IMPA - Instituto Nacional de Matemática Pura e Aplicada

Some results on Hydrodynamical Limit of Exclusion Process in Non-homogeneous Medium

Tertuliano Franco

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Advisor: Claudio Landim

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Abstract

We present here three results concerned on hydrodynamical limit of exclusion process. The first one: for conductances driven by any increasing function W, the time evolution of the spatial density of particles is given by a parabolic partial equation associated to a symmetric operator $\frac{d}{dx}\frac{d}{dW}$, expressing a large class of non-homogeneous cases. The second one is about a d-dimensional case, where slow bonds (bonds of conductance of order N^{-1}) models a membrane slowing down the passage of particles between two regions. It is also proved the hydrodynamical limit of such case. At last, the third result: for the one-dimensional case with finite slow bonds of parameter $N^{-\beta}$, the hydrodynamical limit has three different behaviors depending if $\beta \in [0,1)$, $\beta = 1$ or $\beta \in (1,\infty)$.

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À minha família, Xykão, Mãemosa, Moara.

Contents

Chapter 1

Introduction

The subject of this PhD thesis is the hydrodynamical limit of exclusion process in non-homogeneous medium. The goal of this introduction is to clarify what does it mean and what kind of results we present here.

This work consists of three parts corresponding to three different papers (Chapters 2, 3 and 4). Each chapter is self-contained and can be read independently. In the beginning of each chapter we give the references about the respective paper and also the collaborators, to whom I am very much grateful. Namely, Ana Patrícia Gonçalves, Claudio Landim, Adriana Neumann and Glauco Valle.

First of all, the motivation of this work. The exclusion process, to be described ahead, is a well-known random dynamics, having a extensive literature associated, both in Mathematics and Physics. The hydrodynamical limit is the name given to the convergence of the time trajectory of the spatial density of particles of a interacting particle system, when re-scaling time and space in a suitable way. Here, the particle system considered is the exclusion process.

In the language of Probability Theory, the hydrodynamical limit is a law of large numbers for the trajectory (in time) of the spatial density of particles. Re-scaling in a suitable way both space and time, this trajectory of the spatial density of particles (which is random) converges in distribution to a solution of a partial differential equation. Being such solution unique, therefore deterministic, also holds the convergence in probability. Of course, this occurs only under a hypothesis concerning the limit of the spatial density of particles at time zero, which must be the initial condition for the PDE.

This study of time-evolution of spatial density of particles is of clear interest in Physics (specially in Statistical Mechanics). Besides, there are obvious relations with PDE's. Indeed, by means of Probability techniques, the three papers here present proofs of existence of solutions for the respective PDE's, for instance. On the other hand, tools from PDE's are also required: uniqueness is needed in the method we have utilized, and it is proved here invoking Analysis tools. Particular motivations of each model (Chapters 2, 3 and 4) are given in the introduction of the respective chapter. Much more can be said about the motivations, but we turn now to details of our work.

About the exclusion process: Let $\mathbb{T}_N^d$ be the d-dimensional discrete torus with N^d sites, or else, $(\mathbb{Z}/N\mathbb{Z})^d$. Each site of $\mathbb{T}_N^d$ is allowed to have one or no particle. Then, the space state will be the space of configurations $\eta \in \{0,1\}^{\mathbb{T}_N}$. Remark: the name exclusion comes from this rule of at most one particle per site. We say that two sites $x, y \in T_N^d$ are neighbors (denoted by $x \sim y$) if $|x - y| = 1$ in the norm of the sum of coordinates. For each pair of neighbors x, y, we associate a number $\xi_{x,y}^N = \xi_{y,x}^N > 0$, usually called *conductance*. The dynamics of the exclusion process can be described as follows. For each site $x \in T_N^d$, we associate an independent Poisson clock (Poisson Process) of parameter

$$\sum_{y;\, y\sim x} \xi_{x,y}^N \,.$$

When this clock rings, if there is no particle at x, nothing happens. If there is a particle at x, this particle choose a neighbor y with probability proportional to $\xi_{x,y}^N$. If this site y is empty, the particle moves to there. If y is occupied, nothing happens.

The process can be also characterized in terms of a generator L_N (generator of the corresponding semi-group of the Markov Process) given by

$$L_N f(\eta) \;=\; \sum_{x,y;\; x\sim y} \eta(x)\,(1-\eta(y))\,\xi_{x,y}^N\,[f(n^{x,y}) - f(\eta)]\,,$$

where $\eta^{x,y}$ is the configuration obtained exchanging the values of η at x and y, and f is a real-valued function of the configuration. Since $\xi_{x,y} = \xi_{y,x}$, the generator can be rewritten as

$$L_N f(\eta) \;=\; \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} \xi_{x,x+e_j}^N [f(\eta^{x,x+e_j}) - f(\eta)]\,.$$

which is a form of the generator often presented.

The (spatial) non-homogeneity cited in the title of this thesis comes from the choice of the conductances $\xi_{x,y}^N$. There is a natural embedding of the discrete torus $\mathbb{T}_N$ into the continuous d-dimensional torus $\mathbb{T}^d = [0,1)^d$ given by

$$x \longmapsto x/N \in [0,1)^d\,.$$

If $\xi_{x,y}^N$ is constant, the exclusion process will be spatially homogeneous with well known hydrodynamical behavior driven by the heat equation, see [17]. If $\xi_{x,y}^N$ depends on the position of x, y (as embedded in the continuous torus), we will say the process is (spatially) non-homogeneous. A fundamental remark: since $\xi_{x,y}^N = \xi_{y,x}^N$, the exclusion process with conductances can be interpreted as the non-homogeneous version of the simple symmetric exclusion process. Because of this symmetry $\xi_{x,y}^N = \xi_{y,x}^N$, the Bernoulli product measures (with constant parameter) are invariant (and in fact reversible) for the dynamics, for any choice of the conductances.

As can be see in this thesis (but not only here), the choice of the conductances may modify the hydrodynamical limit (the macroscopic behavior of the system), which will follow a partial differential equation depending on the conductances. Non-homogeneity does not necessarily imply a macroscopical effect. For instance, in the case $\beta < 1$ of Chapter 4, the partial differential equation obtained is the same one would obtain in the homogeneous case. In the same spirit, [5] consider random conductances in such a way the hydrodynamical equation depends only on a average of conductances. Recently, much attention has been raised to such subject, as we can see in [6], [7], [15], [26], [1], [14], [24], [23] and many others not cited here. Notice not all papers just cited are concerned about the type of exclusion process described above, [23] deals with random walks and [24] deals with totally asymmetric exclusion process, which are related topics. And all papers deal with non-homogeneous medium.

In all three chapters (2, 3 and 4) the scale will be diffusive. In words, we will concerned about η_t, which denotes the configuration of particles at time tN^2. Besides, $\eta_t(x)$ will denote the occupation of the site x at this time tN^2. This Markov Process η_t will have as generator $N^2 L_N$, where L_N has been defined above. The way to characterize the spatial density follows the classical one, is to say, to consider the *empirical measure* defined by

$$\pi_t^N \quad := \quad \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_t(x) \delta_{x/N} \, ,$$

which is a random positive measure on the continuous torus $\mathbb{T}^d$, with total mass bounded by one. Notice the definition is intuitive: if there is a particle at the site x in the corresponding time, $\eta_t(x) = 1$ and a delta of Dirac measure is putted there (as embedded in the continuous torus). The factor N^{-d} guarantees the total mass will be bounded by one. Then, we consider the trajectories (in time) given by

$$\pi_t^N \, , \; {}_{0 \leq t \leq T} \, .$$

Such space of trajectories has a metric, called the *Skorohod Metric* and, under such metric, it is a Polish Space (complete separable metric space). Supposing that π_0^N converges in distribution to $\gamma(u)du$, the hydrodynamical limit consists of proving the following convergence in distribution:

$$\pi_t^N \, , \; {}_{0 \leq t \leq T} \stackrel{N \to \infty}{\longrightarrow} \rho(t,u)du \, , \; {}_{0 \leq t \leq T} \, ,$$

where $\rho(t,u)$ will be a solution of some PDE with initial condition γ, what is called the *hydrodynamical equation.*

In the next paragraphs, we briefly describe the content of each chapter.

Chapter 2 deals with the one-dimensional case where conductances are related to a strictly increasing function $W : [0,1) \to \mathbb{R}$ in the way

$$\xi_{x,x+1}^N \quad = \quad \frac{1}{N\Big[W(\frac{x+1}{N}) - W(\frac{x}{N})\Big]} \, .$$

Furthermore, the exchange rate between x and $x+1$ also is a particular function of the presence of particles in $x-1$ and $x+2$. Such choice is the right one in order to observe a

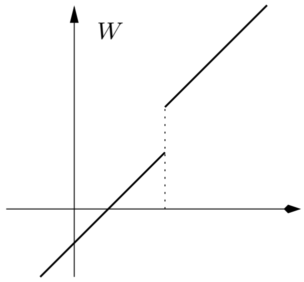

Figure 1.1: *Function W needed to obtain the case $\beta = 1$. Roughly speaking, identity with a discontinuity*

macroscopical effect of conductances. Given W, the hydrodynamical equation will be

$$\begin{cases} \partial_t \rho = \frac{d}{dx}\frac{d}{dW}\Phi(\rho) \\ \rho(0,\cdot) = \gamma(\cdot) \end{cases} \tag{1.0.1}$$

where $\frac{d}{dx}\frac{d}{dW}$ is an operator depending on W. The function Φ appearing in the hydrodynamical equation comes from this influence of sites $x-1$ and $x+2$ in the exchange rate between x and $x+1$. The complete discussion and technical details are given in Chapter 2. Such model includes a wide class of cases and also the case $\beta = 1$ of Chapter 4 as a particular case.

Chapter 3 deals with d-dimensional exclusion process. The problem considered there is about conductances which models a membrane slowing down a passage of particles between a smooth surface dividing the continuous torus in two regions Λ and $\Lambda^{\complement}$. The conductances close to this surface are of order N^{-1} times a projection into the exterior normal vector to the surface. Such projection into the normal vector was strictly necessary in the proof and has a physical interpretation. The surface slows down the passage of particles, and the passage of particles into the surface becomes even harder as the direction becomes closer to the tangent to the surface. The hydrodynamical equation is obtained and involves an operator $\mathcal{L}_\Lambda$, which has a nice geometrical interpretation.

Chapter 4 is also about one-dimensional exclusion process (as the Chapter 2). There, all edges have conductances equal to 1, except finite edges, which have conductances equal to $N^{-\beta}$, $\beta \in [0,\infty)$. As said before, the case $\beta = 1$ is a particular case of Chapter 2. For only one edge with conductance N^{-1}, take, for example, W as in Figure 1.1 and recall (1.0.1). In this Chapter 4, by a simple proof, we arrive at the same result of expect by Chapter 2 in the case $\beta = 1$. The cases $\beta \in [0,1)$ and $\beta \in (0,\infty)$ there considered show that $\beta = 1$ is sharp, what is natural. In the case $\beta < 1$ the slows bonds (edges with conductance $N^{-\beta}$) have no influence in the macroscopical behavior. In the case $\beta > 1$, the passage of particles between the slow bond is so small that implies no passage in the limit, which is given by the heat equation with *Neumann*'s boundary condition (isolated boundary).

Finally, some open questions and works in progress. The one-dimensional case has already quite general results. The hydrodynamical limit of exclusion process in non-homogeneous media in more dimensions still has natural unsolved questions. What should be the equivalent of Chapter 2 of this thesis in more dimensions? In [26], a generalization is this direction was made. There, conductances are taken in such way the generator could be, in certain way, decomposed in the sum of d generators of the form considered here in the Chapter 2. The Chapter 3 in this thesis is also a work in d-dimensions, but not the same line of [26]. As another example of open question, but not so general as before, what should be the hydrodynamical limit in the same line of Chapter 3, but with a denumerable quantity of smooth curves $\partial\Lambda$? It is possible to avoid the hypothesis about projection in the normal vector to the surface?

Since the hydrodynamical limit is a law of large numbers, it gives raise to other natural questions: large deviations and central limit theorem. Large deviations in one dimension with a slow bond in the case $\beta = 1$, is a work in progress with the author, C. Landim and Adriana Neumann. Fluctuation in equilibrium for general W in one dimensional has been considered by [8]. Fluctuations for the three cases of β in the real line and the CLT for the tagged particle is a work in progress with the author and P. Gonçalves.

Chapter 2

Hydrodynamic limit of gradient exclusion processes with conductances

Joint work with Claudio Landim (IMPA). Published in the Archive for Rational Mechanics and Analysis, v.195, p.409 - 439, 2010.

2.1 Abstract

Fix a strictly increasing right continuous with left limits function $W : \mathbb{R} \to \mathbb{R}$ and a smooth function $\Phi : [l, r] \to \mathbb{R}$, defined on some interval $[l, r]$ of $\mathbb{R}$, such that $0 < b \leq \Phi' \leq b^{-1}$. On the diffusive time scale, the evolution of the empirical density of exclusion processes with conductances given by W is described by the unique weak solution of the non-linear differential equation $\partial_t \rho = (d/dx)(d/dW)\Phi(\rho)$. We also present some properties of the operator $(d/dx)(d/dW)$.

2.2 Introduction

Recently, attention has been raised to the hydrodynamic behavior of interacting particle systems with random conductances [23, 16, 5, 7]. In [7], for instance, the authors considered the nearest-neighbor, one-dimensional exclusion process on $N^{-1}\mathbb{Z}$ in which a particle jumps from x/N (resp. $(x+1)/N$) to $(x+1)/N$ (resp. x/N) at rate $\{N[W((x+1)/N) - W(x/N)]\}^{-1}$, for a double sided α–stable subordinator W, $0 < \alpha < 1$. Their main result can be restated as follows. On the diffusive time scale, as the parameter $N \uparrow \infty$, the empirical density evolves according to the solution of the differential equation

$$\partial_t \rho \;=\; \frac{d}{dx}\frac{d}{dW}\rho \;. \tag{2.2.1}$$

In contrast with usual homogenization phenomena, the entire noise survives in the limit and the differential operator itself depends on the specific realization of the Levy process W. Moreover, the differential equation introduces a derivative with respect to a strictly increasing function W which may have jumps. In fact, in the Levy case, the set of discontinuities is dense in $\mathbb{R}$.

While the operators $(d/dW)(d/dx)$ have attracted much attention, being closely related to the so-called gap diffusions or quasi-diffusions when W has no jumps [22], the operator $(d/dx)(d/dW)$ have not been examined yet in the case where W exhibit jumps. We refer to [20, 21, 13] for recent results on the operators $(d/dx)(d/dW)$ in the case where W are increasing continuous functions.

As we shall see below, non-linear versions of the partial differential equation (2.2.1) appear naturally as scaling limits of interacting particle systems in inhomogeneous media. They may model diffusions in which permeable membranes, at the points of the discontinuities of W, tend to reflect particles, creating space discontinuities in the solutions.

We present in this paper a gradient exclusion process whose macroscopic evolution is described by the nonlinear differential equation

$$\partial_t \rho \;=\; \frac{d}{dx}\frac{d}{dW}\Phi(\rho)\;,$$

where Φ is a smooth function strictly increasing in the range of ρ (for a definition of Φ and a discussion about, see Theorem 2.3.2). To prove this result we examine in details the operator $(d/dx)(d/dW)$ in $L^2(\mathbb{T})$, where $\mathbb{T}$ is the one-dimensional torus. We prove in Theorem 2.3.1 that $(d/dx)(d/dW)$, defined in an appropriate domain, is non-positive, self-adjoint and dissipative. It is, in particular, the infinitesimal generator of a reversible Markov process. We also prove that the eigenvalues of $-(d/dx)(d/dW)$ are countable and have finite multiplicity, the associated eigenvectors forming a complete orthonormal system.

2.3 Notation and Results

We examine the hydrodynamic behavior of a one-dimensional exclusion process with conductances given by a strictly increasing function. Let $\mathbb{T}_N$ be the one-dimensional discrete torus with N points. Distribute particles on $\mathbb{T}_N$ in such a way that each site of $\mathbb{T}_N$ is occupied by at most one particle. Denote by η the configurations of the state space $\{0,1\}^{\mathbb{T}_N}$ so that $\eta(x) = 0$ if site x is vacant and $\eta(x) = 1$ if site x is occupied.

Fix $a > -1/2$ and a *strictly increasing* right continuous with left limits (càdlàg) function $W : \mathbb{R} \to \mathbb{R}$, periodic in the sense that $W(u+1) - W(u) = W(1) - W(0)$ for all u in $\mathbb{R}$. To simplify notation assume that W vanishes at the origin, $W(0) = 0$. For $0 \le x \le N-1$, let

$$c_{x,x+1}(\eta) \;=\; 1 \;+\; a\{\eta(x-1) + \eta(x+2)\}\;,$$

where all sums are modulo N, and let

$$\xi_x \;=\; \frac{1}{N[W((x+1)/N) - W(x/N)]}$$

with the convention that $\xi_{N-1} = \{N[W(1) - W(1 - [1/N])]\}^{-1}$.

The stochastic evolution can be described as follows. At rate $\xi_x c_{x,x+1}(\eta)$ the occupation variables $\eta(x)$, $\eta(x+1)$ are exchanged. Note that if W is differentiable at x/N, the rate at which particles are exchanged is of order 1, while if W is discontinuous, the rate is of order $1/N$. To understand the dynamics, assume that W is discontinuous at some point x/N and smooth on the intervals $(x/N, x/N+\epsilon)$, $(x/N-\epsilon, x/N)$. In this case, the rate at which particles cross the bond $\{x-1, x\}$ is of order $1/N$, while in a neighborhood of size N of this bond, particles jump at rate 1. In particular, a particle at site $x-1$ jumps to x at rate $1/N$ and jumps to $x-2$ at rate 1. Particles rebound therefore at the bond $\{x-1, x\}$. However, since time will be scaled diffusively and since on a time interval of length N^2 a particle spends a time of order N at site x, particles will be able to cross the slower bond $\{x-1, x\}$. This bond may model a membrane which obstructs the passage of particles.

The effect of the factor $c_{x,x+1}(\eta)$ is less dramatic. If the parameter a is positive, the presence of particles at the neighbor sites of the bond $\{x, x+1\}$ speeds up the exchange by a factor of order one.

The dynamics informally presented above describes a Markov evolution. The generator L_N of this Markov process acts on functions $f : \{0,1\}^{\mathbb{T}_N} \to \mathbb{R}$ as

$$(L_N f)(\eta) \;=\; \sum_{x\in\mathbb{T}_N} \xi_x\, c_{x,x+1}(\eta)\, \{f(\sigma^{x,x+1}\eta) - f(\eta)\}\;, \tag{2.3.1}$$

where $\sigma^{x,x+1}\eta$ is the configuration obtained from η by exchanging the variables $\eta(x)$, $\eta(x+1)$:

$$(\sigma^{x,x+1}\eta)(y) \;=\; \begin{cases} \eta(x+1) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+1, \\ \eta(y) & \text{otherwise.} \end{cases} \tag{2.3.2}$$

A simple computation shows that the Bernoulli product measures $\{\nu^N_\alpha : 0 \le \alpha \le 1\}$ are invariant, in fact reversible, for the dynamics. The measure ν^N_α is obtained by placing a particle at each site, independently from the other sites, with probability α. Thus, ν^N_α is a product measure over $\{0,1\}^{\mathbb{T}_N}$ with marginals given by

$$\nu^N_\alpha\{\eta : \eta(x) = 1\} \;=\; \alpha$$

for x in $\mathbb{T}_N$. We will often omit the index N of ν^N_α.

Denote by $\{\eta_t : t \ge 0\}$ the Markov process on $\{0,1\}^{\mathbb{T}_N}$ associated to the generator L_N *speeded up* by N^2. Let $D(\mathbb{R}_+, \{0,1\}^{\mathbb{T}_N})$ be the path space of càdlàg trajectories with values in $\{0,1\}^{\mathbb{T}_N}$ endowed with the Skorohod topology. For a measure μ_N on $\{0,1\}^{\mathbb{T}_N}$, denote by $\mathbb{P}_{\mu_N}$ the probability measure on $D(\mathbb{R}_+, \{0,1\}^{\mathbb{T}_N})$ induced by the initial state μ_N and the Markov process $\{\eta_t : t \ge 0\}$. Expectation with respect to $\mathbb{P}_{\mu_N}$ is denoted by $\mathbb{E}_{\mu_N}$.

Denote by $\mathbb{T}$ the one-dimensional torus $[0,1)$. A sequence of probability measures $\{\mu_N : N \ge 1\}$ on $\{0,1\}^{\mathbb{T}_N}$ is said to be associated to a profile $\rho_0 : \mathbb{T} \to [0,1]$ if

$$\lim_{N\to\infty} \mu_N \Big\{ \Big| \frac{1}{N} \sum_{x\in\mathbb{T}_N} H(x/N)\eta(x) - \int H(u)\rho_0(u)du \Big| > \delta \Big\} \;=\; 0 \tag{2.3.3}$$

for every $\delta > 0$ and every continuous functions $H : \mathbb{T} \to \mathbb{R}$.

2.3.1 The operator $\mathcal{L}_W$

Denote by $\langle \cdot, \cdot \rangle$ the inner product of $L^2(\mathbb{T})$:

$$\langle f, g \rangle \;=\; \int_{\mathbb{T}} f(u)\, g(u)\, du\,.$$

Let $\mathcal{D}_W$ be the set of functions f in $L^2(\mathbb{T})$ such that

$$f(x) \;=\; a \;+\; bW(x) \;+\; \int_{(0,x]} W(dy) \int_0^y \mathfrak{f}(z)\, dz$$

for some function $\mathfrak{f}$ in $L^2(\mathbb{T})$ such that

$$\int_0^1 \mathfrak{f}(z)\, dz \;=\; 0\,, \quad \int_{(0,1]} W(dy) \Big\{ b + \int_0^y \mathfrak{f}(z)\, dz \Big\} \;=\; 0\,.$$

Define the operator $\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T})$ by $\mathcal{L}_W f = \mathfrak{f}$. Formally,

$$\mathcal{L}_W f \;=\; \frac{d}{dx} \frac{d}{dW} f\,,$$

where the generalized derivative d/dW is defined as

$$\frac{df}{dW}(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{W(x+\epsilon) - W(x)}\,,$$

if the above limit exists and is finite.

Denote by $\mathbb{I}$ the identity operator in $L^2(\mathbb{T})$.

Theorem 2.3.1. *The operator $\mathcal{L}_W : \mathcal{D}_W \to \mathbb{L}^2(\mathbb{T})$ enjoys the following properties.*

(a) *$\mathcal{D}_W$ is dense in $\mathbb{L}^2(\mathbb{T})$;*

(b) *The operator $\mathbb{I} - \mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T})$ is bijective;*

(c) *$\mathcal{L}_W : \mathcal{D}_W \to \mathbb{L}^2(\mathbb{T})$ is self-adjoint and non-positive:*

$$\langle -\mathcal{L}_W f, f \rangle \;\geq\; 0\,;$$

(d) *$\mathcal{L}_W$ is dissipative;*

(e) *The eigenvalues of the operator $-\mathcal{L}_W$ form a countable set $\{\lambda_n : n \geq 0\}$. All eigenvalues have finite multiplicity, $0 = \lambda_0 \leq \lambda_1 \leq \cdots$, and $\lim_{n\to\infty} \lambda_n = \infty$;*

(f) *The eigenvectors $\{f_n\}$ form a complete orthonormal system.*

In view of (a), (b), (d), by the Hille-Yosida theorem, $\mathcal{L}_W$ is the generator of a strongly continuous contraction semi-group semigroup $\{P_t : t \geq 0\}$, $P_t : L^2(\mathbb{T}) \to L^2(\mathbb{T})$. Moreover, $\mathcal{D}_W$ is a core for $\mathcal{L}_W$.

Denote by $\{G_\lambda : \lambda > 0\}$, $G_\lambda : L^2(\mathbb{T}) \to L^2(\mathbb{T})$, the semi-group of resolvents associated to the operator $\mathcal{L}_W$: $G_\lambda = (\lambda - \mathcal{L}_W)^{-1}$. In terms of the semi-group $\{P_t\}$, $G_\lambda = \int_0^\infty e^{-\lambda t} P_t\, dt$.

2.3.2 Discrete approximation of the operator $\mathcal{L}_W$.

Consider a random walk in $N^{-1}\mathbb{T}_N$ which jumps from x/N (resp. $(x+1)/N$) to $(x+1)/N$ (resp. x/N) at rate $N^2\xi_x = N/\{W((x+1)/N) - W(x/N)\}$. The generator $\mathbb{L}_N$ of this Markov process writes

$$(\mathbb{L}_N f)(x/N) \;=\; N^2\xi_x\{f((x+1)/N) - f(x/N)\} \;+\; N^2\xi_{x-1}\{f((x-1)/N) - f(x/N)\}\;.$$

Denote by $\{P_t^N : t \ge 0\}$ the semigroup associated to the generator $\mathbb{L}_N$ and by $\{G_\lambda^N : \lambda \ge 0\}$ the resolvents. By Lemma 4.5 (i) in [7], for every continuous function $H : \mathbb{T} \to \mathbb{R}$, $P_t^N H$ converges to $P_t H$ in $L^1(\mathbb{T})$, and therefore in $L^2(\mathbb{T})$, as $N \uparrow \infty$. Moreover, it follows from Lemma 4.5 (iii) in [7] that for every continuous function $H : \mathbb{T} \to \mathbb{R}$ and for every $\lambda > 0$,

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{T}_N} \big|(G_\lambda^N H)(x/N) - (G_\lambda H)(x/N)\big| \;=\; 0\;. \tag{2.3.4}$$

The same results holds in $L^1(\mathbb{T})$ and $L^2(\mathbb{T})$.

Note that in [7], the function W is of pure jump type, while here it is any strictly increasing càdlàg function. One can check, however, that the proof applies to the present general case.

2.3.3 The hydrodynamic equation

For a positive integer $m \ge 1$, denote by $C^m(\mathbb{T})$ the space of continuous functions $H : \mathbb{T} \to \mathbb{R}$ with m continuous derivatives. Fix $l < r$ and a smooth function $\Phi : [l,r] \to \mathbb{R}$ whose derivative is bounded below by a strictly positive constant and bounded above by a finite constant:

$$0 \;<\; B^{-1} \le \Phi'(u) \;\le\; B$$

for u in $[l,r]$. Consider a bounded density profile $\gamma : \mathbb{T} \to [l,r]$. A bounded function $\rho : \mathbb{R}_+ \times \mathbb{T} \to [l,r]$ is said to be a weak solution of the parabolic differential equation

$$\begin{cases} \partial_t \rho \;=\; \mathcal{L}_W \Phi(\rho) \\ \rho(0,\cdot) \;=\; \gamma(\cdot) \end{cases} \tag{2.3.5}$$

if for all functions H in $C^1(\mathbb{T})$, all $t > 0$ and all $\lambda > 0$,

$$\langle \rho_t, G_\lambda H\rangle \;-\; \langle \gamma, G_\lambda H\rangle \;=\; \int_0^t \langle \Phi(\rho_s), \mathcal{L}_W G_\lambda H\rangle\, ds\;. \tag{2.3.6}$$

We prove in Section 2.7 uniqueness of weak solutions. Existence follows from the tightness of the sequence of probability measures $\mathbb{Q}_{\mu_N}^{W,N}$ introduced in Section 2.5.

Theorem 2.3.2. *Fix a continuous initial profile $\rho_0 : \mathbb{T} \to [0,1]$ and consider a sequence of probability measures μ_N on $\{0,1\}^{\mathbb{T}_N}$ associated to ρ_0. Then, for any $t \ge 0$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big\{ \Big|\frac{1}{N}\sum_{x\in\mathbb{T}_N} H(x/N)\eta_t(x) - \int H(u)\rho(t,u)du\Big| > \delta\Big\} \;=\; 0$$

for every $\delta > 0$ and every continuous functions H. Here, ρ is the unique weak solution of the non-linear equation (2.3.5) with $l = 0$, $r = 1$, $\gamma = \rho_0$ and $\Phi(\alpha) = \alpha + a\alpha^2$.

Remark 2.3.3. *The specific form of the rates $c_{x,x+1}$ is not important, but three conditions must be fulfilled. The rates have to be strictly positive, they may not depend on the occupation variables $\eta(x)$, $\eta(x+1)$, and the induced process has to be* gradient. *(cf. Chapter 7 in [17] for the definition of gradient processes).*

We may define rates $c_{x,x+1}$ to obtain any polynomial Φ of the form $\Phi(\alpha) = \alpha + \sum_{2\le j\le m} a_j\alpha^j$, $m \ge 1$, such that $1 + \sum_{2\le j\le m} ja_j > 0$. For $m = 3$, for instance, let

$$\hat{c}_{x,x+1}(\eta) \;=\; c_{x,x+1}(\eta) \;+\; b\Big\{\eta(x-2)\eta(x-1) + \eta(x-1)\eta(x+2) + \eta(x+2)\eta(x+3)\Big\}\;,$$

where $c_{x,x+1}$ is the rate defined at the beginning of Section 2 and a, b are such that $1+2a+3b > 0$. An elementary computation shows that these rates satisfy the above three conditions and that $\Phi(\alpha) = 1 + a\alpha^2 + b\alpha^3$.

Denote by π_t^N the empirical measure at time t. This is the measure on $\mathbb{T}$ obtained by rescaling space by N and by assigning mass N^{-1} to each particle:

$$\pi_t^N \;=\; \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta_t(x)\,\delta_{x/N}\;,$$

where δ_u is the Dirac measure concentrated on u.

Theorem 2.3.2 states that the empirical measure π_t^N converges, as $N\uparrow\infty$, to an absolutely continuous measure $\pi(t,du) = \rho(t,u)du$, whose density ρ is the solution of (2.3.5). In Sections 2.5, 2.6 we prove that ρ has finite energy: for all $t > 0$,

$$\int_0^t ds \int_{\mathbb{T}} \Big\{\frac{d}{dW}\Phi(\rho(s,u))\Big\}^2 dW \;<\; \infty\;.$$

The derivative $d/dW\,\Phi(\rho(s,u))$ must be understood in the generalized sense. Details are given in Section 2.6.

2.3.4 Outline of the proof

We present in this subsection a sketch of the proof which clarifies the relation between the stochastic evolution and the operator $\mathcal{L}_W$.

Fix a density profile $\rho_0 : \mathbb{T}\to\mathbb{R}$ and a sequence of measures $\{\mu^N : N\ge 1\}$ associated to ρ_0 in the sense (2.3.3). Recall the definition of the empirical measure π_t^N introduced above. We prove in Section 2.5 that the sequence of random measures $\{\pi_t^N : t\ge 0\}_{N\ge 1}$ is pre-compact and that all its limit points are absolutely continuous measures $\pi(t,du) = \rho(t,u)du$ with density ρ positive and bounded by 1.

To prove that all limit points are measures $\pi(t,du) = \rho(t,u)du$ whose density ρ are solutions of (2.3.5), assume, without loss of generality, that $\{\pi_t^N : t\ge 0\}_{N\ge 1}$ converges, as

$N \uparrow \infty$, to π_t, and fix a test function $H : \mathbb{T} \to \mathbb{R}$. Denote by $\langle \pi_t^N, H\rangle$ the integral of H with respect to the measure π_t^N:

$$\langle \pi_t^N, H\rangle \;=\; \frac{1}{N} \sum_{x\in\mathbb{T}_N} H(x/N)\eta_t(x) \; .$$

In Section 2.5, we prove that the martingale $M_t^{H,N}$, defined by

$$M_t^{H,N} \;=\; \langle \pi_t^N, H\rangle \;-\; \langle \pi_0^N, H\rangle \;-\; \int_0^t N^2 L_N \langle \pi_s^N, H\rangle \, ds \; ,$$

vanishes as $N \uparrow 0$. By assumption, $\langle \pi_t^N, H\rangle$, $\langle \pi_0^N, H\rangle$ converge to $\langle \pi_t, H\rangle$, $\langle \rho_0, H\rangle$, respectively. On the other hand, an elementary computation shows that

$$\begin{aligned} N^2 L_N \langle \pi^N, H\rangle \;&=\; \frac{1}{N} \sum_{x\in\mathbb{T}_N} (\mathbb{L}_N H)(x/N)\, \eta(x) \\ &+\; \frac{a}{N} \sum_{x\in\mathbb{T}_N} \big\{ (\mathbb{L}_N H)((x+1)/N) + (\mathbb{L}_N H)(x/N) \big\} \, (\tau_x h_1)(\eta) \\ &-\; \frac{a}{N} \sum_{x\in\mathbb{T}_N} (\mathbb{L}_N H)(x/N) (\tau_x h_2)(\eta) \; . \end{aligned}$$

In this formula, $\mathbb{L}_N$ is the generator of the random walk introduced in Subsection 2.3.2; $\{\tau_x : x \in \mathbb{Z}\}$ represents the group of translations in the configuration space so that $(\tau_x\eta)(y) = \eta(x + y)$ for x, y in $\mathbb{Z}$, where the sum is understood modulo N; and $h_1(\eta) = \eta(0)\eta(1)$, $h_2(\eta) = \eta(-1)\eta(1)$.

Recall the definition of the rates ξ_x to note that the generator $\mathbb{L}_N$ is a discrete approximation of the differential operator $\mathcal{L}_W$. In particular, one expects $\mathbb{L}_N H$ to converge to $\mathcal{L}_W H$ for a class of test functions. On the other hand, by local ergodicity of the dynamics, $\tau_x h_j(\eta_t)$ should be close to its expected value under the invariant measure with density given by the density profile $\rho(t, \cdot)$: $\tau_x h_j(\eta_t) \sim E_{\nu_{\rho(t,x/N)}}[h_j] = \rho(t, x/N)^2$.

Since the martingale vanishes in the limit, and since we assumed π^N to converge to $\pi(t, du) = \rho(t, u)du$, for a class of test functions H,

$$\langle \pi_t, H\rangle \;-\; \langle \rho_0, H\rangle \;=\; \int_0^t ds \int_{\mathbb{T}} \Phi(\rho(s, u)) \, (\mathcal{L}_W H)(u) \, du \; ,$$

which is the weak formulation (2.3.6) of the differential equation (2.3.5) if we replace the test function H by $G_\lambda H$. It remains to prove uniqueness of weak solutions of (2.3.5) to conclude the proof.

2.4 The operator $\mathcal{L}_W$

We examine in this section properties of the operator $\mathcal{L}_W$ introduced in the previous section. Recall that we denote by $\langle \cdot, \cdot\rangle$ the inner product of the Hilbert space $L^2(\mathbb{T})$ and by $\| \cdot \|$ its norm.

Let $\mathbb{D}(f)$ be the set of discontinuity points of a function $f : \mathbb{T} \to \mathbb{R}$. Denote by $C_W(\mathbb{T})$ the set of càdlàg functions $f : \mathbb{T} \to \mathbb{R}$ such that $\mathbb{D}(f) \subset \mathbb{D}(W)$. $C_W(\mathbb{T})$ is provided with the usual sup norm $\| \cdot \|_\infty$.

All functions in $C_W(\mathbb{T})$ are bounded. In fact, it is easy to prove that for each fixed f in $C_W(\mathbb{T})$ and $\epsilon > 0$, there exists $n \geq 1$ and $0 \leq z_1 < z_2 < \cdots < z_n < 1$ such that

$$|f(x) - f(y)| \;\leq\; \epsilon \text{ for all } z_k \leq x, y < z_{k+1},\ 1 \leq k \leq n, \tag{2.4.1}$$

where $z_{n+1} = z_1$.

Define the generalized derivative $\frac{d}{dW}$ as follows

$$\frac{df}{dW}(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{W(x+\epsilon) - W(x)}\,, \tag{2.4.2}$$

if the above limit exists and is finite. Denote by $\mathfrak{D}_W$ the set of functions f in $C_W(\mathbb{T})$ such that $\frac{df}{dW}(x)$ is well defined and differentiable, and $\frac{d}{dx}\big(\frac{df}{dW}\big)$ belongs to $C_W(\mathbb{T})$. Define the operator $\mathfrak{L}_W : \mathfrak{D}_W \to C_W(\mathbb{T})$ by

$$\mathfrak{L}_W f \;=\; \frac{d}{dx}\frac{d}{dW} f \;=\; \frac{d}{dx}\left(\frac{df}{dW}\right) .$$

By [3, Lemma 0.9 in Appendix], given a right continuous function f and a continuous function h,

$$\frac{df}{dW}(x) \;=\; h(x)$$

for all x in $\mathbb{T}$ if and only if

$$f(b) - f(a) \;=\; \int_{(a,b]} h(y) dW(y) \tag{2.4.3}$$

for all $a < b$. Note that the function h has integral equal to zero, $\int_{\mathbb{T}} h\, dW = 0$, because $f(1) = f(0)$.

It follows from this observation and the definition of the operator $\mathfrak{L}_W$ that $\mathfrak{D}_W$ is the set of functions f in $C_W(\mathbb{T})$ such that

$$f(x) \;=\; a \;+\; bW(x) \;+\; \int_{(0,x]} dW(y) \int_0^y g(z)\, dz \tag{2.4.4}$$

for some function g in $C_W(\mathbb{R})$ and two real numbers a, b such that

$$bW(1) \;+\; \int_{\mathbb{T}} dW(y) \int_0^y g(z)\, dz \;=\; 0\,, \qquad \int_{\mathbb{T}} g(z)\, dz \;=\; 0\,. \tag{2.4.5}$$

The first requirement corresponds to the boundary condition $f(1) = f(0)$ and the second one to the boundary condition $(df/dW)(1) = (df/dW)(0)$. Equivalently, (2.4.5) follows from the conditions

$$\int_{\mathbb{T}} \frac{df}{dW}\, dW \;=\; 0\,, \qquad \int_{\mathbb{T}} \frac{d}{dx}\frac{df}{dW}\, dx \;=\; 0\,. \tag{2.4.6}$$

One can check that the function g, as well as the constants a, b, are unique.

Lemma 2.4.1. *The following statements hold.*

(a) *The set* $\mathfrak{D}_W$ *is dense in* $L^2(\mathbb{T})$.

(b) *The operator* $\mathfrak{L}_W : \mathfrak{D}_W \to L^2(\mathbb{T})$ *is symmetric and non-positive. More precisely,*

$$\langle \mathfrak{L}_W f, g\rangle \;=\; -\int_{\mathbb{T}} \frac{df}{dW}\,\frac{dg}{dW}\,dW$$

for all f, g *in* $\mathfrak{D}_W$.

(c) $\mathfrak{L}_W$ *satisfies a Poincaré inequality: There exists a finite constant* C_0 *such that*

$$\|f\|^2 \;\le\; C_0\langle -\mathfrak{L}_W f, f\rangle \;+\; \Big(\int_{\mathbb{T}} f(x)\,dx\Big)^2$$

for all functions f *in* $\mathfrak{D}_W$.

(d) *The Green function* G *of the boundary-value problem*

$$\begin{cases} \mathfrak{L}_W u \;=\; 0 \text{ in } (0,1), \\ u(0) = u(1) = 0\,, \end{cases}$$

is given by

$$G(x,y) \;=\; \begin{cases} -\dfrac{[W(y)-W(0)]\,[W(1)-W(x)]}{W(1)-W(0)} & 0\le y\le x\le 1\,, \\[2ex] -\dfrac{[W(1)-W(y)]\,[W(x)-W(0)]}{W(1)-W(0)} & 0\le x\le y\le 1\,. \end{cases}$$

Proof. Since the continuous functions are dense in $L^2(\mathbb{T})$, to prove (a) it is enough to show that for each continuous function $f:\mathbb{T}\to\mathbb{R}$ and $\epsilon>0$, there exists g in $\mathfrak{D}_W$ such that $\|f-g\|\le\epsilon$.

Fix therefore a continuous function $f:\mathbb{T}\to\mathbb{R}$ and $\epsilon>0$. There exists $\delta>0$ such that $|f(y)-f(x)|\le\epsilon$ if $|x-y|\le\delta$. Choose an integer $n\ge\delta^{-1}$ and consider the function $g:\mathbb{T}\to\mathbb{R}$ defined by

$$g(x) \;=\; \sum_{j=0}^{n-1} \frac{f([j+1]/n)-f(j/N)}{W([j+1]/n)-W(j/N)}\mathbf{1}\{(j/n,(j+1)/n]\}(x)\,,$$

where $\mathbf{1}\{A\}$ stands for the indicator of the set A. Let $G:\mathbb{T}\to\mathbb{R}$ be given by $G(x)=f(0)+\int_{(0,x]} g(y)W(dy)$. By definition of g, $G(j/n)=f(j/n)$ for $0\le j<n$. Thus, by our choice of n and by definition of G, for $j/n\le x\le (j+1)/n$,

$$\big|G(x)-f(x)\big| \;\le\; \big|G(x)-G(j/n)\big| \;+\; \big|f(x)-f(j/n)\big| \;\le\; 2\epsilon\,.$$

so that $\|G - f\|_\infty \le 2\epsilon$ if $\|\cdot\|_\infty$ stands for the sup norm. Note that

$$\int_{(0,1]} g\, dW \;=\; 0\;. \tag{2.4.7}$$

It remains to show that the function G may be approximated in $L^2(\mathbb{T})$ by functions in the domain $\mathfrak{D}_W$. Note that we were free to choose the set $\{0, 1/n, \dots, (n-1)/n\}$ as long as the distance between two consecutive points is bounded by δ. We may therefore assume, without loss of generality, that W is continuous at these points. Denote by $\{H_k : k \ge 1\}$ a sequence of smooth functions $H_k : \mathbb{T} \to \mathbb{R}$ absolutely bounded by $\|g\|_\infty$ and such that $\lim_k H_k(x) = g(x)$ for $xn \notin \mathbb{Z}$. By the dominated convergence theorem,

$$\lim_{k\to\infty} \int_{\mathbb{T}} \big| H_k(y) - g(y) \big|\, dW(y) \;=\; 0\;. \tag{2.4.8}$$

Let $\{F_k : k \ge 1\}$ be the sequence of functions $F_k : \mathbb{T} \to \mathbb{R}$ defined by

$$\begin{aligned} F_k(x) \;&=\; f(0) \;+\; \int_{(0,x]} \Big\{ b_k + \int_0^y H_k'(z)\, dz \Big\} W(dy) \\ &=\; f(0) \;+\; b_k\, W(x) \;+\; \int_{(0,x]} W(dy) \int_0^y H_k'(z)\, dz\;, \end{aligned}$$

where $b_k \;=\; H_k(0) - W(1)^{-1} \int_{(0,1]} H_k(y)\, dW(y)$. By (2.4.7), (2.4.8), F_k converges in the uniform topology to G. On the other hand, in view of (2.4.4) and our choice of b_k, F_k belongs to $\mathfrak{D}_W$ for each $k \ge 1$ because H_k', being continuous, belongs to $C_W(\mathbb{T})$. This concludes the proof of (a).

To prove (b), fix two functions f, g in $\mathfrak{D}_W$ and let $F = df/dW$. F is differentiable with derivative in $C_W(\mathbb{T})$. Fix $\epsilon > 0$ and denote by $\{z_1, \dots, z_n\}$ the finite set given by (2.4.1) for the function g. Adding extra points if necessary, we may assume that $\max_{1\le k\le n} \sup_{z_k \le x, y \le z_{k+1}} |F(y) - F(x)| \;\le\; \epsilon$ because F is continuous. Decomposing the integral over $\mathbb{T}$ on the intervals $[z_k, z_{k+1}]$, we get that

$$\langle \mathfrak{L}_W f, g\rangle \;=\; \int_{\mathbb{T}} \frac{dF}{dx}(x)\, g(x)\, dx \;=\; \sum_{k=1}^n g(z_k)\{F(z_{k+1}) - F(z_k)\} \;\pm\; \epsilon \Big\| \frac{dF}{dx} \Big\|_\infty\;,$$

where $\pm C$ stands for a constant absolutely bounded by C. Changing the order of summations in the last term, in view of (2.4.3), we obtain that the previous sum is equal to

$$-\sum_{k=1}^n \{g(z_k) - g(z_{k-1})\} F(z_k) \;=\; -\sum_{k=1}^n F(z_k) \int_{(z_{k-1}, z_k]} \frac{dg}{dW}(x)\, dW(x)\;.$$

Recall that dg/dW is continuous and that $|F(x) - F(z_k)| \;\le\; \epsilon$ for $z_{k-1} \;\le\; x \;\le\; z_k$. The previous sum is thus equal to

$$-\int_{\mathbb{T}} F(x)\, \frac{dg}{dW}(x)\, dW(x) \;\pm\; \epsilon \Big\| \frac{dg}{dW} \Big\|_\infty [W(1) - W(0)]\;.$$

This proves the first identity from which it follows that $\mathfrak{L}_W$ is symmetric and non-positive.

To prove the Poincaré inequality, fix a function f in $\mathfrak{D}_W$ and observe that by (2.4.3)

$$\begin{aligned}\int_{\mathbb{T}} f(x)^2\,dx \;-\; \Big(\int_{\mathbb{T}} f(x)\,dx\Big)^2 \;&=\; \int_{\mathbb{T}} \Big(\int_{\mathbb{T}} [f(x) - f(y)]\,dy\Big)^2\,dx \\ &=\; \int_{\mathbb{T}} dx \Big(\int_{\mathbb{T}} dy \int_{(y,x]} \frac{df}{dW}(z)\,dW(z)\Big)^2 \;.\end{aligned}$$

To conclude the proof, it remains to apply twice the Schwarz inequality and to change the order of integration. Note that this proof gives $C_0 = W(1) - W(0)$.

An elementary computation permits to check that the Green's function is given by the expression proposed. □

Denote by $\langle \cdot, \cdot \rangle_{1,2}$ the inner product on $\mathfrak{D}_W$ defined by

$$\langle f, g\rangle_{1,2} \;=\; \langle f, g\rangle \;+\; \langle -\mathfrak{L}_W f, g\rangle \;=\; \langle f, g\rangle \;+\; \int_{\mathbb{T}} \frac{df}{dW}\,\frac{dg}{dW}\,dW \;.$$

Let $H_2^1(\mathbb{T})$ be the set of all functions f in $L^2(\mathbb{T})$ for which there exists a sequence $\{f_n : n \geq 1\}$ in $\mathfrak{D}_W$ such that f_n converges to f in $L^2(\mathbb{T})$ and f_n is Cauchy for the inner product $\langle \cdot, \cdot \rangle_{1,2}$. Such sequence $\{f_n\}$ is called admissible for f. For f, g in $H_2^1(\mathbb{T})$, define

$$\langle f, g\rangle_{1,2} \;=\; \lim_{n\to\infty} \langle f_n, g_n\rangle_{1,2} \;, \tag{2.4.9}$$

where $\{f_n\}$, $\{g_n\}$ are admissible sequences for f, g, respectively. By [27, Proposition 5.3.3], this limit exists and does not depend on the admissible sequence chosen. Moreover, $H_2^1(\mathbb{T})$ endowed with the scalar product $\langle \cdot, \cdot \rangle_{1,2}$ just defined is a real Hilbert space.

Denote by $L^2_W(\mathbb{T})$ the Hilbert space generated by the continuous functions endowed with the inner product $\langle \cdot, \cdot \rangle_W$ defined by

$$\langle f, g\rangle_W \;=\; \int_{\mathbb{T}} f(x)\,g(x)\,W(dx)\;.$$

The norm associated to the scalar product $\langle \cdot, \cdot \rangle_W$ is denoted by $\| \cdot \|_W$.

Lemma 2.4.2. *A function f in $L^2(\mathbb{T})$ belongs to $H_2^1(\mathbb{T})$ if and only if there exists F in $L^2_W(\mathbb{T})$ and a finite constant c such that*

$$\int_{(0,1]} F(y)\,dW(y) \;=\; 0 \quad \textit{and} \quad f(x) \;=\; c \;+\; \int_{(0,x]} F(y)\,dW(y)$$

Lebesgue almost surely. We denote the generalized W-derivative F of f by df/dW. For f, g in $H_2^1(\mathbb{T})$,

$$\langle f, g\rangle_{1,2} \;=\; \langle f, g\rangle \;+\; \int_{\mathbb{T}} \frac{df}{dW}\,\frac{dg}{dW}\,dW \;.$$

Proof. Fix f in $H_2^1(\mathbb{T})$. By definition, there exists a sequence $\{f_n : n \ge 1\}$ in $\mathfrak{D}_W$ which converges to f in $L^2(\mathbb{T})$ and which is Cauchy in $H_2^1(\mathbb{T})$. In particular, df_n/dW is Cauchy in $L^2_W(\mathbb{T})$ and therefore converges to some function G in $L^2_W(\mathbb{T})$. By (2.4.6),

$$\int_{\mathbb{T}} \frac{df_n}{dW}\, dW \;=\; 0$$

for all $n \ge 1$ so that $\int_{(0,1]} G\, dW = 0$. Let $g(x) = \int_{(0,x]} G(y) dW(y)$. Since $\mathbf{1}\{(x,y]\}$ belongs to $L^2_W(\mathbb{T})$, for all x, y in $\mathbb{T}$,

$$g(y) - g(x) \;=\; \int_{(x,y]} G\, dW \;=\; \lim_{n\to\infty} \int_{(x,y]} \frac{df_n}{dW}\, dW \;=\; \lim_{n\to\infty} \{f_n(y) - f_n(x)\}\;.$$

We claim that $\int_{\mathbb{T}} \{f_n(y) - f_n(x)\} dx$ converges to $\int_{\mathbb{T}} \{g(y) - g(x)\} dx$ for all y in $\mathbb{T}$. Indeed, on the one hand, for each fixed y, $f_n(y) - f_n(x)$ converges to $g(y) - g(x)$. On the other hand, by Schwarz inequality,

$$[f_n(y) - f_n(x)]^2 \;\le\; [W(1) - W(0)] \int_{\mathbb{T}} \Big(\frac{df_n}{dW}\Big)^2 dW \;\le\; C_0$$

for some finite constant C_0. It remains to apply the dominated convergence theorem to conclude.

Since f_n converges to f in $L^2(\mathbb{T})$, $\int_{\mathbb{T}} f_n(x)\, dx$ converges to $\int_{\mathbb{T}} f(x)\, dx$. By Schwarz inequality, g belongs to $L^2(\mathbb{T})$ so that $\int_{\mathbb{T}} g(x) dx$ is finite. Therefore, for all y in $\mathbb{T}$,

$$\begin{aligned} \lim_{n\to\infty} f_n(y) \;&=\; \lim_{n\to\infty} \Big\{ f_n(y) - \int_{\mathbb{T}} f_n(x)\, dx \Big\} \;+\; \int_{\mathbb{T}} f(x)\, dx \\ &=\; g(y) - \int_{\mathbb{T}} g(x)\, dx \;+\; \int_{\mathbb{T}} f(x)\, dx\;. \end{aligned}$$

Thus f_n converges pointwisely to the above function. As f_n also converges to f in $L^2(\mathbb{T})$, we deduce that $f = c + g$ a.s., and thus in $L^2(\mathbb{T})$, for $c = \int_{\mathbb{T}} f(x)\, dx - \int_{\mathbb{T}} g(x)\, dx$, proving the first statement of the lemma.

The reciprocal is simpler. Let $f = c + \int_{(0,x]} F(y)\, dW(y)$ for some F in $L^2_W(\mathbb{T})$ such that $\int_{(0,1]} F(y)\, dW(y) = 0$. There exists a sequence $\{g_n : n \ge 1\}$ of smooth functions converging to F in $L^2_W(\mathbb{T})$ and such that $\int_{(0,1]} g_n(y)\, dW(y) = 0$. Let $f_n(x) = c + \int_{(0,x]} dW(y) \{g_n(0) + \int_0^y g_n'(z) dz\}$. For each $n \ge 1$, f_n belongs to $\mathfrak{D}_W$ because g_n' is continuous. Schwarz inequality shows that f_n converges to f in $L^2(\mathbb{T})$. Finally, $\{f_n : n \ge 1\}$ is a Cauchy sequence for the inner product $\langle \cdot, \cdot \rangle_{1,2}$ because $df_n/dW = g_n$ converges to F in $L^2_W(\mathbb{T})$. Note that we just proved that the sequence $\{f_n : n \ge 1\}$ is admissible for f

Fix f, g in $H_2^1(\mathbb{T})$ and recall that we denote by df/dW, dg/dW the generalized W-derivatives of f, g, respectively. Denote by $\{f_n : n \ge 1\}$, $\{g_n : n \ge 1\}$ the admissible sequences constructed in the previous paragraph for f and g, respectively. By definition,

$$\langle f, g \rangle_{1,2} \;=\; \lim_{n\to\infty} \langle f_n, g_n \rangle_{1,2} \;=\; \lim_{n\to\infty} \Big\{ \langle f_n, g_n \rangle \;+\; \int_{\mathbb{T}} \frac{df_n}{dW} \frac{dg_n}{dW}\, dW \Big\}\;.$$

Since f_n (resp. g_n) converges to f (resp. g) in $L^2(\mathbb{T})$ and since df_n/dW (resp. dg_n/dW) converges to df/dW (resp. dg/dW) in $L^2_W(\mathbb{T})$, the previous expression is equal to

$$\langle f, g\rangle \;+\; \int_{\mathbb{T}} \frac{df}{dW}\,\frac{dg}{dW}\, dW \;.$$

This conclude the proof of the lemma. □

Lemma 2.4.3. *The embedding $H^1_2(\mathbb{T}) \subset L^2(\mathbb{T})$ is compact.*

Proof. Consider a sequence $\{u_n : n \ge 1\}$ bounded in $H^1_2(\mathbb{T})$. We need to prove the existence of a subsequence $\{u_{n_k} : k \ge 1\}$ which converges in $L^2(\mathbb{T})$.

By the previous lemma, $u_n(x) = c_n + \int_{(0,x]} U_n(y)\, dW(y)$ for some U_n in $L^2_W(\mathbb{T})$ such that $\int_{(0,1]} U_n(y)\, dW(y) = 0$. Moreover, $\|U_n\|_W \le \|u_n\|_{1,2}$. The sequence $\{U_n\}$ is therefore bounded in $L^2_W(\mathbb{T})$. Also, by Schwarz inequality, the sequence $\int_{(0,x]} U_n(y) dW(y)$ is bounded in $L^2(\mathbb{T})$. Since $c_n = u_n(x) - \int_{(0,x]} U_n(y) dW(y)$ and since both sequence of functions on the right hand side are bounded in $L^2(\mathbb{T})$, the sequence of real numbers $\{c_n\}$ is also bounded.

Since $\{U_n\}$ is a bounded sequence in $L^2_W(\mathbb{T})$ and since the sequence of real numbers $\{c_n\}$ is bounded, there exists a subsequence $\{n_k\}$ such that c_{n_k} converges and U_{n_k} converges weakly in $L^2_W(\mathbb{T})$ to a limit denoted by U. As constants belong to $L^2_W(\mathbb{T})$, $\int_{(0,1]} U(y)\, dW(y) = \lim_k \int_{(0,1]} U_{n_k}(y)\, dW(y) = 0$. Moreover, for all x in $\mathbb{T}$, as $\mathbf{1}\{(0,x]\}$ belongs to $L^2_W(\mathbb{T})$,

$$\lim_{k\to\infty} u_{n_k}(x) \;=\; \lim_{k\to\infty} \Big\{ c_{n_k} \;+\; \int_{(0,x]} U_{n_k}(y) dW(y) \Big\} \;=\; c \;+\; \int_{(0,x]} U(y) dW(y)\;,$$

if c stands for the limit of the sequence c_{n_k}. The sequence u_{n_k} thus converges pointwisely to $u(x) = c + \int_{(0,x]} U(y) dW(y)$. Since, by Schwarz inequality, $u_{n_k}(x)^2$ is bounded by $2c^2_{n_k} + 2[W(1) - W(0)]\, \|U_{n_k}\|^2_W$, by the dominated convergence theorem, u_{n_k} converges to u in $L^2(\mathbb{T})$. Note that the limit u belongs to $H^1_2(\mathbb{T})$. □

Let $\mathcal{D}_W$ be the set of functions f in $H^1_2(\mathbb{T})$ for which there exists u in $L^2(\mathbb{T})$ such that

$$\langle f, g\rangle_{1,2} \;=\; \langle f, g\rangle \;+\; \int \frac{df}{dW}\,\frac{dg}{dW}\, dW \;=\; \langle u, g\rangle \tag{2.4.10}$$

for all g in $H^1_2(\mathbb{T})$. By Lemma 2.4.1 (b), $\mathfrak{D}_W \subset \mathcal{D}_W$ and, by definition, $\mathcal{D}_W \subset H^1_2(\mathbb{T})$. The function u is uniquely determined because, by Lemma 2.4.1 (a), $H^1_2(\mathbb{T}) \supset \mathfrak{D}_W$ is dense in $L^2(\mathbb{T})$. By definition of $H^1_2(\mathbb{T})$ and by (2.4.9), it is enough to check (2.4.10) for functions g in $\mathfrak{D}_W$.

Lemma 2.4.4. *The domain $\mathcal{D}_W$ consists of all functions f in $L^2(\mathbb{T})$ such that*

$$f(x) \;=\; a \;+\; bW(x) \;+\; \int_{(0,x]} W(dy) \int_0^y \mathfrak{f}(z)\, dz$$

for some function $\mathfrak{f}$ *in* $L^2(\mathbb{T})$ *such that*

$$\int_0^1 \mathfrak{f}(z)\, dz \;=\; 0\,, \qquad \int_{(0,1]} W(dy)\Big\{b + \int_0^y \mathfrak{f}(z)\, dz\Big\} \;=\; 0\,.$$

Moreover, in this case,

$$-\int \frac{df}{dW}\,\frac{dg}{dW}\, dW \;=\; \langle \mathfrak{f}, g\rangle$$

for all g *in* $H_2^1(\mathbb{T})$*.*

Proof. We first show that any function f in $L^2(\mathbb{T})$ with the properties listed in the statement of the lemma belongs to $\mathcal{D}_W$. Fix such a function and consider a sequence $\{\mathfrak{f}_n : n \ge 1\}$ of smooth functions $\mathfrak{f}_n : \mathbb{T} \to \mathbb{R}$ which converges to $\mathfrak{f}$ in $L^2(\mathbb{T})$ and such that $\int_0^1 \mathfrak{f}_n(z)\, dz = 0$. Let

$$f_n(x) \;=\; a \;+\; \int_{(0,x]} W(dy)\Big\{b_n \;+\; \int_0^y \mathfrak{f}_n(z)\, dz\Big\}\,,$$

where b_n is chosen so that $\int_{(0,1]} W(dy)\{b_n + \int_0^y \mathfrak{f}_n(z)\, dz\} = 0$. Note that f_n belongs to $\mathfrak{D}_W$ for each $n \ge 1$.

As $n \uparrow \infty$, b_n converges to b, f_n converges to f in $L^2(\mathbb{T})$ and $\{f_n\}$ is Cauchy for the $\|\cdot\|_{1,2}$ norm. Thus, f belongs to $H_2^1(\mathbb{T})$ and $\{f_n\}$ is an admissible sequence for f.

Fix g in $\mathfrak{D}_W$. We claim that

$$\langle f, g\rangle_{1,2} \;=\; \langle f, g\rangle \;-\; \langle \mathfrak{f}\,, g\rangle\,.$$

Indeed, as g belongs to $\mathfrak{D}_W$, by (2.4.9), $\langle f, g\rangle_{1,2} = \lim_n \langle f_n, g\rangle_{1,2}$ because the sequence $\{g_n : n \ge 1\}$ constant equal to g is admissible for g. By definition of the inner product $\langle \cdot, \cdot\rangle_{1,2}$ and since $\mathfrak{L}_W f_n = \mathfrak{f}_n$, $\langle f_n, g\rangle_{1,2} = \langle f_n, g\rangle + \langle -\mathfrak{L}_W f_n, g\rangle = \langle f_n, g\rangle + \langle -\mathfrak{f}_n, g\rangle$. Since f_n, $\mathfrak{f}_n$ converge in $L^2(\mathbb{T})$ to f, $\mathfrak{f}$, respectively, the claim is proved. In particular (2.4.10) holds with $u = f - \mathfrak{f}$. This proves that f belongs to $\mathcal{D}_W$ and the identity claimed.

Conversely, assume that f belongs to $\mathcal{D}_W$ and satisfy (2.4.10) for some u in $L^2(\mathbb{T})$. Thus, there exists v (equal to $f - u$) in $L^2(\mathbb{T})$ such that

$$-\int \frac{df}{dW}\,\frac{dg}{dW}\, dW \;=\; \langle v, g\rangle \tag{2.4.11}$$

for all g in $\mathfrak{D}_W$. Taking $g = 1$ in this equation we obtain that $\int_0^1 v(x)\, dx = 0$.

Since f belongs to $H_2^1(\mathbb{T})$, by Lemma 2.4.2, $f(x) = c + \int_{(0,x]} F(y) dW(y)$ for some function F in $L^2_W(\mathbb{T})$ such that $\int_{(0,1]} F(y) dW(y) = 0$. To prove the lemma we need to show that

$$F(y) \;=\; b \;+\; \int_0^y \mathfrak{f}(z)\, dz$$

for some finite constant b and some function $\mathfrak{f}$ in $L^2(\mathbb{T})$ such that $\int_0^1 \mathfrak{f}(z)\, dz = 0$.

Fix g in $\mathfrak{D}_W$ so that

$$g(x) \;=\; a \;+\; \int_{(0,x]} G(y)\, dW(y)$$

for some continuous function $G : \mathbb{T} \to \mathbb{R}$ such that $\int_0^1 G(y)\, dW(y) = 0$. Since the integral of v (resp. G) with respect to the Lebesgue measure (resp. the measure dW) vanishes, changing the order of integration, we obtain that

$$\int_0^1 v(x)\, g(x)\, dx \;=\; -\int_{(0,1]} G(y) \int_0^y v(x)\, dx\, dW(y)\;.$$

Therefore, in view of (2.4.11),

$$\int_{(0,1]} G(y) \int_0^y v(x)\, dx\, dW(y) \;=\; \int_{(0,1]} G(y)\, F(y)\, dW(y)$$

for all functions g in $\mathfrak{D}_W$. The proof of Lemma 2.4.1 (a) shows that the set $\{dg/dW : g \in \mathfrak{D}_W\}$ is dense in $L^2_{W,0} = \{H \in L^2_W(\mathbb{T}) : \int H dW = 0\}$. In particular, $F(y) = c + \int_0^y v(x)\, dx$ for some finite constant c. This concludes the proof of the lemma. □

Recall that we denote by $\mathbb{I}$ the identity in $L^2(\mathbb{T})$. By Lemma 2.4.1, the symmetric operator $(\mathbb{I} - \mathfrak{L}_W) : \mathfrak{D}_W \to \mathbb{L}^2(\mathbb{T})$, is strongly monotone:

$$\langle (\mathbb{I} - \mathfrak{L}_W) f, f\rangle \;\geq\; \langle f, f\rangle$$

for all f in $\mathfrak{D}_W$. Denote by $\mathcal{A}_1 : \mathcal{D}_W \to \mathbb{L}^2(\mathbb{T})$ its Friedrichs extension, defined as $\mathcal{A}_1 f = u$, where u is the function in $L^2(\mathbb{T})$ given by (2.4.10). By [27, Theorem 5.5.a], $\mathcal{A}_1$ is self-adjoint, bijective and

$$\langle \mathcal{A}_1 f, f\rangle \;\geq\; \langle f, f\rangle \tag{2.4.12}$$

for all f in $\mathcal{D}_W$. Note that the Friedrichs extension of the strongly monotone operator $(\lambda\mathbb{I} - \mathfrak{L}_W)$, $\lambda > 0$, is $\mathcal{A}_\lambda = (\lambda - 1)\mathbb{I} + \mathcal{A}_1 : \mathcal{D}_W \to \mathbb{L}^2(\mathbb{T})$.

Define $\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T})$ by $\mathcal{L}_W = \mathbb{I} - \mathcal{A}_1$. In view of (2.4.10), $\mathcal{L}_W f = u$ if and only if

$$-\int \frac{df}{dW}\,\frac{dg}{dW}\, dW \;=\; \langle u, g\rangle$$

for all g in $H^1_2(\mathbb{T})$. In particular by Lemma 2.4.1 (b) $\mathcal{L}_W f = \mathfrak{L}_W f$ for all f in $\mathfrak{D}_W$. Moreover, if a function f in $\mathcal{D}_W$ is represented as in Lemma 2.4.3, $\mathcal{L}_W f = \mathfrak{f}$. This identity together with the identification of the space $\mathcal{D}_W$ provides the alternative definition of the operator $\mathcal{L}_W$ presented just before the statement of Theorem 2.3.1.

Proof of Theorem 2.3.1. It follows from Lemma 2.4.1 (a) that the domain $\mathcal{D}_W$ is dense in $L^2(\mathbb{T})$ because $\mathfrak{D}_W \subset \mathcal{D}_W$. This proves (a).

By definition, $\mathbb{I} - \mathcal{L}_W = \mathcal{A}_1 : \mathcal{D}_W \to \mathbb{L}^2(\mathbb{T})$, which have been shown to be bijective. This proves (b).

The self-adjointness of $\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T})$ follows from the one of $\mathcal{A}_1$ and the definition of $\mathcal{L}_W$ as $\mathbb{I} - \mathcal{A}_1$. Moreover, from (2.4.12) we obtain that $\langle -\mathcal{L}_W f, f\rangle \geq 0$ for all f in $\mathcal{D}_W$.

To prove (d), fix a function g in $\mathcal{D}_W$, $\lambda > 0$ and let $f = (\lambda\mathbb{I} - \mathcal{L}_W)g$. Taking the scalar product with respect to g on both sides of this equation, we obtain that

$$\lambda\langle g, g\rangle \;+\; \langle -\mathcal{L}_W g, g\rangle \;=\; \langle g, f\rangle \;\leq\; \langle g, g\rangle^{1/2}\,\langle f, f\rangle^{1/2}\;.$$

Since g belongs to $\mathcal{D}_W$, by (c), the second term on the left hand side is positive. Thus, $\|\lambda g\| \leq \|f\| = \|(\lambda\mathbb{I} - \mathcal{L}_W)g\|$.

We have already seen that the operator $(\mathbb{I} - \mathfrak{L}_W) : \mathfrak{D}_W \to L^2(\mathbb{T})$ is symmetric and strongly monotone. By Lemma 2.4.3, the embedding $H_2^1(\mathbb{T}) \subset L^2(\mathbb{T})$ is compact. Therefore, by [27, Theorem 5.5.c], the Friedrichs extension of $(\mathbb{I} - \mathfrak{L}_W)$, denoted by $\mathcal{A}_1 : \mathcal{D}_W \to L^2(\mathbb{T})$, satisfies claims (e) and (f) with $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_n \uparrow \infty$. In particular, the operator $-\mathcal{L}_W = \mathcal{A}_1 - \mathbb{I}$ has the same property with $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_n \uparrow \infty$. Since 0 is an eigenvalue of $-\mathcal{L}_W$ associated at least to the constants, (e) and (f) are in force. □

It follows also from [27, Theorem 5.5.c] that f_n belongs to $H_2^1(\mathbb{T})$ for all n.

2.4.1 Random walk with conductances

Recall that $\mathbb{T}_N$ stands for the discrete one-dimensional torus with N points and recall the definition of the sequence $\{\xi_x : 0 \leq x \leq N-1\}$. Consider the random walk $\{X_t^N : t \geq 0\}$ on $N^{-1}\mathbb{T}_N$ which jumps over the bond $\{x/N, (x+1)/N\}$ at rate $N^2\xi_x = N/\{W((x+1)/N) - W(x/N)\}$. The generator $\mathbb{L}_N$ of this Markov process writes

$$(\mathbb{L}_N f)(x/N) \;=\; N^2\xi_x\{f((x+1)/N) - f(x/N)\} \;+\; N^2\xi_{x-1}\{f((x-1)/N) - f(x/N)\}\;.$$

The counting measure m_N on $N^{-1}\mathbb{T}_N$ is reversible for this process. Denote by $\{P_t^N : t \geq 0\}$ (resp. $\{G_\lambda^N : \lambda > 0\}$) the semigroup (resp. the resolvent) associated to the generator $\mathbb{L}_N$:

$$G_\lambda^N H \;=\; \int_0^\infty dt\, e^{-\lambda t} P_t^N H$$

for $H : N^{-1}\mathbb{T}_N \to \mathbb{R}$.

Fix a function $H : N^{-1}\mathbb{T}_N \to \mathbb{R}$. For $\lambda > 0$, let $H_\lambda^N = G_\lambda^N H$ be the solution of the resolvent equation

$$\lambda H_\lambda^N \;-\; \mathbb{L}_N H_\lambda^N \;=\; H\;.$$

Taking the scalar product on both sides of this equation with respect to H_λ^N, we obtain that for all $N \geq 1$

$$\begin{aligned}\frac{1}{N}\sum_{x\in\mathbb{T}_N} H_\lambda^N(x/N)^2 \;&\leq\; \frac{1}{\lambda^2}\frac{1}{N}\sum_{x\in\mathbb{T}_N} H(x/N)^2\;,\\ \frac{1}{N}\sum_{x\in\mathbb{T}_N} \xi_x(\nabla_N H_\lambda^N)(x/N)^2 \;&\leq\; \frac{1}{\lambda}\frac{1}{N}\sum_{x\in\mathbb{T}_N} H(x/N)^2\;,\end{aligned} \tag{2.4.13}$$

where ∇_N stands for the discrete derivative: $(\nabla_N H)(x/N) = N[H((x+1)/N) - H(x/N)]$.

On the other hand, if $H : \mathbb{T} \to \mathbb{R}$ is a continuous function and we denote also by H its restriction to $N^{-1}\mathbb{T}_N$, by [7, Lemma 4.6],

$$\lim_{\lambda\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{T}_N} \big|\lambda H_\lambda^N(x/N) - H(x/N)\big| \;=\; 0 \;. \tag{2.4.14}$$

Note that in [7], the function W is of pure jump type, while here it is any strictly increasing càdlàg function. One can check, however, that the proof applies to our general case.

2.5 Scaling limit

Let $\mathcal{M}$ be the space of positive measures on $\mathbb{T}$ with total mass bounded by one endowed with the weak topology. Recall that $\pi_t^N \in \mathcal{M}$ stands for the empirical measure at time t. This is the measure on $\mathbb{T}$ obtained by rescaling space by N and by assigning mass N^{-1} to each particle:

$$\pi_t^N \;=\; \frac{1}{N} \sum_{x\in\mathbb{T}_N} \eta_t(x)\, \delta_{x/N} \;, \tag{2.5.1}$$

where δ_u is the Dirac measure concentrated on u. For a continuous function $H : \mathbb{T} \to \mathbb{R}$, $\langle \pi_t^N, H\rangle$ stands for the integral of H with respect to π_t^N:

$$\langle \pi_t^N, H\rangle \;=\; \frac{1}{N} \sum_{x\in\mathbb{T}_N} H(x/N)\eta_t(x) \;.$$

This notation is not to be mistaken with the inner product in $L^2(\mathbb{T})$ introduced earlier. Also, when π_t has a density ρ, $\pi(t, du) = \rho(t, u)du$, we sometimes write $\langle \rho_t, H\rangle$ for $\langle \pi_t, H\rangle$.

Fix $T > 0$. Let $D([0, T], \mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi : [0, T] \to \mathcal{M}$ endowed with the Skorohod topology. For each probability measure μ_N on $\{0, 1\}^{\mathbb{T}_N}$, denote by $\mathbb{Q}^{W,N}_{\mu_N}$ the measure on the path space $D([0, T], \mathcal{M})$ induced by the measure μ_N and the process π_t^N introduced in (2.5.1).

Fix a continuous profile $\rho_0 : \mathbb{T} \to [0, 1]$ and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on $\{0, 1\}^{\mathbb{T}_N}$ associated to ρ_0 in the sense (2.3.3). Let $\mathbb{Q}_W$ be the probability measure on $D([0, T], \mathcal{M})$ concentrated on the deterministic path $\pi(t, du) = \rho(t, u)du$, where ρ is the unique weak solution of (2.3.5) with $\gamma = \rho_0$, $l = 0$, $r = 1$ and $\Phi(\alpha) = \alpha + a\alpha^2$.

Proposition 2.5.1. *As $N \uparrow \infty$, the sequence of probability measures $\mathbb{Q}^{W,N}_{\mu_N}$ converges in the uniform topology to $\mathbb{Q}_W$.*

The proof of this result is divided in two parts. In Subsection 2.5.1, we show that the sequence $\{\mathbb{Q}^{W,N}_{\mu_N} : N \geq 1\}$ is tight and in Subsection 2.5.2 we characterize the limit points of this sequence.

Proof of Theorem 2.3.2. Since $\mathbb{Q}^{W,N}_{\mu_N}$ converges in the uniform topology to $\mathbb{Q}_W$, a measure which is concentrated on a deterministic path, for each $0 \leq t \leq T$ and each continuous function $H : \mathbb{T} \to \mathbb{R}$, $\langle \pi_t^N, H\rangle$ converges in probability to $\int_{\mathbb{T}} du\, \rho(t, u)\; H(u)$, where ρ is the unique weak solution of (2.3.5) with $l = 0$, $r = 1$, $\gamma = \rho_0$ and $\Phi(\alpha) = \alpha + a\alpha^2$. □

2.5.1 Tightness

Tightness of the sequence $\{\mathbb{Q}^{W,N}_{\mu_N} : N \geq 1\}$ is proved as in [16, 7]. by considering first the auxiliary $\mathcal{M}$-valued Markov process $\{\Pi^{\lambda,N}_t : t \geq 0\}$, $\lambda > 0$, defined by

$$\Pi^{\lambda,N}_t(H) \;=\; \langle \pi^N_t, G^N_\lambda H\rangle \;=\; \frac{1}{N}\sum_{x\in\mathbb{Z}} \big(G^N_\lambda H\big)(x/N)\eta_t(x) \;,$$

H in $C(\mathbb{T})$, where $\{G^N_\lambda : \lambda > 0\}$ is the resolvent associated to the random walk $\{X^N_t : t \geq 0\}$ introduced in Section 2.4.

We first prove tightness of the process $\{\Pi^{\lambda,N}_t : 0 \leq t \leq T\}$ for every $\lambda > 0$ and then show that $\{\Pi^{\lambda,N}_t : 0 \leq t \leq T\}$ and $\{\pi^N_t : 0 \leq t \leq T\}$ are not far apart if λ is large.

It is well known [17] that to prove tightness of $\{\Pi^{\lambda,N}_t : 0 \leq t \leq T\}$ it is enough to show tightness of the real-valued processes $\{\Pi^{\lambda,N}_t(H) : 0 \leq t \leq T\}$ for a set of smooth functions $H : \mathbb{T} \to \mathbb{R}$ dense in $C(\mathbb{T})$ for the uniform topology.

Fix a smooth function $H : \mathbb{T} \to \mathbb{R}$. Denote by the same symbol the restriction of H to $N^{-1}\mathbb{T}_N$. Let $H^N_\lambda = G^N_\lambda H$ so that

$$\lambda H^N_\lambda \;-\; \mathbb{L}_N H^N_\lambda \;=\; H \;. \tag{2.5.2}$$

Keep in mind that $\Pi^{\lambda,N}_t(H) = \langle \pi^N_t, H^N_\lambda\rangle$ and denote by $M^{N,\lambda}_t$ the martingale defined by

$$M^{N,\lambda}_t \;=\; \Pi^{\lambda,N}_t(H) \;-\; \Pi^{\lambda,N}_0(H) \;-\; \int_0^t ds\, N^2 L_N \langle \pi^N_s, H^N_\lambda\rangle \;. \tag{2.5.3}$$

Clearly, tightness of $\Pi^{\lambda,N}_t(H)$ follows from tightness of the martingale $M^{N,\lambda}_t$ and tightness of the additive functional $\int_0^t ds\, N^2 L_N\langle \pi^N_s, H^N_\lambda\rangle$.

An elementary computation shows that the quadratic variation $\langle M^{N,\lambda}\rangle_t$ of the martingale $M^{N,\lambda}_t$ is given by

$$\frac{1}{N^2}\sum_{x\in\mathbb{T}_N} \xi_x \,[(\nabla_N H^N_\lambda)(x/N)]^2 \int_0^t c_{x,x+1}(\eta_s)\,[\eta_s(x+1) - \eta_s(x)]^2\, ds \;.$$

In particular, by (2.4.13),

$$\langle M^{N,\lambda}\rangle_t \;\leq\; \frac{C_0 t}{N^2}\sum_{x\in\mathbb{T}_N} \xi_x\,[(\nabla_N H^N_\lambda)(x/N)]^2 \;\leq\; \frac{C(H)t}{\lambda N}$$

for some finite constant $C(H)$ which depends only on H. Thus, by Doob inequality, for every $\lambda > 0$, $\delta > 0$

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big[\sup_{0\leq t\leq T} \big|M^{N,\lambda}_t\big| \;>\; \delta\Big] \;=\; 0 \;. \tag{2.5.4}$$

In particular, the sequence of martingales $\{M^{N,\lambda}_t : N \geq 1\}$ is tight for the uniform topology.

It remains to examine the additive functional of the decomposition (2.5.3). A long and elementary computations shows that $N^2 L_N \langle \pi^N, H_\lambda^N \rangle$ is equal to

$$\begin{aligned} & \frac{1}{N} \sum_{x \in \mathbb{T}_N} (\mathbb{L}_N H_\lambda^N)(x/N)\, \eta(x) \\ & \quad + \frac{a}{N} \sum_{x \in \mathbb{T}_N} \big\{ (\mathbb{L}_N H_\lambda^N)((x+1)/N) + (\mathbb{L}_N H_\lambda^N)(x/N) \big\}\, (\tau_x h_1)(\eta) \\ & \quad\quad - \frac{a}{N} \sum_{x \in \mathbb{T}_N} (\mathbb{L}_N H_\lambda^N)(x/N) (\tau_x h_2)(\eta)\ , \end{aligned}$$

where $\{\tau_x : x \in \mathbb{Z}\}$ is the group of translations so that $(\tau_x \eta)(y) = \eta(x+y)$ for x, y in $\mathbb{Z}$ and the sum is understood modulo N. Also, h_1, h_2 are the cylinder functions

$$h_1(\eta) \;=\; \eta(0)\eta(1)\ , \quad h_2(\eta) \;=\; \eta(-1)\eta(1)\ .$$

Since H_λ^N is the solution of the resolvent equation (2.5.2), we may replace $\mathbb{L}_N H_\lambda^N$ by $U_\lambda^N = \lambda H_\lambda^N - H$ in the previous formula. In particular, for all $0 \le s < t \le T$,

$$\Big| \int_s^t dr\, N^2 L_N \langle \pi_r^N, H_\lambda^N \rangle \Big| \;\le\; \frac{(1+3|a|)(t-s)}{N} \sum_{x \in \mathbb{T}_N} |U_\lambda^N(x/N)|\ .$$

It follows from the first estimate in (2.4.13) and from Schwarz inequality that the right hand side is bounded above by $C(H,a)(t-s)$ uniformly in N, where $C(H,a)$ is a finite constant depending only on a and H. This proves that the additive part of the decomposition (2.5.3) is tight for the uniform topology and therefore that the sequence of processes $\{\Pi_t^{\lambda,N} : N \ge 1\}$ is tight.

Lemma 2.5.2. *The sequence of measures* $\{\mathbb{Q}_{\mu^N}^{W,N} : N \ge 1\}$ *is tight for the uniform topology.*

Proof. It is enough to show that for every smooth function $H : \mathbb{T} \to \mathbb{R}$ and every $\epsilon > 0$, there exists $\lambda > 0$ such that

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N} \Big[\sup_{0 \le t \le T} |\, \Pi_t^{\lambda,N}(\lambda H) - \langle \pi_t^N, H \rangle \,| > \epsilon \Big] \;=\; 0$$

because in this case the tightness of π_t^N follows from the tightness of $\Pi_t^{\lambda,N}$. Since there is at most one particle per site the expression inside the absolute value is less than or equal to

$$\frac{1}{N} \sum_{x \in \mathbb{T}_N} \big| \lambda H_\lambda^N(x/N) - H(x/N) \big|\ .$$

By (2.4.14) this expression vanishes as $N \uparrow \infty$, $\lambda \uparrow \infty$. □

2.5.2 Uniqueness of limit points

We prove in this subsection that all limit points $\mathbb{Q}^*$ of the sequence $\mathbb{Q}^{W,N}_{\mu_N}$ are concentrated on absolutely continuous trajectories $\pi(t, du) = \rho(t, u)du$, whose density $\rho(t, u)$ is a weak solution of the hydrodynamic equation (2.3.5) with $l = 0$, $r = 1$, $\gamma = \rho_0$ and $\Phi(\theta) = \theta + a\theta^2$.

Let $\mathbb{Q}^*$ be a limit point of the sequence $\mathbb{Q}^{W,N}_{\mu_N}$ and assume, without loss of generality, that $\mathbb{Q}^{W,N}_{\mu_N}$ converges to $\mathbb{Q}^*$.

Since there is at most one particle per site, it is clear that $\mathbb{Q}^*$ is concentrated on trajectories $\pi_t(du)$ which are absolutely continuous with respect to the Lebesgue measure, $\pi_t(du) = \rho(t, u)du$, and whose density ρ is non-negative and bounded by 1.

Fix a function $H : \mathbb{T} \to \mathbb{R}$ continuously differentiable and $\lambda > 0$. Recall the definition of the martingale $M^{N,\lambda}_t$ introduced in the previous subsection. By (2.5.4), for every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big[\sup_{0\le t\le T} \big|M^{N,\lambda}_t\big| > \delta\Big] = 0 .$$

The martingale $M^{N,\lambda}_t$ can be written in terms of the empirical measure as

$$\langle \pi^N_t, G^N_\lambda H\rangle - \langle \pi^N_0, G^N_\lambda H\rangle - \int_0^t ds\, N^2 L_N \langle \pi^N_s, G^N_\lambda H\rangle .$$

Therefore, for fixed $0 < t \le T$ and $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{Q}^{W,N}_{\mu_N}\Big[\Big|\langle \pi^N_t, G^N_\lambda H\rangle - \langle \pi^N_0, G^N_\lambda H\rangle - \int_0^t ds\, N^2 L_N \langle \pi^N_s, G^N_\lambda H\rangle\Big| > \delta\Big] = 0 .$$

Since there is at most one particle per site, by (2.3.4), we may replace $G^N_\lambda H$ by $G_\lambda H$ in the expressions $\langle \pi^N_t, G^N_\lambda H\rangle$, $\langle \pi^N_0, G^N_\lambda H\rangle$ above.

On the other hand, the expression $N^2 L_N \langle \pi^N_s, G^N_\lambda H\rangle$ has been computed in the previous subsection. Recall that $\mathbb{L}_N G^N_\lambda H = \lambda G^N_\lambda H - H$. As before, we may replace $G^N_\lambda H$ by $G_\lambda H$. Let $U_\lambda = \lambda G_\lambda H - H$. Since $E_{\nu_\alpha}[h_j] = \alpha^2$, $j = 1, 2$, in view of (2.4.13) and by Corollary 2.5.4, for every $t > 0$, $\lambda > 0$, $\delta > 0$, $j = 1, 2$,

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} \mathbb{P}_{\mu_N}\Big[\Big| \int_0^t ds\, \frac{1}{N} \sum_{x\in\mathbb{T}_N} U_\lambda(x/N) \Big\{ \tau_x h_j(\eta_s) - \big[\eta^{\varepsilon N}_s(x)\big]^2 \Big\} \Big| > \delta \Big] = 0 .$$

Since $\eta^{\varepsilon N}_s(x) = \varepsilon^{-1} \pi^N_s([x/N, x/N + \varepsilon])$, we obtain from the previous considerations that

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} \mathbb{Q}^{W,N}_{\mu_N}\Big[\Big| \langle \pi^N_t, G_\lambda H\rangle - \\ - \langle \pi^N_0, G_\lambda H\rangle - \int_0^t ds \Big\langle \Phi\big(\varepsilon^{-1}\pi^N_s([\cdot, \cdot + \varepsilon])\big), U_\lambda \Big\rangle \Big| > \delta \Big] = 0 .$$

Since H is a smooth function, $G_\lambda H$ and U_λ can be approximated in $L^1(\mathbb{T})$ by continuous functions. Since we assumed that $\mathbb{Q}^{W,N}_{\mu_N}$ converges in the uniform topology to $\mathbb{Q}^*$, we have that

$$\lim_{\varepsilon\to 0} \mathbb{Q}^* \Big[\Big| \langle \pi_t, G_\lambda H\rangle \; - \\ - \langle \pi_0, G_\lambda H\rangle \; - \; \int_0^t ds \, \Big\langle \Phi\big(\varepsilon^{-1}\pi_s([\cdot, \cdot + \varepsilon])\big) \, , \, U_\lambda \Big\rangle \Big| > \delta \Big] \; = \; 0 \; .$$

As $\mathbb{Q}^*$ is concentrated on absolutely continuous paths $\pi_t(du) = \rho(t,u)du$ with positive density bounded by 1, $\varepsilon^{-1}\pi_s([\cdot, \cdot + \varepsilon])$ converges in $L^1(\mathbb{T})$ to $\rho(s,u)$ as $\varepsilon \downarrow 0$. Thus,

$$\mathbb{Q}^* \Big[\Big| \langle \pi_t, G_\lambda H\rangle \; - \; \langle \pi_0, G_\lambda H\rangle \; - \; \int_0^t ds \, \langle \Phi(\rho_s) \, , \, \mathcal{L}_W G_\lambda H\rangle \Big| > \delta \Big] \; = \; 0$$

because $U_\lambda = \mathcal{L}_W G_\lambda H$. Letting $\delta \downarrow 0$, we see that $\mathbb{Q}^*$ a.s.

$$\langle \pi_t, G_\lambda H\rangle \; - \; \langle \pi_0, G_\lambda H\rangle \; = \; \int_0^t ds \, \langle \Phi(\rho_s) \, , \, \mathcal{L}_W G_\lambda H\rangle \; .$$

This identity can be extended to a countable set of times t. Taking this set to be dense, by continuity of the trajectories π_t, we obtain that it holds for all $0 \le t \le T$. In the same way, it holds for any countable family of continuous functions. Taking a countable set of continuous functions, dense for the uniform topology, we extend this identity to all continuous function H because $G_\lambda H_n$ converges to $G_\lambda H$ in $L^1(\mathbb{T})$ if H_n converges to H in the uniform topology. Similarly, we can show that it holds for all $\lambda > 0$, since, for any continuous function H, $G_{\lambda_n} H$ converges to $G_\lambda H$ in $L^1(\mathbb{T})$, as $\lambda_n \to \lambda$.

Proof of Proposition 2.5.1. In the previous subsection we showed that the sequence of probability measures $\mathbb{Q}^{W,N}_{\mu_N}$ is tight for the uniform topology. We just proved that all limit points of this sequence are concentrated on weak solutions of the parabolic equation (2.3.5). The statement of the proposition follows from the uniqueness of weak solutions proved in Section 2.7. □

2.5.3 Replacement lemma

Denote by $H_N(\mu_N|\nu_\alpha)$ the entropy of a probability measure μ_N with respect to a stationary state ν_α. We refer to [17, Section A1.8] for a precise definition. By the explicit formula given in [17, Theorem A1.8.3], we see that there exists a finite constant K_0, depending only on α, such that

$$H_N(\mu_N|\nu_\alpha) \; \le \; K_0 N \tag{2.5.5}$$

for all measures μ_N.

Denote by $\langle \cdot, \cdot \rangle_{\nu_\alpha}$ the scalar product of $L^2(\nu_\alpha)$ and denote by I_N^ξ the convex and lower semicontinuous [17, Corollary A1.10.3] functional defined by

$$I_N^\xi(f) \; = \; \langle -L_N \sqrt{f} \, , \, \sqrt{f} \rangle_{\nu_\alpha} \; ,$$

for all probability densities f with respect to ν_α (i.e., $f \ge 0$ and $\int f d\nu_\alpha = 1$). An elementary computation shows that

$$
\begin{aligned}
I_N^\xi(f) \;&=\; \sum_{x\in\mathbb{T}_N} I_{x,x+1}^\xi(f)\;, \quad \text{where}\\
I_{x,x+1}^\xi(f) \;&=\; (1/2)\,\xi_x \int c_{x,x+1}(\eta)\big\{\sqrt{f(\sigma^{x,x+1}\eta)} - \sqrt{f(\eta)}\big\}^2\, d\nu_\alpha\;.
\end{aligned}
$$

By [17, Theorem A1.9.2], if $\{S_t^N : t \ge 0\}$ represents the semi-group associated to the generator $N^2 L_N$,

$$
H_N(\mu_N S_t^N|\nu_\alpha) \;+\; N^2 \int_0^t I_N^\xi(f_s^N)\, ds \;\le\; H_N(\mu_N|\nu_\alpha)\;,
$$

provided f_s^N stands for the Radon-Nikodym derivative of $\mu_N S_s^N$ with respect to ν_α.

For a local function $g : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, let $\tilde g : [0,1] \to \mathbb{R}$ be the expected value of g under the stationary states:

$$
\tilde g(\alpha) \;=\; E_{\nu_\alpha}[g(\eta)]\;.
$$

For $\ell \ge 1$, let $\eta^\ell(x)$ be the density of particles on the interval $\{x, \dots, x+\ell-1\}$:

$$
\eta^\ell(x) \;=\; \frac{1}{\ell}\sum_{y=x}^{x+\ell-1}\eta(y)\;.
$$

Lemma 2.5.3. *Fix a function $F : N^{-1}\mathbb{T}_N \to \mathbb{R}$. There exists a finite constant C_0, depending only on a, g and W, such that*

$$
\begin{aligned}
&\frac{1}{N}\sum_{x\in\mathbb{T}_N} F(x/N)\int \{\tau_x g(\eta) - \tilde g(\eta^{\varepsilon N}(x))\}\, f(\eta)\nu_\alpha(d\eta)\\
&\quad\le\; \frac{C_0}{\varepsilon N^2}\sum_{x\in\mathbb{T}_N}\big|F(x/N)\big| \;+\; \frac{C_0\varepsilon}{\delta N}\sum_{x\in\mathbb{T}_N} F(x/N)^2 \;+\; \delta N I_N^\xi(f)
\end{aligned}
$$

for all $\delta > 0$ and all probability density f with respect to ν_α.

Proof. Any local function can be written as a linear combination of functions of type $\prod_{x\in A}\eta(x)$, for finite sets $A's$. It is therefore enough to prove the lemma for such functions. We prove the result for $g(\eta) = \eta(0)\eta(1)$. The general case can be handled in a similar way.

We estimate first

$$
\frac{1}{N}\sum_{x\in\mathbb{T}_N} F(x/N)\int \eta(x)\Big\{\eta(x+1) - \frac{1}{\varepsilon N}\sum_{y=x}^{x+\varepsilon N-1}\eta(y)\Big\} f(\eta)\nu_\alpha(d\eta) \tag{2.5.6}
$$

in terms of the functional $I_N^\xi(f)$. The integral can be rewritten as

$$\frac{1}{\varepsilon N}\sum_{y=x+2}^{x+\varepsilon N-1}\sum_{z=x+1}^{y-1}\int \eta(x)\{\eta(z)-\eta(z+1)\}f(\eta)\nu_\alpha(d\eta) \;+\; O(\frac{1}{\varepsilon N})\;,$$

where the remainder comes from the contribution $y=x$. Writing last integral as twice the same expression and performing the change of variables $\eta' = \sigma^{z,z+1}\eta$ in one of them, the previous integral becomes

$$(1/2)\int \eta(x)\{\eta(z)-\eta(z+1)\}\left\{f(\eta)-f(\sigma^{z,z+1}\eta)\right\}\nu_\alpha(d\eta)\;.$$

Since $a-b=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$, by Schwarz inequality the previous expression is less than or equal to

$$\begin{aligned}&\frac{A}{16(1-2a^-)\xi_z}\int \eta(x)\{\eta(z)-\eta(z+1)\}^2\left\{\sqrt{f(\eta)}+\sqrt{f(\sigma^{z,z+1}\eta)}\right\}^2\nu_\alpha(d\eta)\\ &\quad+\frac{\xi_z}{A}\int c_{z,z+1}(\eta)\left\{\sqrt{f(\eta)}-\sqrt{f(\sigma^{z,z+1}\eta)}\right\}^2\nu_\alpha(d\eta)\end{aligned}$$

for every $A>0$. In this formula we used the fact that $c_{z,z+1}$ is bounded below by $1-2a^-$. Since f is a density with respect to ν_α, the first expression is bounded by $A/4(1-2a^-)\xi_z$, while the second one is equal to $2A^{-1}I^\xi_{z,z+1}(f)$. Adding together all previous estimates, we obtain that (2.5.6) is less than or equal to

$$\frac{1}{\varepsilon N^2}\sum_{x\in\mathbb{T}_N}\left|F(x/N)\right| \;+\; \frac{A}{4(1-2a^-)N}\sum_{x\in\mathbb{T}_N}F(x/N)^2\sum_{z=x+1}^{x+\varepsilon N}\xi_z^{-1} \;+\; \frac{2\varepsilon}{A}\sum_{z\in\mathbb{T}_N}I^\xi_{z,z+1}(f)\;.$$

By definition of the sequence $\{\xi_z\}$, $\sum_{x+1\le z\le x+\varepsilon N}\xi_z^{-1}\le N[W(1)-W(0)]$. Thus, choosing $A=2\varepsilon N^{-1}\delta^{-1}$, for some $\delta>0$, we obtain that the previous sum is bounded above by

$$\frac{1}{\varepsilon N^2}\sum_{x\in\mathbb{T}_N}\left|F(x/N)\right| \;+\; \frac{C_0\varepsilon}{\delta N}\sum_{x\in\mathbb{T}_N}F(x/N)^2 \;+\; \delta N I_N^\xi(f)\;.$$

Up to this point we have replaced $\eta(x)\eta(x+1)$ by $\eta(x)\eta^{\varepsilon N}(x)$. The same arguments permit to replace this latter expression by $[\eta^{\varepsilon N}(x)]^2$, which concludes the proof of the lemma. □

Corollary 2.5.4. *Fix a cylinder function g and a sequence of functions $\{F_N : N\ge 1\}$, $F_N : N^{-1}\mathbb{T}_N\to\mathbb{R}$ such that*

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{x\in\mathbb{T}_N}F_N(x/N)^2 \;<\; \infty\;.$$

Then, for any $t>0$ and any sequence of probability measures $\{\mu_N : N\ge 1\}$ on $\{0,1\}^{\mathbb{T}_N}$,

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu_N}\Big[\Big|\int_0^t\frac{1}{N}\sum_{x\in\mathbb{T}_N}F_N(x/N)\left\{\tau_x g(\eta_s)-\tilde g(\eta_s^{\varepsilon N}(x))\right\}\Big|\Big] \;=\; 0\;.$$

Proof. Fix $0 < \alpha < 1$. By the entropy and Jensen inequalities, the expectation appearing in the statement of the lemma is bounded above by

$$\frac{H_N(\mu_N|\nu_\alpha)}{\gamma N} + \frac{1}{\gamma N}\log \mathbb{E}_{\nu_\alpha}\Big[\exp\Big\{\gamma\Big|\int_0^t \sum_{x\in\mathbb{T}_N} F_N(x/N)\,\{\tau_x g(\eta_s) - \tilde g(\eta_s^{\varepsilon N}(x))\}\,ds\,\Big|\Big\}\Big]$$

for all $\gamma > 0$. In view of (2.5.5), to prove the corollary it is enough to show that the second term vanishes as $N\uparrow\infty$ and then $\varepsilon\downarrow 0$ for every $\gamma>0$. We may remove the absolute value inside the exponential because $e^{|x|}\le e^x+e^{-x}$ and because $\limsup_{N\to\infty} N^{-1}\log\{a_N+b_N\}\le \max\{\limsup_{N\to\infty}N^{-1}\log a_N, \limsup_{N\to\infty}N^{-1}\log b_N\}$. Thus, to prove the corollary, we need to show that

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log \mathbb{E}_{\nu_\alpha}\Big[\exp\Big\{\gamma\int_0^t\sum_{x\in\mathbb{T}_N}F_N(x/N)\{\tau_x g(\eta_s)-\tilde g(\eta_s^{\varepsilon N}(x))\}\Big\}\Big] \;=\; 0$$

for every $\gamma > 0$.

By Feynman-Kac formula, for each fixed N the previous expression is bounded above by

$$t\sup_f\Big\{\int\frac{\gamma}{N}\sum_{x\in\mathbb{T}_N}F_N(x/N)\{\tau_x g(\eta)-\tilde g(\eta^{\varepsilon N}(x))\}f(\eta)\,d\nu_\alpha \;-\; NI_N^{\xi}(f)\Big\}\;,$$

where the supremum is carried over all density functions f with respect to ν_α. Letting $\delta=1$ in Lemma 2.5.3, we obtain that the previous expression is less than or equal to

$$\frac{C_0\gamma}{\varepsilon N^2}\sum_{x\in\mathbb{T}_N}\big|F_N(x/N)\big| \;+\; \frac{C_0\gamma\varepsilon}{N}\sum_{x\in\mathbb{T}_N}F_N(x/N)^2$$

for some finite constant C_0 which depends on g and W. By assumption on the sequence $\{F_N\}$, for every $\gamma>0$, this expression vanishes as $N\uparrow\infty$ and then $\varepsilon\downarrow 0$. This concludes the proof of the lemma. □

2.6 Energy estimate

We prove in this section that any limit point $\mathbb{Q}_W^*$ of the sequence $\mathbb{Q}_{\mu_N}^{W,N}$ is concentrated on trajectories $\rho(t,u)du$ with finite energy. Though not needed in the proof of uniqueness of weak solutions, this estimate plays an important role in the proof of a large deviations principle.

Let $\mathbb{Q}_W^*$ be a limit point of the sequence $\mathbb{Q}_{\mu_N}^{W,N}$ and assume without loss of generality that the sequence $\mathbb{Q}_{\mu_N}^{W,N}$ converges to $\mathbb{Q}_W^*$. Denote by ∂_u the partial derivative of a function with respect to the space variable. Let $L_W^2([0,T]\times\mathbb{T})$ be the Hilbert space of measurable functions $H:[0,T]\times\mathbb{T}\to\mathbb{R}$ such that

$$\int_0^T ds\int_{\mathbb{T}} dW(u)\,H(s,u)^2 \;<\;\infty\;,$$

endowed with the scalar product $\langle\!\langle H, G\rangle\!\rangle_W$ defined by

$$\langle\!\langle H, G\rangle\!\rangle_W \;=\; \int_0^T ds \int_{\mathbb{T}} dW(u)\, H(s,u)\, G(s,u)\ .$$

Proposition 2.6.1. *The measure $\mathbb{Q}_W^*$ is concentrated on paths $\rho(t,u)du$ with the property that there exists a function in $L_W^2([0,T]\times\mathbb{T})$, denoted by $d\Phi/dW$, such that*

$$\int_0^T ds \int_{\mathbb{T}} du\, (\partial_u H)(s,u)\, \Phi(\rho(s,u)) \;=\; -\int_0^T ds \int_{\mathbb{T}} dW(u)\, (d\Phi/dW)(s,u)\, H(s,u)$$

for all functions H in $C^{0,1}([0,T]\times\mathbb{T})$.

The previous result follows from the next lemma. Recall the definition of the constant K_0 given in (2.5.5).

Lemma 2.6.2. *There exists a finite constant K_1, depending only on a, such that*

$$E_{\mathbb{Q}_W^*}\Big[\sup_H \Big\{ \int_0^T ds \int_{\mathbb{T}} du\, (\partial_u H)(s,u)\, \Phi(\rho(s,u)) - K_1 \int_0^T ds \int_{\mathbb{T}} H(s,u)^2\, dW(u)\Big\}\Big] \;\le\; K_0\ ,$$

where the supremum is carried over all functions H in $C^{0,1}([0,T]\times\mathbb{T})$.

Proof of Proposition 2.6.1. Denote by $\ell : C^{0,1}([0,T]\times\mathbb{T}) \to \mathbb{R}$ the linear functional defined by

$$\ell(H) \;=\; \int_0^T ds \int_{\mathbb{T}} du\, (\partial_u H)(s,u)\, \Phi(\rho(s,u))\ .$$

Since $C^{0,1}([0,T]\times\mathbb{T})$ is dense in $L_W^2([0,T]\times\mathbb{T})$, by Lemma 2.6.2, ℓ is $\mathbb{Q}_W^*$-almost surely finite in $L_W^2([0,T]\times\mathbb{T})$. In particular, by Riesz representation theorem, there exists a function G in $L_W^2([0,T]\times\mathbb{T})$ such that

$$\ell(H) \;=\; -\int_0^T ds \int_{\mathbb{T}} dW(u)\, H(s,u)\, G(s,u)\ .$$

This concludes the proof of the proposition. □

The proof of Lemma 2.6.2 relies on the following result. For a finite constant K_1, a smooth function $H\colon \mathbb{T}\to\mathbb{R}$, $\delta>0$, $\varepsilon>0$ and a positive integer N, define $W_N(\varepsilon,\delta,H,\eta)$ by

$$\begin{aligned} W_N(\varepsilon,\delta,H,\eta) \;&=\; \sum_{x\in\mathbb{T}_N} H(x/N)\frac{1}{\varepsilon N}\Big\{\Phi(\eta^{\delta N}(x)) - \Phi(\eta^{\delta N}(x+\varepsilon N))\Big\} \\ &-\; \frac{K_1}{\varepsilon N}\sum_{x\in\mathbb{T}_N} H(x/N)^2\{W([x+\varepsilon N+1]/N) - W(x/N)\}\ . \end{aligned}$$

Lemma 2.6.3. *Consider a sequence $\{H_\ell,\, \ell \geq 1\}$ dense in $C^{0,1}([0,T]\times\mathbb{T})$. There exists a finite constant K_1 such that*

$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\Big[\max_{1\le i\le k}\Big\{\int_0^T W_N(\varepsilon,\delta,H_i(s,\cdot),\eta_s)\,ds\Big\}\Big]\ \le\ K_0\ .$$

For every $k\geq 1$ and every $\varepsilon > 0$.

Proof. It follows from the replacement lemma that in order to prove the lemma we just need to show that

$$\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\Big[\max_{1\le i\le k}\Big\{\int_0^T W_N(\varepsilon,H_i(s,\cdot),\eta_s)\,ds\Big\}\Big]\ \le\ K_0\ ,$$

where

$$\begin{aligned} W_N(\varepsilon,H,\eta) \ &=\ \frac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N} H(x/N)\big\{\tau_x g(\eta)-\tau_{x+\varepsilon N} g(\eta)\big\}\\ &-\ \frac{K_1}{\varepsilon N}\sum_{x\in\mathbb{T}_N} H(x/N)^2\{W([x+\varepsilon N+1]/N)-W(x/N)\}\ , \end{aligned}$$

and $g(\eta)=\eta(0)+2a\eta(0)\eta(1)$.

By the entropy and the Jensen inequality, for each fixed N, the previous expectation is bounded above by

$$\frac{H(\mu^N|\nu_\alpha)}{N}\ +\ \frac{1}{N}\log\mathbb{E}_{\nu_\alpha}\Big[\exp\Big\{\max_{1\le i\le k}\Big\{N\int_0^T ds\,W_N(\varepsilon,H_i(s,\cdot),\eta_s)\Big\}\Big\}\Big]\ .$$

By (2.5.5), the first term is bounded by K_0. Since $\exp\{\max_{1\le j\le k} a_j\}$ is bounded above by $\sum_{1\le j\le k}\exp\{a_j\}$ and since $\limsup_N N^{-1}\log\{a_N+b_N\}$ is less than or equal to the maximum of $\limsup_N N^{-1}\log a_N$ and $\limsup_N N^{-1}\log b_N$, the limit, as $N\uparrow\infty$, of the second term of the previous expression is less than or equal to

$$\max_{1\le i\le k}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}_{\nu_\alpha}\Big[\exp\Big\{N\int_0^T ds\,W_N(\varepsilon,H_i(s,\cdot),\eta_s)\Big\}\Big]\ .$$

We now prove that for each fixed i the above limit is non-positive.

Fix $1\le i\le k$. By Feynman–Kac formula and the variational formula for the largest eigenvalue of a symmetric operator, for each fixed N, the previous expression is bounded above by

$$\int_0^T ds\,\sup_f\Big\{\int W_N(\varepsilon,H_i(s,\cdot),\eta)f(\eta)\nu_\alpha(d\eta)-NI_N^\xi(f)\Big\}\ .$$

In this formula the supremum is taken over all probability densities f with respect to ν_α.

It remains to rewrite $\eta(x)\eta(x+1)-\eta(x+\varepsilon N)\eta(x+\varepsilon N+1)$ as $\eta(x)\{\eta(x+1)-\eta(x+\varepsilon N+1)\}+\eta(x+\varepsilon N+1)\{\eta(x)-\eta(x+\varepsilon N)\}$ and to repeat the arguments presented in the proof of Lemma 2.5.3 to conclude. □

Proof of Lemma 2.6.2. Assume without loss of generality that $\mathbb{Q}^{W,N}_{\mu_N}$ converges to $\mathbb{Q}^*_W$. Consider a sequence $\{H_\ell,\, \ell \ge 1\}$ dense in $C^{0,1}([0,T]\times\mathbb{T})$. By Lemma 2.6.3, for every $k \ge 1$

$$\limsup_{\delta\to 0} E_{\mathbb{Q}^*_W}\Big[\max_{1\le i\le k}\Big\{\frac{1}{\varepsilon}\int_0^T ds\int_{\mathbb{T}} du\, H_i(s,u)\Big\{\Phi(\rho^\delta_s(u)) - \Phi(\rho^\delta_s(u+\varepsilon))\Big\} \\ - \frac{K_1}{\varepsilon}\int_0^T ds\int_{\mathbb{T}} du\, H_i(s,u)^2\,[W(u+\varepsilon)-W(u)]\Big\}\Big] \;\le\; K_0\,,$$

where $\rho^\delta_s(u) = (\rho_s * \iota_\delta)(u)$ and ι_δ is the approximation of the identity $\iota_\delta(\cdot) = (2\delta)^{-1}\mathbf{1}\{[-\delta,\delta]\}(\cdot)$.

Letting $\delta \downarrow 0$, changing variables and then letting $\varepsilon \downarrow 0$, we obtain that

$$E_{\mathbb{Q}^*_W}\Big[\max_{1\le i\le k}\Big\{\int_0^T ds\int_{\mathbb{T}} (\partial_u H_i)(s,u)\Phi(\rho(s,u))\,du \\ - K_1\int_0^T ds\int_{\mathbb{T}} H_i(s,u)^2 dW(u)\Big\}\Big] \;\le\; K_0\,.$$

To conclude the proof it remains to apply the monotone convergence theorem and recall that $\{H_\ell,\, \ell\ge 1\}$ is a dense sequence in $C^{0,1}([0,T]\times\mathbb{T})$ for the norm $\|H\|_\infty + \|(\partial_u H)\|_\infty$. □

2.7 Uniqueness of weak solutions of (2.3.5)

Recall that we denote by $\langle\cdot,\cdot\rangle$ the inner product of the Hilbert space $L^2(\mathbb{T})$ and that $\{G_\lambda : \lambda > 0\}$ stands for the resolvents associated to $\mathcal{L}_W$.

Let ρ be a weak solution of the hydrodynamic equation (2.3.5). Since ρ, $\Phi(\rho)$ are bounded, the smooth functions are dense in $L^2(\mathbb{T})$ and $\mathcal{L}_W G_\lambda = -\mathbb{I} + \lambda G_\lambda$ are bounded operators, for any function H in $L^2(\mathbb{T})$,

$$\langle \rho_t, G_\lambda H\rangle \;-\; \langle \gamma, G_\lambda H\rangle \;=\; \int_0^t \langle \Phi(\rho_s), \mathcal{L}_W G_\lambda H\rangle\, ds$$

for all $t > 0$ and all $\lambda > 0$.

Let $\rho : \mathbb{R}_+\times\mathbb{T} \to [l,r]$ be a weak solution of (2.3.5). We claim that

$$\langle \rho_t\,,\, G_\lambda\rho_t\rangle \;-\; \langle \rho_0\,,\, G_\lambda \rho_0\rangle \;=\; 2\int_0^t \langle \Phi(\rho_s)\,,\, \mathcal{L}_W G_\lambda \rho_s\rangle\, ds \tag{2.7.1}$$

for all $t > 0$ and $\lambda > 0$.

To prove this claim, fix $\lambda > 0$, $t > 0$ and consider a partition $0 = t_0 < t_1 < \cdots < t_n = t$ of the interval $[0,t]$ so that

$$\langle \rho_t\,,\, G_\lambda\rho_t\rangle \;-\; \langle \rho_0\,,\, G_\lambda\rho_0\rangle \;=\; \sum_{k=0}^{n-1}\langle \rho_{t_{k+1}}\,,\, G_\lambda\rho_{t_{k+1}}\rangle \;-\; \langle \rho_{t_{k+1}}\,,\, G_\lambda\rho_{t_k}\rangle \\ +\; \sum_{k=0}^{n-1}\langle \rho_{t_{k+1}}\,,\, G_\lambda\rho_{t_k}\rangle \;-\; \langle \rho_{t_k}\,,\, G_\lambda\rho_{t_k}\rangle\,.$$

We handle the first term, the second one being similar. Since G_λ is self-adjoint in $L^2(\mathbb{T})$, since $\rho_{t_{k+1}}$ belongs to $L^2(\mathbb{T})$ and since ρ is a weak solution of (2.3.5),

$$\langle \rho_{t_{k+1}} \, , \, G_\lambda \rho_{t_{k+1}} \rangle \; - \; \langle \rho_{t_{k+1}} \, , \, G_\lambda \rho_{t_k} \rangle \; = \; \int_{t_k}^{t_{k+1}} \langle \Phi(\rho_s) \, , \, \mathcal{L}_W G_\lambda \rho_{t_{k+1}} \rangle \, ds \; .$$

Add and subtract on the right hand side $\langle \Phi(\rho_s) \, , \, \mathcal{L}_W G_\lambda \rho_s \rangle$. The time integral of this term is exactly the expression announced in (2.7.1) and the remainder is given by

$$\int_{t_k}^{t_{k+1}} \Big\{ \langle \Phi(\rho_s) \, , \, \mathcal{L}_W G_\lambda \rho_{t_{k+1}} \rangle \; - \; \langle \Phi(\rho_s) \, , \, \mathcal{L}_W G_\lambda \rho_s \rangle \Big\} \, ds \; .$$

Since $\mathcal{L}_W G_\lambda = -\mathbb{I} + \lambda G_\lambda$, where $\mathbb{I}$ is the identity, and since G_λ is self-adjoint, we may rewrite the previous difference as

$$- \Big\{ \langle \Phi(\rho_s) \, , \, \rho_{t_{k+1}} \rangle \; - \; \langle \Phi(\rho_s) \, , \, \rho_s \rangle \Big\} \; + \; \lambda \Big\{ \langle G_\lambda \Phi(\rho_s) \, , \, \rho_{t_{k+1}} \rangle \; - \; \langle G_\lambda \Phi(\rho_s) \, , \, \rho_s \rangle \Big\} \; .$$

The time integral between t_k and t_{k+1} of the second term is equal to

$$\lambda \int_{t_k}^{t_{k+1}} ds \int_s^{t_{k+1}} \langle \mathcal{L}_W G_\lambda \Phi(\rho_s) \, , \, \Phi(\rho_r) \rangle \, dr$$

because ρ is a weak solution of (2.3.5) and $\Phi(\rho_s)$ belongs to $L^2(\mathbb{T})$. It follows from the boundedness of the operator $\mathcal{L}_W G_\lambda$ and from the boundedness of $\Phi(\rho)$ that this expression is of order $(t_{k+1} - t_k)^2$.

To conclude the proof of claim (2.7.1) it remains to show that

$$\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \Big\{ \langle \Phi(\rho_s) \, , \, \rho_{t_{k+1}} \rangle \; - \; \langle \Phi(\rho_s) \, , \, \rho_s \rangle \Big\} \, ds$$

vanishes as the mesh of the partition tends to 0. Fix $\varepsilon > 0$ and choose β large enough for

$$\int_0^t ds \int_{\mathbb{T}} \Big\{ \beta G_\beta \Phi(\rho(s,u)) - \Phi(\rho(s,u)) \Big\}^2 \, du \; \le \; \varepsilon \; .$$

This is possible because $\Phi(\rho)$ is bounded, $\{\beta G_\beta : \beta > 0\}$ are uniformly bounded operators, and $\beta G_\beta \Phi(\rho(s,\cdot))$ converges to $\Phi(\rho(s,\cdot))$ in $L^2(\mathbb{T})$, as $\beta \uparrow \infty$, for all $0 \le s \le t$.

Paying a price of order $\sqrt{\varepsilon}$, because $l \le \rho \le r$, we may replace $\Phi(\rho_s)$ in the penultimate formula by $\beta G_\beta \Phi(\rho_s)$. After this replacement, since ρ is weak solution, we may rewrite the sum as

$$\beta \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} ds \int_s^{t_{k+1}} \langle \mathcal{L}_W G_\beta \Phi(\rho_s) \, , \, \Phi(\rho_r) \rangle \, dr \; .$$

We have already seen that this expression vanishes as the mesh of the partition tends to 0. This proves (2.7.1).

Recall the definition of the constant B given at the beginning of Subsection 2.3.3

Lemma 2.7.1. *Fix two density profiles γ^1, $\gamma^2 : \mathbb{T} \to [l,r]$ and denote by ρ^1, ρ^2 weak solutions of* (2.3.5) *with initial value γ^1, γ^2, respectively. Then,*

$$\Big\langle \rho_t^1 - \rho_t^2 \,,\, G_\lambda\big[\rho_t^1 - \rho_t^2\big] \Big\rangle \;\le\; \Big\langle \gamma^1 - \gamma^2 \,,\, G_\lambda\big[\gamma^1 - \gamma^2\big] \Big\rangle\, e^{B\lambda t/2}$$

for all $\lambda > 0$, $t > 0$. In particular, there exists at most one weak solution of (2.3.5).

Proof. Fix two density profiles γ^1, $\gamma^2 : \mathbb{T} \to [l,r]$. Let ρ^1, ρ^2 be two weak solutions with initial value γ^1, γ^2, respectively. By (2.7.1), for any $\lambda > 0$,

$$\begin{aligned} &\Big\langle \rho_t^1 - \rho_t^2 \,,\, G_\lambda\big[\rho_t^1 - \rho_t^2\big] \Big\rangle \;-\; \Big\langle \gamma^1 - \gamma^2 \,,\, G_\lambda\big[\gamma^1 - \gamma^2\big] \Big\rangle \;= \\ &-2\int_0^t \langle \Phi(\rho_s^1) - \Phi(\rho_s^2) \,,\, \rho_s^1 - \rho_s^2 \rangle\, ds \;+\; 2\lambda \int_0^t \Big\langle \Phi(\rho_s^1) - \Phi(\rho_s^2) \,,\, G_\lambda\big[\rho_s^1 - \rho_s^2\big] \Big\rangle\, ds\,. \end{aligned}$$

By Schwarz inequality, the second term on the right hand side is bounded above by

$$\frac{1}{A}\int_0^t \Big\langle \Phi(\rho_s^1) - \Phi(\rho_s^2) \,,\, G_\lambda\big[\Phi(\rho_s^1) - \Phi(\rho_s^2)\big] \Big\rangle\, ds \;+\; A\lambda^2 \int_0^t \Big\langle \rho_s^1 - \rho_s^2 \,,\, G_\lambda\big[\rho_s^1 - \rho_s^2\big] \Big\rangle\, ds$$

for every $A > 0$. Since the operator G_λ is bounded by λ^{-1}, and since Φ' is bounded by B, the first term of the previous expression is less than or equal to

$$\frac{B}{A\lambda}\int_0^t \Big\langle \rho_s^1 - \rho_s^2 \,,\, \Phi(\rho_s^1) - \Phi(\rho_s^2) \Big\rangle\, ds\,.$$

Choosing $A = B/2\lambda$, this expression cancels with the first term on the right hand side of the first formula. In particular, the left hand side of this formula is bounded by

$$\frac{B\lambda}{2}\int_0^t \Big\langle \rho_s^1 - \rho_s^2 \,,\, G_\lambda\big[\rho_s^1 - \rho_s^2\big] \Big\rangle\, ds\,.$$

It remains to recall Gronwall's inequality to conclude. $\square$

Chapter 3

Hydrodynamic limit for a type of exclusion processes with slow bonds in dimension ≥ 2

Joint work with Adriana Neumann (IMPA) and Glauco Valle (UFRJ). To appear in the Journal of Applied Probability 48.2 (June 2011).

3.1 Abstract

Let Λ be a connected closed region with smooth boundary contained in the d-dimensional continuous torus $\mathbb{T}^d$. In the discrete torus $N^{-1}\mathbb{T}_N^d$, we consider a nearest neighbor symmetric exclusion process where occupancies of neighboring sites are exchanged at rates depending on Λ in the following way: if both sites are in Λ or $\Lambda^{\complement}$, the exchange rate is one; If one site is in Λ and the other one is in $\Lambda^{\complement}$ and the direction of the bond connecting the sites is e_j, then the exchange rate is defined as N^{-1} times the absolute value of the inner product between e_j and the normal exterior vector to $\partial\Lambda$. We show that this exclusion type process has a non-trivial hydrodynamical behavior under diffusive scaling and, in the continuum limit, particles are not blocked or reflected by $\partial\Lambda$. Thus the model represents a system of particles under hard core interaction in the presence of a permeable membrane which slows down the passage of particles between two complementar regions.

3.2 Introduction

The exclusion process is a continuous time interacting particle system where particles move as independent random walks on a graph except for the exclusion rule that prevents two particles from occupying the same site, or vertex. In the symmetric case, the process evolves as follows: to each bond we associate a waiting exponential time, which are independent of the waiting time for any other bond; at the waiting time the occupancies of the sites connected by the bond are exchanged; the parameter of the exchange times, or exchange rate, depends only on the bond. The specification of the exchange rates determines the environment for the exclusion process. In our case, as the underlying graph, we consider the discrete torus with N^d points and nearest neighbor bonds. The variable N is the scaling parameter.

This paper studies the hydrodynamical behavior of symmetric exclusion processes in non-homogeneous environments, where the non-homogeneity is due to the presence of slow bonds. While a usual bond has exchange rate one, a slow bond has a lower exchange rate. With respect to the scaling parameter, we assume that a slow bond has exchange rate of order N^{-1}.

When the environment is homogeneous, the exclusion process has a well-known hydrodynamical behavior under diffusive scaling. About non-homogeneous environments, results have been obtained in several cases, even when the environment is random and consists only of slow bonds. For one dimensional processes, in [7], the exchange rate over a bond $[\frac{x}{N}, \frac{x+1}{N}]$ is given by $[N(W(x+1/N) - W(x/N))]^{-1}$, where W is an α-stable subordinator of a Lévy Process. They obtain a quenched hydrodynamic limit. In papers previous to [7], for example [5] and [23], the randomness or non-homogeneity did not survive in the continuum limit. Another one-dimensional result, following [7], was obtained in [9], for more general, but non-random, increasing functions W. The techniques used in those papers were strongly based on theorems about convergence of one dimensional continuous time stochastic processes. In fact, even the d-dimensional case treated in [26] consists of a class of non-homogeneous environments that could be decomposed, in a proper sense, in d one-dimensional cases. Recently, different approaches have been searched to deal with d-dimensional environments, see [6] and [15].

We now describe the exclusion processes we are concerned with. Let $\{e_j : j = 1, ..., d\}$ be the canonical basis of $\mathbb{R}^d$ and $\Lambda \subset \mathbb{T}^d$ be a simple connected region with smooth boundary $\partial\Lambda$. If the bond $[\frac{x}{N}, \frac{x+e_j}{N}] \in N^{-1}\mathbb{T}^d_N$ has vertices in each of the regions Λ and $\Lambda^{\complement}$, its exchange rate is defined as N^{-1} times the absolute value of the inner product between e_j and the normal exterior vector to $\partial\Lambda$. For others edges, the exchange rate is defined as one. This means that the slow bonds are among those crossing the boundary of Λ. We call this process the exclusion process with slow bonds over $\partial\Lambda$.

We can interpret $\partial\Lambda$ as a permeable membrane, which slows down the passage of particles between the regions Λ and $\Lambda^{\complement}$. For this type of exclusion process, the membrane does not completely prevent the passage of particles, and still survives in the continuum limit, appearing explicitly in the hydrodynamic equation. The exchange rate of particles for a bond crossing $\partial\Lambda$ is smaller if the bond is close to a tangent line of $\partial\Lambda$. Note that this assumption has physical meaning, take for example cases of reflections in several physical models: partial reflection of light crossing a medium with different refraction indexes, mechanical systems where particles try to cross some interface, etc. However the direction of the speed of particles is not changed as usually occur in physical reflection. Our definition of the exchange rates also allows a strong convergence result for the empirical measures associated to the exclusion process making simpler the proof of the hydrodynamic limit.

The hydrodynamical equation of the exclusion process with slow bonds over $\partial\Lambda$ is a parabolic partial differential equation $\partial_t \rho = \mathcal{L}_\Lambda \rho$, where the operator $\mathcal{L}_\Lambda$ is a sort of d-dimensional Krein-Feller operator. Without the presence of slow bonds, the operator $\mathcal{L}_\Lambda$ would be replaced by the laplacian operator acting on C^2 functions and the hydrodynamical equation is therefore the heat equation. Here, the existence of the membrane modifies the domain, and thus the operator itself. In fact, we observe that the proper domain for $\mathcal{L}_\Lambda$ contains functions that are discontinuous over $\partial\Lambda$. Geometrically, $\mathcal{L}_\Lambda$ glues the discontinuity of a function around $\partial\Lambda$ and then behaves like the laplacian.

One possible approach to prove the hydrodynamic limit for the exclusion process with slow bonds over $\partial\Lambda$ is through Gamma convergence. In [15], this approach and the conditions for it to hold are discussed, see also [5]. There, the coersiveness condition would require some kind of Rellich-Kondrachov Theorem (namely, the compact embedding in L^2 of some sort of Sobolev space supporting an extension of $\mathcal{L}_\Lambda$, see [4]). In the method presented here, we go in this direction, but instead of reach the hypotheses in [15], we have used similar analytical tools in order to obtain a short and simple proof of uniqueness of the hydrodynamic equation. We also show that the extension of $\mathcal{L}_\Lambda$ satisfies the Hille-Yoshida Theorem. On the other hand, the convergence from discrete to continuous that we present here is made in a very direct way, and it was inspired by the convergence of the discrete laplacian to the continuous laplacian.

The paper is presented as follows: In Section 3.3, we define the model and state all results contained in the paper; Section 3.4 is devoted to prove the results concerning the continuous operator $\mathcal{L}_\Lambda$; In Section 3.5, the hydrodynamic limit is proved.

3.3 Notation and Results

Let $\mathbb{T}^d$ be the d-dimensional torus, which is $[0,1)^d$ with periodic boundary conditions, and $\mathbb{T}_N^d$ be the discrete torus with N^d points, i.e., $\{0, ..., N-1\}^d$ with periodic boundary conditions. We denote by $\eta = (\eta(x))_{x\in\mathbb{T}_N^d}$ a typical configuration in the state space $\Omega_N = \{0,1\}^{\mathbb{T}_N^d}$, for which, $\eta(x) = 0$ means that site x is vacant, and $\eta(x) = 1$ that site x is occupied. If a bond of $N^{-1}\mathbb{T}_N^d$ has vertices $\frac{x}{N}$ and $\frac{y}{N}$, it will be denoted by $[\frac{x}{N}, \frac{y}{N}]$.

Recall that $\{e_j : j = 1, ..., d\}$ is the canonical basis of $\mathbb{R}^d$. The symmetric nearest neighbor exclusion process with exchange rates $\xi^N_{x,y} > 0$, $x, y \in \mathbb{T}_N^d$, $|x-y| = 1$, is a Markov process with configuration space Ω_N, whose generator L_N acts on functions $f : \Omega_N \to \mathbb{R}$ as

$$(L_N f)(\eta) = \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} \xi^N_{x,x+e_j} \Big[f(\eta^{x,x+e_j}) - f(\eta) \Big] , \tag{3.3.1}$$

where $\eta^{x,x+e_j}$ is the configuration obtained from η by exchanging the variables $\eta(x)$ and $\eta(x+e_j)$:

$$(\eta^{x,x+e_j})(y) = \begin{cases} \eta(x+e_j), & \text{if } y = x\,, \\ \eta(x), & \text{if } y = x+e_j\,, \\ \eta(y), & \text{otherwise.} \end{cases}$$

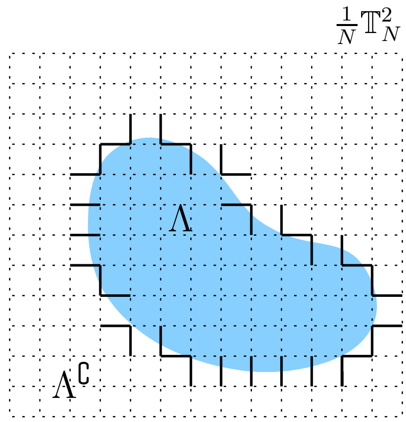

Figure 3.1: The darker region corresponds to Λ. The bolded bonds have exchanges rates $\frac{|\vec{\zeta}_{x,j}\cdot e_j|}{N}$, any other bond has exchange rate 1.

Let ν^N_α, $\alpha \in [0,1]$, be the Bernoulli product measure Ω_N, i.e., the product measure whose marginals have Bernoulli distribution with parameter α. Then $\{\nu^N_\alpha : 0 \leq \alpha \leq 1\}$ is a family of invariant, in fact reversible, measures for any symmetric exclusion process.

Now, fix a simple connected region $\Lambda \subset \mathbb{T}^d$ with smooth boundary $\partial\Lambda$. Denote by $\vec{\zeta}(u)$ the normal unitary exterior vector to the smooth surface $\partial\Lambda$ in the point $u \in \partial\Lambda$. If $\frac{x}{N} \in \Lambda$ and $\frac{x+e_j}{N} \in \Lambda^\complement$, or $\frac{x}{N} \in \Lambda^\complement$ and $\frac{x+e_j}{N} \in \Lambda$, we define $\vec{\zeta}_{x,j}$ as a vector $\vec{\zeta}(u)$ evaluated in an arbitrary but fixed point $u \in \partial\Lambda \cap [x, x+e_j]$. The exclusion process with slow bonds over $\partial\Lambda$ is a symmetric nearest neighbor exclusion process with exchange rates $\xi^N_{x,x+e_j} = \xi^N_{x+e_j,x}$ given by

$$\begin{cases} \dfrac{|\vec{\zeta}_{x,j}\cdot e_j|}{N}\,, & \text{if } \frac{x}{N} \in \Lambda \text{ and } \frac{x+e_j}{N} \in \Lambda^\complement\,, \text{ or } \frac{x}{N} \in \Lambda^\complement \text{ and } \frac{x+e_j}{N} \in \Lambda\,, \\ 1\,, & \text{otherwise,} \end{cases} \tag{3.3.2}$$

for $j = 1, \ldots, d$, and for every $x \in \mathbb{T}^d_N$. In this case, the exchange rate of a bond crossing the boundary $\partial\Lambda$ is also of order N^{-1}, but it depends on the angle of incidence: the crossing of $\partial\Lambda$ by a particle gets harder to happen as the direction of entrance gets closer to the tangent plane to the surface $\partial\Lambda$.

From now on, the rates in the definition of L_N will always be given by (3.3.2). Denote by $\{\eta^N_t : t \geq 0\}$ a Markov process with state space Ω_N and generator L_N speeded up by N^2. Let $D(\mathbb{R}_+, \Omega_N)$ be the Skorohod space of càdlàg trajectories taking values in Ω_N. For a measure μ on Ω_N, denote by $\mathbb{P}^N_\mu$ the probability measure on $D(\mathbb{R}_+, \Omega_N)$ induced by the initial state μ and the Markov process $\{\eta^N_t : t \geq 0\}$. The expectation with respect to $\mathbb{P}^N_\mu$ is going to be denoted by $\mathbb{E}^N_\mu$.

A sequence of probability measures $\{\mu_N : N \geq 1\}$ is said to be associated to a profile

$\gamma : \mathbb{T}^d \to [0,1]$ if μ_N is a probability measure on Ω_N, for every N, and

$$\lim_{N\to\infty} \mu_N \left\{ \Big| \tfrac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} H(\tfrac{x}{N})\eta(x) - \int H(u)\gamma(u)du \Big| > \delta \right\} \;=\; 0 \tag{3.3.3}$$

for every $\delta > 0$, and every continuous function $H : \mathbb{T}^d \to \mathbb{R}$.

The exclusion process with slow bonds over $\partial\Lambda$ has a related random walk on $N^{-1}\mathbb{T}_N^d$ that describes the evolution of the system with a single particle. Thus particles in the exclusion process evolve independently as such random walk except for the hard core interaction. To simplify notation later, we introduce here the generator of this random walk, which is given by

$$(\mathbb{L}_N H)(\tfrac{x}{N}) = \sum_{j=1}^{d} \Big\{ \xi^N_{x,x+e_j} \Big[H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N}) \Big] + \xi^N_{x,x-e_j} \Big[H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N}) \Big] \Big\}\,, \tag{3.3.4}$$

for every $H : N^{-1}\mathbb{T}_N^d \to \mathbb{R}$ and every $x \in \mathbb{T}_N^d$. We will not differentiate the notation for functions H defined on $\mathbb{T}^d$ and on $N^{-1}\mathbb{T}_N^d$.

3.3.1 The Operator $\mathcal{L}_\Lambda$

Here we define the operator $\mathcal{L}_\Lambda$ and state its main properties. First, its domain is defined as a set of functions that are two times continuously differentiable inside and outside Λ and satisfy some additional conditions related to their behavior at $\partial\Lambda$. Such conditions are imposed in order to have good properties of $\mathcal{L}_\Lambda$ that allows us to conclude the uniqueness of solutions of the hydrodynamic equation, and obtain a strong convergence result for the empirical measures in the proof of the hydrodynamic limit. The necessity of these conditions are going to be made clear later in the text.

Definition 3.3.1. *Recall that $\vec{\zeta}$ denotes the normal exterior vector to the surface $\partial\Lambda$. The domain $\mathfrak{D}_\Lambda \subset L^2(\mathbb{T}^d)$ will be the set of functions $H \in L^2(\mathbb{T}^d)$, such that $H(u) = h(u) + \lambda \mathbf{1}_\Lambda(u)$, where:*

- ***(i)*** $\lambda \in \mathbb{R}$;
- ***(ii)*** $h \in C^2(\mathbb{T}^d)$;
- ***(iii)*** $\nabla h|_{\partial\Lambda}(u) = -\lambda\,\vec{\zeta}(u)$.

Now, we define the operator $\mathcal{L}_\Lambda : \mathfrak{D}_\Lambda \to L^2(\mathbb{T}^d)$ by

$$\mathcal{L}_\Lambda H \;=\; \Delta h\,.$$

Geometrically, the operator $\mathcal{L}_\Lambda$ removes the discontinuity around the surface $\partial\Lambda$ and then acts like the laplacian operator.

Remark 3.3.1. *It is not entirely obvious why there exist functions $h \in C^2(\mathbb{T}^d)$ such that $\nabla h|_{\partial\Lambda}(u) = -\lambda\,\vec{\zeta}(u)$, for $\lambda \neq 0$. For an example of such a function, consider firstly $g : \mathbb{T}^d \to \mathbb{R}$ defined by*

$$g(u) \;=\; \begin{cases} \lambda\, dist(u, \partial\Lambda)\,, & if\;\; u \in \Lambda^{\complement}\,, \\ -\lambda\, dist(u, \partial\Lambda)\,, & if\;\; u \in \Lambda\,. \end{cases}$$

Since $\partial\Lambda$ has no self intersection and is smooth, it is simple to check that there exists a sufficiently small $\varepsilon > 0$ such that

$$V = \{u \in \mathbb{T}^d :\; dist(u, \partial\Lambda) < \varepsilon\,\}$$

has smooth boundary and without self intersection. Thus, the function g is smooth in the open neighborhood V of $\partial\Lambda$, and satisfies the condition $\nabla g|_{\partial\Lambda}(u) = -\lambda\,\vec{\zeta}(u)$. However, g is not differentiable in the space $\mathbb{T}^d$. To solve this problem, it is enough to multiply g by $\sum_i \Phi_i$, where $\{\Phi_i\}$ is a partition of unity such that the support of any Φ_i is contained in V and $\sum_i \Phi_i(u) = 1$ for all $u \in U \subset V$, U an open set containing $\partial\Lambda$. Finally, the function

$$h(u) \;=\; g(u) \sum_i \Phi_i(u)$$

satisfies the required conditions.

For the next result we need to introduce some notation. We denote by $\mathbb{I}$ the identity operator in $L^2(\mathbb{T}^d)$ and by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and $\|\cdot\|$ its usual inner product and norm:

$$\langle\!\langle f, g \rangle\!\rangle \;=\; \int_{\mathbb{T}^d} f(u)\, g(u)\, du \quad \text{and} \quad \|f\| = \sqrt{\langle\!\langle f, f \rangle\!\rangle}\;, \quad f,\; g \in L^2(\mathbb{T}^d)\,.$$

Theorem 3.3.2. *There exists a Hilbert space $(\mathcal{H}^1_\Lambda, \langle\!\langle \cdot, \cdot \rangle\!\rangle_{1,\Lambda})$ which is compactly embedded in $L^2(\mathbb{T}^d)$ such that $\mathfrak{D}_\Lambda \subset \mathcal{H}^1_\Lambda$ and $\mathcal{L}_\Lambda$ can be extended to $\mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^d)$ in such a way that the extension enjoys the following properties:*

(a) The domain $\mathcal{H}^1_\Lambda$ is dense in $L^2(\mathbb{T}^d)$;

(b) The operator $\mathcal{L}_\Lambda$ is self-adjoint and non-positive: $\langle\!\langle H, -\mathcal{L}_\Lambda H \rangle\!\rangle \geq 0$, for all H in $\mathcal{H}^1_\Lambda$;

(c) The operator $\mathbb{I} - \mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^d)$ is bijective and $\mathfrak{D}_\Lambda$ is a core for it;

(d) The operator $\mathcal{L}_\Lambda$ is dissipative, i.e.,

$$\|\mu H - \mathcal{L}_\Lambda H\| \geq \mu \|H\|\,,$$

for all $H \in \mathcal{H}^1_\Lambda$ and $\mu > 0$;

(e) The eigenvalues of $-\mathcal{L}_\Lambda$ form a countable set $0 = \mu_0 \leq \mu_1 \leq \cdots$ with $\lim_{n\to\infty} \mu_n = \infty$, and all these eigenvalues have finite multiplicity;

(f) There exists a complete orthonormal basis of $L^2(\mathbb{T}^d)$ composed of eigenvectors of $-\mathcal{L}_\Lambda$.

In view of (a), (c) and (d), by the Hille-Yoshida Theorem, $\mathcal{L}_\Lambda$ is the generator of a strongly continuous contraction semigroup in $L^2(\mathbb{T}^d)$.

The space $\mathcal{H}^1_\Lambda$ will be defined in Section 3.4. The name has been chosen in analogy to the notation used for Sobolev spaces.

3.3.2 The hydrodynamic equation

Consider a bounded Borel measurable profile $\rho_0 : \mathbb{T}^d \to \mathbb{R}$. A bounded function $\rho : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ is said to be a weak solution of the parabolic differential equation

$$\begin{cases} \partial_t \rho \;=\; \mathcal{L}_\Lambda \rho \\ \rho(0,\cdot) \;=\; \rho_0(\cdot)\,, \end{cases} \tag{3.3.5}$$

if for all functions H in $\mathcal{H}^1_\Lambda$ and all $t > 0$, ρ satisfies the integral equation

$$\langle\langle \rho_t, H \rangle\rangle \;-\; \langle\langle \rho_0, H \rangle\rangle - \int_0^t \langle\langle \rho_s, \mathcal{L}_\Lambda H \rangle\rangle \, ds \;=\; 0, \tag{3.3.6}$$

where ρ_t is the notation for $\rho(t,\cdot)$. We prove in Subsection 3.5.3 the uniqueness of weak solutions of (3.3.5). Existence follows from the convergence result for the empirical measures associated to the diffusively rescaled exclusion processes with slow bonds over Λ, this is discussed in Section 3.5. Here we do not use time dependent test functions as usual in the definition of weak solution, but we have a well posed problem and we do not need a solution in a stronger sense to prove the hydrodynamic limit which is the next stated theorem.

Theorem 3.3.3. *Fix a Borel measurable initial profile $\gamma : \mathbb{T}^d \to [0,1]$ and consider a sequence of probability measures μ_N on Ω_N associated to γ. Then, for any $t \geq 0$,*

$$\lim_{N\to\infty} \mathbb{P}^N_{\mu_N} \Big\{ \Big| \tfrac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} H(x/N)\, \eta_t(x) - \int_{\mathbb{T}^d} H(u)\rho(t,u)du \Big| > \delta \Big\} \;=\; 0\,,$$

for every $\delta > 0$ and every function $H \in C(\mathbb{T}^d)$, where ρ is the unique weak solution of the differential equation (3.3.5) *with $\rho_0 = \gamma$.*

3.4 The operator $\mathcal{L}_\Lambda$

We begin by studying properties of $\mathcal{L}_\Lambda$ defined on the domain $\mathfrak{D}_\Lambda$ and we consider the extension afterwards.

Lemma 3.4.1. *The domain $\mathfrak{D}_\Lambda$ is dense in $L^2(\mathbb{T}^d)$.*

Proof. It is enough to prove that there exists a subset of $\mathfrak{D}_\Lambda$ which is dense in $L^2(\mathbb{T}^d)$. All smooth functions with support contained in $\mathbb{T}^d \backslash \partial\Lambda$ belong to $\mathfrak{D}_\Lambda$, which is clearly a dense subset of $L^2(\mathbb{T}^d)$, since $\partial\Lambda$ is a smooth zero Lebesgue measure surface that divides $\mathbb{T}^d \backslash \partial\Lambda$ in two disjoint open regions. □

From now on, we use ℓ_d to denote the d-dimensional Lebesgue measure on $\mathbb{T}^d$.

Lemma 3.4.2. *The operator $-\mathcal{L}_\Lambda : \mathfrak{D}_\Lambda \to L^2(\mathbb{T}^d)$ is symmetric and non-negative. Furthermore, it satisfies a Poincaré inequality, which means that there exists a finite constant $C > 0$ such that*

$$\|H\|^2 \leq C \, \langle\!\langle -\mathcal{L}_\Lambda H, H \rangle\!\rangle + \Big(\int_{\mathbb{T}^d} H(x)\, dx \Big)^2 \tag{3.4.1}$$

for all functions $H \in \mathfrak{D}_\Lambda$.

Proof. Let $H, G \in \mathfrak{D}_\Lambda$. Write $H = h + \lambda_h \mathbf{1}_\Lambda$ and $G = g + \lambda_g \mathbf{1}_\Lambda$, as in Definition 3.3.1. By the first Green identity and condition (iii) in Definition 3.3.1, we have that

$$\begin{aligned} \lambda_h \int_\Lambda \Delta g \, du &= \lambda_h \int_{\partial\Lambda} (\nabla g \cdot \vec{\zeta}) \, dS \;=\; -\lambda_h \, \lambda_g \, \mathrm{Vol}_{d-1}(\partial\Lambda) \\ &= \lambda_g \int_{\partial\Lambda} (\nabla h \cdot \vec{\zeta}) \, dS \;=\; \lambda_g \int_\Lambda \Delta h \, du \,, \end{aligned} \tag{3.4.2}$$

where dS is a infinitesimal element of volume of $\partial\Lambda$ and $\mathrm{Vol}_{d-1}(\partial\Lambda)$ is its $(d-1)$-dimensional volume. Thus,

$$\begin{aligned} \langle\!\langle H, -\mathcal{L}_\Lambda G \rangle\!\rangle &= \langle\!\langle h + \lambda_h \mathbf{1}_\Lambda, -\Delta g \rangle\!\rangle = -\int_{\mathbb{T}^d} h \, \Delta g \, du - \lambda_h \int_\Lambda \Delta g \, du \\ &= -\int_{\mathbb{T}^d} g \, \Delta h \, du - \lambda_g \int_\Lambda \Delta h \, du \;=\; \langle\!\langle -\mathcal{L}_\Lambda H, G \rangle\!\rangle \,. \end{aligned}$$

For the non-negativeness, using (3.4.2) above,

$$\begin{aligned} \langle\!\langle H, -\mathcal{L}_\Lambda H \rangle\!\rangle &= -\int_{\mathbb{T}^d} h \, \Delta h \, du - \lambda_h \int_\Lambda \Delta h \, du \\ &= \int_{\mathbb{T}^d} |\nabla h|^2 \, du + \lambda_h^2 \, \mathrm{Vol}_{d-1}(\partial\Lambda) \;\geq\; 0 \,. \end{aligned}$$

It remains to prove the Poincaré inequality. Write

$$\|H\|^2 - \Big(\int_{\mathbb{T}^d} H(x)\, dx \Big)^2 = \int_{\mathbb{T}^d} \Big[H(u) - \int_{\mathbb{T}^d} H(v)\, dv \Big]^2 du \,,$$

which can be rewritten as

$$\int_{\mathbb{T}^d} \Big[\Big(h(u) - \int_{\mathbb{T}^d} h(v)\, dv \Big) + \lambda_h \Big(\mathbf{1}_\Lambda(u) - \ell_d(\Lambda) \Big) \Big]^2 du \,.$$

Now apply the inequality $(a+b)^2 \leq 2\,(a^2 + b^2)$ to the previous expression to obtain that it is bounded by

$$2 \int_{\mathbb{T}^d} \Big(h(u) - \int_{\mathbb{T}^d} h(v)\, dv \Big)^2 du + 2\, \lambda_h^2 \Big(\ell_d(\Lambda) - (\ell_d(\Lambda))^2 \Big) \,.$$

By the usual Poincaré inequality, see [4], the last expression is less than or equal to

$$2\,C_1 \int_{\mathbb{T}^d} |\nabla h(u)|^2\,du + 2\,\lambda_h^2\Big(\ell_d(\Lambda) - (\ell_d(\Lambda))^2\Big)\,.$$

Choosing a constant $C_2 > 0$ such that $\ell_d(\Lambda) - (\ell_d(\Lambda))^2 \le C_2 \mathrm{Vol}_{d-1}(\partial\Lambda)$, the previous expression is bounded above by

$$2\,\max\{C_1, C_2\}\,\langle\!\langle -\mathcal{L}_\Lambda H, H\rangle\!\rangle\,,$$

which finishes the proof with $C = 2\,\max\{C_1, C_2\}$. □

Denote by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{1,\Lambda}$ the inner product on $\mathfrak{D}_\Lambda$ defined by

$$\langle\!\langle F, G\rangle\!\rangle_{1,\Lambda} \;=\; \langle\!\langle F, G\rangle\!\rangle \;+\; \langle\!\langle F, -\,\mathcal{L}_\Lambda G\rangle\!\rangle\,.$$

Let $\mathcal{H}^1_\Lambda$ be the set of all functions F in $L^2(\mathbb{T}^d)$ for which there exists a sequence $\{F_n : n \ge 1\}$ in $\mathfrak{D}_\Lambda$ such that F_n converges to F in $L^2(\mathbb{T}^d)$ and F_n is Cauchy for the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{1,\Lambda}$. Such sequence $\{F_n\}$ is called admissible for F. For F, G in $\mathcal{H}^1_\Lambda$, define

$$\langle\!\langle F, G\rangle\!\rangle_{1,\Lambda} \;=\; \lim_{n\to\infty} \langle\!\langle F_n, G_n\rangle\!\rangle_{1,\Lambda}\,, \tag{3.4.3}$$

where $\{F_n\}$, $\{G_n\}$ are admissible sequences for F, G, respectively. By [27, Proposition 5.3.3], the limit exists and does not depend on the admissible sequence chosen. Moreover, $\mathcal{H}^1_\Lambda$ endowed with the scalar product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{1,\Lambda}$ just defined is a real Hilbert space. From now on, we consider $\mathcal{H}^1_\Lambda$ with the norm induced by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{1,\Lambda}$, unless we mention that we are going to use the L^2-norm.

Lemma 3.4.3. *The embedding $\mathcal{H}^1_\Lambda \subset L^2(\mathbb{T}^d)$ is compact.*

Proof. Let $\{H_n\}$ a bounded sequence in $\mathcal{H}^1_\Lambda$. Fix $\{F_n\}$ as a sequence in $\mathfrak{D}_\Lambda$ such that $\|F_n - H_n\| \to 0$ and $\{F_n\}$ is also bounded in $\mathcal{H}^1_\Lambda$. Thus, to get a convergent subsequence of $\{H_n\}$, it is sufficient to find a convergent subsequence of $\{F_n\}$ in $L^2(\mathbb{T}^d)$. Write $F_n = f_n + \lambda_n \mathbf{1}_\Lambda$, with $f_n \in C^2(\mathbb{T}^d)$. Then,

$$\langle\!\langle F_n, F_n\rangle\!\rangle_{1,\Lambda} = \langle\!\langle f_n + \lambda_n\mathbf{1}_\Lambda, f_n + \lambda_n\mathbf{1}_\Lambda\rangle\!\rangle + \langle\!\langle f_n + \lambda_n\mathbf{1}_\Lambda, -\Delta f_n\rangle\!\rangle\,.$$

Expanding the right hand side and using (3.4.2), we get that

$$\langle\!\langle F_n, F_n\rangle\!\rangle_{1,\Lambda} = \|f_n\|^2 + \lambda_n^2\ell_d(\Lambda) + 2\lambda_n \int_\Lambda f_n(u)\,du + \|\nabla f_n\|^2 + \lambda_n^2\,\mathrm{Vol}_{d-1}(\partial\Lambda)\,,$$

which is greater or equal to

$$\|f_n\|^2 + \lambda_n^2\ell_d(\Lambda) - \lambda_n^2 - \ell_d(\Lambda)\int_\Lambda f_n^2(u)\,du + \|\nabla f_n\|^2 + \lambda_n^2\,\mathrm{Vol}_{d-1}(\partial\Lambda)$$

$$= \Big(\ell_d(\Lambda) - 1 + \mathrm{Vol}_{d-1}(\partial\Lambda)\Big)\,\lambda_n^2 + (1-\ell_d(\Lambda))\int_\Lambda f_n^2(u)\,du + \int_{\Lambda^\complement} f_n^2(u)\,du + \|\nabla f_n\|^2$$

$$\geq \Big(\mathrm{Vol}_{d-1}(\partial\Lambda) - \ell_d(\Lambda^\complement)\Big)\,\lambda_n^2 + (1-\ell_d(\Lambda))\,\|f_n\|^2 + \|\nabla f_n\|^2\,.$$

If we put $\tilde{f}_n = f_n + \lambda_n$, and write $F_n = \tilde{f}_n - \lambda_n \mathbf{1}_{\Lambda^\complement}$, an analogous computation shows that $\langle\!\langle F_n, F_n\rangle\!\rangle_{1,\Lambda}$ is greater or equal than

$$\Big(\mathrm{Vol}_{d-1}(\partial\Lambda) - \ell_d(\Lambda)\Big)\,\lambda_n^2 + (1-\ell_d(\Lambda^\complement))\,\|\tilde{f}_n\|^2 + \|\nabla\tilde{f}_n\|^2\,.$$

By the classical isoperimetric inequality on the torus (see [2, Lemma 4.6] for the statement and a direct proof), we have that

$$\max\{\,\mathrm{Vol}_{d-1}(\partial\Lambda) - \ell_d(\Lambda^\complement)\,,\,\mathrm{Vol}_{d-1}(\partial\Lambda) - \ell_d(\Lambda)\,\} > 0\,.$$

Since $\{\langle\!\langle F_n, F_n\rangle\!\rangle_{1,\Lambda}\}$ is a bounded sequence, we conclude that $\{\lambda_n\}$ is bounded, as well the sequence $\{\|f_n\|^2 + \|\nabla f_n\|^2\}$. By the Rellich-Kondrachov Compactness Theorem, see [4, Theorem 5.7.1], $\{f_n\}$ has a convergent subsequence in $L^2(\mathbb{T}^d)$. From this subsequence, choosing a convergent subsequence of $\{\lambda_n\}$ finishes the proof. □

Lemma 3.4.4. *The image of* $\mathbb{I} - \mathcal{L}_\Lambda : \mathfrak{D}_\Lambda \to L^2(\mathbb{T}^d)$ *is dense in* $L^2(\mathbb{T}^d)$.

Proof. By a similar argument to the one found in Lemma 3.4.1, it is enough to show that any smooth function f with support contained in $\mathbb{T}^d\backslash\partial\Lambda$ belongs to $(\mathbb{I} - \mathcal{L}_\Lambda)(\mathfrak{D}_\Lambda)$. Therefore, we need to find a function h in $C^2(\mathbb{T}^d)$ with support in $\mathbb{T}^d\backslash\partial\Lambda$ such that

$$h - \Delta h \,=\, f\,. \tag{3.4.4}$$

From the classical theory of second-order elliptic equations, e.g., see [4, Theorem 5.7.1], there exists $h \in C^2$ satisfying (3.4.4). □

Proof of Theorem 3.3.2. **(a)** Since $\mathfrak{D}_\Lambda \subset \mathcal{H}^1_\Lambda$, it follows from Lemma 3.4.1 that $\mathcal{H}^1_\Lambda$ is dense in $L^2(\mathbb{T}^d)$.

(b) Denote $\mathbb{I} - \mathcal{L}_\Lambda = \mathcal{A} : \mathfrak{D}_\Lambda \to \mathbb{L}^2(\mathbb{T}^d)$. From Lemma 3.4.2, $\mathcal{A}$ is linear, symmetric and strongly monotone on the Hilbert space $L^2(\mathbb{T}^d)$. By strongly monotone, we mean that there exists $c > 0$ such that

$$\langle\!\langle \mathcal{A}\,H, H\rangle\!\rangle \geq\, c\,\|H\|^2\,, \quad \forall H \in \mathfrak{D}_\Lambda\,.$$

In this case, $\mathcal{A}$ satisfies the inequality above with $c = 1$. By [27, Theorem 5.5.a], in the conditions above, the Friedrichs extension $\mathcal{A} : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$ is self-adjoint, bijective and strongly monotone. By an abuse of notation, define now the extension $\mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$ as $(\mathbb{I}-\mathcal{A})$. Since $\mathbb{I}$ and $\mathcal{A}$ are self-adjoint in $\mathcal{H}^1_\Lambda$, this property is inherited by $\mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$.

For non-positiveness, note that

$$\langle\!\langle -\,\mathcal{L}_\Lambda\,H, H\rangle\!\rangle = \langle\!\langle -(\mathbb{I} - \mathcal{A})H, H\rangle\!\rangle = -\langle\!\langle H, H\rangle\!\rangle + \langle\!\langle \mathcal{A}\,H, H\rangle\!\rangle \geq 0\,.$$

(c) As mentioned in the proof of (b) above, the Friedrichs extension $\mathcal{A} : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$ is bijective. So it remains to show that $\mathfrak{D}_\Lambda$ is a core of $\mathcal{A} : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$. For any operator B, denote by $\mathcal{G}(B)$ the graphic of B. Then $\mathfrak{D}_\Lambda$ is a core for $\mathcal{A}$, if the closure of $\mathcal{G}(\mathcal{A}|_{\mathfrak{D}_\Lambda})^{L^2\times L^2}$ in $L^2 \times L^2$ is equal to $\mathcal{G}(\mathcal{A})$. Since $\mathcal{A}$ is self-adjoint, $\mathcal{A}$ is a closed operator, or else, $\mathcal{G}(\mathcal{A})$ is a closed set. Thus the closure of $\mathcal{G}(\mathcal{A}|_{\mathfrak{D}_\Lambda})$ is a subset of $\mathcal{G}(\mathcal{A})$. Let $H \in \mathcal{H}^1_\Lambda$, from Lemma 3.4.4, there exists a sequence $\{H_n\}$ in $\mathfrak{D}_\Lambda$ such that $\mathcal{A}\, H_n$ converges to $\mathcal{A}\, H$ in L^2. Hence, as proved in [27, Theorem 5.5.a], $\mathcal{A}^{-1}$ is a bounded linear operator, and H_n converges to H in L^2, which yields that the closure of $\mathcal{G}(\mathcal{A}|_{\mathfrak{D}_\Lambda})$ contains $\mathcal{G}(\mathcal{A})$.

(d) Fix a function H in $\mathcal{H}^1_\Lambda$ and $\mu > 0$. Put $G = (\mu\mathbb{I} - \mathcal{L}_\Lambda)H$. Taking the inner product with respect to H on both sides of this equality, we obtain that

$$\mu \langle\!\langle H, H \rangle\!\rangle \;+\; \langle\!\langle -\mathcal{L}_\Lambda H, H \rangle\!\rangle \;=\; \langle\!\langle H, G \rangle\!\rangle \;\le\; \langle\!\langle H, H \rangle\!\rangle^{1/2} \langle\!\langle G, G \rangle\!\rangle^{1/2} \,.$$

Since H belongs to $\mathcal{H}^1_\Lambda$, by (b), the second term on the left hand side is positive. Therefore, $\mu\|H\| \le \|G\| = \|(\mu\mathbb{I} - \mathcal{L}_\Lambda)H\|$.

(e) and (f) We have seen that the operator $(\mathbb{I} - \mathcal{L}_\Lambda) : \mathfrak{D}_\Lambda \to L^2(\mathbb{T})$ is symmetric and strongly monotone. By Lemma 3.4.3 , the embedding $\mathcal{H}^1_\Lambda \subset L^2(\mathbb{T}^d)$ is compact. Therefore, by [27, Theorem 5.5.c], the Friedrichs extension $\mathcal{A} : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^d)$, satisfies claims (e) and (f) with $1 \le \lambda_1 \le \lambda_2 \le \cdots$, $\lambda_n \uparrow \infty$. In particular, the operator $-\mathcal{L}_\Lambda = (\mathcal{A} - \mathbb{I})$ has the same property with $0 \le \mu_1 \le \mu_2 \le \cdots$, $\mu_n \uparrow \infty$. Since 0 is an eigenvalue of $-\mathcal{L}_\Lambda$, a constant function is an eigenfunction with eigenvalue 0, then (e) and (f) also hold. □

3.5 Scaling Limit

Let $\mathcal{M}$ be the space of positive Radon measures on $\mathbb{T}^d$ with total mass bounded by one endowed with the weak topology. For a measure $\pi \in \mathcal{M}$ and a measurable π-integrable function $H : \mathbb{T}^d \to \mathbb{R}$, we denote by $\langle \pi, H \rangle$ the integral of H with respect to π.

Recall that $\{\eta^N_t : t \ge 0\}$ denote a Markov process with state space Ω_N and generator L_N speeded up by N^2. Let $\pi^N_t \in \mathcal{M}$ be the empirical measure at time t associated to $\{\eta^N_t : t \ge 0\}$, which is the random measure in $\mathcal{M}$ given by

$$\pi^N_t \;=\; \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \eta^N_t(x)\, \delta_{x/N} \,, \tag{3.5.1}$$

where δ_u is the Dirac measure concentrated on u.

Note that

$$\langle \pi^N_t, H \rangle \;=\; \tfrac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} H(\tfrac{x}{N}) \eta^N_t(x) \,,$$

for the empirical measures, and $\langle \pi, H \rangle = \langle\!\langle \rho, H \rangle\!\rangle$, for absolutely continuous measures π with L^2 bounded density ρ, and $H \in L^2(\mathbb{T}^d)$.

Fix $T > 0$. Let $D([0,T], \mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi : [0,T] \to \mathcal{M}$ endowed with the *Skorohod* topology. Then, the $\mathcal{M}$-valued process $\{\pi^N_t : t \ge 0\}$ is a random element of $D([0,T],\mathcal{M})$ whose distribution is determined by the initial distribution of $\{\eta_t^N : t \geq 0\}$. For each probability measure μ on Ω_N, denote by $\mathbb{Q}_\mu^{\Lambda,N}$ the distribution of $\{\pi_t^N : t \geq 0\}$ on the path space $D([0,T],\mathcal{M})$, when η_0^N has distribution μ.

Proposition 3.5.1. *Fix a Borel measurable profile $\gamma : \mathbb{T}^d \to [0,1]$ and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on Ω_N associated to γ in the sense of* (3.3.3). *Then there exists a unique weak solution ρ of* (3.3.5) *with initial condition γ and the sequence of probability measures $\mathbb{Q}_{\mu_N}^{\Lambda,N}$ converges weakly to $\mathbb{Q}_\Lambda^\gamma$ as $N \uparrow \infty$, where $\mathbb{Q}_\Lambda^\gamma$ is the probability measure on $D([0,T],\mathcal{M})$ concentrated on the deterministic path $\pi(t,du) = \rho(t,u)du$.*

It is straightforward to obtain Theorem 3.3.3 as a corollary of the previous proposition. The proof of Proposition 3.5.1 follows directly from the uniqueness of weak solutions of (3.3.5), proved in Subsection 3.5.3, and the next two results:

Proposition 3.5.2. *For any sequence $\{\mu_N : N \geq 1\}$ of probability measures with μ_N concentrated on Ω_N, the sequence of measures $\{\mathbb{Q}_{\mu_N}^{\Lambda,N} : N \geq 1\}$ is tight.*

Proposition 3.5.3. *Fix a Borel measurable profile $\gamma : \mathbb{T}^d \to [0,1]$ and consider a sequence $\{\mu_N : N \geq 1\}$ of probability measures on Ω_N associated to γ in the sense of* (3.3.3). *Then any limit point of $\mathbb{Q}_{\mu_N}^{\Lambda,N}$ is concentrated on absolutely continuous trajectories that are weak solutions of* (3.3.5) *with initial condition γ.*

Proof of Proposition 3.5.1. By Proposition 3.5.2, the set of measures $\{\mathbb{Q}_{\mu_N}^{\Lambda,N} : N \geq 1\}$ is tight. Since the Skorohod space $D([0,T],\mathcal{M})$ is Polish, by Prohorov's Theorem, tightness is equivalent to relative compactness (for the weak convergence). By the relative compactness, in order to prove the convergence of the sequence $(Q_{\mu_N}^{\Lambda,N})_{N\geq 1}$ to the probability measure $\mathbb{Q}_\Lambda^\gamma$, it is enough to show that any convergent subsequence of $(Q_{\mu_N}^{\Lambda,N})_{N\geq 1}$ has limit equal to $\mathbb{Q}_\Lambda^\gamma$. Let $\mathbb{Q}^*$ be a limit of a convergent subsequence. By Proposition 3.5.3, $\mathbb{Q}^*$ is concentrated on trajectories $\pi(t,du) = \rho(t,u)\,du$ such that $\rho(t,u)$ is a weak solution of (3.3.5) with initial condition γ. Uniqueness of weak solutions of (3.3.5) proved in Section 3.5.3 implies that $\mathbb{Q}^* = \mathbb{Q}_\Lambda^\gamma$. □

In Subsection 3.5.1, we prove Proposition 3.5.2 and in Subsection 3.5.2 we prove Proposition 3.5.3. As a consequence, we have the existence of solutions of (3.3.5) with initial condition γ. We complete the proof in Subsection (3.5.3) showing the uniqueness of weak solutions of (3.3.5).

3.5.1 Tightness

Here we prove Proposition 3.5.2. Let $D([0,T],\mathbb{R})$ be the space of $\mathbb{R}$-valued càdlàg trajectories with domain $[0,T]$ endowed with the *Skorohod* topology. To prove tightness of $\{\pi_t^N : 0 \leq t \leq T\}$ in $D([0,T],\mathcal{M})$, it is enough to show tightness in $D([0,T],\mathbb{R})$ of the real-valued processes $\{\langle \pi_t^N, H\rangle : 0 \leq t \leq T\}$ for a set of functions $H : \mathbb{T}^d \to \mathbb{R}$ which is dense in the space of continuous real functions on $\mathbb{T}^d$ endowed with the uniform topology, see [17]. Furthermore, if a sequence of distributions in $D([0,T],\mathbb{R})$ endowed with the uniform topology is tight, then it is also tight in $D([0,T],\mathbb{R})$ endowed with the Skorohod topology. Here we prove tightness of $\{\langle \pi_t^N, H\rangle : 0 \le t \le T\}$ in $D([0,T],\mathbb{R})$, endowed with the uniform topology, for $H \in C^2(\mathbb{T}^d)$.

Fix $H \in C^2(\mathbb{T}^d)$. By definition $\{\langle \pi_t^N, H\rangle : 0 \le t \le T\}$ is tight in $D([0,T],\mathbb{R})$ endowed with the uniform topology if, for the boundedness,

$$\lim_{m\to\infty} \sup_N \mathbb{P}^N_{\mu_N}\left[\sup_{0\le t\le T} |\langle \pi_t^N, H\rangle| > m\right] = 0\,, \tag{3.5.2}$$

and, for the equicontinuity,

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}^N_{\mu_N}\left[\sup_{|t-s|\le\delta} |\langle \pi_t^N, H\rangle - \langle \pi_s^N, H\rangle| > \varepsilon\right] = 0\,, \text{ for all } \varepsilon > 0\,. \tag{3.5.3}$$

The limit in (3.5.2) is trivial since

$$|\langle \pi_t^N, H\rangle| \le \sup_{u\in\mathbb{T}^d} |H(u)|\,.$$

So we only need to prove (3.5.3). By Dynkyn's formula (see appendix in [17]),

$$M_t^N = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t N^2 L_N \langle \pi_s^N, H\rangle ds \tag{3.5.4}$$

is a martingale. By the previous expression, (3.5.3) follows from

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}^N_{\mu_N}\left[\sup_{|t-s|\le\delta} |M_t^N - M_s^N| > \varepsilon\right] = 0\,, \text{ for all } \varepsilon > 0\,, \tag{3.5.5}$$

and

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}^N_{\mu_N}\left[\sup_{0\le t-s\le\delta} \left|\int_s^t N^2 L_N \langle \pi_r^N, H\rangle dr\right| > \varepsilon\right] = 0\,, \text{ for all } \varepsilon > 0\,. \tag{3.5.6}$$

Indeed, we show the stronger results below:

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{E}^N_{\mu^N}\left[\sup_{|t-s|\le\delta} |M_t^N - M_s^N|\right] = 0\,, \tag{3.5.7}$$

and

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{E}^N_{\mu^N}\left[\sup_{0\le t-s\le\delta} \left|\int_s^t N^2 L_N \langle \pi_r^N, H\rangle dr\right|\right] = 0\,. \tag{3.5.8}$$

To verify (3.5.7), we use the quadratic variation of M_t^N that we denote by $\langle M_t^N\rangle$. By Doob's inequality, we have that

$$\begin{aligned}\mathbb{E}_{\mu^N}^N\Big[\sup_{|t-s|\le\delta}|M_t^N-M_s^N|\Big] &\le 2\,\mathbb{E}_{\mu^N}^N\Big[\sup_{0\le t\le T}|M_t^N|\Big]\\ &\le 2\,\mathbb{E}_{\mu^N}^N\Big[\sup_{0\le t\le T}|M_t^N|^2\Big]^{\frac{1}{2}} \le 4\,\mathbb{E}_{\mu^N}^N\big[\langle M_T^N\rangle\big]^{\frac{1}{2}}.\end{aligned}$$

Since

$$\langle M_t^N\rangle=\int_0^t N^2[L_N\langle\pi_s^N,H\rangle^2-2\langle\pi_s^N,H\rangle L_N\langle\pi_s^N,H\rangle]ds\,,$$

we obtain by a straightforward computation that

$$\langle M_t^N\rangle=\int_0^t N^2\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\xi_{x,x+e_j}^N\tfrac{1}{N^{2d}}\Big[(\eta_s(x)-\eta_s(x+e_j))(H(\tfrac{x+e_j}{N})-H(\tfrac{x}{N}))\Big]^2ds\,.$$

Therefore, since $\xi_{x,x+e_j}^N\le 1$,

$$\begin{aligned}\langle M_t^N\rangle &\le \frac{T}{N^{2d-2}}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\xi_{x,x+e_j}^N\Big[H(\tfrac{x+e_j}{N})-H(\tfrac{x}{N})\Big]^2\\ &\le \frac{Td}{N^d}\Big(\sup_{u\in\mathbb{T}^d}|\nabla H(u)\cdot e_j|\Big)^2. \qquad (3.5.9)\end{aligned}$$

Thus, M_t^N converges to zero in L^2 and (3.5.7) holds.

We finish the proof by verifying (3.5.8). Write

$$N^2L_N\langle\pi_s^N,H\rangle=\tfrac{1}{N^{d-2}}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\xi_{x,x+e_j}^N((\eta_s(x)-\eta_s(x+e_j))\big(H(\tfrac{x+e_j}{N})-H(\tfrac{x}{N})\big)$$

$$=\tfrac{1}{N^{d-2}}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\eta_s(x)\Big[\xi_{x,x+e_j}^N\Big(H(\tfrac{x+e_j}{N})-H(\tfrac{x}{N})\Big)+\xi_{x,x-e_j}^N\Big(H(\tfrac{x-e_j}{N})-H(\tfrac{x}{N})\Big)\Big].$$

Define $\Gamma_N\subset\mathbb{T}_N^d$ as the set of vertices whose have some adjacent edge with exchange rate not equal to one. Then $N^2L_N\langle\pi_s^N,H\rangle$ is equal to the sum of

$$\tfrac{1}{N^{d-2}}\sum_{j=1}^d\sum_{x\notin\Gamma_N}\eta_s(x)\Big[H(\tfrac{x+e_j}{N})+H(\tfrac{x-e_j}{N})-2H(\tfrac{x}{N})\Big] \qquad (3.5.10)$$

and

$$+\tfrac{1}{N^{d-2}}\sum_{j=1}^d\sum_{x\in\Gamma_N}\eta_s(x)\Big[\xi_{x,x+e_j}^N\Big(H(\tfrac{x+e_j}{N})-H(\tfrac{x}{N})\Big)+\xi_{x,x-e_j}^N\Big(H(\tfrac{x-e_j}{N})-H(\tfrac{x}{N})\Big)\Big]. \qquad (3.5.11)$$

By the Taylor expansion (remember $H \in C^2$), the absolute value of the summand in (3.5.10) is bounded by $N^{-2} \sup_{u\in\mathbb{T}^d} |\Delta H(u)|$. Considering the factor N^{-d+2} in front of the sum, we conclude that the expression (3.5.10) is bounded in absolute value by $d \sup_{u\in\mathbb{T}^d} |\Delta H(u)|$.

Since there are in order of N^{d-1} vertices in Γ_N, and $\xi_{x,x+e_j} \leq 1$, the absolute value of the expression (3.5.11) is bounded by

$$\tfrac{1}{N^{d-2}} \sum_{j=1}^{d} \sum_{x\in\Gamma_N} \Big[|H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})| + |H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N})| \Big] \leq 2d \sup_{u\in\mathbb{T}^d} |\nabla H(u)\cdot e_j| \,.$$

By the boundedness of expressions (3.5.10) and (3.5.11), there exists $C > 0$, depending only on H, such that $|N^2 L_N \langle \pi_s^N, H\rangle| \leq C$, which yields

$$\Big| \int_r^t N^2 L_N \langle \pi_s^N, H\rangle ds \Big| \leq C(t-r)\,,$$

and (3.5.8) holds.

3.5.2 Characterization of limit points

Let $\gamma : \mathbb{T}^d \to [0,1]$ be a Borel measurable profile and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on Ω_N associated to γ in the sense of (3.3.3). We prove Proposition 3.5.3 in this subsection, i.e., that all limit points $\mathbb{Q}^*$ of the sequence $\mathbb{Q}^{\Lambda,N}_{\mu_N}$ are concentrated on absolutely continuous trajectories $\pi(t,du) = \rho(t,u)du$, whose density $\rho(t,u)$ is a weak solution of the hydrodynamic equation (3.3.5) with γ as initial condition.

Let $\mathbb{Q}^*$ be a limit point of the sequence $\mathbb{Q}^{\Lambda,N}_{\mu_N}$ and assume, without loss of generality, that $\mathbb{Q}^{\Lambda,N}_{\mu_N}$ converges to $\mathbb{Q}^*$.

Since there is at most one particle per site, $\mathbb{Q}^*$ is concentrated on trajectories $\pi_t(du)$ which are absolutely continuous with respect to the Lebesgue measure, $\pi_t(du) = \rho(t,u)du$, and whose density ρ is non-negative and bounded by 1, see [17, Chapter 4].

We shall prove the following result:

Lemma 3.5.4. *Any limit point $\mathbb{Q}^*$ of $\mathbb{Q}^{\Lambda,N}_{\mu_N}$ is concentrated on absolutely continuous trajectories $\pi_t(du) = \rho(t,u)du$ such that, for any $H \in \mathfrak{D}_\Lambda$,*

$$\langle\!\langle \rho_t, H\rangle\!\rangle - \langle\!\langle \gamma, H\rangle\!\rangle \;=\; \int_0^t \langle\!\langle \rho_s\,,\, \mathcal{L}_\Lambda H\rangle\!\rangle\, ds\,. \tag{3.5.12}$$

By the previous lemma we can show Proposition 3.5.3.

Proof of Proposition 3.5.3. It just remains to extend the equality (3.5.12) to functions $H \in \mathcal{H}^1_\Lambda$. By Theorem 3.3.2, the set $\mathfrak{D}_\Lambda$ is a core for the Friedrichs extension $\mathbb{I} - \mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^d)$. Thus, for any $H \in \mathcal{H}^1_\Lambda$, there exists a sequence $H_n \in \mathfrak{D}_\Lambda$ such that $H_n \to H$ and $(\mathbb{I} - \mathcal{L}_\Lambda)\, H_n \to (\mathbb{I} - \mathcal{L}_\Lambda)\, H$, both in $L^2(\mathbb{T}^d)$. This implies that $\mathcal{L}_\Lambda\, H_n \to \mathcal{L}_\Lambda\, H$ in $L^2(\mathbb{T}^d)$. Replacing H_n in equality (3.5.12), and taking the limit as $n \to \infty$ finishes the proof. □

The remain of this section is devoted to the proof of Lemma 3.5.4. Fix a function $H \in \mathfrak{D}_\Lambda$ and define the martingale M_t^N by

$$\langle \pi_t^N, H\rangle \;-\; \langle \pi_0^N, H\rangle \;-\; \int_0^t N^2 L_N \langle \pi_s^N, H\rangle \, ds\,. \tag{3.5.13}$$

We claim that, for every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}^N_{\mu_N}\Big[\sup_{0\le t\le T}\Big|M_t^N\Big| \;>\; \delta\Big] \;=\; 0\,. \tag{3.5.14}$$

For $H \in C^2$, this follows from Chebyshev inequality and the estimates done in the proof of tightness, where we have shown that

$$\lim_{N\to\infty} \mathbb{E}^N_\mu\Big[\sup_{0\le t\le T}|M_t^N|\Big] \;\le\; \lim_{N\to\infty} \mathbb{E}^N_\mu\Big[\sup_{0\le t\le T}\langle M_t^N\rangle\Big]^{\frac{1}{2}} = 0\,. \tag{3.5.15}$$

For $H = h + \lambda\, \mathbf{1}_\Lambda$ in $\mathfrak{D}_\Lambda$, the first inequality in (3.5.9) is still valid and

$$\begin{aligned}
\langle M_t^N\rangle \;&\le\; \tfrac{T}{N^{2d-2}}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}_N^d}\xi^N_{x,x+e_j}\Big[H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})\Big]^2\\
&=\; \tfrac{T}{N^{2d-2}}\sum_{j=1}^{d}\sum_{x\notin\Gamma_N}\Big[h(\tfrac{x+e_j}{N}) - h(\tfrac{x}{N})\Big]^2 \qquad (3.5.16)\\
&\quad + \tfrac{T}{N^{2d-2}}\sum_{j=1}^{d}\sum_{x\in\Gamma_N}\xi^N_{x,x+e_j}\Big[H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})\Big]^2, \qquad (3.5.17)
\end{aligned}$$

where Γ_N is also defined in the proof of tightness. The expression (3.5.16) goes to zero as N increases, since the function h is Lipschitz. For the expression in (3.5.17), let $x \in \Gamma_N$. If $\frac{x}{N} \in \Lambda$ and $\frac{x+e_j}{N} \in \Lambda^{\complement}$, then $\xi^N_{x,x+e_j} \le \frac{1}{N}$. The same occurs if $\frac{x}{N} \in \Lambda^{\complement}$ and $\frac{x+e_j}{N} \in \Lambda$. If $\frac{x}{N}, \frac{x+e_j}{N}$ both belong to Λ or $\Lambda^{\complement}$, the exchange rate $\xi^N_{x,x+e_j}$ is one, but $|H(\frac{x+e_j}{N}) - H(\frac{x}{N})| = |h(\frac{x+e_j}{N}) - h(\frac{x}{N})| \le \frac{1}{N}\sup_{u\in\mathbb{T}^d}|\nabla H(u)\cdot e_j|$. In both cases, the expression (3.5.17) is of order $O(N^{-d})$. Therefore, from (3.5.15), we obtain (3.5.14). □

The next step is to show that we can replace $N^2\mathbb{L}_N$ by the continuous operator $\mathcal{L}_\Lambda$ in the martingale formula (3.5.13) and that the resulting expression still converges to zero in probability. This will follow from the ensuing proposition. Recall the definition of $\mathbb{L}_N$ given in (3.3.4).

Proposition 3.5.5. *For any $H \in \mathfrak{D}_\Lambda$,*

$$\lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\Big|N^2\mathbb{L}_N H(\tfrac{x}{N}) - \mathcal{L}_\Lambda H(\tfrac{x}{N})\Big| = 0\,. \tag{3.5.18}$$

Proof. As usual, put $H = h + \lambda\,\mathbf{1}_\Lambda$, where $h \in C^2(\mathbb{T}^d)$. Rewrite the sum in (3.5.18) as

$$\frac{1}{N^d}\sum_{x\notin\Gamma_N}\Big|N^2\mathbb{L}_N H(\tfrac{x}{N}) - \mathcal{L}_\Lambda H(\tfrac{x}{N})\Big| + \frac{1}{N^d}\sum_{x\in\Gamma_N}\Big|N^2\mathbb{L}_N H(\tfrac{x}{N}) - \mathcal{L}_\Lambda H(\tfrac{x}{N})\Big|\,.$$

The first term above is equal to

$$\frac{1}{N^d}\sum_{x\notin\Gamma_N}\Big|N^2\Big(h(\tfrac{x+e_j}{N}) + h(\tfrac{x-e_j}{N}) - 2h(\tfrac{x}{N})\Big) - \Delta h(\tfrac{x}{N})\Big|\,,$$

which converges to zero because $h \in C^2$. The second one is less than or equal to the sum of

$$\frac{1}{N^d}\sum_{x\in\Gamma_N}|\Delta h(\tfrac{x}{N})| \tag{3.5.19}$$

and

$$\frac{1}{N^{d-1}}\sum_{x\in\Gamma_N}\sum_{j=1}^{d}\Big|N\xi^N_{x,x+e_j}(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})) + N\xi^N_{x,x-e_j}(H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N}))\Big|\,. \tag{3.5.20}$$

Since there are $O(N^{d-1})$ terms in Γ_N, the expression in (3.5.19) converges to zero as $N\to\infty$. Since $\partial\Lambda$ is smooth, the quantity of points $x\in\Gamma_N$ for which both $\xi^N_{x,x+e_j}$ and $\xi^N_{x,x-e_j}$ are different of one is negligible. Therefore, we must only worry about points $x\in\Gamma_N$ such that, for some j, only one of $\xi^N_{x,x+e_j}$ and $\xi^N_{x,x-e_j}$ is of order N^{-1}. This occurs in one of the following four cases: $\frac{x}{N}\in\Lambda$, $\frac{x-e_j}{N}\in\Lambda$ and $\frac{x+e_j}{N}\in\Lambda^\complement$; $\frac{x}{N}\in\Lambda$, $\frac{x-e_j}{N}\in\Lambda^\complement$ and $\frac{x+e_j}{N}\in\Lambda$; $\frac{x}{N}\in\Lambda^\complement$, $\frac{x-e_j}{N}\in\Lambda$ and $\frac{x+e_j}{N}\in\Lambda^\complement$; $\frac{x}{N}\in\Lambda^\complement$, $\frac{x-e_j}{N}\in\Lambda^\complement$ and $\frac{x+e_j}{N}\in\Lambda$. The analysis of these cases are analogous, thus we only consider the first one. Suppose $\frac{x}{N}\in\Lambda$, $\frac{x-e_j}{N}\in\Lambda$ and $\frac{x+e_j}{N}\in\Lambda^\complement$. In this case, the summand in (3.5.20) can be rewritten as

$$\begin{aligned}&N\xi^N_{x,x+e_j}(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})) + N\xi^N_{x,x-e_j}(H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N}))\\ &= |\vec{\zeta}_{x,j}\cdot e_j|\,[H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})]\; + N\,[H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N})]\,,\end{aligned}$$

which becomes uniformly (in $x\in\Gamma_N$) close to

$$-\lambda|\vec{\zeta}_{x,j}\cdot e_j|\;\mathrm{sgn}\Big(\vec{\zeta}_{x,j}\cdot e_j\Big) - \tfrac{\partial h}{\partial u_j}(\tfrac{x}{N}) = -\lambda\,\vec{\zeta}_{x,j}\cdot e_j - \tfrac{\partial h}{\partial u_j}(\tfrac{x}{N})\,.$$

The condition $\nabla h|_{\partial\Lambda}(u) = -\lambda\,\vec{\zeta}(u)$, which was imposed in the definition of $\mathfrak{D}_\Lambda$, implies that

$$\lim_{N\to\infty} N\xi^N_{x,x+e_j}(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})) + N\xi^N_{x,x-e_j}(H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N})) = 0\,.$$

Therefore, the terms in (3.5.20) converge uniformly to zero, and the same holds for the whole sum. □

Corollary 3.5.6. *For $H \in \mathfrak{D}_\Lambda$ and for every $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{Q}^{\Lambda,N}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi^N_t, H\rangle \;-\; \langle \pi^N_0, H\rangle \;-\; \int_0^t \langle \pi^N_s, \mathcal{L}_\Lambda H\rangle\, ds\,\Big| \;>\; \delta\Big] \;=\; 0\,.$$

Proof. By a simple calculation, the martingale defined in (3.5.13) can be rewritten as

$$M^N_t = \langle \pi^N_t, H\rangle \;-\; \langle \pi^N_0, H\rangle \;-\; \int_0^t \langle \pi^N_s, N^2\mathbb{L}_N\, H\rangle\, ds\,.$$

The result follows from Proposition 3.5.5 and expression (3.5.14). □

At this point we have all the ingredients needed to prove Lemma 3.5.4, which says that, under $\mathbb{Q}^*$, with probability one, (3.5.12) holds for any $H \in \mathfrak{D}_\Lambda$. In order to prove this, it is enough to show that, for any $\delta > 0$, and any $H \in \mathfrak{D}_\Lambda$,

$$\mathbb{Q}^*\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\; ds\Big| \;>\; \delta\Big] \;=\; 0\,. \tag{3.5.21}$$

So let H be a fixed function in $\mathfrak{D}_\Lambda$. The idea to estimate the probability in (3.5.21) is to apply Portmanteau's Theorem to replace Q^* by $\mathbb{Q}^{\Lambda,N}_{\mu_N}$ and then use Corollary 3.5.6. But to obtain an appropriate inequality we need the set

$$\Big\{\sup_{0\le t\le T}\Big|\langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\; ds\Big| \;>\; \delta\Big\}$$

to be open in $D([0,T],\mathcal{M})$. In order to guarantee this, we need H to be continuous which is not the case. To solve this problem, we use approximations of H by smooth functions.

For $\varepsilon > 0$, define

$$(\partial\Lambda)^\varepsilon = \{u \in \mathbb{T}^d;\, \operatorname{dist}(u,\partial\Lambda) \le \varepsilon\}\,.$$

Let H^ε be a smooth function which coincides with H on $\mathbb{T}^d\backslash(\partial\Lambda)^\varepsilon$ and $\sup_{\mathbb{T}^d}|H^\varepsilon| \le \sup_{\mathbb{T}^d}|H|$. Fix $\delta > 0$. By the triangular inequality,

$$\begin{aligned}
&\mathbb{Q}^*\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\; ds\Big| \;>\; \delta\Big]\\
&\le \mathbb{Q}^*\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H^\varepsilon\rangle \;-\; \langle \pi_0, H^\varepsilon\rangle \;-\; \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\; ds\Big| \;>\; \delta/3\Big]\\
&+ 2\,\mathbb{Q}^*\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H^\varepsilon - H\rangle\Big| \;>\; \delta/3\Big]\,.
\end{aligned} \tag{3.5.22}$$

Recall that $\mathbb{Q}^*$ is concentrated on trajectories $\pi_t(du) = \rho(t,u)du$ whose density ρ is non-negative and bounded above by 1. Then, under $\mathbb{Q}^*$,

$$\begin{aligned}
\sup_{0\le t\le T}|\langle \pi_t, H^\varepsilon - H\rangle| \;&\le\; \sup_{0\le t\le T}\int_{(\partial\Lambda)^\varepsilon} \rho(t,u)\,|H^\varepsilon(u) - H(u)|\,du\\
&\le\; 2\,\ell_d((\partial\Lambda)^\varepsilon)\,\sup_{u\in\mathbb{T}^d}|H(u)|\,.
\end{aligned}$$

Therefore, for small enough ε, the second probability in the right hand side of inequality (3.5.22) is null. So it remains to show that

$$\mathbb{Q}^*\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H^\varepsilon\rangle \;-\; \langle \pi_0, H^\varepsilon\rangle \;-\; \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\, ds\Big| \;>\; \delta/3\Big] = 0\,.$$

If G_1, G_2, G_3 are continuous functions, the application from $D([0,T],\mathcal{M})$ to $\mathbb{R}$ that associates to a trajectory $\{\pi_t, 0\le t\le T\}$ the number

$$\sup_{0\le t\le T}\Big|\langle \pi_t, G_1\rangle \;-\; \langle \pi_0, G_2\rangle \;-\; \int_0^t \langle \pi_s, G_3\rangle\, ds\Big|$$

is continuous in the Skorohod metric. Notice that H^ε and $\mathcal{L}_\Lambda H$ are continuous functions. By Portmanteau's Theorem,

$$\mathbb{Q}^*\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H^\varepsilon\rangle \;-\; \langle \pi_0, H^\varepsilon\rangle \;-\; \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\, ds\Big| \;>\; \delta/3\Big]$$

$$\le \varliminf_{N\to\infty} \mathbb{Q}^{\Lambda,N}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi^N_t, H^\varepsilon\rangle \;-\; \langle \pi^N_0, H^\varepsilon\rangle \;-\; \int_0^t \langle \pi^N_s, \mathcal{L}_\Lambda H\rangle\, ds\Big| \;>\; \delta/3\Big]\,, \qquad (3.5.23)$$

since $\mathbb{Q}^{\Lambda,N}_{\mu_N}$ converges weakly to $\mathbb{Q}^*$ and the above set is open.

Now we replace H^ε by H. This may be confusing to the reader, however the previous introduction of H^ε was a necessary step in the proof. From this point, to deal with the right hand side in (3.5.23), we need Corollary 3.5.6. Hence H^ε should be replaced by H.

By definition,

$$\sup_{0\le t\le T}\Big|\langle \pi^N_t, H^\varepsilon - H\rangle\Big| \le \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}\Big|H^\varepsilon(x/N) - H(x/N)\Big|$$

$$\le \Big(\ell_d((\partial\Lambda)^\varepsilon) + O(\tfrac{1}{N})\Big)\, 2\sup_{u\in\mathbb{T}}|H(u)|\,,$$

because H^ε coincides with H in $\mathbb{T}\backslash(\partial\Lambda)^\varepsilon$. Using the same argument as before, we obtain

$$\varliminf_{N\to\infty} \mathbb{Q}^{\Lambda,N}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H^\varepsilon\rangle \;-\; \langle \pi_0, H^\varepsilon\rangle \;-\; \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\, ds\Big| \;>\; \delta/3\Big]$$

$$\le \varlimsup_{N\to\infty} \mathbb{Q}^{\Lambda,N}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\, ds\Big| \;>\; \delta/9\Big]$$

$$+2\varlimsup_{N\to\infty} \mathbb{Q}^{\Lambda,N}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H^\varepsilon - H\rangle\Big| \;>\; \delta/9\Big]\,.$$

Again, for small enough ε, the second probability in the sum above is null. From Corollary 3.5.6, we finally conclude that (3.5.21) holds. Therefore $\mathbb{Q}^*$ is concentrated on absolutely continuous paths $\pi_t(du) = \rho(t,u)du$ with positive density bounded by 1, and $\mathbb{Q}^*$ a.s.

$$\langle\!\langle \rho_t, H\rangle\!\rangle - \langle\!\langle \rho_0, H\rangle\!\rangle \;=\; \int_0^t \langle\!\langle \rho_s\,,\, \mathcal{L}_\Lambda H\rangle\!\rangle\, ds\,,$$

for any $H\in\mathfrak{D}_\Lambda$. Hence we have proved Lemma 3.5.4.

3.5.3 Uniqueness of weak solutions

Now, we prove that the solution of (3.3.5) is unique. It suffices to check that the only solution of (3.3.5) with $\rho_0 \equiv 0$ is $\rho \equiv 0$, because of the linearity of $\mathcal{L}_\Lambda$. Let $\rho : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ be a weak solution of the parabolic differential equation

$$\begin{cases} \partial_t \rho &= \mathcal{L}_\Lambda \rho \\ \rho(0,\cdot) &= 0\,. \end{cases}$$

By definition,

$$\langle\!\langle \rho_t, H \rangle\!\rangle = \int_0^t \langle\!\langle \rho_s, \mathcal{L}_\Lambda H \rangle\!\rangle \, ds \,, \tag{3.5.24}$$

for all functions H in $\mathcal{H}^1_\Lambda$ and all $t > 0$. From the Theorem 3.3.2, the operator $-\mathcal{L}_\Lambda$ has countable eigenvalues $\{\mu_n : n \geq 0\}$ and eigenvectors $\{F_n\}$. All eigenvalues have finite multiplicity, $0 = \mu_0 \leq \mu_1 \leq \cdots$, and $\lim_{n\to\infty} \mu_n = \infty$. Besides, the eigenvectors $\{F_n\}$ form a complete orthonormal system in the $L^2(\mathbb{T}^d)$. Define

$$R(t) = \sum_{n\in\mathbb{N}} \frac{1}{n^2(1+\mu_n)} \langle\!\langle \rho_t, F_n \rangle\!\rangle^2,$$

for all $t > 0$. Notice that $R(0) = 0$ and $R(t)$ is well defined because ρ_t belongs to $L^2(\mathbb{T}^d)$. Since ρ satisfy (3.5.24), we have that $\frac{d}{dt}\langle\!\langle \rho_t, F_n \rangle\!\rangle^2 = -2\mu_n \langle\!\langle \rho_t, F_n \rangle\!\rangle^2$. Then

$$(\tfrac{d}{dt} R)(t) = -\sum_{n\in\mathbb{N}} \frac{2\mu_n}{n^2(1+\mu_n)} \langle\!\langle \rho_t, F_n \rangle\!\rangle^2 \,,$$

because $\sum_{n\leq N} \frac{-2\mu_n}{n^2(1+\mu_n)} \langle\!\langle \rho_t, F_n \rangle\!\rangle^2$ converges uniformly to $\sum_{n\in\mathbb{N}} \frac{-2\mu_n}{n^2(1+\mu_n)} \langle\!\langle \rho_t, F_n \rangle\!\rangle^2$, as N increases to infinity. Thus $R(t) \geq 0$ and $(\frac{d}{dt} R)(t) \leq 0$, for all $t > 0$ and $R(0) = 0$. From this, we obtain $R(t) = 0$ for all $t > 0$. Since $\{F_n\}$ is a complete orthonormal system, $\langle\!\langle \rho_t, \rho_t \rangle\!\rangle = 0$, for all $t > 0$, which implies $\rho \equiv 0$.

Chapter 4

Hydrodynamical behavior of symmetric exclusion with slow bonds of parameter $N^{-\beta}$

Joint work with Adriana Neumann (IMPA) and Ana Patrícia Gonçalves (Universidade do Minho).

4.1 Abstract

We consider the exclusion process in the one-dimensional discrete torus with N points, where all the bonds have conductance one, except a finite number of slow bonds, with conductance $N^{-\beta}$, with $\beta \in [0, \infty)$. We prove that the time evolution of the empirical density of particles, in the diffusive scaling, has a distinct behavior according to the range of the parameter β. If $\beta \in [0, 1)$, the hydrodynamic limit is given by the usual heat equation. If $\beta = 1$, it is given by a parabolic equation involving an operator $\frac{d}{dx}\frac{d}{dW}$, where W is the Lebesgue measure on the torus plus the sum of the Dirac measure supported on each macroscopic point related to the slow bond. If $\beta \in (1, \infty)$, it is given by the heat equation with Neumann's boundary conditions, meaning no passage through the slow bonds in the continuum.

4.2 Introduction

An important subject in Statistical Physics is the characterization of the hydrodynamical behavior of interacting particle systems in random or inhomogeneous media. One relevant and puzzling problem is to consider particle systems with slow bonds and to analyze the macroscopic effect on the hydrodynamic profiles, depending on the *strength* at these bonds. The problem we address in this paper is the complete characterization of the hydrodynamic limit scenario for the exclusion process with a finite number of slow bonds. Depending on the strength at the slow bonds, one observes a phase transition that goes from smooth profiles to the development of singularities in the continuum.

We begin by giving a brief and far from complete review about some results on the subject, all of them related to the exclusion process. In [5], by taking suitable random conductances $\{c_k : k \geq 1\}$, such that $\{c_k^{-1} : k \geq 1\}$ satisfy a Law of Large Numbers, it was proved that the randomness of the media does not survive in the macroscopic time evolution of the density of particles. In [7], the authors consider conductances driven by an α-stable subordinator W, and in this case, the randomness survives in the continuum, by replacing in the hydrodynamical equation the usual Laplacian by a generalized operator $\frac{d}{dx}\frac{d}{dW}$, which results in the weak heat equation. In the same line of such quenched result, [9] shows the analogous behavior, but for a general strictly increasing function W. All the previous works are restricted to the one-dimensional setting, and strongly based on convergence results for diffusions or random walks in one-dimensional inhomogeneous media, see [25]. In [26], there is a generalization of [9] for a suitable d-dimensional setting, in some sense decomposable into d one-dimensional cases. General sufficient conditions for the hydrodynamical limit of exclusion process in inhomogeneous media were established in [15]. All the above works have in common the association of the exponential clock with the *bonds*, having the Bernoulli product measure as invariant measure, and being close, in some sense, to the symmetric simple exclusion process.

In [24], the totally asymmetric simple exclusion process is considered to have a single bond with smaller clock parameter. Such "slow bond", not only slows down the passage of particles across it, but it also has a macroscopical impact since it disturbs the hydrodynamic profile. Somewhat intermediate between the symmetric and asymmetric case, in [1] is considered a single asymmetric bond in the exclusion process. This unique asymmetric bond gives rise to a flux in the torus and also influences the macroscopic evolution of the density of particles. In the symmetric case, [12] obtained a d-dimensional result for a model in which the slow bonds are close to a smooth surface.

As a consequence of the above results, one can observe the recurrent phenomena about the distinct characteristics of slow bonds in symmetric and asymmetric settings. In the asymmetric case, e.g. [24] and [1], the slow bond parameter does not need to be rescaled, in order to have a macroscopic influence. Nevertheless, in the symmetric case, from [7], [9] and [12] we see that the slow bond must have parameter of order N^{-1} in order to have macroscopical impact.

In this paper, we make precise this last statement for the following model. Consider the state space of configurations with at most one particle per site in the discrete torus. To each bond is associated an exponential clock. When this clock rings, the occupancies of the sites connected by the bond are exchanged. All the bonds have clock parameter equal to 1, except k finite bonds, chosen in such a way that these bonds correspond to k fixed macroscopic points $b_1, \ldots, b_k$. The conductances in these slow bonds are given by $N^{-\beta}$, with $\beta \in [0, +\infty)$ and the scale here is diffusive in all bonds.

If $\beta = 1$, the time evolution of the density of particles $\rho(t, \cdot)$ is described by the partial differential equation

$$\left\{ \begin{array}{rcl} \partial_t \rho & = & \frac{d}{dx}\frac{d}{dW}\rho \\ \rho(0, \cdot) & = & \gamma(\cdot) \end{array} \right. ,$$

where the operator $\frac{d}{dx}\frac{d}{dW}$ is defined in subsection 4.3.1 and W is the Lebesgue measure on the torus plus the sum of the Dirac measure in each of the $\{b_i : i = 1, ..., k\}$. This result is a particular case of both the results in [9] and [12]. For the sake of completeness, we present here a simpler proof of it. It is relevant to mention the interpretation of such partial differential equation as a weak version of

$$\begin{cases} \partial_t \rho \; = \; \partial_u^2 \rho \\ \partial_u \rho_t(1) \; = \; \partial_u \rho_t(0) \; = \; \rho_t(1) - \rho_t(0) \; , \\ \rho(0, \cdot) \; = \; \gamma(\cdot) \end{cases}$$

where 0 and 1 mean the left and right side of a macroscopic point b_i related to a slow bond. This equation says that ρ is discontinuous at each macroscopic point $\{b_i : i = 1, ..., k\}$ with passage of energy at such point and is governed by the *Fourier's Law: the rate of passage of energy is proportional to the difference of temperature.* Such interpretation comes from the natural domain C_W of the operator $\frac{d}{dx}\frac{d}{dW}$, defined in subsection 4.7.2. It is easy to verify that all the functions in the domain C_W satisfy the above boundary condition, for more details see [9] and [11].

If $\beta \in [0, 1)$, the conductances in these slow bonds do not converge to zero sufficiently fast in order to appear in the hydrodynamical limit. As a consequence, there is no macroscopical influence of the slow bonds in the continuum and we obtain the hydrodynamical equation as the usual heat equation. The proof of last result is based on the *Replacement Lemma*, and the range parameter of β is sharp in the sense that, it only works for $\beta \in [0, 1)$.

As β increases, the conductance at the slow bonds decreases and the passage of particles through these bonds becomes more difficult. In fact, for $\beta \in (1, +\infty)$, the clock parameters go to zero faster than at the critical value $\beta = 1$ and each slow bond gives rise to a barrier in the continuum. Macroscopically this phenomena gives rise to the usual heat equation with Neumann's boundary conditions at each macroscopic point $\{b_i : i = 1, ..., k\}$, which means here that the spatial derivative of ρ at each $\{b_i : i = 1, ..., k\}$ equals to zero and, physically, this represents an *isolated boundary*. Moreover, the uniqueness of weak solutions of such equation says explicitly that the macroscopic evolution of the density of particles is independent for each interval $[b_i, b_{i+1}]$, however the passage of particles in the discrete torus through the slow bonds is still possible. The proof of this result is also based on the *Replacement Lemma* and requires sharp energy estimates.

Since the regime $\beta = 1$ was already known from previous works, the main contribution of this article is the complete characterization of the three distinct behaviors for the time evolution of the empirical density of particles, exhibiting a phase transition depending on the parameter of the conductance at the slow bonds. From our knowledge, no similar phenomena were exploited for the hydrodynamic limit of interacting particle systems. Moreover, for the regime $\beta \in (1, \infty)$ the density evolves according to the heat equation with Neumann's boundary conditions, which has a meaningful physical interpretation. This the other great novelty developed in this paper. So far, partial differential equations with Dirichlet's boundary conditions could be approached by e.g. studying interacting particle systems in contact with reservoirs. Here, by considering partial differential equations with Neumann's boundary conditions, we give a step towards extending the set of treatable partial differential equations by the hydrodynamic limit theory. Besides all the mentioned achievements, we also prove that the regime $\beta = 1$ is critical, since the other two regimes have positive Lebesgue measure on the line.

In order to achieve our goal, the main difficulties appear in the characterization of limit points for each regime of β. We overcome this difficulty by developing a suitable *Replacement Lemma*, which allows us to replace product of site occupancies by functions of the empirical measure in the continuum limit. Furthermore, that Lemma is also crucial for characterizing the behavior near the slow bonds.

Our result can also be extended to non-degenerate exclusion type models as introduced in [14]. In such models, particles interact with hard core exclusion and the rate of exchange between two consecutive sites is influenced by the number of particles in the vicinity of the exchanging sites. The jump rate is strictly positive, so that all the configurations are erdogic, in the sense that a move to an unoccupied site can always occur. It was shown in [14] that the hydrodynamical equation for such models is given by a non-linear partial equation. Having established the *Replacement Lemma*, the extension of our results to these models is almost standard [9]. We also believe that our method is robust enough to fit other models such as independent random walks, the zero-range process, the generalized exclusion process, when a finite number of slow bonds is present.

The present work is divided as follows. In section 4.3, we introduce notation and state the main result, namely Theorem 4.3.1. In section 4.4 we make precise the scaling limit and sketch the proof of Theorem 4.3.1. In section 4.5, we prove tightness for any range of the parameter β. In section 4.6, we prove the *Replacement Lemma* and we establish the energy estimates, which are fundamental for characterizing the limit points and the uniqueness of weak solutions of the partial differential equations considered here. In section 4.7 we characterize the limit points as weak solutions of the corresponding partial differential equations. Finally, uniqueness of weak solutions is refereed to section 4.8.

4.3 Notation and Results

Let $\mathbb{T}_N = \{1, \ldots, N\}$ be the one-dimensional discrete torus with N points. At each site, we allow at most one particle. Therefore, we will be concerned about the space state $\{0,1\}^{\mathbb{T}_N}$. Configurations will be denoted by the Greek letter η, so that $\eta(x) = 1$, if the site x is occupied, otherwise $\eta(x) = 0$.

We define now the exclusion process with space state $\{0,1\}^{\mathbb{T}_N}$ and with conductance $\{\xi^N_{x,x+1}\}_x$ at the bond of vertices $x, x+1$. The dynamics of this Markov process can be described as follows. To each bond of vertices $x, x+1$, we associate an exponential clock of parameter $\xi^N_{x,x+1}$. When this clock rings, the value of η at the vertices of this bond are exchanged. This process can also be characterized in terms of its infinitesimal generator $\mathcal{L}_N$, which acts on local functions $f : \{0,1\}^{\mathbb{T}_N} \to \mathbb{R}$ as

$$\mathcal{L}_N f(\eta) = \sum_{x\in\mathbb{T}_N} \xi^N_{x,x+1} \Big[f(\eta^{x,x+1}) - f(\eta) \Big] \, ,$$

where $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the variables $\eta(x)$ and $\eta(x+1)$:

$$(\eta^{x,x+1})(y) = \begin{cases} \eta(x+1), & \text{if } y = x\,, \\ \eta(x), & \text{if } y = x+1\,, \\ \eta(y), & \text{otherwise.} \end{cases}$$

The Bernoulli product measures $\{\nu^N_\alpha : 0 \le \alpha \le 1\}$ are invariant and in fact, reversible, for the dynamics introduced above. Namely, ν^N_α is a product measure on $\{0,1\}^{\mathbb{T}_N}$ with marginal at site x in $\mathbb{T}_N$ given by $\nu^N_\alpha\{\eta : \eta(x) = 1\} \;=\; \alpha$.

Denote by $\mathbb{T}$ the one-dimensional continuous torus $[0,1)$. The exclusion process with a slow bond at each point $b_1 \dots, b_k \in \mathbb{T}$ is defined with the following conductances:

$$\xi^N_{x,x+1} \;=\; \begin{cases} N^{-\beta}, & \text{if } \; \{b_1,\dots,b_k\} \cap (\frac{x}{N}, \frac{x+1}{N}] \neq \varnothing\,, \\ \\ 1, & \text{otherwise}\,. \end{cases}$$

The conductances are chosen in such a way that particles cross bonds at rate one, except k particular bonds in which the dynamics is slowed down by a factor $N^{-\beta}$, with $\beta \in [0,\infty)$. Each one of these particular bonds contains the macroscopic point $b_i \in \mathbb{T}$; or b_i coincides with some vertex $\frac{x}{N}$ and the slow bond is chosen as the bond to the left of $\frac{x}{N}$. To simplify notation, we denote by Nb_i the left vertex of the slow bond containing b_i.

Denote by $\{\eta_t := \eta_{tN^2} : t \ge 0\}$ the Markov process on $\{0,1\}^{\mathbb{T}_N}$ associated to the generator $\mathcal{L}_N$ *speeded up* by N^2. Although η_t depends on N and β, we are not indexing it on that in order not to overload notation. Let $D(\mathbb{R}_+, \{0,1\}^{\mathbb{T}_N})$ be the path space of càdlàg trajectories with values in $\{0,1\}^{\mathbb{T}_N}$. For a measure μ_N on $\{0,1\}^{\mathbb{T}_N}$, denote by $\mathbb{P}^\beta_{\mu_N}$ the probability measure on $D(\mathbb{R}_+, \{0,1\}^{\mathbb{T}_N})$ induced by the initial state μ_N and the Markov process $\{\eta_t : t \ge 0\}$ and denote by $\mathbb{E}^\beta_{\mu_N}$ the expectation with respect to $\mathbb{P}^\beta_{\mu_N}$.

Definition 4.3.1. *A sequence of probability measures $\{\mu_N : N \ge 1\}$ on $\{0,1\}^{\mathbb{T}_N}$ is said to be associated to a profile $\rho_0 : \mathbb{T} \to [0,1]$ if for every $\delta > 0$ and every continuous functions $H : \mathbb{T} \to \mathbb{R}$*

$$\lim_{N\to\infty} \mu_N \Big\{ \eta : \Big| \tfrac{1}{N} \sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\, \eta(x) - \int_{\mathbb{T}} H(u)\, \rho_0(u) du \Big| > \delta \Big\} \;=\; 0. \tag{4.3.1}$$

Now we introduce an operator which corresponds to the generator of the random walk in $\mathbb{T}_N$ with conductance $\xi^N_{x,x+1}$ at the bond of vertices $x, x+1$. This operator is given on

$H : \mathbb{T} \to \mathbb{R}$ by

$$\mathbb{L}_N H(\tfrac{x}{N}) = \xi^N_{x,x+1}\left[H\Big(\tfrac{x+1}{N}\Big) - H\Big(\tfrac{x}{N}\Big)\right] + \xi^N_{x-1,x}\left[H\Big(\tfrac{x-1}{N}\Big) - H\Big(\tfrac{x}{N}\Big)\right]. \tag{4.3.2}$$

We will not differentiate the notation for functions H defined on $\mathbb{T}$ and on $\mathbb{T}_N$. The indicator function of a set A will be written by $\mathbf{1}_A(u)$, which is one when $u \in A$ and zero otherwise.

4.3.1 The Operator $\frac{d}{dx}\frac{d}{dW}$

Given the points $b_1, \dots, b_k \in \mathbb{T}$, define the measure $W(du)$ in the torus $\mathbb{T}$ by

$$W(du) \;=\; du + \delta_{b_1}(du) + \cdots + \delta_{b_k}(du)\,,$$

so that W is the Lebesgue measure on the torus $\mathbb{T}$ plus the sum of the Dirac measure in each of the $\{b_i : i = 1, ..., k\}$.

Let $\mathcal{H}^1_W$ be the set of functions F in $L^2(\mathbb{T})$ such that for $x \in \mathbb{T}$

$$F(x) \;=\; a \;+\; bW(x) \;+\; \int_{(0,x]} \Big(\int_0^y f(z)\,dz\Big) W(dy),$$

for some function f in $L^2(\mathbb{T})$ such that

$$\int_0^1 f(x)\,dx \;=\; 0\,, \qquad \int_{(0,1]} \Big(b + \int_0^y f(z)\,dz\Big) W(dy) \;=\; 0\,. \tag{4.3.3}$$

Define the operator

$$\frac{d}{dx}\frac{d}{dW} : \mathcal{H}^1_W \to L^2(\mathbb{T})$$
$$\frac{d}{dx}\frac{d}{dW} F = f.$$

4.3.2 The hydrodynamical equations

Consider a continuous density profile $\gamma : \mathbb{T} \to [0,1]$. Denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\mathbb{T})$, by ρ_t a function $\rho(t,\cdot)$ and for an integer n denote by $C^n(\mathbb{T})$ the set of continuous functions from $\mathbb{T}$ to $\mathbb{R}$ and with continuous derivatives of order up to n. For $\mathcal{I}$ an interval of $\mathbb{T}$, here and in the sequel, for n and m integers, we use the notation $C^{n,m}([0,T] \times \mathcal{I})$ to denote the set of functions defined on the domain $[0,T] \times \mathcal{I}$, that are of class C^n in time and C^m in space.

Definition 4.3.2. *A bounded function $\rho : [0,T] \times \mathbb{T} \to \mathbb{R}$ is said to be a weak solution of the parabolic differential equation*

$$\begin{cases} \partial_t \rho \;=\; \partial_u^2 \rho \\ \rho(0,\cdot) \;=\; \gamma(\cdot) \end{cases} \tag{4.3.4}$$

if, for $t \in [0,T]$ and $H \in C^2(\mathbb{T})$, $\rho(t,\cdot)$ satisfies the integral equation

$$\langle \rho_t, H\rangle \;-\; \langle \gamma, H\rangle \;-\; \int_0^t \langle \rho_s, \partial_u^2 H\rangle \, ds \;=0.$$

Definition 4.3.3. *A bounded function $\rho : [0,T]\times\mathbb{T} \to \mathbb{R}$ is said to be a weak solution of the parabolic differential equation*

$$\begin{cases} \partial_t \rho \;=\; \dfrac{d}{dx}\dfrac{d}{dW}\rho \\ \rho(0,\cdot) \;=\; \gamma(\cdot) \end{cases} \tag{4.3.5}$$

if, for $t \in [0,T]$ and $H \in \mathcal{H}_W^1$, $\rho(t,\cdot)$ satisfies the integral equation

$$\langle \rho_t, H\rangle \;-\; \langle \gamma, H\rangle - \int_0^t \Big\langle \rho_s, \frac{d}{dx}\frac{d}{dW}H\Big\rangle \, ds \;=\; 0\,.$$

Following the notation of [4], denote by $L^1(0,T;\mathcal{H}^1(a,b))$ the space of functions $\varrho \in L^2([0,T]\times[a,b])$ for which there exists a function in $L^2([0,T]\times[a,b])$, denoted by $\partial_u\varrho$, satisfying

$$\int_0^T \int_a^b (\partial_u H)(s,u)\, \varrho(s,u)\, du\, ds \;=\; -\int_0^T \int_a^b H(s,u)\, (\partial_u \varrho)(s,u)\, du\, ds\,,$$

for any $H \in C^{0,1}([0,T]\times[a,b])$ with compact support in $[0,T]\times(a,b)$.

Definition 4.3.4. *Let $[b_i, b_{i+1}] \subset \mathbb{T}$. A bounded function $\rho : [0,T]\times[b_i,b_{i+1}] \to \mathbb{R}$ is said to be a weak solution of the parabolic differential equation with Neumann's boundary conditions in the cylinder $[0,T]\times[b_i,b_{i+1}]$*

$$\begin{cases} \partial_t \rho \;=\; \partial_u^2 \rho \\ \rho(0,\cdot) \;=\; \gamma(\cdot) \\ \partial_u\rho(t,b_i) = \partial_u\rho(t,b_{i+1}) = 0, \;\forall t\in[0,T] \end{cases} \tag{4.3.6}$$

if, for $t \in [0,T]$ and $H \in C^{1,2}([0,T]\times[b_i,b_{i+1}])$, $\rho(t,\cdot)$ satisfies the integral equation

$$\begin{aligned} &\int_{b_i}^{b_{i+1}} \rho(t,u)\, H(t,u)\, du - \int_{b_i}^{b_{i+1}} \gamma(u)\, H(0,u)\, du \\ &- \int_0^t \int_{b_i}^{b_{i+1}} \rho(s,u)\, \{\partial_u^2 H(s,u) + \partial_s H(s,u)\}\, du\, ds \\ &+ \int_0^t \partial_u H(s,b_{i+1})\, \rho(s,b_{i+i}^-)\, ds - \int_0^t \partial_u H(s,b_i)\, \rho(s,b_i^+)\, ds \;=\; 0 \end{aligned} \tag{4.3.7}$$

and $\rho(t,\cdot)$ belongs to $L^1(0,T;\mathcal{H}^1(b_i,b_{i+1}))$.

For classical results about Sobolev spaces, we refer the reader to [4] and [19]. Since in Definition 4.3.4 we impose $\rho \in L^1(0,T;\mathcal{H}^1(b_i,b_{i+1}))$, the integrals are well-defined at the boundary. Heuristically, in order to establish an integral equation for the weak solution of the heat equation with Neumann's boundary conditions as above, one should multiply (4.3.6) by a test function H and perform twice a formal integration by parts to arrive at (4.3.7).

We are now in position to state the main result of this paper:

Theorem 4.3.1. *Fix $\beta \in [0,\infty)$. Consider the exclusion process with k slow bonds corresponding to macroscopic points $b_1,\dots,b_k \in \mathbb{T}$ and with conductance $N^{-\beta}$ at each one of these slow bonds.*

Fix a continuous initial profile $\gamma : \mathbb{T} \to [0,1]$. Let $\{\mu_N : N \geq 1\}$ be a sequence of probability measures on $\{0,1\}^{\mathbb{T}_N}$ associated to γ. Then, for any $t \in [0,T]$, for every $\delta > 0$ and every $H \in C(\mathbb{T})$, it holds that

$$\lim_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big\{\eta_\cdot : \Big|\tfrac{1}{N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\,\eta_t(x) - \int_{\mathbb{T}} H(u)\,\rho(t,u)du\Big| > \delta\Big\} \;=\; 0\,,$$

where :

- *if $\beta \in [0,1)$, $\rho(t,\cdot)$ is the unique weak solution of* (4.3.4)*;*
- *if $\beta = 1$, $\rho(t,\cdot)$ is the unique weak solution of* (4.3.5)*;*
- *if $\beta \in (1,\infty)$, in each cylinder $[0,T]\times[b_i,b_{i+1}]$, $\rho(t,\cdot)$ is the unique weak solution of* (4.3.6).

Remark 4.3.2. *The assumption that all slow bonds have exactly the same conductance is not necessary at all. In fact, last result is true when considering each slow bond containing the macroscopic point b_i with conductance $N^{-\beta_i}$. In that case, we would obtain a parabolic differential equation with the behavior at each $[b_i,b_{i+1}]$ given by the regime of the corresponding β_i as above. Another straightforward generalization is to consider conductances not exactly equal to $N^{-\beta}$, but of order $N^{-\beta}$, in the sense that the quotient with $N^{-\beta}$ converges to one. For sake of clarity, we present the proof under the conditions of Theorem 4.3.1.*

4.4 Scaling Limit

Let $\mathcal{M}$ be the space of positive measures on $\mathbb{T}$ with total mass bounded by one, endowed with the weak topology. Let $\pi^N_t \in \mathcal{M}$ be the empirical measure at time t associated to η_t, namely, it is the measure on $\mathbb{T}$ obtained by re-scaling space by N and by assigning mass N^{-1} to each particle:

$$\pi^N_t \;=\; \tfrac{1}{N}\sum_{x\in\mathbb{T}_N} \eta_t(x)\,\delta_{x/N}\,, \tag{4.4.1}$$

where δ_u is the Dirac measure concentrated on u. For an integrable function $H : \mathbb{T} \to \mathbb{R}$, $\langle \pi_t^N, H \rangle$ stands for the integral of H with respect to π_t^N:

$$\langle \pi_t^N, H \rangle \;=\; \tfrac{1}{N} \sum_{x \in \mathbb{T}_N} H(\tfrac{x}{N})\, \eta_t(x)\,.$$

This notation is not to be mistaken with the inner product in $L^2(\mathbb{R})$. Also, when π_t has a density ρ, namely when $\pi(t,du) = \rho(t,u)du$, we sometimes write $\langle \rho_t, H\rangle$ for $\langle \pi_t, H\rangle$.

Fix $T > 0$. Let $D([0,T],\mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi : [0,T] \to \mathcal{M}$ endowed with the *Skorohod* topology. For each probability measure μ_N on $\{0,1\}^{\mathbb{T}_N}$, denote by $\mathbb{Q}_{\mu_N}^{\beta,N}$ the measure on the path space $D([0,T],\mathcal{M})$ induced by the measure μ_N and the empirical process π_t^N introduced in (4.4.1).

Fix a continuous profile $\gamma : \mathbb{T} \to [0,1]$ and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on $\{0,1\}^{\mathbb{T}_N}$ associated to γ. Let $\mathbb{Q}^\beta$ be the probability measure on $D([0,T],\mathcal{M})$ concentrated on the deterministic path $\pi(t,du) = \rho(t,u)du$, where:

- if $\beta \in [0,1)$, $\rho(t,\cdot)$ is the unique weak solution of (4.3.4);
- if $\beta = 1$, $\rho(t,\cdot)$ is the unique weak solution of (4.3.5);
- if $\beta \in (1,\infty)$, in each cylinder $[0,T] \times [b_i, b_{i+1}]$, $\rho(t,\cdot)$ is the unique weak solution of (4.3.6).

Proposition 4.4.1. *As $N \uparrow \infty$, the sequence of probability measures $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$ converges weakly to $\mathbb{Q}^\beta$.*

The proof of this result is divided into three parts. In the next section, we show that the sequence $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$ is tight, for any $\beta \in [0,\infty)$. In section 4.7 we characterize the limit points of this sequence for each regime of the parameter β. Uniqueness of weak solutions is presented in section 4.8 and this implies the uniqueness of limit points of the sequence $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$. In the fifth section, we prove a suitable *Replacement Lemma* for each regime of β, which is crucial in the task of characterizing limit points and uniqueness.

4.5 Tightness

Proposition 4.5.1. *For any fixed $\beta \in [0,\infty)$, the sequence of measures $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$ is tight in the Skorohod topology of $D([0,T],\mathcal{M})$.*

Proof. In order to prove tightness of $\{\pi_t^N : 0 \leq t \leq T\}$ it is enough to show tightness of the real-valued processes $\{\langle \pi_t^N, H\rangle : 0 \leq t \leq T\}$ for $H \in C(\mathbb{T})$. In fact, c.f. [17] it is enough to show tightness of $\{\langle \pi_t^N, H\rangle : 0 \leq t \leq T\}$ for a dense set of functions in $C(\mathbb{T})$ with respect to the uniform topology. For that purpose, fix $H \in C^2(\mathbb{T})$. By Dynkin's formula,

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t N^2 \mathcal{L}_N \langle \pi_s^N, H\rangle\, ds\,, \tag{4.5.1}$$

is a martingale with respect to the natural filtration $\mathcal{F}_t := \sigma(\eta_s : s \leq t)$. In order to prove tightness of $\{\langle \pi_t^N, H\rangle : N \geq 1\}$, we prove tightness of the sequence of the martingales and the integral terms in the decomposition above. We start by the former.

We begin by showing that the $L^2(\mathbb{P}^\beta_{\mu_N})$-norm of the martingale above vanishes as $N \to +\infty$. The quadratic variation of $M_t^N(H)$ is given by

$$\langle M^N(H)\rangle_t \;=\; \int_0^t \sum_{x\in\mathbb{T}_N} \xi^N_{x,x+1}\Big[(\eta_s(x)-\eta_s(x+1))(H(\tfrac{x+1}{N})-H(\tfrac{x}{N}))\Big]^2 ds. \tag{4.5.2}$$

It is easy to show that $\langle M^N(H)\rangle_t \;\leq\; \frac{T}{N}\|\partial_u H\|^2_\infty$. Here and in the sequel we use the notation $\|H\|_\infty := \sup_{u\in\mathbb{T}} |H(u)|$.

Thus, $M_t^N(H)$ converges to zero as $N \to +\infty$ in $L^2(\mathbb{P}^\beta_{\mu_N})$. Notice that above we used the trivial bound $\xi^N_{x,x+1} \leq 1$. By Doob's inequality, for every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}^\beta_{\mu_N}\left[\sup_{0\leq t\leq T} |M_t^N(H)| > \delta\right] = 0\,, \tag{4.5.3}$$

which implies tightness of the sequence of martingales $\{M_t^N(H); N \geq 1\}$. Now, we need to examine tightness of the integral term in (4.5.1).

Denote by Γ_N the subset of sites $x \in \mathbb{T}_N$ such that x has some adjacent slow bond, namely, $\xi^N_{x,x+1} = N^{-\beta}$ or $\xi^N_{x-1,x} = N^{-\beta}$. The term $N^2\mathcal{L}_N\langle \pi_s^N, H\rangle$ appearing inside the integral in (4.5.1) is explicitly given by

$$\begin{aligned} &N\sum_{x\notin\Gamma_N} \eta_s(x)\Big[H(\tfrac{x+1}{N})+H(\tfrac{x-1}{N})-2H(\tfrac{x}{N})\Big] \\ &+N\sum_{x\in\Gamma_N} \eta_s(x)\Big[\xi^N_{x,x+1}\{H(\tfrac{x+1}{N})-H(\tfrac{x}{N})\}+\xi^N_{x-1,x}\{H(\tfrac{x-1}{N})-H(\tfrac{x}{N})\}\Big]\,. \end{aligned}$$

By Taylor expansion on H, the absolute value of the first sum above is bounded by $\|\partial_u^2 H\|_\infty$. Since there are at most $2k$ elements in Γ_N, $\xi_{x,x+1} \leq 1$ and by the exclusion rule, the absolute value of the second sum above is bounded by $2\,k\|\partial_u H\|_\infty$. Therefore, there exists a constant $C := C(H,k) > 0$, such that $|N^2\mathcal{L}_N\langle \pi_s^N, H\rangle| \leq C$, which yields

$$\left|\int_r^t N^2\mathcal{L}_N\langle \pi_s^N, H\rangle ds\right| \leq C|t-r|\,.$$

By Proposition 4.1.6 of [17], last inequality implies tightness of integral term. This concludes the proof. □

4.6 Replacement Lemma and Energy Estimates

In this section, we obtain fundamental results that allow us to replace the mean occupation of a site by the mean density of particles in a small macroscopic box around this site. This result implies that the limit trajectories must belong to some Sobolev space, this will be clear later. Before proceeding we introduce some tools that we use in the sequel.

Denote by $H_N(\mu_N|\nu_\alpha)$ the entropy of a probability measure μ_N with respect to the invariant state ν_α. For a precise definition and properties of the entropy, we refer the reader to [17]. In Proposition 5.0.2 in the Appendix we review a classical result saying that there exists a finite constant $K_0 := K_0(\alpha)$, such that

$$H_N(\mu_N|\nu_\alpha) \;\leq\; K_0 N\,, \tag{4.6.1}$$

for any probability measure $\mu_N \in \{0,1\}^{\mathbb{T}_N}$.

Denote by $\langle\cdot,\cdot\rangle_{\nu_\alpha}$ the scalar product of $L^2(\nu_\alpha)$ and denote by $\mathfrak{D}_N$ the Dirichlet form of f, which is the convex and lower semicontinuous functional (see Corollary A1.10.3 of [17]) defined by

$$\mathfrak{D}_N(f) \;=\; \langle -L_N\sqrt{f}\,,\,\sqrt{f}\rangle_{\nu_\alpha}\,,$$

where f is a probability density with respect to ν_α (i.e. $f \geq 0$ and $\int f d\nu_\alpha = 1$). An elementary computation shows that

$$\mathfrak{D}_N(f) \;=\; \sum_{x\in\mathbb{T}_N} \frac{\xi^N_{x,x+1}}{2}\int \Big(\sqrt{f(\eta^{x,x+1})}-\sqrt{f(\eta)}\Big)^2 d\nu_\alpha\,.$$

By Theorem A1.9.2 of [17], if $\{S^N_t : t\geq 0\}$ stands for the semi-group associated to the generator $N^2\mathcal{L}_N$, then

$$H_N(\mu_N S^N_t|\nu_\alpha) \;+\; N^2\int_0^t \mathfrak{D}_N(f^N_s)\,ds \;\leq\; H_N(\mu_N|\nu_\alpha)\,,$$

provided f^N_s stands for the Radon-Nikodym derivative of $\mu_N S^N_s$ (the distribution of η_s starting from μ_N) with respect to ν_α.

4.6.1 Replacement Lemma

Now, we define the local density of particles, which corresponds to the mean occupation in a box around a given site. We represent this empirical density in the box of size ℓ around a given site x by $\eta^\ell(x)$. For $\beta\in[0,1)$, this box can be chosen in the usual way, but for $\beta\in[1,\infty)$, this box must avoid the slow bond. From this point on, we denote the integer part of εN, namely $\lfloor\varepsilon N\rfloor$, simply by εN.

Definition 4.6.1. *For $\beta\in[0,1)$, define the empirical density by*

$$\eta^{\varepsilon N}(x) \;=\; \tfrac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\eta(y)\,.$$

Definition 4.6.2. *For $\beta \in [1,\infty)$, if x is such that $\{Nb_1,\dots,Nb_k\}\cap\{x,\dots,x+\varepsilon N\}=\varnothing$, then the empirical density is defined by*

$$\eta^{\varepsilon N}(x) \;=\; \tfrac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\eta(y)\,.$$

Otherwise, if, let us say, $Nb_i \in \{x,\dots,x+\varepsilon N\}$ for some $i=1,..,k$, then the empirical density is defined by

$$\eta^{\varepsilon N}(x) \;=\; \tfrac{1}{\varepsilon N}\sum_{y=Nb_i-\varepsilon N+1}^{Nb_i}\eta(y)\,.$$

Since we are considering a finite number of slow bonds, the distance between two consecutive macroscopic points related to two consecutive slow bonds is at least ε, for ε sufficiently small. As a consequence, we can suppose, without lost of generality that in the previous definition, b_i is unique.

Lemma 4.6.1. *Fix $\beta\in[0,1)$. Let f be a density with respect to the invariant measure ν_α. Then,*

$$\int\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\alpha(d\eta) \;\leq\; 2(kN^{\beta-1}+\varepsilon)+N\,\mathfrak{D}_N(f)\,.$$

Proof. From Definition 4.6.1 we have that

$$\int\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\alpha(d\eta)=\int\Big\{\tfrac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}(\eta(x)-\eta(y))\Big\}f(\eta)\,\nu_\alpha(d\eta)\,.$$

Writing $\eta(x)-\eta(y)$ as a telescopic sum, last expression becomes equal to

$$\int\Big\{\tfrac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}(\eta(z)-\eta(z+1))\Big\}f(\eta)\,\nu_\alpha(d\eta)\,.$$

Rewriting the expression above as twice the half and making the transformation $\eta\mapsto\eta^{z,z+1}$ (for which the probability ν_α is invariant) it becomes as:

$$\tfrac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\int\{\eta(z)-\eta(z+1)\}(f(\eta)-f(\eta^{z,z+1}))\,\nu_\alpha(d\eta)\,.$$

Since $(a-b)=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ and by Cauchy-Schwarz's inequality, for any $A>0$, we bound the previous expression from above by

$$\tfrac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\frac{A}{\xi^N_{z,z+1}}\int\{\eta(z)-\eta(z+1)\}^2\Big(\sqrt{f(\eta)}+\sqrt{f(\eta^{z,z+1})}\Big)^2\,\nu_\alpha(d\eta)$$
$$+\tfrac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\frac{\xi^N_{z,z+1}}{A}\int\Big(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\Big)^2\,\nu_\alpha(d\eta)\,.$$

The second sum above is bounded by

$$\tfrac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z\in\mathbb{T}_N}\frac{\xi^N_{z,z+1}}{A}\int\Big(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\Big)^2\nu_\alpha(d\eta)=\tfrac{1}{A}\mathfrak{D}_N(f)\,.$$

On the other hand, since f is a density, the first sum is bounded from above by

$$\tfrac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\frac{4A}{\xi^N_{z,z+1}}\leq\tfrac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}2A(kN^\beta+\varepsilon N)=2A(kN^\beta+\varepsilon N)\,.$$

Notice that the term kN^β comes from the existence of k slow bonds. Choosing $A=\frac{1}{N}$, the proof ends. □

Lemma 4.6.2 (Replacement Lemma)**.** *Fix $\beta\in[0,1)$. Let $b\in\mathbb{T}$ and let x be the right (or left) vertex of the bond containing the macroscopic point b. Then,*

$$\varlimsup_{\varepsilon\to0}\varlimsup_{N\to\infty}\mathbb{E}^\beta_{\mu_N}\Big[\Big|\int_0^t\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}\,ds\Big|\Big]\;=\;0\,.$$

Proof. From Jensen's inequality and the definition of entropy, for any $\gamma\in\mathbb{R}$ (which will be chosen large), the expectation appearing on the statement of the Lemma is bounded from above by

$$\frac{H_N(\mu_N|\nu_\alpha)}{\gamma N}+\frac{1}{\gamma N}\log\mathbb{E}_{\nu_\alpha}\Big[\exp\Big\{\gamma\,N\Big|\int_0^t\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}\,ds\Big|\Big\}\Big]\,. \tag{4.6.2}$$

By Proposition 5.0.2, $H_N(\mu_N|\nu_\alpha)\leq K_0\,N$, so that it remains to focus on the second summand above. Since $e^{|x|}\leq e^x+e^{-x}$ and

$$\varlimsup_N\tfrac{1}{N}\log(a_N+b_N)=\max\Big\{\varlimsup_N\tfrac{1}{N}\log a_N,\varlimsup_N\tfrac{1}{N}\log b_N\Big\}\,, \tag{4.6.3}$$

we can remove the modulus inside the exponential. By Feynman-Kac's formula, see Lemma A1.7.2 of [17] and Proposition 5.0.3, the expectation in (4.6.2) is less than or equal to

$$t\sup_{f\text{ density}}\Big\{\int\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\alpha(d\eta)-N\,\mathfrak{D}_N(f)\Big\}\,ds\,.$$

Applying Lemma 4.6.1 and recalling that γ is arbitrary large, the proof finishes. □

The next two results are concerned with both cases $\beta=1$ and $\beta\in(1,\infty)$.

Lemma 4.6.3. *Fix $\beta\in[1,\infty)$. Let f be a density with respect to the invariant measure ν_α. Then,*

$$\int\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\alpha(d\eta)\leq N\mathfrak{D}_N(f)+4\varepsilon\,.$$

Moreover, given a function $H:\mathbb{T}\to\mathbb{R}$:

$$\tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\int H(\tfrac{x}{N})\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\alpha(d\eta)\leq N\mathfrak{D}_N(f)+\tfrac{4\varepsilon}{N}\sum_{x\in\mathbb{T}_N}\Big(H(\tfrac{x}{N})\Big)^2\,.$$

Proof. Recall Definition 4.6.2. Let first x be a site such that there is no slow bond connecting two sites in $\{x, \dots, x+\varepsilon N\}$. In this case,

$$\begin{aligned}&\int H(\tfrac{x}{N})\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\alpha(d\eta)\\ =&\int H(\tfrac{x}{N})\Big\{\tfrac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}(\eta(x)-\eta(y))\Big\}f(\eta)\nu_\alpha(d\eta)\,,\end{aligned}$$

and following the same arguments as in Lemma 4.6.1, we bound the previous expression from above by

$$\begin{aligned}&\tfrac{(H(\frac{x}{N}))^2}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\int\frac{A}{\xi^N_{z,z+1}}\{\eta(z)-\eta(z+1)\}^2\Big(\sqrt{f(\eta)}+\sqrt{f(\eta^{z,z+1})}\Big)^2\nu_\alpha(d\eta)\\ &+\frac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\int\frac{\xi^N_{z,z+1}}{A}\{\eta(z)-\eta(z+1)\}^2\Big(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\Big)^2\nu_\alpha(d\eta)\,.\end{aligned}$$

Since $\xi^N_{z,z+1}=1$ for all $z\in\{x,\dots,x+\varepsilon N-1\}$, it yields the boundedness of the previous expression by

$$2\varepsilon NA\left(H(\tfrac{x}{N})\right)^2+\frac{\mathfrak{D}_N(f)}{A}\,.$$

Let now x be a site such that $Nb_i\in\{x,\dots,x+\varepsilon N\}$ for some $i=1,\dots,k$. In this case,

$$\begin{aligned}&\int H(\tfrac{x}{N})\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\alpha(d\eta)\\ =&\int H(\tfrac{x}{N})\frac{1}{\varepsilon N}\sum_{y=Nb_i-\varepsilon N+1}^{Nb_i}\Big\{\eta(x)-\eta(y)\Big\}f(\eta)\nu_\alpha(d\eta)\end{aligned}\tag{4.6.4}$$

Now we split last summation into two cases, $y>x$ and $y<x$ and then we proceed by writing $\eta(x)-\eta(y)$ as a telescopic sum as in Lemma 4.6.1. Then, by the same arguments of Lemma 4.6.1 and since $\xi^N_{z,z+1}=1$ for all z in the range $\{Nb_i-\varepsilon N+1,\dots,Nb_i-1\}$, we bound the previous expression by

$$4\varepsilon NA\Big(H(\tfrac{x}{N})\Big)^2+\frac{\mathfrak{D}_N(f)}{A}\,.$$

Now the first claim of the Lemma follows by taking the particular case $H(\frac{x}{N})=1$ and choosing $A=\frac{1}{N}$.

Finally, if in (4.6.4) we sum over $x\in\mathbb{T}_N$ and then divide by N, one concludes the second claim of the Lemma. □

Lemma 4.6.4 (Replacement Lemma)**.** *Fix $\beta\in[1,\infty)$. Therefore,*

$$\varlimsup_{\varepsilon\to0}\varlimsup_{N\to\infty}\mathbb{E}^\beta_{\mu_N}\Big[\Big|\int_0^t\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}\,ds\Big|\Big]\;=\;0\,.$$

Moreover, given a function $H : \mathbb{T} \to \mathbb{R}$ satisfying

$$\varlimsup_{N\to\infty} \tfrac{1}{N} \sum_{x\in\mathbb{T}_N} \Big(H(\tfrac{x}{N})\Big)^2 \;<\; \infty\,,$$

also holds

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{N\to\infty} \mathbb{E}^{\beta}_{\mu_N}\Big[\Big|\int_0^t \tfrac{1}{N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}\,ds\Big|\Big] \;=\; 0\,.$$

Proof. The proof follows exactly the same arguments in Lemma 4.6.2. Therefore, is sufficient to show that the expressions

$$t\sup_{f \text{ density}}\Big\{\int\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)d\nu_\alpha - N\mathfrak{D}_N(f)\Big\}$$

and

$$t\sup_{f \text{ density}}\Big\{\int \tfrac{1}{N}\sum_x H(\tfrac{x}{N})\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)d\nu_\alpha - N\mathfrak{D}_N(f)\Big\}\,,$$

vanish as $N\to+\infty$, which is an immediate consequence of Lemma 4.6.3. □

In the next subsection, we will need the following variation of Lemma 4.6.3:

Lemma 4.6.5. *Let $H : \mathbb{T}\to\mathbb{R}$ and let f be a density with respect to ν_α. Then,*

$$\int \frac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\Big\{\eta(x)-\eta(x+\varepsilon N)\Big\}f(\eta)\,\nu_\alpha(d\eta)$$

$$\le N\mathfrak{D}_N(f) + \frac{2}{\varepsilon N}\sum_{x\in\mathbb{T}_N}\Big(H(\tfrac{x}{N})\Big)^2\Big\{\varepsilon + N^{\beta-1}\sum_{i=1}^{k}\mathbf{1}_{[b_i,b_i+\varepsilon)}(\tfrac{x}{N})\Big\}\,.$$

The proof of last Lemma follows the same steps as above and for that reason will be omitted. Nevertheless, we sketch the idea of the proof. One begins by writing $\eta(x)-\eta(x+\varepsilon N)$ as a telescopic sum and proceeding as in Lemma 4.6.3. The only relevant difference in this case is that is not possible to avoid the slow bonds inside the telescopic sum, and therefore the upper bound depends on β.

4.6.2 Energy Estimates

We prove in this subsection that any limit point $\mathbb{Q}^{\beta}_{*}$ of the sequence $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N\ge 1\}$ is concentrated on trajectories $\rho(t,u)du$ with finite energy meaning that $\rho(t,u)$ belongs to some Sobolev space. For $\beta\in[0,1)$, this result is an immediate consequence of uniqueness of weak solutions of the heat equation. The case $\beta=1$ is a particular case of the one considered in [9]. Therefore, we will treat here the remaining case $\beta\in(1,\infty)$. Such result will play an important role in the uniqueness of weak solutions of (4.3.6).

Let $\mathbb{Q}^{\beta}_{*}$ be a limit point of $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N\ge 1\}$ and assume without lost of generality that whole sequence converges weakly to $\mathbb{Q}^{\beta}_{*}$.

Proposition 4.6.6. *The measure $\mathbb{Q}_*^\beta$ is concentrated on paths $\pi(t,u) = \rho(t,u)du$. Moreover, there exists a function in $L^2([0,T]\times\mathbb{T})$, denoted by $\partial_u\rho$, such that*

$$\int_0^T \int_{\mathbb{T}} (\partial_u H)(s,u)\,\rho(s,u)\,du\,ds \;=\; -\int_0^T \int_{\mathbb{T}} H(s,u)\,(\partial_u\rho)(s,u)\,du\,ds\,,$$

for all H in $C^{0,1}([0,T]\times\mathbb{T})$ whose support is contained in $[0,T]\times(\mathbb{T}\backslash\{b_1,\dots,b_k\})$.

The previous result follows from next Lemma. Recall the definition of the constant K_0 given in (4.6.1).

Lemma 4.6.7.

$$E_{\mathbb{Q}_*^\beta}\Big[\sup_H\Big\{\int_0^T \int_{\mathbb{T}} (\partial_u H)(s,u)\,\rho(s,u)\,du\,ds \\ -\;2\int_0^T\int_{\mathbb{T}}\Big(H(s,u)\Big)^2\,du\,ds\Big\}\Big] \;\le\; K_0\,,$$

where the supremum is carried over all functions H in $C^{0,1}([0,T]\times\mathbb{T})$ with support contained in $[0,T]\times(\mathbb{T}\backslash\{b_1,\dots,b_k\})$.

We start by showing Proposition 4.6.6 assuming last result. Later and independently we will prove the previous Lemma.

of Proposition 4.6.6. Denote by $\ell : C^{0,1}([0,T]\times\mathbb{T})\to\mathbb{R}$ the linear functional defined by

$$\ell(H) \;=\; \int_0^T \int_{\mathbb{T}} (\partial_u H)(s,u)\,\rho(s,u)\,du\,ds.$$

Since the set of functions $H\in C^{0,1}([0,T]\times\mathbb{T})$ with support contained in $[0,T]\times(\mathbb{T}\backslash\{b_1,\dots,b_k\})$ is dense in $L^2([0,T]\times\mathbb{T})$ and since by Lemma 4.6.7, ℓ is a $\mathbb{Q}_*^\beta$-a.s. bounded functional in $C^{0,1}([0,T]\times\mathbb{T})$, we can extend it to a $\mathbb{Q}_*^\beta$-a.s. bounded functional in $L^2([0,T]\times\mathbb{T})$. In particular, by the Riesz Representation Theorem, there exists a function G in $L^2([0,T]\times\mathbb{T})$ such that

$$\ell(H) \;=\; -\int_0^T \int_{\mathbb{T}} H(s,u)\,G(s,u)\,du\,ds\,.$$

This finishes the proof. □

For a smooth function $H\colon\mathbb{T}\to\mathbb{R}$, $\varepsilon>0$ and a positive integer N, define $V_N(\varepsilon,H,\eta)$ by

$$V_N(\varepsilon,H,\eta) \;=\; \tfrac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\{\eta(x)-\eta(x+\varepsilon N)\} - \tfrac{2}{N}\sum_{x\in\mathbb{T}_N}\Big(H(\tfrac{x}{N})\Big)^2\,.$$

In order to prove Lemma 4.6.7, we need the following technical result:

Lemma 4.6.8. *Consider $H_1, \dots, H_k$ functions in $C^{0,1}([0,T]\times\mathbb{T})$ with support contained in $[0,T]\times(\mathbb{T}\backslash\{b_1,\dots,b_k\})$. Hence, for every $\varepsilon>0$:*

$$\varlimsup_{\delta\to 0}\varlimsup_{N\to\infty}\mathbb{E}^{\beta}_{\mu^N}\Big[\max_{1\le i\le k}\Big\{\int_0^T V_N(\varepsilon,H_i(s,\cdot),\eta_s^{\delta N})\,ds\Big\}\Big]\;\le\;K_0\,. \tag{4.6.5}$$

Proof. It follows from Lemma 4.6.4 that in order to prove (4.6.5), we just need to show that

$$\varlimsup_{N\to\infty}\mathbb{E}^{\beta}_{\mu^N}\Big[\max_{1\le i\le k}\Big\{\int_0^T V_N(\varepsilon,H_i(s,\cdot),\eta_s)\,ds\Big\}\Big]\;\le\;K_0\,.$$

By the entropy and the Jensen's inequality, for each fixed N, the previous expectation is less than or equal to

$$\frac{H(\mu^N|\nu_\alpha)}{N}\;+\;\frac{1}{N}\log\mathbb{E}_{\nu_\alpha}\Big[\exp\Big\{\max_{1\le i\le k}N\int_0^T V_N(\varepsilon,H_i(s,\cdot),\eta_s)ds\Big\}\Big]\,.$$

By (4.6.1), the first term above is bounded by K_0. Since $\exp\{\max_{1\le j\le k}a_j\}$ is bounded from above by $\sum_{1\le j\le k}\exp\{a_j\}$ and by (4.6.3), the limit as $N\uparrow\infty$, of the second term of the previous expression is less than or equal to

$$\max_{1\le i\le k}\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}_{\nu_\alpha}\Big[\exp\Big\{N\int_0^T V_N(\varepsilon,H_i(s,\cdot),\eta_s)ds\Big\}\Big]\,.$$

We now prove that, for each fixed i the limit above is nonpositive.

Fix $1\le i\le k$. By Feynman-Kac's formula and the variational formula for the largest eigenvalue of a symmetric operator, for each fixed N, the previous expectation is bounded from above by

$$\int_0^T\sup_f\Big\{\int V_N(\varepsilon,H_i(s,\cdot),\eta)f(\eta)\nu_\alpha(d\eta)-N\mathfrak{D}_N(f)\Big\}\,ds.$$

In last formula the supremum is taken over all probability densities f with respect to ν_α. By assumption, each of the functions $\{H_i: i=1,\dots,k\}$ vanishes in a neighborhood of each $b_i\in\mathbb{T}$. This together with Lemma 4.6.5, imply that the previous expression has nonpositive limsup. This is enough to conclude. □

We define now an approximation of the identity in the continuous torus given by

$$\iota_\varepsilon(u,v)=\begin{cases}\frac{1}{\varepsilon}\mathbf{1}_{(v,v+\varepsilon)}(u)\,, & \text{if } v\in\mathbb{T}\backslash\cup_{i=1}^k(b_i-\varepsilon,b_i)\,,\\[1ex] \frac{1}{\varepsilon}\mathbf{1}_{(b_1-\varepsilon,b_1)}(u)\,, & \text{if } v\in(b_1-\varepsilon,b_1)\,,\\ \quad\vdots & \quad\vdots\\ \frac{1}{\varepsilon}\mathbf{1}_{(b_k-\varepsilon,b_k)}(u)\,, & \text{if } v\in(b_k-\varepsilon,b_k)\,.\end{cases} \tag{4.6.6}$$

The convolution of a measure π with ι_ε is defined by

$$(\pi * \iota_\varepsilon)(v) \;=\; \int \iota_\varepsilon(u,v)\, \pi(du)\,.$$

For a function ρ, the convolution $\rho * \iota_\varepsilon$ is understood as the convolution of the measure $\rho(u)\, du$ with ι_ε. Recall Definition 4.6.2. At this point, an important remark is the equality

$$\eta_t^{\varepsilon N}(x) = (\pi_t^N * \iota_\varepsilon)(\tfrac{x}{N})\,, \tag{4.6.7}$$

which is of straightforward verification.

of Lemma 4.6.7. Consider a sequence $\{H_i : i \geq 1\}$ dense (with respect to the norm $\|H\|_\infty + \|\partial_u H\|_\infty$) in the subset of $C^{0,1}([0,T]\times\mathbb{T})$ of functions with support contained in $[0,T]\times(\mathbb{T}\backslash\{b_1,\dots,b_k\})$.

Recall that we suppose that $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N\geq 1\}$ converges to $\mathbb{Q}^\beta_*$. By (4.6.5) and (4.6.7), for every $k \geq 1$,

$$\varlimsup_{\delta\to 0} E_{\mathbb{Q}^\beta_*}\Big[\max_{1\leq i\leq k}\Big\{\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{T}} H_i(s,u)\Big\{\rho_s^\delta(u)-\rho_s^\delta(u+\varepsilon)\Big\}\, du\, ds \\ -\; 2\int_0^T\int_{\mathbb{T}} (H_i(s,u))^2\, du\, ds\Big\}\Big] \;\leq\; K_0\,,$$

where $\rho_s^\delta(u) = (\rho_s * \iota_\delta)(u)$ as defined above. Letting $\delta\downarrow 0$, performing a change of variables and then letting $\varepsilon\downarrow 0$, we obtain that

$$E_{\mathbb{Q}^\beta_*}\Big[\max_{1\leq i\leq k}\Big\{\int_0^T\int_{\mathbb{T}}(\partial_u H_i)(s,u)\rho(s,u)\; du\, ds \\ -2\int_0^T\int_{\mathbb{T}}(H_i(s,u))^2\; du\, ds\Big\}\Big] \;\leq\; K_0\,.$$

To conclude the proof it remains to apply the Monotone Convergence Theorem and recall that $\{H_i : i\geq 1\}$ is a dense sequence (with respect to the norm $\|H\|_\infty + \|\partial_u H\|_\infty$) in the subset of functions of $C^{0,1}([0,T]\times\mathbb{T})$ with support contained in $[0,T]\times(\mathbb{T}\backslash\{b_1\dots,b_k\})$. □

Remark 4.6.9. *In terms of Sobolev spaces, we have just proved that, for $\beta\in(1,\infty)$, Q^β_*-almost surely, the limit trajectory $\rho(t,u)du$ is such that $\rho(t,u)$ belongs to $L^1(0,T;\mathcal{H}^1(b_i,b_{i+1}))$, in each cylinder $[0,T]\times(b_i,b_{i+1})$. Notice that in view of the presence of slow bonds and of Lemma 4.6.5 is it not possible to obtain the same result considering the whole space $L^1(0,T;\mathcal{H}^1(\mathbb{T}))$.*

4.7 Characterization of Limit Points

We prove in this section that all limit points $\mathbb{Q}_*^\beta$ of the sequence $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$ are concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure: $\pi(t, du) = \rho(t, u)du$, whose density $\rho(t, u)$ is a weak solution of the hydrodynamic equation (4.3.4), (4.3.5) or (4.3.6), for each corresponding value of β.

Let $\mathbb{Q}_*^\beta$ be a limit point of the sequence $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$ and assume, without lost of generality, that $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$ converges to $\mathbb{Q}_*^\beta$. The existence of $\mathbb{Q}_*^\beta$ is guaranteed by Proposition 4.5.1.

Since there is at most one particle per site, it is easy to show that $\mathbb{Q}_*^\beta$ is concentrated on trajectories $\pi_t(du)$ which are absolutely continuous with respect to the Lebesgue measure, $\pi_t(du) = \rho(t, u)du$ and whose density $\rho()t, \cdot$ is non-negative and bounded by 1 (for more details see [17]). We distinguish the regime of β in different subsections below. In all the cases, we will make use of the martingale $M_t^N(H)$ defined in (4.5.1). By a simple change of variables, the integral term in (4.5.1) can be rewritten as a function of the empirical measure, such that:

$$M_t^N(H) \;=\; \langle \pi_t^N, H\rangle \;-\; \langle \pi_0^N, H\rangle \;-\; \int_0^t \langle \pi_s^N, N^2\, \mathbb{L}_N H\rangle \, ds\,, \tag{4.7.1}$$

where $\mathbb{L}_N$ was defined in (4.3.2).

We notice here that, for any choice of H, $M_t^N(H)$ is a martingale. In due course we impose extra conditions on H in order to identify the density $\rho(t, \cdot)$ as a weak solution of the corresponding weak equation depending on the regime of the parameter β.

4.7.1 Characterization of Limit Points for $\beta \in [0, 1)$

Here, we want to show that $\rho(t, \cdot)$ is a weak solution of (4.3.4). Let $H \in C^2(\mathbb{T})$. We begin by claiming that

$$\mathbb{Q}_*^\beta\Big[\, \pi_\cdot : \langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \partial_u^2 H\rangle \, ds \;=\; 0, \, \forall t \in [0, T] \Big] \;=\; 1. \tag{4.7.2}$$

In order to prove last claim, it is enough to show that, for every $\delta > 0$:

$$\mathbb{Q}_*^\beta\Big[\, \pi_\cdot : \sup_{0\leq t\leq T} \Big| \langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \partial_u^2 H\rangle \, ds \,\Big| \;>\; \delta \,\Big] = 0.$$

By Portmanteau's Theorem and Proposition 5.0.4, last probability is bounded from above by

$$\varliminf_{N\to\infty} \mathbb{Q}_{\mu_N}^{\beta,N}\Big[\, \pi_\cdot : \sup_{0\leq t\leq T} \Big| \langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \partial_u^2 H\rangle \, ds \,\Big| \;>\; \delta \,\Big]$$

since the supremum above is a continuous function in the Skorohod metric. Adding and subtracting $\langle \pi_s^N, N^2\, \mathbb{L}_N H\rangle$ in the integral term above and recalling the definition of $\mathbb{Q}_{\mu_N}^{\beta,N}$, the previous expression is bounded from above by

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi^N_t, H\rangle \,-\, \langle \pi^N_0, H\rangle \,-\, \int_0^t \langle \pi^N_s, N^2\,\mathbb{L}_N H\rangle\,ds\Big| \,>\, \delta/2\Big]$$
$$+\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \langle \pi^N_s, \partial_u^2 H - N^2\,\mathbb{L}_N H\rangle\,ds\Big| \,>\, \delta/2\Big]\,.$$

By (4.7.1) and (4.5.3), the first term in last expression is null. By the definition of Γ_N given in Section 4.5, together with the exclusion rule, the second term in last expression becomes bounded by

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\tfrac{T}{N}\sum_{x\notin\Gamma_N}\Big|\partial_u^2 H\Big(\frac{x}{N}\Big) - N^2\,\mathbb{L}_N H\Big(\frac{x}{N}\Big)\Big| \,>\, \delta/4\Big]$$
$$+\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \tfrac{1}{N}\sum_{x\in\Gamma_N}\Big\{\partial_u^2 H(\tfrac{x}{N}) - N^2\,\mathbb{L}_N H(\tfrac{x}{N})\Big\}\,\eta_s(x)\,ds\Big| \,>\, \delta/4\Big]\,.$$

Outside Γ_N, the operator $N^2\,\mathbb{L}_N$ coincides with the discrete Laplacian and since $H\in C^2(\mathbb{T})$, the first term in last expression is zero. Recall there are $2k$ elements in Γ_N. Applying the triangular inequality, the second expression in the previous sum becomes bounded by

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\tfrac{2kT}{N}\|\partial_u^2 H\|_\infty \,>\, \delta/8\Big]$$
$$+\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\sum_{x\in\Gamma_N}\int_0^t N\,\mathbb{L}_N H(\tfrac{x}{N})\,\eta_s(x)\,ds\Big| \,>\, \delta/8\Big]\,.$$

For large N, the first probability vanishes. Now we deal with the second term. We associate to each slow bond containing a point b_i, a unique pair of sites in Γ_N, namely Nb_i and Nb_i+1. By the triangular inequality, in order to show that the second expression above is zero, it is sufficient to verify that

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \{N\,\mathbb{L}_N H(\tfrac{Nb_i}{N})\,\eta_s(Nb_i)$$
$$+N\,\mathbb{L}_N H(\tfrac{Nb_i+1}{N})\,\eta_s(Nb_i+1)\}\,ds\Big| \,>\, \delta/8k\Big] \,=0,$$

for each $i=1,\dots,k$. The expression inside the integral above can be explicitly written as

$$\Big\{N\,[H(\tfrac{Nb_i-1}{N}) - H(\tfrac{Nb_i}{N})] + N^{1-\beta}\,[H(\tfrac{Nb_i+1}{N}) - H(\tfrac{Nb_i}{N})]\Big\}\,\eta_s(Nb_i)$$
$$+\Big\{N^{1-\beta}\,[H(\tfrac{Nb_i}{N}) - H(\tfrac{Nb_i+1}{N})] + N\,[H(\tfrac{Nb_i+2}{N}) - H(\tfrac{Nb_i+1}{N})]\Big\}\,\eta_s(Nb_i+1)\,.$$

Since H is smooth and $\beta\in[0,1)$, the terms inside the parenthesis involving $N^{1-\beta}$ converge to zero and the terms involving N converge to plus or minus the space derivative of H at b_i. Therefore, again by the triangular inequality, it remains to show that, for any $\delta>0$,

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \partial_u H(b_i)\Big\{\eta_s(Nb_i) \,-\, \eta_s(Nb_i+1)\Big\}\,ds\Big| \,>\, \delta\Big] \qquad (4.7.3)$$

equals to zero. The integral inside the probability above is continuous as a function of time t. Moreover, it has a Lipschitz constant bounded by $|\partial_u H(b_i)|$. If $\partial_u H(b_i) = 0$, then there is nothing to do. Otherwise, let $t_0 = 0 < t_1 < \cdots < t_n = T$ be a partition of $[0,T]$ with mesh bounded by $\delta(|2\partial_u H(b_i)|)^{-1}$. Notice the partition is fixed, depending only on the function H. By the triangular inequality, (4.7.3) is bounded by

$$\sum_{j=0}^{n} \varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\Big|\int_0^{t_j} \partial_u H(b_i)\Big\{\eta_s(Nb_i) \;-\; \eta_s(Nb_i+1)\Big\}\,ds\Big| \;>\; \delta/2\Big]\,.$$

Therefore, we just need to prove that, for any $\delta > 0$ and any $t \in [0,T]$

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\Big|\int_0^{t} \Big\{\eta_s(Nb_i) \;-\; \eta_s(Nb_i+1)\Big\}\,ds\Big| \;>\; \delta\Big] \;=0.$$

Applying Markov's inequality, we bound the previous probability by

$$\delta^{-1}\,\mathbb{E}^{\beta}_{\mu_N}\Big[\Big|\int_0^{t} \Big\{\eta_s(Nb_i) \;-\; \eta_s(Nb_i+1)\Big\}\,ds\Big|\Big]\,.$$

Now, in order to conclude it is enough to do the following. First add and subtract the empirical mean in the box of size εN around Nb_i and Nb_i+1. Then, by the triangular inequality and since $|\eta_s^{\varepsilon N}(x) - \eta_s^{\varepsilon N}(x+1)| \leq \frac{2}{\varepsilon N}$, the term involving the two empirical means vanishes. For the other two terms, we invoke Lemma 4.6.2. This finishes the claim.

Proposition 4.7.1. *For $\beta \in [0,1)$, any limit point of $\mathbb{Q}^{\beta,N}_{\mu_N}$ is concentrated in absolutely continuous paths $\pi_t(du) = \rho(t,u)\,du$, with positive density $\rho(t,\cdot)$ bounded by 1, such that $\rho(t,\cdot)$ is a weak solution of* (4.3.4).

Proof. Let $\{H_i : i \geq 1\}$ be a countable dense set of functions on $C^2(\mathbb{T})$, with respect to the norm $\|H\|_\infty + \|\partial_u^2 H\|_\infty$. Provided by (4.7.2) and intercepting a countable number of sets of probability one, is straightforward to extend (4.7.2) for all functions $H \in C^2(\mathbb{T})$ simultaneously. □

4.7.2 Characterization of Limit Points for $\beta = 1$

The idea in this case is to show that $\rho(t,\cdot)$ is an integral solution of (4.3.5) for a small domain of functions and then extend this set to $\mathcal{H}^1_W$.

Let $\mathcal{C}_W \subset \mathcal{H}^1_W$ be the set of functions H in $L^2(\mathbb{T})$ such that for $x \in \mathbb{T}$

$$H(x) \;=\; a \;+\; bW(x) \;+\; \int_{(0,x]} \Big(\int_0^y h(z)dz\Big) W(dy),$$

for some function h in $C(\mathbb{T})$ satisfying

$$\int_0^1 h(x)\,dx \;=\; 0\;, \quad \int_{(0,1]} \Big(b + \int_0^y h(z)\,dz\Big) W(dy) \;=\; 0\;.$$

Now, fix a function $H \in \mathcal{C}_W$ and define the martingale $M_t^N(H)$ as in (4.5.1). We aim that, for every $\delta > 0$, the result in (4.5.3) holds for $H \in \mathcal{C}_W$, In fact, this was already shown, for $H \in C^2(\mathbb{T})$, in the proof of Proposition 4.5.1. By (4.5.2), for $t \in [0,T]$

$$\langle M^N(H)\rangle_t \leq T \sum_{x\in\mathbb{T}_N} \xi^N_{x,x+1}\Big[H(\tfrac{x+1}{N}) - H(\tfrac{x}{N})\Big]^2.$$

Since $H \in \mathcal{C}_W$, then H is differentiable with bounded derivative, except at the points $b_1, \dots, b_k$. Therefore, for any pair $x, x+1$ such that there is no b_i between $\frac{x}{N}$ and $\frac{x+1}{N}$, the following inequality holds

$$\xi^N_{x,x+1}\Big[\,H(\tfrac{x+1}{N}) - H(\tfrac{x}{N})\,\Big]^2 \leq \frac{1}{N^2}\|\partial_u^2 H\|^2_\infty.$$

On the other hand, if there is some $\{b_i : i = 1, .., k\}$ in the interval $[\frac{x}{N}, \frac{x+1}{N})$, then $\xi^N_{x,x+1} = N^{-\beta}$ and in this case we get to:

$$\xi^N_{x,x+1}\Big[\,H(\tfrac{x+1}{N}) - H(\tfrac{x}{N})\,\Big]^2 \leq \frac{4}{N^{2\beta}}\|H\|^2_\infty\,.$$

Since there are only finite k slow bonds, we conclude that, for any $\beta > 0$ fixed, the quadratic variation of $M_t^N(H)$ vanishes as $N \to \infty$. Now, Doob's inequality is enough to conclude. As above, by a simple change of variables, we may rewrite the martingale $M_t^N(H)$ in terms of the empirical measure as in (4.7.1). Now we want to analyze the integral term in the martingale decomposition (4.7.1).

Lemma 4.7.2. *For any $H \in \mathcal{C}_W$,*

$$\lim_{N\to\infty} \tfrac{1}{N} \sum_{x\in\mathbb{T}_N} \Big|\, N^2\mathbb{L}_N H(\tfrac{x}{N}) - \tfrac{d}{dx}\tfrac{d}{dW} H(\tfrac{x}{N}) \,\Big| \;=\; 0\,.$$

Proof. Recall the definition of the set Γ_N given in section 4.5 and rewrite the previous sum as

$$\tfrac{1}{N} \sum_{x\notin\Gamma_N} \Big|\, N^2\mathbb{L}_N H(\tfrac{x}{N}) - \tfrac{d}{dx}\tfrac{d}{dW} H(\tfrac{x}{N})\Big| + \tfrac{1}{N} \sum_{x\in\Gamma_N} \Big|\, N^2\mathbb{L}_N H(\tfrac{x}{N}) - \tfrac{d}{dx}\tfrac{d}{dW} H(\tfrac{x}{N})\Big|. \tag{4.7.4}$$

Outside $b_1, \dots, b_k$, the operator $\frac{d}{dx}\frac{d}{dW}$ coincides with the Laplacian, and outside Γ_N, the discrete operator $N^2\,\mathbb{L}_N$ coincides with the discrete Laplacian. Hence, the first term above is equal to

$$\tfrac{1}{N} \sum_{x\notin\Gamma_N} \Big|\, N^2\Big(H(\tfrac{x+1}{N}) + H(\tfrac{x-1}{N}) - 2H(\tfrac{x}{N})\Big) - \partial_u^2 H(\tfrac{x}{N})\,\Big|\,.$$

It is easy to verify that $H \in C^2(\mathbb{T}\backslash\{b_1, \dots, b_k\})$ and has bounded derivatives. Thus, by a Taylor expansion on H, it follows that the previous sum converges to zero as $N \to +\infty$. On the other hand, the second sum in (4.7.4) is bounded by the sum of

$$\tfrac{1}{N} \sum_{x\in\Gamma_N} \Big|\, \tfrac{d}{dx}\tfrac{d}{dW} H(\tfrac{x}{N})\,\Big|$$

and

$$\sum_{x\in\Gamma_N}\Big|\,N\xi^N_{x,x+1}\Big[H(\tfrac{x+1}{N})-H(\tfrac{x}{N})\Big]+N\xi^N_{x-1,x}\Big[H(\tfrac{x-1}{N})-H(\tfrac{x}{N})\Big]\,\Big|\,.$$

Since $H\in C_W$, $\frac{d}{dx}\frac{d}{W}H$ is a continuous function, therefore bounded. Since Γ_N has k elements, the first sum above converges to zero as $N\to+\infty$. It remains to analyze the second sum above, where now the definition of the domain C_W is crucial. For each $x\in\Gamma_N$, one of the conductances above is equal to N^{-1}. Let us suppose that $\xi^N_{x,x+1}=N^{-1}$ and $\xi^N_{x-1,x}=1$, the other case being completely analogous. In this case, there exists some $b_i\in(\frac{x}{N},\frac{x+1}{N}]$. From the definition of C_W and the measure W, the function H has a discontinuity at b_i of size

$$\int_0^{b_i}h(dz)\,dz\,.$$

Besides that, the function H has also sided-derivatives at b_i of the same value. With this in mind, is easy to see that

$$[H(\tfrac{x+1}{N})-H(\tfrac{x}{N})]+N[H(\tfrac{x-1}{N})-H(\tfrac{x}{N})]$$

converges to zero as $N\to\infty$. Recalling there are finite $2k$ elements in Γ_N, we finish the proof of the Lemma. □

Now, fix $H\in C_W$ and take a continuous function H^ε which coincides with H in $\mathbb{T}\setminus\cup_{i=1}^k(b_i-\varepsilon,b_i+\varepsilon)$ and that $\|H^\varepsilon\|_\infty\le\|H\|_\infty$. The choice of ε will be determined later. Notice that

$$\sup_{0\le t\le T}|\langle\pi_t,H^\varepsilon-H\rangle|\le\sup_{0\le t\le T}\int_{(a-\varepsilon,a+\varepsilon)}\rho(t,u)\,|H^\varepsilon(u)-H(u)|\,du\le 4\,k\,\varepsilon\,\|H\|_\infty\,.$$

For every $\delta>0$,

$$\mathbb{Q}^\beta_*\Big[\pi.:\sup_{0\le t\le T}\Big|\langle\pi_t,H\rangle\;-\;\langle\pi_0,H\rangle\;-\;\int_0^t\langle\pi_s,\tfrac{d}{dx}\tfrac{d}{dW}H\rangle\,ds\Big|\;>\;\delta\Big]\tag{4.7.5}$$

$$\le\mathbb{Q}^\beta_*\Big[\pi.:\sup_{0\le t\le T}\Big|\langle\pi_t,H^\varepsilon\rangle\;-\;\langle\pi_0,H^\varepsilon\rangle\;-\;\int_0^t\langle\pi_s,\tfrac{d}{dx}\tfrac{d}{dW}H\rangle\,ds\Big|\;>\;\delta/3\Big]$$
$$+2\,\mathbb{Q}^\beta_*\Big[\pi.:\sup_{0\le t\le T}\Big|\langle\pi_t,H^\varepsilon-H\rangle\Big|\;>\;\delta/3\Big]\,.$$

By a suitable choice of ε, the second probability in the sum above is null. Since H^ε and $\frac{d}{dx}\frac{d}{dW}H$ are continuous, by the Portmanteau's Theorem and Proposition 5.0.4, it holds that

$$\mathbb{Q}^\beta_*\Big[\pi:\sup_{0\le t\le T}\Big|\langle\pi_t,H^\varepsilon\rangle\;-\;\langle\pi_0,H^\varepsilon\rangle\;-\;\int_0^t\langle\pi_s,\tfrac{d}{dx}\tfrac{d}{dW}H\rangle\,ds\Big|\;>\;\delta/3\Big]$$
$$\le\;\varliminf_{N\to\infty}\;\mathbb{Q}^{\beta,N}_{\mu_N}\Big[\pi:\sup_{0\le t\le T}\Big|\langle\pi_t,H^\varepsilon\rangle\;-\;\langle\pi_0,H^\varepsilon\rangle\;-\;\int_0^t\langle\pi_s,\tfrac{d}{dx}\tfrac{d}{dW}H\rangle\,ds\Big|\;>\;\delta/3\Big]$$
$$=\;\varliminf_{N\to\infty}\;\mathbb{P}^\beta_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle\pi^N_t,H^\varepsilon\rangle\;-\;\langle\pi^N_0,H^\varepsilon\rangle\;-\;\int_0^t\langle\pi^N_s,\tfrac{d}{dx}\tfrac{d}{dW}H\rangle\,ds\Big|\;>\;\delta/3\Big]\,.$$

Notice the last equality is just the definition of the measure $\mathbb{Q}^{\beta,N}_{\mu_N}$. By the exclusion rule, it holds that $\sup_{0\le t\le T}\big|\langle \pi^N_t, H^\varepsilon - H\rangle\big| \le 4\,k\,\varepsilon\|H\|_\infty$, since H^ε coincides with H in $\mathbb{T}\backslash\cup_{i=1}^k(b_i-\varepsilon, b_i+\varepsilon)$. Adding and subtracting $\langle \pi^N_s, N^2\,\mathbb{L}_N H\rangle$, $\langle\pi^N_t, H\rangle$ and $\langle \pi^N_0, H\rangle$, we obtain that

$$\begin{aligned}
&\varlimsup_{N\to\infty}\ \mathbb{P}^\beta_{\mu_N}\Big[\sup_{0\le t\le T}|\langle\pi^N_t, H^\varepsilon\rangle \;-\; \langle\pi^N_0, H^\varepsilon\rangle \;-\; \int_0^t \langle\pi^N_s, \tfrac{d}{dx}\tfrac{d}{dW}H\rangle\,ds| \;>\; \delta/3\Big]\\
&\le \varlimsup_{N\to\infty}\ \mathbb{P}^\beta_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle\pi^N_t, H\rangle \;-\; \langle\pi^N_0, H\rangle \;-\; \int_0^t \langle\pi^N_s, N^2\,\mathbb{L}_N H\rangle\,ds\Big| \;>\; \delta/12\Big]\\
&+ \varlimsup_{N\to\infty}\ \mathbb{P}^\beta_{\mu_N}\Big[\tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\Big|N^2\mathbb{L}_N H(\tfrac{x}{N}) - \tfrac{d}{dx}\tfrac{d}{dW}H(\tfrac{x}{N})\Big| \;>\; \delta/12\Big]\\
&+2\,\varlimsup_{N\to\infty}\ \mathbb{P}^\beta_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle\pi^N_t, H^\varepsilon - H\rangle\Big| \;>\; \delta/12\Big]\,.
\end{aligned}$$

With another suitable choice of ε, the third probability in the sum above is null. Lemma 4.7.2 implies that the second probability above is zero for N sufficiently large. Recall we proved that (4.5.3) holds for $H\in\mathcal{C}_W$, so that the first term in the sum above is zero. Finally, from the previous computations we conclude that (4.7.5) is zero for any $\delta>0$. Therefore, $\mathbb{Q}^\beta_*$ is concentrated on absolutely continuous paths $\pi_t(du)=\rho(t,u)\,du$ with positive density bounded by 1 and for any fixed $H\in\mathcal{C}_W$, $\mathbb{Q}^\beta_*$ a.s.

$$\langle\rho_t, H\rangle - \langle\rho_0, H\rangle \;=\; \int_0^t \Big\langle \rho_s\,,\, \tfrac{d}{dx}\tfrac{d}{dW}H\Big\rangle\,ds\,,\qquad \text{for all } t\in[0,T]\,. \tag{4.7.6}$$

Proposition 4.7.3. *For $\beta=1$, any limit point of $\mathbb{Q}^{\beta,N}_{\mu_N}$ is concentrated in absolutely continuous paths $\pi_t(du)=\rho(t,u)\,du$, with positive density $\rho(t,\cdot)$ bounded by* 1*, such that $\rho(t,\cdot)$ is a weak solution of* (4.3.5).

Proof. By a density argument, (4.7.6) also holds, Q^β_* a.s., for all $H\in C_W$ simultaneously. It remains to extend (4.7.6) for $H\in\mathcal{H}^1_W$. For that purpose fix $H\in\mathcal{H}^1_W$. Thus, for $x\in\mathbb{T}$

$$H(x) = \alpha + \int_{(0,x]}\Big(\beta + \int_0^y h(z)\,dz\Big)\,W(dy)\,,$$

with $\alpha,\beta\in\mathbb{R}$, $h\in L^2(\mathbb{T})$ satisfying (4.3.3). Let $h_n\in C(\mathbb{T})$ converging to $h\in L^2(\mathbb{T})$. Define

$$H_n(x) = \alpha_n + \int_{(0,x]}\Big(\beta_n + \int_0^y h_n(z)\,dz\Big)\,W(dy)\,,$$

where $\alpha_n\to\alpha$ and $\beta_n\to\beta$. By the Dominated Convergence Theorem, it follows that that H_n converges uniformly to H. Therefore (4.7.6) is true for all $H\in\mathcal{H}^1_W$. □

4.7.3 Characterization of Limit Points for $\beta\in(1,\infty)$

In this regime of the parameter β, Proposition 4.6.6 says that Q^β_* is concentrated on trajectories absolutely continuous with respect to the Lebesgue measure $\pi_t(du)=\rho(t,u)\,du$ such that, for each interval (b_i, b_{i+1}), $\rho(t,\cdot)$ belongs to $L^1(0,T;\mathcal{H}^1(b_i,b_{i+1}))$. It is well known that the Sobolev space $\mathcal{H}^1(a,b)$ has special properties: all its elements are absolutely continuous functions with bounded variation, c.f. [4] and [19], therefore with lateral limits well-defined. Such property is inherited by $L^1(0,T;\mathcal{H}^1(b_i,b_{i+1}))$ in the sense that we can integrate in time the lateral limits. Therefore, $Q^\beta_* a.s.$, for each $i=1,\dots,k$ and for any $t\in[0,T]$:

$$\int_0^t \rho(s,b_i^+)\,ds<\infty \quad \text{and} \quad \int_0^t \rho(s,b_{i+1}^-)\,ds\,<\infty.$$

To simplify notation, in this subsection we denote $a=b_i$ and $b=b_{i+1}$. Fix $h\in C^2(\mathbb{T})$ and define $H:[0,T]\times\mathbb{T}\to\mathbb{R}$ by $H(t,u) \;=\; h(t,u)\,\mathbf{1}_{[a,b]}(u)$.

Recall that $\pi_t(du)=\rho(t,u)du$. We begin by claiming that

$$\begin{aligned}\mathbb{Q}^\beta_*\Big[\pi_\cdot:\ &\langle\rho_t,H_t\rangle\;-\;\langle\rho_0,H_0\rangle\;-\;\int_0^t\langle\rho_s,\partial_u^2H_s+\partial_sH_s\rangle\,ds\\ &-\int_0^t\partial_uH(s,a^+)\,\rho(s,a^+)\,ds+\int_0^t\partial_uH(s,b^-)\,\rho(s,b^-)\,ds=0,\forall t\in[0,T]\Big]\;=\;1\,.\end{aligned}\tag{4.7.7}$$

In order to prove (4.7.7), its enough to show that, for every $\delta>0$

$$\begin{aligned}\mathbb{Q}^\beta_*\Big[\pi:\ \sup_{0\le t\le T}\Big|&\langle\rho_t,H_t\rangle\;-\;\langle\rho_0,H_0\rangle\;-\;\int_0^t\langle\rho_s,\partial_u^2H_s+\partial_sH_s\rangle\,ds\\ &-\int_0^t\partial_uH(s,a^+)\,\rho(s,a^+)\,ds+\int_0^t\partial_uH(s,b^-)\,\rho(s,b^-)\,ds\Big|\;>\;\delta\Big]\;=0\,.\end{aligned}$$

Since the boundary integrals are not well-defined in the whole Skorohod space $D([0,T],\mathcal{M})$, we cannot use directly Portmanteau's Theorem. To avoid this technical obstacle, fix $\varepsilon>0$, which will be taken small later. Adding and subtracting the convolution of $\rho(t,u)$ with ι_ε, the probability above is less than or equal to the sum of

$$\begin{aligned}\mathbb{Q}^\beta_*\Big[\pi_\cdot:\ \sup_{0\le t\le T}\Big|&\langle\rho_t,H_t\rangle\;-\;\langle\rho_0,H_0\rangle\;-\;\int_0^t\langle\rho_s,\partial_u^2H_s+\partial_sH_s\rangle\,ds\\ &-\int_0^t\partial_uH(s,a^+)\,(\rho_s*\iota_\varepsilon)(a)\,ds+\int_0^t\partial_uH(s,b^-)\,(\rho_s*\iota_\varepsilon)(b-\varepsilon)\,ds\Big|>\delta/2\Big]\end{aligned}\tag{4.7.8}$$

and

$$\begin{aligned}\mathbb{Q}^\beta_*\Big[\pi:\ \sup_{0\le t\le T}\Big|&\int_0^t\partial_uH(s,a^+)\,(\rho_s*\iota_\varepsilon)(a)\,ds-\int_0^t\partial_uH(s,b^-)\,(\rho_s*\iota_\varepsilon)(b-\varepsilon)\,ds\\ &-\int_0^t\partial_uH(s,a^+)\,\rho(s,a^+)\,ds+\int_0^t\partial_uH(s,b^-)\,\rho(s,b^-)\,ds\Big|\;>\;\delta/2\Big]\,.\end{aligned}$$

where ι_ε and the convolution $\rho*\iota_\varepsilon$ were defined in (4.6.6). The convolutions above are suitable averages of ρ around the boundary points a and b. Therefore, as $\varepsilon\downarrow0$, the set inside the previous probability decreases to a set of null probability. It remains to deal with (4.7.8).

By Portmanteau's Theorem, Proposition 5.0.4 and the exclusion rule, (4.7.8) is bounded from above by

$$\varlimsup_{N\to\infty} \; \mathbb{Q}^{\beta,N}_{\mu_N}\Big[\pi_\cdot : \sup_{0\le t\le T}\Big|\langle \pi_t, H\rangle \;-\; \langle \pi_0, H_0\rangle \;-\; \int_0^t \langle \pi_s, \partial_u^2 H_s + \partial_s H_s\rangle\, ds$$
$$-\int_0^t \partial_u H(s,a^+)\,(\pi_s * \iota_\varepsilon)(a)\,ds + \int_0^t \partial_u H(s,b^-)\,(\pi_s * \iota_\varepsilon)(b-\varepsilon)\,ds\Big| \;>\; \delta/2\Big]\,.$$

Now, by the definition of $\mathbb{Q}^{\beta,N}_{\mu_N}$, we can rewrite the previous expression as

$$\varlimsup_{N\to\infty} \; \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi^N_t, H_t\rangle \;-\; \langle \pi^N_0, H_0\rangle \;-\; \int_0^t \langle \pi^N_s, \partial_u^2 H_s + \partial_s H_s\rangle\, ds$$
$$-\int_0^t \partial_u H(s,a^+)\,\eta^{\varepsilon N}_s(Na+1)\,ds + \int_0^t \partial_u H(s,b^-)\,\eta^{\varepsilon N}_s(Nb)\,ds\Big| \;>\; \delta/2\Big]\,.$$

If we consider the discrete torus as embedded in the continuous torus, $Na+1$ is the closest site to the right of a and Nb is the closest site to the left of b. The next step is to add and subtract $\langle \pi^N_s, N^2\,\mathbb{L}_N H\rangle$ and the previous probability becomes now bounded from above by the sum of

$$\overline{\lim_{N\to\infty}} \; \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi^N_t, H_t\rangle \;-\; \langle \pi^N_0, H_0\rangle - \int_0^t \langle \pi^N_s, N^2\,\mathbb{L}_N H_s + \partial_s H_s\rangle\, ds\,\Big| \;>\; \delta/4\Big]$$

and

$$\overline{\lim_{N\to\infty}} \; \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \langle \pi^N_s, N^2\,\mathbb{L}_N H_s\rangle\, ds \;-\; \int_0^t \langle \pi^N_s, \partial_u^2 H_s\rangle\, ds$$
$$-\int_0^t \partial_u H(s,a^+)\,\eta^{\varepsilon N}_s(Na+1)\,ds + \int_0^t \partial_u H(s,b^-)\,\eta^{\varepsilon N}_s(Nb)\,ds\Big| \;>\; \delta/4\Big]\,.$$

Repeating similar computations as performed in section 4.5 we can show (4.5.3) for a test function H that depends also on time. Therefore the first probability above is null. Now we focus on showing that the second probability above is null. Recalling the definition of $H(s,\cdot)$ above, we have that $H(s,\cdot)$ is zero outside the interval $[a,b]$. Besides that, for the set of vertices $\{Na+2,\dots,Nb-1\}$, the discrete operator $N^2\,\mathbb{L}_N$ coincides with the discrete Laplacian, which applied to $H(s,\cdot)$ converges uniformly to the continuous Laplacian of $H(s,\cdot)$. Hence, by the triangular inequality, it is enough to show that, for any $\delta>0$:

$$\begin{aligned}\varlimsup_{N\to\infty}\ \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\tfrac{1}{N}\int_0^t \{N^2\,\mathbb{L}_N H_s(\tfrac{Na}{N})-\partial_u^2 H_s(\tfrac{Na}{N})\}\,\eta_s(Na)\,ds\\ +\tfrac{1}{N}\int_0^t \{N^2\,\mathbb{L}_N H_s(\tfrac{Na+1}{N})-\partial_u^2 H_s(\tfrac{Na+1}{N})\}\,\eta_s(Na+1)\,ds\\ +\tfrac{1}{N}\int_0^t \{N^2\,\mathbb{L}_N H_s(\tfrac{Nb}{N})-\partial_u^2 H_s(\tfrac{Nb}{N})\}\,\eta_s(Nb)\,ds\\ +\tfrac{1}{N}\int_0^t \{N^2\,\mathbb{L}_N H_s(\tfrac{Nb+1}{N})-\partial_u^2 H_s(\tfrac{Nb+1}{N})\}\,\eta_s(Nb+1)\,ds\\ -\int_0^t \partial_u H(s,a^+)\,\eta_s^{\varepsilon N}(Na+1)\,ds+\int_0^t \partial_u H(s,b^-)\,\eta_s^{\varepsilon N}(Nb)\,ds\Big|\ >\delta\Big]=0.\end{aligned}$$

Since $h\in C^2(\mathbb{T})$, the term involving the Laplacian above is bounded. Now, by the triangular inequality, it is sufficient to show that, for any $\delta>0$:

$$\begin{aligned}\varlimsup_{N\to\infty}\mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t N\mathbb{L}_N H_s(\tfrac{Na}{N})\,\eta_s(Na)ds+\int_0^t N\mathbb{L}_N H_s(\tfrac{Na+1}{N})\,\eta_s(Na+1)ds\\ +\int_0^t N\,\mathbb{L}_N H_s(\tfrac{Nb}{N})\,\eta_s(Nb)\,ds+\int_0^t N\,\mathbb{L}_N H_s(\tfrac{Nb+1}{N})\,\eta_s(Nb+1)\,ds\\ -\int_0^t \partial_u H(s,a^+)\,\eta_s^{\varepsilon N}(Na+1)\,ds+\int_0^t \partial_u H(s,b^-)\,\eta_s^{\varepsilon N}(Nb)\,ds\ \Big|\ >\ \delta\Big]\ =0.\end{aligned}$$

For each one of the four vertices appearing inside the previous probability, the operator $\mathbb{L}_N$ has two conductances, one equals to $N^{-\beta}$ and the other equals to 1. Since $\beta>1$, the terms involving $N^{-\beta}$ converge to zero. The terms involving the conductances equal to 1, converge to plus or minus the lateral space derivatives of H. Recall from definition of H that $\partial_u H(s,a^-)=\partial_u H(s,b^+)=0$ for all $0\le s\le t$. From this, it remains to show that for any $\delta>0$

$$\begin{aligned}\varlimsup_{N\to\infty}\ \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \partial_u H(s,a^+)\,\eta_s(Na+1)\,ds-\int_0^t \partial_u H(s,b^-)\,\eta_s(Nb)\,ds\\ -\int_0^t \partial_u H(s,a^+)\,\eta_s^{\varepsilon N}(Na+1)\,ds+\int_0^t \partial_u H(s,b^-)\,\eta_s^{\varepsilon N}(Nb)\,ds\ \Big|\ >\ \delta\Big],\end{aligned}$$

is null. Last expression is bounded from above by

$$\begin{aligned}&\varlimsup_{N\to\infty}\ \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \partial_u H(s,a^+)\Big\{\eta_s(Na+1)-\eta_s^{\varepsilon N}(Na+1)\Big\}\,ds\Big|\ >\ \delta/2\Big]\\ +&\varlimsup_{N\to\infty}\ \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \partial_u H(s,b^-)\Big\{\eta_s(Nb)-\eta_s^{\varepsilon N}(Nb)\Big\}\,ds\Big|\ >\ \delta/2\Big].\end{aligned}$$

The integral inside the probability above is a continuous function of the time t. Moreover, it has a bounded Lipschitz constant. The same argument as the one used in (4.7.3) together with Lemma 4.6.4 imply that the previous expression converges to zero when $\varepsilon \downarrow 0$, which proves (4.7.7).

Proposition 4.7.4. *For $\beta \in (1,\infty)$, any limit point of $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N \geq 1\}$ is concentrated in absolutely continuous paths $\pi_t(du) = \rho(t,u)\,du$, with positive density $\rho(t,\cdot)$ bounded by 1, such that $\rho(t,\cdot)$ is a weak solution of (4.3.6) in each cylinder $[0,T]\times[b_i,b_{i+1}]$.*

Proof. Given (4.7.7), it remains to extend the result for all functions H and all cylinders $[0,T]\times[b_i,b_{i+1}]$ simultaneously. Intercepting a countable number of sets of probability one and applying a density argument as in Proposition 4.7.1, the statement follows. $\square$

4.8 Uniqueness of Weak Solutions

The uniqueness of weak solutions of (4.3.4) is standard and we refer to [17] for a proof. It remains to prove uniqueness of weak solutions of the parabolic differential equations (4.3.5) and (4.3.6). In both cases, by linearity it suffices to check the uniqueness for $\gamma(\cdot) \equiv 0$. Notice that existence of weak solutions of (4.3.4), (4.3.5) and (4.3.6) is guaranteed by tightness of the process as proved in section 4.5, together with the characterization of limit points as proved in section 4.7.

4.8.1 Uniqueness of weak solutions of (4.3.5)

Let $\rho : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$ be a weak solution of (4.3.5) with $\gamma \equiv 0$. By Definition 4.3.3, for all $H \in \mathcal{H}^1_W$ and all $t > 0$

$$\langle \rho_t, H\rangle \;=\; \int_0^t \Big\langle \rho_s, \frac{d}{dx}\frac{d}{dW}H\Big\rangle\, ds\;. \tag{4.8.1}$$

From Theorem 1 of [9], the operator $-\frac{d}{dx}\frac{d}{dW}$ has a countable number eigenvalues $\{\lambda_n : n \geq 0\}$ and eigenvectors $\{F_n : n \geq 0\}$. All eigenvalues have finite multiplicity, $0 = \lambda_0 \leq \lambda_1 \leq \cdots$ and $\lim_{n\to\infty}\lambda_n = \infty$. Moreover, the eigenvectors $\{F_n : n \geq 0\}$ form a complete orthonormal system in $L^2(\mathbb{T})$. For $t > 0$, define

$$R(t) = \sum_{n\in\mathbb{N}} \frac{1}{n^2(1+\lambda_n)}\langle \rho_t, F_n\rangle^2.$$

Notice that $R(0) = 0$ and since ρ_t belongs to $L^2(\mathbb{T})$, $R(t)$ is well defined for all $t \geq 0$. By (4.8.1), it follows that $\frac{d}{dt}\langle \rho_t, F_n\rangle^2 = -2\lambda_n\langle \rho_t, F_n\rangle^2$. Thus

$$(\tfrac{d}{dt}R)(t) = -\sum_{n\in\mathbb{N}} \frac{2\lambda_n}{n^2(1+\lambda_n)}\langle \rho_t, F_n\rangle^2\,,$$

because $\sum_{n\leq N}\frac{-2\lambda_n}{n^2(1+\lambda_n)}\langle \rho_t, F_n\rangle^2$ converges uniformly to $\sum_{n\in\mathbb{N}}\frac{-2\lambda_n}{n^2(1+\lambda_n)}\langle \rho_t, F_n\rangle^2$, as N increases to infinity. Therefore $R(t) \geq 0$ and $(\frac{d}{dt}R)(t) \leq 0$, for all $t > 0$ and since $R(0) = 0$, it follows that $R(t) = 0$ for all $t > 0$. As a consequence of $\{F_n : n \geq 0\}$ being a complete orthonormal system, it follows that $\langle \rho_t, \rho_t\rangle = 0$, which is enough to conclude.

4.8.2 Uniqueness of weak solutions of (4.3.6)

At first, we begin with an auxiliary Lemma on integration by parts.

Lemma 4.8.1. *Let $\rho(t,\cdot)$ be a function in the Sobolev space $L^1(0,T;\mathcal{H}^1(a,b))$. Then, for any $H \in C^{0,1}([0,T]\times[a,b])$:*

$$\int_0^T \int_a^b \rho(s,u)\,\partial_u H(s,u)\,du\,ds$$
$$= -\int_0^T \int_a^b \partial_u \rho(s,u)\,H(u,s)\,du\,ds + \int_0^T \Big\{\rho(s,b)\,H(s,b) - \rho(s,a)\,H(s,a)\Big\}\,ds\,.$$

Notice the partial derivative in ρ is the weak derivative, while the partial derivative in H is the usual one. Besides that, the function H is smooth, but possibly not null at the boundary $[0,T]\times\{a,b\}$, and therefore is not valid the integration by parts in the sense of $L^1(0,T;\mathcal{H}^1(a,b))$, which has no boundary integrals.

Proof. Fix $\varepsilon > 0$ and write $H = H^\varepsilon + (H - H^\varepsilon)$, where H^ε coincides with H in the region $[0,T]\times(a+\varepsilon, b-\varepsilon)$, has compact support contained in $[0,T]\times(a,b)$ and belongs to $C^{0,1}([0,T]\times(a,b))$. By the assumptions on H^ε, we have that

$$\int_0^T \int_a^b \rho(s,u)\,\partial_u H(s,u)\,du\,ds$$
$$= -\int_0^T \int_a^b \partial_u \rho(s,u)\,H^\varepsilon(s,u)\,du\,ds + \int_0^T \int_a^b \rho(s,u)\partial_u(H - H^\varepsilon)(s,u)\,du\,ds\,.$$

Last result is a consequence of H^ε having compact support strictly contained in the open set (a,b). Let $f_\varepsilon : [a,b] \to \mathbb{R}$ be the function such that $f(u) = 1$ if $u \in (a+\varepsilon, b-\varepsilon)$, $f(a) = f(b) = 0$, and interpolated linearly otherwise. The decomposition $H = H\,f^\varepsilon + H(1-f^\varepsilon)$ can be done, but now the function $H\,f^\varepsilon$ does not have the properties as required above for H^ε. Nevertheless, taking a suitable approximating sequence of functions H^ε, it follows that

$$\int_0^T \int_a^b \rho(s,u)\,\partial_u H(s,u)\,du\,ds$$
$$= -\int_0^T \int_a^b \Big\{\partial_u \rho(s,u) H(s,u) f^\varepsilon(u) + \rho(s,u)\partial_u\Big(H(s,u)(1 - f^\varepsilon(u))\Big)\Big\}\,du\,ds.$$

Taking the limit as $\varepsilon \downarrow 0$ yields the statement of the Lemma. □

Let $\rho(t,\cdot)$ be a weak solution of (4.3.6) with $\gamma \equiv 0$. Provided by Lemma 4.8.1, for any function $H \in C^{1,2}([0,T]\times(b_i, b_{i+1}))$,

$$\int_{b_i}^{b_{i+1}} \rho_t(u) H(t,u)\,du + \int_0^t \int_{b_i}^{b_{i+1}} \Big\{\partial_u \rho_s(u)\partial_u H(s,u) - \rho_s(u)\partial_s H(s,u)\Big\} du\,ds = 0.$$

From this point, uniqueness is a particular case of a general result in [18], namely Theorem III.4.1. In sake of completeness, we sketch an adaptation of it to our particular case. Denote by $W^1_{2,T} = W^1_{2,T}([0,T]\times(a,b))$ the space of functions with one weak derivative in space and time, both belonging to $L^2([0,T]\times(a,b))$ and vanishing at time T. By extending the previous equality to $H\in W^1_{2,T}$ it follows that

$$\int_0^T\int_{b_i}^{b_{i+1}}\Big\{\partial_u\rho_s(u)\,\partial_u H(s,u)-\rho_s(u)\,\partial_s H(s,u)\Big\}\,du\,ds \;=\; 0\,. \tag{4.8.2}$$

It is not difficult to show that the function

$$H(s,u) \;=\; -\int_s^T \rho(r,u)\,dr$$

belongs to $W^1_{2,T}$. Replacing last function in (4.8.2), then we can rewrite (4.8.2) as

$$\int_0^T\int_{b_i}^{b_{i+1}}\Big\{\frac{1}{2}\partial_s(\partial_u H(s,u))^2-(\partial_s H(s,u))^2\Big\}\,du\,ds \;=\; 0\,.$$

By Fubini's Theorem we get to

$$\frac{1}{2}\int_{b_i}^{b_{i+1}}\Big\{(\partial_u H(T,u))^2\;-(\partial_u H(0,u))^2\Big\}\,du-\int_0^T\int_{b_i}^{b_{i+1}}(\partial_s H(s,u))^2\,du\,ds \;=\; 0\,.$$

By the definition of H, its weak space derivative vanishes at time T, so that the first integral above is null. Therefore, $\partial_s H$ is identically null, and by the definition of H above, this implies that ρ vanishes, finishing the proof.

Chapter 5

Appendix

Proposition 5.0.2. *Denote by $H_N(\mu_N|\nu_\alpha)$ the entropy of a probability measure μ_N with respect to a stationary state ν_α. Then, there exists a finite constant $K_0 := K_0(\alpha)$ such that $H_N(\mu_N|\nu_\alpha) \;\leq\; K_0 N$, for all probability measures μ_N.*

Proof. Recall that ν_α is Bernoulli product of parameter α. By the explicit formula given in Theorem A1.8.3 of [17],

$$\begin{aligned}
H_N(\mu_N|\nu_\alpha) &= \sum_{\eta\in\{0,1\}^{\mathbb{T}_N}} \mu_N(\eta)\,\log\frac{\mu_N(\eta)}{\nu_\alpha(\eta)}\\
&\leq \sum_{\eta\in\{0,1\}^{\mathbb{T}_N}} \mu_N(\eta)\,\log\frac{1}{\nu_\alpha(d\eta)}\\
&\leq \sum_{\eta\in\{0,1\}^{\mathbb{T}_N}} \mu_N(\eta)\,\log\frac{1}{[\alpha\wedge(1-\alpha)]^N}\\
&= N\left(-\log[\alpha\wedge(1-\alpha)]\right).
\end{aligned}$$

□

Proposition 5.0.3. *Assume that L is a reversible generator with respect to an invariant measure ν in a countable space-state E, and $V : \mathbb{R}_+\times E \to \mathbb{R}$ is a bounded function. Notice that $L + V_t$ will be a symmetric operator in $L^2(\nu)$. Denote by Γ_t the largest eigenvalue of $L + V_t$:*

$$\Gamma_t = \sup_{\langle f,f\rangle_\nu=1}\Big\{\langle V_t, f^2\rangle_\nu + \langle Lf, f\rangle_\nu\Big\}.$$

Then, the supremum above can be taken over only positive functions f, or else,

$$\Gamma_t = \sup_{f \text{ density}}\Big\{\langle V_t, (\sqrt{f})^2\rangle_\nu + \langle L\sqrt{f}, \sqrt{f}\rangle_\nu\Big\}.$$

Proof. It follows from the expression of the Dirichlet form (see [17]),

$$\langle Lf, f\rangle_\nu = -\tfrac{1}{2}\sum_{x,y\in E} \nu(x)\, L(x,y)[f(y)-f(x)]^2\,,$$

and the inequality $||f(y)| - |f(x)|| \le |f(y) - f(x)|$. □

Proposition 5.0.4. *If G_1, G_2, G_3 are continuous functions defined in the torus $\mathbb{T}$, the application from $D([0,T],\mathcal{M})$ to $\mathbb{R}$ that associates to a trajectory $\{\pi_t : 0 \le t \le T\}$ the number*

$$\sup_{0 \le t \le T} \Big| \langle \pi_t, G_1 \rangle \; - \; \langle \pi_0, G_2 \rangle \; - \; \int_0^t \langle \pi_s, G_3 \rangle \, ds \Big|$$

is continuous for the Skorohod metric in $D([0,T],\mathcal{M})$.

Proof. If G is a continuous function in the torus, the application $\pi \mapsto \langle \pi, G \rangle$ is a continuous application from $\mathcal{M}$ to $\mathbb{R}$ in the weak topology. From this observation and the definition of the Skorohod metric as an infimum under reparametrizations (c.f. [17]), the statement follows. □

Chapter 6

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