Instituto Nacional de Matemática Pura e Aplicada

Partial Crossed Product Description of the Cuntz-Li Algebras

PhD Thesis by

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À minha mãe

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Abstract

In this text, we study three algebras: Cuntz-Li, ring and Bost-Connes algebras. The Cuntz-Li algebras $\mathfrak{A}[R]$, presented in [12], are C^*-algebras associated to an integral domain R with finite quotients. We show that $\mathfrak{A}[R]$ is a partial group algebra of the group $K \rtimes K^\times$ with suitable relations, where K is the field of fractions of R. We identify the spectrum of these relations and we show that it is homeomorphic to the profinite completion of R. By using partial crossed product theory, we reconstruct some results proved by Cuntz and Li. Among them, we prove that $\mathfrak{A}[R]$ is simple by showing that the action is topologically free and minimal. In [33], Li generalized the Cuntz-Li algebras for more general rings and called it ring C^*-algebras. Here, we propose a new extension for the Cuntz-Li algebras. Unlike ring C^*-algebras, our construction takes into account the zero-divisors of the ring in definition of the multiplication operators. In [6], Bost and Connes constructed a C^*-dynamical system having the Riemann ζ-function as partition function. We conclude this work proving that the C^*-algebra $C_{\mathbb{Q}}$ underlying the Bost-Connes system has a partial crossed product structure.

Keywords: Cuntz-Li algebras, ring C^*-algebras, Bost-Connes algebra, partial group algebra, partial crossed product.

Contents

Chapter 1

Introduction

Sixteen years ago, motivated by the work of Julia [24], Bost and Connes constructed a C^*-dynamical system having the Riemann ζ-function as partition function [6]. The C^*-algebra of the Bost-Connes system, denoted by $C_{\mathbb{Q}}$, is a Hecke C^*-algebra obtained from the inclusion of the integers into the rational numbers. In [29], Laca and Raeburn showed that $C_{\mathbb{Q}}$ can be realized as a semigroup crossed product and, in [30], they characterized the primitive ideal space of $C_{\mathbb{Q}}$.

In [2], [9] and [25], by observing that the construction of $C_{\mathbb{Q}}$ is based on the inclusion of the integers into the rational numbers, Arledge, Cohen, Laca and Raeburn generalized the construction of Bost and Connes. They replaced the field $\mathbb{Q}$ by an algebraic number field K and $\mathbb{Z}$ by the ring of integers of K. Many of the results obtained for $C_{\mathbb{Q}}$ were generalized to arbitrary algebraic number fields (at least when the ideal class group of the field is $h = 1$) [26], [27].

Recently, a new construction appeared. In [10], Cuntz defined two new C^*-algebras: $\mathcal{Q}_{\mathbb{N}}$ and $\mathcal{Q}_{\mathbb{Z}}$. Both algebras are simple and purely infinite and $\mathcal{Q}_{\mathbb{N}}$ can be seen as a C^*-subalgebra of $\mathcal{Q}_{\mathbb{Z}}$. These algebras encode the additive and multiplicative structure of the semiring $\mathbb{N}$ and of the ring $\mathbb{Z}$. Cuntz showed that the algebra $\mathcal{Q}_{\mathbb{N}}$ is, essentially, the algebra generated by $C_{\mathbb{Q}}$ and one unitary operator. In [40], Yamashita realized $\mathcal{Q}_{\mathbb{N}}$ as the C^*-algebra of a topological higher-rank graph.

The next step was given by Cuntz and Li. In [12], they generalized the construction of $\mathcal{Q}_{\mathbb{Z}}$ by replacing $\mathbb{Z}$ by a unital commutative ring R (which is an integral domain with finite quotients by nonzero principal ideals and which is not a field). This algebra was called $\mathfrak{A}[R]$.

Cuntz and Li showed that $\mathfrak{A}[R]$ is simple and purely infinite and they related a C^*-subalgebra of its with the generalized Bost-Connes systems (when R is the ring of integers in an algebraic number field having $h = 1$ and, at most, one real place). In [33], Li extended the construction of $\mathfrak{A}[R]$ to an arbitrary unital ring and called it ring C^*-algebras.

The main aim of this text is to show that the Cuntz-Li algebra $\mathfrak{A}[R]$ can be seen as a partial crossed product. We show that $\mathfrak{A}[R]$ is $*$-isomorphic to a partial group algebra. By using the relationship between partial group algebras and partial crossed products, we see that $\mathfrak{A}[R]$ is a partial crossed product. Our second purpose is to present an alternative generalization of the Cuntz-Li algebras for more general rings, different from that introduced by Li in [33]. The last goal of this text is to find a partial crossed product description of the Bost-Connes algebra $C_{\mathbb{Q}}$. To present these results, we divide this thesis in five chapters.

In Chapter 2, we define the algebras studied here. In the first section, we introduce the Cuntz-Li algebras following the original [12] and we exhibit the main results proved there by them. In the second section, we deal with the ring C^*-algebras, which are the extensions of the Cuntz-Li algebras for arbitrary unital rings proposed by Li in [33]. We finish this chapter defining the Bost-Connes algebra, following [6].

In Chapter 3, we review the theory used to tackle the mentioned algebras. In the first section, we define partial actions and partial representations. Following, we construct the partial crossed product associated to a partial action. In the last two sections, we exhibit the partial group algebra, a C^*-algebra which is universal with respect to partial representations.

The Chapter 4 is dedicated to study the Cuntz-Li algebras $\mathfrak{A}[R]$ under a new look. First, we show that $\mathfrak{A}[R]$ is $*$-isomorphic to a partial group algebra of the group $K \rtimes K^\times$ with suitable relations, where K is the field of fractions of the ring R. Following, we see that $\mathfrak{A}[R]$ is a partial crossed product by the group $K \rtimes K^\times$. We characterize the spectrum of the commutative algebra arising in the crossed product and show that this spectrum is homeomorphic to $\hat{R}$ (the profinite completion of R). Furthermore, we show that the partial action is topologically free and minimal. By using that the group $K \rtimes K^\times$ is amenable, we conclude that $\mathfrak{A}[R]$ is simple.

In Chapter 5, we present our definition for the Cuntz-Li algebras in more general cases. In the first section, we develop elementary algebraic properties about annihilators of ideals. These properties allow us to define multiplication operators for zero-divisors, which are not included in Li's construction. We deduce some properties of our definition which are closely related to the original Cuntz-Li algebras.

In the last chapter, we show that the Bost-Connes algebra $C_{\mathbb{Q}}$ is $*$-isomorphic to a partial crossed product. We use the partial crossed product obtained in Chapter 4 in case $R = \mathbb{Z}$ as a starting point for the proof. We show that $C_{\mathbb{Q}}$ is a C^*-subalgebra of that partial crossed product.

Before we start the main content of the text, we standardize certain notations and terminology. For a given set X, the identity function on X will be denoted by Id_X. In this thesis, all groups considered are discrete, unless we say otherwise. In general, we use G to denote a group and r, s and t to represent its elements. We reserve the letter e to represent the unit of the group. The next notation, unconventional, will be designed to simplify formulas and proofs. Given a logical statement P, the symbol $[P]$ will represent the value 1 if the sentence P is true and 0 if P is a false sentence. For example, $[s = t] = 1$ if $s = t$ and $[s = t] = 0$ if $s \neq t$.

Chapter 2

Cuntz-Li, Ring and Bost-Connes C^*-algebras

In this chapter, we present the C^*-algebras which will be studied in this thesis. First, we define the Cuntz-Li algebras and exhibit their main properties. Following, we introduce the ring C^*-algebras, which are a generalization of the Cuntz-Li algebras. The last section is dedicated to the Bost-Connes algebra.

2.1 Cuntz-Li Algebras

In [10], Cuntz has defined a C^*-algebra, denoted by $\mathcal{Q}_{\mathbb{Z}}$, which encodes the ring structure of $\mathbb{Z}$. Such construction has been generalized by Cuntz and Li in [12], where they replace $\mathbb{Z}$ by an integral domain (satisfying certain properties). In this section, following [12], we define such C^*-algebra and present the main results obtained by Cuntz and Li.

Throughout this section, R will be an integral domain (unital commutative ring without zero divisors) with the property that the quotient $R/(m)$ is finite, for all $m \neq 0$ in R. In addition, we exclude the case where R is a field. We denote by $R^\times$ the set $R \backslash \{0\}$ and by R^* the set of units in R. Examples of such rings are the rings of integers in an algebraic number field and polynomial rings on a finite field.

Definition 2.1.1. [12, Definition 1] The **Cuntz-Li algebra of** R, denoted by $\mathfrak{A}[R]$, is the universal[1] C^*-algebra generated by isometries $\{s_m \mid m \in R^\times\}$ and unitaries $\{u^n \mid n \in R\}$

[1]For universal C^*-algebras on sets of generators and relations, see the original [3] or even [4, Apêndice A]

subject to the relations

(CL1) $s_m s_{m'} = s_{mm'}$;

(CL2) $u^n u^{n'} = u^{n+n'}$;

(CL3) $s_m u^n = u^{mn} s_m$;

(CL4) $\displaystyle\sum_{l+(m)\in R/(m)} u^l s_m s_m^* u^{-l} = 1$;

for all $m, m' \in R^\times$ and $n, n' \in R$.

We denote by e_m the range projection of s_m, namely $e_m = s_m s_m^*$. Relations (CL1) and (CL2) tell us that the operations of R are preserved by s and u. Intuitively, (CL3) encodes the distributivity of the ring. The relation (CL4) represents the fact that R is the disjoint union of the cosets for a given m. These facts will be clear after the next definition.

Note that if $l + (m) = l' + (m)$, say $l' = l + km$, then

$$u^{l'} s_m s_m^* u^{-l'} = u^{l+km} s_m s_m^* u^{-l-km} \overset{(CL2)}{=} u^l u^{km} s_m s_m^* u^{-km} u^{-l} \overset{(CL3)}{=}$$

$$u^l s_m u^k u^{-k} s_m^* u^{-l} = u^l s_m s_m^* u^{-l},$$

which shows that the sum in (CL4) is independent of the choice of l.

As in other similar constructions, there is a *reduced* version of $\mathfrak{A}[R]$. Consider the Hilbert space $\ell^2(R)$ and let $\{\xi_r \mid r \in R\}$ be its canonical basis. For $m \in R^\times$, define the linear operator S_m on $\ell^2(R)$ such that $S_m(\xi_r) = \xi_{mr}$. Clearly, S_m is bounded and

$$S_m^*(\xi_r) = \begin{cases} \xi_{r/m}, & \text{if } r \in (m), \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we have $S_m^* S_m(\xi_r) = S_m^*(\xi_{mr}) = \xi_r$, i.e., S_m is an isometry. For $n \in R$, let U^n be the linear operator on $\ell^2(R)$ such that $U^n(\xi_r) = \xi_{n+r}$. It's easy to see that U^n is bounded, unitary and that $(U^n)^* = U^{-n}$. Denote by $\mathcal{B}(\ell^2(R))$ the C^*-algebra of the bounded linear operators on $\ell^2(R)$.

Definition 2.1.2. [12, Section 2] The **reduced Cuntz-Li algebra of** R, denoted by $\mathfrak{A}_r[R]$, is the C^*-subalgebra of $\mathcal{B}(\ell^2(R))$ generated by the operators $\{S_m \mid m \in R^\times\}$ and $\{U^n \mid n \in R\}$.

or [34, Apêndice A].

We claim that $\{S_m \mid m \in R^\times\}$ and $\{U^n \mid n \in R\}$ satisfy (CL1)-(CL4). Indeed, (CL1) and (CL2) are obvious and $S_m U^n(\xi_r) = S_m(\xi_{n+r}) = \xi_{mn+mr} = U^{mn}(\xi_{mr}) = U^{mn}S_m(\xi_r)$ shows (CL3). To see (CL4), observe that

$$U^n S_m S_m^* U^{-n}(\xi_r) = \begin{cases} \xi_r, \text{ if } r \in n + (m), \\ 0, \text{ otherwise,} \end{cases}$$

i.e., $U^n S_m S_m^* U^{-n}$ is the projection onto $\overline{\text{span}}\{\xi_r \mid r \in n+(m)\}$. Since R is the disjoint union of $n+(m)$ with n ranging over all classes modulo m, then (CL4) is satisfied. It follows from the universal property of $\mathfrak{A}[R]$ that there exists a (surjective) $*$-homomorphism $\mathfrak{A}[R] \longrightarrow \mathfrak{A}_r[R]$.

From now on, we shall exhibit the results about $\mathfrak{A}[R]$ proved by Cuntz and Li. The first lemma, which will be used in Chapter 4, will be proved here.

Denote by P the set of projections $\{u^n e_m u^{-n} \mid m \in R^\times,\ n \in R\}$ in $\mathfrak{A}[R]$. The next result shows that span(P) is a commutative $*$-algebra.

Lemma 2.1.3. *[12, Lemma 1]*

(i) For all $m, m' \in R^\times$,

$$e_m = \sum_{l+(m')\in R/(m')} u^{ml} e_{mm'} u^{-ml};$$

(ii) The projections in P commute;

(iii) The product of elements in P are in span(P).

Proof. Since

$$e_m = s_m 1 s_m^* \overset{\text{(CL4)}}{=} s_m \left(\sum_{l+(m')\in R/(m')} u^l s_{m'} s_{m'}^* u^{-l} \right) s_m^* \overset{\text{(CL1),(CL3)}}{=} \sum_{l+(m')\in R/(m')} u^{ml} e_{mm'} u^{-ml},$$

we have *(i)*. By (CL4), we see that $u^q e_p u^{-q} = u^{q'} e_p u^{-q'}$ if $q + (p) = q' + (p)$, and $u^q e_p u^{-q}$ and $u^{q'} e_p u^{-q'}$ are orthogonal if $q + (p) \neq q' + (p)$. To see *(ii)* and *(iii)*, let $u^n e_m u^{-n}$ and $u^{n'} e_{m'} u^{-n'}$ be in P. We use *(i)* to write

$$u^n e_m u^{-n} = \sum_{l+(m')\in R/(m')} u^{n+ml} e_{mm'} u^{-n-ml}$$

and

$$u^{n'} e_{m'} u^{-n'} = \sum_{l'+(m)\in R/(m)} u^{n'+m'l'} e_{mm'} u^{-n'-m'l'}.$$

From these equalities, it's easy to see that $u^n e_m u^{-n}$ and $u^{n'} e_{m'} u^{-n'}$ commute and that $u^n e_m u^{-n} u^{n'} e_{m'} u^{-n'}$ are in span(P). ☐

By the above lemma, $\overline{\text{span}}(P)$ is a commutative C^*-subalgebra of $\mathfrak{A}[R]$, which will be denoted by $\mathfrak{D}[R]$. In the next result, Cuntz and Li exhibit a standard form for the elements in $\mathfrak{A}[R]$.

Lemma 2.1.4. *[12, Lemma 2]* $\mathfrak{A}[R] = \overline{\text{span}}\{s_{m''}^* u^n s_m s_m^* u^{-n'} s_{m'} \mid m, m', m'' \in R^\times,\ n, n' \in R\}$.

This lemma allows us to know a bounded linear operator whose domain is $\mathfrak{A}[R]$ from its behavior in the elements of the form $s_{m''}^* u^n s_m s_m^* u^{-n'} s_{m'}$, as in proposition below.

Proposition 2.1.5. *[12, Proposition 1] There is a faithful conditional expectation*[2] $\Theta : \mathfrak{A}[R] \longrightarrow \mathfrak{D}[R]$ *characterized by*

$$\Theta(s_{m''}^* u^n s_m s_m^* u^{-n'} s_{m'}) = [m' = m''][n = n'] s_{m'}^* u^n s_m s_m^* u^{-n} s_{m'},$$

where $[T]$ *represents* 1 *if the sentence* T *is true and* 0 *if* T *is false.*

The next three theorems are the main results proved by Cuntz and Li about $\mathfrak{A}[R]$.

Theorem 2.1.6. *[12, Theorem 1]* $\mathfrak{A}[R]$ *is simple and purely infinite.*[3]

As a corollary, we obtain that the canonical $*$-homomorphism $\mathfrak{A}[R] \longrightarrow \mathfrak{A}_r[R]$ is, in fact, a $*$-isomorphism.

There exists a natural partial order on $R^\times$ given by the divisibility: we say that $m \leq m'$ if there exists $r \in R$ such that $m' = mr$. Whenever $m \leq m'$, we can consider the canonical projection $p_{m,m'} : R/(m') \longrightarrow R/(m)$. Since $(R^\times, \leq)$ is a directed set, we can consider the inverse limit

$$\hat{R} = \varprojlim \{R/(m),\ p_{m,m'}\},$$

which is the **profinite completion of** R. In this text, we shall use the following concrete description of $\hat{R}$:

$$\hat{R} = \left\{ (r_m + (m))_m \in \prod_{m \in R^\times} R/(m) \;\middle|\; p_{m,m'}(r_{m'} + (m')) = r_m + (m),\ \text{if } m \leq m' \right\}.$$

Give to $R/(m)$ the discrete topology, to $\prod_{m \in R^\times} R/(m)$ the product topology and to $\hat{R}$ the induced topology of $\prod_{m \in R^\times} R/(m)$. With the operations defined componentwise, $\hat{R}$ is a

[2]See [4, Definição B.2.30 and page 191] for faithful conditional expectation.

[3]See [38, page 86] or [37, Definition 2.3] for purely infinite C^*-algebras.

compact topological ring. Since R is not a field, there exists a canonical inclusion of R into $\hat{R}$ given by $r \longmapsto (r + (m))_m$ (to see injectivity, take $r \neq 0$, m non-invertible and note that $r \notin (rm)$).

Theorem 2.1.7. *[12, Observation 1] The sprectrum of $\mathfrak{D}[R]$ is homeomorphic to $\hat{R}$ and the corresponding $*$-isomorphism (via Gelfand representation[4]) $\mathfrak{D}[R] \longrightarrow C(\hat{R})$ is given by $u^n e_m u^{-n} \longmapsto 1_{(n,m)}$, where $1_{(n,m)}$ represents the characteristic function of the subset $\{(r_{m'} + (m'))_{m'} \in \hat{R} \mid r_m + (m) = n + (m)\}$ of $\hat{R}$.*

Consider the semidirect product $R \rtimes R^\times$, which is a semigroup under the operation $(n, m)(n', m') = (n + mn', mm')$. Cuntz and Li have shown that there exist a action α by endomorphisms of the semigroup $R \rtimes R^\times$ on $C(\hat{R})$ given by $\alpha_{(n,m)}(1_{(n',m')}) = 1_{(n+mn',mm')}$. By using the theory of crossed products by semigroups developed by Adji, Laca, Nilsen and Raeburn in [1], Cuntz and Li have constructed the crossed product C^*-algebra $C(\hat{R}) \rtimes_\alpha R \rtimes R^\times$, which appears in the theorem below.

Theorem 2.1.8. *[12, Remark 3] $\mathfrak{A}[R]$ is $*$-isomorphic to $C(\hat{R}) \rtimes_\alpha R \rtimes R^\times$.*

We will return to the Cuntz-Li algebras in Chapter 4, in which we will study $\mathfrak{A}[R]$ under a new look. Almost all results exhibited here will be proved there by using the partial crossed products theory.

2.2 Ring C^*-algebras

The Cuntz-Li algebras, presented in the previous section, are C^*-algebras associated to an integral domain. It's natural to ask whether it is possible to extend this construction to a larger class of rings or, even, to all rings. In [33], Li has answered affirmatively this question, extending this construction to an arbitrary unital ring. He called such algebras ring C^*-algebras. In this section, we reproduce the main definitions and results obtained by Li in [33].

A first attempt to extend the construction to more general rings, would be to check if the definitions of the operators S_m and U^n in $\mathcal{B}(\ell^2(R))$ are still valid in case R is an arbitrary ring. For U^n, it's easy to see that the above definition remains valid. However, if R is not

[4] See [35, Theorem 2.1.10] or [39, Theorem 3.3.6] for Gelfand representation.

a domain, S_m may not be a bounded operator. Indeed, if R is an infinite ring where the product of any two elements are 0, then S_m does not define a bounded operator. To solve this problem, Li considers operators S_m only in case m is not a zero-divisor. Thus, the algebra $\mathfrak{A}_r[R]$ is perfectly well-defined.

However, when we try to define the full algebra $\mathfrak{A}[R]$, another problem arises. Although the relations (CL1)-(CL3) in Definition 2.1.1 remain valid in this case, the relation (CL4) may not make sense in the language of universal C^*-algebras. We allow only finite sums as relations; situation that will be violated if the quotient R/mR is not finite.

A similar problem occurred while attempting to generalize the Cuntz-Krieger algebras [11]. In [21], Exel and Laca have extended these algebras to infinite matrices by finding "all" finite relations that are consequences of the infinite relations. Here, to solve this problem, Li added new generators to the algebra, as the definitions below.

Let R be a unital ring and denote by $R^\times$ the set of elements in R which are not zero-divisors. Let $\mathcal{C}$ be a subset of the power set $\mathcal{P}(R)$ such that: (i) $R \in \mathcal{C}$, (ii) $\mathcal{C}$ is closed under finite unions, finite intersections and complements, (iii) if $n \in R$, $m \in R^\times$ and $X \in \mathcal{C}$, then $n+mX \in \mathcal{C}$ (we refer to this property by saying that $\mathcal{C}$ is closed under affine transformations).

Consider the Hilbert space $\ell^2(R)$ and denote by $\{\xi_r \mid r \in R\}$ its canonical basis. We already saw that, for $m \in R^\times$, $S_m(\xi_r) = \xi_{mr}$ defines an isometry and, for $n \in R$, $U^n(\xi_r) = \xi_{n+r}$ defines a unitary in $\mathcal{B}(\ell^2(R))$. Furthermore, for each $X \in \mathcal{C}$, we can define a projection E_X in $\mathcal{B}(\ell^2(R))$ such that $E_X(\xi_r) = [r \in X]\xi_r$.

Definition 2.2.1. [33, Definition 3.1] The **reduced ring C^*-algebra of R with respect to $\mathcal{C}$**, denoted by $\mathfrak{A}_r[R,\mathcal{C}]$, is the C^*-subalgebra of $\mathcal{B}(\ell^2(R))$ generated by the operators $\{S_m \mid m \in R^\times\}$, $\{U^n \mid n \in R\}$ and $\{E_X \mid X \in \mathcal{C}\}$.

As before, (CL1)-(CL3) of Definition 2.1.1 are satisfied by S_m and U^n. In addition, we can verify some relations involving the projections E_X:

(i) $E_R(\xi_r) = [r \in R]\xi_r = \xi_r = \mathrm{Id}_{\ell^2(R)}(\xi_r)$;

(ii) $E_{X\cap Y}(\xi_r) = [r \in X \cap Y]\xi_r = [r \in X][r \in Y]\xi_r = E_X E_Y(\xi_r)$;

(iii) If X and Y belong $\mathcal{C}$ and are disjoint, then $E_{X\cup Y}(\xi_r) = [r \in X \cup Y](\xi_r) = ([r \in X]+[r \in Y])\xi_r = E_X(\xi_r) + E_Y(\xi_r)$.

(iv) $U^n S_m E_X S_m^* U^{-n}(\xi_r) = U^n S_m E_X S_m^*(\xi_{r-n}) = [r-n=mk,\ k\in R]U^n S_m E_X(\xi_k) = [r-n=mk,\ k\in R][k\in X]U^n S_m(\xi_k) = [r-n=mk,\ k\in R][k\in X]U^n(\xi_{r-n}) = [r-n=mk,\ k\in R][k\in X]\xi_r = [r\in n+mX]\xi_r = E_{n+mX}(\xi_r)$.

These relations motivate the definition below.

Definition 2.2.2. [33, Definition 3.2] The **full ring C^*-algebra of R with respect to $\mathcal{C}$**, denoted by $\mathfrak{A}[R,\mathcal{C}]$, is the universal C^*-algebra generated by isometries $\{s_m \mid m \in R^\times\}$, unitaries $\{u^n \mid n \in R\}$ and projections $\{e_X \mid X \in \mathcal{C}\}$ subject to the relations

(L1) $s_m s_{m'} = s_{mm'}$;

(L2) $u^n u^{n'} = u^{n+n'}$;

(L3) $s_m u^n = u^{mn} s_m$;

(L4) $u^n s_m e_X s_m^* u^{-n} = e_{n+mX}$;

(L5) $e_R = 1$;

(L6) $e_{X\cap Y} = e_X e_Y$;

(L7) $e_{X\dot\cup Y} = e_X + e_Y$.

The next step is to check that this definition actually extends Definition 2.1.1 in case R is an integral domain with finite quotients by nonzero principal ideals. First, we note that if R is an integral domain, then $R^\times = R\backslash\{0\}$ as before. Therefore, the operators s_m are indexed by the same set. However, the operators e_X are not present in definition 2.1.1. Li justifies their presence in the next two results.

Proposition 2.2.3. *[33, Remark 3.7] Let R be a unital ring and $\mathcal{C}$ the smallest family of subsets of R which contains R and is closed under finite unions, finite intersections, complements and affine transformations. Then*

$$\mathfrak{A}[R,\mathcal{C}] = C^*\left(\{s_m \mid m\in R^\times\}\cup\{u^n \mid n\in R\}\right),$$

i.e., the generators e_X don't add new elements to $\mathfrak{A}[R,\mathcal{C}]$.

Proposition 2.2.4. *[33, Lemma 3.8] Let R be an integral domain with finite quotients by non-zero principal ideals and let $\mathcal{F}$ be any family of nontrivial ideals of R. If $\mathcal{C}$ is the smallest* *family of subsets of R which contains $\mathcal{F} \cup \{R\}$ and is closed under finite unions, finite intersections, complements and affine transformations, then the natural map $\mathfrak{A}[R] \longrightarrow \mathfrak{A}[R, \mathcal{C}]$ sending generators to generators exists and is a $*$-isomorphism.*

The first result tell us that the "undesirable" generators e_X don't increase the size of $\mathfrak{A}[R]$ when $\mathcal{C}$ is the smallest family generated by R. Indeed, these generators only add new relations to the algebra. The second result confirms that $\mathfrak{A}[R, \mathcal{C}]$ really extends Definition 2.1.1.

In Chapter 5, we propose another generalization to the Cuntz-Li algebras. We have found a very satisfactory way to include generators S_m when m is a zero-divisor.

2.3 Bost-Connes Algebra

In [6], Bost and Connes constructed a C^*-dynamical system which revealed deep connections between Operators Algebras and Number Theory. The most remarkable result is the appearance of the Riemann ζ-function as partition function of the KMS states of the dynamical system. In this section, we introduce the Bost-Connes algebra, namely, the underlying C^*-algebra of the Bost-Connes dynamical system.

Consider the quotient $\mathbb{Q}/\mathbb{Z}$ as an additive group.

Definition 2.3.1. [6, Proposition 18] The **Bost-Connes algebra**, denoted by $C_{\mathbb{Q}}$, is the universal C^*-algebra generated by isometries $\{\mu_m \mid m \in \mathbb{N}^*\}$ and unitaries $\{e_\gamma \mid \gamma \in \mathbb{Q}/\mathbb{Z}\}$ subject to the relations

(BC1) $\mu_m \mu_{m'} = \mu_{mm'}$;

(BC2) $\mu_m \mu_{m'}^* = \mu_{m'}^* \mu_m$, if $(m, m') = 1$;

(BC3) $e_\gamma e_{\gamma'} = e_{\gamma+\gamma'}$;

(BC4) $e_\gamma \mu_m = \mu_m e_{m\gamma}$;

(BC5) $\mu_m e_\gamma \mu_m^* = \frac{1}{m} \sum e_\delta$, where the sum is taken over all $\delta \in \mathbb{Q}/\mathbb{Z}$ such that $m\delta = \gamma$.

It's easy to see that if $\gamma = \frac{n'}{m'} + \mathbb{Z}$, then the sum in (BC5) is indexed by the set $\left\{\frac{n'}{mm'} + \mathbb{Z}, \frac{n'+m'}{mm'} + \mathbb{Z}, \ldots, \frac{n'+(m-1)m'}{mm'} + \mathbb{Z}\right\}$. From this, one can see that (BC5) is independent of the representation of γ in $\mathbb{Q}/\mathbb{Z}$ (this verification in (BC3) and (BC4) is trivial).

In Proposition 2.8 of [29], Laca and Raeburn deduced a curious fact: the relations (BC2) and (BC4) are consequences of the other three relations. In other words, we may remove these relations without modify the definition.

Originally, Bost and Connes have defined $C_{\mathbb{Q}}$ as the C^*-algebra of a certain Hecke pair. However, this equivalent definition (the equivalence is proved in [6, Proposition 18]) is more appropriate for our purposes. Again in [29], Laca and Raeburn showed that $C_{\mathbb{Q}}$ is the crossed product of $C^*(\mathbb{Q}/\mathbb{Z})$ (the group C^*-algebra[5] of $\mathbb{Q}/\mathbb{Z}$) by the multiplicative semigroup $\mathbb{N}^*$ with a certain action of endomorphisms. We return to the Bost-Connes algebra in Chapter 6, where a similar result is obtained: we show that $C_{\mathbb{Q}}$ is a partial crossed product of (a C^*-algebra isomorphic to) $C^*(\mathbb{Q}/\mathbb{Z})$ by $\mathbb{Q}_+^*$.

[5] See [8] for group C^*-algebra.

Chapter 3

Partial Crossed Products and Partial Group Algebras

In this chapter, we present the basic definitions and results concerning partial crossed products and partial group algebras. First, we define partial actions and partial representations. Hereafter, we construct the partial crossed product and exhibit its equivalent forms. In the last two sections, we introduce the partial group algebra and we obtain a characterization of it as a partial crossed product.

These theories are developed in [15], [16], [17], [18] and [22]. For more detailed texts, we recommend [4] and [34] (only in Portuguese).

3.1 Partial Actions and Partial Representations

Definition 3.1.1. [18, Definition 1.2] A **partial action** α of a (discrete) group G on a C^*-algebra $\mathcal{A}$ is a collection $(\mathcal{D}_t)_{t\in G}$ of ideals of $\mathcal{A}$ and $*$-isomorphisms $\alpha_t : \mathcal{D}_{t^{-1}} \longrightarrow \mathcal{D}_t$ such that

(PA1) $\mathcal{D}_e = \mathcal{A}$, where e represents the identity element of G;

(PA2) $\alpha_t^{-1}(\mathcal{D}_t \cap \mathcal{D}_{s^{-1}}) \subseteq \mathcal{D}_{(st)^{-1}}$;

(PA3) $\alpha_s \circ \alpha_t(x) = \alpha_{st}(x), \quad \forall\, x \in \alpha_t^{-1}(\mathcal{D}_t \cap \mathcal{D}_{s^{-1}})$.

The triple $(\mathcal{A}, G, \alpha)$ is called a **partial dynamical system**. In the above definition, if we replace the C^*-algebra $\mathcal{A}$ by a locally compact Hausdorff space X, the ideals $\mathcal{D}_t$ by open sets X_t and the $*$-isomorphisms α_t by homeomorphisms $\theta_t : X_{t^{-1}} \longrightarrow X_t$, we obtain a **partial action** θ of the group G on the space X.

Remark 3.1.2. Applying item (iii) with $s = t = e$ and using item (i), we see that $\alpha_e = \mathrm{Id}_{\mathcal{A}}$. Also by item (iii), with $s = t^{-1}$, we conclude that $\alpha_t^{-1} = \alpha_{t^{-1}}$. Furthermore, the inclusion in item (ii) is equivalent to $\alpha_t(\mathcal{D}_{t^{-1}} \cap \mathcal{D}_s) = \mathcal{D}_t \cap \mathcal{D}_{ts}$. To see this, apply item (ii) with $(ts)^{-1}$ in place s and use the fact that α_t is an $*$-isomorphism from $\mathcal{D}_{t^{-1}}$ to $\mathcal{D}_t$ to conclude that $\alpha_t(\mathcal{D}_{t^{-1}} \cap \mathcal{D}_s) \supseteq \mathcal{D}_t \cap \mathcal{D}_{ts}$. The reverse inclusion is obtained from (ii) with t^{-1} in place t, s^{-1} in place s and using that $\alpha_{t^{-1}}^{-1} = \alpha_t$.

Remark 3.1.3. Suppose that α is a partial action such that each ideal $\mathcal{D}_t$ is unital with unit 1_t. In this case, $\mathcal{D}_{t^{-1}} \cap \mathcal{D}_s$ and $\mathcal{D}_t \cap \mathcal{D}_{ts}$ are unital with units $1_{t^{-1}}1_s$ and 1_t1_{ts}, respectively. By previous remark, α_t is a $*$-isomorphism from $\mathcal{D}_{t^{-1}} \cap \mathcal{D}_s$ to $\mathcal{D}_t \cap \mathcal{D}_{ts}$. Since $*$-isomorphism take units on units, then $\alpha_t(1_{t^{-1}}1_s) = 1_t1_{ts}$.

Example 3.1.4. *Let G be a group, Y a locally compact Hausdorff space, X an open set of Y and ρ an action of G on Y. If we define, for each $t \in G$, $X_t = X \cap \rho_t(X)$ and*

$$\begin{aligned} \theta_t : X_{t^{-1}} &\longrightarrow X_t \\ x &\longmapsto \rho_t(x), \end{aligned}$$

then θ is a partial action of G on X (see [4, Exemplo 2.1.15]).

Example 3.1.5. *Let θ be a partial action of a group G on a locally compact Hausdorff space X. Define, for each $t \in G$, $\mathcal{D}_t = C_0(X_t)$ and*

$$\begin{aligned} \alpha_t : \mathcal{D}_{t^{-1}} &\longrightarrow \mathcal{D}_t \\ f &\longmapsto f \circ \theta_{t^{-1}}. \end{aligned}$$

If we identify $C_0(X_t)$ with the functions in $C_0(X)$ which vanish outside of X_t, then α defines a partial action of G on $C_0(X)$. We say that α is the **partial action induced** *by θ (see [4, Exemplo 2.1.18]).*

The previous example associates a partial action α on a C^*-algebra from a partial action θ on a topological space. In this case, we can extract useful informations about α by analysing θ. The most important for us, which will be seen in the next section, is the fact that we can use θ to classify the ideals in the crossed product associated to α. For this, we need some definitions.

Definition 3.1.6. [22, Definition 2.1] We say that a partial action θ on a space X is **topologically free** if, for all $t \in G \backslash \{e\}$, the set $F_t = \{x \in X_{t^{-1}} \mid \theta_t(x) = x\}$ has empty interior.

Definition 3.1.7. [22, Definition 2.7] Let θ be a partial action on a space X. We say that a subset V of X is **invariant** under θ if $\theta_t(V \cap X_{t^{-1}}) \subseteq V$, for every $t \in G$.

Definition 3.1.8. [22, Definition 2.7] Let α be a partial action on a C^*-algebra $\mathcal{A}$. We say that an ideal $\mathcal{I}$ of $\mathcal{A}$ is **invariant** under α if $\alpha_t(\mathcal{I} \cap \mathcal{D}_{t^{-1}}) \subseteq \mathcal{I}$, for every $t \in G$.

It's easy to see that if V is an open θ-invariant subset of X, then $C_0(V)$ is an α-invariant ideal of $C_0(X)$, where α is the partial action induced by θ.

Definition 3.1.9. [22, Definition 2.8] We say that a partial action θ on a space X is **minimal** if there are no invariant open subsets of X other than $\emptyset$ and X.

Proposition 3.1.10. *A partial action θ is minimal if, and only if, every $x \in X$ has dense orbit, namely $\mathcal{O}_x = \{\theta_t(x) \mid t \in G \text{ for which } x \in X_{t^{-1}}\}$ is dense in X.*

From now on, we change the subject to partial representations. At the end of this section, we return to talk about partial actions.

Definition 3.1.11. [18, Definition 6.2] A **partial representation** π of a (discrete) group G into a unital C^*-algebra $\mathcal{B}$ is a map $\pi : G \longrightarrow \mathcal{B}$ such that, for all $s, t \in G$,

(PR1) $\pi(e) = 1$;

(PR2) $\pi(t^{-1}) = \pi(t)^*$;

(PR3) $\pi(s)\pi(t)\pi(t^{-1}) = \pi(st)\pi(t^{-1})$.

It's noteworthy that, under (PR2) and (PR3), $\pi(s^{-1})\pi(s)\pi(t) = \pi(s^{-1})\pi(st)$ is valid too.

Example 3.1.12. *Consider the Hilbert space $\ell^2(\mathbb{N}^*)$ and denote by $\{\xi_n\}_{n \in \mathbb{N}^*}$ its canonical basis. Let S be the shift operator on $\ell^2(\mathbb{N}^*)$, i.e., $S(\xi_n) = \xi_{n+1}$. Then $\pi : \mathbb{Z} \longrightarrow \mathcal{B}(\ell^2(\mathbb{N}^*))$ given by*

$$\pi(n) = \begin{cases} S^n, & \text{if } n \geq 0 \\ (S^*)^{|n|}, & \text{if } n < 0, \end{cases}$$

is a partial representation of the additive group $\mathbb{Z}$ into $\mathcal{B}(\ell^2(\mathbb{N}^*))$ (see [4, Exemplo 3.1.7]).

The next proposition exhibits useful properties about partial representations.

Proposition 3.1.13. *[15, page 15] Let $\pi : G \longrightarrow \mathcal{B}$ be a partial representation of a group G into a unital C^*-algebra $\mathcal{B}$ and denote $\pi(t)\pi(t)^*$ by ε_t. For all $s, t \in G$, we have:*

(i) $\pi(t)$ *is a partial isometry, i.e.,* $\pi(t)\pi(t)^*\pi(t) = \pi(t)$;

(ii) ε_t *is a projection;*

(iii) $\pi(t)\varepsilon_s = \varepsilon_{ts}\pi(t)$;

(iv) $\varepsilon_s\varepsilon_t = \varepsilon_t\varepsilon_s$;

(v) $\pi(t)\pi(s) = \varepsilon_t\varepsilon_{ts}\pi(ts)$;

(vi) $\pi(t_1)\pi(t_2)\cdots\pi(t_n) = \varepsilon_{t_1}\varepsilon_{t_1t_2}\cdots\varepsilon_{t_1t_2\cdots t_n}\pi(t_1t_2\cdots t_n)$, *for all* $t_1, \dots, t_n \in G$.

Definition 3.1.14. [4, Definição 4.1.1] Let α be a partial action of a group G on a C^*-algebra $\mathcal{A}$, $\pi : G \longrightarrow \mathcal{B}$ be a partial representation of G into a unital C^*-algebra $\mathcal{B}$ and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a $*$-homomorphism. We say that the pair (φ, π) is **α-covariant** if:

(COV1) $\varphi(\alpha_t(x)) = \pi(t)\varphi(x)\pi(t^{-1})$, for all $t \in G$ e $x \in \mathcal{D}_{t^{-1}}$;

(COV2) $\varphi(x)\pi(t)\pi(t^{-1}) = \pi(t)\pi(t^{-1})\varphi(x)$, for all $x \in \mathcal{A}$ e $t \in G$.

This definition will be used later as a way to characterize the partial crossed product.

3.2 Partial Crossed Products

Throughout this section, we fix a partial action α of a group G on a C^*-algebra $\mathcal{A}$. Denote by $\mathcal{L}$ the direct sum $\oplus_{t\in G}\mathcal{D}_t$. With the operations defined componentwise, $\mathcal{L}$ is a vector space. If we denote by $a\delta_t$ the element of $\mathcal{L}$ whose entry t is a and which is 0 in the other entries, then every element of $\mathcal{L}$ can be written as a finite sum $\sum_{t\in G} a_t\delta_t$, where $a_t \in \mathcal{D}_t$. If we require that a_t is nonzero, then this representation is unique. We define a multiplication in $\mathcal{L}$ by $(a_s\delta_s)(a_t\delta_t) = \alpha_s(\alpha_{s^{-1}}(a_s)a_t)\delta_{st}$. It can be shown that $\mathcal{L}$ is an associative algebra with these operations (see [15, Corollary 3.4]). Furthermore, we can view $\mathcal{L}$ as a normed $*$-algebra with an involution and a norm given by $(a_t\delta_t)^* = \alpha_{t^{-1}}(a_t^*)\delta_{t^{-1}}$ and $||\sum_{t\in G} a_t\delta_t|| = \sum_{t\in G} ||a_t||$.

Definition 3.2.1. The **full partial crossed product** (or simply, crossed product) of $\mathcal{A}$ by G through α, denoted by $\mathcal{A}\rtimes_\alpha G$, is the enveloping C^*-algebra[1] of $\mathcal{L}$ (see [4, Proposição 2.2.31]).

[1]See [4, Exemplo A.2.8] for enveloping C^*-algebras.

It can be shown that there is an injective $*$-homomorphism $\mathcal{L} \longrightarrow \mathcal{A}\rtimes_\alpha G$ (see [4, Corolário 2.2.32]). In other words, $\mathcal{A}\rtimes_\alpha G$ is the completion of $\mathcal{L}$ under a certain C^*-norm. There is another characterization of $\mathcal{A}\rtimes_\alpha G$ as the universal C^*-algebra for α-covariant representations, according to the next proposition.

Proposition 3.2.2. *Let α be a partial action of a group G on a C^*-algebra $\mathcal{A}$, $\pi : G \longrightarrow \mathcal{B}$ be a partial representation of G into a unital C^*-algebra $\mathcal{B}$ and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a $*$-homomorphism such that the pair (φ, π) is α-covariant. Then there exists a unique $*$-homomorphism $\varphi \times \pi : \mathcal{A}\rtimes_\alpha G \longrightarrow \mathcal{B}$ such that*

$$(\varphi \times \pi)(a_t\delta_t) = \varphi(a_t)\pi(t), \quad \forall\, t \in G,\ \forall\, a_t \in \mathcal{D}_t$$

(see [4, Corolário 4.1.5]).

In addition to the full crossed product, there exists the **reduced crossed product**, denoted by $\mathcal{A}\rtimes_{\alpha,r}G$. It can also be defined as the completion of $\mathcal{L}$ under a certain C^*-norm (not equal to the previous one, in general). For a formal definition of $\mathcal{A}\rtimes_{\alpha,r}G$ see [4, Definição 2.2.36].

There is a natural surjective $*$-homomorphism $\mathcal{A}\rtimes_\alpha G \longrightarrow \mathcal{A}\rtimes_{\alpha,r}G$ which is the identity on $\mathcal{L}$. When this $*$-homomorphism is injective, we say that the dynamical system $(\mathcal{A}, G, \alpha)$ is **amenable**. It's a fact that if G is an amenable group,[2] then $(\mathcal{A}, G, \alpha)$ is amenable (see [17, Theorem 4.7]).

We can identify $\mathcal{A}$ as a C^*-subalgebra of $\mathcal{A}\rtimes_{\alpha,r}G$ and of $\mathcal{A}\rtimes_\alpha G$ through the injective $*$-homomorphisms $\mathcal{A} \longrightarrow \mathcal{A}\rtimes_{\alpha,r}G$ and $\mathcal{A} \longrightarrow \mathcal{A}\rtimes_\alpha G$ both given by $a \longmapsto a\delta_e$. There exists a faithful conditional expectation $E : \mathcal{A} \rtimes_{\alpha,r} G \longrightarrow \mathcal{A}$ given by $E(a\delta_t) = a$ if $t = e$, and $E(a\delta_t) = 0$ if $t \neq e$. When the dynamical system is amenable, the full and reduced crossed products are $*$-isomorphic and, in this case, there exists a faithful conditional expectation of $\mathcal{A} \rtimes_\alpha G$ onto $\mathcal{A}$.

Henceforth, we consider that $\mathcal{A} = C_0(X)$ and that α is induced by a partial action θ on X. The next results are valid for the reduced crossed product only. However, when the dynamical system is amenable, we can replace the reduced by the full crossed product.

Proposition 3.2.3. *[22, Theorem 2.6] Suppose that θ is topologically free. If $\mathcal{J}$ is an ideal in $\mathcal{A}\rtimes_{\alpha,r}G$ with $\mathcal{J} \cap \mathcal{A} = \{0\}$, then $\mathcal{J} = \{0\}$.*

[2] See [4, Definição B.3.2] for amenable groups.

Proposition 3.2.4. *[22, Corollary 2.9] If θ is topologically free and minimal, then $\mathcal{A}\rtimes_{\alpha,r}G$ is simple.*

3.3 Partial Group Algebras

Let G be a discrete group, let $\mathcal{G}$ be the set G without the group operations and denote the elements in $\mathcal{G}$ by $[t]$ (namely, $\mathcal{G} = \{[t] \mid t \in G\}$).

Definition 3.3.1. [18, Definition 6.4 and Theorem 6.5] The **partial group algebra of** G, denoted by $C^*_{\mathrm{p}}(G)$, is defined to be the universal C^*-algebra generated by the set $\mathcal{G}$ subject to the relations

$$\mathcal{R}_{\mathrm{p}} = \{[e] = 1\} \cup \{[t^{-1}] = [t]^*\}_{t\in G} \cup \{[s][t][t^{-1}] = [st][t^{-1}]\}_{s,t\in G}.$$

Observe that the relations in $\mathcal{R}_{\mathrm{p}}$ correspond to the partial representation axioms (PR1), (PR2) and (PR3). Sometimes, we will refer to a relation in $\mathcal{R}_{\mathrm{p}}$ by indicating the corresponding axiom. For example, if we use $[t^{-1}] = [t]^*$, we refer to it through the axiom (PR2).

Just as the C^*-algebra of G is universal with respect to unitary representations of G, the partial group algebra of G is universal with respect to partial representations.

Proposition 3.3.2. *[18, Definition 6.4 and Theorem 6.5] If $\pi : G \longrightarrow \mathcal{B}$ is a partial representation of G into a unital C^*-algebra $\mathcal{B}$, then there exists a unique $*$-homomorphism $\psi : C^*_{\mathrm{p}}(G) \longrightarrow \mathcal{B}$ such that $\psi([t]) = \pi(t)$ for all $t \in G$.*

Now, we will study an important C^*-subalgebra of $C^*_{\mathrm{p}}(G)$. For each $t \in G$, denote $[t][t^{-1}]$ by ε_t and denote by $\mathcal{A}_G$ the C^*-subalgebra of $C^*_{\mathrm{p}}(G)$ generated by $\{\varepsilon_t\}_{t\in G}$. By Proposition 3.1.13, $\mathcal{A}_G$ is a commutative C^*-algebra generated by projections. Denote by $\mathcal{C}_G$ the universal C^*-algebra generated by a set of projections $\{e_t\}_{t\in G}$ subject to the relations that e_s commutes with e_t, for all $s, t \in G$. The next result shows that the commuting relations between the projections in $\mathcal{A}_G$ are sufficient to characterize it.

Proposition 3.3.3. *The map $\mathcal{C}_G \longrightarrow \mathcal{A}_G$ which sends e_t on ε_t is a $*$-isomorphism (see [4, Proposição 4.4.7 and Corolário 4.4.10]).*

There is another way of understanding $\mathcal{A}_G$. Since $\mathcal{A}_G$ is commutative then, by the Gelfand representation, $\mathcal{A}_G$ is $*$-isomorphic to $C(\hat{\mathcal{A}}_G)$, where $\hat{\mathcal{A}}_G$ denotes the spectrum of $\mathcal{A}_G$. Let's characterize $\hat{\mathcal{A}}_G$. Consider the natural bijection between $\mathcal{P}(G)$ and $\{0,1\}^G$, where $\mathcal{P}(G)$ is the power set of G. With the product topology, $\{0,1\}^G$ is a compact Hausdorff space. Give to $\mathcal{P}(G)$ the topology of $\{0,1\}^G$. Denote by X_G the subset of $\mathcal{P}(G)$ of the subsets ξ of G such that $e \in \xi$. Clearly, with the induced topology of $\mathcal{P}(G)$, X_G is a compact space.

Proposition 3.3.4. *[18, Proposition 6.6] The spectrum of $\mathcal{A}_G$ is homeomorphic to X_G through the map $\hat{\mathcal{A}}_G \ni \phi \longmapsto \{t \in G \mid \phi(\varepsilon_t) = 1\} \in X_G$.*

As a corollary, we have $\mathcal{A}_G \cong C(X_G)$. It's important for us to explicit the map that defines the $*$-isomorphism. Indeed, by using the above proposition, it's not hard to see that $\mathcal{A}_G \ni \varepsilon_t \longmapsto 1_t \in C(X_G)$, where 1_t represents the characteristic function of the subset $\{\xi \in X_G \mid t \in \xi\}$ of X_G.

These characterizations of $\mathcal{A}_G$ enable us to find equivalent formulations for $C^*_{\mathrm{p}}(G)$. For each $t \in G$, $\varepsilon_t\mathcal{A}_G$ is an ideal of $\mathcal{A}_G$ and the map

$$\begin{aligned} \bar{\alpha}_t : \varepsilon_{t^{-1}}\mathcal{A}_G &\longrightarrow \varepsilon_t\mathcal{A}_G \\ x &\longmapsto [t]x[t^{-1}] \end{aligned}$$

is a $*$-isomorphism; defining a partial action $\bar{\alpha}$ on $\mathcal{A}_G$ (see [4, Corolário 4.1.16]).

Proposition 3.3.5. *[18, Definition 6.4 and Theorem 6.5] There is a $*$-isomorphism $C^*_{\mathrm{p}}(G) \longrightarrow \mathcal{A}_G \rtimes_{\bar{\alpha}} G$ given by $[t] \longmapsto \varepsilon_t\delta_t$.*

Next, we will find a partial action on X_G. For each $t \in G$, denote by X_t the open subset $\{\xi \in X_G \mid t \in \xi\}$ of X_G. The map

$$\begin{aligned} \theta_t : X_{t^{-1}} &\longrightarrow X_t \\ \xi &\longmapsto t\xi \end{aligned}$$

is a homeomorphism, where $t\xi = \{ts \mid s \in \xi\}$. It defines a partial action θ on X_G (see [4, Proposição 4.4.3]). Denote by α the partial action induced by θ on $C(X_G)$.

Proposition 3.3.6. *[18, Definition 6.4 and Theorem 6.5] There is a $*$-isomorphism $C^*_{\mathrm{p}}(G) \longrightarrow C(X_G) \rtimes_{\alpha} G$ given by $[t] \longmapsto 1_t\delta_t$.*

We finish this section presenting a useful property about α. Note that the set $\{1_t\}_{t\in G}$ in $C(X_G)$ separates points in X_G and that $1_e = 1$. Hence, by Stone-Weierstrass theorem [39, Theorem A.6.9], the C^*-algebra generated by $\{1_t\}_{t\in G}$ is $C(X_G)$. Since the ideal $\mathcal{D}_t$ of α is $C(X_t) \cong 1_t C(X_G)$, then $C(X_t)$ is generated by $\{1_t 1_s\}_{s\in G}$. These informations are used when we need to prove some property involving $C(X_G)$ or $C(X_t)$. In general, to prove a property on the generators it is enough to ensure that the property is valid on the whole C^*-algebra.

3.4 Partial Group Algebras with Relations

In this section, we define a generalized version of the partial group algebra. Let G, $\mathcal{G}$ and $\mathcal{R}_{\mathrm{p}}$ be as in the previous section. Let $\mathcal{R}$ be a set of relations on $\mathcal{G}$ such that every relation is of the form

$$\sum_i \lambda_i \prod_j \varepsilon_{t_{ij}} = 0,$$

where $\lambda_i \in \mathbb{C}$ and $\varepsilon_t = [t][t^{-1}]$ as before.

Definition 3.4.1. [22, Definition 4.3] The **partial group algebra of G with relations $\mathcal{R}$**, denoted by $C^*_{\mathrm{p}}(G, \mathcal{R})$, is defined to be the universal C^*-algebra generated by the set $\mathcal{G}$ with the relations $\mathcal{R}_{\mathrm{p}} \cup \mathcal{R}$.

Given a partial representation π of G, we can extend π naturally to sums of products of elements in $\mathcal{G}$. If this extension satisfies the relations $\mathcal{R}$, we say that π is a **partial representation that satisfies $\mathcal{R}$**. The next result presents the universal property of $C^*_{\mathrm{p}}(G, \mathcal{R})$.

Proposition 3.4.2. *[22, Definition 4.3] If $\pi : G \longrightarrow \mathcal{B}$ is a partial representation of G into a unital C^*-algebra $\mathcal{B}$ that satisfies $\mathcal{R}$, then there exists a unique $*$-homomorphism $\psi : C^*_{\mathrm{p}}(G, \mathcal{R}) \longrightarrow \mathcal{B}$ such that $\psi([t]) = \pi(t)$ for all $t \in G$.*

In analogy to the previous section, we will exhibit characterizations of $C^*_{\mathrm{p}}(G, \mathcal{R})$ as partial crossed products. Denote by $\mathcal{A}_{(G,\mathcal{R})}$ the (commutative) C^*-subalgebra of $C^*_{\mathrm{p}}(G, \mathcal{R})$ generated by $\{\varepsilon_t\}_{t\in G}$. As before, the maps

$$\begin{aligned} \bar{\alpha}_t : \varepsilon_{t^{-1}}\mathcal{A}_{(G,\mathcal{R})} &\longrightarrow \varepsilon_t \mathcal{A}_{(G,\mathcal{R})} \\ x &\longmapsto [t]x[t^{-1}] \end{aligned}$$

define a partial action $\bar{\alpha}$ of G on $\mathcal{A}_{(G,\mathcal{R})}$.

Proposition 3.4.3. *[22, Theorem 4.4] There is a $*$-isomorphism $C^*_{\mathrm{p}}(G, \mathcal{R}) \longrightarrow \mathcal{A}_{(G,\mathcal{R})} \rtimes_{\bar{\alpha}} G$ given by $[t] \longmapsto \varepsilon_t \delta_t$.*

We can use this proposition to define a conditional expectation on $C_{\mathrm{p}}^*(G,\mathcal{R})$. If we transport the natural conditional expectation on $\mathcal{A}_{(G,\mathcal{R})} \rtimes_{\bar{\alpha}} G$ to $C_{\mathrm{p}}^*(G,\mathcal{R})$, we obtain $E : C_{\mathrm{p}}^*(G,\mathcal{R}) \longrightarrow \mathcal{A}_{(G,\mathcal{R})}$ given by $E([t_1][t_2]\cdots[t_k]) = [t_1][t_2]\cdots[t_k]$ if $t_1 t_2 \cdots t_k = e$ and $E([t_1][t_2]\cdots[t_k]) = 0$ otherwise.

Denote by $\mathcal{J}_{\mathcal{R}}$ the smallest (closed) ideal of $\mathcal{A}_{(G,\mathcal{R})}$ which contains

$$\left\{ [t] \left(\sum_i \lambda_i \prod_j \varepsilon_{t_{ij}} \right) [t^{-1}] \;\middle|\; \sum_i \lambda_i \prod_j \varepsilon_{t_{ij}} \in \mathcal{R} \text{ and } t \in G \right\}.$$

It is noteworthy that, by using item (iii) of Proposition 3.1.13,

$$[t] \left(\sum_i \lambda_i \prod_j \varepsilon_{t_{ij}} \right) [t^{-1}] = \varepsilon_t \sum_i \lambda_i \prod_j \varepsilon_{tt_{ij}}$$

and, hence, it belongs to $\mathcal{A}_{(G,\mathcal{R})}$. There is a natural surjective $*$-homomorphism $\mathcal{A}_G \longrightarrow \mathcal{A}_{(G,\mathcal{R})}$, where $\mathcal{A}_G$ is as in previous section, which sends ε_t on ε_t (obviously, the first one is in $\mathcal{A}_G$ and the last in $\mathcal{A}_{(G,\mathcal{R})}$). The kernel of this $*$-homomorphism is exactly $\mathcal{J}_{\mathcal{R}}$ and, therefore, $\mathcal{A}_{(G,\mathcal{R})} \cong \mathcal{A}_G/\mathcal{J}_{\mathcal{R}}$ (see [4, page 111]).

Now, we will find a concrete realization of $\mathcal{A}_{(G,\mathcal{R})}$. Let $C(X_G)$ and 1_t be as in previous section. By an abuse of notation, we also denote by $\mathcal{R}$ the subset of $C(X_G)$ given by the functions $\sum_i \lambda_i \prod_j 1_{t_{ij}}$, where $\sum_i \lambda_i \prod_j e_{t_{ij}} = 0$ is a relation in (the original) $\mathcal{R}$.

Definition 3.4.4. [22, Definition 4.2] The **spectrum of the relations** $\mathcal{R}$ is defined to be the compact Hausdorff space

$$\Omega_{\mathcal{R}} = \{\xi \in X_G \,|\, f(t^{-1}\xi) = 0,\ \forall\, f \in \mathcal{R},\ \forall\, t \in \xi\}.$$

Proposition 3.4.5. *There is a $*$-isomorphism $\mathcal{A}_{(G,\mathcal{R})} \longrightarrow C(\Omega_{\mathcal{R}})$ given by $\varepsilon_t \longmapsto 1_t$ (see [4, page 113]).*

Denote by Ω_t the subset $\{\xi \in \Omega_{\mathcal{R}} \mid t \in \xi\}$ of $\Omega_{\mathcal{R}}$. It can be shown that, if we restrict the domain of the homeomorphism $\theta_t : X_{t^{-1}} \longrightarrow X_t$ (defined in the section above) to $\Omega_{t^{-1}}$, we obtain a homeomorphism from $\Omega_{t^{-1}}$ onto Ω_t. Thus we have a partial action (also denoted by) θ of G on $\Omega_{\mathcal{R}}$ (see [4, page 108]). Let α be the partial action on $C(\Omega_{\mathcal{R}})$ induced by θ. The theorem below is the most important result concerning partial group algebras.

Proposition 3.4.6. *[22, Theorem 4.4] There is a $*$-isomorphism $C_{\mathrm{p}}^*(G,\mathcal{R}) \longrightarrow C(\Omega_{\mathcal{R}}) \rtimes_\alpha G$ given by $[t] \longmapsto 1_t\delta_t$, where 1_t denotes the characteristic function of Ω_t.*

Chapter 4

Characterizations of the Cuntz-Li Algebras

In this chapter, we show that the Cuntz-Li algebras $\mathfrak{A}[R]$ presented in Section 2.1 can be seen as partial group algebras with relations. By using Theorem 3.4.6, we obtain a characterization of $\mathfrak{A}[R]$ as a partial crossed product. With the theory presented in Chapter 3, we recover many of the results proved by Cuntz and Li in [12]. Among then, we will prove Proposition 2.1.5, a part of Theorem 2.1.6 and Theorem 2.1.7.

The results of this chapter are in [5].

4.1 Partial Group Algebra Description of $\mathfrak{A}[R]$

As in Section 2.1, let R be an integral domain which is not a field and with the property that the quotient $R/(m)$ is finite, for all $m \neq 0$ in R. Denote by K the field of fractions of R and consider the semidirect product $K \rtimes K^\times$. The elements of $K \rtimes K^\times$ will be denoted by a pair (u, w), where $u \in K$ and $w \in K^\times$. Recall that $(u, w)(u', w') = (u + u'w, ww')$ and $(u, w)^{-1} = (-u/w, 1/w)$. We denote by $[u, w]$ an element of set $K \rtimes K^\times$ without the group operations (as the set $\mathcal{G}$ associated to G in Section 3.3).[1] Also as in Section 3.3, denote $[t][t^{-1}]$ by ε_t. Consider the sets of relations

$$\mathcal{R}_1 = \left\{\varepsilon_{(n,1)} = 1 \mid n \in R\right\}, \quad \mathcal{R}_2 = \left\{\varepsilon_{\left(0,\frac{1}{m}\right)} = 1 \mid m \in R^\times\right\},$$

[1] Sometimes, we work with the element $(u, w)^{-1}$ or the element $(u_1, w_1)(u_2, w_2)$. For these elements, our corresponding notations will be $[(u, w)^{-1}]$ and $[(u_1, w_1)(u_2, w_2)]$.

$$\mathcal{R}_3 = \left\{ \sum_{l+(m)\in R/(m)} \varepsilon_{(l,m)} = 1 \;\middle|\; m \in R^\times \right\}$$

and $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$. Our goal is to construct the partial group algebra $C^*_{\mathrm{p}}(K \rtimes K^\times, \mathcal{R})$. However, the relations in $\mathcal{R}_3$ apparently depend on a choice of l. Observe that, under the relations $\mathcal{R}_1$ and $\mathcal{R}_{\mathrm{p}}$ (see Sections 3.3 and 3.4), the sum in $\mathcal{R}_3$ is independent of this choice. Indeed, if $l+(m) = l'+(m)$, say $l' = l + km$,

$$\varepsilon_{(l',m)} = \varepsilon_{(l+km,m)} = [l+km, m][(l+km,m)^{-1}] = [(l,m)(k,1)][(k,1)^{-1}(l,m)^{-1}] \overset{\mathcal{R}_1}{=}$$

$$[(l,m)(k,1)]\varepsilon_{(-k,1)}[(k,1)^{-1}(l,m)^{-1}] = [(l,m)(k,1)][(k,1)^{-1}][k,1][(k,1)^{-1}(l,m)^{-1}] \overset{\text{(PR3)}}{=}$$

$$[l,m][k,1][(k,1)^{-1}][k,1][(k,1)^{-1}][(l,m)^{-1}] = [l,m]\varepsilon_{(k,1)}\varepsilon_{(k,1)}[(l,m)^{-1}] \overset{\mathcal{R}_1}{=} \varepsilon_{(l,m)}.$$

Thus, we can consider the partial group algebra $C^*_{\mathrm{p}}(K \rtimes K^\times, \mathcal{R})$. We will show that this algebra is $*$-isomorphic to $\mathfrak{A}[R]$.

Proposition 4.1.1. *There exists a $*$-homomorphism $\Psi : \mathfrak{A}[R] \longrightarrow C^*_{\mathrm{p}}(K \rtimes K^\times, \mathcal{R})$ such that $\Psi(u^n) = [n,1]$ and $\Psi(s_m) = [0,m]$.*

Proof. We need to show that $[n,1]$ is a unitary (for $n \in R$), that $[0,m]$ is an isometry (for $m \in R^\times$) and that the relations (CL1)-(CL4) of Definition 2.1.1 are satisfied. From $\mathcal{R}_1$ and (PR2), we have $[n,1][n,1]^* \overset{\text{(PR2)}}{=} [n,1][(n,1)^{-1}] = \varepsilon_{(n,1)} \overset{\mathcal{R}_1}{=} 1$ and $[n,1]^*[n,1] = e_{(-n,1)} = 1$, i.e., $[n,1]$ is a unitary. Similarly, from $\mathcal{R}_2$ and (PR2) we see that $[0,m]$ is an isometry. By using this fact,

$$\Psi(s_m s_{m'}) = [0,m][0,m'] = [0,m][0,m'][0,m']^*[0,m'] \overset{\text{(PR3)}}{=}$$

$$[(0,m)(0,m')][0,m']^*[0,m'] = [0,mm'][0,m']^*[0,m'] = [0,mm'] = \Psi(s_{mm'}),$$

hence (CL1) is satisfied. We can prove (CL2) in the same way. To show (CL3), note that

$$\Psi(s_m u^n) = [0,m][n,1] = [0,m][n,1][n,1]^*[n,1] \overset{\text{(PR3)}}{=}$$

$$[(0,m)(n,1)][n,1]^*[n,1] = [mn,m][n,1]^*[n,1] = [mn,m],$$

because $[n,1]$ is a unitary. On the other hand,

$$\Psi(u^{mn} s_m) = [mn,1][0,m] = [mn,1][mn,1]^*[mn,1][0,m] \overset{\text{(PR3)}}{=}$$

$$[mn,1][mn,1]^*[(mn,1)(0,m)] = [mn,1][mn,1]^*[mn,m] = [mn,m].$$

Finally, (CL4) follows from $\mathcal{R}_3$ and

$$\Psi(u^l s_m s_m^* u^{-l}) = [l,1][0,m][0,m]^*[-l,1] \overset{\text{(PR3),(PR2)}}{=} [(l,1)(0,m)][(0,m)^{-1}][-l,1] =$$

$$[l,m][0,1/m][-l,1][-l,1]^*[-l,1] \overset{\text{(PR3)}}{=} [l,m][(0,1/m)(-l,1)][-l,1]^*[-l,1] =$$

$$[l,m][(l,m)^{-1}][-l,1]^*[-l,1] = [l,m][(l,m)^{-1}] = \varepsilon_{(l,m)}.$$

□

Now, we will construct an inverse for Ψ. For this, we will define a partial representation of $K \rtimes K^\times$ into $\mathfrak{A}[R]$ that satisfies $\mathcal{R}$ and use the universal property of $C^*_{\mathrm{p}}(K \rtimes K^\times, \mathcal{R})$ in Proposition 3.4.2. In the next claim, note that every element in $K \rtimes K^\times$ can be written under the form $\left(\frac{n}{m'}, \frac{m}{m'}\right)$, where $n \in R$ and $m, m' \in R^\times$.

Claim 4.1.2. *The map $\pi : K \rtimes K^\times \longrightarrow \mathfrak{A}[R]$ given by $\pi\left(\left(\frac{n}{m'}, \frac{m}{m'}\right)\right) = s^*_{m'} u^n s_m$ is independent of the representation of $\left(\frac{n}{m'}, \frac{m}{m'}\right)$.*

Proof. Let $\left(\frac{n}{m'}, \frac{m}{m'}\right) = \left(\frac{q}{p'}, \frac{p}{p'}\right)$, i.e., $pm' = p'm$ and $m'q = p'n$. Hence,

$$s^*_{p'} u^q s_p = s^*_{p'} s^*_{m'} s_{m'} u^q s_p \overset{\text{(CL3)}}{=} s^*_{p'} s^*_{m'} u^{m'q} s_{m'} s_p \overset{\text{(CL1)}}{=} s^*_{p'm'} u^{m'q} s_{m'p} \overset{\text{(CL1)}}{=}$$

$$s^*_{m'} s^*_{p'} u^{p'n} s_{p'} s_m \overset{\text{(CL3)}}{=} s^*_{m'} s^*_{p'} s_{p'} u^n s_m = s^*_{m'} u^n s_m.$$

□

Before showing that π is a partial representation that satisfies $\mathcal{R}$, we observe that $s_1 = 1$ and $u^0 = 1$ in $\mathfrak{A}[R]$. Indeed, both are idempotent and have a left inverse.

Proposition 4.1.3. *The map π defined above is a partial representation of $K \rtimes K^\times$ that satisfies $\mathcal{R}$.*

Proof. First, we will show that π is a partial representation. Since $\pi((0,1)) = s_1^* u^0 s_1 = 1$, we have (PR1). Observe that

$$\pi\left(\left(\frac{n}{m'}, \frac{m}{m'}\right)^{-1}\right) = \pi\left(\left(\frac{-n}{m}, \frac{m'}{m}\right)\right) = s^*_m u^{-n} s_{m'} = \pi\left(\left(\frac{n}{m'}, \frac{m}{m'}\right)\right)^*,$$

which shows (PR2). To see (PR3), let $s = \left(\frac{q}{p'}, \frac{p}{p'}\right)$ and $t = \left(\frac{n}{m'}, \frac{m}{m'}\right)$. We have $st = \left(\frac{m'q+pn}{p'm'}, \frac{pm}{p'm'}\right)$ and, therefore,

$$\pi(st)\pi(t^{-1}) = \pi(st)\pi(t)^* = (s^*_{p'm'} u^{m'q+pn} s_{pm})(s^*_m u^{-n} s_{m'}) \overset{\text{(CL1),(CL2)}}{=}$$

$$s_{p'}^* s_{m'}^* u^{m'q} u^{pn} s_p s_m s_m^* u^{-n} s_{m'} \overset{\text{(CL3)}}{=} s_{p'}^* u^q s_{m'}^* s_p u^n s_m s_m^* u^{-n} s_{m'} =$$

$$s_{p'}^* u^q s_{m'}^* s_p \underbrace{u^n s_m s_m^* u^{-n}}_{\in P} \underbrace{s_{m'} s_{m'}^*}_{\in P} s_{m'} \overset{\text{Lemma 2.1.3}}{=} s_{p'}^* u^q s_{m'}^* s_p s_{m'} s_{m'}^* u^n s_m s_m^* u^{-n} s_{m'} \overset{\text{(CL1)}}{=}$$

$$s_{p'}^* u^q s_{m'}^* s_{m'} s_p s_{m'}^* u^n s_m s_m^* u^{-n} s_{m'} = (s_{p'}^* u^q s_p)(s_{m'}^* u^n s_m)(s_m^* u^{-n} s_{m'}) = \pi(s)\pi(t)\pi(t^{-1}).$$

This shows that π is a partial representation. It remains to show that the extension of π satisfies the relations in $\mathcal{R}$. Since

$$\pi(\varepsilon_{(n,1)}) = \pi([n,1][-n,1]) = (s_1^* u^n s_1)(s_1^* u^{-n} s_1) = u^n u^{-n} = u^0 = 1,$$

the relations in $\mathcal{R}_1$ are satisfied. For $\mathcal{R}_2$, observe that

$$\pi(\varepsilon_{(0,1/m)}) = \pi([0,1/m][0,m]) = (s_m^* u^0 s_1)(s_1^* u^0 s_m) = s_m^* s_m = 1.$$

As a conclusion,

$$\pi\left(\sum_{l+(m)\in R/(m)} \varepsilon_{(l,m)}\right) = \sum_{l+(m)\in R/(m)} s_1^* u^l s_m s_m^* u^{-l} s_1 = \sum_{l+(m)\in R/(m)} u^l s_m s_m^* u^{-l} \overset{\text{(CL4)}}{=} 1$$

shows that $\mathcal{R}_3$ is satisfied. □

Remark 4.1.4. We can define π for a general representation of a element in $K \rtimes K^\times$ by $\pi\left(\left(\frac{n}{m''}, \frac{m}{m'}\right)\right) = s_{m''}^* u^n s_{m'}^* s_{m''} s_m$.

By the universal property of $C_{\mathrm{p}}^*(K \rtimes K^\times, \mathcal{R})$ and by the above proposition, there exists a $*$-homomorphism $\Phi : C_{\mathrm{p}}^*(K \rtimes K^\times, \mathcal{R}) \longrightarrow \mathfrak{A}[R]$ such that $\Phi\left(\left[\frac{n}{m'}, \frac{m}{m'}\right]\right) = s_{m'}^* u^n s_m$.

Theorem 4.1.5. *Ψ and Φ are inverses of each other.*

Proof. It is enough to prove that the two relevant compositions agree with the identity on the generators. Thus, $\Phi(\Psi(u^n)) = \Phi([n,1]) = s_1^* u^n s_1 = u^n$ and $\Phi(\Psi(s_m)) = \Phi([0,m]) = s_1^* u^0 s_m = s_m$. On the other hand,

$$\Psi\left(\Phi\left(\left[\frac{n}{m'}, \frac{m}{m'}\right]\right)\right) = \Psi(s_{m'}^* u^n s_m) = \left[0, 1/m'\right][n,1][0,m] =$$

$$\left[0,1/m'\right][n,1][n,1]^*[n,1][0,m] \overset{\text{(PR3)}}{=} \left[0,1/m'\right][n,1][n,1]^*[n,m] \overset{\mathcal{R}_1}{=} \left[0,1/m'\right][n,m] =$$

$$\left[0,1/m'\right]\left[0,1/m'\right]^*\left[0,1/m'\right][n,m] \overset{\text{(PR3)}}{=} \left[0,1/m'\right]\left[0,1/m'\right]^*\left[\frac{n}{m'}, \frac{m}{m'}\right] \overset{\mathcal{R}_2}{=} \left[\frac{n}{m'}, \frac{m}{m'}\right].$$

□

This theorem shows that $\mathfrak{A}[R]$ is a partial group algebra. We can use it to define a faithful conditional expectation on $\mathfrak{A}[R]$. Since that additive group K and the multiplicative group $K^{\rtimes}$ are abelian (hence solvable), then $K \rtimes K^{\times}$ is solvable. In [23], Theorem 1.2.1 asserts that every abelian group is amenable and Theorem 1.2.6 says that if a group G has a normal subgroup N such that N and G/N are amenable then G is amenable. By using these results, we see that every solvable group is amenable and, hence so is $K \rtimes K^{\times}$. Therefore, the conditional expectation on $C^*_{\mathrm{p}}(K \rtimes K^{\times}, \mathcal{R})$ defined in Section 3.4 is faithful. The next proposition shows that, under the $*$-isomorphism Ψ, the conditional expectations E on $C^*_{\mathrm{p}}(K \rtimes K^{\times}, \mathcal{R})$ and Θ on $\mathfrak{A}[R]$ (Proposition 2.1.5) are the same.

Proposition 4.1.6. *$E \circ \Psi = \Psi \circ \Theta$.*

Proof. First of all, observe that $\left(\frac{n}{m''}, \frac{m}{m''}\right)\left(\frac{-n'}{m}, \frac{m'}{m}\right) = (0,1)$ if, and only if, $m' = m''$ and $n = n'$. By using the Kronecker delta notation, we have

$$E \circ \Psi(s^*_{m''} u^n s_m s^*_m u^{-n'} s_{m'}) = E\left(\left[\frac{n}{m''}, \frac{m}{m''}\right]\left[\frac{-n'}{m}, \frac{m'}{m}\right]\right) =$$

$$\delta_{m',m''}\delta_{n,n'}\left[\frac{n}{m'}, \frac{m}{m'}\right]\left[\frac{-n}{m}, \frac{m'}{m}\right].$$

On the other hand

$$\Psi \circ \Theta(s^*_{m''} u^n s_m s^*_m u^{-n'} s_{m'}) = \Psi(\delta_{m',m''}\delta_{n,n'} s^*_{m'} u^n s_m s^*_m u^{-n} s_{m'}) =$$

$$\delta_{m',m''}\delta_{n,n'}\left[\frac{n}{m'}, \frac{m}{m'}\right]\left[\frac{-n}{m}, \frac{m'}{m}\right].$$

□

4.2 Partial Crossed Product Description of $\mathfrak{A}[R]$

We already know that $\mathfrak{A}[R]$ is a partial crossed product. Indeed, every partial group algebra is a partial crossed product (Theorems 3.4.3 and 3.4.6). From now on, our goal is to study $\mathfrak{A}[R]$ by this way.

First of all, we will find a concrete realisation of the spectrum of the relations $\mathcal{R}$ (Definition 3.4.4), which will be denoted by Ω. As in Section 2.1, consider the profinite completion $\hat{R}$ of R. A similar construction can be obtained extending the divisibility order in $R^{\times}$ to $K^{\times}$. For $w, w' \in K^{\times}$, we say that $w \leq w'$ if there exists $r \in R$ such that $w' = wr$. Denote by

(w) the fractional ideal generated by w, namely $(w) = wR \subseteq K$. As before, if $w \leq w'$, we can consider the canonical projection[2] $p_{w,w'} : (R + (w'))/(w') \longrightarrow (R + (w))/(w)$. Similarly to $\hat{R}$, we consider the inverse limit

$$\hat{R}_K = \varprojlim\{(R + (w))/(w),\ p_{w,w'}\} \cong$$

$$\left\{ (u_w + (w))_w \in \prod_{w \in K^\times} (R + (w))/(w) \;\middle|\; p_{w,w'}(u_{w'} + (w')) = u_w + (w),\ \text{if } w \leq w' \right\}.$$

It is a compact topological ring too. In fact, $\hat{R}_K$ is naturally isomorphic to $\hat{R}$ as a topological ring. We will show that Ω is homeomorphic to $\hat{R}_K$ (hence, homeomorphic to $\hat{R}$). We use $\hat{R}_K$ instead of $\hat{R}$ because it simplifies our proofs.

Define

$$\begin{aligned} \rho : \hat{R}_K &\longrightarrow \mathcal{P}(K \rtimes K^\times) \\ (u_w + (w))_w &\longmapsto \{(u_w + rw, w) \mid w \in K^\times,\ r \in R\}. \end{aligned}$$

Note that the definition is independent of the choice of u_w in $u_w + (w)$.

Claim 4.2.1. *$\rho(\hat{R}_K) \subseteq \Omega$.*

Proof. Let $(u_w + (w))_w \in \hat{R}_K$. By the definition of $\hat{R}_K$, if $w \leq w'$, then $u_{w'} = u_w + kw$ for some $k \in R$. Denote $\rho((u_w + (w))_w)$ by ξ. Clearly, $(0, 1) \in \xi$. By Definition 3.4.4, we need to show that $f(t^{-1}\xi) = 0$, for all $f \in \mathcal{R}$ and $t \in \xi$. Fix $t = (u_w + rw, w) \in \xi$. Let $f = 1_{(n,1)} - 1$ in $\mathcal{R}_1$ and note that $f(t^{-1}\xi) = 0$ is equivalent to $t(n, 1) \in \xi$. Since $t(n, 1) = (u_w + rw, w)(n, 1) = (u_w + (r+n)w, w)$, we have $t(n, 1) \in \xi$. Now, let $f = 1_{(0,1/m)} - 1$ in $\mathcal{R}_2$. Similarly, we must show that $t(0, 1/m) \in \xi$. Observe that $t(0, 1/m) = (u_w + rw, w)(0, 1/m) = (u_w + rw, w/m)$. Since $w/m \leq w$, then $t(0, 1/m) = (u_{w/m} + k(w/m) + rw, w/m) = (u_{w/m} + (k+rm)(w/m), w/m) \in \xi$. To conclude, fix $m \in R^\times$ and let $f = \sum_{l+(m)} 1_{(l,m)} - 1$ in $\mathcal{R}_3$. We must show that there exists one, and only one class $l + (m)$ such that $t(l, m) \in \xi$. Indeed, $t(l, m) = (u_w + rw, w)(l, m) = (u_w + (l + r)w, wm) = (u_{wm} + (l + r - k)w, wm)$ and, for it belongs to ξ, we must have $(l + r - k)w \in (wm)$. Hence, $l \equiv k - r \mod m$, in other words, there exists only one class $l + (m)$ such that $t(l, m) \in \xi$. Since $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, the proof is completed. □

Proposition 4.2.2. *$\rho : \hat{R}_K \longrightarrow \Omega$ is a homeomorphism.*

[2]By the second isomorphism theorem, it could be $p_{w,w'} : R/(R \cap (w')) \longrightarrow R/(R \cap (w))$.

Proof.

Injectivity. Let $(u_w+(w))_w, (v_w+(w))_w \in \hat{R}_K$ such that $\rho((u_w+(w))_w) = \rho(v_w+(w))_w)$. By the definition of ρ, the elements in $\rho((u_w+(w))_w)$ whose second component equals w are of the form $(u_w + rw, w)$. Since $(v_w, w) \in \rho((v_w + (w))_w)$ and, therefore, $(v_w, w) \in \rho((u_w + (w))_w)$, we must have $v_w = u_w + rw$ for some $r \in R$. This show that $(u_w + (w))_w = (v_w + (w))_w$.

Surjectivity. Let $\xi \in \Omega$. The relations in $\mathcal{R}_1$ and $\mathcal{R}_2$ together implies that if $t \in \xi$, then $t(q/p, 1/p) \in \xi$ for all $q \in R$ and $p \in R^\times$ (fix t and apply $f(t^{-1}\xi) = 0$ for various f). For each $m \in R^\times$, let $f = \sum_{l+(m)} 1_{(l,m)} - 1$ in $\mathcal{R}_3$ and apply $f(t^{-1}\xi) = 0$ with $t = (0,1)$ to see that there exists only one class $l + (m)$ such that $(l, m) \in \xi$. Denote this class by $u_m + (m)$. Since $t(0, 1/p) \in \xi$ if $t \in \xi$, then $p_{m,mp}(u_{mp} + (mp)) = (u_m + (m))$. From this, we can define unambiguously $u_w + (w) = u_m + (w)$ for $w = m/m' \in K^\times$. One can see that the classes $u_w + (w)$ are compatible with the projections $p_{w,w'}$ by using that $t(q/p, 1/p) \in \xi$ if $t \in \xi$. Hence, we have constructed $(u_w + (w))_w \in \hat{R}_K$. We claim that $\rho((u_w + (w))_w) = \xi$. Since $(u_w, w) \in \xi$, $(u_w, w)(q, 1) = (u_w+qw, w)$ must belongs to ξ. This shows that $\rho((u_w+(w))_w) \subseteq \xi$. Suppose, by contradiction, $\rho((u_w+(w))_w) \neq \xi$. Hence, there exists $s \in \xi$ such that $s \notin \rho((u_w+(w))_w)$. If we write $s = (n'/m', m/m')$, then $s \notin \rho((u_w + (w))_w)$ is equivalent to $n' - m'u_m \notin (m)$. Let $t = (u_m, 1/m')$, $s' = (u_m, m/m')$ and note that both belong to $\rho((u_w + (w))_w)$ (hence, belong to ξ). Since $t^{-1}s = (-m'u_m, m')(n'/m', m/m') = (n' - m'u_m, m)$, $t^{-1}s' = (0, m)$ and $n' - m'u_m \notin (m)$, then $f(t^{-1}\xi) \neq 0$ if $f = \sum_{l+(m)} 1_{(l,m)} - 1$, which contradicts the fact that $\xi \in \Omega$. Hence, $\rho((u_w + (w))_w) = \xi$.

To conclude the proof, observe that $\hat{R}_K$ and Ω are compact Hausdorff, therefore it suffices to show that ρ (or ρ^{-1}) is continuous to conclude that ρ is a homeomorphism. We will prove that ρ^{-1} is continuous by showing that $\pi_w \circ \rho^{-1}$ is continuous for all $w \in K^\times$, where $\pi_w : \hat{R}_K \longrightarrow (R + (w))/(w)$ is the canonical projection. Since $(R + (w))/(w)$ is discrete, it suffices to show that $\rho \circ \pi_w^{-1}(\{u_w + (w)\})$ is an open set of Ω, for all $u_w + (w) \in (R+(w))/(w)$. To see this, note that

$$\rho \circ \pi_w^{-1}(\{u_w + (w)\}) = \{\xi \in \Omega \mid (u_w, w) \in \xi\},$$

which is an open set of Ω (recall that the topology on Ω is induced by the product topology of $\{0,1\}^{K \rtimes K^\times}$). □

According Section 3.4, there exists a partial action of $K \rtimes K^\times$ on Ω. By the above proposition, we can define this partial action on $\hat{R}_K$. Let $\hat{R}_t = \rho^{-1}(\Omega_t)$, where $\Omega_t = \{\xi \in$

$\Omega \mid t \in \xi\}$, and θ_t be the homeomorphism between $\hat{R}_{t^{-1}}$ and $\hat{R}_t$. It's easy to see that

$$\hat{R}_{(u,w)} = \{(u_{w'} + (w'))_{w'} \in \hat{R}_K \mid u_w + (w) = u + (w)\}$$

and

$$\theta_{(u,w)}((u_{w'} + (w'))_{w'}) = (u + wu_{w'} + (ww'))_{ww'} = (u + wu_{w^{-1}w'} + (w'))_{w'},$$

i.e., $\theta_{(u,w)}$ acts on $\hat{R}_{(u,w)^{-1}}$ by the affine transformation corresponding to (u, w). The next proposition, whose proof is trivial, will be useful later.

Proposition 4.2.3. *We have that*

(i) $\hat{R}_{(u,w)} = \emptyset \iff u \notin R + (w)$;

(ii) $\hat{R}_{(u,w)} = \hat{R}_K \iff R \subseteq u + (w)$.

Now, we describe the topology on $\hat{R}_K$. For $w \in K^\times$ and $C_w \subseteq (R + (w))/(w)$, we define the open set

$$V_w^{C_w} = \{(u_{w'} + (w'))_{w'} \in \hat{R}_K \mid u_w + (w) \in C_w\}.$$

Clearly, if $w \leq w'$, then $V_w^{C_w} = V_{w'}^{C_{w'}}$, where $C_{w'} = \{u+(w') \in (R+(w'))/(w') \mid u+(w) \in C_w\}$. From the product topology, we know that the finite intersections of open sets $V_w^{C_w}$ form a basis for the topology on $\hat{R}_K$. By taking a common multiple of the w's in the intersection, we see that every basic open set is of the form $V_w^{C_w}$ (since $V_w^{C_1} \cap V_w^{C_2} = V_w^{C_1 \cap C_2}$). Furthermore, if $C_w \neq \emptyset$, r is a non-invertible element in R (it always exists) and $V_w^{C_w} = V_{wr}^{C_{wr}}$, then C_{wr} has, at least, two elements. Indeed, let $u + (w) \in C_w$ and $r_1, r_2 \in R$ such that $r_1 + (r) \neq r_2 + (r)$. It's easy to see that $u + wr_1 + (wr)$ and $u + wr_2 + (wr)$ are in C_{wr} and that $u + wr_1 + (wr) \neq u + wr_2 + (wr)$. This says that, if $V_w^{C_w}$ is non-empty, we can suppose that C_w has more than one element.

Proposition 4.2.4. *The partial action θ on $\hat{R}_K$ is topologically free (Definition 3.1.6).*

Proof. We need to show that $F_t = \{x \in \hat{R}_{t^{-1}} \mid \theta_t(x) = x\}$ has empty interior, for all $t \in K \rtimes K^\times \setminus \{(0, 1)\}$. We shall consider two cases: $t = (u, 1)$ and $t = (u, w)$, $w \neq 1$.

Case 1. If $u \notin R$, then Proposition 4.2.3 says that $\hat{R}_{t^{-1}} = \emptyset$. So, we can suppose $u \in R$. If $F_t \neq \emptyset$, then equation $\theta_t(x) = x$ implies that $u \in (m)$ for every $m \in R^\times$. Since R is not a field, then $u = 0$. This show that $F_t = \emptyset$ if $t = (u, 1)$ and $u \neq 0$.

Case 2. Let $t = (u, w)$ such that $w \neq 1$ and $u \in R + (w)$ (if $u \notin R + (w)$, then $\hat{R}_{t^{-1}} = \emptyset$).

Let V be a non-empty open set contained in $\hat{R}_{t^{-1}}$. We will show that there exists $x \in V$ such that $\theta_t(x) \neq x$. By shrinking V if necessary, we can suppose that $V = V_{w'}^{C_{w'}}$. Furthermore, we can assume that $C_{w'}$ has more than one element. Let $u_1 + (w')$ and $u_2 + (w')$ be distinct elements of $C_{w'}$ which, by definition, can be written such that u_1 and u_2 are in R. Therefore, $(u_1 + (w''))_{w''}$ and $(u_1 + (w''))_{w''}$ belong to $\hat{R}_K$ and, since $V = V_{w'}^{C_{w'}}$, belong to V. Note that $u_1 + (w')$ and $u_2 + (w')$ be distinct is equivalent to $u_1 - u_2 \notin (w')$. Suppose, by contradiction, $\theta_t(x) = x$ for all $x \in V$. Since $(u_i + (w''))_{w''} \in V$, $i = 1, 2$, then

$$\theta_{(u,w)}((u_i + (w''))_{w''}) = (u_i + (w''))_{w''} \implies (u + wu_i + (w''))_{w''} = (u_i + (w''))_{w''}.$$

By choosing $w'' = (w-1)w'$ (note that $w \neq 1$), we see that $u + (w-1)u_i \in ((w-1)w')$, for $i = 1, 2$. By subtracting the equations (for different i's), we have $(w-1)(u_1 - u_2) \in ((w-1)w')$ and, therefore $u_1 - u_2 \in (w')$; which is a contradiction! This show that F_t has empty interior. □

Proposition 4.2.5. *The partial action θ is minimal (Definition 3.1.9).*

Proof. We will prove that every $x \in \hat{R}_K$ has dense orbit (Proposition 3.1.10) by showing that if V is a non-empty open set, then there exists $t \in K \rtimes K^\times$ such that $x \in \hat{R}_{t^{-1}}$ and $\theta_t(x) \in V$. Let $x = (u_w + (w))_w \in \hat{R}_K$ and $V = V_{w'}^{C_{w'}}$ be non-empty. Take $u' + (w') \in C_{w'}$ and observe that we can suppose, without loss of generality, $u' \in R$ and $u_{w'} \in R$. Let $t = (u' - u_{w'}, 1)$. By Proposition 4.2.3, $\hat{R}_{t^{-1}} = \hat{R}_K$ and, hence, $x \in \hat{R}_{t^{-1}}$. To conclude, note that $\theta_t(x) = \theta_{(u'-u_{w'},1)}((u_w + (w))_w) = (u' - u_{w'} + u_w + (w))_w \in V$. □

Following, we summarize the results of this section.

Theorem 4.2.6. *The algebra $\mathfrak{A}[R]$ is $*$-isomorphic to the partial crossed product $C(\hat{R}_K) \rtimes_\alpha K \rtimes K^\times$, where α is the partial action induced by θ. The $*$-isomorphism is given by $u^n \longmapsto 1\delta_{(n,1)}$ and $s_m \longmapsto 1_{(0,m)}\delta_{(0,m)}$, where $1_{(0,m)}$ is the characteristic function of $\hat{R}_{(0,m)}$.*

The theorem above is a consequence of Theorems 3.4.6 and 4.1.5.

Theorem 4.2.7. *$\mathfrak{A}[R]$ is simple.*

Proof. Since $K \rtimes K^\times$ is amenable, then Proposition 3.2.4 is valid for the full crossed product. Therefore, by Propositions 4.2.4 and 4.2.5, we conclude that $C(\hat{R}_K) \rtimes_\alpha K \rtimes K^\times$ is simple. The result follows from the previous theorem. □

In Section 2.1 we see that there exists a surjective $*$-homomorphism $\mathfrak{A}[R] \longrightarrow \mathfrak{A}_r[R]$. By using that $\mathfrak{A}[R]$ is simple, we obtain the following consequence.

Corollary 4.2.8. $\mathfrak{A}[R] \cong \mathfrak{A}_r[R]$.

In [10], Cuntz defined two C^*-algebras: $\mathcal{Q}_{\mathbb{Z}}$ and $\mathcal{Q}_{\mathbb{N}}$. The algebra $\mathcal{Q}_{\mathbb{N}}$ is a C^*-subalgebra of $\mathcal{Q}_{\mathbb{Z}}$, which is nothing but $\mathfrak{A}[R]$ when $R = \mathbb{Z}$. In [31] and [7], Brownlowe, an Huef, Laca and Raeburn showed that $\mathcal{Q}_{\mathbb{N}}$ is a partial crossed product by using a boundary quotient of the Toeplitz (or Wiener-Hopf) algebra of the quasi-lattice ordered group $(\mathbb{Q} \rtimes \mathbb{Q}_+^\times, \mathbb{N} \rtimes \mathbb{N}^\times)$ (see [36] and [28] for Toeplitz algebras of quasi-lattice ordered groups). We observe that our techniques are different from theirs. We don't use Nica's construction [36] (indeed, our group $K \rtimes K^\times$ is not a quasi-lattice, in general). From our results, in the particular case $R = \mathbb{Z}$, we see that $\mathcal{Q}_{\mathbb{Z}}$ is a partial crossed product by the group $\mathbb{Q} \rtimes \mathbb{Q}^\times$. From this, it's immediate that $\mathcal{Q}_{\mathbb{N}}$ is a partial crossed product by $\mathbb{Q} \rtimes \mathbb{Q}_+^\times$ (as in [7]).

Chapter 5

Generalized Cuntz-Li Algebras

In Section 2.1, we introduced the Cuntz-Li algebras and, in Section 2.2, we exhibited the ring C^*-algebras, which are the generalization proposed by Li for that. In this chapter, we propose a new generalization for the Cuntz-Li algebras which, in our view, better encodes the multiplicative structure of the ring.

We begin with some algebraic preliminaries in the first section, where we develop basic properties about the annihilator of an ideal. Next, we present our generalization of the Cuntz-Li algebras.

5.1 Algebraic Preliminaries

In this section, we fix R a unital commutative ring.

Definition 5.1.1. The **annihilator** of an ideal I in R, denoted by $\mathrm{Ann}(I)$ or $I^\perp$, is defined to be the ideal $\{r \in R \mid ry = 0,\ \forall\, y \in I\}$.

Definition 5.1.2. We say that an ideal I is **non-degenerate** if $I \cap I^\perp = \{0\}$. We say that I is **essential** if $I^\perp = \{0\}$.

We show some elementary properties involving ideals and annihilators which will be useful later.

Proposition 5.1.3. *Let I and J be ideals of R. Then:*

(i) $II^\perp = \{0\}$;

(ii) $I^\perp$ *is the maximal ideal* K *such that* $IK = \{0\}$;

(iii) $(I \cap I^\perp)^2 = \{0\}$;

(iv) $IJ = I \cap J$ *if* $I + J = R$;

(v) $I \subseteq J \implies J^\perp \subseteq I^\perp$;

(vi) $I \subseteq J \implies I^{\perp\perp} \subseteq J^{\perp\perp}$;

(vii) $I \subseteq I^{\perp\perp}$;

(viii) $I^\perp = I^{\perp\perp\perp}$.

Proof.

(i) Trivial.

(ii) Let K be an ideal such that $IK = \{0\}$. Thus, for all $k \in K$, $ky = 0$ for all $y \in I$. It follows from definition of $I^\perp$ that $k \in I^\perp$.

(iii) It is a consequence of *(i)*.

(iv) $IJ \subseteq I \cap J = (I \cap J)R = (I \cap J)(I + J) \subseteq IJ + IJ = IJ$.

(v) It is clear from definition of annihilator.

(vi) Apply the previous item twice.

(vii) If $r \in I$ then, by definition of $I^\perp$, $ry = 0$ for all $y \in I^\perp$. It says that $r \in I^{\perp\perp}$.

(viii) The inclusion "$\subseteq$" follows from the previous item. On the other hand, let $r \in I^{\perp\perp\perp}$. We need to show that $ry = 0$ for all $y \in I$. But this is a consequence from definition of $I^{\perp\perp\perp}$ and from the fact that $I \subseteq I^{\perp\perp}$.

□

Let $m \in R$ and consider the linear map $p_m : R \longrightarrow R$ given by multiplication by m, i.e., $p_m(r) = mr$. If m is a zero divisor, then p_m is not injective. We look for a (good) ideal I of R such that $p_m : I \longrightarrow R$ is injective. The next two propositions give the right choice in case $(m)^\perp$ is non-degenerate.

Proposition 5.1.4. *If $p_m : I \longrightarrow R$ is injective, then $I \subseteq (m)^{\perp\perp}$.*

Proof. Let $r \in I$ and $y \in (m)^{\perp}$. We will show that $ry = 0$. By definition of $(m)^{\perp}$, $ym = 0$ and therefore, $rym = 0$. Since I is an ideal and $p_m : I \longrightarrow R$ is injective, then $rym = 0$ implies $ry = 0$. □

Proposition 5.1.5. *$p_m : (m)^{\perp\perp} \longrightarrow R$ is injective if, and only if, $(m)^{\perp}$ is non-degenerate, i.e., $(m)^{\perp} \cap (m)^{\perp\perp} = \{0\}$.*

Proof. ($\Rightarrow$) Let $r \in (m)^{\perp} \cap (m)^{\perp\perp}$. By definition of $(m)^{\perp}$, we have $rm = 0$. Since $r \in (m)^{\perp\perp}$ and $p_m : (m)^{\perp\perp} \longrightarrow R$ is injective, then $rm = 0$ implies $r = 0$.
($\Leftarrow$) Let $r \in (m)^{\perp\perp}$ such that $rm = 0$. Thus, $r \in (m)^{\perp}$ and, hence, $r \in (m)^{\perp} \cap (m)^{\perp\perp}$. It follows from the hypothesis that $r = 0$, i.e., $p_m : (m)^{\perp\perp} \longrightarrow R$ is injective. □

Our concern with the injectivity of p_m will become clear in next section. For now, let's see some sufficient conditions for p_m to be injective. First, we obtain conditions on each m and, afterwards, we derive conditions on the ring R such that p_m is injective for all m.

Proposition 5.1.6. *If any of the following situations occur, then $p_m : (m)^{\perp\perp} \longrightarrow R$ is injective.*

(i) *$(m)^{\perp}$ is non-degenerate;*

(ii) *$(m)^{\perp} + (m)^{\perp\perp} = R$;*

(iii) *$(m)^{\perp} \cap (m)^{\perp\perp}$ is idempotent.*

Proof. Item *(i)* has already been shown and *(ii)* and *(iii)* are consequence of *(i)*, *(iii)* and *(iv)* of the Proposition 5.1.3. □

Definition 5.1.7. We say that the ring R is **semiprime** if $\{0\}$ is the only nilpotent ideal of R.

Proposition 5.1.8. *The following are equivalent:*

(i) *Every non-zero ideal of R is non-degenerate;*

(ii) *Every non-zero ideal of R is either idempotent or non-degenerate;*

(iii) R *is semiprime;*

(iv) *For all ideal* I *of* R *such that* $I^2 = 0$*, we have* $I = 0$*;*

(v) R *has no nilpotent elements other than* 0*;*

(vi) *The nilradical of* R *is* $\{0\}$*.*

In this case, $p_m : (m)^{\perp\perp} \longrightarrow R$ *is injective for all* $m \in R$*.*

Proof. The equivalence among *(iv)*, *(v)* and *(vi)* is clear and, for the equivalence among *(i)*, *(ii)* and *(iii)*, see [15, Proposition 2.6] or [4, Proposição 2.2.17]. Furthermore, the implications $(iii) \Rightarrow (iv) \Rightarrow (i)$ are trivial. It follows from the previous proposition that $p_m : (m)^{\perp\perp} \longrightarrow R$ is injective for all $m \in R$. □

We finish this section with a proposition which will be used later.

Proposition 5.1.9. *Let* $m, m' \in R$ *and suppose* R *semiprime.*

(i) *If* $r \in (m')^{\perp\perp}$ *and* $m'r \in (m)^{\perp\perp}$*, then* $r \in (m)^{\perp\perp}$*;*

(ii) $(m)^{\perp\perp} \cap (m')^{\perp\perp} = (mm')^{\perp\perp}$*.*

Proof. Since R is semiprime, $p_m : (m)^{\perp\perp} \longrightarrow R$ and $p_{m'} : (m')^{\perp\perp} \longrightarrow R$ are injective.

(i) If $y \in (m)^{\perp}$, then

$$ym = 0 \quad \Longrightarrow \quad rymm' = 0 \quad \overset{rym' \in (m)^{\perp\perp}}{\Longrightarrow} \quad rym' = 0 \quad \overset{ry \in (m')^{\perp\perp}}{\Longrightarrow}$$
$$ry = 0 \quad \overset{y \in (m)^{\perp} \text{ arbitrary}}{\Longrightarrow} \quad r \in (m)^{\perp\perp}.$$

(ii) The inclusion "$\supseteq$" follows from Proposition 5.1.3*(vi)* since $(m) \supseteq (mm')$ and $(m') \supseteq (mm')$. Let $r \in (m)^{\perp\perp} \cap (m')^{\perp\perp}$ and $y \in (mm')^{\perp}$. Hence,

$$ymm' = 0 \quad \Longrightarrow \quad rymm' = 0 \quad \overset{rym \in (m')^{\perp\perp}}{\Longrightarrow} \quad rym = 0 \quad \overset{ry \in (m)^{\perp\perp}}{\Longrightarrow}$$
$$ry = 0 \quad \overset{y \in (mm')^{\perp} \text{ arbitrary}}{\Longrightarrow} \quad r \in (mm')^{\perp\perp}.$$

□

5.2 Definition of the Algebra

In this section, we introduce our generalization for the Cuntz-Li algebras of more general rings than those considered by Cuntz and Li in [12]. We extend the definition for unital commutative semiprime rings. Although our extension does not cover the entire category of the unital rings as done by Li in [33], we believe that our approach is more consistent in the cases covered by the two approaches.

Throughout this section, let R be a unital commutative *semiprime* ring. As before, consider the Hilbert space $\ell^2(R)$ and let $\{\xi_r \mid r \in R\}$ be its canonical basis. Again, consider the unitary operator U^n in $\mathcal{B}(\ell^2(R))$ given by $U^n(\xi_r) = \xi_{n+r}$. In the original Cuntz-Li algebras, the operators S_m are defined for each nonzero $m \in R$ and, in the extension of Li in [33], we have operators S_m if m is not a zero-divisor; here we will define an operator S_m for all $m \in R$. For $m \in R$, define the linear operator S_m on $\ell^2(R)$ by $S_m(\xi_r) = [r \in (m)^{\perp\perp}]\xi_{mr}$, where $[T]$ represents 1 if the sentence T is true and 0 if T is false. Since p_m is injective on $(m)^{\perp\perp}$ by Proposition 5.1.8, we obtain that S_m is bounded. We claim that $S_m^*(\xi_r) = [r \in m(m)^{\perp\perp}]\xi_{m^{-1}r}$, where $m^{-1}r$ denotes the unique element k in $(m)^{\perp\perp}$ such that $mk = r$.[1] Indeed,

$$\langle S_m(\xi_r), \xi_{r'}\rangle = [r \in (m)^{\perp\perp}][mr = r'] = [m^{-1}r' \in (m)^{\perp\perp}][m^{-1}r' = r] =$$

$$[r' \in m(m)^{\perp\perp}][m^{-1}r' = r] = \langle \xi_r, S_m^*(\xi_{r'})\rangle.$$

Furthermore, we have that S_m is a partial isometry since

$$S_m S_m^* S_m(\xi_r) = [r \in (m)^{\perp\perp}]S_m S_m^*(\xi_{mr}) = [r \in (m)^{\perp\perp}]S_m(\xi_r) = [r \in (m)^{\perp\perp}]\xi_{mr} = S_m(\xi_r).$$

So far, everything is working fine. But the crucial question is whether the operators S_m encode the multiplicative structure of the ring, i.e., whether $S_m S_{m'} = S_{mm'}$ is valid. The answer is affirmative and is shown below.

Claim 5.2.1. *For all $m, m' \in R$, $S_m S_{m'} = S_{m'} S_m = S_{mm'}$.*

Proof. Observe that $S_m S_{m'}(\xi_r) = [r \in (m')^{\perp\perp}]S_m(\xi_{m'r}) = [r \in (m')^{\perp\perp}][m'r \in (m)^{\perp\perp}]\xi_{mm'r}$. On the other hand, $S_{mm'}(\xi_r) = [r \in (mm')^{\perp\perp}]\xi_{mm'r}$. The result follows from both items of Proposition 5.1.9. □

[1] Note that the expression $m^{-1}r$ does not make sense when $r \notin m(m)^{\perp\perp}$. However, in this case, the boolean expression $[r \in m(m)^{\perp\perp}]$ has value 0. Thus, we adopt the convention that when the boolean value is 0, the rest of the expression is ignored.

Before introducing our definition for the Cuntz-Li algebra of R, we need to remember some basic facts. Given a Hilbert space H, we have in $\mathcal{B}(H)$ the ideal $\mathcal{K}(H)$ of the compact operators, which can be obtained from the closure in $\mathcal{B}(H)$ of the set of finite-rank operators. These facts can be found in [35, Section 2.4]. Furthermore, the lemma below will be useful.

Lemma 5.2.2. *Let H a Hilbert space with orthonormal basis $\{\xi_i\}_{i\in I}$. For each $i, j \in I$, consider the rank-one operator $\xi_i \otimes \xi_j$ on H given by $\xi_i \otimes \xi_j(\xi_k) = \langle \xi_k, \xi_j\rangle \xi_i = [k = j]\xi_i$. Then the C^*-algebra generated by the set $\{\xi_i \otimes \xi_j\}_{i,j\in I}$ is $\mathcal{K}(H)$.*

Proof. For each finite subset F of I let p_F the orthogonal projection onto the subspace of H generated by $\{\xi_i\}_{i\in F}$. Let $N \in \mathcal{K}(H)$ and consider the net $\{p_F N p_F\}_{F\subset I}$. Since $p_F = \sum_{i\in F} \xi_i \otimes \xi_i$ and $\xi_i \otimes \xi_i N \xi_j \otimes \xi_j = \langle N(\xi_j), \xi_i\rangle \xi_i \otimes \xi_j$, then $p_F N p_F$ is in $\operatorname{span}\{\xi_i \otimes \xi_j \mid i, j \in I\}$. The proof will be complete if we show that $\{p_F N p_F\}_{F\subset I}$ converges to N. Without loss of generality, we can suppose N self-adjoint and $||N|| \leq 1$. Denote by B the unit ball in H and fix $\epsilon > 0$. By compacity of N, we can choose $\chi_1, \ldots, \chi_n \in H$ such that, for all $\chi \in N(B)$, $||\chi - \chi_k|| < \epsilon^2/9$ for some k. Choose a finite subset F of I such that, for all k, $||\chi_k - p_F(\chi_k)|| < \epsilon^2/9$. We claim that, for all $T \in \mathcal{B}(H)$ such that $||T|| \leq 1$, $||(1 - p_F)NT|| \leq \epsilon^2/3$. Indeed, for $\xi \in B$ choose k such that $||NT(\xi) - \chi_k|| < \epsilon^2/9$ (such k exists because $T(\xi) \in B$) and observe that

$$||(1 - p_F)NT(\xi)|| \leq ||NT(\xi) - \chi_k|| + ||\chi_k - p_F(\chi_k)|| + ||p_F(\chi_k) - p_F NT(\xi)|| < \epsilon^2/3.$$

This shows that $||(1 - p_F)NT|| \leq \epsilon^2/3$. Finally, note that

$$||N - p_F N p_F||^2 = ||(N - p_F N p_F)(N - p_F N p_F)^*|| =$$

$$||N^2 - p_F N p_F N - N p_F N p_F + p_F N p_F N p_F|| \leq$$

$$||(1 - p_F)N^2|| + ||p_F N(1 - p_F)N|| + ||(1 - p_F)N p_F N p_F|| \leq \epsilon^2.$$

□

Consider the operator S_0. Since $(0)^{\perp\perp} = \{0\}$, then

$$U^n S_0 U^{-n'}(\xi_r) = U^n S_0(\xi_{r-n'}) = [r - n' = 0]U^n(\xi_0) = [r = n']\xi_n = \xi_n \otimes \xi_{n'}(\xi_r).$$

By the lemma above, the C^*-algebra in $\mathcal{B}(\ell^2(R))$ generated by the set $\{U^n S_0 U^{-n'}\}_{n,n'\in R}$ is $\mathcal{K}(\ell^2(R))$. Now, we are ready to define the Cuntz-Li algebra of R.

Definition 5.2.3. The **reduced Toeplitz-Cuntz-Li algebra of** R, denoted by $\mathcal{T}\mathfrak{A}'_r[R]$, is the C^*-subalgebra of $\mathcal{B}(\ell^2(R))$ generated by the operators $\{S_m \mid m \in R\}$ and $\{U^n \mid n \in R\}$. We define the **reduced Cuntz-Li algebra of** R to be the quotient $\mathcal{T}\mathfrak{A}'_r[R]/\mathcal{K}(\ell^2(R))$ and we denote it by $\mathfrak{A}'_r[R]$.

By the comments above, $\mathcal{K}(\ell^2(R))$ is contained in $\mathcal{T}\mathfrak{A}'_r[R]$. Thus, the quotient $\mathcal{T}\mathfrak{A}'_r[R]/\mathcal{K}(\ell^2(R))$ makes sense. Now, we will show that our definition actually extends that in 2.1.2.

Proposition 5.2.4. *Suppose that R is an integral domain with finite quotients which is not a field, as in Section 2.1. Then $\mathfrak{A}'_r[R]$ is $*$-isomorphic to $\mathfrak{A}_r[R]$.*

Proof. Since R is not a field, then $\mathrm{card}(R) = \infty$ (indeed, the elements a, a^2, a^3, $\ldots$, are different if a is nonzero and non-invertible). Thus, $1 \in \mathcal{B}(\ell^2(R))$ is not a compact operator and, hence, $\mathfrak{A}_r[R] \not\subseteq \mathcal{K}(\ell^2(R))$. By simplicity of $\mathfrak{A}_r[R]$ (Theorem 2.1.6), we must have $\mathfrak{A}_r[R] \cap \mathcal{K}(\ell^2(R)) = \{0\}$. Furthermore, we have $\mathcal{T}\mathfrak{A}'_r[R] = \mathfrak{A}_r[R] + \mathcal{K}(\ell^2(R))$ because the generators of $\mathcal{T}\mathfrak{A}'_r[R]$ are the generators of $\mathfrak{A}_r[R]$ together S_0 and any operator generated from S_0 is compact. Finally, by using the second isomorphism theorem,

$$\mathfrak{A}'_r[R] = \mathcal{T}\mathfrak{A}'_r[R]/\mathcal{K}(\ell^2(R)) = (\mathfrak{A}_r[R] + \mathcal{K}(\ell^2(R)))/\mathcal{K}(\ell^2(R)) \cong$$

$$\mathfrak{A}_r[R]/(\mathfrak{A}_r[R] \cap \mathcal{K}(\ell^2(R))) = \mathfrak{A}_r[R]/\{0\} \cong \mathfrak{A}_r[R].$$

□

The next proposition exhibits some properties of the operators S_m and U^n in $\mathcal{T}\mathfrak{A}'_r[R]$. Obviously, the equalities between operators are valid in $\mathfrak{A}'_r[R]$ too.

Proposition 5.2.5.

(i) *$S_m S_m^*$ is the projection onto $\overline{\mathrm{span}}\{\xi_r \mid r \in m(m)^{\perp\perp}\}$;*

(ii) *$U^n S_m S_m^* U^{-n}$ is the projection onto $\overline{\mathrm{span}}\{\xi_r \mid r \in n + m(m)^{\perp\perp}\}$;*

(iii) *$U^n S_m S_m^* U^{-n}$ and $U^{n'} S_m S_m^* U^{-n'}$ are equal if $n - n' \in m(m)^{\perp\perp}$ and orthogonal otherwise;*

(iv) *In the strong operator topology, we have $\sum U^l S_m S_m^* U^{-l} = S_m^* S_m$, where the sum is taken over all cosets $l + m(m)^{\perp\perp}$ in $(m)^{\perp\perp}/m(m)^{\perp\perp}$;*

(v) $S_m^* S_m$ *is the projection onto* $\overline{\operatorname{span}}\{\xi_r \mid r \in (m)^{\perp\perp}\}$*;*

(vi) $U^n S_m^* S_m U^{-n}$ *is the projection onto* $\overline{\operatorname{span}}\{\xi_r \mid r \in n + (m)^{\perp\perp}\}$*;*

(vii) $U^n S_m^* S_m U^{-n}$ *and* $U^{n'} S_m^* S_m U^{-n'}$ *are equal if* $n - n' \in (m)^{\perp\perp}$ *and orthogonal otherwise;*

(viii) *In the strong operator topology, we have* $\displaystyle\sum_{l+(m)^{\perp\perp} \in R/(m)^{\perp\perp}} U^l S_m^* S_m U^{-l} = 1$*;*

(ix) $S_m U^n = U^{mn} S_m$ *if* $n \in (m)^{\perp\perp}$*;*

(x) $S_m^* S_m S_{m'}^* S_{m'} = S_{mm'}^* S_{mm'}$.

Proof.

(i) $S_m S_m^*(\xi_r) = [r \in m(m)^{\perp\perp}] S_m(\xi_{m^{-1}r}) = [r \in m(m)^{\perp\perp}][m^{-1}r \in (m)^{\perp\perp}]\xi_r = [r \in m(m)^{\perp\perp}]\xi_r$.

(ii) $U^n S_m S_m^* U^{-n}(\xi_r) = U^n S_m S_m^*(\xi_{r-n}) = [r-n \in m(m)^{\perp\perp}] U^n(\xi_{r-n}) = [r-n \in m(m)^{\perp\perp}]\xi_r = [r \in n + m(m)^{\perp\perp}]\xi_r$.

(iii) It follows from *(ii)* and from the fact that $n + m(m)^{\perp\perp} = n' + m(m)^{\perp\perp}$ if $n - n' \in m(m)^{\perp\perp}$ and $(n + m(m)^{\perp\perp}) \cap (n' + m(m)^{\perp\perp}) = \emptyset$ otherwise.

(iv) Since $(m)^{\perp\perp}$ is the disjoint union of its cosets modulo $m(m)^{\perp\perp}$, the result follows from *(ii)* and *(iii)*.

(v) $S_m^* S_m(\xi_r) = [r \in (m)^{\perp\perp}] S_m^*(\xi_{mr}) = [r \in (m)^{\perp\perp}][mr \in m(m)^{\perp\perp}]\xi_r = [r \in (m)^{\perp\perp}]\xi_r$.

(vi), (vii), (viii) Similar to *(ii), (iii)* and *(iv)*.

(ix) Let $n \in (m)^{\perp\perp}$. Thus, $S_m U^n(\xi_r) = S_m(\xi_{r+n}) = [r + n \in (m)^{\perp\perp}](\xi_{mr+mn}) = [r \in (m)^{\perp\perp}](\xi_{mr+mn})$. On the other hand, $U^{mn} S_m(\xi_r) = [r \in (m)^{\perp\perp}] U^{mn}(\xi_{mr}) = [r \in (m)^{\perp\perp}](\xi_{mr+mn})$.

(x) By *(v)*, $S_m^* S_m S_{m'}^* S_{m'}$ is the projection onto $\overline{\operatorname{span}}\{\xi_r \mid r \in (m)^{\perp\perp} \cap (m')^{\perp\perp}\}$ and $S_{mm'}^* S_{mm'}$ is the projection onto $\overline{\operatorname{span}}\{\xi_r \mid r \in (mm')^{\perp\perp}\}$. The result follows from Proposition 5.1.9*(ii)*.

□

The relations *(iv)* and *(ix)* above generalize (CL3) and (CL4) in Definition 2.1.1 in a very satisfactory way. This together with Proposition 5.2.4 credits our definition as a good candidate for extension of the Cuntz-Li algebras.

We finish this chapter talking about the next steps to be taken in this project. First, we need to find the correct definition for the full version of this algebra. There are many new relations involving the generators (as seen in the above proposition); to find which of them should appear in the full version and to know whether the set of relations is complete probably will be a difficult task. Furthermore, we need to solve the problem of relations with infinite sums, as in *(iv)* and *(viii)*. The second step is to find a tool to study the algebra. It is unlikely that the theory of partial group algebras applies to this case. Indeed, the group $K \rtimes K^\times$ (see Chapter 4) does not make sense if R is not an integral domain. We conjecture that the theory of *tight* representations (see [19] and [20]) applies to this case. The last step is to extend the construction for noncommutative rings. Apparently, slight modifications in the ideals (considering left ideals and left annihilators) could solve the problem. To finish, we do not see a way to extend the definition for non-semiprime rings.

Chapter 6

Bost-Connes Algebra as Partial Crossed Product

In this chapter, we show that the Bost-Connes algebra $C_{\mathbb{Q}}$ (Definition 2.3.1) is $*$-isomorphic to a partial crossed product. In the first section, we present the partial action from which we construct the crossed product and we develop some properties which are used in the proofs. In the last section, we exhibit the $*$-isomorphism.

6.1 Preliminaries

In Chapter 4, for each integral domain R with finite quotients, we constructed a partial action θ of the group $K \rtimes K^{\times}$ on $\hat{R}_K$, where K is the field of fractions of R and $\hat{R}_K$ is (homeomorphic to) the profinite completion of R. When we take $R = \mathbb{Z}$, we obtain a partial action of $\mathbb{Q} \rtimes \mathbb{Q}^*$ on $\hat{\mathbb{Z}}_{\mathbb{Q}}$. There is a natural embedding of the multiplicative group $\mathbb{Q}^*_+$ in $\mathbb{Q} \rtimes \mathbb{Q}^*$ which sends w to $(0, w)$. If we restrict θ to the subgroup $\{0\} \rtimes \mathbb{Q}^*_+$ and if we identify it with $\mathbb{Q}^*_+$, then we get a partial action θ of $\mathbb{Q}^*_+$ on $\hat{\mathbb{Z}}_{\mathbb{Q}}$. From now on, fix such θ and the induced partial action α of $\mathbb{Q}^*_+$ on $C(\hat{\mathbb{Z}}_{\mathbb{Q}})$. At the end of this chapter, we show that $C_{\mathbb{Q}}$ is $*$-isomorphic to $C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_{\alpha} \mathbb{Q}^*_+$.

Let's analyse the action θ. Although θ is an action of $\mathbb{Q}^*_+$, we need to remember that an element of $w \in \mathbb{Q}^*_+$ acts as $(0, w)$. Hence, according to Chapter 4,

$$\hat{\mathbb{Z}}_{\mathbb{Q}} = \left\{ (u_w + (w))_w \in \prod_{w \in \mathbb{Q}^*} (\mathbb{Z} + (w))/(w) \;\middle|\; p_{w,w'}(u_{w'} + (w')) = u_w + (w), \text{ if } w \leq w' \right\},$$

$$\hat{\mathbb{Z}}_w = \{(u_{w'} + (w'))_{w'} \in \hat{\mathbb{Z}}_{\mathbb{Q}} \mid u_w + (w) = 0 + (w)\}$$

and

$$\begin{aligned} \theta_w : \hat{\mathbb{Z}}_{1/w} &\longrightarrow \hat{\mathbb{Z}}_w \\ (u_{w'} + (w'))_{w'} &\longmapsto (wu_{w^{-1}w'} + (w'))_{w'}. \end{aligned}$$

We will need some properties about α too. According to Sections 3.3 and 3.4, $C(\hat{\mathbb{Z}}_{\mathbb{Q}})$ is generated by $\{1_{(u,w)}\}_{(u,w)\in\mathbb{Q}\rtimes\mathbb{Q}^*}$, where $1_{(u,w)}$ is the characteristic function of the set $\hat{\mathbb{Z}}_{(u,w)} = \{(u_{w'} + (w'))_{w'} \in \hat{\mathbb{Z}}_{\mathbb{Q}} \mid u_w + (w) = u + (w)\}$. Furthermore, the ideals $C(\hat{\mathbb{Z}}_w) \cong 1_{(0,w)}C(\hat{\mathbb{Z}}_{\mathbb{Q}})$ are generated by $\{1_{(0,w)}1_{(u,w')}\}_{(u,w')\in\mathbb{Q}\rtimes\mathbb{Q}^*}$. The functions $1_{(u,w)}$ play an important role in the construction of the isomorphism. Let's see some of their properties.

Proposition 6.1.1. *Let $u, u' \in \mathbb{Q}$, $w \in \mathbb{Q}^*$, $n, n' \in \mathbb{Z}$ and $m, m' \in \mathbb{Z}^*$.*

(P1) $1_{(u,w)} = 1 \iff \mathbb{Z} \subseteq u + (w)$;

(P2) $1_{(u,w)} = 0 \iff u \notin \mathbb{Z} + (w)$;

(P3) $1_{(n,m)}1_{(n,mm')} = 1_{(n,mm')}$;

(P4) $1_{(n,m/m')} = 1_{(n,m)}$ *if* $(m, m') = 1$;

(P5) $1_{(u,w)} = 1_{(u',w)}$ *if* $u + (w) = u' + (w)$;

(P6) $1_{(u,w)}1_{(u',w)} = 0$ *if* $u + (w) \neq u' + (w)$;

(P7) $1_{(n,m)} = \sum_{l+(m')\in\mathbb{Z}/(m')} 1_{(n+lm,mm')}$;

(P8) If $1_{(u,w)} \neq 0$*, then there exists* $n \in \mathbb{Z}$ *and* $m \in \mathbb{Z}^*$ *such that* $1_{(u,w)} = 1_{(n,m)}$.

Proof. There are two ways to show these properties: we can use the definition of $\hat{\mathbb{Z}}_{\mathbb{Q}}$ or we can use the $*$-isomorphism $\mathfrak{A}[\mathbb{Z}] \cong C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_\alpha \mathbb{Q} \rtimes \mathbb{Q}^*$ and check them in $\mathfrak{A}[\mathbb{Z}]$. We have chosen the first one.

(P1), (P2) Follows from Proposition 4.2.3.

(P3) We need to show that $\hat{\mathbb{Z}}_{(n,mm')} \subseteq \hat{\mathbb{Z}}_{(n,m)}$. Indeed, if $(u_w + (w), (w))_w \in \hat{\mathbb{Z}}_{(n,mm')}$, then $u_{mm'} + (mm') = n + (mm')$. By using the definition of $\hat{\mathbb{Z}}_{\mathbb{Q}}$, we have $u_m + (m) = n + (m)$ and, hence, $(u_w + (w), (w))_w \in \hat{\mathbb{Z}}_{(n,m)}$.

(P4) If $(m, m') = 1$, then $\mathbb{Z} \cap \frac{m}{m'}\mathbb{Z} = m\mathbb{Z}$. Thus, $\dfrac{\mathbb{Z} + \frac{m}{m'}\mathbb{Z}}{\frac{m}{m'}\mathbb{Z}} \cong \dfrac{\mathbb{Z}}{\mathbb{Z} \cap \frac{m}{m'}\mathbb{Z}} = \dfrac{\mathbb{Z}}{m\mathbb{Z}} \cong \dfrac{\mathbb{Z} + m\mathbb{Z}}{m\mathbb{Z}}$, which says that $\hat{\mathbb{Z}}_{(n,m/m')} = \hat{\mathbb{Z}}_{(n,m)}$.

(P5), (P6) Trivial.

(P7) We need to show that the union $\bigcup_{l+(m') \in \mathbb{Z}/(m')} \hat{\mathbb{Z}}_{(n+lm,mm')}$ is disjoint and equal to $\hat{\mathbb{Z}}_{(n,m)}$. The previous item shows that the union is disjoint. It's clear that $\hat{\mathbb{Z}}_{(n+lm,mm')} \subseteq \hat{\mathbb{Z}}_{(n,m)}$. Conversely, if $(u_w + (w), (w))_w \in \hat{\mathbb{Z}}_{(n,m)}$, i.e., $u_m + (m) = n + (m)$, then we must have $u_{mm'} + (mm') = n + lm + (mm')$ for some l.

(P8) If $1_{(u,w)} \neq 0$, by item *(P2)* there exist $n, k \in \mathbb{Z}$ such that $u = n + kw$. By item *(P5)*, $1_{(u,w)} = 1_{(n,w)}$ and, writing $w = m/m'$ with $(m, m') = 1$, follows from item *(P4)* that $1_{(u,w)} = 1_{(n,m)}$.

□

Now, let's see elementary properties of the partial crossed product $C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_\alpha \mathbb{Q}_+^*$.

Proposition 6.1.2.

(P9) $(1_{(0,w)}\delta_w)^* = 1_{(0,1/w)}\delta_{1/w}$ *and, for* $f \in C(\hat{\mathbb{Z}}_{\mathbb{Q}})$, $(f\delta_1)^* = f^*\delta_1$;

(P10) $(1_{(0,w)}\delta_w)(1_{(0,w')}\delta_{w'}) = 1_{(0,w)}1_{(0,ww')}\delta_{ww'}$ *and, for* $f \in C(\hat{\mathbb{Z}}_{\mathbb{Q}})$ *and* $g \in C(\hat{\mathbb{Z}}_w)$, $(f\delta_1)(g\delta_w) = fg\delta_w$.

Proof. Both items follows from definitions in Section 3.2 and from Remark 3.1.3. □

We recall that the the Bost-Connes algebra $C_{\mathbb{Q}}$ is generated by isometries $\{\mu_m\}_{m \in \mathbb{N}^*}$ and unitaries $\{e_\gamma\}_{\gamma \in \mathbb{Q}/\mathbb{Z}}$. Here, we use $e(\gamma)$ instead of e_γ and for $\gamma = n/m + \mathbb{Z}$, we write simply $\gamma = n/m$. Below, we present some useful properties about $C_{\mathbb{Q}}$.

Proposition 6.1.3.

(P11) $\mu_m\mu_m^*\mu_{m'}\mu_{m'}^* = \mu_{m'}\mu_{m'}^*\mu_m\mu_m^*$;

(P12) $\mu_m\mu_m^*e_\gamma = e_\gamma\mu_m\mu_m^*$;

(P13) $\mu_m\mu_m^* = \dfrac{1}{m}\displaystyle\sum_{l+(m)\in\mathbb{Z}/(m)} e\left(\frac{lm'}{m}\right)$, *for all* $m' \in \mathbb{Z}^*$ *such that* $(m, m') = 1$. *In particular,* $\left(\dfrac{1}{m}\displaystyle\sum_{l+(m)\in\mathbb{Z}/(m)} e\left(\frac{lm'}{m}\right)\right)\mu_m = \mu_m$ *if* $(m, m') = 1$.

Proof. From (BC5) taking $\gamma = 0$, we see

$$\mu_m\mu_m^* = \frac{1}{m}\sum_{l+(m)\in\mathbb{Z}/(m)} e\left(\frac{l}{m}\right),$$

from which *(P11)* and *(P12)* follows. Since $lm' + (m)$ take all values in $\mathbb{Z}/(m)$ when $l + (m)$ varies in $\mathbb{Z}/(m)$ in case $(m, m') = 1$, then we have *(P13)*. □

At a certain stage, we will need a $*$-homomorphism whose domain is $C(\hat{\mathbb{Z}}_{\mathbb{Q}})$. Since it's a hard work to get it directly, we will exhibit a new look for $C(\hat{\mathbb{Z}}_{\mathbb{Q}})$. In [29, page 336], Laca and Raeburn showed that the dual $\widehat{\mathbb{Q}/\mathbb{Z}}$ of the group $\mathbb{Q}/\mathbb{Z}$ is homeomorphic to $\hat{\mathbb{Z}}_{\mathbb{Q}}$. Thus, from group C^*-algebras theory[1], the group C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z})$ is $*$-isomorphic to $C(\hat{\mathbb{Z}}_{\mathbb{Q}})$ through the Fourier transform. Since that $C^*(\mathbb{Q}/\mathbb{Z})$ is universal with respect to unitary representations of $\mathbb{Q}/\mathbb{Z}$, now we have a good way to construct $*$-homomorphisms from $C(\hat{\mathbb{Z}}_{\mathbb{Q}})$. We summarize it in proposition below. For $x \in \mathbb{C}$, we denote e^x by $\exp(x)$ since the letter e is overloaded.

Proposition 6.1.4. *There is a $*$-isomorphism $C^*(\mathbb{Q}/\mathbb{Z}) \longrightarrow C(\hat{\mathbb{Z}}_{\mathbb{Q}})$ given by*

$$i(\gamma) \longmapsto \sum_{l+(m)\in\mathbb{Z}/(m)} \exp\left(-\frac{ln}{m}\cdot 2\pi i\right) 1_{(l,m)},$$

where $i(\gamma)$ represents the unitary canonical image of γ in $C^*(\mathbb{Q}/\mathbb{Z})$. It's inverse is given by

$$1_{(n/m', m/m')} \longmapsto \frac{1}{m}\sum_{l+(m)\in\mathbb{Z}/(m)} \exp\left(\frac{nl}{m}\cdot 2\pi i\right) i\left(\frac{lm'}{m}\right).$$

To complete our list of properties, we present two elementary facts.

Proposition 6.1.5.

(P14) For $m, m' \in \mathbb{Z}^*$, the map

$$\begin{aligned} \mathbb{Z}/(m)\times\mathbb{Z}/(m') &\longrightarrow \mathbb{Z}/(mm') \\ (l+(m), l'+(m')) &\longmapsto l + l'm + (mm') \end{aligned}$$

is a bijection;

[1] See [8] for group C^*-algebras.

(P15) For $m \in \mathbb{Z}^*$ and $k \in \mathbb{Z}$,

$$\sum_{l+(m)\in\mathbb{Z}/(m)} \exp\left(\frac{kl}{m}\cdot 2\pi i\right) = \begin{cases} m, & \text{if } k \in (m), \\ 0, & \text{otherwise.} \end{cases}$$

Now, we are ready to begin the proof that $C_{\mathbb{Q}}$ and $C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_\alpha \mathbb{Q}_+^*$ are $*$-isomorphic.

6.2 The $*$-isomorphism between $C_{\mathbb{Q}}$ and $C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_\alpha \mathbb{Q}_+^*$

First, we will construct a $*$-homomorphism $\Phi : C_{\mathbb{Q}} \longrightarrow C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_\alpha \mathbb{Q}_+^*$. For this, we will find a representation of μ_m and $e(\gamma)$ in $C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_\alpha \mathbb{Q}_+^*$ that satisfies the relations (BC1)-(BC5) in definition 2.3.1 and we will use the universal property of $C_{\mathbb{Q}}$. For $m \in \mathbb{N}^*$, define $\Phi(\mu_m) = 1_{(0,m)}\delta_m$ and for $\gamma = \frac{n}{m} \in \mathbb{Q}/\mathbb{Z}$, set

$$\Phi(e(\gamma)) = \sum_{l+(m)\in\mathbb{Z}/(m)} \exp\left(-\frac{ln}{m}\cdot 2\pi i\right) 1_{(l,m)}\delta_1.$$

By Proposition 6.1.4, Φ is well-defined on $e(\gamma)$.

Proposition 6.2.1. *$\Phi(\mu_m)$ is an isometry, $\Phi(\gamma)$ is a unitary and Φ satisfies the relations (BC1)-(BC5) in Definition 2.3.1.*

Proof. Since

$$\Phi(\mu_m)^*\Phi(\mu_m) = (1_{(0,m)}\delta_m)^*(1_{(0,m)}\delta_m) \overset{(P9)}{=} (1_{(0,1/m)}\delta_{1/m})(1_{(0,m)}\delta_m) \overset{(P10)}{=}$$

$$1_{(0,1/m)}1_{(0,1/m)}\delta_1 = 1_{(0,1/m)}\delta_1 \overset{(P1)}{=} 1\delta_1,$$

we see that $\Phi(\mu_m)$ is an isometry. By Proposition 6.1.4, we obtain that $\Phi(\gamma)$ is a unitary and that (BC3) is satisfied. In Proposition 2.8 of [29], Laca and Raeburn showed that the relations (BC2) and (BC4) are unnecessary and, hence, it remains to show that (BC1) and (BC5) are satisfied. Since

$$\Phi(\mu_m)\Phi(\mu_{m'}) = (1_{(0,m)}\delta_m)(1_{(0,m')}\delta_{m'}) \overset{(P10)}{=} 1_{(0,m)}1_{(0,mm')}\delta_{mm'} \overset{(P3)}{=} 1_{(0,mm')}\delta_{mm'} = \Phi(\mu_{mm'}),$$

we have (BC1). Handling the left side of (BC5), we have

$$\Phi(\mu_{m'})\Phi\left(e\left(\frac{n}{m}\right)\right)\Phi(\mu_{m'})^* \overset{(P9)}{=}$$

$$(1_{(0,m')}\delta_{m'}) \left(\sum_{l+(m)\in\mathbb{Z}/(m)} \exp\left(-\frac{ln}{m}\cdot 2\pi i\right) 1_{(l,m)}\delta_1 \right) (1_{(0,1/m')}\delta_{1/m'}) \overset{(P1),(P3),(P5),(P10)}{=}$$

$$\sum_{l+(m)\in\mathbb{Z}/(m)} \exp\Big(-\frac{ln}{m}\cdot 2\pi i\Big)\, 1_{(lm',mm')}\delta_1.$$

In developing the right side below, the sets on which the sums are computed are understood. For example, a sum on $k+(m)$ means $k+(m)\in\mathbb{Z}/(m)$. Thus,

$$\frac{1}{m'}\sum_{m'\delta=n/m}\Phi(e(\delta)) = \frac{1}{m'}\sum_{k+(m')}\Phi\Big(e\Big(\frac{n+km}{mm'}\Big)\Big) =$$

$$\frac{1}{m'}\sum_{k+(m')}\sum_{k'+(mm')}\exp\Big(-\frac{k'(n+km)}{mm'}\cdot 2\pi i\Big)\, 1_{(k',mm')}\delta_1 \overset{(P14)}{=}$$

$$\frac{1}{m'}\sum_{k+(m')}\sum_{l+(m)}\sum_{l'+(m')}\exp\Big(-\frac{(l'+lm')(n+km)}{mm'}\cdot 2\pi i\Big)\, 1_{(l'+lm',mm')}\delta_1 =$$

$$\frac{1}{m'}\sum_{l+(m)}\exp\Big(-\frac{ln}{m}\cdot 2\pi i\Big)\sum_{l'+(m')}\exp\Big(-\frac{l'n}{mm'}\cdot 2\pi i\Big)\sum_{k+(m')}\exp\Big(-\frac{kl'}{m'}\cdot 2\pi i\Big)\, 1_{(l'+lm',mm')}\delta_1.$$

By *(P15)*, the sum on $k+(m')$ is nonzero except when $l'\in(m')$. In this case, taking $l'=pm'$, we have

$$\frac{1}{m'}\sum_{m'\delta=n/m}\Phi(e(\delta)) = \frac{1}{m'}\sum_{l+(m)}\exp\Big(-\frac{ln}{m}\cdot 2\pi i\Big)\exp\Big(-\frac{pn}{m}\cdot 2\pi i\Big)\, m' 1_{(pm'+lm',mm')}\delta_1 =$$

$$\sum_{l+(m)}\exp\Big(-\frac{(l+p)n}{m}\cdot 2\pi i\Big)\, 1_{((l+p)m',mm')}\delta_1 = \sum_{l+(m)\in\mathbb{Z}/(m)}\exp\Big(-\frac{ln}{m}\cdot 2\pi i\Big)\, 1_{(lm',mm')}\delta_1.$$

□

This proposition ensures the existence of the desired $*$-homomorphism $\Phi : C_{\mathbb{Q}} \longrightarrow C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_\alpha \mathbb{Q}_+^*$. Now, we will present an inverse for Φ. A natural way to construct a $*$-homomorphism whose domain is a partial crossed product is to use Proposition 3.2.2, i.e., is to find a covariant pair. In our case, we need a partial representation $\pi : \mathbb{Q}_+^* \longrightarrow C_{\mathbb{Q}}$ and a $*$-homomorphism $\varphi : C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \longrightarrow C_{\mathbb{Q}}$ such that (φ,π) is α-covariant (Definition 3.1.14).

Define $\pi : \mathbb{Q}_+^* \longrightarrow C_{\mathbb{Q}}$ by $\pi\big(\frac{m}{m'}\big) = \mu_{m'}^*\mu_m$. We claim that π is well-defined. Indeed $\pi\big(\frac{md}{m'd}\big) = \mu_{m'd}^*\mu_{md} \overset{(BC1)}{=} \mu_{m'}^*\mu_d^*\mu_d\mu_m = \mu_{m'}^*\mu_m = \pi\big(\frac{m}{m'}\big)$.

Proposition 6.2.2. *π is a partial representation.*

Proof.

(PR1) $\pi(1) = \mu_1^*\mu_1 = 1$.

(PR2) $\pi\Big(\big(\frac{m}{m'}\big)^{-1}\Big) = \pi\Big(\frac{m'}{m}\Big) = \mu_m^*\mu_{m'} = (\mu_{m'}^*\mu_m)^* = \pi\big(\frac{m}{m'}\big)^*$.

(PR3) Let $s = \frac{p}{p'}$ and $t = \frac{m}{m'}$. Thus,

$$\pi(st)\pi(t^{-1}) = \mu^*_{m'p'}\mu_{mp}\mu^*_m\mu_{m'} \overset{\text{(BC1)}}{=} \mu^*_{p'}\mu^*_{m'}\mu_p\mu_m\mu^*_m\mu_{m'} = \mu^*_{p'}\mu^*_{m'}\mu_p\mu_m\mu^*_m\mu_{m'}\mu^*_{m'}\mu_{m'}$$

$$\overset{(P11)}{=} \mu^*_{p'}\mu^*_{m'}\mu_p\mu_{m'}\mu^*_{m'}\mu_m\mu^*_m\mu_{m'} \overset{\text{(BC1)}}{=} \mu^*_{p'}\mu^*_{m'}\mu_{m'}\mu_p\mu^*_{m'}\mu_m\mu^*_m\mu_{m'} =$$

$$\mu^*_{p'}\mu_p\mu^*_{m'}\mu_m\mu^*_m\mu_{m'} = \pi(s)\pi(t)\pi(t^{-1}).$$

□

Now, our goal is to find a $*$-homomorphism from $C(\hat{\mathbb{Z}}_{\mathbb{Q}})$ to $C_{\mathbb{Q}}$. Since that the natural map $\mathbb{Q}/\mathbb{Z} \ni \gamma \longmapsto e(\gamma) \in C_{\mathbb{Q}}$ is obviously a unitary representation of $\mathbb{Q}/\mathbb{Z}$, there is a $*$-homomorphism from $C^*(\mathbb{Q}/\mathbb{Z})$ to $C_{\mathbb{Q}}$ which sends $i(\gamma)$ to $e(\gamma)$. By Proposition 6.1.4, there is a $*$-homomorphism $\varphi : C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \longrightarrow C_{\mathbb{Q}}$ such that

$$\varphi(1_{(n/m',m/m')}) = \frac{1}{m}\sum_{l+(m)\in\mathbb{Z}/(m)} \exp\left(\frac{nl}{m}\cdot 2\pi i\right) e\left(\frac{lm'}{m}\right).$$

Proposition 6.2.3. *The pair (φ, π) is α-covariant.*

Proof. Let $t = \frac{m}{m'} \in \mathbb{Q}^*_+$. Without loss of generality, we can assume $(m, m') = 1$. Thus, $\pi(t)\pi(t^{-1}) = \mu^*_{m'}\mu_m\mu^*_m\mu_{m'} \overset{\text{(BC2)}}{=} \mu_m\mu^*_{m'}\mu_{m'}\mu^*_m = \mu_m\mu^*_m$. Hence, follows from *(P12)* that (COV2) is satisfied. By *(P8)* and since the set $\{1_{(0,m'/m)}1_{(u,w)}\}_{(u,w)\in\mathbb{Q}\rtimes\mathbb{Q}^*}$ generates the ideal $C(\hat{Z}_{t^{-1}})$ as seen in section 6.1, it suffices to prove (COV1) with $x = 1_{(0,m'/m)}1_{(n,m'')}$, where $n \in \mathbb{Z}$ and $m'' \in \mathbb{Z}^*$. Whereas $\alpha_t(1_{(0,m'/m)}1_{(n,m'')}) = 1_{(mn/m',mm''/m')}1_{(0,m/m')}$ by Remark 3.1.3, then

$$\varphi(\alpha_t(x)) = \varphi(1_{(mn/m',mm''/m')}1_{(0,m/m')}) =$$

$$\left(\frac{1}{mm''}\sum_{l+(mm'')} \exp\left(\frac{nl}{m''}\cdot 2\pi i\right) e\left(\frac{lm'}{mm''}\right)\right)\left(\frac{1}{m}\sum_{l+(m)} e\left(\frac{lm'}{m}\right)\right) \overset{(P13)}{=}$$

$$\left(\frac{1}{mm''}\sum_{l+(mm'')} \exp\left(\frac{nl}{m''}\cdot 2\pi i\right) e\left(\frac{lm'}{mm''}\right)\right)\mu_m\mu^*_m \overset{\text{(BC4)}}{=}$$

$$\mu_m\left(\frac{1}{mm''}\sum_{l+(mm'')} \exp\left(\frac{nl}{m''}\cdot 2\pi i\right) e\left(\frac{lm'}{m''}\right)\right)\mu^*_m \overset{(P14)}{=}$$

$$\mu_m\left(\frac{1}{mm''}\sum_{l'+(m)}\sum_{l''+(m'')} \exp\left(\frac{n(l''+l'm'')}{m''}\cdot 2\pi i\right) e\left(\frac{(l''+l'm'')m'}{m''}\right)\right)\mu^*_m =$$

$$\mu_m\left(\frac{1}{mm''}\sum_{l'+(m)}\sum_{l''+(m'')} \exp\left(\frac{nl''}{m''}\cdot 2\pi i\right) e\left(\frac{l''m'}{m''}\right)\right)\mu^*_m =$$

$$\mu_m \left(\frac{1}{m''} \sum_{l''+(m'')} \exp\left(\frac{nl''}{m''} \cdot 2\pi i \right) e\left(\frac{l''m'}{m''} \right) \right) \mu_m^*.$$

On the other hand,

$$\pi(t)\varphi(x)\pi(t^{-1}) \overset{\text{(BC2)}}{=} \mu_m\mu_{m'}^* \left(\frac{1}{m''} \sum_{l+(m'')} \exp\left(\frac{nl}{m''} \cdot 2\pi i \right) e\left(\frac{l}{m''} \right) \right) \left(\frac{1}{m'} \sum_{l'+(m')} e\left(\frac{l'm}{m'} \right) \right) \mu_{m'}\mu_m^*$$

$$\overset{(P13)}{=} \mu_m\mu_{m'}^* \left(\frac{1}{m''} \sum_{l+(m'')} \exp\left(\frac{nl}{m''} \cdot 2\pi i \right) e\left(\frac{l}{m''} \right) \right) \mu_{m'}\mu_m^* \overset{\text{(BC4)}}{=}$$

$$\mu_m \left(\frac{1}{m''} \sum_{l+(m'')} \exp\left(\frac{nl}{m''} \cdot 2\pi i \right) e\left(\frac{lm'}{m''} \right) \right) \mu_m^*,$$

which shows (COV1). Hence, (φ, π) is α-covariant. □

By Proposition 3.2.2, there exists a $*$-homomorphism $\varphi \times \pi : C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_\alpha \mathbb{Q}_+^* \longrightarrow C_{\mathbb{Q}}$ such that $\varphi \times \pi(x\delta_t) = \varphi(x)\pi(t)$. The next theorem is the main goal of this chapter.

Theorem 6.2.4. *The $*$-homomorphisms Φ and $\varphi \times \pi$ are inverses of each other. In particular, $C_{\mathbb{Q}} \cong C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_\alpha \mathbb{Q}_+^*$.*

Proof. It's enough to verify that $(\varphi \times \pi) \circ \Phi = \mathrm{Id}_{C_{\mathbb{Q}}}$ and $\Phi \circ (\varphi \times \pi) = \mathrm{Id}_{C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_\alpha \mathbb{Q}_+^*}$ on the generators. By Proposition 6.1.4, we have $(\varphi \times \pi) \circ \Phi(e(\gamma)) = e(\gamma)$ and since

$$(\varphi \times \pi) \circ \Phi(\mu_m) = \varphi \times \pi(1_{(0,m)}\delta_m) = \varphi(1_{(0,m)})\pi(m) = \left(\frac{1}{m} \sum_{l+(m) \in \mathbb{Z}/(m)} e\left(\frac{l}{m} \right) \right) \mu_m \overset{(P13)}{=} \mu_m,$$

one side is complete. On the other hand, it suffices to show that $\Phi \circ (\varphi \times \pi) = \mathrm{Id}_{C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_\alpha \mathbb{Q}_+^*}$ on $1_s 1_t \delta_t$, where $t \in \mathbb{Q}_+^*$ and $s \in \mathbb{Q} \rtimes \mathbb{Q}^*$. Let $t = m/m'$ and $s = (n, m'')$, where $n \in \mathbb{Z}$ and $m, m', m'' \in \mathbb{Z}^*$ with $(m, m') = 1$ (we can choose such s because of *(P8)*). Thus,

$$\Phi \circ (\varphi \times \pi)(1_s 1_t \delta_t) = \Phi(\varphi(1_s)\varphi(1_t)\pi(t)) =$$

$$\Phi\left(\left(\frac{1}{m''} \sum_{l+(m'')} \exp\left(\frac{nl}{m''} \cdot 2\pi i \right) e\left(\frac{l}{m''} \right) \right) \left(\frac{1}{m} \sum_{l+(m)} e\left(\frac{lm'}{m} \right) \right) \mu_{m'}^* \mu_m \right) \overset{(P13),(BC2)}{=}$$

$$\Phi\left(\left(\frac{1}{m''} \sum_{l+(m'')} \exp\left(\frac{nl}{m''} \cdot 2\pi i \right) e\left(\frac{l}{m''} \right) \right) \mu_m \mu_{m'}^* \right) =$$

$$\left(\frac{1}{m''} \sum_{l+(m'')} \exp\left(\frac{nl}{m''} \cdot 2\pi i \right) \sum_{l'+(m'')} \exp\left(-\frac{l'l}{m''} \cdot 2\pi i \right) 1_{(l',m'')}\delta_1 \right) 1_{(0,m)}\delta_m 1_{(0,1/m')}\delta_{1/m'} \overset{(P4),(P10)}{=}$$

$$\left(\frac{1}{m''} \sum_{l'+(m'')} \sum_{l+(m'')} \exp\left(\frac{(n-l')l}{m''} \cdot 2\pi i \right) 1_{(l',m'')} \right) 1_{(0,m/m')} \delta_{m/m'}.$$

As before, by *(P15)* we must have $n - l' \in (m'')$. Taking $l' = n + km''$, we have

$$\Phi \circ (\varphi \times \pi)(1_s 1_t \delta_t) = \left(\frac{1}{m''} \sum_{l'+(m'')} \sum_{l+(m'')} \exp\left(\frac{(n-l')l}{m''} \cdot 2\pi i \right) 1_{(l',m'')} \right) 1_{(0,m/m')} \delta_{m/m'} =$$

$$1_{(n+km'',m'')} 1_{(0,m/m')} \delta_{m/m'} \overset{(P5)}{=} 1_{(n,m'')} 1_{(0,m/m')} \delta_{m/m'} = 1_s 1_t \delta_t.$$

□

A continuation of this project involves to prove that the generalized Bost-Connes algebras (see [2]) are partial crossed products too. Furthermore, we hope the available tools in the partial crossed products theory can recover, in a natural way, the connections between these algebras and the Number Theory. To conclude, the procedure presented in this chapter gives rise to many new algebras. Indeed, we obtain $C_{\mathbb{Q}}$ by restricting the group $\mathbb{Q} \rtimes \mathbb{Q}^*$ to $\mathbb{Q}^*_+$ in the partial crossed product $C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_\alpha \mathbb{Q} \rtimes \mathbb{Q}^* \cong \mathfrak{A}[\mathbb{Z}]$. If we replace $\mathbb{Z}$ by an integral domain (as in Chapter 4) and $\mathbb{Q}^*_+$ for an arbitrary subgroup of $K \rtimes K^\times$, we obtain new algebras, which may be interesting to study.

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