Simple Cocycles over Lorenz Attractors

Ph.D. Thesis

Mohammad Fanaee

Advisor: Marcelo Viana

May 03, 2010

ABSTRACT. We show that Lyapunov exponents of typical Hölder continuous fiber bunched linear cocycles over Lorenz attractors have multiplicity one: the exceptional set has infinite codimension. It is described in terms of rather explicit geometric conditions on sufficient simplicity criterion exhibited by Avila and Viana in [5].

Acknowledgments

I would like to thanks
prof. Marcelo Viana, advisor of my PhD thesis, for all supports during my studies at IMPA,
J. Santamaria for useful conversations on this work,
the referees: E. Pujals, C. Moreira, M. Pacifico and A. Arbieto, and
all of my professors and colleagues at IMPA.

This work is supported by a doctoral grant from CNPq-TWAS.

Contents

CHAPTER 1

Introduction

A linear cocycle over a flow $f^t : \Lambda \to \Lambda$ is a flow $F^t : \Lambda \times \mathbb{C}^d \to \Lambda \times \mathbb{C}^d$ of the form

$$F^t(x, v) = (f^t(x), A^t(x)v)$$

where each $A^t(x) : \mathbb{C}^d \to \mathbb{C}^d$ is a linear isomorphism. The Lyapunov exponents are the exponential rates

$$\lambda(x, v) = \lim_{t \to +\infty} \frac{1}{t} \log ||A^t(x)v||, \;\; v \neq 0.$$

By Oseledets [14] this limit exists for every $v \in \mathbb{C}^d$ on a full measure set of $x \in \Lambda$, relative to any invariant probability measure m. There are at most d Lyapunov exponents; they are constant on orbits and vary measurably with the base point. Thus Lyapunov exponents are constant if m is ergodic.

One problem is to characterize when these exponents are different from zero. Another main problem is to know when all exponents have multiplicity one meaning that the subspace of vectors $v \in \mathbb{C}$ that share the same value of $\lambda(x, v)$ has dimension one. In this case we say that the Lyapunov spectrum is *simple*.

There has been much recent progress on these problems, specially when the base dynamics is hyperbolic. See [7,18,4] for the first question and [8,5] for the second one.

Here, we extend the theory to the case when the base dynamics is a geometric Lorenz attractor. This class of systems was introduced in [19,11] as a geometric model for the behavior of the famous Lorenz equations [12]. Recently, it was shown by Tucker [15] that these equations have all main features predicted by the geometric models, as we recall next.

A geometric Loyrenz flow in 3-dimensions admits a cross section S and a Poincaré transformation $P : S \backslash \Gamma \to S$ defined outside a curve Γ which is contained in the intersection of S with the stable manifold of some hyperbolic equilibrium. Trajectories through Γ just converge to the equilibrium and the other trajectories through S eventually return to S. Their accumulation set is the so-called geometric Lorenz attractor. Moreover, there is an invariant splitting

$$E^s \oplus E^{cu}$$

of the tangent bundle where the uniformly contracting bundle E^s has dimension 1, and the volume-expanding bundle E^{cu}, which contains the flow direction has dimension 2. Another important feature is that the Poincaré transformation admits an invariant contracting foliation $\mathcal{F}$ through which the dynamics can be reduced to that of a map on the interval (leaf space of $\mathcal{F}$).

1. Linear cocycles

A *linear cocycle* over a bijection $f : N \to N$ is a transformation $F : N \times \mathbb{C}^d \to N \times \mathbb{C}^d$ satisfying $f \circ \pi = \pi \circ F$ which acts by linear isomorphisms $A(x)$ on fibers. So, the cocycle has the form

$$F(x, v) = (f(x), A(x)v)$$

where

$$A : N \to \mathrm{GL}(d, \mathbb{C}).$$

Conversely, any $A : N \to \mathrm{GL}(d, \mathbb{C})$ defines a linear cocycle over f. Note that $F^n(x, v) = (f^n(x), A^n(x)v)$, where

$$A^n(x) = A(f^{n-1}(x)) \ \ldots \ A(f(x))A(x),$$

$$A^{-n}(x) = (A^n(f^{-n}(x)))^{-1},$$

for any $n \geq 1$, and $A^0(x) = \mathrm{id}$.

Let μ be a probability measure invariant by f. If $x \mapsto \max\{0, \log \parallel A(x) \parallel\}$ is μ-integrable then Oseledets [14] states that there exist a Lyapunov splitting

$$E_1(x) \oplus \ldots \oplus E_k(x), \ 1 \leq k = k(x) \leq d,$$

and Lyapunov exponents $\lambda_1(x) > \ldots > \lambda_k(x)$,

$$\lambda_i(x) = \lim_{n \to +\infty} \frac{1}{n} \log \parallel A^n(x)v_i \parallel, \ v_i \in E_i(x), \ 1 \leq i \leq k,$$

at μ-almost every point. Lyapunov exponents are invariant, uniquely defined at almost every x and vary measurably with the base point x. Thus, Lyapunov exponents are constant when μ is ergodic.

We recall that, in a differentiable structure, for any $r \in \mathbb{N} \cup \{0\}$ and $0 \leq \rho \leq 1$, the $C^{r,\rho}$ topology is defined by

$$||A||_{r,\rho} = \max_{0 \leq i \leq r} \sup_x ||D^i A(x)|| + \sup_{x \neq y} \frac{||D^r A(x) - D^r A(y)||}{\mathrm{d}(x, y)^\rho}$$

(for $\rho = 0$ omit the last term) and then

$$\mathcal{C}^{r,\rho}(N, d, \mathbb{C}) = \{A : N \to \mathrm{GL}(d, \mathbb{C}) : \ ||A||_{r,\rho} < +\infty\}$$

is a Banach space. We assume that $r + \rho > 0$ which implies $\eta-$Hölder continuity:

$$\parallel A(x) - A(y) \parallel \leq \parallel A \parallel_{0,\eta} \mathrm{d}(x, y)^\eta,$$

with

$$\eta = \begin{cases} \rho & r = 0 \\ 1 & r \geq 1. \end{cases}$$

2. Fiber bunching condition

In general, suppose that N is endowed with a metric d for which
(i) $\mathrm{d}(f(y), f(z)) \leq \theta(x)\mathrm{d}(y, z)$, for all $y, z \in W^s_{\mathrm{loc}}(x)$,
(ii) $\mathrm{d}(f^{-1}(y), f^{-1}(z)) \leq \theta(x)\mathrm{d}(y, z)$, for all $y, z \in W^u_{\mathrm{loc}}(x)$,
where $0 < \theta(x) \leq \theta < 1$, for all $x \in N$.

Let A be an η-Hölder continuous linear cocycle over f.

Definition 1.1. *A is fiber bunched if there exists some constant* $\tau \in (0,1)$ *such that*

$$||A(x)|| \; ||A(x)^{-1}|| \theta(x)^{\eta} < \tau,$$

for any $x \in N$.

Remark 1.1. *Fiber bunching is an open condition in* $C^{r,\rho}(N, d, \mathbb{C})$*: if A is a fiber bunched linear cocycle then any linear cocycle B sufficiently* C^0 *close to A is also fiber bunched, by definition.*

Bonatti and Viana [8] showed that generic dominated linear cocycles over any hyperbolic transformation have simple spectrum. Avila and Viana [5] proved a sufficient condition for simplicity of Lyapunov spectrum over any shift structure when the corresponding invariant measure has product structure. Assuming an induced smooth structure on complete shift space we prove that

Theorem 1. Lyapunov exponents of typical Hölder continuous fiber bunched linear cocycle over complete shift map have multiplicity one: the set of exceptional cocycles has infinite codimention, i.e. it is locally contained in finite unions of closed submanifolds with arbitrarily high codimension.

3. Suspension flows

Consider a suspension flow $f^t : \Lambda \to \Lambda$ of $f : N \to N$ and let $T : N \to \mathbb{R}$ be the corresponding return time to N. Assume that $A^t : \Lambda \to \mathrm{GL}(d, \mathbb{C})$ is a linear cocycle over f^t, and define

$$A_f(x) = A^{T(x)}(x),$$

for any $x \in N$. Note that $A_f : N \to \mathrm{GL}(d, \mathbb{C})$ is a linear cocycle over f.

Then we define, for any $r \in \mathbb{N} \cup \{0\}$ and $0 \leq \rho \leq 1$ with $r + \rho > 0$, the Banach space

$$\mathcal{C}^{r,\rho}(\Lambda, d, \mathbb{C}) = \{A^t : \Lambda \to \mathrm{GL}(d, \mathbb{C}) : \; ||A_f||_{r,\rho} < +\infty\}.$$

It turns out that, A^t is η-Hölder continuous if the corresponding discrete time linear cocycle A_f is Hölder.

Let A^t be an η-Hölder continuous linear cocycle over f^t.

Definition 1.2. A^t *is fiber bunched if the corresponding linear cocycle* A_f *is a fiber bunched linear cocycle over* f.

Note that fiber bunching is an open condition in $\mathcal{C}^{r,\rho}(\Lambda, d, \mathbb{C})$, by definition.

Theorem 2. Lyapunov exponents of typical Hölder continuous fiber bunched linear cocycles over a Lorenz attractor have multiplicity one.

CHAPTER 2

Lorenz Attractors

Attractors of flows present important features with respect to the discrete time case when they involve singularities interacting with regular orbits.

The geometric model of a Lorenz attractor is historically the first example of a C^1 robust singular attractor as a rigorous model for the behavior of the Lorenz attractor. An attractor for a smooth flow f^t on a manifold M is a compact invariant transitive set Λ admitting an open neighborhood U such that $f^t(\bar{U}) \subset U$, for all $t > 0$, and

$$\Lambda = \bigcap_{t>0} f^t(U).$$

The attractor is singular if it contains some singularity of the vector field.

From measure theoretic view point, a Lorenz flow provides a robust class of expansive attractors that are not hyperbolic sets but exhibit some non-uniformly hyperbolic behavior: the attractor supports a unique physical probability measure m for which the tangent bundle splits into three 1-dimensional invariant subspaces

$$E^s \oplus E^X \oplus E^u,$$

at m-almost every point depending measurably on the base point, where E^s is the stable direction, E^X is the direction of the Lorenz flow f^t, and

$$\lim_{t\to+\infty} \frac{1}{t} \log ||Df^t|_{E^u}|| > 0.$$

1. Lorenz equations

The system of equations

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= bx - y - xz, \\ \dot{z} &= xy - cx, \end{aligned} \tag{1}$$

proposed by Lorenz [12] loosely related to fluid convection and weather prediction. W. Tucker [15] showed that the system (1) exhibits a robust attractor Λ, for classical parameters $a = 10,\ b = 28,\ c = 8/3$.

The system of equations (1) is symmetric with respect to the $z-$axis. The singularity $\mathbf{0}$ has real eigenvalues $\alpha_{ss} < \alpha_s < 0 < -\alpha_{ss} < \alpha_u$ with $\alpha_s + \alpha_u > 0$. There are also two symmetric saddles σ_1, σ_2 with a real negative and two conjugate complex eigenvalues where the complex eigenvalues have positive real parts. The character of this flow is strongly dissipative, in particular, any maximally positively invariant subset has zero volume.

Numerical simulations show that there exists an open set U homeomorphic to a 2-torus where $\bigcap_{t>0} f^t(U)$ is an attracting set and the origin is the only singularity contained in U. Indeed, a very general view of the orbit of a generic point in

U is that the trajectory starts spiraling around one of the singularities, say σ_2, and suddenly jumps to the other singularity, σ_1, and starts spiraling around the other. This process repeats endlessly and then implies the "butterfly" appearance of Lorenz attractor.

2. Geometric model

To construct a geometric Lorenz attractor, we should analyze the dynamics in a neighborhood of $\mathbf{0}$ imitating the effect of the pair of saddles. By construction, there is a cross section S intersecting the stable manifold of 0 along a curve Γ that separates S into 2 connected components. We denote the corresponding Poincaré transformation

$$P : S \backslash \Gamma \to S.$$

Note that the future trajectories of points in Γ do not came back to S.

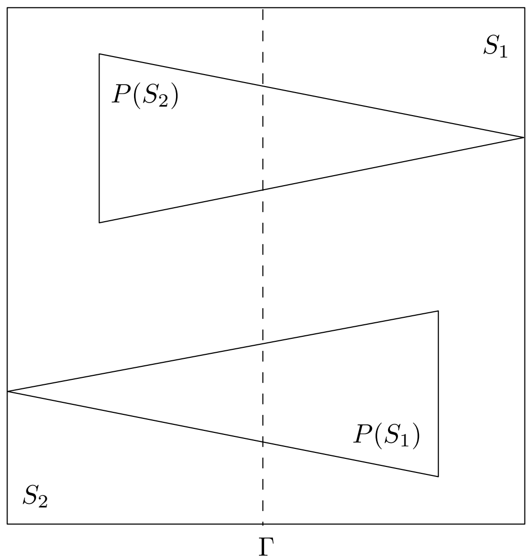

FIGURE 2.1. Poincaré transformation

We consider the smooth foliation $\mathcal{F}$ of S into curves having Γ as a leaf which are invariant and uniformly contracted by forward iterates of P. Indeed, every leaf $\mathcal{F}_{(x,y)}$ is mapped by P completely inside the leaf $\mathcal{F}_{P(x,y)}$, and $P|_{\mathcal{F}_{(x,y)}}$ is a uniform contraction. Henceforth, P must have the form

$$P(x, y) = (g(x), h(x, y))$$

which by effect of saddles and singularity, we can assume that h is a contraction along its second coordinate. The map g is uniformly expanding with derivative tending to infinity as one approaches to Γ. We assume that $|g'| \geq \theta^{-1} > \sqrt{2}$ and since the rate of contraction of h on the second coordinate should be much higher than the expansion of g, we can take $|\partial_y h| \leq \theta < 1$.

Let π be the canonical projection of section S into $\mathcal{F}$, i.e. π assigns to each point of S the leaf that contained it. By invariance of $\mathcal{F}$, one dimensional Lorenz map

$$g : (\mathcal{F} \backslash \Gamma) \to \mathcal{F}$$

is uniquely defined so that

$$\begin{array}{ccc} S\backslash\Gamma & \xrightarrow{P} & S \\ \pi\downarrow & & \downarrow\pi \\ \mathcal{F}\backslash\Gamma & \xrightarrow[g]{} & \mathcal{F} \end{array}$$

commutes, i.e. $g \circ \pi = \pi \circ P$ on $S\backslash\Gamma$.

One may identify quotient space $S/\mathcal{F}$ with a compact interval as $I = [-1, 1]$, and so

$$g : [-1, 1]\backslash\{0\} \to [-1, 1]$$

is smooth on $I\backslash\{0\}$ with a discontinuity and infinite left and right derivatives at 0. Note that the symmetry of the Lorenz equations implies $g(-x) = -g(x)$.

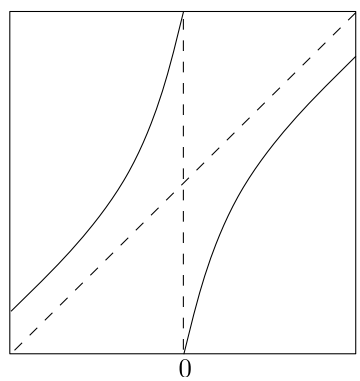

FIGURE 2.2. Lorenz 1-dimensioanl map

2.1. The attractor. The geometric Lorenz attractor Λ is characterized as follows. Note that the restriction of g to both $\{x < 0\}$ and $\{x > 0\}$ admits continuous extensions to the point 0. Hence, g may be considered as an extension to a 2-valued map at 0 and continuous on both $\{x \leq 0\}$ and $\{x \geq 0\}$. Correspondingly, the restriction of the Poincaré transformation to each of the connected components of $S\backslash\Gamma$ admits a continuous extension to the closure, each one collapsing the curve Γ to a single point. Thus, P may also be considered as a 2-valued transformation defined on the whole cross section and continuous on the closure of each of the connected components. Let

$$\Lambda_P = \bigcap_{n \geq 0} P^n(S) \subset S.$$

We define Λ to be the saturation of Λ_P by the Lorenz flow, that is, the orbits of its points. Therefore, orbits in Λ intersect the cross section infinitely often, both forward and backward.

This attractor has a complicated fractal structure that can be described as "a cantor book with uncountably many pages" joined along a spine corresponding to the unstable manifold of the singularity point $\mathbf{0}$ (see figure 4). Notice that the unstable manifold accumulates on itself, and so the geometry of Λ is indeed very complex.

Dynamical properties of Λ may be deduced from corresponding properties for the quotient map h. More important, a quotient map with similar properties exists for all nearby vector fields, and so such properties are robust for these flows.

3. Physical probability measure

The existence of a unique absolutely continuous invariant probability μ_g which is ergodic and $0 < \frac{d\mu_g}{dm} < +\infty$ for Lorenz one-dimensional map g is well-known.

One may construct an invariant probability measure μ_P on Λ_P, as the lifting of μ_g. Indeed, we may think of μ_g as a probability measure on Borel subsets of $\mathcal{F}$. Since P is uniformly contracting on leaves of $\mathcal{F}$, one concludes that the sequence

$$(P_*^n \mu_g)_{n\geq 1},$$

of push-forwards is weak*-Cauchy: given any continuous $\varphi : S \to \mathbb{R}$,

$$\int \varphi \; d(P_*^n \mu_g) = \int (\varphi \circ P^n) \; d\mu_g, \; n \geq 1,$$

is a Cauchy sequence in $\mathbb{R}$. Define the probability measure μ_P as the weak*-limit of this sequence that is

$$\int \varphi \; d\mu_P = \lim_{n\to+\infty} \int \varphi \; d(P_*^n \mu_g),$$

for each continuous function φ. Thus μ_P is invariant under P, and it is a physical probability measure on Borel subsets of Λ_P which is ergodic.

Later, as the Poincaré transformation maybe extended to the Lorenz flow through a suspension construction, the invariant probability μ_P corresponds to an ergodic physical probability measure μ_X on Λ: Denote by $R : S\backslash\Gamma \to (0, +\infty)$ the *first return time* to S defined by

$$P(x) = f^{R(x)}(x).$$

The first return time R is Lebesgue integrable, since $P(x) \approx |\log(\mathrm{d}(x, \Gamma))|$, for x close to Γ. This follows that

$$\int R \; d\mu_P < +\infty.$$

Let $\sim$ be an equivalence relation on $S \times \mathbb{R}$ defined as $(x, R(x)) \sim (P(x), 0)$. Set $\tilde{S} = (S \times \mathbb{R})/\sim$ and define the finite measure

$$\tilde{\mu} = \pi_*(\mu_P \times dt)$$

where $\pi : S \times \mathbb{R} \to \tilde{S}$ is the quotient map and dt is Lebesgue measure in $\mathbb{R}$. Define $\phi : \tilde{S} \to M$ as $\phi(x, t) = f^t(x)$, and let

$$m = \phi_* \tilde{\mu}.$$

One may check also that

$$\frac{1}{T} \int_0^T \varphi(f^t(x)) \; dt \; \to \; \int \varphi \; dm$$

as $T \to +\infty$, for any continuous function $\varphi : M \to \mathbb{R}$, and Lebesgue almost every $x \in \phi(\tilde{S})$.

CHAPTER 3

Symbolic Structure

Suppose that $N = \mathbb{N}^{\mathbb{Z}}$, the full shift space with countably many symbols, and $f : N \to N$ the shift map

$$f((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}.$$

A cylinder of N is any subset

$$[a_k, ...; a_0; ..., a_l] = \{x : \; x_j = \iota_j, \; j = k, ..., l\}$$

of N. We endowed N with topology generated by cylinders. The local stable and local unstable sets of any $x \in N$ are defined as

$$W^s_{\text{loc}}(x) = \{y : \; x_n = y_n, \; n \geq 0\}$$

and

$$W^u_{\text{loc}}(x) = \{y : \; x_n = y_n, \; n < 0\}.$$

Let $N_u = \mathbb{N}^{\{n\geq 0\}}$ and $N_s = \mathbb{N}^{\{n<0\}}$. The map

$$x \mapsto (x_s, x_u)$$

is a homeomorphism form N onto $N_s \times N_u$ where $x_s = \pi_s(x)$ and $x_u = \pi_u(x)$, for natural projections $\pi_s : N \to N_s$ and $\pi_u : N \to N_u$. We also consider the maps $f_s : N_s \to N_s$ and $f_u : N_u \to N_u$ defined by

$$f_u \circ \pi_u = \pi_u \circ f,$$

$$f_s \circ \pi_s = \pi_s \circ f^{-1}.$$

Assume that μ_f is an ergodic probability measure for f. Let $\mu_s = (\pi_s)_*\mu_f$ and $\mu_u = (\pi_u)_*\mu_f$ be the images of μ_f under the natural projections. It is easy to see that μ_s and μ_u are ergodic probabilities for f_s and f_u, respectively. Notice that μ_s and μ_u are positive on cylinders, by definition.

We say that μ_f has product structure if there exists a measurable density function $\omega : N \to (0, +\infty)$ such that

$$\mu_f = \omega(x)(\mu_s \times \mu_u).$$

Notation 3.1. *For simplicity, we omit u in the notation of f_u, π_u, N_u and μ_u, and represent these objects by $\hat{f}$, $\hat{\pi}$, $\hat{N}$ and $\hat{\mu}$.*

1. Markov structure in dimension 1

Now consider the Lorenz map $g : I\backslash\{0\} \to I$.

Theorem 3.1. [D06] *There exists a return map $\hat{g}$, an interval $\hat{I} = (-\delta, \delta)$, $0 < \delta < 1$, and a partition $\{\hat{I}(l) : \ l \in \mathbb{N}\}$ to subintervals of $\hat{I}$, Lebesgue mod 0, for which $\hat{g}$ maps any $\hat{I}(l)$ diffeomorphically onto $\hat{I}$, and the return time $\hat{r}$ is Lebesgue integrable. Moreover, there exists a constant $0 < c < 1$ such that, for all x, y in any $\hat{I}(l)$,*

$$\log \frac{|\hat{g}'(x)|}{|\hat{g}'(y)|} \leq c^{n(x,y)}$$

where $n(x,y) = \min\{n : \ \hat{g}^n(x) \in \hat{I}(l_i), \ \hat{g}^n(y) \in \hat{I}(l_j), \ i \neq j\}$.

Remark 3.1. *Note that, as Lorenz map g is uniformely expanding, the intesection of $(\hat{g}^{-n}(J(l_n)))$ over all $n \geq 0$ consists of exactly one point.*

Therefore $\hat{g}$ maybe seen as the shift map on $\hat{\Sigma}$: there exists a conjugation between $\hat{f}$ and $\hat{g}$ presented by the next commutting diagram

$$\begin{array}{ccc} \hat{\Sigma} & \xrightarrow{\hat{f}} & \hat{\Sigma} \\ {\scriptstyle\hat{\phi}}\big\downarrow & & \big\downarrow{\scriptstyle\hat{\phi}} \\ \hat{I} & \xrightarrow[\hat{g}]{} & \hat{I} \end{array}$$

where the bijection $\hat{\phi}$ maybe defined as

$$\hat{\phi} : (l_n)_{n\geq 0} \mapsto \bigcap_{n\geq 0} \hat{g}^{-n}(\hat{I}(l_n)).$$

2. Markov structure in dimension 2

Now, we consider the bi-dimensional domain $\hat{S} = \pi^{-1}(\hat{I}) \subset S$ and corresponding to the Markov partition of $\hat{I}$ define a Markov partition $\{\hat{S}(l) = \pi^{-1}(\hat{I}(l)) : \ l \in \mathbb{N}\}$ of $\hat{S}$. The return time is defined as

$$r(x) = \hat{r}(\pi(x)).$$

Hence, there exists a return map $\hat{P}$ to $\hat{S}$ as

$$\hat{P}(x) = P^{r(x)}(x),$$

for any $x \in \hat{S}$. Moreover

$$\hat{g} \circ \pi = \pi \circ \hat{P}.$$

Let

$$\Lambda_{\hat{P}} = \bigcap_{n\geq 0} \hat{P}^n(\hat{S}).$$

So $\Lambda_{\hat{P}}$ is homeomorphically equal to N. Indeed, since $\bigcap_{n\in\mathbb{Z}} \hat{P}^{-n}(\hat{S}(l_n))$ consists of exactly one point, one may define a bijection $\phi : N \to \Lambda_{\hat{P}}$ as

$$\phi : (l_n)_{n\in\mathbb{Z}} \mapsto \bigcap_{n\in\mathbb{Z}} \hat{P}^{-n}(\hat{S}(l_n))$$

which implies the commuting diagram

$$\begin{array}{ccc} N & \xrightarrow{f} & N \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ \Lambda_{\hat{P}} & \xrightarrow[\hat{P}]{} & \Lambda_{\hat{P}}. \end{array}$$

3. Lifting the probability measure

The normalized restriction $\hat{\mu}$ of μ_g to the domain of $\hat{g}$ is an absolutely continuous ergodic probability for $\hat{g}$ and then for $\hat{f}$, by conjugacy.

As the natural extension of $\hat{f}$ realized as the complete shift map f on N, the *lift* μ of $\hat{\mu}$ is the unique f-invariant probability measure on N such that

$$\hat{\pi}_* \mu = \hat{\mu}.$$

Indeed, let $\mu(E) = \hat{\mu}(\hat{\pi}(E))$, for any cylinder $E \subset N$. Using properties of $\hat{\mu}$ and applying Approximation Theorem, one may define μ on any Borelian subset of N. Finally, Extension Theorem for probabilities concludes a unique extended probability μ on Borelians of N for which, by construction, $\hat{\pi}_*\mu = \hat{\mu}$. Then μ is an absolutely continuous ergodic probability for f.

Proposition 3.1. *The lift probability μ has product structure. Moreover, the density function ω is continuous and, bounded from zero and infinity*

Proof. Note that by Theorem 3.1, for all $\hat{x}, \hat{y}$ in the same cylinder

$$\log \frac{J\hat{f}(\hat{x})}{J\hat{f}(\hat{y})} \le c^{n(x,y)}.$$

The rest of proof is based on 4 main steps stated with more details in [5] *Step 1.* If $\hat{x}, \hat{y} \in \hat{N}$ then for any $x \in W^s_{\mathrm{loc}}(\hat{x})$ and $y \in W^u_{\mathrm{loc}}(x) \cap W^s_{\mathrm{loc}}(\hat{y})$, the limit

$$J_{\hat{x},\hat{y}}(x) = \lim_{n\to\infty} \frac{J\hat{f}^n(\hat{x}^n)}{J\hat{f}^n(\hat{y}^n)},$$

where $\hat{x}^n = \hat{\pi}(f^{-n}(x)),\ \hat{y}^n = \hat{\pi}(f^{-n}(y))$, exists uniformly on $\hat{x}, \hat{y}, x$. Moreover,

$$(\hat{x}, \hat{y}, x) \mapsto J_{\hat{x},\hat{y}}(x)$$

is continuous and uniformly bounded from zero and infinity.

Indeed, we observe that

$$\log \frac{J\hat{f}^n(\hat{x}^n)}{J\hat{f}(\hat{y}^n)} \le \sum_{i=1}^{n} \log \frac{J\hat{f}(\hat{x}^i)}{J\hat{f}(\hat{y}^i)}.$$

Since $\hat{x}^i$ and $\hat{y}^i$ are in the same cylinder, the series is uniformly bounded by $\sum_i c^{n(\hat{x}^i,\hat{y}^i)}$. But $n(\hat{x}^i, \hat{y}^i)$ is strictly increasing that implies uniform convergence of the series.

Step 2. If $\{\mu_{\hat{x}} :\ \hat{x} \in \hat{N}\}$ be an integration of μ then, for μ-almost every $\hat{x} \in \hat{N}$,

$$\mu_{\hat{x}}(\xi_n) = \frac{1}{J\hat{f}^n(\hat{x}^n)},$$

for every cylinder $\xi_n = [x_{-n}, ..., x_{-1}],\ n \geq 1$, and any $x \in \xi_n \times \{\hat{x}\}$.
Step 3. Given any disintegration, by the last step, one may find a disintegration $\{\mu_{\hat{x}} :\ \hat{x} \in \hat{N}\}$ of μ so that

$$\mu_{\hat{y}} = J_{\hat{x},\hat{y}} \mu_{\hat{x}}.$$

Step 4. Fixing any $\hat{x}_0 \in \hat{N}$, one may define

$$\hat{\omega}(x_s, x_u) = J_{\hat{x}_0, x_u}(x_s, x_u),$$

for every $x = (x_s, x_u) \in N$. By Step 2, $\mu_{x_u} = \hat{\omega}(x_s, x_u)$, for any $x_u \in \hat{N}$.

The lift measure μ projects to $\hat{\mu} = \mu_u$, but the projection μ_s to N_s is given by

$$\mu_s = \mu_{\hat{x}_0} \int_{\hat{N}} \hat{\omega}(x_s, x_u)\ d\hat{\mu}.$$

Therefore

$$\mu = \omega(x_s, x_u) \mu_s \times \mu_u$$

where

$$\omega(x_s, x_u) = \frac{1}{\int_{\hat{N}} \hat{\omega}(x_s, x_u)\ d\hat{\mu}}\ \hat{\omega}(x_s, x_u).$$

As conditional probabilities in the proof of the last proposition vary continuously with the base point so the density function ω is continuous. Also, ω is bounded from zero and infinity. □

4. Suspending the bi-lateral shift by the flow

The saturation of N by the Lorenz flow f^t, by ergodicity of μ, has full measure in Λ since it is invariant and has positive measure. Now on, by Λ we mean this full measure subset. Henceforth a return time to N is defined as

$$T : N \to \mathbb{R}$$

$$T(x) = \sum_{j=0}^{r(x)-1} R(P^j(x)),$$

for any $x \in N$.

Lemma 3.1. *T is integrable with respect to μ.*

Proof. For almost every x,

$$\int T(x)\ dm = \int r(x)[\frac{1}{r(x)} \sum_{j=0}^{r(x)-1} R(P^j(x))]\ dm$$

converges to

$$\int r(x)(\int R\ dm)\ dm < +\infty$$

which implies

$$\int T\ dm < +\infty.$$

The proof is now completed by absolute continuity.

□

Now, corresponding to any linear cocycle A^t over Λ we take the linear cocycle A_f on N by

$$A_f(x) = A^{T(x)}(x),$$

for any $x \in N$.

Proposition 3.2. *Lyapunov spectrum of A^t is simple if and only if Lyapunov spectrum of A_f is simple.*

Proof. The Lyapunov exponents of A_f are obtained by multiplying those of A^t by the average return time

$$s_n(x) = \sum_{j=0}^{n-1} T(\hat{P}^j(x)), \; x \in N.$$

Indeed, given any non zero vector v,

$$\lim_{n\to+\infty} \frac{1}{n} \log ||A_f^n(x)v|| = \lim_{n\to+\infty} \frac{1}{n} \log ||A^{s_n(x)}(x)v||$$

which, for μ-almost every x, this is equal to

$$\lim_{n\to+\infty} \frac{1}{n} s_n(x) \lim_{m\to+\infty} \frac{1}{m} \log ||A^m(x)v||.$$

But $\frac{1}{n}s_n(x)$ converges to $\int T \; d\mu < +\infty$. The proof is now completed.

□

CHAPTER 4

Fiber Bunched Linear Cocycles

Note that if A^t is a Hölder continuous fiber bunched linear cocycle over Λ then, by definition,

$$||A_f(x)||\ ||A_f(x)^{-1}||\theta^{r(x)\eta} < \tau,$$

for any $x \in N$ where $0 < \theta(x) = \theta^{r(x)} \leq \theta < 1$.

In general, assume an induced differentiable structure on N and let A be an η-Hölder linear cocycle over f. We notice that all of the following results hold, up to appropriate adjustments, under weaker condition that the last definition expressed for some power > 1.

Notation 4.1. *Set*

$$\theta^n(x) = \theta(f^{n-1}(x))\ \ldots\ \theta(x),\ \ n \geq 1.$$

Lemma 4.1. *If A is fiber bunched then there exists some constant $C > 0$ such that*

$$\| A^n(y) \| \| A^n(z)^{-1} \| \theta^n(x)^\eta \leq C\tau^n,$$

for any $y, z \in W^s_{\mathrm{loc}}(x)$, and all $n \geq 1$.

Proof. Sub-multiplicativity of norms implies that

$$\| A^n(y) \| \| A^n(z)^{-1} \| \leq \prod_{j=0}^{n-1} \| A(f^j(y)) \|\ \| A(f^j(z))^{-1} \| .$$

By regularity of cocycle A, there is $C_1 > 0$ such that

$$\| A(f^j(y)) \| / \| A(f^j(x)) \| \leq \exp(C_1 \mathrm{d}(f^j(x), f^j(y))^\eta) \leq \exp(C_1 \theta^j(x)^\eta \mathrm{d}(x,y)^\eta).$$

It is similar for $\| A(f^j(z))^{-1} \| / \| A(f^j(x))^{-1} \|$. So, the right hand side in lemma is bounded above by

$$\exp[C_1 \sum_{j=0}^{n-1} \theta^j(x)^\eta (\mathrm{d}(x,y)^\eta + \mathrm{d}(x,z)^\eta)] \prod_{j=0}^{n-1} \| A(f^j(x)) \|\ \| A(f^j(x))^{-1} \| \theta^{n\eta}.$$

Since $\theta(x) < \theta < 1$, the first factor is bounded by some uniform constant $C > 0$, and fiber bunching implies that the second one is bounded by τ^n. The proof is now completed.

□

1. Holonomy maps

Set $H^n_{x,y} = A^n(y)^{-1} A^n(x)$.

Definition 4.1. *A cocycle A admits s-holonomy if*

$$H^s_{x,y} = \lim_{n\to+\infty} H^n_{x,y}$$

exists for any pair of points x, y in the same local stable set. u-holonomy is defined in a similar way, when $n \to -\infty$, for pairs of points in the same local unstable set.

Proposition 4.1. *If A is fiber bunched then, for all x and any $y \in W^s_{\text{loc}}(x)$, s-holonomy $H^s_{x,y}$ exists, where*
(a) $H^s_{x,y} = H^s_{z,y}.H^s_{x,z}$, for any $z \in W^s_{\text{loc}}(x)$, and $H^s_{y,x}.H^s_{x,y} = \text{id}$,
(b)$H^s_{f^j(x),f^j(y)} = A^j(y) \circ H^s_{x,y} \circ A^j(x)^{-1}$, for all $j \geq 1$.

Proof. We have

$$\| H^{n+1}_{x,y} - H^n_{x,y} \| \leq \| A^n(x)^{-1} \| \, \| A(f^n(x))^{-1}A(f^n(y)) - \text{id} \| \, \| A^n(y) \| .$$

By regularity of A, there is $C_2 > 0$ such that the middle factor is bounded by

$$C_2 \text{d}(f^n(x), f^n(y))^\eta \leq C_2 \theta^n(x)^\eta \text{d}(x,y)^\eta,$$

and hence, by the last lemma

$$\| H^{n+1}_{x,y} - H^n_{x,y} \| \leq CC_2\tau^n \text{d}(x,y)^\eta. \tag{2}$$

As $\tau < 1$, this implies that $H_n(x,y)$ is a Cauchy sequence, uniformly on x, y, and therefore, it is uniformly convergent. This proves the first part of proposition. (a) follows immediately from definition, and

$$A^n(f^j(y))^{-1}A^n(f^j(x)) = A^j(y)A^{n+j}(y)^{-1}A^{n+j}(x)A^j(x)^{-1} \tag{3}$$

proves (b). The proof is now completed.

□

Remark 4.1. *The s-holonomy $H^s_{x,y}$ vary continuously on (x,y) in the sense that the map*

$$(x,y) \to H^s_{x,y}$$

is continuous on $W^s_n = \{(x,y) : \ f^n(y) \in W^s_{\text{loc}}(x)\}$, for every $n \geq 0$. It is, in fact, a direct consequence of the uniform limit on (3) when $(x,y) \in W^s_0$, for instance. The general case $n > 0$ follows immediately, by (b) of the last proposition .

Indeed, as the constants $C, \bar{C}$ may be taken uniformly on $\mathcal{U}$, the Cauchy estimate in (3) is also locally uniform on A. Therefore, one may consider this notion of dependence:

$$(A,x,y) \to H^s_{A,x,y}$$

is continuous on $\mathcal{C}^{r,\rho}(M,d,\mathbb{C}) \times W^s_n$, for all $n \geq 0$.

There exist dual expressions of last results for u-holonomies, for points in $W^u_{\text{loc}}(x)$.

2. Dependence of holonomies

Notice that $\mathcal{C}^{r,\rho}(N,d,\mathbb{C})$ is the Banach space of all $C^{r,\rho}$ maps from N to the space of all $d \times d$ invertible matrices, and so the tangent space at each point $A \in \mathcal{C}^{r,\rho}(N,d,\mathbb{C})$ is naturally identified with that Banach space.

Recall fiber bunching is an open condition and the constants in Lemma 4.1 and Proposition 4.1 may be taken uniform on some neighborhood $\mathcal{U}$ of A when A is fiber bunched.

Proposition 4.2. *If A is fiber bunched then the map*

$$B \mapsto H^s_{B,x,y}$$

is of class C^1 on $\mathcal{U}$, for any $y \in W^s_{\text{loc}}(x)$, and

$$\partial_B H^s_{B,x,y}(\dot{B}) = \sum_{i=0}^{+\infty} B^i(y)^{-1}[H^s_{B,f^i(x),f^i(y)} B(f^i(x))^{-1}\dot{B}(f^i(x)) - B(f^i(y))^{-1}\dot{B}(f^i(y)))H^s_{B,f^i(x),f^i(y)}]B^i(x).$$

Proof. First, we show that the expression of $\partial_B H^s_{B,x,p}$ is well-defined. Let $i \geq 0$.

(4) $H^s_{B,f^i(x),f^i(y)} B(f^i(x))^{-1}\dot{B}(f^i(x)) - B(f^i(y))^{-1}\dot{B}(f^i(y))H^s_{B,f^i(x),f^i(y)}$

may be written as

$$(H^s_{B,f^i(x),f^i(y)} - \text{Id})B(f^i(x))^{-1}\dot{B}(f^i(x)) + B(f^i(y))^{-1}\dot{B}(f^i(y))(\text{Id} - H^s_{B,f^i(x),f^i(y)}) + B(f^i(x))^{-1}\dot{B}(f^i(x)) - B(f^i(y))^{-1}\dot{B}(f^i(y)).$$

By last proposition and Remark , there is some uniform $\bar{C} > o$ such that the first term is bounded by

$$\bar{C}\text{d}(f^i(x), f^i(y))^\eta \parallel B(f^i(x))^{-1} \parallel \parallel \dot{B}(f^i(x)) \parallel .$$

It is the same for second term. The third one is equal to

$$B(f^i(x))^{-1}[\dot{B}(f^i(x)) - \dot{B}(f^i(y))] + [B(f^i(x))^{-1} - B(f^i(y))^{-1}]\dot{B}(f^i(y)),$$

and since B^{-1} and $\dot{B}$ are Hï¿½lder continuous, using (3), it is bounded by

$$[||B^{-1}||_{0,0}\eta(\dot{B})+\eta(B^{-1})||\dot{B}||_{0,0}]\text{d}(f^i(x), f^i(y))^\eta \leq \parallel B^{-1} \parallel_{0,\eta} \parallel \dot{B} \parallel_{0,\eta} \text{d}(f^i(x), f^i(y))^\eta.$$

Hence (5) is bounded by

$$(2\bar{C} + 1)C_3 \parallel \dot{B} \parallel_{0,\eta} \text{d}(f^i(x), f^i(y))^\eta \leq (2\bar{C} + 1)C_3 \parallel \dot{B} \parallel_{0,\eta} \theta^i(x)^\eta \text{d}(x, y)^\eta$$

where $C_3 = \sup\{\parallel B^{-1} \parallel_{0,\eta},\ B \in \mathcal{U}\}$. So, the ith term in the expression of $\partial_B h^s_{B,y,z}(\dot{B})$ is bounded by

(5) $C_4 \parallel \dot{B} \parallel_{0,\eta} \theta^i(x)^\eta \text{d}(x, y)^\eta ||B^i(p)^{-1}||||B^i(x)|| \leq C_4\tau^i \text{d}(x, y)^\eta \parallel \dot{B} \parallel_{0,\eta}$,

by fiber bunching hypothesis where $C_4 = (2\bar{C} + 1)C_3$. Therefore, as $\tau < 1$, the series (5) does converge, uniformly.

Now, we derivate $H^s_{B,x,y}$. By definition, $H^s_{B,x,y}$ is the uniform limit of $H^n_{B,x,y} = B^n(y)^{-1}B^n(x)$ when $n \to \infty$. Indeed, $H^n_{B,x,y}$ is a differentiable function of B with derivative $\partial_B H^n_{B,x,y}(\dot{B})$ equal to

$$\sum_{i=0}^{n-1} B^i(y)^{-1}[H^{n-i}_{B,f^i(x),f^i(y)} B(f^i(x))^{-1}\dot{B}(f^i(x)) - B(f^i(y))^{-1}\dot{B}(f^i(y))H^{n-i}_{B,f^i(x),f^i(y)}]B^i(x),$$

for all $\dot{B} \in T_B\mathcal{C}^{r,\rho}(N, d, \mathbb{C})$ and any $n \geq 1$.

It suffices to show that $\partial_B H^n_{B,x,y}$ converges uniformly to $\partial_B H^s_{B,x,y}$ as $n \to \infty$. By (2), for any $\tau_0 \in (\tau, 1)$,

$$\parallel H^s_{B,x,y} - H^n_{B,x,y} \parallel \leq CC_2 \sum_{i=n}^{\infty} \tau^i \text{d}(x, y)^\eta \leq C_5\tau^n \text{d}(x, y)^\eta$$

$$\leq C_5\tau_0^n \mathrm{d}(x,y)^\eta,$$

for some uniform constant $C_5 > 0$. Then, for all $0 \leq i \leq n$,

$$\| H^s_{B,f^i(x),f^i(y)} - H^{n-i}_{B,f^i(x),f^i(y)} \| \leq C_5\tau_0^{(n-i)}\mathrm{d}(f^i(x),f^i(y))^\eta$$
$$\leq C_5\tau_0^{(n-i)}\theta^i(x)^\eta \mathrm{d}(x,y)^\eta.$$

It follows, by Lemma 4.1, that the difference between the ith terms in the expressions of $\partial_B H^s_{B,x,y}$ and $\partial_B H^n_{B,x,y}$ is bounded by

$$2C_3C_5\tau_0^{n-i}\theta^i(x)^\eta \mathrm{d}(x,y)^\eta ||B^i(y)^{-1}||||B^i(x)|| \leq 2C_3C_5\tau_0^{n-i}\tau^i\mathrm{d}(x,y)^\eta.$$

Combining with , $\| \partial_B H^s_{B,x,y} - \partial_B H^n_{B,x,y} \|$ is bounded by

$$\{2C_3C_5\tau_0^n \sum_{i=0}^{n-1}(\tau_0^{-1}\tau)^i + C_4\sum_{i=n}^{+\infty}\tau^i\}\mathrm{d}(x,y)^\eta \| \dot{B} \|_{0,\eta} .$$

Since τ, τ_0 and $(\tau_0^{-1}\tau)$ are strictly less that 1, therefore the series tends uniformly to 0 as $n \to \infty$. The proof is now completed.

□

There exists the dual of the last proposition

Proposition 4.3. *If A is fiber bunched then*

$$\mathcal{U} \ni B \mapsto H^u_{B,x,y}$$

is of class C^1, and, for any $y \in W^u_{\mathrm{loc}}(x)$,

$$\partial_B H^u_{B,x,y}(\dot{B}) = -\sum_{i=1}^{+\infty} B^{-i}(y)^{-1}[H^u_{B,f^{-i}(x),f^{-i}(y)}B(f^{-i}(x))^{-1}\dot{B}(f^{-i}(x))$$
$$-B(f^{-i}(y))^{-1}\dot{B}(f^{-i}(y))H^u_{B,f^{-i}(x),f^{-i}(y)}]B^{-i}(x).$$

CHAPTER 5

Perturbation Tools

Consider the ergodic complete shift system (f, μ) where μ has product structure and let A be a linear cocycle over f. Suppose that p is a periodic point of f, and q a homoclinc point of p, i.e. $q \in W^u_{\text{loc}}(p)$ and there is some multiple $m \geq 1$ of $\text{per}(p)$ such that $f^m(q) \in W^s_{\text{loc}}(p)$. We define the *transition map*

$$\Psi_{A,p,q} : \mathbb{C}^d_p \to \mathbb{C}^d_p$$

by

$$\Psi_{A,p,q} = H^s_{f^m(q),p} A^m(q) H^u_{p,q} \in \text{GL}(d, \mathbb{C}).$$

Definition 5.1. *A is pinching at p if all eigenvalues of $A^{\text{per}(p)}(p)$ have distinct absolute values. A is twisting at p, q if, for any pair of invariant subspaces E_1, E_2 of $A^{\text{per}(p)}(p)$ with $\dim E_1 + \dim E_2 = d$,*

$$\Psi_{A,p,q}(E_1) \cap E_2 = \{\mathbf{0}\}.$$

A cocycle A is simple if there exist some periodic point p and some homoclinic point q of p such that A is both pinching at p and twisting at p, q.

Theorem 5.1. [5] *If A is simple then the Lyapunov spectrum of A is simple.*

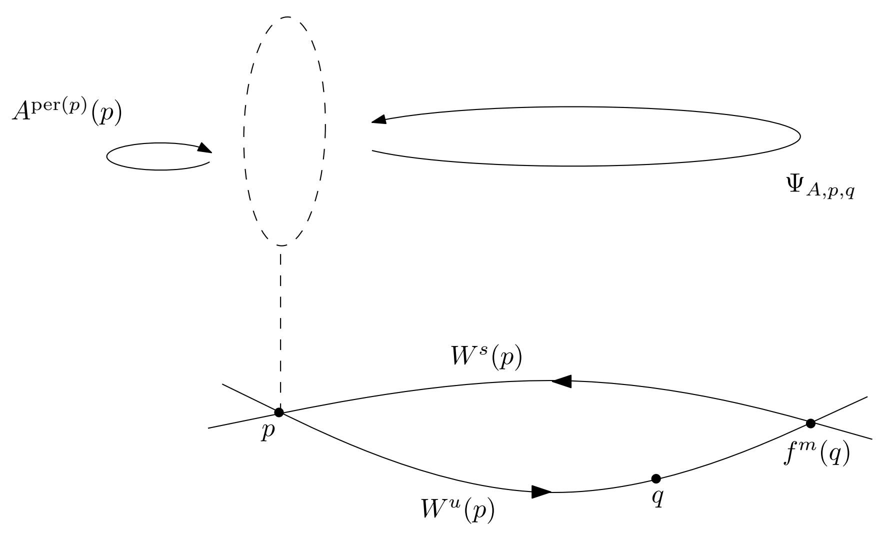

FIGURE 5.1. Pinching and twisting

We remember that a submersion $\mathcal{S}_1 \to \mathcal{S}_2$ means that all elements of $\mathcal{S}_2$ are regular values which implies that every non-empty pre-image of any element of $\mathcal{S}_2$

is a submanifold of $\mathcal{S}_1$ with codimension equal to the dimension of $\mathcal{S}_2$. In a more general case, the pre-image of any submanifold of $\mathcal{S}_2$ is a submanifold of $\mathcal{S}_1$ with the same codimension.

1. Perturbation along periodic orbits

As we mentioned before, the tangent space at any $B \in \mathcal{C}^{r,\rho}(N,d,\mathbb{C})$ is identified naturally with the space of all $C^{r,\rho}$ maps on N into the space of linear maps in $\mathbb{C}^d$. Indeed, we may give the tangent vectors $\dot{B}$ as $C^{r,\rho}$ maps which assign to every point of N a linear map on $\mathbb{C}^d$.

Proposition 5.1. *Let p be a periodic point of f then the application*

$$A \mapsto A^{\mathrm{per}(p)}(p) \in \mathrm{GL}(d,\mathbb{C}),$$

is a submersion at any $A \in \mathcal{C}^{r,\rho}(N,d,\mathbb{C})$, even restricted to tangent vectors supported in some neighborhood of p.

Proof. Assume that p is a fixed point of f. It is easy to see that

$$\partial_A A(p)(\dot{A}) = \dot{A}(p).$$

Fix a neighborhood U_p of p such that p is the unique point of its orbit in U_p. Let $\alpha : N \to [0,1]$ be a $C^{r,\rho}$ function vanishing outside U_p, and $\alpha(p) = 1$. For any $\mathcal{A} \in \mathrm{GL}(d,\mathbb{C})$, define $\dot{\mathcal{A}} \in T_B\mathcal{C}^{r,\rho}(N,d,\mathbb{C})$ as

$$\dot{\mathcal{A}}(x) = \mathcal{A}A(p)^{-1}\alpha(x)A(x).$$

Note that $\dot{\mathcal{A}}$ is supported on U_p, and $\dot{\mathcal{A}}(p) = \mathcal{A}$. Hence $\partial_A A(p)(\dot{\mathcal{A}}) = \mathcal{A}$, as we have claimed. It is similar when $\mathrm{per}(p) > 1$ where in this case

$$\partial_A A^{\mathrm{per}(p)}(p)(\dot{A}) = A(f^{\mathrm{per}(p)-1}(p)) \ \dots\ \dot{A}(p) + \ \dots\ + \dot{A}(f^{\mathrm{per}(p)-1}(p)) \ \dots\ A(p)$$

which, for tangent vectors supported on U_p, reduces to

$$A(f^{\mathrm{per}(p)-1}(p)) \ \dots\ \dot{A}(p).$$

The proof is now completed.

□

2. Perturbation along homoclinic orbits

Assume that p is a periodic point of f and q some homoclinic point of p. The derivative of $\Psi_{B,p,q} = H^s_{f^m(q),p}.B^m(q).H^u_{p,q}$ at a vector $\dot{B}$ is given by

$$\begin{aligned}&\partial_B H^s_{f^m(q),p}(\dot{B}).B^m(q).H^u_{p,q} +\\ &H^s_{f^m(q),p}\partial_B B^m(q)(\dot{B})H^u_{p,q} +\\ &H^s_{f^m(q),p}B^m(q)\partial_B H^u_{p,q}(\dot{B})\end{aligned} \tag{6}$$

where

$$\partial_B B^m(q)(\dot{B}) = B(f^{m-1}(q)) \ \dots\ \dot{B}(q) + \ \dots\ + \dot{B}(f^{m-1}(q)) \ \dots\ B(q),$$

by definition.

Proposition 5.2. *The application*

$$\mathcal{U} \ni B \mapsto \Psi_{B,p,q}$$

is a submersion, even restricted to tangent vectors $\dot{B}$ supported on a neighborhood of q, for any periodic point p and each homoclinic pint q of p.

Proof. Without loose of generality, we assume that p is a fixed point of f, and $m = 1$. Let U_q be any neighborhood of q which is disjoint from the orbit of p and $\{f^j(q) : \; j \neq 0\}$. So, the expression in (6) reduces to

$$H^s_{f(q),p}\partial_B B(q)(\dot{B})H^u_{p,q} = H^s_{f(q),p}\dot{B}(q)H^u_{p,q}.$$

Thus, $\partial_B \Psi_{B,p,q}$ is given by

$$\dot{B} \mapsto H^s_{f(q),p}\dot{B}(q)H^u_{p,q},$$

for any vector $\dot{B}$ supported on U_q. We claim that

$$\Phi(\dot{B}) = H^s_{f(q),p}\dot{B}(q)H^u_{p,q}$$

is surjective on $T_B\mathcal{C}^{r,\rho}(M, d, \mathbb{C})$.

Let $\beta : N \to [0, 1]$ be a $C^{r,\rho}$ function vanishing outside U_q, where $\beta(q) = 1$. For any $\mathcal{B} \in \mathrm{GL}(d, \mathbb{C})$, define $\dot{\mathcal{B}} \in T_B\mathcal{C}^{r,\rho}(M, d, \mathbb{C})$ as

$$\dot{\mathcal{B}}(w) = (H^s_{B,f(q),p})^{-1}\mathcal{B}B(q)^{-1}\beta(w)B(w)(H^u_{B,p,q})^{-1}.$$

Note that $\dot{\mathcal{B}}(q) = {H^s_{B,f(q),p}}^{-1}\mathcal{B}{H^u_{B,p,q}}^{-1}$, and so $\Phi(\dot{\mathcal{B}}) = \mathcal{B}$, as we have claimed. The proof is now completed. □

3. The main perturbation

Now, we consider the main perturbation including both periodic and homoclinic orbits.

Proposition 5.3. *If $A \in \mathcal{C}^{r,\rho}(N, d, \mathbb{C})$ is fiber bunched then the application*

$$\Theta : \mathcal{U} \to \mathrm{GL}(d, \mathbb{C})^2$$

$$\Theta(B) = (B(p), \Psi_{B,p,q}),$$

$B \in \mathcal{U}$, is a submersion, even restricted to the subspace of tangent vectors $\dot{B}$ supported on some neighborhoods of p, q.

Proof. Take U_p such that $U_p \cap \mathrm{orb}(p) = \{p\}$, $U_p \cap \mathrm{orb}(q) = \emptyset$, and similarly U_q so that $U_q \cap \mathrm{orb}(q) = \{q\}$, $U_q \cap \mathrm{orb}(p) = \emptyset$.

First note that, if $\dot{B}$ is a tangent vector supported on $U_p \cup U_q$, so, there exist two tangent vectors $\dot{B}_1$ supported on U_p, and $\dot{B}_2$ supported on U_q such that $\dot{B} = \dot{B}_1 + \dot{B}_2$. Indeed, we may assume that

$$\dot{B}_1(x) = \begin{cases} \dot{B}(x) & x \in U_P \\ \mathbf{0} & x \notin U_P \end{cases}$$

and

$$\dot{B}_2(x) = \begin{cases} \dot{B}(x) & x \in U_q \\ \mathbf{0} & x \notin U_q. \end{cases}$$

So

$$\partial_B\Theta(B)(\dot{B}) = \partial_B\Theta(B)(\dot{B}_1) + \partial_B\Theta(B)(\dot{B}_2)$$

which is equal to

$$(\partial_B B(p)(\dot{B}_1), \partial_B\Psi_{B,p,q}(\dot{B}_1)) + (\partial_B B(\dot{B}_2), \partial_B\Psi_{B,p,q}(\dot{B}_2)) =$$

$$(\dot{B}_1(p), \partial_B\Psi_{B,p,q}(\dot{B}_1)) + (\mathbf{0}, \partial_B\Psi_{B,p,q}(\dot{B}_2))$$

By Propositions 5.1 and 5.2, for any $(\mathcal{B}_1,\mathcal{B}_2)\in \mathrm{GL}(d,\mathbb{C})^2$, there exist tangent vectors $\dot{\mathcal{B}}_1$ supported on U_p, and then $\dot{\mathcal{B}}_2$ supported on U_q such that

$$\partial_B \Psi_{B,p,q}(\dot{\mathcal{B}}_2)=\mathcal{B}_2-\partial_B\Psi_{B,p,q}(\dot{\mathcal{B}}_1),$$

and therefore

$$\partial_B\Theta(B)(\dot{\mathcal{B}})=(\mathcal{B}_1,\mathcal{B}_2)$$

where $\dot{\mathcal{B}}=\dot{\mathcal{B}}_1+\dot{\mathcal{B}}_2$ is supported on $U_p\cup U_q$.

□

4. Proof of Theorem 1

4.1. Pinching. Let Z be the subset of matrices $A\in \mathrm{GL}(d,\mathbb{C})$ whose eigenvalues are not all distinct in norm. Z is closed and contained in a finite union of closed submanifolds of $\mathrm{GL}(d,\mathbb{C})$ with codimention ≥ 1.

Proposition 5.1 follows that the subset of cocycles $B\in\mathcal{U}$ for which $B^{\mathrm{per}(p)}(p)\in Z$ is closed and contained in a finite union of closed submanifolds with codimention ≥ 1.

For any $l\geq 1$, consider periodic points $p_1,...,p_l$. We imply that the subset of linear cocycles $B\in\mathcal{U}$ where $B^{\mathrm{per}(p_i)}(p_i)\in Z$ is closed and contained in a finite union of closed submanifolds with codimention $\geq l$.

4.2. Twisting. The subset Y of all pairs of matrices (A,B) such that there exist B-invariant subspaces E_1,E_2 with $\dim E_1+\dim E_2=d$ where $A(E_1)\cap E_2\neq\{\mathbf{0}\}$, is closed and contained in a finite union of closed submanifolds of positive codimention. Indeed, Fixing E_1,E_2, the application

$$\mathrm{GL}(d,\mathbb{C})\ni A\mapsto A(E_1)\in \mathrm{Grass}(\dim E_1,d)$$

is a submersion. In the other hand,

$$\{A:\ A(E_1)\text{ do not intersect transversally }E_2\}$$

is a submanifold with codimension≥ 1, since

$$\{E\in \mathrm{Grass}(\dim E_1,d):\ E\text{ do not intersect transversally }E_2\}$$

is a submanifold of positive codimension. Now, for any fixed matrix B, the set Y is contained in a finite number of submanifolds of positive codimension. So, Y is contained in a finite number of submanifolds of positive codimension in $\mathrm{GL}(\mathbb{C},d)^2$.

Therefore, by Proposition 5.3, the subset of cocycles $B\in\mathcal{U}$ so that

$$(B^{\mathrm{per}(p)}(p),\Psi_{B,p,q})\in Y$$

is closed and contained in a finite union of closed submanifolds with positive codimension.

Given $l\geq 1$, if $q_1,...,q_l$ are some homoclinic points of periodic points $p_1,...,p_l$, respectively, then the subset of cocycles $B\in\mathcal{U}$ for which

$$(B^{\mathrm{per}(p_i)}(p_i),\Psi_{B,p_i,q_i})\in Y$$

is closed and contained in a finite union of closed submanifolds with codimension $\geq l$.

4.3. Real valued cocycles. All results in [5] and Perturbation arguments of this section are valid for cocycles with values in $\mathrm{GL}(d,\mathbb{R})$. But, in this case there is the possibility of existence of pairs of complex conjugate eigenvalues. Indeed, the subset of matrices whose eigenvalues are not all distinct in norm has non-empty interior in $\mathrm{GL}(d,\mathbb{R})$.

The way to hypass this, is treated in [7] and [8]:
Excluding a codimension 1 subset of cocycles, one may assume that
(i) all the eigenvalues of $B^{\mathrm{per}(p)}(p)$ are real and have distinct norms, except for $c \geq 0$ pairs of complex conjugate eigenvalues,
(ii) $\Psi_{A,p,q}(E_1) \cap E_2 = \{\mathbf{0}\}$, for any direct sums E_1 and E_2 of eigenspaces of $B^{\mathrm{per}(p)}(p)$ with $\dim E_1 + \dim E_2 \leq d$.

Avoiding another subset of positive codimention, we can choose a new periodic point $\hat{p}$ so that all the eigenvalues of $B^{\mathrm{per}(\hat{p})}(\hat{p})$ are real and distinct.

Now, in this way, for any $l \geq 1$, avoiding a codimension l subset of cocycles, one may suppose that periodic points $\hat{p}_1, ..., \hat{p}_l$ are defined.

The proof of Theorem 1 is now completed.

5. Proof of Theorem 2

Let A^t be a linear cocycle over Λ. We define a neighborhood $\mathcal{V}$ of A^t as the subset of all cocycles B^t over Λ for which $B_f \in \mathcal{U}$.

Proposition 5.4. *The application*

$$\mathcal{V} \ni B^t \mapsto B_f \in \mathcal{U}$$

is a submersion.

Proof. By definition,

$$\partial_{B^t} B_f(\dot{B}_t) = \dot{B}_f.$$

Let $\dot{B} \in \mathcal{C}^{r,\rho}(N,d,\mathbb{C})$. Then the suspension $\dot{B}^t$ of $\dot{B}$ is defined by

$$\dot{B}^t(X^s(x)) = (\mathrm{id}, t+s),\ 0 < t+s \leq T(x),$$

identifying $(\mathrm{id}, T(x))$ with $(\dot{B}(x), 0)$, for any $x \in N$, setting $\dot{B}^0 = \mathrm{id}$. $\dot{B}^t$ is an η-Hölder linear cocycle over Λ for which $\dot{B}_f(x) = (\dot{B}(x), 0)$. This shows that the derivative is surjective. The proof is now completed. ☐

The proof of Theorem 2 is now completed.

CHAPTER 6

Final Remarks

1. Non-vanishing exponents

Bonatti, Gomez-Mont and Viana in [7] proved that typical Hölder continuous fiber bunched linear cocycles over any hyperbolic transformation have some nonzero Lyapunov exponent. Viana in [18] improved the last result, in particular, removing the fiber bunching condition. Therefore, by Proposition 5.4 we have also proved

Theorem 6.1. *Typical Hölder continuous linear cocycles over a Lorenz attractor have non-zero Lyapunov exponents.*

2. Singular-hyperbolic attractors

[13] introduce a notion of singular hyperbolic sets and flows that includes the geometric Lorenz models as special cases. They prove that every robust attractor for a flow in dimension 3 is singular hyperbolic.

Singular hyperbolicity has many important consequences, for instance, [3] prove that singular hyperbolic flows admit convenient cross sections and invariant foliations for the corresponding Poincaré transformation that allow us to reduce the dynamics to dimension 1. Moreover, by [2] every singular hyperbolic attractor admits a Markov structure.

It would be interesting to know whether our results extend to this class of systems:

Problem 1. *Typical Hölder continuous fiber bunched linear cocycles over a singular-hyperbolic attractor are simple?*

and then removing the fiber bunching condition

Problem 2. *Typical Hölder continuous linear cocycles over a singular-hyperbolic attractor have non-zero Lyapunov exponents?*

Bibliography

[1] V. Araujo and M. Pacifico, Three dimensional flows, Springer-Verlag (2010).

[2] A. Arroyo, E. Pujals, Dynamical properties of singular-hyperbolic attractors, Discrete and Continuous Dynamical Systems 19 (2007) 67–87.

[3] V. Araujo, M. Pacifico, E. Pujals and M. Viana, Singular-hyperbolic attractors, Transactions of the American Mathematical Society 361 (2009) 2431-2485.

[4] A. Avila, J. Santamaria and M. Viana, Cocycles over partially hyperbolic maps, preprint IMPA.

[5] A. Avila and M. Viana, Simplicity of Lyapunov spectra: a sufficient condition, Potugaliae Matematica 64 (2007) 311-376.

[6] C. Bonatti, L. Diaz and M. Viana, Dynamics Beyond Uniform Hyperbolicity: A Global Geometric and Probabilistic Perspective, Encyclopedia of Mathematical Sciences 102 Springer-Verlag (2004).

[7] C. Bonatti, X. Gomez-Mont and M. Viana, Généreicité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices, Annales de l'Institut Henri Poincaré 20 (2003) 579-624.

[8] C. Bonatti and M. Viana, Lyapunov exponents with multiplicity 1 for deterministic products of matrices, Ergodic Theory and Dynamical Systems 24 (2004) 1295-1330.

[9] K. Díaz-Ordaz, Decay of correlation for non-Hölder observables for one-dimensional expanding Lorenz-like maps, Discrete and Continuous Dynamical Systems 15 (2006) 159-176.

[10] H. Furstenberg, Non-commuting random products, Transactions of the American Mathematical Society 108 (1963) 377-428.

[11] J. Guckenheimer and R. Williams, Structural stability of Lorenz attractors, Publications Mathématiques de l'IHÉS 50 (1979) 59-72.

[12] E. Lorenz, Deterministic non periodic flow, Journal of the Atmospheric Sciences, 20 (1963) 130-141.

[13] C. Morales, M. Pacifico and E. Pujals, Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers, Annals of Mathematics 160 (2004) 357-432.

[14] V. Oseledets, A multiplicative ergodic theorem, Transactions of the Moscow Mathematical Society 19 (1968) 197-231.

[15] W. Tuker, The Lorenz attractor exists, Comptes Rendus de l'AcadÃ©mie des Sciences Paris. Série I. Mathématique 328 (1999) 1197-1202.

[16] M. Viana, Stochastic dynamics of deterministic systems, Brazilian Mathematics Colloquium IMPA (1997).

[17] M. Viana, What's new on Lorenz strange attractors?, The Mathematical Intelligencer 22 (2000), 6-19

[18] M. Viana, Almost all cocycles over any hyperbolic system have non-vanishing Lyapunov exponents, Annals of Mathematics 167 (2008) 643-680.

[19] R. Williams, The structure of the Lorenz attractor, Publications Mathématiques de l'IHÉS 50 (1979) 73-99.