Instituto Nacional de Matemática Pura e Aplicada

Scaling limits: d-dimensional models with conductances, velocity, reservatories and random environment

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Tese apresentada para obtenção do título de Doutor em Ciências

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Rio de Janeiro
Junho, 2010

Resumo

Nesta tese consideramos três modelos de processo de exclusão em dimensão $d \geq 1$: Processo de Exclusão com Condutâncias, com Condutâncias em Ambiente Aleatório e com Bordos e Velocidades. Para o primeiro, obtemos o limite hidrodinâmico, no segundo obtemos limite hidrodinâmico e as flutuações no equilíbrio, e no último provamos o princípio dos grandes desvios.

Keywords: Exclusion Processes, Boundary Driven Exclusion Processes, Hydrodynamic Limit, Equilibrium Fluctuations, Large Deviations, Conductances, Random Environment, Homogenization.

Aos amores de minha vida:
minha mãe, minha esposa Tatiana,
a meus filhos Arthur (Tuco) e Estela (Teia)
e em memória do meu pai, meu grande herói.

Agradecimentos

Agradeço a Deus por ter me abençoado com a intensidade de virtudes superior aos meus defeitos de modo que consegui finalizar com sucesso o programa de Doutorado. Isto também, somente foi possível, pela força, amizade, companheirismo e incentivo que inúmeras pessoas sempre depositaram em mim durante toda a minha formação acadêmica. Em especial, destaco a minha amada e companheira esposa Tatiana, minha querida mãe Maria da Penha e ao estimado amigo professor Carlos Roberto Alves dos Santos, meu sincero muito obrigado.

Agradeço a meu orientador de Mestrado e Doutorado, professor Claudio Landim, pelas oportunidades, os ensinamentos, paciência, pela segurança e competência com que conduziu toda minha orientação. Seguramente, tenho nele o exemplo de conduta a seguir em minha vida profissional.

Tive a felicidade de usufruir de um programa de doutorado sanduiche no Courant - NYU, de forma que tenho a agradecer a excelente recepção e suporte com que professores, alunos e funcionários deste renomado instituto me acolheram. Em especial, sou muito grato ao meu orientador no exterior, professor S.R.S. Varadhan, por ter me propiciado inesquecíveis e valiosos momentos de ensinamento de conduta como matemático e pessoa e ao amigo Antonio Carlos Auffinger (Tuca) pelo apoio e companheirismo desde os tempos de graduação.

Agradeço aos parceiros de pesquisa Alexandre de Bustamante Simas e Jonathan Farfan que, efetivamente, contribuiram para o sucesso desta etapa de minha vida, dividimos momentos de descontração, concentração, tristeza e alegria. A matemática ficou mais prazerosa com as inúmeras e intermináveis discurssões no IMPA, em casa, nos ônibus, na praia,...

Agradeço a todos os professores que contribuiram para minha formação. Em especial, destaco no CEFETES os professores Oscar Rezende. Na UFES os professores José Gilvan, Valmecir Bayer, Ademir Sartim, Luzia Casati e Luiz Fernando Camargo (também orientador de Iniciação Científica, que além do rigor matemático, me instruía em minha formação pessoal), sou muito grato. No IMPA, os professores Fernado Codá, Carlos Gustavo Moreira (Gugu), Manfredo do Carmo, Milton Jara que com sua brilhante capacidade de conduzir as aulas tornavam a Matemática mais bela e inspiradora.

Agradeço aos funcionários e alunos do IMPA que, durante minha estada, fizeram com que o ambiente de trabalho fosse agradável e prazeroso. De fato, um local que fiz muitas amizades. Em especial, agradeço a Alexandre Oliveira (Xandão), Fátima Russo, José Paulo Fahl (Paulinho), Fábio do Santos, aos funcionários do ensino e a turma do futebol de quinta, obrigado pela acolhida. Um agradecimento especial a Jean Silva, Jefferson Melo, Fernando Marek, Etereldes, Thiago Fassarela (Mocha) e a turma da Probabilidade, obrigado pelo companheirismo.

Agradeço aos professores Leandro Pimentel, Rob Morris, Serguei Popov e Vladas Sidoravicius por terem aceitado participar de minha banca de defesa de tese contribuindo com sugestões, críticas e elogios.

Agradeço ao CNPq pelo financiamento de minha bolsa de doutorado no país e no exterior.

Contents

Introduction

O limite hidrodinâmico permite obter uma descrição das características termodinâmicas (por exemplo, temperatura, densidade, pressão) de sistemas infinitos assumindo que a dinâmica das partículas é estocástica. Seguindo a abordagem da mecânica estatística introduzida por Boltzmann, deduzimos o comportamento macroscópico de um sistema a partir da iteração microscópica entre as partículas. Considera-se a dinâmica microscópica consistindo de caminhos aleatórios sobre um grafo submetida a alguma iteração local, denominado *sistema de partículas interagentes* introduzido por Spitzer [36], veja também [24]. Ademais, esta abordagem justifica rigorosamente um método algumas vezes utilizado pelos físicos para estabelecer equações diferenciais parciais que descrevem a evolução de características termodinâmicas de um fluido. Assim, a existência de soluções fracas de tais EDPs podem ser vistas como um dos objetivos do limite hidrodinâmico.

Um conhecido sistema de partículas interagentes é o processo de exclusão simples. Informalmente é um processo onde apenas uma partícula por sitio é permitida (dai o nome exclusão), e o salto das partículas somente ocorrem para os vizinhos próximos. Nesta tese consideramos o processo de exclusão simples sobre o toro discreto d-dimensional, $\mathbb{T}_N^d$, e obtemos o comportamento hidrodinâmico nos seguintes modelos:

No capítulo 1, consideramos o processo de exclusão com condutâncias induzida por uma classe de funções W e obtemos que, sobre uma escala difusiva, a evolução das densidades empíricas do processo de exclusão sobre o toro d-dimensional, $\mathbb{T}^d$, é descrita pela única solução fraca da equação diferencial parcial generalizada não-linear

$$\partial_t \rho = \sum_{k=1}^{d} \partial_{x_k} \partial_{W_k} \Phi(\rho), \tag{0.0.1}$$

Onde a função $\Phi : [l, r] \to \mathbb{R}$ é fixada e suave, definida sobre um intervalo $[l, r]$ de $\mathbb{R}$. Esta função está associada a um fator na taxa de salto das partículas no processo microscópico e depende das configurações do sistema. O adjetivo generalizada decorre do termo ∂_{W_k} cuja definição e referências são dadas na Seção 1.2. Em Particular, se considerarmos $W_k(x) = x_k$, obtemos que (0.0.1) é a equação do calor. Para a prova do limite hidrodinâmico, nós também obtemos algumas propriedades do operador elíptico do lado direito de (0.0.1).

Ultimamente, a evolução de processos de exclusão uni-dimensional com condutâncias tem atraído atenção [13, 14, 18, 21]. Um dos propósitos desta tese é estender esta análise para dimensões maiores. Este processo pode, por exemplo, modelar difusões de partículas em um meio com membranas permeáveis, nos pontos de descontinuidade de W, tendendo a refletir partículas, criando espaços de descontinuidade nos perfis de densidade. Nas primeiras linhas do capítulo 1, encontra-se uma detalhamento maior desta aplicação e da real conexão deste operador com os famosos operadores diferenciais de Féller.

No capítulo 2, consideramos um processo de exclusão com condutâncias em ambiente aleatório e obtemos o limite hidrodinâmico. A condutância é a mesma considerada no capítulo 1, no entanto a novidade neste capítulo não se resume ao ambiente aleatório. Isto porque a prova do comportamento hidrodinâmico no capítulo 1 é baseada em estimativas do semigrupo e resolventes entre o processo original e um corrigido. O elo entre o casos $d = 1$ [18, 14] e $d \geq 1$ é então estabelecido via uma classe especial de funções W, a saber:

$$W(x_1, \ldots, x_d) = \sum_{k=1}^{d} W_k(x_k) \qquad x \in \mathbb{R},$$

onde cada W_k é da forma considerada em [18]. Enquanto no capítulo 2, usando as propriedades obtidas do operador elíptico em (0.0.1), construímos o espaço W-Sobolev, o qual consiste das funções f tendo gradiente generalizado fraco $\nabla_W f = (\partial_{W_1} f, \ldots, \partial_{W_d} f)$. Obtemos varias propriedades para este espaço, que são análogas aos clássicos resultados para espaços de Sobolev. Equações W-generalizada elíptica e parabólica são consideradas, alcançando resultados de existência e unicidade de soluções fracas para estas equações. Resultados de homogenização para uma classe de operadores aleatórios são investigados, finalmente, como primeira aplicação desta teoria desenvolvida, nos provamos o limite hidrodinâmico para o processo em questão. Em particular, substituindo a análise de semigrupos e resolventes feita no capítulo 1, por homogenização.

A motivação para este enfoque foi o artigo [20]. Nele os autores consideram um processo de exclusão gradiente em ambiente aleatório e usam a teoria de homogenização, desenvolvida em [31], para obterem o limite hidrodinâmico e flutuações.

No capítulo 3, nos obtemos as flutuações do equilíbrio para o processo considerado no capítulo 2. Esta foi a segunda aplicação da teoria previamente desenvolvida. Nos obtemos que a distribuição empírica é governada pela única solução de uma equação diferencial estocástica, tomando valores em um certo espaço Frechet Nuclear.

No capítulo 4, nos provamos os grandes desvios dinâmicos para um processo boundary driven, i.e. um sistema que possui dois reservatórios infinitos de partículas na fronteira com partículas que podem ter diferentes velocidades. Este resultado baseia-se na recente abordagem introduzida em [15].

Cada capítulo desta tese resultou em um artigo, os quais salvo alguns cortes para evitar excessivas repetições, são os próprios artigos. Em particular cada início de capítulo tem uma pequena introdução que complementa esta. Ressalto que o capítulo 2 é um trabalho conjunto com Alexandre Bustamante de Simas e os capítulos 3 e 4 são em parceria com Jonathan Farfan e Alexandre Bustamante de Simas.

Chapter 1

Hydrodynamic limit of a d-dimensional exclusion process with conductances

The evolution of one-dimensional exclusion processes with random conductances has attracted some attention recently [21, 13, 14, 18]. The purpose of this chapter is to extend this analysis to higher dimension.

Let $W : \mathbb{R}^d \to \mathbb{R}$ be a function such that $W(x_1, \ldots, x_d) = \sum_{k=1}^{d} W_k(x_k)$, where $d \geq 1$ and each function $W_k : \mathbb{R} \to \mathbb{R}$ is strictly increasing, right continuous with left limits (càdlàg) , and periodic in the sense that $W_k(u+1) - W_k(u) = W_k(1) - W_k(0)$ for all $u \in \mathbb{R}$. Informally, the exclusion process with conductances associated to W is an interacting particle systems on the d-dimensional discrete torus $N^{-1}\mathbb{T}_N^d$, in which at most one particle per site is allowed, and only nearest-neighbor jumps are permitted. Moreover, the jump rate in the direction e_j is given by the reciprocal of the increments of W with respect to the jth coordinate.

We show that, on the diffusive scale, the macroscopic evolution of the empirical density of exclusion processes with conductances W is described by the nonlinear differential equation

$$\partial_t \rho \;=\; \sum_{k=1}^{d} \partial_{x_k} \partial_{W_k} \Phi(\rho) \;, \tag{1.0.1}$$

where Φ is a smooth function, strictly increasing in the range of ρ, and such that $0 < b \leq \Phi' \leq b^{-1}$. Furthermore, we denote by ∂_{W_k} the generalized derivative with respect to W_k, see [8, 18] and a revision in Section 1.2. The partial differential equation (1.0.1) appears naturally as, for instance, scaling limits of interacting particle systems in inhomogeneous media. It may model diffusions in which permeable membranes, at the points of discontinuities of W, tend to reflect particles, creating space discontinuities in the density profiles.

The proof of hydrodynamic limit relies strongly on some properties of the differential operator $\sum_{k=1}^{d} \partial_{x_k} \partial_{W_k}$ presented in Theorem 1.1.2. We prove, among other properties: that the operator $\sum_{k=1}^{d} \partial_{x_k} \partial_{W_k}$, defined on an appropriate domain, is non-positive, self-adjoint and dissipative; that its eigenvalues are countable and have finite multiplicity; and that the associated eigenvectors form a complete orthonormal system.

There is a wide literature on the so-called Feller's generalized diffusion operator $(d/du)(d/dv)$. Where, typically, u and v are strictly increasing functions with v (but not necessarily u) being continuous. It provides general diffusions operators and an appreciable simplification of the theory of second-order differential operators (see, for instance, [16, 17, 26]). The operator $(d/dx)(d/du)$, considered in [18], is the formal adjoint of $(d/du)(d/dv)$ in the particular case $v(x) = x$ (as in [17]). The goal of this work is to extend this adjoint operator to higher dimensions and provide some results regarding this extension.

This chapter is organized as follows: in Section 1.1 we state the main results of the chapter; in Section 1.2 we prove the main properties of the operator $\mathcal{L}_W = \sum_{k=1}^{d} \partial_{x_k} \partial_{W_k}$; in Section 1.3 we prove the convergence of random walks with random conductances to Markov processes with generator given by $\mathcal{L}_W$; in Section 1.4 we prove the scaling limit of the exclusion process with conductances given by W; and, finally, in Section 1.5 we show that the unique solution of (1.0.1) has finite energy.

1.1 Notation and Results

We examine the hydrodynamic behavior of a d-dimensional exclusion process, with $d \geq 1$, with conductances induced by a special class of functions $W : \mathbb{R}^d \to \mathbb{R}$ such that:

$$W(x_1, \ldots, x_d) = \sum_{k=1}^{d} W_k(x_k) \tag{1.1.1}$$

where $W_k : \mathbb{R} \to \mathbb{R}$ are *strictly increasing* right continuous functions with left limits (càdlàg), and periodic in the sense that

$$W_k(u + 1) - W_k(u) = W_k(1) - W_k(0)$$

for all $u \in \mathbb{R}$ and $k = 1, \ldots, d$. To keep notation simple, we assume that W_k vanishes at the origin, that is, $W_k(0) = 0$.

Denote by $\mathbb{T}^d = [0, 1)^d$ the d-dimensional torus and by $e_1, \ldots, e_d$ the canonical basis of $\mathbb{R}^d$. For this class of functions we have:

- $W(0) = 0$;
- W is strictly increasing on each coordinate:

  $$W(x + ae_j) > W(x)$$

  for all $1 \leq j \leq d$, $a > 0, x \in \mathbb{R}^d$;
- W is continuous from above:

  $$W(x) = \lim_{y \to x,\ y \geq x} W(y),$$

  where we say that $y \geq x$ if $y_j \geq x_j$ for all $1 \leq j \leq d$;
- W is defined on the torus $\mathbb{T}^d$:

  $$W(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_d) = W(x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_d) - W(e_j),$$

  for all $1 \leq j \leq d$, $(x_1, \ldots, x_{j-1}, x_{j+1}, ..., x_d) \in \mathbb{T}^{d-1}$.

Unless explicitly stated W belongs to this class. Let $\mathbb{T}^d_N = (\mathbb{Z}/N\mathbb{Z})^d = \{0, \ldots, N - 1\}^d$ be the d-dimensional discrete torus with N^d points. Distribute particles throughout $\mathbb{T}^d_N$ in such a way that each site of $\mathbb{T}^d_N$ is occupied at most by one particle. Denote by η the configurations of the state space $\{0, 1\}^{\mathbb{T}^d_N}$, so that $\eta(x) = 0$ if site x is vacant and $\eta(x) = 1$ if site x is occupied.

Fix $b > -1/2$ and W. For $x = (x_1, \ldots, x_d) \in \mathbb{T}^d_N$ let

$$c_{x,x+e_j}(\eta) \ = \ 1 \ + \ b\{\eta(x - e_j) + \eta(x + 2\ e_j)\}\ ,$$

where all sums are modulo N, and let

$$\xi_{x,x+e_j} \ = \ \frac{1}{N[W((x + e_j)/N) - W(x/N)]} \ = \ \frac{1}{N[W_j((x_j + 1)/N) - W_j(x_j/N)]}.$$

We now describe the stochastic evolution of the process. Let $x = (x_1, \ldots, x_d) \in \mathbb{T}^d_N$. At rate $\xi_{x,x+e_j} c_{x,x+e_j}(\eta)$ the occupation variables $\eta(x)$, $\eta(x+e_j)$ are exchanged. If W is differentiable at $x/N \in [0, 1)^d$, the rate at which particles are exchanged is of order 1 for each direction, but if some W_j is discontinuous at x_j/N , it no longer holds. In fact, assume, to fix ideas, that W_j is discontinuous at x_j/N, and smooth on the segments $(x_j/N, x_j/N + \varepsilon e_j)$ and $(x_j/N - \varepsilon e_j, x_j/N)$. Assume, also, that W_k is differentiable in a neighborhood of x_k/N for $k \neq j$. In this case, the rate at which particles jump over the bonds $\{y - e_j, y\}$, with $y_j = x_j$, is of order $1/N$, whereas in a neighborhood of size N of these bonds, particles jump at rate 1. Thus, note that a particle at site $y - e_j$ jumps to y at rate $1/N$ and jumps at rate 1 to each one of the $2d-1$ other options. Particles, therefore, tend to avoid the bonds $\{y - e_j, y\}$. However, since time will be scaled diffusively, and since on a time interval of length N^2 a particle spends a time of order N at each site y, particles will be able to cross the slower bond $\{y - e_j, y\}$.

Then, this process models membranes that obstruct passages of particles. Note that these membranes are $(d-1)$-dimensional hyperplanes embedded in a d-dimensional environment. Moreover, if we consider W_j having more than one discontinuity point for more than one j, these membranes will be more sophisticated manifolds, for instance, unions of $(d-1)$-dimensional boxes.

The effect of the factor $c_{x,x+e_j}(\eta)$ is the following: if the parameter b is positive, the presence of particles in the neighboring sites of the bond $\{x, x+e_j\}$ speeds up the exchange rate by a factor of order one, and if the parameter b is negative, the presence of particles in the neighboring sites slows down the exchange rate also by a factor of order one.

The dynamics informally presented describes a Markov evolution. The generator L_N of this Markov process acts on functions $f : \{0,1\}^{\mathbb{T}_N^d} \to \mathbb{R}$ as

$$L_N f(\eta) \;=\; \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_N^d} \xi_{x,x+e_j}\, c_{x,x+e_j}(\eta)\, \{f(\sigma^{x,x+e_j}\eta) - f(\eta)\}\,, \tag{1.1.2}$$

where $\sigma^{x,x+e_j}\eta$ is the configuration obtained from η by exchanging the variables $\eta(x)$ and $\eta(x+e_j)$:

$$(\sigma^{x,x+e_j}\eta)(y) \;=\; \begin{cases} \eta(x+e_j) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+e_j, \\ \eta(y) & \text{otherwise.} \end{cases} \tag{1.1.3}$$

A straightforward computation shows that the Bernoulli product measures $\{\nu_\alpha^N : 0 \le \alpha \le 1\}$ are invariant, and in fact reversible, for the dynamics. The measure ν_α^N is obtained by placing a particle at each site, independently from the other sites, with probability α. Thus, ν_α^N is a product measure over $\{0,1\}^{\mathbb{T}_N^d}$ with marginals given by

$$\nu_\alpha^N\{\eta : \eta(x) = 1\} \;=\; \alpha,$$

for x in $\mathbb{T}_N^d$. For more details see [23, chapter 2]. We will often omit the index N on ν_α^N.

Denote by $\{\eta_t : t \ge 0\}$ the Markov process on $\{0,1\}^{\mathbb{T}_N^d}$ associated to the generator L_N *speeded up* by N^2. Let $D(\mathbb{R}_+, \{0,1\}^{\mathbb{T}_N^d})$ be the path space of càdlàg trajectories with values in $\{0,1\}^{\mathbb{T}_N^d}$. For a measure μ_N on $\{0,1\}^{\mathbb{T}_N^d}$, denote by $\mathbb{P}_{\mu_N}$ the probability measure on $D(\mathbb{R}_+, \{0,1\}^{\mathbb{T}_N^d})$ induced by the initial state μ_N, and the Markov process $\{\eta_t : t \ge 0\}$. Expectation with respect to $\mathbb{P}_{\mu_N}$ is denoted by $\mathbb{E}_{\mu_N}$.

1.1.1 The hydrodynamic equation

Fix $W = \sum_{k=1}^{d} W_k$ as in (1.1.1). In [18] it was shown that there exist self-adjoint operators $\mathcal{L}_{W_k} : \mathcal{D}_{W_k} \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$. The domain $\mathcal{D}_{W_k}$ is completely characterized in the following proposition:

Proposition 1.1.1. *The domain $\mathcal{D}_{W_k}$ consists of all functions f in $L^2(\mathbb{T})$ such that*

$$f(x) \;=\; a \;+\; bW_k(x) \;+\; \int_{(0,x]} W_k(dy) \int_0^y \mathfrak{f}(z)\, dz$$

for some function $\mathfrak{f}$ in $L^2(\mathbb{T})$ that satisfies

$$\int_0^1 \mathfrak{f}(z)\, dz \;=\; 0 \quad \text{and} \quad \int_{(0,1]} W_k(dy) \Big\{ b + \int_0^y \mathfrak{f}(z)\, dz \Big\} \;=\; 0\,.$$

The proof and further details can be found in [18]. Further, the set $\mathcal{A}_{W_k}$ of the eigenvectors of $\mathcal{L}_{W_k}$ forms a complete orthonormal system in $L^2(\mathbb{T})$. Let

$$\mathcal{A}_W \;=\; \{f : \mathbb{T}^d \to \mathbb{R}; f(x_1, \ldots, x_d) = \prod_{k=1}^{d} f_k(x_k), f_k \in \mathcal{A}_{W_k},\; k = 1, \ldots, d\}, \tag{1.1.4}$$

and denote by $span(A)$ the space of finite linear combinations of the set A, and let $\mathbb{D}_W := span(\mathcal{A}_W)$. Define the operator $\mathbb{L}_W : \mathbb{D}_W \to L^2(\mathbb{T}^d)$ as follows: for $f = \prod_{k=1}^d f_k \in \mathcal{A}_W$, we have

$$\mathbb{L}_W(f)(x_1, \dots x_d) = \sum_{k=1}^{d} \prod_{j=1, j\neq k}^{d} f_j(x_j)\mathcal{L}_{W_k} f_k(x_k), \tag{1.1.5}$$

and then extend to $\mathbb{D}_W$ by linearity.

Lemma 1.2.2, in Section 1.2, shows that: $\mathbb{L}_W$ is symmetric and non-positive; $\mathbb{D}_W$ is dense in $L^2(\mathbb{T}^d)$; and the set $\mathcal{A}_W$ forms a complete, orthonormal, countable system of eigenvectors for the operator $\mathbb{L}_W$. Let $\mathcal{A}_W = \{h_k\}_{k\geq 0}$, $\{\alpha_k\}_{k\geq 0}$ be the corresponding eigenvalues of $-\mathbb{L}_W$, and consider

$$\mathcal{D}_W = \{v = \sum_{k=1}^{\infty} v_k h_k \in L^2(\mathbb{T}^d); \sum_{k=1}^{\infty} v_k^2 \alpha_k^2 < +\infty\}. \tag{1.1.6}$$

Define the operator $\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ by

$$-\mathcal{L}_W v = \sum_{k=1}^{+\infty} \alpha_k v_k h_k \tag{1.1.7}$$

The operator $\mathcal{L}_W$ is clearly an extension of the operator $\mathbb{L}_W$, and we present in Theorem 1.1.2 some properties of this operator.

Theorem 1.1.2. *The operator $\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ enjoys the following properties:*

(a) *The domain $\mathcal{D}_W$ is dense in $L^2(\mathbb{T}^d)$. In particular, the set of eigenvectors $\mathcal{A}_W = \{h_k\}_{k\geq 0}$ forms a complete orthonormal system;*

(b) *The eigenvalues of the operator $-\mathcal{L}_W$ form a countable set $\{\alpha_k\}_{k\geq 0}$. All eigenvalues have finite multiplicity, and it is possible to obtain a re-enumeration $\{\alpha_k\}_{k\geq 0}$ such that*

$$0 = \alpha_0 \leq \alpha_1 \leq \cdots \quad \textit{and} \quad \lim_{n\to\infty} \alpha_n = \infty;$$

(c) *The operator $\mathbb{I} - \mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ is bijective;*

(d) *$\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ is self-adjoint and non-positive:*

$$\langle -\mathcal{L}_W f, f\rangle \;\geq\; 0;$$

(e) *$\mathcal{L}_W$ is dissipative.*

In view of (a), (b) and (d), we may use Hille-Yosida theorem to conclude that $\mathcal{L}_W$ is the generator of a strongly continuous contraction semigroup $\{P_t : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)\}_{t\geq 0}$.

Denote by $\{G_\lambda : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)\}_{\lambda>0}$ the semigroup of resolvents associated to the operator $\mathcal{L}_W$: $G_\lambda = (\lambda - \mathcal{L}_W)^{-1}$. G_λ can also be written in terms of the semigroup $\{P_t\,; t \geq 0\}$:

$$G_\lambda = \int_0^\infty e^{-\lambda t} P_t\, dt.$$

In Section 1.3 we derive some properties and obtain some results for these operators.

The hydrodynamic equation is, roughly, a PDE that describes the time evolution of the thermodynamical quantities of the model in a fluid. A sequence of probability measures $\{\mu_N : N \geq 1\}$ on $\{0,1\}^{\mathbb{T}_N^d}$ is said to be associated to a profile $\rho_0 : \mathbb{T}^d \to [0,1]$ if

$$\lim_{N\to\infty} \mu_N \left\{ \Big|\frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} H(x/N)\eta(x) - \int H(u)\rho_0(u)du\Big| > \delta \right\} \;=\; 0 \tag{1.1.8}$$

for every $\delta > 0$, and every continuous function $H : \mathbb{T}^d \to \mathbb{R}$. For details, see [23, chapter 3].

For a positive integer $m \geq 1$, denote by $C^m(\mathbb{T}^d)$ the space of continuous functions $H : \mathbb{T}^d \to \mathbb{R}$ with m continuous derivatives. Fix $l < r$, and a smooth function $\Phi : [l, r] \to \mathbb{R}$, whose derivative is bounded below by a strictly positive constant and bounded above by a finite constant, that is,

$$0 \; < \; B^{-1} \leq \Phi'(x) \; \leq \; B,$$

for all $x \in [l, r]$. Let $\gamma : \mathbb{T}^d \to [l, r]$ be a bounded density profile, and consider the parabolic differential equation

$$\begin{cases} \partial_t \rho \; = \; \mathcal{L}_W \Phi(\rho) \\ \rho(0, \cdot) \; = \; \gamma(\cdot) \end{cases} . \tag{1.1.9}$$

A bounded function $\rho : \mathbb{R}_+ \times \mathbb{T}^d \to [l, r]$ is said to be a weak solution of the parabolic differential equation (1.1.9) if

$$\langle \rho_t, G_\lambda H \rangle \; - \; \langle \gamma, G_\lambda H \rangle \; = \; \int_0^t \langle \Phi(\rho_s), \mathcal{L}_W G_\lambda H \rangle \, ds$$

for every continuous function $H : \mathbb{T}^d \to \mathbb{R}$, all $t > 0$ and all $\lambda > 0$.

Existence of these weak solutions follows from tightness of the sequence of probability measures $\mathbb{Q}^{W,N}_{\mu_N}$ introduced in Section 1.4. The proof of uniquenesses of weak solutions is analogous to [18].

Theorem 1.1.3. *Fix a continuous initial profile* $\rho_0 : \mathbb{T}^d \to [0, 1]$*, and consider a sequence of probability measures* μ_N *on* $\{0, 1\}^{\mathbb{T}^d_N}$ *associated to* ρ_0*, in the sense of* (1.1.8)*. Then, for any* $t \geq 0$*,*

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N} \left\{ \Big| \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} H(x/N) \eta_t(x) - \int H(u) \rho(t, u) \, du \Big| > \delta \right\} \; = \; 0$$

for every $\delta > 0$ *and every continuous function* H*. Here,* ρ *is the unique weak solution of the non-linear equation* (1.1.9) *with* $l = 0$*,* $r = 1$*,* $\gamma = \rho_0$*, and* $\Phi(\alpha) = \alpha + a\alpha^2$*.*

Remark 1.1.4. *As noted in [18, remark 2.3], the specific form of the rates* $c_{x,x+e_i}$ *is not important, but two conditions must be fulfilled: the rates must be strictly positive, although they may not depend on the occupation variables* $\eta(x)$*,* $\eta(x + e_i)$*; but they have to be chosen in such a way that the resulting process is* gradient*. (cf. Chapter 7 in [23] for the definition of gradient processes).*

We may define rates $c_{x,x+e_i}$ *to obtain any polynomial* Φ *of the form* $\Phi(\alpha) = \alpha + \sum_{2 \leq j \leq m} a_j \alpha^j$*,* $m \geq 1$*, with* $1 + \sum_{2 \leq j \leq m} j a_j > 0$*. Let, for instance,* $m = 3$*. Then the rates*

$$\begin{aligned} \hat{c}_{x,x+e_i}(\eta) \; &= \; c_{x,x+e_i}(\eta) \; + \\ & b_1 \left\{ \eta(x - 2e_i)\eta(x - e_i) + \eta(x - e_i)\eta(x + 2e_i) + \eta(x + 2e_i)\eta(x + 3e_i) \right\}, \end{aligned}$$

satisfy the above three conditions, where $c_{x,x+e_i}$ *is the rate defined at the beginning of Section 2 and* b*,* b_1 *are such that* $1 + 2b + 3b_1 > 0$*. An elementary computation shows that* $\Phi(\alpha) = \alpha + b\alpha^2 + b_1\alpha^3$*.*

In Section 1.5 we prove that any limit point $\mathbb{Q}^*_W$ of the sequence $\mathbb{Q}^{W,N}_{\mu_N}$ is concentrated on trajectories $\rho(t, u)du$, with finite energy in the following sense: for each $1 \leq j \leq d$, there is a Hilbert space $L^2_{x_j \otimes W_j}$, associated to W_j, such that

$$\int_0^t ds \, \| \frac{d}{dW_j} \Phi(\rho(s, .)) \|^2_{x_j \otimes W_j} < \infty \; ,$$

where $\|.\|_{x_j \otimes W_j}$ is the norm in $L^2_{x_j \otimes W_j}$, and d/dW_j is the derivative, which must be understood in the generalized sense.

1.2 The operator $\mathcal{L}_W$

The operator $\mathcal{L}_W : \mathcal{D}_W \subset L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ is a natural extension, for the d-dimensional case, of the self-adjoint operator obtained for the one-dimensional case in [18]. We begin by presenting one of the main results obtained in [18], and we then present the necessary modifications to conclude similar results for the d-dimensional case.

1.2.1 Some remarks on the one-dimensional case

Let $\mathbb{T} \subset \mathbb{R}$ be the one-dimensional torus. Denote by $\langle \cdot, \cdot \rangle$ the inner product of $L^2(\mathbb{T})$:

$$\langle f, g \rangle \;=\; \int_{\mathbb{T}} f(u)\, g(u)\, du\,.$$

Let $W_1 : \mathbb{R} \to \mathbb{R}$ be a *strictly increasing* right continuous function with left limits (càdlàg), and periodic in the sense that $W_1(u+1) - W_1(u) = W_1(1) - W_1(0)$ for all u in $\mathbb{R}$.

Let $\mathcal{D}_{W_1}$ be the set of functions f in $L^2(\mathbb{T})$ such that

$$f(x) \;=\; a \;+\; bW_1(x) \;+\; \int_{(0,x]} W_1(dy) \int_0^y \mathfrak{f}(z)\, dz,$$

for $a, b \in \mathbb{R}$ and some function $\mathfrak{f}$ in $L^2(\mathbb{T})$ that satisfies:

$$\int_0^1 \mathfrak{f}(z)\, dz \;=\; 0\,, \qquad \int_{(0,1]} W_1(dy)\Big(b + \int_0^y \mathfrak{f}(z)\, dz\Big) \;=\; 0.$$

Define the operator $\mathcal{L}_{W_1} : \mathcal{D}_{W_1} \to L^2(\mathbb{T})$ by $\mathcal{L}_{W_1} f = \mathfrak{f}$. Formally

$$\mathcal{L}_{W_1} f \;=\; \frac{d}{dx}\,\frac{d}{dW_1}\, f\,, \tag{1.2.1}$$

where the generalized derivative d/dW_1 is defined as

$$\frac{df}{dW_1}(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{W_1(x+\epsilon) - W_1(x)}\,, \tag{1.2.2}$$

if the above limit exists and is finite.

Theorem 1.2.1. *Denote by $\mathbb{I}$ the identity operator in $L^2(\mathbb{T})$. The operator $\mathcal{L}_{W_1} : \mathcal{D}_{W_1} \to L^2(\mathbb{T})$ enjoys the following properties:*

(a) *$\mathcal{D}_{W_1}$ is dense in $L^2(\mathbb{T})$;*

(b) *The operator $\mathbb{I} - \mathcal{L}_{W_1} : \mathcal{D}_{W_1} \to L^2(\mathbb{T})$ is bijective;*

(c) *$\mathcal{L}_{W_1} : \mathcal{D}_{W_1} \to L^2(\mathbb{T})$ is self-adjoint and non-positive:*

$$\langle -\mathcal{L}_{W_1} f, f \rangle \;\geq\; 0;$$

(d) *$\mathcal{L}_{W_1}$ is dissipative i.e., for all $g \in \mathcal{D}_W$ and $\lambda > 0$, we have*

$$\|\lambda g\| \leq \|(\lambda \mathbb{I} - \mathcal{L}_{W_1})g\|;$$

(e) *The eigenvalues of the operator $-\mathcal{L}_W$ form a countable set $\{\lambda_n : n \geq 0\}$. All eigenvalues have finite multiplicity, $0 = \lambda_0 \leq \lambda_1 \leq \cdots$, and $\lim_{n\to\infty} \lambda_n = \infty$;*

(f) *The eigenvectors $\{f_n\}_{n \geq 0}$ of the operator $\mathcal{L}_W$ form a complete orthonormal system.*

The proof can be found in [18].

1.2.2 The d-dimensional case

Consider W as in (1.1.1). Let $\mathcal{A}_{W_k}$ be the countable complete orthonormal system of eigenvectors of the operator $\mathcal{L}_{W_k} : \mathcal{D}_{W_k} \subset L^2(\mathbb{T}) \to \mathbb{R}$ given in Theorem 1.2.1.

Let $\mathcal{A}_W$ be as in (1.1.4), and let the operator $\mathbb{L}_W : \mathbb{D}_W := span(\mathcal{A}_W) \to L^2(\mathbb{T}^d)$ be as in (1.1.5). By Fubini's theorem, the set $\mathcal{A}_W$ is orthonormal in $L^2(\mathbb{T}^d)$, and the constant functions are eigenvectors of the operator $\mathcal{L}_{W_k}$. Moreover, $\mathcal{A}_{W_k} \subset \mathcal{A}_W$, in the sense that $f_k(x_1, \ldots, x_d) = f_k(x_k)$, $\;f_k \in \mathcal{A}_{W_k}$.

By (1.2.1), the operators $\mathcal{L}_{W_k}$ can be formally extended to functions defined on $\mathbb{T}^d$ as follows: given a function $f : \mathbb{T}^d \to \mathbb{R}$, we define $\mathcal{L}_{W_k} f$ as

$$\mathcal{L}_{W_k} f \;=\; \partial_{x_k} \partial_{W_k} f, \tag{1.2.3}$$

where the generalized derivative ∂_{W_k} is defined by

$$\partial_{W_k} f(x_1, \dots, x_k, \dots, x_d) = \lim_{\epsilon \to 0} \frac{f(x_1, \dots, x_k + \epsilon, \dots, x_d) - f(x_1, \dots, x_k, \dots, x_d)}{W_k(x_k + \epsilon) - W_k(x_k)} \;, \tag{1.2.4}$$

if the above limit exists and is finite. Hence, by (1.1.5), if $f \in \mathbb{D}_W$

$$\mathbb{L}_W f = \sum_{k=1}^{d} \mathcal{L}_{W_k} f. \tag{1.2.5}$$

Note that if $f = \prod_{k=1}^d f_k$, where $f_k \in \mathcal{A}_{W_k}$ is an eigenvector of $\mathcal{L}_{W_k}$ associated to the eigenvalue λ_k, then f is an eigenvector of $\mathbb{L}_W$, with eigenvalue $\sum_{k=1}^d \lambda_k$.

Lemma 1.2.2. *The following statements hold:*

(a) *The set* $\mathbb{D}_W$ *is dense in* $L^2(\mathbb{T}^d)$*;*

(b) *The operator* $\mathbb{L}_W : \mathbb{D}_W \to L^2(\mathbb{T}^d)$ *is symmetric and non-positive:*

$$\langle -\mathbb{L}_W f, f \rangle \;\geq\; 0.$$

Proof. The strategy to prove the above Lemma is the following. We begin by showing that the set

$$\mathcal{S} \;=\; span(\{f \in L^2(\mathbb{T}^d); f(x_1, \dots, x_d) = \prod_{k=1}^{d} f_k(x_k), f_k \in \mathcal{D}_{W_k}\})$$

is dense in

$$\mathbb{S} = span(\{f \in L^2(\mathbb{T}^d); f(x_1, \dots, x_d) = \prod_{k=1}^{d} f_k(x_k), f_k \in L^2(\mathbb{T})\}).$$

We then show that $\mathbb{D}_W$ is dense in $\mathcal{S}$. Since $\mathbb{S}$ is dense in $L^2(\mathbb{T}^d)$, item (a) follows.

We now prove item (a) rigorously. Since $\mathcal{S}$ is a vector space, we only have to show that we can approximate the functions $\prod_{k=1}^d f_k \in L^2(\mathbb{T}^d)$, where $f_k \in \mathcal{D}_{W_k}$, by functions of $\mathbb{D}_W$. By Theorem 1.2.1, the set $\mathcal{D}_{W_k}$ is dense in $L^2(\mathbb{T})$, thus, there exists a sequence $(f_n^k)_{n\in\mathbb{N}}$ converging to f_k in $L^2(\mathbb{T})$. Thus, let

$$f_n(x_1, \dots, x_d) = \prod_{k=1}^{d} f_n^k(x_k).$$

By the triangle inequality and Fubini's theorem, the sequence (f_n) converges to $\prod_{k=1}^d f_k$. Fix $\epsilon > 0$, and let

$$h(x_1, \dots, x_d) = \prod_{k=1}^{d} h_k(x_k), \quad h_k \in \mathcal{D}_{W_k}.$$

Since, for each $k = 1 \dots, d$, $\mathcal{A}_{W_k} \subset \mathcal{D}_{W_k}$ is a complete orthonormal set, there exist sequences $g_j^k \in \mathcal{A}_{W_k}$, and $\alpha_j^k \in \mathbb{R}$, such that

$$\|h_k - \sum_{j=1}^{n(k)} \alpha_j^k g_j^k\|_{L^2(\mathbb{T})} < \delta \;,$$

where $\delta = \epsilon / dM^{d-1}$ and $M := 1 + sup_{k=1:n} \|h_k\|$. Let

$$g(x_1, \dots, x_d) = \prod_{k=1}^{d} \sum_{j=1}^{n(k)} \alpha_j^k g_j^k(x_k) \in \mathbb{D}_W.$$

An application of the triangle inequality, and Fubini's theorem, yields $\|h - g\| < \epsilon$. This proves (a).

To prove (b), let

$$f(x_1, \ldots, x_d) = \prod_{k=1}^{d} f_k(x_k) \text{ and } g(x_1, \ldots, x_d) = \prod_{k=1}^{d} g_k(x_k)$$

be functions belonging to $\mathcal{A}_W$. We have that

$$\langle f, \mathbb{L}_W g \rangle = \langle \prod_{k=1}^{d} f_k, \sum_{k=1}^{d} \prod_{j=1, j\neq k}^{d} g_j \mathcal{L}_{W_k} g_k \rangle = \sum_{k=1}^{d} \langle \prod_{j=1, j\neq k}^{d} f_j g_j, f_k \mathcal{L}_{W_k} g_k \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{T}^d)$. Since, by Theorem 1.2.1, $\mathcal{L}_{W_k}$ is self-adjoint, we have

$$\sum_{k=1}^{d} \langle \prod_{j=1, j\neq k}^{d} f_j g_j, g_k \mathcal{L}_{W_k} f_k \rangle = \langle \mathcal{L}_W f, g \rangle.$$

In particular, the operator $\mathcal{L}_{W_k}$ is non-positive, and, therefore,

$$\langle f, \mathbb{L}_W f \rangle = \sum_{k=1}^{d} \langle \prod_{j=1, j\neq k}^{d} f_j^2, f_k \mathcal{L}_{W_k} f_k \rangle \leq 0.$$

Item (b) follows by linearity. □

Lemma 1.2.2 implies that the set $\mathcal{A}_W$ forms a complete, orthonormal, countable, system of eigenvectors for the operator $\mathbb{L}_W$.

Let $\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ be the operator defined in (1.1.7). The operator $\mathcal{L}_W$ is clearly an extension of the operator $\mathbb{L}_W$. Formally, by (1.2.5),

$$\mathcal{L}_W f = \sum_{k=1}^{d} \mathcal{L}_{W_k} f, \tag{1.2.6}$$

where

$$\mathcal{L}_{W_k} f = \partial_{x_k} \partial_{W_k} f.$$

We are now in conditions to prove Theorem 1.1.2.

Proof of Theorem 1.1.2. By Lemma 1.2.2, $\mathbb{D}_W$ is dense in $L^2(\mathbb{T}^d)$. Since $\mathbb{D}_W \subset \mathcal{D}_W$, we conclude that $\mathcal{D}_W$ is dense in $L^2(\mathbb{T}^d)$.

If α_k are eigenvalues of $-\mathcal{L}_W$, we may find eigenvalues λ_j, associated to some $f_j \in \mathcal{A}_{W_j}$, such that $\alpha_k = \sum_{j=1}^{d} \lambda_j$. By item (e) of Theorem 1.2.1, (b) follows.

Let $\{\alpha_k\}_{k\geq 0}$ be the set of eigenvalues of $-\mathcal{L}_W$. Then, the set of eigenvalues of $\mathbb{I} - \mathcal{L}_W$ is $\{\gamma_k\}_{k\geq 0}$, where $\gamma_k = \alpha_k + 1$, and the eigenvectors are the same as the ones of $\mathcal{L}_W$. By item (b), we have

$$1 = \gamma_0 \leq \gamma_1 \leq \cdots \text{ and } \lim_{n\to\infty} \gamma_n = \infty .$$

Thus, $\mathbb{I} - \mathcal{L}_W$ is injective. For

$$v = \sum_{k=1}^{+\infty} v_k h_k \in L^2(\mathbb{T}^d) \text{ , such that } \sum_{k=1}^{\infty} v_k^2 < +\infty ,$$

let

$$u = \sum_{k=1}^{+\infty} \frac{v_k}{\gamma_k} h_k.$$

Then $u \in \mathcal{D}_W$ and $(\mathbb{I} - \mathcal{L}_W)u = v$. Hence, item (c) follows.

Let $\mathcal{L}_W^* : \mathcal{D}_{W^*} \subset L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ be the adjoint of $\mathcal{L}_W$. Since $\mathcal{L}_W$ is symmetric, we have $\mathcal{D}_W \subset \mathcal{D}_{W^*}$. So, to show the equality of the operators it suffices to show that $\mathcal{D}_{W^*} \subset \mathcal{D}_W$. Given

$$\varphi = \sum_{k=1}^{+\infty} \varphi_k h_k \in \mathcal{D}_{W^*},$$

let $\mathcal{L}_{W*}\varphi = \psi \in L^2(\mathbb{T}^d)$. Therefore, for all $v = \sum_{k=1}^{+\infty} v_k h_k \in \mathcal{D}_W$,

$$\langle v, \psi \rangle = \langle v, \mathcal{L}_{W*}\varphi \rangle = \langle \mathcal{L}_W v, \varphi \rangle = \sum_{k=1}^{+\infty} -\alpha_k v_k \varphi_k.$$

Hence

$$\psi = \sum_{k=1}^{+\infty} -\alpha_k \varphi_k h_k.$$

In particular,

$$\sum_{k=1}^{+\infty} \alpha_k^2 \varphi_k^2 < +\infty \ \text{ and } \ \varphi \in \mathcal{D}_W.$$

Thus, $\mathcal{L}_W$ is self-adjoint. Let $v = \sum_{k=1}^{+\infty} v_k h_k \in \mathcal{D}_W$. From item (b), $\alpha_k \geq 0$, and

$$\langle -\mathcal{L}_W v, v \rangle = \sum_{k=1}^{+\infty} \alpha_k v_k^2 \geq 0.$$

Therefore $\mathcal{L}_W$ is non-positive, and item (d) follows.

Fix a function g in $\mathcal{D}_W$, $\lambda > 0$, and let $f = (\lambda\mathbb{I} - \mathcal{L}_W)g$. Taking inner product, with respect to g, on both sides of this equation, we obtain

$$\lambda\langle g, g \rangle \ + \ \langle -\mathcal{L}_W g, g \rangle \ = \ \langle g, f \rangle \ \leq \ \langle g, g \rangle^{1/2} \langle f, f \rangle^{1/2} \, .$$

Since g belongs to $\mathcal{D}_W$, by (d), the second term on the left hand side is non-negative. Thus, $\|\lambda g\| \leq \|f\| = \|(\lambda\mathbb{I} - \mathcal{L}_W)g\|$. □

1.3 Random walk with conductances

Recall the decomposition obtained in (1.2.6) for the operator $\mathcal{L}_W$. In next subsection, we present the discrete version $\mathbb{L}_N$ of $\mathcal{L}_W$ and we describe, informally, the Markovian dynamics generated by $\mathbb{L}_N$.

1.3.1 Discrete approximation of the operator $\mathcal{L}_W$

Consider the random walk $\{X_t^N\}_{t\geq 0}$ in $\frac{1}{N}\mathbb{T}_N^d$, which jumps from x/N (resp. $(x+e_j)/N$) to $(x+e_j)/N$ (resp. x/N) with rate

$$N^2 \xi_{x,x+e_j} = N/\{W_j((x_j+1)/N) - W_j(x_j/N)\}.$$

The generator $\mathbb{L}_N$ of this Markov process acts on local functions $f : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$ as

$$\mathbb{L}_N f(x/N) \ = \ \sum_{j=1}^{d} \mathbb{L}_N^j f(x/N), \tag{1.3.1}$$

where

$$\begin{aligned} \mathbb{L}_N^j f(x/N) \quad &= \quad N^2 \big\{ \xi_{x,x+e_j}[f((x+e_j)/N) - f(x/N)] \\ &+ \quad \xi_{x-e_j,x}[f((x-e_j)/N) - f(x/N)] \big\}. \end{aligned}$$

Note that $\mathbb{L}_N^j f(x/N)$ is, in fact, a discrete version of the operator $\mathcal{L}_{W_j}$. The counting measure m_N on $\mathbb{T}_N^d$ is reversible for this process. The following estimate is a key ingredient for proving the results in Section 1.4:

Lemma 1.3.1. *Let f be a function on $\frac{1}{N}\mathbb{T}_N^d$. Then, for each $j = 1, \ldots, d$:*

$$\frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \Big(\mathbb{L}_N^j f(x/N) \Big)^2 \le \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \Big(\mathbb{L}_N f(x/N) \Big)^2.$$

Proof. Let X_{N^d} be the linear space of functions f on $\frac{1}{N^d}\mathbb{T}_N^d$ over the field $\mathbb{R}$. Note that the dimension of X_{N^d} is N^d. Denote by $\langle \cdot, \cdot \rangle_{N^d}$ the following inner product in X_{N^d}:

$$\langle f, g \rangle_{N^d} = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} f(x/N) g(x/N).$$

For each $j = 1, \ldots, d$, consider the linear operators $\mathcal{L}_N^j$ on X_N (i.e., $d = 1$) given by

$$\mathcal{L}_N^j f = \partial_x^N \partial_{W_j}^N f,$$

where ∂_x^N and $\partial_{W_j}^N$ are the difference operators:

$$\begin{aligned} \partial_x^N f(x/N) &= N[f((x+1)/N) - f(x/N)] \qquad \text{and} \\ \partial_{W_j}^N f(x/N) &= \frac{f((x+1)/N) - f(x/N)}{W_j((x+1)/N) - W_j(x/N)}. \end{aligned}$$

The operators $\mathcal{L}_N^j$ are symmetric and non-positive. In fact, a simple computation shows that

$$\langle \mathcal{L}_N^j f, g \rangle_N = - \sum_{x \in \mathbb{T}_N} \Big(W_j((x+1)/N) - W_j(x/N) \Big) \partial_{W_j}^N f(x/N) \partial_{W_j}^N g(x/N).$$

Using the spectral theorem, we obtain an orthonormal basis $\mathcal{A}_N^j = \{h_1^j, \ldots, h_N^j\}$ of X_N formed by the eigenvectors of $\mathcal{L}_N^j$, i.e.,

$$\mathcal{L}_N^j h_i^j = \alpha_i^j h_i^j \qquad \text{and} \qquad \langle h_i^j, h_k^j \rangle_N = \delta_{i,k} \ ,$$

where $\delta_{i,k}$ is the Kronecker's delta, which equals 0 if $i \neq k$, and equals 1 if $i = k$. Since $\mathcal{L}_N^j$ is non-positive, we have that the eigenvalues α_i^j are non-positive: $\alpha_i^j \le 0$, $j = 1, \ldots, d$ and $i = 1, \ldots, N$.

Let $\mathcal{A}_N = \{\phi_1, \ldots, \phi_{N^d}\} \subset X_{N^d}$ be set of functions of the form $\phi_i(x_1, \ldots, x_d) = \prod_{j=1}^d h^j(x_j)$, with $h^j \in \mathcal{A}_N^j$.

Let α^j be the eigenvalue of h^j, i.e., $\mathcal{L}_N^j h^j = \alpha^j h^j$. The linear operator $\mathbb{L}_N$ on X_{N^d}, defined in (1.3.1), is such that $\mathbb{L}_N^j \phi_i = \alpha^j \phi_i$ and $\mathbb{L}_N \phi_i = \sum_{j=1}^d \alpha^j \phi_i$. Furthermore, if $\phi_i(x_1, \ldots, x_d) = \prod_{j=1}^d h^j(x_j)$ and $\phi_k(x_1, \ldots, x_d) = \prod_{j=1}^d g^j(x_j)$, $\phi_i, \phi_k \in \mathcal{A}_N$, we have that

$$\langle \phi_i, \phi_k \rangle_{N^d} = \prod_{j=1}^d \langle h^j, g^j \rangle_N = \delta_{i,k},$$

for $i, k = 1, \ldots, N^d$. So, the set $\mathcal{A}_N$ is an orthonormal basis of X_{N^d} formed by the eigenvectors of $\mathbb{L}_N$ and $\mathbb{L}_N^j$. In particular, for each $f \in X_{N^d}$, there exist $\beta_i \in \mathbb{R}$ such that $f = \sum_{i=1}^{N^d} \beta_i \phi_i$. Thus,

$$\frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \Big(\mathbb{L}_N^j f(x/N) \Big)^2 = \|\mathbb{L}_N^j f\|_{N^d}^2 = \|\mathbb{L}_N^j \sum_{i=1}^{N^d} \beta_i \phi_i\|_{N^d}^2 = \sum_{i=1}^{N^d} (\alpha_i^j \beta_i)^2 \le$$
$$\sum_{i=1}^{N^d} (\sum_{j=1}^d \alpha_i^j)^2 (\beta_i)^2 = \|\mathbb{L}_N f\|_{N^d}^2 = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \Big(\mathbb{L}_N f(x/N) \Big)^2,$$

where $\alpha_i^j \le 0$ is the eigenvalue of the operator $\mathbb{L}_N^j$ associated to the eigenvector ϕ_i. This concludes the proof of the lemma. □

1.3.2 Semigroups and resolvents.

In this subsection we introduce families of semigroups and resolvents associated to the generators $\mathbb{L}_N$ and $\mathcal{L}_W$. We present some properties and results regarding the convergence of these operators.

Denote by $\{P_t^N : t \geq 0\}$ (resp. $\{G_\lambda^N : \lambda > 0\}$) the semigroup (resp. the resolvent) associated to the generator $\mathbb{L}_N$, by $\{P_t^{N,j} : t \geq 0\}$ the semigroup associated to the generator $\mathbb{L}_N^j$, by $\{P_t^j : t \geq 0\}$ the semigroup associated to the generator $\mathcal{L}_{W_j}$ and by $\{P_t : t \geq 0\}$ (resp. $\{G_\lambda : \lambda > 0\}$) the semigroup (resp. the resolvent) associated to the generator $\mathcal{L}_W$.

Since the jump rates from x/N (resp. $(x+e_j)/N$) to $(x+e_j)/N$ (resp. x/N) are equal, P_t^N is symmetric: $P_t^N(x,y) \;=\; P_t^N(y,x)$.

Using the decompositions (1.3.1) and (1.2.6), we obtain

$$P_t^N(x,y) = \prod_{j=1}^d P_t^{N,j}(x_j,y_j) \;\text{ and }\; P_t(x,y) = \prod_{j=1}^d P_t^j(x_j,y_j).$$

By definition, for every $H : N^{-1}\mathbb{T}_N^d \to \mathbb{R}$,

$$G_\lambda H \;=\; \int_0^\infty dt\, e^{-\lambda t} P_t H \;=\; (\lambda \mathbb{I} - \mathcal{L}_W)^{-1} H,$$

where $\mathbb{I}$ is the identity operator.

Lemma 1.3.2. *Let $H : \mathbb{T}^d \to \mathbb{R}$ be a continuous function. Then*

$$\lim_{N\to+\infty} \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} |P_t^N H(x/N) - P_t H(x/N)| = 0. \tag{1.3.2}$$

Proof. If $H : \mathbb{T}^d \to \mathbb{R}$ has the form $H(x_1,\ldots,x_d) = \prod_{j=1}^d H_j(x_j)$, we have

$$P_t^N H(x) \;=\; \prod_{j=1}^d P_t^{N,j} H_j(x_j) \;\text{ and }\; P_t H(x) \;=\; \prod_{j=1}^d P_t^j H_j(x_j). \tag{1.3.3}$$

Now, for any continuous function $H : \mathbb{T}^d \to \mathbb{R}$, and any $\epsilon > 0$, we can find continuous functions $H_{j,k} : \mathbb{T} \to \mathbb{R}$, such that $H' : \mathbb{T}^d \to \mathbb{R}$, which is given by

$$H'(x) = \sum_{j=1}^m \prod_{k=1}^d H_{j,k}(x_k),$$

satisfies $\|H' - H\|_\infty \leq \epsilon$. Thus,

$$\frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} |P_t^N H(x/N) - P_t H(x/N)| \;\leq\; 2\epsilon + \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} |P_t^N H'(x/N) - P_t H'(x/N)|.$$

By (1.3.3) and similar identities for $P_t H'$ and $P_t^{N,j} H'$, the sum on the right hand side in the previous inequality is less than or equal to

$$\frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^m \Big| \prod_{k=1}^d P_t^{N,k} H_{j,k}(x_k/N) - \prod_{k=1}^d P_t^k H_{j,k}(x_k/N) \Big| \;\leq$$
$$\frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^m C_j \sum_{k=1}^d |P_t^{N,k} H_{j,k}(x_k/N) - P_t^k H_{j,k}(x_k/N)|,$$

where C_j is a constant that depends on the product $\prod_{k=1}^d H_{j,k}$. The previous expressions can be rewritten as

$$\sum_{j=1}^m C_j \sum_{k=1}^d \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^{d-1}} \sum_{i=1}^N |P_t^{N,k} H_{j,k}(i/N) - P_t^k H_{j,k}(i/N)| \; = \; \sum_{j=1}^m C_j \sum_{k=1}^d \frac{1}{N} \sum_{i=1}^N |P_t^{N,k} H_{j,k}(i/N) - P_t^k H_{j,k}(i/N)|.$$

Moreover, by [14, Lemma 4.5 item iii], when $N \to \infty$, the last expression converges to 0.

□

Corollary 1.3.3. *Let $H : \mathbb{T}^d \to \mathbb{R}$ be a continuous function. Then*

$$\lim_{N\to+\infty} \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} |G_\lambda^N H(x/N) - G_\lambda H(x/N)| = 0. \tag{1.3.4}$$

Proof. By the definition of resolvent, for each N, the previous expression is less than or equal to

$$\int_0^\infty dt\; e^{-\lambda t} \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} |P_t^N H(x/N) - P_t H(x/N)|.$$

Corollary now follows from the previous lemma. □

Let $f_N : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$ be any function. Then, whenever needed, we consider $f : \mathbb{T}^d \to \mathbb{R}$ as the extension of f_N to $\mathbb{T}^d$ given by:

$$f(y) = f_N(x), \text{ if } x \in \mathbb{T}_N^d,\; y \geq x \text{ and } \|y - x\|_\infty < \frac{1}{N}.$$

Let $H : \mathbb{T}^d \to \mathbb{R}$ be a continuous function. Then the extension of $P_t^N H : \mathbb{T}_N^d \to \mathbb{R}$ to $\mathbb{T}^d$ belongs to $L^1(\mathbb{T}^d)$, and by symmetry of the transition probability $P_t^N(x,y)$ we have

$$\int_{\mathbb{T}^d} du P_t^N H(u) \; = \; \frac{1}{N^d} \sum_{x\in\mathbb{T}^d} H(x/N). \tag{1.3.5}$$

The next Lemma shows that H can be approximated by $P_t^N H$. As an immediate consequence, we obtain an approximation result involving the resolvent.

Lemma 1.3.4. *Let $H : \mathbb{T}^d \to \mathbb{R}$ be a continuous function. Then,*

$$\lim_{t\to 0} \overline{\lim_{N\to+\infty}} \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} |P_t^N H(x/N) - H(x/N)| = 0, \tag{1.3.6}$$

and

$$\lim_{\lambda\to+\infty} \overline{\lim_{N\to+\infty}} \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} |\lambda G_\lambda^N H(x/N) - H(x/N)| = 0. \tag{1.3.7}$$

Proof. Fix $\epsilon > 0$, and consider H' as in the proof of Lemma 1.3.2. Thus,

$$\frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} |P_t^N H(x/N) - H(x/N)| \;\leq\; 2\epsilon + \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} |P_t^N H'(x/N) - H'(x/N)|,$$

where the second term on the right hand side is less than or equal to

$$C_0 \sup_{j,k} \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} |P_t^{N,k} H_{j,k}(x_k/N) - H_{j,k}(x_k/N)|,$$

with C_0 being a constant that depends on H'. By [14, Lemma 4.6], the last expression converges to 0, when $N \to \infty$, and then $t \to 0$. This proves the first equality.

To obtain the second limit, note that, by definition of the resolvent, the second expression is less than or equal to

$$\int_0^\infty dt \lambda e^{-\lambda t} \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} |P_t^N H(x/N) - H(x/N)|.$$

By (1.3.5) the sum is uniformly bounded in t and N. Furthermore, it vanishes as $N \to \infty$ and $t \to 0$. This proves the second part.

□

Fix a function $H : \mathbb{T}_N^d \to \mathbb{R}$. For $\lambda > 0$, let $H_\lambda^N = G_\lambda^N H$ be the solution of the resolvent equation

$$\lambda H_\lambda^N - \mathbb{L}_N H_\lambda^N \;=\; H. \tag{1.3.8}$$

Taking inner product on both sides of this equation with respect to H_λ^N,we obtain

$$\lambda \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} (H_\lambda^N(x/N))^2 \;-\; \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} H_\lambda^N(x/N) \mathbb{L}_N H_\lambda^N$$
$$= \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} H_\lambda^N(x/N) H(x/N).$$

A simple computation shows that the second term on the left hand side is equal to

$$\frac{1}{N^d} \sum_{j=1}^d \sum_{x \in \mathbb{T}_N^d} \xi_{x,x+e_j} [\nabla_{N,j} H_\lambda^N(x/N)]^2,$$

where $\nabla_{N,j} H(x/N) = N[H((x+e_j)/N) - H(x/N)]$ is the discrete derivative of the function H in the direction of the vector e_j. In particular, by Schwarz inequality,

$$\begin{aligned} \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} H_\lambda^N(x/N)^2 \;&\le\; \frac{1}{\lambda^2} \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} H(x/N)^2 \;\text{ and} \\ \frac{1}{N^d} \sum_{j=1}^d \sum_{x \in \mathbb{T}_N^d} \xi_{x,x+e_j} [\nabla_{N,j} H_\lambda^N(x/N)]^2 \;&\le\; \frac{1}{\lambda} \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} H(x/N)^2 \,. \end{aligned} \tag{1.3.9}$$

1.4 Scaling limit

Let $\mathcal{M}$ be the space of positive measures on $\mathbb{T}^d$ with total mass bounded by one, and endowed with the weak topology. Recall that $\pi_t^N \in \mathcal{M}$ stands for the empirical measure at time t. This is the measure on $\mathbb{T}^d$ obtained by rescaling space by N, and by assigning mass $1/N^d$ to each particle:

$$\pi_t^N \;=\; \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_t(x)\, \delta_{x/N} \;, \tag{1.4.1}$$

where δ_u is the Dirac measure concentrated in u.

For a continuous function $H : \mathbb{T}^d \to \mathbb{R}$, $\langle \pi_t^N, H \rangle$ stands for the integral of H with respect to π_t^N:

$$\langle \pi_t^N, H \rangle \;=\; \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} H(x/N) \eta_t(x) \;.$$

This notation is not to be mistaken with the inner product in $L^2(\mathbb{T}^d)$ introduced earlier. Also, when π_t has a density ρ, $\pi(t, du) = \rho(t, u) du$, we sometimes write $\langle \rho_t, H \rangle$ for $\langle \pi_t, H \rangle$.

For a local function $g : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$, let $\tilde{g} : [0,1] \to \mathbb{R}$ be the expected value of g under the stationary states:

$$\tilde{g}(\alpha) \;=\; E_{\nu_\alpha}[g(\eta)] \;.$$

For $\ell \geq 1$ and d-dimensional integer $x = (x_1, \dots, x_d)$, denote by $\eta^\ell(x)$ the empirical density of particles in the box $\mathbb{B}_+^\ell(x) = \{(y_1, \dots, y_d) \in \mathbb{Z}^d\, ; 0 \leq y_i - x_i < \ell\}$:

$$\eta^\ell(x) \;=\; \frac{1}{\ell^d} \sum_{y \in \mathbb{B}_+^\ell(x)} \eta(y) \;.$$

Fix $T > 0$, and let $D([0,T], \mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi : [0,T] \to \mathcal{M}$ endowed with the *uniform* topology. For each probability measure μ_N on $\{0,1\}^{\mathbb{T}_N^d}$, denote by $\mathbb{Q}_{\mu_N}^{W,N}$ the measure on the path space $D([0,T], \mathcal{M})$ induced by the measure μ_N and the process π_t^N introduced in (1.4.1).

Fix a continuous profile $\rho_0 : \mathbb{T}^d \to [0,1]$, and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on $\{0,1\}^{\mathbb{T}_N^d}$ associated to ρ_0 in the sense (1.1.8). Further, we denote by $\mathbb{Q}_W$ be the probability measure on $D([0,T], \mathcal{M})$ concentrated on the deterministic path $\pi(t, du) = \rho(t,u)du$, where ρ is the unique weak solution of (1.1.9) with $\gamma = \rho_0$, $l_k = 0$, $r_k = 1$, $k = 1, \dots, d$, and $\Phi(\alpha) = \alpha + a\alpha^2$.

In subsection 1.4.1 we show that the sequence $\{\mathbb{Q}_{\mu_N}^{W,N} : N \geq 1\}$ is tight, and in subsection 1.4.2 we characterize the limit points of this sequence.

1.4.1 Tightness

The proof of tightness of sequence $\{\mathbb{Q}_{\mu_N}^{W,N} : N \geq 1\}$ is motivated by [21, 18]. We consider, initially, the auxiliary $\mathcal{M}$-valued Markov process $\{\Pi_t^{\lambda,N} : t \geq 0\}$, $\lambda > 0$, defined by

$$\Pi_t^{\lambda,N}(H) \;=\; \langle \pi_t^N, G_\lambda^N H \rangle \;=\; \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \big(G_\lambda^N H\big)(x/N)\eta_t(x),$$

for H in $C(\mathbb{T}^d)$, where $\{G_\lambda^N : \lambda > 0\}$ is the resolvent associated to the random walk $\{X_t^N : t \geq 0\}$ introduced in Section 1.3.

We first prove tightness of the process $\{\Pi_t^{\lambda,N} : 0 \leq t \leq T\}$ for every $\lambda > 0$, and we then show that $\{\lambda \Pi_t^{\lambda,N} : 0 \leq t \leq T\}$ and $\{\pi_t^N : 0 \leq t \leq T\}$ are not far apart if λ is large.

It is well-known [23, proposition 4.1.7] that to prove tightness of $\{\Pi_t^{\lambda,N} : 0 \leq t \leq T\}$ it is enough to show tightness of the real-valued processes $\{\Pi_t^{\lambda,N}(H) : 0 \leq t \leq T\}$ for a set of smooth functions $H : \mathbb{T}^d \to \mathbb{R}$ dense in $C(\mathbb{T}^d)$ for the uniform topology.

Fix a smooth function $H : \mathbb{T}^d \to \mathbb{R}$. Denote by the same symbol the restriction of H to $N^{-1}\mathbb{T}_N^d$. Let $H_\lambda^N = G_\lambda^N H$, and keep in mind that $\Pi_t^{\lambda,N}(H) = \langle \pi_t^N, H_\lambda^N \rangle$. Denote by $M_t^{N,\lambda}$ the martingale defined by

$$M_t^{N,\lambda} \;=\; \Pi_t^{\lambda,N}(H) \;-\; \Pi_0^{\lambda,N}(H) \;-\; \int_0^t ds\, N^2 L_N \langle \pi_s^N, H_\lambda^N \rangle \;. \tag{1.4.2}$$

Clearly, tightness of $\Pi_t^{\lambda,N}(H)$ follows from tightness of the martingale $M_t^{N,\lambda}$ and tightness of the additive functional $\int_0^t ds\, N^2 L_N \langle \pi_s^N, H_\lambda^N \rangle$.

A simple computation shows that the quadratic variation $\langle M^{N,\lambda} \rangle_t$ of the martingale $M_t^{N,\lambda}$ is given by:

$$\frac{1}{N^{2d}} \sum_{j=1}^d \sum_{x \in \mathbb{T}^d} \xi_{x,x+e_j} [\nabla_{N,j} H_\lambda^N(x/N)]^2 \int_0^t c_{x,x+e_j}(\eta_s)\, [\eta_s(x+e_j) - \eta_s(x)]^2\, ds \;.$$

In particular, by (1.3.9),

$$\langle M^{N,\lambda} \rangle_t \;\leq\; \frac{C_0 t}{N^{2d}} \sum_{j=1}^d \sum_{x \in \mathbb{T}_N^d} \xi_{x,x+e_j} \, [(\nabla_{N,j} H_\lambda^N)(x/N)]^2 \;\leq\; \frac{C(H)t}{\lambda N^d},$$

for some finite constant $C(H)$ which depends only on H. Thus, by Doob inequality, for every $\lambda > 0$, $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N} \Big[\sup_{0 \leq t \leq T} \big| M_t^{N,\lambda} \big| \;>\; \delta \Big] \;=\; 0 \;. \tag{1.4.3}$$

In particular, the sequence of martingales $\{M_t^{N,\lambda} : N \geq 1\}$ is tight for the uniform topology.

It remains to be examined the additive functional of the decomposition (1.4.2). The generator of the exclusion process L_N can be decomposed in terms of generators of the random walks $\mathbb{L}_N^j$. By (1.3.1) and a long but simple computation, we obtain that $N^2 L_N \langle \pi^N, H_\lambda^N \rangle$ is equal to

$$\sum_{j=1}^{d} \Big\{ \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} (\mathbb{L}_N^j H_\lambda^N)(x/N)\, \eta(x) \\ + \frac{b}{N^d} \sum_{x \in \mathbb{T}_N^d} \big[(\mathbb{L}_N^j H_\lambda^N)((x+e_j)/N) + (\mathbb{L}_N^j H_\lambda^N)(x/N) \big] (\tau_x h_{1,j})(\eta) \\ - \frac{b}{N^d} \sum_{x \in \mathbb{T}_N^d} (\mathbb{L}_N^j H_\lambda^N)(x/N) (\tau_x h_{2,j})(\eta) \Big\} ,$$

where $\{\tau_x : x \in \mathbb{Z}^d\}$ is the group of translations, so that $(\tau_x \eta)(y) = \eta(x+y)$ for x, y in $\mathbb{Z}^d$, and the sum is understood modulo N. Also, $h_{1,j}$, $h_{2,j}$ are the cylinder functions

$$h_{1,j}(\eta) \;=\; \eta(0)\eta(e_j) \; , \quad h_{2,j}(\eta) \;=\; \eta(-e_j)\eta(e_j) \; .$$

For all $0 \leq s < t \leq T$, we have

$$\Big| \int_s^t dr\, N^2 L_N \langle \pi_r^N, H_\lambda^N \rangle \Big| \;\leq\; \frac{(1+3|b|)(t-s)}{N^d} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_N^d} |\mathbb{L}_N^j H_\lambda^N(x/N)| \; ,$$

from Schwarz inequality and Lemma 1.3.1, the right hand side of the previous expression is bounded above by

$$(1+3|b|)(t-s)d \sqrt{\frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \Big(\mathbb{L}_N H_\lambda^N(x/N) \Big)^2} .$$

Since H_λ^N is the solution of the resolvent equation (1.3.8), we may replace $\mathbb{L}_N H_\lambda^N$ by $U_\lambda^N = \lambda H_\lambda^N - H$ in the previous formula. In particular, It follows from the first estimate in (1.3.9), that the right hand side of the previous expression is bounded above by $dC(H,b)(t-s)$ uniformly in N, where $C(H,b)$ is a finite constant depending only on b and H. This proves that the additive part of the decomposition (1.4.2) is tight for the uniform topology and therefore that the sequence of processes $\{\Pi_t^{\lambda,N} : N \geq 1\}$ is tight.

Lemma 1.4.1. *The sequence of measures $\{\mathbb{Q}_{\mu^N}^{W,N} : N \geq 1\}$ is tight for the uniform topology.*

Proof. It is enough to show that for every smooth function $H : \mathbb{T} \to \mathbb{R}$, and every $\epsilon > 0$, there exists $\lambda > 0$ such that

$$\lim_{N \to \infty} \mathbb{P}_{\mu^N} \Big[\sup_{0 \leq t \leq T} |\, \Pi_t^{\lambda,N}(\lambda H) - \langle \pi_t^N, H \rangle \,| > \epsilon \Big] \;=\; 0,$$

since, in this case, tightness of π_t^N follows from tightness of $\Pi_t^{\lambda,N}$. Since there is at most one particle per site, the expression inside the absolute value is less than or equal to

$$\frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \big| \lambda H_\lambda^N(x/N) - H(x/N) \big| \; .$$

By Lemma 1.3.4, this expression vanishes as $N \uparrow \infty$ and then $\lambda \uparrow \infty$. □

1.4.2 Uniqueness of limit points

We prove in this subsection that all limit points $\mathbb{Q}^*$ of the sequence $\mathbb{Q}_{\mu^N}^{W,N}$ are concentrated on absolutely continuous trajectories $\pi(t,du) = \rho(t,u)du$, whose density $\rho(t,u)$ is a weak solution of the hydrodynamic equation (1.1.9) with $l = 0 < r = 1$ and $\Phi(\alpha) = \alpha + a\alpha^2$.

Let $\mathbb{Q}^*$ be a limit point of the sequence $\mathbb{Q}^{W,N}_{\mu_N}$ and assume, without loss of generality, that $\mathbb{Q}^{W,N}_{\mu_N}$ converges to $\mathbb{Q}^*$.

Since there is at most one particle per site, it is clear that $\mathbb{Q}^*$ is concentrated on trajectories $\pi_t(du)$ which are absolutely continuous with respect to the Lebesgue measure, $\pi_t(du) = \rho(t,u)du$, and whose density ρ is non-negative and bounded by 1.

Fix a continuously differentiable function $H : \mathbb{T}^d \to \mathbb{R}$, and $\lambda > 0$. Recall the definition of the martingale $M_t^{N,\lambda}$ introduced in the previous section. By (1.4.2) and (1.4.3), for fixed $0 < t \le T$ and $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{Q}^{W,N}_{\mu_N} \left[\left| \langle \pi_t^N, G_\lambda^N H\rangle \; - \; \langle \pi_0^N, G_\lambda^N H\rangle \; - \; \int_0^t ds\, N^2 L_N \langle \pi_s^N, G_\lambda^N H\rangle \right| \; > \; \delta \right] \; = \; 0.$$

Since there is at most one particle per site, we may use Corollary 1.3.3 to replace $G_\lambda^N H$ by $G_\lambda H$ in the expressions $\langle \pi_t^N, G_\lambda^N H\rangle$, $\langle \pi_0^N, G_\lambda^N H\rangle$ above. On the other hand, the expression $N^2 L_N \langle \pi_s^N, G_\lambda^N H\rangle$ has been computed in the previous subsection. Since $E_{\nu_\alpha}[h_{i,j}] = \alpha^2$, $i = 1$, 2 and $j = 1, \dots, d$, Lemma 1.3.1 and the estimate (1.3.9), permit us use Corollary 1.4.4 to obtain, for every $t > 0$, $\lambda > 0$, $\delta > 0$, $i = 1$, 2,

$$\lim_{\varepsilon\to 0} \overline{\lim_{N\to\infty}} \mathbb{P}_{\mu_N} \left[\Big| \int_0^t ds \, \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \mathbb{L}_N^j H_\lambda^N(x/N) \left\{ \tau_x h_{i,j}(\eta_s) - \left[\eta_s^{\varepsilon N}(x)\right]^2 \right\} \Big| \; > \; \delta \right] \; = \; 0.$$

Recall that $\mathbb{L}_N G_\lambda^N H = \lambda G_\lambda^N H - H$. As before, we may replace $G_\lambda^N H$ by $G_\lambda H$. Let $U_\lambda = \lambda G_\lambda H - H$. Since $\eta_s^{\varepsilon N}(x) = \varepsilon^{-d} \pi_s^N(\prod_{j=1}^d [x_j/N, x_j/N + \varepsilon e_j])$, we obtain, from the previous considerations, that

$$\lim_{\varepsilon\to 0} \overline{\lim_{N\to\infty}} \mathbb{Q}^{W,N}_{\mu_N} \Big[\Big| \langle \pi_t^N, G_\lambda H\rangle \; - \; \langle \pi_0^N, G_\lambda H\rangle \; - \; \int_0^t ds \, \Big\langle \Phi\big(\varepsilon^{-d}\pi_s^N(\prod_{j=1}^d [\cdot, \cdot + \varepsilon e_j])\big) \, , \, U_\lambda \Big\rangle \Big| > \delta \Big] \; = \; 0 \, .$$

Since H is a smooth function, $G_\lambda H$ and U_λ can be approximated, in $L^1(\mathbb{T}^d)$, by continuous functions. Since we assumed that $\mathbb{Q}^{W,N}_{\mu_N}$ converges in the uniform topology to $\mathbb{Q}^*$, we have that

$$\lim_{\varepsilon\to 0} \mathbb{Q} * \Big[\Big| \langle \pi_t, G_\lambda H\rangle \; - \; \langle \pi_0, G_\lambda H\rangle \; - \; \int_0^t ds \; \Big\langle \Phi\big(\varepsilon^{-d}\pi_s(\prod_{j=1}^d [\cdot, \cdot + \varepsilon e_j])\big) \, , \, U_\lambda \Big\rangle \Big| > \delta \Big] \; = \; 0 \, .$$

Using the fact that $\mathbb{Q}^*$ is concentrated on absolutely continuous paths $\pi_t(du) = \rho(t,u)du$, with positive density bounded by 1, $\varepsilon^{-d}\pi_s(\prod_{j=1}^d [\cdot, \cdot + \varepsilon e_j])$ converges in $L^1(\mathbb{T}^d)$ to $\rho(s,.)$ as $\varepsilon \downarrow 0$. Thus,

$$\mathbb{Q}^* \left[\Big| \langle \pi_t, G_\lambda H\rangle \; - \; \langle \pi_0, G_\lambda H\rangle \; - \; \int_0^t ds \, \langle \Phi(\rho_s) \, , \, \mathcal{L}_W G_\lambda H\rangle \Big| > \delta \right] \; = \; 0,$$

because $U_\lambda = \mathcal{L}_W G_\lambda H$. Letting $\delta \downarrow 0$, we see that, $\mathbb{Q}^*$ a.s.,

$$\langle \pi_t, G_\lambda H\rangle \; - \; \langle \pi_0, G_\lambda H\rangle \; = \; \int_0^t ds \, \langle \Phi(\rho_s) \, , \, \mathcal{L}_W G_\lambda H\rangle \, .$$

This identity can be extended to a countable set of times t. Taking this set to be dense, by continuity of the trajectories π_t, we obtain that it holds for all $0 \le t \le T$. In the same way, it holds for any countable family of continuous functions H. Taking a countable set of continuous functions, dense for the uniform topology, we extend this identity to all continuous functions H, because $G_\lambda H_n$ converges to $G_\lambda H$ in $L^1(\mathbb{T}^d)$, if H_n converges to H in the uniform topology. Similarly, we can show that it holds for all $\lambda > 0$, since, for any continuous function H, $G_{\lambda_n} H$ converges to $G_\lambda H$ in $L^1(\mathbb{T}^d)$, as $\lambda_n \to \lambda$.

Proposition 1.4.2. *As $N \uparrow \infty$, the sequence of probability measures $\mathbb{Q}^{W,N}_{\mu_N}$ converges in the uniform topology to $\mathbb{Q}_W$.*

Proof. In the previous subsection we showed that the sequence of probability measures $\mathbb{Q}^{W,N}_{\mu_N}$ is tight for the uniform topology. Moreover, we just proved that all limit points of this sequence are concentrated on weak solutions of the parabolic equation (1.1.9). The proposition now follows from a straightforward adaptation of the uniquenesses of weak solutions proved in [18] for the d-dimensional case. □

Proof of Theorem 1.1.3. Since $\mathbb{Q}^{W,N}_{\mu_N}$ converges in the uniform topology to $\mathbb{Q}_W$, a measure which is concentrated on a deterministic path. For each $0 \le t \le T$ and each continuous function $H : \mathbb{T}^d \to \mathbb{R}$, $\langle \pi^N_t, H\rangle$ converges in probability to $\int_{\mathbb{T}} du\, \rho(t,u)\ H(u)$, where ρ is the unique weak solution of (1.1.9) with $l_k = 0$, $r_k = 1$, $\gamma = \rho_0$ and $\Phi(\alpha) = \alpha + a\alpha^2$. □

1.4.3 Replacement lemma

We will use some results from [23, Appendix A1]. Denote by $H_N(\mu_N|\nu_\alpha)$ the relative entropy of a probability measure μ_N with respect to a stationary state ν_α, see [23, Section A1.8] for a precise definition. By the explicit formula given in [23, Theorem A1.8.3], we see that there exists a finite constant K_0, depending only on α, such that

$$H_N(\mu_N|\nu_\alpha) \ \le\ K_0 N^d, \tag{1.4.4}$$

for all measures μ_N.

Denote by $\langle\cdot,\cdot\rangle_{\nu_\alpha}$ the inner product of $L^2(\nu_\alpha)$ and denote by I^ξ_N the convex and lower semicontinuous [23, Corollary A1.10.3] functional defined by

$$I^\xi_N(f) \ =\ \langle -L_N\sqrt{f}\,,\ \sqrt{f}\rangle_{\nu_\alpha}\ ,$$

for all probability densities f with respect to ν_α (i.e., $f \ge 0$ and $\int f d\nu_\alpha = 1$). By [23, proposition A1.10.1], an elementary computation shows that

$$I^\xi_N(f) \ =\ \sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d_N} I^\xi_{x,x+e_j}(f)\ , \quad \text{where}$$
$$I^\xi_{x,x+e_j}(f) \ =\ (1/2)\,\xi_{x,x+e_j}\int c_{x,x+e_j}(\eta)\left\{\sqrt{f(\sigma^{x,x+e_j}\eta)} - \sqrt{f(\eta)}\right\}^2 d\nu_\alpha\ .$$

By [23, Theorem A1.9.2], if $\{S^N_t : t \ge 0\}$ stands for the semigroup associated to the generator N^2L_N,

$$H_N(\mu_N S^N_t|\nu_\alpha) \ + 2\ N^2\int_0^t I^\xi_N(f^N_s)\, ds \ \le\ H_N(\mu_N|\nu_\alpha)\ ,$$

where f^N_s stands for the Radon-Nikodym derivative of $\mu_N S^N_s$ with respect to ν_α.

Recall the definition of $\mathbb{B}^\ell_+(x)$ in begin of this section. For each $y \in \mathbb{B}^\ell_+(x)$, such that $y_1 > x_1$, let

$$\Lambda^\ell_{x+e_1,y} = (z^y_k)_{0\le k\le M(y)} \tag{1.4.5}$$

be a path from $x + e_1$ to y such that:

1. $\Lambda^\ell_{x+e_1,y}$ begins at $x + e_1$ and ends at y, i.e.:
$$z^y_0 \ =\ x + e_1 \quad\text{and}\quad z^y_{M(y)} = y;$$
2. The distance between two consecutive sites of the $\Lambda^\ell_{x+e_1,y} = (z^y_k)_{0\le k\le M(y)}$ is equal to 1, i.e.:
$$z^y_{k+1} = z^y_k + e_j;\ \text{for some } j = 1\dots,d \quad\text{and for all}\quad k = 0,\dots,M(y)-1;$$
3. $\Lambda^\ell_{x+e_1,y}$ is injective:
$$z^y_i \ne z^y_j \quad\text{for all } 0 \le i < j \le M(y);$$
4. The path begins by jumping in the direction of e_1. Furthermore, the jump in the direction of e_{j+1} is only allowed when it is not possible to jump in the direction of e_j, for $j = 1,\dots,d-1$.

Lemma 1.4.3. *Fix a function $F : N^{-1}\mathbb{T}_N^d \to \mathbb{R}$. There exists a finite constant $C_0 = C_0(a, g, W)$, depending only on a, g and W, such that*

$$\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} F(x/N)\int\{\tau_x g(\eta) - \tilde{g}(\eta^{\varepsilon N}(x))\}\, f(\eta)\nu_\alpha(d\eta)$$
$$\leq \frac{C_0}{\varepsilon N^{d+1}}\sum_{x\in\mathbb{T}_N^d}\big|F(x/N)\big| \;+\; \frac{C_0\varepsilon}{\delta N^d}\sum_{x\in\mathbb{T}_N^d}F(x/N)^2 \;+\; \frac{\delta}{N^{d-2}}I_N^{\xi}(f),$$

for all $\delta > 0$, $\varepsilon > 0$ and all probability densities f with respect to ν_α.

Proof. Any local function can be written as a linear combination of functions in the form $\prod_{x\in A}\eta(x)$, for finite sets $A's$. It is therefore enough to prove the Lemma for such functions. We will only prove the result for $g(\eta) = \eta(0)\eta(e_1)$. The general case can be handled in a similar way.

We begin by estimating

$$\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}F(x/N)\int\eta(x)\{\eta(x+e_1) - \frac{1}{(\varepsilon N)^d}\sum_{y\in\mathbb{B}_+^{N\varepsilon}(x)}\eta(y)\}f(\eta)\nu_\alpha(d\eta) \tag{1.4.6}$$

in terms of the functional $I_N^{\xi}(f)$. The integral in (1.4.6) can be rewritten as:

$$\frac{1}{(N\varepsilon)^d}\sum_{y\in\mathbb{B}_+^{N\varepsilon}(x)}\int\eta(x)[\eta(x+e_1) - \eta(y)]f(\eta)\nu_\alpha(d\eta).$$

For each $y \in \mathbb{B}_+^{N\varepsilon}(x)$, such that $y_1 > x_1$, let $\Lambda^{\ell}_{x+e_1,y} = (z_k^y)_{0\le k\le M(y)}$ be a path like the one in (1.4.5). Then, by property (1) of $\Lambda^{\ell}_{x+e_1,y}$ and using telescopic sum we have the following:

$$\eta(x+e_1) - \eta(y) = \sum_{k=0}^{M(y)-1}[\eta(z_k^y) - \eta(z_{k+1}^y)].$$

We can, therefore, bound (1.4.6) above by

$$\frac{1}{N^d}\frac{1}{(N\varepsilon)^d}\sum_{x\in\mathbb{T}_N^d}\sum_{y\in\mathbb{B}_+^{N\varepsilon}(x)}\sum_{k=0}^{M(y)-1}\int F(x/N)\eta(x)[\eta(z_k^y) - \eta(z_{k+1}^y)]f(\eta)\nu_\alpha(d\eta) +$$
$$\frac{1}{\varepsilon N^{d+1}}\sum_{x\in\mathbb{T}_N^d}\big|F(x/N)\big|,$$

where the last term in the previous expression comes from the contribution of the points $y \in \mathbb{B}_+^{N\varepsilon}(x)$, such that $y_1 = x_1$. Recall that, by property (2) of $\Lambda^{\ell}_{x+e_1,y}$, we have that $z_{k+1}^y = z_k^y + e_j$, for some $j = 1, \dots, d$.

For each term of the form

$$\int F(x/N)\eta(x)\{\eta(z) - \eta(z+e_j)\}f(\eta)\nu_\alpha(d\eta)$$

we can use the change of variables $\eta' = \sigma^{z,z+e_j}\eta$ to write the previous integral as

$$(1/2)\int F(x/N)\eta(x)\{\eta(z) - \eta(z+e_j)\}\,\big\{f(\eta) - f(\sigma^{z,z+e_j}\eta)\big\}\,\nu_\alpha(d\eta)\,.$$

Since $a - b = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$ and $\sqrt{ab} \le a + b$, by Schwarz inequality the previous expression is less than or equal to

$$\frac{A}{4(1-2a^-)\xi_{z,z+e_j}}\int F(x/N)^2\eta(x)\{\eta(z) - \eta(z+e_j)\}^2\times$$
$$\times\Big\{\sqrt{f(\eta)} + \sqrt{f(\sigma^{z,z+e_j}\eta)}\Big\}^2\,\nu_\alpha(d\eta) +$$
$$+\frac{\xi_{z,z+e_j}}{A}\int c_{z,z+e_j}(\eta)\Big\{\sqrt{f(\eta)} - \sqrt{f(\sigma^{z,z+e_j}\eta)}\Big\}^2\,\nu_\alpha(d\eta)$$

for every $A > 0$. In this formula we used the fact that $c_{z,z+e_j}(\eta)$ is bounded below by $1 - 2a^-$, where $a^- = \max\{-a, 0\}$. Since f is a density with respect to ν_α, the first expression is bounded above by $A/(1-2a^-)\xi_{z,z+e_j}$, whereas the second one is equal to $2A^{-1} I^{\xi}_{z,z+e_j}(f)$.

So, using all the previous calculations together with properties (3) and (4) of the path $\Lambda^{\ell}_{x+e_1,y}$, we obtain that (1.4.6) is less than or equal to

$$\frac{1}{\varepsilon N^{d+1}} \sum_{x\in\mathbb{T}^d_N} \big|F(x/N)\big| \;+\; \frac{A}{(1-2a^-)N^d} \sum_{x\in\mathbb{T}^d_N} F(x/N)^2 \sum_{j=1}^{d}\sum_{k=1}^{\varepsilon N} \xi^{-1}_{x+(k-1)e_j, x+ke_j} \;+$$

$$\frac{2\varepsilon}{AN^{d-1}} \sum_{j=1}^{d} \sum_{x\in\mathbb{T}^d_N} I^{\xi}_{x,x+e_j}(f)\;.$$

By definition of the sequence $\{\xi_{x,x+e_j}\}$, $\sum_{k=1}^{\varepsilon N} \xi^{-1}_{x+ke_j,e_j} \le N[W_j(1) - W_j(0)]$. Thus, choosing $A = 2\varepsilon N^{-1}\delta^{-1}$, for some $\delta > 0$, we obtain that the previous sum is bounded above by

$$\frac{C_0}{\varepsilon N^{d+1}} \sum_{x\in\mathbb{T}^d_N} \big|F(x/N)\big| \;+\; \frac{C_0\varepsilon}{\delta N^d} \sum_{x\in\mathbb{T}^d_N} F(x/N)^2 \;+\; \frac{\delta}{N^{d-2}} I^{\xi}_N(f)\;.$$

Up to this point we have succeeded to replace $\eta(x)\eta(x+e_1)$ by $\eta(x)\eta^{\varepsilon N}(x)$. The same arguments permit to replace this latter expression by $[\eta^{\varepsilon N}(x)]^2$, which concludes the proof of the Lemma. □

Corollary 1.4.4. *Fix a cylinder function g, and a sequence of functions $\{F_N : N \ge 1\}$, $F_N : N^{-1}\mathbb{T}^d_N \to \mathbb{R}$ such that*

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} F_N(x/N)^2 \;<\; \infty\;.$$

Then, for any $t > 0$ and any sequence of probability measures $\{\mu_N : N \ge 1\}$ on $\{0,1\}^{\mathbb{T}^d_N}$,

$$\overline{\lim_{\varepsilon\to 0}}\; \overline{\lim_{N\to\infty}}\; \mathbb{E}_{\mu_N}\Big[\Big|\int_0^t \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} F_N(x/N)\left\{\tau_x g(\eta_s) - \tilde{g}(\eta^{\varepsilon N}_s(x))\; ds\right\}\Big|\Big] \;=\; 0\;.$$

Proof. Fix $0 < \alpha < 1$. By the entropy and Jensen inequalities, the expectation appearing in the statement of the Lemma is bounded above by

$$\frac{1}{\gamma N^d} \log \mathbb{E}_{\nu_\alpha}\Big[\exp\Big\{\gamma\Big|\int_0^t ds \sum_{x\in\mathbb{T}^d_N} F_N(x/N)\left\{\tau_x g(\eta_s) - \tilde{g}(\eta^{\varepsilon N}_s(x))\right\}\Big|\Big\}\Big] + \frac{H_N(\mu_N|\nu_\alpha)}{\gamma N^d},$$

for all $\gamma > 0$. In view of (1.4.4), in order to prove the corollary it is enough to show that the first term vanishes as $N \uparrow \infty$, and then $\varepsilon \downarrow 0$, for every $\gamma > 0$. We may remove the absolute value inside the exponential by using the elementary inequalities $e^{|x|} \le e^x + e^{-x}$ and $\overline{\lim}_{N\to\infty} N^{-1} \log\{a_N + b_N\} \le \max\{\overline{\lim}_{N\to\infty} N^{-1} \log a_N, \overline{\lim}_{N\to\infty} N^{-1} \log b_N\}$. Thus, to prove the corollary, it is enough to show that

$$\overline{\lim_{\varepsilon\to 0}}\; \overline{\lim_{N\to\infty}}\; \frac{1}{N^d} \log \mathbb{E}_{\nu_\alpha}\Big[\exp\Big\{\gamma \int_0^t ds \sum_{x\in\mathbb{T}^d_N} F_N(x/N)\{\tau_x g(\eta_s) - \tilde{g}(\eta^{\varepsilon N}_s(x))\}\Big\}\Big] = 0,$$

for every $\gamma > 0$.

By Feynman-Kac formula, for each fixed N the previous expression is bounded above by

$$t\gamma \sup_f \left\{\int \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} F_N(x/N)\{\tau_x g(\eta) - \tilde{g}(\eta^{\varepsilon N}(x))\} f(\eta)\, d\nu_\alpha \;-\; \frac{1}{N^{d-2}} I^{\xi}_N(f)\right\},$$

where the supremum is carried over all density functions f with respect to ν_α. Letting $\delta = 1$ in Lemma 1.4.3, we obtain that the previous expression is less than or equal to

$$\frac{C_0\gamma t}{\varepsilon N^{d+1}} \sum_{x\in\mathbb{T}^d_N} \big|F_N(x/N)\big| \;+\; \frac{C_0\gamma\varepsilon t}{N^d} \sum_{x\in\mathbb{T}^d_N} F_N(x/N)^2,$$

for some finite constant C_0 which depends on a, g and W. By assumption on the sequence $\{F_N\}$, for every $\gamma > 0$, this expression vanishes as $N \uparrow \infty$ and then $\varepsilon \downarrow 0$. This concludes the proof of the Lemma. □

1.5 Energy estimate

We prove in this section that any limit point $\mathbb{Q}_W^*$ of the sequence $\mathbb{Q}_{\mu_N}^{W,N}$ is concentrated on trajectories $\rho(t,u)du$ having finite energy. A more comprehensive treatment of energies can be found in [34].

Denote by ∂_{x_j} the partial derivative of a function with respect to the j-th coordinate, and by $C^{0,1_j}([0,T]\times\mathbb{T}^d)$ the set of continuous functions with continuous partial derivative in the j-th coordinate. Let $L^2_{x_j\otimes W_j}([0,T]\times\mathbb{T}^d)$ be the Hilbert space of measurable functions $H:[0,T]\times\mathbb{T}^d\to\mathbb{R}$ such that

$$\int_0^T ds \int_{\mathbb{T}^d} d(x_j\otimes W_j)\, H(s,u)^2 \;<\; \infty\;,$$

where $d(x_j\otimes W_j)$ represents the product measure in $\mathbb{T}^d$ obtained from Lesbegue's measure in $\mathbb{T}^{d-1}$ and the measure induced by W_j:

$$d(x_j\otimes W_j) \;=\; dx_1\dots dx_{j-1}\, dW_j\, dx_{j+1}\dots dx_d\;,$$

endowed with the inner product $\langle\!\langle H,G\rangle\!\rangle_{x_j\otimes W_j}$ defined by

$$\langle\!\langle H,G\rangle\!\rangle_{x_j\otimes W_j} \;=\; \int_0^T ds\int_{\mathbb{T}^d} d(x_j\otimes W_j)\, H(s,u)\, G(s,u)\;.$$

Let $\mathbb{Q}_W^*$ be a limit point of the sequence $\mathbb{Q}_{\mu_N}^{W,N}$, and assume, without loss of generality, that the sequence $\mathbb{Q}_{\mu_N}^{W,N}$ converges to $\mathbb{Q}_W^*$.

Proposition 1.5.1. *The measure $\mathbb{Q}_W^*$ is concentrated on paths $\rho(t,x)dx$ with the property that for all $j=1,\dots,d$ there exists a function in $L^2_{x_j\otimes W_j}([0,T]\times\mathbb{T}^d)$, denoted by $d\Phi/dW_j$, such that*

$$\int_0^T ds\int_{\mathbb{T}^d} dx\,(\partial_{x_j}H)(s,x)\,\Phi(\rho(s,x)) \;=\; -\int_0^T ds\int_{\mathbb{T}} d(x_j\otimes W_j(x))\,(d\Phi/dW_j)(s,x)\,H(s,x),$$

for all functions H in $C^{0,1_j}([0,T]\times\mathbb{T}^d)$.

The previous proposition follows from the next Lemma. Recall the definition of the constant K_0 given in (1.4.4).

Lemma 1.5.2. *There exists a finite constant K_1, depending only on a, such that*

$$E_{\mathbb{Q}_W^*}\left[\sup_H\left\{\int_0^T ds\int_{\mathbb{T}^d} dx\;(\partial_{x_j}H)(s,x)\,\Phi(\rho(s,x)) - K_1\int_0^T ds\int_{\mathbb{T}^d} H(s,x)^2\, d(x_j\otimes W_j(x))\right\}\right] \;\le\; K_0\;,$$

where the supremum is carried over all functions $H\in C^{0,1_j}([0,T]\times\mathbb{T}^d)$.

Proof of Proposition 1.5.1. Denote by $\ell: C^{0,1_j}([0,T]\times\mathbb{T}^d)\to\mathbb{R}$ the linear functional defined by

$$\ell(H) \;=\; \int_0^T ds\int_{\mathbb{T}^d} dx\,(\partial_{x_j}H)(s,x)\,\Phi(\rho(s,x))\;.$$

Since $C^{0,1}([0,T]\times\mathbb{T}^d)$ is dense in $L^2_{x_j\otimes W_j}([0,T]\times\mathbb{T}^d)$, by Lemma 1.5.2, ℓ is $\mathbb{Q}^*_W$-almost surely finite in $L^2_{x_j\otimes W_j}([0,T]\times\mathbb{T}^d)$. In particular, by Riesz representation theorem, there exists a function G in $L^2_{x_j\otimes W_j}([0,T]\times\mathbb{T}^d)$ such that

$$\ell(H) \;=\; -\int_0^T ds \int_{\mathbb{T}^d} d(x_j\otimes W_j(x))\, H(s,x)\, G(s,x)\;.$$

This concludes the proof of the proposition. □

For a smooth function $H\colon \mathbb{T}^d\to\mathbb{R}$, $\delta>0$, $\varepsilon>0$ and a positive integer N, define $W^j_N(\varepsilon,\delta,H,\eta)$ by

$$\begin{aligned} W^j_N(\varepsilon,\delta,H,\eta) \;&=\; \sum_{x\in\mathbb{T}^d_N} H(x/N)\frac{1}{\varepsilon N}\left\{\Phi(\eta^{\delta N}(x))-\Phi(\eta^{\delta N}(x+\varepsilon N e_j))\right\}\\ &-\; \frac{K_1}{\varepsilon N}\sum_{x\in\mathbb{T}^d_N} H(x/N)^2\{W_j([x_j+\varepsilon N+1]/N)-W_j(x_j/N)\}\;. \end{aligned}$$

The proof of Lemma 1.5.2 relies on the following result:

Lemma 1.5.3. *Consider a sequence $\{H_\ell,\ \ell\ge 1\}$ dense in $C^{0,1}([0,T]\times\mathbb{T}^d)$. For every $k\ge 1$, and every $\varepsilon>0$,*

$$\overline{\lim_{\delta\to 0}}\ \overline{\lim_{N\to\infty}}\ \mathbb{E}_{\mu^N}\left[\max_{1\le i\le k}\left\{\int_0^T W^j_N(\varepsilon,\delta,H_i(s,\cdot),\eta_s)\,ds\right\}\right]\;\le\; K_0\;.$$

Proof. It follows from the replacement lemma that in order to prove the Lemma we just need to show that

$$\overline{\lim_{N\to\infty}}\ \mathbb{E}_{\mu^N}\left[\max_{1\le i\le k}\left\{\int_0^T W^j_N(\varepsilon,H_i(s,\cdot),\eta_s)\,ds\right\}\right]\;\le\; K_0\;,$$

where

$$\begin{aligned} W^j_N(\varepsilon,H,\eta) \;&=\; \frac{1}{\varepsilon N}\sum_{x\in\mathbb{T}^d_N} H(x/N)\left\{\tau_x g(\eta)-\tau_{x+\varepsilon N e_j} g(\eta)\right\}\\ &-\; \frac{K_1}{\varepsilon N}\sum_{x\in\mathbb{T}^d_N} H(x/N)^2\{W_j([x_j+\varepsilon N+1]/N)-W_j(x_j/N)\}\;, \end{aligned}$$

and $g(\eta)=\eta(0)+a\eta(0)\eta(e_j)$.

By the entropy and Jensen's inequalities, for each fixed N, the previous expectation is bounded above by

$$\frac{H(\mu^N|\nu_\alpha)}{N^d}\;+\;\frac{1}{N^d}\log\mathbb{E}_{\nu_\alpha}\left[\exp\left\{\max_{1\le i\le k}\left\{N^d\int_0^T ds\, W^j_N(\varepsilon,H_i(s,\cdot),\eta_s)\right\}\right\}\right]\;.$$

By (1.4.4), the first term is bounded by K_0. Since $\exp\{\max_{1\le j\le k} a_j\}$ is bounded above by $\sum_{1\le j\le k}\exp\{a_j\}$, and since $\overline{\lim}_N N^{-d}\log\{a_N+b_N\}$ is less than or equal to the maximum of $\overline{\lim}_N N^{-d}\log a_N$ and $\overline{\lim}_N N^{-d}\log b_N$, the limit, as $N\uparrow\infty$, of the second term in the previous expression is less than or equal to

$$\max_{1\le i\le k}\ \overline{\lim_{N\to\infty}}\ \frac{1}{N^d}\log\mathbb{E}_{\nu_\alpha}\left[\exp\left\{N^d\int_0^T ds\, W^j_N(\varepsilon,H_i(s,\cdot),\eta_s)\right\}\right]\;.$$

We now prove that, for each fixed i, the above limit is non-positive for a convenient choice of the constant K_1.

Fix $1\le i\le k$. By Feynman–Kac formula and the variational formula for the largest eigenvalue of a symmetric operator, the previous expression is bounded above by

$$\int_0^T ds\,\sup_f\left\{\int W^j_N(\varepsilon,H_i(s,\cdot),\eta)f(\eta)\nu_\alpha(d\eta)-\frac{1}{N^{d-2}}I^\xi_N(f)\right\}\;,$$

for each fixed N. In this formula the supremum is taken over all probability densities f with respect to ν_α.

To conclude the proof, rewrite

$$\eta(x)\eta(x+e_j) - \eta(x+\varepsilon N e_j)\eta(x+(\varepsilon N+1)e_j)$$

as

$$\eta(x)\{\eta(x+e_j) - \eta(x+(\varepsilon N+1)e_j)\} + \eta(x+(\varepsilon N+1)e_j)\{\eta(x) - \eta(x+\varepsilon N e_j)\},$$

and repeat the arguments presented in the proof of Lemma 1.4.3. □

Proof of Lemma 1.5.2. Assume without loss of generality that $\mathbb{Q}^{W,N}_{\mu_N}$ converges to $\mathbb{Q}^*_W$. Consider a sequence $\{H_\ell,\, \ell \geq 1\}$ dense in $C^{0,1_j}([0,T]\times\mathbb{T}^d)$. By Lemma 1.5.3, for every $k \geq 1$

$$\begin{aligned}\varlimsup_{\delta\to 0} E_{\mathbb{Q}^*_W}\Bigg[\max_{1\leq i\leq k}\Bigg\{\frac{1}{\varepsilon}\int_0^T ds \int_{\mathbb{T}^d} dx\, H_i(s,x)\ \{\Phi(\rho^\delta_s(x)) - \Phi(\rho^\delta_s(x+\varepsilon e_j))\}\\ -\ \frac{K_1}{\varepsilon}\int_0^T ds\int_{\mathbb{T}^d} dx\, H_i(s,x)^2\,[W_j(x_j+\varepsilon) - W_j(x_j)]\Bigg\}\Bigg]\ \leq\ K_0\ ,\end{aligned}$$

where $\rho^\delta_s(x) = (\rho_s * \iota_\delta)(x)$ and ι_δ is the approximation of the identity $\iota_\delta(\cdot) = (\delta)^{-d}\mathbf{1}\{[0,\delta]^d\}(\cdot)$.

Letting $\delta \downarrow 0$, changing variables, and then letting $\varepsilon \downarrow 0$, we obtain that

$$\begin{aligned}E_{\mathbb{Q}^*_W}\Bigg[\max_{1\leq i\leq k}\Bigg\{\int_0^T ds\int_{\mathbb{T}^d}(\partial_{x_j}H_i)(s,x)\Phi(\rho(s,x))\,dx\\ -\ K_1\int_0^T ds\int_{\mathbb{T}^d} H_i(s,x)^2 d(x_j\otimes W_j(x))\Bigg\}\Bigg]\ \leq\ K_0\ .\end{aligned}$$

To conclude the proof, we apply the monotone convergence theorem, and recall that $\{H_\ell,\, \ell\geq 1\}$ is a dense sequence in $C^{0,1_j}([0,T]\times\mathbb{T}^d)$ for the norm $\|H\|_\infty + \|(\partial_{x_j}H)\|_\infty$. □

Chapter 2

W-Sobolev spaces: Theory, Homogenization and Applications

The space of functions that admit differentiation in a weak sense has been widely studied in the mathematical literature. The usage of such spaces provides a wide application to the theory of partial differential equations (PDE), and to many other areas of pure and applied mathematics. These spaces have become associated with the name of the late Russian mathematician S. L. Sobolev, although their origins predate his major contributions to their development in the late 1930s. In theory of PDEs, the idea of Sobolev space allows one to introduce the notion of weak solutions whose existence, uniqueness, regularities, and well-posedness are based on tools of functional analysis.

In classical theory of PDEs, two important classes of equations are: elliptic and parabolic PDEs. They are second-order PDEs, with some constraints (coerciveness) in the higher-order terms. The elliptic equations typically model the flow of some chemical quantity within some region, whereas the parabolic equations model the time evolution of such quantities. Consider the following particular classes of elliptic and parabolic equations:

$$\sum_{i=1}^{d} \partial_{x_i}\partial_{x_i} u(x) = g(x), \quad \text{and} \quad \begin{cases} \partial_t u(t,x) = \sum_{i=1}^{d} \partial_{x_i}\partial_{x_i} u(t,x), \\ u(0,x) = g(x), \end{cases} \tag{2.0.1}$$

for $t \in (0,T]$ and $x \in D$, where D is some suitable domain, and g is a function. Sobolev spaces are the natural environment to treat equations like (2.0.1) - an elegant exposition of this fact can be found in [11].

Consider the following generalization of the above equations:

$$\sum_{i=1}^{d} \partial_{x_i}\partial_{W_i} u(x) = g(x), \quad \text{and} \quad \begin{cases} \partial_t u(t,x) = \sum_{i=1}^{d} \partial_{x_i}\partial_{W_i} u(t,x), \\ u(0,x) = g(x), \end{cases} \tag{2.0.2}$$

where ∂_{W_i} stands for the generalized derivative operator, and for each i, W_i is a one-dimensional strictly increasing (not necessarily continuous) function, as in Chapter 1. Note that if $W_i(x_i) = x_i$, we obtain the equations in (2.0.1). This notion of generalized derivative has been studied by several authors in the literature, see for instance, [8, 16, 25, 26, 27]. We also call attention to [8] since it provides a detailed study of such notion. The equations in (2.0.2) have the same physical interpretation as the equations in (2.0.1). However, the latter covers more general situations. For instance, [18] and chapter 1 argue that these equations may be used to model a diffusion of particles within a region with membranes induced by the discontinuities of the functions W_i. Unfortunately, the standard Sobolev spaces are not suitable for being used as the space of weak solutions of equations in the form of (2.0.2).

One of our goals in this work is to define and obtain some properties of a space, which we call *W-Sobolev space*. This space lets us formalize a notion of weak generalized derivative in such a way that, if a function is W-differentiable in the strong sense, it will also be differentiable in the weak sense, with their derivatives coinciding. Moreover, the W-Sobolev space will coincide with the standard Sobolev space if $W_i(x_i) = x_i$ for all i. With this in mind, we will be able to define weak solutions of equations in

(2.0.2). We will prove that there exist weak solutions for such equations, and also, for some cases, the uniqueness of such weak solutions. Some analogous to classical results of Sobolev spaces are obtained, such as Poincaré's inequality and Rellich-Kondrachov's compactness theorem.

Besides the treatment of elliptic and parabolic equations in terms of these W-Sobolev spaces, we are also interested in studying *Homogenization* and *Hydrodynamic Limits*. The study of homogenization is motivated by several applications in mechanics, physics, chemistry and engineering. For example, when one studies the thermal or electric conductivity in heterogeneous materials, the macroscopic properties of crystals or the structure of polymers, are typically described in terms of linear or non-linear PDEs for medium with periodic or quasi-periodic structure, or, more generally, stochastic.

We will consider stochastic homogenization. In the stochastic context, several works on homogenization of operators with random coefficients have been published (see, for instance, [30, 31] and references therein). In homogenization theory, only the stationarity of such random field is used. The notion of stationary random field is formulated in such a manner that it covers many objects of non-probabilistic nature, e.g., operators with periodic or quasi-periodic coefficients.

The focus of our approach is to study the asymptotic behavior of effective coefficients for a family of random difference schemes, whose coefficients can be obtained by the discretization of random high-contrast lattice structures. In this sense, we want to extend the theory of homogenization of random operators developed in [31], as well as to prove its main Theorem (Theorem 2.16) to the context in which we have weak generalized derivatives.

Lastly, as an application of all the theory developed for W-Sobolev spaces, elliptic operators, parabolic equations and homogenization, we prove a hydrodynamic limit for a *process with conductances in random environments*. Hydrodynamic limit for process with conductances have been obtained in [18] for the one-dimensional setup and in Chapter 1 for the d-dimensional setup. However, with the tools developed in our present Chapter, the proof of the hydrodynamic limit on a more general setup (in random environments) turns out to be simpler and much more natural. Furthermore, the proof of this hydrodynamic limit also provides an existence theorem for the generalized parabolic equations such as the one in (2.0.2).

The random environment we considered is governed by the coefficients of the discrete formulation of the model (the process on the lattice). It is possible to obtain other formulations of random environments, for instance, in [14] they proved a hydrodynamic limit for a gradient process with conductances in a random environment whose randomness consists of the random choice of the conductances. The hydrodynamic limit for a gradient process without conductances on the random environment we are considering was proved in [20]. We would like to mention that in [13] a process evolving on a percolation cluster (a lattice with some bonds removed randomly) was considered and the resulting process turned out to be non-gradient. However, the homogenization tools facilitated the proof of the hydrodynamic limit, which made the proof much simpler than the usual proof of hydrodynamic limit for non-gradient processes (see for instance [23, Chapter 7]).

We now describe the organization of the Chapter. In Section 2.1 we define the W-Sobolev spaces and obtain some results, namely, approximation by smooth functions, Poincaré's inequality, Rellich-Kondrachov theorem (compact embedding), and a characterization of the dual of the W-Sobolev spaces. In Section 2.2 we define the W-generalized elliptic equations, and what we call by weak solutions. We then obtain some energy estimates and use them together with Lax-Milgram's theorem to conclude results regarding existence, uniqueness and boundedness of such weak solutions. In Section 2.3 we define the W-generalized parabolic equations, their weak solutions, and prove uniquenesses of these weak solutions. Moreover, a notion of energy is also introduced in this Section. Section 2.4 consists in obtaining discrete analogous results to the ones of the previous sections. This Section serves as preamble for the subsequent sections. In Section 2.5 we define the random operators we are interested and obtain homogenization results for them. Finally, Section 2.6 concludes the Chapter with an application that is interesting for both probability and theoretical physics, which is the hydrodynamic limit for a process in random environments with conductances. This application uses results from all the previous sections and provides a proof for existence of weak solutions of W-generalized parabolic equations.

2.1 W-Sobolev spaces

This Section is devoted to the definition and derivation of properties of the W-Sobolev spaces. We begin by introducing some notation, stating some known results, and giving a precise definition of these spaces. Poincaré's inequality, Rellich-Kondrachov theorem and a characterization of the dual space of these Sobolev spaces are also obtained.

Fix a function $W : \mathbb{R}^d \to \mathbb{R}$ as in Chapter 1:

$$W(x_1, \dots, x_d) = \sum_{k=1}^{d} W_k(x_k),$$

where each $W_k : \mathbb{R} \to \mathbb{R}$ is a *strictly increasing* right continuous function with left limits (càdlàg), periodic in the sense that for all $u \in \mathbb{R}$

$$W_k(u+1) - W_k(u) = W_k(1) - W_k(0).$$

Let $L^2_{x^k \otimes W_k}(\mathbb{T}^d)$ be the Hilbert space of measurable functions $H : \mathbb{T}^d \to \mathbb{R}$ such that

$$\int_{\mathbb{T}^d} d(x^k \otimes W_k)\, H(x)^2 \;<\; \infty,$$

where $d(x^k \otimes W_k)$ represents the product measure in $\mathbb{T}^d$ obtained from Lesbegue's measure in $\mathbb{T}^{d-1}$ and the measure induced by W_k in $\mathbb{T}$:

$$d(x^k \otimes W_k) \;=\; dx_1 \cdots dx_{k-1}\, dW_k\, dx_{k+1} \cdots dx_d.$$

Denote by $\langle H, G \rangle_{x^k \otimes W_k}$ the inner product of $L^2_{x^k \otimes W_k}(\mathbb{T}^d)$:

$$\langle H, G \rangle_{x^k \otimes W_k} \;=\; \int_{\mathbb{T}^d} d(x^k \otimes W_k)\, H(x)\, G(x)\ ,$$

and by $\| \cdot \|_{x^k \otimes W_k}$ the norm induced by this inner product.

Recall the definition of the operator $\mathbb{L}_W : \mathbb{D}_W \to L^2(\mathbb{T}^d)$ given in (1.1.5).

Lemma 2.1.1. *Let $f, g \in \mathbb{D}_W$, then for $i = 1, \dots, d$,*

$$\int_{\mathbb{T}^d} \big(\partial_{x_i} \partial_{W_i} f(x)\big) g(x)\ dx \;=\; -\int_{\mathbb{T}^d} (\partial_{W_i} f)(\partial_{W_i} g) d(x^i \otimes W_i).$$

In particular,

$$\int_{\mathbb{T}^d} \mathbb{L}_W f(x) g(x)\ dx \;=\; -\sum_{i=1}^{d} \int_{\mathbb{T}^d} (\partial_{W_i} f)(\partial_{W_i} g) d(x^i \otimes W_i).$$

Proof. Let $f, g \in \mathbb{D}_W$. By Fubini's theorem

$$\int_{\mathbb{T}^d} \mathcal{L}_{W_i} f(x) g(x) dx = \int_{\mathbb{T}^{d-1}} \left[\int_{\mathbb{T}} \mathcal{L}_{W_i} f(x) g(x) dx_i \right] dx^i,$$

where dx^i is the Lebesgue product measure in $\mathbb{T}^{d-1}$ on the coordinates $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d$.

An application of [18, Lemma 3.1 (b)] and again Fubini's theorem concludes the proof of this Lemma. □

Let $L^2_{x^j \otimes W_j, 0}(\mathbb{T}^d)$ be the closed subspace of $L^2_{x^j \otimes W_j}(\mathbb{T}^d)$ consisting of the functions that have zero mean with respect to the measure $d(x^j \otimes W_j)$:

$$\int_{\mathbb{T}^d} f d(x^j \otimes W_j) = 0.$$

Finally, using the characterization of the functions in $\mathcal{D}_{W_j}$ given in Proposition 1.1.1, and the definition of $\mathbb{D}_W$, we have that the set $\{\partial_{W_j} h; h \in \mathbb{D}_W\}$ is dense in $L^2_{x^j \otimes W_j, 0}(\mathbb{T}^d)$.

2.1.1 The W-Sobolev space

We define the Sobolev space of W-generalized derivatives as the space of functions $g \in L^2(\mathbb{T}^d)$ such that for each $i = 1, \ldots, d$ there exist fuctions $G_i \in L^2_{x^i \otimes W_i,0}(\mathbb{T}^d)$ satisfying the following integral by parts identity.

$$\int_{\mathbb{T}^d} \left(\partial_{x_i}\partial_{W_i} f\right)\, g\, dx \; = \; - \int_{\mathbb{T}^d} (\partial_{W_i} f)\; G_i d(x^i \otimes W_i), \tag{2.1.1}$$

for every function $f \in \mathbb{D}_W$. We denote this space by $\tilde{H}_{1,W}(\mathbb{T}^d)$. A standard measure-theoretic argument allows one to prove that for each function $g \in \tilde{H}_{1,W}(\mathbb{T}^d)$ and $i = 1, \ldots, d$, we have a unique function G_i that satisfies (2.1.1). Note that $\mathbb{D}_W \subset \tilde{H}_{1,W}(\mathbb{T}^d)$. Moreover, if $g \in \mathbb{D}_W$ then $G_i = \partial_{W_i} g$. For this reason for each function $g \in \tilde{H}_{1,W}$ we denote G_i simply by $\partial_{W_i} g$, and we call it the ith *generalized weak derivative* of the function g with respect to W.

Lemma 2.1.2. *The set $\tilde{H}_{1,W}(\mathbb{T}^d)$ is a Hilbert space with respect to the inner product*

$$\langle f, g\rangle_{1,W} \; = \; \langle f, g\rangle + \sum_{i=1}^{d} \int_{\mathbb{T}^d} (\partial_{W_i} f)(\partial_{W_i} g)\; d(x^i \otimes W_i) \tag{2.1.2}$$

Proof. Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\tilde{H}_{1,W}(\mathbb{T}^d)$, and denote by $\|\cdot\|_{1,W}$ the norm induced by the inner product (2.1.2). By the definition of the norm $\|\cdot\|_{1,W}$, we obtain that $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{T}^d)$ and that $(\partial_{W_i} g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2_{x^i\otimes W_i,0}(\mathbb{T}^d)$ for each $i = 1, \ldots, d$. Therefore, there exist functions $g \in L^2(\mathbb{T}^d)$ and $G_i \in L^2_{x^i\otimes W_i,0}(\mathbb{T}^d)$ such that $g = \lim_{n\to\infty} g_n$, and $G_i = \lim_{n\to\infty} \partial_{W_i} g_n$. It remains to be proved that G_i is, in fact, the ith generalized weak derivative of g with respect to W. But this follows from a simple calculation: for each $f \in \mathbb{D}_W$ we have

$$\begin{aligned}
\int_{\mathbb{T}^d} \left(\partial_{x_i}\partial_{W_i} f\right) g dx \;&=\; \lim_{n\to\infty} \int_{\mathbb{T}^d} \left(\partial_{x_i}\partial_{W_i} f\right) g_n dx \\
&=\; -\lim_{n\to\infty} \int_{\mathbb{T}^d} (\partial_{W_i} f)(\partial_{W_i} g) d(x^i \otimes W_i) \\
&=\; -\int_{T^d} (\partial_{W_i} f) G_i d(x^i \otimes W_i),
\end{aligned}$$

where we used Hölder's inequality to pass the limit through the integral sign. □

2.1.2 Approximation by smooth functions and the energetic space

We will now obtain approximation of functions in the Sobolev space $\tilde{H}_{1,W}(\mathbb{T}^d)$ by functions in $\mathbb{D}_W$. Note that the functions in $\mathbb{D}_W$ can be seen as smooth, in the sense that one may apply the operator $\mathbb{L}_W$ to these functions in the strong sense.

Let us introduce $\langle\cdot,\cdot\rangle_{1,W}$ the inner product on $\mathbb{D}_W$ defined by

$$\langle f, g\rangle_{1,W} \; = \; \langle f, g\rangle \; + \; \langle -\mathbb{L}_W f, g\rangle, \tag{2.1.3}$$

and note that by Lemma 2.1.1,

$$\langle f, g\rangle_{1,W} \; = \; \langle f, g\rangle \; + \; \sum_{i=1}^{d} \int_{\mathbb{T}^d} (\partial_{W_i} f)(\partial_{W_i} g) d(x^i \otimes W_i).$$

Let $H_{1,W}(\mathbb{T})$ be the set of all functions f in $L^2(\mathbb{T}^d)$ for which there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{D}_W$ such that f_n converges to f in $L^2(\mathbb{T}^d)$ and f_n is a Cauchy sequence for the inner product $\langle\cdot,\cdot\rangle_{1,W}$. Such sequence $(f_n)_{n\in\mathbb{N}}$ is called *admissible* for f.

For f, g in $H_{1,W}(\mathbb{T}^d)$, define

$$\langle f, g\rangle_{1,W} \; = \; \lim_{n\to\infty} \langle f_n, g_n\rangle_{1,W}\;, \tag{2.1.4}$$

where $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ are admissible sequences for f, and g, respectively. By [40, Proposition 5.3.3], this limit exists and does not depend on the admissible sequence chosen; the set $\mathbb{D}_W$ is dense in $H_{1,W}$;

and the embedding $H_{1,W} \subset L^2(\mathbb{T}^d)$ is continuous. Moreover, $H_{1,W}(\mathbb{T}^d)$ endowed with the inner product $\langle \cdot, \cdot \rangle_{1,W}$ just defined is a Hilbert space. Denote $\|\cdot\|_{1,W}$ the norm in $H_{1,W}$ induced by $\langle \cdot, \cdot \rangle_{1,W}$. The space $H_{1,W}(\mathbb{T}^d)$ is called energetic space. For more details on the theory of energetic spaces see [40, Chapter 5].

Note that $H_{1,W}$ is the space of functions that can be approximated by functions in $\mathbb{D}_W$ with respect to the norm $\| \cdot \|_{1,W}$. The following Proposition shows that this space is, in fact, the Sobolev space $\tilde{H}_{1,W}(\mathbb{T}^d)$.

Proposition 2.1.3 (Approximation by smooth functions)**.** *We have the equality of the sets*

$$\tilde{H}_{1,W}(\mathbb{T}^d) = H_{1,W}(\mathbb{T}^d).$$

In particular, we can approximate any function f in the Sobolev space $\tilde{H}_{1,W}(\mathbb{T}^d)$ by functions in $\mathbb{D}_W$.

Proof. Fix $g \in H_{1,W}(\mathbb{T}^d)$. By definition, there exists a sequence g_n in $\mathbb{D}_W$ such that g_n converges to g in $L^2(\mathbb{T}^d)$ and g_n is Cauchy for the inner product $\langle \cdot, \cdot \rangle_{1,W}$. So, for each $i = 1, \dots, d$ there exists functions $G_i \in L^2_{x^i \otimes W_i,0}(\mathbb{T}^d)$ such that $\partial_{W_i} g_n$ converges to G_i in $L^2_{x^i \otimes W_i,0}(\mathbb{T}^d)$. Applying the Hölder's inequality, we deduce that for every $f \in \mathbb{D}_W$

$$\int_{\mathbb{T}^d} \big(\partial_{x_i}\partial_{W_i} f\big) g \; dx \;=\; \lim_{n\to\infty} \int_{\mathbb{T}^d} \big(\partial_{x_i}\partial_{W_i} f\big) g_n \; dx.$$

By Lemma 2.1.1, we obtain

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{T}^d} \big(\partial_{x_i}\partial_{W_i} f\big) g_n dx \;&=\; \lim_{n\to\infty} \int_{\mathbb{T}^d} (\partial_{W_i} f)(\partial_{W_i} g_n) \; d(x^i \otimes W_i) \\ &=\; - \int_{\mathbb{T}^d} (\partial_{W_i} f) G_i \; d(x^i \otimes W_i). \end{aligned}$$

Then, $g \in \tilde{H}_{1,W}(\mathbb{T}^d)$ and therefore $H_{1,W}(\mathbb{T}^d) \subset \tilde{H}_{1,W}(\mathbb{T}^d)$.

We will now prove that $H_{1,W}(\mathbb{T}^d)$ is dense in $\tilde{H}_{1,W}(\mathbb{T}^d)$, and since both of them are complete, they are equal. Note that since $\mathbb{D}_W$ is dense in $L^2(\mathbb{T}^d)$ and $\mathbb{D}_W \subset H_{1,W}(\mathbb{T}^d)$, we have that $H_{1,W}(\mathbb{T}^d)$ is also dense in $L^2(\mathbb{T}^d)$.

Therefore, given a function $g \in \tilde{H}_{1,W}(\mathbb{T}^d)$, we can approximate g by a sequence of functions $(f_n)_{n\in\mathbb{N}}$ in $H_{1,W}(\mathbb{T}^d)$ with respect to the $L^2(\mathbb{T}^d)$ norm. Let $F_{i,n}$ be the ith generalized weak derivative of f_n with respect to W. We have, therefore, for each $h \in \mathbb{D}_W$

$$\lim_{n\to\infty} \int_{\mathbb{T}^d} (\partial_{W_i} h)(F_{i,n} - G_i) d(x^i \otimes W_i) = - \lim_{n\to\infty} \int_{\mathbb{T}^d} \big(\partial_{x_i}\partial_{W_i} h\big)(f_n - g) dx = 0.$$

Denote by $\mathcal{F}_{i,n} : L^2_{x^i \otimes W_i,0}(\mathbb{T}^d) \to \mathbb{R}$ the sequence of bounded linear functionals induced by $F_{i,n} - G_i$:

$$\mathcal{F}_{i,n}(h) := \int_{\mathbb{T}^d} h[F_{i,n} - G_i] d(x^i \otimes W_i),$$

for $h \in L^2_{x^i \otimes W_i,0}(\mathbb{T}^d)$. We then note that, since the set $\{\partial_{W_i} h; h \in \mathbb{D}_W\}$ is dense in $L^2_{x^i \otimes W_i,0}(\mathbb{T}^d)$, $\mathcal{F}_{i,n}$ converges to 0 pointwisely. By Banach-Steinhaus' Theorem, $\mathcal{F}_{i,n}$ converges strongly to 0, and, thus, $F_{i,n}$ converges to G_i in $L^2_{x^i \otimes W_i,0}(\mathbb{T}^d)$, for each $i = 1, \dots, d$. Therefore, f_n converges to g in $L^2(\mathbb{T}^d)$ and $\partial_{W_i} f_n$ converges to G_i in $L^2_{x^i \otimes W_i,0}(\mathbb{T}^d)$ for each i, i.e., f_n converges to g with the norm $\| \cdot \|_{1,W}$, and the density of $H_{1,W}(\mathbb{T}^d)$ in $\tilde{H}_{1,W}(\mathbb{T}^d)$ follows. □

The next Corollary shows an analogous of the classic result for Sobolev spaces with dimension $d = 1$, which states that every function in the one-dimensional Sobolev space is absolutely continuous.

Corollary 2.1.4. *A function f in $L^2(\mathbb{T})$ belongs to the Sobolev space $\tilde{H}_{1,W}(\mathbb{T})$ if and only if there exists F in $L^2_W(\mathbb{T})$ and a finite constant c such that*

$$\int_{(0,1]} F(y)\, dW(y) \;=\; 0 \quad \text{and} \quad f(x) \;=\; c \,+\, \int_{(0,x]} F(y)\, dW(y)$$

Lebesgue almost surely.

Proof. In [18] the energetic extension $H_{1,W}(\mathbb{T})$ has the characterization given in Corollary 2.1.4. By Proposition 2.1.3 we have that these spaces coincide, and hence the proof follows. □

From Proposition 2.1.3, we may use the notation $H_{1,W}(\mathbb{T}^d)$ for the Sobolev space $\tilde{H}_{1,W}(\mathbb{T}^d)$. Another interesting feature we have on this space, which is very useful in the study of elliptic equations, is the Poincaré inequality:

Corollary 2.1.5 (Poincaré Inequality)**.** *For all $f \in H_{1,W}(\mathbb{T}^d)$ there exists a finite constant C such that*

$$\begin{aligned} \left\| f - \int_{\mathbb{T}^d} f\, dx \right\|^2_{L^2(\mathbb{T}^d)} &\leq C \sum_{i=1}^{n} \int_{\mathbb{T}^d} (\partial_{W_i} f)^2\, d(x^i \otimes W_i) \\ &:= C \|\nabla_W f\|^2_{L^2_W(\mathbb{T}^d)}. \end{aligned}$$

Proof. We begin by introducing some notations. For $x, y \in \mathbb{T}^d$, $i = 0, \dots, d$ and $t \in \mathbb{T}$, denote

$$z(x, y, i) = (x_1, \dots, x_{d-i}, y_{d-i+1}, \dots, y_d) \in \mathbb{T}^d$$

and

$$z(x, y, t, i) = (x_1, \dots, x_{d-i}, t, y_{d-i+2}, \dots, y_d) \in \mathbb{T}^d.$$

With this notation, we may write $f(x) - f(y)$ as the telescopic sum

$$f(x) - f(y) = \sum_{i=1}^{d} f(z(x, y, i-1)) - f(z(x, y, i)).$$

We are now in conditions to prove this Lemma. Let $f \in \mathbb{D}_W$, then

$$\begin{aligned} &\left\| f - \int_{\mathbb{T}^d} f dx \right\|^2_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} \Big[\int_{\mathbb{T}^d} f(x) - f(y) dy \Big]^2 dx \\ &= \int_{\mathbb{T}^d} \Big[\int_{\mathbb{T}^d} \sum_{i=1}^{d} \int_{y_i}^{x_i} \partial_{W_i} f(z(x, y, t, i)) dW_i(t) dy \Big]^2 dx \\ &\leq \int_{\mathbb{T}^d} \Big[\int_{\mathbb{T}^d} \sum_{i=1}^{d} \int_{\mathbb{T}} \Big| \partial_{W_i} f(z(x, y, t, i)) \Big| dW_i(t) dy \Big]^2 dx \\ &\leq \int_{\mathbb{T}^d} \Big[\sum_{i=1}^{d} \int_{\mathbb{T}^{d-i+1}} \Big| \partial_{W_i} f(z(x, y, t, i)) \Big| dW_{d-i}(t) \otimes y_{d-i+1} \otimes \cdots \otimes y_d \Big]^2 dx \\ &\leq C \int_{\mathbb{T}^d} \sum_{i=1}^{d} \int_{\mathbb{T}^{d-i+1}} \Big| \partial_{W_i} f(z(x, y, t, i)) \Big|^2 dW_{d-i}(t) \otimes dy_{d-i+1} \otimes \cdots \otimes dy_d dx \\ &= C \sum_{i=1}^{d} \int_{\mathbb{T}^d} \Big(\partial_{W_i} f \Big)^2 d(x^i \otimes W_i), \end{aligned}$$

where in the next-to-last inequality, we used Jensen's inequality and the elementary inequality $(\sum_i x_i)^2 \leq C \sum_i x_i^2$ for some positive constant C. To conclude the proof, one uses Proposition 2.1.3 to approximate functions in $H_{1,W}(\mathbb{T}^d)$ by functions in $\mathbb{D}_W$. □

2.1.3 A Rellich-Kondrachov theorem

In this subsection we prove an analogous of the Rellich-Kondrachov theorem for the W-Sobolev spaces. We begin by stating this result in dimension 1, whose proof can be found in [18, Lemma 3.3].

Lemma 2.1.6. *Fix some $k \in \{1, \dots, d\}$. The embedding $H_{1,W_k}(\mathbb{T}) \subset L^2(\mathbb{T})$ is compact.*

Recall that they proved this result for the energetic extension, but in view of Proposition 2.1.3, this result holds for our Sobolev space $H_{1,W_k}(\mathbb{T})$.

Proposition 2.1.7 (Rellich-Kondrachov). *The embedding $H_{1,W}(\mathbb{T}^d) \subset \mathbb{L}^2(\mathbb{T}^d)$ is compact.*

Proof. We will outline the strategy of the proof. Using the definition of the set $\mathbb{D}_W$ and the fact that it is dense in $H_{1,W}(\mathbb{T}^d)$, it is enough to show this fact for sequences in $\mathbb{D}_W$. From this point, the main tool is Lemma 2.1.6 and Cantor's diagonal method to obtain converging subsequences.

We begin by noting that by Proposition 2.1.3, it is enough to prove that the embed $\mathbb{D}_W \subset L^2(\mathbb{T}^d)$ is compact.

Let $C > 0$ and consider a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathbb{D}_W$, with $\|v_n\|_{1,W} \leq C$ for all $n \in \mathbb{N}$. We have, by definition of $\mathbb{D}_W$ (see the definition at the beginning of Section 2.1), that each v_n can be expressed as a finite linear combination of elements in $\mathcal{A}_W$. Furthermore, each element in $\mathcal{A}_W$ is a product of elements in $\mathcal{A}_{W_k}$ for $k = 1, \ldots, d$. Therefore, we can write v_n as

$$v_n = \sum_{j=1}^{N(n)} \alpha_j^n \prod_{k=1}^{d} g_{k,j}^n = \sum_{j=1}^{N(n)} \alpha_j^n g_j^n,$$

where $g_{k,j}^n \in \mathcal{A}_{W_k}, \alpha_j^n \in \mathbb{R}, g_j^n = \prod_{k=1}^d g_{j,k}^n$, and $N(n)$ is chosen such that $N(n) \geq n$ (we can complete with zeros if necessary). Recall that these functions $g_{k,j}^n$ have $\|g_{k,j}^n\|_{L^2(\mathbb{T})} = 1$, and hence, $\|g_j^n\|_{L^2(\mathbb{T}^d)} = 1$. Moreover, the set $\{g_1^n, \ldots, g_{N(n)}^n\}$ is orthogonal in $L^2(\mathbb{T}^d)$.

From orthogonality, we obtain that

$$\sum_{j=1}^{N(n)} (\alpha_j^n)^2 \leq C^2, \qquad \text{uniformly in } n \in \mathbb{N}.$$

Note that the uniform boundedness of v_n in $H_{1,W}(\mathbb{T}^d)$ implies the uniform boundedness of $\|g_{k,j}^n\|_{1,W_k}$, for all $k = 1, \ldots, d$, $j = 1, \ldots, N(n)$ and $n \in \mathbb{N}$. Our goal now is to apply Lemma 2.1.6 to our current setup.

Consider the sequence of functions $\alpha_1^n g_{1,1}^n$ in $H_{1,W_1}(\mathbb{T})$. By Lemma 2.1.6, this sequence has a converging subsequence, and we call the limit point $\alpha_1 g_{1,1}$. Repeat this step $d-1$ times for the sequences $g_{k,1}^n$ in $H_{1,W_k}(\mathbb{T})$, for $k = 2, \ldots, d$, considering in each step a subsequence of the previous step, to obtain converging subsequences, and call their limit points $g_{k,1}$. At the end of this procedure, we obtain a converging subsequence of $\prod_{k=1}^d \alpha_1^n g_{1,k}^n$, with limit point $\prod_{k=1}^d \alpha_1 g_{1,k} \in L^2(\mathbb{T}^d)$, which we will denote by $\alpha_1 g_1$.

In the jth step, in which we want to obtain the limit point $\alpha_j g_j$, we repeat the previous idea, with the sequences $\alpha_j^n g_{j,1}^n$ and $g_{j,k}^n$, with $n \leq j$ and $k = 2, \ldots, d$. We note that it is always necessary to consider a subsequence of all the previous steps.

This procedure provides limiting functions $\alpha_j g_j$, for all $j \in \mathbb{N}$. From now on, we use the notation v_n to mean the diagonal sequence obtained to ensure the convergence of the functions $\alpha_j^n g_j^n$ to $\alpha_j g_j$. We claim that the function

$$v = \sum_{j=1}^{\infty} \alpha_j g_j$$

is well-defined and belongs to $L^2(\mathbb{T}^d)$. To prove this claim, note that the set $\{g_k\}_{k\in\mathbb{N}}$ is orthonormal by the continuity of the inner product. Suppose that there exists $N \in \mathbb{N}$ such that

$$\sum_{j=1}^{N} (\alpha_j)^2 > C^2.$$

We have that the sequence of functions

$$v_n^N := \sum_{j=1}^{N} \alpha_j^n g_j^n$$

converges to

$$v^N := \sum_{j=1}^{N} \alpha_j g_j.$$

Since $\|v_n^N\| \leq C$ uniformly in $n \in \mathbb{N}$, this yields a contradiction. Therefore $v \in L^2(\mathbb{T}^d)$ with the bound $\|v\| \leq C$.

It remains to be proved that v_n has a subsequence that converges to v. Choose N so large that $\|v - v^N\| < \epsilon/3$, $\|v_n^N - v^N\| < \epsilon/3$ and $\|v_n^N - v_n\| < \epsilon/3$, and use the triangle inequality to conclude the proof.

□

2.1.4 The space $H_W^{-1}(\mathbb{T}^d)$

Let $H_W^{-1}(\mathbb{T}^d)$ be the dual space to $H_{1,W}(\mathbb{T}^d)$, that is, $H_W^{-1}(\mathbb{T}^d)$ is the set of bounded linear functionals on $H_{1,W}(\mathbb{T}^d)$. Our objective in this subsection is to characterize the elements of this space. This proof is based on the characterization of the dual of the standard Sobolev space in $\mathbb{R}^d$ (see [11]).

We will write $(\cdot,\cdot)$ to denote the pairing between $H_W^{-1}(\mathbb{T}^d)$ and $H_{1,W}(\mathbb{T}^d)$.

Lemma 2.1.8. *$f \in H_W^{-1}(\mathbb{T}^d)$ if and only if there exist functions $f_0 \in L^2(\mathbb{T}^d)$, and $f_k \in L^2_{x^k \otimes W_k,0}(\mathbb{T}^d)$, such that*

$$f = f_0 - \sum_{i=1}^{d} \partial_{x_i} f_i, \qquad (2.1.5)$$

in the sense that for $v \in H_{1,W}(\mathbb{T}^d)$

$$(f,v) = \int_{\mathbb{T}^d} f_0 v dx + \sum_{i=1}^{d} \int_{\mathbb{T}^d} f_i (\partial_{W_i} v) d(x^i \otimes W_i).$$

Furthermore,

$$\|f\|_{H_W^{-1}} = \inf\left\{ \left(\int_{\mathbb{T}^d} \sum_{i=0}^{d} |f_i|^2 dx \right)^{1/2} ; \quad f \text{ satisfies } (2.1.5) \right\}.$$

Proof. Let $f \in H_W^{-1}(\mathbb{T}^d)$. Applying the Riesz Representation Theorem, we deduce the existence of a unique function $u \in H_{1,W}(\mathbb{T}^d)$ satisfying $(f,v) = \langle u, v\rangle_{1,W}$, for all $v \in H_{1,W}(\mathbb{T}^d)$, that is

$$\int_{\mathbb{T}^d} uv dx + \sum_{j=1}^{d} \int_{\mathbb{T}^d} (\partial_{W_j} u)(\partial_{W_j} v) d(x^j \otimes W_j) = (f,v), \qquad \text{for all } \ v \in H_{1,W}(\mathbb{T}^d). \qquad (2.1.6)$$

This establishes the first claim of the Lemma for $f_0 = u$ and $f_i = \partial_{W_i} u$, for $i = 1,\dots,d$.

Assume now that $f \in H_W^{-1}(\mathbb{T}^d)$,

$$(f,v) = \int_{\mathbb{T}^d} g_0 v dx + \sum_{i=1}^{d} \int_{\mathbb{T}^d} g_i (\partial_{W_i} v) d(x^i \otimes W_i), \qquad (2.1.7)$$

for $g_0, g_1, \dots, g_d \in L^2_{x^j \otimes W_j,0}(\mathbb{T}^d)$. Setting $v = u$ in (2.1.6), using (2.1.7), and applying the Cauchy-Schwartz inequality twice, we deduce

$$\|u\|_{1,W}^2 \leq \int_{\mathbb{T}^d} g_0^2 dx + \sum_{i=1}^{d} \int_{\mathbb{T}^d} \partial_{W_i} g_i^2 d(x^i \otimes W_i). \qquad (2.1.8)$$

From (2.1.6) it follows that

$$|(f,v)| \leq \|u\|_{1,W}$$

if $\|v\|_{1,W} \leq 1$. Consequently

$$\|f\|_{H_W^{-1}} \leq \|u\|_{1,W}.$$

Setting $v = u/\|u\|_{1,W}$ in (2.1.6), we deduce that, in fact,

$$\|f\|_{H_W^{-1}} = \|u\|_{1,W}.$$

The result now follows from the above expression and equation (2.1.8). □

2.2 W-Generalized elliptic equations

This subsection investigates the solvability of uniformly elliptic generalized partial differential equations defined below. Energy methods within Sobolev spaces are, essentially, the techniques exploited.

Let $A = (a_{ii}(x))_{d\times d}$, $x \in \mathbb{T}^d$, be a diagonal matrix function such that there exists a constant $\theta > 0$ satisfying

$$\theta^{-1} \leq a_{ii}(x) \leq \theta, \tag{2.2.1}$$

for every $x \in \mathbb{T}^d$ and $i = 1, \dots, d$. To keep notation simple, we write $a_i(x)$ to mean $a_{ii}(x)$.

Our interest lies on the study of the problem

$$T_\lambda u = f, \tag{2.2.2}$$

where $u : \mathbb{T}^d \to \mathbb{R}$ is the unknown function and $f : \mathbb{T}^d \to \mathbb{R}$ is given. Here T_λ denotes the generalized elliptic operator

$$T_\lambda u \ := \ \lambda u - \nabla A \nabla_W u \ := \ \lambda u - \sum_{i=1}^{d} \partial_{x_i}\Big(a_i(x)\partial_{W_i} u\Big). \tag{2.2.3}$$

The bilinear form $B[\cdot,\cdot]$ associated with the elliptic operator T_λ is given by

$$B[u,v] = \lambda\langle u, v\rangle + \sum_{i=1}^{d} \int a_i(x)(\partial_{W_i} u)(\partial_{W_i} v)\, d(W_i \otimes x_i), \tag{2.2.4}$$

where $u, v \in H_{1,W}(\mathbb{T}^d)$.

Let $f \in H_W^{-1}(\mathbb{T}^d)$. A function $u \in H_{1,W}(\mathbb{T}^d)$ is said to be a weak solution of the equation $T_\lambda u = f$ if

$$B[u,v] \ = \ (f, v) \ \text{ for all } \ v \in H_{1,W}(\mathbb{T}^d).$$

Recall a classic result from linear functional analysis, which provides in certain circumstances the existence and uniqueness of weak solutions of our problem, and whose proof can be found, for instance, in [11]. Let $\mathcal{H}$ be a Hilbert space endowed with inner product $<\cdot,\cdot>$ and norm $|||\cdot|||$. Also, $(\cdot,\cdot)$ denotes the pairing of $\mathcal{H}$ with its dual space.

Theorem 2.2.1 (Lax-Milgram Theorem)**.** *Assume that $B : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ is a bilinear mapping on Hilbert space $\mathcal{H}$, for which there exist constants $\alpha > 0$ and $\beta > 0$ such that for all $u, v \in \mathcal{H}$,*

$$|B[u,v]| \leq \alpha|||u|||\cdot|||v||| \ \text{ and } \ B[u,u] \geq \beta|||u|||^2.$$

Let $f : \mathcal{H} \to \mathbb{R}$ be a bounded linear functional on $\mathcal{H}$. Then there exists a unique element $u \in \mathcal{H}$ such that

$$B[u,v] = (f, v),$$

for all $v \in \mathcal{H}$.

Return now to the specific bilinear form $B[\cdot,\cdot]$ defined in (2.2.4). Our goal now is to verify the hypothesis of Lax-Milgram Theorem for our setup. We consider the cases $\lambda = 0$ and $\lambda > 0$ separately. We begin by analyzing the case in which $\lambda = 0$.

Let $H_{1,W}^{\perp}(\mathbb{T}^d)$ be the set of functions in $H_{1,W}(\mathbb{T}^d)$ which are orthogonal to the constant functions:

$$H_{1,W}^{\perp}(\mathbb{T}^d) \ = \ \{f \in H_{1,W}(\mathbb{T}^d); \int_{\mathbb{T}^d} f\, dx = 0\}.$$

The space $H_{1,W}^{\perp}(\mathbb{T}^d)$ is the natural environment to treat elliptic operators with Neumann condition.

Proposition 2.2.2 (Energy estimates for $\lambda = 0$)**.** *Let B be the bilinear form on $H_{1,W}(\mathbb{T}^d)$ defined in (2.2.4) with $\lambda = 0$. There exist constants $\alpha > 0$ and $\beta > 0$ such that for all $u, v \in H_{1,W}(\mathbb{T}^d)$,*

$$|B[u,v]| \leq \alpha\|u\|_{1,W}\ \|v\|_{1,W}$$

and for all $u \in H_{1,W}^{\perp}$

$$B[u,u] \geq \beta\|u\|_{1,W}^2.$$

Proof. By (2.2.1), the computation of the upper bound α easily follows. For the lower bound β, we have for $u \in H^{\perp}_{1,W}(\mathbb{T}^d)$,

$$\|u\|^2_{1,W} = \int_{\mathbb{T}^d} u^2\, dx + \sum_{i=1}^{d} \int_{\mathbb{T}^d} \Big(\partial_{W_i} u\Big)^2 d(x^i \otimes W_i).$$

Using Poincaré's inequality and (2.2.1), we obtain a constant $C > 0$ such that the previous expression is bounded above by

$$C \int_{\mathbb{T}^d} \Big(\partial_{W_i} u\Big)^2 d(x^i \otimes W_i) \leq CB[u,u].$$

The lemma follows from the previous estimates. □

Corollary 2.2.3. *Let $f \in L^2(\mathbb{T}^d)$. There exists a weak solution $u \in H_{1,W}(\mathbb{T}^d)$ for the equation*

$$\nabla A \nabla_W u \;=\; f \tag{2.2.5}$$

if and only if

$$\int_{\mathbb{T}^d} f dx \;=\; 0.$$

In this case, we have uniquenesses of the weak solutions if we disregard addition by constant functions. Also, let u be the unique weak solution of (2.2.5) *in $H^{\perp}_{1,W}(\mathbb{T}^d)$. Then*

$$\|u\|_{1,W} \leq C\|f\|_{L^2(\mathbb{T}^d)},$$

for some constant C independent of f.

Proof. Suppose that there exists a weak solution $u \in H_{1,W}(\mathbb{T}^d)$ of (2.2.5). Since the function $v \equiv 1 \in H_{1,W}(\mathbb{T}^d)$, we have by definition of weak solution that

$$\int_{\mathbb{T}^d} f dx \;=\; B[u,v] \;=\; 0.$$

Now, let $f \in L^2(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} f dx = 0$. Consider the bilinear form B, defined in (2.2.4) with $\lambda = 0$, on the Hilbert space $H^{\perp}_{1,W}(\mathbb{T}^d)$. By Proposition 2.2.2, B satisfies the hypothesis of the Lax-Milgram's Theorem. Further, f defines the bounded linear functional in $H^{\perp}_{1,W}(\mathbb{T}^d)$ given by $(f,g) = \langle f,g\rangle$ for every $g \in H^{\perp}_{1,W}(\mathbb{T}^d)$. Then, an application of Lax-Milgram's Theorem yields that there exists a unique $u \in H^{\perp}_{1,W}(\mathbb{T}^d)$ such that

$$B[u,v] = \langle f,v\rangle \text{ for all } v \in H^{\perp}_{1,W}(\mathbb{T}^d).$$

Moreover, by Proposition 2.2.2, there is a $\beta > 0$ such that

$$\beta\|u\|^2_{1,W} \leq B[u,u] = \langle f,u\rangle \leq \|f\|_{L^2(\mathbb{T}^d)}\|u\|_{L^2(\mathbb{T}^d)} \leq \|f\|_{L^2(\mathbb{T}^d)}\|u\|_{1,W}.$$

The existence of weak solutions and the bound C in the statement of the Corollary follows from the previous expression. □

We now analyze the case in which $\lambda > 0$.

Proposition 2.2.4 (Energy estimates for $\lambda > 0$)**.** *Let $f \in L^2(\mathbb{T}^d)$. There exists a unique weak solution $u \in H_{1,W}(\mathbb{T}^d)$ for the equation*

$$\lambda u - \nabla A \nabla_W u \;=\; f, \qquad \lambda > 0. \tag{2.2.6}$$

This solution enjoys the following bounds

$$\|u\|_{1,W} \leq C\|f\|_{L^2(\mathbb{T}^d)}$$

for some constant $C > 0$ independent of f, and

$$\|u\| \leq \lambda^{-1}\|f\|_{L^2(\mathbb{T}^d)}.$$

Proof. Let $\beta = min\{\lambda, \theta^{-1}\} > 0$ and $\alpha = max\{\lambda, \theta\} < \infty$, where θ is given in (2.2.1). An elementary computation shows that

$$B[u,v]| \leq \alpha \|u\|_{1,W} \ \|v\|_{1,W} \quad \text{and} \quad B[u,u] \geq \beta \|u\|_{1,W}^2.$$

By Lax-Milgram's Theorem, there exists a unique solution $u \in H_{1,W}(\mathbb{T}^d)$ of (2.2.6). Note that

$$\beta \|u\|_{1,W}^2 \leq B[u,u] = \langle f, u \rangle \leq \|f\|_{L^2(\mathbb{T}^d)} \|u\|_{L^2(\mathbb{T}^d)} \leq \|f\|_{L^2(\mathbb{T}^d)} \|u\|_{1,W},$$

and therefore $\|u\|_{1,W} \leq C\|f\|_{L^2(\mathbb{T}^d)}$ for some constant $C > 0$ independent of f. The computation to obtain the other bound is analogous. □

Remark 2.2.5. *Let $\mathbb{L}_W^A : \mathbb{D}_W \to \mathbb{L}^2(\mathbb{T}^d)$ be given by $\mathbb{L}_W^A = \nabla A \nabla_W$. This operator has the properties stated in Theorem 1.1.2. We now outline the main steps to prove it. We may prove an analogous of Lemma 1.2.2 for the operator $\mathbb{L}_W^A$. Using the bounds on the diagonal matrix A and Proposition 2.1.7 (Rellich-Kondrachov), we conclude that the energetic extension of the space induced by this operator has compact embedding in $L^2(\mathbb{T}^d)$. The previous results together with [39, Theorems 5.5.a and 5.5.c] implies that $\mathbb{L}_W^A$ has a self-adjoint extension $\mathcal{L}_W^A$, which is dissipative and non-positive, and its eigenvectors form a complete orthonormal set in $L^2(\mathbb{T}^d)$. Furthermore, the set of eigenvalues of this extension is countable and its elements can be ordered resulting in a non-increasing sequence that tends to $-\infty$.*

Remark 2.2.6. *Let $\mathcal{L}_W^A$ be the self-adjoint extension given in Remark 2.2.5, and $\mathcal{D}_W^A$ its domain. For $\lambda > 0$ the operator $\lambda \mathbb{I} - \mathcal{L}_W^A : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ is bijective. Therefore, the equation*

$$\lambda u - \nabla A \nabla_W u = f,$$

has strong solution in $\mathbb{D}_W$ if and only if $f \in (\lambda \mathbb{I} - \mathcal{L}_W^A)(\mathbb{D}_W)$, where $\mathbb{I}$ is the identity operator and $(\lambda \mathbb{I} - \mathcal{L}_W^A)(\mathbb{D}_W)$ stands for the range of $\mathbb{D}_W$ under the operator $\lambda \mathbb{I} - \mathcal{L}_W^A$. Moreover, this strong solution coincides with the weak solution obtained in Proposition 2.2.4.

2.3 W-Generalized parabolic equations

In this Section, we study a class of W-generalized PDEs that involves time: the parabolic equations. The parabolic equations are often used to describe in physical applications the time-evolution of the density of some quantity, say a chemical concentration within a region. The motivation of this generalization is to enlarge the possibility of such applications, for instance, these equations may be used to model a diffusion of particles within a region with membranes (see Chapter 1 and [18]).

We begin by introducing the class of W-generalized parabolic equations we are interested. Then, we define what is meant by weak solution of such equations, using the W-Sobolev spaces, and prove uniquenesses of these weak solutions. In Section 2.6, we obtain existence of weak solutions of these equations.

Fix $T > 0$ and let $(B, \|\cdot\|_B)$ be a Banach space. We denote by $L^2([0,T], B)$ the Banach space of measurable functions $U : [0,T] \to B$ for which

$$\|U\|_{L^2([0,T],B)}^2 := \int_0^T \|U_t\|_B^2 dt < \infty.$$

Let $A = A(t,x)$ be a diagonal matrix satisfying the ellipticity condition (2.2.1) for all $t \in [0,T]$, $\Phi : [l,r] \to \mathbb{R}$ be a continuously differentiable function such that

$$B^{-1} < \Phi'(x) < B,$$

for all x, where $B > 0$, $l, r \in \mathbb{R}$ are constants. We will consider the equation

$$\begin{cases} \partial_t u = \nabla A \nabla_W \Phi(u) & \text{in } (0,T] \times \mathbb{T}^d, \\ u = \gamma & \text{in } \{0\} \times \mathbb{T}^d. \end{cases} \tag{2.3.1}$$

where $u : [0,T] \times T^d \to \mathbb{R}$ is the unknown function and $\gamma : \mathbb{T}^d \to \mathbb{R}$ is given.

We say that a function $\rho = \rho(t,x)$ is a weak solution of the problem (2.3.1) if:

- For every $H \in \mathbb{D}_W$ the following integral identity holds

$$\int_{\mathbb{T}^d} \rho(t,x)H(x)dx - \int_{\mathbb{T}^d} \gamma(x)H(x)dx = \int_0^t \int_{\mathbb{T}^d} \Phi(\rho(s,x))\nabla A \nabla_W H(x)dx\, ds$$

- $\Phi(\rho(\cdot,\cdot))$ and $\rho(\cdot,\cdot)$ belong to $L^2([0,T], H_{1,W}(\mathbb{T}^d))$:

$$\int_0^T \|\Phi(\rho(s,x))\|^2_{L^2(\mathbb{T}^d)} + \|\nabla_W \Phi(\rho(s,x))\|^2_{L^2_W(\mathbb{T}^d)} ds < \infty,$$

  and

$$\int_0^T \|\rho(s,x)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla_W \rho(s,x)\|^2_{L^2_W(\mathbb{T}^d)} ds < \infty.$$

Consider the energy in jth direction of a function $u(s,x)$ as

$$\mathcal{Q}_j(u) = \sup_{H \in \mathbb{D}_W} \Big\{ 2\int_0^T \int_{\mathbb{T}^d} (\partial_{x_j} \partial_{W_j} H)(s,x)\, u(s,x) dx\, ds \\ - \int_0^T ds \int_{\mathbb{T}^d} [\partial_{W_j} H(s,x)]^2 d(x^j \otimes W_j) \Big\},$$

and the total energy of a function $u(s,x)$ as

$$\mathcal{Q}(u) = \sum_{j=1}^d \mathcal{Q}_j(u).$$

The notion of energy is important in probability theory and is often used in large deviations of Markov processes. We also use this notion to prove the hydrodynamic limit in Section 2.6. The following lemma shows the connection between the functions of finite energy and functions in the Sobolev space.

Lemma 2.3.1. *A function $u \in L^2([0,T], L^2(\mathbb{T}^d))$ has finite energy if and only if u belongs to $L^2([0,T], H_{1,W}(\mathbb{T}^d))$. In the case the energy is finite, we have*

$$\mathcal{Q}(u) = \int_0^T \|\nabla_W u\|^2_{L^2_W(\mathbb{T}^d)} dt.$$

Proof. Consider functions $U \in L^2([0,T], L^2_{x^j \otimes W_j,0}(\mathbb{T}^d))$ as trajectories in $L^2_{x^j \otimes W_j,0}(\mathbb{T}^d)$, that is, consider a trajectory $\boldsymbol{U} : [0,T] \to L^2_{x^j \otimes W_j,0}(\mathbb{T}^d)$ and define $U(s,x)$ as $U(s,x) := [\boldsymbol{U}(s)](x)$.

Let $u \in L^2([0,T], L^2(\mathbb{T}^d))$ and recall that the set $\{\partial_{W_j} H; H \in \mathbb{D}_W\}$ is dense in $L^2_{x^j \otimes W_j,0}(\mathbb{T}^d)$. Then the set $\{\partial_{W_j} H(s,x); H \in L^2([0,T], \mathbb{D}_W)\}$ is dense in $L^2([0,T], L^2_{x^j \otimes W_j,0}(\mathbb{T}^d))$. Suppose that u has finite energy, and let $H \in L^2([0,T], \mathbb{D}_W)$, then

$$\mathcal{F}_j(\partial_{W_j} H) = \int_0^T \int_{\mathbb{T}^d} (\partial_{x_j} \partial_{W_j} H)(s,x)\, u(s,x) dx\, ds$$

is a bounded linear functional in $L^2([0,T], L^2_{x^j \otimes W_j,0}(\mathbb{T}^d))$. Consequently, by Riesz's representation theorem, there exists a function $G_j \in L^2([0,T], L^2_{x^j \otimes W_j,0}(\mathbb{T}^d))$ such that

$$\mathcal{F}_j(\partial_{W_j} H) = \int_0^T \int_{\mathbb{T}^d} (\partial_{W_j} H)(x)\, G_j(s,x) dx\, ds,$$

for all $H \in L^2([0,T], \mathbb{D}_W)$.

From the uniqueness of the generalized weak derivative, we have that $G_j(s,x) = -\partial_{W_j} u(s,x)$.

Now, suppose u belongs to $L^2([0,T], H_{1,W}(\mathbb{T}^d))$ and let $H \in L^2([0,T], \mathbb{D}_W)$. Then, we have

$$2\int_0^T \int_{\mathbb{T}^d} (\partial_{x_j}\partial_{W_j} H)(s,x)\, u(s,x) dx\, ds - \int_0^T ds \int_{\mathbb{T}^d} \big(\partial_{W_j} H(s,x)\big)^2 d(x^j \otimes W_j) =$$
$$-2\int_0^T \int_{\mathbb{T}^d} \partial_{W_j} H(s,x) \partial_{W_j} u(s,x) d(x^j \otimes W_j) - \int_0^T \int_{\mathbb{T}^d} \big(\partial_{W_j} H(s,x)\big)^2 d(x^j \otimes W_j)$$

We can rewrite the right-hand side of the above expression as

$$-2\langle \partial_{W_j} H, 2\partial_{W_j} u + \partial_{W_j} H\rangle_{x^j \otimes W_j}. \tag{2.3.2}$$

A simple calculation shows that, for a Hilbert space $\mathcal{H}$ with inner product $<\cdot,\cdot>$, the following inequality holds:

$$- <v, u+v> \leq \frac{1}{4} <u,u>,$$

for all $u, v \in \mathcal{H}$, and we have equality only when $v = -1/2u$.

Therefore, by the previous estimates and (2.3.2)

$$2\int_0^T \int_{\mathbb{T}^d} (\partial_{x_j}\partial_{W_j} H)(s,x)\, u(s,x) dx\, ds - \int_0^T ds \int_{\mathbb{T}^d} \big(\partial_{W_j} H(s,x)\big)^2 d(x^j \otimes W_j) \leq$$
$$\int_0^T \int_{\mathbb{T}^d} \big(\partial_{W_j} u(s,x)\big)^2 d(x^j \otimes W_j).$$

By the definition of energy, we have for each $j = 1, \ldots, d$,

$$\mathcal{Q}_j(u) \leq \int_0^T \int_{\mathbb{T}^d} \big(\partial_{W_j} u(s,x)\big)^2 d(x^j \otimes W_j).$$

Hence, the total energy is finite. Using the fact that $L^2([0,T], \mathbb{D}_W)$ is dense in $L^2([0,T], H_{1,W}(\mathbb{T}^d))$, we have that

$$\begin{aligned} \mathcal{Q}(u) &= \sum_{j=1} \int_0^T \|\partial_{W_j} u\|^2_{x^j \otimes W_j} dt \\ &= \int_0^T \|\nabla_W u\|^2_{L^2_W(\mathbb{T}^d)} dt. \end{aligned}$$

□

2.3.1 Uniqueness of weak solutions of the parabolic equation

Recall that we denote by $\langle \cdot, \cdot \rangle$ the inner product of the Hilbert space $L^2(\mathbb{T}^d)$. Fix $H, G \in L^2(\mathbb{T}^d)$, $\lambda > 0$, and denote by H_λ and G_λ in $H_{1,W}(\mathbb{T}^d)$ the unique weak solutions of the elliptic equations

$$\lambda H_\lambda - \nabla A \nabla_W H_\lambda = H,$$

and

$$\lambda G_\lambda - \nabla A \nabla_W G_\lambda = G,$$

respectively. Then, we have the following symmetry property

$$\langle G_\lambda, H \rangle = \langle G, H_\lambda \rangle.$$

In fact, both terms in the previous equality are equal to

$$\lambda \int_{\mathbb{T}^d} H_\lambda G_\lambda + \sum_{j=1}^d a_{jj} \int_{\mathbb{T}^d} (\partial_{W_j} H_\lambda)(\partial_{W_j} G_\lambda) d(x^j \otimes W_j).$$

Let $\rho : \mathbb{R}_+ \times \mathbb{T} \to [l,r]$ be a weak solution of the parabolic equation (2.3.1). Since ρ, $\Phi(\rho) \in L^2([0,T], H_{1,W}(\mathbb{T}^d))$, and the set $\mathbb{D}_W$ is dense in $H_{1,W}(\mathbb{T}^d)$, we have for every H in $H_{1,W}(\mathbb{T}^d)$,

$$\langle \rho_t, H\rangle \;-\; \langle \gamma, H\rangle \;=-\;\sum_{j=1}^{d} a_{jj}\int_0^t \langle \partial_{W_j}\Phi(\rho_s), \partial_{W_j} H\rangle_{x^j\otimes W_j}\, ds \tag{2.3.3}$$

for all $t > 0$.

Denote by $\rho_s^\lambda \in H_{1,W}(\mathbb{T}^d)$ the unique weak solution of the elliptic equation

$$\lambda\rho_s^\lambda - \nabla A \nabla_W \rho_s^\lambda = \rho(s,\cdot). \tag{2.3.4}$$

We claim that

$$\langle \rho_t\,,\, \rho_t^\lambda\rangle \;-\; \langle \rho_0\,,\, \rho_0^\lambda\rangle \;=\; -2\sum_{j=1}^{d} a_{jj}\int_0^t \langle \partial_{W_j}\Phi(\rho_s)\,,\, \partial_{W_j}\rho_s^\lambda\rangle_{x^j\otimes W_j}\, ds \tag{2.3.5}$$

for all $t > 0$.

To prove this claim, fix $t > 0$ and consider a partition $0 = t_0 < t_1 < \cdots < t_n = t$ of the interval $[0,t]$. Using the telescopic sum, we obtain

$$\begin{aligned}\langle \rho_t\,,\, \rho_t^\lambda\rangle \;-\; \langle \rho_0\,,\, \rho_0^\lambda\rangle \;&=\; \sum_{k=0}^{n-1}\langle \rho_{t_{k+1}}\,,\, \rho_{t_{k+1}}^\lambda\rangle \;-\; \langle \rho_{t_{k+1}}\,,\, \rho_{t_k}^\lambda\rangle \\ &+\; \sum_{k=0}^{n-1}\langle \rho_{t_{k+1}}\,,\, \rho_{t_k}^\lambda\rangle \;-\; \langle \rho_{t_k}\,,\, \rho_{t_k}^\lambda\rangle\;.\end{aligned}$$

We handle the first term, the second one being similar. From the symmetric property of the weak solutions, $\rho_{t_{k+1}}^\lambda$ belongs to $H_{1,W}(\mathbb{T}^d)$ and since ρ is a weak solution of (2.3.1),

$$\langle \rho_{t_{k+1}}\,,\, \rho_{t_{k+1}}^\lambda\rangle \;-\; \langle \rho_{t_{k+1}}\,,\, \rho_{t_k}^\lambda\rangle \;=\; -\sum_{j=1}^{d} a_{jj}\int_{t_k}^{t_{k+1}} \langle \partial_{W_j}\Phi(\rho_s)\,,\, \partial_{W_j}\rho_{t_{k+1}}^\lambda\rangle\, ds\;.$$

Add and subtract $\langle \partial_{W_j}\Phi(\rho_s)\,,\, \partial_{W_j}\rho_s^\lambda\rangle$ inside the integral on the right hand side of the above expression. The time integral of this term is exactly the expression announced in (2.3.5) and the remainder is given by

$$\sum_{j=1}^{d} a_{jj}\int_{t_k}^{t_{k+1}} \Big\{\langle \partial_{W_j}\Phi(\rho_s)\,,\, \partial_{W_j}\rho_s^\lambda\rangle \;-\; \langle \partial_{W_j}\Phi(\rho_s)\,,\, \partial_{W_j}\rho_{t_{k+1}}^\lambda\rangle\Big\}\, ds\;.$$

Since ρ_s^λ is the unique weak solution of the elliptic equation (2.3.4), and the weak solution has the symmetric property, we may rewrite the previous difference as

$$\Big\{\langle \Phi(\rho_s)\,,\, \rho_{t_{k+1}}\rangle \;-\; \langle \Phi(\rho_s)\,,\, \rho_s\rangle\Big\} \;-\; \lambda\Big\{\langle \Phi(\rho_s)^\lambda\,,\, \rho_{t_{k+1}}\rangle \;-\; \langle \Phi(\rho_s)^\lambda\,,\, \rho_s\rangle\Big\}\;.$$

The time integral between t_k and t_{k+1} of the second term is equal to

$$-\lambda\int_{t_k}^{t_{k+1}} ds\int_s^{t_{k+1}} \langle \partial_{W_j}\Phi(\rho_s)^\lambda\,,\, \partial_{W_j}\Phi(\rho_r)\rangle\, dr$$

because ρ is a weak solution of (2.3.1) and $\Phi(\rho_s)$ belongs to $H_{1,W}(\mathbb{T}^d)$. It follows from the boundedness of the weak solution given in Proposition 2.2.4 and from the boundedness of the $L^2_{x^j\otimes W_j}(\mathbb{T}^d)$ norm of $\partial_{W_j}\Phi(\rho)$ obtained in expression (2.3.3), that this expression is of order $(t_{k+1} - t_k)^2$.

To conclude the proof of claim (2.3.5) it remains to be shown that

$$\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \Big\{\langle \Phi(\rho_s)\,,\, \rho_{t_{k+1}}\rangle \;-\; \langle \Phi(\rho_s)\,,\, \rho_s\rangle\Big\}\, ds$$

vanishes as the mesh of the partition tends to 0. Using, again, the fact that ρ is a weak solution, we may rewrite the sum as

$$-\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} ds \int_s^{t_{k+1}} \langle \partial_{W_j}\Phi(\rho_s)\,,\,\partial_{W_j}\Phi(\rho_r)\rangle\, dr\ .$$

We have that this expression vanishes as the mesh of the partition tends to 0 from the boundedness of the $L^2_{x^j\otimes W_j}(\mathbb{T}^d)$ norm of $\partial_{W_j}\Phi(\rho)$. This proves (2.3.5).

Recall the definition of the constant B given at the beginning of this Section.

Lemma 2.3.2. *Fix $\lambda > 0$, two density profiles γ^1, $\gamma^2 : \mathbb{T} \to [l, r]$ and denote by ρ^1, ρ^2 weak solutions of* (2.3.1) *with initial value γ^1, γ^2, respectively. Then,*

$$\Big\langle \rho_t^1 - \rho_t^2\,,\,\rho_t^{1,\lambda} - \rho_t^{2,\lambda}\Big\rangle \;\le\; \Big\langle \gamma^1 - \gamma^2\,,\,\gamma^{1,\lambda} - \gamma^{2,\lambda}\Big\rangle\, e^{B\lambda t/2}$$

for all $t > 0$. In particular, there exists at most one weak solution of (2.3.1).

Proof. We begin by showing that if there exists $\lambda > 0$ such that

$$\langle H, H^\lambda\rangle = 0,$$

then $H = 0$. In fact, we would have the following

$$\int_{\mathbb{T}^d} \lambda (H^\lambda)^2 dx + \sum_{j=1}^d a_{jj}\int_{\mathbb{T}^d} \left(\partial_{W_j} H^\lambda\right)^2 d(x^j\otimes W_j) = \int_{\mathbb{T}^d} H H^\lambda dx = 0,$$

which implies that $\|H^\lambda\|_{H_{1,W}(\mathbb{T}^d)} = 0$, and hence $H_\lambda = 0$, which yields $H = 0$.

Fix two density profiles γ^1, $\gamma^2 : \mathbb{T}^d \to [l, r]$. Let ρ^1, ρ^2 be two weak solutions with initial values γ^1, γ^2, respectively. By (2.3.5), for any $\lambda > 0$,

$$\begin{aligned}&\Big\langle \rho_t^1 - \rho_t^2\,,\,\rho_t^{1,\lambda} - \rho_t^{2,\lambda}\Big\rangle - \Big\langle \gamma^1 - \gamma^2\,,\,\gamma^{1,\lambda} - \gamma^{2,\lambda}\Big\rangle \;=\\ &-2\textstyle\int_0^t \langle \Phi(\rho_s^1) - \Phi(\rho_s^2)\,,\,\rho_s^1 - \rho_s^2\rangle\, ds \;+\; 2\lambda \int_0^t \Big\langle \Phi(\rho_s^1) - \Phi(\rho_s^2)\,,\,\rho_s^{1,\lambda} - \rho_s^{2,\lambda}\Big\rangle\, ds\ .\end{aligned} \tag{2.3.6}$$

Define the inner product in $H_{1,W}(\mathbb{T}^d)$

$$\langle u, v\rangle_\lambda = \langle u, v^\lambda\rangle.$$

This is, in fact, an inner product, since $\langle u, v\rangle_\lambda = \langle v, u\rangle_\lambda$ by the symmetric property, and if $u \neq 0$, then $\langle u, u\rangle_\lambda > 0$:

$$\int_{\mathbb{T}^d} u u_\lambda dx = \lambda \int_{\mathbb{T}^d} u_\lambda^2 dx + \sum_{j=1}^d a_{jj}\int_{\mathbb{T}^d}\left(\partial_{W_j} u^\lambda\right)^2 d(x^j\otimes W_j).$$

The linearity of this inner product can be easily verified.

Then, we have

$$2\lambda \int_0^t \Big\langle \Phi(\rho_s^1) - \Phi(\rho_s^2)\,,\,\rho_s^{1,\lambda} - \rho_s^{2,\lambda}\Big\rangle\, ds = 2\lambda\int_0^t \Big\langle \Phi(\rho_s^1) - \Phi(\rho_s^2)\,,\,\rho_s^1 - \rho_s^2\Big\rangle_\lambda ds.$$

By using the Cauchy-Schwartz inequality twice, the term on the right hand side of the above formula is bounded above by

$$\frac{1}{A}\int_0^t \Big\langle \Phi(\rho_s^1) - \Phi(\rho_s^2)\,,\,\Phi(\rho_s^1)^\lambda - \Phi(\rho_s^2)^\lambda\Big\rangle\, ds \;+\; A\lambda^2\int_0^t \Big\langle \rho_s^1 - \rho_s^2\,,\,\rho_s^{1,\lambda} - \rho_s^{2,\lambda}\Big\rangle\, ds$$

for every $A > 0$. From Proposition 2.2.4, we have that $\|u^\lambda\| \le \lambda^{-1}\|u\|$, and since Φ' is bounded by B, the first term of the previous expression is less than or equal to

$$\frac{B}{A\lambda}\int_0^t \Big\langle \rho_s^1 - \rho_s^2\,,\,\Phi(\rho_s^1) - \Phi(\rho_s^2)\Big\rangle\, ds\ .$$

Choosing $A = B/2\lambda$, this expression cancels with the first term on the right hand side of (2.3.6). In particular, the left hand side of this formula is bounded by

$$\frac{B\lambda}{2}\int_0^t \Big\langle \rho_s^1 - \rho_s^2\,,\, \rho_s^{1,\lambda} - \rho_s^{2,\lambda}\Big\rangle\, ds\ .$$

To conclude, recall Gronwall's inequality. □

Remark 2.3.3. *Let $\mathcal{L}_W^A : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ be the self-adjoint extension given in Remark 2.2.5. For $\lambda > 0$, define the resolvent operator $G_\lambda^A = (\lambda\mathbb{I} - \mathcal{L}_W^A)^{-1}$. Following the Chapter 1 and [18], another possible definition of weak solution of equation* (2.3.1) *is given as follows: a bounded function $\rho : [0,T]\times\mathbb{T}^d \to [l,r]$ is said to be a weak solution of the parabolic differential equation* (2.3.1) *if*

$$\langle \rho_t, G_\lambda^A h\rangle \ -\ \langle \gamma, G_\lambda^A h\rangle \ =\ \int_0^t \langle \Phi(\rho_s), \mathcal{L}_W^A G_\lambda^A h\rangle\, ds \tag{2.3.7}$$

for every continuous function $h : \mathbb{T}^d \to \mathbb{R}$, $t \in [0,T]$, and all $\lambda > 0$. We claim that this definition of weak solution coincides with our definition introduced at the beginning of Section 2.3. Indeed, for continuous $h : \mathbb{T}^d \to \mathbb{R}$, $G_\lambda^A h$ belongs to $\mathcal{D}_W$. Since $\mathbb{D}_W$ is dense in $\mathcal{D}_W$ with respect to the $H_{1,W}(\mathbb{T}^d)$-norm, it follows that our definition implies the current definition. Conversely, since the set of continuous functions is dense in $L^2(\mathbb{T}^d)$, the identity (2.3.7) *is valid for all $h \in L^2(\mathbb{T}^d)$. Therefore, for each $H \in \mathcal{D}_W$ we have*

$$\langle \rho_t, H\rangle \ -\ \langle \gamma, H\rangle \ =\ \int_0^t \langle \Phi(\rho_s), \mathcal{L}_W^A H\rangle\, ds.$$

In particular, the above identity holds for every $H \in \mathbb{D}_W$, and therefore the integral identity in our definition of weak solutions holds.

It remains to be checked that the weak solution of the current definition belongs to $L^2([0,T], H_{1,W}(\mathbb{T}^d))$. This follows from the fact that there exists at most one weak solution satisfying (2.3.7)*, that this unique solution has finite energy, and from Lemma 2.3.1. A proof of the fact that there exists at most one solution satisfying* (2.3.7)*, and that this unique solution has finite energy, can be found in [18].*

Finally, the integral identity of our definition of weak solution has an advantage regarding the integral identity (2.3.7)*, due to the fact that we do not need the resolvent operator G_λ^A for any λ. Moreover, we have an explicit characterization of our test functions.*

2.4 W-Generalized Sobolev spaces: Discrete version

We will now establish some of the results obtained in the above sections to the discrete version of the W-Sobolev space. Our motivation to obtain these results is that they will be useful when studying homogenization in Section 2.5. We begin by introducing some definitions and notations.

Fix W as in (1.1.1) and functions f, g defined on $N^{-1}\mathbb{T}_N^d$. Consider the following difference operators: $\partial_{x_j}^N$, which is the standard difference operator,

$$\partial_{x_j}^N f\left(\frac{x}{N}\right) \ =\ N\left[f\left(\frac{x+e_j}{N}\right) - f\left(\frac{x}{N}\right)\right]\ ,$$

and $\partial_{W_j}^N$, which is the W_j-difference operator:

$$\partial_{W_j}^N f\left(\frac{x}{N}\right) \ =\ \frac{f\left(\frac{x+e_j}{N}\right) - f\left(\frac{x}{N}\right)}{W\left(\frac{x+e_j}{N}\right) - W\left(\frac{x}{N}\right)},$$

for $x \in \mathbb{T}_N^d$. We introduce the following scalar product

$$\langle f,g\rangle_N \ := \ \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} f(x)g(x),$$

$$\langle f,g\rangle_{W_j,N} := \frac{1}{N^{d-1}}\sum_{x\in\mathbb{T}_N^d} f(x)g(x)\big(W((x+e_j)/N)-W(x/N)\big),$$

$$\langle f,g\rangle_{1,W,N} \ := \langle f,g\rangle_N \ + \ \sum_{j=1}^d \langle \partial^N_{W_j} f, \partial^N_{W_j} g\rangle_{W_j,N},$$

and its induced norms

$$\|f\|^2_{L^2(\mathbb{T}_N^d)} = \langle f,f\rangle_N, \quad \|f\|^2_{L^2_{W_j}(\mathbb{T}_N^d)} = \langle f,f\rangle_{W_j,N} \text{ and } \|f\|^2_{H_{1,W}(\mathbb{T}_N^d)} = \langle f,f\rangle_{1,W,N}.$$

These norms are natural discretizations of the norms introduced in the previous sections. Note that the properties of the Lebesgue's measure used in the proof of Corollary 2.1.5, also holds for the normalized counting measure. Therefore, we may use the same arguments of this Corollary to prove its discrete version.

Lemma 2.4.1 (Discrete Poincaré Inequality)**.** *There exists a finite constant C such that*

$$\left\| f - \frac{1}{N^d}\sum_{x\in\mathbb{T}^d} f \right\|_{L^2(\mathbb{T}_N^d)} \ \leq \ C\|\nabla^N_W f\|_{L^2_W(\mathbb{T}_N^d)},$$

where

$$\|\nabla_W f\|^2_{L^2_W(\mathbb{T}_N^d)} = \sum_{j=1}^d \|\partial^N_{W_j} f\|^2_{L^2_{W_j}(\mathbb{T}_N^d)},$$

for all $f: N^{-1}\mathbb{T}_N^d \to \mathbb{R}$.

Let A be a diagonal matrix satisfying (2.2.1). We are interested in studying the problem

$$T^N_\lambda u = f, \tag{2.4.1}$$

where $u: N^{-1}\mathbb{T}_N^d \to \mathbb{R}$ is the unknown function, $f: N^{-1}\mathbb{T}_N^d \to \mathbb{R}$ is given, and T^N_λ denotes the discrete generalized elliptic operator

$$T^N_\lambda u \ := \ \lambda u - \nabla^N A \nabla^N_W u, \tag{2.4.2}$$

with

$$\nabla^N A\nabla^N_W u \ := \ \sum_{i=1}^d \partial^N_{x_i}\Big(a_i(x/N)\partial^N_{W_i} u\Big).$$

The bilinear form $B^N[\cdot,\cdot]$ associated with the elliptic operator T^N_λ is given by

$$\begin{aligned} B^N[u,v] \ &= \ \lambda\langle u,v\rangle_N \ + \\ + \tfrac{1}{N^{d-1}}\textstyle\sum_{i=1}^d \sum_{x\in\mathbb{T}_N^d} &a_i(x/N)(\partial^N_{W_i}u)(\partial^N_{W_i}v)[W_i((x_i+1)/N)-W_i(x_i/N)], \end{aligned} \tag{2.4.3}$$

where $u,v: N^{-1}\mathbb{T}_N^d \to \mathbb{R}$.

A function $u: N^{-1}\mathbb{T}_N^d \to \mathbb{R}$ is said to be a weak solution of the equation $T^N_\lambda u = f$ if

$$B^N[u,v] \ = \ \langle f,v\rangle_N \ \text{ for all } \ v: N^{-1}\mathbb{T}_N^d \to \mathbb{R}.$$

We say that a function $f: N^{-1}\mathbb{T}_N^d \to \mathbb{R}$ belongs to the discrete space of functions orthogonal to the constant functions $H^\perp_N(\mathbb{T}_N^d)$ if

$$\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} f(x/N) = 0.$$

The following results are analogous to the weak solutions of generalized elliptic equations for this discrete version. We remark that the proofs of these lemmas are identical to the ones in the continuous case. Furthermore, the weak solution for the case $\lambda = 0$ is unique in $H^\perp_N(\mathbb{T}_N^d)$.

Lemma 2.4.2. *The equation*

$$\nabla^N A \nabla_W^N u = f,$$

has weak solution $u : N^{-1}\mathbb{T}_N^d \to \mathbb{R}$ *if and only if*

$$\frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} f(x) = 0.$$

In this case we have uniqueness of the solution disregarding addition by constants. Moreover, if $u \in H_N^{\perp}(\mathbb{T}_N^d)$ *we have the bound*

$$\|u\|_{H_{1,W}(\mathbb{T}_N^d)} \leq C\|f\|_{L^2(\mathbb{T}_N^d)}, \text{ and } \|u\|_{L^2(\mathbb{T}_N^d)} \leq \lambda^{-1}\|f\|_{L^2(\mathbb{T}_N^d)},$$

where $C > 0$ *does not depend on* f *nor* N*.*

Lemma 2.4.3. *Let* $\lambda > 0$*. There exists a unique weak solution* $u : N^{-1}\mathbb{T}_N^d \to \mathbb{R}$ *of the equation*

$$\lambda u - \nabla^N A \nabla_W^N u = f. \tag{2.4.4}$$

Moreover,

$$\|u\|_{H_{1,W}(\mathbb{T}_N^d)} \leq C\|f\|_{L^2(\mathbb{T}_N^d)}, \text{ and } \|u\|_{L^2(\mathbb{T}_N^d)} \leq \lambda^{-1}\|f\|_{L^2(\mathbb{T}_N^d)},$$

where $C > 0$ *does not depend neither on* f *nor* N*.*

Remark 2.4.4. *Note that in the set of functions in* $\mathbb{T}_N^d$ *we have a "Dirac measure" concentrated in a point* x *as a function: the function that takes value* N^d *in* x *and zero elsewhere. Therefore, we may integrate these weak solutions with respect to this function to obtain that every weak solution is, in fact, a strong solution.*

2.4.1 Connections between the discrete and continuous Sobolev spaces

Given a function $f \in H_{1,W}(\mathbb{T}^d)$, we can define its restriction f_N to the lattice $N^{-1}\mathbb{T}_N^d$ as

$$f_N(x) = f(x) \text{ if } x \in N^{-1}\mathbb{T}_N^d.$$

However, given a function $f : N^{-1}\mathbb{T}_N^d \to \mathbb{R}$ it is not straightforward how to define an extension belonging to $H_{1,W}(\mathbb{T}^d)$. To do so, we need the definition of W-interpolation, which we give below.

Let $f_N : N^{-1}\mathbb{T}_N \to \mathbb{R}$ and $W : \mathbb{R} \to \mathbb{R}$, a *strictly increasing* right continuous function with left limits (càdlàg), and periodic. The W-interpolation f_N^* of f_N is given by:

$$\begin{aligned} f_N^*(x+t) \quad &:= \quad \frac{W((x+1)/N) - W((x+t)/N)}{W((x+1)/N) - W(x/N)} f(x) + \\ &+ \quad \frac{W((x+t)/N) - W(x/N)}{W((x+1)/N) - W(x/N)} f(x+1) \end{aligned}$$

for $0 \leq t < 1$. Note that

$$\frac{\partial f_N^*}{\partial W}(x+t) = \frac{f(x+1) - f(x)}{W((x+1)/N) - W(x/N)} = \partial_W^N f(x).$$

Using the standard construction of d-dimensional linear interpolation, it is possible to define the W-interpolation of a function $f_N : \mathbb{T}_N^d \to \mathbb{R}$, with $W(x) = \sum_{i=1}^d W_i(x_i)$ as defined in (1.1.1).

We now establish the connection between the discrete and continuous Sobolev spaces by showing how a sequence of functions defined in $\mathbb{T}_N^d$ can converge to a function in $H_{1,W}(\mathbb{T}^d)$.

We say that a family $f_N \in L^2(\mathbb{T}_N^d)$ converges strongly (resp. weakly) to the function $f \in L^2(\mathbb{T}^d)$ as $N \to \infty$ if f_N^* converges strongly (resp. weakly) to the function f. From now on we will omit the symbol " * " in the W-interpolated function, and denoting them simply by f_N.

The convergence in $H_W^{-1}(\mathbb{T}^d)$ can be defined in terms of duality. Namely, we say that a functional f_N on $\mathbb{T}_N^d$ converges to $f \in H_W^{-1}(\mathbb{T}^d)$ strongly (resp. weakly) if for any sequence of functions $u_N : \mathbb{T}_N^d \to \mathbb{R}$ and $u \in H_{1,W}(\mathbb{T}^d)$ such that $u_N \to u$ weakly (resp. strongly) in $H_{1,W}(\mathbb{T}^d)$, we have

$$(f_N, u_N)_N \longrightarrow (f, u), \quad \text{as } N \to \infty.$$

Remark 2.4.5. *Suppose in Lemma 2.4.3 that $f \in L^2(\mathbb{T}^d)$, and let u be a weak solution of the problem* (2.4.4)*, then we have the following bound*

$$\|u\|_{H_{1,W}(\mathbb{T}_N^d)} \leq C\|f\|_{L^2(\mathbb{T}^d)},$$

since $\|f\|_{L^2(\mathbb{T}_N^d)} \to \|f\|_{L^2(\mathbb{T}^d)}$ as $N \to \infty$.

2.5 Homogenization

In this "brief" Section we prove a homogenization result for the W-generalized differential operator. We follow the approach considered in [31]. The study of homogenization is motivated by several applications in mechanics, physics, chemistry and engineering. The focus of our approach is to study the asymptotic behavior of effective coefficients for a family of random difference schemes whose coefficients can be obtained by the discretization of random high-contrast lattice structures.

This Section is structured as follows: in subsection 6.1 we define the concept of H-convergence together with some properties; subsection 6.2 deals with a description of the random environment along with some definitions, whereas the main result is proved in subsection 6.3.

2.5.1 H-convergence

We say that the diagonal matrix $A^N = (a_{jj}^N)$ H-converges to the diagonal matrix $A = (a_{jj})$, denoted by $A^N \xrightarrow{H} A$, if, for every sequence $f^N \in H_W^{-1}(\mathbb{T}_N^d)$ such that $f^N \to f$ as $N \to \infty$ in $H_W^{-1}(\mathbb{T}^d)$, we have

- $u_N \to u_0$ weakly in $H_{1,W}(\mathbb{T}^d)$ as $N \to \infty$,
- $a_{jj}^N \partial_{W_j}^N u_N \to a_{jj}\partial_{W_j} u_0$ weakly in $L^2_{x^j\otimes W_j}(\mathbb{T}^d)$ for each $j = 1, \dots, d$,

where $u_N : \mathbb{T}_N^d \to \mathbb{R}$ is the solution of the problem

$$\lambda u_N - \nabla^N A^N \nabla_W^N u_N = f_N,$$

and $u_0 \in H_{1,W}(\mathbb{T}^d)$ is the solution of the problem

$$\lambda u_0 - \nabla A \nabla_W u_0 = f.$$

The notion of convergence used in both items above was defined in subsection 2.4.1.

We now obtain a property regarding H-convergence.

Proposition 2.5.1. *Let $A^N \xrightarrow{H} A$, as $N \to \infty$, with u_N being the solution of*

$$\lambda u_N - \nabla^N A^N \nabla_W^N u_N = f,$$

where $f \in H_W^{-1}(\mathbb{T}^d)$ is fixed. Then, the following limit relations hold true:

$$\frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} u_N^2(x) \to \int_{\mathbb{T}^d} u_0^2(x)dx,$$

and

$$\frac{1}{N^{d-1}} \sum_{j=1}^{d} \sum_{x\in\mathbb{T}_N^d} a_{jj}^N(x)(\partial_{W_j}^N u_N(x))^2 \left[W_j((x_j+1)/N) - W_j(x_j/N)\right]$$

$$\to \sum_{j=1}^{d} \int_{\mathbb{T}^d} a_{jj}(x)(\partial_{W_j} u_0(x))^2 d(x^j \otimes W_j),$$

as $N \to \infty$.

Proof. We begin by noting that

$$\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} f(u_N-u_0)\to 0, \tag{2.5.1}$$

as $N\to\infty$ since u_N-u_0 converges weakly to 0 in $H_{1,W}(\mathbb{T}^d)$. On the other hand, we have

$$\begin{aligned}\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} f(u_N-u_0) &= \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}(\lambda u_N-\nabla^N A^N\nabla_W^N u_N)(u_N-u_0)\\ &= \frac{\lambda}{N^d}\sum_{x\in\mathbb{T}_N^d}u_N^2-\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}u_N\nabla^N A^N\nabla_W^N u_N\\ &- \frac{\lambda}{N^d}\sum_{x\in\mathbb{T}_N^d}u_N u_0+\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}u_0\nabla^N A^N\nabla_W^N u_N.\end{aligned}$$

Using the weak convergences of u_N and $a_{jj}\partial_{W_j}^N u_N$, and the convergence in (2.5.1), we obtain, after a summation by parts in the above expressions,

$$\begin{aligned}&\frac{\lambda}{N^d}\sum_{x\in\mathbb{T}_N^d}u_N^2+\frac{1}{N^{d-1}}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}a_{jj}^N(\partial_{W_j}^N u_N)^2[W_j((x_j+1)/N)-W_j(x_j)]\\ &\qquad\stackrel{N\to\infty}{\longrightarrow}\lambda\int_{\mathbb{T}^d}u_0^2dx+\sum_{j=1}^d\int_{\mathbb{T}^d}a_{jj}(\partial_{W_j}u_0)^2d(x^j\otimes W_j).\end{aligned} \tag{2.5.2}$$

By Lemma 2.4.3, the sequence u_N is $\|\cdot\|_{1,W}$ bounded uniformly. Suppose, now, that u_N does not converge to u_0 in $L^2(\mathbb{T}^d)$. That is, there exist $\epsilon>0$ and a subsequence (u_{N_k}) such that

$$\|u_{N_k}-u_0\|_{L^2(\mathbb{T}^d)}>\epsilon,$$

for all k. By Rellich-Kondrachov Theorem (Proposition 2.1.7), we have that there exists $v\in L^2(\mathbb{T}^d)$ and a further subsequence (also denoted by u_{N_k}) such that

$$u_{N_k}\stackrel{k\to\infty}{\longrightarrow}v,\quad \text{in } L^2(\mathbb{T}^d).$$

This implies that

$$u_{N_k}\to v,\quad \text{weakly in } L^2(\mathbb{T}^d),$$

but this is a contradiction, since

$$u_{N_k}\to u_0,\quad \text{weakly in } L^2(\mathbb{T}^d),$$

and $\|v-u_0\|_{L^2(\mathbb{T}^d)}\geq\epsilon$. Therefore, $u_N\to u_0$ in $L^2(\mathbb{T}^d)$. The proof thus follows from expression (2.5.2). □

This Proposition shows that even though the H-convergence only requires weak convergence in its definition, it yields a convergence in the strong sense (convergence in the L^2-norm).

2.5.2 Random environment

In this subsection we introduce the statistically homogeneous rapidly oscillating coefficients that will be used to define the random W-generalized difference elliptic operators, where the W-generalized difference elliptic operator was given in Section 2.4.

Let $(\Omega,\mathcal{F},\mu)$ be a standard probability space and $\{T_x:\Omega\to\Omega; x\in\mathbb{Z}^d\}$ be a group of $\mathcal{F}$-measurable and ergodic transformations which preserve the measure μ:

- $T_x:\Omega\to\Omega$ is $\mathcal{F}$-measurable for all $x\in\mathbb{Z}^d$,
- $\mu(T_x\mathbf{A})=\mu(\mathbf{A})$, for any $\mathbf{A}\in\mathcal{F}$ and $x\in\mathbb{Z}^d$,
- $T_0=I$, $T_x\circ T_y=T_{x+y}$,

- For any $f \in L^1(\Omega)$ such that $f(T_x\omega) = f(\omega)$ μ-a.s for each $x \in \mathbb{Z}^d$, is equal to a constant μ-a.s.

The last condition implies that the group T_x is ergodic.

Let us now introduce the vector-valued $\mathcal{F}$-measurable functions $\{a_j(\omega); j = 1, \ldots, d\}$ such that there exists $\theta > 0$ with

$$\theta^{-1} \leq a_j(w) \leq \theta,$$

for all $\omega \in \Omega$ and $j = 1, \ldots, d$. Then, define the diagonal matrices A^N whose elements are given by

$$a_{jj}^N(x) := a_j^N = a_j(T_{Nx}\omega)\ ,\ \ x \in T_N^d\ ,\ \ j = 1, \ldots, d. \tag{2.5.3}$$

2.5.3 Homogenization of random operators

Let $\lambda > 0$, f_N be a functional on the space of functions $h_N : \mathbb{T}_N^d \to \mathbb{R}$, $f \in H_W^{-1}(\mathbb{T}^d)$ (see also, subsection 2.1.4), u_N be the unique weak solution of

$$\lambda u_N - \nabla^N A^N \nabla_W^N u_N = f_N,$$

and u_0 be the unique weak solution of

$$\lambda u_0 - \nabla A \nabla_W u_0 = f. \tag{2.5.4}$$

For more details on existence and uniqueness of such solutions see Sections 2.2 and 2.4.

We say that the diagonal matrix A is a *homogenization* of the sequence of random matrices A^N if the following conditions hold:

- For each sequence $f_N \to f$ in $H_W^{-1}(\mathbb{T}^d)$, u_N converges weakly in $H_{1,W}$ to u_0, when $N \to \infty$;

- $a_i^N \partial_{W_i}^N u^N \to a_i \partial_{W_i} u$, weakly in $L^2_{x^i \otimes W_i}(\mathbb{T}^d)$ when $N \to \infty$.

Note that homogenization is a particular case of H-convergence.

We will now state and prove the main result of this Section.

Theorem 2.5.2. *Let A^N be a sequence of ergodic random matrices, such as the one that defines our random environment. Then, almost surely, $A^N(\omega)$ admits a homogenization, where the homogenized matrix A does not depend on the realization ω.*

Proof. Fix $f \in H^{-1}(\mathbb{T}^d)$, and consider the problem

$$\lambda u_N - \nabla^N A^N \nabla_W^N u_N = f.$$

Using Lemma 2.4.3 and Remark 2.4.5, there exists a unique weak solution u_N of the problem above, such that its $H_{1,W}^N$ norm is uniformly bounded in N. That is, there exists a constant $C > 0$ such that

$$\|u_N\|_{H_{1,W}(\mathbb{T}_N^d)} \leq C\|f\|_{L^2(\mathbb{T}^d)}.$$

Thus, the $L^2(\mathbb{T}_N^d)$-norm of $a_i^N \partial_{W_i}^N u_N$ is uniformly bounded.

From W-interpolation (see subsection 2.4.1) and the fact that $H_{1,W}(\mathbb{T}^d)$ is a Hilbert space (Lemma 2.1.2), there exists a convergent subsequence of u_N (which we will also denote by u_N) such that

$$u_N \to u_0, \qquad \text{weakly in} \quad H_{1,W}(\mathbb{T}^d),$$

and

$$a_i^N \partial_{W_i}^N u_N \to v_0 \qquad \text{weakly in} \quad L^2(\mathbb{T}^d), \tag{2.5.5}$$

as $N \to \infty$; v_0 being some function in $L^2_{x^i \otimes W_i}(\mathbb{T}^d)$.

First, observe that the weak convergence in $H_{1,W}(\mathbb{T}^d)$ implies that

$$\partial_{W_i}^N u_N \stackrel{N\to\infty}{\longrightarrow} \partial_{W_i} u \quad \text{weakly in} \quad L^2_{x^i \otimes W_i}(\mathbb{T}^d). \tag{2.5.6}$$

From Birkhoff's ergodic theorem, we obtain the almost sure convergence, as N tends to infinity, of the random coefficients:

$$a_i^N \longrightarrow a_i, \tag{2.5.7}$$

where $a_i = E[a_i^{N_0}]$, for any $N_0 \in \mathbb{N}$.

From convergences in (2.5.5), (2.5.6) and (2.5.7), we obtain that

$$v_0 = a_i \partial_{W_i} u_0,$$

where, from the weak convergences, u_0 clearly solves problem (2.5.4).

To conclude the proof it remains to be shown that we can pass from the subsequence to the sequence. This follows from uniquenesses of weak solutions of the problem (2.5.4). □

Remark 2.5.3. *At first sight, one may think that we are dealing with a very special class of matrices A (diagonal matrices). Nevertheless, the random environment for random walks proposed in [31, Section 2.3], which is also exactly the same random environment employed in [20], results in diagonal matrices. This is essentially due to the fact that in symmetric nearest-neighbor interacting particle systems (for example, the zero-range dynamics considered in [20]), a particle at a site $x \in \mathbb{T}_N^d$ may jump to the sites $x \pm e_j$, $j = 1, \ldots, d$. In such a case, the jump rate from x to $x + e_j$ determines the jth element of the diagonal matrix.*

Remark 2.5.4. *Note that if $u \in \mathbb{D}_W$ is a strong solution (or weak, in view of Remark 2.4.4) of*

$$\lambda u - \nabla A \nabla_W u = f$$

and u_N is strong solution of the discrete problem

$$\lambda u_N - \nabla^N A^N \nabla_W^N u_N = f$$

then, the homogenization theorem also holds, that is, u_N also converges weakly in $H_{1,W}$ to u.

2.6 Hydrodynamic limit of processes with conductances in random environment

Lastly, as an application of all the theory developed in the previous sections, we prove a hydrodynamic limit for a *process with conductances in random environments*. Hydrodynamic limits for processes with conductances have been obtained in [18] for the one-dimensional setup and in Chapter 1 for the d-dimensional setup. However, the proof given here is much simpler and more natural, in view of the theory developed here, than the proofs given in [18] and Chapter 1. Furthermore, the proof of this hydrodynamic limit also provides an existence theorem for the W-generalized parabolic equations in (2.3.1).

The hydrodynamic limit allows one to deduce the macroscopic behavior of the system from the microscopic interaction among particles. Moreover, this approach justifies rigorously a method often used by physicists to establish the partial differential equations that describe the evolution of the thermodynamic characteristics of a fluid.

This Section is structured as follows: in subsection 7.1 we present the model, derive some properties and fix the notations; subsection 7.2 deals with the hydrodynamic equation; finally, subsections 7.3 and 7.4 are devoted to the proof of the hydrodynamic limit.

2.6.1 The exclusion processes with conductances in random environments

Fix a typical realization $\omega \in \Omega$ of the random environment defined in Section 2.5. For each $x \in \mathbb{T}_N^d$ and $j = 1, \ldots, d$, define the symmetric rate $\xi_{x,x+e_j} = \xi_{x+e_j,x}$ by

$$\xi_{x,x+e_j} = \frac{a_j^N(x)}{N[W((x+e_j)/N) - W(x/N)]} = \frac{a_j^N(x)}{N[W_j((x_j+1)/N) - W_j(x_j/N)]}. \tag{2.6.1}$$

where $a_j^N(x)$ is given by (2.5.3), and $e_1, \ldots, e_d$ is the canonical basis of $\mathbb{R}^d$. Also, let $b > -1/2$ and recall that

$$c_{x,x+e_j}(\eta) \;=\; 1 \;+\; b\{\eta(x-e_j) + \eta(x+2\,e_j)\}\;,$$

where all sums are modulo N.

Distribute particles on $\mathbb{T}_N^d$ in such a way that each site of $\mathbb{T}_N^d$ is occupied at most by one particle. Denote by η the configurations of the state space $\{0,1\}^{\mathbb{T}_N^d}$ so that $\eta(x) = 0$ if site x is vacant, and $\eta(x) = 1$ if site x is occupied.

The exclusion process with conductances in a random environment is a continuous-time Markov process $\{\eta_t : t \geq 0\}$ with state space $\{0,1\}^{\mathbb{T}_N^d} = \{\eta : \mathbb{T}_N^d \to \{0,1\}\}$, whose generator L_N acts on functions $f : \{0,1\}^{\mathbb{T}_N^d} \to \mathbb{R}$ as

$$(L_N f)(\eta) \;=\; \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_N^d} \xi_{x,x+e_j} c_{x,x+e_j}(\eta) \,\{f(\sigma^{x,x+e_j}\eta) - f(\eta)\}\;,$$

where $\sigma^{x,x+e_j}\eta$ is the configuration obtained from η by exchanging the variables $\eta(x)$ and $\eta(x+e_j)$:

$$(\sigma^{x,x+e_j}\eta)(y) \;=\; \begin{cases} \eta(x+e_j) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+e_j, \\ \eta(y) & \text{otherwise.} \end{cases}$$

We consider the Markov process $\{\eta_t : t \geq 0\}$ on the configurations $\{0,1\}^{\mathbb{T}_N^d}$ associated to the generator L_N in the diffusive scale, i.e., L_N is speeded up by N^2.

We now describe the stochastic evolution of the process. After a time given by an exponential distribution, a random choice of a point $x \in \mathbb{T}_N^d$ is made. At rate $\xi_{x,x+e_j}$ the occupation variables $\eta(x)$, $\eta(x+e_j)$ are exchanged. Note that only nearest neighbor jumps are allowed. The conductances are induced by the function W, whereas the random environment is given by the matrix $A^N := (a_{jj}^N(x))_{d\times d}$.

The dynamics informally presented describes a Markov evolution. A computation shows that the Bernoulli product measures $\{\nu_\alpha^N : 0 \leq \alpha \leq 1\}$ are invariant, in fact reversible, for the dynamics.

Consider the random walk $\{X_t\}_{t\geq 0}$ of a particle in $\mathbb{T}_N^d$ induced by the generator L_N given as follows. Let $\xi_{x,x+e_j}$ given by (2.6.1). If the particle is on a site $x \in \mathbb{T}_N^d$, it will jump to $x+e_j$ with rate $N^2\xi_{x,x+e_j}$. Furthermore, only nearest neighbor jumps are allowed. The generator $\mathbb{L}_N$ of the random walk $\{X_t\}_{t\geq 0}$ acts on functions $f : \mathbb{T}_N^d \to \mathbb{R}$ as

$$\mathbb{L}_N f\left(\frac{x}{N}\right) \;=\; \sum_{j=1}^{d} \mathbb{L}_N^j f\left(\frac{x}{N}\right),$$

where,

$$\mathbb{L}_N^j f\Big(\frac{x}{N}\Big) = N^2\Big\{\xi_{x,x+e_j}\Big[f\Big(\frac{x+e_j}{N}\Big) - f\Big(\frac{x}{N}\Big)\Big] + \xi_{x-e_j,x}\Big[f\Big(\frac{x-e_j}{N}\Big) - f\Big(\frac{x}{N}\Big)\Big]\Big\}$$

It is not difficult to see that the following equality holds:

$$\mathbb{L}_N f(x/N) = \sum_{j=1}^{d} \partial_{x_j}^N (a_j^N \partial_{W_j}^N f)(x) \;:=\; \nabla^N A^N \nabla_W^N f(x). \tag{2.6.2}$$

Note that several properties of the above operator have been obtained in Section 2.4. The counting measure m_N on $N^{-1}\mathbb{T}_N^d$ is reversible for this process. This random walk plays an important role in the proof of the hydrodynamic limit of the process η_t, as we will see in subsection 7.3.

Recall that $D(\mathbb{R}_+, \{0,1\}^{\mathbb{T}_N^d})$ is the path space of càdlàg trajectories with values in $\{0,1\}^{\mathbb{T}_N^d}$. For a measure μ_N on $\{0,1\}^{\mathbb{T}_N^d}$, denote by $\mathbb{P}_{\mu_N}$ the probability measure on $D(\mathbb{R}_+, \{0,1\}^{\mathbb{T}_N^d})$ induced by the initial state μ_N and the Markov process $\{\eta_t : t \geq 0\}$. Expectation with respect to $\mathbb{P}_{\mu_N}$ is denoted by $\mathbb{E}_{\mu_N}$.

2.6.2 The hydrodynamic equation

Let $A = (a_{jj})_{d\times d}$ be a diagonal matrix with $a_{jj} > 0, j = 1, \ldots, d$, and consider the operator

$$\nabla A \nabla_W := \sum_{j=1}^{d} a_{jj} \partial_{x_j} \partial_{W_j}$$

defined on $\mathbb{D}_W$.

A sequence of probability measures $\{\mu_N : N \geq 1\}$ on $\{0,1\}^{\mathbb{T}_N^d}$ is said to be associated to a profile $\rho_0 : \mathbb{T}^d \to [0,1]$ if

$$\lim_{N\to\infty} \mu_N \left\{ \Big| \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} H(x/N)\eta(x) - \int H(u)\rho_0(u)du \Big| > \delta \right\} \;=\; 0 \tag{2.6.3}$$

for every $\delta > 0$ and every function $H \in \mathbb{D}_W$.

Let $\gamma : \mathbb{T}^d \to [l, r]$ be a bounded density profile and consider the parabolic differential equation

$$\begin{cases} \partial_t \rho \;=\; \nabla A \nabla_W \Phi(\rho) \\ \rho(0,\cdot) \;=\; \gamma(\cdot) \end{cases} , \tag{2.6.4}$$

where the function $\Phi : [l, r] \to \mathbb{R}$ is given as in the beginning of Section 1.5, and $t \in [0, T]$, for $T > 0$ fixed.

Recall, from Section 2.3, that a bounded function $\rho : [0, T] \times \mathbb{T}^d \to [l, r]$ is said to be a weak solution of the parabolic differential equation (1.1.9) if the following conditions hold. $\Phi(\rho(\cdot,\cdot))$ and $\rho(\cdot,\cdot)$ belong to $L^2([0,T], H_{1,W}(\mathbb{T}^d))$, and we have the integral identity

$$\int_{\mathbb{T}^d} \rho(t,u)H(u)du - \int_{\mathbb{T}^d} \rho(0,u)H(u)du = \int_0^t \int_{\mathbb{T}^d} \Phi(\rho(s,u))\nabla A \nabla_W H(u)du\, ds ,$$

for every function $H \in \mathbb{D}_W$ and all $t \in [0, T]$.

Existence of such weak solutions follow from the tightness of the process proved in subsection 2.6.3, and from the energy estimate obtained in Lemma 1.5.2. Uniquenesses of weak solutions was proved in subsection 2.3.1.

Theorem 2.6.1. *Fix a continuous initial profile $\rho_0 : \mathbb{T}^d \to [0,1]$ and consider a sequence of probability measures μ_N on $\{0,1\}^{\mathbb{T}_N^d}$ associated to ρ_0, in the sense of* (2.6.3)*. Then, for any $t \geq 0$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N} \left\{ \Big| \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} H(x/N)\eta_t(x) - \int H(u)\rho(t,u)\, du \Big| > \delta \right\} \;=\; 0$$

for every $\delta > 0$ and every function $H \in \mathbb{D}_W$. Here, ρ is the unique weak solution of the non-linear equation (1.1.9) *with $l = 0$, $r = 1$, $\gamma = \rho_0$ and $\Phi(\alpha) = \alpha + a\alpha^2$.*

Let $\mathcal{M}$ be the space of positive measures on $\mathbb{T}^d$ with total mass bounded by one endowed with the weak topology. Recall that $\pi_t^N \in \mathcal{M}$ stands for the empirical measure at time t. This is the measure on $\mathbb{T}^d$ obtained by rescaling space by N and by assigning mass $1/N^d$ to each particle:

$$\pi_t^N \;=\; \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \eta_t(x)\, \delta_{x/N} , \tag{2.6.5}$$

where δ_u is the Dirac measure concentrated on u.

For a function $H : \mathbb{T}^d \to \mathbb{R}$, $\langle \pi_t^N, H\rangle$ stands for the integral of H with respect to π_t^N:

$$\langle \pi_t^N, H\rangle \;=\; \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} H(x/N)\eta_t(x) .$$

This notation is not to be mistaken with the inner product in $L^2(\mathbb{T}^d)$ introduced earlier. Also, when π_t has a density ρ, $\pi(t,du) = \rho(t,u)du$.

Fix $T > 0$ and let $D([0,T],\mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi : [0,T] \to \mathcal{M}$ endowed with the *uniform* topology. For each probability measure μ_N on $\{0,1\}^{\mathbb{T}_N^d}$, denote by $\mathbb{Q}_{\mu_N}^{W,N}$ the measure on the path space $D([0,T],\mathcal{M})$ induced by the measure μ_N and the process π_t^N introduced in (2.6.5).

Fix a continuous profile $\rho_0 : \mathbb{T}^d \to [0,1]$ and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on $\{0,1\}^{\mathbb{T}_N^d}$ associated to ρ_0 in the sense (2.6.3). Further, we denote by $\mathbb{Q}_W$ the probability measure on $D([0,T],\mathcal{M})$ concentrated on the deterministic path $\pi(t,du) = \rho(t,u)du$, where ρ is the unique weak solution of (2.6.4) with $\gamma = \rho_0$, $l_k = 0$, $r_k = 1$, $k = 1,\ldots,d$ and $\Phi(\alpha) = \alpha + b\alpha^2$.

In subsection 2.6.3 we show that the sequence $\{\mathbb{Q}_{\mu_N}^{W,N} : N \geq 1\}$ is tight, and in subsection 2.6.4 we characterize the limit points of this sequence.

2.6.3 Tightness

The goal of this subsection is to prove tightness of sequence $\{\mathbb{Q}_{\mu_N}^{W,N} : N \geq 1\}$. We will do it by showing that the set of equicontinuous paths of the empirical measures (2.6.5) has probability close to one.

Fix $\lambda > 0$ and consider, initially, the auxiliary $\mathcal{M}$-valued Markov process $\{\Pi_t^{\lambda,N} : t \geq 0\}$ defined by

$$\Pi_t^{\lambda,N}(H) \;=\; \langle \pi_t^N, H_\lambda^N \rangle \;=\; \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} H_\lambda^N(x/N)\eta_t(x),$$

for H in $\mathbb{D}_W$, where H_λ^N is the unique weak solution in $H_{1,W}(\mathbb{T}_N^d)$ (see Section 2.4) of

$$\lambda H_\lambda^N - \nabla^N A^N \nabla_W^N H_\lambda^N = \lambda H - \nabla A \nabla_W H,$$

with the right-hand side being understood as the restriction of the function to the lattice $\mathbb{T}_N^d$ (see subsection 2.4.1).

We first prove tightness of the process $\{\Pi_t^{\lambda,N} : 0 \leq t \leq T\}$,then we show that $\{\Pi_t^{\lambda,N} : 0 \leq t \leq T\}$ and $\{\pi_t^N : 0 \leq t \leq T\}$ are not far apart.

It is well known [23] that to prove tightness of $\{\Pi_t^{\lambda,N} : 0 \leq t \leq T\}$ it is enough to show tightness of the real-valued processes $\{\Pi_t^{\lambda,N}(H) : 0 \leq t \leq T\}$ for a set of smooth functions $H : \mathbb{T}^d \to \mathbb{R}$ dense in $C(\mathbb{T}^d)$ for the uniform topology.

Fix a smooth function $H : \mathbb{T}^d \to \mathbb{R}$. Keep in mind that $\Pi_t^{\lambda,N}(H) = \langle \pi_t^N, H_\lambda^N \rangle$, and denote by $M_t^{N,\lambda}$ the martingale defined by

$$M_t^{N,\lambda} \;=\; \Pi_t^{\lambda,N}(H) \;-\; \Pi_0^{\lambda,N}(H) \;-\; \int_0^t ds\, N^2 L_N \langle \pi_s^N, H_\lambda^N \rangle \;. \tag{2.6.6}$$

Clearly, tightness of $\Pi_t^{\lambda,N}(H)$ follows from tightness of the martingale $M_t^{N,\lambda}$ and tightness of the additive functional $\int_0^t ds\, N^2 L_N \langle \pi_s^N, H_\lambda^N \rangle$.

A long computation, albeit simple, shows that the quadratic variation $\langle M^{N,\lambda} \rangle_t$ of the martingale $M_t^{N,\lambda}$ is given by:

$$\frac{1}{N^{2d-1}} \sum_{j=1}^d \sum_{x\in\mathbb{T}^d} [\partial_{W,j}^N H_\lambda^N(x/N)]^2 [W((x+e_j)/N) - W(x/N)] \times$$
$$\times \int_0^t c_{x,x+e_j}(\eta_s)\, [\eta_s(x+e_j) - \eta_s(x)]^2\, ds \;.$$

In particular, by Lemma 2.4.3,

$$\langle M^{N,\lambda} \rangle_t \;\leq\; \frac{C_0 t}{N^{2d-1}} \sum_{j=1}^d \| H_\lambda^N \|_{W_j,N}^2 \;\leq\; \frac{C(H)t}{\lambda N^d},$$

for some finite constant $C(H)$, which depends only on H. Thus, by Doob inequality, for every $\lambda > 0$, $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N} \left[\sup_{0\leq t\leq T} \left| M_t^{N,\lambda} \right| \;>\; \delta \right] \;=\; 0 \;. \tag{2.6.7}$$

In particular, the sequence of martingales $\{M_t^{N,\lambda} : N \geq 1\}$ is tight for the uniform topology.

It remains to be examined the additive functional of the decomposition (2.6.6). The generator of the exclusion process L_N can be decomposed in terms of the generator of the random walk $\mathbb{L}_N$. A simple computation, we obtain that $N^2 L_N \langle \pi^N, H_\lambda^N \rangle$ is equal to

$$\sum_{j=1}^{d} \Big\{ \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} (\mathbb{L}_N^j H_\lambda^N)(x/N)\, \eta(x) \\ + \frac{b}{N^d} \sum_{x \in \mathbb{T}_N^d} \big[(\mathbb{L}_N^j H_\lambda^N)((x+e_j)/N) + (\mathbb{L}_N^j H_\lambda^N)(x/N) \big] (\tau_x h_{1,j})(\eta) \\ - \frac{b}{N^d} \sum_{x \in \mathbb{T}_N^d} (\mathbb{L}_N^j H_\lambda^N)(x/N) (\tau_x h_{2,j})(\eta) \Big\} ,$$

where $\{\tau_x : x \in \mathbb{Z}^d\}$ is the group of translations, so that $(\tau_x \eta)(y) = \eta(x+y)$ for x, y in $\mathbb{Z}^d$, and the sum is understood modulo N. Also, $h_{1,j}$, $h_{2,j}$ are the cylinder functions

$$h_{1,j}(\eta) \;=\; \eta(0)\eta(e_j) \;, \quad h_{2,j}(\eta) \;=\; \eta(-e_j)\eta(e_j) \;.$$

For all $0 \leq s < t \leq T$, we have

$$\Big| \int_s^t dr\, N^2 L_N \langle \pi_r^N, H_\lambda^N \rangle \Big| \;\leq\; \frac{(1+3|b|)(t-s)}{N^d} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_N^d} |\mathbb{L}_N^j H_\lambda^N(x/N)| \;,$$

from Schwarz inequality and Lemma 1.3.1, the right hand side of the previous expression is bounded above by

$$(1+3|b|)(t-s)d \quad \sqrt{\frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \Big(\mathbb{L}_N H_\lambda^N(x/N) \Big)^2}.$$

Since H_λ^N is the weak solution of the discrete equation, we have by Remark 2.4.4 that it is also a strong solution. Then, we may replace $\mathbb{L}_N H_\lambda^N$ by $U_\lambda^N = \lambda H_\lambda^N - H$ in the previous formula. In particular, It follows from the estimate given in Lemma 2.4.3, that the right hand side of the previous expression is bounded above by $dC(H,b)(t-s)$ uniformly in N, where $C(H,b)$ is a finite constant depending only on b and H. This proves that the additive part of the decomposition (2.6.6) is tight for the uniform topology and therefore that the sequence of processes $\{\Pi_t^{\lambda,N} : N \geq 1\}$ is tight.

Lemma 2.6.2. *The sequence of measures* $\{\mathbb{Q}_{\mu^N}^{W,N} : N \geq 1\}$ *is tight for the uniform topology.*

Proof. Fix $\lambda > 0$. It is enough to show that for every function $H \in \mathbb{D}_W$ and every $\epsilon > 0$, we have

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N} \left[\sup_{0 \leq t \leq T} |\, \Pi_t^{\lambda,N}(H) - \langle \pi_t^N, H \rangle \,| > \epsilon \right] \;=\; 0,$$

whence tightness of π_t^N follows from tightness of $\Pi_t^{\lambda,N}$. By Chebyshev's inequality, the last expression is bounded above by

$$\mathbb{E}_{\mu_N} \left[\sup_{0 \leq t \leq T} |\, \Pi_t^{\lambda,N}(H) - \langle \pi_t^N, H \rangle \,|^2 \right] \leq 2 \| H_\lambda^N - H \|_N^2,$$

since there exists at most one particle per site. By Theorem 2.5.2 and Proposition 2.5.1, $\|H_\lambda^N - H\|_N^2 \to 0$ as $N \to \infty$, and the proof follows. □

2.6.4 Uniqueness of limit points

We prove in this subsection that all limit points $\mathbb{Q}^*$ of the sequence $\mathbb{Q}_{\mu_N}^{W,N}$ are concentrated on absolutely continuous trajectories $\pi(t,du) = \rho(t,u)du$, whose density $\rho(t,u)$ is a weak solution of the hydrodynamic equation (1.1.9) with $l = 0$, $r = 1$ and $\Phi(\alpha) = \alpha + a\alpha^2$.

We now state a result necessary to prove the uniqueness of limit points. Recall that, for a local function $g : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$, $\tilde{g} : [0,1] \to \mathbb{R}$ be the expected value of g under the stationary states:

$$\tilde{g}(\alpha) \;=\; E_{\nu_\alpha}[g(\eta)]\;.$$

For $\ell \geq 1$ and d-dimensional integer $x = (x_1, \dots, x_d)$, denote by $\eta^\ell(x)$ the empirical density of particles in the box $\mathbb{B}^\ell_+(x) = \{(y_1, \dots, y_d) \in \mathbb{Z}^d\,; 0 \leq y_i - x_i < \ell\}$:

$$\eta^\ell(x) \;=\; \frac{1}{\ell^d} \sum_{y \in \mathbb{B}^\ell_+(x)} \eta(y)\;.$$

Let $\mathbb{Q}^*$ be a limit point of the sequence $\mathbb{Q}^{W,N}_{\mu_N}$ and assume, without loss of generality, that $\mathbb{Q}^{W,N}_{\mu_N}$ converges to $\mathbb{Q}^*$.

Since there is at most one particle per site, it is clear that $\mathbb{Q}^*$ is concentrated on trajectories $\pi_t(du)$ which are absolutely continuous with respect to the Lebesgue measure, $\pi_t(du) = \rho(t,u)du$, and whose density ρ is non-negative and bounded by 1.

Fix a function $H \in \mathbb{D}_W$ and $\lambda > 0$. Recall the definition of the martingale $M^{N,\lambda}_t$ introduced in the previous section. From (2.6.7) we have, for every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left[\sup_{0\leq t\leq T} \left|M^{N,\lambda}_t\right| \,>\, \delta\right] \;=\; 0\;,$$

and from (1.4.2), for fixed $0 < t \leq T$ and $\delta > 0$, we have

$$\lim_{N\to\infty} \mathbb{Q}^{W,N}_{\mu_N}\left[\left|\langle \pi^N_t, H^N_\lambda\rangle \;-\; \langle \pi^N_0, H^N_\lambda\rangle \;-\; \int_0^t ds\, N^2 L_N \langle \pi^N_s, H^N_\lambda\rangle\right| \,>\, \delta\right] \;=\; 0.$$

Note that the expression $N^2 L_N \langle \pi^N_s, H^N_\lambda\rangle$ has been computed in the previous subsection in terms of generator $\mathbb{L}_N$. On the other hand, $\mathbb{L}_N H^N_\lambda = \lambda H^N_\lambda - \lambda H + \nabla A \nabla_W H$. Since there is at most one particle per site, we may apply Theorem 2.5.2 to replace $\langle \pi^N_t, H^N_\lambda\rangle$ and $\langle \pi^N_0, H^N_\lambda\rangle$ by $\langle \pi_t, H\rangle$ and $\langle \pi_0, H\rangle$, respectively, and replace $\mathbb{L}_N H^N_\lambda$ by $\nabla A \nabla_W H$ plus a term that vanishes as $N \to \infty$.

Since $E_{\nu_\alpha}[h_{i,j}] = \alpha^2$, $i = 1$, 2 and $j = 1, \dots, d$, we have by Proposition 1.4.4 that, for every $t > 0$, $\lambda > 0$, $\delta > 0$, $i = 1$, 2,

$$\lim_{\varepsilon\to 0} \varlimsup_{N\to\infty} \mathbb{P}_{\mu_N}\Big[\Big|\int_0^t ds\, \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} \mathbb{L}^j_N H^N_\lambda(x/N)\times \times\Big\{\tau_x h_{i,j}(\eta_s) - \big[\eta^{\varepsilon N}_s(x)\big]^2\Big\}\Big| \,>\, \delta\Big] \;=\; 0.$$

Since $\eta^{\varepsilon N}_s(x) = \varepsilon^{-d}\pi^N_s(\prod_{j=1}^d [x_j/N, x_j/N + \varepsilon e_j])$, we obtain, from the previous considerations, that

$$\lim_{\varepsilon\to 0} \varlimsup_{N\to\infty} \mathbb{Q}^{W,N}_{\mu_N}\Big[\Big|\langle \pi_t, H\rangle - \langle \pi_0, H\rangle \;-\; \int_0^t ds\,\Big\langle \Phi\big(\varepsilon^{-d}\pi^N_s(\prod_{j=1}^d [\cdot, \cdot + \varepsilon e_j])\big)\,,\, \nabla A \nabla_W H\Big\rangle\Big| > \delta\Big] \;=\; 0\;.$$

Using the fact that $\mathbb{Q}^{W,N}_{\mu_N}$ converges in the uniform topology to $\mathbb{Q}^*$, we have that

$$\lim_{\varepsilon\to 0} \mathbb{Q}*\Big[\Big|\langle \pi_t, G_\lambda H\rangle \;-\; \langle \pi_0, G_\lambda H\rangle \;-\; \int_0^t ds\;\Big\langle \Phi\big(\varepsilon^{-d}\pi_s(\prod_{j=1}^d [\cdot, \cdot + \varepsilon e_j])\big)\,,\, \nabla A \nabla_W H\Big\rangle\Big| > \delta\Big] \;=\; 0\;.$$

Recall that $\mathbb{Q}^*$ is concentrated on absolutely continuous paths $\pi_t(du) = \rho(t,u)du$ with positive density bounded by 1. Therefore, $\varepsilon^{-d}\pi_s(\prod_{j=1}^d [\cdot, \cdot + \varepsilon e_j])$ converges in $L^1(\mathbb{T}^d)$ to $\rho(s,.)$ as $\varepsilon \downarrow 0$. Thus,

$$\mathbb{Q}^*\left[\left|\langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t ds\,\langle \Phi(\rho_s)\,,\, \nabla A \nabla_W H\rangle\right| > \delta\right] \;=\; 0.$$

Letting $\delta \downarrow 0$, we see that, $\mathbb{Q}^*$ a.s.,

$$\int_{\mathbb{T}^d} \rho(t,u)H(u)du - \int_{\mathbb{T}^d} \rho(0,u)H(u)du = \int_0^t \int_{\mathbb{T}^d} \Phi(\rho(s,u))\nabla A\nabla_W H(u)du\, ds\ .$$

This identity can be extended to a countable set of times t. Taking this set to be dense we obtain, by continuity of the trajectories π_t, that it holds for all $0 \le t \le T$.

From Lemma 1.5.2, we may conclude that all limit points have, almost surely, finite energy, and therefore, by Lemma 2.3.1, $\Phi(\rho(\cdot,\cdot)) \in L^2([0,T], H_{1,W}(\mathbb{T}^d))$. Analogously, it is possible to show that $\rho(\cdot,\cdot)$ has finite energy and hence it belongs to $L^2([0,T], H_{1,W}(\mathbb{T}^d))$.

Proposition 2.6.3. *As $N \uparrow \infty$, the sequence of probability measures $\mathbb{Q}^{W,N}_{\mu_N}$ converges in the uniform topology to $\mathbb{Q}_W$.*

Proof. In the previous subsection, we showed that the sequence of probability measures $\mathbb{Q}^{W,N}_{\mu_N}$ is tight for the uniform topology. Moreover, we just proved that all limit points of this sequence are concentrated on weak solutions of the parabolic equation (2.6.4). The proposition now follows from the uniqueness proved in subsection 2.3.1. □

Proof of Theorem 2.6.1. Since $\mathbb{Q}^{W,N}_{\mu_N}$ converges in the uniform topology to $\mathbb{Q}_W$, a measure which is concentrated on a deterministic path, for each $0 \le t \le T$ and each continuous function $H : \mathbb{T}^d \to \mathbb{R}$, $\langle \pi^N_t, H\rangle$ converges in probability to $\int_{\mathbb{T}^d} du \rho(t,u)H(u)$, where ρ is the unique weak solution of (2.6.4) with $l_k = 0$, $r_k = 1$, $\gamma = \rho_0$ and $\Phi(\alpha) = \alpha + b\alpha^2$. □

Chapter 3

Equilibrium fluctuations for exclusion processes with conductances in random environments

In this Chapter we study the equilibrium fluctuations for exclusion processes with conductances in random environments, which can be viewed as a central limit theorem for the empirical distribution of particles when the system starts from an equilibrium measure.

Let $W : \mathbb{R}^d \to \mathbb{R}$ be a function such that $W(x_1, \dots, x_d) = \sum_{k=1}^{d} W_k(x_k)$, where $d \geq 1$ and each function $W_k : \mathbb{R} \to \mathbb{R}$ is strictly increasing, right continuous with left limits (càdlàg), and periodic in the sense that $W_k(u+1) - W_k(u) = W_k(1) - W_k(0)$, for all $u \in \mathbb{R}$. The inverse of the increments of the function W will play the role of conductances in our system.

The random environment that we considered is governed by the coefficients of the discrete formulation of the model on the lattice. Moreover, we will assume the underlying random field is ergodic, stationary and satisfies an ellipticity condition.

The purpose of this Chapter is to study the density fluctuation field of this system as $N \to \infty$, and also the influence of the randomness in this limit. For any realization of the random environment, the scaling limit depends on the randomness only through some constants which depend on the distribution of the random transition rates, but not on the particular realization of the random environment.

The evolution of one-dimensional exclusion processes with random conductances has attracted some attention recently [12, 13, 14, 18, 21], with the hydrodynamic limit proved in [21] being also obtained in [12], independently. In all of these papers, a hydrodynamic limit was proved. The hydrodynamic limit may be interpreted as a law of large numbers for the empirical density of the system. Our goal is to go beyond the hydrodynamic limit and provide a new result for such processes, which is the equilibrium fluctuations and can be seen as a central limit theorem for the empirical density of the process.

To prove the equilibrium fluctuations, we would like to call attention to the main tools we needed: (i) the theory of nuclear spaces and (ii) homogenization of differential operators. The first one followed the classical approach of Kallianpur and Perez-Abreu [22] and Gel'fand and Vilenkin [19]. Nuclear spaces are very suitable to attain existence and uniqueness of solutions for a general class of stochastic differential equations. Furthermore, tightness of processes on such spaces was established by Mitoma [29]. A wide literature on these spaces can be found cited inside the fourth volume of the amazing collection by Gel'fand [19]. The second tool is motivated by several applications in mechanics, physics, chemistry and engineering. We will consider stochastic homogenization. In the stochastic context, several works on homogenization of operators with random coefficients have been published (see, for instance, [30, 31] and references therein). In homogenization theory, only the stationarity of such random fields is used. The notion of stationary random field is formulated in such a manner that it covers many objects of non-probabilistic nature, e.g., operators with periodic or quasi-periodic coefficients. We follow the approach given in Chapter 2, which was introduced by [31].

The focus of our approach is to study the asymptotic behavior of effective coefficients for a family of random difference schemes, whose coefficients can be obtained by the discretization of random high-contrast lattice structures. Furthermore, the introduction of a corrected empirical measure was needed. The corrected empirical measure was used in the literature, for instance, by [21, 18, 20] and also Chapters 1 and 2. It can be understood as a version of Tartar's compensated compactness lemma in the context of particle systems. In this situation, the averaging due to the dynamics and the inhomogeneities introduced by the random media factorize after introducing the corrected empirical process, in such a way that we can average them separately. It is noteworthy that we managed to prove an equivalence between the asymptotic behavior with respect to both the corrected empirical measure and the uncorrected one. This equivalence was helpful in the sense that whenever the calculation with the corrected empirical measure turned cumbersome, we changed to a calculation with respect to the uncorrected one, and the other way around. This whole approach made the proof more simpler than the usual one with respect solely to the corrected empirical measure developed in the articles mentioned above.

We now describe the organization of the Chapter. In Section 3.1 we state the main results of the article; in Section 3.2 we define the nuclear space needed in our context; in Section 3.3 we recall some results obtained in [34] about homogenization, and then we prove the equilibrium fluctuations by showing that the density fluctuation field converges to a process that solves the martingale problem. We also show that the solution of the martingale problem corresponds to a generalized Ornstein-Uhlenbeck process. In Section 3.4 we prove tightness of the density fluctuation field, as well as tightness of other related quantities. In Section 3.5 we prove the Boltzmann-Gibbs principle, which is a key result for proving the equilibrium fluctuations. Finally, the Appendix contains some known results about nuclear spaces and stochastic differential equations evolving on topologic dual of such spaces.

3.1 Notation and results

Fix a function $W : \mathbb{R}^d \to \mathbb{R}$ as (1.1.1):

$$W(x_1, \ldots, x_d) = \sum_{k=1}^{d} W_k(x_k),$$

where each $W_k : \mathbb{R} \to \mathbb{R}$ is a *strictly increasing* right continuous function with left limits (càdlàg), periodic in the sense that for all $u \in \mathbb{R}$

$$W_k(u + 1) - W_k(u) = W_k(1) - W_k(0).$$

Recall in subsection 1.2 the definitions and properties of the generalized gradient of a function f:

$$\nabla_W f = (\partial_{W_1} f, \ldots, \partial_{W_d} f) \,.$$

We now recall the random environment introduced in Chapter 2. The statistically homogeneous rapidly oscillating coefficients that will be used to define the random rates of the exclusion process with conductances of which we want to study the equilibrium fluctuations. Let $(\Omega, \mathcal{F}, \mu)$ be a standard probability space and $\{T_x : \Omega \to \Omega; x \in \mathbb{Z}^d\}$ be an ergodic group of $\mathcal{F}$-measurable transformations which preserve the measure μ:

- $T_x : \Omega \to \Omega$ is $\mathcal{F}$-measurable for all $x \in \mathbb{Z}^d$,
- $\mu(T_x \mathbf{A}) = \mu(\mathbf{A})$, for any $\mathbf{A} \in \mathcal{F}$ and $x \in \mathbb{Z}^d$,
- $T_0 = I\,,\ \ T_x \circ T_y = T_{x+y}$,
- Any $f \in L^1(\Omega)$ such that $f(T_x\omega) = f(\omega)$ μ-a.s. for each $x \in \mathbb{Z}^d$, is equal to a constant μ-a.s..

The last condition implies that the group T_x is ergodic.

Let the vector-valued $\mathcal{F}$-measurable functions $\{a_j(\omega); j = 1, \ldots, d\}$ be such that satisfies an ellipticity condition: there exists $\theta > 0$ such that

$$\theta^{-1} \le a_j(\omega) \le \theta,$$

for all $\omega \in \Omega$ and $j = 1, \ldots, d$. Then, the diagonal matrices A^N whose elements are given by

$$a_{jj}^N(x) := a_j^N = a_j(T_{Nx}\omega) \ , \ \ x \in T_N^d \ , \ \ j = 1, \ldots, d. \tag{3.1.1}$$

Fix a typical realization $\omega \in \Omega$ of the random environment. For each $x \in \mathbb{T}_N^d$ and $j = 1, \ldots, d$, remember the symmetric rate $\xi_{x,x+e_j} = \xi_{x+e_j,x}$ by

$$\xi_{x,x+e_j} \ = \ \frac{a_j^N(x)}{N[W((x+e_j)/N) - W(x/N)]} \ = \ \frac{a_j^N(x)}{N[W_j((x_j+1)/N) - W_j(x_j/N)]}, \tag{3.1.2}$$

where $e_1, \ldots, e_d$ is the canonical basis of $\mathbb{R}^d$.

Distribute particles on $\mathbb{T}_N^d$ in such a way that each site of $\mathbb{T}_N^d$ is occupied at most by one particle. Denote by η the configurations of the state space $\{0,1\}^{\mathbb{T}_N^d}$ so that $\eta(x) = 0$ if site x is vacant, and $\eta(x) = 1$ if site x is occupied.

The exclusion process with conductances in a random environment is the continuous-time Markov process $\{\eta_t : t \geq 0\}$ with state space $\{0,1\}^{\mathbb{T}_N^d} = \{\eta : \mathbb{T}_N^d \to \{0,1\}\}$, whose generator L_N acts on functions $f : \{0,1\}^{\mathbb{T}_N^d} \to \mathbb{R}$ as

$$(L_N f)(\eta) \ = \ \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_N^d} \xi_{x,x+e_j} c_{x,x+e_j}(\eta) \left\{ f(\sigma^{x,x+e_j}\eta) - f(\eta) \right\} , \tag{3.1.3}$$

We consider the Markov process $\{\eta_t : t \geq 0\}$ on the configurations $\{0,1\}^{\mathbb{T}_N^d}$ associated to the generator L_N in the diffusive scale, i.e., L_N is speeded up by N^2. A describe of the stochastic evolution of the process can be found in Section 2.6.

Consider the random walk $\{X_t\}_{t \geq 0}$ of a particle in $\mathbb{T}_N^d$ induced by the generator $\mathbb{L}_N$ given as follows. Let $\xi_{x,x+e_j}$ given by (3.1.2). If the particle is on a site $x \in \mathbb{T}_N^d$, it will jump to $x + e_j$ with rate $N^2 \xi_{x,x+e_j}$. Furthermore, only nearest neighbor jumps are allowed. The generator $\mathbb{L}_N$ of the random walk $\{X_t\}_{t \geq 0}$ acts on functions $f : \mathbb{N}^{-1} T_N^d \to \mathbb{R}$ as

$$\mathbb{L}_N f\left(\frac{x}{N}\right) \ = \ \sum_{j=1}^{d} \mathbb{L}_N^j f\left(\frac{x}{N}\right),$$

where,

$$\mathbb{L}_N^j f\Big(\frac{x}{N}\Big) = N^2 \Big\{ \xi_{x,x+e_j} \Big[f\Big(\frac{x+e_j}{N}\Big) - f\Big(\frac{x}{N}\Big) \Big] + \xi_{x-e_j,x} \Big[f\Big(\frac{x-e_j}{N}\Big) - f\Big(\frac{x}{N}\Big) \Big] \Big\}$$

It is not difficult to see that the following equality holds:

$$\mathbb{L}_N f(x/N) = \sum_{j=1}^{d} \partial_{x_j}^N (a_j^N \partial_{W_j}^N f)(x) \ := \ \nabla^N A^N \nabla_W^N f(x), \tag{3.1.4}$$

where, $\partial_{x_j}^N$ is the standard difference operator:

$$\partial_{x_j}^N f\left(\frac{x}{N}\right) \ = \ N\left[f\left(\frac{x+e_j}{N}\right) - f\left(\frac{x}{N}\right) \right] ,$$

and $\partial_{W_j}^N$ is the W_j-difference operator:

$$\partial_{W_j}^N f\left(\frac{x}{N}\right) \ = \ \frac{f\left(\frac{x+e_j}{N}\right) - f\left(\frac{x}{N}\right)}{W\left(\frac{x+e_j}{N}\right) - W\left(\frac{x}{N}\right)},$$

for $x \in \mathbb{T}_N^d$. Several properties of the above operator have been obtained in Chapter 2.

Now we state a central limit theorem for the empirical measure, starting from an equilibrium measure ν_ρ. Fix $\rho > 0$ and denote by $\mathcal{S}_W(\mathbb{T}^d)$ the generalized Schwartz space on $\mathbb{T}^d$, whose definition as well as some properties are given in Section 3.2.

Denote by $Y_\cdot^N$ the *density fluctuation field*, which is the bounded linear functional acting on functions $G \in \mathcal{S}_W(\mathbb{T}^d)$ as

$$Y_t^N(G) = \frac{1}{N^{d/2}} \sum_{x \in \mathbb{T}_N^d} G(x)[\eta_t(x) - \rho]. \tag{3.1.5}$$

Let $D([0,T], X)$ be the path space of càdlàg trajectories with values in a metric space X. In this way we have defined a process in $D([0,T], \mathcal{S}'_W(\mathbb{T}^d))$, where $\mathcal{S}'_W(\mathbb{T}^d)$ is the topologic dual of the space $\mathcal{S}_W(\mathbb{T}^d)$.

Theorem 3.1.1. *Consider the fluctuation field $Y_\cdot^N$ defined above. Then, $Y_\cdot^N$ converges weakly to the unique $\mathcal{S}'_W(\mathbb{T}^d)$-solution, $Y_t \in D([0,T], \mathcal{S}'_W(\mathbb{T}^d))$, of the stochastic differential equation*

$$dY_t = \phi'(\rho)\nabla A \nabla_W Y_t dt + \sqrt{2\chi(\rho)\phi'(\rho)A}\, dN_t, \tag{3.1.6}$$

where $\chi(\rho) = \rho(1-\rho)$, $\phi(\rho) = \rho + b\rho^2$, and ϕ' is the derivative of ϕ, $\phi'(\rho) = 1 + 2b\rho$; further A is a constant diagonal matrix with jth diagonal element given by $a_j := E(a_j^N)$, for any $N \in \mathbb{N}$; and N_t is a $\mathcal{S}'_W(\mathbb{T}^d)$-valued mean-zero martingale, with quadratic variation

$$\langle N(G) \rangle_t = t \sum_{j=1}^d \int_{\mathbb{T}^d} \left[\partial_{W_j} G(x)\right]^2 d(x^j \otimes W_j),$$

where $d(x^j \otimes W_j)$ is the product measure $dx_1 \otimes \cdots \otimes dx_{j-1} \otimes dW_j \otimes dx_{j+1} \otimes \cdots \otimes dx_d$. Furthermore, N_t is a Gaussian process with independent increments. More precisely, for each $G \in S_W(\mathbb{T}^d)$, $N_t(G)$ is a time deformation of a standard Brownian motion.

The proof of this theorem is given in Section 3.3.

Remark 3.1.2. *The process Y_t is known in the literature as the generalized Ornstein-Uhlenbeck process with characteristics $\phi'(\rho)\nabla A \nabla_W$ and $\sqrt{2\chi(\rho)\phi'(\rho)A}\nabla_W$.*

3.2 The space $\mathcal{S}_W(\mathbb{T}^d)$

Recall the properties of the operator $\mathcal{L}_W$ introduced in Section 1.2. In this Section we build the countably Hilbert nuclear space $\mathcal{S}_W(\mathbb{T}^d)$, which is associated the the self-adjoint operator $\mathcal{L}_W = \nabla\nabla_W$. This space, as we shall see, is a natural environment to attain existence and uniqueness of solutions of the stochastic differential equation (3.1.6). Several lemmas are obtained to fulfill the conditions to ensure existence and uniqueness of such solutions. The reader is referred to Appendix.

Let $\{\varphi_j\}_{j\geq 1}$ be the complete orthonormal set of the eigenvectors of the operator $\mathcal{L} = \mathbb{I} - \mathcal{L}_W$, and $\{\lambda_j\}_{j\geq 1}$ the associated eigenvalues. Note that $\lambda_j = 1 + \alpha_j$.

Consider the following increasing sequence $\|\cdot\|_n$, $n \in \mathbb{N}$, of Hilbertian norms:

$$\langle f, g \rangle_n = \sum_{k=1}^\infty \langle \mathbb{P}_k f, \mathbb{P}_k g \rangle \lambda_k^{2n} k^{2n},$$

where we denote by $\mathbb{P}_k$ the orthogonal projection on the linear space generated by the eigenvector φ_k.

So,

$$\|f\|_n^2 = \sum_{k=1}^\infty \|\mathbb{P}_k f\|^2 \lambda_k^{2n} k^{2n},$$

where $\|\cdot\|$ is the $L^2(\mathbb{T}^d)$ norm.

Consider the Hilbert spaces $\mathcal{S}_n$ which are obtained by completing the space $\mathbb{D}_W$ with respect to the inner product $\langle \cdot, \cdot \rangle_n$.

The set

$$\mathcal{S}_W(\mathbb{T}^d) = \bigcap_{n=0}^\infty \mathcal{S}_n$$

endowed with the metric (3.6.2) is our *countably Hilbert space*, and even more, it is a countably Hilbert nuclear space, see the Appendix for further details. In fact, fixed $n \in \mathbb{N}$ and $m > n + 1/2$, we have that $\{\frac{1}{(j\lambda_j)^m}\varphi_j\}_{j\geq 1}$ is a complete orthonormal set in $\mathcal{S}_m$. Therefore,

$$\sum_{j=1}^{\infty} \| \frac{1}{(j\lambda_j)^m}\varphi_j\|_n^2 \leq \sum_{j=1}^{\infty} \frac{1}{j^{2(m-n)}} < \infty,$$

where the above formula corresponds to formula (3.6.3) in Appendix.

Lemma 3.2.1. *Let $\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T}^d)$ be the operator obtained in Theorem 1.1.2. We have*

(a) *$\mathcal{L}_W$ is the generator of a strongly continuous contraction semigroup $\{P_t : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)\}_{t\geq 0}$;*

(b) *$\mathcal{L}_W$ is a closed operator;*

(c) *For each $f \in L^2(\mathbb{T}^d)$, $t \mapsto P_t f$ is a continuous function from $[0,\infty)$ to $L^2(\mathbb{T}^d)$;*

(d) *$\mathcal{L}_W P_t f = P_t \mathcal{L}_W f$ for each $f \in \mathcal{D}_W$ and $t \geq 0$;*

(e) *$(\mathbb{I} - \mathcal{L}_W)^n P_t f = P_t(\mathbb{I} - \mathcal{L}_W)^n f$ for each $f \in \mathbb{D}_W$, $t \geq 0$ and $n \in \mathbb{N}$;*

Proof. Item (a) follows from Theorem 1.1.2 and Hille-Yosida's theorem. Items (b), (c) and (d) follows from item (a), see, for instance, [10, chapter 1]. Item (e) follows from item (d) and from the fact that $\mathcal{L}_W f = \mathbb{L}_W f$ if $f \in \mathbb{D}_W$. □

The next Lemma permits to conclude that the semigroup $\{P_t : t \geq 0\}$ acting on the domain $\mathcal{S}_W(\mathbb{T}^d)$ is a $C_{0,1}$-semigroup, whose definition is recalled in Appendix 3.6.2.

Lemma 3.2.2. *Let $\{P_t : t \geq 0\}$ be the semigroup whose infinitesimal generator is $\mathcal{L}_W$. Then for each $q \in \mathbb{N}$ we have:*

$$\|P_t f\|_q \leq \|f\|_q,$$

for all $f \in \mathcal{S}_W(\mathbb{T}^d)$. In particular, $\{P_t : t \geq 0\}$ is a $C_{0,1}$-semigroup.

Proof. Let $f \in \mathbb{D}_W$, then

$$f = \sum_{j=1}^{k} \beta_j \varphi_j,$$

for some $k \in \mathbb{N}$, and some constants $\beta_1, \ldots, \beta_k$. A simple calculation shows that

$$P_t f = \sum_{j=1}^{k} \beta_j e^{t(1-\lambda_j)} \varphi_j.$$

Therefore, for $f \in \mathbb{D}_W$:

$$\begin{aligned} \|P_t f\|_n^2 &= \| \sum_{j=1}^{k} \beta_j e^{t(1-\lambda_j)} \varphi_j \|_n \\ &= \sum_{j=1}^{k} \|\beta_j e^{t(1-\lambda_j)} \varphi_j\|^2 \lambda_j^{2n} j^{2n} \\ &\leq \sum_{j=1}^{k} \|\beta_j \varphi_j\|^2 \lambda_j^{2n} j^{2n} = \|f\|_n^2. \end{aligned}$$

Since $\mathbb{D}_W$ is dense in $\mathcal{S}_W(\mathbb{T}^d)$, we conclude the proof of the lemma. □

Lemma 3.2.3. *The operator $\mathcal{L}_W$ belongs to $\mathcal{L}(\mathcal{S}_W(\mathbb{T}^d), \mathcal{S}_W(\mathbb{T}^d))$, the space of linear continuous operators from $\mathcal{S}_W(\mathbb{T}^d)$ into $\mathcal{S}_W(\mathbb{T}^d)$.*

Proof. Let $f \in \mathcal{S}_W(\mathbb{T}^d)$, and $\{\varphi_j\}_{j\geq 1}$ be the complete orthonormal set of eigenvectors of $\mathcal{L}_W$, with $\{(1-\lambda_j)\}_{j\geq 1}$ being their respectively eigenvalues. We have that

$$f = \sum_{j=1}^{\infty} \beta_j \varphi_j, \quad \text{with} \quad \sum_{j=1}^{\infty} \beta_j^2 < \infty.$$

We also have that

$$\mathcal{L}_W f = \sum_{j=1}^{\infty} (1-\lambda_j)\beta_j \varphi_j.$$

For every $n \in \mathbb{N}$:

$$\begin{aligned}
\|\mathcal{L}_W f\|_n^2 &= \sum_{k=1}^{\infty} \|\mathbb{P}_k(\mathcal{L}_W f)\|^2 \lambda_k^{2n} k^{2n} = \sum_{k=1}^{\infty} \|\beta_k (1-\lambda_k)\varphi_k\|^2 \lambda_k^{2n} k^{2n} \\
&= \sum_{k=1}^{\infty} \|\beta_k \varphi_k\|^2 (1-\lambda_k)^2 \lambda_k^{2n} k^{2n} \\
&\leq 2\sum_{k=1}^{\infty} \|\mathbb{P}_k f\|^2 \lambda_k^{2n} k^{2n} + 2\sum_{k=1}^{\infty} \|\mathbb{P}_k f\|^2 \lambda_k^{2(n+1)} k^{2(n+1)} \\
&= 2(\|f\|_n + \|f\|_{n+1}).
\end{aligned}$$

Therefore, by the definition of $\mathcal{S}_W(\mathbb{T}^d)$, $\mathcal{L}_W f$ belongs to $\mathcal{S}_W(\mathbb{T}^d)$. Furthermore, $\mathcal{L}_W$ is continuous from $\mathcal{S}_W(\mathbb{T}^d)$ to $\mathcal{S}_W(\mathbb{T}^d)$. □

3.3 Equilibrium Fluctuations

We begin by stating some results on homogenization of differential operators obtained in Chapter 2, which will be very useful along this section.

Let $L^2_{x^i \otimes W_i}(\mathbb{T}^d)$ be the space of square integrable functions with respect to the product measure $d(x^i \otimes W_i) = dx_1 \otimes \cdots \otimes dx_{i-1} \otimes dW_i \otimes dx_{i+1} \otimes \cdots \otimes dx_d$, and $H_{1,W}(\mathbb{T}^d)$ be the Sobolev space of functions with W-generalized derivatives. More precisely, $H_{1,W}(\mathbb{T}^d)$ is the space of functions $g \in L^2(\mathbb{T}^d)$ such that for each $i = 1, \ldots, d$ there exist functions $G_i \in L^2_{x^i \otimes W_i, 0}(\mathbb{T}^d)$ satisfying the following integral by parts identity.

$$\int_{\mathbb{T}^d} \left(\partial_{x_i} \partial_{W_i} f\right) g \, dx \;=\; -\int_{\mathbb{T}^d} \left(\partial_{W_i} f\right) G_i d(x^i \otimes W_i), \tag{3.3.1}$$

for every function $f \in \mathcal{S}_W(\mathbb{T}^d)$, where $L^2_{x^j \otimes W_j, 0}(\mathbb{T}^d)$ is the closed subspace of $L^2_{x^j \otimes W_j}(\mathbb{T}^d)$ consisting of the functions that have zero mean with respect to the measure $d(x^j \otimes W_j)$:

$$\int_{\mathbb{T}^d} f d(x^j \otimes W_j) = 0.$$

. We denote G_i simply by $\partial_{W_i} g$. See [34] for further details and properties of this space.

Let $\lambda > 0$, f be a functional on $H_{1,W}(\mathbb{T}^d)$, u_N be the unique weak solution of

$$\lambda u_N - \nabla^N A^N \nabla^N_W u_N = f,$$

and u_0 be the unique weak solution of

$$\lambda u_0 - \nabla A \nabla_W u_0 = f. \tag{3.3.2}$$

For more details on existence and uniqueness of such solutions see [34].

In this context, we say that the diagonal matrix $A = \{a_{jj}\} = \{a_j\}$ is a *homogenization* of the sequence of random matrices A^N, denoted by $A^N \xrightarrow{H} A$, if the following conditions hold:

- u_N converges weakly in $H_{1,W}(\mathbb{T}^d)$ to u_0, when $N \to \infty$;

- $a_i^N \partial_{W_i}^N u^N \to a_i \partial_{W_i} u$, weakly in $L^2_{x^i \otimes W_i}(\mathbb{T}^d)$ when $N \to \infty$.

Theorem 3.3.1. *Let A^N be a sequence of ergodic random matrices, such as the one that defines our random environment. Then, almost surely, $A^N(\omega)$ admits a homogenization, where the homogenized matrix A does not depend on the realization ω.*

The following proposition regards the convergence of energies:

Proposition 3.3.2. *Let $A^N \xrightarrow{H} A$, as $N \to \infty$, with u_N being the solution of*

$$\lambda u_N - \nabla^N A^N \nabla_W^N u_N = f,$$

where f is a fixed functional on $H_{1,W}(\mathbb{T}^d)$. Then, the following limit relations hold true:

$$\frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} u_N^2(x) \to \int_{\mathbb{T}^d} u_0^2(x) dx,$$

and

$$\frac{1}{N^{d-1}} \sum_{j=1}^d \sum_{x \in \mathbb{T}_N^d} a_{jj}^N(x) (\partial_{W_j}^N u_N(x))^2 \left[W_j((x_i+1)/N) - W_j(x_i/N) \right]$$
$$\to \sum_{j=1}^d \int_{\mathbb{T}^d} a_{jj}(x) (\partial_{W_j} u_0(x))^2 d(x^j \otimes W_j),$$

as $N \to \infty$.

The proofs of these results can be found in Chapter 2.

3.3.1 Martingale Problem

We say that $Y_t \in \mathcal{S}'_W(\mathbb{T}^d)$ solves the martingale problem with initial condition Y_0 if for any $G \in \mathcal{S}_W(\mathbb{T}^d)$

$$M_t(G) = Y_t(G) - Y_0(G) - \phi'(\rho) \int_0^t Y_s(\nabla A \nabla_W G) ds \tag{3.3.3}$$

is a martingale with quadratic variation

$$\langle M_t(G) \rangle = 2t\chi(\rho)\phi'(\rho) \sum_{j=1}^d \int_{\mathbb{T}^d} a_{jj} \left(\partial_{W_j} G \right)^2 d(x^j \otimes W_j). \tag{3.3.4}$$

Observe that if Y_t is the generalized Ornstein-Uhlenbeck process with characteristics $\phi'(\rho)\nabla A \nabla_W$ and $\sqrt{2\chi(\rho)\phi'(\rho)A}\nabla_W$, then Y_t solves the martingale problem above.

Recall the definition of the density fluctuation field $Y_\cdot^N$ given in (3.1.5), and denote by Q_N the distribution in $D([0,T], \mathcal{S}_W(\mathbb{T}^d))$ induced by $Y_\cdot^N$, with initial distribution ν_ρ. Our goal is to show that any limit point of $Y_\cdot^N$ solves the martingale problem. To this end, let us introduce the *corrected density fluctuation field*:

$$Y_t^{N,\lambda}(G) = \frac{1}{N^{d/2}} \sum_{x \in \mathbb{T}^d} G_N^\lambda(x) \left[\eta_t(x) - \rho \right],$$

where G_N^λ is the weak solution of the equation

$$\lambda G_N^\lambda - L_N G_N^\lambda = \lambda G - \nabla A \nabla_W G, \tag{3.3.5}$$

that, via homogenization, converges to G which is the trivial solution of the problem

$$\lambda G - \nabla A \nabla_W G = \lambda G - \nabla A \nabla_W G.$$

The processes $Y_\cdot^N$ and $Y_\cdot^{N,\lambda}$ have the same asymptotic behavior, as we will see. But some calculations are simpler with one of them than with the other. In this way, we have defined two processes in $D([0,T],\mathcal{S}'_W(\mathbb{T}^d))$.

Given a process $Y_\cdot$ in $D([0,T],\mathcal{S}'_W(\mathbb{T}^d))$, and for $t\geq 0$, let $\mathcal{F}_t$ be the σ-algebra generated by $Y_s(H)$ for $s\leq t$ and $H\in\mathcal{S}_W(\mathbb{T}^d)$. Furthermore, set $\mathcal{F}_\infty=\sigma\Big(\bigcup_{t\geq 0}\mathcal{F}_t\Big)$. Denote by Q_N^λ the distribution on $D([0,T],\mathcal{S}'_W(\mathbb{T}^d))$ induced by the corrected density fluctuation field $Y_\cdot^{N,\lambda}$ and initial distribution ν_ρ.

Theorem 3.1.1 is a consequence of the following result about the corrected fluctuation field.

Theorem 3.3.3. *Let Q be the probability measure on $D([0,T],\mathcal{S}'_W(\mathbb{T}^d))$ corresponding to the generalized Ornstein-Uhlenbeck process of mean zero and characteristics $\phi'(\rho)\nabla\cdot A\nabla_W$ and $\sqrt{2\chi(\rho)\phi'(\rho)A}\nabla_W$. Then the sequence $\{Q_N^\lambda\}_{N\geq 1}$ converges weakly to the probability measure Q.*

Note also that the above theorem implies that any limit point of $Y_\cdot^N$ solves the martingale problem (3.3.3)-(3.3.4).

Before proving the Theorem 3.3.3, we will state and prove a lemma. This lemma shows that tightness of $Y_t^{N,\lambda}$ follows from tightness of Y_t^N, and even more, that they have the same limit points. So we can derive our main theorem from Theorem 3.3.3.

Lemma 3.3.4. *For all $t\in[0,T]$ and $G\in\mathcal{S}_W(\mathbb{T}^d)$, $\lim_{N\to\infty}E_{\nu_\rho}\big[Y_t^N(G)-Y_t^{N,\lambda}(G)\big]^2=0$.*

Proof. By convergence of energies, we have that $\lim_{N\to\infty}G_N^\lambda=G$ in $L^2_N(\mathbb{T}^d)$, i.e.

$$\|G_N^\lambda-G\|_N^2:=\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}[G_N^\lambda(x/N)-G(x/N)]^2\to 0,\qquad\text{as}\qquad N\to\infty. \tag{3.3.6}$$

Since ν_ρ is a product measure we obtain

$$E_{\nu_\rho}\big[Y_t^N(G)-Y_t^{N,\lambda}(G)\big]^2 =$$

$$= E_{\nu_\rho}\Big[\frac{1}{N^d}\sum_{x,y\in\mathbb{T}_N^d}[G_N^\lambda(x/N)-G(x/N)][G_N^\lambda(y/N)-G(y/N)](\eta_t(x)-\rho)(\eta_t(y)-\rho)\Big] =$$

$$= E_{\nu_\rho}\Big[\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}[G_N^\lambda(x/N)-G(x/N)]^2(\eta_t(x)-\rho)^2\Big]\leq\frac{C(\rho)}{N^d}\sum_{x\in\mathbb{T}_N^d}[G_N^\lambda(x/N)-G(x/N)]^2,$$

where $C(\rho)$ is a constant that depend on ρ. By (3.3.6) the last expression vanishes as $N\to\infty$. □

Proof of Theorem 3.3.3

Consider the martingale

$$M_t^N(G)=Y_t^N(G)-Y_0^N(G)-\int_0^t N^2L_NY_s^N(G)ds \tag{3.3.7}$$

associated to the original process and the martingale

$$M_t^{N,\lambda}(G)=Y_t^{N,\lambda}(G)-Y_0^{N,\lambda}(G)-\int_0^t N^2L_NY_s^{N,\lambda}(G)ds \tag{3.3.8}$$

associated to the corrected process.

A long, albeit simple, computation shows that the quadratic variation of the martingale $M_t^{N,\lambda}(G)$, $\langle M^{N,\lambda}(G)\rangle_t$, is given by:

$$\frac{1}{N^{d-1}}\sum_{j=1}^d\sum_{x\in\mathbb{T}^d}a_{jj}^N[\partial_{W_j}^N G_N^\lambda(x/N)]^2[W((x+e_j)/N)-W(x/N)]\times \tag{3.3.9}$$

$$\times\int_0^t c_{x,x+e_j}(\eta_s)\,[\eta_s(x+e_j)-\eta_s(x)]^2\,ds\,.$$

Is not difficult see that the quadratic variation of the martingale $M_t^N(G)$, $\langle M^N(G)\rangle_t$, has the expression (3.3.9) with G replacing G_N^λ. Further,

$$\begin{aligned}&E_{\nu_\rho}\big[c_{x,x+e_j}(\eta)\,[\eta_s(x+e_j)-\eta_s(x)]^2\big] =\\ &\quad E_{\nu_\rho}[1+b(\eta(x-e_j)+\eta(x))]E_{\nu_\rho}[(\eta(x+e_j)-\eta(x))^2] =\\ &\qquad 2(1+2b\rho)\rho(1-\rho) = 2\phi'(\rho)\chi(\rho).\end{aligned}$$

Lemma 3.3.5. *Fix $G \in \mathcal{S}_W(\mathbb{T}^d)$ and $t>0$, and let $\langle M^{N,\lambda}(G)\rangle_t$ and $\langle M^N(G)\rangle_t$ be the quadratic variations of the martingales $M_t^{N,\lambda}(G)$ and $M_t^N(G)$, respectively. Then,*

$$\lim_{N\to\infty} E_{\nu_\rho}\big[\langle M^{N,\lambda}(G)\rangle_t - \langle M^N(G)\rangle_t\big]^2 \;=\; 0. \tag{3.3.10}$$

Proof. Fix $G \in \mathcal{S}_W(\mathbb{T}^d)$ and $t>0$. A straightforward calculation shows that

$$E_{\nu_\rho}\big[\langle M^{N,\lambda}(G)\rangle_t - \langle M^N(G)\rangle_t\big]^2 \leq$$

$$\{k^2t^2\frac{1}{N^{d-1}}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d} a_{jj}^N[\big(\partial_{W_j}^N G_N^\lambda(x/N)\big)^2 - \big(\partial_{W_j}^N G(x/N)\big)^2][W(\frac{x+e_j}{N}) - W(\frac{x}{N})]\}^2,$$

where the constant k comes from the integral term. By the convergence of energies (Proposition 2.5.1), the last term vanishes as $N\to\infty$. □

Lemma 3.3.6. *Let $G \in \mathcal{S}_W(\mathbb{T}^d)$ and $d \geq 1$. Then*

$$\begin{aligned}\lim_{N\to\infty} E_{\nu_\rho}\Big[\frac{1}{N^{d-1}}\int_0^t ds \sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d} a_{jj}^N\big(\partial_{W_j}^N G(x/N)\big)^2[W((x+e_j)/N) - W(x/N)]\times\\ \times\big[c_{x,x+e_j}(\eta_s)\,[\eta_s(x+e_j)-\eta_s(x)]^2 - 2\chi(\rho)\phi'(\rho)\big]\Big]^2 \;=\; 0.\end{aligned}$$

Proof. Fix $G \in \mathcal{S}_W(\mathbb{T}^d)$ and $d>1$. The term in the previous expression is less than or equal to

$$\frac{t^2\theta^4 C(\rho)}{N^{d-1}}\|\nabla_W^N G\|_{W,N,4}^4, \tag{3.3.11}$$

where

$$\|\nabla_W^N G\|_{W,N,4}^4 := \frac{1}{N^{d-1}}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d}\big(\partial_{W_j}^N G(x/N)\big)^4[W((x+e_j)/N) - W(x/N)].$$

Thus, since for $G \in \mathcal{S}_W(\mathbb{T}^d)$, $\|\nabla_W^N G\|_{W,N,4}^4$ is bounded, the term in (3.3.11) converges to zero as $N\to\infty$. The case $d=1$ follows from calculations similar to the ones found in Lemma 12 of [28]. □

So, by Lemma 3.3.5 and 3.3.6, $\langle M^{N,\lambda}(G)\rangle_t$ is given by

$$\frac{2t\chi(\rho)\phi'(\rho)}{N^{d-1}}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d} a_{jj}^N\big(\partial_{W_j}^N G_N^\lambda(x/N)\big)^2[W((x+e_j)/N) - W(x/N)]$$

plus a term that vanishes in $L^2_{\nu_\rho}(\mathbb{T}^d)$ as $N\to\infty$. By the convergence of energies, Proposition 2.5.1, it converges, as $N\to\infty$, to

$$2t\chi(\rho)\phi'(\rho)\sum_{j=1}^{d}\int_{\mathbb{T}^d} a_{jj}^N\big(\partial_{W_j}G(x)\big)^2 dx^j\otimes W_j.$$

Our goal now consists in showing that it is possible to write the integral part of the martingale as the integral of a function of the density fluctuation field plus a term that goes to zero in $L^2_{\nu_\rho}(\mathbb{T}^d)$. After some simple computations, we obtain that

$$
\begin{aligned}
N^2 L_N Y_s^{N,\lambda}(G) \;=\; & \sum_{j=1}^{d} \Big\{ \frac{1}{N^{d/2}} \sum_{x\in T_N^d} \mathbb{L}_N^j G_N^\lambda(x/N)\, \eta_s(x) \\
& + \frac{b}{N^{d/2}} \sum_{x\in T_N^d} \big[\mathbb{L}_N^j G_N^\lambda((x+e_j)/N) + \mathbb{L}_N^j G_N^\lambda(x/N) \big]\, (\tau_x h_{1,j})(\eta_s) \\
& - \frac{b}{N^{d/2}} \sum_{x\in T_N^d} \mathbb{L}_N^j G_N^\lambda(x/N) (\tau_x h_{2,j})(\eta_s) \Big\} \;,
\end{aligned}
$$

where $\{\tau_x : x \in \mathbb{Z}^d\}$ is the group of translations, so that $(\tau_x \eta)(y) = \eta(x+y)$ for x, y in $\mathbb{Z}^d$, and the sum is understood modulo N. Also, $h_{1,j}$, $h_{2,j}$ are the cylinder functions

$$
h_{1,j}(\eta) \;=\; \eta(0)\eta(e_j) \;, \quad h_{2,j}(\eta) \;=\; \eta(-e_j)\eta(e_j) \;.
$$

Note that inside the expression $N^2 L_N Y_s^{N,\lambda}$ we may replace $\mathbb{L}_N^j G_N^\lambda$ by $a_j \partial_{x_j} \partial_{W_j} G$. Indeed, the expression

$$
\begin{aligned}
E_{\nu(\rho)} \Big\{ \int_0^t & \sum_{j=1}^{d} \frac{1}{N^{d/2}} \sum_{x\in \mathbb{T}_N^d} \Big[\mathbb{L}_N^j G_N^\lambda(x/N) - a_j \partial_{x_j} \partial_{W_j} G(x/N) \Big] \big(\eta_s(x) - \rho \big) + \\
& + \frac{b}{N^{d/2}} \sum_{x\in \mathbb{T}_N^d} \Big[\mathbb{L}_N^j G_N^\lambda((x+e_j)/N) - a_j \partial_{x_j} \partial_{W_j} G((x+e_j)/N) + \\
& \qquad \mathbb{L}_N^j G_N^\lambda(x/N) - a_j \partial_{x_j} \partial_{W_j} G(x/N) \Big] \big((\tau_x h_{1,j})(\eta_s) - \rho^2 \big) - \\
& - \frac{b}{N^{d/2}} \sum_{x\in \mathbb{T}_N^d} \Big[\mathbb{L}_N^j G_N^\lambda(x/N) - a_j \partial_{x_j} \partial_{W_j} G(x/N) \Big] \big((\tau_x h_{2,j})(\eta_s) - \rho^2 \big) \Big\}^2 .
\end{aligned}
$$

is less than or equal to

$$
C(\rho, b) \int_0^t \frac{1}{N^d} \sum_{x\in \mathbb{T}^d} \big[L_N G_N^\lambda(x/N) - \nabla A \nabla_W G(x/N) \big]^2 .
$$

Now, recall that G_N^λ is solution of the equation (3.3.5), and therefore, the previous expression is less than or equal to

$$
\frac{t\, C(\rho, b)}{\lambda^2} \| G_N^\lambda - G \|_N^2 ,
$$

thus, by homogenization and energy estimates in Theorem 3.3.1 and Proposition 3.3.2, respectively, the last expression converges to zero as $N \to \infty$.

By the Boltzmann Gibbs principle, Theorem 3.5.1 below, we can replace $(\tau_x h_{i,j})(\eta_s) - \rho^2$ by $2\rho[\eta_s(x) - \rho]$ for $i = 1, 2$. Doing so, the martingale (3.3.8) can be written as

$$
M_t^{N,\lambda}(G) = Y_t^{N,\lambda}(G) - Y_0^{N,\lambda}(G) - \int_0^t \frac{1}{N^{d/2}} \sum_{x\in \mathbb{T}^d} \nabla A \nabla_W G(x/N) \phi'(\rho) \big(\eta_s - \rho \big) ds, \tag{3.3.12}
$$

plus a term that vanishes in $L^2_{\nu_\rho}(\mathbb{T}^d)$ as $N \to \infty$.

Notice that, by (3.1.5), the integrand in the previous expression is a function of the density fluctuation field Y_t^N. By Lemma 3.3.4, we can replace the term inside the integral of the above expression by a term which is a function of the corrected density fluctuation field $Y_t^{N,\lambda}$.

From the results of Section 3.4, the sequence $\{Q_N^\lambda\}_{N \geq 1}$ is tight and let Q^λ be a limit point of it. Let Y_t be the process in $D([0,T], \mathcal{S}'_W(\mathbb{T}^d))$ induced by the canonical projections under Q^λ. Taking the limit as $N \to \infty$, under an appropriate subsequence, in expression (3.3.12), we obtain that

$$
M_t^\lambda(G) = Y_t(G) - Y_0(G) - \int_0^t Y_s(\phi'(\rho) \nabla \cdot A \nabla_W G) ds, \tag{3.3.13}
$$

where M_t^λ is some $\mathcal{S}'_W(\mathbb{T}^d)$-valued process, in fact, a martingale. To see this, note that for a measurable set U with respect to the canonical σ-algebra $\mathcal{F}_t$, $E_{Q_N^\lambda}[M_t^{N,\lambda}(G)\mathbf{1}_U]$ converges to $E_{Q^\lambda}[M_t^\lambda(G)\mathbf{1}_U]$. Since $M_\cdot^{N,\lambda}(G)$ is a martingale, $E_{Q_N^\lambda}[M_T^{N,\lambda}(G)\mathbf{1}_U] = E_{Q_N^\lambda}[M_t^{N,\lambda}(G)\mathbf{1}_U]$. Taking a further subsequence if necessary, this last term converges to $E_{Q^\lambda}[M_t^\lambda(G)\mathbf{1}_U]$, which proves that $M_\cdot^\lambda(G)$ is a martingale for any $G \in \mathcal{S}_W(\mathbb{T}^d)$. Since all the projections of M_t^λ are martingales, we conclude that M_t^λ is a $\mathcal{S}'_W(\mathbb{T}^d)$-valued martingale.

Now, we need obtain the quadratic variation $\langle M^\lambda(G)\rangle_t$ of the martingale $M_t^\lambda(G)$. A simple application of Tchebyshev's inequality shows that $\langle M^{N,\lambda}(G)\rangle_t$ converges in probability to

$$2t\chi(\rho)\phi'(\rho)\sum_{j=1}^d \int_{\mathbb{T}^d} a_j \Big[\partial_{W_j} G\Big]^2 d(x^j \otimes W_j),$$

where $\chi(\rho)$ stands for the static compressibility given by $\chi(\rho) = \rho(1-\rho)$. By Doob-Meyer's decomposition theorem, we need to prove that

$$N_t^\lambda(G) := M_t^\lambda(G)^2 - 2t\chi(\rho)\phi'(\rho)\sum_{j=1}^d \int_{\mathbb{T}^d} a_j \Big[\partial_{W_j} G\Big]^2 d(x^j \otimes W_j)$$

is a martingale. The same argument we used above applies now if we can show that $\sup_N E_{Q_N^\lambda}[M_T^{N,\lambda}(G)^4] < \infty$ and $\sup_N E_{Q_N^\lambda}[\langle M^{N,\lambda}(G)\rangle_T^2] < \infty$. Both bounds follows easily from the explicit form of $\langle M^{N,\lambda}(G)\rangle_t$ and (3.3.12).

On the other hand, by a standard central limit theorem, Y_0 is a Gaussian field with covariance

$$E\big[Y_0(G)Y_0(H)\big] = \chi(\rho)\int_{\mathbb{T}^d} G(x)H(x)dx.$$

Therefore, by Theorem 3.3.7, Q^λ is equal to the probability distribution Q of a generalized Ornstein-Uhlenbeck process in $D([0,T], \mathcal{S}'_W(\mathbb{T}^d))$ (and it does not depend on λ). By existence and uniqueness of the generalized Ornstein-Uhlenbeck processes (also due to Theorem 3.3.7), the sequence $\{Q_N^\lambda\}_{N\geq 1}$ has at most one limit point, and from tightness, it does have a unique limit point. This concludes the proof of Theorem 3.3.3.

3.3.2 Generalized Ornstein-Uhlenbeck Processes

In this subsection we show that the generalized Ornstein-Uhlenbeck process obtained as the solution martingale problem which we are interested, is also a $\mathcal{S}'_W(\mathbb{T}^d)$-solution of a stochastic differential equation, and then we apply the theory in Appendix to conclude that there is at most one solution of the martingale problem. Moreover, we also conclude that this process is a Gaussian process.

Theorem 3.3.7. *Let Y_0 be a Gaussian field on $\mathcal{S}'_W(\mathbb{T}^d)$. Then the unique $\mathcal{S}'_W(\mathbb{T}^d)$-solution, Y_t, of the stochastic differential equation*

$$dY_t = \phi'(\rho)\nabla A\nabla_W Y_t dt + \sqrt{2\chi(\rho)\phi'(\rho)A}\,dN_t, \tag{3.3.14}$$

solves the martingale problem (3.3.3)-(3.3.4) with initial condition Y_0, where N_t is a mean-zero $\mathcal{S}'_W(\mathbb{T}^d)$-valued martingale with quadratic variation given by

$$\langle N(G)\rangle_t = t\sum_{j=1}^d \int_{\mathbb{T}^d} \big[\partial_{W_j} G\big]^2 d(x^j \otimes W_j).$$

Moreover, Y_t is a Gaussian process.

Proof. In view of the definition of solutions of stochastic differential equations (see Appendix), Y_t is a $\mathcal{S}'_W(\mathbb{T}^d)$-solution of (3.3.14). In fact, by hypothesis Y_t satisfies the integral identity (3.3.3), and is also an additive functional of a Markov process.

We now check the conditions in Proposition 3.6.1 to ensure uniqueness of $\mathcal{S}'_W(\mathbb{T}^d)$-solutions of (3.3.14). Since by hypothesis Y_0 is a Gaussian field, condition 1 is satisfied, and since the martingale M_t has quadratic variation given by (3.3.4), we use Remark 3.6.2 to conclude that condition 2 holds. Condition 3 follows from Lemmas 3.2.2 and 3.2.3. Therefore Y_t is unique.

Finally, by Blumenthal's 0-1 law for Markov processes, M_t and Y_0 are independent, since for measurable sets A and B,

$$\begin{aligned} &P(Y_0 \in A, M_t \in B) = E(\mathbf{1}_{Y_0\in A}\mathbf{1}_{M_t\in B}) = \\ &\quad E[E(\mathbf{1}_{Y_0\in A}\mathbf{1}_{M_t\in B}|\mathcal{F}_{0+})] = E[\mathbf{1}_{Y_0\in A}E(\mathbf{1}_{M_t\in B}|\mathcal{F}_{0+})] = \\ &\quad E[\mathbf{1}_{Y_0\in A}P(M_t \in B)] = P(Y_0 \in A)P(M_t \in B). \end{aligned}$$

Applying Lévy's martingale characterization of Brownian motions, the quadratic variation of M_t, given by (3.3.4), yields that M_t is a time deformation of a Brownian motion. Therefore, M_t is a Gaussian process with independent increments. Since Y_0 is a Gaussian field, we apply Proposition 3.6.3 to conclude that Y_t is a Gaussian process in $D([0,T], \mathcal{S}'_W(\mathbb{T}^d))$. □

3.4 Tightness

In this section we prove tightness of the density fluctuation field $\{Y^N_\cdot\}_N$ introduced in Section 1.1. We begin by stating Mitoma's criterion [29]:

Proposition 3.4.1. *Let Φ_∞ be a nuclear Fréchet space and Φ'_∞ its topological dual. Let $\{Q^N\}_N$ be a sequence of distributions in $D([0,T], \Phi'_\infty)$, and for a given function $G \in \Phi_\infty$, let $Q^{N,G}$ be the distribution in $D([0,T],\mathbb{R})$ defined by $Q^{N,G}\,[y \in D([0,T],\mathbb{R}); y(\cdot) \in A] = Q^N\,[Y \in D([0,T], \Phi'_\infty); Y(\cdot)(G) \in A]$. Therefore, the sequence $\{Q^N\}_N$ is tight if and only if $\{Q^{N,G}\}_N$ is tight for any $G \in \Phi_\infty$.*

From Mitoma's criterion, $\{Y^N_\cdot\}_N$ is tight if and only if $\{Y^N_\cdot(G)\}_N$ is tight for any $G \in \mathcal{S}_W(\mathbb{T}^d)$, since $\mathcal{S}_W(\mathbb{T}^d)$ is a nuclear Fréchet space. By Dynkin's formula and after some manipulations, we see that

$$\begin{aligned} Y^N_t(G) \;=\; & Y^N_0(G) \int_0^t \sum_{j=1}^d \Big\{ \frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}^d_N} \mathbb{L}^j_N G_N(x/N)\, \eta_s(x) \\ & + \frac{b}{N^{d/2}} \sum_{x\in\mathbb{T}^d_N} \big[\mathbb{L}^j_N G_N((x+e_j)/N) + \mathbb{L}^j_N G_N(x/N)\big]\, (\tau_x h_{1,j})(\eta_s) \\ & - \frac{b}{N^{d/2}} \sum_{x\in\mathbb{T}^d_N} \mathbb{L}^j_N G_N(x/N)(\tau_x h_{2,j})(\eta_s) \Big\} ds \;+ M^N_t(G), \end{aligned} \tag{3.4.1}$$

where $M^N_t(G)$ is a martingale of quadratic variation

$$\begin{aligned} \langle M^N(G)\rangle_t = \frac{1}{N^{d-1}} \sum_{j=1}^d \sum_{x\in\mathbb{T}^d} a^N_{jj} [\partial^N_{W_j} G_N(x/N)]^2 [W((x+e_j)/N) - W(x/N)] \times \\ \times \int_0^t c_{x,x+e_j}(\eta_s)\, [\eta_s(x+e_j) - \eta_s(x)]^2\, ds\;. \end{aligned}$$

In order to prove tightness for the sequence $\{Y^N_\cdot(G)\}_N$, it is enough to prove tightness for $\{Y^N_0(G)\}_N$, $\{M^N_\cdot(G)\}_N$ and the integral term in (3.4.1). The easiest one is the initial condition: from the usual central limit theorem, $Y^N_0(G)$ converges to a normal random variable of mean zero and variance $\chi(\rho)\int G(x)^2dx$, where $\chi(\rho) = \rho(1-\rho)$. For the other two terms, we use *Aldous' criterion*:

Proposition 3.4.2 (Aldous' criterion)**.** *A sequence of distributions $\{P^N\}$ in the path space $D([0,T],\mathbb{R})$ is tight if:*

i) For any $t \in [0,T]$ the sequence $\{P^N_t\}$ of distributions in $\mathbb{R}$ defined by $P^N_t(A) = P^N\,[y \in D([0,T],\mathbb{R}) : y(t) \in A]$ is tight, *ii) For any $\epsilon > 0$,*

$$\lim_{\delta>0} \overline{\lim_{n\to\infty}} \sup_{\substack{\tau\in\Upsilon_T \\ \theta\leq\delta}} P^N\big[y \in D([0,T],\mathbb{R}) : |y(\tau+\theta) - y(\tau)| > \epsilon\big] = 0,$$

where Υ_T is the set of stopping times bounded by T and $y(\tau+\theta) = y(T)$ if $\tau+\theta > T$.

Now we prove tightness of the martingale term. By the optional sampling theorem, we have

$$\begin{aligned}
Q_N\big[\big|M^N_{\tau+\theta}(G) - M^N_\tau(G)\big| > \epsilon\big] &\leq \frac{1}{\epsilon^2} E_{Q_N}\big[\langle M^N_{\tau+\theta}(G)\rangle - \langle M^N_\tau(G)\rangle\big] \\
&= \frac{1}{\epsilon^2}\big[\langle M^N_{\tau+\theta}(G)\rangle - \langle M^N_\tau(G)\rangle\big] \\
&= \frac{1}{\epsilon^2 N^{d-1}} \sum_{j=1}^d \sum_{x\in\mathbb{T}^d_N} a_{jj}(x)[\partial^N_{W_j} G(x/N)]^2 [W((x+e_j)/N) - W(x)] \\
&\times \int_t^{t+\delta} c_{x,x+e_j}(\eta_s)[\eta_s(x+e_j) - \eta_s(x)]^2 ds \\
&\leq \frac{\delta}{\epsilon^2}(1+2|b|)\theta \frac{1}{N^{d-1}} \sum_{j=1}^d \sum_{x\in\mathbb{T}^d_N} [\partial^N_{W_j} G(x/N)]^2 [W((x+e_j)/N) - W(x)] \qquad (3.4.2) \\
&\leq \frac{\delta}{\epsilon^2}(1+2|b|)\theta(\|\nabla_W G\|^2_W + \delta),
\end{aligned}$$

for N sufficiently large, since the rightmost term on (3.4.2) converges to $\|\nabla_W G\|^2_W$, as $N \to \infty$, where

$$\|\nabla_W G\|^2_W = \sum_{i=1}^d \int_{\mathbb{T}^d} \Big(\partial_{W_i} f\Big)^2 d(x^i \otimes W_i).$$

Therefore, the martingale $M^N_t(G)$ satisfies the conditions of Aldous' criterion. The integral term can be handled in a similar way:

$$\begin{aligned}
E_{Q_N}\Big[& \Big(\int_\tau^{\tau+\delta} \frac{1}{N^{d/2}} \sum_{j=1}^d \sum_x \Big\{\mathbb{L}^j_N G(x/N)(\eta_t - \rho) \\
&+ b[\mathbb{L}^j_N G((x+e_j)/N) + \mathbb{L}^j_N G(x/N)](\tau_x h_1 - \rho^2) \\
&- b\mathbb{L}^j_N G(x/N)(\tau_x h_2 - \rho^2)\Big)^2 dt\Big] \\
&\leq \delta C(b) \frac{1}{N^d} \sum_{j=1}^d \sum_{x\in\mathbb{T}^d_N} \Big(\mathbb{L}^j_N G(x/N)\Big)^2 \\
&\leq \delta C(G,b),
\end{aligned}$$

where $C(b)$ is a constant that depends on b, and $C(G,b)$ is a constant that depends on $C(b)$ and on the function $G \in \mathcal{S}_W(\mathbb{T}^d)$. Therefore, we conclude, by Mitoma's criterion, that the sequence $\{Y^N_\cdot\}_N$ is tight. Thus, the sequence of $\mathcal{S}'_W(\mathbb{T}^d)$-valued martingales $\{M^N_\cdot\}_N$ is also tight.

3.5 Boltzmann-Gibbs Principle

We show in this section that the martingales $M^N_t(G)$ introduced in Section 3.3 can be expressed in terms of the fluctuation field Y^N_t. This replacement of the cylinder function $(\tau_x h_{i,j})(\eta_s) - \rho^2$ by $2\rho[\eta_s(x) - \rho]$ for $i = 1, 2$, constitutes one of the main steps toward the proof of equilibrium fluctuations.

Recall that $(\Omega, \mathcal{F}, \mu)$ is a standard probability space where we consider the vector-valued $\mathcal{F}$-measurable functions $\{a_j(\omega); j = \dots, d\}$ that form our random environment (see Sections 1.1 and 3.3 for more details).

Take a function $f : \Omega \times \{0,1\}^{\mathbb{T}_N^d} \to \mathbb{R}$. Fix a realization $\omega \in \Omega$, let $x \in \mathbb{T}_N^d$, and define

$$f(x, \eta) = f(x, \eta, \omega) =: f(T_{Nx}\omega, \tau_x \eta),$$

where $\tau_x \eta$ is the shift of η to x: $\tau_x \eta(y) = \eta(x + y)$.

We say that f is local if there exists $R > 0$ such that $f(\omega, \eta)$ depends only on the values of $\eta(y)$ for $|y| \le R$. On this case, we can consider f as defined in all the space $\Omega \times \{0,1\}^{\mathbb{T}_N^d}$ for $N \ge R$.

We say that f is Lipschitz if there exists $c = c(\omega) > 0$ such that for all x, $|f(\omega, \eta) - f(\omega, \eta')| \le c|\eta(x) - \eta'(x)|$ for any $\eta, \eta' \in \{0,1\}^{\mathbb{T}_N^d}$ such that $\eta(y) = \eta'(y)$ for any $y \neq x$. If the constant c can be chosen independently of ω, we say that f is uniformly Lipschitz.

Theorem 3.5.1. *(Boltzmann-Gibbs principle)*

For every $G \in \mathcal{S}_W(\mathbb{T}^d)$, every $t > 0$ and every local, uniformly Lipschitz function $f : \Omega \times \{0,1\}^{\mathbb{T}_N^d} \to \mathbb{R}$, it holds

$$\lim_{N \to \infty} E_{\nu_\rho} \Big[\int_0^t \frac{1}{N^{d/2}} \sum_{x \in \mathbb{T}_N^d} G(x) V_f(x, \eta_s) ds \Big]^2 = 0, \tag{3.5.1}$$

where

$$V_f(x, \eta) = f(x, \eta) - E_{\nu_\rho}\big[f(x, \eta)\big] - \partial_\rho E\Big[\int f(x, \eta) d\nu_\rho(\eta)\Big]\big(\eta(x) - \rho\big).$$

Here, E denotes the expectation with respect to μ, the random environment.

Let $f : \Omega \times \{0,1\}^{\mathbb{T}_N^d} \to \mathbb{R}$ be a local, uniformly Lipschitz function and take $f(x, \eta) = f(\theta_{Nx}\omega, \tau_x \eta)$. Fix a function $G \in \mathcal{S}_W(\mathbb{T}^d)$ and an integer K that shall increase to ∞ after N. For each N, we subdivide $\mathbb{T}_N^d$ into non-overlapping boxes of linear size K. Denote them by $\{B_i, 1 \le i \le M^d\}$, where $M = [\frac{N}{K}]$. More precisely,

$$B_i = y_i + \{1, \dots, K\}^d,$$

where $y_i \in \mathbb{T}_N^d$, and $B_i \cap B_r = \emptyset$ if $i \neq r$. We assume that the points y_i have the same relative position on the boxes.

Let B_0 be the set of points that are not included in any B_i, then $|B_0| \le dKN^{d-1}$. If we restrict the sum in the expression that appears inside the integral in (3.5.1) to the set B_0, then its $L^2_{\nu_\rho}(\mathbb{T}^d)$-norm clearly vanishes as $N \to +\infty$, since f is local, ν_ρ is an invariant product measure, and V_f has mean zero with respect to ν_ρ.

Let Λ_{s_f} be the smallest cube centered at the origin that contains the support of f and define s_f as the radius of Λ_{s_f}. Denote by B_i^0 the interior of the box B_i, namely the sites x in B_i that are at a distance at least $s_f + 2$ from the boundary:

$$B_i^0 = \{x \in B_i, d(x, \mathbb{T}_N^d \setminus B_i) > s_f + 2\}.$$

Denote also by B^c the set of points that are not included in any B_i^0. By construction, it is easy to see that $|B^c| \le dN^d(\frac{c(f)}{K} + \frac{K}{N})$, where $c(f)$ is a constant that depends on f.

We have that for continuous $H : \mathbb{T}^d \to \mathbb{R}$,

$$\frac{1}{N^{d/2}} \sum_{x \in \mathbb{T}_N^d} H(x) V_f(x, \eta_t) = \frac{1}{N^{d/2}} \sum_{x \in B^c} H(x) V_f(x, \eta_t) +$$
$$+ \frac{1}{N^{d/2}} \sum_{i=1}^{M^d} \sum_{x \in B_i^0} \Big[H(x) - H(y_i)\Big] V_f(x, \eta_t) + \frac{1}{N^{d/2}} \sum_{i=1}^{M^d} H(y_i) \sum_{x \in B_i^0} V_f(x, \eta_t).$$

Note that we may take H continuous, since the continuous functions are dense in $L^2(\mathbb{T}^d)$. The first step is to prove that

$$\lim_{K \to \infty} \lim_{N \to \infty} E_{\nu_\rho} \Big[\int_o^t \frac{1}{N^{d/2}} \sum_{x \in B^c} H(x) V_f(x, \eta_t) ds \Big]^2 = 0.$$

As ν_ρ is an invariant product measure and V_f has mean zero with respect to the measure ν_ρ, the last expectation is bounded above by

$$\frac{t^2}{N^d}\sum_{\substack{x,y\in B^c\\|x-y|\le 2s_f}} H(x)H(y)E_{\nu_\rho}\big[V_f(x,\eta)V_f(y,\eta)\big].$$

Since V_f belongs to $L^2_{\nu_\rho}(\mathbb{T}^d)$ and $|B^c|\le dN^d(\frac{c(f)}{K}+\frac{K}{N})$, the last expression vanishes by taking first $N\to+\infty$ and then $K\to+\infty$.

From the continuity of H, and applying similar arguments, one may show that

$$\lim_{N\to\infty}\mathbb{E}_{\nu_\rho}\Big[\int_0^t\frac{1}{N^{d/2}}\sum_{i=1}^{M^d}\sum_{x\in B_i^0}\big[H(x)-H(y_i)\big]V_f(x,\eta_t)ds\Big]^2=0.$$

In order to conclude the proof it remains to be shown that

$$\lim_{K\to\infty}\lim_{N\to\infty}E_{\nu_\rho}\Big[\int_0^t\frac{1}{N^{d/2}}\sum_{i=1}^{M^d}H(y_i)\sum_{x\in B_i^0}V_f(x,\eta_t)ds\Big]^2=0. \tag{3.5.2}$$

To this end, recall proposition A 1.6.1 of [23]:

$$E_{\nu_\rho}\left[\int_0^t V(\eta_s)ds\right]\le 20\theta t\|V\|_{-1}^2, \tag{3.5.3}$$

where $\|\cdot\|_{-1}$ is given by

$$\|V\|_{-1}^2=\sup_{F\in L^2(\nu_\rho)}\left\{2\int V(\eta)F(\eta)d\nu_\rho-\langle F,L_NF\rangle_\rho\right\},$$

and $\langle\cdot,\cdot\rangle_\rho$ denotes the inner product in $L^2(\nu_\rho)$.

Let $\tilde L_N$ be the generator of the exclusion process without the random environment, and without the conductances (that is, taking $a(\omega)\equiv 1$, and $W_j(x_j)=x_j$, for $j=1,\dots,d$, in (1.1.2)), and also without the diffusive scaling N^2:

$$\tilde L_N g(\eta)=\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}c_{x,x+e_j}(\eta)\big[g(\eta^{x,x+e_j})-g(\eta)\big],$$

for cylindric functions g on the configuration space $\{0,1\}^{\mathbb{T}_N^d}$.

For each $i=1,..,M^d$ denote by ζ_i the configuration $\{\eta(x),x\in B_i\}$ and by $\tilde L_{B_i}$ the restriction of the generator $\tilde L_N$ to the box B_i, namely:

$$\tilde L_{B_i}h(\eta)=\sum_{\substack{x,y\in B_i\\|x-y|=1/N}}c_{x,y}(\eta)\big[h(\eta^{x,y})-h(\eta)\big].$$

We would like to emphasize that we introduced the generator $\tilde L_N$ because it is translation invariant.

Now we introduce some notation. Let $L^2(P\otimes\nu_\rho)$ the set of measurable functions g such that $E[\int g(\omega,\eta)^2d\nu_\rho]<\infty$. Fix a local function $h:\Omega\times\{0,1\}^{\mathbb{T}_N^d}\to\mathbb{R}$ in $L^2(P\otimes\nu_\rho)$, measurable with respect to $\sigma(\eta(x),x\in B_1)$, and let h_i be the translation of h by y_i-y_1: $h_i(x,\eta)=h(\theta_{(y_i-y_1)N}\omega,\tau_{y_i-y_1}\eta)$. Consider

$$V_{H,h}^N(\eta)=\frac{1}{N^{d/2}}\sum_{i=1}^{M^d}H(y_i)\tilde L_{B_i}h_i(\zeta_i).$$

The strategy of the proof (3.5.2) is the following: we show that $V_{H,h}^N$ vanishes in some sense as $N\to\infty$, and then, that the difference between V_f and $V_{H,h}^N$ also vanishes, as $N\to\infty$. The result follows a simple triangle inequality. The first part is done by obtaining estimates on boxes, whereas the second part mainly considers the projections of V_f on some appropriate Hilbert spaces, plus ergodicity of the environment.

Let

$$L_{W,B_i}h(\eta) = \sum_{j=1}^{d} \sum_{x \in B_i} c_{x,x+e_j}(\eta) \frac{N a_j(x)}{W(x+e_j) - W(x)} [h(\eta^{x,x+e_j}) - h(\eta)].$$

Note that the following estimate holds

$$\sum_{i=1}^{M^d} \langle h, -L_{W,B_i}h \rangle_\rho \leq \langle h, -L_N h \rangle_\rho.$$

Furthermore,

$$\langle f, -\tilde{L}_{B_i}h \rangle \leq \max_{1 \leq k \leq d} \frac{\{W_k(1) - W_k(0)\}}{N} \theta \langle h, -L_{W,B_i}h \rangle_\rho.$$

Using the Cauchy-Schwartz inequality, we have, for each i,

$$\langle \tilde{L}_{B_i}h_i, F \rangle_\rho \leq \frac{1}{2\gamma_i} \langle -\tilde{L}_{B_i}h_i, h_i \rangle_\rho + \frac{\gamma_i}{2} \langle F, -\tilde{L}_{B_i}F \rangle_\rho,$$

where γ_i is a positive constant.

Therefore,

$$2 \int V_{H,h}^N(\eta) F(\eta) d\nu_\rho \leq \frac{2}{N^{d/2}} \sum_{i=1}^{M^d} H(y_i) \left[\frac{1}{2\gamma_i} \langle -\tilde{L}_{B_i}h_i, h_i \rangle_\rho + \frac{\gamma_i}{2} \langle F, -\tilde{L}_{B_i}F \rangle_\rho \right]. \tag{3.5.4}$$

Choose

$$\gamma_i = \frac{N^{1+d/2}}{\theta \max_{1 \leq k \leq d}\{W_k(1) - W_k(0)\} |H(y_i)|},$$

and observe that the generator L_N is already speeded up by the factor N^2. We, thus, obtain

$$\frac{2}{N^{d/2}} \sum_{i=1}^{M^d} H(y_i) \frac{\gamma_i}{2} \langle F, -\tilde{L}_{B_i}F \rangle_\rho \leq \langle F, -\tilde{L}_N F \rangle_\rho.$$

The above bound and (3.5.4) allow us to use inequality (3.5.2) on $V_{H,h}^N$, with the generator L_{W,B_i}. Therefore, we have that the expectation in (3.5.3) with $V_{H,h}^N$ is bounded above by

$$\frac{20\theta t}{N^{d/2}} \sum_{i=1}^{M^d} \frac{|H(y_i)|}{\gamma_i} \langle -\tilde{L}_{B_i}h_i, h_i \rangle_\rho,$$

which in turn is less than or equal to

$$\frac{20t \|H\|_\infty M^d \theta^2}{N^{d+1} \max_{1 \leq k \leq d}\{W_k(1) - W_k(0)\}} \sum_{i=1}^{M^d} \frac{1}{M^d} \langle -\tilde{L}_{B_i}h_i, h_i \rangle_\rho.$$

By Birkhoff's ergodic theorem, the sum in the previous expression converges to a finite value as $N \to \infty$. Therefore, this whole expression vanishes as $N \to \infty$. This concludes the first part of the strategy of the proof.

To conclude the proof of the theorem it is enough to show that

$$\lim_{K \to \infty} \inf_{h \in L^2(\nu_\rho \otimes P)} \lim_{N \to \infty} E_{\nu_\rho} \Big[\int_0^t \frac{1}{N^{d/2}} \sum_{i=1}^{M^d} H(y_i) \Big\{ \sum_{x \in B_i^0} V_f(x, \eta_s) - \tilde{L}_{B_i}h_i(\zeta_i(s)) \Big\} \Big]^2 = 0.$$

To this end, observe that the expectation in the previous expression is bounded by

$$\frac{t^2}{N^d}\sum_{i=1}^{M^d}||H||_\infty^2 E_{\nu_\rho}\Big(\sum_{x\in B_i^0} V_f(x,\eta)-\tilde{L}_{B_i}h_i(\zeta_i)\Big)^2,$$

because the measure ν_ρ is invariant under the dynamics and the supports of $V_f(x,\eta)-\tilde{L}_{B_i}h_i(\zeta_i)$ and $V_f(y,\eta)-\tilde{L}_{B_r}h_r(\zeta_r)$ are disjoint for $x\in B_i^0$ and $y\in B_r^0$, with $i\neq r$.

By the ergodic theorem, as $N\to\infty$, this expression converges to

$$\frac{t^2}{K^d}||H||_\infty^2 E\Big[\int\Big(\sum_{x\in B_1^0} V_f(x,\eta)-\tilde{L}_{B_1}h(\omega,\eta)\Big)^2 d\nu_\rho\Big]. \tag{3.5.5}$$

So, it remains to be shown that

$$\lim_{K\to\infty}\frac{t^2}{K^d}||H||_\infty^2 \inf_{h\in L^2(\nu_\rho\otimes P)} E\Big[\int\Big(\sum_{x\in B_1^0} V_f(x,\eta)-\tilde{L}_{B_1}h(\omega,\eta)\Big)^2 d\nu_\rho\Big]=0.$$

Denote by $R(\tilde{L}_{B_1})$ the range of the generator $\tilde{L}_{B_1}$ in $L^2(\nu_\rho\otimes P)$ and by $R(\tilde{L}_{B_1})^\perp$ the space orthogonal to $R(\tilde{L}_{B_1})$. The infimum of (3.5.5) over all $h\in L^2(\nu_\rho\otimes P)$ is equal to the projection of $\sum_{x\in B_1^0}V_f(x,\eta)$ into $R(\tilde{L}_{B_1})^\perp$.

The set $R(\tilde{L}_{B_1})^\perp$ is the space of functions that depend on η only through the total number of particles on the box B_1. So, the previous expression is equal to

$$\lim_{K\to\infty}\frac{t^2||H||_\infty^2}{K^d}E\Big[\int\Big(E_{\nu_\rho}\Big[\sum_{x\in B_1^0}V_f(x,\eta)\Big|\eta^{B_1}\Big]\Big)^2 d\nu_\rho\Big], \tag{3.5.6}$$

where $\eta^{B_1}=K^{-d}\sum_{x\in B_1}\eta(x)$.

Let us call this last expression $\mathcal{I}_0$. Define $\psi(x,\rho)=E_{\nu_\rho}[f(\theta_x\omega)]$. Notice that $V_f(x,\eta)=f(x,\eta)-\psi(x,\rho)-E[\partial_\rho\psi(x,\rho)]\big(\eta(x)-\rho\big)$, since in the last term the partial derivative with respect to ρ commutes with the expectation with respect to the random environment. In order to estimate the expression (3.5.6), we use the elementary inequality $(x+y)^2\le 2x^2+2y^2$. Therefore, we obtain $\mathcal{I}_0\le 4(\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3)$, where

$$\mathcal{I}_1=\frac{1}{K^d}E\Big[\int\Big(\sum_{x\in B_1^0}E_{\nu_\rho}\big[f(x,\eta)|\eta^{B_1}\big]-\psi(x,\eta^{B_1})\Big)^2 d\nu_\rho\Big],$$

$$\mathcal{I}_2=\frac{1}{K^d}E\Big[\int\Big(\sum_{x\in B_1^0}\psi(x,\eta^{B_1})-\psi(x,\rho)-\partial_\rho\psi(x,\rho)[\eta^{B_1}-\rho]\Big)^2 d\nu_\rho\Big],$$

and

$$\mathcal{I}_3=\frac{1}{K^d}E\Big[E_{\nu_\rho}\Big[\Big(\sum_{x\in B_1^0}\big(\partial_\rho\psi(x,\rho)-E[\partial_\rho\psi(x,\rho)]\big)\,[\eta^{B_1}-\rho]\Big)^2\Big]\Big].$$

Recall the equivalence of ensembles (see Lemma A.2.2.2 in [23]):

Lemma 3.5.2. *Let $h:\{0,1\}^{\mathbb{T}_N^d}\to\mathbb{R}$ be a local uniformly Lipschitz function and $S\in\{1,\dots,N\}$. Then, there exists a constant C that depends on h only through its support and its Lipschitz constant, such that*

$$\Big|E_{\nu_\rho}[h(\eta)|\eta^S]-E_{\nu_{\eta^S}}[h(\eta)]\Big|\le\frac{C}{S^d},$$

and

$$\eta^S(x)=\frac{1}{S^d}\sum_{y\in\Lambda_S}\eta(y),$$

with $\Lambda_S=\{0,\dots,S-1\}^d$.

Applying Lemma 3.5.2, we get

$$\frac{1}{K^d}E\Big[\int\Big(\sum_{x\in B_1^0}E_{\nu_\rho}\big[f(x,\eta)|\eta^{B_1}\big]-\psi(x,\eta^{B_1})\Big)^2d\nu_\rho\Big]\le\frac{C}{K^d},$$

which vanishes as $K\to\infty$.

Using a Taylor expansion for $\psi(x,\rho)$, we obtain that

$$\frac{1}{K^d}E\Big[\int\Big(\sum_{x\in B_1^0}\psi(x,\eta^{B_1})-\psi(x,\rho)-\partial_\rho\psi(x,\rho)[\eta^{B_1}-\rho]\Big)^2d\nu_\rho\Big]\le\frac{C}{K^d},$$

and also goes to 0 as $K\to\infty$.

Finally, we see that

$$\mathcal{I}_3=E_{\nu_\rho}\big[(\eta(0)-\rho)^2\big]\cdot E\Big[\Big(\frac{1}{K^d}\sum_{x\in B_1^0}(\partial_\rho\psi(x,\rho)-E[\partial_\rho\psi(x,\rho)]\Big)^2\Big],$$

and it goes to 0 as $K\to\infty$ by the L^2-ergodic theorem. This concludes the proof of Theorem 3.5.1.

3.6 Appendix: Stochastic differential equations on nuclear spaces

3.6.1 Countably Hilbert nuclear spaces

In this subsection we introduce countably Hilbert nuclear spaces which will be the natural environment for the study of the stochastic evolution equations obtained from the martingale problem. We will begin by recalling some basic definitions on these spaces. To this end, we follow the ideas of Kallianpur and Perez-Abreu [22] and Gel'fand and Vilenkin [19].

Let Φ be a (real) linear space, and let $\|\cdot\|_r,\ \ r\in\mathbb{N}$ be an increasing sequence of Hilbertian norms. Define Φ_r as the completion of Φ with respect to $\|\cdot\|_r$. Since for $n\le m$

$$\|f\|_n\le\|f\|_m,\qquad\text{for all } f\in\Phi,\tag{3.6.1}$$

we have,

$$\Phi_m\subset\Phi_n,\ \ \text{for all}\ \ m\ge n.$$

Let

$$\Phi_\infty=\bigcap_{r=1}^{\infty}\Phi_r.$$

Then Φ_∞ is a Fréchet space with respect to the metric

$$\rho(f,g)=\sum_{r=1}^{\infty}2^{-r}\frac{\|f-g\|_r}{1+\|f-g\|_r},\tag{3.6.2}$$

and (Φ_∞,ρ) is called a countably Hilbert space.

A countably Hilbert space Φ_∞ is called *nuclear* if for each $n\ge0$, there exists $m>n$ such that the canonical injection $\pi_{m,n}:\Phi_m\to\Phi_n$ is Hilbert-Schmidt, i.e., if $\{f_j\}_{j\ge1}$ is a complete orthonormal system in Φ_m, we have

$$\sum_{j=1}^{\infty}\|f_j\|_n^2<\infty.\tag{3.6.3}$$

We now characterize the topologic dual Φ'_∞ of the countably Hilbert nuclear space Φ_∞ in terms of the topologic dual of the auxiliary spaces Φ_n.

Let Φ'_n be the dual (Hilbert) space of Φ_n, and for $\phi\in\Phi'_n$ let

$$\|\phi\|_{-n}=\sup_{\|f\|_n\le1}|\phi[f]|,$$

where $\phi[f]$ means the value of ϕ at f. Equation (3.6.1) implies that

$$\Phi'_n \subset \Phi'_m \text{ for all } m \geq n.$$

Let Φ'_∞ be the topologic dual of Φ_∞ with respect to the strong topology, which is given by the complete system of neighborhoods of zero given by sets of the form, $\{\phi \in \Phi'_\infty : \|\phi\|_B < \epsilon\}$, where $\|\phi\|_B = \sup\{|\phi[f]| : f \in B\}$ and B is a bounded set in Φ_∞. So,

$$\Phi'_\infty = \bigcup_{r=1}^{\infty} \Phi'_r.$$

3.6.2 Stochastic differential equations

The aim of this subsection is to recall some results about existence and uniqueness of stochastic evolution equations in nuclear spaces.

We denote by $\mathcal{L}(\Phi_\infty, \Phi_\infty)$ (resp. $\mathcal{L}(\Phi'_\infty, \Phi'_\infty)$) the class of continuous linear operators from Φ_∞ to Φ_∞ (resp.Φ'_∞ to Φ'_∞).

A family $\{S(t) : t \geq 0\}$ of the linear operators on Φ_∞ is said to be a $C_{0,1}$-semigroup if the following three conditions are satisfied:

- $S(t_1)S(t_2) = S(t_1 + t_2)$ for all $t_1, t_2 \geq 0$, $S(0) = I$;
- The map $t \to S(t)f$ is Φ_∞-continuous for each $f \in \Phi_\infty$;
- For each $q \geq 0$ there exist numbers $M_q > 0, \sigma_q > 0$ and $p \geq q$ such that

$$\|S(t)f\|_q \leq M_q\, e^{\sigma_q t}\|f\|_p \quad \text{for all} \quad f \in \Phi_\infty, \;\; t > 0.$$

Let A in $\mathcal{L}(\Phi_\infty, \Phi_\infty)$ be infinitesimal generator of the semigroup $\{S(t) : t \geq 0\}$ in $\mathcal{L}(\Phi_\infty, \Phi_\infty)$. The relations

$$\begin{aligned} \phi[S(t)f] &:= (S'(t)\phi)[f] \text{ for all } t \geq 0,\ f \in \Phi_\infty \text{ and } \phi \in \Phi'_\infty; \\ \phi[Af] &:= (A'\phi)[f] \text{ for all } f \in \Phi_\infty \text{ and } \phi \in \Phi'_\infty; \end{aligned}$$

define the infinitesimal generator A' in $\mathcal{L}(\Phi'_\infty, \Phi'_\infty)$ of the semigroup $\{S'(t) : t \geq 0\}$ in $\mathcal{L}(\Phi'_\infty, \Phi'_\infty)$.

Let $(\Sigma, \mathcal{U}, P)$ be a complete probability space with a right continuous filtration $(\mathcal{U}_t)_{t\geq 0}, \mathcal{U}_0$ containing all the P-null sets of $\mathcal{U}$, and $M = (M_t)_{t\geq 0}$ be a Φ'_∞-valued martingale with respect to $\mathcal{U}_t$, i.e., for each $f \in \Phi_\infty$, $M_t[f]$ is a real-valued martingale with respect to $\mathcal{U}_t$, $t \geq 0$. We are interested in results of existence and uniqueness of the following Φ'_∞-valued stochastic evolution equation:

$$\begin{aligned} d\xi_t &= A'\xi_t dt + dM_t, \quad t > 0, \\ \xi_0 &= \gamma, \end{aligned} \tag{3.6.4}$$

where γ is a Φ'_∞-valued random variable, and A is the infinitesimal generator of a $C_{0,1}$-semigroup on Φ_∞.

We say that $\xi = (\xi_t)_{t\geq 0}$ is a Φ'_∞-solution of the stochastic evolution equation (3.6.4) if the following conditions are satisfied:

- ξ_t is Φ'_∞-valued, progressively measurable, and $\mathcal{U}_t$-adapted;
- the following integral identity holds:

$$\xi_t[f] = \gamma[f] + \int_0^t \xi_s[Af]ds + M_t[f],$$

  for all $f \in \Phi_\infty$, $t \geq 0$ a.s..

It is proved in [22, Corollary 2.2] the following result on existence and uniqueness of solutions of the stochastic differential equation (3.6.4):

Proposition 3.6.1. *Assume the conditions below:*

1. γ *is a* Φ'_∞*-valued* $\mathcal{U}_0$*-measurable random element such that, for some* $r_0 > 0$, $E|\gamma|^2_{-r_0} < \infty$;
2. $M = (M_t)_{t\geq 0}$ *is a* Φ'_∞*-valued martingale such that* $M_0 = 0$ *and, for each* $t \geq 0$ *and* $f \in \Phi$, $E(M_t[f])^2 < \infty$;
3. A *is a continuous linear operator on* Φ_∞, *and is the infinitesimal generator of a* $C_{0,1}$*-semigroup* $\{S(t) : t \geq 0\}$ *on* Φ_∞.

Then, the Φ'_∞*-valued homogeneous stochastic evolution equation* (3.6.4) *has a unique solution* $\xi = (\xi_t)_{t\geq 0}$ *given explicitly by the "evolution solution":*

$$\xi_t = S'(t)\gamma + \int_0^t S'(t-s)dM_s.$$

Remark 3.6.2. *The statement* $E(M_t[f])^2 < \infty$ *in condition 2 of Proposition 3.6.1 is satisfied if* $E(M_t[f])^2 = tQ(f,f)$, *where* $f \in \Phi_\infty$, *and* $Q(\cdot,\cdot)$ *is a positive definite continuous bilinear form on* $\Phi_\infty \times \Phi_\infty$.

We now state a proposition, whose proof can be found in Corollary 2.1 of [22], that gives a sufficient condition for the solution ξ_t of the equation (3.6.4) be a Gaussian process.

Proposition 3.6.3. *Assume* γ *is a* Φ'_∞*-valued Gaussian element independent of the* Φ'_∞*-valued Gaussian martingale with independent increments* M_t. *Then, the solution* $\xi = (\xi_t)$ *of* (3.6.4) *is a* Φ'_∞*-valued Gaussian process.*

Chapter 4

Dynamical large deviations for a boundary driven stochastic lattice gas model with many conserved quantities

In the last years there has been considerable progress in understanding stationary non equilibrium states: diffusive systems in contact with different reservoirs at the boundary imposing a gradient on the conserved quantities of the system. In these systems there is a flow of matter through the system and the dynamics is not reversible. The main difference with respect to equilibrium (reversible) states is the following: in equilibrium, the invariant measure, which determines the thermodynamic properties, is given for free by the Gibbs distribution specified by the Hamiltonian; on the other hand, in non equilibrium states the construction of the stationary state requires the solution of a dynamical problem. One of the most striking typical property of these systems is the presence of long-range correlations. For the symmetric simple exclusion this was already shown in a pioneering paper by Spohn [37]. We refer to [5, 7] for two recent reviews on this topic.

We discuss this issue in the context of stochastic lattice gases in a box of linear size N with birth and death processes at the boundary modeling the reservoirs. We consider the case when there are many thermodynamic variables: the local density denoted by ρ, and the local momentum denoted by p_k, $k = 1, \ldots, d$, d being the dimension of the box.

Let the set of possible velocities, $\mathcal{V}$, be a finite subset of $\mathbb{R}^d$, and for a point $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, let $\tilde{x} = (x_2, \ldots, x_d)$. The model which we will study can be informally described as follows: fix a velocity $v \in \mathcal{V}$, an integer $N \geq 1$, and boundary densities $0 < \alpha_v(\cdot) < 1$ and $0 < \beta_v(\cdot) < 1$; at any given time, each site of the set $\{1, \ldots, N-1\} \times \{0, \ldots, N-1\}^{d-1}$ is either empty or occupied by one particle at velocity v. In the bulk, each particle attempts to jump at any of its neighbors at the same velocity, with a weakly asymmetric rate. To respect the exclusion rule, the particle jumps only if the target site at the same velocity v is empty; otherwise nothing happens. At the boundary, sites with first coordinates given by 1 or $N-1$ have particles being created or removed in such a way that the local densities are $\alpha_v(\tilde{x})$ and $\beta_v(\tilde{x})$: at rate $\alpha_v(\tilde{x}/N)$ a particle is created at $\{1\} \times \{\tilde{x}\}$ if the site is empty, and at rate $1 - \alpha_v(\tilde{x})$ the particle at $\{1\} \times \{\tilde{x}\}$ is removed if the site is occupied, and at rate $\beta_v(\tilde{x})$ a particle is created at $\{N-1\} \times \{\tilde{x}\}$ if the site is empty, and at rate $1 - \beta_v(\tilde{x})$ the particle at $\{N-1\} \times \{\tilde{x}\}$ is removed if the site is occupied. Superposed to this dynamics, there is a collision process which exchange velocities of particles in the same site in a way that momentum is conserved. Similar models have been studied by [1, 9, 32]. In fact, the model we consider here is based on the model of Esposito et al. [9] which was used to derive the Navier-Stokes equation. It is also noteworthy that the derivation of hydrodynamic limits and macroscopic fluctuation theory for a system with two conserved quantities have been studied in [4].

The hydrodynamic limit for the above model has been proved in [33]. The hydrodynamic equation is derived from the underlying stochastic dynamics through an appropriate scaling limit in which the microscopic time and space coordinates are rescaled diffusively. The hydrodynamic equation thus represents the law of large numbers for the empirical density of the stochastic lattice gas. The convergence has to be understood in probability with respect to the law of the stochastic lattice gas. Once it is established a natural question is to consider large deviations from the hydrodynamic limit.

In this Chapter thus provides a derivation of the dynamical large deviations for this model. As usual, the main difficulty appears in the proof of the lower bound where one needs to show that any trajectory λ_t, $0 \le t \le T$, with finite rate function, $I_T(\lambda) < \infty$, can be approximated by a sequence of regular trajectories $\{\lambda^n : n \ge 1\}$ such that

$$\lambda^n \longrightarrow \lambda \quad \text{and} \quad I_T(\lambda^n) \longrightarrow I_T(\lambda) \ . \tag{4.0.1}$$

To avoid this difficulty, we follow the method introduced in [15]. It is well known that if $I_T(\lambda) < \infty$, then there exists an external field H associated to λ, in the sense that λ solves a hydrodynamic equation perturbed by the external field H. The strategy of [15] is to approximate the external field H by a sequence of smooth functions, H_n, and then to show that the corresponding weak solutions of the hydrodynamical equations perturbed by H_n converge to λ in the sense (4.0.1).

The main difference of our proof with respect to theirs, is that their proof of the convergence (4.0.1) relied on some energy estimates that we were not able to achieve due to the presence of velocities. Therefore, we had to overcome this problem by taking an alternative approach at that part. More specific details are given in Section 4.4.

The Chapter is organized as follows: in Section 4.1 we establish the notation and state the main results of the article; in Section 4.2, we review the hydrodynamics for this model, that was obtained in [33]; in Section 4.3, several properties of the rate function are derived; Section 4.4 proves the $I_T(\cdot|\gamma)$-density, which is a key result for proving the lower bound; finally, in Section 4.5 the proofs of the upper and lower bounds of the dynamical large deviations are given.

4.1 Notation and Results

Fix a positive integer $d \ge 1$, and denote by D^d the open set $(0,1) \times \mathbb{T}^{d-1}$, where $\mathbb{T}^k$ is the k-dimensional torus $(\mathbb{R}/\mathbb{Z})^k = [0,1)^k$, and by Γ the boundary of D^d: $\Gamma = \{(u_1, \dots, u_d) \in [0,1] \times \mathbb{T}^{d-1}; u_1 = 0 \text{ or } 1\}$.

For an open subset Λ of $\mathbb{R} \times \mathbb{T}^{d-1}$, $\mathcal{C}^m(\Lambda)$, $1 \le m \le +\infty$, stands for the space of m-continuously differentiable real functions defined on Λ. Let $\mathcal{C}_0^m(\Lambda)$ (resp. $\mathcal{C}_c^m(\Lambda)$), $1 \le m \le +\infty$, be the subset of functions in $\mathcal{C}^m(\Lambda)$ which vanish at the boundary of Λ (resp. with compact support in Λ).

For each integer $N \ge 1$, denote by $\mathbb{T}_N^{d-1} = (\mathbb{Z}/N\mathbb{Z})^{d-1} = \{0, \dots, N-1\}^{d-1}$, the discrete $(d-1)$-dimensional torus of length N. Let $D_N^d = \{1, \dots, N-1\} \times \mathbb{T}_N^{d-1}$ be the cylinder in $\mathbb{Z}^d$ of length $N-1$ and basis $\mathbb{T}_N^{d-1}$ and let $\Gamma_N = \{(x_1, \dots, x_d) \in \mathbb{Z} \times \mathbb{T}_N^{d-1} ; x_1 = 1 \text{ or } (N-1)\}$ be the boundary of D_N^d.

Let $\mathcal{V} \subset \mathbb{R}^d$ be a finite set of velocities $v = (v_1, \dots, v_d)$. Assume that $\mathcal{V}$ is invariant under reflexions and permutations of the coordinates:

$$(v_1, \dots, v_{i-1}, -v_i, v_{i+1}, \dots, v_d) \text{ and } (v_{\sigma(1)}, \dots, v_{\sigma(d)}) \tag{4.1.1}$$

belong to $\mathcal{V}$ for all $1 \le i \le d$, and all permutations σ of $\{1, \dots, d\}$, provided $(v_1, \dots, v_d)$ belongs to $\mathcal{V}$.

On each site of D_N^d, at most one particle for each velocity is allowed. We denote: the number of particles with velocity v at x, $v \in \mathcal{V}$, $x \in D_N^d$, by $\eta(x,v) \in \{0,1\}$; the number of particles in each velocity v at a site x by $\eta_x = \{\eta(x,v); v \in \mathcal{V}\}$; and a configuration by $\eta = \{\eta_x; x \in D_N^d\}$. The set of particle configurations is $X_N = \left(\{0,1\}^{\mathcal{V}}\right)^{D_N^d}$.

On the interior of the domain, the dynamics consists of two parts: (i) each particle of the system evolves according to a nearest neighbor weakly asymmetric random walk with exclusion among particles of the same velocity, and (ii) binary collision between particles of different velocities. Let $p(x,v)$ be an irreducible probability transition function of finite range, and mean velocity v:

$$\sum_x x p(x,v) = v.$$

The jump law and the waiting times are chosen so that the jump rate from site x to site $x+y$ for a particle with velocity v is

$$P_N(y,v) = \frac{1}{2} \sum_{j=1}^{d} (\delta_{y,e_j} + \delta_{y,-e_j}) + \frac{1}{N} p(y,v),$$

where $\delta_{x,y}$ stands for the Kronecker delta, which equals one if $x = y$ and 0 otherwise, and $\{e_1, \dots, e_d\}$ is the canonical basis in $\mathbb{R}^d$.

4.1.1 The boundary driven exclusion process

Our main interest is to examine the stochastic lattice gas model given by the generator $\mathcal{L}_N$ which is the superposition of the boundary dynamics with the collision and exclusion:

$$\mathcal{L}_N = N^2\{\mathcal{L}_N^b + \mathcal{L}_N^c + \mathcal{L}_N^{ex}\}, \tag{4.1.2}$$

where $\mathcal{L}_N^b$ stands for the generator which models the part of the dynamics at which a particle at the boundary can enter or leave the system, $\mathcal{L}_N^c$ stands for the generator which models the collision part of the dynamics and lastly, $\mathcal{L}_N^{ex}$ models the exclusion part of the dynamics. Let f be a function on X_N. The generator of the exclusion part of the dynamics, $\mathcal{L}_N^{ex}$, is given by

$$(\mathcal{L}_N^{ex} f)(\eta) = \sum_{v\in\mathcal{V}} \sum_{x,z\in D_N^d} \eta(x,v)[1-\eta(z,v)]P_N(z-x,v)\left[f(\eta^{x,z,v}) - f(\eta)\right],$$

where

$$\eta^{x,y,v}(z,w) = \begin{cases} \eta(y,v) & \text{if } w = v \text{ and } z = x, \\ \eta(x,v) & \text{if } w = v \text{ and } z = y, \\ \eta(z,w) & \text{otherwise.} \end{cases}$$

The generator of the collision part of the dynamics, $\mathcal{L}_N^c$, is given by

$$(\mathcal{L}_N^c f)(\eta) = \sum_{y\in D_N^d} \sum_{q\in\mathcal{Q}} p(y,q,\eta)\left[f(\eta^{y,q}) - f(\eta)\right],$$

where $\mathcal{Q}$ is the set of all collisions which preserve momentum:

$$\mathcal{Q} = \{q = (v,w,v',w') \in \mathcal{V}^4; v + w = v' + w'\},$$

the rate $p(y,q,\eta)$ is given by

$$p(y,q,\eta) = \eta(y,v)\eta(y,w)[1-\eta(y,v')][1-\eta(y,w')],$$

and for $q = (v_0, v_1, v_2, v_3)$, the configuration $\eta^{y,q}$ after the collision is defined as

$$\eta^{y,q}(z,u) = \begin{cases} \eta(y, v_{j+2}) & \text{if } z = y \text{ and } u = v_j \text{ for some } 0 \le j \le 3, \\ \eta(z,u) & \text{otherwise,} \end{cases}$$

where the index of v_{j+2} should be taken modulo 4. Particles of velocities v and w at the same site collide at rate one and produce two particles of velocities v' and w' at that site.

Finally, the generator of the boundary part of the dynamics is given by

$$\begin{aligned}(\mathcal{L}_N^b f)(\eta) \;=\; & \sum_{\substack{x\in D_N^d \\ x_1=1}} \sum_{v\in\mathcal{V}} [\alpha_v(\tilde{x}/N)[1-\eta(x,v)] + (1-\alpha_v(\tilde{x}/N))\eta(x,v)][f(\sigma^{x,v}\eta) - f(\eta)] \\ & + \sum_{\substack{x\in D_N^d \\ x_1=N-1}} \sum_{v\in\mathcal{V}} [\beta_v(\tilde{x}/N)[1-\eta(x,v)] + (1-\beta_v(\tilde{x}/N))\eta(x,v)][f(\sigma^{x,v}\eta) - f(\eta)],\end{aligned}$$

where $\tilde{x} = (x_2, \dots, x_d)$,

$$\sigma^{x,v}\eta(y,w) = \begin{cases} 1 - \eta(x,w), & \text{if } w = v \text{ and } y = x, \\ \eta(y,w), & \text{otherwise.} \end{cases},$$

and for every $v \in \mathcal{V}$, $\alpha_v, \beta_v \in C^2(\mathbb{T}^{d-1})$. We also assume that, for every $v \in \mathcal{V}$, α_v and β_v have images belonging to some compact subset of $(0,1)$. The functions α_v and β_v, which affect the birth and death rates at the two boundaries, represent the densities of the reservoirs.

Note that time has been speeded up diffusively in (4.1.2). Let $\{\eta(t); t \ge 0\}$ be the Markov process with generator $\mathcal{L}_N$, and let $D(\mathbb{R}_+, X_N)$ be the set of right continuous functions with left limits taking values on X_N. For a probability measure μ on X_N, denote by $\mathbb{P}_\mu$ the measure on the path space $D(\mathbb{R}_+, X_N)$ induced by $\{\eta(t); t \ge 0\}$ and the initial measure μ. Expectation with respect to $\mathbb{P}_\mu$ is denoted by $\mathbb{E}_\mu$.

4.1.2 Mass and momentum

For each configuration $\xi \in \{0,1\}^{\mathcal{V}}$, denote by $I_0(\xi)$ the mass of ξ and by $I_k(\xi)$, $k = 1, \ldots, d$, the momentum of ξ:

$$I_0(\xi) = \sum_{v \in \mathcal{V}} \xi(v), \quad I_k(\xi) = \sum_{v \in \mathcal{V}} v_k \xi(v).$$

Set $\boldsymbol{I}(\xi) := (I_0(\xi), \ldots, I_d(\xi))$. Assume that the set of velocities is chosen in such a way that the unique quantities conserved by the random walk dynamics described above are mass and momentum: $\sum_{x \in D_N^d} \boldsymbol{I}(\eta_x)$. Two examples of sets of velocities satisfying these conditions can be found at [9].

For each chemical potential $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_d) \in \mathbb{R}^{d+1}$, denote by m_λ the measure on $\{0,1\}^{\mathcal{V}}$ given by

$$m_\lambda(\xi) = \frac{1}{Z(\boldsymbol{\lambda})} \exp\{\lambda \cdot \boldsymbol{I}(\xi)\}, \tag{4.1.3}$$

where $Z(\boldsymbol{\lambda})$ is a normalizing constant. Note that $m_{\boldsymbol{\lambda}}$ is a product measure on $\{0,1\}^{\mathcal{V}}$, i.e., that the variables $\{\xi(v); v \in \mathcal{V}\}$ are independent under $m_{\boldsymbol{\lambda}}$.

Denote by $\mu_{\boldsymbol{\lambda}}^N$ the product measure on X_N, with marginals given by

$$\mu_{\boldsymbol{\lambda}}^N\{\eta; \eta(x, \cdot) = \xi\} = m_{\boldsymbol{\lambda}}(\xi),$$

for each ξ in $\{0,1\}^{\mathcal{V}}$ and $x \in D_N^d$. Note that $\{\eta(x,v); x \in D_N^d, v \in \mathcal{V}\}$ are independent variables under $\mu_{\boldsymbol{\lambda}}^N$, and that the measure $\mu_{\boldsymbol{\lambda}}^N$ is invariant for the exclusion process with periodic boundary condition.

The expectation under $\mu_{\boldsymbol{\lambda}}^N$ of the mass and momentum are given by

$$\begin{aligned} \rho(\boldsymbol{\lambda}) &:= E_{\mu_{\boldsymbol{\lambda}}^N}[I_0(\eta_x)] = \sum_{v \in \mathcal{V}} \theta_v(\boldsymbol{\lambda}), \\ p_k(\boldsymbol{\lambda}) &:= E_{\mu_{\boldsymbol{\lambda}}^N}[I_k(\eta_x)] = \sum_{v \in \mathcal{V}} v_k \theta_v(\boldsymbol{\lambda}). \end{aligned}$$

In this formula $\theta_v(\boldsymbol{\lambda})$ denotes the expected value of the density of particles with velocity v under $m_{\boldsymbol{\lambda}}$:

$$\theta_v(\boldsymbol{\lambda}) := E_{m_{\boldsymbol{\lambda}}}[\xi(v)] = \frac{\exp\left\{\lambda_0 + \sum_{k=1}^d \lambda_k v_k\right\}}{1 + \exp\left\{\lambda_0 + \sum_{k=1}^d \lambda_k v_k\right\}}.$$

Denote by $(\rho, \boldsymbol{p})(\boldsymbol{\lambda}) := (\rho(\boldsymbol{\lambda}), p_1(\boldsymbol{\lambda}), \ldots, p_d(\boldsymbol{\lambda}))$ the map that associates the chemical potential to the vector of density and momentum. It is possible to prove that $(\rho, \boldsymbol{p})$ is a diffeomorphism onto $\mathfrak{U} \subset \mathbb{R}^{d+1}$, the interior of the convex envelope of $\left\{\boldsymbol{I}(\xi); \xi \in \{0,1\}^{\mathcal{V}}\right\}$. Denote by $\Lambda = (\Lambda_0, \ldots, \Lambda_d) : \mathfrak{U} \to \mathbb{R}^{d+1}$ the inverse of $(\rho, \boldsymbol{p})$. This correspondence allows one to parameterize the invariant states by the density and momentum: for each $(\rho, \boldsymbol{p})$ in $\mathfrak{U}$ we have a product measure $\nu_{\rho,\boldsymbol{p}}^N = \mu_{\Lambda(\rho,\boldsymbol{p})}^N$ on X_N.

4.1.3 Dynamical large deviations

Fix $T > 0$, let $\mathcal{M}_+$ be the space of finite positive measures on D^d endowed with the weak topology, and let $\mathcal{M}$ be the space of bounded variation signed measures on D^d endowed with the weak topology. Let $\mathcal{M}_+ \times \mathcal{M}^d$ be the cartesian product of these spaces endowed with the product topology, which is metrizable. Let also $\mathcal{M}^0$ be the subset of $\mathcal{M}_+ \times \mathcal{M}^d$ of all absolutely continuous measures with respect to the Lebesgue measure satisfying:

$$\mathcal{M}^0 = \left\{\pi \in \mathcal{M}_+ \times \mathcal{M}^d; \pi(du) = (\rho, \boldsymbol{p})(u)du, \quad \text{and } (\rho, \boldsymbol{p}) \in \mathfrak{U}, \text{ a.e.}\right\}.$$

Note that if $(\rho, \boldsymbol{p}) \in \mathfrak{U}$, then $0 \le \rho(u) \le |\mathcal{V}|$, $|p_k(u)| \le \check{v}|\mathcal{V}|, k = 1, \ldots, d$, where $\check{v} = \max_{v \in \mathcal{V}} v_1$. Let $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ be the set of right continuous functions with left limits taking values on $\mathcal{M}_+ \times \mathcal{M}^d$ endowed with the Skorohod topology. $\mathcal{M}^0$ is a closed subset of $\mathcal{M}_+ \times \mathcal{M}^d$ and $D([0,T], \mathcal{M}^0)$ is a closed subset of $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$. For a measure $\pi \in \mathcal{M}$, denote by $\langle \pi, G \rangle$ the integral of a function G with respect to π.

Let $\Omega_T = (0,T)\times D^d$ and $\overline{\Omega_T} = [0,T]\times \overline{D^d}$. For $1 \le m,n \le +\infty$, denote by $\mathcal{C}^{m,n}(\overline{\Omega_T})$ the space of functions $G = G_t(u) : \overline{\Omega_T} \to \mathbb{R}$ with m continuous derivatives in time and n continuous derivatives in space. We also denote by $\mathcal{C}_0^{m,n}(\overline{\Omega_T})$ (resp. $\mathcal{C}_c^\infty(\Omega_T)$) the set of functions in $\mathcal{C}^{m,n}(\overline{\Omega_T})$ (resp. $\mathcal{C}^{\infty,\infty}(\overline{\Omega_T})$) which vanish at $[0,T]\times\Gamma$ (resp. with compact support in Ω_T).

Let the energy $\mathcal{Q} : D([0,T],\mathcal{M}^0) \to [0,\infty]$ be given by

$$\mathcal{Q}(\pi) = \sum_{k=0}^d \sum_{i=1}^d \sup_{G\in\mathcal{C}_c^\infty(\Omega_T)} \Big\{ 2\int_0^T dt\, \langle p_{k,t}, \partial_{u_i} G_t\rangle - \int_0^T dt \int_{D^d} G(t,u)^2\, du \Big\}.$$

where $p_{k,t}(u) = p_k(t,u)$ and $p_{0,t}(u) = \rho(t,u)$.

Let $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$ be the set of vector valued function $G = (G^0,\dots,G^d) : [0,T]\times D^d \to \mathbb{R}^{d+1}$, with each coordinate G_k in $C_0^{1,2}\left(\overline{\Omega_T}\right)$, $k = 0,\dots,d$. For each $G \in \mathfrak{C}_0^{1,2}(\overline{\Omega_T})$ and each measurable function $\gamma = (\rho_0, \boldsymbol{p}_0) : \overline{D^d} \to \mathfrak{U}$, let $\hat{J}_G = \hat{J}_{G,\gamma,T} : D([0,T],\mathcal{M}^0) \to \mathbb{R}$ be the functional given by

$$\begin{aligned}
\hat{J}_G(\pi) &= \int_{D^d} G(T,u)\cdot(\rho,\boldsymbol{p})(T,u)du - \int_{D^d} G(0,u)\cdot(\rho_0,\boldsymbol{p}_0)(u)du \\
&- \int_0^T dt \int_{D^d} du \left\{ (\rho,\boldsymbol{p})(t,u)\cdot\partial_t G(t,u) + \frac{1}{2}(\rho,\boldsymbol{p})(t,u)\cdot\sum_{i=1}^d \partial_{u_i}^2 G(t,u)\right\} \\
&+ \frac{1}{2}\int_0^T dt \int_{\{1\}\times\mathbb{T}^{d-1}} dS\ b(\tilde{u})\cdot\partial_{u_1} G(t,u) - \frac{1}{2}\int_0^T dt \int_{\{0\}\times\mathbb{T}^{d-1}} dS\ a(\tilde{u})\cdot\partial_{u_1} G(t,u) \\
&+ \int_0^T dt \int_{D^d} du\ \sum_{v\in\mathcal{V}} \tilde{v}\cdot\chi_v(\rho,\boldsymbol{p}) \sum_{i=1}^d v_i \partial_{u_i} G(t,u) \\
&- \int_0^T dt \int_{D^d} du \sum_{v\in\mathcal{V}} \left(\sum_{k=0}^d v_k \partial_{u_i} G_t^k(u) \right)^2 \chi_v(\rho,\boldsymbol{p}),
\end{aligned}$$

where $\chi(r) = r(1-r)$ is the static compressibility, $\chi_v(\cdot) = \chi(\theta_v(\Lambda(\cdot)))$, for $u = (u_1,\dots,u_d)\in\mathbb{R}^d$, $\tilde{u} = (u_2,\dots,u_d)$, $\pi_t(du) = (\rho,\boldsymbol{p})(t,u)du$, and dS is the Lebesgue measure on $\mathbb{T}^{d-1}$. Define $J_G = J_{G,\gamma,T} : D([0,T],\mathcal{M}_+\times\mathcal{M}^d) \to \mathbb{R}$ by

$$J_G(\pi) = \begin{cases} \hat{J}_G(\pi), & \text{if } \pi\in D([0,T],\mathcal{M}^0), \\ +\infty, & \text{otherwise}. \end{cases}$$

We define the rate functional $I_T(\cdot|\gamma) : D([0,T],\mathcal{M}_+\times\mathcal{M}^d) \to [0,+\infty]$ as

$$I_T(\pi|\gamma) = \begin{cases} \sup\limits_{G\in\mathfrak{C}_0^{1,2}(\overline{\Omega_T})} \big\{J_G(\pi)\big\}, & \text{if } \mathcal{Q}(\pi) < \infty, \\ +\infty, & \text{otherwise}. \end{cases}$$

We now present the main result of this article, whose proof is given in Section 4.5, which is the dynamical large deviations for this boundary driven exclusion process with many conserved quantities.

Theorem 4.1.1. *Fix $T > 0$ and a measurable function $\gamma = (\rho_0,\boldsymbol{p}_0) : D^d \to \mathfrak{U}$. Consider a sequence η^N of configurations in X_N associated to γ in the sense that:*

$$\lim_{N\to\infty} \langle \pi_0^N(\eta^N), G\rangle = \int_{D^d} G(u)\rho_0(u)\, du,$$

and

$$\lim_{N\to\infty} \langle \pi_k^N(\eta^N), G\rangle = \int_{D^d} G(u)p_k(u)\, du, \quad k = 1,\dots,d,$$

for every continuous function $G : \overline{D^d} \to \mathbb{R}$. Then, the measure $Q_{\eta^N} = \mathbb{P}_{\eta^N}(\pi^N)^{-1}$ on $D([0,T],\mathcal{M}_+\times\mathcal{M}^d)$ satisfies a large deviation principle with speed N^d and rate function $I_T(\cdot|\gamma)$. Namely, for each closed set $\mathcal{C}\subset D([0,T],\mathcal{M}_+\times\mathcal{M}^d)$,

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d}\log Q_{\eta^N}(\mathcal{C}) \le -\inf_{\pi\in\mathcal{C}} I_T(\pi|\gamma)$$

and for each open set $\mathcal{O} \subset D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$,

$$\varliminf_{N\to\infty} \frac{1}{N^d} \log Q_{\eta^N}(\mathcal{O}) \geq - \inf_{\pi\in\mathcal{O}} I_T(\pi|\gamma) \; .$$

Moreover, the rate function $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets.

4.2 Hydrodynamics

Fix $T > 0$ and let $(B, \|\cdot\|_B)$ be a Banach space. We denote by $L^2([0,T], B)$ the Banach space of measurable functions $U : [0,T] \to B$ for which

$$\|U\|^2_{L^2([0,T],B)} = \int_0^T \|U_t\|^2_B dt < \infty.$$

Moreover, we denote by $H^1(D^d)$ the Sobolev space of measurable functions in $L^2(D^d)$ that have generalized derivatives in $L^2(D^d)$.

For $x = (x_1, \tilde{x}) \in \{0,1\} \times \mathbb{T}^{d-1}$, let

$$d(x) = \begin{cases} a(\tilde{x}) = \sum_{v\in\mathcal{V}} (\alpha_v(\tilde{x}), v_1\alpha_v(\tilde{x}), \ldots, v_d\alpha_v(\tilde{x})), & \text{if} \;\; x_1 = 0, \\ b(\tilde{x}) = \sum_{v\in\mathcal{V}} (\beta_v(\tilde{x}), v_1\beta_v(\tilde{x}), \ldots, v_d\beta_v(\tilde{x})), & \text{if} \;\; x_1 = 1. \end{cases} \tag{4.2.1}$$

Fix a bounded density profile $\rho_0 : D^d \to \mathbb{R}_+$, and a bounded momentum profile $\boldsymbol{p}_0 : D^d \to \mathbb{R}^d$. A bounded function $(\rho, \boldsymbol{p}) : [0,T] \times D^d \to \mathbb{R}_+ \times \mathbb{R}^d$ is a weak solution of the system of parabolic partial differential equations

$$\begin{cases} \partial_t(\rho, \boldsymbol{p}) + \sum_{v\in\mathcal{V}} \tilde{v} \, [v \cdot \nabla \chi_v(\rho, \boldsymbol{p})] = \frac{1}{2}\Delta(\rho, \boldsymbol{p}), \\ (\rho, \boldsymbol{p})(0, \cdot) = (\rho_0, \boldsymbol{p}_0)(\cdot) \text{ and } (\rho, \boldsymbol{p})(t, x) = d(x), x \in \{0,1\} \times \mathbb{T}^{d-1}, \end{cases} \tag{4.2.2}$$

if for every vector valued function $H \in \mathfrak{C}_0^{1,2}(\overline{\Omega_T})$, we have

$$\int_{D^d} H(T,u) \cdot (\rho, \boldsymbol{p})(T,u) du - \int_{D^d} H(0,u) \cdot (\rho_0, \boldsymbol{p}_0)(u) du$$
$$= \int_0^T dt \int_{D^d} du \left\{ (\rho, \boldsymbol{p})(t,u) \cdot \partial_t H(t,u) + \frac{1}{2} (\rho, \boldsymbol{p})(t,u) \cdot \sum_{i=1}^d \partial^2_{u_i} H(t,u) \right\}$$
$$- \frac{1}{2} \int_0^T dt \int_{\{1\}\times\mathbb{T}^{d-1}} dS \; b(\tilde{u}) \cdot \partial_{u_1} H(t,u) + \frac{1}{2} \int_0^T dt \int_{\{0\}\times\mathbb{T}^{d-1}} dS \; a(\tilde{u}) \cdot \partial_{u_1} H(t,u)$$
$$- \int_0^T dt \int_{D^d} du \sum_{v\in\mathcal{V}} \tilde{v} \cdot \chi_v(\rho, \boldsymbol{p}) \sum_{i=1}^d v_i \partial_{u_i} H(t,u).$$

We say that that the solution $(\rho, \boldsymbol{p})$ has finite energy if its components belong to $L^2([0,T], H^1(D^d))$:

$$\int_0^T ds \left(\int_{D^d} \|\nabla \rho(s,u)\|^2 du \right) < \infty,$$

and

$$\int_0^T ds \left(\int_{D^d} \|\nabla p_k(s,u)\|^2 du \right) < \infty,$$

for $k = 1, \ldots, d$, where ∇f represents the generalized gradient of the function f.

In [33] the following theorem was proved:

Theorem 4.2.1. *Let $(\mu^N)_N$ be a sequence of probability measures on X_N associated to the profile $(\rho_0, \boldsymbol{p}_0)$ in the sense of Theorem 4.1.1. Then, for every $t \geq 0$, for every continuous function $H : D^d \to \mathbb{R}$ vanishing at the boundary Γ, and for every $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N} \left[\left| \frac{1}{N^d} \sum_{x \in D_N^d} H\left(\frac{x}{N}\right) I_0(\eta_x(t)) - \int_{D^d} H(u)\rho(t,u)du \right| > \delta \right] = 0,$$

and for $1 \leq k \leq d$

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N} \left[\left| \frac{1}{N^d} \sum_{x \in D_N^d} H\left(\frac{x}{N}\right) I_k(\eta_x(t)) - \int_{D^d} H(u)p_k(t,u)du \right| > \delta \right] = 0,$$

where $(\rho, \boldsymbol{p})$ has finite energy and is the unique weak solution of equation (4.2.2).

4.3 The rate function $I_T(\cdot|\gamma)$

We examine in this section the rate function $I_T(\cdot|\gamma)$. The main result, presented in Theorem 4.3.6 below, states that $I_T(\cdot|\gamma)$ has compact level sets. The proof relies on two ingredients. The first one, stated in Lemma 4.3.2, is an estimate of the energy and of the H_{-1} norm of the time derivative of a trajectory in terms of the rate function. The second one, stated in Lemma 4.3.5, establishes that sequences of trajectories, with rate function uniformly bounded, which converge weakly in L^2 converge in fact strongly. We follow the strategy introduced in [15].

Let V be an open neighborhood of D^d, and consider, for each $v \in \mathcal{V}$, smooth functions $\kappa_k^v : V \to (0,1)$ in $C^2(V)$, for $k = 0, \ldots, d$. We assume that the restriction of $\kappa = \sum_{v\in\mathcal{V}}(\kappa_0^v, v_1\kappa_1^v, \ldots, v_d\kappa_d^v)$ to $\{0\}\times\mathbb{T}^{d-1}$ equals the vector valued function $a(\cdot)$ defined in (4.2.1), and that the restriction of κ to $\{1\}\times\mathbb{T}^{d-1}$ equals the vector valued function $b(\cdot)$, also defined in (4.2.1), in the sense that $\kappa(x) = d(x_1, \tilde{x})$ if $x \in \{0,1\}\times\mathbb{T}^{d-1}$.

Let $L^2(D^d)$ be the Hilbert space of functions $G : D^d \to \mathbb{R}$ such that $\int_{D^d} |G(u)|^2 du < \infty$ equipped with the inner product

$$\langle G, F\rangle_2 = \int_\Omega G(u)\, F(u)\, du\ ,$$

and the norm of $L^2(D^d)$ is denoted by $\|\cdot\|_2$.

Recall that $H^1(D^d)$ is the Sobolev space of functions G with generalized derivatives $\partial_{u_1} G, \ldots, \partial_{u_d} G$ in $L^2(D^d)$. $H^1(D^d)$ endowed with the scalar product $\langle\cdot,\cdot\rangle_{1,2}$, defined by

$$\langle G, F\rangle_{1,2} = \langle G, F\rangle_2 + \sum_{j=1}^d \langle \partial_{u_j} G\, ,\, \partial_{u_j} F\rangle_2\ ,$$

is a Hilbert space. The corresponding norm is denoted by $\|\cdot\|_{1,2}$.

Recall also that we denote by $\mathcal{C}_c^\infty(D^d)$ the set of infinitely differentiable functions $G : D^d \to \mathbb{R}$, with compact support in D^d. Denote by $H_0^1(D^d)$ the closure of $C_c^\infty(D^d)$ in $H^1(D^d)$. Since D^d is bounded, by Poincaré's inequality, there exists a finite constant C such that for all $G \in H_0^1(D^d)$

$$\|G\|_2^2 \ \leq\ C \sum_{j=1}^d \langle \partial_{u_j} G\, ,\, \partial_{u_j} G\rangle_2\ .$$

This implies that, in $H_0^1(D^d)$

$$\|G\|_{1,2,0} \ =\ \Big\{ \sum_{j=1}^d \langle \partial_{u_j} G\, ,\, \partial_{u_j} G\rangle_2 \Big\}^{1/2}$$

is a norm equivalent to the norm $\|\cdot\|_{1,2}$. Moreover, $H_0^1(D^d)$ is a Hilbert space with inner product given by

$$\langle G\, ,\, J\rangle_{1,2,0} \ =\ \sum_{j=1}^d \langle \partial_{u_j} G\, ,\, \partial_{u_j} J\rangle_2\ .$$

To assign boundary values along the boundary Γ of D^d to any function G in $H^1(D^d)$, recall, from the trace Theorem ([39], Theorem 21.A.(e)), that there exists a continuous linear operator $\mathrm{Tr} : H^1(D^d) \to L^2(\Gamma)$, called trace, such that $\mathrm{Tr}(G) = G\big|_\Gamma$ if $G \in H^1(D^d) \cap \mathcal{C}(\overline{D^d})$. Moreover, the space $H^1_0(D^d)$ is the space of functions G in $H^1(D^d)$ with zero trace ([39], Appendix (48b)):

$$H^1_0(D^d) = \left\{ G \in H^1(D^d);\ \mathrm{Tr}(G) = 0 \right\} .$$

Finally, denote by $H^{-1}(D^d)$ the dual of $H^1_0(D^d)$. $H^{-1}(D^d)$ is a Banach space with norm $\|\cdot\|_{-1}$ given by

$$\|v\|^2_{-1} = \sup_{G \in \mathcal{C}^\infty_c(D^d)} \left\{ 2\langle v, G\rangle_{-1,1} - \int_{D^d} \|\nabla G(u)\|^2 du \right\} ,$$

where $\langle v, G\rangle_{-1,1}$ stands for the values of the linear form v at G.

For each $G \in \mathcal{C}^\infty_c(\Omega_T)$ and each integer $1 \le i \le d$, let $\mathcal{Q}^G_{i,k} : D([0,T], \mathcal{M}^0) \to \mathbb{R}$ be the functional given by

$$\mathcal{Q}^G_{i,k}(\pi) = 2 \int_0^T dt\ \langle \pi^k_t, \partial_{u_i} G_t\rangle - \int_0^T dt \int_{D^d} du\ G(t,u)^2 ,$$

where $\pi = (\pi^0, \pi^1, \dots, \pi^d)$. Recall, from subsection 2.2, that the energy $\mathcal{Q}(\pi)$ is given by

$$\mathcal{Q}(\pi) = \sum_{k=0}^d \sum_{i=1}^d \mathcal{Q}_{i,k}(\pi), \quad \text{with} \quad \mathcal{Q}_{i,k}(\pi) = \sup_{G \in \mathcal{C}^\infty_c(\Omega_T)} \mathcal{Q}^G_{i,k}(\pi) .$$

The functional $\mathcal{Q}^G_{i,k}$ is convex and continuous in the Skorohod topology. Therefore $\mathcal{Q}_{i,k}$ and $\mathcal{Q}$ are convex and lower semicontinuous. Furthermore, it is well known that a measure $\pi(t,du) = (\rho, \boldsymbol{p})(t,u)du$ in $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ has finite energy, $\mathcal{Q}(\pi) < \infty$, if and only if its density ρ and its momentum $\boldsymbol{p}$ belong to $L^2([0,T], H^1(D^d))$. In which case

$$\mathcal{Q}(\pi) \ := \ \sum_{k=0}^d \int_0^T dt \int_{D^d} du\ \|\nabla p_{k,t}(u)\|^2 \ < \ \infty,$$

where $p_{0,t}(u) = \rho(t,u)$.

Let $D_\gamma = D_{\gamma,d}$ be the subset of $C([0,T], \mathcal{M}^0)$ consisting of all paths $\pi(t,du) = (\rho, \boldsymbol{p})(t,u)du$ with initial profile $\gamma(\cdot) = (\rho_0, \boldsymbol{p}_0)(\cdot)$, finite energy $\mathcal{Q}(\pi)$ (in which case ρ_t and $\boldsymbol{p}_t$ belong to $H^1(D^d)$ for almost all $0 \le t \le T$ and so $\mathrm{Tr}(\rho_t)$ is well defined for those t) and such that $\mathrm{Tr}(\rho_t) = d_0$ and $\mathrm{Tr}(p_{k,t}) = d_k$, $k = 1, \dots, d$, for almost all t in $[0,T]$, where $d(\cdot) = (d_0(\cdot), d_1(\cdot), \dots, d_d(\cdot))$.

Lemma 4.3.1. *Let π be a trajectory in $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ such that $I_T(\pi|\gamma) < \infty$. Then π belongs to D_γ.*

Proof. Fix a path π in $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ with finite rate function, $I_T(\pi|\gamma) < \infty$. By definition of I_T, π belongs to $D([0,T], \mathcal{M}^0)$. Denote its density and momentum by $(\rho, \boldsymbol{p})$: $\pi(t,du) = (\rho, \boldsymbol{p})(t,u)du$.

The proof that $(\rho, \boldsymbol{p})(0, \cdot) = \gamma(\cdot)$ is similar to the one of Lemma 3.5 in [6], and the proof that $\mathrm{Tr}(\rho_t) = d_0$, $\mathrm{Tr}(p_{k,t}) = d_k$, $k = 1, \dots, d$, is similar to the one found in Lemma 4.1 in [15].

We deal now with the continuity of π. We claim that there exists a positive constant C_0 such that, for any $g \in [C^\infty_c(D^d)]^{d+1}$, and any $0 \le s < r < T$,

$$|\langle \pi_r, g\rangle - \langle \pi_s, g\rangle| \ \le \ C_0 (r-s)^{1/2} \left\{ C_1 + I_T(\pi|\gamma) + \|g\|^2_{1,2,0} + (r-s)^{1/2} \|\Delta g\|_1 \right\} . \tag{4.3.1}$$

Indeed, for each $\delta > 0$, let $\psi^\delta : [0,T] \to \mathbb{R}$ be the function given by

$$(r-s)^{1/2} \psi^\delta(t) = \begin{cases} 0 & \text{if } 0 \le t \le s \ \text{ or } \ r + \delta \le t \le T , \\ \frac{t-s}{\delta} & \text{if } s \le t \le s + \delta , \\ 1 & \text{if } s + \delta \le t \le r , \\ 1 - \frac{t-r}{\delta} & \text{if } r \le t \le r + \delta , \end{cases}$$

and let $G^\delta_\epsilon(t,u) = \psi^\delta_\epsilon(t) g(u)$, where $\psi^\delta_\epsilon(\cdot)$ is the standard ϵ-mollification of $\psi^\delta(\cdot)$. Since G^δ_ϵ is in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$, we have

$$\begin{aligned}(r-s)^{1/2} \lim_{\delta\to 0}\lim_{\epsilon\to 0} J_{G^\delta_\epsilon}(\pi) &= \langle \pi_r, g\rangle - \langle \pi_s, g\rangle - \int_s^r dt\, \langle \pi_t, \Delta g\rangle \\ &+ \int_r^s dt \int_{D^d} du \sum_{v\in\mathcal{V}} \tilde{v}\cdot \chi_v(\rho,\boldsymbol{p}) \sum_{i=1}^d v_i \partial_{u_i} g(u) \\ &- \frac{1}{(r-s)^{1/2}} \int_s^r dt \int_{D^d} du \sum_{v\in\mathcal{V}} \left(\sum_{k=0}^d v_k \partial_{u_i} g^k(u)\right)^2 \chi_v(\rho,\boldsymbol{p}).\end{aligned}$$

To conclude the proof, we observe that the left-hand side is bounded by $(r-s)^{1/2} I_T(\pi|\gamma)$, that χ is positive and bounded above on $[0,1]$ by $1/4$, and finally, we use the elementary inequality $2ab \le a^2+b^2$. □

Denote by $L^2([0,T], H^1_0(D^d))^*$ the dual of $L^2([0,T], H^1_0(D^d))$. By Proposition 23.7 in [39], $L^2([0,T], H^1_0(D^d))^*$ corresponds to $L^2([0,T], H^{-1}(D^d))$ and for v in $L^2([0,T], H^1_0(D^d))^*$, G in $L^2([0,T], H^1_0(D^d))$,

$$\langle\!\langle v, G\rangle\!\rangle_{-1,1} = \int_0^T \langle v_t, G_t\rangle_{-1,1}\, dt\,, \tag{4.3.2}$$

where the left hand side stands for the value of the linear functional v at G. Moreover, if we denote by $|\!|\!| v |\!|\!|_{-1}$ the norm of v,

$$|\!|\!| v |\!|\!|_{-1}^2 = \int_0^T \|v_t\|_{-1}^2\, dt\,.$$

Fix a path $\pi(t,du) = (\rho,\boldsymbol{p})(t,u)du$ in D_γ and suppose that for $k = 0,\ldots,d$

$$\sup_{G\in\mathcal{C}_c^\infty(\Omega_T)} \left\{ 2\int_0^T dt\, \langle p_{k,t}, \partial_t G_t\rangle_2 - \int_0^T dt \int_{D^d} du\, \|\nabla G_t\|^2 \right\} < \infty\,. \tag{4.3.3}$$

In this case, for each k, $\partial_t p_k : C_c^\infty(\Omega_T) \to \mathbb{R}$ defined by

$$\partial_t p_k(G) = -\int_0^T \langle p_{k,t}, \partial_t G_t\rangle_2\, dt$$

can be extended to a bounded linear operator $\partial_t p_k : L^2([0,T], H^1_0(D^d)) \to \mathbb{R}$. It belongs therefore to $L^2([0,T], H^1_0(D^d))^* = L^2([0,T], H^{-1}(D^d))$. In particular, there exists $v^k = \{v^k_t; 0 \le t \le T\}$ in $L^2([0,T], H^{-1}(D^d))$, which we denote by $v^k_t = \partial_t p_{k,t}$, such that for any G in $L^2([0,T], H^1_0(D^d))$,

$$\langle\!\langle \partial_t p_k, G\rangle\!\rangle_{-1,1} = \int_0^T \langle \partial_t p_{k,t}, G_t\rangle_{-1,1}\, dt\,.$$

Moreover,

$$\begin{aligned}|\!|\!| \partial_t p_k |\!|\!|_{-1}^2 &= \int_0^T \|\partial_t p_{k,t}\|_{-1}^2\, dt \\ &= \sup_{G\in\mathcal{C}_c^\infty(\Omega_T)} \left\{ 2\int_0^T dt\, \langle p_{k,t}, \partial_t G_t\rangle_2 - \int_0^T dt \int_{D^d} du\, \|\nabla G_t\|^2 \right\}.\end{aligned}$$

Denote by $\langle\!\langle \partial_t(\rho,\boldsymbol{p}), \cdot\rangle\!\rangle_{-1,1} : L^2([0,T], [H^1_0(D^d)]^{d+1}) \to \mathbb{R}$ the linear functional given by

$$\langle\!\langle \partial_t(\rho,\boldsymbol{p}), G\rangle\!\rangle_{-1,1} = \sum_{k=0}^d \langle\!\langle \partial_t p_k, G^k\rangle\!\rangle_{-1,1},$$

with $G = (G^0,\ldots,G^d)$, and

$$|\!|\!| \partial_t(\rho,\boldsymbol{p}) |\!|\!|_{-1}^2 = \sum_{k=0}^d |\!|\!| \partial_t p_k |\!|\!|_{-1}^2.$$

Let W be the set of paths $\pi(t,du) = (\rho,\boldsymbol{p})(t,u)du$ in D_γ such that (4.3.3) holds, i.e., such that $\partial_t p_k$ belongs to $L^2\big([0,T], H^{-1}(D^d)\big)$. For G in $L^2\big([0,T],[H^1_0(D^d)]^{d+1}\big)$, let $\mathbb{J}_G : W \to \mathbb{R}$ be the functional given by

$$\begin{aligned}
\mathbb{J}_G(\pi) \;&=\; \langle\!\langle \partial_t(\rho,\boldsymbol{p}), G\rangle\!\rangle_{-1,1} + \frac{1}{2}\int_0^T dt \int_{D^d} du \sum_{i=1}^d \partial_{u_i}(\rho,\boldsymbol{p})(t,u)\cdot \partial_{u_i} G(t,u)\\
&+\; \int_0^T dt \int_{D^d} du \sum_{v\in\mathcal{V}} \tilde{v}\cdot \chi_v(\rho,\boldsymbol{p}) \sum_{i=1}^d v_i \partial_{u_i} G(t,u)\\
&-\; \int_0^T dt \int_{D^d} du \sum_{v\in\mathcal{V}} \big(\tilde{v}\cdot\partial_{u_i} G_t(u)\big)^2 \chi_v(\rho,\boldsymbol{p}),
\end{aligned}$$

Note that $\mathbb{J}_G(\pi) = J_G(\pi)$ for every G in $C^\infty_c(\Omega_T)\times[\mathcal{C}^\infty_c(D^d)]^d$. Moreover, since $\mathbb{J}_\cdot(\pi)$ is continuous in $L^2\big([0,T],[H^1_0(D^d)]^{d+1}\big)$ and since $\mathcal{C}^\infty_c(\Omega_T)$ is dense in $\mathcal{C}^{1,2}_0(\overline{\Omega_T})$ and in $L^2([0,T],\, H^1_0(D^d))$, for every π in W,

$$I_T(\pi|\gamma) = \sup_{G\in C^\infty_c(\Omega_T)\times[\mathcal{C}^\infty_c(D^d)]^d} \mathbb{J}_G(\pi) \;=\; \sup_{G\in L^2\left([0,T],[H^1_0(D^d)]^{d+1}\right)} \mathbb{J}_G(\pi)\,. \tag{4.3.4}$$

Lemma 4.3.2. *There exists a constant $C_0 > 0$ such that if the density and momentum $(\rho,\boldsymbol{p})$ of some path $\pi(t,du) = (\rho,\boldsymbol{p})(t,u)du$ in $D([0,T],\mathcal{M}^0)$ has generalized gradients, $\nabla\rho$ and ∇p_k, $k = 1,\dots,d$. Then*

$$\|\partial_t(\rho,\boldsymbol{p})\|^2_{-1} \;\le\; C_0\left\{I_T(\pi|\gamma) + \mathcal{Q}(\pi)\right\}\,, \tag{4.3.5}$$

$$\sum_{k=0}^d \int_0^T dt \int_{D^d} du\, \|\nabla p_k(t,u)\|^2 \;\le\; C_0\left\{I_T(\pi|\gamma) + 1\right\}\,, \tag{4.3.6}$$

where $p_0 = \rho$.

Proof. Fix a path $\pi(t,du) = (\rho,\boldsymbol{p})(t,u)du$ in $D([0,T],\mathcal{M}^0)$. In view of the discussion presented before the lemma, we need to show that the left hand side of (4.3.3) is bounded by the right hand side of (4.3.5). Such an estimate follows from the definition of the rate function $I_T(\cdot|\gamma)$ and from the elementary inequality $2ab \le Aa^2 + A^{-1}b^2$.

To prove (4.3.6), observe that

$$\begin{aligned}
I_T(\pi|\gamma) \;&\ge\; J_G(\pi) = \partial_t\pi(G) + \frac{1}{2}\int_0^T dt \int_{D^d} du \sum_{i=1}^d \partial_{u_i}(\rho,p)\cdot\partial_{u_i} G\\
&+\; \int_0^T dt \int_{D^d} du \sum_{v\in\mathcal{V}} \chi_v(\rho,p) \sum_{i=1}^d \tilde{v}\cdot(v_i\partial_{u_i} G)\\
&-\; \int_0^T dt \int_{D^d} du \sum_{v\in\mathcal{V}}\sum_{i=1}^d \big(\tilde{v}\cdot\partial_{u_i} G\big)^2 \chi_v(\rho,p)\\
&\ge\; \partial_t\pi(G) + \frac{1}{2}\int_0^T dt \int_{D^d} du \sum_{i=1}^d \partial_{u_i}(\rho,p)\cdot\partial_{u_i} G - C\int_0^T dt \int_{D^d} du \sum_{k=0}^d \|\nabla G^k\|^2,
\end{aligned}$$

where C is constant obtained from the elementary inequality $2ab \le a^2 + b^2$, the fact that $\mathcal{V}$ is finite, and that χ is bounded above by $1/4$ in $[0,1]$.

Recall the definition of the function κ given at the beginning of Section 4.3. Now, consider $G = K(\pi - \kappa)$, $K > 0$ being a constant, and note that $\pi - \kappa$ belongs to $L^2([0,T], H^1_0(D^d))$, which implies that it may be approximated by C^∞_c functions. Therefore $|\partial_t\pi(G)| = K|\langle\pi_T, \pi_T/2 - \kappa\rangle - \langle\pi_0, \pi_0/2 - \kappa\rangle|$, which is bounded from above by some constant C_1. We, then, obtain that

$$\begin{aligned}
I(\pi) \;&\ge\; \int_0^T dt \int_{D^d} du \Big\{ -C_1 + \frac{K}{2}\sum_{k=0}^d \|\nabla p_k\|^2 - \frac{K}{2}\sum_{i=1}^d \partial_{u_i}(\rho,p)\cdot\partial_{u_i}\kappa - CK^2 \sum_{k=0}^d \|\nabla(p_k - \kappa_k)\|^2\Big\}\\
&\ge\; \int_0^T dt \int_{D^d} du \Big\{\Big(K/4 - 2CK^2\Big)\sum_{k=0}^d \|\nabla p_k\|^2 - \frac{K}{4}\sum_{k=0}^d \|\nabla\kappa_k\|^2 - 2CK^2\sum_{k=0}^d \|\nabla\kappa_k\|^2 - C_1\Big\},
\end{aligned}$$

where in the last inequality we used the Cauchy-Schwartz inequality and the elementary inequality $2ab \le a^2 + b^2$. The proof thus follows from choosing a suitable K, the estimate given in (4.3.5), and the fact we have a fixed smooth function κ. □

Corollary 4.3.3. *The density $(\rho, \boldsymbol{p})$ of a path $\pi(t, du) = (\rho, \boldsymbol{p})(t, u)du$ in $D([0,T], \mathcal{M}^0)$ is the weak solution of the equation* (4.2.2) *and initial profile γ if and only if the rate function $I_T(\pi|\gamma)$ vanishes. Moreover, if any of the above conditions hold, π has finite energy ($\mathcal{Q}(\pi) < \infty$).*

Proof. On the one hand, if the density $(\rho, \boldsymbol{p})$ of a path $\pi(t, du) = (\rho, \boldsymbol{p})(t, u)du$ in $D([0,T], \mathcal{M}^0)$ is the weak solution of equation (4.2.2) with initial condition is γ, in the formula of $\hat{J}_G(\pi)$, the linear part in G vanishes which proves that the rate functional $I_T(\pi|\gamma)$ vanishes. On the other hand, if the rate functional vanishes, the path $(\rho, \boldsymbol{p})$ belongs to $L^2([0,T], [H^1(D^d)]^{d+1})$ and the linear part in G of $J_G(\pi)$ has to vanish for all functions G. In particular, $(\rho, \boldsymbol{p})$ is a weak solution of (4.2.2). Moreover, if the rate function is finite, by the previous lemma, π has finite energy. Accordingly, if π is a weak solution, we have from Theorem 4.2.1 that it has finite energy. □

For each $q > 0$, let E_q be the level set of $I_T(\pi|\gamma)$ defined by

$$E_q = \left\{\pi \in D([0,T], \mathcal{M}_+ \times \mathcal{M}^d); I_T(\pi|\gamma) \le q\right\} .$$

By Lemma 4.3.1, E_q is a subset of $C([0,T], \mathcal{M}^0)$. Thus, from the previous lemma, it is easy to deduce the next result.

Corollary 4.3.4. *For every $q \ge 0$, there exists a finite constant $C(q)$ such that*

$$\sup_{\pi \in E_q} \Big\{ \|\partial_t(\rho, \boldsymbol{p})\|_{-1}^2 \; + \; \sum_{k=0}^{d} \int_0^T dt \int_{D^d} du \; \|\nabla p_k(t,u)\|^2 \Big\} \; \le \; C(q) \, .$$

Next result together with the previous estimates provide the compactness needed in the proof of the lower semicontinuity of the rate function.

Lemma 4.3.5. *Let $\{\rho^n; n \ge 1\}$ be a sequence of functions in $L^2(\Omega_T)$ such that uniformly on n,*

$$\int_0^T dt \, \|\rho_t^n\|_{1,2}^2 + \int_0^T dt \, \|\partial_t \rho_t^n\|_{-1}^2 \le C$$

for some positive constant C. Suppose that $\rho \in L^2(\Omega_T)$ and that $\rho^n \to \rho$ weakly in $L^2(\Omega_T)$. Then $\rho^n \to \rho$ strongly in $L^2(\Omega_T)$.

Proof. Since $H^1(D^d) \subset L^2(D^d) \subset H^{-1}(D^d)$ with compact embedding $H^1(D^d) \to L^2(D^d)$, from Corollary 8.4, [35], the sequence $\{\rho_n\}$ is relatively compact in $L^2\big([0,T], L^2(D^d)\big)$. Therefore the weak convergence implies the strong convergence in $L^2\big([0,T], L^2(D^d)\big)$. □

Theorem 4.3.6. *The functional $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets.*

Proof. We have to show that, for all $q \ge 0$, E_q is compact in $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$. Since $E_q \subset C([0,T], \mathcal{M}^0)$ and $C([0,T], \mathcal{M}^0)$ is a closed subset of $D([0,T], \mathcal{M})$, we just need to show that E_q is compact in $C([0,T], \mathcal{M}^0)$.

We will show first that E_q is closed in $C([0,T], \mathcal{M}^0)$. Fix $q \in \mathbb{R}$ and let $\{\pi^n; \, n \ge 1\}$ be a sequence in E_q converging to some π in $C([0,T], \mathcal{M}^0)$. Then, for all $G \in \mathcal{C}(\overline{\Omega_T}) \times [\mathcal{C}(\overline{D^d})]^d$,

$$\lim_{n \to \infty} \int_0^T dt \, \langle \pi_t^n, G_t \rangle = \int_0^T dt \, \langle \pi_t, G_t \rangle \, .$$

Notice that this means that $\pi^{n,k} \to \pi^k$ weakly in $L^2(\Omega_T)$, for each $k = 0, \dots, d$, which together with Corollary 4.3.4 and Lemma 4.3.5 imply that $\pi^{n,k} \to \pi^k$ strongly in $L^2(\Omega_T)$. From this fact and the definition of J_G it is easy to see that, for all G in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$,

$$\lim_{n \to \infty} J_G(\pi_n) = J_G(\pi) \, .$$

This limit, Corollary 4.3.4 and the lower semicontinuity of $\mathcal{Q}$ permit us to conclude that $\mathcal{Q}(\pi) \le C(q)$ and that $I_T(\pi|\gamma) \le q$.

We prove now that $\overline{E_q}$ is relatively compact. To this end, it is enough to prove that for every continuous function $G : \overline{D^d} \to \mathbb{R}$, and every $k = 0, \dots, d$,

$$\lim_{\delta\to 0} \sup_{\pi\in E_q} \sup_{\substack{0\le s,r\le T\\ |r-s|<\delta}} |\langle \pi_r^k, G\rangle - \langle \pi_s^k, G\rangle| = 0 . \tag{4.3.7}$$

Since $E_q \subset C([0,T], \mathcal{M}^0)$, we may assume by approximations of G in $L^1(D^d)$ that $G \in \mathcal{C}_c^\infty(D^d)$. In which case, (4.3.7) follows from (4.3.1). □

We conclude this section with an explicit formula for the rate function $I_T(\cdot|\gamma)$. For each $\pi(t, du) = (\rho, \boldsymbol{p})(t, u)du$ in $D([0,T], \mathcal{M}^0)$, denote by $H_0^1(\pi)$ the Hilbert space induced by $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$ endowed with the inner product $\langle\cdot,\cdot\rangle_\pi$ defined by

$$\langle H, G\rangle_\pi = \sum_{v\in\mathcal{V}} \sum_{i=1}^{d} \int_0^T dt \int_{D^d} du \chi_v(\rho, \boldsymbol{p})[\tilde{v}\cdot\partial_{u_i} H][\tilde{v}\cdot\partial_{u_i} G] . \tag{4.3.8}$$

Induced means that we first declare two functions F, G in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$ to be equivalent if $\langle F-G, F-G\rangle_\pi = 0$, and then we complete the quotient space with respect to the inner product $\langle\cdot,\cdot\rangle_\pi$. The norm of $H_0^1(\pi)$ is denoted by $\|\cdot\|_\pi$.

Fix a path π in $D([0,T], \mathcal{M}^0)$ and a function H in $H_0^1(\pi)$. A measurable function $\lambda : [0,T]\times D^d \to \mathbb{R}_+ \times \mathbb{R}^d$ is said to be a weak solution of the nonlinear boundary value parabolic equation

$$\begin{cases} \partial_t\lambda & + \sum_{i=1}^d \sum_{v\in\mathcal{V}} \tilde{v}\partial_{u_i}\left[\chi_v(\lambda)(v_i - \tilde{v}\cdot\partial_{u_i} H)\right] = \frac{1}{2}\Delta\lambda, \\ \lambda(0,\cdot) & = \gamma(\cdot) \\ \lambda(t,x) & = d(x), x\in\{0,1\}\times\mathbb{T}^{d-1}, \end{cases} \tag{4.3.9}$$

if it satisfies the following two conditions:

(i) For $k = 0, \dots, d$, λ_k belongs to $L^2\left([0,T], H^1(D^d)\right)$:

$$\int_0^T ds \Big(\int_{D^d} \| \nabla\lambda_k(s,u) \|^2 du \Big) < \infty ;$$

(ii) For every function $G(t,u) = G_t(u)$ in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$,

$$\begin{aligned} &\int_{D^d} G(T,u)\cdot\lambda(T,u)du - \int_{D^d} G(0,u)\cdot\gamma(u)du \\ &= \int_0^T dt \int_{D^d} du \left\{ \lambda(t,u)\cdot\partial_t G(t,u) + \frac{1}{2}\lambda(t,u)\cdot\sum_{i=1}^d \partial_{u_i}^2 G(t,u) \right\} \\ &-\frac{1}{2}\int_0^T dt \int_{\{1\}\times\mathbb{T}^{d-1}} dS\ b(\tilde{u})\cdot\partial_{u_1} G(t,u) + \frac{1}{2}\int_0^T dt \int_{\{0\}\times\mathbb{T}^{d-1}} dS\ a(\tilde{u})\cdot\partial_{u_1} G(t,u) \\ &- \int_0^T dt \int_{D^d} du \sum_{v\in\mathcal{V}} \tilde{v}\cdot\chi_v(\lambda) \sum_{i=1}^d v_i\partial_{u_i} G(t,u), \\ &+ \sum_{v\in\mathcal{V}} \sum_{i=1}^d \int_0^T dt \int_{D^d} du \chi_v(\lambda)[\tilde{v}\cdot\partial_{u_i} H][\tilde{v}\cdot\partial_{u_i} G]. \end{aligned}$$

Uniqueness of solutions of equation (1.3.9) follows from the same arguments of the uniqueness proved in [33].

Lemma 4.3.7. *Assume that $\pi(t, du) = (\rho, \boldsymbol{p})(t, u)du$ in $D([0,T], \mathcal{M}^0)$ has finite rate function: $I_T(\pi|\gamma) < \infty$. Then, there exists a function H in $H_0^1(\pi)$ such that $(\rho, \boldsymbol{p})$ is a weak solution to* (4.3.9)*. Moreover,*

$$I_T(\pi|\gamma) = \frac{1}{4}\|H\|_\pi^2 . \tag{4.3.10}$$

The proof of this lemma is similar to the one of Lemma 10.5.3 in [3] and is therefore omitted.

4.4 $I_T(\cdot|\gamma)$-Density

The main result of this section, stated in Theorem 4.4.5, asserts that any trajectory λ_t, $0 \le t \le T$, with finite rate function, $I_T(\lambda|\gamma) < \infty$, can be approximated by a sequence of smooth trajectories $\{\lambda^n; n \ge 1\}$ such that

$$\lambda^n \longrightarrow \lambda \quad \text{and} \quad I_T(\lambda^n|\gamma) \longrightarrow I_T(\lambda|\gamma) \; .$$

This is one of the main steps in the proof of the lower bound of the large deviations principle for the empirical measure. The proof is mainly based on the regularizing effects of the hydrodynamic equation. This strategy was introduced in [15].

A subset A of $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ is said to be $I_T(\cdot|\gamma)$-dense if for every π in $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ such that $I_T(\pi|\gamma) < \infty$, there exists a sequence $\{\pi^n; n \ge 1\}$ in A such that π^n converges to π and $I_T(\pi^n|\gamma)$ converges to $I_T(\pi|\gamma)$.

Let Π_1 be the subset of $D([0,T], \mathcal{M}^0)$ consisting of paths $\pi(t, du) = (\rho, \boldsymbol{p})(t, u)du$ whose density $(\rho, \boldsymbol{p})$ is a weak solution of the hydrodynamic equation (4.2.2) in the time interval $[0, \delta]$ for some $\delta > 0$.

Lemma 4.4.1. *The set Π_1 is $I_T(\cdot|\gamma)$-dense.*

Proof. Fix $\pi(t, du) = (\rho, \boldsymbol{p})(t, u)du$ in $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ such that $I_T(\pi|\gamma) < \infty$. By Lemma 4.3.1, π belongs to $C([0,T], \mathcal{M}^0)$. For each $\delta > 0$, let $(\rho^\delta, \boldsymbol{p}^\delta)$ be the path defined as

$$(\rho^\delta, \boldsymbol{p}^\delta)(t, u) = \begin{cases} \tau(t, u) & \text{if } 0 \le t \le \delta \, , \\ \tau(2\delta - t, u) & \text{if } \delta \le t \le 2\delta \, , \\ (\rho, \boldsymbol{p})(t - 2\delta, u) & \text{if } 2\delta \le t \le T \, , \end{cases}$$

where τ is the weak solution of the hydrodynamic equation (4.2.2) starting at γ. It is clear that $\pi^\delta(t, du) = (\rho^\delta, \boldsymbol{p}^\delta)(t, u)du$ belongs to D_γ, because so do π and τ and that $\mathcal{Q}(\pi^\delta) \le \mathcal{Q}(\pi) + 2\mathcal{Q}(\tau) < \infty$. Moreover, π^δ converges to π as $\delta \downarrow 0$ because π belongs to $\mathcal{C}([0,T], \mathcal{M}^0)$. By the lower semicontinuity of $I_T(\cdot|\gamma)$, $I_T(\pi|\gamma) \le \underline{\lim}_{\delta\to 0} I_T(\pi^\delta|\gamma)$. Then, in order to prove the lemma, it is enough to prove that $I_T(\pi|\gamma) \ge \overline{\lim}_{\delta\to 0} I_T(\pi^\delta|\gamma)$. To this end, decompose the rate function $I_T(\pi^\delta|\gamma)$ as the sum of the contributions on each time interval $[0, \delta]$, $[\delta, 2\delta]$ and $[2\delta, T]$. The first contribution vanishes because π^δ solves the hydrodynamic equation in this interval. On the time interval $[\delta, 2\delta]$, $\partial_t \rho_t^\delta = -\partial_t \tau_{2\delta - t} = -\frac{1}{2}\Delta \tau_{2\delta-t} + \sum_{v\in\mathcal{V}} \tilde{v}[v \cdot \nabla \chi_v(\tau_{2\delta-t})] = -\frac{1}{2}\Delta(\rho_t^\delta, \boldsymbol{p}_t^\delta) + \sum_{v\in\mathcal{V}} \tilde{v}[v \cdot \nabla \chi_v(\rho_t^\delta, \boldsymbol{p}_t^\delta)]$. In particular, the second contribution is equal to

$$\sup_{G \in \mathfrak{C}_0^{1,2}(\overline{\Omega_T})} \sum_{i=1}^d \Big\{ \int_0^\delta ds \int_{D^d} du \; \partial_{u_i}(\rho, \boldsymbol{p}) \cdot \partial_{u_i} G - \sum_{v\in\mathcal{V}} \int_0^\delta dt \; \int_{D^d} du \chi_v(\rho, \boldsymbol{p}) [\tilde{v} \cdot \partial_{u_i} G]^2 \Big\}$$

which, by Lemma 4.5.5 is bounded from above, and therefore this last expression converges to zero as $\delta \downarrow 0$. Finally, the third contribution is bounded by $I_T(\pi|\gamma)$ because π^δ in this interval is just a time translation of the path π. □

Recall the definition of the set $\mathfrak{U}$ given at the ending of subsection 4.1.2. Let Π_2 be the set of all paths π in Π_1 with the property that for every $\delta > 0$ there exists $\epsilon > 0$ such that, for $k = 0, \ldots, d$, $d(\pi_t^k(\cdot), \partial\mathfrak{U}) \ge \epsilon$ for all $t \in [\delta, T]$, where $\partial\mathfrak{U}$ stands for the boundary of $\mathfrak{U}$.

We begin by proving an auxiliary lemma.

Lemma 4.4.2. *Let $\pi, \lambda \in \mathfrak{U}$, and let $\pi^\epsilon = (1-\epsilon)\pi + \epsilon\lambda$, $0 \le \epsilon \le 1$. Then, for all $v \in \mathcal{V}$, we have*

$$\theta_v(\Lambda(\pi^\epsilon)) = (1-\epsilon)\theta_v(\Lambda(\pi)) + \epsilon\theta_v(\Lambda(\lambda)).$$

Proof. Fix some $\lambda \in \mathfrak{U}$. Observe that

$$\Bigg(\sum_{v\in\mathcal{V}} \theta_v(\Lambda(\lambda)), \sum_{v\in\mathcal{V}} v_1 \theta_v(\Lambda(\lambda)), \ldots, \sum_{v\in\mathcal{V}} v_d \theta_v(\Lambda(\lambda)) \Bigg) = (\lambda_0, \lambda_1, \ldots, \lambda_d)$$

is a linear system with $d+1$ equations and $|\mathcal{V}|$ unknowns (given by $\theta_v(\Lambda(\lambda))$, for $v \in \mathcal{V}$). Therefore, any solution of this linear system can be expressed as a linear combination of λ_i, $i = 0, 1, \ldots, d$. The proof follows from this fact. □

Remark 4.4.3. *In the particular case when $d = 1$ and the set of velocities is $\mathcal{V} = \{v, -v\} \subset \mathbb{R}$, a simple computation gives the unique solution*

$$\theta_v(\Lambda(\lambda_0, \lambda_1)) = \frac{\lambda_0}{2} + \frac{\lambda_1}{2v} \quad \text{and} \quad \theta_{-v}(\Lambda(\lambda_0, \lambda_1)) = \frac{\lambda_0}{2} - \frac{\lambda_1}{2v}.$$

Lemma 4.4.4. *The set Π_2 is $I_T(\cdot|\gamma)$-dense.*

Proof. By Lemma 4.4.1, it is enough to show that each path $\pi(t, du) = (\rho, \boldsymbol{p})(t, u)du$ in Π_1 can be approximated by paths in Π_2. Fix π in Π_1 and let τ be as in the proof of the previous lemma. For each $0 < \varepsilon < 1$, let $(\rho^\varepsilon, \boldsymbol{p}^\varepsilon) = (1-\varepsilon)(\rho, \boldsymbol{p}) + \varepsilon\tau$, $\pi^\varepsilon(t, du) = (\rho^\varepsilon, \boldsymbol{p}^\varepsilon)(t, u)du$. Note that $\mathcal{Q}(\pi^\varepsilon) < \infty$ because $\mathcal{Q}$ is convex and both $\mathcal{Q}(\pi)$ and $\mathcal{Q}(\tau)$ are finite. Hence, π^ε belongs to D_γ since both ρ and τ satisfy the boundary conditions. Moreover, It is clear that π^ε converges to π as $\varepsilon \downarrow 0$. By the lower semicontinuity of $I_T(\cdot|\gamma)$, in order to conclude the proof, it is enough to show that

$$\overline{\lim_{N\to\infty}} I_T(\pi^\varepsilon|\gamma) \leq I_T(\pi|\gamma) \,. \tag{4.4.1}$$

By Lemma 4.3.7, there exists $H \in H_0^1(\pi)$ such that $(\rho, \boldsymbol{p})$ solves the equation (4.3.9). Let $P_{i,v}(\pi) = \chi_v(\rho, \boldsymbol{p})\Big(\tilde{v} \cdot \partial_{u_i} H - v_i\Big)$, and note that $P_{i,v}(\tau) = -v_i\chi_v(\tau)$. Let also

$$P^\epsilon_{i,v} = (1-\epsilon)P_{i,v}(\pi) + \epsilon P_{i,v}(\tau).$$

Observe that, by Lemma 4.3.7,

$$I_T(\pi|\gamma) = \frac{1}{4}\|H\|^2_\pi,$$

and that, using the definition of $\|\cdot\|_\pi$ in (4.3.8),

$$\begin{aligned}
\frac{1}{4}\|H\|^2_\pi &= \frac{1}{4}\sum_{v\in\mathcal{V}}\sum_{i=1}^d \int_0^T dt \int_{D^d} du \chi_v(\rho, \boldsymbol{p})(\tilde{v}\cdot\partial_{u_i}H)^2 \\
&= \frac{1}{4}\sum_{v\in\mathcal{V}}\sum_{i=1}^d \int_0^T dt \int_{D^d} du \frac{(P_{i,v}(\pi) + v_i\chi_v(\rho, \boldsymbol{p}))^2}{\chi_v(\rho, \boldsymbol{p})}.
\end{aligned}$$

A simple computation shows that

$$\mathbb{J}_G(\pi^\epsilon) = \sum_{v\in\mathcal{V}}\sum_{i=1}^d \int_0^T \int_{D^d} [P^\epsilon_{i,v} + \chi_v(\rho^\epsilon, \boldsymbol{p}^\epsilon)v_i](\tilde{v}\cdot\partial_{u_i}G) - \chi_v(\rho^\epsilon, \boldsymbol{p}^\epsilon)(\tilde{v}\cdot\partial_{u_i}G)^2$$

$$= \frac{1}{4}\sum_{v\in\mathcal{V}}\sum_{i=1}^d \int_0^T dt \int_{D^d} du \frac{[P^\epsilon_{i,v} + \chi_v(\rho^\epsilon, \boldsymbol{p}^\epsilon)v_i]^2}{\chi_v(\rho^\epsilon, \boldsymbol{p}^\epsilon)} - \left(\frac{1}{2}\frac{P^\epsilon_{i,v} + \chi_v(\rho^\epsilon, \boldsymbol{p}^\epsilon)}{\sqrt{\chi_v(\rho^\epsilon, \boldsymbol{p}^\epsilon)}} - \sqrt{\chi_v(\rho, \boldsymbol{p})}(\tilde{v}\cdot\partial_{u_i}G)\right)^2.$$

Let

$$A_\epsilon = \frac{1}{4}\sum_{v\in\mathcal{V}}\sum_{i=1}^d \int_0^T dt \int_{D^d} du \frac{[P^\epsilon_{i,v} + \chi_v(\rho^\epsilon, \boldsymbol{p}^\epsilon)v_i]^2}{\chi_v(\rho^\epsilon, \boldsymbol{p}^\epsilon)},$$

and

$$B_\epsilon(G) = \int_0^T dt \int_{D^d} du \left(\frac{1}{2}\frac{P^\epsilon_{i,v} + \chi_v(\rho^\epsilon, \boldsymbol{p}^\epsilon)}{\sqrt{\chi_v(\rho^\epsilon, \boldsymbol{p}^\epsilon)}} - \sqrt{\chi_v(\rho, \boldsymbol{p})}(\tilde{v}\cdot\partial_{u_i}G)\right).$$

This implies that

$$I_T(\pi^\epsilon|\gamma) = \sup_G \mathbb{J}_G(\pi^\epsilon) = \sup_G \left\{A_\epsilon - B_\epsilon(G)^2\right\} = A_\epsilon - \inf_G B_\epsilon(G)^2 \leq A_\epsilon,$$

where the supremum and infimum are taken over in G in $C_c^\infty(\Omega_T) \times [\mathcal{C}_c^\infty(D^d)]^d$.

It remains to be shown that A_ϵ is uniformly integrable in ϵ. However, this is a simple consequence of Lemma 4.4.2. □

Let Π be the subset of Π_2 consisting of all those paths π which are solutions of the equation (4.3.9) for some $H \in \mathfrak{C}_0^{1,2}(\overline{\Omega_T})$.

Theorem 4.4.5. *The set* Π *is* $I_T(\cdot|\gamma)$*-dense.*

Proof. By the previous lemma, it is enough to show that each path π in Π_2 can be approximated by paths in Π. Fix $\pi(t, du) = (\rho, \boldsymbol{p})(t, u)du$ in Π_2. By Lemma 4.3.7, there exists $H \in H_0^1(\pi)$ such that $(\rho, \boldsymbol{p})$ solves the equation (4.3.9). Since π belongs to $\Pi_2 \subset \Pi_1$, $(\rho, \boldsymbol{p})$ is the weak solution of (4.2.2) in some time interval $[0, 2\delta]$ for some $\delta > 0$. In particular, $\tilde{v} \cdot \partial_{u_i} H = 0$ a.e in $[0, 2\delta] \times D^d$, $i = 1, \dots, d$, $v \in \mathcal{V}$. This implies, by equation (4.1.1), that $\nabla H^k = 0$ a.e. in $[0, 2\delta] \times D^d$, $k = 0, \dots, d$. On the other hand, since π belongs to Π_1, there exists $\epsilon > 0$ such that, for $k = 0, \dots, d$, $d(\pi_t^k(\cdot), \partial\mathfrak{U}) \geq \epsilon$ for $\delta \leq t \leq T$. Therefore,

$$\int_0^T dt \int_{D^d} \|\nabla H_t^k(u)\|^2 \, du \; < \; \infty \, , \qquad k = 0, \dots, d. \tag{4.4.2}$$

Since H belongs to $H_0^1(\pi)$, there exists a sequence of functions $\{H^n = (H^{n,1}, \dots, H^{n,d});\ n \geq 1\}$ in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$ converging to H in $H_0^1(\pi)$. We may assume of course that $\nabla H_t^{n,k} \equiv 0$ in the time interval $[0, \delta]$, $k = 0, \dots, d$. In particular,

$$\lim_{n\to\infty} \int_0^T dt \int_{D^d} du \, \|\nabla H_t^{n,k}(u) - \nabla H_t^k(u)\|^2 = 0 \, , \qquad k = 0, \dots, d. \tag{4.4.3}$$

For each integer $n > 0$, let $(\rho^n, \boldsymbol{p}^n)$ be the weak solution of (4.3.9) with H^n in place of H and set $\pi^n(t, du) = (\rho^n, \boldsymbol{p}^n)(t, u)du$. By (4.3.10) and since χ is bounded above in $[0, 1]$ by $1/4$, we have that

$$I_T(\pi^n|\gamma) = \frac{1}{2} \sum_{v\in\mathcal{V}} \sum_{i=1}^d \int_0^T dt \, \langle \chi_v(\rho_t^n, \boldsymbol{p}_t^n), (\tilde{v} \cdot \partial_{u_i} H_t^n)^2 \rangle_2 \leq C_0 \sum_{v\in\mathcal{V}} \sum_{i=1}^d \int_0^T dt \int_{D^d} du \, (\tilde{v} \cdot \partial_{u_i} H_t^n(u))^2 \, .$$

In particular, by (4.4.2) and (4.4.3), $I_T(\pi^n|\gamma)$ is uniformly bounded on n. Thus, by Theorem 4.3.6, the sequence π^n is relatively compact in $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$.

The sequence π^n has a subsequence converging to some π^0 in $D([0, T], \mathcal{M}^0)$. To keep notation simple, we will assume that the sequence π^n converges to π^0. For every G in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$,

$$\int_{D^d} G(T, u) \cdot (\rho_t^n, \boldsymbol{p}_t^n)(T, u)du - \int_{D^d} G(0, u) \cdot \gamma(u)du$$

$$= \int_0^T dt \int_{D^d} du \left\{ (\rho_t^n, \boldsymbol{p}_t^n)(t, u) \cdot \partial_t G(t, u) + \frac{1}{2} (\rho_t^n, \boldsymbol{p}_t^n)(t, u) \cdot \sum_{i=1}^d \partial_{u_i}^2 G(t, u) \right\}$$

$$-\frac{1}{2} \int_0^T dt \int_{\{1\}\times\mathbb{T}^{d-1}} dS \; b(\tilde{u}) \cdot \partial_{u_1} G(t, u) + \frac{1}{2} \int_0^T dt \int_{\{0\}\times\mathbb{T}^{d-1}} dS \; a(\tilde{u}) \cdot \partial_{u_1} G(t, u)$$

$$- \int_0^T dt \int_{D^d} du \; \sum_{v\in\mathcal{V}} \tilde{v} \cdot \chi_v(\rho_t^n, \boldsymbol{p}_t^n) \sum_{i=1}^d v_i \partial_{u_i} G(t, u),$$

$$+ \sum_{v\in\mathcal{V}} \sum_{i=1}^d \int_0^T dt \; \int_{D^d} du \chi_v(\rho_t^n, \boldsymbol{p}_t^n)[\tilde{v} \cdot \partial_{u_i} H^n][\tilde{v} \cdot \partial_{u_i} G].$$

Letting $n \to \infty$ in this equation, we obtain the same equation with π^0 and H in place of π^n and H^n, respectively, if

$$
\begin{aligned}
&\lim_{n\to\infty}\int_0^T dt\int_{D^d} du\ \sum_{v\in\mathcal{V}}\tilde v\cdot\chi_v(\rho_t^n,\boldsymbol{p}_t^n)\sum_{i=1}^d v_i\partial_{u_i}G(t,u)\\
&=\int_0^T dt\int_{D^d} du\ \sum_{v\in\mathcal{V}}\tilde v\cdot\chi_v(\rho_t^0,\boldsymbol{p}_t^0)\sum_{i=1}^d v_i\partial_{u_i}G(t,u),\\
&\lim_{n\to\infty}\sum_{v\in\mathcal{V}}\sum_{i=1}^d\int_0^T dt\ \int_{D^d} du\chi_v(\rho_t^n,\boldsymbol{p}_t^n)[\tilde v\cdot\partial_{u_i}H^n][\tilde v\cdot\partial_{u_i}G]\\
&=\sum_{v\in\mathcal{V}}\sum_{i=1}^d\int_0^T dt\ \int_{D^d} du\chi_v(\rho_t^0,\boldsymbol{p}_t^0)[\tilde v\cdot\partial_{u_i}H][\tilde v\cdot\partial_{u_i}G].
\end{aligned}
\tag{4.4.4}
$$

We prove the second claim, the first one being simpler. Note first that we can replace H^n by H in the previous limit, because χ is bounded in $[0,1]$ by $1/4$, and (4.4.3) holds. Now, $(\rho^n,\boldsymbol{p}^n)$ converges to $(\rho^0,\boldsymbol{p}^0)$ weakly in $L^2(\Omega_T)\times[L^2(D^d)]^d$ because π^n converges to π^0 in $D([0,T],\mathcal{M}^0)$. Since $I_T(\pi^n|\gamma)$ is uniformly bounded, by Corollary 4.3.4 and Lemma 4.3.5, $(\rho^n,\boldsymbol{p}^n)$ converges to $(\rho^0,\boldsymbol{p}^0)$ strongly in $L^2(\Omega_T)\times[L^2(D^d)]^d$ which implies (4.4.4). In particular, since (4.4.2) holds, by uniqueness of weak solutions of equation (4.3.9), $\pi^0=\pi$ and we are done. □

4.5 Large deviations

We prove in this section Theorem 4.1.1, which is the dynamical large deviations principle for the empirical measure of boundary driven stochastic lattice gas model with many conserved quantities. The proof uses some of the ideas introduced in [15].

4.5.1 Superexponential estimates

It is well known that one of the main steps in the derivation of the upper bound is a super-exponential estimate which allows the replacement of local functions by functionals of the empirical density in the large deviations regime.

Let κ be as in the beginning of Section 4.3. Note that since ν_κ^N is not the invariant state, there are no reasons for $\langle -N^2\mathcal{L}_N f,f\rangle_{\nu_\kappa^N}$ to be positive. The next statement shows that this expression is almost positive.

For each function $f:X_N\to\mathbb{R}$, let $D_{\nu_\kappa^N}(f)$ be

$$D_{\nu_\kappa^N}(f)=D^{ex}_{\nu_\kappa^N}(f)+D^{c}_{\nu_\kappa^N}(f)+D^{b}_{\nu_\kappa^N}(f),$$

where

$$
\begin{aligned}
D^{ex}_{\nu_\kappa^N}(f)&=\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{x+z\in D_N^d}P_N(z-x,v)\int\left[\sqrt{f(\eta^{x,z,v})}-\sqrt{f(\eta)}\right]^2\nu_\kappa^n(d\eta),\\
D^{c}_{\nu_\kappa^N}(f)&=\sum_{q\in\mathcal{Q}}\sum_{x\in D_N^d}\int p(x,q,\eta)\left[\sqrt{f(\eta^{x,q})}-\sqrt{f(\eta)}\right]^2\nu_\kappa^N(d\eta),
\end{aligned}
$$

and

$$
\begin{aligned}
D^{b}_{\nu_\kappa^N}(f)=&\sum_{v\in\mathcal{V}}\sum_{x\in\{1\}\times\mathbb{T}_N^{d-1}}\int[\alpha_v(\tilde x/N)(1-\eta(x,v))+(1-\alpha_v(\tilde x/N))\eta(x,v)]\times\\
&\times\left[\sqrt{f(\sigma^{x,v}\eta)}-\sqrt{f(\eta)}\right]^2\nu_\kappa^N(d\eta)+\\
&+\sum_{v\in\mathcal{V}}\sum_{x\in\{N-1\}\times\mathbb{T}_N^{d-1}}\int[\beta_v(\tilde x/N)(1-\eta(x,v))+(1-\beta_v(\tilde x/N))\eta(x,v)]\times\\
&\times\left[\sqrt{f(\sigma^{x,v}\eta)}-\sqrt{f(\eta)}\right]^2\nu_\kappa^N(d\eta).
\end{aligned}
$$

Proposition 4.5.1. *There exist constants $C_1 > 0$ and $C_2 = C_2(\alpha, \beta) > 0$ such that for every density f with respect to ν_κ^N, we have*

$$< \mathcal{L}_N \sqrt{f}, \sqrt{f} >_{\nu_\kappa^N} \le -C_1 D_{\nu_\kappa^N}(f) + C_2 N^{d-2}.$$

The proof of this proposition is elementary and is thus omitted.

Further, we may choose κ for which there exists a constant $\theta > 0$ such that:

$$\begin{aligned} \kappa(u_1, \tilde{u}) = d(0, \tilde{u}) &\qquad \text{if } 0 \le u_1 \le \theta\,, \\ \kappa(u_1, \tilde{u}) = d(1, \tilde{u}) &\qquad \text{if } 1 - \theta \le u_1 \le 1\,, \end{aligned}$$

for all $\tilde{u} \in \mathbb{T}^{d-1}$. In that case, for every N large enough, ν_κ^N is reversible for the process with generator $\mathcal{L}_N^b$ and then $\langle -N^2 \mathcal{L}_N^b f, f \rangle_{\nu_\kappa^N}$ is positive.

Fix $L \ge 1$ and a configuration η, let $\boldsymbol{I}^L(x, \eta) := \boldsymbol{I}^L(x) = (I_0^L(x), \dots, I_d^L(x))$ be the average of the conserved quantities in a cube of the length L centered at x:

$$\boldsymbol{I}^L(x) = \frac{1}{|\Lambda_L|} \sum_{z \in x + \Lambda_L} \boldsymbol{I}(\eta_z),$$

where, $\Lambda_L = \{-L, \dots, L\}^d$ and $|\Lambda_L| = (2L+1)^d$ is the discrete volume of box Λ_L.

For each $G \in \mathcal{C}(\overline{\Omega_T}) \times [C(\overline{D^d})]^d$, and each $\varepsilon > 0$, let

$$V_{N\varepsilon}^{G,1}(s, \eta) = \frac{1}{N^d} \sum_{k=0}^{d} \sum_{i,j=1}^{d} \sum_{x \in D_N^d} \partial_{u_i} G^k(s, x/N) \left[\tau_x \tilde{V}_{N\varepsilon}^{j,k} \right] ,$$

where

$$\begin{aligned} \tilde{V}_{N\varepsilon}^{j,k}(\eta) = &\frac{1}{(2\ell+1)^d} \sum_{y \in \Lambda_{N\varepsilon}} \sum_{v \in \mathcal{V}} v_k \sum_{z \in \mathbb{Z}^d} p(z, v) z_j \tau_y(\eta(0, v)[1 - \eta(z, v)]) \\ &- \sum_{v \in \mathcal{V}} v_j v_k \chi_v(\boldsymbol{I}^\ell(0)), \end{aligned}$$

and let

$$\begin{aligned} V_{N\varepsilon}^{G,2}(s, \eta) = &\frac{1}{2N^d} \sum_{v \in \mathcal{V}} \sum_{x \in D_N^d} \sum_{i=1}^{d} \sum_{j,k=0}^{d} v_k v_j \partial_{u_i}^N G_t^j(x/N) \partial_{u_i}^N G_t^k(x/N) \times \\ &\times \Big\{ \eta(x, v)[1 - \eta(x + e_i, v)] + \eta(x, v)[1 - \eta(x - e_i, v)] - 2\chi_v(\boldsymbol{I}^\ell(0)) \Big\} \end{aligned}$$

Let, again, $G \in \mathcal{C}(\overline{\Omega_T}) \times [C(\overline{D^d})]^d$, and consider the quantities

$$V_N^-(s, \eta, G) = \frac{1}{N^{d-1}} \sum_{k=0}^{d} \sum_{\tilde{x} \in \mathbb{T}_N^{d-1}} G_k(s, \tilde{x}/N) \Big(I_k(\eta_{(1, \tilde{x})}(s)) - \sum_{v \in \mathcal{V}} v_k \alpha_v(\tilde{x}/N) \Big),$$

$$V_N^+(s, \eta, G) = \frac{1}{N^{d-1}} \sum_{k=0}^{d} \sum_{\tilde{x} \in \mathbb{T}_N^{d-1}} G_k(s, \tilde{x}/N) \Big(I_k(\eta_{(N-1, \tilde{x})}(s)) - \sum_{v \in \mathcal{V}} v_k \beta_v(\tilde{x}/N) \Big),$$

Proposition 4.5.2. *Fix $G \in \mathcal{C}(\overline{\Omega_T}) \times [\mathcal{C}(\overline{D^d})]^d$, H in $\mathcal{C}([0,T] \times \Gamma) \times [\mathcal{C}(\Gamma)]^d$, a cylinder function Ψ and a sequence $\{\eta^N; N \ge 1\}$ of configurations with η^N in X_N. For every $\delta > 0$,*

$$\begin{aligned} \overline{\lim_{\varepsilon \to 0}} \, \overline{\lim_{N \to \infty}} \, \frac{1}{N^d} \log \mathbb{P}_{\eta^N} \Big[\Big| \int_0^T V_{N\varepsilon}^{G,j}(s, \eta_s)\, ds \Big| > \delta \Big] &= -\infty\,, \\ \overline{\lim_{N \to \infty}} \, \frac{1}{N^d} \, \mathbb{P}_{\eta^N} \Big[\Big| \int_0^T V_N^{\pm}(s, \eta, G) \Big| > \delta \Big] &= -\infty\,, \end{aligned}$$

for $j = 1, 2$.

The proof of the above proposition follows from Proposition 4.5.1, the replacement lemmas proved in [33], and the computation presented in [3], p. 78, for nonreversible processes.

For each $\varepsilon > 0$ and π in $\mathcal{M}_+ \times \mathcal{M}^d$, for $k = 0, \dots, d$, denote by $\Xi_\varepsilon(\pi_k) = \pi_k^\varepsilon$ the absolutely continuous measure obtained by smoothing the measure π_k:

$$\Xi_\varepsilon(\pi_k)(dx) \;=\; \pi_k^\varepsilon(dx) \;=\; \frac{1}{U_\varepsilon} \frac{\pi_k(\mathbf{\Lambda}_\varepsilon(x))}{|\mathbf{\Lambda}_\varepsilon(x)|}\, dx\ ,$$

where $\mathbf{\Lambda}_\varepsilon(x) = \{y \in D^d; |y - x| \le \varepsilon\}$, $|A|$ stands for the Lebesgue measure of the set A, and $\{U_\varepsilon; \varepsilon > 0\}$ is a strictly decreasing sequence converging to 1: $U_\varepsilon > 1$, $U_\varepsilon > U_{\varepsilon'}$ for $\varepsilon > \varepsilon'$, $\lim_{\varepsilon \downarrow 0} U_\varepsilon = 1$. Let

$$\pi^{N,\varepsilon} \;=\; \Big(\Xi_\varepsilon(\pi_0^N), \Xi_\varepsilon(\pi_1^N), \dots, \Xi_\varepsilon(\pi_d^N)\Big).$$

A simple computation shows that $\pi^{N,\varepsilon}$ belongs to $\mathcal{M}^0$ for N sufficiently large because $U_\varepsilon > 1$, and that for each continuous function $H : D^d \to \mathbb{R}^{d+1}$,

$$\langle \pi^{N,\varepsilon}, H\rangle \;=\; \frac{1}{N^d} \sum_{x \in D_N^d} H(x/N) \cdot \boldsymbol{I}^{\varepsilon N}(x) \;+\; O(N,\varepsilon)\ ,$$

where $O(N,\varepsilon)$ is absolutely bounded by $C_0\{N^{-1} + \varepsilon\}$ for some finite constant C_0 depending only on H.

For each H in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$ consider the exponential martingale M_t^H defined by

$$\begin{aligned} M_t^H \;&=\; \exp\Big\{ N^d \Big[\big\langle \pi_t^N, H_t \big\rangle - \big\langle \pi_0^N, H_0 \big\rangle \\ &\qquad - \frac{1}{N^d} \int_0^t e^{-N^d \langle \pi_s^N, H_s\rangle} \left(\partial_s + N^2 \mathcal{L}_N\right) e^{N^d \langle \pi_s^N, H_s\rangle}\, ds \Big] \Big\}\ . \end{aligned}$$

Recall from subsection 2.2 the definition of the functional $\hat{J}_H$. An elementary computation shows that

$$M_T^H = \exp\Big\{ N^d \Big[\hat{J}_H(\pi^{N,\varepsilon}) + \mathbb{V}_{N,\varepsilon}^H + c_H^1(\varepsilon) + c_H^2(N^{-1}) \Big] \Big\}\ . \tag{4.5.1}$$

In this formula,

$$\begin{aligned} \mathbb{V}_{N,\varepsilon}^H \;&=\; -\int_0^T V_{N\varepsilon}^{G,1}(s,\eta)\, ds - \sum_{i=1}^d \int_0^T V_{N\varepsilon}^{G,2}(s,\eta)\, ds \\ &\quad +\; V_N^+(s,\eta,\partial_{u_1} H) \;-\; V_N^-(s,\eta,\partial_{u_1} H) \;+\; \langle \pi_0^N, H_0\rangle - \langle \gamma, H_0\rangle\ ; \end{aligned}$$

and $c_H^j : \mathbb{R}_+ \to \mathbb{R}$, $j = 1, 2$, are functions depending only on H such that $c_H^j(\delta)$ converges to 0 as $\delta \downarrow 0$. In particular, the martingale M_T^H is bounded by $\exp\big\{C(H,T)N^d\big\}$ for some finite constant $C(H,T)$ depending only on H and T. Therefore, Proposition 4.5.2 holds for $\mathbb{P}_{\eta^N}^H = \mathbb{P}_{\eta^N} M_T^H$ in place of $\mathbb{P}_{\eta^N}$.

4.5.2 Energy estimates

To exclude paths with infinite energy in the large deviations regime, we need an energy estimate. We state first the following technical result.

Lemma 4.5.3. *There exists a finite constant C_0, depending on T, such that for every G in $C_c^\infty(\Omega_T)$, every integer $1 \le i \le d$, $0 \le k \le d$, and every sequence $\{\eta^N; N \ge 1\}$ of configurations with η^N in X_N,*

$$\varlimsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}_{\eta^N} \Big[\exp\Big\{ N^d \int_0^T dt\, \langle \pi_t^{N,k}, \partial_{u_i} G\rangle \Big\} \Big] \;\le\; C_0 \Big\{ 1 + \int_0^T \|G_t\|_2^2\, dt \Big\}\ .$$

The proof of this proposition follows from Lemma 3.8 in [33], and the fact that $d\delta_{\eta^N}/d\nu_\kappa^N \le C^{N^d}$, for some positive constant $C = C(\kappa)$.

For each G in $\mathcal{C}_c^\infty(\Omega_T)$ and each integer $1 \le i \le d$, let $\tilde{\mathcal{Q}}_{i,k}^G : D([0,T], \mathcal{M}_+ \times \mathcal{M}^d) \to \mathbb{R}$ be the function given by

$$\tilde{\mathcal{Q}}_{i,k}^G(\pi) = \int_0^T dt\, \langle \pi_t^k, \partial_{u_i} G_t\rangle - C_0 \int_0^T dt \int_{D^d} du\, G(t,u)^2\ .$$

Notice that

$$\sup_{G\in\mathcal{C}_c^\infty(\Omega_T)} \left\{\tilde{\mathcal{Q}}_{i,k}^G(\pi)\right\} = \frac{\mathcal{Q}_{i,k}(\pi)}{4C_0}\,. \tag{4.5.2}$$

Fix a sequence $\{G_r; r\geq 1\}$ of smooth functions dense in $L^2([0,T],H^1(D^d))$. For any positive integers m,l, let

$$B_{m,l}^k = \Big\{\pi\in D([0,T],\mathcal{M}_+\times\mathcal{M}^d);\ \max_{\substack{1\leq j\leq m\\ 1\leq i\leq d}} \tilde{\mathcal{Q}}_{i,k}^{G_j}(\pi)\leq l\Big\}\,.$$

Since, for fixed G in $\mathcal{C}_c^\infty(\Omega_T)$ and $1\leq i\leq d$ integer, the function $\tilde{\mathcal{Q}}_{i,k}^G$ is continuous, $B_{m,l}$ is a closed subset of $D([0,T],\mathcal{M})$.

Lemma 4.5.4. *There exists a finite constant C_0, depending on T, such that for any positive integers r,l and any sequence $\{\eta^N; N\geq 1\}$ of configurations with η^N in X_N,*

$$\varlimsup_{N\to\infty}\frac{1}{N^d}\log Q_{\eta^N}\left[(B_{m,l}^k)^c\right]\leq -l+C_0,$$

where $k=0,\ldots,d$.

Proof. For integers $1\leq k\leq r$ and $1\leq i\leq d$, by Chebychev inequality and by Lemma 4.5.3,

$$\varlimsup_{N\to\infty}\frac{1}{N^d}\log \mathbb{P}_{\eta^N}\left[\tilde{\mathcal{Q}}_{i,k}^{G_m}>l\right]\leq -l+C_0\,.$$

Hence, from

$$\varlimsup_{N\to\infty}\frac{1}{N^d}\log(a_N+b_N)\leq \max\left\{\varlimsup_{N\to\infty}\frac{1}{N^d}\log a_N,\ \varlimsup_{N\to\infty}\frac{1}{N^d}\log b_N\right\}\,, \tag{4.5.3}$$

we obtain the desired inequality. □

Lemma 4.5.5. *There exists a finite constant C_0, depending on T, such that for every G in $C_c^\infty(\Omega_T)\times[C_c^\infty(D^d)]^d$, and every sequence $\{\eta^N; N\geq 1\}$ of configurations with η^N in X_N,*

$$\varlimsup_{N\to\infty}\frac{1}{N^d}\log \mathbb{E}_{\nu_\kappa^N}\Big[\exp\Big\{N^d\int_0^T\sum_{i=1}^d\sum_{k=0}^d dt\,\langle\pi_t^{N,k},\partial_{u_i}G^k\rangle\Big\}\Big]\ \leq\ C_0\Big\{1+\int_0^T\|G_t\|_\pi^2\,dt\Big\}\,.$$

In particular, we have that if $(\rho,\boldsymbol{p})$ is the solution of (4.2.2), then

$$\sup_{G\in\mathfrak{C}_0^{1,2}(\overline{\Omega_T})}\sum_{i=1}^d\Big\{\int_0^T ds\int_{D^d}du\,\partial_{u_i}(\rho,\boldsymbol{p})\cdot\partial_{u_i}G-\sum_{v\in\mathcal{V}}\int_0^T dt\int_{D^d}du\chi_v(\rho,\boldsymbol{p})[\tilde{v}\cdot\partial_{u_i}G]^2\Big\},$$

is finite, and vanishes if $T\to 0$.

Proof. Applying Feynman-Kac's formula and using the same arguments of Lemma 3.3 in [33], we have that

$$\frac{1}{N^d}\log E_{\nu_\kappa^N}\left[\exp\left\{N\int_0^T ds\sum_{i=1}^d\sum_{k=0}^d\sum_{x\in D_N^d}(I_k(\eta_x(s))-I_k(\eta_{x-e_i}(s)))\partial_{u_i}G^k(s,x/N)\right\}\right]$$

is bounded above by

$$\frac{1}{N^d}\int_0^T\lambda_s^N ds,$$

where λ_s^N is equal to

$$\sup_f\Big\{\Big\langle N\sum_{i,k}\sum_{x\in D_N^d}(I_k(\eta(x))-I_k(\eta(x-e_i)))\partial_{u_i}G^k(s,x/N),f\Big\rangle_{\nu_\kappa^N}+N^2<\mathcal{L}_N\sqrt{f},\sqrt{f}>_{\nu_\kappa^N}\Big\},$$

where the supremum is taken over all densities f with respect to ν_κ^N. By Proposition 4.5.1, the expression inside brackets is bounded above by

$$CN^d - \frac{N^2}{2} D_{\nu_\kappa^N}(f) + \sum_{i,k} \sum_{x \in D_N^d} \left\{ N \partial_{u_i} G^k(s, x/N) \int [I_k(\eta_x) - I_k(\eta_{x-e_i})] f(\eta) \nu_\kappa^N(d\eta) \right\}.$$

We now rewrite the term inside the brackets as

$$\sum_{v \in \mathcal{V}} \sum_{i=1}^{d} \sum_{x \in D_N^d} \left\{ \int N(\tilde{v} \cdot \partial_{u_i} G(s, x/N)) [\eta(x,v) - \eta(x - e_i, v)] f(\eta) \nu_\kappa^N(d\eta) \right\}.$$

Writing $\eta(x,v) - \eta(x-e_i,v) = \eta(x,v)[1 - \eta(x-e_i,v)] - \eta(x-e_i,v)[1-\eta(x,v)]$, and applying the same arguments in Lemma 3.8 of [33], we obtain that

$$N(\tilde{v} \cdot \partial_{u_i} G(s,x/N)) \int [\eta(x,v) - \eta(x-e_i,v)] f(\eta) \nu_\kappa^N(d\eta)$$

$$\begin{aligned}
\leq\ & (\tilde{v} \cdot \partial_{u_i} G(s,x/N))^2 \int \eta(x,v)[1 - \eta(x-e_i,v)] f(\eta^{x-e_i,x,v}) d\nu_\kappa^N \\
+\ & \frac{1}{4} \int f(\eta^{x-e_i,x,v}) \left[N \left(1 - \frac{\gamma_{x-e_i}, v}{\gamma_{x,v}} \right) \right]^2 \nu_\kappa^N(d\eta) \\
+\ & N^2 \int \frac{1}{2} [\sqrt{f(\eta^{x-e_i,x,v})} - \sqrt{f(\eta)}]^2 \nu_\kappa^N(d\eta) \\
+\ & 2(\tilde{v} \cdot \partial_{u_i} G(s,x/N))^2 \int \eta(x,v)[1-\eta(x-e_i,v)](\sqrt{f(\eta)} + \sqrt{f(\eta^{x-e_i,x,v})})^2 \nu_\kappa^N(d\eta),
\end{aligned}$$

we have that $(\sqrt{f(\eta)} + \sqrt{f(\eta^{x-e_i,x,v})})^2 \leq 2(f(\eta) + f(\eta^{x-e_i,x,v}))$. An application of the replacement lemma (Lemma 3.7 in [33]) concludes the proof. □

4.5.3 Upper Bound

Fix a sequence $\{F_j; j \geq 1\}$ of smooth functions dense in $\mathcal{C}(\overline{D^d})$ for the uniform topology, with positive coordinates. For $j \geq 1$ and $\delta > 0$, let

$$D_{j,\delta} = \left\{ \pi \in D([0,T], \mathcal{M}_+ \times \mathcal{M}^d);\ |\langle \pi_t^k, F_j \rangle| \leq \check{v}^k |\mathcal{V}| \int_{D^d} F_j(x)\, dx + C_j \delta\ ,\ k = 0, \dots, d,\ 0 \leq t \leq T \right\},$$

where $\check{v}^0 = 1$ and $\check{v}^k = \check{v}$, $C_j = \|\nabla F_j\|_\infty$ and ∇F is the gradient of F. Clearly, the set $D_{j,\delta}$, $j \geq 1$, $\delta > 0$, is a closed subset of $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$. Moreover, if

$$E_{m,\delta} = \bigcap_{j=1}^{m} D_{j,\delta}\ ,$$

we have that $D([0,T], \mathcal{M}^0) = \cap_{n \geq 1} \cap_{m \geq 1} E_{m,1/n}$. Note, finally, that for all $m \geq 1$, $\delta > 0$,

$$\pi^{N,\varepsilon} \text{ belongs to } E_{m,\delta} \text{ for } N \text{ sufficiently large.} \tag{4.5.4}$$

Fix a sequence of configurations $\{\eta^N; N \geq 1\}$ with η^N in X_N and such that $\pi^N(\eta^N)$ converges to $\gamma(u)du$ in $\mathcal{M}_+ \times \mathcal{M}^d$. Let A be a subset of $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$,

$$\frac{1}{N^d} \log \mathbb{P}_{\eta^N} \left[\pi^N \in A \right] = \frac{1}{N^d} \log \mathbb{E}_{\eta^N} \left[M_T^H\, (M_T^H)^{-1}\, \mathbf{1}\{\pi^N \in A\} \right].$$

Maximizing over π^N in A, we get from (4.5.1) that the last term is bounded above by

$$- \inf_{\pi \in A} \hat{J}_H(\pi^\varepsilon) + \frac{1}{N^d} \log \mathbb{E}_{\eta^N} \left[M_T^H\, e^{-N^d \mathbb{V}_{N,\varepsilon}^H} \right] - c_H^1(\varepsilon) - c_H^2(N^{-1}).$$

Since $\pi^N(\eta^N)$ converges to $\gamma(u)du$ in $\mathcal{M}_+ \times \mathcal{M}^d$ and since Proposition 4.5.2 holds for $\mathbb{P}^H_{\eta^N} = \mathbb{P}_{\eta^N} M_T^H$ in place of $\mathbb{P}_{\eta^N}$, the second term of the previous expression is bounded above by some $C_H(\varepsilon, N)$ such that

$$\overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ C_H(\varepsilon, N) = 0 \,.$$

Hence, for every $\varepsilon > 0$, and every H in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$,

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbb{P}_{\eta^N}[A] \le - \inf_{\pi\in A} \hat{J}_H(\pi^\varepsilon) + C'_H(\varepsilon)\,, \tag{4.5.5}$$

where $\lim_{\varepsilon\to 0} C'_H(\varepsilon) = 0$. Let

$$B_{r,l} = \Big\{\pi \in D([0,T], \mathcal{M}_+ \times \mathcal{M}^d);\ \max_{\substack{1\le j\le r\\ 1\le i\le d}} \sum_{k=0}^{d} \tilde{\mathcal{Q}}_{i,k}^{G_j}(\pi) \le l\Big\},$$

and, for each $H \in \mathfrak{C}_0^{1,2}(\overline{\Omega_T})$, each $\varepsilon > 0$ and any $r, l, m, n \in \mathbb{Z}_+$, let $J_{H,\varepsilon}^{r,l,m,n} : D([0,T], \mathcal{M}_+ \times \mathcal{M}^d) \to \mathbb{R} \cup \{\infty\}$ be the functional given by

$$J_{H,\varepsilon}^{r,l,m,n}(\pi) = \begin{cases} \hat{J}_H(\pi^\varepsilon) & \text{if } \pi \in B_{r,l} \cap E_{m,1/n}\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

This functional is lower semicontinuous because so is $\hat{J}_H \circ \Xi_\varepsilon$ and because $B_{r,l}$, $E_{m,1/n}$ are closed subsets of $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$.

Let $\mathcal{O}$ be an open subset of $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$. By Lemma 4.5.4, (4.5.3), (4.5.4) and (4.5.5),

$$\begin{aligned} \overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q_{\eta^N}[\mathcal{O}] &\le \max\Big\{ \overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q_{\eta^N}[\mathcal{O} \cap B_{r,l} \cap E_{m,1/n}]\,, \\ &\qquad\qquad \overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q_{\eta^N}[(B_{r,l})^c]\Big\} \\ &\le \max\Big\{ - \inf_{\pi\in\mathcal{O}\cap B_{r,l}\cap E_{m,1/n}} \hat{J}_H(\pi^\varepsilon) + C'_H(\varepsilon)\,,\ -l + C_0\Big\} \\ &= - \inf_{\pi\in\mathcal{O}} L_{H,\varepsilon}^{r,l,m,n}(\pi)\,, \end{aligned}$$

where

$$L_{H,\varepsilon}^{r,l,m,n}(\pi) = \min\Big\{ J_{H,\varepsilon}^{r,l,m,n}(\pi) - C'_H(\varepsilon)\,,\ l - C_0\Big\}\,.$$

In particular,

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q_{\eta^N}[\mathcal{O}] \le - \sup_{H,\varepsilon,r,l,m,n} \inf_{\pi\in\mathcal{O}} L_{H,\varepsilon}^{r,l,m,n}(\pi)\,.$$

Note that, for each $H \in \mathfrak{C}_0^{1,2}(\overline{\Omega_T})$, each $\varepsilon > 0$ and $r, l, m, n \in \mathbb{Z}_+$, the functional $L_{H,\varepsilon}^{r,l,m,n}$ is lower semicontinuous. Then, by Lemma A2.3.3 in [23], for each compact subset $\mathcal{K}$ of $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$,

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q_{\eta^N}[\mathcal{K}] \le - \inf_{\pi\in\mathcal{K}} \sup_{H,\varepsilon,r,l,m,n} L_{H,\varepsilon}^{r,l,m,n}(\pi)\,.$$

By (4.5.2) and since $D([0,T], \mathcal{M}^0) = \cap_{n\ge 1} \cap_{m\ge 1} E_{m,1/n}$,

$$\overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{l\to\infty}}\ \overline{\lim_{r\to\infty}}\ \overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\ L_{H,\varepsilon}^{r,l,m,n}(\pi) = \begin{cases} \hat{J}_H(\pi) & \text{if } \mathcal{Q}(\pi) < \infty \text{ and } \pi \in D([0,T], \mathcal{M}^0)\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

This result and the last inequality imply the upper bound for compact sets because $\hat{J}_H$ and J_H coincide on $D([0,T], \mathcal{M}^0)$. To pass from compact sets to closed sets, we have to obtain exponential tightness for the sequence $\{Q_{\eta^N}\}$. This means that there exists a sequence of compact sets $\{\mathcal{K}_n;\ n \ge 1\}$ in $D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q_{\eta^N}({\mathcal{K}_n}^c) \le -n\,.$$

The proof presented in [2] for the non interacting zero range process is easily adapted to our context.

4.5.4 Lower Bound

The proof of the lower bound is similar to the one in the convex periodic case. We just sketch it and refer to [23], Section 10.5. Fix a path π in Π and let $H \in \mathfrak{C}_0^{1,2}(\overline{\Omega_T})$ be such that π is the weak solution of equation (4.3.9). Recall from the previous section the definition of the martingale M_t^H and denote by $\mathbb{P}_{\eta^N}^H$ the probability measure on $D([0,T], X_N)$ given by $\mathbb{P}_{\eta^N}^H[A] = \mathbb{E}_{\eta^N}[M_T^H \mathbf{1}\{A\}]$. Under $\mathbb{P}_{\eta^N}^H$ and for each $0 \le t \le T$, the empirical measure π_t^N converges in probability to π_t. Further,

$$\lim_{N\to\infty} \frac{1}{N^d} H\left(\mathbb{P}_{\eta^N}^H \middle| \mathbb{P}_{\eta^N}\right) = I_T(\pi|\gamma)\,,$$

where $H(\mu|\nu)$ stands for the relative entropy of μ with respect to ν. From these two results we can obtain that for every open set $\mathcal{O} \subset D([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ which contains π,

$$\varliminf_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^N}\left[\mathcal{O}\right] \ge -I_T(\pi|\gamma)\,.$$

The lower bound follows from this and the $I_T(\cdot|\gamma)$-density of Π established in Theorem 4.4.5.

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