C^2-Iterated Function Systems on the Circle and Symbolic Blender-like

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Contents

1 Introduction

In the 60s and 70s a good statistical and topological description was given of hyperbolic dynamics, giving examples of structurally stable chaotic systems. The defintion of hyperbolicity involves contracting and expanding directions in the tangent space. Relaxing the definition gives rise to partial hyperbolicity and systems with dominated decomposition. Conversely, robust dynamical phenomena lead to conditions on the tangent space, aumenting the importance of studying such systems [12]. However, many questions remain in the open regarding how do the known theorems for hyperbolic dynamics adapt to the case of partial hyperbolicity. For example, hyperbolic dynamical systems can be broken down into a finite number of dynamically indecomposable sets (spectral decomposition), but it is uknown if this holds at least in a dense subset of partially hyperbolic systems.

Probably the simplest way to create a partially hyperbolic system out of a hyperbolic one is to form a direct product of a hyperbolic one with the identity map. That is a map of the form $F \times Id_N$ on the manifold $M \times N$ where F has a hyperbolic set in M. It is a natural question to ask what are the dynamics for diffeomorphisms nearby. This question was adressed in many works from the topological and ergodic perspectives, see for example [6], [1], [5], [14], [11].

In [1] it was shown that nearby $F \times Id_N$ there exist robustly transitive non-hyperbolic diffeomorphisms. For the proof the notion of a blender was used. Blenders are hyperbolic sets with the extra property that a projection of the stable set of the blender has a larger topological dimension then that of the stable sub-bundle. Blenders appear in a variety of settings such as in producing robust heterodimensional cycles [2], robustly transitivite sets in symplectic dynamics [9] and as criteria for stable ergodicity [13].

The main property of the blender can be related to the study of iterated function systems [3]. An iterated function system (IFS) with respect to a set of diffeomorphisms on a manifold is the set of all their finite forward compositions. Important notions in dynamics like minimality, transitivity, and spectral decomposition can be extended to IFS (see the next section).

By the theorems of [6], diffeomorphisms nearby $F \times Id_N$ are conjugated to skew-products of the form $(x, y) \to (F(x), G_x(y))$ where F is again the hyperbolic map. As dynamics on hyperbolic sets are conjugated to a shift map on a symbolic space, it makes sense to work with symbolic skew-products of the form $(\theta, y) \to (\tau(\theta), G_\theta(y))$ acting on the space $\Lambda \times N$ where Λ is a symbolic space of a finite number of symbols and N is the manifold.

The symbolic skew-products with the maps G_θ contractions were explored in [9] where symbolic blenders were defined and were used to create robustly transitive sets in the Hamiltonian setting. The interplay between dynamics of iterated function sytems and the symbolic skew-products was an important tool. Symbolic blenders become minimal sets with non-empty interior in the language of IFS.

The work of [5] studied iterated function systems of diffeomorphisms on the circle, and there was given an example of a robustly minimal IFS in the topology of IFS. The robust properties in the IFS topology were then translated to robust properties in the topology of symbolic skew-products. But for this, the space of symbolic skew-products required the additional assumption that the fibers $G_\theta(y)$ have α-Holder dependence with respect to the sequence. That is

$$d(G_\theta, G_\sigma) < C \cdot d(\theta, \sigma)^\alpha.$$

The constant α had a relationship with how close the fiber maps G_θ are to the identity. This Holder constraint does not create a problem in going from symbolic skew-products back to diffeomorphisms on an actual manifold. Diffeomorphisms nearby $F \times Id_N$ are actually conjugated to skew-products with Holder dependence on the fibers [4].

The above discussion motivates the study of dynamics of IFS on its own right, the connections with symbolic skew-products and partially hyperbolic sets. In this work firstly are studied iterated function systems on the circle. We searched for minimal sets with non-empty interior in the IFS context, as this will translate to sets with the blender property for the symbolic skew-products.

Next will be given a brief description of the results. Consider an IFS given by a generic pair of diffeomorphisms on the circle. The extra assumption is that the maps are close to the identity in the C^2 topology. With respect to this pair the following is proven.

- Existence of minimal sets with non-empty interior (see theorems 2.1 and 2.4).
- Simple criteria based on the combinatorics of periodic points under which the minimal set is the whole circle, thus giving new examples of robustly minimal IFS on the circle (theorem 2.6).
- When the pair has hyperbolic fixed points we give a complete description of the dynamics, in particular spectral decomposition and specification of minimal/transitive sets (theorem 2.8).

It is interesting to note that minimal Cantor sets for these maps do not exist. This mimics the classical Denjoy theory for the dynamics of a single diffeomorphism on the circle where the Cantor sets are excluded in the C^2 topology.

In the context of symbolic skew-products we

- Define symbolic blender-like sets which are sets carrying the key topological property of the symbolic blender but are not necessarily hyperbolic (definition 2.9).
- Give sufficient conditions on IFS for existence of symbolic blender-like in the topology of Holder skew-products (theorem 2.10).
- Prove that these sufficient conditions are satisfied for the generic pair of IFS on the circle, C^2 close to the identity. This shows the abundace of symbolic blender-like sets (theorem 2.11 and corollary 2.12).

The next section states the definitions and the main theorems. The following sections deal separetely with the proofs of each of the theorems.

This work was done in collaboration with Pablo Barrientos under the orientation of Enrique Pujals.

2 Definitions and Main Results

Let $f_1, \dots, f_n$ be maps of a manifold M, possibly with boundary. We will be mainly concerned when M is the closed interval or the circle $\mathbb{S}^1$. In particular denote by $Diff_+^r(\mathbb{S}^1)$ the set of orientation-preserving diffeomorphisms of the circle.

An iterated function system $< f_1 \dots f_n >$ is the set of all finite forward compositions of the maps f_i. That is

$$< f_1 \dots f_n >= \{h; h = f_{j_k}^{l_k} \circ \dots \circ f_{j_1}^{l_1}, j_i \in \{1, \dots, n\}\}.$$

An orbit of a point x is

$$Orb(x) = \{h(x); h \in < f_1 \dots f_n >\}.$$

The next set of definitions generalizes for IFS the usual notions of dynamical systems.

A set Λ is **minimal** for $< f_1, \dots, f_n >$ if for all $x \in \Lambda$, $\Lambda \subset \overline{Orb(x)}$. This is equivalent to saying that for all $x \in \Lambda$ and open set $U \subset \Lambda$, there exists h in $< f_1 \dots f_n >$ with $h(x) \in U$.

A set Λ with the induced topology is called **transitive** if for any two open sets U, V in Λ, there exists h in $< f_1 \dots f_n >$ with $h(U) \cap V \neq \emptyset$. A transitive set Λ is **maximal** if for any transitive set $\Xi \not\subseteq \Lambda$, $\Lambda \cup \Xi$ is not transitive.

Observe that the notions of minimality and transitivity as defined above do not require the set to be closed or invariant under the IFS. It can happen that $\Lambda \not\subseteq \cup_{i=1}^n f_i(\Lambda)$, see for example the K^{su} set defined below. This is one of the difficulties, when the manifold is the circle, in applying methods from the well-developed theory of group actions of diffeomorphisms on $\mathbb{S}^1$ [10], since they depend on the invariance of the minimal sets.

Observe that if Λ is minimal for $< f_1, \dots, f_n >$, then it is transitive for $< f_1^{-1}, \dots, f_n^{-1} >$. The $< f_1 \dots f_n >$ is called minimal when the whole circle or interval is minimal. It is called **robustly minimal** if there exists neighborhoods U_j of f_j in the relevant topology such that for all $\phi_j \in U_j$, $< \phi_1 \dots \phi_n >$ is minimal.

Minimal sets of uniformly contracting iterated function systems were studied in [7] and [9]. Examples of robustly minimal IFS of diffeomorphisms on the circle are given in [5]. A similar question on surfaces for volume-preserving IFS was adressed in [8].

Given a possibly infinite sequence $\theta = \{\theta_j\} \in \{1, \dots, n\}^{\mathbb{N}}$ we will use the notation

$$f_\theta^j(x) = f_{\theta_{j-1}} \circ \dots \circ f_{\theta_0}(x).$$

The forward or **ω-limit** of a point x with respect to a sequence θ is defined as

$$\omega_\theta(x) = \{y \mid \text{there exists } j_k \text{ such that } f_\theta^{j_k}(x) \to y\}.$$

The forward limit of $< f_1 \dots f_n >$ is defined by

$$\omega = \omega(< f_1 \dots f_n >) = \overline{\bigcup_{\theta, x} \omega_\theta(x)}.$$

Similarly we define the backward or α-**limit** of $< f_1 \dots f_n >$ as

$$\alpha = \alpha(< f_1 \dots f_n >) = \omega(< f_1^{-1} \dots f_n^{-1} >).$$

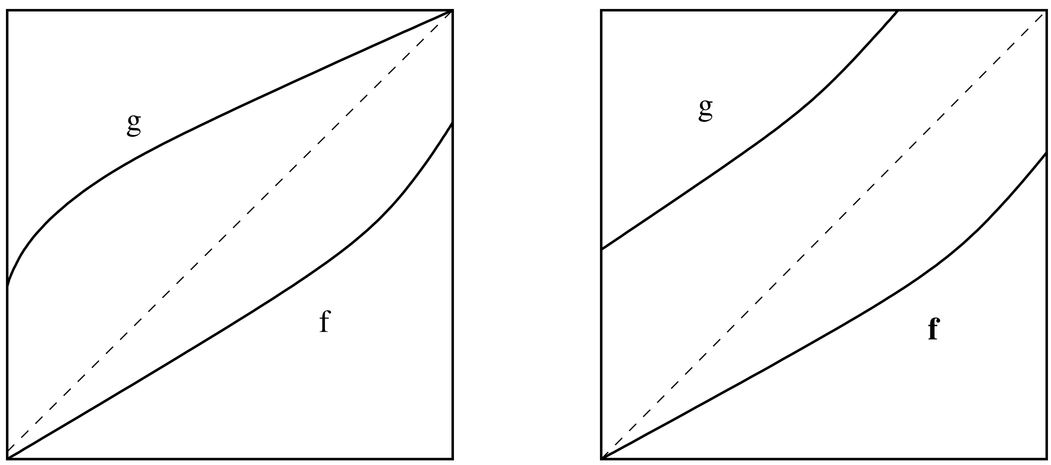
On a compact manifold the system $< f_1 \dots f_n >$ has **spectral decomposition** if the limit set, $L = \alpha \cup \omega$, can be written as a finite union of maximal transitive sets.

A diffeomorphism f on the circle is called Morse-Smale if the set of periodic points is non-empty and all the periodic points are hyperbolic. Morse-Smale maps form an open and dense set in $Diff^r(\mathbb{S}^1)$.

From the motivation in the introduction our main objective is to search for minimal sets with non-empty interior in the simplest setting under iteration of just two maps on the circle. These sets will play the role similar to that of symoblic blenders for the skew-products.

First lets describe geometrically the sets that will be proven to be minimal, have non-empty interior and will make up the pieces of the spectral decomposition.

Let $f, g \in Diff^2(\mathbb{S}^1)$ and we will define the following sets of type K^{**}, where $** = ss, su$, or uu. Apriori lets suppose that there are no periodic points of f or g in the interior of K^{**} sets. For now the periodic points do not have to be hyperbolic and so they can attract from one side and repel from the other (semi-attractor, semi-repeller). For simplicity assume that p is a fixed point of f, q is a fixed point of g.



$\mathbf{K^{ss}}$ $\mathbf{K^{su}}$

- A K^{ss} type set is the interval $[p, q]$ where p is a (semi) attractor for f and q is in the basin of attraction of p. Similarly q is a (semi) attractor for g and p is in the basin of attraction of q.

- A K^{uu} type set is the interval $[p, q]$ where where p is a (semi) repeller for f and q is in the basin of repulsion of p. Similarly q is a (semi) repeller for g and p is in the basin of repulsion of q.

- The K^{su} type is the semi-open interval $[p,q)$ where p is a (semi) attractor (resp. (semi) repeller) for the map f, q is a (semi) repeller (resp. (semi) attractor) for the same map, and $g((p,q)) \cap (p,q) \neq \emptyset$.

If the maps g, f are periodic with periods m, n, the K^{**} sets are defined as above with respect to the maps f^m, g^n.

The first theorem that will be proved gives sufficient conditions under which K^{**} sets are minimal.

Theorem 2.1. *There exists an $\epsilon > 0$ ($\epsilon \geq 0.14$) such that if f and g are ϵ-close to the identity in the C^2 topology, then K^{ss} and K^{su} sets are minimal for $< f, g >$ and K^{uu} is minimal for $< f^{-1}, g^{-1} >$. Moreover $K^{**} \subset \overline{(Per(< f, g >)}$.*

This theorem actually forms part of the proof of a theorem of Duminy from the 70s which has to do with the dynamics of groups of diffeomorphisms on the circle [10]. The orbit of a point under the action of a group is related to the study of co-dimension one foliations. For group actions the inverses of the functions can enter in the compositions. For example compositions like $f_1^j \circ f_0^{-k} \circ f_1^l$ are possible whereas for IFS (or semi-group actions) no.

Therefore, it is natural to expect a stronger result for group actions, which for our purposes can be written in the following manner.

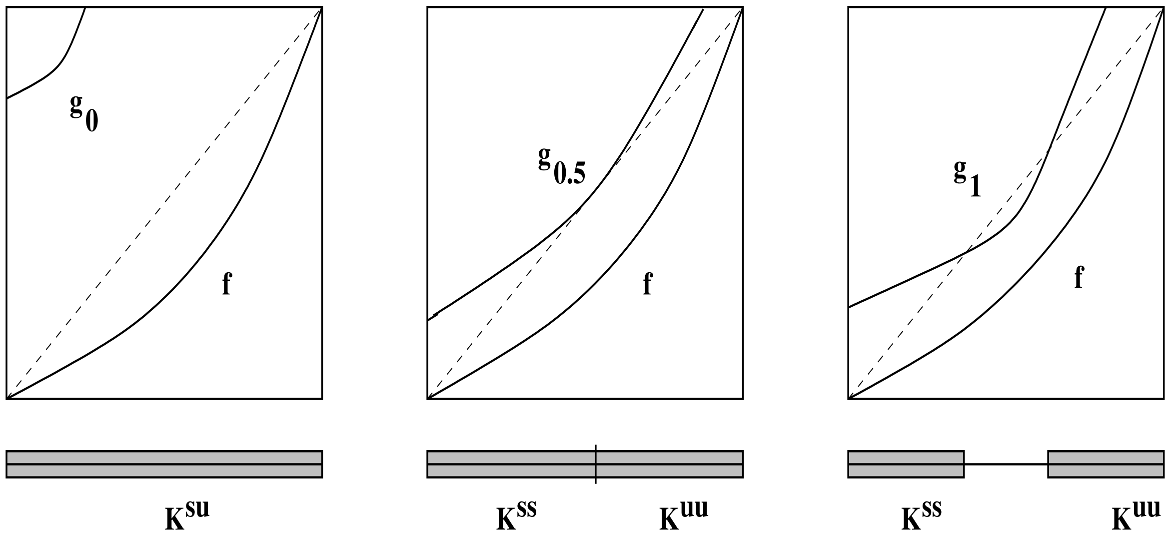
Theorem 2.2. ***(Duminy)*** *There exists an $\epsilon > 0$ such that if f and g are ϵ-close to the identity in the C^2 topology, and one of the maps has a finite number of periodic points, then or S^1 is minimal for the group action or there is a finite orbit.*

We will not prove Duminys theorem but to go from theorem 2.1 to the second theorem 2.2 basically one can create the geometry of a K^{**} set by making the necessary compositions using the inverses of the functions. Then appling theorem 2.1 gives minimality of K^{**}. For group actions minimality of an interval actually implies minimality of the whole circle [10].

The proof of theorem 2.1 that we will give is similar to the original proof of Duminy in the main ideias but somewhat different in the organization. It is organized as to help the reader see the parallels with the proof of the more difficult result that follows.

A few words about the proof. It is based on finding an expanding return map for backward iterations. The harder part is to estimate the derivative. The C^2 condition is necessary for the bounded distortion, and the maps being close to the identity garantees that the derivative of the return map is greater than one. After this to prove minimality, it is enough to pre-iterate any interval by the return map so it grows enough to capture an attractor. This last argument was already used in [9] and will permute several of the results.

Observe that theorem 2.1 does not require the points to be hyperbolic. One of the applications of this, which will not be persued here, is related to bifurcation of transitive sets as indicated in the next figure. Here f is a fixed map and g_t, $t \in [0,1]$,



is a one-parameter family such that all of the maps satisfy the hypothesis of theorem 2.1.

A minimal set for $< f, g >$ with non-empty interior will be called **blender-like**. The name comes from the connections with skew-products and symbolic blenders. Then we can restate theorem 2.1.

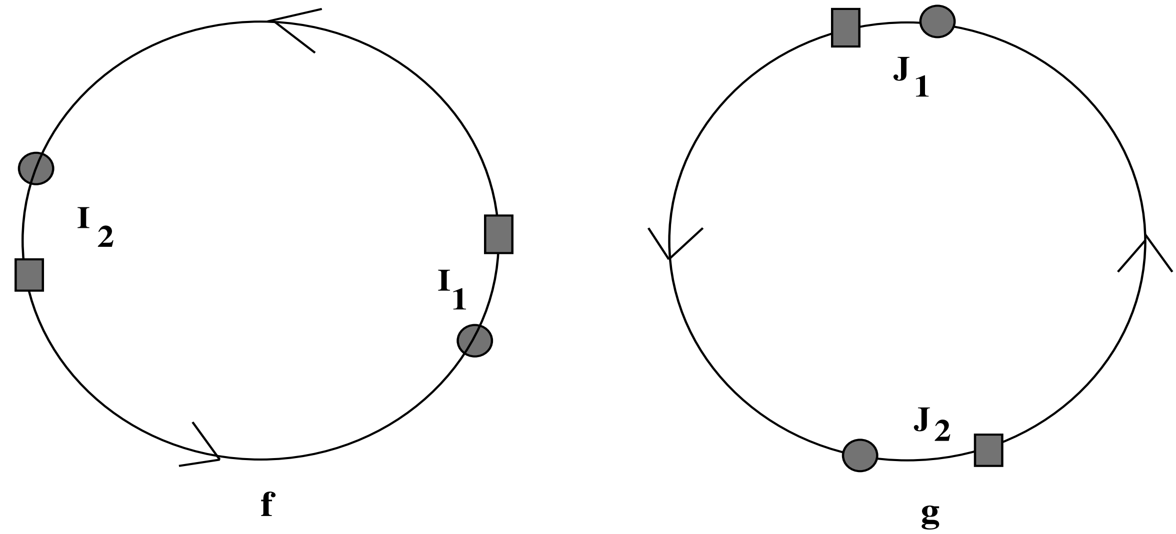
Theorem 2.3. *There exists an $\epsilon > 0$ ($\epsilon \geq 0.14$) such that if f and g are ϵ-close to the identity in the C^2 topology and there is a K^{ss} or K^{su} set then $< f, g >$ has a blender-like.*

As the following example shows, not necessarily with f, g Morse-Smale with fixed points there is a K^{**} set for $< f, g >$.

Example 1. Let f, g be as in the figure. Then there is no K^{ss} or K^{uu} sets because there is no attractor-attractor or repeller-repeller pairs of fixed points. We also ask that I_i is contained in a fundamental domain of g, J_i is contained in a fundamental domain of f. Then there is also no K^{su} type sets for the following reason. There exists a K^{su} type set with an attractor q for the map g if and only if $f(q) \cap B_g(q) \neq \emptyset$ where B_g denotes the basin of attraction. The previous conditions on the fundamental domains prevents this from happening.

Then the question is if for f, g C^2-close to the identity generically there exists a blender-like. This is answered affirmatively by the next theorem, which can be thought of as the parallel of Duminys theorem in the context of IFS or semi-groups.

Theorem 2.4. *Let f, g be C^2, orientation-preserving, Morse-Smale diffeomorphisms of the circle with no periodic points in common. There exists an $\epsilon > 0$ ($\epsilon \geq 0.14$) such that if f, g are ϵ-close to the identity in the C^2 topology, then there is a blender-like set for $< f, g >$. Moreover the blender-like contains a fundamental domain of f (or g) and is contained in $\overline{(Per(< f, g >)}$.*



The proof uses the main ideas of theorem 2.1. The key new step is supposing that there is no K^{ss} type set permits one to create a global expanding return map by going around the whole circle inductively through the basins of attraction of f or g. One of the difficulties is that the derivative of the return map also has to be computed in an inductive manner.

When the maps f, g are not necessarily orientation-preserving consider f^2, g^2, which become orientation-preserving. Then result is the same at the cost of making ϵ smaller.

Corollary 2.5. *Let f, g be C^2 Morse-Smale diffeomorphisms of the circle with no periodic points in common. There exists an $\epsilon > 0$ ($\epsilon \geq 0.06$) such that if f,g are ϵ-close to the identity in the C^2 topology, then there is a blender-like set for $< f, g >$. Moreover the blender-like is contained in $\overline{(Per(< f, g >))}$.*

It turns out that having a K^{ss} set is the only obstruction to the whole circle being minimal.

Theorem 2.6. *Let f, g be C^2, orientation-preserving, Morse-Smale diffeomorphisms of the circle with no periodic points in common. There exists an $\epsilon > 0$ ($\epsilon \geq 0.38$) such that if f, g are ϵ-close to the identity in the C^2 topology and there is no K^{ss} set, then the system $< f, g >$ is robustly minimal.*

Supposing that f, g have fixed points, the existence of a K^{ss} set implies $< f, g >$ is not minimal. This is because for any $x \in K^{ss}$, $Orb(x) \subset K^{ss}$ and therefore it is impossible to visit the whole circle. With this observation we obtain a complete characterization of minimal IFS in our setting.

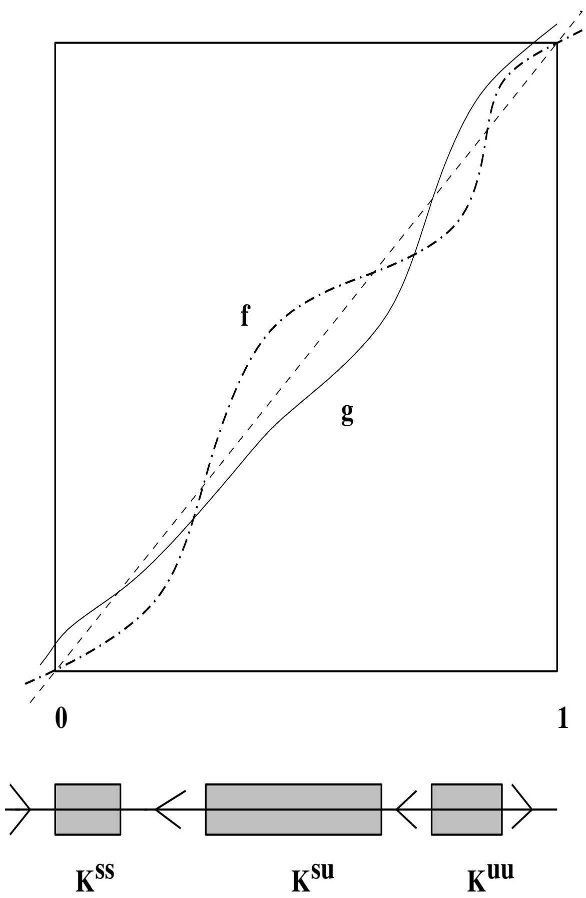
Corollary 2.7. *Let f, g be C^2, orientation-preserving, Morse-Smale diffeomorphisms of the circle both having fixed points which are not in common. There exists an $\epsilon > 0$ ($\epsilon \geq 0.38$) such that if f,g are ϵ-close to the identity in the C^2 topology, then $< f, g >$ is robustly minimal if and only if there is no K^{ss} set.*

Observe that the dichotomy in the theorem only depends on the combinatorics of the periodic points. In the case the periodic points of f, g are fixed, we can completely describe the global toplogical dynamics of the IFS.

Theorem 2.8. *Let f, g be orientation-preserving, Morse-Smale diffeomorphisms of the circle, both with fixed points which are not in common. There exists an $\epsilon > 0$ ($\epsilon \geq 0.14$) such that if f, g are ϵ-close to the identity in the C^2 topology then $< f, g >$ has spectral decomposition.*

Specifically if $\mathbb{S}^1$ is not minimal, $L(< g_1, g_2 >) = \cup_{i=1}^{n} B_i$, where each B_i is either a $K^{ss}, \overline{K^{su}}$, or K^{uu} set or is a single fixed point of f or g.

The next diagram gives an example of how spectral decomposition of an IFS may look like. Here the circle is the interval $[0, 1]$ with the endpoints identified.



Now lets move on to symbolic skew-products. Consider a skew-product of the form

$$\Psi : \Sigma_k \times M \to \Sigma_k \times M, \Psi(\theta, x) = (\tau(\theta), \psi_\theta(x))$$

where M is a Riemannian manifold with a metric d_M, τ is the shift over a space of k symbols $\Sigma_k = \{1, \ldots, k\}^{\mathbb{Z}}$,

$$\tau : \Sigma_k \to \Sigma_k$$

$$\theta = (\ldots, \theta_{-1}, \theta_0; \theta_1, \ldots) \mapsto (\ldots, \theta_{-1}, \theta_0, \theta_1; \ldots)$$

and $\Sigma_k = \{1, \ldots, k\}^{\mathbb{Z}}$ has the metric

$$d(\theta, \sigma)_{\Sigma_k} = \sum_{i \in \mathbb{Z}} \frac{\mid \theta_i - \sigma_i \mid}{k^i}.$$

The metric d in $\Sigma_k \times M$ is the product metric. The functions ψ_θ are taken to be diffeomorphisms of the manifold M. The local and global unstable manifolds of the shift with respect to a sequence θ are

$$W^u_{loc}(\theta; \tau) = \{(\sigma_i); \forall i \leq 0; \sigma_i = \theta_i\},$$

$$W^u(\theta, \tau) = \bigcup_{i \geq 0} \tau^i(W^u_{loc}(\tau^{-i}(\theta); \tau) = \{(\sigma_i); \exists k, \forall i \leq k, \sigma_i = \theta_i\}.$$

The unstable manifold for a point (θ, p) with repect to Ψ is

$$W^u(\theta, p) = \{(\sigma, q); d(\Psi^n(\theta, p), \Psi^n(\sigma, q)) \to 0 \text{ as } n \to -\infty\}.$$

The stable manifolds are defined in a similar manner for τ^{-1}, Ψ^{-1}. Observe that

$$\Psi^n(\theta, p) = (\tau^n(\theta), \psi_{\tau^{n-1}(\theta)} \circ \cdots \circ \psi_\theta(p))\}$$

and similarly for Ψ^{-n}.

We will make the following additional assumption. Suppose also that the fiber maps only depend on the local unstable manifold of the shift, $W^u_{loc}(\theta, \tau)$. That is $\phi_\theta = \phi_\xi$ if $\theta_i = \xi_i$ for all $i \leq 0$. This implies that the local unstable manifold of (θ, p), $W^u_{loc}(\theta, p)$, contains the set $W^u_{loc}(\theta; \tau) \times \{p\}$. Define the local and global strong unstable manifold to be

$$W^{uu}_{loc}(\theta, p) \equiv W^u_{loc}(\theta; \tau) \times \{p\},$$
$$W^{uu}(\theta, p) = \cup_{n \geq 0} \Psi^n(W^{uu}_{loc}(\Psi^{-n}(\theta, p))).$$

Denote by $\mathcal{H}^r(M)$ the set of C^r locally constant skew-products that only depend on the zeroth coordinate of the sequence. That is for $\Psi \in \mathcal{H}^r(M)$, $\psi_\theta = \psi_{\theta_0}$. We will write Ψ as $\tau \ltimes < \psi_1, \ldots, \psi_k >$.

To connect the dynamics of IFS with symbolic skew-products consider

$$\Psi = \tau \ltimes < \psi_1, \ldots, \psi_k > \in \mathcal{H}^r(M)$$

and the relevant IFS given by $< \psi_1, \ldots, \psi_k >$. Let (θ, p) be a fixed point of Ψ and let $\sigma \in W^u_{loc}(\theta; \tau)$. As

$$\Psi^{-n}(\tau^n(\theta), \psi_{\tau^{n-1}(\theta)} \circ \cdots \circ \psi_\theta(p))) = (\sigma, p) \in (W^u_{loc}(\theta), p)$$

and (θ, p) is a fixed point,

$$d(\Psi^{-n}(\Psi^n(\sigma, p)), \Psi^{-n}(\theta, p)) = d((\sigma, p), (\theta, p)).$$

And so for all n,

$$d(\Psi^{-k}(\Psi^n(\sigma, p)), \Psi^{-k}(\theta, p)) \to 0, k \to \infty.$$

Therefore we obtain $\Psi^n(\sigma, p) \in W^{uu}(\theta, p)$. Let π_M be the projection onto M,then

$$\pi_M(\Psi^n(\sigma, p)) \in Orb(p)$$

where the orbit of p, $Orb(p)$, is with respect to the IFS. In conclusion, the relation between IFS and skew-products is given by

$$Orb(p) = \pi_M(W^{uu}(\theta, p)) \subset \pi_M(W^u(\theta, p)).$$

The above notation and commentaries are taken from section 2.3 of [9].

Next we will give a definition of a symbolic-blender like but for a restricted set of skew-products $\mathcal{S}_k^{\alpha,r}(M)$ which will be described afterwards. The definition is the same as that of symbolic blender in [9] and preserves the main topoligical property of blenders (see (ii) in the definition). The difference being that in that paper the authors worked with uniformely contracting IFS and with hyperbolic sets. By definition blenders are hyperbolic sets [3]. As here there is no assumptions on hyperbolicity, we use the name blender-like.

Given a periodic point $(\theta_\Gamma, p_\Gamma)$ of Γ with period n. Denote by $\mathcal{O}(\theta_\Gamma, p_\Gamma)$ the orbit of the point, $\bigcup_{i=0}^{n-1} \Gamma^i(\theta_\Gamma, p_\Gamma)$, and let

$$W^{uu}(\mathcal{O}(\theta_\Gamma, p_\Gamma)) = \bigcup_{i=0}^{n-1} W^{uu}(\Gamma^i(\theta_\Gamma, p_\Gamma)).$$

Definition 2.9. ***Symbolic Blender-like***
Let B be an open set in M. The set $\mathbb{B} = \Sigma_k \times B$ is a symbolic cs-blender-like of $\Gamma \in \mathcal{S}_k^{\alpha,r}(M)$ if there exists a neighborhood Ω of Γ and a periodic point of Γ, $(\theta_\Gamma, p_\Gamma)$, such that

(i) For any $\Psi \in \Omega$, there exists the continuation (θ_Ψ, p_Ψ) of $(\theta_\Gamma, p_\Gamma)$

(ii) Given a sequence ξ and an open set $U \subset B$,

$$W^{uu}(\mathcal{O}(\theta_\Psi, p_\Psi)) \cap (W^s_{loc}(\xi, \tau) \times U) \neq \emptyset.$$

A symbolic cu-blender-like for Γ is a cs-blender-like for Γ^{-1}.

Observe that the symbolic blender-like is robust by definition. To give examples of existence of these sets in the symbolic skew-products we would need the additional assumption that the inverses of the fiber maps have Holder dependence on the sequences.

Denote by $\mathcal{S}_k^{\alpha,r}(M)$ the set of the above skew-products that satisfies the following.

1. Ψ^{-1} has α-Holder dependence on the the fibers. There exists a (minimal) constant C_Ψ such that for all sequences θ, ξ with $\theta_0 = \xi_0$

$$d_{C^0}(\psi_\theta^{-1}, \psi_\xi^{-1}) \leq C_\Psi d_{\Sigma_k}(\theta, \xi)^\alpha \leq C_\Psi (1/k^N)^\alpha$$

if $\theta_j = \xi_j$ for $0 \leq \mid j \mid \leq N-1$.

2. Each fiber map is C^r-diffeomorphism of the manifold M and $sup_{\theta\in\Sigma_k}\{D^j\psi_\theta\} < \infty$ for $0 \leq j \leq r$.

A distance in this space is given by

$$d(\Phi,\Psi)_{\mathcal{S}_k^{\alpha,r}(M)} = max\{\mid C_\Phi - C_\Psi \mid, sup_{\theta\in\Sigma_k}\{d_M(\phi_\theta,\psi_\theta)_{C^r}\}\}$$

where C_Φ, C_Ψ are the constants from the Holder dependence.

The set $\mathcal{H}^r(M)$ is contained in $\mathcal{S}_k^{\alpha,r}(M)$ for all α and the distance with respect to the set $\mathcal{H}^r(M)$ is

$$d(\Phi,\Psi)_{\mathcal{H}^r(M)} = max_{j=1,\ldots,k}\{d_M(\phi_j,\psi_j)_{C^r}\}.$$

The next theorem states sufficient conditions for obtaining symbolic blender-like sets. The two properties that appear in the hypothesis of the theorem, covering and minimality, are similar to the ones used in [9] where symbolic blenders were obtained for locally constant skew-products. The Holder topology and the additional hypothesis of the fiber maps close to the identity allows the existence of blenders in the bigger set $\mathcal{S}_k^{\alpha,r}(M)$.

Theorem 2.10. *Let c be such that $(1-c)k^\alpha = 1$ and $\mathcal{B}(Id,c)_{\mathcal{S}_k^{\alpha,r}(M)}$ be a ball of radius c about the identity map $Id = (\tau, Id)$ in $\mathcal{S}_k^{\alpha,r}(M)$, $r > 1$. Consider*

$$\Gamma \in \mathcal{H}^r(M) \cap \mathcal{B}(Id,c)_{\mathcal{S}_k^{\alpha,r}(M)},$$

where $\Gamma = \tau \ltimes <\gamma_1,\ldots,\gamma_k>$. Suppose there exists a bounded open set $B \subset M$, a finite number of bounded closed sets U_i and the respective maps $H_i \in <\gamma_1^{-1},\ldots,\gamma_k^{-1}>$ such that

(i) Covering property:

$$\overline{B} \subset \bigcup_{i=1}^{k} int(U_i),$$

with $H_i(U_i) \subset B$ and $DH_i > 1$ in U_i.

(ii) Periodic point with minimal orbit: there exists a hyperbolic periodic point $p_\Gamma \in B$ of $<\gamma_1,\ldots,\gamma_k>$ such that $B \subset Orb(p_\Gamma)$.

Then B is a cs-symbolic blender-like set in $\mathcal{S}_k^{\alpha,r}(M)$ for Γ.

The condition on c also appears in [5] where the authors had a similar objective in mind: to transfer properties of IFS to robust properties in the space of symbolic skew-products. Some of the steps that appear in our proof have resembling counterparts in the technical lemmas of that paper.

The expanding return maps used in the proofs of theorems 2.1 and 2.4 have an infinite number of branches and this creates the problem for perturbations in $\mathcal{S}_k^{\alpha,r}(S^1)$. But we can perform a reduction on the number of branches for a generic pair, thus achieving the required hypothesis of theorem 2.10.

Theorem 2.11. *There exists a generic set G in $Diff^r(S^1), r \geq 2$, such that for $f, g \in G \cap B(Id, 0.06)$ the following conditions are satisfied.*

(i) There exists an open minimal set B such that $\overline{B} \subset \overline{Per < f, g >}$.

(ii) There is a finite set of closed intervals U_j such that $\overline{B} \subset \bigcup_{j=1}^{m} int(U_j)$. To each U_j there is an associated map $H_j \in < f^{-1}, g^{-1} >$ such that $DH_j > 1$ in U_j and $H_j(U_j) \subset B$.

A combination of theorems 2.10 and 2.11 will yield the last result.

Corollary 2.12. *Consider $\mathcal{B}(Id, \lambda)_{\mathcal{H}^r(S^1)}$ to be a ball of radius λ about the identity map $Id = (\tau, Id)$ in $\mathcal{H}^r(S^1)$. For a given α let c be such that $(1 - c)k^\alpha = 1$, and $\lambda = min\{c, 0.06\}$.*

There exists a generic set $\Lambda \subset \mathcal{H}^r(S^1)$ for $r \geq 2$ such that for

$$\Gamma \in \mathcal{B}(Id, \lambda)_{\mathcal{H}^r} \cap \Lambda$$

Γ has cs-symbolic blender-like in $\mathcal{S}_k^{\alpha,r}(S^1)$, $r \geq 1$.

This ends the statement of the results. The theorems are proven in order in each of the following sections.

3 Minimality of K^{**}

Let f_0, f_1 be C^2 diffeomorphisms of the circle. Suppose that there exists a K^{ss} or K^{su} set for the IFS $< f_0^{n_0}, f_1^{n_1} >$ (where n_i are the periods). We want to show the following.

Theorem 3.1. *There exists an $\epsilon > 0$ ($\epsilon \geq 0.14$) such that if f_0 and f_1 are ϵ-close to the identity in the C^2 topology, then K^{**} is minimal for $< f_0^{n_0}, f_1^{n_1} >$. Moreover $K^{**} \subset \overline{(Per(< f_0, f_1 >)}$.*

Proof. We first deal with the case when f_i have fixed points. For simplicity we assume that $I = [0, 1], f_0(0) = 0, f_0 < Id, f_1 > Id$ in $(0, 1)$. In the K^{ss} case we can suppose that the overlap condition holds, that is $f_0(I) \cap f_1(I) \neq \emptyset$. This is true if $Df_0 + Df_1 > 1$ or $\mid f_i - Id \mid_{C^1} < 0.5$.

Step 1: Creating a Return Map

We will create a return map in the fundamental domain of f_0,

$$D = (f_1(0), f_0^{-1}(f_1(0))].$$

Here enters the overlap condition, since it is necessary to be able to take the inverse for f_0. Let l be such that $f_1^l(0) \in D$, $f_1^{l+1}(0) \notin D$. We can write D as

$$D = \biguplus_{k=1}^{l} J_k$$

where $J_k = (f_1^k(0), f_1^{k+1}(0)]$ for $k < l, J_l = (f_1^l(0), f_0^{-1}(f_1(0))]$

Consider then $f_1^{-k}(J_k) = (0, c_k]$, for some c_k with $c_k \leq f_1(0)$. Then there exists m_k, the first time that $f_0^{m_k}(f_1(0))$ is in the interior of $f_1^{-k}(J_k) = (0, c_k]$. Let

$$J_{ki} = (f_0^{m_k+i}(f_1(0)), f_0^{m_k+i-1}(f_1(0))], i > 0$$

$$J_{k0} = (f_0^{m_k}(f_1(0)), c_k].$$

Then $(0, c_k] = \biguplus_{i=0}^{\infty} J_{ki}$ and $D = f_0^{-(m_k+i)}(J_{ki})$ for $i > 0$, $f_0^{-(m_k)}(J_{k0}) \subset D$ for $i = 0$.

Define $I_{ki} = f_1^k(J_{ki})$, then $D = \biguplus I_{ki}$. Setting $h_{ki} = f_1^k \circ f_0^{m_k+i}$, obtain that $h_{ki}^{-1}(I_{ki}) = D$ for $i > 0$, $h_{ki}^{-1}(I_{ki}) \subset D$ for $i = 0$. See the figure for the geometry of the construction.

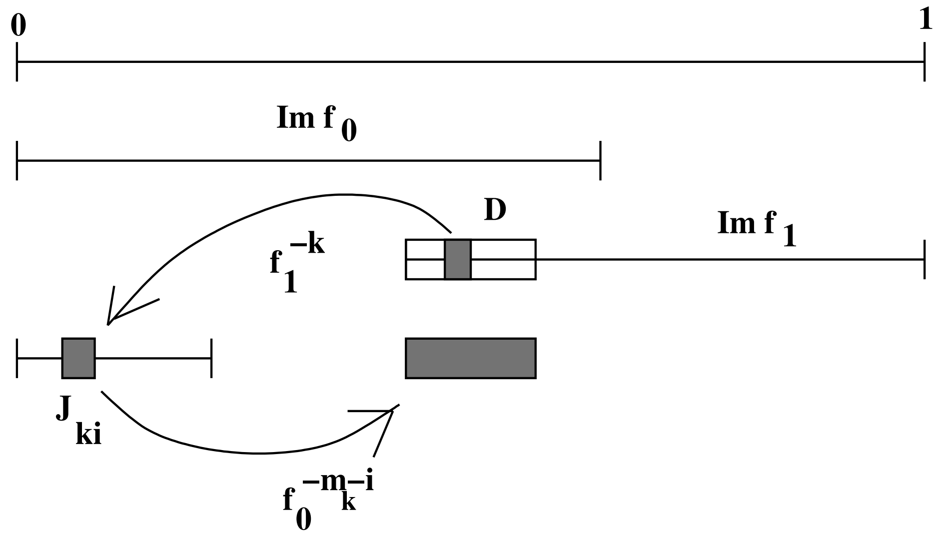
Now we can define a return map in D with an infinite number of branches by $H : D \to D, H = h_{ki}^{-1}$ in I_{ki}. There is a finite number of accumulation points of the branches, given by the set $\{f_1^k(0)\}_{1 \leq k \leq l}$.

Step 2: Bounded Distortion

In order to estimate the derivative of the maps h_{ki}^{-1}, firstly we would need a bounded distortion estimate.

Let c_{f_i} be the distortion constants of f_i and $c = max\{c_{f_i}\}$, where

$$c_{f_i} = max\{\frac{D^2 f_i(x)}{Df_i(x)}\},$$



then $e^{-C} \leq \dfrac{Df_i(x)}{Df_i(y)} \leq e^C$. Since for difeomorphisms there is always a point with derivative 1, $e^{-c_{f_i}} \leq Df_i(x) \leq e^{c_{f_i}}$.

Lemma 3.2. *For $x, y \in I_{ki}$,*

$$e^{-c} \leq \frac{Dh_{ki}^{-1}(x)}{Dh_{ki}^{-1}(y)} \leq e^c$$

.

Call $U_j = f_1^{-j}(I_{ki}), 0 \leq j \leq k$ and $U_{kj} = f_0^{-j} \circ f_1^{-k}(I_{ki}), 1 \leq j < m_k + i$. By the construction, these intervals are all disjoint. The proof of the lemma is then the classical bounded distortion argument.

Proof.

$$log \frac{Dh_{ki}^{-1}(x)}{Dh_{ki}^{-1}(y)} = log(Dh_{ki}^{-1}(x)) - log(Dh_{ki}^{-1}(y))$$

$$= log[\prod_{j=0}^{m_k+i-1} Df_0^{-1}(f_0^{-j}(f_1^{-k}(x))) \cdot \prod_{j=0}^{k-1} Df_1^{-1}(f_1^{-j}(x))] -$$

$$log[\prod_{j=0}^{m_k+i-1} Df_0^{-1}(f_0^{-j}(f_1^{-k}(y))) \cdot \prod_{j=0}^{k-1} Df_1^{-1}(f_1^{-j}(y))]$$

$$\leq \sum_{j=0}^{m_k+i-1} \mid log[Df_0^{-1}(f_0^{-j}(f_1^{-k}(x)))] - log[Df_0^{-1}(f_0^{-j}(f_1^{-k}(y)))] \mid$$

$$+ \sum_{j=0}^{k-1} \mid log[Df_1^{-1}(f_1^{-j}(x)))] - log[Df_1^{-1}(f_1^{-j}(y)))] \mid$$

$$\leq c \cdot \sum_{j=0}^{m_k+i-1} \mid f_0^{-j}(f_1^{-k}(x)) - f_0^{-j}(f_1^{-k}(y)) \mid + c \cdot \sum_{j=0}^{k-1} \mid f_1^{-j}(x)) - f_1^{-j}(y)) \mid$$

$$= c \cdot \sum_{j=1}^{m_k+i-1} \mid U_{kj} \mid + c \cdot \sum_{j=0}^{k} \mid U_j \mid \leq c$$

where the last inequality is a consequence of the disjointess of the intervals. Since this holds for all $x, y \in I_{ki}$, we can invert the fraction to obtain the bounded constant from below. □

For $J \subset I_{ki}$, by the mean value theorem, there exists $x \in J$ and $y \in I_{ki}$ such that $|h_{ki}^{-1}(J)| = Dh_{ki}^{-1}(x) \cdot |J|$ and $|h_{ki}^{-1}(I_{ki})| = Dh_{ki}^{-1}(y) \cdot |I_{ki}|$. Therefore from the bounded distortion lemma,

$$\frac{|h_{ki}^{-1}(J)|}{|J|} \geq e^{-c} \frac{|h_{ki}^{-1}(I_{ik})|}{|I_{ki}|}$$

Step 3: Estimation of the Derivative for the Return Map

We will show that $DH^{-1} > (1-\epsilon)^3 \cdot \dfrac{e^{-3c}}{3\epsilon}$, where $\mid f_i - Id \mid_{C^1} < \epsilon$. Let $k = min\{Df_i\}$. Then the bounds for the return map, H, calculated above appear as

$$DH \geq \frac{k^3}{3(1-k)} \cdot e^{-3c} \geq \frac{e^{-3c}}{3(1-e^{-c})} \cdot e^{-3c} = \frac{e^{-6c}}{3(1-e^{-c})}$$

$$\geq \frac{e^{\frac{-6\epsilon}{1+\epsilon}}}{3(1-e^{\frac{-\epsilon}{1+\epsilon}})}$$

In particular, $DH > 1$ if $\epsilon < 0.17$.

Given $x \in I_{ki}$, consider the ball of radius r, $B_r(x)$, then by the previous calculation

$$Dh_{ki}^{-1}(x) = lim_{r\to 0} \frac{|h_{ki}^{-1}(B_r(x))|}{|B_r(x)|} \geq e^{-c} \frac{|h_{ki}^{-1}(I_{ki})|}{|I_{ki}|}.$$

Therefore we have to estimate $\dfrac{|h_{ki}^{-1}(I_{ki})|}{|I_{ki}|}$. This is easier to do first for the case when $i > 0$ because then $h_{ki}^{-1}(I_{ki}) = D$.

The whole idea behind the estimations is to move the two intervals in question (and so appear the bounded distorion constants) as to get in the situation of comparing $\dfrac{\mid f_0(c) - c \mid}{c}$ for some c. This on the other hand is greater than $1/\epsilon$ where $\mid f_i - Id \mid_{C^1} < \epsilon$.

Lemma 3.3. *For $i > 0$,*

$$\frac{|h_{ki}^{-1}(I_{ki})|}{|I_{ki}|} > \frac{e^{-c}}{\epsilon}$$

and therefore $Dh_{ki}^{-1} > \dfrac{e^{-2c}}{\epsilon}$, where $\mid f_i - Id \mid_{C^1} < \epsilon$

Proof. As J_k is contained in a fundamental domain of f_1 and $I_{ki} \subset J_k$, we have that

$$\frac{\mid J_k \mid}{\mid I_{ki} \mid} \geq e^{-c} \frac{\mid f_1^{-k}(J_k) \mid}{\mid f_1^{-k}(I_{ki}) \mid}.$$

Since $i > 0$, $J_k \subset D = h_{ki}^{-1}(I_{ki})$ and so

$$\frac{|h_{ki}^{-1}(I_{ki})|}{|I_{ki}|} \geq \frac{|J_k|}{|I_{ki}|} \geq e^{-c}\frac{|f_1^{-k}(J_k)|}{|f_1^{-k}(I_{ki})|}.$$

Writing $f_1^{-k}(J_k) = (0, c_k]$, remember that $f_1^{-k}(I_{ki}) = (f_0^{m_k+i}(f_1(0)), f_0^{m_k+i-1}(f_1(0))]$ with $f_0^{m_k+i-1}(f_1(0)) \leq c_k$ (since $i > 0$). Obtain then

$$\frac{|h_{ki}^{-1}(I_{ki})|}{|I_{ki}|} \geq e^{-c}\frac{|c_k|}{|f_0^{m_k+i}(f_1(0)) - f_0^{m_k+i-1}(f_1(0))|}$$

$$\geq e^{-c}\frac{|f_0^{m_k+i-1}(f_1(0))|}{|f_0^{m_k+i}(f_1(0)) - f_0^{m_k+i-1}(f_1(0))|} = e^{-c}\frac{|c|}{|f_0(c) - c|}$$

By the mean value theorem, $f_0(c) = Df_0(z)c$ for some z and so

$$= e^{-c}\frac{1}{|\ 1 - Df_0(z)\ |} > \frac{e^{-c}}{\epsilon}.$$

□

For the case $i = 0$ we dont have necessarilly that $J_k \subset D = h_{k0}^{-1}(I_{k0})$, what creates the difficulty in the estimate.

Lemma 3.4. *For $i = 0$, $Dh_{k0}^{-1} > (1-\epsilon)^3 \cdot \dfrac{e^{-3c}}{3\epsilon}$*

Proof. We will distinguish two cases.

Case 1: $h_{k0}^{-1}(I_{k0}) \cap I_{k0} \neq \emptyset$. In this case $(J_k - I_{k0}) \subset h_{k0}^{-1}(I_{k0})$, then we can proceed as before

$$\frac{|h_{k0}^{-1}(I_{ki})|}{|I_{k0}|} \geq \frac{|J_k - I_{k0}|}{|I_{k0}|} \geq e^{-c}\frac{|f_1^{-k}(J_k - I_{k0})|}{|f_1^{-k}(I_{k0})|}$$

$$= e^{-c}\frac{|f_0^{m_k}(f_1(0))|}{|c_k - f_0^{m-k}(f_1(0))|} \geq e^{-c}\frac{|f_0^{m_k}(f_1(0))|}{|f_0^{m_k-1}(f_1(0)) - f_0^{m-k}(f_1(0))|}$$

$$= e^{-c}\frac{|f_0(c)|}{|c - f_0(c)|} \geq (1-\epsilon)e^{-c}\frac{|c|}{|c - f_0(c)|} > \frac{(1-\epsilon)e^{-c}}{\epsilon}$$

Case 2: $h_{k0}^{-1}(I_{k0}) \cap I_{k0} = \emptyset$. First note that $h_{k0}^{-1} = f_0 \circ h_{k1}^{-1}$ and so

$$Dh_{k0}^{-1}(x) = Df_0(h_{k1}^{-1}(x))Dh_{k1}^{-1}(x) > (1-\epsilon)Dh_{k1}^{-1}(x)$$

where $x \in I_{k0}$. We will estimate Dh_{k1}^{-1} but in I_{k0}.

The intervals defined previously $U_j = f_1^{-j}(I_{k0}), 1 \leq l \leq k$ and $U_{kj} = f_0^{-j} \circ f_1^{-k}(I_{k0}), 1 \leq j < m_k$ are disjoint. The hypothesis $h_{k0}^{-1}(I_{k0}) \cap I_{k0} = \emptyset$ implies U_{kj} are disjoint for $j \leq m_k$. Then the bounded distortion argument can be applied to the function $h_k^{-1}1 = f_0^{-1} \circ h_{k0}^{-1}$ on the interval I_{k0}. We obtain that for $x, y \in I_{k0}$

$$e^c \geq \frac{Dh_{k1}^{-1}(x)}{Dh_{k1}^{-1}(y)} \geq e^{-c}$$

Let $I = I_{k0} \cup I_{k1}$. If both x, y are in I_{k0} or both are in I_{k1}, then the bounded distortion estimate above holds. If $x \in I_{k0}$ and $y \in I_{k1}$ letting $z = f_0^{m_k}(f_1(0))$,

$$\frac{Dh_{k1}^{-1}(x)}{Dh_{k1}^{-1}(y)} = \frac{Dh_{k1}^{-1}(x)}{Dh_{k1}^{-1}(z)} \cdot \frac{Dh_{k1}^{-1}(z)}{Dh_{k1}^{-1}(y)} \geq e^{-2c}.$$

Then as before,

$$\mid Dh_{k1}^{-1} \mid \geq e^{-2c} \frac{\mid h_{k1}^{-1}(I) \mid}{\mid I \mid}$$

Now as $J_k \subset D \subset h_{k1}^{-1}(I)$ and $I \subset J_k$ we have

$$\frac{|h_{k1}^{-1}(I)|}{|I|} \geq \frac{|J_k \,|}{|I|} \geq e^{-c} \frac{|f_1^{-k}(J_k)|}{|f_1^{-k}(I)|}$$

Using that $I = (f_0^{m_k+1}(f_1(0)), c_k]$

$$= e^{-c} \frac{|c_k \,|}{|c_k - f_0^{m_k+1}(f_1(0))|} \geq e^{-c} \frac{|f_0^{m_k+1}(f_1(0)) \,|}{|\, f_0^{m_k-1}(f_1(0)) - f_0^{m_k+1}(f_1(0))|}$$

$$= e^{-c} \frac{|f_0^2(c) \,|}{|\, f_0^2(c) - c|}$$

As $f_0^2(c) = Df_0^2(z)c$ for some z, then

$$e^{-c} \frac{|Df_0^2(z) \,|}{|\, Df_0^2(z) - 1|} \geq e^{-c} \frac{((1-\epsilon)^2}{((1+\epsilon)^2 - 1)} \geq e^{-c} \frac{(1-\epsilon)^2}{3\epsilon}.$$

In conclusion, $Dh_{k1}^{-1} \geq (1-\epsilon)^2 \cdot \dfrac{e^{-3c}}{3\epsilon}$ and so $Dh_{k0}^{-1} > (1-\epsilon)^3 \cdot \dfrac{e^{-3c}}{3\epsilon}$. □

Step 4: End of Proof (for maps with fixed points)

Lemma 3.5. *To show minimality of the interval $I = (0, 1)$, it is enough to prove that for any open interval $J \subset I$, there exists $h_1, h_2 \in< f_0, f_1 >$ such that $h_2(0) \in h_1^{-1}(J)$.*

Proof. Take any point $x \in I$ and an open interval $J \subset I$. To show minimality it is sufficient to prove that there exists $h \in< f_0, f_1 >$ with $h(x) \in J$. By hypothesis there exists $h_1, h_2 \in< f_0, f_1 >$ such that $h_2(0) \in h_1^{-1}(J)$. Since 0 is a global attractor in I, there exists h_3 with $h_3(x)$ so close to 0 such that $h_3 \circ h_2(x) \in h_1^{-1}(J)$. Then $h_1 \circ h_3 \circ h_2(x) \in J$. □

No given $J \subset I$ it is not hard to see that there exists $h \in< f_0, f_1 >$ with $h^{-1}(J) \cap D \neq \emptyset$. We can suppose $h_1^{-1}(J) \subset D$. Now there are two options or $h_1^{-1}(J) \subset I_{kj}$ for some I_{kj}, or $h_1^{-1}(J)$ contains a point of the form $f_0^{m_k+i}(f_1(0))$. In the second case we are done (by the above lemma), in the first case we can iterate by the return map H and consider $H \circ h_1^{-1}(J)$.

Now repeating the argument or $H \circ h_1^{-1}(J) \subset I_{kj}$ for some I_{kj}, or $H \circ h_1^{-1}(J)$ contains $f_0^{m_k+i}(f_1(0))$ and here we are done. Continuing in this manner we will obtain that $H^n \circ h_1^{-1}(J) \subset D$ and as $\mid H^n \circ h_1^{-1}(J) \mid > \lambda^n \mid h_1^{-1}(J) \mid, \lambda > 1$, then at some point $H \circ h_1^{-1}(J)$ will contain a point of the form $f_0^{m_k+i}(f_1(0))$.

Observe that this in particular shows that the orbit under $< f_0, f_1 >$ of the attractor at 0 is dense in I.

Finally to show that $I \subset \overline{(Per(< f_0, f_1 >)}$, we will use that the orbit of the attracto 0 is dense. So given $J \subset I$, open, there exists $h \in < f_0, f_1 >$ with $h(0) \in J$. As 0 is the attractor, there exists a k such that $f_0^k \circ h(J) \subset J$ and $D(f_0^k \circ h) < 1$ in J. Then there exists a fixed point in J for the map $f_0^k \circ h \in < f_0, f_1 >$.

Step 5: Bounds for Periodic f_i

When there are periodic points we are led to consider the system $< f_0^{n_0}, f_1^{n_1} >$ where n_i are the periods. But the distortion constant of $f_i^{n_i}$ is $n_i \cdot c_{f_i}$, and if we apply the above estimates obtain that $DH \geq \dfrac{e^{-6nc}}{3(1-e^{-nc})}$ where $n = max\{n_i\}$, that is depends on the period. This is bad if we are looking for a uniform neighborhood of the Id.

The objective of this section is to show that actually $DH \geq \dfrac{e^{-8c}}{3(1-e^{-c})}$, independent of the periods n_i. With respect to f_i being ϵ-close to the identity, we have

$$DH \geq \frac{e^{\frac{-8\epsilon}{1+\epsilon}}}{3(1-e^{\frac{-\epsilon}{1+\epsilon}})}$$

If $\epsilon \leq 0.14$, then $DH > 1$.

To extend Duminys lemma for periodic Morse-Smale we will need the following properties of Morse-Smale dynamics on the circle.

Lemma 3.6. *Let f be a periodic Morse-Smale with period j, and as above $C = max\{\dfrac{D^2 f(x)}{Df(x)}\}$. Then (i) $e^{-C} \leq Df^j(x) \leq e^C$.*
(ii) Suppose I is a fundamental domain of f^j, $I = (f^{2j}(x), f^j(x))$, x is not a one of the periodic points. Then $f^m(I) \cap f^n(I) = \emptyset$ for all $m \neq n$ (not necessarily multiples of j)

Proof. Let $J = (p, q)$ where p, q is an attractor-repeller pair, fixed for f^j. Then $f^k(J) \cap f^l(J) = \emptyset$ for all $0 \leq k, l < j$. On the contrary, $f^{k-l}(J) \cap J \neq \emptyset$ and therefore p or $q \in J$. The disjointness of the intervals implies $e^{-C} \leq \dfrac{Df^j(x)}{Df^j(y)} \leq e^C$ for all $x, y \in J$. Since there is always a point of derivative one in J, we have that $e^{-C} \leq Df^j(x) \leq e^C$.

In a similar manner for the second part, suppose $f^m(I) \cap f^n(I) = \emptyset$, then $f^{m-n}(I) \cap I \neq \emptyset$. As j is the minimal period $m - n = ij$ for some i. So $f^{ij}(I) \cap I \neq \emptyset$, a contradiction since I is a fundamental domain for f^j. □

We will indicate the modifications required in estimating the return map derivative in steps 2 and 3. The return maps h_{ki} are with respect to the system $< f_0^{n_0}, f_1^{n_1} >=<$

$g_0, g_1 >$. Let c_{f_i} ($c = max\{c_{f_i}\}$) be the distortion constants of the original maps f_i and not $f_i^{n_i}$.

The statement of the bounded distortion lemma takes form as

Lemma 3.7. *For $x, y \in I_{ki}$,*

$$e^{-2c} \leq \frac{Dh_{ki}^{-1}(x)}{Dh_{ki}^{-1}(y)} \leq e^{2c}$$

.

Let $U_j = f_1^{-j}(I_{ki}), 0 \leq j \leq kn_1$ and $U_{kj} = f_0^{-j} \circ f_1^{-k}(I_{ki}), 1 \leq j < (m_k + i)n_0$. As I_{ki} is contained in a fundamental domain of f_1 and $f_1^{-kn_1}(I_{ki})$ is contained in a fundamental domain of f_0. Then lemma3.6 says that U_j are disjoint with respect to each other and U_{kj} are disjoint with respect to each other.

Proof.

$$log\frac{Dh_{ki}^{-1}(x)}{Dh_{ki}^{-1}(y)} = log[\prod_{j=0}^{m_k+i-1} Dg_0^{-1}(g_0^{-j}(g_1^{-k}(x))) \cdot \prod_{j=0}^{k-1} Dg_1^{-1}(g_1^{-j}(x))] -$$

$$log[\prod_{j=0}^{m_k+i-1} Dg_0^{-1}(g_0^{-j}(f_g^{-k}(y))) \cdot \prod_{j=0}^{k-1} Dg_1^{-1}(g_1^{-j}(y))]$$

$$= log[\prod_{j=0}^{n_0(m_k+i)-1} Df_0^{-1}(f_0^{-j}(f_1^{-n_1 k}(x))) \cdot \prod_{j=0}^{n_1(k-1)} Df_1^{-1}(f_1^{-j}(x))] -$$

$$log[\prod_{j=0}^{n_0(m_k+i)-1} Df_0^{-1}(f_0^{-j}(f_1^{-n_1 k}(y))) \cdot \prod_{j=0}^{n_1(k-1)} Df_1^{-1}(f_1^{-j}(y))]$$

$$\leq \sum_{j=0}^{n_0(m_k+i)-1} \mid log[Df_0^{-1}(f_0^{-j}(f_1^{-n_1 k}(x)))] - log[Df_0^{-1}(f_0^{-j}(f_1^{-n_1 k}(y)))] \mid$$

$$+ \sum_{j=0}^{n_1(k-1)} \mid log[Df_1^{-1}(f_1^{-j}(x)))] - log[Df_1^{-1}(f_1^{-j}(y)))] \mid$$

$$\leq c \cdot \sum_{j=0}^{n_0(m_k+i)-1} \mid f_0^{-j}(f_1^{-n_1 k}(x)) - f_0^{-j}(f_1^{-n_1 k}(y)) \mid + c \cdot \sum_{j=0}^{n_1(k-1)} \mid f_1^{-j}(x)) - f_1^{-j}(y)) \mid$$

$$= c \cdot \sum_{j=1}^{n_0(m_k+i)-1} \mid U_{kj} \mid + c \cdot \sum_{j=0}^{n_1 k} \mid U_j \mid \leq 2c$$

where the last inequality is a consequence of the disjointess of U_j and U_{kj}. □

The following is the analogue of lemma 3.3 of part 3,

Lemma 3.8. *For $i > 0$,*

$$\frac{|h_{ki}^{-1}(I_{ki})|}{|I_{ki}|} > \frac{e^{-c}}{\epsilon}$$

and therefore $Dh_{ki}^{-1} > \dfrac{e^{-3c}}{\epsilon}$, where $\mid f_i^{n_i} - Id \mid_{C^1} < \epsilon$

Proof. As J_k is contained in a fundamental domain of $f_1^{n_1}$ and $I_{ki} \subset J_k$, using lemma 3.6 we still have that

$$\frac{\mid J_k \mid}{\mid I_{ki} \mid} \geq e^{-c} \frac{\mid f_1^{-n_1 k}(J_k) \mid}{\mid f_1^{-n_1 k}(I_{ki}) \mid}.$$

The rest of the proof is the same, and using the bounded distortion estimate from above we obtain the result. □

Finally we will make the necessary changes when $i = 0$ in lemma 3.4

Lemma 3.9. *For $i = 0$, $Dh_{k0}^{-1} > (1 - \epsilon)^3 \cdot \dfrac{e^{-5c}}{3\epsilon}$*

Proof. For the case $h_{k0}^{-1}(I_{k0}) \cap I_{k0} \neq \emptyset$, we can proceed as in 3.4 with the same observations that were made in the last lemma 3.8, to obtain

$$\frac{|h_{k0}^{-1}(I_{ki})|}{|I_{k0}|} > \frac{(1-\epsilon)e^{-c}}{\epsilon}$$

When $h_{k0}^{-1}(I_{k0}) \cap I_{k0} \neq \emptyset$, using the new bounded distortione estimate with the same notation as before to obtain for $x, y \in I = I_{k0} \cup I_{k1}$

$$\frac{Dh_{k1}^{-1}(x)}{Dh_{k1}^{-1}(y)} \geq e^{-4c}.$$

As $J_k \subset D \subset h_{k1}^{-1}(I)$ and $I \subset J_k$ and J_k is contained in a fundamental domain of $f_1^{n_1}$ by lemma 3.6 we have as in the original inequality

$$\frac{|h_{k1}^{-1}(I)|}{|I|} \geq \frac{|J_k\mid}{|I|} \geq e^{-c} \frac{|f_1^{-k}(J_k)|}{|f_1^{-k}(I)|}.$$

The rest of the proof is the same giving the result. □

Letting $k = min\{Df_i\}$, the bounds for the return map become

$$DH \geq \frac{k^3}{3(1-k)} \cdot e^{-5c} \geq \frac{e^{-5c}}{3(1-e^{-c})} \cdot e^{-3c} = \frac{e^{-8c}}{3(1-e^{-c})}$$

The proof of minimality of K^{**} for periodic f_i is the same as for f_i with fixed points. This ends the proof of Duminys lemma. □

4 Blender-like Sets

Theorem 4.1. *Let f, g be C^2,orientation-preserving, Morse-Smale diffeomorphisms of the circle with no periodic points in common. There exists an $\epsilon > 0$ ($\epsilon \geq 0.14$) such that if f,g are ϵ-close to the identity in the C^2 topology, then there is a blender-like set for $< f, g >$. Moreover the blender-like contains a fundamental domain of f (or g) and is contained in $\overline{(Per(< f, g >)}$.*

Observation: From step 4 of the proof of the theorem, one can see that the blender-like set is of the form $\overline{B} = [p_1, g^{-n_g}(p_1)]$ where p_1 is an attractor for f and n_g is the period of g. If n_f is the period of f then $f^{n_f}(\overline{B}) \cap \overline{B} \neq \emptyset$ and $g^{n_g}(\overline{B}) \cap \overline{B} \neq \emptyset$. This will become important in the next section.

When the diffeomorphisms are not necessarily orientation-preserving, the result becomes

Corollary 4.2. *Let f, g be C^2 Morse-Smale diffeomorphisms of the circle with no periodic points in common. There exists an $\epsilon > 0$ ($\epsilon \geq 0.06$) such that if f,g are ϵ-close to the identity in the C^2 topology, then there is a blender-like set for $< f, g >$. Moreover the blender-like is contained in $\overline{(Per(< f, g >)}$.*

Proof. The proof will be similar to that of Duminys lemma (a local case) when the blender-like set (K^{**}) was obtained from the local geometries of the two functions. Here the expanding return map will be of global character. From the beginning we will assume that there is no K^{ss} blender-like set for $< f, g >$ and will obtain a different type of blender-like.

Lets suppose that the periodic points of f and g are actually fixed points. To find the candidate for the return map we will define a cycle.

Definition 4.3. *Denote by p_i the attractors of f, q_i the attractors of g. Define a partial order on the attracting points by $p_i \prec q_j \Leftrightarrow p_i \in B_g(q_j)$, where B_g denotes the basin of attraction for g, with similar definitions for $q_i \prec p_j$. A sequence of attractors forms a **cycle** when we have $p_{i_1} \prec q_{i_2} \prec p_{i_3} \cdots \prec q_{i_{n-1}} \prec p_{i_n}$ and $p_{i_1} = p_{i_n}$.*

Since f,g are Morse-Smale with no fixed points in common and have a finite number of attractors, there always exists at least one cycle, and renumbering the points we can suppose we have a sequence of the form $p_1 \prec q_2 \prec p_3 \cdots \prec q_n \prec p_{n+1} = p_1$.

Step 1: Creating a Return Map

The return map will be created in a fundamental domain of g, $D = (p_1, g^{-1}(p_1)]$, by going inductively around the circle through the cycle.

Lemma 4.4. *D can be written as*

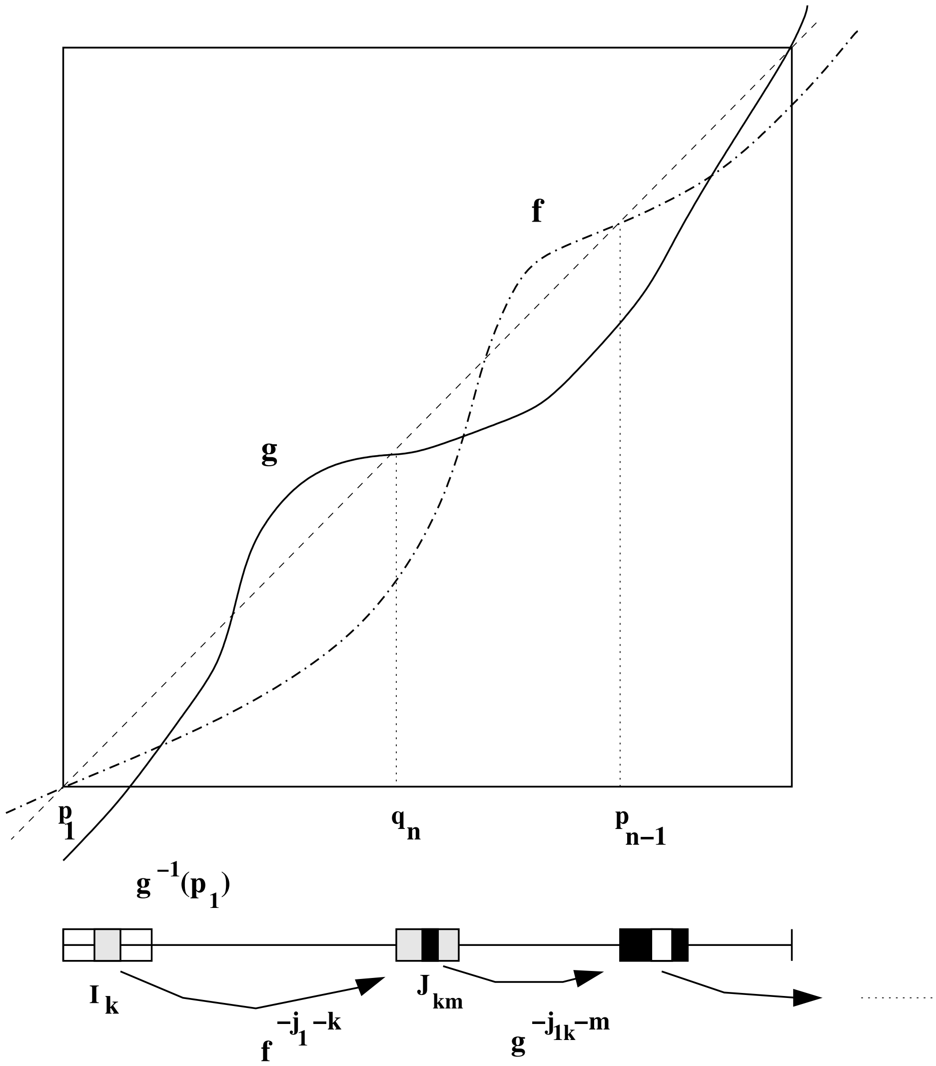
$$D = \biguplus_{i_1,\ldots,i_n \geq 0} I_{i_1,\ldots,i_n}$$

where each $I_{i_1,\ldots,i_n}$ is a right-closed interval such that:

(i) To each interval, $I_{i_1\ldots i_n}$, there is an associated map $h_{i_1\ldots i_n} \in < f, g >$ with $h^{-1}_{i_1\ldots i_n}(I_{i_1\ldots i_n}) \subset I$ when $i_n = 0$ and $h^{-1}_{i_1\ldots i_n}(I_{i_1\ldots i_n}) = I$ for $i_n > 0$.

(ii) If a point $c \neq g^{-1}(p_1)$ is the endpoint of $I_{i_1\ldots i_n}$, then it is on the orbit of the attracting points of the cycle. That is, there exists $h \in < f, g >$ and p_i (or q_i) of the cycle such that $h(p_i) = c$.

Proof. Denote by p_1^-, p_1^+, the repelling points of f closest to p_1, $p_1^- < p_1 < p_1^+$ (here we are looking at the lifts of f,g on the real line, with a small abuse of notation). We can suppose, without losing generality, that $q_n \in (p_1, p_1^+)$. If there is no K^{ss} blender-like set from the geometry of the functions, we cannot have an attractor-attractor pair for the system $< f, g >$. Therefore, the fixed point of g in (p_1, p_1^+) closest to p_1 is a repeller, which implies that the fundamental domain for g, $[p_1, g^{-1}(p_1)]$, is in $[p_1, p_1^+]$.



To create the expanding return map we will divide $I = [p_1, g^{-1}(p_1)]$ inductively. If j_1 is the first time that $f^{j_1}(q_n) \in (p_1, g^{-1}(p_1))$, then $D = (p_1, g^{-1}(p_1)] = \biguplus_{k=0}^{\infty} I_k$, where

$$I_0 = (f^{j_1}(q_n), g^{-1}(p_1)], I_k = (f^{j_1+k}(q_n), f^{j_1+k-1}(q_n)], k > 0.$$

Letting $h_k = f^{j_1+k}$, we have that $h_k^{-1}(I_k) = (q_n, c_k]$ for $c_k = f^{-1}(q_n)$ when $k > 0$ and $c_0 = f^{-j_1} \circ g^{-1}(p_1) \in [q_n, f^{-1}(q_n)]$.

Now consider q_n and as before let q_n^-, q_n^+ be the repelling points of g closest to q_n, $q_n^- < q_n < q_n^+$. Since g has to have a repeller as the fixed point closest to p_1 (not to create a K^{ss} set), $q_n^- \in (p_1, p_1^+)$. As $p_{n-1} \in B_g(q_n)$, then $p_{n-1} \in (q_n, q_n^+)$. Observe that $[q_n, c_k]$ is contained in a fundamental domain of f on the basin of p_1, and so $p_1 < c_k < p_1^+ < p_{n-1}$. In conclusion, for all k there is the following order on the real line: $q_n < c_k < p_{n-1} < q_n^+$.

This completes the first step of the induction and now we proceed with the inductive hypothesis:

(i) Suppose that $D = \biguplus_{i_1,\ldots,i_k \geq 0} I_{i_1\ldots i_k}$, where $I_{i_1\ldots i_k}$ are right-closed intervals and there exist the corresponding functions $h_{i_1\ldots i_k}$. The functions satisfy $h_{i_1\ldots i_k}^{-1}(I_{i_1\ldots i_k}) = (q_{n-k+1}, c_{i_1\ldots i_k}]$ (q_{n-k+1} can be substituted for p_{n-k+1} depending on k), with $c_{i_1\ldots i_k} = f^{-1}(q_{n-k+1})$ for $i_k > 0$ and $c_{i_1\ldots i_{k-1},0} \in [q_{n-k+1}, f^{-1}(q_{n-k+1})]$.

(ii) If a point, $c \neq g^{-1}(p_1)$ is the endpoint of $I_{i_1\ldots i_k}$, then there exists $h \in < f, g >$ and p_i (or q_i) of the cycle such that $h(p_i) = c$.

(iii) There is the following order on the real line: $q_{n-k+1} < c_{i_1\ldots i_k} < p_{n-k} < q_{n-k+1}^+$.

For each $h_{i_1\ldots i_k}^{-1}(I_{i_1\ldots i_k}) = (q_{n-k+1}, c_{i_1\ldots i_k}]$ there exists a $j_{i_1\ldots i_k}$ such that $g^{j_{i_1\ldots i_k}}(p_{n-k})$ is the first time that $f^{j_{i_1\ldots i_k}}(p_{n-k}) \in (q_{n-k+1}, c_{i_1\ldots i_k})$. Then

$$(q_{n-k+1}, c_{i_1\ldots i_k}] = \biguplus_{l=0}^{\infty} J_{i_1\ldots i_k,l}$$

where $J_{i_1\ldots i_k,0} = (g^{j_{i_1\ldots i_k}}(p_{n-k}), c_{i_1\ldots i_k}]$ and

$$J_{i_1\ldots i_k,l} = (g^{j_{i_1\ldots i_k}+l}(p_{n-k}), g^{j_{i_1\ldots i_k}+(l-1)}(p_{n-k})], l > 0.$$

Define $I_{i_1\ldots i_k,l} \subset I_{i_1\ldots i_k}$ by $h_{i_1\ldots i_k}(J_{i_1\ldots i_k,l})$. Then $D = \biguplus_{i_1,\ldots,i_{k+1}\geq 0} I_{i_1,\ldots,i_{k+1}}$. When $l > 0$,

$$I_{i_1\ldots i_k,l} = (h_{i_1\ldots i_k} \circ g^{j_{i_1\ldots i_k}+l}(p_{n-k}), h_{i_1\ldots i_k} \circ g^{j_{i_1\ldots i_k}+(l-1)}(p_{n-k})]$$

and so the endpoints of the interval are images of the attracting points of the cycle. When $l = 0$,

$$I_{i_1\ldots i_k,0} = (h_{i_1\ldots i_k} \circ g^j_{i_1\ldots i_k}(p_{n-k}), h_{i_1\ldots i_k}(c_{i_1\ldots i_k})].$$

By the inductive hypothesis $h_{i_1\ldots i_k}(c_{i_1\ldots i_k})$ is the endpoint of $I_{i_1\ldots i_k}$ and therefore $h_{i_1\ldots i_k}(c_{i_1\ldots i_k}) = h(p_i), h(q_i)$, or $g^{-1}(p_1)$ for some $h \in < f, g >$.

Let $h_{i_1\ldots i_k l} = h_{i_1\ldots i_k} \circ g^{j_{i_1\ldots i_k}+l}$, we have by construction that $h_{i_1\ldots i_{k+1}}^{-1}(I_{i_1,\ldots i_{k+1}}) = (p_{n-k}, c_{i_1\ldots i_{k+1}}]$ for some $c_{i_1\ldots i_{k+1}}$.

This shows the first three parts of the the inductive step and to complete the induction we have to show the order of points on the real line satisfies $p_{n-k} < c_{i_1\ldots i_{k+1}} < q_{n-k-1} < p_{n-k}^+$.

Since $q_{n-k-1} \in B_f(p_{n-k})$, then $q_{n-k-1} \in [p^-_{n-k}, p_{n-k}]$ or $q_{n-k-1} \in [p_{n-k}, p^+_{n-k}]$. As there is no K^{ss} (no attractor-attractor pairs) and $p_{n-k} \in [q_{n-k+1}, q^+_{n-k+1}]$, then $p^-_{n-k} \in [q_{n-k+1}, q^+_{n-k+1}]$, and so $q_{n-k-1} \in [p_{n-k}, p^+_{n-k}]$. Observing that

$$h^{-1}_{i_1 \dots i_k}(I_{i_1 \dots i_k}) \subset [q_{n-k+1}, q^+_{n-k+1}],$$

we have

$$h^{-1}_{i_1 \dots i_{k+1}}(I_{i_1 \dots i_{k+1}}) = g^{-j_{i_1 \dots i_k} - i_{k+1}} \circ h^{-1}_{i_1 \dots i_k}(I_{i_1 \dots i_{k+1}}) \subset [q_{n-k+1}, q^+_{n-k+1}].$$

Therefore $p_{n-k} < c_{i_1 \dots i_{k+1}} < q^+_{n-k+1} < q_{n-k-1} < p^+_{n-k}$, which ends the inductive process.

Going through the n steps of the cycle we conclude the lemma. □

Step 2: Bounded Distortion

We can write the maps $h^{-1}_{i_1 \dots i_n}$ as

$$h^{-1}_{i_1 \dots i_n} = g^{-k_{i_n}} \circ f^{-k_{i_n}} \dots g^{-k_{i_2}} \circ f^{-k_{i_1}}.$$

Fixing the indexes $i_1 \dots i_n$, set

$$J_{i_1 \dots i_m, l} = f^{-l} \circ g^{-k_{i_m}} \circ \dots g^{-k_{i_2}} \circ f^{-k_{i_1}}(I_{i_1 \dots i_n}) = f^{-l} \circ h^{-1}_{i_1 \dots i_m}(I_{i_1 \dots i_n})$$

with $l \leq k_{i_{m+1}}$, for $m < n-1$ and $l < k_{i_{m+1}}$ when $m = n-1$. Observe that when $m = 0$, $J_l = f^{-l}(I_{i_1 \dots i_n})$. By construction $J_{i_1, \dots, i_m}$ is contained in a fundamental domain of f (or g). Thus for a fixed set $\{i_1, \dots, i_m\}$, the intervals $J_{i_1, \dots, i_m, l} = f^{-l}(J_{i_1, \dots, i_m})$ are disjoint.

Let c_f, c_g be the distortion constants of f, g, and $c = max\{c_f, c_g\}$.

Lemma 4.5. *For $x, y \in I_{i_1 \dots i_n}$,*

$$e^{-nc} \leq \frac{Dh^{-1}_{i_1 \dots i_n}(x)}{Dh^{-1}_{i_1 \dots i_n}(y)} \leq e^{nc}$$

.

Proof.

$$log \frac{Dh^{-1}_{i_1 \dots i_n}(x)}{Dh^{-1}_{i_1 \dots i_n}(y)} = log(Dh^{-1}_{i_1 \dots i_n}(x)) - log(Dh^{-1}_{i_1 \dots i_n}(y))$$

$$= log[\prod_{l=0}^{k_{i_n}-1} Dg^{-1}(g^{-l} \circ h^{-1}_{i_1 \dots i_{n-1}}(x)) \cdot \prod_{l=0}^{k_{i_{n-1}}-1} Df^{-1}(f^{-l} \circ h^{-1}_{i_1 \dots i_{n-2}}(x)) \dots$$

$$\dots \prod_{l=0}^{k_{i_1}-1} Df^{-1}(f^{-l}(x))]$$

$$-log[\prod_{l=0}^{k_{i_n}-1} Dg^{-1}(g^{-l} \circ h^{-1}_{i_1 \dots i_{n-1}}(y)) \cdot \prod_{l=0}^{k_{i_{n-1}}-1} Df^{-1}(f^{-l} \circ h^{-1}_{i_1 \dots i_{n-2}}(y)) \dots$$

$$\cdots \prod_{l=0}^{k_{i_1}-1} Df^{-1}(f^{-l}(y))]$$

$$\leq \sum_{l=0}^{k_{i_n}-1} \mid log[Dg^{-1}(g^{-l} \circ h^{-1}_{i_1 \ldots i_{n-1}}(x))] - log[Dg^{-1}(g^{-l} \circ h^{-1}_{i_1 \ldots i_{n-1}}(y))] \mid$$

$$+ \sum_{l=0}^{k_{i_{n-1}}-1} \mid log[Df^{-1}(f^{-l} \circ h^{-1}_{i_1 \ldots i_{n-2}}(x))] - log[Df^{-1}(f^{-l} \circ h^{-1}_{i_1 \ldots i_{n-2}}(x))] \mid$$

$$+ \cdots + \sum_{l=0}^{k_{i_1}-1} \mid log[Df^{-1}(f^{-l}(x))] - log[Df^{-1}(f^{-l}(y))] \mid$$

$$\leq c \cdot \sum_{l=0}^{k_{i_n}-1} \mid g^{-l} \circ h^{-1}_{i_1 \ldots i_{n-1}}(x) - g^{-l} \circ h^{-1}_{i_1 \ldots i_{n-1}}(y) \mid$$

$$+c \cdot \sum_{l=0}^{k_{i_{n-1}}-1} \mid f^{-l} \circ h^{-1}_{i_1 \ldots i_{n-2}}(x) - f^{-l} \circ h^{-1}_{i_1 \ldots i_{n-2}}(x) \mid$$

$$+ \cdots + c \cdot \sum_{l=0}^{k_{i_1}-1} \mid f^{-l}(x) - f^{-l}(y) \mid$$

$$\leq c \cdot \sum_{l=0}^{k_{i_n}-1} \mid J_{i_1 \ldots i_{n-1},l} \mid +c \cdot \sum_{l=0}^{k_{i_{n-1}}-1} \mid J_{i_1 \ldots i_{n-2},l} \mid + \cdots + c \cdot \sum_{l=0}^{k_{i_1}-1} \mid J_l \mid \leq nc$$

where the last inequality is a consequence of the disjointess of the intervals $J_{i_1 \ldots i_k,l}$. Since this holds for all $x, y \in I_{i_1 \ldots i_n}$, we can invert the fraction to obtain the bounded constant from below. □

For $J \subset I_{i_1 \ldots i_n}$, by the mean value theorem, there exists $x \in J$ and $y \in I_{i_1 \ldots i_n}$ such that $|h^{-1}_{i_1 \ldots i_n}(J)| = Dh^{-1}_{i_1 \ldots i_n}(x) \cdot |J|$ and $|h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n})| = Dh^{-1}_{i_1 \ldots i_n}(y) \cdot |I_{i_1 \ldots i_n}|$. Therefore from the bounded distortion lemma,

$$\frac{|h^{-1}_{i_1 \ldots i_n}(J)|}{|J|} \geq e^{-c} \frac{|h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n})|}{|I_{i_1 \ldots i_n}|}$$

Step 3: Estimation of the Derivative for the Return Map

The objective is to prove that

$$Dh^{-1}_{i_1 \ldots i_n}(x) > e^c(1-\epsilon) \cdot (e^{-3c}(1-\epsilon)/\epsilon)^n,$$

where $\mid f_i - Id \mid_{C^1} < \epsilon$ and n is the length of the cycle. This will be greater than 1, if $e^{-3c}\dfrac{(1-\epsilon)^2}{\epsilon} > 1$, the importance being that this is independent of the length of the cycle n. Letting $k = min\{Df, Dg\}$, then the bounds for the return map become

$$e^{-3c}\frac{(1-\epsilon)^2}{\epsilon} \geq \frac{k^2}{1-k} \cdot e^{-3c} \geq \frac{e^{-2c}}{1-e^{-c}} \cdot e^{-3c} = \frac{e^{-5c}}{1-e^{-c}}$$

$$\geq \frac{e^{\frac{-5\epsilon}{1+\epsilon}}}{1-e^{\frac{-\epsilon}{1+\epsilon}}}$$

In particular, $Dh^{-1}_{i_1\dots i_n} > 1$ if $\epsilon \leq 0.38$.

As in the proof of Duminys lemma, we have to estimate $\frac{|h^{-1}_{i_1\dots i_n}(I_{i_1\dots i_n})|}{|I_{i_1\dots i_n}|}$ as then

$$Dh^{-1}_{i_1\dots i_n}(x) = lim_{r\to 0}\frac{|h^{-1}_{i_1\dots i_n}(B_r(x))|}{|B_r(x)|} \geq e^{-nc}\frac{|h^{-1}_{i_1\dots i_n}(I_{i_1\dots i_n})|}{|I_{i_1\dots i_n}|}.$$

Many of the estimates that follow, apart from induction on the length of the cycle, are analogous to lemmas 3.3 and 3.4 in the proof of Duminys proposition.

Lemma 4.6.

$$\frac{\mid h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k})\mid}{\mid h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_{k+1}})\mid} > \frac{1-\epsilon}{\epsilon}$$

where f, g are ϵ C^1-close to the identity.

Proof. We have that $h^{-1}(I_{i_1\dots i_k}) = (q_{n-k+1}, c_{i_1\dots i_k}]$ and $h^{-1}(I_{i_1\dots i_{k+1}}) \subset J_{i_1\dots i_{k+1}}$ where

$$J_{i_1\dots i_{k+1}} = (g^{j_{i_1\dots i_k}+i_{k+1}}(p_{n-k}), g^{j_{i_1\dots i_k}+i_{k+1}-1}(p_{n-k})], l > 0,$$

$$J_{i_1\dots i_k,0} = (g^{j_{i_1\dots i_k}}(p_{n-k}), c_{i_1\dots i_k}].$$

In the case of $i_{k+1} > 0$, we obtain

$$\frac{\mid h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k})\mid}{\mid h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_{k+1}})\mid} = \frac{\mid c_{i_1\dots i_k} - q_{n-k+1}\mid}{\mid g^{j_{i_1\dots i_k}+i_{k+1}}(p_{n-k}) - g^{j_{i_1\dots i_k}+i_{k+1}-1}(p_{n-k})\mid}$$

$$\geq \frac{\mid g^{j_{i_1\dots i_k}+i_{k+1}-1}(p_{n-k}) - q_{n-k+1}\mid}{\mid g^{j_{i_1\dots i_k}+i_{k+1}}(p_{n-k}) - g^{j_{i_1\dots i_k}+i_{k+1}-1}(p_{n-k})\mid}$$

$$= \frac{\mid c - q_{n-k+1}\mid}{\mid g(c) - c\mid} > \frac{1}{\epsilon}$$

The last inequality follows from the fact that

$$\mid g(c) - c \mid = \mid\mid c - q \mid - \mid g(c) - q_{n-k+1} \mid\mid = \mid c - q_{n-k+1} \mid (\mid 1 - Dg(z) \mid)$$

as $\mid g(c) - q_{n-k+1} \mid = \mid g(c) - g(q_{n-k+1}) \mid = Dg(z) \mid c - q_{n-k+1} \mid$ for some z. And so $\mid g(c) - c \mid > \epsilon \mid c - q_{n-k+1} \mid$.

In the case $i_{k+1} = 0$, we obtain

$$\frac{\mid h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k})\mid}{\mid h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k,0})\mid} = \frac{\mid c_{i_1\dots i_k} - q_{n-k+1}\mid}{\mid g^{j_{i_1\dots i_k}}(p_{n-k}) - c_{i_1\dots i_k}\mid}$$

$$\geq \frac{\mid g^{j_{i_1\dots i_k}}(p_{n-k}) - q_{n-k+1}\mid}{\mid g^{j_{i_1\dots i_k}}(p_{n-k}) - g^{j_{i_1\dots i_k}-1}(p_{n-k})\mid}$$

$$= \frac{\mid g(c) - q_{n-k+1}\mid}{\mid g(c) - c\mid} = Dg(z)\frac{\mid c - q_{n-k+1}\mid}{\mid g(c) - c\mid} > (1-\epsilon)\frac{\mid c - q_{n-k+1}\mid}{\mid g(c) - c\mid} > \frac{1-\epsilon}{\epsilon}$$

□

Lemma 4.7. *For* $1 \leq k \leq (n-1)$,

$$\frac{\mid h^{-1}_{i_1 \ldots i_{k-1}}(I_{i_1 \ldots i_k}) \mid}{\mid h^{-1}_{i_1 \ldots i_{k-1}}(I_{i_1 \ldots i_n}) \mid} \geq e^{-c} \frac{\mid h^{-1}_{i_1 \ldots i_k}(I_{i_1 \ldots i_k}) \mid}{\mid h^{-1}_{i_1 \ldots i_k}(I_{i_1 \ldots i_n}) \mid}$$

Proof. We may assume $h^{-1}_{i_1 \ldots i_k} = g^{-j_{i_1 \ldots i_{k-1}} - i_k} \circ h^{-1}_{i_1 \ldots i_{k-1}}$ and that $h^{-1}_{i_1 \ldots i_{k-1}}(I_{i_1 \ldots i_k})$ as well as $h^{-1}_{i_1 \ldots i_{k-1}}(I_{i_1 \ldots i_n})$ are contained in a fundamental domain of g. Then the lemma follows from the classical bounded distortion argument. □

Lemma 4.8.

$$\frac{\mid h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n}) \mid}{\mid I_{i_1 \ldots i_n} \mid} > (e^{-c}(1-\epsilon)/\epsilon)^{n-1} \frac{\mid h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n}) \mid}{\mid I_{i_1} \mid}$$

Proof.

$$\frac{\mid h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n}) \mid}{\mid I_{i_1 \ldots i_n} \mid} = \frac{\mid h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n}) \mid}{\mid I_{i_1} \mid} \cdot \frac{\mid I_{i_1} \mid}{\mid I_{i_1 \ldots i_n} \mid}$$

Which by lemma 4.7 is

$$\geq e^{-c} \frac{\mid h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n}) \mid}{\mid I_{i_1} \mid} \cdot \frac{\mid h^{-1}_{i_1}(I_{i_1}) \mid}{\mid h^{-1}_{i_1}(I_{i_1 \ldots i_n}) \mid}$$

Multiplying by $\frac{\mid h^{-1}_{i_1}(I_{i_1 i_2}) \mid}{\mid h^{-1}_{i_1}(I_{i_1 i_2}) \mid}$ and repeating the argument gives

$$\geq e^{-2c} \frac{\mid h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n}) \mid}{\mid I_{i_1} \mid} \cdot \frac{\mid h^{-1}_{i_1}(I_{i_1}) \mid}{\mid h^{-1}_{i_1}(I_{i_1 \ldots i_n}) \mid} \cdot \frac{\mid h^{-1}_{i_1 i_2}(I_{i_1 i_2}) \mid}{\mid h^{-1}_{i_1 i_2}(I_{i_1 \ldots i_{n-1}}) \mid}$$

Again repeating everything inductively in total (n-1) times obtain

$$\geq e^{-c(n-1)} \frac{\mid h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n}) \mid}{\mid I_{i_1} \mid} \cdot \frac{\mid h^{-1}_{i_1}(I_{i_1}) \mid}{\mid h^{-1}_{i_1}(I_{i_1 i_2}) \mid} \cdot \frac{\mid h^{-1}_{i_1 i_2}(I_{i_1 i_2}) \mid}{\mid h^{-1}_{i_1 i_2}(I_{i_1 i_2 i_3}) \mid} \cdot$$

$$\cdots \frac{\mid h^{-1}_{i_1 \ldots i_{n-1}}(I_{i_1 \ldots i_{n-1}}) \mid}{\mid h^{-1}_{i_1 \ldots i_{n-1}}(I_{i_1 \ldots i_n}) \mid}$$

Now apply lemma 4.6 to conclude that

$$> (e^{-c}(1-\epsilon)/\epsilon)^{n-1} \frac{\mid h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n}) \mid}{\mid I_{i_1} \mid}$$

□

Now we have to estimate $\frac{\mid h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n}) \mid}{\mid I_{i_1} \mid}$

Lemma 4.9. *When* $i_n > 0$

$$\frac{\mid h^{-1}_{i_1 \ldots i_k}(I_{i_1 \ldots i_k}) \mid}{\mid I_{i_1} \mid} > \frac{1-\epsilon}{\epsilon}$$

Proof. When $i_n > 0$, $h^{-1}_{i_1 \ldots i_k}(I_{i_1 \ldots i_k}) = D = (p_1, g^{-1}(p_1)]$,

$$I_0 = (f^{j_1}(q_n), g^{-1}(p_1)], I_{i_1} = (f^{j_1+i_1}(q_n), f^{j_1+i_1-1}(q_n)], i_1 > 0.$$

So the calculations as in the above lemma 4.6 hold to obtain

$$\frac{\mid h^{-1}_{i_1 \ldots i_k}(I_{i_1 \ldots i_k}) \mid}{\mid I_{i_1} \mid} > \frac{1-\epsilon}{\epsilon}.$$

□

Thus when $i_n > 0$, putting all the calculations together,

$$Dh^{-1}_{i_1 \ldots i_n}(x) > e^{-nc} \cdot (e^{-c}(1-\epsilon)/\epsilon)^{n-1} \cdot \frac{1-\epsilon}{\epsilon} = e^c \cdot (e^{-2c}(1-\epsilon)/\epsilon)^n.$$

When $i_n = 0$,

$$h^{-1}_{i_1 \ldots i_{n-1},0}(I_{i_1 \ldots i_{n-1},0}) = (p_1, c_{i_1,\ldots,i_{n-1},0}] \subset (p_1, g^{-1}(p_1)]$$

and not necessarily equal. There are two options, depending if

$$h^{-1}_{i_1 \ldots i_{n-1},0}(I_{i_1 \ldots i_{n-1},0}) \cap I_{i_1} \neq \emptyset$$

or

$$h^{-1}_{i_1 \ldots i_{n-1},0}(I_{i_1 \ldots i_{n-1},0}) \cap I_{i_1} = \emptyset$$

Lemma 4.10. *When $i_n = 0$ and*

$$h^{-1}_{i_1 \ldots i_{n-1},0}(I_{i_1 \ldots i_{n-1},0}) \cap I_{i_1} \neq \emptyset$$

then

$$\frac{|h^{-1}_{i_1 \ldots i_{n-1},0}(I_{i_1 \ldots i_{n-1},0})|}{|I_{i_1}|} > \frac{1-\epsilon}{\epsilon}.$$

Thus

$$Dh^{-1}_{i_1 \ldots i_{n-1},0}(x) > e^c \cdot (e^{-2c}(1-\epsilon)/\epsilon)^n.$$

Proof. In this case $f^{j_1+i_1}(q_n) \in I_{i_1}$ and

$$f^{j_1+i_1}(q_n) \in h^{-1}_{i_1 \ldots i_{n-1},0}(I_{i_1 \ldots i_{n-1},0})$$

Then

$$\frac{|h^{-1}_{i_1 \ldots i_{n-1},0}(I_{i_1 \ldots i_{n-1},0})|}{|I_{i_1}|} = \frac{|c_{i_1 \ldots i_{n-1},0} - p_1|}{|I_{i_1}|}$$

$$\geq \frac{|f^{j_1+i_1}(q_n) - p_1|}{|f^{j_1+i_1}(q_n) - f^{j_1+i_1-1}(q_n)|} > \frac{1-\epsilon}{\epsilon}$$

by the same calculation as in lemma 4.6. □

Now lets proceed to the second case when

$$h^{-1}_{i_1 \dots i_{n-1},0}(I_{i_1 \dots i_{n-1},0}) \cap I_{i_1} = \emptyset.$$

Since $g^{-1} \circ h^{-1}_{i_1 \dots i_{n-1},0} = h^{-1}_{i_1 \dots i_{n-1},1}$, and $\| f - Id \|_{C^1} < \epsilon$, we have

$$Dh^{-1}_{i_1 \dots i_{n-1},0}(x) = Dg(h^{-1}_{i_1 \dots i_{n-1},1}(x)) \cdot Dh^{-1}_{i_1 \dots i_{n-1},1}(x) > (1-\epsilon) \cdot Dh^{-1}_{i_1 \dots i_{n-1},1}(x).$$

We will estimate $Dh^{-1}_{i_1 \dots i_{n-1},1}(x)$ but in $I = I_{i_1 \dots i_{n-1},0} \cup I_{i_1 \dots i_{n-1},1}$. The usefulness of this is that

$$h^{-1}_{i_1 \dots i_{n-1},1}(I) \supset h^{-1}_{i_1 \dots i_{n-1},1}(I_{i_1 \dots i_{n-1},1}) = (p_1, g^{-1}(p_1)] = D$$

The bounded distortion argument of step 2 and lemmas 4.7, and 4.8 would have to be repeated with respect to the interval I.

Lemma 4.11. *When*

$$h^{-1}_{i_1 \dots i_{n-1},0}(I_{i_1 \dots i_{n-1},0}) \cap I_{i_1} = \emptyset$$

and for $x, y \in I$

$$\frac{Dh^{-1}_{i_1 \dots i_{n-1},1}(x)}{Dh^{-1}_{i_1 \dots i_{n-1},1}(y)} \geq e^{-2nc}$$

Proof. Firstly lets remember the notation used:

$$h^{-1}_{i_1 \dots i_n} = g^{-k_{i_n}} \circ f^{-k_{i_n}} \dots g^{-k_{i_2}} \circ f^{-k_{i_1}},$$

$$J_{i_1 \dots i_m, l} = f^{-l} \circ g^{-k_{i_m}} \circ \dots g^{-k_{i_2}} \circ f^{-k_{i_1}}(I_{i_1 \dots i_n}),$$

$= f^{-l} \circ h^{-1}_{i_1 \dots i_m}(I_{i_1 \dots i_n})$ with $l \leq k_{i_{m+1}}$, for $m < n-1$ and $l < k_{i_{m+1}}$ when $m = n-1$. The intervals $J_{i_1 \dots i_m, l}$ were proven to be disjoint.

When $h^{-1}_{i_1 \dots i_{n-1},0}(I_{i_1 \dots i_{n-1},0}) \cap I_{i_1 \dots i_{n-1},0} = \emptyset$, this implies $J_{i_1 \dots i_m, l}$ are disjoint for $l \leq k_{i_{m+1}}$, when $m < n-1$ and for $l \leq k_{i_{m+1}}$ when $m = n-1$. Then the bounded distortion argument can be applied to the function $g^{-1} \circ h^{-1}_{i_1 \dots i_{n-1},0}$ on the interval $I_{i_1 \dots i_{n-1},0}$.

Since $g^{-1} \circ h^{-1}_{i_1 \dots i_{n-1},0} = h^{-1}_{i_1 \dots i_{n-1},1}$, we obtain for $x, y \in I_{i_1 \dots i_{n-1},0}$

$$e^{nc} \geq \frac{Dh^{-1}_{i_1 \dots i_{n-1},1}(x)}{Dh^{-1}_{i_1 \dots i_{n-1},1}(y)} \geq e^{-nc}$$

If both x, y are in $I_{i_1 \dots i_{n-1},0}$ or both are in $I_{i_1 \dots i_{n-1},1}$, then the bounded distortion estimates hold as above. If $x \in I_{i_1 \dots i_{n-1},0}$ and $y \in I_{i_1 \dots i_{n-1},1}$, take $z \in \overline{I_{i_1 \dots i_{n-1},0}} \cap \overline{I_{i_1 \dots i_{n-1},1}}$. Then

$$\frac{Dh^{-1}_{i_1 \dots i_{n-1},1}(x)}{Dh^{-1}_{i_1 \dots i_{n-1},1}(y)} = \frac{Dh^{-1}_{i_1 \dots i_{n-1},1}(x)}{Dh^{-1}_{i_1 \dots i_{n-1},1}(z)} \cdot \frac{Dh^{-1}_{i_1 \dots i_{n-1},1}(z)}{Dh^{-1}_{i_1 \dots i_{n-1},1}(y)} \geq e^{-2nc}.$$

□

Lemma 4.6 does not have to be modified. Lemma 4.7 will hold for I in the sense that for $1 \le k \le (n-1)$

$$\frac{\mid h^{-1}_{i_1 \dots i_{k-1}}(I_{i_1 \dots i_k}) \mid}{\mid h^{-1}_{i_1 \dots i_{k-1}}(I) \mid} \ge e^{-c} \frac{\mid h^{-1}_{i_1 \dots i_k}(I_{i_1 \dots i_k}) \mid}{\mid h^{-1}_{i_1 \dots i_k}(I) \mid}.$$

This is because by construction of the return map $h^{-1}_{i_1 \dots i_{k-1}}(I)$ is contained inside the fundamental domain of g (or f) for $1 \le k \le (n-1)$.

As a consequence lemma 4.8 holds:

$$\frac{\mid h^{-1}_{i_1 \dots i_{n-1},1}(I) \mid}{\mid I \mid} > (e^{-c}(1-\epsilon)/\epsilon)^{n-1} \frac{\mid h^{-1}_{i_1 \dots i_{n-1},1}(I) \mid}{\mid I_{i_1} \mid}$$

As $I_{i_1} \subset [f^{j_1+i_1}(q_n), f^{j_1+i_1-1}(q_n)]$ and $h^{-1}_{i_1 \dots i_{n-1},1}(I) \supset D = (p_1, g^{-1}(p_1)]$, by the same calculations as in the previous steps,

$$\frac{h^{-1}_{i_1 \dots i_{n-1},1}(I)}{I_{i_1}} \ge \frac{\mid g^{-1}(p_1) - p_1 \mid}{\mid f^{j_1+i_1}(q_n) - f^{j_1+i_1-1}(q_n) \mid} > \frac{1-\epsilon}{\epsilon}$$

Therefore now we can estimate the derivative of $Dh^{-1}_{i_1 \dots i_{n-1},1}(x)$ for all $x \in I$,

$$Dh^{-1}_{i_1 \dots i_{n-1},1}(x) \ge e^{-2nc} \frac{\mid h^{-1}_{i_1 \dots i_{n-1},1}(I) \mid}{\mid I \mid}$$

$$> e^{-2nc} \cdot (e^{-c}(1-\epsilon)/\epsilon)^{n-1} \cdot \frac{1-\epsilon}{\epsilon} = e^{c} \cdot (e^{-3c}(1-\epsilon)/\epsilon)^{n}.$$

Finally,

$$Dh^{-1}_{i_1 \dots i_{n-1},0}(x) > e^{c}(1-\epsilon) \cdot (e^{-3c}(1-\epsilon)/\epsilon)^{n}$$

and so this estimate holds for all the return maps

$$Dh^{-1}_{i_1 \dots i_n}(x) > e^{c}(1-\epsilon) \cdot (e^{-3c}(1-\epsilon)/\epsilon)^{n}$$

Step 4: Minimality of the Interval and Density of Periodic Points

This is very similar to the end of proof of Duminys lemma.

Lemma 4.12. *To prove minimality of the interval $(p_1, g^{-1}(p_1))$, it is sufficient to show that for any open interval $J \subset (p_1, g^{-1}(p_1))$, there exists q_i (or p_i), the attractor of the minimal cycle, and maps $h_1, h_2 \in < f, g >$ such that $h_1(q_i) \in h_2^{-1}(J)$.*

Proof. Let $x, J \in (p_1, g^{-1}(p_1))$, where J is open, and x a point. As q_i is part of the cycle there exists a map $h_3 \in < f, g >$ such that $h_3(p_1)$ is arbitrary close to q_i, and in particular $h_1 \circ h_3(p_1) \in h_2^{-1}(J)$. Since p_1 is a global attractor in $[p_1, g^{-1}(p_1)]$, there exists a j such that $f^j \circ h_1 \circ h_3(x) \cap h_2^{-1}(J) \ne \circ$ and so $h_2 \circ f^j \circ h_1 \circ h_3(x) \cap J \ne \circ$. Since the point x and the interval J were arbitrary, this shows minimality. □

To end the proof, we will show that last the lemma holds. Let $\overline{A}$ be the set of endpoints of the intervals $I_{i_1 \dots i_n}$. Remember that if $c \in \overline{A}$, then there exists $h \in < f, g >$ and a point $q_i(p_i)$ of the cycle, such that $h(q_i) = c$. Take $J \subset [p_1, g^{-1}(p_1)]$ and we can assume $J \subset I_{i_1 \dots i_n}$ for some $i_1, \dots i_n$. On the contrary $J \cap \overline{A} \neq \circ$ and we are done. Since $\mid Dh^{-1}_{i_1 \dots i_n}(x) \mid > \lambda > 1$,then $\mid h^{-1}_{i_1 \dots i_n}(J) \mid > \lambda \cdot J$. Now or $h^{-1}_{i_1 \dots i_n}(J) \cap \overline{A} \neq \circ$ (and we are done) or $h^{-1}_{i_1 \dots i_n}(J) \subset I_{j_1 \dots j_n}$ for some $j_1 \dots j_n$. In the second case we have $\mid h^{-1}_{j_1 \dots j_n} \circ h^{-1}_{i_1 \dots i_n}(J) \mid > \lambda^2 \cdot J$. Proceeding in this manner, as $\lambda^n \to \infty$, the pre-images of J will have to swallow $c \in \overline{A}$ at some point.

Observe that as p_1 can be attracted arbitrary close to any of the points in the set $\overline{A}$, we actually have that the orbit of p_1 is dense in $(p_1, g^{-1}(p_1))$ and so the closed interval $[p_1, g^{-1}(p_1)]$ is minimal.

To show that $[p_1, g^{-1}(p_1)] \subset \overline{(Per(< f, g >)}$, we will use that the orbit of the attractor p_1 is dense. Given $J \subset [p_1, g^{-1}(p_1)]$, open, there exists $h \in < f, g >$ with $h(p_1) \in J$. As p_1 is the attractor, there exists a k such that $f^k \circ h(J) \subset J$ and $D(f^k \circ h) < 1$ in J. Then there exists a fixed point in J for the map $f^k \circ h \in < f, g >$.

Step 5: Bounds for Periodic f, g

Let f, g be periodic with periods p, q. Then the return maps $h^{-1}_{i_1 \dots i_n}$ are found with respect to $< f^p, g^q >$. Let $S = f^p, T = g^q$. The bounds on the derivatives will actually be the same as for maps with fixed points,

$$Dh^{-1}_{i_1 \dots i_n}(x) > e^c(1 - \epsilon) \cdot (e^{-3c}(1 - \epsilon)/\epsilon)^n$$

where c is $max\{c_f, c_g\}$.

Lemma 4.13. *For $x, y \in I_{i_1 \dots i_n}$,*

$$e^{-nc} \leq \frac{Dh^{-1}_{i_1 \dots i_n}(x)}{Dh^{-1}_{i_1 \dots i_n}(y)} \leq e^{nc}$$

.

Proof.

$$log \frac{Dh^{-1}_{i_1 \dots i_n}(x)}{Dh^{-1}_{i_1 \dots i_n}(y)} \leq \sum_{l=0}^{k_{i_n}-1} \mid log[DT^{-1}(T^{-l} \circ h^{-1}_{i_1 \dots i_{n-1}}(x))] - log[DT^{-1}(T^{-l} \circ h^{-1}_{i_1 \dots i_{n-1}}(y))] \mid$$

$$+ \sum_{l=0}^{k_{i_{n-1}}-1} \mid log[DS^{-1}(S^{-l} \circ h^{-1}_{i_1 \dots i_{n-2}}(x))] - log[DS^{-1}(S^{-l} \circ h^{-1}_{i_1 \dots i_{n-2}}(x))] \mid$$

$$+ \dots + \sum_{l=0}^{k_{i_1}-1} \mid log[DS^{-1}(S^{-l}(x))] - log[DS^{-1}(S^{-l}(y))] \mid$$

$$\geq \sum_{l=0}^{q(k_{i_n})-1} \mid log[Dg^{-1}(g^{-l} \circ h^{-1}_{i_1 \dots i_{n-1}}(x))] - log[Dg^{-1}(g^{-l} \circ h^{-1}_{i_1 \dots i_{n-1}}(y))] \mid$$

$$+\sum_{l=0}^{p(k_{i_{n-1}})-1} \mid log[Df^{-1}(f^{-l}\circ h^{-1}_{i_1\ldots i_{n-2}}(x))] - log[Df^{-1}(f^{-l}\circ h^{-1}_{i_1\ldots i_{n-2}}(x))] \mid$$

$$+\cdots+\sum_{l=0}^{p(k_{i_1})-1} \mid log[Df^{-1}(f^{-l}(x))] - log[Df^{-1}(f^{-l}(y))] \mid$$

$$\geq c\cdot\sum_{l=0}^{q(k_{i_n})-1} \mid g^{-l}\circ h^{-1}_{i_1\ldots i_{n-1}}(x) - g^{-l}\circ h^{-1}_{i_1\ldots i_{n-1}}(y) \mid$$

$$+c\cdot\sum_{l=0}^{p(k_{i_{n-1}})-1} \mid f^{-l}\circ h^{-1}_{i_1\ldots i_{n-2}}(x) - f^{-l}\circ h^{-1}_{i_1\ldots i_{n-2}}(x) \mid$$

$$+\cdots+c\cdot\sum_{l=0}^{p(k_{i_1})-1} \mid f^{-l}(x) - f^{-l}(y) \mid$$

$$\leq c\cdot\sum_{l=0}^{q(k_{i_n})-1} \mid J_{i_1\ldots i_{n-1},l}\mid + c\cdot\sum_{l=0}^{p(k_{i_{n-1}})-1} \mid J_{i_1\ldots i_{n-2},l}\mid + \cdots + c\cdot\sum_{l=0}^{p(k_{i_1})-1} \mid J_l$$

where $J_{i_1\ldots i_m,l}$ is defined as before to be $f^{-l}\circ h^{-1}_{i_1\ldots i_m}(I_{i_1\ldots i_n})$. Since $h^{-1}_{i_1\ldots i_m}(I_{i_1\ldots i_n})$ is contained in a fundamental domain of f (or g), lemma 3.6 implies that the intervals $J_{i_1\ldots i_m,l}$ are disjoint, and so the last inequality is again as before less then nc. □

The other lemma that used the bounded distortion constant was lemma 4.7

Lemma 4.14. *For* $1 \leq k \leq (n-1)$,

$$\frac{\mid h^{-1}_{i_1\ldots i_{k-1}}(I_{i_1\ldots i_k})\mid}{\mid h^{-1}_{i_1\ldots i_{k-1}}(I_{i_1\ldots i_n})\mid} \geq e^{-c}\frac{\mid h^{-1}_{i_1\ldots i_k}(I_{i_1\ldots i_k})\mid}{\mid h^{-1}_{i_1\ldots i_k}(I_{i_1\ldots i_n})\mid}$$

Proof. As before

$$h^{-1}_{i_1\ldots i_k} = T^{-j_{i_1\ldots i_{k-1}}-i_k}\circ h^{-1}_{i_1\ldots i_{k-1}}$$
$$= h^{-1}_{i_1\ldots i_k} = g^{-q(j_{i_1\ldots i_{k-1}}-i_k)}\circ h^{-1}_{i_1\ldots i_{k-1}}$$

and $h^{-1}_{i_1\ldots i_{k-1}}(I_{i_1\ldots i_k})$ as well as $h^{-1}_{i_1\ldots i_{k-1}}(I_{i_1\ldots i_n})$ are contained in a fundamental domain of g. So again lemma 3.6 will imply the disjointness of the intervals $g^{-l}\circ h^{-1}_{i_1\ldots i_{k-1}}$ for $0 \leq l \leq q(j_{i_1\ldots i_{k-1}} + i_k)$. Then the classical bounded distortion argument ends the proof. □

The rest of the proof goes on exactly as in the case of fixed points to give the same bounds on the return map derivative. Combining this with lemma 3.6 that $e^{-c} \leq Df^p(x) \leq e^c$ (same for g), obtain that $Dh^{-1}_{i_1\ldots i_n} > 1$ if $\frac{e^{-5c}}{1-e^{-c}} > 1$ or $\epsilon \leq 0.38$.

This completes the proof of the theorem □

Next we will prove corollary 4.2

Proof. If f, g are not necessarily orientation-preserving, consider $f \circ f$ and $g \circ g$ which are orientation-preserving. Then the return maps are created with respect to the IFS $< f^2, g^2 >$. If the maximum distortion of f, g is c, the maximum distortion of f^2, g^2 is $2c$. The worst estimation on the derivative of the return map from the proofs of theorems 2.1 and 2.4 is given in step 5 in the proof of theorem 2.1. Substituting $2c$ it becomes

$$DH \geq \frac{e^{-16c}}{3(1 - e^{-2c})}.$$

Then $DH > 1$ if f, g are 0.06-close to the identity in the C^2 topology. The rest of the proof of the corollary is the same as in theorems 2.1 and 2.4 applied to the system $< f^2, g^2 >$. □

5 Robustly Minimal IFS in S^1

Theorem 5.1. *Let f, g be C^2 Morse-Smale diffeomorphisms of the circle with no periodic points in common. There exists an $\epsilon > 0$ ($\epsilon \geq 0.38$) such that if f, g are ϵ-close to the identity in the C^2 topology, then if there is no K^{ss} set the system $< f, g >$ is robustly minimal.*

To prove the theorem we would need the following proposition which is based on the combinatorics of periodic points on the circle.

Proposition 5.2. *There is an interval K^{ss} for the iterated function system* IFS(f_0, f_1) *if and only if there is an interval K^{uu} for* IFS(f_0, f_1).

Proof. Suppose there exists a K^{uu} set but no K^{ss}. Let R_i, A_i denote the repellers and attractors of f_1 and $K^{uu} = [R, R_1]$ where R is a repeller of f_0. Then we can subdivide the circle in the following manner,

$$S^1 = [R, R_1] \cup [R_1, A_1] \cup [A_1, R_2] \cup [R_2, A_2] \cdots \cup [A_n, R].$$

We will show that in the interior of each of this closed intervals f_0 has an even number of fixed points. Denote this set by F. Then $\sharp F$ is even and the total number of fixed points of f_0 is $\sharp F \cup \{R\}$ is odd. This will contradict the fact that for any Morse-Smale on the circle the number of fixed points is even.

The interval $[R, R_1]$ does not have any fixed points of f_0 in its interior by definition. Let $[R, R^+]$ denote an repeller-attractor interval for f_0. If R^+ belongs to the interval of the form (A_i, R_{i+1}) then $[A_i, R^+]$ form an attractor-attractor pair for the system $IFS(f_0, f_1)$, and a K^{ss} set. Therefore $R^+ \in (R_i, A_i)$. The hypothesis of f_0 and f_1 having no fixed points in common is used here to guarantee $R^+ \neq R_i$ or A_j.

Again since there is no K^{ss} set there has to be a repeller $\overline{R}$ of f_0 in (R_i, A_i) closest to A_i. Consider $[R^+, \overline{R}] \subset (R_i, A_i)$. Then in $[R^+, \overline{R}]$, f_0 has an even number of fixed points and therefore the same holds for (R_i, A_i). For $j < i$ the intervals $[R_j, A_j], [A_j, R_{j+1}]$ do not contain any fixed points of f_0.

In this manner we proceed inductively now considering $[\overline{R}, \overline{R^+}]$ as the repeller-attractor pair, where $\overline{R^+} \in (R_k, A_k)$ with $k > i$. The same reasoning applies and this ends the proof. □

Proposition 5.3. *Suppose that there is no K^{ss} (or K^{uu}) set for the system $IFS(f_0, f_1)$ and there exists an interval I such that $f_i(I) \cap I \neq \emptyset$. Then for all $x \in S^1$ there exists $h_1 \in IFS(f_0, f_1)$ and $h_2^{-1} \in IFS(f_0^{-1}, f_1^{-1})$ with $h_1(x) \in I$ and $h_2^{-1}(x) \in I$.*

Proof. As before let p_i the attractors of f, q_i the attractors of g. In theorem 2.4 we defined a partial order relation on the attracting points by $p_i \prec q_j \Leftrightarrow p_i \in B_g(q_j)$, where B_g denotes the basin of attraction for g, with similar definitions for $q_i \prec p_j$. The no existence of K^{ss} sets was important in the inductive creation of the return map, in particular the order of points on the real line $p_{n-k} < c_{i_1 \ldots i_{k+1}} < q_{n-k-1} < p^+_{n-k}$ was crucial. Here we need a similar order.

Suppose that $x \in B_f(p_1)$. From now on we will work with the lifts of f, g, which we will denote by the same letters. Looking at the lifts on the real line with the same partial order relation as was defined on the circle, we can form an arbitrary long chain on the real line starting from p_1,

$$p_1 \prec q_2 \prec \cdots \prec p_n \prec \dots$$

We may assume $p_1 < q_2$ on the real line

Lemma 5.4. *There is the following order on the real line*

$$p_1 < q_2 < \cdots < p_n < \dots$$

Proof. The proof is by induction. Supposing that $q_{n-1} < p_n$, lets show that $p_n < q_{n+1}$. By hypothesis $p_n \in B_g(q_{n+1})$, then $p_n \in [q_{n+1}^-, q_{n+1}]$ or $p_n \in [q_{n+1}^+, q_{n+1}]$. If $p_n \in [q_{n+1}^+, q_{n+1}]$, as there is no K^{ss} (no attractor-attractor pairs) then necessarily $p_n^- \in [q_{n+1}^+, q_{n+1}]$, and so $[p_n^-, p_n] \subset (q_{n+1}^+, q_{n+1})$. As by the inductve hypothesis $q_{n-1} < p_n$, then $q_{n-1} \in [p_n^-, p_n] \subset (q_{n+1}^+, q_{n+1})$, a contradiction. Therefore $p_n \in [q_{n+1}^-, q_{n+1}]$ and $p_n < q_{n+1}$. □

Lemma 5.5. *There exists a sequence $h_k \in < f, g >$ such that $h_k(x) \to \infty$ (here we are looking at the lifts on the real line) $h_{k+1}(x) > h_k(x)$, and $h_{k+1} = f \circ h_k$(or $g \circ h_k$).*

Proof. Since the order $p_k < q_{k+1} < p_{k+2}$ holds for all k, we can attract x inductively to p_1 then to q_2, etc... In this manner given k there exists $h_k \in < f, g >$ such that $h_k \in B_f(p_k)$. As $p_k \in B_g(q_{k+1})$, then $g^m \circ h_k(x)$ is arbitrary close to q_{k+1} and since $p_k < q_{k+1}$ we have that $g^{m-1} \circ h_k(x) < g^m \circ h_k(x)$. Let $h_{k+l} = g^l \circ h_k$ for $l \leq m$. Because of the increasing order of the attractors $p_k < q_{k+1}^- < p_{k+1}$, $p_k \to \infty$, the same holds for h_k with $h_{k+1}(x) > h_k(x)$. □

From the lemma we can assume there exists a sequence of functions $h_k(x) \to \infty$. If $h_k(x) \notin I$ for any k, there exists a k such that $h_k(x) < I$ and $h_{k+1}(x) > I$. Suppose that $h_{k+1}(x) = f \circ h_k(x)$. Then $I \subset (h_k(x), f \circ h_k(x))$ and as f is an increasing function on the real line $f(I) > I$ contradicting that $f(I) \cap I \neq \emptyset$. Therefore there exists k such that $h_k \in I$.

Observing that $f(I) \cap I \neq \emptyset$ if and only if $f^{-1}(I) \cap I \neq \emptyset$ and there exists a K^{ss} set if and only there exists K^{uu}, repeating the argument we obtain h_2 with $h_2^{-1}(x) \in I$. □

Corollary 5.6. *If under the above hypothesis I is minimal, then $< f, g >$ is minimal (as well as $< f^{-1}, g^{-1} >$).*

Proof. Take any $x, y \in S^1$. It is enough to show that given $\epsilon > 0$ there exists $h \in < f, g >$ such that $h(x) \in B_\epsilon(y)$. By the proposition there exists h_1 with $h_1(x) \in I$, and h_2 with $h_2^{-1}(B_\epsilon(y)) \cap I \neq \emptyset$. Since I is minimal, take h_3 such that $h_3(x) \in h_2^{-1}(B_\epsilon(y)) \cap I$. Then $h_2 \circ h_3 \circ h_1(x) \in B_\epsilon(y)$. □

To prove theorem 5.1 it is enough to observe that by theorem 4.1 and the observation that follows there exists a blender-like set satisfying the hypothesis of the last corollary. That $\epsilon \geq 0.38$ again comes from step 3 of theorem 4.1 and the fact that we are not actually worried about the minimality of K^{ss} (theorem 2.1). The robustez comes from the fact that not having a K^{ss} set is a robust property under the condition that that fixed points are not in common. This ends the proof.

6 Spectral Decomposition

First we will deal with spectral decomposition on the real line, and afterwards pass on the circle considering the lifts to the real line. For now let g_1, g_2 be diffeomorphisms of the real line. We say g_i is Morse-Smale if the set of fixed points of both g_1 and g_2 is not empty and all the fixed points are hyperbolic.

Definition 6.1. *Spectral Decomposition for IFS on the real line The IFS $< g_1, g_2 >$ has spectral decompostion if the limit set*

$$L(< g_1, g_2 >) = \cup_{i=1}^{\infty} B_i \cup \{\pm\infty\}$$

where B_i transitive sets, and given any compact set $K \subset \mathbb{R}$

$$L(< g_1, g_2 >) \cap K = \cup_{j=1}^{n} (B_{j_i} \cap K) \text{ (a finite union)}$$

To state the theorem we have to enlarge the set of different types of K^{**}. Now $**$ can also be s or u where considering $I = [a, \infty)$, K^s is defined as

$$g_{i_1}(a) = a, g_{i_1} < Id \text{ in } (a, \infty) \text{ and } g_{i_2} > Id \text{ in } I.$$

Symmetrically denote as well by K^{ss} $I = (-\infty, a]$ where the relevant definition becomes

$$g_{i_1}(a) = a, g_{i_1} > Id \text{ in } (-\infty, a] \text{ and } g_{i_2} < Id \text{ in } I.$$

The fixed point a is an attractor of g_{i_1}. A K^u set is a K^s set for $< g_1^{-1}, g_2^{-1} >$.

The proof of Duminys lemma is exactly the same for K^s sets and so these are minimal if $| g_i - Id |_{C^2} \leq 0.14$.

Theorem 6.2. *Let g_1, g_2 be Morse-Smale diffeomorphisms of the real line with no fixed points in common. There exists an $\epsilon > 0$ ($\epsilon \geq 0.14$) such that if g_i are ϵ-close to the identity in the C^2 topology then $< g_1, g_2 >$ has spectral decomposition.*

Specifically $L(< g_1, g_2 >) = \cup_{i=1}^{\infty} B_i \cup \{\pm\infty\}$, where each B_i is either a K^{**} set or is a single fixed point of g_i.

Proof. By Duminys lemma for ($\epsilon \leq 0.14$) K^s, K^{ss} sets are minimal, and $\overline{K^{su}}, K^u, K^{uu}$ sets are transitive.

If z is in the ω-limit of $< g_1, g_2 >$, then z can be approximated by points of the form $y_l = \lim_{k \to +\infty} g_{\sigma}^{j_k}(x_l)$. It enough to show that if $y = \lim_{k \to +\infty} g_{\sigma}^{j_k}(x)$ then y is in some K^{**}, a fixed point of the maps g_i, or is $\{\pm\infty\}$. Let $\{p_i\}_{i \in \mathcal{Z}}$ be the ordered set of fixed points of both g_1 and g_2, which by hypothesis is not empty.

If $y \neq \{\pm\infty\}$, we can assume that $y \in (p_i, p_{i+1})$ where by abuse of notation p_i, p_{i+1} can take on the values of $-\infty, +\infty$ respectively. Also suppose that $I = [p_i, p_{i+1}]$ is not a K^{**} type and that p_i is an attractor for g_1 (the other cases are handled similar). Then there are three options:

(1) p_{i+1} is a repeller for g_1

(2) $p_{i+1} = \{\infty\}$

(3) p_{i+1} is repeller for g_2.

The following lemma says that for $I = [p_i, p_{i+1}]$, if a point leaves it, then it can never come back.

Lemma 6.3. *Let $x \in I = [p_i, p_{i+1}]$ but $g_l(x) \notin I$ ($l = 1$ or 2). Then $g_\sigma^j \circ g_l(x) \notin I$ for all σ, j.*

Proof. The point p_i is a fixed attractor of g_1 and suppose that $g_l(x) < p_i$ (the proof for the other case is the same). Necessarily $g_l = g_2$ because if $y > p_i$ then $g_1(y) > p_i$. As well by the assumptions $g_2 < Id$ in $[p_1, p_2)$. Let c be the closest attractor of g_2 to the left of p_i and if it does not exist,let $c = \infty$. If $y \leq c$, then $g_2(y) \leq c$ and if $y \in [c, p_i)$ then $g_2(y) < y$. Putting the above observations together conclude that for $y < p_i$, $g_l(y) < p_i, l = 1, 2$, and this proves the lemma. ☐

To take care of the first case, p_{i+1} is a repeller for g_2 implies that both g_1 and g_2 are below the identity line. Therefore if $y = \lim_{k\to+\infty} g_\sigma^{j_k}(x)$ and $g_\sigma^{j_{k-1}}(x)$ is in (p_i, p_{i+1}), it follows that $g_l \circ g_\sigma^{j_{k-1}}(x) < g_\sigma^{j_{k-1}}(x)$ for $l = 1, 2$.

The interval $[p_i, y]$ only contains a finite number of fundamental domains of g_2. So if the map g_2 appears an infinite number of times in the sequence σ then at some point $g_\sigma^{j_k}(x) < p_i$. But then by lemma 6.3 $g_l \circ g_\sigma^{j_k}(x) < p_i$ for $l = 1, 2$ and therefore $g_\sigma^{j_n}(x) \notin (p_i, p_{i+1}$ for $n > k$, a contradiction with that $y = \lim_{k\to+\infty} g_\sigma^{j_k}(x) \in (p_i, p_{i+1})$

Then for the sequence $\sigma = \{\sigma_l\}$, there exists an m such that $\sigma_l = 1$ for $l > m$. So we may assume that $y = \lim_{l\to+\infty} g_1^l \circ g_\sigma^{j_k}(x)$ where $g_\sigma^{j_k}(x) \in (p_i, p_{i+1})$. As p_i is the attractor for g_1 in I, consequently $y = p_i$, a contradiction as y is in the interior of I.

For case (2), $p_{i+1} = \infty$, since (p_i, p_{i+1}) is not a K^s set, this implies $g_2 < Id$ in (p_i, p_{i+1}). As in the first case if $g_\sigma^{j_{k-1}}(x)$ is in (p_i, p_{i+1}), then $g_l \circ g_\sigma^{j_{k-1}}(x) < g_\sigma^{j_{k-1}}(x)$ for $l = 1, 2$ and the rest of the proof goes on as for the first case.

If p_{i+1} is a repeller for g_1 (case 3) there are two sub-cases, $g_2(I) < Id$ or $g_2(I) > Id$. When $g_2(I) < Id$ the argument is again the same as in case (1). If $g_2(I) > Id$ since I is not a K^{su} set, then $g_2((p_i, p_{i+1})) \cap (p_i, p_{i+1}) = \emptyset$.

Remembering that $y = \lim_{k\to+\infty} g_\sigma^{j_k}(x)$ and take $g_\sigma^{j_k}(x)$ in (p_i, p_{i+1}). Suppose that in the sequence σ the function g_2 appears after the index j_k and will arrive at a contradiction.

Let the index $l > j_k$ be the first time that 2 appears in the sequence. Then

$$g_\sigma^l(x) = g_2 \circ g_\sigma^{l-1}(x) = g_2 \circ g_1^{l-j_k} \circ g_\sigma^{j_k}(x)$$

Since p_1 is an attrctor for g_1, $g_1^{l-j_k} \circ g_\sigma^{j_k}(x) \in (p_i, p_{i+1})$. As $g_2(I) \cap I = \emptyset$ this implies $g_2 \circ g_1^{l-j_k} \circ g_\sigma^{j_k}(x) \neq I$. By lemma 6.3 $g_\sigma^n(x) \neq I$ for all $n > l$, contradiction with that $y = \lim_{k\to+\infty} g_\sigma^{j_k}(x) \in I$.

Therefore can conclude that the number 2 never appears in the sequence σ for σ_l with $l > j_k$. Then $g_\sigma^l(x) = g_1^{l-j_k} \circ g_\sigma^{j_k}(x)$ as p_i is a attractor for g_1 in I implies $y = \lim_{l\to+\infty} g_1^l \circ g_\sigma^{j_k}(x) = p_1$, again a contradiction, and this ends the proof. ☐

The theorem for IFS on the circle takes form as

Theorem 6.4. *Let g_1, g_2 be Morse-Smale diffeomorphisms of the circle, both with fixed points but which are not in common. There exists an $\epsilon > 0$ ($\epsilon \geq 0.14$) such that if g_i are ϵ-close to the identity in the C^2 topology then $< g_1, g_2 >$ has spectral decomposition.*

Specifically $L(< g_1, g_2 >) = \cup_{i=1}^{n} B_i$, where each B_i is either a $K^{ss}, \overline{K^{su}}$, or K^{uu} set or is a single fixed point of g_i.

Proof. By corollary 2.7 there is no K^{ss} type set if and only if $< g_1, g_2 >$ is minimal and in which case the spectral decomposition is the whole circle. Therefore we can suppose that there is a K^{ss} type set.

Considering G_i as the lifts of g_i to the real line, the system $< G_1, G_2 >$ satisfies the hypothesis of the anterior theorem 6.2, and so there is the spectral decomposition

$$L(< G_1, G_2 >) = \cup_{i=1}^{\infty} B_i \cup \{\pm\infty\}.$$

The sets B_i cannot be of type K^s, K^u as g_i have fixed points. The lifts of the fixed points to the real line will repeat periodically going to $\pm\infty$.

If B_j is a K^{ss} type set then it is invariant in the sense that $G_i(B_j) \subset B_j$. This means that if $B_j = [a, b], B_k = [c, d]$ are K^{ss} sets and $a \leq x \leq d$, then $a \leq g_{\sigma}^{j_k}(x) \leq d$.

The lift of a K^{ss} type set on the circle will give and infinite number of K^{ss} type sets on the real line, call B_{j_k}, which go of to $\pm\infty$. Therefore given $x \in \mathbb{R}$ there exists $B_{j_l} = [a, b]$ and $B_{j_k} = [c, d]$ of K^{ss} type such that $a \leq x \leq d$. Then for any y with $y = \lim_{k\to+\infty} g_{\sigma}^{j_k}(x)$, $a \leq y \leq d$. This shows that $\{\pm\infty\}$ is not part of the spectral decomposition $L(< G_1, G_2 >)$.

From the above discussion we may conclude that $L(< G_1, G_2 >) = \cup_{i=1}^{\infty} B_i$ where B_i is of type $K^{ss}, \overline{K^{su}}$, or K^{uu} or is fixed point of G_i. Projecting these sets on the circle we obtain a finite number sets of sets each containing one of the fixed points of g_i and the same result for $L(< g_1, g_2 >)$. □

7 Symbolic Blender-like

Theorem 7.1. *Let c be such that $(1-c)k^{\alpha} = 1$ and $\mathcal{B}(Id, c)_{\mathcal{S}_k^{\alpha,r}(M)}$ be a ball of radius c about the identity map $Id = (\tau, Id)$ in $\mathcal{S}_k^{\alpha,r}(M)$. Consider*

$$\Gamma \in \mathcal{H}^r(M) \cap \mathcal{B}(Id, c)_{\mathcal{S}_k^{\alpha,r}(M)},$$

where $\Gamma = \tau \ltimes < \gamma_1, \ldots, \gamma_k >$. Suppose there exists a bounded open set $B \subset M$, a finite number of bounded closed sets U_i and the respective maps $H_i \in < \gamma_1^{-1}, \ldots, \gamma_k^{-1} >$ such that

(i) Covering property:

$$\overline{B} \subset \bigcup_{i=1}^{k} int(U_i),$$

with $H_i(U_i) \subset B$ and $DH_i > 1$ in U_i.

(ii) Periodic point with minimal orbit: there exists a hyperbolic periodic point $p_\Gamma \in B$ of $< \gamma_1, \ldots, \gamma_k >$ such that $B \subset Orb(p_\Gamma)$.

Then B is a cs-symbolic blender-like set in $\mathcal{S}_k^{\alpha,r}(M)$ for Γ.

First lets prepare the notation.

Denote by $\xi(\theta_n, \ldots, \theta_1)$ a sequence that satisfies $\{\xi \in \Sigma_k; \xi_{-j} = \theta_j, j \leq n\}$. And by $\xi(0, \theta_1, \ldots, \theta_n)$ that satisfies $\{\xi \in \Sigma_k; \xi_j = \theta_j, j \leq n\}$. Sometimes instead of a single index θ_j we will use blocks of a sequence.

The following notation will be handy, $\circ_{i=1}^{n} \gamma_{\tau^i(\xi)} = \gamma_{\tau^n(\xi)} \circ \cdots \circ \gamma_{\tau(\xi)}$

Each map H_i can be written as $\gamma_{\xi_j}^{-1} \circ \cdots \circ \gamma_{\xi_1}^{-1}$ and let v_i denote the block $\{\xi_j \ldots \xi_1\}$ respective to each map. Then

$$H_i = \circ_{j=1}^{|v_i|} \gamma_{\tau^{-j}(\xi(v_i))}^{-1}$$

for all sequences $\xi(v_i)$.

As $DH_i > 1$ in U_i, let σ be the minimum over the expanding constants of H_i in U_i.

Take a neighborhood Ω of Γ such that for all $\Psi \in \Omega$ and all sequences of the form $\xi(v_i)$,

$$\circ_{j=1}^{|v_i|} \psi_{\tau^{-j}(\xi(v_i))}^{-1}(U_i) \subset B$$

and $\circ_{j=1}^{|v_i|} \psi_{\tau^{-j}(\xi(v_i))}^{-1}$ are expanding in U_i with expansion at least σ.

By hypothesis p_Γ is a periodic point in B and may write

$$\gamma_{\theta_n} \circ \cdots \circ \gamma_{\theta_1}(p_\Gamma) = p_\Gamma.$$

Take the sequence θ to be periodic with the block $\{\theta_1, \ldots, \theta_n\}$. Then (θ, p_Γ) is a periodic point for Γ.

Since p_Γ is hyperbolic, we may assume that the neighborhood Ω is such that for all $\Psi \in \Omega$ there is continuation of p_Γ given by $p_\Psi \in B$ such that (θ, p_Ψ) is a periodic point of Ψ.

Let L be the Lebesgue number of the open cover

$$\overline{B} \subset \bigcup_{i=1}^{l} int(U_i).$$

By minimality of B for $< \gamma_1, \gamma_2 >$, the orbit of p_Γ is dense in B. Then there exists a finite set of functions $\{h_i\}$ in $< \gamma_1, \gamma_2 >$ so that the set of points $\{h_i(p_\Gamma)\}$ is $L/8$ dense in B.

Each map h_i can be written as $\gamma_{\xi^i_j} \circ \cdots \circ \gamma_{\xi^i_1}$ Designating by w_i the block $\{\xi^i_j \dots \xi^i_1\}$ then

$$h_i = \circ_{j=0}^{|w_i|-1} \gamma_{\tau^j(\xi(0,w_i))}$$

for all sequences $\xi(0, w_i)$.

Also let Ω be so that for $\Psi \in \Omega$ and an open interval $V \subset B$ with $\mid V \mid > \rho > L/8$ there exists a block w_i for which

$$\circ_{j=0}^{|w_i|-1} \psi_{\tau^j(\xi(0,w_i))}(p_\Psi) \in V$$

for all sequences $\xi(0, w_i)$.

Since $\Gamma \in \mathcal{H}^r$, $C_\Gamma = 0$. As well there exists a c_Γ,

$$\Gamma \in \mathcal{B}(Id, c_\Gamma)_{\mathcal{S}^{\alpha,r}_k(M)}$$

with $(1 - c_\Gamma)k^\alpha > 1$.

Therefore we can assume that the neighborhood Ω of Γ is contained in

$$\mathcal{B}(Id, c_\Gamma)_{\mathcal{S}^{\alpha,r}_k(M)}$$

and is small enough as to satisfy for all $\Psi \in \Omega$

$$C_\Psi \sum_{j=0}^{\infty} \frac{1}{((1 - c_\Gamma)k^\alpha)^j} < L/8.$$

Proposition 7.2. *For $\Psi \in \Omega$, given $B(x, r)$ (a ball of radius r around x) in B there is a sequence of blocks $\{v_{\theta_n}, \dots, v_{\theta_1}\}$ and a block w_i (depending on $B(x, r)$), such that*

$$\circ_{j=0}^{|w_i|-1} \psi_{\tau^j(\zeta(0,w_i))}(p_\Psi) \in \circ_{j=1}^{\beta} \psi^{-1}_{\tau^{-j}(\xi(v_{\theta_n},\dots,v_{\theta_1}))}(B(x, r))$$

for all sequences $\zeta(0, w_i)$, $\xi(v_{\theta_n}, \dots, v_{\theta_1})$ where $\beta = \sum_{j=1}^{n} \mid v_{\theta_j} \mid$

Proof.

Lemma 7.3. *Consider two sequences $\zeta(\theta_n, \dots, \theta_1)$ and $\xi(\theta_n, \dots, \theta_1)$ with the extra assumption that $W^s_{loc}(\zeta(\theta_n, \dots, \theta_1)) = W^s_{loc}(\xi(\theta_n, \dots, \theta_1))$. Then for $1 \leq i \leq n$,*

$$d(\circ_{j=1}^{i} \psi^{-1}_{\tau^{-j}(\zeta(\theta_n,\dots,\theta_1))}(x), \circ_{j=1}^{i} \psi^{-1}_{\tau^{-j}(\xi(\theta_n,\dots,\theta_1))}(x)) < C_\Psi (\frac{1}{k^\alpha})^{n-i} \sum_{j=0}^{i-1} \frac{1}{((1 - c_\Gamma)k^\alpha)^j}$$

Proof. The proof is by induction, for $i = 1$ by the Holder-continuity hypothesis we have that

$$d(\psi_{\tau^{-1}(\zeta(\theta_n,\ldots,\theta_1))}(x), \psi_{\tau^{-1}(\xi(\theta_n,\ldots,\theta_1)(x))}) \leq C_\Psi(\frac{1}{k^\alpha})^{n-1}$$

Supposing that the formula is valid at step $i-1$ and lets show that it is also valid for step i. Applying the triangle inequality gives,

$$\begin{aligned} & d(\circ_{j=1}^{i}\psi^{-1}_{\tau^{-j}(\zeta(\theta_n,\ldots,\theta_1))}(x), \circ_{j=1}^{i}\psi^{-1}_{\tau^{-j}(\xi(\theta_n,\ldots,\theta_1))}(x)) \\ \leq\ & d(\circ_{j=1}^{i}\psi^{-1}_{\tau^{-j}(\zeta(\theta_n,\ldots,\theta_1))}(x), \psi^{-1}_{\tau^{-i}(\xi(\theta_n,\ldots,\theta_1)} \circ_{j=1}^{i-1} \psi^{-1}_{\tau^{-j}(\zeta(\theta_n,\ldots,\theta_1))}(x)) \\ & + d(\psi^{-1}_{\tau^{-i}(\zeta(\theta_n,\ldots,\theta_1)} \circ_{j=1}^{i-1} \psi^{-1}_{\tau^{-j}(\xi(\theta_n,\ldots,\theta_1))}(x), \circ_{j=1}^{i}\psi^{-1}_{\tau^{-j}(\xi(\theta_n,\ldots,\theta_1))}(x)) \end{aligned}$$

Now using that the inverses of the functions expand at most $(1-c_\Gamma)^{-1}$ and setting $y = \circ_{j=1}^{i-1}\psi_{\tau^{-j}(\xi(\theta_n,\ldots,\theta_1))}(x)$ obtain

$$< (1-c_\Gamma)^{-1} d(\circ_{j=1}^{i-1}\psi^{-1}_{\tau^{-j}(\zeta(\theta_n,\ldots,\theta_1))}(x), \circ_{j=1}^{i-1}\psi^{-1}_{\tau^{-j}(\xi(\theta_n,\ldots,\theta_1))}(x))$$

$$+ d(\psi^{-1}_{\tau^{-i}(\zeta(\theta_n,\ldots,\theta_1)}(y), \psi^{-1}_{\tau^{-i}(\xi(\theta_n,\ldots,\theta_1)}(y))$$

From the induction on the first term and the Holder-continuity on the second,

$$< (1-c_\Gamma)^{-1} C_\Psi(\frac{1}{k^\alpha})^{n-i+1} \sum_{j=0}^{i-2} \frac{1}{((1-c_\Gamma)k^\alpha)^j} + \epsilon(\frac{1}{k^\alpha})^{n-i}$$

$$= C_\Psi(\frac{1}{k^\alpha})^{n-i} \sum_{j=0}^{i-2} \frac{(1-c_\Gamma)^{-1}}{k^\alpha} \frac{1}{((1-c_\Gamma)k^\alpha)^j} + C_\Psi(\frac{1}{k^\alpha})^{n-i}$$

$$= C_\Psi(\frac{1}{k^\alpha})^{n-i} \sum_{j=1}^{i-1} \frac{1}{((1-c_\Gamma)k^\alpha)^j} + C_\Psi(\frac{1}{k^\alpha})^{n-i}$$

$$= C_\Psi(\frac{1}{k^\alpha})^{n-i} \sum_{j=0}^{i-1} \frac{1}{((1-c_\Gamma)k^\alpha)^j}$$

which ends the proof of the lemma. □

Observe that for the case when $i = n$,

$$d(\circ_{j=1}^{i}\psi^{-1}_{\tau^{-j}(\zeta(\theta_n,\ldots,\theta_1))}(x), \circ_{j=1}^{i}\psi^{-1}_{\tau^{-j}(\xi(\theta_n,\ldots,\theta_1))}(x)) < C_\Psi \sum_{j=0}^{n-1} \frac{1}{((1-c_\Gamma)k^\alpha)^j} < L/8. \quad (1)$$

To prove the proposition, if $r \geq L/2$ then by the initial assumptions there exists a w_i such that

$$\circ_{j=0}^{|w_i|-1}\psi_{\tau^j(\xi(0,w_i))}(p_\Psi) \in B(x,r),$$

which concludes the proposition. So let $r < L/2$ and then we have the following lemma.

Lemma 7.4. *Consider* $0 < r < L/2$ *and* $x \in B$ *such that* $B(x,r) \subset B$. *There exists a sequence of blocks* $\{v_{\theta_n}, \ldots, v_{\theta_1}\}$ *and a specific sequence* $\zeta(v_{\theta_n}, \ldots, v_{\theta_1})$ *such that*

$$diam(\circ_{j=1}^{m_n} \psi^{-1}_{\tau^{-j}(\zeta(v_{\theta_n},\ldots,v_{\theta_1}))}(B(x,r))) > L/2$$

for this sequence. And for all sequences of the form $\xi(v_{\theta_n}, \ldots, v_{\theta_1})$

$$\circ_{j=1}^{m_n} \psi^{-1}_{\tau^{-j}(\xi(v_{\theta_n},\ldots,v_{\theta_1}))}(B(x,r))) \subset B$$

where $m_n = \sum_{i=1}^{n} \mid v_{\theta_i} \mid$.

Proof. Observe that as $r < L/2$,

$$B(x,r) \subset U_{\theta_1}$$

for some θ_1. Consider

$$\circ_{j=1}^{|v_{\theta_1}|} \psi^{-1}_{\tau^{-j}(\xi(v_{\theta_1}))}(B(x,r)) \subset B.$$

Lets distinguish two cases

(i) Either for all sequences $\xi(v_{\theta_1})$

$$diam(\circ_{j=1}^{|v_{\theta_1}|} \psi^{-1}_{\tau^{-j}(\xi(v_{\theta_1}))}(B(x,r))) < L/2$$

(ii) Or there exists a sequence $\zeta(v_{\theta_1})$ such that

$$diam(\circ_{j=1}^{|v_{\theta_1}|} \psi^{-1}_{\tau^{-j}(\xi(v_{\theta_1}))}(B(x,r))) \geq L/2.$$

Assuming the first case call Ξ_1 the set of all sequences of the form $\xi(v_{\theta_1})$ and let

$$\mathcal{A}_1 = \{\circ_{j=1}^{|v_{\theta_i}|} \psi^{-1}_{\tau^{-j}(\zeta)}(B(x,r)); \zeta \in \Xi\}.$$

The next goal is to prove that $diam(\mathcal{A}_1) < L$ and so $\mathcal{A}_1 \subset U_{\theta_2}$.

This follows from the fact that

$$diam(\circ_{j=1}^{|v_{\theta_1}|} \psi^{-1}_{\tau^{-j}(\xi(v_{\theta_1}))}(B(x,r))) < L/2$$

combined with lemma 7.3 which states

$$d(\circ_{j=1}^{|v_{\theta_1}|} \psi^{-1}_{\tau^{-j}(\zeta(v_{\theta_1}))}(x), \circ_{j=1}^{|v_{\theta_1}|} \psi^{-1}_{\tau^{-j}(\xi(v_{\theta_1}))}(x)) < C_\Psi \sum_{j=0}^{|v_{\theta_1}|-1} \frac{1}{((1-c_\Gamma)k^\alpha)^j} < L/8$$

for all sequences $\zeta(v_{\theta_1}), \xi(v_{\theta_1})$.

Suppose at step n we have constructed a sequence $\theta_n, \ldots, \theta_1$ with the additional hypothesis that

$$diam(\circ_{j=1}^{m_l} \psi^{-1}_{\tau^{-j}(\xi(v_{\theta_l},\ldots,v_{\theta_1}))}(B(x,r))) < L/2$$

for all sequences of the form $\xi(v_{\theta_l},\dots,v_{\theta_1})$ with $1\le l\le n$ and $m_l=\sum_{i=1}^{l}\mid v_{\theta_i}\mid$.

Define the sets Ξ_l the set of all sequences of the form $\xi(v_{\theta_l}\dots v_{\theta_1})$ and

$$\mathcal{A}_l=\{\circ_{j=1}^{m_l}\psi^{-1}_{\tau^{-j}(\zeta)}(B(x,r));\zeta\in\Xi_l\}.$$

where $m_l=\sum_{i=1}^{l}\mid v_{\theta_i}\mid$ and $1\le l\le n$

By induction assume $\mathcal{A}_l\subset U_{\theta_{l+1}}$ for $1\le l\le n-1$.

Now lets produce a sequence of length $n+1$ observing that $diam(\mathcal{A}_n)<L$. As in the case of $\mathcal{A}_1$, this follows from the inductive hypothesis and lemma 7.3 from which

$$d(\circ_{j=1}^{m_n}\psi^{-1}_{\tau^{-j}(\zeta(v_{\theta_n},\dots,v_{\theta_1}))}(x),\circ_{j=1}^{m_n}\psi^{-1}_{\tau^{-j}(\xi(v_{\theta_n},\dots,v_{\theta_1}))}(x))<C_\Psi\sum_{j=0}^{m_n-1}\frac{1}{((1-c_\Gamma)k^\alpha)^j}<L/8$$

for all sequences $\zeta(v_{\theta_1}),\xi(v_{\theta_1})$.

Therefore $\mathcal{A}_n\subset U_j$ for some U_j. Let $\theta_{n+1}=j$ and assume again the hypothesis that

$$diam(\circ_{j=1}^{m_{n+1}}\psi^{-1}_{\tau^{-j}(\xi(v_{\theta_{n+1}},\dots,v_{\theta_1}))}(B(x,r)))<L/2 \tag{2}$$

for all sequences of the form $\xi(v_{\theta_{n+1}},\dots,v_{\theta_1})$ with $m_{n+1}=\sum_{i=1}^{n+1}\mid v_{\theta_i}\mid$.

Lets prove that this process cannot go on forever. As $\mathcal{A}_n\subset U_{\theta_{n+1}}$ for all n, and by the initial conditions the maps $\circ_{j=1}^{|v_i|}\psi^{-1}_{\tau^{-j}(\xi(v_i))}$ are expanding in U_i with expansion at least σ, implies

$$diam(\circ_{j=1}^{m_n}\psi^{-1}_{\tau^{-j}(\xi(v_{\theta_{n+1}},\dots,v_{\theta_1}))}(B(x,r)))>\sigma^n r.$$

For a sequence of size n, big enough so that $\sigma^n r>L/2$ this would contradict the hypothesis (eq. 2).

Therefore there exists a sequence of blocks $\{v_n,\dots,v_1\}$ and a specific sequence $\zeta(v_n,\dots,v_1)$ such that

$$diam(\circ_{j=1}^{m_n}\psi^{-1}_{\tau^{-j}(\zeta(v_{\theta_n},\dots,v_{\theta_1}))}(B(x,r)))>L/2.$$

□

Lemma 7.5. *Consider the sequence of blocks $\{v_{\theta_n},\dots,v_{\theta_1}\}$ given by previous lemma 7.4. There exists a point $z\in B$ and a ball $B(z,L/4)$ so that*

$$B(z,L/4)\subset\circ_{j=1}^{m_n}\psi^{-1}_{\tau^{-j}(\xi(v_{\theta_n},\dots,v_{\theta_1}))}(B(x,r)))$$

for all sequences of the form $\xi(v_{\theta_n},\dots,v_{\theta_1})$.

Proof. Let $\overline{\theta}=v_{\theta_n}\dots v_{\theta_1}$ be the concatenation of the block from lemma 7.4. Consider the boundary of the set

$$\circ_{j=1}^{m_n}\psi^{-1}_{\tau^{-j}(\xi(\overline{\theta}))}(B(x,r))$$

with respect to a given sequence $\xi(\overline{\theta})$. Since the fiber maps are diffeomorphisms the boundary is a connected set given by

$$F_{\xi(\overline{\theta})} = \{\circ_{j=1}^{m_n} \psi^{-1}_{\tau^{-j}(\xi(\overline{\theta}))}(y) : y \in \partial(B(x,r))\}.$$

With respect to the specific sequence of lemma 7.4, $\zeta(\overline{\theta})$, by lemma 7.3 and equation 1

$$F_{\xi(\overline{\theta})} \subset \{y \in B : \exists z \in F_{\zeta(\overline{\theta})}, d(y,z) < L/8\}.$$

for all sequences $\xi(\overline{\theta})$. Thus

$$diam(\bigcap_{\xi(\overline{\theta})} \circ_{j=1}^{m_n} \psi^{-1}_{\tau^{-j}(\xi(\overline{\theta}))}(B(x,r))) \geq L/2 - 2L/8 = L/4.$$

Therefore there exists a point z such that

$$B(z, L/4) \subset \circ_{j=1}^{m_n} \psi^{-1}_{\tau^{-j}(\xi(v_{\theta_n},\dots,v_{\theta_1}))}(B(x,r)))$$

for all sequences of the form $\xi(v_{\theta_n}, \dots, v_{\theta_1})$, which concludes the proof. □

To end the proof of the proposition, by the initial hypothesis there exists a block w_i such that

$$\circ_{i=0}^{|w_i|-1} \psi_{\tau^j(\zeta(0,w_i))}(p_\Psi) \in \overline{B(\circ_{j=1}^{m_n} \psi^{-1}_{\tau^{-j}(\zeta(v_{\theta_n},\dots,v_{\theta_1}))}(z), L/4)} \subset$$

$$\circ_{i=1}^{\beta} \psi^{-1}_{\tau^{-j}(\xi(v_{\theta_n},\dots,v_{\theta_1}))}(B(x,r))$$

for all sequences $\zeta(0, w_i)$, $\xi(v_{\theta_n}, \dots, v_{\theta_1})$. □

Now we are ready to prove theorem 2.10.

Proof. To show that B is a symbolic blender-like, we have to show that for a given sequence ξ and an open set $U \subset B$,

$$W^u(\theta, p_\Psi) \cap (W^s_{loc}(\xi, \tau) \times U) \neq \emptyset.$$

For a fixed $x \in U$ by proposition 7.2 there exists a sequence of blocks $\{v_{\mu_n}, \dots, v_{\mu_1}\}$ and a block w_i such that for any sequences of the form $\xi(v_{\mu_n}, \dots, v_{\mu_1})$ and $\zeta(0, w_i)$,

$$\circ_{j=0}^{|w_i|-1} \psi_{\tau^j(\zeta(0,w_i))}(p_\Psi) \in \circ_{j=1}^{\beta} \psi^{-1}_{\tau^{-j}(\xi(v_{\mu_n},\dots,v_{\mu_1}))}(U).$$

For a block sequence v_i let v_i^{-1} represent the sequence written in reverse order. Rearranging the last equation obtain that for any sequence of the form $\zeta(0, w_i, v^{-1}_{\mu_n}, \dots, v^{-1}_{\mu_1})$

$$\circ_{j=0}^{\beta+|w_i|-1} \psi_{\tau^j(\zeta(0,w_i,v^{-1}_{\mu_n},\dots,v^{-1}_{\mu_1}))}(p_\Psi) \in U.$$

Define a sequence η by

$$\eta = \{\dots, \theta_1, \theta_0, w_i, v^{-1}_{\mu_n}, \dots, v^{-1}_{\mu_1}, (\xi_j)_{j\geq 0}\}$$

centered at ξ_0. Then $\eta \in W^s_{loc}(\xi,\tau)$ and is of the form $\zeta(0, w_i, v^{-1}_{\mu_n}, \dots, v^{-1}_{\mu_1})$. Therefore

$$\circ_{j=0}^{\beta+|w_i|-1} \psi_\eta(p_\Psi) \in U.$$

And so

$$(\eta, \circ_{i=1}^{\beta} \psi_{\tau^{i-\beta-1}(\zeta(v_{\theta_n},\dots,v_{\theta_1}))}(p_\Psi)) \in W^s_{loc}(\xi,\tau) \times U.$$

It follows that

$$\Psi^{-\beta-|w_i|+1}(\eta, \circ_{j=0}^{\beta+|w_i|-1} \psi_\eta(p_\Psi)) = (\tau^{-\beta-|w_i|+1}(\xi), p_\Psi) \in (W^u_{loc}(\theta), p_\Psi).$$

Therefore

$$(\eta, \circ_{i=1}^{\beta} \psi_{\tau^{i-\beta-1}(\zeta(v_{\theta_n},\dots,v_{\theta_1}))}(p_\Psi)) \in W^{uu}(\mathcal{O}(\theta, p_\Psi)) \cap (W^s_{loc}(\xi,\tau) \times U).$$

Thus the set B is a cs-symbolic blender like and so the proof is complete. $\square$

8 Reduction on the Number of Branches of Return Maps

Theorem 8.1. *There exists a generic set G in $Diff^r(S^1), r \geq 2$, such that for $f, g \in G \cap B(Id, 0.06)$ the following conditions are satisfied.*

(i) There exists an open minimal set B such that $\overline{B} \subset \overline{Per < f, g >}$.

(ii) There is a finite set of closed intervals U_j such that $\overline{B} \subset \bigcup_{j=1}^{m} int(U_j)$. To each U_j there is an associated map $H_j \in < f^{-1}, g^{-1} >$ such that $DH_j > 1$ in U_j and $H_j(U_j) \subset B$.

Corollary 8.2. *Consider $\mathcal{B}(Id, \lambda)_{\mathcal{H}^r(S^1)}$ to be a ball of radius λ about the identity map $Id = (\tau, Id)$ in $\mathcal{H}^r(S^1)$. For a given α let c be such that $(1-c)k^{\alpha} = 1$, and $\lambda = min\{c, 0.06\}$.*

There exists a generic set $\Lambda \subset \mathcal{H}^r(S^1)$ for $r \geq 2$ such that for

$$\Gamma \in \mathcal{B}(Id, \lambda)_{\mathcal{H}^r} \cap \Lambda$$

Γ has cs-symbolic blender-like in $\mathcal{S}_k^{\alpha,r}(S^1)$, $r \geq 1$.

First lets obtain the corollary from the theorem.

Proof. (corollary 8.2) Let the set $\Lambda \subset \mathcal{H}^r(S^1)$ be defined as

$$\Gamma = \tau \ltimes < \gamma_1, \ldots, \gamma_k > \in \Lambda \text{ if } \gamma_i \in G \text{ for } i = 1, \ldots, k.$$

As G is generic in $Diff^r(S^1)$, Λ is generic in $\mathcal{H}^r$. As $\gamma_i \in B(Id, 0.06)$ the previous theorem may be applied to the system $< \gamma_1, \gamma_2 >$. The corollary then follows by using theorem 2.10. □

The generic set G comes from the next proposition.

Definition 8.3. *With respect to an IFS $< f_1, f_2 >$ a pair of functions (h_1, h_2) with $h_i \in < f_1, f_2 >$ is said to be reducible to a pair $(\overline{h_1}, \overline{h_2})$ if there exists a sequence of functions $f_{j_1}, \ldots, f_{j_k}$ such that $h_1 = f_{j_k} \circ \cdots \circ f_{j_1} \circ \overline{h_1}$ and $h_2 = f_{j_k} \circ \cdots \circ f_{j_1} \circ \overline{h_2}$.*

Proposition 8.4. *For f_1, f_2 Morse-Smale in $Diff^r(S^1)$ denote by n_i the period of f_i and by $\{p_j\}$ the set of periodic points of both f_i.*

There exists a generic set G in $Diff^r(S^1), r \geq 1$, contained in Morse-Smale, such that for $f_i \in G$ and $h_1, h_2 \in < f_1^{n_1}, f_2^{n_2} >$ with $h_1(p_k) = h_2(p_j)$ we have that (h_1, h_2) is reducible to $(f_i^{mn_i}, Id)$. Also $p_j = p_k$ is a fixed point of the same function $f_i^{n_i}$.

Proof. The proof is by induction on the length $\mid h \mid$, where the length is the number of compositions of functions $f_i^{n_i}$. Let G_l be the set such that the lemma holds for h of length $\leq l$. We will show that G_l is open and dense in $Diff^r(S^1)$ for all $l \geq 0$, and so $\cap G_l$ is generic.

As Morse-Smale functions are open and dense in $Diff^r(S^1)$, we can suppose f_i are Morse-Smale and perturbing f_i if necessary that f_i have no periodic points in common. As this last condition is open we obtain that f_i are in G_1.

By induction suppose f_i are in G_l and there exists $\overline{h_1}, \overline{h_2} \in < f_1^{n_1}, f_2^{n_2} >$ of length $l+1$ with $\overline{h_1}(p_k) = \overline{h_2}(p_j)$. We can suppose that there are the following two cases

(i) $\overline{h_1} = f_1^{n_1} \circ h_1$ and $\overline{h_2} = f_1^{n_1} \circ h_2$ or

(ii) $\overline{h_1} = f_1^{n_1} \circ h_1$ and $\overline{h_2} = f_2^{n_1} \circ h_2$

for some functions h_1, h_2 of length l.

In case (i) the pair $(\overline{h_1}, \overline{h_2})$ is reducible to (h_1, h_2) and here we can apply the induction hypothesis.

For case (ii) we may assume that or $h_1(p_k)$ is not a fixed point of $f_1^{n_1}$ or $h_2(p_j)$ is not a fixed point of $f_2^{n_2}$. On the contrary $\overline{h_1}(p_k) = \overline{h_2}(p_j)$ gives $f_1^{n_1}(p_k) = f_2^{n_2}(p_j)$ contradicting that the two functions do not have fixed points in common.

So suppose that $h_1(p_k)$ is not a fixed point of $f_1^{n_1}$. Then we can perturb f_1, arbitrary small around the point $f_1^{n_1-1}(h_1(p_k))$ such that $f_1^{n_1}(p_k) \neq \overline{h_2}(p_j)$ and without affecting the periodic points of f_i. The perturbation is the standard $f_\epsilon = f_i + \epsilon\phi$ where ϕ is a bump function with support small enough as to not affect the rest of periodic points, and ϵ controlling the size of the perturbation.

After the perturbation and with abuse of notation we may assume that we are working with maps $f_i^{n_i}$ such that for the sequence of functions given by $\overline{h_1}, \overline{h_2}$, we have

$$\overline{h_1}(p_k) \neq \overline{h_2}(p_j).$$

Since there are a finite number periodic points and a finite number of functions of length $l+1$, then in a limited number of perturbations we will obtain maps f_i arbitrary close to the original maps and satisfying $\overline{h_1}(p_k) \neq \overline{h_2}(p_j)$ for all $\overline{h_i}$ of length $l+1$. Since this conditions is open this completes the inductive step. □

Now we will prove theorem 8.1

Proof. Assume that f, g are in $G \cap B(Id, 0.06)$, G as above. By corollary 2.5 we may assume that f, g are orientation preserving. By the theorems 2.1, 2.4 there is a fundamental domain D of f (or g) which is minimal and $D \subset \overline{Per < f, g >}$. In D there was constructed the backwards expanding return map with an infinite number of branches.

To reduce the return map to a finite number of branches the idea is to inductively take out the accumulation points of the branches by throwing them into the interior of D via some other map.

Lemma 8.5. *We may assume that the domain of the return map is $D = (p, g^{-1}(p)]$. Then $\overline{D} = \bigcup_{j=1}^{m} L_j$, where L_j are closed intervals, $L_j \subset int(D)$ for $2 \leq j \leq n-1$. To each interval there is an associated map $H_j \in < f, g >$ such that $H_j(L_j) \subset D$, $H_1(L_1)$ and $H_n(L_n) \subset int(D)$. Also $DH_j > 1$ in L_j.*

Proof. Lets deal with the more complicated case when there is no K^{ss} set and the blender-like was constructed in theorem 2.4 inductively. The reader is referred to step 1 of the proof for the notation. We will proceed as well inductively, and as will become clearer, the induction is done on the indexes i_k of the intervals $I_{i_1 \dots i_n}$.

The domain of the return map H is $D = (p_1, g^{-1}(p_1)]$ and since by theorem 2.6 $< f^{-1}, g^{-1} >$ is minimal there exists $h^{-1} \in < f^{-1}, g^{-1} >$ such that $h^{-1}(p_1) \in int(D)$. Lets show $h^{-1}(p_1) \neq c$, where c is a discontinuity point of the return map. By construction of the return map there exists $\overline{h}$ and a periodic point of say g, q_j, such that $c = \overline{h}(q_j)$. Then $\overline{h}^{-1} \circ h^{-1}(p_1) = q_j$, which contradicts that $f, g \in G$ and proposition 8.4.

Therefore $h^{-1}(p_1) \in int(I_{i_1 \dots i_n})$ for some $i_1 \dots i_n$. By the same reasons $H^m \circ h^{-1}(p_1)$ never hits the endpoints of any interval of the form $I_{i_1 \dots i_n}$. This means that the map $H^m \circ h^{-1}(p_1)$ is well defined for all m. There exists m big enough and l_1 such that $L_1 = [p_1, l_1]$ satisfies $H^m \circ h^{-1}(L_1) \subset int(D)$ and $D(H^m \circ h^{-1}) > \lambda > 1$ in L_1.

There exists n_1 the first time that $f^{j_1+n_1}(q_n) \in int(L_1)$. Then

$$\overline{D} = L_1 \bigcup_{0 \leq i_1 \leq n_1, i_2, \dots, i_n \geq 0} I_{i_1 \dots i_n}.$$

Define R_1 on these intervals, which may overlap by, by

$$R_1 = H^m \circ h^{-1} \text{ in } L_1,\ R_1 = H = h^{-1}_{i_1 \dots i_n} \text{ for } 0 \leq i_1 \leq n_1, \text{ and } i_2, \dots, i_n \geq 0.$$

Suppose by the inductive hypothesis that at step k there is the following.

(i) Closed intervals $L_1, \dots, L_k$, $L_j \subset int(D)$ for $2 \leq j$ and the associated maps $H_j \in < f^{-1}, g^{-1} >$ with $DH_j > \lambda > 1$ in L_j.

(ii) We can write $\overline{D}$ as

$$\overline{D} = \bigcup_{j=1}^{k} L_j \cup \bigcup I_{i_1 \dots i_n}.$$

where the last union is taken over indexes $i_1 \dots i_n$ that satisfy $0 \leq i_j \leq n_j$ for $j \leq k$, and $i_{k+1}, \dots, i_n \geq 0$.

(iii) As a consequence of the first two points, there is the return map R_k defined in the intervals L_j and $I_{i_1 \dots i_n}$ (which overlap) for the above indexes with $DR_k > \lambda > 1$.

$$R_k = H_j \text{ in } L_j,\ R_k = h^{-1}_{i_1 \dots i_n} \text{ for } 0 \leq i_j \leq n_j \text{ for } j \leq k, \text{ and } i_{k+1}, \dots, i_n \geq 0.$$

The objective now is to limit the index i_{k+1} superiorly. Consider the point q_{n-k+1}, by theorem and minimality of S^1 there exists $h^{-1} \in < f^{-1}, g^{-1} >$ such that $h^{-1}(q_{n-k+1}) \in int(D)$. Since f, g are in G, by proposition 8.4 and the same reasons as in the first step of the induction $h^{-1}(q_{n-k+1})$ is not one of the discontinuity points of the original return map H. The same holds for the iterates $H^m \circ h^{-1}(q_{n-k+1})$.

With respect to the intervals $h^{-1}_{i_1 \dots i_k}(I_{i_1 \dots i_k}) = (q_{n-k+1}, c_{i_1 \dots i_k}]$, consider

$$c = inf_{\{0 \leq i_j \leq n_j\}} \{ \mid c_{i_1 \dots i_k} - q_{n-k+1} \mid \}$$

Define t to be

$$t = inf_{\{0 \le i_j \le n_j\}} \{Dh^{-1}_{i_1 \dots i_k}(x); x \in [q_{n-k+1}, q_{n-k+1} + c]\}$$

Take the number m of iterates by H big enough so that

$$D(H^m \circ h^{-1})(q_{n-k+1}) > \lambda^m/t > \lambda > 1.$$

Then there exists an interval $[q_{n-k+1}, l_{k+1}]$ such that the same is satisfied for all x in the interval. We may suppose that $l_{k+1} \le q_{n-k+1} + c$ and define

$$L_{k+1} = h_{i_1 \dots i_k}([q_{n-k+1}, l_{k+1}]),$$

and the corresponding map

$$H_{k+1} = H^m \circ h^{-1} \circ h^{-1}_{i_1 \dots i_k}.$$

Observe that $L_{k+1} \subset int(D)$. The derivative of H_{k+1} is

$$DH_{k+1} = D(H^m \circ h^{-1}) \cdot Dh^{-1}_{i_1 \dots i_k} > \lambda^m/t \cdot t > \lambda.$$

Let n_{k+1} be the first time that $g^{j_{i_1 \dots i_k} + n_{k+1}}(p_{n-k})$ belongs to (q_{n-k+1}, l_{k+1}). For $i_j \le n_j$ for $j \le k$ and $i_{k+1} > n_{k+1}$, the interval $I_{i_1 \dots i_n}$ is contained in L_{k+1}. Then

$$\overline{D} = \bigcup_{j=1}^{k+1} L_j \cup \bigcup I_{i_1 \dots i_n}$$

where the last union is taken over indexes $i_1 \dots i_n$ that satisfy $0 \le i_j \le n_j$ for $j \le k+1$, and $i_{k+2}, \dots, i_n \ge 0$.

Define the return map R_{k+1} in these intervals that overlap as

$$H_j \text{ for } x \in L_j \text{ and } h^{-1}_{i_1 \dots i_n} \text{ for } x \in I_{i_1 \dots i_n}$$
$$\text{with } 0 \le i_j \le n_j \text{ for } j \le k+1, \text{ and } i_{k+2}, \dots, i_n \ge 0.$$

This completes the induction. Going through the n steps of the cycle almost completes the proof of the lemma in the case of blender-like when f, g have no K^{ss} set. The last step is to obtain that the final interval L_m is contained in $int(D)$, which can be done by repeating the same process as for L_1.

For the case when the fundamental domain D, in which the return map is constructed, is part of a K^{ss} set (see the proof of theorem 2.1), the set K^{ss} is not necessarily minimal for $< f^{-1}, g^{-1} >$ (it is transitive). In the above induction the minimality of $< f^{-1}, g^{-1} >$ was important for throwing points into the interior of D. In the K^{ss} case we will use the geometry of the functions to accomplish this.

Lets suppose K^{ss} is of the form $[a, b]$, where a is an attractor for f and b is the attractor for g, f, g both with fixed points. The domain D is given by $D = (g(a), f^{-1}(g(a))]$. What is needed is to find $h \in < f, g >$ such that $h^{-1}(g(a)) \in int(D)$.

Consider j such that $g^{-j} \circ f^{-1}(g(a)) \in [a, g(a)]$. Since f, g are in G, proposition 8.4 implies $g^{-j} \circ f^{-1}(g(a))$ is actually in the interior of $f^k(D)$ for some k. Therefore, $f^{-k} \circ g^{-j} \circ f^{-1}(g(a))$ is in the interior of D. The rest of the proof is similar as in the case of the cycle. □

To end the proof of the proposition, first extend the closed intervals L_1 and L_m to closed intervals U_1, U_m such that $H_1(U_1), H_m(U_m) \subset int(D)$ and $DH_1, DH_m > \lambda > 1$ in U_1, U_m respectively. Set

$$D_1 = U_1 \bigcup U_m \bigcup_{j=1}^{n-1} L_j.$$

Then D_1 is a closed connected interval and $\overline{D} \subset int(D_1)$.

For $2 \leq j \leq m-1$ extend L_j to closed intervals $U_j \subset int(D)$, such that $H(U_j) \subset int(D_1)$ and $DH_j > \lambda > 1$ in U_j.

Consider for $2 \leq j \leq n-1$

$$K_{1j} = H_j^{-1}(H_j(U_j) \cap U_1), K_{mj} = H_j^{-1}(H_j(U_j) \cap U_m)$$

with the associated maps defined as

$$H_{1j} = H_1 \circ H_j, H_{mj} = H_m \circ H_j.$$

As $H_1(U_1)$ and $H_m(U_m)$ are in the interior of D obtain that

$$H_{ij}(K_{ij}) \subset int(D).$$

Let V_{ij1}, V_{ij2} denote the two closed connected components of $U_j - int(K_{ij})$. Then

$$\bigcup_{i,j,k} int(V_{ijk}) \bigcup_{i,j} int(K_{ij}) = \bigcup_{i=1}^{m} int(U_i) \supset \overline{D}$$

and

$$H_j(V_{ijk}) \subset int(D), H_{ij}(K_{ij}) \subset int(D)$$

with $DH_j > 1$ in V_{ijk}, $DH_{ij} > 1$ in K_{ij}. Reordering and renaming the intervals and the return maps we obtain the theorem. □

References

[1] C. Bonatti and L. J. Diaz, *Persistence of nonhyperbolic transitive diffeomorphisms*, Annals of Math. **2** (1996), no. 143, 357–396.

[2] ———, *Robust heterodimensional cycles and c1-generic dynamics*, J. Inst. Math. Jussieu **3** (2008), no. 7, 469–525.

[3] C. Bonatti, L. J. Diaz, and M. Viana, *Dynamics beyond uniform hyperbolicity*, Encyclopedia of mathematical sciences, Springer, 2005.

[4] A. S Gorodetski, *Regularity of central leaves of partially hyperbolic sets and its applicationsaslo*, Izvestia RAN (The Newsletter of the Russian Academy of Science) **70** (2006), no. 6, 52–78.

[5] A. S. Gorodetski and Yu. S. Ilyashenko, *Certain properties of skew products over a horseshoe and a solenoid*, Proc. of Steklov Institute (2000), no. 231, 96–118.

[6] M. Hirsh, C. Pugh, and M. Shub, *Invariant manifolds*, Lecture Notes in Math. **583** (1977).

[7] J Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. Jour. (1981), no. 30, 713–747.

[8] A. Koropecki and M. Nassiri, *Transitivity of generic semigroups of area-preserving surface diffeomorphisms*, Math. Zeitschrift **266** (2010), 707–719.

[9] M. Nassirri and E. R. Pujals, *Robust transitivity in hamiltonian dynamics*, Ann. Scient. Éc. Norm. Sup., To be published.

[10] A. Navas, *Grupos de difeomorfismos del circulo*, Ensaios Matematicos **13** (2007).

[11] V. Nitica and A Török, *An open dense set of stably ergodic diffeomorphisms in a neighborhood of a non-ergodic one*, Topology **40** (2001), 259–278.

[12] E. R. Pujals and M. Sambarino, *Homoclinic bifurcations, dominated splitting, and robust transitivity*, Handbook of dynamical systems, Elsevier B. V., Amsterdam, Vol. 1B 2006, pp. 327–378.

[13] F. Rodriguez-Hertz, M. A. Rodriguez-Hertz, A. Tahzibi, and R. Ures, *A criterion for ergodicity of non-uniformly hyperbolic diffeomorphisms*, Electron. Res. Announc. Math. Sci. **14** (2001), 74–81.

[14] M. Shub and A. Wilkinson, *Pathological foliations and removable zero exponents*, Invent. Math. **139** (2000), no. no. 3, 495–508.