Instituto Nacional de Matemática Pura e Aplicada

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Upper Bounds for the Dimension of Moduli Spaces of Algebraic Curves with Prescribed Weierstrass Semigroups

Tese de Doutorado

Tese Apresentada à Banca Examinadora do Instituto Nacional de Matemática Pura e Aplicada como Requerimento Parcial para Obtenção do Grau de Doutor em Ciências

Orientador: Prof. Karl-Otto Stöhr

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Abstract

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In this thesis we investigate the dimensions of the moduli spaces of pointed algebraic curves with prescribed Weierstrass semigroups. We present an implementable method to obtain upper bounds for the dimension of moduli spaces of pointed algebraic curves with prescribed symmetric semigroup. In the shown examples and families of symmetric semigroups the upper bounds produced by our method are optimal, or we get the exact dimensions of the moduli spaces or better bounds than those given by Deligne's Formula or Eisenbud–Harris expected dimensions.

Keywords

Weierstrass points. Gorenstein Curves. Symmetric Semigroups. Moduli Space of Curves.

Contents

Chapter 1

Introduction

We present an implementable method to obtain upper bounds for the dimension of moduli spaces of pointed algebraic curves with prescribed symmetric semigroup. In the shown examples and families of symmetric semigroups the upper bounds produced by our method are optimal, or we get the exact dimensions of the moduli spaces or better bounds than those given by Deligne's Formula or Eisenbud–Harris expected dimensions.

Let C be an integral projective algebraic curve of genus $g \geq 2$ defined over an algebraically closed field $\mathbf{k}$ of characteristic zero. For each smooth point $p \in C$, there is associated a subsemigroup $\mathcal{H}$ of the nonnegative integers $\mathbb{N}$ which is formed by pole orders of meromorphic functions of C which are holomorphic on $C \setminus \{p\}$. By the Riemann–Roch Theorem (see [26] for the singular and smooth cases) the cardinality of $\mathbb{N} \setminus \mathcal{H}$ is exactly g. The semigroup $\mathcal{H}$ is called a *Weierstrass semigroup* if it is different from $\{0, g+1, g+2, \ldots\}$. The sequence $\ell_1 < \ldots < \ell_g$ of elements of $\mathbb{N} \setminus \mathcal{H}$, is the *gap sequence*.

A numerical semigroup $\mathcal{H}$ is *realizable* if there is a smooth curve possessing a point whose Weierstrass semigroup is $\mathcal{H}$. The question was posed by Hurwitz of which numerical semigroups are realizable remains open. As is well known, there are numerical semigroups which are not realizable. We refer to [29] for symmetric nonrealizable semigroups. On the other hand, symmetric semigroups are realized by irreducible, possibly singular, Gorenstein curves; see [27].

Let $\mathscr{M}_{g,1}$ be the moduli space of pointed smooth projective curves of genus g. Given a numerical subsemigroup $\mathcal{H}$ of $\mathbb{N}$, let us consider in $\mathscr{M}_{g,1}$ the locally closed subscheme $\mathscr{M}_{\mathcal{H}}$ parameterizing irreducible curves whose Weierstrass semigroup is equal to $\mathcal{H}$ at the base point.

In the 80's Eisenbud and Harris considered, among others, the question: "What are the dimensions of $\mathscr{M}_{\mathcal{H}}$, or when they are reducible, of their components?"; see [8]. Gatto and Ponza [10] pointed as a "difficult problem to study the dimension, when it is nonempty, of the closure of the locus $W(\ell_1, \ell_2, \dots, \ell_g)$". We recall that $W(\ell_1, \ell_2, \dots, \ell_g)$ is the sublocus of isomorphism classes of curves possessing a point with gap sequence $\ell_1, \ell_2, \dots, \ell_g$.

We are, in fact, especially interested in the dimension of $\mathscr{M}_{\mathcal{H}}$ when $\mathcal{H}$ is symmetric. We mention briefly a few relevant results in this direction.

By considering irreducible Gorenstein curves, Stöhr [27] constructed a compactification of $\mathscr{M}_{\mathcal{H}}$, denoted by $\overline{\mathscr{M}_{\mathcal{H}}}$, when $\mathcal{H}$ is symmetric. Its boundary was built by $\mathbb{G}_m(\mathbf{k})$-orbits of nonsmooth integral Gorenstein curves. He constructed the moduli space by deforming curves canonically embedded, analyzing their ideals and their syzygies, he made in a rather explicitly way. Thus we have a promising approach and a rich source of examples. The approach can also be carried out for quasi-symmetric semigroups by considering reducible curves, see [19] and [20].

As regard to the dimension of $\mathscr{M}_{\mathcal{H}}$, Eisenbud and Harris [9, p. 496] proved that the *weight of a semigroup*, $w(\mathcal{H}) = \sum \ell_i - i$, gives an upper bound for the codimension of any component of $\mathscr{M}_{\mathcal{H}}$. Since $\dim \mathscr{M}_{g,1} = 3g-2$, it follows that $3g - 2 - w(\mathcal{H}) \leq \dim \mathscr{M}_{\mathcal{H}}$, this lower bound being called the *expected dimension* of $\mathscr{M}_{\mathcal{H}}$. They also proved that if $w(\mathcal{H}) \leq g-2$, then there is a point $(C, p) \in \mathscr{M}_{g,1}$ such that $\dim \mathscr{M}_{\mathcal{H}} = 3g - 2 - w(\mathcal{H})$ in a neighborhood of (C, p), see [9], Theorem 5.4. On the other hand for certain, in particular, for certain families of symmetric semigroups the lower bound $3g - 2 - w(\mathcal{H})$ can be negative[1] and then it does not provide any information.

In [4], Thm 2.27, Deligne considered the local ring $\mathcal{O}$ at a singular point of a reduced projective algebraic curve, and E an irreducible component of the semiuniversal deformation of $\mathrm{Spec}(\mathcal{O})$. Assuming that the fiber of the

[1]See for example the families in Sections 4.1 and 4.2 of this thesis.

deformation above the generic point of E is smooth, he gave a formula for the dimension of E. Taking into account Pinkham's theorems in [21] we may see that Deligne's formula provides an upper bound: $\dim \mathscr{M}_{\mathcal{H}} \leq 2g + [End(\mathcal{H}) : \mathcal{H}] - 2$, where $End(\mathcal{H}) = \{n \in \mathbb{N} \mid n + h \in \mathcal{H},\ \forall\, h \in \mathcal{H} \setminus \{0\}\}$. We give more details about this in Section 3.1 of this thesis.

Rim and Vitulli, see [23], Section 5, proved that if $\mathcal{H}$ is *negatively graded* then $\mathcal{H}$ is realizable and the dimension of $\mathscr{M}_{\mathcal{H}}$ is equal to $2g + [End(\mathcal{H}) : \mathcal{H}] - 2$. A semigroup is negatively graded if the first cohomology module of the cotangent complex associated to the monomial curve induced by $\mathcal{H}$ has no elements of positive degree.

By studying ideals of codimension three, Waldi, see [30] Korollar 2, described the moduli space $\mathscr{M}_{\mathcal{H}}$ when $\mathcal{H}$ is generated by at most four elements. Recently Nakano and Mori, in [16] and [15], constructed explicitly $\mathscr{M}_{\mathcal{H}}$ when the genus is low ($2 \leq g \leq 5$). They are able to compute the dimension of $\mathscr{M}_{\mathcal{H}}$, to prove its irreducibility and its rationality if $\mathcal{H}$ is generated by at most four elements.

Our main goal is to use [27] to obtain informations about the dimension of $\overline{\mathscr{M}_{\mathcal{H}}}$, *a fortiori* about the dimension of $\mathscr{M}_{\mathcal{H}}$, because $\mathscr{M}_{\mathcal{H}}$ is an open set of $\overline{\mathscr{M}_{\mathcal{H}}}$. In order to avoid trivial cases and those already treated we deal with symmetric semigroups of multiplicity at least six.

The strategy we suggest for achieving such a goal is as follows. Stöhr's compactification $\overline{\mathscr{M}_{\mathcal{H}}}$ is isomorphic to the quotient of an affine algebraic set $\mathcal{X}$ by a $G_m(\mathbf{k})$-action. The vertex of $\mathcal{X}$ belongs to all of its irreducible components; see Proposition 3.6. Thus the dimension of $\overline{\mathscr{M}_{\mathcal{H}}}$ is the local dimension of $\mathcal{X}$ at the vertex minus one. We construct a space, denoted by $\mathcal{Q}_{\mathcal{H}}$, which is given by the zero locus of suitable quadratic forms and contains the quadratic approximation of $\mathcal{X}$ at the vertex. Hence $\mathcal{Q}_{\mathcal{H}}$ provides, by taking into account the tangent cone of $\mathcal{X}$ at the vertex, an upper bound for the dimension of $\overline{\mathscr{M}_{\mathcal{H}}}$, *a fortiori* for the dimension of $\mathscr{M}_{\mathcal{H}}$, see Theorem 3.8.

The thesis is organized as follows. In Chapter 2 we introduce the main objects and notations. Also we summarize without proofs the construction of Stöhr's, which is fundamental for this thesis.

Since Deligne's upper bound plays an important role, Section 1 of Chapter

3 is devoted to recall how to obtain it. To this end, some knowledge on the theory of deformations of curves will be needed.

In section 3.2 we describe how to construct $\mathcal{Q}_{\mathcal{H}}$ and we show that it provides an upper bound for $\overline{\mathscr{M}}_{\mathcal{H}}$. It is also shown that we do not need to construct the moduli space $\overline{\mathscr{M}_{\mathcal{H}}}$ to get $\mathcal{Q}_{\mathcal{H}}$, implying in a tremendous simplification of computations.

Section 3.3 is devoted to the examples. We show for four numerical examples that the upper bound given by $\mathcal{Q}_{\mathcal{H}}$ is really good. In three examples, two of genus 8 and one of genus 9, we get the exact dimension of the moduli space and, in another of genus 9, a better bound than that given by Deligne's formula. However we do not know the dimension of the moduli space correspondig to the latter.

In Chapter 4 we apply our method to one-parameter families of symmetric semigroups. By considering a minimal system of generators for $\mathcal{H}$, we adapt, for two families, Stöhr's Theorem to construct the moduli space $\overline{\mathscr{M}_{\mathcal{H}}}$. The first family consists of semigroups of multiplicity five, $\mathcal{H} :=< 5, 2 + 5\tau, 3 + 5\tau, 4 + 5\tau >$, with $\tau \geq 1$. By analyzing two syzygies, we show explicitly that the moduli space $\overline{\mathscr{M}_{\mathcal{H}}}$ is a weighted projective space isomorphic to $\mathbb{P}(T^{1,-}(B_{\mathcal{H}}))$ with dimension $7\tau + 4$; see Corollary 4.4 for the statement and explanations of notation. As a final illustration of the method, we consider a family of semigroups of multiplicity six, namely $\mathcal{H} :=< 6, 2+6\tau, 3+6\tau, 4+6\tau, 5+6\tau >$, with $\tau \geq 1$. By analyzing five syzygies, we construct explicitly $\mathcal{Q}_{\mathcal{H}}$ and we are able to compute its dimension. In this way, we obtain the upper bound $8\tau + 5$ for $\mathscr{M}_{\mathcal{H}}$, improving the upper bound given by Deligne's Formula which is $12\tau + 1$; see Theorem 4.8. The main idea of the proof of this theorem is to view $\mathcal{Q}_{\mathcal{H}}$ as a closed subset of an affine space over a suitable Artinian $\mathbf{k}$-algebra. For these two families of symmetric semigroups Eisenbud–Harris expected dimensions become negative for $\tau \geq 3$.

Chapter 2

Moduli Space of Curves with Symmetric Weierstrass Semigroups

In this chapter we introduce the main objects and notations. We recall certain results derived from [27] of a compactification of $\mathscr{M}_{\mathcal{H}}$, which is a fundamental tool for this thesis.

A *numerical semigroup* $\mathcal{H}$ is a subsemigroup of the additive semigroup $\mathbb{N}$ of nonnegative integers whose greatest common divisor of its elements is 1. Thus, in a numerical semigroup $\mathcal{H}$ there are only a finite number of elements of $\mathbb{N}$ missing in $\mathcal{H}$. These elements are called *gaps* of $\mathcal{H}$, denoted by ℓ_i, and the elements of $\mathcal{H}$ are the *nongaps*. The number of gaps is the *genus* of $\mathcal{H}$ and denoted by $g = g(\mathcal{H})$. We denote by $n_0 < n_1 < \cdots < n_{g-1}$ the first g nongaps. The least positive integer $m := n_1$ in $\mathcal{H}$ is the *multiplicity* of $\mathcal{H}$.

A numerical semigroup $\mathcal{H}$ of genus g is called *symmetric* if the largest gap ℓ_g is equal to $2g-1$. Equivalently, an integer n belongs to $\mathcal{H}$ if and only if $\ell_g - n$ does not belong to $\mathcal{H}$, thus:

$$\ell_j = 2g - 1 - n_{g-j}\,, \quad \text{for } j = 1, \ldots, g\,. \tag{2.1}$$

Let us consider a nonsingular point P on an irreducible projective curve C of genus g. The *Weierstrass semigroup* of (C, P) is the set of pole orders at P of all meromorphic functions which are holomorphic on $C \setminus \{P\}$. By the

Riemann–Roch Theorem, the Weierstrass semigroup of (C, P) is a numerical semigroup of genus g.

In [27] Stöhr introduced the following curve: Let $\mathcal{H}$ be a numerical symmetric semigroup of genus g with $n_{g-2} = 2g-3$. Take the following curve, called the *canonical monomial curve:*

$$C_0 := \{(u^{\ell_g - 1}v^{n_0} : u^{\ell_{g-1}-1}v^{n_1} : \ldots : u^{\ell_1 - 1}v^{n_{g-1}}) \mid (u:v) \in \mathbb{P}^1(\mathbf{k})\}\,. \qquad (2.2)$$

C_0 is a rational algebraic curve of degree $2g-2$ in $\mathbb{P}^{g-1}(\mathbf{k})$ and arithmetical genus g. It has a unique singular point at $(1:0\ldots:0)$ of multiplicity m and singularity degree g. The function field of C_0 is generated by z, where z is defined by $(u^{\ell_g-1}v^{n_0} : u^{\ell_{g-1}-1}v^{n_1} : \ldots : u^{\ell_1-1}v^{n_{g-1}}) \mapsto u/v$. The point $P := (0:\ldots:0:1)$ is nonsingular and the differentials $z^{\ell_i - i}dz$ form a basis of the space of holomorphic differentials on C_0. We conclude that C_0 is a canonical Gorenstein[1] curve of arithmetical genus g whose Weierstrass semigroup at P is $\mathcal{H}$.

Let us fix a symmetric semigroup $\mathcal{H}$ of genus $g > 4$ and denote by $\overline{\mathscr{M}_{\mathcal{H}}}$ the following set:

$\overline{\mathscr{M}_{\mathcal{H}}}$:= set of isomorphism classes of pointed irreducible projective Gorenstein curves whose Weierstrass semigroup at the base point is $\mathcal{H}$.

Let C be an irreducible projective Gorenstein curve C of genus $g > 4$ defined over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $P \in C$ be a smooth point with symmetric Weierstrass semigroup $\mathcal{H}$.

Let $N = \{n_0, n_1, \ldots, n_{g-1}\}$ be the set of the first g nongaps of C at P, it is called the *canonical system* of generators for $\mathcal{H}$. For each n_j there is a meromorphic function x_{n_j} with a pole at P of order n_j and does not have other poles. Hence, $\{x_{n_0}, \ldots, x_{n_{g-1}}\}$ is a basis of the vector space $H^0(C, (2g-2)P)$.

We assume that $l_2 = 2$ i.e. $n_{g-2} = 2g-3$, so the Gorenstein curve C is nonhyperelliptic and $x_{n_0}, \ldots, x_{n_{g-1}}$ induce an embedding in the projective space $\mathbb{P}^{g-1}$. Then, we can suppose that $C \subset \mathbb{P}^{g-1}$ is a canonical Gorenstein curve and $P = (0:\ldots:0:1)$.

In the light of Petri's analysis of the canonical ideal we obtain explicitly a P-Hermitian basis of the space $H^0(C, k(2g-2)P)$ with $k > 1$.

[1]For more about singular curves, in particular, Gorenstein curves, we refer to [26] or [28]

Let τ be the largest integer such that $n_\tau = \tau n_1$. Then the following quadratic functions form a P-hermitian basis of $H^0(C, 2(2g-2)P)$.

$$\begin{array}{ll} x_{n_0}x_{n_j} & (j = 0, \dots, g-1) \\ x_{n_{\tau+1}}x_{\ell_{g-\tau-1}+kn_1} & (k = 1, \dots, \tau - 1) \\ x_{n_i}x_k & (k = 2g - n_1, \dots, 2g-2, \quad i = 1, \dots, \tau) \\ x_{n_j}x_i & (j = \tau+1, \dots, g-1, \quad i = n_{g-j-1} - \ell_j, \dots, n_{g-1})\,. \end{array} \tag{2.3}$$

For each $s \in N + N$, we write $s = a_{s0} + b_{s0} = a_{s1} + b_{s1} = \ldots = a_{s\nu_s} + b_{s\nu_s}$ where a_{si}, b_{si} are nongaps satisfying $a_{s0} < a_{s1} < \ldots < a_{s\nu_s}$, $a_{si} \le b_{si}$ and ν_s is maximal. For convenience we denote $a_s = a_{s0}$ and $b_s = b_{s0}$. One can see that $\{x_{a_s}x_{b_s}\}$ is the basis in 2.3.

For each $n \ge 3$ the following meromorphic functions form a P-hermitian basis of $H^0\,(C, n(2g-2)P)$.

$$\begin{array}{ll} x_{n_0}^{n-1}x_{n_j} & (j = 0, \dots, g-1) \\ x_{n_0}^{n-3-i}x_{n_1}x_{2g-n_1}x_{n_{g-2}}x_{n_{g-1}}^i & (i = 0, \dots, n-3) \\ x_{n_0}^{n-2-i}x_{a_s}x_{b_s}x_{n_{g-1}}^i & (i = 0, \dots, n-2 \quad s = 2g, \dots, 4g-4)\,. \end{array} \tag{2.4}$$

Since $\{x_{a_s}x_{b_s}\}$ is a basis of the vector space $H^0(C, 2(2g-2)P)$, for each $x_{a_{si}}x_{b_{si}}$ $(i = 1, \dots, \nu_s)$ there are constants $c_{sir} \in \mathbf{k}$, after multiplying eventually x_{a_s}'s by suitable constants, such that:

$$x_{a_{si}}x_{b_{si}} = x_{a_s}x_{b_s} + \sum_{r<s} c_{sir}x_{a_r}x_{b_r}\,. \tag{2.5}$$

Let $I(C)$ be the canonical ideal of the Gorenstein curve $C \subset \mathbb{P}^{g-1}$. Thus $I(C) \subset \mathbf{k}[X_{n_0}, \dots, X_{n_{g-1}}]$ is a homogeneous ideal, $I(C) = \oplus_{n=2}^{\infty} I_n(C)$, where $I_n(C)$ denotes the vector space of all n-forms vanishing identically on the canonical curve C. The following $\frac{1}{2}(g-2)(g-3)$ quadratic forms

$$F_{si} := X_{a_{si}}X_{b_{si}} - X_{a_s}X_{b_s} - \sum_{r<s} c_{sir}X_{a_r}X_{b_r} \tag{2.6}$$

form a basis of the vector space $I_2(C)$, because dim $I_n(C) = \binom{n+g-1}{n}$ and the polynomials F_{si} are linearly independent.

Now we invert the above considerations. Let $\mathcal{H}$ be a numerical symmetric semigroup of genus $g > 4$, such that $3 < n_1 < g$ and $\mathcal{H} \neq < 4, 5 >$. With these conditions on the symmetric semigroup we may have the existence of non gaps a and b with $n_{g-1} + n_i = a + b$ and $n_i < a \leq b < n_{g-1}$; see [17, Thm 1.7].

Now we introduce the following quadratic forms

$$F_{si} := X_{a_{si}} X_{b_{si}} - X_{a_s} X_{b_s} - \sum_{r<s} c_{sir} X_{a_r} X_{b_r} \quad (s \in N+N,\ i = 1, \ldots, \nu_s) \tag{2.7}$$

where the coefficients belong to $\mathbf{k}$ and r ranges over the elements of $N+N$ smaller than s. We also consider the following quadratics forms

$$F_{si}^{(0)} := X_{a_{si}} X_{b_{si}} - X_{a_s} X_{b_s} \quad (s \in N+N,\ i = 1, \ldots, \nu_s)\,. \tag{2.8}$$

The ideal of C_0 is minimally generated by these quadratic forms, cf. [27] Lemma 2.2.

We define the *weight* of X_{n_i} to be n_i. Thus, the quadratic forms in (2.8) are *isobaric*, i.e., all monomials have the same weight.

We ask for the conditions on the coefficients c_{sir} of the forms F_{si} for which the intersection of the $\frac{1}{2}(g-2)(g-3)$ quadratic hypersurfaces “$F_{si} = 0$“ in the projective space $\mathbb{P}^{g-1}$ is a canonical Gorenstein curve having at $P = (0 : \ldots : 0 : 1)$ the Weierstrass semigroup $\mathcal{H}$.

A (first) syzygy between the quadratic forms $F_{si}^{(0)}$, say $\sum_{si} B_{si} F_{si}^{(0)} = 0$, is *homogeneous* of degree n (respectively, *isobaric* of weight w) when the polynomials B_{si} are homogeneous of degree $n-2$ (respectively, isobaric of weight $w-s$). The syzygy is called linear when the polynomials B_{si} are linear forms. A syzygy is *trivial* when it comes from a trivial relation like $BF - FB = 0$.

Now we summarize certain results from [27], culminating with the construction of $\overline{\mathscr{M}_{\mathcal{H}}}$.

There are $\frac{1}{2}(g-2)(g-5)$ linear isobaric syzygies which are fundamental to the construction of the moduli space $\overline{\mathscr{M}_{\mathcal{H}}}$.

Syzygy Lemma (cf. [27] Lemma 2.3 page 199)**.** *For each $F_{si}^{(0)}$ different from $F_{2g-2+n,1}^{(0)}$ $(n = n_0, n_1, \ldots, n_{g-3})$ there is a linear isobaric syzygy of the form*

$$X_{2g-2} F_{si}^{(0)} + \sum b_{sijs'i'} X_{n_j} F_{s'i'}^{(0)} = 0$$

where $j \in \{0, 1, \ldots, g-1\}$, $s' \in N+N$, $i' = 1, \ldots, \nu_{s'}$, $n_j + s' = 2g-2+s$ and $n_j < 2g-2$ whenever $F_{s'i'}^{(0)}$ is different from $F_{2g-2+n,1}^{(0)}$ $(n = n_0, n_1, \ldots, n_{g-3})$.

The conditions on the coefficients c_{sir} of F_{si} which we are searching are given by the the following theorem, in particular (b).

Theorem 2.1 (cf. [27] Lemma 2.1, Proposition 2.5 and Theorem 2.6)**.** *Let us assume that the symmetric semigroup satisfies the additional condition*[2] $m+1 < n_2 < 2m-1$. *Let I be the ideal generated by the $\frac{1}{2}(g-2)(g-3)$ quadratic forms F_{si}, $(s \in N+N, i = 1, \dots, \nu_s)$. Then the following statements are equivalent:*

a) The quadratic forms F_{si} cut out a nondegenerate canonical Gorenstein genus-g curve C whose ideal $I(C)$ is equal to I, and $P = (0 : \dots : 0 : 1)$ is a nonsingular point of C whose Weierstrass semigroup is $\mathcal{H}$;

b) The remainders of the $\frac{1}{2}(g-2)(g-5)$ polynomials, induced by the Syzygy Lemma, $X_{2g-2}F_{si} + \sum b_{sijs'i'} X_{n_j} F_{s'i'}$ divided by the $F_{s,i}$ are zero.

We note that canonical curves in $\mathbb{P}^{g-1}$ are isomorphic if and only if they are projectively equivalent. The isomorphisms are induced by linear transformations of the type $X_{n_j} \mapsto \sum_{j=0}^{g-1} z_{ij} X_{n_i}$ where $z_{ij} \in \mathbf{k}$, or equivalently, by a $g \times g$ matrix (z_{ij}).

Since we fixed a P-hermitian basis of the vector space $H^0(C, n(2g-2)P)$, the matrix (z_{ij}) is lower triangular, i.e., $z_{ij} = 0$ whenever $i < j$. From $c_{sis} = 1$ it follows that $z_{jj} = z^{n_j}$ for $j = 0, \dots, g-1$, where z is a nonzero constant.

Proposition 2.2 (cf. [27] Proposition 3.1)**.** *One can normalize $\frac{1}{2}g(g-1)$ coefficients c_{sir} to zero.*

After this Proposition the only freedom left us is to transform $c_{sir} \mapsto z^{s-r} c_{sir}$, where z belongs to the multiplicative group $\mathbb{G}_m(\mathbf{k})$ of the constant field $\mathbf{k}$.

Stöhr's Construction. *The isomorphisms classes of the projective irreducible pointed Gorenstein curves whose Weierstrass semigroup is $\mathcal{H}$ correspond bijectively to the orbits of the equivariant $\mathbb{G}_m(\mathbf{k})$-action $(z, c_{sir}) \mapsto z^{s-r} c_{sir}$ on the algebraic set of the vectors of constants c_{sir} normalized by Proposition 2.2 and satisfying the polynomial equations of Theorem 2.1 (b).*

[2] This is not a necessary condition and we do it just to simplify the statements, otherwise we should introduce the exceptional monomials, see [27] Lemma 2.1.

Chapter 3

On the Dimension of $\mathscr{M}_{\mathcal{H}}$

3.1 A Formula of Deligne

This section is devoted to recall the upper bound of Deligne's for the dimension of $\mathscr{M}_{\mathcal{H}}$. For this purpose we will need some knowledge of theory of deformations of curves, that we will not develop here. For a complete presentation on deformation theory we refer to [24] and [25].

Through the Kodaira–Spencer map and relations among three moduli spaces Deligne generalizes a result by Rim[1] in the following way.

Let C be a reduced projective algebraic curve and $q \in C$. Let us write $\mathcal{O}$ for the local ring of C at q and $\tilde{\mathcal{O}}$ for its normalization. Let $\delta := \dim_{\mathbf{k}} \tilde{\mathcal{O}}/\mathcal{O}$ the degree of singularity of C at q. We denote by $D(\tilde{\mathcal{O}})$ the module of $\mathbf{k}$-derivations of $\tilde{\mathcal{O}}$, $D(\mathcal{O})$ that of $\mathcal{O}$ and set $m_1 := \dim_{\mathbf{k}} \dfrac{D(\tilde{\mathcal{O}})}{D(\tilde{\mathcal{O}}) \cap D(\mathcal{O})} - \dim_{\mathbf{k}} \dfrac{D(\mathcal{O})}{D(\tilde{\mathcal{O}}) \cap D(\mathcal{O})}$.

Deligne's Formula (cf. [4], Theorem 2.27)**.** *Let E be an irreducible component of the semiuniversal deformation of* $\mathrm{Spec}(\mathcal{O})$*. Suppose that the fiber above the generic point of E is smooth. Then*

$$\dim E = 3\delta - m_1\,. \tag{3.1}$$

We are interested in a particular case of the above theorem. We shall make the connection between the semiuniversal deformation of $\mathrm{Spec}(\mathcal{O})$ and the moduli space $\mathscr{M}_{\mathcal{H}}$. This is done by considering *monomial curves*.

[1] See [22], page 268, Thm 2.7.

By a *monomial curve* we mean an affine algebraic curve given by a numerical semigroup. In more details, given a numerical semigroup $\mathcal{H}$ and $\{m_1, \ldots, m_r\}$ a set of generators for $\mathcal{H}$, a parametrization of the affine monomial curve associated to $\mathcal{H}$ is:

$$C = \{(t^{m_1}, \ldots, t^{m_r}),\ \ t \in \mathbf{k}\}\,.$$

Let $B := B_{\mathcal{H}}$ be the subring of the polynomial ring $\mathbf{k}[t]$ generated by the monomials t^h with $h \in \mathcal{H}$, where t is a transcendent over $\mathbf{k}$. The *semigroup algebra* B is the coordinate ring of C.

Observe that on a monomial curve there is a natural $\mathbb{G}_m(\mathbf{k})$-action. For a more detailed presentation on monomial curves and their deformations we refer to [3], [21] and [23]. Note that the curve (2.2) is a projectivization of a monomial curve.

In the remainder of this section we assume $\mathcal{H} \neq \{0, g+1, g+2, \ldots\}$. We will compute the right side of the expression (3.1) when C is a monomial curve.

Lemma 3.1. *Let $\mathcal{H}$ be a numerical semigroup of multiplicity m and genus g. Set $B := B_{\mathcal{H}}$ then:*

1. *The integral closure $\overline{B}$ of B in its total ring of fractions is equal to $\mathbf{k}[t]$ and $g = \dim \overline{B}/B$;*

2. *B is smooth over $\mathbf{k}$ if and only if $\mathcal{H} = \mathbb{N}$. If not, B has an isolated singularity at $\mathbf{0}$ and $m = e(B_{\mathbf{0}})$, where $\mathbf{0}$ is the maximal ideal of B generated by the t^h, $h \in \mathcal{H} - \{0\}$ and $e(B_{\mathbf{0}})$ is the multiplicity of the local ring.*

Proof. The proof is straightforward. □

If C is a monomial curve and $\mathcal{O}$ is the local ring at its singular point, then $D(\mathcal{O}) \subset D(\tilde{\mathcal{O}})$ and $m_1 = \dim_{\mathbf{k}} \dfrac{D(\tilde{\mathcal{O}})}{D(\mathcal{O})}$. Instead of working with the local rings we can work with the algebra B and its integral closure $\overline{B} = \mathbf{k}[t]$. We may see that $m_1 = [D(\mathbf{k}[t]) : D(B)]$, where $D(\mathbf{k}[t]) = Der_{\mathbf{k}}(\mathbf{k}[t], \mathbf{k}[t])$ and $D(B) = Der_{\mathbf{k}}(B, B)$. The module $Der_{\mathbf{k}}(B, B)$ is a graded module and the homogeneous part of degree s is described by:

$$Der_{\mathbf{k}}(B,B)_s = \begin{cases} \mathbf{k}\, t^{s+1}\frac{\partial}{\partial t}, & \text{if } s \in \mathrm{End}(H)\,; \\ 0, & \text{other wise}\,. \end{cases}$$

where $\mathrm{End}(\mathcal{H}) = \{n \in \mathbb{N} \,|\, n+h \in \mathcal{H},\ \forall\, h \in \mathcal{H} - \{0\}\}$. Thus we see that

$$[D(\mathbf{k}[t]) : D(B)] = 1 + [\mathbb{N} : \mathrm{End}(\mathcal{H})] = 1 + g - [\mathrm{End}(\mathcal{H}) : \mathcal{H}]$$

and we are ready to verify:

Lemma 3.2. *If C is a monomial curve associated to a semigroup $\mathcal{H}$. Then*

$$3\delta - m_1 = 2g + [\mathrm{End}(\mathcal{H}) : \mathcal{H}] - 1\,.$$

Now we make the connection between the semiuniversal deformation in Deligne's Formula and the moduli space $\mathscr{M}_{\mathcal{H}}$. We denote by $\mathscr{M}_{g,1}$ the coarse moduli space of smooth projective curves of genus g with a base point. For the precise definition and details about coarse moduli spaces, in particular, about $\mathscr{M}_{g,1}$, we refer to [11].

Let $\mathcal{H}$ be a numerical semigroup of genus $g > 1$ and fix a minimal system of generators of $\mathcal{H}$. Let $\mathscr{M}_{\mathcal{H}}$ be the subscheme of $\mathscr{M}_{g,1}$ defined by:

$$\mathscr{M}_{\mathcal{H}} = \{[C,p] \in \mathscr{M}_{g,1} | \mathcal{H}_{C,p} = \mathcal{H}\}$$

where $\mathcal{H}_{C,p}$ denotes the Weierstrass semigroup of C at p.

Let us write (S,R) for the semiuniversal deformation[2] of $B_{\mathcal{H}}$. Pinkham[3] proved that there is an ideal N of R such that by taking $R' = R/NR$ and $S' = S/NS$, it follows that (S',R') is an infinitesimal deformation of B and the set

$$U = \{x \in \mathrm{Spec}(R') | \text{the fiber above } x \text{ in } S' \text{ is smooth}\}$$

is invariant by the $G_m(\mathbf{k})$ action on S'. Using this he proved the following.

Theorem 3.3 (cf. [21] Theorem 13.9)**.** *There exists a morphism $U \to \mathscr{M}_{g,1}$ that factors through the quotient $\overline{U}$ of U by the action of $\mathbb{G}_m(\mathbf{k})$, inducing a bijection between $\overline{U}$ and $\mathscr{M}_{\mathcal{H}}$.*

---

[2]The semiuniversal deformation exists because $B = B_{\mathcal{H}}$ has an isolated singularity.

[3]See [21], chapter I, section 2, for the general case and chapter IV, section 13, for monomials curves.

By forgetting the condition of smoothness of the generic fiber, Deligne's Formula provides $\dim U \leq 3g - m_1$, and from Lemma 3.2 and the above Theorem 3.3 we get

$$\dim \mathscr{M}_{\mathcal{H}} \leq 2g + [\mathrm{End}(\mathcal{H}) : \mathcal{H}] - 2\,. \tag{3.2}$$

Definition 3.4. *The above upper bound is called Deligne's upper bound for the dimension of* $\mathscr{M}_{\mathcal{H}}$.

Remark 3.5. *It is straightforward to verify that* $\mathcal{H}$ *is symmetric if and only if* $\mathrm{End}(\mathcal{H}) = \mathcal{H} \cup \{l_g\}$. *Thus, if we suppose that* $\mathcal{H}$ *is symmetric, then*

$$\dim \mathscr{M}_{\mathcal{H}} \leq 2g - 1\,. \tag{3.3}$$

Deligne's upper bound is sharp. For each monomial curve C, rather, for each semigroup algebra $B = B_{\mathcal{H}}$ we consider the first cohomology module $T^1(B) = T^1(B|\mathbf{k}, B)$ of the cotangent complex. Let $n_1 < \cdots < n_r$ be a system of generators of $\mathcal{H}$. We can write B as the quotient of the polynomial ring $P = \mathbf{k}[X_1, \ldots, X_r]$ by sending X_i to t^{n_i}; denote by $\mathcal{I}$ the kernel of this map. By the theory of Lichtenbaum and Schlessinger [12] we have:

$$T^1(B) \cong coker\left(Der_{\mathbf{k}}(P, B) \to Hom_B\left(\frac{\mathcal{I}}{\mathcal{I}^2}, B\right)\right),$$

and $T^1(B)$ is a graded module. A semigroup $\mathcal{H}$ is *negatively graded* if $T^1(B)_s$ is zero for every $s \geq 0$.

In [23], Rim–Vitulli proved that if $\mathcal{H}$ is negatively graded then $\mathcal{H}$ is realizable and $\dim \mathscr{M}_{\mathcal{H}} = 2g - [\mathrm{End}(\mathcal{H}) : \mathcal{H}] - 2$. Also in [23], the semigroups that are negatively graded are completely listed; [23, Thm 4.7].

There are only two families of symmetric semigroups which are negatively graded, namely:

$$\{0, g, g+1, \ldots, 2g-2, 2g, 2g+1, \ldots\} \text{ and } \{0, g-1, g+1, \ldots, 2g-2, 2g, 2g+1, \ldots\} \tag{3.4}$$

Thus, if we are concerned with the dimension of $\mathscr{M}_{\mathcal{H}}$, when $\mathcal{H}$ is symmetric, then we need not be concerned with the above two families.

3.2 Description of the method

We are concerned with the dimension of $\mathscr{M}_{\mathcal{H}}$, when $\mathcal{H}$ is symmetric. Though we are out of the range of Eisenbud–Harris Theorem, we can use Stöhr's Construction to obtain information about the dimension of $\overline{\mathscr{M}_{\mathcal{H}}}$, *a fortiori* about that of $\mathscr{M}_{\mathcal{H}}$. In this way we proceed as follows.

Let $\mathcal{H} = \{n_0, n_1, \ldots, n_{g-1}, \ldots\}$ be a numerical semigroup of genus g. We assume that $\ell_3 = 3$, $\ell_{g-1} \geq g$, $\ell_g = 2g-1$ and $\mathbb{N} - \mathcal{H} \neq \{1, 2, 3, 6, 7, 11\}$ i.e. $3 < n_1 < g$ and $\mathcal{H} \neq < 4, 5 >$. We denote by $\mathcal{X}$ the algebraic set formed by the vectors of constants c_{sir} normalized according to Proposition 2.2 and satisfying the polynomial equations induced by Theorem 2.1 (b). From Chapter 2 it follows:

$$\overline{\mathscr{M}_{\mathcal{H}}} = \mathcal{X}/\mathbb{G}_m(\mathbf{k})$$

where the $\mathbb{G}_m(\mathbf{k})$-action is defined by $(z, c_{sir}) \mapsto z^{s-r} c_{sir}$.

Now we recall some basic terminology on quasi-cones, which are affine algebraic sets with a $\mathbb{G}_m(\mathbf{k})$-action. For more information we refer to [7].

Let $r = (r_0, \ldots, r_n)$ be a vector of integer positive numbers whose greatest common divisor is one. We denote by $\mathcal{S}_r$ the polynomial algebra $\mathbf{k}[T_0, \ldots, T_n]$ over a field $\mathbf{k}$, graded by the condition weight$(T_i) = r_i$.

A polynomial $f \in \mathcal{S}_r$ is called *isobaric* if all its monomials have the same weight. An ideal $\mathcal{I}$ of $\mathcal{S}_r$ is *quasi-homogeneous* if it is generated by isobaric polynomials. Any polynomial F has a unique expression $F = F_k + \ldots + F_u$, where each F_i is isobaric of weight i. As in the homogeneous case, it is simple to verify that if $\mathcal{I}$ is quasi-homogeneous and $F = F_k + \ldots + F_u \in \mathcal{I}$, then each F_i belongs to $\mathcal{I}$.

The multiplicative group $\mathbb{G}_m(\mathbf{k})$ acts on $\mathbb{A}^{n+1} = \mathrm{Spec}(\mathcal{S}_r)$ in the following way:

$$(T_0, \ldots, T_n) \mapsto (z^{r_0} T_0, \ldots, z^{r_n} T_n), \; z \in \mathbb{G}_m(\mathbf{k}) \, .$$

A closed subset V of $\mathbb{A}^{n+1}$ is a *quasi-cone* if its ideal is quasi-homogeneous. If V is a quasi-cone, then the action of $\mathbb{G}_m(\mathbf{k})$ on V is effective. The point $\mathbf{0} = (0, \ldots, 0) \in \mathbb{A}^{n+1}$ is the *vertex* of V. Note that the vertex belongs to the closure of each orbit of V.

The space $\mathbb{P}(r) = \mathrm{Proj}(\mathcal{S}_r)$ is called the *weighted projective space of type* r.

If $r_0 = \ldots = r_n = 1$ then $\mathbb{P}(r)$ is the usual projective space $\mathbb{P}^n$. In this way, $\overline{\mathscr{M}_{\mathcal{H}}}$ is a *weighted projective algebraic set*, where the weight of each c_{sir} is $s-r$.

Proposition 3.6. *Suppose that V is a quasi-cone. Then the vertex belongs to all of its irreducible components.*

Proof. The basic idea of the proof is that each irreducible component of V is also a quasi-cone, because it is invariant with respect the $G_m(\mathbf{k})$-action on V. It is similar to the case of cones. □

The algebraic set $\mathcal{X}$ is a quasi-cone and the vertex $\mathbf{0}$ corresponds to the canonical monomial curve C_0 defined in (2.2).

By Proposition 3.6 and since the dimension of an irreducible algebraic set is its local dimension at any point (cf. [2, Theorem 11.25]), we may see that:

$$\dim \overline{\mathscr{M}_{\mathcal{H}}} = \dim \mathcal{X} - 1 = \dim_{\mathbf{0}} \mathcal{X} - 1\,. \tag{3.5}$$

The quasi-cone $\mathcal{X}$ is the zero locus of isobaric polynomials, say $H_1, \ldots, H_r$. Each H_i can be taken without linear terms, because we can always eliminate them. The vertex $\mathbf{0}$ can be a singular point, see examples in the next two sections. Thus, the linear approximation of $\mathcal{X}$ near $\mathbf{0}$, which is, the Zariski tangent space, is somewhat coarse and does not provide refined information about the local dimension.

An important part of the local intrinsic study at a singular point of an algebraic set X can be done by applying the global extrinsic theory of the *tangent cone* to X at this point.

Let $X \subset \mathbb{A}^N$ be an affine algebraic set and $p \in X$. For each $f \in \mathcal{I}(X)$, we denote by f_p the Taylor expansion of f around p and f_p^{min} the leading term of f_p. Here we use the usual degree. The tangent cone to X at p is the affine algebraic set $C_p(X)$ defined by all f_p^{min} for $f \in \mathcal{I}(X)$.

The algebra of regular functions of $C_p(X)$ is isomorphic to $\bigoplus_{i=0}^{\infty} m_p^i/{m_p}^{i+1}$, where m_p is the maximal ideal in the local ring $\mathcal{O}_{X,p}$. The graded ring $\bigoplus_{i=0}^{\infty} m_p^i/{m_p}^{i+1}$ is generated by its first graded part $m_p/{m_p}^2$. The algebra of regular functions of the Zariski tangent space $T_p(X)$ is $\bigoplus_{i=1}^{\infty} Symm^i(m_p/{m^2}_p)$. Therefore the tangent cone can be considered a subvariety of $T_p(X)$.

A very classical result tells us that the local dimension of X at p is the global dimension of $C_p(X)$. For a proof we refer to [13, Theorem 13.9]. In our particular case we get:

$$\dim \overline{\mathscr{M}_{\mathcal{H}}} = \dim \mathcal{X} - 1 = \dim C_{\mathbf{0}}(\mathcal{X}) - 1\,. \tag{3.6}$$

As regard to tangent cone computations, F. Mora [14] presents an algorithm to compute the equations of the tangent cone of any variety. The algorithm computes a standard basis of a given ideal. It is a variant of the Buchberger algorithm to compute Gröbner basis with suitable modifications. The complexity of the algorithm is unknown, but in certain cases it has the same complexity as Buchberger's.

Since $\mathcal{X}$ is the zero locus of $H_1, \ldots, H_t$, it follows that $\mathcal{I}(\mathcal{X}) = \sqrt{(H_1, \ldots, H_t)}$, and asking for the dimension, we could say that:

$$\dim \mathcal{X} = \dim C_{\mathbf{0}}(\mathcal{X}) \leq \dim V\left(\langle H_1^{\min}, \ldots, H_t^{\min}\rangle\right). \tag{3.7}$$

Here we will make a suitable simplification. We will consider a very simple algebraic set instead of $V(H_1^{\min}, \ldots, H_r^{\min})$. So we introduce the following quadratic forms:

$$H_i^{(2)} = \begin{cases} H_i^{\min}\ , \text{ if } \ deg\, H_i^{\min} = 2\,; \\ 0\ , \text{ otherwise.} \end{cases}$$

Then:

$$\dim \mathcal{X} \leq \dim V\left(\langle H_1^{\min}, \ldots, H_t^{\min}\rangle\right) \leq \dim V\left(\left\langle H_1^{(2)}, \ldots, H_t^{(2)}\right\rangle\right). \tag{3.8}$$

We will see in numerical examples in the next section and also with families of semigroups in Chapter 4, that this simplification allows for a tremendous simplification of computations.

Notation 3.7. *We denote by $\mathcal{Q}_{\mathcal{H}}$ the quasi-cone defined by the zero locus of all quadratic forms $H_i^{(2)}$.*

Note that $\mathcal{Q}_{\mathcal{H}}$ is given by isobaric quadratic forms, i.e, isobaric and homogeneous polynomials of degree two. Moreover, $\mathcal{Q}_{\mathcal{H}}$ contains the quadratic approximation of $\mathcal{X}$, which is the zero locus of all quadratic forms that vanish on $\mathcal{X}$, which contains the tangent cone.

If we put together the three previous formulas (3.5), (3.6) and (3.8) we get:

Theorem 3.8. *Let $\mathcal{H}$ be a symmetric semigroup of genus $g > 4$ such that $3 < n_1 < g$ and $\mathcal{H} \neq < 4, 5 >$. Then*

$$\dim \overline{\mathscr{M}_{\mathcal{H}}} \leq \dim \mathcal{Q}_{\mathcal{H}} - 1\,. \tag{3.9}$$

So we also have the same upper bound for the dimension of the moduli space $\mathscr{M}_{\mathcal{H}}$, i.e., $\dim \mathscr{M}_{\mathcal{H}} \leq \dim \mathcal{Q}_{\mathcal{H}} - 1$.

Now, we describe how to obtain $\mathcal{Q}_{\mathcal{H}}$ without constructing the moduli space $\overline{\mathscr{M}_{\mathcal{H}}}$.

As in Chapter 2, we fix a symmetric semigroup $\mathcal{H}$ of genus g such that $3 < m < g$ and $\mathcal{H} \neq < 4, 5 >$.

We fix a Hermitian basis of the vector space $H^0(C_0, n(2g-2)P)$ formed by monomials, as in (2.3) and (2.4). We denote by Γ_n the vector space in $\mathbf{k}[X_0, \ldots, X_{g-1}]$ generated by the lifting of this basis, and $\Gamma = \oplus \Gamma_n$.

Let N be the set of first g nongaps and we take the quadratic forms

$$F_{si} := X_{a_{si}} X_{b_{si}} - X_{a_s} X_{b_s} - \sum_{r<s} c_{sir} X_{a_r} X_{b_r} \quad (s \in N + N,\ i = 1, \ldots, \nu_s)$$

where the coefficients belongs to $\mathbf{k}$, r ranges over the elements of $N+N$ smaller than s, the integers a_s, b_s, a_{si} and b_{si} are defined by $s = a_s + b_s = a_{s1} + b_{s1} = \ldots = a_{s\nu_s} + b_{s\nu_s}$ with a_{si}, b_{si} nongaps satisfying $a_s < a_{s1} < \ldots < a_{s\nu_s}$ and ν_s is maximal. Also we consider the quadratic forms that generate the ideal of the canonical monomial curve C_0

$$F_{si}^{(0)} := X_{a_{si}} X_{b_{si}} - X_{a_s} X_{b_s} \quad (s \in N + N,\ i = 1, \ldots, \nu_s)\,.$$

To apply the division algorithm we equip the additive semigroup $\mathbb{N}^g$ of the exponents of the monomials in $X_{n_0}, \ldots, X_{n_{g-1}}$ with the following total ordering: $(j_0, \ldots, j_{g-1}) \geq (i_0, \ldots, i_{g-1})$ if and only if the first nonzero entry of the vector

$$\left(\sum_{k=0}^{g-1} j_k - i_k, \sum_{k=0}^{g-1} n_k (j_k - i_k), i_0 - j_0, i_{g-1} - j_{g-1}, \ldots, i_1 - j_1 \right) \tag{3.10}$$

is positive.

The first entry of the previous vector tells us about the total degree, the second entry tells us about the weight and the others about the partial degrees. Note that the monomials of the forms F_{si} appear in decreasing order.

From Proposition 2.2 we normalize to zero $\frac{1}{2}g(g-1)$ coefficients c_{sir} of the the quadratic forms F_{si}.

We take the syzygies given by the Syzygy Lemma. Thus for each $F_{si}^{(0)}$ different from $F_{2g-2+n,1}^{(0)}$ $(n = n_0, n_1, \ldots, n_{g-3})$ we have

$$X_{2g-2}F_{si}^{(0)} + \sum b_{sijs'i'}X_{n_j}F_{s'i'}^{(0)} = 0$$

where $j \in \{0, 1, \ldots, g-1\}$, $s' \in N + N$, $i' = 1, \ldots, \nu_{s'}$, $n_j + s' = 2g - 2 + s$ and $n_j < 2g - 2$ whenever $F_{s'i'}^{(0)}$ is different from $F_{2g-2+n,1}^{(0)}$ with $n = n_0, n_1, \ldots, n_{g-3}$.

We consider the quadratic forms:

$$X_{2g-2}F_{si} + \sum b_{sijs'i'}X_{n_j}F_{s'i'}\,. \tag{3.11}$$

Let us take the homomorphism $\mathbf{k}[X_{n_0}, \ldots, X_{n_{g-1}}] \to \mathbf{k}[t]$ induced by $X_{n_i} \mapsto t^{n_i}$ $(i = 0, \ldots, g-1)$. For each quadratic form in (3.11) we take its image. Then we ask for the linear conditions on the constants c_{sir} that make this image identically zero. We have the linearizations of all the quadratic forms (3.11).

The weight of c_{sir} is defined by $s - r$. Thus, for each weight we solve a linear system, writing certain coefficients as a linear combination of other of the same weight. If we take the $\mathbf{k}$-algebra $\mathbf{k}[c_{sir}]$ given by the linearizations, then $\mathrm{Spec}(\mathbf{k}[c_{sir}])$ is the ambient space where the algebraic set $\mathcal{X}$ and $\mathcal{Q}_{\mathcal{H}}$ are. In fact, the vector space generated by the linearizations is in bijection with $T^{1,-}(B_{\mathcal{H}})$; see [27], page 212.

We are only interested in the quadratic relations of the linearizations. Then we take a form induced by the Syzygy Lemma, say:

$$S_{si} := X_{2g-2}F_{si} + \sum b_{sijs'i'}X_{n_j}F_{s'i'} = X_{2g-2}(F_{si} - F_{si}^{(0)}) + \sum b_{sijs'i'}X_{n_j}(F_{s'i'} - F_{s'i'}^{0}).$$

We will work with increasing weights on the coefficients c_{sir}. Starting with weight one, we take a monomial of S_{si} whose weight is $s - 1$. It is equivalent to say that its coefficient has weight one. If this monomial belongs to the basis of Γ_3 we do not have any division to do. Otherwise there is a form F_{rj} whose initial form divides this monomial, we take the form with the largest exponent. We do it for each monomial whose weight is $s - 1$. We set

$$S_{si}^{1} := \pi_2\left(S_{si} - \sum \alpha_{sirj}^{(1)}F_{rj}\right)$$

where the coefficients α_{sirj} depend on certain coefficients c_{sir} such that $s-r=1$, and π_2 is the projection map on the polynomials in c_{sir} that annihilates the terms of degree bigger than 2. Then we work with $S_{si}^{(1)}$ instead of S_{si} and apply the same procedure successively for increasing weights. We do this for all S_{si}, at the end we will have polynomials $S_{si}^{(w_{si})}$, where w_{si} is the weight of S_{si}.

Remark 3.9. *The quasi-cone $\mathcal{Q}_{\mathcal{H}}$ is set of the vectors of constants c_{sir} normalized by Proposition 2.2 and satisfying the polynomial equations $S_{si}^{(w_{si})}=0$.*

3.3 Numerical Examples

In this section we apply our method to four symmetric semigroups, two in genus eight and two more in genus nine. Although the moduli space $\overline{\mathscr{M}_{\mathcal{H}}}$ can be constructed, we show that our method works very well when compared with Deligne's bound $(2g-1)$, with the expected dimension by Eisenbud–Harris $(3g-2-w(\mathcal{H}))$, or even with the exact dimension of the moduli space, when we know it.

Even for the computer these examples can be heavy. We used the Maple software to compute the isobaric equations of $\mathcal{X}$ and $\mathcal{Q}_{\mathcal{H}}$, following their constructions presented in this thesis above.

3.3.1 Two examples in genus 8

Let us consider the symmetric semigroup of genus 8

$$\mathcal{H}_1 := \{0,6,8,10,11,12,13,14,16,17,18,\dots\}.$$

The canonical system of generators for $\mathcal{H}_1$ is $\{0,6,8,10,11,12,13,14\}$, the sequence of gaps is $1,2,3,4,5,7,9,15$ and the C_0 curve is:

$$C_0=\{(u^{14}:u^8v^6:u^6v^8:u^4v^{10}:u^3v^{11}:u^2v^{12}:uv^{13}:v^{14})\mid (u:t)\in\mathbb{P}^1\}.$$

We fix the P-hermitian basis $\{x_0,x_6,x_8,x_{10},x_{11},x_{12},x_{13},x_{14}\}$ of $H^0(C_0,14P)$, where $P=(0:\dots:0:1)\in\mathbb{P}^7$ and the order of pole of each x_i is i.

We draw a table of the sums $n_i + n_j$, where $0 \le i \le j \le g-1$. For each sum n_i+n_j there is a meromorphic function, namely $x_{n_i}x_{n_j}$, on C_0 whose pole order at P is exactly $n_i + n_j$ and does not have other poles.

0+0	0+6	0+8	0+10	0+11	0+12 6+6	0+13
0+14 6+8	6+10 8+8	6+11	6+12 8+10	6+13 8+11	6+14 8+12 10+10	8+13 10+11
8+14 10+12 11+11	10+13 11+12	10+14 11+13 12+12	11+14 12+13	12+14 13+13	13+14	14+14

The P-hermitian basis for $H^0(C_0, 28P)$ is given by the functions

$$\begin{cases} {x_0}^2,\ x_0x_6,\ x_0x_8,\ x_0x_{10},\ x_0x_{11},\ x_0x_{12},\ x_0x_{13},\ x_0x_{14}, \\ x_6x_{10},\ x_6x_{11},\ x_6x_{12},\ x_6x_{13},\ x_6x_{14}, \\ x_8x_{13},\ x_8x_{14},\ x_{10}x_{13}, \\ x_{10}x_{14},\ x_{11}x_{14},\ x_{12}x_{14},\ x_{13}x_{14},\ {x_{14}}^2. \end{cases}$$

These functions are precisely the $x_{n_i}x_{n_j}$ where $n_i + n_j$ appears on top of the entry that corresponds to the order of pole $n_i + n_j$.

The P-hermitian basis for $H^0(C_0, 42P)$ is:

$$\begin{cases} x_0^2 x_{n_j}, \quad (j = 0, \ldots, 7) \\ x_6x_{10}x_{13}, \\ x_0x_{a_s}x_{b_s},\ x_{a_s}x_{b_s}x_{14} \quad (s = 16, \ldots, 28) \end{cases}$$

with each $a_s + b_s$ on the top of each entry of the previous table.

Using the table we see the generators of the ideal of C_0 and the fifteen quadratic forms $F_{s,i}$. Though write lony two quadratic forms, it is very simple to write the others using the table:

$$F_{12,1} := {X_6}^2 - X_0X_{12} - c_{12,1,11}X_0X_{11} - c_{12,1,10}X_0X_{10} - c_{12,1,8}X_0X_8 - \\ -c_{12,1,6}X_0X_6 - c_{12,1,0}{X_0}^2\,;$$

$$F_{20,2} := {X_{10}}^2 - X_6X_{14} - c_{20,2,19}X_6X_{13} - c_{20,2,18}X_6X_{12} - \ldots - c_{20,2,0}{X_0}^2\,.$$

From Proposition 2.2 we can normalize to zero 28 coefficients of the quadratic forms $F_{s,i}$. Thus, all the coefficients of $F_{12,1}$ and $F_{14,1}$ are normalized to zero. To see this we just take the following change of variables:

$$X_{12} \longmapsto X_{12} - c_{12,1,11}X_{11} - \ldots - c_{12,1,0}X_0 \,;$$
$$X_{14} \longmapsto X_{14} - c_{14,1,13}X_{13} - \ldots - c_{14,1,0}X_0 \,.$$

And then, we ask for all linear transformations of the type $X_{n_i} \mapsto \sum_{n_j<n_i} \alpha_{n_j} X_{n_j}$ that maintain the normalizations $F_{12,1} = {X_6}^2 - X_0X_{12}$ and $F_{14,1} = X_6X_8 - X_0X_{14}$. In this way we can normalize 16 more coefficients as follows:

- from $F_{16,1}$: $c_{16,1,14} = c_{16,1,12} = c_{16,1,10} = c_{16,1,8} = c_{16,1,6} = 0$;
- from $F_{18,1}$: $c_{18,1,16} = 0$;
- from $F_{19,1}$: $c_{19,1,18} = c_{19,1,17} = c_{19,1,16} = c_{19,1,14} = c_{19,1,12} = c_{19,1,8} = c_{19,1,6} = 0$;
- from $F_{22,2}$: $c_{22,2,21} = c_{22,2,19} = c_{22,2,17} = 0$.

Now we write the nine cubic forms induced by the Syzygy Lemma.

$$\begin{aligned}
S_1 &:= X_{14}F_{12,1} - X_{12}F_{14,1} + X_6F_{20,1}\\
S_2 &:= X_{14}F_{16,1} - X_{10}F_{20,1} + X_8F_{22,1}\\
S_3 &:= X_{14}F_{18,1} - X_{12}F_{20,1} + X_8F_{24,2}\\
S_4 &:= X_{14}F_{19,1} - X_{13}F_{20,1} + X_8F_{25,1}\\
S_5 &:= X_{14}F_{20,2} - X_{14}F_{20,1} - X_{12}F_{22,1} + X_{10}F_{24,2}\\
S_6 &:= X_{14}F_{21,1} - X_{13}F_{22,1} + X_{10}F_{25,1}\\
S_7 &:= X_{14}F_{22,2} - X_{14}F_{22,1} - X_{12}F_{24,1} + X_{11}F_{25,1}\\
S_8 &:= X_{14}F_{23,1} - X_{13}F_{24,1} + X_{11}F_{26,1}\\
S_9 &:= X_{14}F_{24,2} - X_{14}F_{24,1} - X_{12}F_{26,1} + X_{13}F_{25,1}
\end{aligned}$$

For each S_i we solve the equation $S_i(t^0, t^6, t^8, t^{10}, t^{11}, t^{12}, t^{13}, t^{14}) = 0$. This means that we solve 26 linear systems, choosing the coefficients c_{sir} that we want to deal with.

One can solve these linear systems in a way that the coefficients c_{sir} of the solutions depend on the following 18 coefficients:

$c_{16,1,13}$, $c_{16,1,11}$,

$c_{18,1,17}$, $c_{18,1,14}$, $c_{18,1,12}$, $c_{18,1,11}$, $c_{18,1,10}$, $c_{18,1,8}$, $c_{18,1,6}$,

$c_{19,1,10}$,

$c_{22,2,20}$, $c_{22,2,18}$, $c_{22,2,16}$, $c_{22,2,14}$, $c_{22,2,12}$, $c_{22,2,10}$, $c_{22,2,8}$, $c_{22,2,6}$.

We already know that the equations of $\mathcal{Q}_{\mathcal{H}_1}$, and also the equations of the quasi-cone $\mathcal{X}$, involve these 18 coefficients, so $\mathcal{Q}_{\mathcal{H}_1}, \mathcal{X} \subseteq \mathbb{A}^{18}$. We also deduce that $\dim T^{1,-}(B_{\mathcal{H}_1}) = 18$.

Now we will apply the division algorithm to obtain the equations that will define $\mathcal{Q}_{\mathcal{H}}$. We have to organize our divisions. We organize them by increasing weights on the c_{sir}, for each form S_i and for each weight, starting with 1, we take the monomial of S_i that corresponds to this weight and divide by one or more $F_{s,i}$, if necessary, to obtain a monomial that belongs to the base Γ_3.

We just deal with the first three cubic forms S_1, S_2 and S_3, and we do not explore completely S_3. We want to illustrate the method, how to construct $\mathcal{Q}_{\mathcal{H}_1}$ and then compute its dimension.

- **Starting with S_1:** We already have that $F_{12,1} = {X_6}^2 - X_0X_{12}$ and $F_{14,1} = X_6X_8 - X_0X_{14}$. Thus, need not do any division, getting

$$S_1 = 0 \ \text{ and so } \ F_{20,1} = X_8X_{12} - X_6X_{14}.$$

- **S_2:** From $S_1 = 0$ we have $F_{20,1} = X_8X_{12} - X_6X_{14}$. We readily see that all monomials of $X_{14}(F_{16,1} - F^{(0)}_{16,1})$ belong to Γ_3. So we need take care of the monomials of $X_8(F_{22,1} - F^{(0)}_{22,1})$. We need not take care of the monomials whose coefficients have weights one on two. Since they are the only ones with this weight, it follows that the coefficients $c_{22,1,21}$ and $c_{22,1,20}$ will be zero. In fact, we need only deal with weights 3, 5 and 16. Thus

$$S_2 + (c_{22,1,19}X_{13} + c_{22,1,17}X_{11} + c_{22,1,6}X_0)F_{14,1} = 0$$

which implies, respecting the linearization,

$$F_{22,1} = X_{10}X_{12} - X_8X_{14} + c_{16,1,13}X_6X_{13} + c_{16,1,11}X_6X_{11} + c_{16,1,0}X_0X_6.$$

- **S_3:** We already know that $F_{20,1} = F^{(0)}_{20,1}$, thus

$$S_3 = X_{14}(F_{18,1} - F^{(0)}_{18,1}) + X_8(F_{24,2} - F^{(0)}_{24,2}).$$

weight 1: The monomial $c_{18,1,17}X_{14}X_6X_{11}$ belongs to Γ_3, whence

$$S_3^{(1)} = S_3 + c_{24,2,23}(X_{13}F_{18,1} + X_6F_{25,1})\,.$$

Here we needed to divide twice. With a simple division it is not possible that the remainder, in weight one, belong to Γ_3.

weight 2: We work with $S_3^{(1)}$ instead of S_3. The monomials that not belong to Γ_3 are precisely $-c_{24,2,23}c_{18,1,17}X_{13}X_6X_{11}$, and $-c_{24,2,22}{X_8}^2X_{14}$. Thus

$$S_3^{(2)} = \pi_3\left(S_3^{(1)} + c_{24,2,23}c_{18,1,17}X_6F_{24,1} + c_{24,2,22}X_{14}F_{16,1}\right)$$

and we can write:

$$S_3^{(2)} = S_3^{(1)} + c_{24,2,23}c_{18,1,17}X_6F_{24,1}^{(0)} + c_{24,2,22}X_{14}F_{16,1}\,.$$

Here we see the simplification. Now we work with $S_3^{(2)}$, and continue with this procedure which ends when we reach the weight 30.

Exploring all the syzygies, we have that $\mathcal{Q}_{\mathcal{H}_1}$ is the intersection of five quadrics in $\mathbb{A}^{18}$ given by the zeros of the isobaric quadratic polynomials:

$$\begin{gathered}
c_{16,1,11}c_{18,1,8} - c_{16,1,13}c_{18,1,6} + 2c_{18,1,11}c_{18,1,10}\,,\\
c_{16,1,13}c_{22,2,8} + c_{16,1,11}c_{22,2,10} - 2c_{19,1,10}c_{18,1,10}\,,\\
c_{18,1,11}c_{22,2,10} + c_{19,1,10}c_{18,1,8} - c_{16,1,13}c_{22,2,6}\,,\\
c_{18,1,11}c_{22,2,8} - c_{16,1,11}c_{22,2,6} + c_{19,1,10}c_{18,1,6}\,,\\
c_{18,1,6}c_{22,2,10} + c_{18,1,8}c_{22,2,8} - 2c_{18,1,10}c_{22,2,6}.
\end{gathered}$$

Now computations using the Maple software show that dim $\mathcal{Q}_{\mathcal{H}_1} = 15$, and so dim $\overline{\mathscr{M}_{\mathcal{H}_1}} \leq$ dim $\mathcal{Q}_{\mathcal{H}_1} - 1 = 14$.

On the other hand, Deligne's upper bound gives dim $\mathscr{M}_{\mathcal{H}_1} \leq 2g - 1 = 15$. The weight of $\mathcal{H}_1$ is $w(\mathcal{H}_1) = 10$, whence $3g - 2 - w(\mathcal{H}_1) = 12$. One can verify that the dimension of $\overline{\mathscr{M}_{\mathcal{H}_1}}$ is 14. This symmetric semigroup is realizable; see in [18] the family of symmetric semigroups N_3.

By using the Maple software, we see that the moduli variety $\overline{\mathscr{M}_{\mathcal{H}_1}}$ is given by 38 isobaric equations and the number of terms on each equation is at least

15, so we have at least 570 terms to deal. My computer[4] takes 14.19 seconds and uses 7.31 Mb of RAM memory to produce all 38 isobaric equations. With the same computer and applying the above method, the five equations of $\mathcal{Q}_{\mathcal{H}}$ take 3.48 seconds and use 5.68 Mb of RAM.

For the second example, set

$$\mathcal{H}_2 = \{0, 7, 9, 10, 11, 12, 13, 14, 16, 17, 18, \dots\}.$$

This symmetric semigroup is negatively graded, it is of the type $\{0, g-1, g+1, \dots, 2g-2, 2g+1, \dots\}$ with $g = 8$. Then we already know which it is realizable and $\dim \mathscr{M}_{\mathcal{H}_2} = 2g - 1 = 15$. The weight of $\mathcal{H}$ is $w(\mathcal{H}_2) = 8$, thus the Eisenbud–Harris lower bound gives $3g - 2 - w(\mathcal{H}_2) = 14$. We will compute the upper bound for $\overline{\mathscr{M}_{\mathcal{H}_2}}$ given by $\mathcal{Q}_{\mathcal{H}}$.

The canonical system of generators for $\mathcal{H}_2$ is $\{0, 7, 9, 10, 11, 12, 13, 14\}$ and then we draw the table of all $n_i + n_j$ $(0 \leq i \leq j \leq 7)$

0+0	0+7	0+9	0+10	0+11	0+12	0+13
0+14 7+7	7+9	7+10	7+11 9+9	7+12 9+10	7+13 9+11 10+10	7+14 9+12 10+11
9+13 10+12 11+11	9+14 10+13 11+12	10+14 11+13 12+12	11+14 12+13	12+14 13+13	13+14	14+14

We fix the P-hermitian basis $\{x_0, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\}$ of $H^0(C_0, 14P)$, where $P = (0 : \dots : 0 : 1) \in \mathbb{P}^7$. The basis for the vector space $H^0(C_0, 28P)$ is:

$$\begin{cases} x_0x_0,\ x_0x_7,\ x_0x_9,\ x_0x_{10},\ x_0x_{11},\ x_0x_{12},\ x_0x_{13},\ x_0x_{14}, \\ x_7x_9,\ x_7x_{10},\ x_7x_{11},\ x_7x_{12},\ x_7x_{13},\ x_7x_{14},\ x_9x_{13}, \\ x_9x_{14},\ x_{10}x_{14},\ x_{11}x_{14},\ x_{12}x_{14},\ x_{13}x_{14},\ x_{14}x_{14} \end{cases}$$

and the P-hermitian basis for $H^0(C_0, 42P)$ is

$$\begin{cases} x_0^2 x_{n_j}, \quad (j = 0, \dots, g-1) \\ x_7x_9x_{13}, \\ x_0x_{a_s}x_{b_s},\ x_{a_s}x_{b_s}x_{14} \quad (s = 16, \dots, 28) \end{cases}$$

[4]My computer is a AMD Athlon X2 dual Core, with 2 Gb RAM

with each $a_s + b_s$ on top of each entry of the above table.

Thus, we have 15 quadratic forms $F_{s,i}$ given by the above table. Again, by Proposition 2.2, we can normalize to zero 28 coefficients of the quadratic forms F_{si}, so we get:

- $F_{14,1} = X_7^2 - X_0 X_{14}$;
- on $F_{18,1}$: $c_{18,1,17} = c_{18,1,16} = 0$;
- on $F_{19,1}$: $c_{19,1,18} = c_{19,1,17} = c_{19,1,16} = c_{19,1,14} = c_{19,1,9} = c_{19,1,7} = 0$;
- on $F_{20,1}$: $c_{20,1,19} = c_{20,1,18} = c_{20,1,17} = c_{20,1,16} = c_{20,1,14} = c_{20,1,13} = c_{20,1,11} = c_{20,1,9} = c_{20,1,7} = 0$;
- on $F_{20,2}$: $c_{20,2,19} = c_{20,2,18} = c_{20,2,17} = c_{20,2,16} = 0$.

The nine cubic forms induced by the Syzygy Lemma are:

$$
\begin{aligned}
S_1 &:= X_{14}F_{18,1} - X_{11}F_{21,1} + X_9F_{23,2}\\
S_2 &:= X_{14}F_{19,1} - X_{12}F_{21,2} + X_{10}F_{23,2}\\
S_3 &:= X_{14}F_{20,1} - X_{13}F_{21,1} + X_9F_{25,1}\\
S_4 &:= X_{14}F_{20,2} - X_{13}F_{21,2} + X_{10}F_{24,1}\\
S_5 &:= X_{14}F_{21,2} - X_{14}F_{21,1} - X_{12}F_{23,1} + X_{10}F_{25,1}\\
S_6 &:= X_{14}F_{22,1} - X_{13}F_{23,1} + X_{10}F_{26,1}\\
S_7 &:= X_{14}F_{22,2} - X_{13}F_{23,2} + X_{11}F_{25,1}\\
S_8 &:= X_{14}F_{23,2} - X_{14}F_{23,1} - X_{13}F_{24,1} + X_{11}F_{26,1}\\
S_9 &:= X_{14}F_{24,2} - X_{14}F_{24,1} - X_{13}F_{25,1} + X_{12}F_{26,1}
\end{aligned}
$$

From the previous nine cubic forms we deduce that $\dim T^{1,-}(\mathcal{H}_2) = 23$ and $T^{1,-}(\mathcal{H}_2)$ is isomorphic to the vector space generated by:

$c_{18,1,14},\ c_{18,1,13},\ c_{18,1,12},\ c_{18,1,10},\ c_{18,1,9},\ c_{18,1,7}$

$c_{19,1,12},\ c_{19,1,11},\ c_{19,1,10},$

$c_{20,1,10},$

$c_{20,2,14},\ c_{20,2,12},\ c_{20,2,10},\ c_{20,2,9},\ c_{20,2,7},$

$c_{21,1,20},\ c_{21,1,19},\ c_{21,1,18},\ c_{21,1,17},\ c_{21,1,16},\ c_{21,1,14},\ c_{21,1,9},\ c_{21,1,7}.$

The quasi-cone $\mathcal{Q}_{\mathcal{H}_2}$ is in $\mathbb{A}^{23}$ and is given by 14 quadratic forms. To symplify displaying then, we make a simple change of variables:

$$c_{18,1,i} \mapsto a_{18-i}, \quad c_{19,1,i} \mapsto b_{19-i}, \quad c_{20,1,i} \mapsto c_{20-i}, \quad c_{20,2,i} \mapsto d_{20-i}, \quad c_{21,1,i} \mapsto e_{21-i}.$$

Thus the 14 quadratic forms that define $\mathcal{Q}_{\mathcal{H}_2}$ are:

$$\begin{gathered}
a_5 d_{11} + a_8 d_8 - b_8^2 + d_{10} a_6 - 2 b_7 b_9 + b_7 a_9\,, \\
a_8 a_9 + b_7 c_{10} - b_8 a_9 - e_{12} a_5 - 2 a_8 b_9 - d_{11} a_6 + b_8 b_9\,, \\
a_6 e_{12} - b_8 d_{10} - a_9 b_9 - d_8 c_{10} + b_8 c_{10} + 2 b_9^2 + b_7 d_{11}\,, \\
b_7 a_{11} - a_9 b_9 - a_5 d_{13} - a_8 d_{10} + b_7 d_{11} + a_6 e_{12} - b_8 c_{10} + a_8 c_{10} + b_9^2\,, \\
a_8 d_{11} - b_8 d_{11} + a_5 e_{14} + a_9 c_{10} - b_9 c_{10} - b_8 a_{11}\,, \\
d_{13} a_6 - d_8 a_{11} - b_9 c_{10} - b_7 e_{12} + 2 b_8 d_{11} + b_8 a_{11} - a_8 d_{11} - d_8 d_{11}\,, \\
a_8 e_{12} + 2 d_{11} b_9 - a_9 d_{11} + 2 b_9 a_{11} - a_9 a_{11} - a_6 e_{14} + c_{10}^2\,, \\
a_8 e_{12} + c_{10} d_{10} - d_{11} b_9 - b_8 e_{12}\,, \\
b_9 e_{12} - d_{13} b_8 - c_{10} d_{11} + b_7 e_{14}\,, \\
d_{10} a_{11} + d_{10} d_{11} + b_9 e_{12} + a_8 d_{13} - d_{13} b_8\,, \\
d_{11} a_{11} + c_{10} e_{12} + a_8 e_{14} - b_8 e_{14} + d_{11}^2\,, \\
d_8 e_{14} - b_8 e_{14} - 2 b_9 d_{13} - c_{10} e_{12} - d_{11}^2 + a_9 d_{13} + d_{10} e_{12}\,, \\
d_{11} e_{12} - c_{10} d_{13} + b_9 e_{14} + a_{11} e_{12}\,, \\
e_{12}^2 + d_{11} d_{13} - d_{10} e_{14}\,.
\end{gathered}$$

Then, we verify using Maple that $\dim \mathcal{Q}_{\mathcal{H}_2} = 16$ and so $\dim \overline{\mathscr{M}_{\mathcal{H}_2}} \leq 15 = \dim \mathscr{M}_{\mathcal{H}_2}$.

By computations using Maple, we may see that the moduli variety $\overline{\mathscr{M}_{\mathcal{H}_2}}$ is given by 81 isobaric equations. My computer takes approximately 1 hour and 28 minutes and uses 87,92 Mb of RAM memory to produce them. With the same computer and applying the above method the fourteen equations of $\mathcal{Q}_{\mathcal{H}}$ take 8.30 seconds and use 5.99 Mb of RAM.

3.3.2 Two examples in genus 9

Let us consider the symmetric semigroup

$$\mathcal{H}_3 = \{0, 6, 8, 10, 12, 13, 14, 15, 16, 18, 19, 20, \ldots\}\,.$$

The canonical system of generators is $\{0, 6, 8, 10, 12, 13, 14, 15, 16\}$. We draw the table of all $n_i + n_j$ $(0 \leq i \leq j \leq 8)$.

0+0	0+6	0+8	0+10	0+12 6+6	0+13	0+14 6+8	0+15
0+16 6+10 8+8	6+12 8+10	6+13	6+14 8+12 10+10	6+15 8+13	6+16 8+14 10+12	8+15 10+13	8+16 10+14 12+12
10+15 12+13	10+16 12+14 13+13	12+15 13+14	12+16 13+15 14+14	13+16 14+15	14+16 15+15	15+16	16+16

We fix the P-hermitian basis $\{x_0, x_6, x_8, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}\}$ of $H^0(C_0, 16P)$, where $P = (0 : \ldots : 0 : 1) \in \mathbb{P}^8$. The basis for the vector space $H^0(C_0, 32P)$ is

$$\begin{cases} x_0x_0,\ x_0x_6, \ldots, x_0x_{16}, \\ x_6x_{12},\ x_6x_{13},\ x_6x_{14},\ x_6x_{15},\ x_6x_{16}, \\ x_8x_{15},\ x_8x_{16},\ x_{10}x_{15},\ x_{10}x_{16},\ x_{12}x_{15},\ x_{12}x_{16}, \\ x_{13}x_{16},\ x_{14}x_{16},\ x_{15}x_{16},\ x_{16}x_{16} \end{cases}$$

and the basis for $H^0(C_0, 48P)$ is

$$\begin{cases} x_0^2x_{n_j}, \qquad (j = 0, \ldots, g-1) \\ x_6x_{12}x_{15}, \\ x_0x_{a_s}x_{b_s},\ x_{a_s}x_{b_s}x_{16} \qquad (s = 18, \ldots, 32) \end{cases}$$

with each $a_s + b_s$ on top of each entry of the previous table.

We have 21 quadratic forms F_{si} and, by Proposition 2.2, we can normalize to zero 36 coefficients. Thus we can make the following normalizations:

- $F_{12,1} = X_6^2 - X_0X_{12}$;
- $F_{14,1} = X_6X_8 - X_0X_{14}$;
- $F_{16,1} = X_6X_{10} - X_0X_{16}$;
- in $F_{16,2}$: $c_{16,2,14} = 0$;
- in $F_{21,1}$: $c_{21,1,20} = c_{21,1,19} = c_{21,1,18} = c_{21,1,16} = c_{21,1,15} = c_{21,1,14} = c_{21,1,13} = c_{21,1,12} = c_{21,1,8} = c_{21,1,7} = 0$;

- in $F_{26,2}$: $c_{26,2,25} = c_{26,2,24} = c_{26,2,23} = c_{26,2,22} = c_{26,2,21} = c_{26,2,19} = c_{26,2,16} = 0$.

The fourteen cubic forms induced by the Syzygy Lemma are:

$$
\begin{aligned}
S_1 &:= X_{16}F_{12,1} + X_6F_{22,2} - X_{12}F_{26,1}\\
S_2 &:= X_{16}F_{14,1} + X_8F_{22,1} - X_{14}F_{16,2}\\
S_3 &:= X_{16}F_{16,2} - X_{16}F_{16,1} - X_{10}F_{22,1} + X_8F_{24,1}\\
S_4 &:= X_{16}F_{18,1} - X_{12}F_{22,2} + X_{10}F_{24,2}\\
S_5 &:= X_{16}F_{20,1} + X_{12}F_{24,1} - X_{14}F_{22,2}\\
S_6 &:= X_{16}F_{20,2} + X_{10}F_{26,1} - X_{14}F_{22,2}\\
S_7 &:= X_{16}F_{21,1} - X_{15}F_{22,1} + X_8F_{29,1}\\
S_8 &:= X_{16}F_{22,2} - X_{16}F_{22,1} - X_{14}F_{24,2} + X_{12}F_{26,1}\\
S_9 &:= X_{16}F_{23,1} - X_{15}F_{24,1} + X_{10}F_{29,1}\\
S_{10} &:= X_{16}F_{24,2} - X_{16}F_{24,1} - X_{14}F_{26,1} + X_{12}F_{28,2}\\
S_{11} &:= X_{16}F_{25,1} - X_{15}F_{26,1} + X_{12}F_{29,1}\\
S_{12} &:= X_{16}F_{26,2} - X_{16}F_{26,1} - X_{14}F_{28,1} + X_{13}F_{29,1}\\
S_{13} &:= X_{16}F_{27,1} - X_{15}F_{28,1} + X_{13}F_{30,1}\\
S_{14} &:= X_{16}F_{28,2} + X_{14}F_{30,1} - X_{15}F_{29,1} - X_{16}F_{28,1}
\end{aligned}
$$

By considering the linearization, we deduce that the vector space $T^{1,-}(B_{\mathcal{H}})$ has dimension 19 and depends on:

$$
\begin{aligned}
&c_{16,2,15},\ c_{16,2,13},\ c_{16,2,12},\ c_{16,2,8},\ c_{16,2,6},\\
&c_{18,1,16},\ c_{18,1,14},\ c_{18,1,13},\ c_{18,1,12},\ c_{18,1,8},\ c_{18,1,6},\\
&c_{21,1,10},\\
&c_{26,2,20},\ c_{26,2,18},\ c_{26,2,14},\ c_{26,2,12},\ c_{26,2,10},\ c_{26,2,8},\ c_{26,2,6}.
\end{aligned}
$$

For typographical reasons we make the following chance of variables:

$$c_{16,1,i} \mapsto a_{16-i},\quad c_{18,2,i} \mapsto b_{18-i},\quad c_{21,1,i} \mapsto c_{21-i},\quad c_{26,2,i} \mapsto d_{26-i}.$$

Exploring all the fourteen cubic forms S_i the quasi-cone $\mathcal{Q}_{\mathcal{H}_3}$ is in $\mathbb{A}^{19}$ and induced by the following five isobaric quadratic polynomials:

$$
\begin{gathered}
b_{10}a_3 - a_1b_{12} + a_3a_{10} - b_5a_8\,,\\
a_8c_{11} + a_3d_{16} + a_1d_{18}\,,
\end{gathered}
$$

$$b_5d_{16} + a_{10}c_{11} + b_{10}c_{11} - a_1d_{20}\,,$$
$$b_5d_{18} - c_{11}b_{12} + a_3d_{20}\,,$$
$$a_{10}d_{18} + b_{12}d_{16} + b_{10}d_{18} + a_8d_{20}\,.$$

Using Maple it follows that $\dim \mathcal{Q}_{\mathcal{H}_3} = 16$ and so $\dim \overline{\mathscr{M}_{\mathcal{H}_3}} \leq 15$.

Deligne's bound gives $2g-1 = 17$. The weight of $\mathcal{H}_3$ is equal to 14, whence the Eisenbud-0Harris lower bound is $3g-2-w(\mathcal{H}_3) = 11$.

Using Maple, we may see that the moduli variety $\overline{\mathscr{M}_{\mathcal{H}_3}}$ is given by 99 isobaric equations and my computer takes approximately 48 minutes and 28 seconds and uses 66.11 Mb of RAM memory to compute them. The number of terms of the biggest equation is 44940. With the same computer, the five equations of $\mathcal{Q}_{\mathcal{H}}$ take 9.95 seconds, and employing 5.87 Mb of RAM.

For the last numerical example let us take the symmetric semigroup

$$\mathcal{H}_4 := \{0, 6, 9, 10, 12, 13, 14, 15, 16, 18, 19, 20, \ldots\}.$$

The canonical system of generators of $\mathcal{H}_4$ is $\{0, 6, 9, 10, 12, 13, 14, 15, 16\}$. The table of all $n_i + n_j$ $(0 \leq i \leq j \leq 8)$ is:

0+0	0+6	0+9	0+10	0+12 6+6	0+13	0+14	0+15 6+9
0+16 6+10	6+12 9+9	6+13 9+10	6+14 10+10	6+15 9+12	6+16 9+13 10+12	9+14 10+13	9+15 10+14 12+12
9+16 10+15 12+13	12+14 10+16 13+13	12+15 13+14	12+16 13+15 14+14	13+16 14+15	14+16 15+15	15+16	16+16

We fix the P-hermitian basis $\{x_0, x_6, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}\}$ of $H^0(C_0, 16P)$, where $P = (0 : \ldots : 0 : 1) \in \mathbb{P}^8$. The basis of the vector space $H^0(C_0, 32P)$ is

$$\begin{cases} x_0x_0,\ x_0x_6, \ldots, x_0x_{16}, \\ x_6x_{12},\ x_6x_{13},\ x_6x_{14},\ x_6x_{15},\ x_6x_{16}, \\ x_9x_{14},\ x_9x_{15},\ x_9x_{16},\ x_{12}x_{14},\ x_{12}x_{15},\ x_{12}x_{16}, \\ x_{13}x_{16},\ x_{14}x_{16},\ x_{15}x_{16},\ x_{16}x_{16} \end{cases}$$

and the basis of $H^0(C_0, 48P)$ is

$$\begin{cases} x_0^2 x_{n_j}, & (j = 0, \dots, g-1) \\ x_6 x_{12} x_{15}, & \\ x_0 x_{a_s} x_{b_s},\ x_{a_s} x_{b_s} x_{16} & (s = 18, \dots, 32) \end{cases}$$

with each $a_s + b_s$ on top of each entry of the previous table.

The normalizations are:

- $F_{12,1} = X_6^2 - X_0 X_{12}$;
- $F_{15,1} = X_6 X_9 - X_0 X_{15}$;
- $F_{16,1} = X_6 X_{10} - X_0 X_{16}$;
- $F_{19,1} = X_9 X_{10} - X_6 X_{13} - c_{19,1,14} X_0 X_{14} - c_{19,1,10} X_0 X_{10} - c_{19,1,0} X_0^2$;
- in $F_{18,1}$: $c_{18,1,15} = c_{18,1,9} = 0$;
- in $F_{20,1}$: $c_{20,1,19} = c_{20,1,18} = c_{20,1,16} = c_{20,1,15} = c_{20,1,12} = c_{20,1,6} = 0$;
- in $F_{24,1}$: $c_{24,1,23} = c_{24,1,20} = 0$.

The fourteen cubic forms induced by the Syzygy Lemma are:

$$\begin{aligned}
S_1 &:= X_{16} F_{12,1} - X_{12} F_{16,1} + X_6 F_{22,2} \\
S_2 &:= X_{16} F_{15,1} - X_{15} F_{16,1} + X_6 F_{25,1} \\
S_3 &:= X_{16} F_{18,1} - X_{12} F_{22,1} + X_9 F_{25,2} \\
S_4 &:= X_{16} F_{19,1} - X_{13} F_{22,1} + X_9 F_{26,2} \\
S_5 &:= X_{16} F_{20,1} - X_{14} F_{22,2} + X_{10} F_{26,1} \\
S_6 &:= X_{16} F_{21,1} - X_{15} F_{22,1} + X_9 F_{28,1} \\
S_7 &:= X_{16} F_{22,2} - X_{16} F_{22,1} - X_{13} F_{25,1} + X_{10} F_{28,1} \\
S_8 &:= X_{16} F_{23,1} - X_{14} F_{25,1} + X_{10} F_{29,1} \\
S_9 &:= X_{16} F_{24,1} - X_{15} F_{25,1} + X_{10} F_{30,1} \\
S_{10} &:= X_{16} F_{24,2} - X_{15} F_{25,2} + X_{12} F_{28,1} \\
S_{11} &:= X_{16} F_{25,2} - X_{16} F_{25,1} - X_{15} F_{26,1} + X_{12} F_{29,1} \\
S_{12} &:= X_{16} F_{26,2} - X_{16} F_{26,1} - X_{14} F_{28,1} + X_{13} F_{29,1} \\
S_{13} &:= X_{16} F_{27,1} - X_{15} F_{28,1} + X_{13} F_{30,1}
\end{aligned}$$

$$S_{14} := X_{16}F_{28,2} - X_{16}F_{28,1} - X_{15}F_{29,1} + X_{14}F_{30,1}$$

By considering the linearization we deduce that the vector space $T^{1,-}(B_{\mathcal{H}})$ has dimension equal to 19 and depends on:

$$\begin{array}{cccccc} c_{18,1,16}, & c_{18,1,14}, & c_{18,1,13}, & c_{18,1,12}, & c_{18,1,10}, & c_{18,1,6}, \\ c_{20,1,13}, & c_{20,1,10}, & c_{20,1,9}, & c_{20,1,}, & & \\ c_{24,1,22}, & c_{24,1,21}, & c_{24,1,19}, & c_{24,1,18}, & c_{24,1,16}, & c_{24,1,15}, \\ & & c_{24,1,12}, & c_{24,1,10}, & c_{24,1,9}, & c_{24,1,6} \end{array}$$

We make the following change of variables:

$$c_{18,1,i} \mapsto a_{18-i}, \quad c_{20,1,i} \mapsto b_{20-i}, \quad c_{24,1,i} \mapsto c_{24-i}.$$

Thus, exploring all the fourteen cubic forms we have that $\mathcal{Q}_{\mathcal{H}_4}$ is in $\mathbb{A}^{19}$ and given by the following five isobaric quadratic polynomials:

$$\begin{gathered} a_5b_{10} + a_4b_{11} + b_7a_8 \, , \\ b_7a_{12} - a_5c_{14} - c_{15}a_4 \, , \\ c_{14}a_8 + a_4c_{18} + b_{10}a_{12} \, , \\ a_8c_{15} - a_5c_{18} + a_{12}b_{11} \, , \\ c_{14}b_{11} - b_7c_{18} - b_{10}c_{15} \, , \end{gathered}$$

Then we may see, using Maple software, that $\dim \mathcal{Q}_{\mathcal{H}_4} = 16$, and so $\dim \overline{\mathscr{M}_{\mathcal{H}_4}} \leq 15$.

Deligne's upper bound gives us $2g - 1 = 17$. The weight of $\mathcal{H}_4$ is equal to 13, hence Eisenbud–Harris lower bound is 12. The dimension of the moduli space $\overline{\mathscr{M}_{\mathcal{H}_4}}$ can be computed: 15.

By using the Maple software, we verify that the moduli variety $\overline{\mathscr{M}_{\mathcal{H}_4}}$ is given by 82 isobaric equations. My computer takes approximately 14 seconds and uses 6,06 Mb of RAM memory to compute them. The number of terms of the biggst equation is 325. With the same computer, the five equations of $\mathcal{Q}_{\mathcal{H}}$ takes 8.31 seconds and uses 5.68 Mb of RAM.

Chapter 4

Working Explicitly with Families of Semigroups

Many studies involving Weierstrass points have been done by investigating families of semigroups; see for example [1], [5], [6], [10], [18]. We will illustrate that our method works very satisfactorily with two one-parameter families of symmetric semigroups.

By considering the Apéry sequence, a symmetric semigroup $\mathcal{H}$ of multiplicity m can be generated by $m-1$ elements as follows. Set $a_0 = 0$. If $a_0 < \ldots < a_i$ have been chosen and $i < m-1$, let a_{i+1} be the least integer in $\mathcal{H}$ having m-residue distinct from those of $a_0, \ldots, a_i$. Since $\mathcal{H}$ is symmetric, it follows that $a_{m-1} = l_g + m = 2g - 1 + m$. Then $\mathcal{H}$ is generated by $m, a_1 \ldots, a_{m-2}$, $\mathcal{H} = \bigsqcup a_i + m\mathbb{N}$. Rim–Vitulli considered in [23] the system of m generators, $m, a_1, \ldots, a_{m-2}$, and they called it a *standard basis* for $\mathcal{H}$. By contrast, Stöhr taked into account in [27] the canonical system of generators for $\mathcal{H}$, namely $n_0, \ldots, n_{g-1}$.

We deal with two families of symmetric semigroups where the genus is larger than 6 and depends linearly on the parameter of the family. The first family consists of semigroups of multiplicity five, namely $\mathcal{H} =< 5, 2+5\tau, 3+5\tau, 4+5\tau >$, and the corresponding moduli varieties are described by linear equations; see Corollary 4.4. We can construct the moduli space explicitly and compute its dimension. The second family consists of semigroups of multiplicity six, namely $\mathcal{H} =< 6, 2 + 6\tau, 3 + 6\tau, 4 + 6\tau, 5 + 6\tau >$; so the theorems in [16] and [30] can not be applied. We construct explicitly $\mathcal{Q}_{\mathcal{H}}$ and we are able to compute its dimension. We obtain the upper bound $8\tau + 5$, improving the upper bound given by Deligne's Formula which is $12\,\tau+1$. For the two families the Eisenbud–Harris lower bound becomes negative for large genus.

4.1 A family with multiplicity five

Let us consider the one-parameter family of symmetric semigroups with multiplicity $m = 5$ given by:

$$\mathcal{H} =< 5, 2 + 5\tau, 3 + 5\tau, 4 + 5\tau >, \quad \text{with } \tau \geq 1. \tag{4.1}$$

The genus of $\mathcal{H}$ is $g = 1 + 5\tau$ and the canonical system of generators is

$$\begin{aligned} &0, 5, \ldots, 10\,\tau, 2 + 5\tau, \ldots, 2 + 5(2\tau - 1),\\ &3 + 5\tau, \ldots, 3 + 5(2\tau - 1), 4 + 5\tau, \ldots, 4 + 5(2\tau - 1). \end{aligned}$$

Suppose that C is an integral projective Gorenstein curve of arithmetical genus g and $P \in C$ is a nonsingular point such that the Weierstrass semigroup of (C, P) is $\mathcal{H}$. By the definition of Weierstrass semigroup, for each $n \in \{5, 2 + 5\tau, 3 + 5\tau, 4 + 5\tau\}$ there is a meromorphic function $x_n \in H^0(C, 10\,\tau\,P)$ whose pole order at P is exactly n and does not have other poles. We introduce the following notations for typographical reasons:

$$x := x_5, \;\; y_i := x_{i+5\tau} \quad (i = 2, 3, 4).$$

More generally, for each $n \in \mathcal{H}$ there is a meromorphic function x_n whose pole order at P is exactly n and does not have other poles. We can normalize them in a way that $x_0 = 1$, $x_{n+5} = x_5 x_n$ and $x_{6+10\,\tau} = y_2 y_4$.

The divisor $10\,\tau P$ is canonical and since $\ell_2 = 2$, i.e. $\mathcal{H}$ is nonhyperelliptic, we can identify C with its image under the canonical embedding

$$(x_0 : \ldots : x_{10\,\tau}) : C \hookrightarrow \mathbb{P}^{5\tau}.$$

Thus, we can assume that C is a canonical curve in $\mathbb{P}^{5\tau}$ of arithmetical genus g and $P = (0 : \ldots : 0 : 1)$.

Searching for a monomial basis of the vector space $H^0(C, 10\,n\,\tau P)$ with $n \geq 1$ we obtain the following:

Proposition 4.1. *The following functions form a P-hermitian basis for $H^0(C, 10\, n\, \tau P)$, with $n \geq 1$.*

$$
\left.\begin{array}{l}
\left.\begin{array}{l}
\left.\begin{array}{l} 1, x, \ldots, x^{2\tau} \\ y_i, xy_i, \ldots, x^{\tau-1}y_i \end{array}\right] H^0(C, 10\,\tau P) \\
x^{2\tau+1}, \ldots, x^{4\tau} \\
x^{\tau} y_i, \ldots, x^{3\tau-1} y_i \\
y_2y_4, xy_2y_4, \ldots, x^{2\tau-2}y_2y_4
\end{array}\right] H^0(C, 20\,\tau P) \\
\vdots \\
x^{4\tau+1}, \ldots, x^{2n\tau} \\
x^{3\tau} y_i, \ldots, x^{\tau(2n-1)-1} y_i \\
x^{2\tau-1} y_2y_4, \ldots, x^{\tau(2n-2)-2} y_2y_4
\end{array}\right] H^0(C, 10\, n\, \tau P)
$$

for $i = 2, 3, 4.$

Proof. The pole orders at P of the functions $1, x, \ldots, x^{2n\tau}$ are congruous to 0 modulo 5 and range from 0 to $10\, n\, \tau$. The pole orders at P of the functions $y_i, \ldots, x^{\tau(2n-1)-1} y_i$ are congruous to 2,3 and 4 and range from $2 + 5\tau$ to $10\, n\, \tau - 1$. And the pole orders at P of the functions $y_2y_4, \ldots, x^{\tau(2n-2)-2} y_2y_4$ are congruous to 1 modulo 5 and range from $6 + 5\tau$ to $-4 + 10\, n\, \tau$. These functions belong to $H^0(C, 10\, n\, \tau)$ and are linearly independent because their pole orders at P are pairwise different. Their number is $10\, n\, \tau - 5\tau$. From the Riemann–Roch theorem it follows that $\dim H^0(C, 10 n \tau P) = 10\, n\, \tau - 5\tau$. $\square$

The curve C is nontrigonal and not isomorphic to a plane quintic (see [17]). So, by Petri's analysis, the ideal of C is generated by quadratic relations. In particular, we can ask for quadratic relations between the functions y_2, y_3 and y_4. Note that the five functions y_2^2, y_2y_3, y_3^2, y_3y_4, y_4^2 belong to $H^0(C, 20\, \tau P)$ and are not basis elements. Thus we can write each one of them as a linear combination of the basis elements of Proposition 4.1.

We denote, provisonally, the basis elements of $H^0(C, 10\, \tau P)$ by $x_{a_s} x_{b_s}$, where s is the pole order at P. Then, for each one of the five quadratic functions in $H^0(C, 20\, \tau P)$, namely y_2^2, y_2y_3, y_3^2, y_3y_4, y_4^2, there are constants $c_{i+j,k} \in \mathbf{k}$ such that

$$
y_i y_j = \sum_{0 \leq s \leq i+j+10\,\tau} c_{i+j,k} x_{a_s} x_{b_s} \, ,
$$

where $k+s=i+j+10\,\tau$. After eventually multiplying the functions x_{a_s} by suitable constants, we may assume that $c_{i+j,0}=1$.

Thus in the polynomial $\mathbf{k}$-algebra $\mathbf{k}[X,Y_2,Y_3,Y_4]$ we consider the following five polynomials given by the lifting of our functions $y_iy_j-\sum c_{i+j,k}x_{a_s}x_{b_s}$:

$$\begin{array}{ll}
F_4=Y_2^2-X^{\tau}Y_4- & F_7=Y_3Y_4-X^{\tau+1}Y_2-c_{7,1}Y_2Y_4\\
-c_{4,1}X^{\tau}Y_3-\ldots-c_{4,1+5\tau}Y_3 & -c_{7,2}X^{2\tau+1}-\ldots-c_{7,2+5(2\tau+1)}\\
-c_{4,2}X^{\tau}Y_2-\ldots-c_{4,2+5\tau}Y_2 & -c_{7,3}X^{\tau}Y_4-\ldots-c_{7,3+5\tau}Y_4\\
-c_{4,4}X^{2\tau}-\ldots-c_{4,4+10\,\tau} & -c_{7,4}X^{\tau}Y_3-\ldots-c_{7,4+5\tau}Y_3\\
-c_{4,5}X^{\tau-1}Y_4-\ldots-c_{4,5\tau}Y_4 & -c_{7,5}X^{\tau}Y_2-\ldots-c_{7,5+5\tau}Y_2
\end{array}$$

$$\begin{array}{ll}
F_5=Y_2Y_3-X^{2\tau+1} & F_8=Y_4^2-X^{\tau+1}Y_3-c_2Y_2Y_4-\\
-c_{5,1}X^{\tau}Y_4-\ldots-c_{5,1+5\tau}Y_4 & -c_{8,1}X^{\tau+1}Y_2-\ldots-c_{8,1+5(\tau+1)}Y_2\\
-c_{5,2}X^{\tau}Y_3-\ldots-c_{5,2+5\tau}Y_3 & -c_{8,3}X^{2\tau+1}-\ldots-c_{8,3+5(2\tau+1)}\\
-c_{5,3}X^{\tau}Y_2-\ldots-c_{5,3+5\tau}Y_2 & -c_{8,4}X^{\tau}Y_4-\ldots-c_{8,4+5\tau}Y_4\\
-c_{5,5}X^{2\tau}-\ldots-c_{5,5+10\,\tau} & -c_{8,5}X^{\tau}Y_3-\ldots-c_{8,5+5\tau}Y_3
\end{array}$$

$$\begin{array}{l}
F_6=Y_3^3-Y_2Y_4-\\
-c_{6,1}X^{2\tau+1}-\ldots-c_{6,6+10\,\tau}\\
-c_{6,2}X^{\tau}Y_4-\ldots-c_{6,2+5\tau}Y_4\\
-c_{6,3}X^{\tau}Y_3-\ldots-c_{6,3+5\tau}Y_3\\
-c_{6,4}X^{\tau}Y_2-\ldots-c_{6,4+5\tau}Y_2
\end{array}$$

Now we invert the above considerations. We consider the polynomial ring $\mathbf{k}[X,Y_2,Y_3,Y_4]$ where the weight of X is 5 and that of Y_i is $i+5\tau$ $(i=2,3,4)$. We take the five polynomials F_s for $s=4,5,6,7,8$, displayed above, where the coefficients $c_{s,k}$ belong to $\mathbf{k}$.

We ask for the conditions on the coefficients $c_{s,k}$ for the polynomials F_s to be induced by a Gorenstein curve of arithmetic genus g such that the Weierstrass semigroup of (C,P) is $\mathcal{H}$, where $P=(0:\ldots:0:1)$.

We also introduce the following five polynomials:

$$\begin{array}{lll}
F_4^{(0)}=Y_2^2-X^{\tau}Y_4\,, & F_5^{(0)}=Y_2Y_3-X^{2\tau+1}\,, & F_6^{(0)}=Y_3^2-Y_2Y_4\,,\\
F_7^{(0)}=Y_3Y_4-X^{\tau+1}Y_2\,, & F_8^{(0)}=Y_4^2-X^{\tau+1}Y_3\,. &
\end{array} \tag{4.2}$$

Note that the zero locus, say D_0, of our five isobaric polynomials $F_s^{(0)}$ is a monomial curve on $\mathbb{A}^4$ whose ring of regular functions is the semigroup algebra $B_{\mathcal{H}} = \oplus_{h\in\mathcal{H}}\mathbf{k}t^h$. The curve D_0 can be obtained by projecting in $\mathbb{A}^4$ the affine curve given by the canonical monomial curve C_0 on the local chart "$u = 1$". Therefore, the projectivization of D_0 is isomorphic to C_0. Because the rings of regular functions of D_0 and of the affine curve given by C_0 on the local chart "$u = 1$"are $B_{\mathcal{H}}$. To complete these affine curves we just include, on each curve, a nonsingular point at infinity.

Since a P-hermitian basis was fixed in Proposition 4.1, it follows that we have the freedom to transform:

$$\begin{aligned}
&x \mapsto \alpha_0 x + \alpha_5\\
&y_2 \mapsto \beta_0 y_2 + \beta_2 x^\tau + \ldots + \beta_{2+5\tau}\\
&y_3 \mapsto \gamma_0 y_3 + \gamma_1 y_2 + \gamma_3 x^\tau + \ldots + \gamma_{3+5\tau}\\
&y_4 \mapsto \theta_0 y_4 + \theta_1 y_3 + \theta_2 y_2 + \theta_4 x^\tau + \ldots + \theta_{4+5\tau}
\end{aligned} \tag{4.3}$$

where $\alpha_i, \beta_i, \gamma_i, \theta_i \in \mathbf{k}$ and $\alpha_0, \beta_0, \gamma_0, \theta_0 \in \mathbf{k}^\star$.

Lemma 4.2. *By linear changes of variables we can assume that the coefficients of the polynomials F_i $(i = 4, 5, 6, 7, 8)$ satisfy:*

$$\begin{cases}
c_{4,1} = c_{7,1} = c_{8,2} = c_{7,5} = 0\\
c_{5,2} = \ldots = c_{5,2+5\tau} = 0\\
c_{7,3} = \ldots = c_{7,3+5\tau} = 0\\
c_{7,4} = \ldots = c_{7,4+5\tau} = 0
\end{cases} \tag{4.4}$$

Proof. We use the above transformations (4.3) with $\alpha_0 = \beta_0 = \gamma_0 = \theta_0 = 1$. By exploring the two freedom in weight one, namely γ_1 and θ_1, we can normalize to zero $c_{4,1}$ and $c_{7,1}$. By using the freedoms of weight two of y_4 and of weight five of x, we can suppose that $c_{8,2} = c_{7,5} = 0$. Finally we use the freedom of y_2 of weights from 2 to $2 + 5\tau$, y_3 of weights from 3 to $3 + 5\tau$ and y_4 of weights from 4 to $4 + 5\tau$. $\square$

We pick up the following two important syzygies between the generators of the ideal of D_0:

$$\begin{aligned}
X^{\tau+1}F_4^{(0)} - Y_4 F_5^{(0)} + Y_2 F_7^{(0)} = 0\\
X^{\tau+1}F_6^{(0)} - Y_4 F_7^{(0)} + Y_3 F_8^{(0)} = 0
\end{aligned} \tag{4.5}$$

They are induced by the Syzygy Lemma; see the proof of Theorem 4.3 below. Moreover, we introduce the following two polynomials:

$$\begin{aligned} &X^{\tau+1}F_4 - Y_4F_5 + Y_2F_7 \\ &X^{\tau+1}F_6 - Y_4F_7 + Y_3F_8 \end{aligned} \tag{4.6}$$

Theorem 4.3. *Let $\mathcal{H}$ be the semigroup generated by $5, 2+5\tau, 3+5\tau, 4+5\tau$ with $\tau \geq 1$. Then, the moduli space $\overline{\mathscr{M}_{\mathcal{H}}}$ corresponds bijectively to the orbits of the equivariant $\mathbb{G}_m(\mathbf{k})$-action $(z, c_{s,i}) \mapsto z^i c_{s,i}$ on the algebraic set of the vectors of constants $c_{s,i}$ normalized by (4.4) and satisfying the following two polynomial equations:*

$$X^{\tau+1}F_4 - Y_4F_5 + Y_2F_7 = \sum_{i=1}^{\tau} X^{\tau-i}\left(c_{5,1+5i}F_8 - c_{7,5+5i}F_4\right)$$

$$\begin{aligned} X^{\tau+1}F_6 - Y_4F_7 + Y_3F_8 = & -\sum_{i=0}^{\tau} X^{\tau-i}\left(c_{8,4+5i}F_7 + c_{8,5+5i}F_6\right) - \\ & -\sum_{i=0}^{\tau+1} X^{\tau+1-i}c_{8,1+5i}F_5 \end{aligned}$$

Proof. We need to prove that all relations between the coefficients $c_{s,i}$ are induced only by the two syzygies in (4.5). It is simple to see it is a necessary condition. In the light of the Syzygy Lemma, it is sufficient to prove that all the $\frac{1}{2}(g-2)(g-5)$ syzygies boil down to the only two in (4.5). First, we note that each basis element of $H^0(20\tau P)$ in Proposition 4.1 can be expressed as a product of two basis elements of $H^0(10\,\tau P)$. Set $x_0 := 1$, then we write:

$$\begin{aligned} &x_0x_0,\ x_0x, \ldots,\ x_0x^{2\tau},\ xx^{2\tau}, \ldots,\ x^{2\tau}x^{2\tau}, \\ &y_2y_4,\ y_2(xy_4), \ldots,\ (x^{\tau-1}y_2)(x^{\tau-1}y_4), \\ &x_0y_i,\ x_0(xy_i), \ldots,\ x_0(x^{\tau-1}y_i),\ x(x^{\tau-1}y_i), \ldots,\ x^{2\tau}(x^{\tau-1}y_i), \quad (i = 2,3,4). \end{aligned}$$

There are five kinds of quadratic relations, separated by congruence modulo five on their degrees ($\equiv 0,1,2,3,4$). In this way, each one of the $g-2 = 5\tau-1$ quadratic relations $F^{(0)}_{2g-2+n,1}$ $(n = 0, \ldots, n_{g-3})$ of the Syzygy Lemma can be written as:

$$\begin{array}{ll}
X^{i+1}X^{2\tau-1} - X^iX^{2\tau} = 0 & (i = 0, \dots, 2\tau-2)\,; \\
(X^{\tau-1}Y_2)(X^{\tau-1}Y_3) - X^{2\tau-1}X^{2\tau} = X^{2\tau-2}F_5^{(0)}\,; & \\
(X^iY_3)(X^{\tau-1}Y_4) - (X^iY_2)X^{2\tau} = X^{\tau+i-1}F_7^{(0)} & (i = 0, \dots, \tau-1)\,; \\
(X^iY_4)(X^{\tau-i}Y_4) - (X^iY_3)X^{2\tau} = X^{\tau+i-1}F_8^{(0)} & (i = 0, \dots, \tau-1)\,; \\
(X^{i+1}Y_4)X^{2\tau-1} - (X^iY_4)X^{2\tau} = 0 & (i = 0, \dots, \tau-2)\,.
\end{array}$$

As in previous computations, all $\frac{1}{2}(g-2)(g-3)$ "quadratic"relations are up to powers of X equal to $F_4^{(0)}$, $F_5^{(0)}$, $F_6^{(0)}$, $F_7^{(0)}$, $F_8^{(0)}$ or identically zero. For example, the forms of degree congruous to 4 modulo 5 are:

$$\begin{array}{ll}
X^k(X^lY_4) - X^j(X^iY_4) = 0, & (0 \le i, l \le \tau-1, 0 \le k, j \le 2\tau,\ k+l = i+j)\,; \\
(X^kY_2)(X^lY_2) - X^i(X^{\tau-1}Y_4) = X^{i+\tau-1}F_4^{(0)}, & (0 \le k, l \le \tau-1,\ 1 \le i \le 2\tau \\
& k+l = i+\tau-1)
\end{array}$$

while those of degree congruous to 1 modulo 5 are:

$$\begin{array}{ll}
(X^kY_3)(X^lY_3) - (X^iY_2)(X^jY_4) = X^{i+j}F_6 & (0 \le i, j, k, l \le \tau-1, \\
& i+j = k+l = 0, \dots, 2\tau-2).
\end{array}$$

Thus, the Syzygy Lemma says that we need to find syzygies starting with $X^{2\tau}F_4^{(0)}$ and $X^{2\tau}F_6^{(0)}$. Thus we have

$$\begin{array}{l}
X^{2\tau}F_4^{(0)} - X^{\tau-1}Y_4F_5^{(0)} + X^{\tau-1}Y_2F_7^{(0)} = 0 \\
X^{2\tau}F_6^{(0)} - X^{\tau-1}Y_4F_7^{(0)} + X^{\tau-1}Y_3F_8^{(0)} = 0
\end{array}$$

and diving by $X^{\tau-1}$ we find the two syzygies in (4.5).

Now, the two polynomial equations of the statement are induced by taking the two forms in (4.6) and dividing by the forms F_i, in the sense that all the monomials of the remainders belong to the lift of the basis of $H^0(30\tau P)$ in Proposition 4.1. □

To explore completely the two equations of Theorem 4.3 we introduce a suitable notation.

$$f_i = F_i(t^{-5}, t^{-2-5\tau}, t^{-3-5\tau}, t^{-4-5\tau})t^{i+10\,\tau} \in \mathbf{k}[t] \quad (i = 4, 5, 6, 7, 8).$$

By reordering and grouping the monomials of f_i whose degrees have the same residue modulo 5, the polynomial f_i is a sum of at most four polynomials.

Thus we can write:

$$
\begin{aligned}
f_4 &= f_4^{(1)} + f_4^{(2)} + f_4^{(4)} + f_4^{(5)} \\
f_5 &= f_5^{(1)} + f_5^{(3)} + f_5^{(5)} \\
f_6 &= f_6^{(1)} + f_6^{(2)} + f_6^{(3)} + f_6^{(4)} \\
f_7 &= f_7^{(2)} + f_7^{(5)} \\
f_8 &= f_8^{(1)} + f_8^{(3)} + f_8^{(4)} + f_8^{(5)}
\end{aligned}
$$

We call each $f_j^{(i)}$ a *partial polynomial*. The two polynomial equations of Theorem 4.3 are:

$$
\begin{aligned}
&f_4 - f_5 + f_7 = f_5^{(1)} f_8 - f_7^{(5)} f_4; \\
&f_6 - f_7 + f_8 = -f_8^{(4)} f_7 - f_8^{(5)} f_6 - f_8^{(1)} f_5.
\end{aligned}
\tag{4.7}
$$

For each equation in (4.7) there are associated five equations separated by congruence modulo five. So we get the following ten equations in the partial polynomials:

$$
\begin{aligned}
&f_4^{(1)} - f_5^{(1)} = f_5^{(1)} f_8^{(5)} - f_7^{(5)} f_4^{(1)} \\
&f_4^{(2)} + f_7^{(2)} = f_5^{(1)} f_8^{(1)} - f_7^{(5)} f_4^{(2)} \\
&f_5^{(3)} = 0 \\
&f_4^{(4)} = f_5^{(1)} f_8^{(3)} - f_7^{(5)} f_4^{(4)} \\
&f_4^{(5)} - f_5^{(5)} + f_7^{(5)} = f_5^{(1)} f_8^{(4)} - f_7^{(5)} f_4^{(5)}
\end{aligned}
$$

$$
\begin{aligned}
&f_6^{(1)} + f_8^{(1)} = -f_8^{(4)} f_7^{(2)} - f_8^{(5)} f_6^{(1)} - f_8^{(1)} f_5^{(5)} \\
&f_6^{(2)} - f_7^{(2)} = -f_8^{(5)} f_6^{(2)} - f_8^{(1)} f_5^{(1)} \\
&f_6^{(3)} + f_8^{(3)} = -f_8^{(5)} f_6^{(3)} \\
&f_6^{(4)} + f_8^{(4)} = -f_8^{(4)} f_7^{(4)} - f_8^{(5)} f_6^{(4)} - f_8^{(1)} f_5^{(3)} \\
&f_7^{(5)} - f_8^{(5)} = 0
\end{aligned}
$$

The above system provides the equations between the coefficients $c_{s,i}$ that will describe the moduli space $\overline{\mathscr{M}_{\mathcal{H}}}$. We will solve it by making eliminations, as follows:

- From the last equation we have $f_8^{(5)} = f_7^{(5)}$, then by substituting this in the first one we get $(f_4^{(1)} - f_5^{(1)})(1 - f_7^{(5)}) = 0$ and so $f_4^{(1)} - f_5^{(1)} = 0$.

- From $f_5^{(3)} = 0$, $f_8^{(5)} = f_7^{(5)}$ and the ninth equation, it follows that $f_6^{(4)} + f_8^{(4)}(1 - f_7^{(5)}) = 0$, whence $f_6^{(4)} + f_8^{(6)} = 0$.

- Taking the sum of the second and the seventh equations, we obtain that the seventh equation can be replaced by $(f_4^{(2)} + f_6^{(2)})(1 - f_7^{(2)}) = 0$, and therefore $f_4^{(2)} + f_6^{(2)} = 0$.
- The eighth says that $f_8^{(3)} = f_6^{(3)}(f_7^{(1)} - 1)$. Entering this in the fourth equation we get $f_4^{(4)}(1 - f_7^{(5)}) = -f_5^{(1)} f_6^{(3)}(f_7^{(5)} - 1)$, whence $f_4^{(4)} - f_4^{(1)} f_6^{(3)} = 0$.
- From the fifth equation it follows that $f_5^{(5)} = f_4^{(5)}(1 + f_7^{(5)}) + f_7^{(5)} - f_5^{(1)} f_8^{(4)}$ and from the second $f_7^{(2)} = f_5^{(1)} f_8^{(1)} - f_4^{(2)}(1 + f_7^{(5)})$. Substituting in the sixth equation we get $(f_7^{(5)} + 1)(f_6^{(1)} + f_8^{(1)}) = (f_7^{(5)} + 1)(f_8^{(4)} f_4^{(2)} - f_8^{(1)} f_4^{(5)})$; then $f_6^{(1)} + f_8^{(1)} = f_8^{(4)} f_4^{(2)} - f_8^{(1)} f_4^{(5)}$.

Thus the moduli variety $\overline{\mathscr{M}_{\mathcal{H}}}$ corresponds to the orbits of the equivariant $\mathbb{G}_m(\mathbf{k})$-action on the algebraic set of the vectors of constants c_{si} normalized by (4.4) and satisfying the following polynomial equations:

$$
\begin{aligned}
f_5^{(1)} &= f_4^{(1)} \\
f_7^{(2)} &= f_4^{(1)} f_8^{(1)} - f_4^{(2)}(1 + f_7^{(5)}) \\
f_5^{(3)} &= 0 \\
f_4^{(4)} &= f_4^{(1)} f_6^{(3)} \\
f_5^{(5)} &= f_4^{(5)}(1 + f_7^{(5)}) + f_7^{(5)} + f_4^{(1)} f_6^{(4)} \\
f_6^{(1)} &= -f_8^{(1)}(1 + f_4^{(5)}) - f_6^{(4)} f_4^{(2)} \\
f_6^{(2)} &= -f_4^{(2)} \\
f_8^{(3)} &= -f_6^{(3)}(1 + f_7^{(5)}) \\
f_8^{(4)} &= -f_6^{(4)} \\
f_8^{(5)} &= f_7^{(5)}
\end{aligned}
$$

In each of the above equations the formal degree of the left side is not smaller than the degree of the right side. The partial polynomials involved in the solution are $f_4^{(1)}, f_6^{(1)}, f_4^{(2)}, f_6^{(3)}, f_6^{(4)}, f_4^{(5)}$ and $f_7^{(5)}$. Summarizing, we have:

Corollary 4.4. *Let $\mathcal{H}$ be the symmetric semigroup generated by $5, 2 + 5\tau$, $3 + 5\tau, 4 + 5\tau$ with $\tau \geq 1$. The moduli space $\overline{\mathscr{M}_{\mathcal{H}}}$ is isomorphic to the weighted projective space $\mathbb{P}(T^{1,-}(B_{\mathcal{H}}))$ and its dimension is $7\tau + 4$.*

Note that Deligne's bound $2g - 1 = 10\,\tau + 1$ is reached only for $\tau = 1$, and in this case $\mathcal{H} =< 5, 7, 8, 9 >$ is negatively graded. For other all values of τ the semigroup is not negatively graded.

With a simple computation we see that the weight of the semigroup $\mathcal{H}$ is equal to $\tau(5\tau+1)$. Thus, the lower bound $3g-2-w(\mathcal{H}) = 15\,\tau+1-\tau(5\tau+1)$ is negative for $\tau \geq 3$, and for $\tau = 1, 2$ it provides the values $10, 9$, respectively.

4.2 A family with multiplicity six

We consider the one-parameter family of symmetric semigroups of multiplicity $m = 6$ given by:

$$\mathcal{H} = \langle 6, 2+6\tau, 3+6\tau, 4+6\tau, 5+6\tau \rangle \,, \quad \tau \geq 1. \tag{4.8}$$

The genus of $\mathcal{H}$ is $g = 1+6\tau$ and the canonical system of generators is:

$$\begin{aligned} &0, 6, 12, \ldots, 12\,\tau, 2+6\tau, \ldots, 2+6(2\tau-1), 3+6\tau, \ldots, 3+6(2\tau-1), \\ &4+6\tau, \ldots, 4+6(2\tau-1), 5+6\tau, \ldots, 5+6(2\tau-1). \end{aligned}$$

Suppose that C is an integral projective Gorenstein curve of arithmetic genus g and $P \in C$ is a nonsingular point such that the Weierstrass semigroup of (C, P) is $\mathcal{H}$. By the definition of Weierstrass semigroup, for each $n \in \{6, 2+6\tau, 3+6\tau, 4+6\tau, 5+6\tau\}$ there is a meromorphic function $x_n \in H^0(C, (12\,\tau)P)$, whose pole order at P is exactly n and does not have other poles. We introduce the following notation for typographical reasons:

$$x := x_6, \;\; y_i := x_{i+6\tau} \quad (i = 2, 3, 4, 5).$$

More generally, for each $n \in \mathcal{H}$ there is a meromorphic function x_n whose pole order at P is exactly n and does not have other poles. We can normalize them in a way that $x_0 = 1$, $x_{n+6} = x_6 x_n$ and $x_{7+12\,\tau} = y_2 y_5$.

The divisor $12\,\tau P$ is canonical and since $\ell_2 = 2$, i.e. $\mathcal{H}$ is nonhyperelliptic, we can identify C with its image under the canonical embedding

$$(x_0 : \ldots : x_{12\,\tau}) : C \hookrightarrow \mathbb{P}^{6\tau}.$$

Thus, we can assume that C is a canonical curve in $\mathbb{P}^{6\tau}$ and the Weierstrass point is $P = (0 : \ldots : 0 : 1)$.

Searching for a monomial basis of the vector space $H^0(C, 12n\tau P)$ with $n \geq 1$ we have the following:

Proposition 4.5. *The following functions form a P-Hermitian basis for $H^0(C, 12\,n\tau P)$ with $n \geq 1$.*

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\left.\begin{array}{l}
1, x, \ldots, x^{2\tau} \\
y_i, xy_i, \ldots, x^{\tau-1}y_i
\end{array}\right] H^0(C, 12\,\tau P) \\
x^{2\tau+1}, \ldots, x^{4\tau} \\
x^{\tau}y_i, \ldots, x^{3\tau-1}y_i \\
y_2y_5, xy_2y_5, \ldots, x^{2\tau-2}y_2y_5
\end{array}\right] H^0(C, 24\,\tau P) \\
\vdots \\
x^{4\tau+1}, \ldots, x^{2n\tau} \\
x^{3\tau}y_i, \ldots, x^{\tau(2n-1)-1}y_i \\
x^{2\tau-1}y_2y_5, \ldots, x^{\tau(2n-2)-2}y_2y_5
\end{array}\right] H^0(C, 12\,n\tau P)$$

for $i = 2, 3, 4, 5.$

Proof. By the Riemann–Roch Theorem it follows that $\dim H^0(12n\tau) = 6\tau(2n-1)$, which is the number of the above functions. These functions are linearly independent, because their pole orders at P are pairwise different. □

The curve C is nontrigonal and not isomorphic to a plane quintic (cf. [17]). So, by Petri's analysis, the ideal of C is generated by quadratic relations. In particular, we can ask for the quadratic relations between the nine functions given by the products y_iy_j $(2 \leq i \leq j \leq 5)$ except y_2y_5 because it is a basis element of $H^0(C, 24\,\tau\,P)$. Note that these nine functions belong to $H^0(C, 24\tau P)$ and are not basis elements. Thus we can write each one of them as a linear combination of the basis elements of Proposition 4.5. We denote, provisionally, the basis elements of $H^0(C, 12\,\tau P)$ by $x_{a_s}x_{b_s}$, where s is the pole order at P. Then, for each one of our nine functions y_iy_j there are constants $c_{y_iy_j,k} \in \mathbf{k}$ such that

$$y_iy_j = \sum_{0 \leq s \leq i+j+12\,\tau} c_{y_iy_j,k}x_{a_s}x_{b_s},$$

where $k + s = i + j + 12\,\tau$. After eventually multiplying the functions x_{a_s} by suitable constants, we can assume that $c_{y_iy_j,0} = 1$. In the polynomial $\mathbf{k}$-algebra $\mathbf{k}[X, Y_2, Y_3, Y_4, Y_5]$ there are nine polynomials given by the lifting of our functions $y_iy_j - \sum c_{y_iy_j,k}x_{a_s}x_{b_s}$. We also introduce a more appropriate notation for the constants $c_{y_iy_j,k}$ as follows.

$$\begin{aligned}
G_4 &:= Y_2^2 - X^\tau Y_4 - \\
&- g_{4,1}X^\tau Y_3 - \ldots - g_{4,1+6\tau}Y_3 - \\
&- g_{4,2}X^\tau Y_2 - \ldots - g_{4,2+6\tau}Y_2 - \\
&- g_{4,4}X^{2\tau} - \ldots - g_{4,4+12\,\tau} - \\
&- g_{4,5}X^{\tau-1}Y_5 - \ldots - g_{4,5+6(\tau-1)}Y_5 - \\
&- g_{4,6}X^{\tau-1}Y_4 - \ldots - g_{4,6\tau}Y_4
\end{aligned}$$

$$\begin{aligned}
G_5 &:= Y_2Y_3 - X^\tau Y_5 - \\
&- g_{5,1}X^\tau Y_4 - \ldots - g_{5,1+6\tau}Y_4 - \\
&- g_{5,2}X^\tau Y_3 - \ldots - g_{5,2+6\tau}Y_3 - \\
&- g_{5,3}X^\tau Y_2 - \ldots - g_{5,3+6\tau}Y_2 - \\
&- g_{5,5}X^{2\tau} - \ldots - g_{5,5+12\,\tau} - \\
&- g_{5,6}X^{\tau-1}Y_5 - \ldots - g_{5,6\tau}Y_5
\end{aligned}$$

$$\begin{aligned}
F_6 &:= Y_2Y_4 - X^{2\tau+1} - \\
&- f_{6,1}X^\tau Y_5 - \ldots - f_{6,1+6\tau}Y_5 - \\
&- f_{6,2}X^\tau Y_4 - \ldots - f_{6,2+6\tau}Y_4 - \\
&- f_{6,3}X^\tau Y_3 - \ldots - f_{6,3+6\tau}Y_3 - \\
&- f_{6,4}X^\tau Y_2 - \ldots - f_{6,4+6\tau}Y_2 - \\
&- f_{6,6}X^{2\tau} - \ldots - f_{6,6+12\,\tau}
\end{aligned}$$

$$\begin{aligned}
G_6 &:= Y_3^2 - X^{2\tau+1} - \\
&- g_{6,1}X^\tau Y_5 - \ldots - g_{6,1+6\tau}Y_5 - \\
&- g_{6,2}X^\tau Y_4 - \ldots - g_{6,2+6\tau}Y_4 - \\
&- g_{6,3}X^\tau Y_3 - \ldots - g_{6,3+6\tau}Y_3 - \\
&- g_{6,4}X^\tau Y_2 - \ldots - g_{6,4+6\tau}Y_2 - \\
&- g_{6,6}X^{2\tau} - \ldots - g_{6,6+12\,\tau}
\end{aligned}$$

$$\begin{aligned}
G_7 &:= Y_3Y_4 - Y_2Y_5 - \\
&- g_{7,1}X^{2\tau+1} - \ldots - g_{7,1+6(2\tau+1)} - \\
&- g_{7,2}X^\tau Y_5 - \ldots - g_{7,2+6\tau}Y_5 - \\
&- g_{7,3}X^\tau Y_4 - \ldots - g_{7,3+6\tau}Y_4 - \\
&- g_{7,4}X^\tau Y_3 - \ldots - g_{7,4+6\tau}Y_3 - \\
&- g_{7,5}X^\tau Y_2 - \ldots - g_{7,5+6\tau}Y_2
\end{aligned}$$

$$\begin{aligned}
F_8 &:= Y_3Y_5 - X^{\tau+1}Y_2 - f_{8,1}Y_2Y_5 - \\
&- f_{8,2}X^{2\tau+1} - \ldots - f_{8,2+6(2\tau+1)} - \\
&- f_{8,3}X^\tau Y_5 - \ldots - f_{8,3+6\tau}Y_5 - \\
&- f_{8,4}X^\tau Y_4 - \ldots - f_{8,4+6\tau}Y_4 - \\
&- f_{8,5}X^\tau Y_3 - \ldots - f_{8,5+6\tau}Y_3 - \\
&- f_{8,6}X^\tau Y_2 - \ldots - f_{8,6+6\tau}Y_2
\end{aligned}$$

$$\begin{aligned}
G_8 &:= Y_4^2 - X^{\tau+1}Y_2 - g_{8,1}Y_2Y_5 - \\
&- g_{8,2}X^{2\tau+1} - \ldots - g_{8,2+6(2\tau+1)} - \\
&- g_{8,3}X^\tau Y_5 - \ldots - g_{8,3+6\tau}Y_5 - \\
&- g_{8,4}X^\tau Y_4 - \ldots - g_{8,4+6\tau}Y_4 - \\
&- g_{8,5}X^\tau Y_3 - \ldots - g_{8,5+6\tau}Y_3 - \\
&- g_{8,6}X^\tau Y_2 - \ldots - g_{8,6+6\tau}Y_2
\end{aligned}$$

$$\begin{aligned}
F_9 &:= Y_4Y_5 - X^{\tau+1}Y_3 - f_{9,2}Y_2Y_5 - \\
&- f_{9,1}X^{\tau+1}Y_2 - \ldots - f_{9,1+6(\tau+1)}Y_2 - \\
&- f_{9,3}X^{2\tau+1} - \ldots - f_{9,3+6(2\tau+1)} - \\
&- f_{9,4}X^\tau Y_5 - \ldots - f_{9,4+6\tau}Y_5 - \\
&- f_{9,5}X^\tau Y_4 - \ldots - f_{9,5+6\tau}Y_4 - \\
&- f_{9,6}X^\tau Y_3 - \ldots - f_{9,6+6\tau}Y_3
\end{aligned}$$

$$\begin{aligned}
F_{10} &:= Y_5^2 - X^{\tau+1}Y_4 - f_{10,3}Y_2Y_5 - \\
&- f_{10,1}X^{\tau+1}Y_3 - \ldots - f_{10,1+6(\tau+1)}Y_3 - \\
&- f_{10,2}X^{\tau+1}Y_2 - \ldots - f_{10,2+6(\tau+1)}Y_2 - \\
&- f_{10,4}X^{2\tau+1} - \ldots - f_{10,4+6(2\tau+1)} - \\
&- f_{10,5}X^\tau Y_5 - \ldots - f_{10,5+6\tau}Y_5 - \\
&- f_{10,6}X^\tau Y_4 - \ldots - f_{10,6+6\tau}Y_4
\end{aligned}$$

Now we invert the above considerations. We consider the polynomial ring $\mathbf{k}[X, Y_2, Y_3, Y_4, Y_5]$ where the weight of X is 6 and that of Y_i is $i + 6\tau$ ($i = 2, 3, 4, 5$). We also take nine polynomials F_s and G_s for $s = 4, 5, 6, 7, 8, 9, 10$ as before, where the coefficients $g_{s,i}$, $f_{s,i}$ belong to $\mathbf{k}$.

We ask for the conditions on the coefficients $g_{s,i}$ and $f_{s,i}$ in a way that the polynomials F_s and G_s are induced by a Gorenstein curve of arithmetic genus g such that the Weierstrass semigroup of (C, P) is $\mathcal{H}$, where $P = (0 : \ldots : 0 : 1)$.

Also we introduce the following nine polynomials:

$$\begin{array}{lll} G_4^{(0)} = Y_2^2 - X^\tau Y_4\,, & G_5^{(0)} = Y_2Y_3 - X^\tau Y_5\,, & F_6^{(0)} = Y_2Y_4 - X^{2\tau+1}\,, \\ G_6^{(0)} = Y_3^2 - X^{2\tau+1}\,, & G_7^{(0)} = Y_3Y_4 - Y_2Y_5\,, & F_8^{(0)} = Y_3Y_5 - X^{\tau+1}Y_2\,, \\ G_8^{(0)} = Y_4^2 - X^{\tau+1}Y_2\,, & F_9^{(0)} = Y_4Y_5 - X^{\tau+1}Y_3\,, & F_{10}^{(0)} = Y_5^2 - X^{\tau+1}Y_4\,. \end{array} \tag{4.9}$$

Note that the zero locus, say D_0, of our nine isobaric polynomials $F_s^{(0)}$ is a monomial curve in $\mathbb{A}^5$ whose ring of regular functions is the semigroup algebra $B_{\mathcal{H}} = \oplus_{h \in \mathcal{H}} \mathbf{k} t^h$. The curve D_0 can be obtained by projecting in $\mathbb{A}^5$ the affine curve given by the canonical monomial curve C_0 on the local chart "$u = 1$". Therefore, the projectivization of D_0 is isomorphic to C_0. Because the rings of regular functions of D_0 and of the affine curve given by C_0 on the local chart "$u = 1$"are $B_{\mathcal{H}}$. To complete these affine curves we just include, on each curve, a nonsingular point at infinity.

Since a P-Hermitian basis was fixed in Proposition 4.5, it follows that we have the freedom to transform:

$$\begin{aligned} x &\mapsto \alpha_0 x + \alpha_6 \\ y_2 &\mapsto \beta_0 y_2 + \beta_2 x^\tau + \ldots + \beta_{2+6\tau} \\ y_3 &\mapsto \gamma_0 y_3 + \gamma_1 y_2 + \gamma_3 x^\tau + \ldots + \gamma_{3+6\tau} \\ y_4 &\mapsto \delta_0 y_4 + \delta_1 y_3 + \delta_2 y_2 + \delta_4 x^\tau + \ldots + \delta_{4+6\tau} \\ y_5 &\mapsto \theta_0 y_5 + \theta_1 y_4 + \theta_2 y_3 + \theta_3 y_2 + \theta_5 x^\tau + \ldots + \theta_{5+6\tau}. \end{aligned}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i, \theta_i \in \mathbf{k}$ and $\alpha_0, \beta_0, \gamma_0, \delta_0, \theta_0 \in \mathbf{k}^\star$. By exploring these freedom we obtain:

Lemma 4.6. *By linear changes of variables we may assume that:*

$$\begin{cases} f_{8,1} = g_{8,1} = f_{9,2} = f_{10,3} = 0\,; \\ f_{6,2+6i} = f_{8,3+6i} = f_{9,4+6i} = f_{9,5+6i} = 0\,,\ (i = 0, 1, \ldots, \tau); \\ f_{10,1} = f_{10,2} = f_{9,6} = 0\,. \end{cases} \tag{4.10}$$

In order to simplify the presentation we introduce the following polynomials in $\mathbf{k}[t]$:

$$\begin{aligned} g_i &= G_i(t^{-6}, t^{-2-6\tau}, t^{-3-6\tau}, t^{-4-6\tau}, t^{-5-6\tau})t^{\mathrm{weight}(G_i^{(0)})} \\ f_i &= F_i(t^{-6}, t^{-2-6\tau}, t^{-3-6\tau}, t^{-4-6\tau}, t^{-5-6\tau})t^{\mathrm{weight}(F_i^{(0)})} \end{aligned} \tag{4.11}$$

for all polynomials G_i and F_i. By reordering and grouping the monomials of f_i and g_i whose degrees have the same residue modulo 6, the polynomials f_i and g_i are a sum of at most six polynomials. By using the normalizations (4.10) we write:

$$\begin{aligned} g_4 &= g_4^{(1)} + g_4^{(2)} + g_4^{(4)} + g_4^{(5)} + g_4^{(6)} \\ g_5 &= g_5^{(1)} + g_5^{(2)} + g_5^{(3)} + g_5^{(5)} + g_5^{(6)} \\ f_6 &= f_6^{(1)} + f_6^{(3)} + f_6^{(4)} + f_6^{(6)} \\ g_6 &= g_6^{(1)} + g_6^{(2)} + g_6^{(3)} + g_6^{(4)} + g_6^{(6)} \\ g_7 &= g_7^{(1)} + g_7^{(2)} + g_7^{(3)} + g_7^{(4)} + g_7^{(5)} \\ f_8 &= f_8^{(2)} + f_8^{(4)} + f_8^{(5)} + f_8^{(6)} \\ g_8 &= g_8^{(2)} + g_8^{(3)} + g_8^{(4)} + g_8^{(5)} + g_8^{(6)} \\ f_9 &= f_9^{(1)} + f_9^{(3)} + f_9^{(6)} \\ f_{10} &= f_{10}^{(1)} + f_{10}^{(2)} + f_{10}^{(4)} + f_{10}^{(5)} + f_{10}^{(6)} \end{aligned}$$

The polynomials $g_i^{(j)}$, $f_i^{(j)}$ are called *partial polynomials* of g_i, f_i.

We pick up five important syzygies between the generators of the ideal of the monomial curve D_0, namely:

$$\begin{aligned} &X^{\tau+1}G_4^{(0)} - Y_4F_6^{(0)} + Y_2G_8^{(0)} = 0 \\ &X^{\tau+1}G_5^{(0)} - Y_5F_6^{(0)} + Y_2F_9^{(0)} = 0 \\ &X^{\tau+1}G_6^{(0)} - X^{\tau+1}F_6^{(0)} - Y_4F_8^{(0)} + Y_3F_9^{(0)} = 0 \\ &X^{\tau+1}G_7^{(0)} - Y_5F_8^{(0)} + Y_3F_{10}^{(0)} = 0 \\ &X^{\tau+1}G_8^{(0)} - X^{\tau+1}F_8^{(0)} - Y_5F_9^{(0)} + Y_4F_{10}^{(0)} = 0 \end{aligned} \tag{4.12}$$

They are induced by the Syzygy Lemma; see the proof of Theorem 4.7 below. Futhermore we consider the associated syzygies with respect the polynomials

F_i and G_i:

$$X^{\tau+1}G_4 - Y_4F_6 + Y_2G_8 - \sum_{i=0}^{\tau} X^{\tau-i}(f_{6,1+6i}F_9 + f_{6,3+6i}G_7 + f_{6,4+6i}F_6)+ \\ + \sum_{i=0}^{\tau} X^{\tau-i}(g_{8,4+6i}F_6 + g_{8,5+6i}G_5 + g_{8,6+6i}G_4) = 0$$

$$X^{\tau+1}G_5 - Y_5F_6 + Y_2F_9 - \sum_{i=0}^{\tau} X^{\tau-i}(f_{6,1+6i}F_{10} + f_{6,3+6i}F_8)+ \\ + \sum_{i=1}^{\tau+1} f_{9,1+6i}X^{\tau+1-i}G_4 + \sum_{i=1}^{\tau} f_{9,6+6i}X^{\tau-i}G_5 = 0$$

$$X^{\tau+1}G_6 - X^{\tau+1}F_6 - Y_4F_8 + Y_3F_9 + \sum_{i=1}^{\tau+1} f_{9,1+6i}X^{\tau+1-i}G_5 + \sum_{i=1}^{\tau} f_{9,6+6i}X^{\tau-i}G_6 \\ - \sum_{i=0}^{\tau} X^{\tau-i}(f_{8,4+6i}G_8 + f_{8,5+6i}G_7 + f_{8,6+6i}F_6) = 0$$

$$X^{\tau+1}G_7 - Y_5F_8 + Y_3F_{10} + \sum_{i=1}^{\tau+1} X^{\tau+1-i}(f_{10,1+6i}G_6 + f_{10,2+6i}G_5)+ \\ + \sum_{i=0}^{\tau} X^{\tau-i}(-f_{8,4+6i}F_9 + (f_{10,5+6i} - f_{8,5+6i})F_8 + f_{10,6+6i}G_7) = 0$$

$$X^{\tau+1}G_8 - X^{\tau+1}F_8 - Y_5F_9 + Y_4F_{10} + \sum_{i=1}^{\tau+1} X^{\tau+1-i}(f_{9,2+6i}F_6 + f_{10,1+6i}G_7)+ \\ + \sum_{i=0}^{\tau} X^{\tau-i}(-f_{9,6+6i}F_8 + f_{10,5+6i}F_9 + f_{10,6+6i}G_8) = 0$$

Note that the monomials of each polynomial on the left hand side of the above equations belong to the lift of the basis of Proposition 4.5. Therefore, we have a linear combination of basis elements and then each equation must be identically zero.

Theorem 4.7. *Let $\mathcal{H}$ be the semigroup generated by $6, 2+6\tau, 3+6\tau, 4+6\tau, 5+6\tau$, with $\tau \geq 1$. The moduli space $\overline{\mathscr{M}_{\mathcal{H}}}$ corresponds bijectively to the orbits of the equivariant $\mathbb{G}_m(\mathbf{k})$-action $(z, g_{s,i}) \mapsto z^i g_{s,i}$, $(z, f_{s,i}) \mapsto z^i f_{s,i}$ on the algebraic set of the vectors of constant $g_{s,i}, f_{s,i}$ normalized by (4.10) and satisfying the following five polynomial equations:*

$$\begin{aligned}
&g_4 - f_6 + g_8 = f_6^{(1)} f_9 + f_6^{(3)} g_7 + f_6^{(4)} f_6 - g_8^{(4)} f_6 - g_8^{(5)} g_5 - g_8^{(6)} g_4 \\
&g_5 - f_6 + f_9 = f_6^{(1)} f_{10} + f_6^{(3)} f_8 - f_9^{(1)} g_4 - f_9^{(6)} g_5 \\
&g_6 - f_6 - f_8 + f_9 = f_8^{(4)} g_8 + f_8^{(5)} g_7 + f_8^{(6)} f_6 - f_9^{(1)} g_5 - f_9^{(6)} g_6 \\
&g_7 - f_8 + f_{10} = f_8^{(4)} f_9 + f_8^{(5)} f_8 - f_{10}^{(1)} g_6 - f_{10}^{(2)} g_5 - f_{10}^{(5)} f_8 - f_{10}^{(6)} g_7 \\
&g_8 - f_8 - f_9 + f_{10} = f_9^{(6)} f_8 - f_{10}^{(1)} g_7 - f_{10}^{(2)} f_6 - f_{10}^{(5)} f_9 - f_{10}^{(6)} g_8
\end{aligned}$$

Proof. We need to prove that all relations between the coefficients $g_{s,i}$ and $f_{s,i}$ are induced only by the five syzygies in (4.12). It is clear it is a necessary condition. We use the Syzygy Lemma to prove that all $\frac{1}{2}(g-2)(g-5)$ syzygies boil down to only five. First of all, we write each basis element of $H^0(24\tau P)$ in Proposition 4.5 as a product of two basis elements of $H^0(12\,\tau P)$. Set $x_0 := 1$, then:

$$\begin{aligned}
&x_0x_0, x_0x, \ldots, x_0x^{2\tau}, xx^{2\tau}, \ldots, x^{2\tau}x^{2\tau}, \\
&y_2y_5, y_2(xy_5), \ldots, (x^{\tau-1}y_2)(x^{\tau-1}y_5), \\
&x_0y_i, x_0(xy_i), \ldots, x_0(x^{\tau-1}y_i), x(x^{\tau-1}y_i), \ldots, x^{2\tau}(x^{\tau-1}y_i), \quad (i = 2,3,4,5)
\end{aligned}$$

There are six kinds of quadratic relations, separated by congruence modulo six on their degrees ($\equiv 0,1,2,3,4,5$). In this way, each one of the $g-2 = 6\tau - 1$ quadratic relations $F_{2g-2+n,1}^{(0)}$ ($n = 0, \ldots, n_{g-3}$), of the Syzygy Lemma can be written as:

$$\begin{aligned}
&X^{i+1}X^{2\tau-1} - X^iX^{2\tau} = 0 && (i = 0, \ldots, 2\tau - 2) \\
&(X^{\tau-1}Y_2)(X^{\tau-1}Y_4) - X^{2\tau-1}X^{2\tau} = X^{2\tau-2}F_6^{(0)} && \\
&(X^iY_3)(X^{\tau-1}Y_5) - (X^iY_2)X^{2\tau} = X^{\tau-1+i}F_8^{(0)} && (i = 0, \ldots, \tau - 1) \\
&(X^iY_4)(X^{\tau-1}Y_5) - (X^iY_3)X^{2\tau} = X^{\tau-1+i}F_9^{(0)} && (i = 0, \ldots, \tau - 1) \\
&(X^iY_5)(X^{\tau-1}Y_5) - (X^iY_4)X^{2\tau} = X^{\tau-1+i}F_{10}^{(0)} && (i = 0, \ldots, \tau - 1) \\
&(X^{i+1}Y_5)X^{2\tau-1} - (X^iY_5)X^{2\tau} = 0 && (i = 0, \ldots, \tau - 2)
\end{aligned}$$

Like in previous computations, all $\frac{1}{2}(g-2)(g-3)$ "quadratic"relations are up to powers of X equal to $F_i^{(0)}$'s, $G_i^{(0)}$'s or identically zero. Writing only the $G_i^{(0)}$'s we have:

$$(X^kY_2)(X^lY_2) - X^i(X^{\tau-1}Y_4) = X^{i+\tau-1}G_4^{(0)}, \quad \begin{aligned}&(0 \le k,l \le \tau-1,\ 1 \le i \le 2\tau, \\ &k + l = i + \tau - 1)\end{aligned}$$

$$(X^kY_2)(X^lY_3) - X^i(X^{\tau-1}Y_5) = X^{i+\tau-1}G_5^{(0)}, \quad (0 \le k,l \le \tau-1,\ 1 \le i \le 2\tau,\ k+l = i+\tau-1)$$

$$(X^kY_3)(X^lY_3) - X^jX^{2\tau} = X^{j-1}G_6^{(0)}, \quad (1 \le j \le \tau-1, 0 \le k,l \le \tau-1,\ k+l = j-1)$$

$$(X^kY_3)(X^lY_4) - (X^iY_2)(X^j)Y_5 = X^{i+j}G_7^{(0)}, \quad (0 \le i,j,k,l \le \tau-1,\ 0 \le i+j \le 2\tau-2,\ k+l = i+j)$$

$$(X^kY_4)(X^lY_4) - X^i(X^{\tau-1}Y_2) = X^{i+\tau-1}G_8^{(0)}, \quad (0 \le k,l \le \tau-1,\ 1 \le i \le 2\tau,\ k+l = i+\tau-1)$$

Thus, the Syzygy Lemma says that we need to find syzygies starting with $X^{2\tau}G_i^{(0)}$ $(i = 4,5,6,7,8)$. Then, after dividing by $X^{\tau-1}$, we get the five syzygies of (4.12). The five polynomial equations of the statement are induced by taking the five syzygies between the forms F_i and G_i of the previous page and writing them in terms of partial polynomials. $\square$

We need to explore the five polynomial equations of Theorem 4.7 in order to find the equations that will define $\mathcal{Q}_{\mathcal{H}}$. To this end we will take the thirty equations induced by these five, organizing their degrees by residue modulo six. Among these thirty there are five very simple equations:

$$f_{10}^{(1)} = f_9^{(1)}, \quad g_8^{(3)} = f_6^{(3)}, \quad f_6^{(4)} = 0, \quad f_8^{(5)} = 0, \quad f_{10}^{(6)} = f_8^{(6)}.$$

Entering with these five equations into the remaining ones, we obtain the following twenty five equations:

$$\begin{aligned}
&g_4^{(1)} - f_6^{(1)} = f_6^{(1)}f_9^{(6)} + f_6^{(3)}g_7^{(4)} - g_8^{(4)}f_6^{(3)} - g_8^{(5)}g_5^{(2)} - g_8^{(6)}g_4^{(1)}\\
&g_5^{(1)} - f_6^{(1)} + f_9^{(1)} = f_6^{(1)}f_8^{(6)} + f_6^{(3)}f_8^{(4)} - f_9^{(1)}g_4^{(6)} - f_9^{(6)}g_5^{(1)}\\
&g_6^{(1)} - f_6^{(1)} + f_9^{(1)} = f_6^{(3)}f_8^{(4)} + f_6^{(1)}f_8^{(6)} - f_9^{(1)}g_5^{(6)} - f_9^{(6)}g_6^{(1)}\\
&g_7^{(1)} + f_9^{(1)} = f_8^{(4)}f_9^{(3)} - f_9^{(1)}g_6^{(6)} - f_{10}^{(2)}g_5^{(5)} - f_{10}^{(5)}f_8^{(2)} - f_8^{(6)}g_7^{(1)}
\end{aligned}$$

$$\begin{aligned}
&g_4^{(2)} + g_8^{(2)} = f_6^{(1)}f_9^{(1)} + f_6^{(3)}g_7^{(5)} - g_8^{(5)}g_5^{(3)} - g_8^{(6)}g_4^{(2)}\\
&g_5^{(2)} = f_6^{(1)}f_9^{(1)} - f_9^{(1)}g_4^{(1)} - f_9^{(6)}g_5^{(2)}\\
&g_6^{(2)} - f_8^{(2)} = f_8^{(4)}g_8^{(4)} - f_9^{(1)}g_5^{(1)} - f_9^{(6)}g_6^{(2)}\\
&f_8^{(2)} - g_7^{(2)} - f_{10}^{(2)} = f_9^{(1)}g_6^{(1)} + f_{10}^{(2)}g_5^{(6)} + f_8^{(6)}g_7^{(2)}\\
&g_8^{(2)} - f_8^{(2)} + f_{10}^{(2)} = f_9^{(6)}f_8^{(2)} - f_9^{(1)}g_7^{(1)} - f_{10}^{(2)}f_6^{(6)} - f_{10}^{(5)}f_9^{(3)} - f_8^{(6)}g_8^{(2)}
\end{aligned}$$

$$
\begin{aligned}
&g_5^{(3)} - f_6^{(3)} + f_9^{(3)} = f_6^{(1)} f_{10}^{(2)} + f_6^{(3)} f_8^{(6)} - f_9^{(1)} g_4^{(2)} - f_9^{(6)} g_5^{(3)} \\
&g_6^{(3)} - f_6^{(3)} + f_9^{(3)} = f_8^{(4)} g_8^{(5)} + f_6^{(3)} f_8^{(6)} - f_9^{(1)} g_5^{(2)} - f_9^{(6)} g_6^{(3)} \\
&g_7^{(3)} = -f_9^{(1)} g_6^{(2)} - f_{10}^{(2)} g_5^{(1)} - f_{10}^{(5)} f_8^{(4)} - f_8^{(6)} g_7^{(3)} \\
&f_9^{(3)} - f_6^{(3)} = f_9^{(1)} g_7^{(2)} + f_6^{(1)} f_{10}^{(2)} + f_6^{(3)} f_8^{(6)} \\
&g_4^{(4)} - f_6^{(4)} + g_8^{(4)} = f_6^{(1)} f_9^{(3)} + f_6^{(3)} g_7^{(1)} - g_8^{(4)} f_6^{(6)} - g_8^{(5)} g_5^{(5)} - g_8^{(6)} g_4^{(4)} \\
&g_6^{(4)} - f_8^{(4)} = f_8^{(4)} g_8^{(6)} - f_9^{(1)} g_5^{(3)} - f_9^{(6)} g_6^{(4)} \\
&g_7^{(4)} - f_8^{(4)} + f_{10}^{(4)} = f_8^{(4)} f_9^{(6)} - f_9^{(1)} g_6^{(3)} - f_{10}^{(2)} g_5^{(2)} - f_8^{(6)} g_7^{(4)} \\
&g_8^{(4)} - f_8^{(4)} + f_{10}^{(4)} = f_8^{(4)} f_9^{(6)} - f_9^{(1)} g_7^{(3)} - f_8^{(6)} g_8^{(4)} \\
&g_4^{(5)} + g_8^{(5)} = f_6^{(3)} g_7^{(2)} - g_8^{(4)} f_6^{(1)} - g_8^{(5)} g_5^{(6)} - g_8^{(6)} g_4^{(5)} \\
&g_5^{(5)} = f_6^{(1)} f_{10}^{(4)} + f_6^{(3)} f_8^{(2)} - f_9^{(1)} g_4^{(4)} - f_9^{(6)} g_5^{(5)} \\
&g_7^{(5)} + f_{10}^{(5)} = f_8^{(4)} f_9^{(1)} - f_9^{(1)} g_6^{(4)} - f_{10}^{(2)} g_5^{(3)} - f_{10}^{(5)} f_8^{(6)} - f_8^{(6)} g_7^{(5)} \\
&g_8^{(5)} + f_{10}^{(5)} = -f_9^{(1)} g_7^{(4)} - f_{10}^{(2)} f_6^{(3)} - f_{10}^{(5)} f_9^{(6)} - f_8^{(6)} g_8^{(5)} \\
&g_4^{(6)} - f_6^{(6)} + g_8^{(6)} = f_6^{(3)} g_7^{(3)} - g_8^{(5)} g_5^{(1)} - g_8^{(6)} g_4^{(6)} \\
&g_5^{(6)} - f_6^{(6)} + f_9^{(6)} = f_6^{(1)} f_{10}^{(5)} - g_4^{(5)} f_9^{(1)} - f_9^{(6)} g_5^{(6)} \\
&g_6^{(6)} - f_6^{(6)} - f_8^{(6)} + f_9^{(6)} = f_8^{(4)} g_8^{(2)} + f_8^{(6)} f_6^{(6)} - f_9^{(1)} g_5^{(5)} - f_9^{(6)} g_6^{(6)} \\
&g_8^{(6)} - f_8^{(6)} - f_9^{(6)} + f_8^{(6)} = f_8^{(6)} f_9^{(6)} - f_9^{(1)} g_7^{(5)} - f_{10}^{(5)} f_9^{(1)} - f_8^{(6)} g_8^{(6)}
\end{aligned}
$$

We note that we can eliminate $g_8^{(2)}, f_8^{(2)}, f_9^{(3)}, f_{10}^{(4)}$ and $f_6^{(6)}$ from the equations 10, 12, 18, 22 and 28 respectively, being left with 20 equations instead of 25. We do not do this now, because we are concerned with the equations of $\mathcal{Q}_{\mathcal{H}}$.

To compute the linearization, that corresponds bijectively to the vector space $T^{1,-}(B_{\mathcal{H}})$, we replace in these 25 equations the quadratic terms on the right sides by zeros.

$$
\begin{array}{llll}
f_6^{(1)} = g_4^{(1)} & g_6^{(1)} = g_5^{(1)} & g_7^{(1)} = g_5^{(1)} - g_4^{(1)} & f_9^{(1)} = g_4^{(1)} - g_5^{(1)} \\
g_5^{(2)} = 0 & g_7^{(2)} = -g_4^{(2)} & f_8^{(2)} = g_6^{(2)} & g_8^{(2)} = -g_4^{(2)} \\
f_{10}^{(2)} = g_4^{(2)} + g_6^{(2)} & & & \\
g_5^{(3)} = 0 & g_6^{(3)} = 0 & g_7^{(3)} = 0 & f_9^{(3)} = f_6^{(3)} \\
g_4^{(4)} = -g_7^{(4)} & f_8^{(4)} = g_6^{(4)} & g_8^{(4)} = g_7^{(4)} & f_{10}^{(4)} = -g_7^{(4)} + g_6^{(4)} \\
g_5^{(5)} = 0 & g_7^{(5)} = -g_4^{(5)} & g_8^{(5)} = -g_4^{(5)} & f_{10}^{(5)} = g_4^{(5)} \\
g_5^{(6)} = g_4^{(6)} & f_6^{(6)} = g_4^{(6)} + g_8^{(6)} & g_6^{(6)} = g_4^{(6)} + f_8^{(6)} & f_9^{(6)} = g_8^{(6)}
\end{array}
$$

Entering with the linearization into the right sides of the twenty five equations of the previous page, we get the equations that will define $\mathcal{Q}_{\mathcal{H}}$, namely:

$$
\begin{aligned}
&g_4^{(1)} - f_6^{(1)} = 0\\
&g_5^{(1)} - f_6^{(1)} + f_9^{(1)} = g_4^{(1)} f_8^{(6)} + f_6^{(3)} g_6^{(4)} + g_4^{(6)} g_5^{(1)} - g_4^{(6)} g_4^{(1)} - g_8^{(6)} g_5^{(1)}\\
&g_6^{(1)} - f_6^{(1)} + f_9^{(1)} = g_4^{(1)} f_8^{(6)} + f_6^{(3)} g_6^{(4)} + g_4^{(6)} g_5^{(1)} - g_4^{(6)} g_4^{(1)} - g_8^{(6)} g_5^{(1)}\\
&g_7^{(1)} + f_9^{(1)} = f_6^{(3)} g_6^{(4)} + g_4^{(6)} g_5^{(1)} - g_4^{(6)} g_4^{(1)} - g_4^{(5)} g_6^{(2)}
\end{aligned}
$$

$$
\begin{aligned}
&g_4^{(2)} + g_8^{(2)} = {g_4^{(1)}}^2 - g_4^{(1)} g_5^{(1)} - f_6^{(3)} g_4^{(5)} - g_8^{(6)} g_4^{(2)}\\
&g_5^{(2)} = 0\\
&g_6^{(2)} - f_8^{(2)} = g_6^{(4)} g_7^{(4)} + {g_5^{(1)}}^2 - g_4^{(1)} g_5^{(1)} - g_8^{(6)} g_6^{(2)}\\
&g_7^{(2)} - f_8^{(2)} + f_{10}^{(2)} = {g_5^{(1)}}^2 - g_4^{(1)} g_5^{(1)} - g_4^{(6)} g_4^{(2)} - g_4^{(6)} g_6^{(2)} + f_8^{(6)} g_4^{(2)}\\
&g_8^{(2)} - f_8^{(2)} + f_{10}^{(2)} = {g_5^{(1)}}^2 - 2 g_4^{(1)} g_5^{(1)} + {g_4^{(1)}}^2 - g_4^{(6)} g_4^{(2)} - g_8^{(6)} g_4^{(2)} - g_4^{(6)} g_6^{(2)} -\\
&\qquad\qquad - f_6^{(3)} g_4^{(5)} + f_8^{(6)} g_4^{(2)}
\end{aligned}
$$

$$
\begin{aligned}
&g_5^{(3)} - f_6^{(3)} + f_9^{(3)} = g_4^{(1)} g_6^{(2)} + f_6^{(3)} f_8^{(6)} + g_4^{(2)} g_5^{(1)}\\
&g_6^{(3)} - f_6^{(3)} + f_9^{(3)} = f_6^{(3)} f_8^{(6)} - g_6^{(4)} g_4^{(5)}\\
&g_7^{(3)} = -g_4^{(1)} g_6^{(2)} - g_4^{(2)} g_5^{(1)} - g_6^{(4)} g_4^{(5)}\\
&f_9^{(3)} - f_6^{(3)} = g_4^{(2)} g_5^{(1)} + g_4^{(1)} g_6^{(2)} + f_6^{(3)} f_8^{(6)}
\end{aligned}
$$

$$
\begin{aligned}
&g_4^{(4)} + g_8^{(4)} = f_6^{(3)} g_5^{(1)} - g_7^{(4)} g_4^{(6)}\\
&g_6^{(4)} - f_8^{(4)} = 0\\
&g_7^{(4)} - f_8^{(4)} + f_{10}^{(4)} = g_6^{(4)} g_8^{(6)} - f_8^{(6)} g_7^{(4)}\\
&g_8^{(4)} - f_8^{(4)} + f_{10}^{(4)} = g_6^{(4)} g_8^{(6)} - f_8^{(6)} g_7^{(4)}
\end{aligned}
$$

$$
\begin{aligned}
&g_4^{(5)} + g_8^{(5)} = g_4^{(5)} g_4^{(6)} - f_6^{(3)} g_4^{(2)} - g_7^{(4)} g_4^{(1)} - g_8^{(6)} g_4^{(5)}\\
&g_5^{(5)} = g_4^{(1)} g_6^{(4)} + f_6^{(3)} g_6^{(2)} - g_7^{(4)} g_5^{(1)}\\
&g_7^{(5)} + f_{10}^{(5)} = 0\\
&g_8^{(5)} + f_{10}^{(5)} = g_7^{(4)} g_5^{(1)} - g_7^{(4)} g_4^{(1)} - f_6^{(3)} g_4^{(2)} - f_6^{(3)} g_6^{(2)} - g_8^{(6)} g_4^{(5)} + g_4^{(5)} f_8^{(6)}
\end{aligned}
$$

$$
\begin{aligned}
&g_4^{(6)} - f_6^{(6)} + g_8^{(6)} = g_4^{(5)} g_5^{(1)} - g_8^{(6)} g_4^{(6)}\\
&g_5^{(6)} - f_6^{(6)} + f_9^{(6)} = g_4^{(5)} g_5^{(1)} - g_8^{(6)} g_4^{(6)}\\
&g_6^{(6)} - f_6^{(6)} - f_8^{(6)} + f_9^{(6)} = f_8^{(6)} g_4^{(6)} - g_6^{(4)} g_4^{(2)} - g_8^{(6)} g_4^{(6)}\\
&g_8^{(6)} - f_9^{(6)} = 0
\end{aligned}
$$

Here we see a significant simplification of computations in order that we can solve these systems. The solution is:

$$
\begin{aligned}
&f_6^{(1)} = g_4^{(1)}\\
&g_6^{(1)} = g_5^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
f_9^{(1)} &= g_4^{(1)} - g_5^{(1)} - g_4^{(6)}g_4^{(1)} + g_4^{(1)}f_8^{(6)} + f_6^{(3)}g_6^{(4)} + g_4^{(6)}g_5^{(1)} - g_8^{(6)}g_5^{(1)} \\
g_7^{(1)} &= g_5^{(1)} - g_4^{(1)} + g_8^{(6)}g_5^{(1)} - g_4^{(1)}f_8^{(6)} - g_4^{(5)}g_6^{(2)}
\end{aligned}
$$

$$
\begin{aligned}
g_5^{(2)} &= 0 \\
g_7^{(2)} &= -g_4^{(2)} \\
g_8^{(2)} &= -g_4^{(2)} + {g_4^{(1)}}^2 - g_8^{(6)}g_4^{(2)} - g_4^{(1)}g_5^{(1)} - f_6^{(3)}g_4^{(5)} \\
f_8^{(2)} &= g_6^{(2)} + g_8^{(6)}g_6^{(2)} - g_6^{(4)}g_7^{(4)} - {g_5^{(1)}}^2 + g_4^{(1)}g_5^{(1)} \\
f_{10}^{(2)} &= g_4^{(2)} + g_6^{(2)} + g_8^{(6)}g_6^{(2)} - g_6^{(4)}g_7^{(4)} - g_4^{(6)}g_4^{(2)} - g_4^{(6)}g_6^{(2)} + f_8^{(6)}g_4^{(2)}
\end{aligned}
$$

$$
\begin{aligned}
g_5^{(3)} &= 0 \\
g_7^{(3)} &= -g_4^{(1)}g_6^{(2)} - g_4^{(2)}g_5^{(1)} - g_6^{(4)}g_4^{(5)} \\
f_9^{(3)} &= f_6^{(3)} + g_4^{(2)}g_5^{(1)} + g_4^{(1)}g_6^{(2)} + f_6^{(3)}f_8^{(6)} \\
g_6^{(3)} &= -g_4^{(1)}g_6^{(2)} - g_4^{(2)}g_5^{(1)} - g_6^{(4)}g_4^{(5)}
\end{aligned}
$$

$$
\begin{aligned}
f_8^{(4)} &= g_6^{(4)} \\
g_8^{(4)} &= g_7^{(4)} \\
g_4^{(4)} &= -g_7^{(4)} + f_6^{(3)}g_5^{(1)} - g_7^{(4)}g_4^{(6)} \\
f_{10}^{(4)} &= g_6^{(4)} - g_7^{(4)} + g_6^{(4)}g_8^{(6)} - f_8^{(6)}g_7^{(4)}
\end{aligned}
$$

$$
\begin{aligned}
g_8^{(5)} &= -g_4^{(5)} + g_4^{(5)}g_4^{(6)} - g_8^{(6)}g_4^{(5)} - f_6^{(3)}g_4^{(2)} - g_7^{(4)}g_4^{(1)} \\
g_5^{(5)} &= g_4^{(1)}g_6^{(4)} + f_6^{(3)}g_6^{(2)} - g_7^{(4)}g_5^{(1)} \\
f_{10}^{(5)} &= g_4^{(5)} - g_4^{(5)}g_4^{(6)} + g_7^{(4)}g_5^{(1)} - f_6^{(3)}g_6^{(2)} + g_4^{(5)}f_8^{(6)} \\
g_7^{(5)} &= -g_4^{(5)} + g_4^{(5)}g_4^{(6)} - g_7^{(4)}g_5^{(1)} + f_6^{(3)}g_6^{(2)} - g_4^{(5)}f_8^{(6)}
\end{aligned}
$$

$$
\begin{aligned}
f_9^{(6)} &= g_8^{(6)} \\
g_5^{(6)} &= g_4^{(6)} \\
f_6^{(6)} &= g_4^{(6)} + g_8^{(6)} - g_4^{(5)}g_5^{(1)} + g_8^{(6)}g_4^{(6)} \\
g_6^{(6)} &= g_4^{(6)} + f_8^{(6)} - g_6^{(4)}g_4^{(2)} - g_4^{(5)}g_5^{(1)} + f_8^{(6)}g_4^{(6)}
\end{aligned}
$$

We notice that in these equations the formal degree of each of the five partial polynomials $f_9^{(1)}$, $f_{10}^{(2)}$, $g_6^{(3)}$, $g_8^{(5)}$ and $f_{10}^{(5)}$ is smaller than the formal degree of the corresponding left hand side. Thus the remaining eleven partial polynomials satisfy the following five conditions:

$$\begin{aligned}
&\pi_{7+6\tau}\left(g_4^{(1)}f_8^{(6)} - g_4^{(6)}g_4^{(1)} + f_6^{(3)}g_6^{(4)} + g_4^{(6)}g_5^{(1)} - g_8^{(6)}g_5^{(1)}\right) = 0\\
&\pi_{8+6\tau}\left(g_8^{(6)}g_6^{(2)} - g_6^{(4)}g_7^{(4)} - g_4^{(6)}g_4^{(2)} - g_4^{(6)}g_6^{(2)} + f_8^{(6)}g_4^{(2)}\right) = 0\\
&\pi_{3+6\tau}\left(g_4^{(1)}g_6^{(2)} + g_4^{(2)}g_5^{(1)} + g_6^{(4)}g_4^{(5)}\right) = 0\\
&\pi_{5+6\tau}\left(g_8^{(6)}g_4^{(5)} + f_6^{(3)}g_4^{(2)} + g_7^{(4)}g_4^{(1)} - g_4^{(5)}g_4^{(6)}\right) = 0\\
&\pi_{5+6\tau}\left(g_7^{(4)}g_5^{(1)} - g_4^{(5)}g_4^{(6)} - f_6^{(3)}g_6^{(2)} + g_4^{(5)}f_8^{(6)}\right) = 0
\end{aligned}$$

where π_i denotes the projection operator on the polynomials in t that annihilates the terms of degree smaller than $i+1$. We introduce the following notations: $\mathfrak{f}_8^{(6)} = f_8^{(6)} - g_4^{(6)}$ and $\mathfrak{g}_8^{(6)} = g_8^{(6)} - g_4^{(6)}$. Then, the above conditions are rewritten as follows:

$$\begin{aligned}
&\pi_{7+6\tau}\left(g_4^{(1)}\mathfrak{f}_8^{(6)} + f_6^{(3)}g_6^{(4)} - \mathfrak{g}_8^{(6)}g_5^{(1)}\right) = 0\\
&\pi_{8+6\tau}\left(\mathfrak{g}_8^{(6)}g_6^{(2)} - g_6^{(4)}g_7^{(4)} + \mathfrak{f}_8^{(6)}g_4^{(2)}\right) = 0\\
&\pi_{3+6\tau}\left(g_4^{(1)}g_6^{(2)} + g_4^{(2)}g_5^{(1)} + g_6^{(4)}g_4^{(5)}\right) = 0\\
&\pi_{5+6\tau}\left(\mathfrak{g}_8^{(6)}g_4^{(5)} + f_6^{(3)}g_4^{(2)} + g_7^{(4)}g_4^{(1)}\right) = 0\\
&\pi_{5+6\tau}\left(g_7^{(4)}g_5^{(1)} - f_6^{(3)}g_6^{(2)} + g_4^{(5)}\mathfrak{f}_8^{(6)}\right) = 0
\end{aligned} \tag{4.13}$$

Note that the above solution depends only on the ten partial polynomials: $g_4^{(1)}$, $g_5^{(1)}$, $g_4^{(2)}$, $g_6^{(2)}$, $f_6^{(3)}$, $g_6^{(4)}$, $g_7^{(4)}$, $g_4^{(5)}$, $\mathfrak{g}_8^{(6)}$, $\mathfrak{f}_8^{(6)}$. The number of coefficients of these partial polynomials is $11\tau + 6$.

The goal is to compute the dimension of $\mathcal{Q}_{\mathcal{H}}$. To simplify the arguments we will make some suitable changes of variables. First we call the ten partials polynomials.

$$\begin{aligned}
g_4^{(1)} &= g_{4,1}t + g_{4,7}t^7 + \ldots + g_{4,1+6\tau}t^{1+6\tau}\\
g_5^{(1)} &= g_{5,1}t + g_{5,7}t^7 + \ldots + g_{5,1+6\tau}t^{1+6\tau}\\
g_4^{(2)} &= g_{4,2}t^2 + g_{4,8}t^8 + \ldots + g_{4,2+6\tau}t^{2+6\tau}\\
g_6^{(2)} &= g_{6,2}t^2 + g_{6,8}t^8 + \ldots + g_{6,2+6\tau}t^{2+6\tau}\\
f_6^{(3)} &= f_{6,3}t^3 + f_{6,9}t^9 + \ldots + f_{6,3+6\tau}t^{3+6\tau}\\
g_6^{(4)} &= g_{6,4}t^4 + g_{6,10}t^{10} + \ldots + g_{6,4+6\tau}t^{4+6\tau}\\
g_7^{(4)} &= g_{7,4}t^4 + g_{7,10}t^{10} + \ldots + g_{7,4+6\tau}t^{4+6\tau}\\
g_4^{(5)} &= g_{4,5}t^5 + \ldots + g_{4,5+6(\tau-1)}t^{5+6(\tau-1)}\\
\mathfrak{f}_8^{(6)} &= \mathfrak{f}_{8,6}t^6 + \mathfrak{f}_{8,12}t^{12} + \ldots + \mathfrak{f}_{8,6+6\tau}t^{6+6\tau}\\
\mathfrak{g}_8^{(6)} &= \mathfrak{g}_{8,6}t^6 + \mathfrak{g}_{8,12}t^{12} + \ldots + \mathfrak{g}_{8,6+6\tau}t^{6+6\tau}
\end{aligned}$$

We introduce the following changes and notations:

$$P_i^{(j)} = f_i^{(j)}(\tfrac{1}{t})t^{6+6\tau}, \quad Q_i^{(j)} = g_i^{(j)}(\tfrac{1}{t})t^{6+6\tau},$$
$$P_8^{(6)} = \mathfrak{f}_8^{(6)}(\tfrac{1}{t})t^{6+6\tau}, \quad Q_8^{(6)} = \mathfrak{g}_8^{(6)}(\tfrac{1}{t})t^{6+6\tau}$$

and we transform the index i of the coefficients $g_{s,i}$ and $f_{s,i}$ of the polynomials $P_i^{(j)}$ and $Q_i^{(j)}$ by $i \mapsto 6(1+\tau)-i$. Let us denote $\zeta = t^6$; with a simple change of notation on the coefficients $g_{s,i}$ and $f_{s,i}$, we write:

$$\begin{aligned}
Q_4^{(1)} &= t^5(a_5 + \ldots + a_{5+6\tau}\zeta^\tau), & Q_5^{(1)} &= t^5(b_5 + \ldots + b_{5+6\tau}\zeta^\tau),\\
Q_4^{(2)} &= t^4(c_4 + \ldots + c_{4+6\tau}\zeta^\tau), & Q_6^{(2)} &= t^4(d_4 + \ldots + d_{4+6\tau}\zeta^\tau),\\
P_6^{(3)} &= t^3(e_3 + \ldots + e_{3+6\tau}\zeta^\tau), & Q_6^{(4)} &= t^2(l_2 + \ldots + l_{2+6\tau}\zeta^\tau),\\
Q_7^{(4)} &= t^2(h_2 + \ldots + h_{2+6\tau}\zeta^\tau), & Q_4^{(5)} &= t^7(m_7 + \ldots + m_{1+6\tau}\zeta^{\tau-1}),\\
P_8^{(6)} &= r_0 + \ldots + r_{6\tau}\zeta^\tau, & Q_8^{(6)} &= s_0 + \ldots + s_{6\tau}\zeta^\tau.
\end{aligned}$$

Finally, we introduce the following polynomials, and note that we adjust, again, the index of the above polynomials.

$$\begin{aligned}
a &:= a_0 + \ldots + a_\tau\zeta^\tau, & b &:= b_0 + \ldots + b_\tau\zeta^\tau,\\
c &:= c_0 + \ldots + c_\tau\zeta^\tau, & d &:= d_0 + \ldots + d_\tau\zeta^\tau,\\
e &:= e_0 + \ldots + e_\tau\zeta^\tau, & l &:= l_0 + \ldots + l_\tau\zeta^\tau,\\
h &:= h_0 + \ldots + h_\tau\zeta^\tau, & m &:= m_0 + \ldots + m_{\tau-1}\zeta^{\tau-1},\\
r &:= r_0 + \ldots + r_\tau\zeta^\tau, & s &:= s_0 + \ldots + s_\tau\zeta^\tau.
\end{aligned} \tag{4.14}$$

Theorem 4.8. *Let $\mathcal{H}$ be the symmetric semigroup generated by $6, 2+6\tau$, $3+6\tau, 4+6\tau, 5+6\tau$. Then the quasi-cone $\mathcal{Q}_{\mathcal{H}}$ is given by the coefficients of the polynomials in (4.14) such that $b_\tau = a_\tau$, $d_\tau = -c_\tau$, $s_\tau = 0$ and satisfying the following five congruences:*

$$\begin{aligned}
ar + el - sb &\equiv 0 \mod \zeta^\tau\\
sd - lh + rc &\equiv 0 \mod \zeta^\tau\\
ad + cb + lm &\equiv 0 \mod \zeta^\tau\\
sm + ec + ha &\equiv 0 \mod \zeta^\tau\\
hb - ed + mr &\equiv 0 \mod \zeta^\tau
\end{aligned} \tag{4.15}$$

The codimension of $\mathcal{Q}_{\mathcal{H}}$ in $\mathbb{A}^{11\tau+6}$ is 3τ. Hence, we can conclude:

$$\dim \overline{\mathscr{M}_{\mathcal{H}}} \leq 8\tau + 5.$$

Proof. First we note that the five equations (4.15) induce the same algebraic set in $\mathbb{A}^{11\tau+6}$ as the equations (4.13). We just consider the previous changes and write the equations (4.13) in terms of the polynomials $P_i^{(j)}$ and $Q_i^{(j)}$. As regard to the codimension, we consider the Artinian $\mathbf{k}$-algebra $\mathcal{A} = \dfrac{\mathbf{k}[\zeta]}{\zeta^\tau}$. By considering the polynomials $q_1 = ar + el - sb$, $q_2 = sd - lh + rc$, $q_3 = -ad - cb - lm$, $q_4 = -sm - ec - ha$ and $q_5 = hb - ed + mr$ in $\mathcal{A}$, we take the following affine algebraic set

$$Q = V(q_1, q_2, q_3, q_4, q_5) \subset \mathcal{A}^{\oplus 10}\,.$$

In the open set $a_0 \neq 0$ of $\mathcal{A}^{\oplus 10}$, a is invertible in $\mathcal{A}$ and then

$$r = (el - sb)a^{-1},\ b = (cb + lm)a^{-1},\ h = (sm + ec)a^{-1}$$

with q_2 and q_3 identically zero. In a completely analogous way we reason when one of the constant terms of the ten polynomials is different from zero. Now, if all the constant terms are zero, we can proceed by induction, just making a shift on the index of the constants, by making equal to zero the index of the constants terms. □

We see that Deligne's upper bound is $2g - 1 = 12\,\tau + 1$. It is equal to the upper bound of the above Theorem only for $\tau = 1$, and in this case $\mathcal{H} =< 6, 8, 9, 10, 11 >$ is negatively graded. Thus, for $\tau = 1$ we have $\dim \mathscr{M}_{\mathcal{H}} = 2g - 1 = 13 = \dim \mathcal{Q}_{\mathcal{H}} - 1$, i.e., for $\tau = 1$ the method provides the exact dimension. For all other values of τ the semigroup is not negatively graded. With a simple computation we see that the weight of the semigroup $\mathcal{H}$ is equal to $\tau(6\tau + 1)$. Thus the lower bound $3g - 2 - w(\mathcal{H}) = 18\,\tau + 1 - \tau(6\tau + 1)$ is negative for $\tau \geq 3$, and for $\tau = 1, 2$ it is 12 and 11, respectively.

Below, we summarize the results obtained in this thesis:

$\mathcal{H}$	E–H	$\dim \overline{\mathscr{M}_{\mathcal{H}}}$	$\dim \mathcal{Q}_{\mathcal{H}} - 1$	Deligne
$< 6, 8, 10, 11, 13 >$	12	14	14	15
$< 7, 9, 10, 11, 13 >$	14	15	15	15
$< 6, 8, 10, 13, 15 >$	11	?	15	17
$< 6, 9, 10, 13, 15 >$	12	15	15	17
$< 5, 2 + 5\tau, 3 + 5\tau, 4 + 5\tau >$	$-5\tau^2 + 14\tau + 1$	$7\tau + 4$	$7\tau + 4$	$10\tau + 1$
$< 6, 2 + 6\tau, 3 + 6\tau, 4 + 6\tau, 5 + 6\tau >$	$-6\tau^2 + 17\tau + 1$	?	$8\tau + 5$	$12\tau + 1$

where E–H means Eisenbud–Harris expected dimensions, and Deligne means Deligne's upper bound.

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