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RIGIDITY OF AREA-MINIMIZING HYPERBOLIC SURFACES IN THREE-MANIFOLDS

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Valeu a pena? Tudo vale a pena
Se a alma não é pequena.
Quem quere passar além do Bojador
Tem que passar além da dor.
Deus ao mar o perigo e o abismo deu,
mas foi nele que espelhou o céu.
— Trecho de *Mar Português*, Fernando Pessoa.

Abstract

If M is a three-manifold with scalar curvature greater than or equal to -2 and $\Sigma \subset M$ is a two-sided compact embedded Riemann surface of genus greater than 1 which is locally area-minimizing, then the area of Σ is greater than or equal to $4\pi(g(\Sigma) - 1)$, where $g(\Sigma)$ denotes the genus of Σ. In the equality case, we prove that the induced metric on Σ has constant Gauss curvature equal to -1 and locally M splits along Σ. We also obtain a rigidity result for cylinders $(I \times \Sigma, dt^2 + g_\Sigma)$, where $I = [a, b] \subset \mathbb{R}$ and g_Σ is a Riemannian metric on Σ with constant Gauss curvature equal to -1.

Keywords: Minimal surfaces, constant mean curvature surfaces, scalar curvature, rigidity.

Resumo

Se M é uma variedade tridimensional com curvatura escalar maior ou igual a -2 e $\Sigma \subset M$ é uma superfície de Riemann compacta, mergulhada, com dois lados e gênero maior que 1 que é localmente minimizante de área, então a área de Σ é maior ou igual a $4\pi(g(\Sigma) - 1)$, onde $g(\Sigma)$ denota o gênero de Σ. No caso de igualdade, provamos que a métrica induzida sobre Σ tem curvatura de Gauss constante igual a -1 e localmente M é isométrica a um cilindro sobre Σ. Obtemos também um resultado de rigidez para cilindros $(I \times \Sigma, dt^2 + g_\Sigma)$, onde $I = [a, b] \subset \mathbb{R}$ e g_Σ é um métrica Riemanniana sobre Σ com curvatura de Gauss constante igual a -1.

Palavras-chave: Superfícies mínimas, superfícies com curvatura média constante, curvatura escalar, rigidez.

Contents

Introduction

In this work, we will deal with the relation between minimal surfaces of a Riemannian three-manifold M and the scalar curvature of M. The link between these two concepts is the second variation formula of area. If the minimal surface is stable then this formula provides, using the Gauss equation, a connection between the topology of the stable minimal surface and the scalar curvature of M. This was first observed in [32] by R. Schoen and S. T. Yau. In that paper, they proved the following result.

Recall that a surface Σ is incompressible in a three-manifold M if the fundamental group of Σ injects into that of M.

Theorem 1 (R. Schoen, S. T. Yau)**.** *Let (M^3, g) be a compact orientable three-manifold with nonnegative scalar curvature. If M contains an incompressible compact orientable surface Σ with genus greater than or equal to 1, then M is flat.*

To prove that result, they first show that any such manifold M contains a stable minimal surface of genus equal to that of Σ. Next, they observe, using the second variation formula of area, that if M has positive scalar curvature, then every compact stable minimal surface in M is a two-sphere. The result follows because if M admits a non-flat metric of nonnegative scalar curvature, then M also admits a metric of positive scalar curvature (see [21]).

Remark 1. A very nice consequence of the theorem above is the fact that any metric on the three-torus T^3 with nonnegative scalar curvature is flat. In [33], R. Schoen and S. T. Yau extended this result to dimension $n < 8$. The general case was settled by M. Gromov and H. B. Lawson using spin techniques (see [15, 16]).

It was also observed in [32] that if M is a Riemannian three-manifold with nonnegative scalar curvature and $\Sigma \subset M$ is a stable minimal two-torus then Σ is flat and totally geodesic. Moreover, the normal Ricci curvature and the scalar curvature of M are equal to zero along Σ.

Motivated by the infinitesimal rigidity above, D. Fischer-Colbrie and R. Schoen conjectured in [13] that in the Schoen and Yau's theorem above it is sufficient that M contains an area-minimizing two-torus (not necessarily incompressible). This conjecture was proved in [9], by M. Cai and G. Galloway. They proved that if M has nonnegative scalar curvature and $\Sigma \subset M$ is a two-sided embedded two-torus which is area-minimizing in its isotopy class, then M is flat. This result is obtained as a corollary of the following local statement.

Theorem 2 (M. Cai, G. Galloway)**.** *Let (M^3, g) be a three-manifold with nonnegative scalar curvature. If $\Sigma \subset M$ is a two-sided embedded two-torus in M which is locally area-minimizing, then M is flat in a neighborhood of Σ.*

It follows that the induced metric on Σ is flat and that locally M splits along Σ. The proof of Theorem 2 uses an argument based on a local deformation around Σ to obtain a metric with positive scalar curvature together with the fact that the three-torus does not admit a metric with positive scalar curvature.

The Cai and Galloway's result above is an example of how the existence of an area-minimizing surface in a three-manifold M with lower bounded scalar curvature can influence the geometry of M.

Remark 2. We note that there are similar rigidity results in higher dimensions. More precisely, it was proved in [33], by R. Schoen and S. T. Yau, that if (M^n, g) is a Riemannian manifold of dimension $n \geqslant 4$ with scalar curvature $R_g \geqslant 0$ and $\Sigma^{n-1} \subset M$ is a compact two-sided stable minimal hypersurface, then either Σ admits a conformal metric with positive scalar curvature or Σ is Ricci flat and totally geodesic. In the case where Σ is Ricci flat and totally geodesic, it was proved by M. Cai in [8] that if Σ is locally volume-minimizing, then locally M splits along Σ. We also note that rigidity results for complete non-compact stable minimal hypersurfaces in complete manifolds with nonnegative seccional curvature was obtained in [34].

Recently, H. Bray, S. Brendle and A. Neves studied in [4] the case where M has scalar curvature greater than or equal to 2 and $\Sigma \subset M$ is a locally area-minimizing embedded two-sphere. In their case, the model is the Riemannian manifold $(\mathbb{R} \times S^2, dt^2 + g)$, where g is the standard metric on S^2 with constant Gauss curvature equal to 1. They proved the following result.

Theorem 3 (H. Bray, S. Brendle, A. Neves). *Let (M^3, g) be a three-manifold with scalar curvature $R_g \geqslant 2$. If Σ is an embedded two-sphere which is locally area-minimizing, then Σ has area less than or equal to 4π. Moreover, if equality holds, then Σ with the induced metric has constant Gauss curvature equal to 1 and locally M splits along Σ.*

The proof in [4] is based on a construction of a one-parameter family of constant mean curvature two-spheres. A global result was also obtained using the local one above. More precisely, it was proved that if Σ is area-minimizing in its homotopy class and has area equal to 4π, then the universal cover of M is isometric to $(\mathbb{R} \times S^2, dt^2 + g)$. A similar rigidity result for area-minimizing projective planes was obtained in [3].

Remark 3. The following heuristic argument[1] using the fact that the Hawking mass is a non-decreasing quantity along the inverse mean curvature flow, is interesting since it indicates the rigidity in Theorem 3. If $\Sigma \subset (M^3, g)$ is a surface and $R_g \geqslant \Lambda$, $\Lambda \in \mathbb{R}$, then the Hawking mass of Σ, denoted by $m_H(\Sigma)$, is defined to be

$$m_H(\Sigma) = |\Sigma|^{1/2}\Big(8\pi\chi(\Sigma) - \int_\Sigma (H^2 + \frac{2}{3}\Lambda)\, d\sigma\Big),$$

where H is the mean curvature of Σ and $\chi(\Sigma)$ denotes the Euler characteristic of Σ.

Now, if $\Lambda = 2$ and Σ is a locally area-minimizing two-sphere with area equal to 4π, then Σ attains the maximum possible value of the Hawking mass. Suppose we have a family of two-spheres $\Sigma_t \subset M$, $\Sigma_0 = \Sigma$, that solves the inverse mean curvature flow. It is well known that $m_H(\Sigma_t)$ is non-decreasing along the flow. Since $m_H(\Sigma)$ is the maximum of the Hawking mass, we have that $m_H(\Sigma_t) = m_H(\Sigma)$ for all t and consequently, all two-spheres Σ_t are minimal and have area equal to 4π.

The next natural question is to know what happens when the model case is the Riemannian product manifold $(\mathbb{R} \times \Sigma, dt^2 + g_\Sigma)$, where Σ is a Riemann surface of genus greater than 1 and g_Σ is a Riemannian metric on Σ with constant Gauss curvature equal to -1.

In the present work, we deal with this question. We prove that the analogous result is true in this case. The first theorem of this work is stated below.

Theorem A. *Let (M^3, g) be a Riemannian manifold with scalar curvature $R_g \geqslant -2$. If $\Sigma \subset M$ is a two-sided compact embedded Riemann surface of*

[1]The author would like to thank A. Neves for pointing out this heuristic argument.

genus $g(\Sigma) \geqslant 2$ *which is locally area-minimizing, then*

$$|\Sigma|_g \geqslant 4\pi(g(\Sigma) - 1)$$

where $|\Sigma|_g$ *denotes the area of* Σ *with respect to the induced metric. Moreover, if equality holds, then the induced metric on* Σ*, denoted by* g_Σ*, has constant Gauss curvature equal to* -1 *and* Σ *has a neighborhood which is isometric to* $((-\epsilon, \epsilon) \times \Sigma, dt^2 + g_\Sigma)$*, for some* $\epsilon > 0$*. More precisely, the isometry is given by* $f(t, x) = \exp_x(t\nu(x))$*,* $(t, x) \in (-\epsilon, \epsilon) \times \Sigma$*, where* ν *is the unit normal vector field along* Σ*.*

Remark 4. Note that if $|\Sigma| = 4\pi(g(\Sigma) - 1)$ in Theorem A, then $m_H(\Sigma)$ is the minimum (not the maximum) possible value of the Hawking mass for minimal surfaces of genus equal to $g(\Sigma)$ in three-manifolds with scalar curvature bounded below by $\Lambda = -2$. It is interesting that rigidity still holds despite the failure of the heuristic argument of Remark 3.

We note that a related rigidity result for constant mean curvature surfaces of genus 1 was obtained in [1]. We also refer the reader to the excellent surveys [5] and [14] on rigidity problems associated to scalar curvature.

Let us give an idea of the proof of Theorem A. The area estimate follows from the second variation of area using the Gauss equation, the lower bound of the scalar curvature and the Gauss-Bonnet theorem. In the equality case, we construct, using the implicit function theorem, a one-parameter family of constant mean curvature surfaces, denoted by Σ_t, with $\Sigma_0 = \Sigma$ and all Σ_t having the same genus. The next argument in the proof is the fundamental one. Arguing by contradiction and using the solution of the Yamabe problem for compact manifolds with boundary and the Hopf's maximum principle, we are able to conclude that each Σ_t has the same area. Finally, we obtain from this that Σ has a neighborhood isometric to $((-\epsilon, \epsilon) \times \Sigma, dt^2 + g_\Sigma)$.

If we suppose that Σ minimizes area in its homotopy class, then we obtain global rigidity using a standard continuation argument contained in [4, 9].

Corollary 1. *Let* (M^3, g) *be a complete Riemannian three-manifold with scalar curvature* $R_g \geqslant -2$*. Moreover, suppose that* $\Sigma \subset M$ *is a two-sided compact embedded Riemann surface of genus* $g(\Sigma) \geqslant 2$ *which minimizes area in its homotopy class. Then* Σ *has area greater than or equal to* $4\pi(g(\Sigma) - 1)$ *and if equality holds, then* $(\mathbb{R} \times \Sigma, dt^2 + g_\Sigma)$ *is an isometric covering of* (M^3, g)*, where* g_Σ *is the induced metric on* Σ *which has constant Gauss curvature equal to* -1*. The covering is given by* $f(t, x) = \exp_x(t\nu(x))$*,* $(t, x) \in \mathbb{R} \times \Sigma$*, where* ν *is the unit normal vector along* Σ*.*

Remark 5. We observe that lower bounds for scalar curvature do not imply area estimates like those of Theorem 3 and Theorem A in higher dimensions.

This follows from the fact that for dimensions $n \geqslant 3$ there exist compact Riemannian manifolds (M^n, g) with positive (negative) scalar curvature with volume arbitrarily large (small). To see this, given $n \geqslant 3$ take any compact Riemannian manifold (N^{n-1}, g_N) with constant positive or negative scalar curvature and consider the Riemannian manifold $(M^n = N^{n-1} \times S^1(r), g = g_N + d\theta^2)$, where $(S^1(r), d\theta^2)$ is the circle of radius $r > 0$ in $\mathbb{R}^2$ and $d\theta^2$ is the canonical metric on $S^1(r)$. Note that for any $r > 0$ the scalar curvature of M is always equal to that of N. Finally, in the case where M has constant positive scalar curvature, we can arbitrarily increase the volume of M by increasing the size of the radius r and we can decrease the radius r to arbitrarily reduce the volume of M when M has constant negative scalar curvature.

Next, let us give a motivation for the second theorem of this work. An important result in differential geometry is the positive mass theorem. This theorem states that an asymptotically flat manifold with nonnegative scalar curvature has nonnegative ADM mass. Moreover, if the manifold is not isometric to the Euclidean space, then the ADM mass is positive. This result was first proved by R. Schoen and S. T. Yau in [31], for dimension $n < 8$, using minimal surfaces techniques. In [37], E. Witten gave a proof of this theorem for any spin manifold of any dimension.

In [23], P. Miao observed that the positive mass theorem implies the following rigidity result for the unit ball $B^n \subset \mathbb{R}^n$.

Theorem 4 (P. Miao)**.** *Let g be a smooth Riemannian metric on B^n with nonnegative scalar curvature such that $\partial B^n = S^{n-1}$ with the induced metric has mean curvature greater than or equal to $(n-1)$ and is isometric to S^{n-1} with the standard metric. Then g is isometric to the standard metric of B^n.*

The theorem above was generalized by Y. Shi and L. Tam in [29]. There are some analogous rigidity results for the hyperbolic space (see [25], [2], [36] and [10]). A similar rigidity result for the hemisphere S^n_+ was conjectured by M. Min-Oo in [26]:

Min-Oo's Conjecture. *Let g be a smooth metric on the hemisphere S^n_+ with scalar curvature $R_g \geqslant n(n-1)$ such that the induced metric on ∂S^n_+ agrees with the standard metric on ∂S^n_+ and is totally geodesic. Then g is isometric to the standard metric on S^n_+.*

This conjecture is true for $n = 2$, in which case it follows by a theorem of Toponogov [35] (see also [17]). Recently, counterexamples were constructed by S. Brendle, F. C. Marques and A. Neves in [6] for $n \geqslant 3$. They proved the following result.

Theorem 5 (S. Brendle, F. C. Marques, A. Neves). *Given any integer $n \geqslant 3$, there exists a smooth metric g on the hemisphere S^n_+ with the following properties:*

- *The scalar curvature of g is at least $n(n-1)$ at each point on S^n_+.*
- *The scalar curvature of g is strictly greater than $n(n-1)$ at some point on S^n_+.*
- *The metric g agrees with the standard metric of S^n_+ in a neighborhood of ∂S^n_+.*

We refer the reader to [17, 11, 19] for partial results concerning the Min-Oo's conjecture. In [7], a rigidity result for small geodesic balls in S^n was proved.

The following theorem is the second one of this work and it can be considered as the analogue of Miao's result and Min-Oo's conjecture in our setting. It is a rigidity result for cylinders $([a,b]\times\Sigma, dt^2+g_\Sigma)$, where (Σ, g_Σ) is a Riemann surface of genus greater than 1 and constant Gauss curvature equal to -1.

Recall that a three-manifold is irreducible if every embedded 2-sphere in M bounds an embedded 3-ball in M.

Theorem B. *Let Σ be a compact Riemann surface of genus $g(\Sigma) \geqslant 2$ and g_Σ a metric on Σ with $K_\Sigma \equiv -1$. Let (Ω^3, g) be a compact irreducible connected Riemannian three-manifold with boundary satisfying the following properties:*

- *$R_g \geqslant -2$.*
- *$H_{\partial\Omega} \geqslant 0$. ($H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ with respect to inward normal vector)*
- *Some connected component of $\partial\Omega$ is incompressible in Ω and with the induced metric is isometric to (Σ, g_Σ).*

Moreover, suppose that Ω does not contain any one-sided compact embedded surface. Then (Ω, g) is isometric to $([a,b]\times\Sigma, dt^2+g_\Sigma)$.

Remark 6. We note that the corresponding result for cylinders $[a,b]\times S^2$, where S^2 is the round sphere, does not hold. In fact, consider a rotationally symmetric Delaunay-type metric $g_a = u_a(t)^4(dt^2+g_{S^2})$ on $\mathbb{R}\times S^2$ with constant scalar curvature equal to 2 such that $u_a(0) = a = \min u < 1$ and $u_a'(0) = 0$ (see section 2.2, pg. 16). Since u_a is a periodic function, choose $t_0, t_1 \in \mathbb{R}$ such that $u_a(t_0) = u_a(t_1) = \max u$. We note that

$u_a(t_0) = u_a(t_1) > 1$. The Riemannian manifold $(\Omega = [t_0, t_1] \times S^2, \overline{g})$, where $\overline{g} = u(t_0)^{-4} g$, gives a counterexample. In fact, $R_{\overline{g}} = u(t_0)^4\, 2 > 2$, $H_{\partial\Omega} = 0$ and every component of $\partial\Omega$ is isometric to the round sphere (S^2, g_{S^2}). Moreover, Ω is irreducible and does not contain any one-sided compact embedded surface and every component of $\partial\Omega$ is incompressible in Ω. However, $(\Omega, \overline{g})$ is not isometric to a standard cylinder $([a, b] \times S^2, dt^2 + g_{S^2})$.

The following example justifies the requirement that Ω does not contain any one-sided compact embedded surface.

Example 1. Let $(\widehat{\Sigma}, g_{\widehat{\Sigma}})$ be a compact non-orientable surface with constant Gauss curvature equal to -1. Denote by Σ the orientable double covering of $\widehat{\Sigma}$ and by π the covering map. Next, define $g_\Sigma = \pi^* g_{\widehat{\Sigma}}$ and consider $(M = [-k, k] \times \Sigma, dt^2 + g_\Sigma)$. Take the subgroup $\Gamma = \{id, f\} \subset \mathrm{Iso}(M, g)$, where f is defined by $f(t, x) = (-t, \phi(x))$ and $\phi \in \mathrm{Iso}(\Sigma, g_\Sigma)$ is the non-trivial deck transformation of $\pi : \Sigma \longrightarrow \widehat{\Sigma}$. Now, consider the Riemannian manifold (Ω, g_Ω), where $\Omega = M/\Gamma$ and g_Ω is the quotient metric. Note that Ω is irreducible, $R_{g_\Omega} = -2$, $H_{g_\Omega} = 0$, $\partial\Omega$ is incompressible in Ω and with the induced metric is isometric to (Σ, g_Σ). Finally, observe that $\partial\Omega$ has only one component and that the image of $\{0\} \times \Sigma$ is a one-sided compact embedded surface in Ω.

The Theorems A and B of this thesis were proved in the reference [27], posted on the arXiv in March of 2011. More recently, M. Micallef and V. Moraru posted a paper on the arXiv, [24], where they prove Theorem A with an alternative argument.

CHAPTER 1

Preliminaries

In this chapter, our purpose is to fix notations, to give definitions and to state some facts which will be used throughout this work. In Section 1.1, we first list the definitions of the geometric objects related to a Riemannian manifold and its submanifolds. We also state a very useful formula which is a consequence of the Gauss equation. Next, we recall the first and second variation formulas and the definition of stable minimal surface. Finally, we state the formula for the first variation of the mean curvature of a one-parameter family of surfaces. In Section 1.2, we state the Hopf's maximum principle which will be used in the proof of Theorem A. In Section 1.3, we discuss the Meeks-Simon-Yau's result concerning the existence of area-minimizing surfaces in isotopy classes.

1.1 Terminology and basic facts

Let (M, g) be a Riemannian manifold of dimension n. Sometimes, we will also denote the metric g by $\langle \cdot, \cdot \rangle$. The Riemann curvature tensor of M, denoted by R, is defined to be

$$\mathrm{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where $X, Y, Z \in \mathfrak{X}(M)$ and ∇ is the Levi-Civita connection of (M, g). Here, $\mathfrak{X}(M)$ denotes the space of smooth vector fields on M.

We also define

$$\mathrm{R}(X,Y,Z,W) = \langle \mathrm{R}(X,Y)W, Z\rangle,$$

where $X,Y,Z,W \in \mathfrak{X}(M)$. We will use the notation R_x for the Riemman curvature tensor at $x \in M$.

The Ricci curvature of (M,g) at $x \in M$ in the direction $v \in T_xM$, $|v| = 1$, denoted by $\mathrm{Ric}_x(v,v)$, is defined to be

$$\mathrm{Ric}_x(v,v) = \sum_{i=1}^{n-1} \mathrm{R}_x(v,e_i,v,e_i),$$

where $\{v, e_1, \ldots, e_{n-1}\} \subset T_xM$ is an orthonormal basis.

The scalar curvature of (M,g) at x, denoted by $R_g(x)$, is defined to be

$$R_g(x) = \sum_{i=1}^{n} \mathrm{Ric}_x(e_i,e_i),$$

where $\{e_1, \ldots, e_n\} \subset T_xM$ is an orthonormal basis.

Next, let $\Sigma \subset M$ be a hypersurface and consider $x \in \Sigma$. The second fundamental form of Σ at x, denoted by $(A_\Sigma)_x$, is defined to be

$$(A_\Sigma)_x(X,Y) = (\nabla_X Y)^\perp,$$

where $X,Y \in T_x\Sigma$ and $(\cdot)^\perp$ denotes the component orthogonal to $T\Sigma$ with respect to the metric g.

The mean curvature vector of Σ at $x \in \Sigma$, denoted by $\vec{H}_\Sigma(x)$, is defined to be

$$\vec{H}_\Sigma(x) = \sum_{i=1}^{n-1} (A_\Sigma)_x(e_i,e_i),$$

where $\{e_1, \ldots, e_{n-1}\} \subset T_x\Sigma$ is an orthonormal basis with respect to the induced metric

Let ν be a local unit normal vector field along Σ around $x \in \Sigma$. The mean curvature of Σ at x with respect to ν, denoted by $H_\Sigma(x)$, is defined to be

$$\begin{aligned} H_\Sigma(x) &= \langle \vec{H}_\Sigma(x), \nu(x)\rangle \\ &= -\sum_{i=1}^{n-1} \langle \nabla_{e_i}\nu, e_i\rangle, \end{aligned}$$

where $\{e_1, \ldots, e_{n-1}\} \subset T_x\Sigma$ is an orthonormal basis with respect to the induced metric.

Remark 7. If there is no ambiguity we will denote the second fundamental form, the mean curvature vector and the mean curvature of Σ only by A, $\vec{H}$ and H, respectively.

Denote by R^Σ the Riemann curvature tensor of Σ with respect to the induced metric.

Proposition 1 (Gauss Equation). *Given $x \in \Sigma$, we have*

$$\mathrm{R}^\Sigma_x(e_1, e_2, e_1, e_2) = \mathrm{R}_x(e_1, e_2, e_1, e_2) + \langle A_x(e_1, e_1), A_x(e_2, e_2)\rangle - |A_x(e_1, e_2)|^2,$$

for every orthonormal vectors $e_1, e_2 \in T_x\Sigma$.

It is easy to see that the Gauss equation implies the following relation:

$$R^\Sigma_g = R_g - 2\,\mathrm{Ric}(\nu, \nu) + H^2_\Sigma - |A_\Sigma|^2, \tag{1.1}$$

where R^Σ_g donotes the scalar curvature of Σ.

Now, suppose $\Sigma \subset M$ is compact and let $\Sigma_t \subset M$, $t \in (-\epsilon, \epsilon)$, $\epsilon > 0$, be a smooth normal variation of Σ in M. More precisely, Σ_t is given by $\Sigma_t = \{f(t, x) : x \in \Sigma\}$, where $f : (-\epsilon, \epsilon) \times \Sigma \longrightarrow M$ is a smooth function such that $f(0, x) = x$, $\forall x \in \Sigma$, and $f_t = f(t, \cdot) : \Sigma \longrightarrow M$ is an immersion $\forall t \in (-\epsilon, \epsilon)$, and moreover $\dfrac{\partial f}{\partial t}(0, x) \perp T_x\Sigma$, $\forall x \in T\Sigma$. Denote by X the variational vector field $\dfrac{\partial f}{\partial t}(0, x)$.

Proposition 2 (First variation formula of area). *We have*

$$\frac{d}{dt}|\Sigma_t|\bigg|_{t=0} = -\int_\Sigma \langle \vec{H}, X\rangle\, d\sigma,$$

where $|\Sigma_t|$ and $d\sigma$ denote the area of Σ_t and the area element of Σ with respect to the induced metric, respectively.

We say that Σ is a minimal hypersurface if $\dfrac{d}{dt}|\Sigma_t|\big|_{t=0} = 0$ for every smooth normal variation Σ_t of Σ. This condition is equivalent to $\vec{H} \equiv 0$.

Assume that $\Sigma \subset M$ is minimal and consider a smooth normal variation Σ_t of Σ with variational vector field denoted by X.

Proposition 3 (Second variation formula of area). *We have*

$$\frac{d^2}{dt^2}|\Sigma_t|\bigg|_{t=0} = \int_\Sigma |\nabla^\perp X|^2 - (\mathrm{Ric}(\nu, \nu) + |A|^2)|X|^2\, d\sigma,$$

where $|\nabla^\perp X|^2 = \sum_{i=1}^{n-1}\langle \nabla^\perp_{e_i} X, \nabla^\perp_{e_i} X\rangle$ for any orthonornal basis $\{e_1, \cdots, e_{n-1}\}$ of $T\Sigma$ and $\nabla^\perp_{e_i} X = (\nabla_{e_i} X)^\perp$, for $i = 1, 2, \ldots, n-1$.

We say that a compact minimal hypersurface Σ is stable if

$$\frac{d^2}{dt^2}|\Sigma_t|\bigg|_{t=0} \geqslant 0,$$

for every smooth normal variation Σ_t of Σ.

Suppose Σ is two-sided, that is, a unit normal vector field along Σ can be globally defined. Denote by ν such a normal vector field. In this case, if X is the variational vector field of a smooth normal variation Σ_t of Σ, then we obtain that $X = \phi\nu$, where $\phi \in C^\infty(\Sigma)$. Thus, the stability of Σ is equivalent to

$$\int_\Sigma |\nabla_\Sigma \phi|^2 - (\mathrm{Ric}(\nu,\nu) + |A|^2)\phi^2 \, d\sigma \geqslant 0$$

for every $\phi \in C^\infty(\Sigma)$, where $\nabla_\Sigma \phi$ denotes the gradient of ϕ on Σ with respect to the induced metric.

Remark that if Σ is locally area-minimizing then Σ is a stable minimal hypersurface. We also note that the condition of stability is equivalent to the first eigenvalue of the operator $L = \Delta_\Sigma + \mathrm{Ric}(\nu,\nu) + |A|^2$, called the Jacobi operator Σ, to be nonnegative. Here, Δ_Σ denotes the Laplacian on Σ with respect to the induced metric.

Example 2. Let (Σ, g_Σ) be a compact Riemannian manifold and consider $(M = \mathbb{R} \times \Sigma, g = dt^2 + g_\Sigma)$. It is easy to see that $\Sigma_t = \{t\} \times \Sigma \subset M$ is a stable minimal (in fact, totally geodesic) hypersurface of M, for all $t \in \mathbb{R}$. In this work, we are interested in the case where (Σ, g_Σ) is a compact Riemann surface with constant Gauss curvature equal to -1.

Finally, consider a smooth variation (not necessarily normal) Σ_t of Σ given by $\Sigma_t = \{f(t,x) : x \in \Sigma\}$. Denote by $\nu(t)$ the unit normal vector field along Σ_t and let H_{Σ_t} be the mean curvature of Σ_t with respect to $\nu(t)$.

Proposition 4. *Let* $\rho(t) = \langle \frac{\partial f}{\partial t}(t,x), \nu(t)\rangle$. *We have*

$$\begin{aligned} \frac{d}{dt}H_{\Sigma_t} &= (\Delta_{\Sigma_t} + \mathrm{Ric}(\nu(t),\nu(t)) + |A_{\Sigma_t}|^2)\rho(t) \\ &= L_{\Sigma_t}\rho(t). \end{aligned}$$

Proof. See Theorem 3.2 in [20]. □

1.2 Hopf's maximum principle

In this section we will recall the Hopf's maximum principle which will be used in the proof of Theorem A.

Let $\Omega \subset \mathbb{R}^n$ be an open connected set. Consider a linear differential operator L in Ω of second order as follows.

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x).$$

Suppose the matrix $a_{ij}(x)$ is symmetric for all $x \in \Omega$ and L is uniformly elliptic which means that there exists a constant $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\eta_i\eta_j \geqslant \lambda|\eta|^2, \forall x \in \Omega, \forall \eta \in \mathbb{R}^n.$$

Moreover, we also assume that there exists a constant $C > 0$ such that

$$|a_{ij}(x)|, |b_j(x)|, |c(x)| \leqslant C, \forall x \in \Omega.$$

Theorem 6 (Hopf's maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open connected set and let L be a linear differential operator in Ω of second order as above such that $c(x) \leqslant 0$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $Lu \geqslant 0$. If u attains its maximum $M \geqslant 0$ in Ω, then u is constant equal to M on Ω. Otherwise, if $u(x_0) = M$ at $x_0 \in \partial\Omega$ and $M \geqslant 0$, then the outward normal derivative, if it exists, satisfies $\frac{\partial u}{\partial \nu}(x_0) > 0$, provided x_0 belongs to the boundary of a ball included in Ω. Moreover, if $c(x) \equiv 0$, the same conclusions hold for a maximum $M < 0$.*

Proof. See [28], pg. 44. □

1.3 Existence of area-minimizing surfaces in isotopy classes

Let (M^3, g) be a compact three-manifold and $\Sigma \subset M$ a compact embedded surface.

We say that a compact surface $\widehat{\Sigma} \subset M$ is isotopic to Σ if there exists a smooth isotopy $\phi : [0,1] \times M \longrightarrow M$ such that $\phi(1, \Sigma) = \widehat{\Sigma}$ and $\phi(0, \cdot) = \mathrm{Id}_M$. The isotopy class of Σ, denoted by $\mathcal{I}(\Sigma)$, is defined to be

$$\mathcal{I}(\Sigma) = \{\widehat{\Sigma} : \widehat{\Sigma} \text{ is isotopic to } \Sigma\}.$$

In [22], W. Meeks, L. Simon and S. T. Yau considered the problem of minimizing the area of surfaces in $\mathcal{I}(\Sigma)$. They proved a general existence result using techniques of geometric measure theory (cf. Theorem 1 in [22]).

The Meeks-Simon-Yau's existence result will play an important role in the proof of Theorem A. We will not need it in its full generality. In fact, we will use a particular consequence in the case where M is irreducible and Σ is incompressible in M.

We say that a three-manifold M is irreducible if every embedded 2-sphere bounds an embedded 3-ball in M and we say that a surface $\Sigma \subset M$ is incompressible in M if the fundamental group of Σ injects into that of M.

The following theorem is consequence of the Meeks-Simon-Yau's result in the case where M is irreducible and $\Sigma \subset M$ is incompressible in M.

Theorem 7 (W. Meeks, L. Simon, S. T. Yau)**.** *Let (M^3, g) be an irreducible compact Riemannian three-manifold and consider $\Sigma \subset M$ a connected compact embedded surface which is incompressible in M. Define*

$$\alpha = \inf_{\widehat{\Sigma} \in \mathcal{I}(\Sigma)} |\widehat{\Sigma}|_g.$$

Then either there is a surface $\overline{\Sigma} \in \mathcal{I}(\Sigma)$ such that $|\overline{\Sigma}|_g = \alpha$ or there is a one-sided compact embedded surface $\widetilde{\Sigma}$ of area $\alpha/2$ and such that the boundary of a tubular neighborhood of $\widetilde{\Sigma}$ is in $\mathcal{I}(\Sigma)$.

The above result also holds when $\partial M \neq \emptyset$ in which case we have to assume the mean curvature of ∂M with respect to the inward normal vector is nonnegative (cf. section 6 in [22]).

We note that J. Hass and P. Scott [18] gave a proof of the result above without using geometric measure theory (cf. Theorem 5.1 in [18]).

Let us give an idea of how Theorem 7 follows from the general existence result in [22] using the fact that M is irreducible and $\Sigma \subset M$ is incompressible in M.

Following [22], denote by B_ρ the closed 3-ball of radius $\rho > 0$ and center 0 in $\mathbb{R}^3$. Since M is compact there exist $\rho_0, \mu > 0$ satisfying the following properties:

- For each $x_0 \in M$, the exponential map $\exp_{x_0}$ is a diffeomorphism of B_{ρ_0} onto $\overline{G_{\rho_0}(x_0)}$ satisfying

  $$\left\| D_v(\exp_{x_0}) \right\|, \left\| D_x(\exp_{x_0})^{-1} \right\| \leqslant 2, \, \forall\, v \in B_{\rho_0}, \, \forall\, x \in G_{\rho_0}(x_0).$$

  Here, G_{ρ_0} denotes the geodesic ball with center x_0 and radius ρ_0.

- For each $x_0 \in M$, we have

  $$\sup_{B_{\rho_0}} \left| \frac{\partial g_{ij}}{\partial x_k} \right| \leqslant \frac{\mu}{\rho_0}, \qquad \sup_{B_{\rho_0}} \left| \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l} \right| \leqslant \frac{\mu}{\rho_0^2}$$

for $i, j, k, l = 1, 2, 3$, where $g_{ij}\, dx^i dx^j$ is the metric relative to normal coordinates for $G_{\rho_0}(x_0)$.

The following Lemma is contained in Section 2 of [22]. We state it here for completeness.

Lemma 1. *Let ρ_0 and μ as above. There is a number $\delta \in (0, 1)$ (independent of M and ρ_0) such that if $\Sigma \subset M$ is an embedded compact surface satisfying*

$$|\Sigma \cap G_{\rho_0}(x_0)| < \delta^2 \rho_0^2$$

for each $x_0 \in M$, then there exist a unique compact $K_\Sigma \subset M$ with $\partial K_\Sigma = \Sigma$ and

$$\mathrm{Vol}(K_\Sigma \cap G_{\rho_0}(x_0)) \leqslant \delta^2 \rho_0^3, \qquad x_0 \in M.$$

This K_Σ also satisfies

$$\mathrm{Vol}(K_\Sigma) \leqslant c|\Sigma|^{3/2}, \qquad c = c(\mu).$$

Also, if Σ is diffeomorphic to $S^2 = \partial B_1$, then K_Σ is diffeomorphic to B_1.

Now, let $\Sigma_1, \Sigma_2 \subset M$ be compact embedded surfaces. Consider $\delta > 0$ small such that the conclusion of Lemma 1 above holds. Following section 3 in [22], given $0 < \gamma < \delta^2/9$ we say that Σ_2 is obtained from Σ_1 by $\gamma-$reduction if the following conditions are satisfied (see Figure 1):

- $\Sigma_1 \setminus \Sigma_2$ has closure diffeomorphic to the standard closed annulus $A = \{x \in \mathbb{R}^2 : \frac{1}{2} \leqslant |x| \leqslant 1\}$;
- $\Sigma_2 \setminus \Sigma_1$ has closure consisting of two components D_1, D_2, each diffeomorphic to the 2-disc $D = \{x \in \mathbb{R}^2 : |x| \leqslant 1\}$;
- $A \cup D_1 \cup D_2 = \partial Y$, where Y is homeomorphic to the 3-ball $B = \{x \in \mathbb{R}^3 : |x| \leqslant 1\}$ and $(Y \setminus \partial Y) \cap (\Sigma_1 \cup \Sigma_2) = \emptyset$;
- $\partial A = \partial D_1 \cup \partial D_2$ and $|A| + |D_1| + |D_2| < 2\gamma$;
- In case $\Sigma_1^* \setminus A$ is not connected, each component is either not simply connected or else has area greater than or equal to $\delta^2/2$. Here, Σ_1^* denotes the component of Σ_1 containing A.

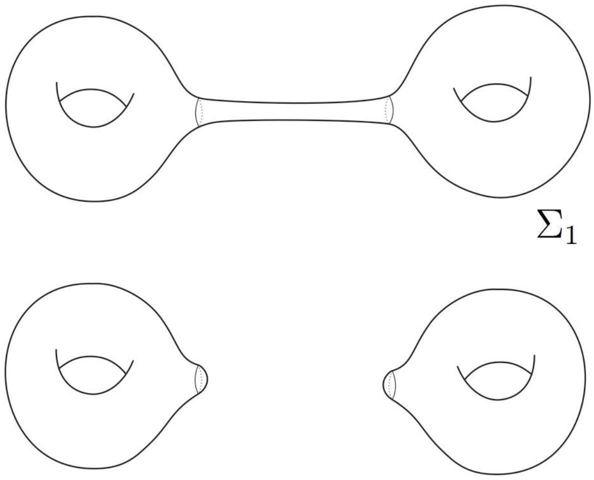

Figura 1

The first observation is that if Σ_1 is irreducible in M and Σ_2 is obtained from Σ_1 by $\gamma-$ reduction, then Σ_2 has genus equal to that of Σ_1. In fact, Σ_2 has two components where one component is a surface homeomorphic to Σ_1 and the other is an embedded 2-sphere. In this case, denote by $\Sigma_2^{(1)}$ the component of Σ_2 which is homeomorphic to Σ_1 and by $\Sigma_2^{(2)}$ the component of Σ_2 which is an embedded 2-sphere. We suppose $D_1 \subset \Sigma_2^{(1)}$ and $D_2 \subset \Sigma_2^{(2)}$. Thus, we have that $\Sigma_2^{(2)} \setminus D_2$ is an embedded 2-disc D and $|\Sigma_2^{(2)} \setminus D_2| \geqslant \delta^2/2$. We also note that

$$\begin{aligned}
|D_1| &< |A| + |D_1| + |D_2| \\
&< 2\gamma \\
&< \frac{2\delta^2}{9} \\
&< \frac{\delta^2}{2} \\
&\leqslant |\Sigma_2^{(2)} \setminus D_2| \\
&< |A| + |\Sigma_2^{(2)} \setminus D_2|.
\end{aligned}$$

Therefore, we conclude that $|\Sigma_2^{(1)}| \leqslant |\Sigma_1|$.

The second observation is that if in addition M is irreducible, then we are able to conclude that $\Sigma_2^{(1)}$ is isotopic to Σ_1. Next, consider a minimizing sequence $\Sigma_k \in \mathcal{I}(\Sigma)$ (cf. section 1 in [22]). By [22] (cf. section 3, pg. 634), there exists $0 < \gamma_0 < \delta^2/9$ such that, after γ_0-reduction, Σ_k yields a strongly γ_0-irreducible (cf. section 3, pg. 630, in [22]) surface $\widetilde{\Sigma}_k$. The point is that in the case where M is irreducible and Σ is incompressible in M, instead of doing γ_0-reduction, we can obtain, by the above observations, a sequence $\widehat{\Sigma}_k$ such that $\widehat{\Sigma}_k$ is strongly γ_0-irreducible, $\widehat{\Sigma}_k \in \mathcal{I}(\Sigma_k) = \mathcal{I}(\Sigma)$ and $|\widehat{\Sigma}_k| \leqslant |\Sigma_k|$ for all k. The inequality $|\widehat{\Sigma}_k| \leqslant |\Sigma_k|$ implies that $\widehat{\Sigma}_k$ is also a minimizing sequence. Finally, to conclude Theorem 7 see Remark 3.27 in [22].

CHAPTER 2

Some Examples and Proofs of the Results

In this chapter, we will give the proofs of the main results of this work. Before doing this, we will consider in Section 2.1 some examples of conformal metrics on $M = \mathbb{R} \times \Sigma$ with constant scalar curvature, where Σ is an orientable connected compact surface. In Section 2.2, we prove Theorem A. In Section 2.3, we give the proof of the rigidity result for cylinders stated in Theorem B.

2.1 Some Examples

In this section, we will discuss a class of examples of conformal metrics on $M^3 = \mathbb{R} \times \Sigma$, where Σ is an orientable connected compact surface.

Let g_Σ be a Riemannian metric on Σ with constant Gauss curvature. We will assume that this constant is equal to $1, 0$ or -1. Thus, $K_\Sigma \equiv 1$ if Σ is a two-sphere, $K_\Sigma \equiv 0$ if Σ is a two-torus and $K_\Sigma \equiv -1$ if Σ has genus greater than 1. Denote by g the Riemannian product metric $dt^2 + g_\Sigma$ on M which has constant scalar curvature R_g equal to $2, 0$ or -2.

First, we consider the case where Σ has genus greater than 1. In this case, $R_g \equiv -2$. For each positive real function $u = u(t)$, define the metric $\overline{g} = u^4(t)\, g$. If we assume that $R_{\overline{g}} \equiv -2$, then u satisfies the following second-order differential linear equation

$$u''(t) + \frac{1}{4}u(t) - \frac{1}{4}u^5(t) = 0.$$

Setting $v'(t) = u(t)$, this equation is equivalent to the Hamiltonian system

$$\begin{cases} u'(t) = v(t), \\ v'(t) = \frac{1}{4}u^5(t) - \frac{1}{4}u(t). \end{cases} \tag{2.1}$$

The Hamiltonian function for the system above is

$$H(u,v) = \frac{1}{8}u^2 - \frac{1}{24}u^6 + \frac{1}{2}v^2.$$

We note that $(1,0)$ is a critical point of H. Moreover, this initial condition correponds to the Riemannian product metric g. Next, observe that

$$\operatorname{Hess} H(1,0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, $(1,0)$ is a saddle point of H. Also observe that $(0,0)$ and $(-1,0)$ are the others critical points of H, where $(0,0)$ is a local strict minimum point and $(-1,0)$ is a saddle point.

From now on, we are only interested in solutions $(u(t), v(t))$ to (2.1) such that $\overline{g} = u(t)^4 g$ is a complete Riemannian metric on M. Using the analysis above, it easy to see that we have the following solutions (see figure 1).

- The solution with initial conditions $u(0) = 1, v(0) = 0$ which we have alrealdy observed to be the Riemannian product metric g;

- Two solutions $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ which are related by $u_2(t) = u_1(-t), v_2(t) = -v_1(-t)$, and such that

$$\lim_{t\to-\infty}(u_1(t), v_1(t)) = (1,0),$$

  and

$$\lim_{t\to+\infty} u_1(t) = \lim_{t\to+\infty} v_1(t) = \infty;$$

- A family of solutions which corresponds to the initial conditions $u(0) = a > 1, v(0) = 0$. For each such solution, we note that $t = 0$ is a strict global minimum of u and that $\lim_{t\to\pm\infty} u(t) = \lim_{t\to+\infty} v(t) = +\infty$ and $\lim_{t\to-\infty} v(t) = -\infty$. For each $a > 1$, denote by g_a the complete Riemannian metric on M corresponding to the initial condition $u(0) = a, v(0) = 0$.

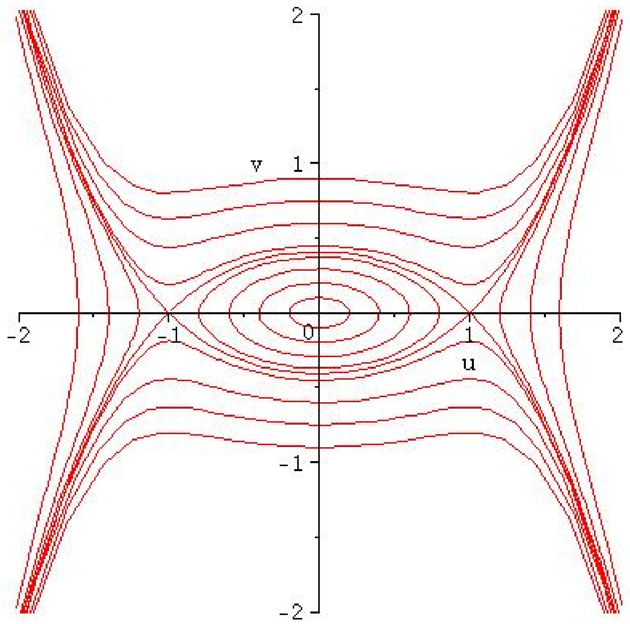

Figure 2

Remark 8. Take $a > 1$ and let g_a the Riemannian metric on $M = \mathbb{R} \times \Sigma$ as above. Consider $\Sigma = \{0\} \times \Sigma \subset (M, g_a)$. We have that the area of Σ with respect to the metric induced by g_a is equal to $4\pi(g(\Sigma) - 1)a^4$. Since $a > 1$, we get $|\Sigma|_{g_a} > 4\pi(g(\Sigma) - 1)$. Also note that Σ is area-minimizing in M because $u(0) = a$ is a strict minimum value of u. This example shows that there exist metrics on M with constant scalar curvature equal to -2 and such that the area-minimizing hyperbolic surface $\Sigma \subset M$ has are greater than $4\pi(g(\Sigma) - 1)$.

Next, suppose that (Σ, g_Σ) is a two-sphere with $K_\Sigma \equiv 1$. In this case, if the metric $\overline{g} = u(t)^4 g$ has constant scalar curvature equal to 2, then $(u(t), v(t))$ satisfies a Hamiltonian system with Hamiltonian function given by

$$H(u, v) = \frac{1}{24}u^6 - \frac{1}{8}u^2 + \frac{1}{2}v^2.$$

The points $(-1, 0), (0, 0)$ and $(1, 0)$ are also critical points of H. The difference is that $(-1, 0)$ and $(1, 0)$ are now local strict minimum points and $(0, 0)$ is a saddle point. Considering only the solutions $(u(t), v(t))$ such that $u(t)$ is positive and $\overline{g} = u(t)^4 g$ defines a complete Riemannian metric on M, we get a one-parameter family of periodic rotationally symmetric metrics $g_a = u_a(t)^4 g$ with constant scalar curvature equal to 2, where $a \in (0, 1]$ and $u_a(t)$ satisfies $u_a(0) = a = \min u$ and $u'_a(0) = 0$ (see Figure 2). These metrics are known as Delaunay-type metrics on $M = \mathbb{R} \times \Sigma$. Note that g_1 corresponds to the metric g.

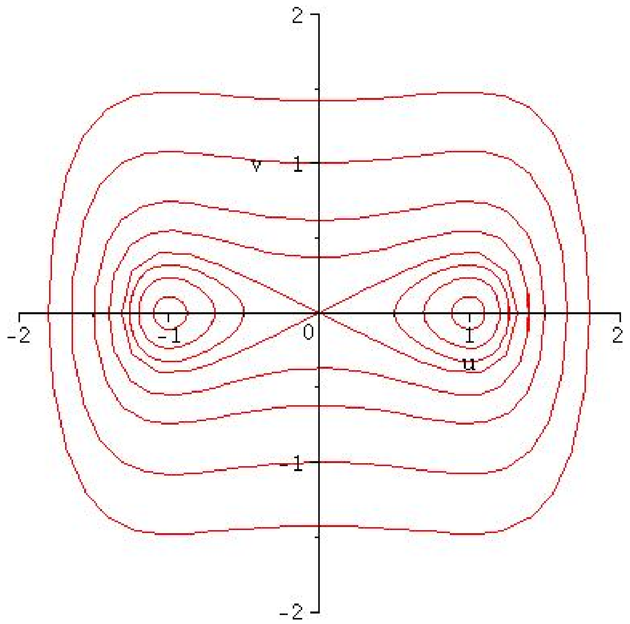

Figure 3

Remark 9. Note also that for each Delaunay-type metric on M with $0 < a < 1$ we have that $\Sigma = \{0\} \times \Sigma \subset (M, g_a)$ is an area-minimizing two-sphere with area less than 4π.

To finish this section, consider the case where (Σ, g_Σ) is a flat two-torus. Thus (M, g) is scalar flat. If $\overline{g} = u(t)^4 g$ is also scalar flat, then $u(t)$ satisfies $u''(t) = 0$. Thus, we get $u(t) = u'(0)t + u(0)$. Note that among these metrics the only ones which contain an area-minimizing two-torus are those of the form $\overline{g} = cg$, where $c > 0$ is constant.

2.2 Proof of Theorem A

In the following, we will give the proof of Theorem A. Let us recall the statement of this result.

Theorem A. *Let (M^3, g) be a Riemannian manifold with scalar curvature $R_g \geqslant -2$. If $\Sigma \subset M$ is a two-sided compact embedded Riemann surface of genus $g(\Sigma) \geqslant 2$ which is locally area-minimizing, then*

$$|\Sigma|_g \geqslant 4\pi(g(\Sigma) - 1), \tag{2.2}$$

where $|\Sigma|_g$ denotes the area of Σ with respect to the induced metric. Moreover, if equality holds, then the induced metric on Σ, denoted by g_Σ, has constant

Gauss curvature equal to -1 *and* Σ *has a neighborhood which is isometric to* $((-\epsilon,\epsilon)\times\Sigma, dt^2+g_\Sigma)$, *for some* $\epsilon>0$. *More precisely, the isometry is given by* $f(t,x)=\exp_x(t\nu(x))$, $(t,x)\in(-\epsilon,\epsilon)\times\Sigma$, *where* ν *is the unit normal vector field along* Σ.

In Subsection 2.2.1, we discuss the area estimate (2.2) and we prove the rigidity statement in Subsection 2.2.2.

2.2.1 Proof of the area estimate

Let ν be the unit normal vector field along Σ. For each function $\phi\in C^\infty(\Sigma)$, we have, by the second variation formula of area and the fact that Σ is locally area-minimizing, that

$$\int_\Sigma (\mathrm{Ric}(\nu,\nu)+|A|^2)\,\phi^2\,d\sigma \leqslant \int_\Sigma |\nabla_\Sigma\phi|^2\,d\sigma,$$

where $d\sigma$ denote the area element of Σ. Choosing $\phi=1$, we obtain

$$\int_\Sigma (\mathrm{Ric}(\nu,\nu)+|A|^2)\,d\sigma \leqslant 0. \tag{2.3}$$

Now, we have by the formula (1.1) that

$$\mathrm{Ric}(\nu,\nu)=\frac{1}{2}R_g-K_\Sigma-\frac{1}{2}|A|^2, \tag{2.4}$$

where K_Σ denotes the Gauss curvature of Σ.

Substituting (2.4) in (2.3), we get

$$\frac{1}{2}\int_\Sigma (R_g+|A|^2)\,d\sigma \leqslant \int_\Sigma K_\Sigma\,d\sigma. \tag{2.5}$$

By the Gauss-Bonnet theorem and the fact that $R_g\geqslant -2$ and $|A|^2\geqslant 0$, we have

$$-|\Sigma|_g\leqslant 4\pi(1-g(\Sigma)).$$

Therefore, $|\Sigma|_g\geqslant 4\pi(g(\Sigma)-1)$.

2.2.2 Equality case

In this subsection we will give the proof of the rigidity statement of Theorem A. We begin with the following infinitesimal rigidity.

Proposition 5. *If* Σ *attains the equality in* (2.2), *then* Σ *is totally geodesic. Moreover,* $\mathrm{Ric}(\nu,\nu)=0$ *and* $R_g=-2$ *on* Σ *and* Σ *has constant Gauss curvature equal to* -1 *with the induced metric.*

Proof. If $|\Sigma|_g = 4\pi(g(\Sigma)-1)$, then it follows from the proof of the inequality (2.2) that the inequalities (2.3) and (2.5) are in fact equalities. Let λ_1 be the first eigenvalue of the Jacobi operator $L = \Delta_\Sigma + \mathrm{Ric}(\nu,\nu) + |A|^2$ of Σ. We have

$$\lambda_1 = \inf_{\int \phi^2=1} \int_\Sigma (|\nabla\phi|^2 - (\mathrm{Ric}(\nu,\nu) + |A|^2)\,\phi^2)\,d\sigma.$$

Since

$$\int_\Sigma (\mathrm{Ric}(\nu,\nu) + |A|^2)\,d\sigma = 0,$$

we obtain that $\lambda_1 = 0$ and that the constant functions are in the kernel of L. Therefore, $\mathrm{Ric}(\nu,\nu) + |A|^2 = 0$ on Σ.

Now, the equality in (2.5) implies that $R_g = -2$ and $A = 0$ on Σ. Finally, by (2.4), we conclude that Σ has constant Gauss curvature equal to -1 with the induced metric. □

The construction in the next proposition is fundamental to conclude the rigidity in Theorem 3. The same construction was used in [1] and [4] to prove similar rigidity results. We prove it here for completeness.

Proposition 6. *If Σ attains the equality in (2.2), then there exists $\epsilon > 0$ and a smooth family $\Sigma_t \subset M$, $t \in (-\epsilon,\epsilon)$, of compact embedded surfaces sastifying:*

- $\Sigma_t = \{\exp_x(w(t,x)\nu(x)) : x \in \Sigma\}$, *where $w : (-\epsilon,\epsilon)\times\Sigma \longrightarrow \mathbb{R}$ is a smooth function such that*

$$w(0,x) = 0, \frac{\partial w}{\partial t}(0,x) = 1 \ \ \text{and} \ \ \int_\Sigma (w(t,\cdot) - t)\,d\sigma = 0.$$

- Σ_t *has constant mean curvature for all $t \in (-\epsilon,\epsilon)$.*

Proof. By the previous proposition, we have $L = \Delta_\Sigma$. Fix $\alpha \in (0,1)$ and consider the Banach spaces $X = \{u \in C^{2,\alpha}(\Sigma) : \int_\Sigma u\,d\sigma = 0\}$ and $Y = \{u \in C^{0,\alpha}(\Sigma) : \int_\Sigma u\,d\sigma = 0\}$. For each real function u defined on Σ, let $\Sigma_u = \{\exp_x(u(x)\nu(x)) : x \in \Sigma\}$, where ν is the unit normal vector field along Σ.

Choose $\epsilon > 0$ and $\delta > 0$ such that Σ_{u+t} is a compact surface of class $C^{2,\alpha}$ for all $(t,u) \in (-\epsilon,\epsilon)\times B(0,\delta)$, where $B(0,\delta) = \{u \in C^{2,\alpha}(\Sigma) : \|u\|_{C^{2,\alpha}} < \delta\}$. Denote by $H_{\Sigma_{u+t}}$ the mean curvature of Σ_{u+t}.

Now, consider the application $\Psi : (-\epsilon,\epsilon)\times B(0,\delta) \subset X \longrightarrow Y$ defined by

$$\Psi(t,u) = H_{\Sigma_{u+t}} - \frac{1}{|\Sigma|}\int_\Sigma H_{\Sigma_{u+t}}\, d\sigma.$$

Notice that $\Psi(0,0) = 0$ because $\Sigma_0 = \Sigma$.

The next step is to compute $D\Psi(0,0)\cdot v$, for $v \in X$. We have

$$\begin{aligned} D\Psi(0,0)\cdot v &= \frac{d\Psi}{ds}(0,sv)\Big|_{s=0} \\ &= \frac{d}{ds}\Big(H_{\Sigma_{sv}} - \frac{1}{|\Sigma|}\int_\Sigma H_{\Sigma_{sv}}\, d\sigma\Big)\Big|_{s=0} \\ &= Lv - \frac{1}{|\Sigma|}\int_\Sigma Lv\, d\sigma \\ &= \Delta_\Sigma v, \end{aligned}$$

where the last equality follows from the fact that $L = \Delta_\Sigma$.

Since $\Delta_\Sigma : X \longrightarrow Y$ is a linear isomorphism, we have, by the implicit function theorem, that there exist $0 < \epsilon_1 < \epsilon$ and $u(t) = u(t,\cdot) \in B(0,\delta)$ for $t \in (-\epsilon_1, \epsilon_1)$ such that

$$u(0) = 0 \ \text{ and } \ \Psi(t, u(t)) = 0, \forall t \in (-\epsilon_1, \epsilon_1).$$

Thus, defining $w(t,x) = u(t,x)+t$, for $(t,x) \in (-\epsilon_1,\epsilon_1)\times\Sigma$, we have that all surfaces $\Sigma_t = \{\exp_x(w(t,x)\nu(x)) : x \in \Sigma\}$ have constant mean curvature. Notice that $w(0,x) = 0$ and $\int_\Sigma (w(t,\cdot) - t)\, d\sigma = 0$ since $w(0,x) = u(0,x) = 0$ and $w(t,\cdot) - t = u(t,\cdot) \in B(0,\delta) = \{u \in C^{2,\alpha}(\Sigma) : \int_\Sigma u\, d\sigma = 0\}$. In order to see that

$$\frac{\partial w}{\partial t}(0,x) = 1, \forall x \in \Sigma,$$

first note that

$$0 = \Psi(t,u(t)) = H_{\Sigma_{w(t,\cdot)}} - \frac{1}{|\Sigma|}\int_\Sigma H_{\Sigma_{w(t,\cdot)}}\, d\sigma, \forall t. \tag{2.6}$$

Define $f(t,x) = \exp_x(w(t,x)\nu(x))$, $x \in \Sigma$. We have that

$$\frac{\partial f}{\partial t}(0,x) = \frac{\partial w}{\partial t}(0,x)\nu(x), \forall x \in \Sigma.$$

Differentiating (2.6) at $t = 0$ and using Proposition 4, we get

$$0 = -\Delta_\Sigma\Big(\frac{\partial w}{\partial t}(0,\cdot)\Big) + \frac{1}{|\Sigma|}\int_\Sigma \Delta_\Sigma\Big(\frac{\partial w}{\partial t}(0,\cdot)\Big)\, d\sigma = -\Delta_\Sigma\Big(\frac{\partial w}{\partial t}(0,\cdot)\Big).$$

Therefore, $\dfrac{\partial w}{\partial t}(0,\cdot)$ is a constant function.

Finally, differentiating $\int_\Sigma (w(t,\cdot)-t)\,d\sigma = 0$ at $t=0$, we obtain that

$$\int_\Sigma \frac{\partial w}{\partial t}(0,\cdot)\,d\sigma = |\Sigma|.$$

Thus, we conclude that $\dfrac{\partial w}{\partial t}(0,x) = 1$, $\forall x \in \Sigma$. □

Let $\nu(t)$ denote the unit normal vector along Σ_t such that $\nu(0)=\nu$. Let H_{Σ_t} be the mean curvature of Σ_t with respect to $\nu(t)$. Thus, we have

$$\frac{d}{dt}|\Sigma_t|_g = -H_{\Sigma_t}\int_{\Sigma_t}\langle \nu(t), \frac{\partial f}{\partial t}(t,\cdot)\rangle\,d\sigma_t, \tag{2.7}$$

where $f(t,x) = \exp_x(w(t,x)\,\nu(x))$, $x \in \Sigma$. Notice that $\dfrac{\partial f}{\partial t}(0,x) = \nu(x)$, so we can suppose, decreasing ϵ if necessarily, that

$$\int_{\Sigma_t}\langle \nu(t), \frac{\partial f}{\partial t}(t,\cdot)\rangle\,d\sigma_t > 0$$

for all $t \in (-\epsilon,\epsilon)$. Moreover, we can assume that $|\Sigma|_g \leqslant |\Sigma_t|_g$ for all $t \in (-\epsilon,\epsilon)$, because Σ is locally area-minimizing.

Before we prove the next proposition, we will recall some facts about the Yamabe problem on manifolds with boundary which was first studied by J. F. Escobar [12]. Let (M^n,g) be a compact Riemannian manifold with boundary $\partial M \neq \emptyset$. It is a basic fact that the existence of a metric $\overline{g}$ in the conformal class of g having scalar curvature equal to $C \in \mathbb{R}$ and the boundary being a minimal hypersurface is equivalent to the existence of a positive smooth function $u \in C^\infty(M)$ satisfying

$$\begin{cases} \Delta_g u - \dfrac{n-2}{4(n-1)}R_g u + \dfrac{n-2}{4(n-1)}Cu^{(n+2)/(n-2)} = 0 \text{ on } M \\ \dfrac{\partial u}{\partial \eta} + \dfrac{n-2}{2(n-1)}H_g u = 0 \text{ on } \partial M \end{cases} \tag{2.8}$$

where η is the outward normal vector with respect to the metric g and H_g is the mean curvature of ∂M with respect to the inward normal vector.

If u is a solution of the equation above, then u is a critical point of the following functional

$$Q_g(\phi) = \frac{\int_M (|\nabla_g \phi|_g^2 + \frac{n-2}{4(n-1)}\,R_g\,\phi^2)\,dv + \frac{n-2}{2(n-1)}\int_{\partial M} H_g\,\phi^2\,d\sigma}{(\int_M |\phi|^{2n/(n-2)}\,dv)^{(n-2)/n}}.$$

The Sobolev quotient $Q(M)$ is then defined by

$$Q(M) = \inf\{Q_g(\phi) : \phi \in C^1(M), \phi \neq 0\}$$

It is a well known fact that $Q(M) \leqslant Q(S^n_+)$, where S^n_+ is the upper standard hemisphere, and if $Q(M) < Q(S^n_+)$, then there exists a smooth minimizer for the functional above. This function turns out to be a positive solution of (2.8), with a constant C that has the same sign as $Q(M)$.

Proposition 7. *There exists $0 < \epsilon_1 < \epsilon$ such that $H_{\Sigma_t} \geqslant 0$ for all $t \in [0, \epsilon_1)$.*

Proof. Suppose, by contradiction, that there exists a sequence $\epsilon_k \to 0$, $\epsilon_k > 0$, such that $H_{\Sigma_{\epsilon_k}} < 0$ for all k. Consider (V_k, g_k), where $V_k = [0, \epsilon_k] \times \Sigma$ and g_k is the pullback of the metric by $f|_{V_k} : V_k \longrightarrow M$. Therefore, V_k is a compact three-manifold with boundary satisfying

- $R_{g_k} \geqslant -2$.
- The mean curvature of ∂V_k with respect to the inward normal vector, denoted by $H_{\partial V_k}$, is nonnegative. More precisely, $\partial V_k = \Sigma \cup \Sigma_{\epsilon_k}$, where Σ is a minimal surface and Σ_{ϵ_k} has positive constant mean curvature with respect to the inward normal vector.
- $|\Sigma|_{g_k} = 4\pi(g(\Sigma) - 1)$.

Claim. *For k suficiently large, we have $Q(V_k) < 0$. In particular, this implies $Q(V_k) < Q(S^3_+)$.*

Proof. By Proposition 5, we have $R_g = -2$ on Σ. Therefore, by continuity, we have $-2 \leqslant R_{g_k} \leqslant -1$ on V_k for k sufficiently large. Choosing $\phi = 1$, we obtain

$$\begin{aligned} Q_{g_k}(\phi) &= \frac{\frac{1}{8}\int_{V_k} R_{g_k}\, dv_k + \frac{1}{4}\int_{\partial V_k} H_{\partial V_k}\, d\sigma_k}{\mathrm{Vol}(V_k)^{1/3}} \\ &\leqslant \frac{-\frac{1}{8}\mathrm{Vol}(V_k) + \frac{1}{4} H_{\Sigma_{\epsilon_k}} |\Sigma_{\epsilon_k}|_{g_k}}{\mathrm{Vol}(V_k)^{1/3}} \end{aligned}$$

Since $\dfrac{\partial f}{\partial t}(0, x) = \nu(x)$ and the stability operator of Σ is equal to Δ_Σ, we obtain that $\dfrac{d}{dt} H_{\Sigma_t}|_{t=0} = 0$. Therefore, we conclude that $H_{\Sigma_{\epsilon_k}} = O(\epsilon_k^2)$ because $H_{\Sigma_0} = H_\Sigma = 0$. Moreover, if $V(t) = [0, t] \times \Sigma$ and $g_t = (f|_{V(t)})^* g$, we have that

$$\begin{aligned}\mathrm{Vol}(V(t)) &= \mathrm{Vol}(V(t), g_t) \\ &= \int_{[0,t]\times\Sigma} (f|_{V(t)})^* \, dv \\ &= \int_{[0,t]\times\Sigma} h(s,x) ds \wedge d\sigma \\ &= \int_0^t \int_\Sigma h(s,x)\, d\sigma \, ds,\end{aligned}$$

where h is defined by $h(s,x) = dv(\frac{\partial f}{\partial s}(s,x), Df(s,x)e_1, Df(s,x)e_2)$ and $\{e_1, e_2\} \subset TM$ is a positive orthonormal basis with respect to the induced metric on Σ. From this, we get

$$\frac{d}{dt}\mathrm{Vol}(V(t))\Big|_{t=0} = \int_\Sigma h(0,x)\, d\sigma.$$

Since $\dfrac{\partial f}{\partial s}(0,x) = \nu(x)$, we have $h(0,x) = 1$. Hence, $\dfrac{d}{dt}\mathrm{Vol}(V(t))|_{t=0} = |\Sigma|_g$. From this, we obtain that $\mathrm{Vol}(V_k) = \epsilon_k |\Sigma|_{g_k} + O(\epsilon_k^2)$. Finally, it is easy to see that for k sufficiently large we have $Q(V_k) \leqslant Q_{g_k}(\phi) < 0$. This concludes the proof of the claim.

□

Next, choose k sufficiently large such that $Q(V_k) < 0$. Thus, we have that there exists a positive function $u \in C^\infty(V_k)$ such that the metric $\overline{g} = u^4 g_k$ satisfies

$$R_{\overline{g}} = C < 0, C \in \mathbb{R}, \text{ on } V_k \text{ and } H_{\overline{g}} = 0 \text{ on } \partial V_k.$$

After scaling the metric $\overline{g}$ if necessary, we can suppose that $C = -2$.

In analytic terms, this means that u solves

$$\begin{cases} \Delta_{g_k} u - \frac{1}{8} R_{g_k} u - \frac{1}{4} u^5 = 0 \text{ on } V_k \\ \frac{\partial u}{\partial \eta} + \frac{1}{4} H_{\partial V_k} u = 0 \text{ on } \partial V_k \end{cases} \tag{2.9}$$

Define $v = u - 1$. By (2.9) and the fact that $R_{g_k} \geqslant -2$, we have that

$$\Delta_{g_k} u + \frac{1}{4} u - \frac{1}{4} u^5 \geqslant 0 \text{ on } V_k.$$

Therefore, we have

$$\Delta_{g_k} v - h(x)\, v \geqslant 0 \quad \text{on} \quad V_k,$$

where $h(x) = \dfrac{1}{4}(u(x) + u(x)^2 + u(x)^3 + u(x)^4)$ is positive.

Now, we consider $x_0 \in V_k$ such that $v(x_0) = \max_{V_k} v$. If $v(x_0) \geqslant 0$, we have, by the Hopf's maximum principle, that either v is constant or $x_0 \in \partial V_k$ with $\dfrac{\partial v}{\partial \eta}(x_0) > 0$. The first possibility does not occur because this implies that u is constant, which is impossible by the fact that the mean curvature of Σ_{ϵ_k} with respect to g_k is positive and with respect to $\overline{g}$ is equal to zero. Therefore, $\dfrac{\partial v}{\partial \eta}(x_0) > 0$. But, since $H_{\partial V_k} \geqslant 0$, (2.9) implies that $\dfrac{\partial v}{\partial \eta}(x_0) = \dfrac{\partial u}{\partial \eta}(x_0) \leqslant 0$ which is a contradiction.

Thus, we obtain that $v(x_0) < 0$ and this implies that $u(x) < 1$ for all $x \in V_k$. From this, we obtain that $|\Sigma|_{\overline{g}} < |\Sigma|_{g_k} = 4\pi(g(\Sigma) - 1)$.

Finally, denote by $\mathcal{I}(\Sigma)$ the isotopy class of Σ in V_k. Observe that Σ is incompressible in V_k. Moreover, we have that V_k is irreducible and does not contain any one-sided compact embedded surface. In fact, the former follows from the fact that if the universal cover of a three-manifold M is irreducible, then M is also irreducible and the latter follows from the fact that M is homeomorphic to a tubular neighborhood of an embedding of Σ in $\mathbb{R}^3$. Since $H_{\overline{g}} = 0$, we can directly apply the version for three-manifolds with boundary of Theorem 7, to obtain a compact embedded surface $\overline{\Sigma} \in \mathcal{I}(\Sigma)$ such that

$$|\overline{\Sigma}|_{\overline{g}} = \inf_{\widehat{\Sigma} \in \mathcal{I}(\Sigma)} |\widehat{\Sigma}|_{\overline{g}}.$$

Therefore, $|\overline{\Sigma}|_{\overline{g}} \leqslant |\Sigma|_{\overline{g}} < 4\pi(g(\Sigma) - 1)$. But this is a contradiction with (2.2), since we have proven, by using the lower bound $R_{\overline{g}} \geqslant -2$ and the second variation of area, that we must have $|\overline{\Sigma}|_{\overline{g}} \geqslant 4\pi(g(\Sigma) - 1)$. This concludes the proof of the proposition. □

We are now in a position to proof the rigidity in Theorem A which is stated in the proposition below.

Proposition 8. *If Σ attains the equality in* (2.2)*, then Σ has a neighborhood which is isometric to $((-\epsilon, \epsilon) \times \Sigma, dt^2 + g_\Sigma)$, where $\epsilon > 0$ and g_Σ is the induced metric on Σ which has constant Gauss curvature equal to -1.*

Proof. Let $\Sigma_t \subset M$, $t \in (-\epsilon, \epsilon)$, be the family of surfaces given by Proposition 6. By Proposition 7 there exists $0 < \epsilon_1 < \epsilon$ such that $\dfrac{d}{dt}|\Sigma_t|_g \leqslant 0$ for all

$t \in [0, \epsilon_1)$. Thus, $|\Sigma_t|_g \leqslant |\Sigma|_g$ for all $t \in [0, \epsilon_1)$ and this implies $|\Sigma_t|_g = |\Sigma|_g$ for all $t \in [0, \epsilon_1)$ because Σ is locally area-minimizing. Therefore, by Proposition 5, we have that Σ_t is totally geodesic and $\mathrm{Ric}(\nu(t), \nu(t)) = 0$ on Σ_t for all $t \in [0, \epsilon_1)$. In particular, we have that all surfaces Σ_t are minimal and the stability operator of Σ_t, denoted by L_{Σ_t}, is equal to Δ_{Σ_t}.

Define $\rho(t) = \rho(t, x) = \langle \nu(t, x), \dfrac{\partial f}{\partial t}(t, x) \rangle$. By Proposition 4, we have

$$L_{\Sigma_t} \rho(t) = \frac{d}{dt} H_{\Sigma_t},$$

so $\Delta_{\Sigma_t} \rho(t) = 0$. Thus, $\rho(t)$ does not depend on x.

Since Σ_t is totally geodesic, we obtain that $\nabla_{\frac{\partial f}{\partial x_i}} \nu(t) = 0$ for all $i = 1, 2$, where (x_1, x_2) are local coordinates on Σ. Moreover, by the fact that $\langle \nu(t), \nu(t) \rangle = 1$ we have that $\nabla_{\frac{\partial f}{\partial t}} \nu(t)$ is tangent to Σ_t. Hence, it follows that

$$\begin{aligned} \langle \nabla_{\frac{\partial f}{\partial t}} \nu(t), \frac{\partial f}{\partial x_i} \rangle &= \frac{\partial}{\partial t} \langle \nu(t), \frac{\partial f}{\partial x_i} \rangle - \langle \nu(t), \nabla_{\frac{\partial f}{\partial t}} (\partial f / \partial x_i) \rangle \\ &= -\langle \nu(t), \nabla_{\frac{\partial f}{\partial x_i}} (\partial f / \partial t) \rangle \\ &= -\frac{\partial}{\partial x_i} \rho(t) \\ &= 0, \end{aligned}$$

for all $i = 1, 2$. Hence, $\nabla_{\frac{\partial f}{\partial t}} \nu(t) = 0$. This means that, for all $x \in \Sigma$, $\nu(t, x)$ is a parallel vector field along the curve $\alpha_x : [0, \epsilon_1) \longrightarrow M$ given by $\alpha_x(t) = f(t, x) = \exp_x(w(t, x)\nu(x))$.

Observe that $D(\exp_x)_{w(t,x)\nu(x)}(\nu(x))$ is also a parallel vector field along the curve α_x. Thus, $\nu(t, x) = D(\exp_x)_{w(t,x)\nu(x)}(\nu(x))$ because $w(0, x) = 1$ by Proposition 6. From this, we conclude that $\rho(t) = \dfrac{\partial w}{\partial t}(t, x)$.

By Proposition 6, we have

$$\int_\Sigma (w(t, x) - t)\, d\sigma = 0,$$

so

$$\int_\Sigma \frac{\partial w}{\partial t}(t, x)\, d\sigma = |\Sigma|_g.$$

Therefore, since $\dfrac{\partial w}{\partial t}(t, x)$ does not depend on x, we get $\dfrac{\partial w}{\partial t}(t, x) = 1$. This implies that $w(t, x) = t$ for all $(t, x) \in [0, \epsilon_1) \times \Sigma$ because $w(0, x) = 0$. Thus, we conclude that $f(t, x) = \exp_x(t\nu(x))$ and, since Σ_t are totally geodesic, the pullback of g by $f|_{[0,\epsilon_1)\times\Sigma}$ is the product metric $dt^2 + g_\Sigma$, where g_Σ is the induced metric on Σ.

Arguing similarly for $t \leqslant 0$, we finish the proof of the proposition. $\square$

Remark 10. We have assumed until now that Σ is embedded. However, we note that the same result holds if Σ is only immersed. The proof is the same. The only difference is that, in this case, $f(t,x) = \exp_x(t\nu(x))$, $(t,x) \in (-\epsilon,\epsilon)\times\Sigma$, is only a local isometry.

In the following proposition we give the proof of the Corollary 1. Suppose Σ minimizes area in its homotopy class and Σ attains the equality in (2.2). Define $f : \mathbb{R}\times\Sigma \longrightarrow M$ by $f(t,x) = \exp_x(t\nu(x))$, where ν is the unit normal vector field along Σ.

Proposition 9. *$f : (\mathbb{R}\times\Sigma, dt^2 + g_\Sigma) \longrightarrow (M,g)$ is an isometric covering.*

Proof. Consider $A = \{t > 0 : f|_{[0,t]\times\Sigma}$ is a local isometry$\}$. By Proposition 8, this set is nonempty. Moreover, A is closed. Let us prove that A is open. Given $t \in A$, consider the immersed surface $\Sigma_t = \{\exp_x(t\nu(x)) : x \in \Sigma\}$ with the metric induced by f. We have that Σ_t is homotopic to Σ and $|\Sigma_t| = |\Sigma|$. Hence, Σ_t minimizes area in its homotopy class and attains the equality in (2.2). Therefore, by Proposition 8, we conclude that there exists $\epsilon > 0$ such that $f|_{[0,t+\epsilon]\times\Sigma}$ is a local isometry. It follows that A is open and consequently $f|_{[0,\infty)\times\Sigma}$ is a local isometry. Arguing similarly for $t < 0$, we conclude that $f : \mathbb{R}\times\Sigma \longrightarrow M$ is a local isometry. Thus, since (M,g) is complete we have that $f : \mathbb{R}\times\Sigma \longrightarrow M$ is an isometric covering. $\square$

2.3 Proof of Theorem B

In this section, we give the proof of Theorem B. At first, we recall the precise statement of this result.

Theorem B. *Let Σ be a compact Riemann surface of genus $g(\Sigma) \geqslant 2$ and g_Σ a metric on Σ with $K_\Sigma \equiv -1$. Let (Ω^3, g) be a compact irreducible connected Riemannian three-manifold with boundary satisfying the following properties:*

- *$R_g \geqslant -2$.*
- *$H_{\partial\Omega} \geqslant 0$. ($H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ with respect to inward normal vector)*
- *Some connected component of $\partial\Omega$ is incompressible in Ω and with the induced metric is isometric to (Σ, g_Σ).*

Moreover, suppose that Ω does not contain any one-sided compact embedded surface. Then (Ω, g) is isometric to $([a,b]\times\Sigma, dt^2 + g_\Sigma)$.

The proof of the above result is as follows. Let $\partial\Omega^{(1)}$ be a connected component of $\partial\Omega$ which is isometric to (Σ, g_Σ). Consider $\alpha = \inf\{|\widehat{\Sigma}|_g : \widehat{\Sigma} \in \mathcal{I}(\partial\Omega^{(1)})\}$, where $\mathcal{I}(\partial\Omega^{(1)})$ is the isotopy class of $\mathcal{I}(\partial\Omega^{(1)})$. By hypothesis, $\partial\Omega^{(1)}$ is incompressible in Ω, $H_{\partial\Omega} \geqslant 0$ and Ω is irreducible and does not contain one-sided compact embedded surfaces. Therefore, we can apply the version for three-manifolds with boundary of Theorem 7, to obtain a compact embedded surface $\overline{\Sigma} \in \mathcal{I}(\partial\Omega^{(1)})$ such that $|\overline{\Sigma}| = \alpha$. Note that $\overline{\Sigma} \in \mathcal{I}(\partial\Omega^{(1)})$ implies $\overline{\Sigma}$ has genus equal to $g(\Sigma)$.

Since all connected components of $\partial\Omega$ have nonnegative mean curvature, it follows from the maximum principle that either $\overline{\Sigma}$ is a boundary component of Ω or $\overline{\Sigma}$ is in the interior of Ω. If $\overline{\Sigma}$ is in the interior of Ω, then we obtain, by Theorem A, that $|\overline{\Sigma}| \geqslant 4\pi(g(\Sigma)-1)$ since $R_g \geqslant -2$ and $\overline{\Sigma}$ has genus equal to $g(\Sigma)$. On the other hand, we have $|\partial\Omega^{(1)}| = 4\pi(g(\Sigma)-1)$ because $\partial\Omega^{(1)}$ is isometric to (Σ, g_Σ). From this, we get $|\overline{\Sigma}| = 4\pi(g(\Sigma)-1)$. Now, if $|\overline{\Sigma}|$ is a boundary component of Ω, then we have that $\overline{\Sigma}$ is a minimal surface because $\overline{\Sigma}$ is area-minimizing and, by hypothesis, $\partial\Omega$ has nonnegative mean curvature with respect to the inward normal vector. This implies, using Theorem A, that $|\overline{\Sigma}| \geqslant 4\pi(g(\Sigma)-1)$. Again we conclude that $|\overline{\Sigma}| = 4\pi(g(\Sigma)-1)$. It follows from the previous arguments that we can suppose $\overline{\Sigma} = \partial\Omega^{(1)}$, that is, $\partial\Omega^{(1)}$ is area-minimizing.

By the proof of the rigidity in Theorem A, we have that there exists $\epsilon > 0$ such that the normal exponential map $f : [0,\epsilon)\times\overline{\Sigma} \longrightarrow \Omega$ defined by $f(t,x) = \exp_x(t\nu(x))$, where ν is the inward normal vector, is an injective local isometry.

Define $l = \sup\{t > 0 : f(t,x) = \exp_x(t\nu)$ is defined on $[0,t)\times\overline{\Sigma}$ and is an injective local isometry$\}$. Since Ω is compact, we have that the normal geodesics to $\overline{\Sigma}$ extend to $t = l$. Thus, f is defined on $[0,l]\times\overline{\Sigma}$. By continuity and the definition of l, we obtain that $f : [0,l]\times\overline{\Sigma} \longrightarrow \Omega$ is a local isometry. In particular, by continuity, the immersion $f : \overline{\Sigma}_l \longrightarrow \Omega$ is totally geodesic, where $\overline{\Sigma}_l = \{l\}\times\overline{\Sigma}$.

Using again the maximum principle, we obtain that either $f(\overline{\Sigma}_l)$ is a boundary component of Ω, different from $\overline{\Sigma}$ because of the injectivity of f on $[0,l)\times\overline{\Sigma}$, or $f(\overline{\Sigma}_l)$ is in the interior of Ω.

Suppose $f(\overline{\Sigma}_l)$ is a boundary component of Ω. Since f is a local isometry on $[0,l]\times\overline{\Sigma}$, we have $\dfrac{\partial f}{\partial t}(l,x)$ is a unit normal vector to $\overline{\Sigma}_l$. It follows from this that $f : \overline{\Sigma}_l \longrightarrow \Omega$ is injective because $f(\overline{\Sigma}_l)$ is a boundary component of Ω. Thus, $f : [0,l]\times\overline{\Sigma} \longrightarrow \Omega$ is an injective local isometry. This implies that

$f([0,l]\times\overline{\Sigma})$ is open in Ω since $f(\overline{\Sigma}_l)$ is a boundary component of Ω. Moreover, $f([0,l]\times\overline{\Sigma})$ is closed in Ω because $[0,l]\times\overline{\Sigma}$ is compact. Therefore, since Ω is connected, we obtain $f([0,l]\times\overline{\Sigma})=\Omega$. It follows that Ω is isometric to $[0,l]\times\overline{\Sigma}$.

Let us analyze the case where $f(\overline{\Sigma}_l)$ is in the interior of Ω. First, we have that $f:\overline{\Sigma}\longrightarrow\Omega$ cannot be injective. In fact, suppose $f:\overline{\Sigma}_l\longrightarrow\Omega$ is injective. Thus, by the rigidity in the Theorem A, there exists $\epsilon>0$ such that $f:[0,l+\epsilon)\times\overline{\Sigma}\longrightarrow\Omega$ is an injective local isometry which is a contradiction because of the maximality of l. Therefore, there exist $x,y\in\overline{\Sigma}$, $x\neq y$, such that $f(l,x)=f(l,y)$. We have $Df(l,x)(T\overline{\Sigma}_l)=Df(l,y)(T\overline{\Sigma}_l)$, since otherwise f would not be injective on $[0,l)\times\overline{\Sigma}$. This implies $\dfrac{\partial f}{\partial t}(l,x)=-\dfrac{\partial f}{\partial t}(t,y)$. Thus, since $f:\overline{\Sigma}_l\longrightarrow\Omega$ is totally geodesic, there exist neighborhoods of x and y in $\overline{\Sigma}_l$, respectively, such that the images by f of these neighborhoods coincide. We conclude that $\widehat{\Sigma}_l=f(\overline{\Sigma}_l)$ is a one-sided embedded compact surface in Ω. But, this is a contradiction because, by hypothesis, Ω does not contain any one-sided embedded compact surface. This concludes the proof of Theorem B.

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