) & : & A & \longrightarrow & \mathcal{C} \\ & & g & \longmapsto & x(g) \end{array}$$

In particular, $\Pi_{\{g\}}(x) = x(g)$. We use the simpler notation $x|_A$ to denote $\Pi_A(x)$.

Definition 5.2.11. If $A \in \mathcal{R}$ is centered at g, we say that $x|_A$ is a *configuration of* x *centered at* g or that $x|_A$ *occurs in* x *centered at* g.

For $A_1, A_2 \in \mathcal{R}$ such that $A_1 \subset A_2$, let $\pi_{A_1}^{A_2} : \Omega_{A_2} \to \Omega_{A_1}$ be the restriction

$$\begin{array}{rcccc} \pi_{A_1}^{A_2}(b) & : & A_1 & \longrightarrow & \mathcal{C} \\ & & g & \longmapsto & b(g) \end{array}$$

As above, when there is no ambiguity, we denote $\pi_{A_1}^{A_2}(b)$ simply by $b|_{A_1}$. It is clear that the diagram below commutes.

$$\begin{tikzcd} \Omega(\mathcal{C}) \arrow[r, "\Pi_{A_2}"] \arrow[dr, "\Pi_{A_1}"'] & \Omega_{A_2}(\mathcal{C}) \arrow[d, "\pi_{A_1}^{A_2}"] \\ & \Omega_{A_1}(\mathcal{C}) \end{tikzcd}$$

These maps will help us to control the patterns to appear in the constructions of Section 5.3.

By a cylinder in $\Omega(\mathcal{C})$ we mean the set of elements of $\Omega(\mathcal{C})$ with some fixed configuration. More specifically, given $b_A \in \Omega^*(\mathcal{C})$, the *cylinder generated by* b_A is the set

$$\mathrm{Cyl}(b_A) \doteq \{x \in \Omega(\mathcal{C})\,;\, x|_A = b_A\}.$$

The family $\mathcal{B}_0 := \{\mathrm{Cyl}(b_A)\,|\, b_A \in \Omega^*(\mathcal{C})\}$ forms a clopen set of cylinders generating $\mathcal{B}$. Hence the set $C_0 = \{\chi_B\,;\, B \in \mathcal{B}_0\}$ of cylinder characteristic functions generates a dense subspace in $C(\Omega(\mathcal{C}))$. Let μ be the probability measure defined by

$$\mu(\mathrm{Cyl}(b_A)) = |\mathcal{C}|^{-|A|},\ \forall\, b_A \in \Omega^*(\mathcal{C})\,,$$

and extended to $\mathcal{B}$ by Caratheódory's Theorem. Above, $|\cdot|$ denotes the number of elements of a set.

Consider the $\mathbb{Z}^d$-action $T : \mathbb{Z}^d \times \Omega(\mathcal{C}) \to \Omega(\mathcal{C})$ defined by

$$T^g(x) = (x(g+h))_{h\in\mathbb{Z}^d}\,,$$

also called the *shift action*. Given $B = \mathrm{Cyl}(b_A)$ and $g \in \mathbb{Z}^d$, let $B+g$ denote the cylinder associated to $b_{A+g} = (\tilde{b}, A+g)$, where $\tilde{b} : A+g \to \{0,1\}$ is defined by $\tilde{b}(h) = b(h-g)$, $\forall\, h \in A+g$. With this notation,

$$\chi_B \circ T^g = \chi_{B+g}\,. \tag{5.2.1}$$

In fact,

$$\begin{array}{rrcl} & \chi_B(T^g x) & = & 1 \\ \iff & T^g x & \in & B \\ \iff & x(g+h) & = & b(h),\ \forall\, h \in A \\ \iff & x(h) & = & \tilde{b}(h),\ \forall\, h \in A+g \\ \iff & x & \in & B+g. \end{array}$$

Definition 5.2.12. A *subshift* of $(\Omega(\mathcal{C}), T)$ is a $\mathbb{Z}^d$-action (X, T), where X is a closed subset of $\Omega(\mathcal{C})$ invariant under T.

5.2.4 Topological entropy

For each subset X of $\Omega(\mathcal{C})$ and $A \in \mathcal{R}$, let

$$\Omega_A(\mathcal{C}, X) = \{x|_A\,;\, x \in X\}$$

denote the set of configurations supported in A which occur in elements of X and $\Omega^*(\mathcal{C}, X)$ the space of all configurations in $\mathbb{Z}^d$ occuring in elements of X,

$$\Omega^*(\mathcal{C}, X) = \bigcup_{A \in \mathcal{R}} \Omega_A(\mathcal{C}, X).$$

Definition 5.2.13. The *topological entropy* of the subshift (X, T) is the limit

$$h(X, T) = \lim_{n \to \infty} \frac{\log |\Omega_{[0,n)^d}(\mathcal{C}, X)|}{n^d}\,, \tag{5.2.2}$$

which always exists and is equal to $\inf_{n \in \mathbb{N}} \frac{1}{n^d} \cdot \log |\Omega_{[0,n)^d}(\mathcal{C}, X)|$.

5.2.5 Frequencies and unique ergodicity

Definition 5.2.14. Given configurations $b_{A_1} \in \Omega_{A_1}(\mathcal{C})$ and $b_{A_2} \in \Omega_{A_2}(\mathcal{C})$, the *set of ocurrences of b_{A_1} in b_{A_2}* is

$$S(b_{A_1}, b_{A_2}) \doteq \{g \in \mathbb{Z}^d\,;\, A_1 + g \subset A_2 \text{ and } \pi^{A_2}_{A_1+g}(b_{A_2}) = b_{A_1+g}\}.$$

The *frequency of b_{A_1} in b_{A_2}* is defined as

$$\mathrm{fr}(b_{A_1}, b_{A_2}) \doteq \frac{|S(b_{A_1}, b_{A_2})|}{|A_2|}\,.$$

Given $F \in \mathcal{F}$ and $h \in \mathbb{Z}^d$, the *set of ocurrences of b_{A_1} in b_{A_2} centered at h modulo F* is

$$S(b_{A_1}, b_{A_2}, h, F) \doteq \{g \in S(b_{A_1}, b_{A_2})\,;\, A_1 + g \text{ is centered at a vertex } \equiv_F h\}$$

and the *frequency of b_{A_1} in b_{A_2} centered at h modulo F* is the quocient

$$\mathrm{fr}(b_{A_1}, b_{A_2}, h, F) \doteq \frac{|S(b_{A_1}, b_{A_2}, h, F)|}{|A_2|}.$$

Observe that if $\bar{F} \subset \mathbb{Z}^d$ is a complete residue set modulo F, then

$$\mathrm{fr}(b_{A_1}, b_{A_2}) = \sum_{g \in \bar{F}} \mathrm{fr}(b_{A_1}, b_{A_2}, g, F).$$

To our purposes, we rewrite Proposition 5.2.8 in a different manner.

Proposition 5.2.15. *A subshift (X,T) is uniquely ergodic if and only if, for every $b_A \in \Omega^*(\mathcal{C})$ and $x \in X$,*

$$\mathrm{fr}(b_A, x) \doteq \lim_{n\to\infty} \mathrm{fr}\left(b_A, x|_{[0,n)^d}\right)$$

exists and is independent of x.

Proof. By approximation, condition (b) of Proposition 5.2.8 holds for $C(X)$ if and only if it holds for $C_0 = \{\chi_B\,;\, B \in \mathcal{B}_0\}$. If $f = \chi_{\mathrm{Cyl}(b_A)}$, (5.2.1) implies that

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} \frac{1}{n^d} \sum_{g\in[0,n)^d} f\left(T^g x\right) \\
&= \lim_{n\to\infty} \frac{1}{n^d} \sum_{g\in[0,n)^d} \chi_{\mathrm{Cyl}\left(b_{A+g}\right)}(x) \\
&= \lim_{n\to\infty} \frac{1}{n^d} \sum_{\substack{g\in[0,n)^d \\ A+g\subset[0,n)^d}} \chi_{\mathrm{Cyl}\left(b_{A+g}\right)}(x) \\
&= \lim_{n\to\infty} \mathrm{fr}\left(b_A, x|_{[0,n)^d}\right) \\
&= \mathrm{fr}(b_A, x),
\end{aligned}$$

where in the third equality we used that, for a fixed $A \in \mathcal{R}$,

$$\lim_{n\to\infty} \frac{|\{g \in [0,n)^d\,;\, A+g \not\subset [0,n)^d\}|}{n^d} = 0.$$

$\square$

Corollary 5.2.16. *A subshift (X,T) is totally uniquely ergodic if and only if, for every $b_A \in \Omega^*(\mathcal{C})$, $x \in X$ and $F \in \mathcal{F}$,*

$$\mathrm{fr}(b_A, x, F) \doteq \lim_{n\to\infty} \mathrm{fr}\left(b_A, x|_{[0,n)^d}, 0, F\right)$$

exists and is independent of x.

So, unique ergodicity is all about constant frequencies. We'll obtain this via the Law of Large Numbers, equidistributing ocurrences of configurations along residue classes of subgroups.

5.2.6 Law of Large Numbers

Intuitively, if A is a subset of $\mathbb{Z}^d$, each letter of $\mathcal{C}$ appears in $x|_A$ with frequency approximately $1/|\mathcal{C}|$, for almost every $x \in \Omega(\mathcal{C})$. This is what the Law of Large Number says. For our purposes, we state this result in a slightly different way. Let $(X, \mathcal{B}, \mu)$ be a probability space and $A \subset \mathbb{Z}^d$ infinite. For each $g \in A$, let $\mathbb{X}_g : X \to \mathbb{R}$ be a random variable.

Theorem 5.2.17. *(Law of Large Numbers) If $(\mathbb{X}_g)_{g\in A}$ is a family of independent and identically distributed random variables such that $\mathbb{E}[\mathbb{X}_g] = m$, for every $g \in A$, then the sequence $\left(\overline{\mathbb{X}}_n\right)_{n\geq 1}$ defined by*

$$\overline{\mathbb{X}}_n = \frac{\sum_{g\in A\cap[0,n)^d} \mathbb{X}_g}{|A \cap [0,n)^d|}$$

converges in probability to m, that is, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\left(\left|\overline{\mathbb{X}}_n - m\right| < \varepsilon\right) = 1.$$

Consider the probability measure space $(X, \mathcal{B}, \mu)$ defined in Subsection 5.2.3. Fixed $w \in \mathcal{C}$, let $\mathbb{X}_g : \Omega(\mathcal{C}) \to \mathbb{R}$ be defined as

$$\begin{aligned} \mathbb{X}_g(x) &= 1\ , \quad \text{if } x(g) = w \\ &= 0\ , \quad \text{if } x(g) \neq w. \end{aligned} \tag{5.2.3}$$

It is clear that $(\mathbb{X}_g)_{g\in\mathbb{Z}^d}$ are independent, identically distributed and satisfy

$$\mathbb{E}[\mathbb{X}_g] = \int_X \mathbb{X}_g(x) d\mu(x) = \frac{1}{|\mathcal{C}|}\ ,\ \forall\, g \in \mathbb{Z}^d.$$

In addition,

$$\overline{\mathbb{X}}_n(x) = \frac{\sum_{g\in[0,n)^d} \mathbb{X}_g(x)}{n^d} = \frac{\left|S\left(w, x|_{[0,n)^d}\right)\right|}{n^d} = \mathrm{fr}\left(w, x|_{[0,n)^d}\right),$$

which implies the

Corollary 5.2.18. *Let $w \in \mathcal{C}$, $g \in \mathbb{Z}^d$, $F \in \mathcal{F}$ and $\varepsilon > 0$.*

(a) The number of elements $b \in \Omega_{[0,n)^d}(\mathcal{C})$ such that

$$\left|\mathrm{fr}(w, b) - \frac{1}{|\mathcal{C}|}\right| < \varepsilon$$

is asymptotic to $|\mathcal{C}|^{n^d}$ as $n \to \infty$.

(b) The number of elements $b \in \Omega_{[0,n)^d}(\mathcal{C})$ such that

$$\left|\mathrm{fr}(w, b, g, F) - \frac{1}{|\mathcal{C}| \cdot (\mathbb{Z}^d : F)}\right| < \varepsilon$$

is asymptotic to $|\mathcal{C}|^{n^d}$ as $n \to \infty$.

(c) The number of elements $b \in \Omega_{[0,n)^d}(\mathcal{C})$ such that

$$\left| \mathrm{fr}(w, b, g, F) - \frac{1}{|\mathcal{C}| \cdot (\mathbb{Z}^d : F)} \right| < \varepsilon$$

for every $w \in \mathcal{C}$ and $g \in \mathbb{Z}^d$ is asymptotic to $|\mathcal{C}|^{n^d}$ as $n \to \infty$.

Proof. (a) The required number is equal to

$$|\mathcal{C}|^{n^d} \cdot \mu\left(\{x \in \Omega(\mathcal{C})\,;\, \left| \mathrm{fr}\left(w, x|_{[0,n)^d}\right) - |\mathcal{C}|^{-1} \right| < \varepsilon\} \right)$$

and is asymptotic to $|\mathcal{C}|^{n^d}$, as the above μ-measure converges to 1.
(b) Take $A = F + g$ and $(\mathbb{X}_h)_{h \in A}$ as in (5.2.3). For any $x \in \Omega(\mathcal{C})$,

$$\begin{aligned} \overline{\mathbb{X}}_n(x) &= \frac{\left| S\left(w, x|_{[0,n)^d}, g, F\right) \right|}{|A \cap [0,n)^d|} \\ &= \mathrm{fr}\left(w, x|_{[0,n)^d}, g, F\right) \cdot (\mathbb{Z}^d : F) + o(1)\,, \end{aligned}$$

because $\left|A \cap [0,n)^d\right|$ is asymptotic to $n^d/(\mathbb{Z}^d : F)$. This implies that for n large

$$\left| \mathrm{fr}\left(w, x|_{[0,n)^d}, g, F\right) - \frac{1}{|\mathcal{C}| \cdot (\mathbb{Z}^d : F)} \right| < \varepsilon \iff \left| \overline{\mathbb{X}}_n(x) - \frac{1}{|\mathcal{C}|} \right| < \varepsilon \cdot (\mathbb{Z}^d : F)$$

and then Theorem 5.2.17 guarantees the conclusion.
(c) As the events are independent, this follows from (b). □

5.3 Main Constructions

Let $\mathcal{C} = \{0, 1\}$. In this section, we construct subshifts (X, T) with topological and ergodic prescribed properties. To this matter, we build a sequence of finite non-empty sets of configurations $\mathcal{C}_k \subset \Omega_{A_k}(\mathcal{C})$, $k \geq 1$, such that:

(i) $A_k = [0, n_k)^d$, where $(n_k)_{k \geq 1}$ is an increasing sequence of positive integers.

(ii) $n_1 = 1$ and $\mathcal{C}_1 = \Omega_{A_1}(\mathcal{C}) \cong \{0, 1\}$.

(iii) $\mathcal{C}_k$ is the concatenation of elements of $\mathcal{C}_{k-1}$, possibly with the insertion of few additional blocks of zeroes and ones.

Given such sequence $(\mathcal{C}_k)_{k \geq 1}$, we consider $X \subset \Omega(\mathcal{C})$ as the set of limits of shifted $\mathcal{C}_k$-patterns as $k \to \infty$, that is, $x \in X$ if there exist sequences $(w_k)_{k \geq 1}$, $w_k \in \mathcal{C}_k$, and $(g_k)_{k \geq 1} \subset \mathbb{Z}^d$ such that

$$x = \lim_{k \to \infty} T^{g_k} w_k.$$

The above limit has an abuse of notation, because T acts in $\Omega(\mathcal{C})$ and $w_k \notin \Omega(\mathcal{C})$. Formally speaking, this means that, for each $g \in \mathbb{Z}^d$, there exists $k_0 \geq 1$ such that

$$x(g) = w_k(g + g_k), \ \forall\, k \geq k_0.$$

By definition, X is invariant under T and, for any k, every $x \in X$ is an infinite concatenation of elements of $\mathcal{C}_k$ and additional blocks of zeroes and ones.

If $\mathcal{C}_k \subset \Omega_{A_k}(\{0,1\})$ and $A \in \mathcal{R}$, $\Omega_A(\mathcal{C}_k)$ is identified in a natural way to a subset of $\Omega_{n_k A}(\{0,1\})$. In some situations, to distinguish this association, we use small letters for $\Omega_A(\mathcal{C}_k)$ and capital letters for $\Omega_{n_k A}(\{0,1\})$[2]. In this situation, if $w \in \Omega_A(\mathcal{C}_k)$ and $g \in A$, the pattern $w(g) \in \mathcal{C}_k$ occurs in $W \in \Omega_{n_k A}(\{0,1\})$ centered at $n_k g$. In other words, if $w_k \in \mathcal{C}_k$, then

$$S(w_k, W, n_k g, F) = n_k \cdot S(w_k, w, g, F). \tag{5.3.1}$$

In each of the next subsections, $(\mathcal{C}_k)_{k\geq 1}$ is constructed with specific combinatorial and statistical properties.

5.3.1 Minimality

The action (X,T) is minimal if and only if, for each $x, y \in X$, every configuration of x is also a configuration of y. For this, suppose $\mathcal{C}_k \subset \Omega_{A_k}(\{0,1\})$ is defined and non-empty.

By the Law of Large Numbers, if l_k is large, every element of $\mathcal{C}_k$ occurs in almost every element of $\Omega_{[0,l_k)^d}(\mathcal{C}_k)$ (in fact, by Corollary 5.2.18, each of them occurs approximately with frequency $1/|\mathcal{C}_k| > 0$). Take any subset $\mathcal{C}_{k+1}$ of $\Omega_{[0,l_k)^d}(\mathcal{C}_k)$ with this property and consider it as a subset of $\Omega_{[0,n_{k+1})^d}(\{0,1\})$, where $n_{k+1} = l_k n_k$.

Let us prove that (X,T) is minimal. Consider $x, y \in X$ and $x|_A$ a finite configuration of x. For large k, $x|_A$ is a subconfiguration of some $w_k \in \mathcal{C}_k$. As y is formed by the concatenation of elements of $\mathcal{C}_{k+1}$, every element of $\mathcal{C}_k$ is a configuration of y. In particular, w_k (and then $x|_A$) is a configuration of y.

5.3.2 Total minimality

The action (X,T) is totally minimal if and only if, for each $x, y \in X$ and $F \in \mathcal{F}$, every configuration $x|_A$ of x centered[3] at 0 also occurs in y centered at some $g \in F$. To guarantee this for every $F \in \mathcal{F}$, we inductively control the ocurrence of subconfigurations centered in finitely many subgroups of $\mathbb{Z}^d$.

[2]For example, $w \in \Omega_A(\mathcal{C}_k)$ and $W \in \Omega_{n_k A}(\{0,1\})$ denote the "same" element.

[3]Because of the T-invariance of X, we can suppose that $x|_A$ is centered in $0 \in \mathbb{Z}^d$. In fact, instead of x, y, we consider $T^g x, T^g y$.

Consider the sequence $(F_k) \subset \mathcal{F}$ defined in Subsection 5.2.2. By induction, suppose $\mathcal{C}_k \subset \Omega_{A_k}(\{0,1\})$ is non-empty satisfying (i), (ii), (iii) and the additional assumption

(iv) $\gcd(n_k, m_k) = 1$ (observe that this holds for $k = 1$).

Take l_k large and $\tilde{\mathcal{C}}_{k+1} \subset \Omega_{[0,l_k m_{k+1})^d}(\mathcal{C}_k)$ non-empty such that

(v) $S(w_k, w|_{[0,l_k m_{k+1}-1)^d}, g, F_k) \neq \emptyset$, for every triple $(w_k, w, g) \in \mathcal{C}_k \times \tilde{\mathcal{C}}_{k+1} \times \mathbb{Z}^d$.

Considering $w|_{[0,l_k m_{k+1}-1)^d}$ as an element of $\Omega_{[0,l_k m_{k+1} n_k - n_k)^d}(\{0,1\})$, (5.3.1) implies that

$$S(w_k, W|_{[0,l_k m_{k+1} n_k - n_k)^d}, n_k g, F_k) \neq \emptyset,\ \forall\,(w_k, w, g) \in \mathcal{C}_k \times \tilde{\mathcal{C}}_{k+1} \times \mathbb{Z}^d.$$

As $\gcd(n_k, m_k) = 1$, the set $n_k\mathbb{Z}^d$ runs over all residue classes modulo F_k and so (the restriction to $[0, l_k m_{k+1} n_k - n_k)^d$ of) every element of $\tilde{\mathcal{C}}_{k+1}$ contains every element of $\mathcal{C}_k$ centered at every residue class modulo F_k.

Obviously, $\mathcal{C}_{k+1}$ must not be equal to $\tilde{\mathcal{C}}_{k+1}$, because m_{k+1} divides $l_k m_{k+1} n_k$. Instead, we take $n_{k+1} = l_k m_{k+1} n_k + 1$ and insert positions B_i, $i = 1, 2, \ldots, d$, next to faces of the cube $[0, l_k m_{k+1} n_k)^d$. These are given by

$$B_i = \{(r_1, \ldots, r_d) \in A_{k+1}\,;\, r_i = l_k m_{k+1} n_k - n_k\}.$$

There is a natural surjection $\Phi : \Omega_{A_{k+1}}(\{0,1\}) \to \Omega_{[0,n_{k+1}-1)^d}(\{0,1\})$ obtained removing the positions $B_1, \ldots, B_d$. More specifically, if

$$\begin{aligned} \delta(r) &= 0\,, \quad \text{if } r < l_k m_{k+1} n_k - n_k\,, \\ &= 1\,, \quad \text{otherwise} \end{aligned}$$

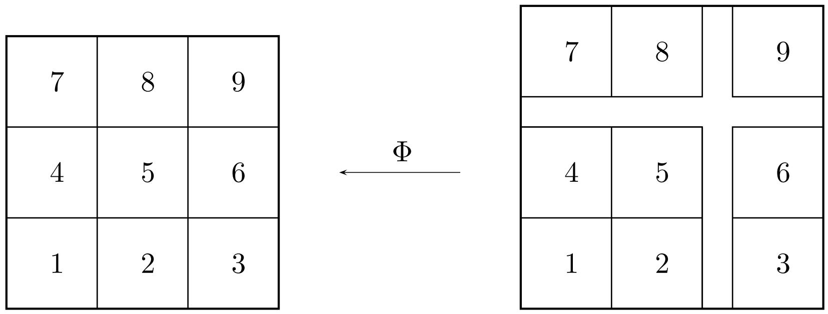
and

$$\Delta(r_1, \ldots, r_d) = (\delta(r_1), \ldots, \delta(r_d)), \tag{5.3.2}$$

the map Φ is given by

$$\Phi(W)(g) = W\left(g + \Delta(g)\right),\ \forall\,(r_1, \ldots, r_d) \in [0, n_{k+1} - 1)^d.$$

We conclude the induction step taking $\mathcal{C}_{k+1} = \Phi^{-1}(\tilde{\mathcal{C}}_{k+1})$.



$\Phi(w_{k+1}) \in \tilde{\mathcal{C}}_{k+1}$ $\qquad$ $w_{k+1} \in \mathcal{C}_{k+1}$

Figure: example of Φ when $d = 2$.

By definition, w_{k+1} and $\Phi(w_{k+1})$ coincide in $[0, n_{k+1} - n_k - 1)^d$, for every $w_{k+1} \in \mathcal{C}_{k+1}$. This implies that every element of $\mathcal{C}_k$ appears in every element of $\mathcal{C}_{k+1}$ centered at every residue class modulo F_k.

Let us prove that (X, T) is totally minimal. Fix elements $x, y \in X$, a subgroup $F \in \mathcal{F}$ and a pattern $x|_A$ of x centered in $0 \in \mathbb{Z}^d$. By the definition of X, $x|_A$ is a subconfiguration of some $w_k \in \mathcal{C}_k$, for k large enough such that $F_k < F$. As y is built concatenating elements of $\mathcal{C}_{k+1}$, w_k occurs in y centered in every residue class modulo F and the same happens to $x|_A$. In particular, $x|_A$ occurs in y centered in some $g \in F$, which is exactly the required condition.

5.3.3 Total strict ergodicity

In addition to the occurrence of configurations in every residue class of subgroups of $\mathbb{Z}^d$, we also control their frequency. Consider a sequence $(d_k)_{k\geq 1}$ of positive real numbers such that $\sum_{k\geq 1} d_k < \infty$. Assume that $\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \mathcal{C}_k$ are non-empty sets satisfying (i), (ii), (iii), (iv) and

(vi) For every $(w_{k-1}, w_k, g) \in \mathcal{C}_{k-1} \times \mathcal{C}_k \times \mathbb{Z}^d$,

$$\mathrm{fr}(w_{k-1}, w_k, g, F_{k-1}) \in \left(\frac{1 - d_{k-1}}{{m_{k-1}}^d \cdot |\mathcal{C}_{k-1}|} \ , \frac{1 + d_{k-1}}{{m_{k-1}}^d \cdot |\mathcal{C}_{k-1}|} \right).$$

Before going to the inductive step, let us make an observation. Condition (vi) also controls the frequency on subgroups F such that $F_{k-1} < F$. In fact, if $\bar{F}_{k-1}$ is a complete residue set modulo F_{k-1},

$$\mathrm{fr}(w_{k-1}, w_k, g, F) = \sum_{\substack{h \in \bar{F}_{k-1} \\ h \equiv_F g}} \mathrm{fr}(w_{k-1}, w_k, h, F_{k-1}) \tag{5.3.3}$$

and, as $\left|\{h \in \bar{F}_{k-1} \,;\, h \equiv_F g\}\right| = (F : F_{k-1})$,

$$\mathrm{fr}(w_{k-1}, w_k, g, F) \in \left(\frac{1 - d_{k-1}}{(\mathbb{Z}^d : F) \cdot |\mathcal{C}_{k-1}|} \ , \frac{1 + d_{k-1}}{(\mathbb{Z}^d : F) \cdot |\mathcal{C}_{k-1}|} \right). \tag{5.3.4}$$

We proceed the same way as in the previous subsection: take l_k large and $\tilde{\mathcal{C}}_{k+1} \subset \Omega_{[0,l_k m_{k+1})^d}(\mathcal{C}_k)$ non-empty such that

$$\mathrm{fr}(w_k, \tilde{w}_{k+1}, g, F_k) \in \left(\frac{1-d_k}{{m_k}^d \cdot |\mathcal{C}_k|} \;\; , \; \frac{1+d_k}{{m_k}^d \cdot |\mathcal{C}_k|} \right) \tag{5.3.5}$$

for every $(w_k, \tilde{w}_{k+1}, g) \in \mathcal{C}_k \times \tilde{\mathcal{C}}_{k+1} \times \mathbb{Z}^d$. Note that the non-emptyness of $\tilde{\mathcal{C}}_{k+1}$ is guaranteed by Corollary 5.2.18. Also, let $n_{k+1} = l_k m_{k+1} n_k + 1$ and $\mathcal{C}_{k+1} = \Phi^{-1}(\tilde{\mathcal{C}}_{k+1})$.

Fix $b_A \in \Omega^*(\mathcal{C})$. Using the big-$O$ notation, we have

$$\mathrm{fr}(b_A, W_{k+1}, g, F) - \mathrm{fr}(b_A, W_{k+1}|_{[0,n_{k+1}-n_k-1)^d}, g, F) = O(1/l_k)\,, \tag{5.3.6}$$

because these two frequencies differ by the frequency of b_A in $[n_{k+1} - n_k - 1, n_{k+1})^d$ and

$$\frac{(n_k+1)^d}{{n_{k+1}}^d} = \left(\frac{n_k+1}{l_k m_{k+1} n_k + 1} \right)^d = O(1/l_k).$$

The same happens to $\mathrm{fr}(w_k, w_{k+1}, g, F)$ and $\mathrm{fr}(w_k, \Phi(w_{k+1}), g, F)$, because $\Delta(g) = 0$ for all $g \in [0, n_{k+1} - n_k - 1)^d$. To simplify citation in the future, we write it down:

$$\mathrm{fr}(w_k, w_{k+1}, g, F) - \mathrm{fr}(w_k, \Phi(w_{k+1}), g, F) = O(1/l_k). \tag{5.3.7}$$

These estimates imply we can assume, taking l_k large enough, that

$$\mathrm{fr}(w_k, w_{k+1}, g, F_k) \in \left(\frac{1-d_k}{{m_k}^d \cdot |\mathcal{C}_k|} \;\; , \; \frac{1+d_k}{{m_k}^d \cdot |\mathcal{C}_k|} \right), \; \forall\, (w_k, w_{k+1}, g) \in \mathcal{C}_k \times \mathcal{C}_{k+1} \times \mathbb{Z}^d.$$

We make a calculation to be used in the next proposition. Fix $b_A \in \Omega^*(\mathcal{C})$ and $F \in \mathcal{F}$. The main (and simple) observation is: if b_A occurs in $W_k \in \mathcal{C}_k$ centered at g and w_k occurs in $\Phi(w_{k+1}) \in \tilde{\mathcal{C}}_{k+1}$ centered at $h \in [0, l_k m_{k+1} - 1)^d$, then b_A occurs in $W_{k+1} \in \mathcal{C}_{k+1}$ centered at $g + n_k h$. This implies that, if $\bar{F}$ is a complete residue set modulo F, the cardinality of $S(b_A, W_{k+1}|_{[0,n_{k+1}-n_k-1)^d}, g, F)$ is equal to

$$\begin{aligned}
& \sum_{\substack{h \in \bar{F} \\ w_k \in \mathcal{C}_k}} \; \sum_{\substack{w_k \text{ occurring in} \\ w_{k+1}|_{[0,l_k m_{k+1}-1)^d} \\ \text{at a vertex } \equiv_F h}} |S(b_A, W_k, g - n_k h, F)| + T \\
= & \sum_{\substack{h \in \bar{F} \\ w_k \in \mathcal{C}_k}} \left| S(w_k, w_{k+1}|_{[0,l_k m_{k+1}-1)^d}, h, F) \right| \cdot |S(b_A, W_k, g - n_k h, F)| + T,
\end{aligned}$$

where T denotes the number of occurrences of b_A in $W_{k+1}|_{[0,n_{k+1}-n_k-1)^d}$ not entirely contained in a concatenated element of $\mathcal{C}_k$. Observe that[4]

$$0 \le T \le d \cdot l_k m_{k+1} \cdot n^{d-1} \cdot n_{k+1} < d n^{d-1} \cdot \frac{{n_{k+1}}^2}{n_k}\;,$$

[4]For each line parallel to a coordinate axis $e_i\mathbb{Z}$ between two elements of $\mathcal{C}_k$ in W_{k+1} or containing a line of ones, there is a rectangle of dimensions $n \times \cdots \times n \times n_{k+1} \times n \times \cdots \times n$ in which b_A is not entirely contained in a concatenated element of $\mathcal{C}_k$.

where n is the length of b_A. Dividing $\left|S(b_A, W_{k+1}|_{[0,n_{k+1}-n_k-1)^d}, g, F)\right|$ by ${n_{k+1}}^d$ and using (5.3.6), (5.3.7), we get

$$\begin{aligned} \mathrm{fr}(b_A, W_{k+1}, g, F) &= \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \sum_{\substack{h\in\bar{F}\\ w_k\in\mathcal{C}_k}} \mathrm{fr}(w_k, w_{k+1}, h, F)\mathrm{fr}(b_A, W_k, g-n_kh, F) \\ &\quad + O(1/l_{k-1})\,. \end{aligned} \tag{5.3.8}$$

We wish to show that $\mathrm{fr}(b_A, x, F)$ does not depend on $x \in X$. For this, define

$$\begin{aligned} \alpha_k(b_A, F) &= \min\left\{\mathrm{fr}(b_A, W_k, g, F)\,;\, W_k \in \mathcal{C}_k, g \in \mathbb{Z}^d\right\} \\ \beta_k(b_A, F) &= \max\left\{\mathrm{fr}(b_A, W_k, g, F)\,;\, W_k \in \mathcal{C}_k, g \in \mathbb{Z}^d\right\}. \end{aligned}$$

The required property is a direct consequence[5] of the next result.

Proposition 5.3.1. *If $b_A \in \Omega^*(\mathcal{C})$ and $F \in \mathcal{F}$, then*

$$\lim_{k\to\infty} \alpha_k(b_A, F) = \lim_{k\to\infty} \beta_k(b_A, F).$$

Proof. By (5.3.3), if l is large such that $F_l < F$, then

$$(F : F_l)\cdot\alpha_k(b_A, F_l) \le \alpha_k(b_A, F) \le \beta_k(b_A, F) \le (F : F_l)\cdot\beta_k(b_A, F_l).$$

This means that we can assume $F = F_l$. We estimate $\alpha_{k+1}(b_A, F)$ and $\beta_{k+1}(b_A, F)$ in terms of $\alpha_k(b_A, F)$ and $\beta_k(b_A, F)$, for $k \ge l$. As b_A and F are fixed, denote the above quantities by α_k and β_k. Take $W_{k+1} \in \mathcal{C}_{k+1}$. By (5.3.8),

$$\begin{aligned} \mathrm{fr}(b_A, W_{k+1}, g, F) &\ge \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \cdot \alpha_k \cdot \sum_{\substack{h\in\bar{F}\\ w_k\in\mathcal{C}_k}} \mathrm{fr}(w_k, w_{k+1}, h, F) + O(1/l_{k-1}) \\ &= \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \cdot \alpha_k + O(1/l_{k-1}) \end{aligned}$$

and, as W_{k+1} and g are arbitrary, we get

$$\alpha_{k+1} \ge \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \cdot \alpha_k + O(1/l_{k-1})\,. \tag{5.3.9}$$

Equality (5.3.8) also implies the upper bound

$$\begin{aligned} \mathrm{fr}(b_A, W_{k+1}, g, F) &\le \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \cdot \beta_k \cdot \sum_{\substack{h\in\bar{F}\\ w_k\in\mathcal{C}_k}} \mathrm{fr}(w_k, w_{k+1}, h, F) + O(1/l_{k-1}) \\ &= \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \cdot \beta_k + O(1/l_{k-1}) \\ \Longrightarrow \qquad \beta_{k+1} &\le \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \cdot \beta_k + O(1/l_{k-1})\,. \end{aligned} \tag{5.3.10}$$

[5] In fact, just take the limit in the inequality $\alpha_k(b_A, F) \le \mathrm{fr}(b_A, x|_{A_k}, 0, F) \le \beta_k(b_A, F)$.

Inequalities (5.3.9) and (5.3.10) show that α_{k+1} and β_{k+1} do not differ very much from α_k and β_k. The same happens to their difference. Consider $w_1, w_2 \in \mathcal{C}_{k+1}$ and $g_1, g_2 \in \mathbb{Z}^d$. Renaming $g - n_k h$ by h in (5.3.8) and considering n_{-k} the inverse of n_k modulo m_l, the difference $\mathrm{fr}(b_A, W_1, g_1, F) - \mathrm{fr}(b_A, W_2, g_2, F)$ is at most

$$\sum_{\substack{h \in \bar{F} \\ w_k \in \mathcal{C}_k}} \mathrm{fr}(b_A, W_k, h, F)\, |\mathrm{fr}(w_k, w_1, n_{-k}(g_1 - h), F) - \mathrm{fr}(w_k, w_2, n_{-k}(g_2 - h), F)| \\ + O(1/l_{k-1})\,.$$

From (5.3.5),

$$\begin{aligned} \mathrm{fr}(b_A, W_1, g_1, F) - \mathrm{fr}(b_A, W_2, g_2, F) &\leq \frac{2d_k}{{m_l}^d \cdot |\mathcal{C}_k|} \sum_{\substack{h \in \bar{F} \\ w_k \in \mathcal{C}_k}} \mathrm{fr}(b_A, W_k, h, F) \\ &\quad + O(1/l_{k-1}) \\ &\leq 2d_k + O(1/l_{k-1})\,, \end{aligned}$$

implying that

$$0 \;\leq\; \beta_{k+1} - \alpha_{k+1} \;\leq\; 2d_k + O(1/l_{k-1})\,. \tag{5.3.11}$$

In particular, $\beta_k - \alpha_k$ converges to zero as $k \to +\infty$. The proposition will be proved if β_k converges. Let us estimate $|\beta_{k+1} - \beta_k|$. On one side, (5.3.10) gives

$$\beta_{k+1} - \beta_k \;\leq\; O(1/l_{k-1})\,. \tag{5.3.12}$$

On the other, by (5.3.9) and (5.3.11),

$$\begin{aligned} \beta_{k+1} - \beta_k &\geq \alpha_{k+1} - \beta_k \\ &\geq \left(\frac{n_{k+1} - 1}{n_{k+1}}\right)^d \cdot \alpha_k - \beta_k + O(1/l_{k-1}) \\ &\geq \left(\frac{n_{k+1} - 1}{n_{k+1}}\right)^d \cdot [\beta_k - 2d_{k-1} - O(1/l_{k-2})] - \beta_k + O(1/l_{k-1}) \end{aligned}$$

which, together with (5.3.12), implies that

$$\begin{aligned} |\beta_{k+1} - \beta_k| &\leq 2d_{k-1} + \beta_k \cdot \left[1 - \left(\frac{n_{k+1} - 1}{n_{k+1}}\right)^d\right] + O(1/l_{k-2}) \\ &= 2d_{k-1} + O(1/l_{k-2})\,. \end{aligned}$$

As $\sum d_k$ and $\sum 1/l_k$ both converge, $(\beta_k)_{k \geq 1}$ is a Cauchy sequence, which concludes the proof.

$\square$

From now on, we consider (X, T) as the dynamical system constructed as above. Note that we have total freedom to choose $\mathcal{C}_k$ with few or many elements. This is what controls the entropy of the system.

5.3.4 Proof of Theorem 5.1.1

By (5.2.2), the topological entropy of the $\mathbb{Z}^d$-action (X, T) satisfies

$$h(X,T) \geq \lim_{k\to\infty} \frac{\log|\mathcal{C}_k|}{{n_k}^d} \cdot$$

Consider a sequence $(\nu_k)_{k\geq 1}$ of positive real numbers. In the construction of $\mathcal{C}_{k+1}$ from $\mathcal{C}_k$, take l_k large enough such that

(vii) $\nu_k \cdot n_{k+1} \geq 1$.

(viii) $|\tilde{\mathcal{C}}_{k+1}| \geq |\mathcal{C}_k|^{(l_k m_{k+1})^d \cdot (1-\nu_k)}$.

These inequalities imply

$$\begin{aligned} \frac{\log|\mathcal{C}_{k+1}|}{{n_{k+1}}^d} &\geq \frac{\log|\tilde{\mathcal{C}}_{k+1}|}{{n_{k+1}}^d} \\ &\geq \frac{(l_k m_{k+1})^d \cdot (1-\nu_k) \cdot \log|\mathcal{C}_k|}{{n_{k+1}}^d} \\ &\geq (1-\nu_k)^{d+1} \cdot \frac{\log|\mathcal{C}_k|}{{n_k}^d} \end{aligned}$$

and then

$$\frac{\log|\mathcal{C}_k|}{{n_k}^d} \geq \prod_{i=1}^{k-1} (1-\nu_i)^{d+1} \cdot \frac{\log|\mathcal{C}_1|}{{n_1}^d} = \prod_{i=1}^{k-1} (1-\nu_i)^{d+1} \cdot \log 2\,.$$

If $\nu \in (0, 1)$ is given and $(\nu_k)_{k\geq 1}$ are chosen also satisfying

$$\lim_{k\to\infty} \prod_{i=1}^{k} (1-\nu_i)^{d+1} = 1-\nu\,,$$

we obtain that $h(X, T) \geq (1 - \nu) \log 2 > 0$. If, instead of $\{0, 1\}$, we take $\mathcal{C}$ with more elements and apply the construction verifying (i) to (viii), the topological entropy of the $\mathbb{Z}^d$-action is at least $(1 - \nu) \log |\mathcal{C}|$. We have thus proved Theorem 5.1.1.

5.4 Proof of Theorems 5.1.2 and 5.1.3

Given a finite alphabet $\mathcal{C}$, consider a configuration $b_{A_1} : A_1 \to \mathcal{C}$ and any $A_2 \subset A_1$ such that $|A_2| \leq \varepsilon |A_1|$. If $b_{A_2} : A_2 \to \mathcal{C}$, the element $w \in \Omega_{A_1}(\mathcal{C})$ defined by

$$\begin{aligned} w(g) &= b_{A_1}(g)\,, \text{ if } g \in A_1 \backslash A_2\,, \\ &= b_{A_2}(g)\,, \text{ if } g \in A_2 \end{aligned}$$

has frequencies not too different from b_{A_1}, depending on how small ε is. In fact, for any $c \in \mathcal{C}$,

$$|S(c, b_{A_1}, g, F)| - |A_2| \leq |S(c, w, g, F)| \leq |S(c, b_{A_1}, g, F)| + |A_2|$$

and then $|\mathrm{fr}(c, b_{A_1}, g, F) - \mathrm{fr}(c, w, g, F)| \leq \varepsilon$.

Definition 5.4.1. The *upper-Banach density* of a set $P \subset \mathbb{Z}^d$ is equal to

$$d^*(P) = \limsup_{n_1,\ldots,n_d \to \infty} \frac{|P \cap [r_1, r_1 + n_1) \times \cdots \times [r_d, r_d + n_d)|}{n_1 \cdots n_d}\,.$$

Consider a set $P \subset \mathbb{Z}^d$ of zero upper-Banach density. We will make l_k grow quickly such that any pattern of $P \cap (A_k + g)$ appears as a subconfiguration in an element of $\mathcal{C}_k$. Let's explain this better. Consider the d-dimensional cubes $(A_k)_{k \geq 1}$ that define (X, T). For each $k \geq 1$, let $\tilde{A}_k \subset A_k$ be the region containing concatenated elements of $\mathcal{C}_{k-1}$. Inductively, they are defined as $\tilde{A}_1 = \{0\}$ and

$$\tilde{A}_{k+1} = \bigcup_{g \in [0, l_k m_{k+1})^d} \left(\tilde{A}_k + n_k g + \Delta(n_k g) \right)\,,\ \forall\, k \geq 1,$$

where Δ is the function defined in (5.3.2).

Lemma 5.4.2. *If $P \subset \mathbb{Z}^d$ has zero upper-Banach density, there exists a totally strictly ergodic $\mathbb{Z}^d$-action (X, T) with the following property: for any $k \geq 1$, $g \in \mathbb{Z}^d$ and $b : P \cap (A_k + g) \to \{0, 1\}$, there exists $w_k \in \mathcal{C}_k$ such that*

$$w_k(h - g) = b(h)\,,\ \forall\, h \in P \cap (A_k + g).$$

Proof. We proceed by induction on k. The case $k = 1$ is obvious, since $\mathcal{C}_1 \cong \{0, 1\}$. Suppose the result is true for some $k \geq 1$ and consider $b : P \cap (A_{k+1} + g_0) \to \{0, 1\}$. By definition, any 0,1 configuration on $A_{k+1} \backslash \tilde{A}_{k+1}$ is admissible, so that we only have to worry about positions belonging to $\tilde{A}_{k+1}$. For each $g \in [0, l_k m_{k+1})^d$, let

$$b^g : P \cap \left(\tilde{A}_k + n_k g + \Delta(n_k g) + g_0 \right) \to \{0, 1\}$$

be the restriction of b to $P\cap\left(\tilde{A}_k+n_kg+\Delta(n_kg)+g_0\right)$. If $\varepsilon>0$ is given and l_k is large enough,

$$\begin{aligned} \frac{\left|P\cap\left(\tilde{A}_{k+1}+g_0\right)\right|}{\left|\tilde{A}_{k+1}+g_0\right|} &< \frac{\varepsilon}{(2n_k)^d} \\ \Longrightarrow \qquad \left|P\cap\left(\tilde{A}_{k+1}+g_0\right)\right| &< \varepsilon\cdot(l_km_{k+1})^d\,, \end{aligned}$$

for any $g_0\in\mathbb{Z}^d$. This implies that $P\cap\left(\tilde{A}_k+n_kg+\Delta(n_kg)+g_0\right)$ is non-empty for at most $\varepsilon\cdot(l_km_{k+1})^d$ values of $g\in[0,l_km_{k+1})^d$. For each of these, the inductive hypothesis guarantees the existence of $w^g\in\mathcal{C}_k$ such that

$$w^g(h-n_kg-\Delta(n_kg)-g_0)=b^g(h)\,,\ \forall\, h\in P\cap\left(\tilde{A}_k+n_kg+\Delta(n_kg)+g_0\right).$$

Take any element $z\in\mathcal{C}_{k+1}$ and define $\tilde{z}\in\Omega_{A_{k+1}}(\{0,1\})$ by

$$\begin{aligned} \tilde{z}(h) &= w^g\left(h-n_kg-\Delta(n_kg)\right)\,,\ \text{if } h\in\tilde{A}_k+n_kg+\Delta(n_kg) \\ &= b(h) \qquad\qquad\qquad\quad\ ,\ \text{if } h\in A_{k+1}\backslash\tilde{A}_{k+1} \\ &= z(h) \qquad\qquad\qquad\quad\ ,\ \text{otherwise.} \end{aligned}$$

If $\varepsilon>0$ is sufficiently small, $\tilde{z}\in\mathcal{C}_{k+1}$. By its own definition, $\tilde{z}$ satisfies the required conditions. $\square$

The above lemma is the main property of our construction. It proves the following stronger statement.

Corollary 5.4.3. *Let (X,T) be the $\mathbb{Z}^d$-action obtained by the previous lemma. For any $b:P\to\{0,1\}$, there is $x\in X$ such that $x|_P=b$. Also, given $x\in X$, $A\in\mathcal{R}$ and $b:P\to\{0,1\}$, there are $\tilde{x}\in X$ and $n\in\mathbb{N}$ such that $\tilde{x}|_A=x|_A$ and $\tilde{x}(g)=b(g)$ for all $g\in P\backslash(-n,n)^d$.*

Proof. The first statement is a direct consequence of Lemma 5.4.2 and a diagonal argument. For the second, remember that x is the concatenation of elements of $\mathcal{C}_k$ and lines of zeroes and ones, for every $k\geq 1$. Consider $k\geq 1$ sufficiently large and $z_k\in\mathcal{C}_k$ such that $x|_A$ occurs in z_k. For any $z\in\mathcal{C}_{k+1}$, there is $g\in\mathbb{Z}^d$ such that $z|_{A_k+g}=z_k$. Constructing $\tilde{z}$ from z making all substitutions described in Lemma 5.4.2, except in the pattern $z|_{A_k+g}$, we still have that $\tilde{z}\in\mathcal{C}_{k+1}$. $\square$

5.4.1 Proof of Theorem 5.1.2

Consider $f:X\to\mathbb{R}$ given by $f(x)=x(0)$. Then

$$\frac{1}{|P\cap(-n,n)^d|}\sum_{g\in P\cap(-n,n)^d} f\left(T^gx\right)=\mathrm{fr}\left(1,x|_{P\cap(-n,n)^d}\right).$$

For each $n \geq 1$, consider the sets

$$\begin{aligned}\Lambda_n &= \bigcup_{k \geq n} \left\{ x \in X\,;\, \mathrm{fr}\left(1, x|_{P\cap(-k,k)^d}\right) < 1/n \right\}\\ \Lambda^n &= \bigcup_{k \geq n} \left\{ x \in X\,;\, \mathrm{fr}\left(1, x|_{P\cap(-k,k)^d}\right) > 1 - 1/n \right\}.\end{aligned}$$

Fixed k and n, the sets $\left\{x \in X\,;\, \mathrm{fr}(1, x|_{P\cap(-k,k)^d}) < 1/n\right\}$ and is clearly open, so that the same happens to Λ_n. It is also dense in X, as we will now prove. Fix $x \in X$ and $\varepsilon > 0$. Let $k_0 \in \mathbb{N}$ be large enough so that $d(x, y) < \varepsilon$ whenever $x|_{(-k_0,k_0)^d} = y|_{(-k_0,k_0)^d}$. Take $y \in X$ such that $y|_{(-k_0,k_0)^d} = x|_{(-k_0,k_0)^d}$ and $y(g) = 0$ for all $g \in P\backslash(-n, n)^d$ as in Corollary 5.4.3. As $\mathrm{fr}(1, y|_{(-k,k)^d})$ approaches to zero as k approaches to infinity, $y \in \Lambda_n$, proving that Λ_n is dense in X. The same argument show that Λ^n is a dense open set. Then

$$X_0 = \bigcap_{n\geq 1} (\Lambda_n \cap \Lambda^n)$$

is a countable intersection of dense open sets, thus residual. For each $x \in X_0$,

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{|P \cap (-n, n)^d|} \sum_{g\in P\cap(-n,n)^d} f(T^g x) &= 0\\ \limsup_{n\to\infty} \frac{1}{|P \cap (-n, n)^d|} \sum_{g\in P\cap(-n,n)^d} f(T^g x) &= 1\,,\end{aligned}$$

which concludes the proof of Theorem 5.1.2.

5.4.2 Proof of Theorem 5.1.3

Choose an infinite set $G = \{g_i\}_{i\geq 1}$ in $\mathbb{Z}^d$ disjoint from P such that $P' = G \cup P \cup \{0\}$ also has zero upper-Banach density and let (X, T) be the $\mathbb{Z}^d$-action given by Lemma 5.4.2 with respect to P', that is: for every $b : P' \to \{0, 1\}$, there exists $x^b \in X$ such that $x^b|_{P'} = b$. Consider

$$X_0 = \left\{ x^b \in X\,;\, b(0) = 0 \text{ and } b(g) = 1,\ \forall\, g \in P \right\}.$$

This is an uncountable set (it has the same cardinality of $2^G = 2^{\mathbb{N}}$) and, for every $x^b \in X_0$ and $g \in P$, the elements $T^g x^b$ and x^b differ at $0 \in \mathbb{Z}^d$, implying that $x^b \notin \overline{\{T^g x^b\,;\, g \in P\}}$. This concludes the proof.

Bibliography

[1] **J. Aaronson**, *An overview of infinite ergodic theory*, Descriptive set theory and dynamical systems, LMS Lecture Notes **277** (2000), 1–29.

[2] **T. Bedford and A. Fisher**, *Analogues of the Lebesgue density theorem for fractal sets of reals and integers*, Proceedings of the London Mathematical Society (3) **64** (1992), no. 1, 95–124.

[3] **V. Bergelson**, *Ergodic Ramsey Theory - an Update*, Ergodic Theory of $\mathbb{Z}^d$-actions, London Math. Soc. Lecture Note Series **228** (1996), 1–61.

[4] **V. Bergelson and A. Leibman**, *Polynomial extensions of van der Waerden's and Szemerédi's theorems*, Journal of AMS **9** (1996), no. 3, 725–753.

[5] **A.S. Besicovitch**, *On existence of subsets of finite measure of sets of infinite measure*, Indagationes Mathematicae **14** (1952), 339–344.

[6] **J. Bourgain**, *Pointwise ergodic theorems for arithmetic sets*, Publ. Math. IHES **69** (1989), 5–45.

[7] **Z. Buczolich and D. Mauldin**, *Divergent square averagesx*, Annals of Mathematics **171** (2010), no. 3, 1479–1530.

[8] **K.L. Chung and P. Erdös**, *Probability limit theorems assuming only the first moment I*, Mem. Amer. Math. Soc. **6** (1951).

[9] **T. Cusick and M. Flahive**, *The Markov and Lagrange spectra*, Mathematical Surveys and Monographs **30**, AMS.

[10] **R.O. Davies**, *Subsets of finite measure in analytic sets*, Indagationes Mathematicae **14** (1952), 488–489.

[11] **K. Falconer**, *The geometry of fractal sets*, Cambridge Tracts in Mathematics, Cambridge (1986).

[12] **A. Fisher**, *Integer Cantor sets and an order-two ergodic theorem*, Ergodic Theory and Dynamical Systems **13** (1993), no. 1, 45–64.

[13] **H. Furstenberg**, *Intersections of Cantor sets and transversality of semigroups*, Problems in analysis (Sympos. Salomon Bochner, Princeton University, 1969), 41–59.

[14] **H. Furstenberg**, *Poincaré recurrence and number theory*, Bull. Amer. Math. Soc. **5** (1981), no. 3, 211–234.

[15] **H. Furstenberg**, *Strict ergodicity and transformations of the torus*, Amer. J. of Math. **83** (1961), 573–601.

[16] **B. Green and T. Tao**, *The primes contain arbitrarily long arithmetic progressions*, Annals of Mathematics **167** (2008), 481–547.

[17] **F. Hahn and Y. Katznelson**, *On the entropy of uniquely ergodic transformations*, Trans. Amer. Math. Soc. **126** (1967), 335–360.

[18] **M. Hall**, *On the sum and product of continued fractions*, Annals of Mathematics **48** (1947), 966–993.

[19] **M. Hochman and P. Shmerkin**, *Local entropy averages and projections of fractal measures*, to appear in Annals of Mathematics.

[20] **A. Katok,** *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Inst. Hautes Études Sci. Publ. Math. **51** (1980), 137–173.

[21] **R. Kaufman**, *On Hausdorff dimension of projections*, Mathematika **15** (1968), 153–155.

[22] **B. Kitchens**, *Symbolic Dynamics: One-Sided, Two-Sided and Countable State Markov Shifts*, Springer (1997).

[23] **Y. Lima**, *$\mathbb{Z}^d$-actions with prescribed topological and ergodic properties*, to appear in Ergodic Theory & Dynamical Systems.

[24] **Y. Lima and C.G. Moreira**, *A Marstrand theorem for subsets of integers*, available at http://arxiv.org/abs/1011.0672.

[25] **Y. Lima and C.G. Moreira**, *A combinatorial proof of Marstrand's theorem for products of regular Cantor sets*, to appear in Expositiones Mathematicae.

[26] **Y. Lima and C.G. Moreira**, *Yet another proof of Marstrand's theorem*, to appear in Bulletin of the Brazilian Mathematical Society.

[27] **J.M. Marstrand**, *Some fundamental geometrical properties of plane sets of fractional dimensions*, Proceedings of the London Mathematical Society **3** (1954), vol. 4, 257–302.

[28] **J.M. Marstrand**, *The dimension of cartesian product sets*, Proceedings of the Cambridge Philosophical Society **50** (1954), 198–202.

[29] **P. Mattila**, *Hausdorff dimension, projections, and the Fourier transform*, Publ. Mat. **48** (2004), no. 1, 3–48.

[30] **P. Mattila and P. Saaranen**, *Ahlfors-David regular sets and bilipschitz maps*, Ann. Acad. Sci. Fenn. Math. **34** (2009), no. 2, 487–502.

[31] **C.G. Moreira**, *A dimension formula for images of cartesian products of regular Cantor sets by differentiable real maps*, in preparation.

[32] **C.G. Moreira**, *Geometric properties of Markov and Lagrange spectra*, available at http://w3.impa.br/∼gugu.

[33] **C.G. Moreira and J.C. Yoccoz**, *Stable Intersections of Cantor Sets with Large Hausdorff Dimension*, Annals of Mathematics **154** (2001), 45–96.

[34] **C.G. Moreira and J.C. Yoccoz**, *Tangences homoclines stables pour des ensembles hyperboliques de grande dimension fractale*, Annales scientifiques de l'ENS 43, fascicule 1 (2010).

[35] **J. Naudts**, *Dimension of discrete fractal spaces*, Journal of Physics A: Math. Gen. **21** (1988), no. 2, 447–452.

[36] **J. Palis**, *Homoclinic orbits, hyperbolic dynamics and dimension of Cantor sets*, Contemporary Mathematics **58** (1987), 203–216.

[37] **J. Palis and F. Takens**, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge studies in advanced mathematics, Cambridge (1993).

[38] **J. Palis and F. Takens**, *Hyperbolicity and the creation of homoclinic orbits*, Annals of Mathematics **125** (1987), no. 2, 337–374.

[39] **J. Palis and J.C. Yoccoz**, *On the Arithmetic Sum of Regular Cantor Sets*, Annales de l'Inst. Henri Poincaré, Analyse Non Lineaire **14** (1997), 439–456.

[40] **R. Pavlov**, *Some counterexamples in topological dynamics*, Ergodic Theory & Dynamical Systems **28** (2008), 1291–1322.

[41] **Y. Peres and P. Shmerkin**, *Resonance between Cantor sets*, Ergodic Theory & Dynamical Systems **29** (2009), 201–221.

[42] **K. Petersen**, *Ergodic Theory*, Cambridge University Press (1983).

[43] **M. Rams**, *Exceptional parameters for iterated function systems with overlaps*, Period. Math. Hungar. **37** (1998), no. 1-3, 111–119.

[44] **M. Rams**, *Packing dimension estimation for exceptional parameters*, Israel J. Math. **130** (2002), 125–144.

[45] **E. Szemerédi**, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245.

[46] **B. Weiss**, *Strictly ergodic models for dynamical systems*, Bull. Amer. Math. Soc. **13** (1985), no. 2, 143–146.

[47] **N. Wiener**, *The ergodic theorem*, Duke Math. J. **5** (1939), no. 1, 1–18.