Wronskian classes in the moduli space of curves

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Chapter 1

Introduction

Over the last few decades the study of algebraic curves experienced significant changes with the introduction of schemes and moduli spaces. For instance, instead of studying a fixed curve, we are also interested in how curves vary in families. And to do so, it is important to have a good parameter space, since, intuitively, a family of curves can be viewed as a continuous choosing of such parameters. It is important as well to understand its geometry.

Such a space, denoted by M_g, the moduli space of smooth curves of genus g (see Section 2.1), is actually a scheme of dimension $3g-3$, which is the dimension predicted by Riemann. It has the property that for every family $\pi : \mathcal{C} \to B$ of smooth genus-g curves, we have a map $B \to M_g$ that takes a (closed) point $b \in B$ to the (closed) point in M_g corresponding to the curve $\pi^{-1}(b)$.

This space can be constructed in several ways, by means of Teichmüller spaces, Hodge theory or Geometric invariant theory (GIT), see [22], chapter 2C. The construction sketched here, in Section 2.1, is via GIT.

Since M_g is not complete it is natural to ask for a compactification. There are several such compactifications. The one we are interested in is the Deligne–Mumford compactification ([6]), the space $\overline{M}_g$. This space is projective and has a modular description: it is the moduli space parametrizing stable curves (Defintion 2.1.1 and Theorem 2.1.2). Such a description is important, because by means of it we have a natural limit for any one-parameter family of smooth curves, in other words, every degeneration has a stable model ([22] chapter 3C).

Moreover, we can describe the boundary Δ of $\overline{M_g}$:

$$\Delta = \Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_{[g/2]}$$

where Δ_0 is the closure of the locus of irreducible nodal curves and Δ_i is the closure of the locus of curves that are the nodal union of a genus-i curve with a genus-$(g-i)$ curve. These Δ_i have codimension 1, and as such can be viewed as elements of the Picard group of $\overline{M}_g$ (see section 2.2).

The space M_g, however, does not admit a universal family, i.e., a family $\pi : \mathcal{C} \to M_g$ such that every other family of smooth curves is a pullback of this family via some map to M_g. In order to have universal families, we need to work in a category larger than that of schemes, the category of stacks.

To study the geometry of $\overline{M}_g$, it is important to understand its Picard group. We actually study the Picard group of the functor (or the Picard group of the stack) $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$ (see Definition 2.2.1). It is an easier group to work with because we have a universal family on which to do the computations. And, we have an isomorphism

$$\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q} \longrightarrow \mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g) \otimes \mathbb{Q}.$$

In the Picard group of the functor we have the boundary classes δ_i, those associated to the Δ_i, and the tautological classe λ defined in Section 2.2. And a theorem by Harer (Theorem 2.2.1) implies that $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g) \otimes \mathbb{Q}$ is freely generated by these classes for $g \geq 3$.

Given this description of the Picard group it is natural to ask: How is the canonical class expressed in terms of this basis? Can the formula for the canonical class can be used to prove that the canonical bundle is (very) ample?

We will present the answer for the first question in Chapter 4: the canonical class is $13\lambda - 2\delta - \delta_1$, where $\delta = \delta_0 + \ldots + \delta_{[g/2]}$; see [22], p. 160 as well. With this description of the class at hand, it turns out that to answer the second question in the affirmative we should be able to exhibit an effective divisor class D such that

$$q(13\lambda - 2\delta - \delta_1) - D \in \mathbb{Q}_+\lambda + \sum \mathbb{Q}_+\delta_i \tag{1.1}$$

for some $q \in \mathbb{Q}_+$. This leads us to ask:

What divisors of the form $a\lambda - b\delta$ are effective? (1.2)

More generally, we can try to characterize the effective cone of $\overline{M}_g$, i.e., the cone in $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{R}$ given by the effective divisors. Unfortunately, we still do not have any characterization for large genus. Nevertheless, several effective divisors are already known in terms of the Harer basis.

At any rate, the existence of a divisor satisfying Equation (1.1) proves that $\overline{M}_g$ is of general type, i.e., has maximum Kodaira dimension, which is weaker than very ampleness for the canonical class. As of now, it is known that $\overline{M}_g$ is of general type for genus $g \geq 22$ and has Kodaira dimension $-\infty$ for $g \leq 16$.

The first effective divisors to be computed by Harris and Mumford in the middle 80's [23] were the so-called Brill–Noether divisors. A Brill–Noether divisor is the closure of the locus in M_g of curves that have a g^r_d, for fixed r, d with $\rho := g - (r+1)(g+r-d) = -1$. Since this divisor satisfies (1.1) for $g \geq 24$, it follows that M_g is of general type for $g \geq 24$ and g odd. Later, Harris and Morrison conjectured a partial answer to Question (1.2): in order that $a\lambda - b\delta$ be effective, it is necessary that $\frac{a}{b} \geq 6 + \frac{12}{g+1}$ (this became known as the slope conjecture, [21]).

Harris's and Mumford's procedure to compute the Brill–Noether divisors was to write the divisors in the form

$$D := a\lambda - b_0\delta_0 - \ldots - b_{[g/2]}\delta_{[g/2]}$$

and intersect them with certain special one-parameter families, for which it was easy to compute their intersections with λ and the δ_i. It remained to compute the intersection of these families with D.

This method, often referred to as the meothod of test curves, was later used by Diaz in [7] and Cukierman in [2] to compute other divisors in $\overline{M}_g$: defined as the closure of the locus of curves possessing a special Weierstrass point of type $g-1$ in the first case and of type $g+1$ in the second case.

The other main tool that Harris and Mumford used was Hurwitz schemes, parametrizing rank-1 linear systems and their degenerations. Later the theory was extended, in a different format, to higher-rank linear systems, though only for curves of compact type, by Eisenbud and Harris [9].

Farkas and Popa found a counterexample to the slope conjecture ([12]). The divisor of $\overline{M}_{10}$ given by the closure of the locus of smooth curves sitting on a $K3$ surface has slope $7 < 6 + \frac{12}{11}$. In loc.cit., and in subsequent papers Farkas computed several divisors, but now, instead of curves that have a g^r_d with $\rho = -1$, the divisors were defined as (the closure of) the locus of curves that have a "special" g^r_d with $\rho = 0$. In the example above, the divisor in $\overline{M}_{10}$ is given by the locus of curves such that the canonical bundle (the only g^9_{18}) has a degenerated Wahl map, i.e., the map $\wedge^2 \mathrm{H}^0(\omega_C) \to \mathrm{H}^0(\omega_C^3)$ is not an isomorphism. Again the method used was the method of test curves.

More recently, in 2008, Cumino, Esteves and Gatto ([4],[5]) recomputed the Diaz and Cukierman divisors, but with a new approach. Instead of using test curves, the calculation was done over a general 1-parameter family of stable curves. Here the main tool was also the theory of limit linear series, but in a slightly more general format, i.e., working for nodal connected curves at the cost of losing some rigidity. This theory is explained in more detail in Section 5.1.

Our aim in this work is to compute the class of a certain effective divisor of $\overline{M}_g$ using the same methods Cumino, Esteves and Gatto used in [4]. This means computing the class over a general 1-parameter family, instead of computing it over test curves. The divisor considered in [4] was the closure of the locus of smooth curves having a special Weierstrass point, and the computation was based on computing the ramification divisor W of the relative dualizing line bundle ω_π of a family $\pi : \mathcal{X} \to T$ of curves (see Definition 5.1.4) and its "derivatives". There are two natural variants of this computation.

The first is: instead of considering the canonical bundle, consider the g^r_d with $\rho(g, d, r) = 0$; then our locus will be the closure of the locus of smooth curves having a g^r_d with a special ramification point. This means that instead of considering the relative dualizing sheaf of the family π, we consider the tautological linear system of the family $\mathcal{X} \times_T \mathcal{G}^r_d(\pi) \to \mathcal{G}^r_d(\pi)$ where $\mathcal{G}^r_d(\pi)$ is the T-scheme parametrizing the g^r_d of the fibers of the family π. This variant has a major difficulty: we do not clearly understand $\mathcal{G}^r_d(\pi)$. Nevertheless, there is work by Khosla ([24]) and Osserman ([32] and [33]) concerning $\mathcal{G}^r_d(\pi)$.

The second is: consider still the canonical bundle, but instead of imposing conditions on one point, impose them on more points (in our case two points). One simple example is the locus of curves having a pair of points (P, Q) with Q having ramification weight 3 in the linear system $\mathrm{H}^0(\omega_C(-aP))$, for some fixed a with $0 < a < g-1$. However, our computations take place now in the double product of the curve, which leads to new difficulties: since we are looking for classes of codimension 3 we might have excess in codimension 2, which is not as easy to remove as a divisorial excess. Moreover, the ramification of $\mathrm{H}^0(\omega_C(-aP))$, as C and P vary, might not form a flat family, which means that imposing conditions on its "derivatives" will not lower the dimension as expected (giving us an excess that is hard to remove). Fortunately, there are good conditions that do not lead us into these difficulties.

The divisor we compute, $\overline{\mathcal{R}_g}$ in $\overline{M}_g$, where $g = 2n$, is defined as the closure of the locus of smooth curves C with a pair of points (P, Q) such that P is a special ramification point of the linear system $\mathrm{H}^0(\omega_C(-nQ))$ and Q is a special ramification point of $\mathrm{H}^0(\omega_C(-nP))$. In fact, $\overline{\mathcal{R}_g}$ is going to be defined more carefully later (Definition 6.1.1), but its support in M_g is the locus above.

The work is organized as follows. In Chapters 2 and 3 we introduce the moduli scheme and the moduli stack; in Chapter 4 we review some theorems on intersection theory; in Chapter 5 we introduce the notions of limit linear systems and ramification schemes. Finally, in Chapter 6 we compute the class of the divisor $\overline{\mathcal{R}_g}$ in $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$.

We will always work over $\mathbb{C}$. A *curve* for us is a connected, projective, reduced scheme of dimension 1 (over $\mathbb{C}$). The *genus* of a curve C is its arithmetic genus, $\mathrm{h}^1(C, \mathcal{O}_C)$. We will often identify a vector bundle with its sheaf of sections; most of the times it will be clear which one we are considering. The category (*schemes*) is the category of schemes over $\mathbb{C}$.

Chapter 2

The moduli space of curves

2.1 Definition and construction

In this section, we are going to introduce the main object we are interested in: the moduli space of curves of a given genus. This is a space parametrizing smooth (or stable for its compactification) curves; in other words, we want that the (closed) points of this space represent curves. Moreover, this space has a structure, of a scheme of finite type. To formalize this, let us define the moduli functor and the notion of representability.

We start by considering a class of objects (these objects could be schemes, sheaves, linear systems, etc.) for which we have the notion of what is a family parametrized by a scheme B, together with the notion of an equivalence relation $\sim$ on the set $S(B)$, which is the set of all families of such objects over B. The notion of family should also include a notion of fiber product, which means that given a morphism $B' \to B$ and a family over B, we have a family over B' in which the fiber over a point $b' \in B'$ is the same as the fiber over its image.

For example, in our case of interest, our objects are smooth curves of a given genus g, a family over B is a flat map $\pi : \mathcal{X} \to B$ whose fibers are smooth (or stable) curves, and two families are equivalent if they are isomorphic over the base. A second example consists of vector bundles over a smooth curve C, where a family over B is a vector bundle over $C \times B$, and two families are equivalent if the bundles are isomorphic.

A moduli functor classifying such a class of objects is the contravariant functor:

$$\begin{array}{rccc} F & : & (schemes) & \to & (sets) \\ & & B & \mapsto & S(B)/\sim \end{array} \tag{2.1}$$

On the other hand, if $\mathcal{M}$ is any scheme, we define the functor of points of $\mathcal{M}$:

$$\begin{array}{rccc} \mathrm{Mor}_{\mathcal{M}} & : & (schemes) & \to & (sets) \\ & & B & \mapsto & \mathrm{Mor}(B, \mathcal{M}) \end{array}$$

We say that F is representable by $\mathcal{M}$ if there is an (natural) isomorphism of functors $\Psi : F \to \mathrm{Mor}_{\mathcal{M}}$. In this case, we call $\mathcal{M}$ the fine moduli space for our class of objects.

If F is representable by $\mathcal{M}$, and $\phi : \mathcal{X} \to B$ is any family in $S(B)$, then $\chi := \Psi(\phi)$ is a morphism from B to $\mathcal{M}$, which intuitively tells us that $\mathcal{M}$ parametrizes our objects, χ mapping a closed point b of B to the point in $\mathcal{M}$ corresponding to the fiber $\mathcal{X}_b$. More precisely, the inverse image by Ψ of the indentity map of $\mathcal{M}$ gives us a family $\pi : \mathcal{C} \to \mathcal{M}$, called the universal family, such that for any morphism $\chi : B \to \mathcal{M}$ we have a fiber diagram:

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{C} \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ B & \xrightarrow{\chi} & \mathcal{M} \end{array}$$

where $\Psi(\phi) = \chi$, Furthermore, since Ψ is an isomorphism, every family over B is a pullback of the universal family by some morphism χ.

Unfortunately, such fine moduli spaces exist only in a few cases. Indeed, if there were a fine moduli space M_g parametrizing smooth genus-g curves, then all families that have isomorphic fibers would be trivial, since the corresponding map χ would be constant, but there are nontrivial families with isomorphic fibers. For instance, let C be a genus-g curve with a nontrivial automorphism group G and E an elliptic curve such that G acts on E by translation, then the family $(C \times E)/G \to E/G$ is an nontrivial family with all fibers isomorphic to C. To be sure, these families have as fibers curves with nontrivial automorphisms, and we do have a fine moduli space parametrizing smooth curves of genus g with no nontrivial automorphisms.

On the other hand, what if instead of an isomorphism Ψ we require only a natural map from F to $\mathrm{Mor}_{\mathcal{M}}$? We would still get that for each family $\phi : \mathcal{X} \to B$ we have a map $\chi = \Psi(\phi) : B \to \mathcal{M}$, and also, given a map $\xi : B' \to B$, and the induced family $\phi' : \mathcal{X} \times_B B' \to B'$, we have $\Psi(\phi') = \chi \circ \xi$. But in this case we may not have uniqueness, allowing completely pathological cases. For instance, we always have a natural map from F to $\mathrm{Mor}_{\mathrm{Spec}(\mathbb{C})}$.

The answer then is, first, to impose that we have a bijection between the closed points of $\mathcal{M}$ and our objects, i.e., that the map

$$\Psi(\mathrm{Spec}(\mathbb{C})) : F(\mathrm{Spec}(\mathbb{C})) \to \mathrm{Mor}(\mathrm{Spec}(\mathbb{C}), \mathcal{M})$$

be a bijection. And second, that $\mathcal{M}$ be universal, in the sense that, given a scheme $\mathcal{M}'$ and a map $\Psi' : F \to \mathrm{Mor}_{\mathcal{M}'}$, we have a map $\mathcal{M} \to \mathcal{M}'$ inducing Ψ' from Ψ. If there exist such M and Ψ we call M the coarse moduli space of the functor F.

One of the few cases where we do have a fine moduli space is the Hilbert scheme $H_{P,r}$, where P is a polynomial and r is a positive integer. This is the moduli space of the functor $\mathrm{Hilb}_{P,r}$, which assigns to a scheme B the set of B-flat closed subschemes of $\mathbb{P}^r \times B$ whose fibers over B have Hilbert polynomial P. The construction of $H_{P,r}$ was first carried out in [19].

Let's now state the main results about the existence of moduli spaces in the case of curves.

Theorem 2.1.1 *There is a coarse moduli space M_g parametrizing smooth curves of genus $g \geq 2$, which contains as an open subscheme the fine moduli space M_g^0 parametrizing smooth curves of genus g without nontrivial automorphisms. The scheme M_g is quasi-projective and has only quotient singularities.*

Since M_g is not complete, it is natural to search for a compactification of it, since we have many more tools to understand the geometry of a projective variety than of a simply quasi-projective one. Fortunately, this compactification can also be viewed as a moduli space:

Definition 2.1.1 *A genus-g, $g \geq 2$,* stable *curve C is a curve whose singularities are only ordinary double points (nodes) and whose smooth rational components E satisfy $\#E \cap \overline{C \backslash E} \geq 3$.*

Theorem 2.1.2 *There is a coarse moduli space $\overline{M}_g$ parametrizing stable curves of genus g. The scheme $\overline{M}_g$ is projective and has only quotient singularities.*

We denote the (closed) point in $\overline{M}_g$ corresponding to a genus-g stable curve C by $[C]$.

Let's give a brief idea how $\overline{M_g}$ is constructed. Given a stable curve C of genus $g \geq 2$, let ω_C denote its dualizing sheaf. Since ω_C^n is very ample for $n \geq 3$, by [6], its sections give us an embedding

$$C \to \mathbb{P}^N$$

of degree $2n(g-1)$, where, by Riemann–Roch,

$$N = \mathrm{h}^0(C, \omega_C^n) - 1 = (2n-1)(g-1) - 1.$$

Seeing C as a subscheme of $\mathbb{P}^N$ we get that $\omega_C^n = \mathcal{O}_C(1)$ (we call such a curve *n-canonically embedded*). The Hilbert polynomial of C is

$$P(T) := 2n(g-1)T + 1 - g.$$

Let now $H := H_{P,N}$ be the Hilbert scheme parametrizing subschemes of $\mathbb{P}^N$ with P as Hilbert polynomial. Let $H' \subset H$ be the open subscheme corresponding to nodal curves. There is a family $\mathcal{U} \subset \mathbb{P}^N \times H'$ over H', which is simply the restriction of the universal family over H. This family $\mathcal{U}$ admits a relative Picard algebraic space $\mathrm{Pic}_{\mathcal{U}/H'}$ over H', for which the sheaves $\omega^n_{\mathcal{U}/H'}$ and $\mathcal{O}_{\mathcal{U}}(1)$ induce a map

$$H' \to \mathrm{Pic}_{\mathcal{U}/H'} \times_{H'} \mathrm{Pic}_{\mathcal{U}/H'}.$$

Define now K as the preimage of the diagonal. Then K is locally closed in H', because so is the diagonal in $\mathrm{Pic}_{\mathcal{U}/H'} \times_{H'} \mathrm{Pic}_{\mathcal{U}/H'}$, and is smooth (see Lemma 3.35 in [22]). We get a family $\nu : \mathcal{V} \to K$ induced by the subscheme $\mathcal{V} \subset \mathbb{P}^N \times K$, the restriction of $\mathcal{U}$ over K.

Since the group of automorphisms $\mathrm{PGL}(N+1)$ of $\mathbb{P}^N$ acts naturally on H, there is an induced action $\mathrm{PGL}(N+1) \times K \to K$. For sufficiently large n, we can show, via GIT, that there is a geometric quotient of K under this action (see [30] for smooth curves and, more generally, [17] for stable curves), which we denote by $\overline{M_g}$.

Now we have to construct the map Ψ between the moduli functor of stable curves and $\mathrm{Mor}_{\overline{M}_g}$. To construct this map, we see first that the family ν is versal, i.e, given a family $\phi : \mathcal{C} \to B$ of genus-g stable curves, there are an open covering U_α of B, and maps $\widetilde{\chi_\alpha} : U_\alpha \to K$ such that the restriction

$$\phi_\alpha : \mathcal{C}_\alpha := \phi^{-1}(U_\alpha) \to U_\alpha$$

is the pullback of ν via $\widetilde{\chi_\alpha}$.

Indeed, let $\phi : \mathcal{C} \to B$ be a family of genus-g stable curves, and consider the n-th power of the relative dualizing sheaf ω_ϕ^n, for $n \geq 3$. Since, by Riemann–Roch, $\mathrm{h}^0(C, \omega_C^n) = (2n-1)(g-1)$ for every stable curve C of genus g, we have that $\phi_*(\omega_\phi^n)$ is a vector bundle of rank $(2n-1)(g-1)$. Choose now an open covering U_α trivializing this bundle; this means that we have an isomorphism $\sigma_\alpha : \mathcal{O}_{U_\alpha}^{\oplus N+1} \to \phi_*(\omega_\phi^n)|_{U_\alpha}$. This isomorphism induces a map $\mathcal{C}_\alpha \to \mathbb{P}^N \times U_\alpha$, which is an embedding because the fibers of ϕ are stable and $n \geq 3$. And since this is a n-canonical embedding by construction, we get a fiber diagram

$$\begin{tikzcd} \mathcal{C}_\alpha \arrow[r] \arrow[d, "\phi_\alpha"'] & \mathcal{V} \arrow[d, "\nu"] \\ U_\alpha \arrow[r, "\widetilde{\chi_\alpha}"] & K \end{tikzcd}$$

from the universal property of the Hilbert scheme.

Composing this map with the quotient map, we get a collection of maps $\chi_\alpha : U_\alpha \to \overline{M_g}$. But on each double intersection $U_\alpha \bigcap U_\beta$ the maps $\widetilde{\chi_\alpha}$ and $\widetilde{\chi_\beta}$ differ only by the choice of the isomorphisms σ_α and σ_β, and this difference is given by an element of $\mathrm{PGL}(N+1)$. So the maps χ_α and χ_β agree where they are both defined, and hence we get a morphism $\chi : B \to \overline{M_g}$ gluing them.

It is easy to see that we have a bijection between the set of families over $\mathrm{Spec}(\mathbb{C})$ (which are just curves) and the set of closed points of $\overline{M_g}$, because if two n-canonically embedded stable curves are isomorphic then their images in $\mathbb{P}^N$ differ by an automorphism of $\mathbb{P}^N$, and if there is an automorphism taking the image of one curve to the image of the other, then these curves are clearly isomorphic.

For the universality just observe that $\nu : \mathcal{V} \to K$ is a family of stable curves, so if M is a scheme with a natural map Ψ_M from the moduli functor to Mor_M, we get a map $\Psi_M(\nu) : K \to M$. Also, composing this map with an automorphism of K given by an element of $\mathrm{PGL}(N+1)$, we still get the same map, since acting on K by an element of $\mathrm{PGL}(N+1)$ does not change the abstract family ν. Then we have a map $\overline{M_g} \to M$, which induces the map Ψ_M.

Given $\overline{M}_g$, we would like to study its properties, for instance, we may try to characterize its canonical bundle. To do this we need first to understand its tangent bundle, for which goal we use deformation theory. The tangent space at a point $[C]$ of M_g^0 consists of the maps from $\mathbb{I} := \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$ to M_g that take the closed point to $[C]$. These maps induce families $\pi : \mathcal{C} \to \mathbb{I}$ with special fiber C (because there is a universal family over M_g^0). These families are called first-order deformations of C, and the space of first-order deformations can be identified with $\mathrm{H}^1(C, T_C)$, where T_C is the tangent bundle of C; see [22] Section 3-B for more details.

This description can be extended to stable curves, but now the space of first-order deformations of a stable curve C is given by $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$, see [6]. Therefore, by duality, the cotangent space at a point $[C]$ of $\overline{M_g}^0$, the locus of stable curves with no nontrivial automorphisms, is $\mathrm{H}^0(C, \Omega^1_C \otimes \omega_C)$. So, if $\pi : \mathcal{C} \to \overline{M}_g^0$ is the universal family, then the canonical bundle over $\overline{M_g}^0$ is simply

$$\bigwedge^{3g-3} \pi_*(\Omega^1_\pi \otimes \omega_\pi).$$

This definition extends nicely to that of the canonical bundle of the stack, i.e., we can still define the tangent bundle using maps from $\mathbb{I}$ to the moduli stack. But over the moduli stack we do have a universal family. To extend the canonical bundle to $\overline{M_g}$ we need to notice that the stack is simply ramified along Δ_1 over $\overline{M}_g$; see Section 3.2.

2.2 The Picard group of $\overline{M_g}$

Let $\overline{M_g}$ be the coarse moduli space parametrizing stable curves. Let $A^1(\overline{M_g})$ be its Chow group of codimension-1 cycles and $\mathrm{Pic}(\overline{M_g})$ its Picard group. Since $\overline{M_g}$ has only finite quotient singularities, we have an isomorphism

$$A^1(\overline{M_g}) \otimes \mathbb{Q} \to \mathrm{Pic}(\overline{M_g}) \otimes \mathbb{Q}.$$

Indeed, if Y is a codimension-1 subvariety of $\overline{M_g}$, there exists $d \in \mathbb{N}$ such that $d[Y]$ is the class of a Cartier divisor.

Another useful notion will be that of the Picard group of the functor $\mathrm{Pic}_{\mathrm{fun}}(\overline{M_g})$, in which instead of looking at the divisors in $\overline{M_g}$ we see them in all families of stable curves:

Definition 2.2.1 *An element $\gamma \in \mathrm{Pic}_{\mathrm{fun}}(\overline{M_g}) \otimes \mathbb{Q}$ is a collection of classes $\gamma_\pi \in \mathrm{Pic}(B) \otimes \mathbb{Q}$ for each family of stable curves $\pi\colon \mathcal{C} \to B$, such that for each fiber product*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{f} & \mathcal{C} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[i]{} & B \end{array} \tag{2.2}$$

we have $\gamma_{\pi'} = i^*(\gamma_\pi)$.

Given this new Picard group, one can ask how it is related to the old one. In fact, it can be proved that $\mathrm{Pic}_{\mathrm{fun}}(\overline{M_g}) \otimes \mathbb{Q}$ is isomorphic to $\mathrm{Pic}(\overline{M_g}) \otimes \mathbb{Q}$ (see [22], Proposition 3.88). The map giving the isomorphism is actually easy to explain: Given a element $\Gamma \in \mathrm{Pic}(\overline{M_g})$ and a family $\phi : \mathcal{C} \to B$ we can pull back Γ to $\mathrm{Pic}(B)$ via the map $\chi : B \to \overline{M_g}$ induced by ϕ.

One last useful incarnation of the Picard group, is the invariant subgroup $\mathrm{Pic}(K)^{\mathrm{PGL}(N+1)} \subset \mathrm{Pic}(K)$, where K is the moduli space of n-canonically embedded stable curves defined in Section 2.1. There is a natural isomorphism (see [23], p.50)

$$\mathrm{Pic}_{\mathrm{fun}}(\overline{M_g}) \to \mathrm{Pic}(K)^{\mathrm{PGL}(N+1)} \tag{2.3}$$

that to a given class $\gamma \in \mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$ associates γ_ν, where $\nu : \mathcal{V} \to K$ is the universal family.

Now, let's define the generators of $\mathrm{Pic}(\overline{M_g}) \otimes \mathbb{Q}$. We begin with the tautological class λ, which is naturally defined in $\mathrm{Pic}_{\mathrm{fun}}(\overline{M_g})$. Given a family $\phi : \mathcal{C} \to B$ of stable curves, let ω_ϕ be its dualizing sheaf; then $\lambda_\phi := \det(\phi_*(\omega_\phi))$.

The other classes are the boundary classes Δ_i; these are naturally defined in $A^1(\overline{M_g})$: Δ_0 is the closure of the locus in $\overline{M_g}$ parametrizing irreducible singular curves, and Δ_i, for $i = 1, \ldots, [g/2]$, is the closure of the locus in $\overline{M_g}$ parametrizing reducible curves which are the union of two curves, one of genus i and the other of genus $g - i$, meeting at a single point. We also denote $\Delta := \bigcup \Delta_i$.

Similarly, we can define the classes $\Delta_{i,K}$ in $\mathrm{Pic}(K)$: $\Delta_{0,K}$ is the closure of the locus in K parametrizing n-canonically embedded irreducible singular curves, and $\Delta_{i,K}$, for $i = 1, \ldots, [g/2]$, is the closure of the locus in K parametrizing n-canonically embedded reducible curves which are the union of two curves, one of genus i and the other of genus $g - i$, meeting at a single point. Clearly, these classes are invariant under the action of $\mathrm{PGL}(N + 1)$, therefore they induce classes $\delta, \delta_0, \ldots, \delta_{[g/2]}$ in $\mathrm{Pic}_{\mathrm{fun}}(\overline{M_g})$. (For the relations among Δ_i and δ_i see [22] page 147.) Then:

Theorem 2.2.1 (Harer) $\mathrm{Pic}_{\mathrm{fun}}(\overline{M_g}) \otimes \mathbb{Q}$ *is freely generated by* $\lambda, \delta_0, \delta_1, \cdots, \delta_{[g/2]}$ *for* $g \geq 3$.

More precisely, we have

Theorem 2.2.2 (Arbarello-Cornalba) $\mathrm{Pic}_{\mathrm{fun}}(\overline{M_g})$ *is freely generated by* $\lambda, \delta_0, \ldots, \delta_{[g/2]}$ *for* $g \geq 3$.

If $g = 2$, $\mathrm{Pic}_{\mathrm{fun}}(\overline{M_g})$ is generated by δ_0 and δ_1, while λ is expressed by Mumford's relation:

$$10\lambda = \delta_0 + 2\delta_1.$$

Furthermore, and this is very important for our calculations, one can show that a class in $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g) \otimes \mathbb{Q}$ is defined by its value on 1-parameter families, i.e., if we have two classes γ, γ' such that for every family $\phi : \mathcal{C} \to B$ over a 1-dimensional scheme B, where we can even assume $\mathcal{C}$ smooth, we have $\gamma_\phi = \gamma'_\phi$ in $\mathrm{Pic}(B) \otimes \mathbb{Q}$ then $\gamma = \gamma'$. It is not even necessarily to consider all families ϕ but just one "sufficiently general".

Chapter 3

The moduli stack of curves

Our purpose in this chapter is to introduce the language of stacks. Although this is not necessary for the understanding of the following chapters, it enables us to properly define the moduli stack of smooth (resp. stable) curves, understand its geometry and its relation to the coarse moduli space. Our goal is simply to give an idea of what is meant by the moduli stack.

Here we follow [6] and [31].

3.1 Grothendieck topologies and sheaves

Before we define stacks, let's first define Grothendieck topologies and algebraic spaces.

Definition 3.1.1 *Let* $\mathbf{C}$ *be a category with fiber product. A* Grothendieck topology τ *over* $\mathbf{C}$ *is the data of, for every object* U *of* $\mathbf{C}$*, a collection* $\tau(U)$ *of families of morphisms* $\{U_i \to U\}_{i\in I}$ *such that:*

1. *If* $U' \to U$ *is an isomorphism then* $\{U' \to U\}$ *is in* $\tau(U)$*.*

2. *If* $\{U_i \to U\}_{i\in I}$ *is in* $\tau(U)$ *and* $\{U_{ij} \to U_i\}_{j\in J_i}$ *is in* $\tau(U_i)$ *for every* $i \in I$*, then* $\{U_{ij} \to U_i \to U\}_{i\in I, j\in J_i}$ *is in* $\tau(U)$*.*

3. *If* $\{U_i \to U\}_{i\in I}$ *is in* $\tau(U)$*, and* $V \to U$ *is a morphism, then* $\{V \times_U U_i \to V\}_{i\in I}$ *is in* $\tau(V)$*.*

The families in $\tau(U)$ are called covering families for U in the topology τ. A category $\mathbf{C}$ with fiber products and a Grothendieck topology τ is called a *site*, and denoted $\mathbf{C}_\tau$.

For example, for the category $(schemes)$, we can define $\tau(U)$ as the collection of the families $\{U_i \to U\}_{i\in I}$ whose morphisms are open embeddings satisfying $\bigcup_{i\in I} U_i = U$; the result is called the big Zarsiki site $(schemes)_{Zar}$. If, instead of imposing the morphisms to be open embeddings, we impose them to be étale morphisms, we get the big étale site $(schemes)_{et}$. And since open embeddings are étale morphisms, we get an inclusion $(schemes)_{Zar} \subset (schemes)_{et}$.

Definition 3.1.2 *A* presheaf *of sets on a category* $\mathbf{C}$ *is a contravariant functor* $\mathcal{F} : \mathbf{C} \to (sets)$*. Morphisms of presheaves are just natural transformations of functors.*

A sheaf is a presheaf with glueing conditions:

Definition 3.1.3 *A* sheaf *on a site* $\mathbf{C}_\tau$ *is a presheaf of sets* $\mathcal{F}$ *with the following properties:*

1. *For every object* U *of* $\mathbf{C}$*, and* $f, g \in \mathcal{F}(U)$ *such that for a covering family* $\{U_i \to U\}_{i \in I}$ *in* $\tau(U)$ *we have* $f|_{U_i} = g|_{U_i}$ *for every* $i \in I$*, we have* $f = g$*.*

2. *For every covering family* $\{U_i \to U\}_{i \in I}$ *in* $\tau(U)$ *and collection* $\{f_i \in \mathcal{F}(U_i)\}_{i \in I}$ *such that* $f_i|_{U_i \times_U U_j} = f_j|_{U_i \times_U U_j}$ *for all* $i, j \in I$*, there exists* $f \in \mathcal{F}(U)$ *such that* $f|_{U_i} = f_i$ *for all* $i \in I$*.*

Theorem 3.1.1 *(*Grothendieck*) For any scheme* X*, the functor of points of* X*,* $h_X := \mathrm{Mor}(-, X)$*, is a sheaf in the étale topology, that is, a sheaf in the site* $(schemes)_{et}$*.*

Proof. See [35].

This theorem and Yoneda Lemma show that the category $(schemes)$ is a full subcategory of the category of sheaves over $(schemes)_{et}$. When no confusion can be made we will still denote by X the sheaf induced of X. This theorem leads us to the notion of representability:

Definition 3.1.4 *A sheaf* $\mathcal{F}$ *over* $(schemes)_{et}$ *is called* representable *if there exists a scheme* X *and a natural isomorphism of functors*

$$\mathcal{F}(-) \cong \mathrm{Mor}(-, X).$$

To end the section, we define an algebraic space.

Definition 3.1.5 *An algebraic space* $\mathcal{X}$ *is a sheaf over* $(schemes)_{et}$ *such that*

1. *For all schemes* Y *and* Z *and all morphisms of sheaves* $y : Y \to \mathcal{X}$ *and* $z : Z \to \mathcal{X}$*, the sheaf* $Y \times_{\mathcal{X}} Z$ *is representable by a scheme.*

2. *There exist a scheme* X*, called an atlas, and a surjective étale morphism* $x : X \to \mathcal{X}$ *(i.e., for all morphisms* $y : Y \to \mathcal{X}$*, where* Y *is a scheme, the projection* $X \times_{\mathcal{X}} Y \to Y$ *is a surjetive étale morphism of schemes).*

Again, the category $(schemes)$ is a full subcategory of the category of algebraic spaces. Algebraic spaces can also be characterized as quotients of a scheme by a smooth or étale equivalence relation. In the next section, we will see that algebraic spaces are also algebraic stacks.

3.2 Algebraic stacks

In this section, we will describe the data that are necessary to define an algebraic stack, what these data are in the special case of the moduli stack of stable curves, and how this moduli stack is related to the moduli space. In fact, we will see that the moduli stack of stable curves is a very special kind of stack, a Deligne–Mumford stack.

We begin by defining a 2-category:

Definition 3.2.1 *A* 2-category **C** *is given by the following data:*

1. *A collection of objects $ob(\mathcal{C})$.*
2. *For each pair of objects $X, Y \in ob(\mathcal{C})$ a category* $\mathrm{Hom}(X, Y)$.
3. *For every $X, Y, Z \in ob(\mathcal{C})$ a functor*

$$\mu_{X,Y,Z} : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

   such that:

   (a) *For each $X \in ob(\mathcal{C})$, there exists an object $id_X \in \mathrm{Hom}(X, X)$ such that*

$$\mu_{X,X,Y}(\mathrm{id}_X, -) = \mu_{X,Y,Y}(-, id_Y) = id_{\mathrm{Hom}(X,Y)}$$

   where $id_{\mathrm{Hom}(X,Y)}$ is the identity functor.

   (b) *For every $X, Y, Z, W \in ob(\mathcal{C})$, we have*

$$\mu_{X,Z,W} \circ (\mu_{X,Y,Z} \times \mathrm{id}_{\mathrm{Hom}(Z,W)}) = \mu_{X,Y,W} \circ (\mathrm{id}_{\mathrm{Hom}(X,Y)} \times \mu_{Y,Z,W}).$$

The objects of $\mathrm{Hom}(X, Y)$ are called 1-morphisms and denoted $f : X \to Y$, while the morphisms of $\mathrm{Hom}(X, Y)$ are called 2-morphisms and denoted $\alpha : f \Rightarrow g$. We can see a category as a 2-category by imposing that the category $\mathrm{Hom}(X, Y)$ has as objects the morphisms between X and Y and as morphisms the identity maps. And a 2-category can be transformed into a category by keeping the objects and imposing that the morphisms are the equivalence classes of isomorphisms of $\mathrm{Hom}(X, Y)$.

Two objects X and Y of a 2-category **C** are equivalent if there exist two 1-morphisms $f : X \to Y$, $g : Y \to X$ and two 2-isomorphisms $\alpha : g \circ f \Rightarrow id_X$ and $\beta : f \circ g \Rightarrow id_Y$, where $g \circ f = \mu_{X,Y,X}(f, g)$ and $f \circ g = \mu_{Y,X,Y}(g, f)$.

The 2-category we are interested in is the 2-category of groupoids. This is the 2-category that will replace the category (*sets*) in the previous section.

Definition 3.2.2 *A* groupoid *is a category whose morphisms are invertible, i.e., isomorphisms.*

Every set can be viewed as a groupoid whose objects are the elements of the set and whose morphisms are the identity morphisms.

Definition 3.2.3 *Let* **Grpds** *be the 2-category of groupoids. Its objects are groupoids, and for each grupoids $\mathcal{G}, \mathcal{H}$,* $\mathrm{Hom}(\mathcal{G}, \mathcal{H})$ *is the category of functors between $\mathcal{G}$ and $\mathcal{H}$ with morphisms being the natural transformations.*

A *prestack* is the analogue of presheaves, replacing the target category (*sets*) by the 2-category **Grpds**:

Definition 3.2.4 *Let* $\mathbf{C}$ *be a category. A prestack* $\mathcal{X}$ *over* $\mathbf{C}$ *is a pseudo-functor*

$$\mathcal{X} : \mathbf{C} \to \mathbf{Grpds}$$

i.e., the following data

1. *For each* $U \in ob(\mathbf{C})$ *an object* $\mathcal{X}(U)$ *in* **Grpds**.

2. *For each morphism* $f : X \to Y$ *in* $\mathbf{C}$ *a functor*

$$f^* := \mathcal{X}(f) : \mathcal{X}(Y) \to \mathcal{X}(X)$$

3. *For each* $f : X \to Y$ *and* $g : Y \to Z$ *in* $\mathbf{C}$*, an ivertible natural transformation*

$$\epsilon_{g,f} : (g \circ f)^* \Rightarrow f^* \circ g^*$$

such that the following diagram is commutative, for every $h : Z \to W$ *in* $\mathbf{C}$*,*

$$\begin{array}{ccc} (h \circ g \circ f)^* & \stackrel{\epsilon_{h,g\circ f}}{\Longrightarrow} & (g \circ f)^* \circ h^* \\ \Downarrow \epsilon_{h\circ g,f} & & \Downarrow \epsilon_{g,f} * \mathrm{id}_{h^*} \\ f^* \circ (h \circ g)^* & \stackrel{\mathrm{id}_{f^*} * \epsilon_{h,g}}{\Longrightarrow} & f^* \circ g^* \circ h^* \end{array}$$

where $\epsilon_{g,f} * \mathrm{id}_{h^*}$ *is the natural transformation that associates to every object* G *in* $\mathcal{X}(Z)$ *the morphism*

$$(\epsilon_{g,f} * \mathrm{id}_{h^*})_G := (\epsilon_{g,f})_{h^*(G)} : (g \circ f)^*(h^*(G)) \longrightarrow f^* \circ g^*(h^*(G))$$

and analogously for $\mathrm{id}_{f^*} * \epsilon_{h,g}$.

A stack is just a prestack with gluing conditions:

Definition 3.2.5 *Let* $\mathbf{C}_\tau$ *be a site. A stack is a prestack* $\mathcal{X}$ *satisfying for every covering family* $\{U_i \to U\}_{i\in I}$ *the following three conditions:*

1. *Given* $x_i \in ob(\mathcal{X}(U_i))$ *and isomorphisms* $\varphi_{ij} : x_i|_{U_{ij}} \to x_j|_{U_{ij}}$ *in* $\mathcal{X}(U_{ij})$ *satisfying*

$$\varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}},$$

there exist $x \in ob(\mathcal{X}(U))$ *and isomorphisms* $\varphi_i : x|_{U_i} \to x_i$ *in* $\mathcal{X}(U_i)$ *such that*

$$\varphi_{ij} \circ \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$$

for all $i, j \in I$.

2. *Given objects $x, y \in ob(\mathcal{X}(U))$ and isomorphisms $\varphi_i : x|_{U_i} \to y|_{U_i}$ such that $\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$, there exists an isomorphism $\eta : x \to y$ such that $\eta|_{U_i} = \varphi_i$.*

3. *Given $x, y \in ob(\mathcal{X}(U))$ and isomorphisms $\varphi : x \to y$ and $\psi : x \to y$ such that $\varphi|_{U_i} = \psi|_{U_i}$, we have $\varphi = \psi$*

For instance, every sheaf $\mathcal{F}$ on $\mathbf{C}_\tau$ can be seen as a stack. Just consider the set $\mathcal{F}(X)$ as a category with just identity morphisms. In particular every algebraic space is a stack, and every scheme is a stack.

In our case, we have the moduli stack of genus-g smooth (resp. stable) curves $\mathcal{M}_g$ (resp. $\widetilde{\mathcal{M}_g}$) over site $(schemes)_{et}$. For an object U, $\mathcal{M}_g(U)$ (resp. $\widetilde{\mathcal{M}_g}(U)$) is the groupoid whose objects are families of smooth (resp. stable) genus-g curves $\pi : \mathcal{C} \to U$ and whose morphisms are isomorphims over U between these families. For a morphism $f : U' \to U$, $\mathcal{M}_g(f)$ is the functor $f^* : \mathcal{M}_g(U) \to \mathcal{M}_g(U')$ given by base change. For morphisms $f : U'' \to U'$ and $g : U' \to U$, the natural transformation $\epsilon_{g,f}$ is given by the natural isomorphisms $\mathcal{C} \times_U U'' \to (\mathcal{C} \times_U U') \times_{U'} U''$. For the fact that $\widetilde{\mathcal{M}_g}$ is in fact a stack over $(schemes)_{et}$ and not just a pre-stack we refer to [20], VIII, 7.8.

We now define morphisms between stacks.

Definition 3.2.6 *Let $\mathbf{C}$ be a category. A 1-morphism between two prestacks $\mathcal{X}$ and $\mathcal{Y}$ over $\mathbf{C}$ is a natural transformation of pseudo-functors of 2-categories $F : \mathcal{X} \to \mathcal{Y}$, i.e., is given by the following data:*

1. *for every $Z \in ob(\mathbf{C})$, a functor $F_Z : \mathcal{X}(Z) \to \mathcal{Y}(Z)$.*

2. *for every morphism $f : Z \to W$ in $\mathbf{C}$, an invertible natural transformation $F_f : \mathcal{Y}(f) \circ F_W \Rightarrow F_Z \circ \mathcal{X}(f)$, which is compatible with the natural transformations*

$$\epsilon_{g,f} : (g \circ f)^* \Rightarrow f^* \circ g^*.$$

Definition 3.2.7 *Let $\mathbf{C}$ be a category. A 2-morphism ψ between two 1-morphism of prestacks $F, G : \mathcal{X} \to \mathcal{Y}$ associates to each $X \in ob(\mathbf{C})$ a 2-morphism $\psi_X : F_X \Rightarrow G_X$ of* **Grpds***, such that, for a morphism $f : X \to Y$, the 2-morphisms ψ_X and ψ_Y are compatible with the natural transformations F_f and G_f.*

The 1- and 2-morphisms between stacks are the the 1- and 2-morphisms between the underlying prestacks. With these two definitions we can see that prestacks (resp. stacks) over a category $\mathbf{C}$ (resp. site $\mathbf{C}_\tau$) together with 1-morphisms and 2-morphisms form a 2-category **Prestacks**($\mathbf{C}$) (resp. **Stacks**($\mathbf{C}_\tau$)). Therefore the category of sheaves is a full 2-subcategory of the 2-category **Stacks**($\mathbf{C}_\tau$).

We have now a Yoneda Lemma for stacks:

Theorem 3.2.1 *Let $\mathcal{X}$ be a stack over $\mathbf{C}_\tau$. Then for any $X \in ob(\mathbf{C})$ there is an equivalence of categories*

$$\begin{array}{rcl} \Theta : \mathrm{Mor}(X, \mathcal{X}) & \to & \mathcal{X}(X) \\ (F : X \to \mathcal{X}) & \mapsto & F(id_X) \end{array}$$

Proof. See [31].

In particular, this theorem implies that a moduli functor is always representable in the category of stacks. Moreover, every morphism $x : X \to \widetilde{\mathcal{M}_g}$ is given by a family $\pi : \mathcal{C} \to X$ of genus-g stable curves, where $\pi \in \Theta(x)$. (Up to isomorphism, there is only one choice of π.) More precisely, to put it in terms of Definition 3.2.6, the morphism x is the collection of functors:

$$\begin{array}{rcl} x_U : Mor(U, X) & \to & \widetilde{\mathcal{M}_g(U)} \\ u : U \to X & \mapsto & \mathcal{C} \times_X U. \end{array}$$

Now, we want to define an algebraic stack, i.e., the analogue of algebraic spaces. Before this, we must introduce the notion of fiber products and representability.

Definition 3.2.8 *Let $\mathcal{X}$, $\mathcal{X}'$ and $\mathcal{I}$ be stacks over a site $\mathbf{C}_\tau$, and $F : \mathcal{X} \to \mathcal{I}$ and $F' : \mathcal{X}' \to \mathcal{I}$ morphisms of stacks. The 2-fiber product $\mathcal{X} \times_{\mathcal{I}} \mathcal{X}'$ is the stack defined on an object U of $\mathbf{C}$ by the category $(\mathcal{X} \times_{\mathcal{I}} \mathcal{X}')(U)$, whose*

- *objects are triples (u, u', ϕ) with $u \in \mathcal{X}(U)$, $u' \in \mathcal{X}'(U)$ and $\phi \in \mathrm{Hom}_{\mathcal{I}(U)}(F(u), F'(u'))$.*
- *morphisms $\mathrm{Hom}((u, u', \phi), (v, v', \psi))$ are the pairs $(f : u \to v, f' : u' \to v')$ such that $\psi \circ F(f) = F'(f') \circ \phi$.*

Definition 3.2.9 *A stack $\mathcal{X}$ over $(schemes)_{et}$ is* representable *by an algebraic space (resp. by a scheme) if there exists an algebraic space (resp. scheme) X, such that $\mathcal{X}$ is isomorphic to the stack associated to X. A morphism of stacks $F : \mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces (resp. schemes) if for each scheme Y and morphism $Y \to \mathcal{Y}$ the fiber product $Y \times_{\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space (resp. scheme).*

In our case, if $x : X \to \widetilde{\mathcal{M}_g}$ and $y : Y \to \widetilde{\mathcal{M}_g}$ are morphisms given by families $\pi_x : \mathcal{C}_x \to X$ and $\pi_y : \mathcal{C}_y \to Y$, then $X \times_{\widetilde{\mathcal{M}_g}} Y(U)$ is given by triples (u, u', ϕ) where $u : U \to X$ and $u' : U \to Y$ are morphisms of schemes, and $\phi \in \mathrm{Hom}_{\widetilde{\mathcal{M}_g(U)}}(u^*(\pi_x), u'^*(\pi_y))$, i.e., ϕ is an isomorphism between the families $u^*(\pi_x)$ and $u'^*(\pi_y)$.

The stack $X \times_{\widetilde{\mathcal{M}_g}} Y$ is actually representable. Indeed, let $h : U \xrightarrow{(u,u')} X \times Y$ and denote by $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ the projections; then $u^* = h^* p_1^*$ and $u'^* = h^* p_2^*$. Also, denote by $\pi_1 : \mathcal{C}_x \times_X (X \times Y) \to X \times Y$ and $\pi_2 : \mathcal{C}_y \times_Y (X \times Y) \to X \times Y$ the projections. Lets construct the scheme $\mathbf{Isom}(X \times Y, p_1^* x, p_2^* y)$ representing $X \times_{\widetilde{\mathcal{M}_g}} Y$. Assume $(U_i)_{i \in I}$ is an open covering of $X \times Y$ that trivializes both $E_1 := \pi_{1*}(\omega_{\pi_1}^n)$ and $E_2 := \pi_{2*}(\omega_{\pi_2}^n)$, for a certain $n >> 0$, and denote by $L := \mathrm{Isom}(\mathbb{P}(E_1), \mathbb{P}(E_2)) \to X \times Y$ the $\mathrm{PGL}(N+1)$-bundle that parametrizes the isomorphisms between $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$. Here, $N = (2n-1)(g-1)-1$.

We can now construct a map $\eta : L|_{U_i} \to K \times K$, where K is the fine moduli space parametrizing n-canonically embedded stable curves. The construction is given as follows: first, fix trivializations

$$\begin{aligned} L|_{U_i} &\cong U_i \times \mathrm{PGL}(N+1) \\ E_1|_{U_i} &\cong U_i \times \mathbb{A}^{N+1} \\ E_2|_{U_i} &\cong U_i \times \mathbb{A}^{N+1}; \end{aligned}$$

then fix a basis α of $E_1|_{U_i}$. For each $g \in \mathrm{PGL}(N+1)$ we have an associated basis $g\alpha$ for $E_2|_{U_i}$; this basis induces a map $U_i \to K \times K$; varying g we get the map η. Define the scheme $\mathbf{Isom}(X \times Y, p_1^*x, p_2^*y)|_{U_i} := \eta^{-1}(\Delta)$. These schemes do not depend on the choice of trivializations and glue together.

To summarize, the construction above implies that every morphism $x : X \to \widetilde{\mathcal{M}_g}$ for a scheme X is representable by schemes.

Definition 3.2.10 *Let $\mathbb{P}$ be a property of morphisms of schemes that is stable under base change and local in the étale topology. A representable morphism $F : \mathcal{X} \to \mathcal{Y}$ of stacks has property $\mathbb{P}$, if for each morphism $Y \to \mathcal{Y}$, with Y a scheme, the induced morphism of schemes $Y \times_{\mathcal{Y}} \mathcal{X} \to Y$ has the property $\mathbb{P}$.*

Examples of properties that are stable under base change and local in the étale topology include: being étale, surjective, smooth, locally of finite type, quasi-compact, open embedding, closed embedding, affine, quasi-affine, proper, unramified and separated.

Now, we are able to define an (Artin) algebraic stack:

Definition 3.2.11 *A stack over (schemes)$_{et}$ is an* Artin algebraic stack *if*

1. *The diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces and quasi-compact.*

2. *There exist a scheme X, called an atlas, and a surjective smooth morphism $X \to \mathcal{X}$.*

Finally, a Deligne–Mumford stack is defined by:

Definition 3.2.12 *A stack over (schemes)$_{et}$ is a* Deligne–Mumford algebraic stack *if*

1. *The diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by schemes, quasi-compact and separated.*

2. *There exist a scheme X, called an atlas, and a surjective étale morphism $X \to \mathcal{X}$.*

In fact, we have a nicer characterization of Deligne–Mumford stacks:

Proposition 3.2.1 *A stack $\mathcal{X}$ is Deligne–Mumford if and only if it is an algebraic stack and the diagonal morphism is unramified and separated.*

Proof. This is [6] Theorem 4.21; a proof can be found in [1].

Since we have a surjective smooth map $K \to \widetilde{\mathcal{M}_g}$, induced by the versal family of n-canonically embedded stable curves, to prove that $\widetilde{\mathcal{M}_g}$ is Deligne–Mumford we need only prove that the diagonal morphism is representable by schemes, quasi-compact, separated and unramified.

Let $(x, x') : X \to \widetilde{\mathcal{M}_g} \times \widetilde{\mathcal{M}_g}$ be a morphism induced by families $\mathcal{C} \to X$ and $\mathcal{C}' \to X$; then we have a Cartesian diagram

$$\begin{tikzcd}
(X \times X) \times_{\widetilde{\mathcal{M}_g} \times \widetilde{\mathcal{M}_g}} \widetilde{\mathcal{M}_g} \arrow[r] \arrow[d] & X \times X \arrow[d, "{(x,x')}"] \\
\widetilde{\mathcal{M}_g} \arrow[r, "\Delta"'] & \widetilde{\mathcal{M}_g} \times \widetilde{\mathcal{M}_g}
\end{tikzcd}$$

But

$$(X \times X) \times_{\widetilde{\mathcal{M}_g} \times \widetilde{\mathcal{M}_g}} \widetilde{\mathcal{M}_g} = X \times_{\widetilde{\mathcal{M}_g}} X = \mathbf{Isom}(X \times X, p_1^* x, p_2^* x').$$

We now have the following Cartesian diagram

$$\begin{tikzcd}
(X \times_{\widetilde{\mathcal{M}_g}} X) \times_{X \times X} X \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\
X \times_{\widetilde{\mathcal{M}_g}} X \arrow[r] \arrow[d] & X \times X \arrow[d, "{(x,x')}"] \\
\widetilde{\mathcal{M}_g} \arrow[r, "\Delta"'] & \widetilde{\mathcal{M}_g} \times \widetilde{\mathcal{M}_g}
\end{tikzcd}$$

which implies that

$$X \times_{\widetilde{\mathcal{M}_g} \times \widetilde{\mathcal{M}_g}} \widetilde{\mathcal{M}_g} = (X \times_{\widetilde{\mathcal{M}_g}} X) \times_{X \times X} X,$$

and hence that Δ is representable by schemes. All we have to prove now is that the map

$$\mathrm{Isom}_X(\mathcal{C}, \mathcal{C}') = \mathbf{Isom}(X \times X, p_1^* x, p_2^* x') \times_{X \times X} X \to X$$

is quasi-compact, separated and unramified; this follows from Theorem 1.11 in [6].

$\widetilde{\mathcal{M}_g}$ being a Deligne–Mumford stack implies that there exist a scheme B and an étale surjective morphism $\Phi : B \to \widetilde{\mathcal{M}_g}$; the family $\mathcal{C} \to B$ induced by this morphism is, therefore, a versal family.

To relate the moduli stack with the moduli space, we have a map $\widetilde{\mathcal{M}_g} \to \overline{M}_g$, given by the functor

$$\begin{array}{rcl}
\widetilde{\mathcal{M}_g}(U) & \to & Mor(U, \overline{M}_g) \\
\pi : \mathcal{C} \to U & \mapsto & \chi(\pi) : U \to \overline{M}_g
\end{array}$$

where $\chi(\pi)$ is the map obtained from the fact that $\overline{M}_g$ is a coarse moduli space. Therefore, we have a map $\eta : B \to \overline{M}_g$, the same map induced by the versal family. Moreover, we can assume that η is finite, see [28].

The map η is, in fact, ramified along Δ_1. To see this, let $b \in B$ be a closed point for which the fiber $C = \mathcal{C}_b$ is a general curve in Δ_1. The tangent space $T_b B$ is given by $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$, because

Φ is étale and surjective. However, we have an action of $\mathrm{Aut}(C) = \mathbb{Z}/2\mathbb{Z}$ on T_bB, and the map Φ is "invariant" by this action, hence the ramification.

3.3 The Picard group of $\widetilde{\mathcal{M}_g}$

In this section, we define the Picard group of the moduli stack of curves. The definition here is basically the definition of $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$, but in the language introduced above. We begin by defining the smooth site of a stack, the site where our sheaves will be defined.

Definition 3.3.1 *The* smooth site $\mathcal{X}_{sm}$ *of a stack* $\mathcal{X}$ *over* $(schemes)_{et}$ *is the category whose:*

1. *objects are pairs* (U, u)*, where* U *is a scheme and* $u : U \to \mathcal{X}$ *is a smooth morphism;*
2. *morphisms are pairs* $(\varphi, \alpha) : (U, u) \to (V, v)$ *where* $\varphi : U \to V$ *is a morphism and* $\alpha : u \Rightarrow v \circ \varphi$ *is a 2-isomorphism;*
3. *coverings are given by smooth coverings of* (U, u)*, i.e., families* $\{U_i \to U\}_{i \in I}$ *whose morphisms are smooth and the union of their images is* U*.*

Definition 3.3.2 *Let* $\mathcal{X}$ *be an algebraic stack. A* sheaf $\mathcal{F}$ *on the smooth site of* $\mathcal{X}$ *is given by the following data:*

1. *For each object* (U, u) *of* $\mathcal{X}_{sm}$ *a sheaf* $\mathcal{F}_{U,u}$ *on* U*.*
2. *For each morphism* $(\varphi, \alpha) : (U, u) \to (V, v)$*a morphism of sheaves*

$$\theta_{\varphi,\alpha} : \varphi^* \mathcal{F}_{V,v} \to \mathcal{F}_{U,u}$$

satifying the cocycle condition, i.e., for morphisms $(\varphi, \alpha) : (U, u) \to (V, v)$ *and* $(\psi, \beta) : (V, v) \to (W, w)$ *we have*

$$\theta_{\varphi,\alpha} \circ \varphi^* \theta_{\psi,\beta} = \theta_{\psi \circ \varphi, \varphi_* \beta \circ \alpha}$$

A morphism of sheaves $h : \mathcal{F} \to \mathcal{F}'$ is just a collection of morphism $h_{U,u} : \mathcal{F}_{U,u} \to \mathcal{F}'_{U,u}$ for all objects (U, u) of $\mathcal{X}_{sm}$ which are compatible with the morphisms $\theta_{\varphi,\alpha}$.

A sheaf over an algebraic stack $\mathcal{X}$ can be identified with a sheaf over some atlas $X \to \mathcal{X}$ together with some "descent data". For example, in the case of the atlas $K \to \widetilde{\mathcal{M}_g}$, a sheaf over $\widetilde{\mathcal{M}_g}$ can be identified with a $\mathrm{PGL}(N+1)$-invarian sheaf over K. A sheaf is called locally free if every $\mathcal{F}_{U,u}$ is locally free, and is called Cartesian if the $\theta_{\varphi,\alpha}$ are isomorphisms.

Now we can define the Picard group of the stack, simply as the isomorphism classes of Cartesian locally free sheaves of rank 1. Note that this definition is pretty much the same definition of $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$, and from the observation above follows Isomorphism (2.3).

Chapter 4

Intersection theory

Here we are going to introduce our main tools, the Thom–Porteous and Grothendieck–Riemann–Roch formulas. Before this, let's recall the basic properties of Chern classes. All that is exposed here can be found in detail in [14] Chapters 3, 14 and 15 (for the definitions and theorems) and in [22] Chapter 3, Section D (for the computations).

Let X be a scheme, and $A_k X$ the Chow group of its k-cycles modulo rational equivalence. Let E be a vector bundle over X of rank e. We write $c_i(E)$ for its i-th Chern class for $i = 0, 1, \ldots$, which is a map:

$$c_i(E) \cap \_ : A_k X \to A_{k-i} X$$

defined for all k by the following five properties:

1. $c_0(E) = 1$
2. If $f \colon X' \to X$ is a flat morphism, then
$$c_i(f^* E) \cap f^* \alpha = f^*(c_i(E) \cap \alpha)$$
for all cycles $\alpha \in A_*(X)$ and all i.
3. (Whitney sum) If
$$0 \to E' \to E \to E'' \to 0$$
is an exact sequence of vector bundles on X, then
$$c_k(E) = \sum_{i+j=k} c_i(E') c_j(E'').$$
4. If E is a line bundle, and D is a Cartier divisor on X with $\mathcal{O}_X(D) \cong E$, then
$$c_1(E) \cap [X] = [D].$$

5. (Projection formula) If $f : X \to Y$ is a proper map, and E is a rank-e vector bundle over Y then for all $\alpha \in A_*X$ we have:

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*(\alpha).$$

Denote by $c_t(E)$ the Chern polynomial of E, defined by

$$c_t(E) := 1 + c_1(E)t + \ldots + c_e(E)t^e.$$

If E admits a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_e = E$, with quotients $E_i/E_{i-1} = L_i$, then, by Whitney sum,

$$c_t(E) = \prod_{i=1}^{e}(1 + c_1(L_i)t),$$

and we say that the $c_1(L_i)$ are the Chern roots of E. However, often E admits no such filtration. Nonetheless, we can factor $c_t(E) = \prod_{1=1}^{e}(1 + \alpha_i t)$ in a formal way, i.e., $c_i(E)$ is the i-th symmetric function of $\alpha_1, \ldots, \alpha_e$. The α_i are also called Chern roots.

We will also write $c_t(F - E)$ for the formal series given by $c_t(F)/c_t(E)$, and write $c_i(F - E)$ for the coefficient of t^i in this series.

Definition 4.0.3 *Let E be a rank-e vector bundle over a scheme X, and $\alpha_1, \ldots, \alpha_e$ its Chern roots. Define the Chern character of E by the formula*

$$\mathrm{ch}(E) := \sum_{i=1}^{e} \exp(\alpha_i)$$

where $\exp(x) = e^x = \sum_{n=0}^{\infty} x^n/n!$.

Since $\mathrm{ch}(E)$ is a symmetric function of the Chern roots of E, it can be written as a function of the Chern classes. Here are the first terms:

$$\mathrm{ch}(E) = e + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)) + \ldots$$

The Chern character is additive on exact sequences and multiplicative on tensor products: For any exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0 \tag{4.1}$$

we have

$$\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E''),$$

and for any two vector bundles E, E' we have

$$\mathrm{ch}(E \otimes E') = \mathrm{ch}(E) \cdot \mathrm{ch}(E').$$

Definition 4.0.4 *Let E be a rank-e vector bundle over a scheme X, and $\alpha_1, \dots, \alpha_e$ its Chern roots. Define the the Todd class of E by the formula*

$$\mathrm{td}(E) := \prod_{i=1}^{e} Q(\alpha_i)$$

where

$$Q(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k},$$

where the B_k are the Bernoulli numbers.

As the Chern character, the Todd class can be written as a function of the Chern classes of E:

$$\mathrm{td}(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1(E)^2 + c_2(E)) + \frac{1}{24}c_1(E)c_2(E) + \dots$$

and for an exact sequence as in (4.1) we get

$$\mathrm{td}(E) = \mathrm{td}(E')\mathrm{td}(E'').$$

When X is smooth we often will identify $c_i(E)$ with the codimension-i cycle $c_i(E) \cap [X]$. Moreover since in this case every coherent sheaf F admits a resolution by vector bundles,

$$0 \to E_n \to E_{n-1} \to \dots \to E_1 \to E_0 \to F \to 0,$$

we can expand the definition of Chern classes to coherent sheaves, defining $c_t(F)$ via the Whitney sum:

$$c_t(F) := \prod_{i=0}^{n} c_t(E_i)^{(-1)^i}$$

The main reason to make the above definitions is the Grothendieck–Riemann–Roch formula, that relates the Chern caracter of the pushforward of a coherent sheaf with the pushforward of its Chern caracter. Before stating the theorem, we need to make one more definition:

Definition 4.0.5 *Let $f : X \to Y$ be a proper morphism, and E a coherent sheaf over X; define then*

$$\mathrm{ch}(f_!(E)) := \sum_{i=0} (-1)^i \mathrm{ch}(R^i f_*(E)).$$

Theorem 4.0.1 (Grothendieck–Riemann–Roch) *Let $f : X \to Y$ be a proper morphism between smooth connected schemes and E a coherent sheaf over X. Then:*

$$\mathrm{ch}(f_!(E)) = f_*(\mathrm{ch}(E) \cdot \mathrm{td}(T_{X/Y}))$$

where $T_{X/Y}$ is the relative tangent sheaf.

Since the Grothendieck–Riemann–Roch formula depends on the Todd class of the relative tangent sheaf, it is useful to understand this relative sheaf for a family of stable curves. In fact, we will only need this for families $\phi : \mathcal{C} \to B$ of stable curves with B unidimensional and both B and $\mathcal{C}$ smooth, because of the observation following Harer Theorem in Chapter 2.

Let ϕ be as above, and denote by Ω_ϕ the relative cotangent sheaf, the cokernel of the natural map

$$\phi^*\Omega_B \to \Omega_X. \tag{4.2}$$

Outside the locus Z of nodes of the fibers, Ω_ϕ is isomorphic to the relative dualizing sheaf ω_ϕ. Let now $P \in Z$ and x, y local coordinates of $\mathcal{C}$ at P, and let t be a local coordinate at the point $\phi(P)$ in B such that the map ϕ is given by $t = xy$. (Such local description is possible because we assumed $\mathcal{C}$ smooth). Then the map (4.2) is locally given by the map

$$\begin{array}{rcl} \mathcal{O}_{\mathcal{C},P}\langle dt\rangle & \to & \mathcal{O}_{\mathcal{C},P}\langle dx, dy\rangle \\ dt & \mapsto & xdy + ydx \end{array}$$

Hence

$$\Omega_{\phi,P} = \frac{\mathcal{O}_{\mathcal{C},P}\langle dx, dy\rangle}{\langle xdy + ydx\rangle},$$

Now, $\omega_{\phi,P}$ can be identified with

$$\mathcal{O}_{\mathcal{C},P}\langle\alpha\rangle$$

where

$$\alpha = \frac{dx}{x} - \frac{dy}{y}.$$

(See [22], p. 157; this follows as well from Rosenlicht's characterization in [34] page 76.) Since $x\alpha = 2dx$ and $y\alpha = -2dy$, we have a map

$$\begin{array}{rcl} \Omega_{\phi,P} & \to & \mathcal{O}_{\mathcal{C},P}\langle\alpha\rangle \\ 2dx & \mapsto & x\alpha \\ 2dy & \mapsto & y\alpha. \end{array}$$

The image of this map is $(x, y)\langle\alpha\rangle$, and thus we conclude that $\Omega_\phi = \mathcal{I}_Z \otimes \omega_\phi$.

Now, let $i : Z \to \mathcal{C}$ be the inclusion. Then, by Grothendieck–Riemman–Roch,

$$\mathrm{ch}(i_*\mathcal{O}_Z) = i_*(\mathrm{ch}(\mathcal{O}_Z)\mathrm{td}(T_{Z/\mathcal{C}})).$$

But, since Z is a finite union of points, all the Chern classes are 0, whence $\mathrm{ch}(\mathcal{O}_Z) = 1$ and $\mathrm{td}(T_{Z/\mathcal{C}}) = 1$, and thus

$$\mathrm{ch}(i_*\mathcal{O}_Z) = [Z].$$

Set $\eta := [Z]$. It follows that

$$\mathrm{ch}(\mathcal{I}_Z) = \mathrm{ch}(\mathcal{O}_\mathcal{C}) - \mathrm{ch}(i_*\mathcal{O}_Z) = 1 - \eta,$$

which gives us

$$\begin{aligned}\mathrm{ch}(\Omega_\phi) &= \mathrm{ch}(\mathcal{I}_Z)\cdot \mathrm{ch}(\omega_\phi)\\ &= (1-\eta)\cdot\left(1+c_1(\omega_\phi)+\frac{c_1(\omega_\phi)^2}{2}\right)\\ &= \left(1+c_1(\omega_\phi)+\left(\frac{c_1(\omega_\phi)^2}{2}-\eta\right)\right).\end{aligned}$$

(The remaining terms of the product expansion are zero because $\dim \mathcal{C} = 2$.) Hence, $c_1(\Omega_\phi) = c_1(\omega_\phi)$ and $c_2(\Omega_\phi) = \eta$, which implies that

$$\mathrm{td}(T_{\mathcal{C}/B}) = 1 - \frac{c_1(\omega_\phi)}{2} + \frac{c_1(\omega_\phi)^2+\eta}{12}. \tag{4.3}$$

Now, define the tautological class κ in $\mathrm{Pic}_{\mathrm{fun}}(\overline{M_g})$, which for a family $\phi : \mathcal{C} \to B$ is given by

$$\kappa_\phi := \phi_*(c_1(\omega_\phi)^2). \tag{4.4}$$

Using the description above, let's write this class in terms of the basis given by Harer's theorem. Using Grothendieck–Riemann–Roch for the sheaf ω_ϕ (with ϕ as above),

$$\mathrm{ch}(\phi_*\omega_\phi) - \mathrm{ch}(R^1\phi_*\omega_\phi) = \phi_*(\mathrm{ch}(\omega_\phi)\cdot \mathrm{td}(T_{\mathcal{C}/B})).$$

Since $R^1\phi_*\omega_\phi = \mathcal{O}_B$ we get

$$g-1+\lambda = \phi_*\left(\left(1+c_1(\omega_\phi)+\frac{c_1(\omega_\phi)^2}{2}\right)\cdot\left(1-\frac{c_1(\omega_\phi)}{2}+\frac{c_1(\omega_\phi)^2+\eta}{12}\right)\right)$$

which then yields $g-1 = \phi_*(\frac{c_1(\omega_\phi)}{2})$ (which is obvious) and

$$\lambda = \phi_*\left(\frac{c_1(\omega_\phi)^2+\eta}{12}\right) = \frac{\kappa+\delta}{12}. \tag{4.5}$$

Thus $\kappa = 12\lambda - \delta$.

Note that (4.3) and (4.5) imply that $\phi_*(\mathrm{td}_2(T_{\mathcal{C}/B})) = \lambda$. Another way to derive this formula, now over a base scheme B of any dimension, is to apply Grothendieck–Riemman–Roch to the bundle $\mathcal{O}_\mathcal{C}$. Using duality, we end up with

$$\begin{aligned}\phi_*(\mathrm{td}(T_{\mathcal{C}/B})) &= \mathrm{ch}(\phi_*(\mathcal{O}_\mathcal{C})) - \mathrm{ch}(\phi_*(\omega_\phi)^\vee)\\ &= (1-g)+\lambda+\dots\end{aligned}$$

which implies that

$$\phi_*(\mathrm{td}_2(T_{\mathcal{C}/B})) = \lambda. \tag{4.6}$$

Another useful application of Grothendieck–Riemann–Roch is the computation of the class of the canonical bundle. By the description in Section 2.1, for a 1-parameter family $\phi : \mathcal{C} \to B$ the canonical class of the stack is given by $c_1(\phi_*(\Omega_\phi \otimes \omega_\phi))$. Since the higher direct images of $\Omega_\phi \otimes \omega_\phi$ are 0 by [6], p. 79, we get by Grothendieck–Riemann–Roch:

$$\begin{aligned}
\mathrm{ch}(\phi_*(\Omega_\phi \otimes \omega_\phi)) &= \phi_*(\mathrm{ch}(\Omega_\phi \otimes \omega_\phi) \cdot \mathrm{td}(T_\phi)) \\
&= \phi_*\left(\mathrm{ch}(\Omega_\phi) \cdot \mathrm{ch}(\omega_\phi) \cdot \left(1 - \frac{c_1(\omega_\phi)}{2} + \frac{c_1(\omega_\phi)^2 + \eta}{12}\right)\right).
\end{aligned}$$

Now, note that

$$\begin{aligned}
\mathrm{ch}(\Omega_\phi) \cdot \mathrm{ch}(\omega_\phi) &= \left(1 + c_1(\omega_\phi) + \left(\frac{c_1(\omega_\phi)^2}{2} - \eta\right)\right)\left(1 + c_1(\omega_\phi) + \frac{c_1(\omega_\phi)^2}{2}\right) \\
&= 1 + 2c_1(\omega_\phi) + 2c_1(\omega_\phi)^2 - \eta,
\end{aligned}$$

whence

$$\begin{aligned}
\mathrm{ch}(\phi_*(\Omega_\phi \otimes \omega_\phi)) &= \phi_*\left((1 + 2c_1(\omega_\phi) + 2c_1(\omega_\phi)^2 - \eta)\left(1 - \frac{c_1(\omega_\phi)}{2} + \frac{c_1(\omega_\phi)^2 + \eta}{12}\right)\right) \\
&= \phi_*\left(1 + \frac{3c_1(\omega_\phi)}{2} + \left(c_1(\omega_\phi)^2 - \eta + \frac{c_1(\omega_\phi)^2 + \eta}{12}\right)\right) \\
&= 3(g-1) + \kappa - \delta + \lambda = 3(g-1) + (13\lambda - 2\delta).
\end{aligned}$$

So $c_1(\phi_*(\Omega_\phi \otimes \omega_\phi)) = 13\lambda - 2\delta$. Since the stack is ramified along Δ_1 over $\overline{M}_g$ we end up with

$$K_{\overline{M}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \ldots - 2\delta_{[g/2]}.$$

Finally, let's state the Thom–Porteous formula, which gives us the class of a degeneration scheme of a map of vector bundles.

Theorem 4.0.2 *Let X be a smooth connected scheme, $\sigma : E \to F$ a morphism of vector bundles of ranks e and f, and $k \le \min(e, f)$. Define $D_k(\sigma)$ as the locus where the map σ has rank $\le k$. If $D_k(\sigma)$ has the expected codimension $(e-k)(f-k)$ then:*

$$[D_k(\sigma)] = \Delta_{f-k}^{(e-k)}(c(F-E)) \cap [X]$$

where

$$\Delta_l^{(d)}(c(F-E)) = \det((c_{l+j-i}(F-E))_{i,j=1,\ldots,d}).$$

In particular, if $e = f$ and $k = e - 1$ then

$$[D_k(\sigma)] = c_1(F) - c_1(E),$$

and if $k = 0$ and $E = \mathcal{O}_X$ then

$$[D_k(\sigma)] = c_f(F).$$

Chapter 5

Limit linear systems

5.1 Limit linear systems

Let C be a smooth connected projective genus-g curve, L a line bundle over C and $V \subset \mathrm{H}^0(L)$ a linear system of dimension $r+1$.

Definition 5.1.1 *Given a point P of C, we can write the orders of vanishing of the sections of V at P increasingly, $a_0, a_1, \ldots, a_r$, and define the* (ramification) weight *of P,*

$$\mathrm{wt}_V(P) := \sum_{i=0}^{r} a_i - i,$$

and the total weight*,*

$$\mathrm{Tw}_V(P) := \sum_{i=0}^{r} a_i.$$

Alternatively, we can look locally at P, where we have a $\mathbb{C}$-linear map $V \to \mathcal{O}_P$, since L is trivial in a neighborhood of P. Let V_P be the image of this map and t a parameter of $\mathcal{O}_P$; V_P is still a $(r+1)$-dimensional vector space. Let $f_0, f_1, \ldots, f_r$ be a basis for V_P; from this basis we can define a Wronskian matrix,

$$W(f_0, f_1, \ldots, f_r) := \left(\frac{\partial^i f_j}{\partial t^i}\right)_{i=0,\ldots,r;j=0,\ldots,r},$$

and from this matrix we define

$$w(f_0, f_1, \ldots, f_r) := \det(W(f_0, f_1, \ldots, f_r)).$$

Then $\mathrm{wt}_V(P) = \mathrm{ord}_P(w(f_0, f_1, \ldots, f_r))$. Indeed, it is easy to see that the order of $w(f_0, f_1, \ldots, f_r)$ at P does not depend on the basis for V_P chosen, nor on the local parameter t, nor the trivialization of L at P, and agrees with $\mathrm{wt}_V(P)$ given in Definition 5.1.1.

We can now define the ramification divisor of V, which is simply

$$R_V := \sum \mathrm{wt}_V(P)P.$$

The first natural question is: can we compute $\deg(R_V)$? Or more specifically, can we find a reasonable equivalence class for $\mathcal{O}_C(R_V)$ in $\mathrm{Pic}(C)$?

Fortunately, we can globalize the second approach as follows. Let $J^i(L)$ be the sheaf of jets of order i (see [11], [26] or [27] for more details on sheaves of jets). This sheaf has the following properties: $J^i(L)$ is a vector bundle of rank $i+1$ with a natural evaluation map

$$\mathrm{H}^0(L) \otimes \mathcal{O}_C \to J^i(L)$$

given locally by a Wronskian matrix involving derivatives up to order i; and we have truncation sequences

$$0 \to \omega_C^{i+1} \otimes L \to J^{i+1}(L) \to J^i(L) \to 0,$$

where ω_C is the canonical bundle and $J^0(L) = L$.

Restricting the evaluation map to V we have a map $V \otimes \mathcal{O}_C \to J^r(L)$ of vector bundles of the same rank $r+1$. Since $V \otimes \mathcal{O}_C$ is trivial this map induces a (determinant) section

$$\sigma\colon \mathcal{O}_C \to \bigwedge^{r+1} J^r(L)$$

whose zero locus is R_V. Hence

$$\mathcal{O}_C(R_V) \cong \bigwedge^{r+1} J^r(L),$$

and since

$$\bigwedge^{r+1} J^r(L) \cong \omega_C^{\binom{r+1}{2}} \otimes L^{r+1},$$

an isomorphism obtained from the truncation sequences, we finally get that

$$\deg R_V = (r+1)r(g-1) + (r+1)\deg L,$$

known as the Plücker formula.

The remaining of this section is dedicated to understanding the behavior of R_V in families, and its “limits” in nodal curves.

Let $\pi\colon \mathcal{X} \to T$ be a family of smooth genus-g curves with $\mathcal{X}$ smooth, $\mathcal{L}$ a line bundle over $\mathcal{X}$, and $\mathcal{V} \subset \pi_*(\mathcal{L})$ a locally free subsheaf of rank $r+1$ over T. Denote by $\mathcal{X}_t$ the fiber over $t \in T$, i.e., $\mathcal{X}_t = \pi^{-1}(t)$. Assume that over any point $t \in T$, the composition below is an injection

$$V_t := \frac{\mathcal{V}_t}{\mathcal{M}_{t,T}\mathcal{V}_t} \longrightarrow \frac{\pi_*(\mathcal{L})_t}{\mathcal{M}_{t,T}\pi_*(\mathcal{L})_t} \longrightarrow \mathrm{H}^0(\mathcal{X}_t, \mathcal{L}|_{\mathcal{X}_t}) \tag{5.1}$$

We say that $\mathcal{V}$ is a relative linear system. Let also $J^i_\pi(\mathcal{L})$ be the *relative* sheaf of jets of order i (see [11], [27] for more details); $J^i_\pi(\mathcal{L})$ restricts to a sheaf of jets on each fiber, and satisfies similar properties to those of the “absolute” sheaf. There are evaluation maps

$$\pi^*\pi_*(\mathcal{L}) \to J^i_\pi(\mathcal{L})$$

and truncation sequences

$$0 \to \omega_\pi^{i+1} \otimes \mathcal{L} \to J_\pi^{i+1}(\mathcal{L}) \to J_\pi^i(\mathcal{L}) \to 0, \tag{5.2}$$

where ω_π is the relative dualizing sheaf.

Definition 5.1.2 *The* ramification divisor $W = W_\mathcal{V}$ *of the relative linear system* $\mathcal{V}$ *is the degeneracy locus of the map*

$$\pi^*\mathcal{V} \to J_\pi^r(\mathcal{L}),$$

whence the divisor of zeros of the induced section

$$w\colon \mathcal{O}_\mathcal{X} \to \bigwedge^{r+1} J_\pi^r(\mathcal{L}) \otimes \left(\bigwedge^{r+1} \pi^*\mathcal{V}\right)^\vee .$$

The divisor W has the property that $W \cap \mathcal{X}_t$ is the ramification divisor R_{V_t} of the linear system

$$V_t \subset \mathrm{H}^0(\mathcal{L}|_{\mathcal{X}_t})$$

induced from (5.1).

Now, suppose the fibers of $\pi : \mathcal{X} \to T$ are nodal, rather than smooth. Denote by $\mathcal{X}_{ns}$ the open locus of nonsingular fibers. We make two natural definitions. Since we still have sheaves of jets for such families (see [11] for a construction of these sheavess) we can define:

Definition 5.1.3 *The degeneration scheme* $W' = W'_\mathcal{V}$ *of* $\mathcal{V}$ *is the degeneracy locus of the map*

$$\pi^*\mathcal{V} \to J_\pi^r(\mathcal{L}).$$

And we can simply define:

Definition 5.1.4 *The ramification divisor* $W = W_\mathcal{V}$ *of* $\mathcal{V}$ *is the closure of* $W' \cap \mathcal{X}_{ns}$

Note that $W' \cap \mathcal{X}_{ns}$ is the ramification divisor of the restriction of $\mathcal{V}$ to $\pi(\mathcal{X}_{ns})$. When $\mathcal{V} = \pi_*(\mathcal{L})$ we say that W is the ramification divisor of the line bundle $\mathcal{L}$.

These two definitions do not coincide in general, but clearly $W \subset W'$. We are more interested in W, but since W' is much easier to compute, it is important to understand how the two definitions differ. Specifically, we want to know the diference $W' - W$ and how W intersects a singular fiber. But, before this, we must understand the formation of W'.

Let $\psi\colon \mathcal{L}' \to \mathcal{L}$ be an injective map of line bundles with degeneracy divisor D (i.e., $\mathcal{L}' = \mathcal{L}(-D)$), $\mathcal{V}' \subset \pi_*(\mathcal{L}')$ a relative linear system of rank $r+1$, and $\mu\colon \mathcal{V}' \to \mathcal{V}$ a map of rank-$(r+1)$ vector bundles with degeneracy scheme Y, such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{V}' & \longrightarrow & \pi_*\mathcal{L}' \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\pi_*\psi} \\ \mathcal{V} & \longrightarrow & \pi_*\mathcal{L}. \end{array}$$

Taking adjoints, and using the naturality of the formation of the sheaves of jets we get another commutative diagram

$$\begin{array}{ccc} \pi^*\mathcal{V}' & \longrightarrow & J^r_\pi(\mathcal{L}') \\ \Big\downarrow & & \Big\downarrow{\scriptstyle J^r_\pi\psi} \\ \pi^*\mathcal{V} & \longrightarrow & J^r_\pi(\mathcal{L}) \end{array}$$

of maps of vector bundles of rank $r+1$. Since $J^r_\pi\psi\colon J^r_\pi(\mathcal{L}') \to J^r_\pi(\mathcal{L})$ has degeneracy divisor $(r+1)D$, as we can see from the truncation sequences, we obtain that Y is also a divisor, and

$$\pi^*Y + W'_{\mathcal{V}} = (r+1)D + W'_{\mathcal{V}'}. \tag{5.3}$$

Definition 5.1.5 *Let C be a nodal genus-g curve. A* smoothing *of C is a flat projective map $p\colon \mathcal{C} \to \Sigma$, where $\Sigma := \mathrm{Spec}(\mathbb{C}[[t]])$ and $\mathcal{C}$ is regular, together with an isomorphism between C and the special fiber.*

Let $p : \mathcal{C} \to \Sigma$ be a smoothing of a nodal genus-g curve C. Let us denote by C_* the generic fiber. Let also $\mathcal{L}$ be a line bundle over $\mathcal{C}$, and $\mathcal{V} \subset p_*(\mathcal{L})$ a relative linear system of rank $r+1$. Since Σ is local, $\mathcal{V}$ is the sheaf associated to a free $\mathbb{C}[[t]]$-module V. Since p is flat and $\Sigma = \mathrm{Spec}(\mathbb{C}[[t]])$, also $L := \mathrm{H}^0(\mathcal{C}, \mathcal{L})$ is a free $\mathbb{C}[[t]]$-module containing V. We denote by V_* and L_* the localizations $V \otimes \mathbb{C}((t))$ and $L \otimes \mathbb{C}((t))$ (so $L_* = \mathrm{H}^0(\mathcal{C}_*, \mathcal{L}|_{\mathcal{C}_*})$). Since we have an injection

$$\frac{V}{tV} \to \frac{L}{tL},$$

we get that $V = V_* \bigcap L$.

Let D be a divisor on $\mathcal{C}$ supported in C. We have an isomorphism

$$\mathrm{H}^0(C_*, \mathcal{L}|_{C_*}) \to \mathrm{H}^0(C_*, \mathcal{L}(D)|_{C_*});$$

let $V(D)_*$ be the image of V_* under it. Define now $V(D) := V(D)_* \cap \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D))$. It gives rise to a new relative linear system $\mathcal{V}(D) \subset p_*(\mathcal{L}(D))$ of the same rank $r+1$. On the other hand, if D is a subscheme of C (the case where D is effective and reduced), we let $V|_D$ denote the image of V under the restriction map $\mathrm{H}^0(\mathcal{C}, \mathcal{L}) \to \mathrm{H}^0(D, \mathcal{L}|_D)$.

Let $C_1, C_2, \ldots, C_n$ be the irreducible components of C. Since C is connected, there exist line bundles $\mathcal{L}_i$ for $i = 1, \ldots, n$ of the form

$$\mathcal{L}_i = \mathcal{L}(\sum_{l=1}^n a_{i,l}C_l) := \mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(\sum_{l=1}^n a_{i,l}C_l)$$

such that the restriction map

$$\mathrm{H}^0(C, \mathcal{L}_i|_C) \to \mathrm{H}^0(C_i, \mathcal{L}_i|_{C_i}) \tag{5.4}$$

is an injection, which implies that the kernel of the restriction map

$$\mathrm{H}^0(\mathcal{C}, \mathcal{L}_i) \to \mathrm{H}^0(C_i, \mathcal{L}_i|_{C_i}) \tag{5.5}$$

is $t\mathrm{H}^0(\mathcal{C}, \mathcal{L}_i)$. We say that $\mathcal{L}_i$ has focus on C_i. More generally, we say that

$$V_i := V\left(\sum_{l=1}^{n} a_{i,l} C_l\right)$$

has focus on C_i if the induced map $V_i|_C \to \mathrm{H}^0(C_i, \mathcal{L}|_{C_i})$ is injective.

Let $\overline{V_i}$ be the image of V_i under the map (5.5). Since the map (5.4) is an injection, $\dim \overline{V_i} = r+1$. This implies that W_i', the degeneration scheme of $\mathcal{V}_i$, the sheaf associated to V_i, does not contain C_i. We call $\overline{V_i} \subset \mathrm{H}^0(C_i, \mathcal{L}_i|_{C_i})$ a limit linear system on C_i.

Let $R_i = R_{\overline{V_i}}$ be the ramification divisor of the linear system $\overline{V_i}$. Let W be the ramification divisor of $\mathcal{V}$. Then W is a divisor and (see [5]):

$$W \cap C = \sum_{i=1}^{n} R_i + \sum_{i<j} \sum_{P \in C_i \cap C_j} (r+1)(r - l_{i,j})P \tag{5.6}$$

where $l_{i,j} = a_{i,i} + a_{j,j} - a_{i,j} - a_{j,i}$.

To finish this section, let's define the special ramification locus. Let $\pi : \mathcal{X} \to T$ be a family of genus-g semistable curves with $\mathcal{X}$ smooth. Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ and $\mathcal{V} \subset \pi_*(\mathcal{L})$ a relative linear system of rank $r+1$. Let W be the ramification divisor of $\mathcal{V}$. Since W is a divisor, because $\mathcal{X}$ is smooth, it induces a section $w : \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}(W)$, and this section induces "derivatives" $w^{(r)} : \mathcal{O}_{\mathcal{X}} \to J_\pi^r(\mathcal{O}_{\mathcal{X}}(W))$. Let S^rW be the zero scheme of the section $w^{(r)}$; we call S^rW the r-th special ramification locus. Note that, on $\mathcal{X}_{ns}$ the support of S^rW is the set of points $P \in \mathcal{X}_{ns}$ such that $\mathrm{wt}_{V_{\pi(P)}}(P) \geq r+1$. When $r=1$ we write $SW = S^1W$. However, on the singular locus, S^rW might have an odd behavior, because W might not be flat over T, and might contain nodes.

5.2 Machinery

Let C be a nodal curve of genus g. Assume $C = E \bigcup F$, where E and F are subcurves, not necessarily irreducible, but with no common components. Write $E \bigcap F = \{P_1, \ldots, P_n\}$. Let $p : \mathcal{C} \to \Sigma$ be a smoothing of C, $\mathcal{L}$ a line bundle on $\mathcal{C}$ and $\mathcal{V} \subset p_*(\mathcal{L})$ a relative linear system of rank $r+1$. Recall the notation in Section 5.1. We will view E and F as divisors of $\mathcal{C}$. Since $p_*(\mathcal{L}(-E)) \subset p_*(\mathcal{L})$ we get an inclusion of modules $L(-E) \subset L$. As before define $V(-E) := V_* \bigcap L(-E)$; then $V(-E)$ induces a rank-$(r+1)$ vector bundle $\mathcal{V}(-E) \subset \pi_*\mathcal{L}(-E)$. This gives us a commutative diagram

$$\begin{array}{ccc} \mathcal{V}(-E) & \to & \pi_*(\mathcal{L}(-E)) \\ \downarrow & & \downarrow \\ \mathcal{V} & \to & \pi_*\mathcal{L} \end{array}$$

From this, as seen in Equation (5.3), we get $W_{\mathcal{V}}' = W_{\mathcal{V}(-E)}' + (r+1)E - p^*Y$, where Y is the degeneracy divisor of $\mathcal{V}(-E) \to \mathcal{V}$. The latter map induces a map $\mu : V(-E) \to V$ of rank-$(r+1)$ free $\mathbb{C}[[t]]$-modules, and we have $p^*Y = \mathrm{ord}_t(\det(\mu))C$. Since $tV \subset V(-E)$, it follows that

$\mathrm{ord}_t(\det(\mu)) = \dim_{\mathbb{C}} \mathrm{coker}(\mu)$. But $\mathrm{coker}(\mu) = V|_E$, which is by definition the image of V via the map $\mathrm{H}^0(\mathcal{C}, \mathcal{L}) \to \mathrm{H}^0(E, \mathcal{L}|_E)$. Furthermore, since we have an exact sequence

$$0 \to V|_F(-P_1 - \ldots - P_n) \to V|_C \to V|_E \to 0, \tag{5.7}$$

we obtain

$$W'_{\mathcal{V}} = W'_{\mathcal{V}(-E)} + \dim_{\mathbb{C}}(V|_F(-P_1 - \ldots - P_n))E - \dim_{\mathbb{C}}(V|_E)F. \tag{5.8}$$

Where by definition

$$V|_F(-P_1 - \ldots - P_n) := V|_F \cap \mathrm{H}^0(F, \mathcal{L}|_F(-P_1 - \ldots - P_n)).$$

In addition, let D be an effective divisor of $\mathcal{C}$ such that D and E have no common components. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^0(\mathcal{C}, \mathcal{L}(-D)) \arrow[r] \arrow[d] & \mathrm{H}^0(E, \mathcal{L}(-D)|_E) \arrow[d] \\
\mathrm{H}^0(\mathcal{C}, \mathcal{L}) \arrow[r] & \mathrm{H}^0(E, \mathcal{L}|_E).
\end{tikzcd}$$

And, letting $V(-D) := V \bigcap \mathrm{H}^0(\mathcal{C}, \mathcal{L}(-D))$, we get that $V(-D)|_E \subset V|_E(-D \cdot E)$, because the vertical maps are injections, since D and E have no common components.

Lemma 5.2.1 *Let $p : \mathcal{C} \to \Sigma$ be a smoothing of a semistable curve C. Let $\mathcal{L}$ be a line bundle on $\mathcal{C}$ and $\mathcal{V} \subset p_*(\mathcal{L})$ a relative linear system. Let E be a subcurve of C, and D an effective divisor of $\mathcal{C}$ without common components with E. Then*

$$V(-D)|_E \subset V|_E(-D \cdot E)$$

Proof. Follows from the previous discussion.

Lemma 5.2.2 *Let C be the nodal union of two smooth curves, $C = X \bigcup Y$, where we identify the point $A \in X$ with the point $B \in Y$. Let $p : \mathcal{C} \to \Sigma$ be a smoothing, $\mathcal{L}$ a line bundle on $\mathcal{C}$ and $\mathcal{V} \subset p_*(\mathcal{L})$ a relative linear system of rank $r + 1$. Let W' be the degeneration scheme and W the ramification divisor of $\mathcal{V}$. Let $V|_X$ and $V|_Y$ the restrictions of $\mathcal{V}$ to X and Y. Assume that for every positive integer i,*

$$\begin{aligned}
\dim_{\mathbb{C}}(V(-iY)|_Y(-B)) + \dim_{\mathbb{C}}(V|_X(-iA)) &\leq r + 1 \\
\dim_{\mathbb{C}}(V(-iX)|_X(-A)) + \dim_{\mathbb{C}}(V|_Y(-iB)) &\leq r + 1.
\end{aligned}$$

Then

$$W' = W + \mathrm{Tw}_{V|_X}(A)Y + \mathrm{Tw}_{V|_Y}(B)X.$$

Proof. Let $\mathcal{L}_i := \mathcal{L}(-iY)$. Let $\mathcal{V}_i := \mathcal{V}(-iY)$ be the induced vector bundle, and W'_i the degeneration scheme of $\mathcal{V}_i$. Denote by $m_Y(i)$ the coefficient of Y in the divisor W'_i. Then, by Equation (5.8) we get

$$m_Y(i) = m_Y(i + 1) + \dim_{\mathbb{C}}(V(-iY)|_X(-A)).$$

From the Exact Sequence (5.7),

$$\dim_{\mathbb{C}}(V(-iY)|_X) = r+1-\dim_{\mathbb{C}}(V(-iY)|_Y(-B)) \geq \dim_{\mathbb{C}}(V|_X(-iA)).$$

where the inequality follows from the hypothesis of the lemma. Therefore, by lemma 5.2.1, we have equality, that is, $V(-iY)|_X = V|_X(-iA)$.

Hence, we get

$$m_Y(0) = m_Y(i) + \sum_{j=1}^{i} \dim_{\mathbb{C}}(V|_X(-jA)).$$

But, for sufficiently large i, $m_Y(i) = 0$ and $\dim_{\mathbb{C}}(V|_X(-iA)) = 0$. Therefore

$$m_Y(0) = \sum_{j=1}^{\infty} \dim_{\mathbb{C}}(V|_X(-jA)) = Tw_{V|_X}(A)$$

The same holds for X, which concludes the proof. $\square$

Lemma 5.2.3 *Let X be a nodal curve which is the union of a smooth genus-$(g-1)$ curve C and a chain $(E_1, E_2, \ldots, E_r)$ of r rational smooth curves. Assume that C intersects only E_1 and E_r, the first at a single point A and the second at a single point B. Let $p\colon \mathcal{C} \to \Sigma$ be a smoothing of X, $\mathcal{L}$ a line bundle on $\mathcal{C}$ and $\mathcal{V} \subset p_*(\mathcal{L})$ a relative linear system of rank $r+1$. Let W' be the degeneration scheme and W the ramification divisor of $\mathcal{V}$. Assume that $\mathcal{L}|_{E_i} = \mathcal{O}_{E_i}(0)$ for every i. Let $\tilde{V} = V|_C(-A-B)$. Assume that*

$$\begin{aligned} \dim_{\mathbb{C}}(\tilde{V}(-iA-(r-i-1)B)) &= 1 \quad \textit{and} \\ \dim_{\mathbb{C}}(\tilde{V}(-(i+1)A-(r-i)B)) &= 0 \end{aligned}$$

for each $i = 0, 1, \ldots, r-1$. Then

$$W' = W + \sum_{i=1}^{r} \frac{i(r+1-i)(r+1)}{2} E_i.$$

Proof. The proof is the same as that given to [4], Thm. 6.1.

For each $i = 1, \ldots, r$, define

$$\mathcal{L}_i := \mathcal{L}\left(-\sum_{j=1}^{r} a_{i,j} E_j\right)$$

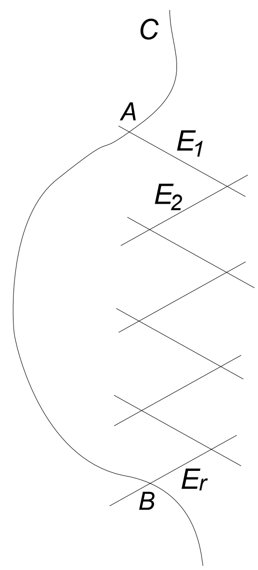
where $a_{i,j} := (r+1)\min\{i,j\} - ij$. Note that

$$\begin{aligned} \mathcal{L}_i|_{E_i} &= \mathcal{O}_{E_i}(r+1) \\ \mathcal{L}_i|_{E_j} &= \mathcal{O}_{E_j} \quad \text{for } j \neq i \\ \mathcal{L}_i|_C &= \mathcal{L}|_C(-(r+1-i)A - iB). \end{aligned}$$

Let $\mathcal{V}_i = \mathcal{V} \cap p_*(\mathcal{L}_i)$. Then

$$V_i|_C(-A-B) \subset V|_C(-(r+1-i)A-iB)(-A-B) = \tilde{V}(-(r-i+1)A-iB) = 0.$$

Figure 5.1: The curve X



Hence, it follows from Exact Sequence (5.7) that $V_i|_X \to V_i|_{E_i}$ is an isomorphism. Keeping the notation of Section 5.1, we set $V_{E_i} := V_i|_{E_i}$. Hence the degeneration scheme of $\mathcal{V}_i$ does not contain E_i.

As in the proof of Lemma 5.2.2, we define a filtration $\mathcal{L}_{i,j}$ of $\mathcal{L}$. For each $i = 1, \ldots, r$ and each $j = 0, 1, \ldots, i(r+1-i)-1$ let k, k', l, l' be integers such that

$$j = ki + l = k'(r+1-i) + l', \quad 0 \leq k, l' \leq r-i, \quad 0 \leq k', l \leq i-1,$$

and define

$$\begin{aligned} c_{i,j,m} &:= km + \max\{0, l-i+m+1\}, \quad m = 1, \ldots, i, \\ c'_{i,j,m} &:= k'(r+1-m) + \max\{0, l'+i-m+1\}, \quad m = i, \ldots, r. \end{aligned}$$

Now, let

$$D_{i,j} := \sum_{m=1}^{i} c_{i,j,m} E_m + \sum_{m=i+1}^{r} c'_{i,j,m} E_m,$$

and put $\mathcal{L}_{i,j} := \mathcal{L}(-D_{i,j})$. Notice that $\mathcal{L}_i = \mathcal{L}_{i,i(r+1-i)-1}$. Denote $\mathcal{V}_{i,j} := \mathcal{V} \cap p_*(\mathcal{L}_{i,j})$.

For each $i = 1, \ldots, r$, set $D_{i,-1} = 0$. Also, for each j let $F_{i,j} = D_{i,j} - D_{i,j-1}$. With this notation, $\mathcal{L}_{i,j} = \mathcal{L}_{i,j-1}(-F_{i,j})$ and

$$F_{i,j} = E_{i-l} + E_{i-l+1} + \ldots + E_i + \ldots + E_{i+l'}.$$

An inductive argument shows now that

$$\mathcal{L}_{i,j}|_C = \begin{cases} \mathcal{L}|_C(-kA-k'B) & \text{if } l \neq i-1 \text{ and } l' \neq r-i, \\ \mathcal{L}|_C(-(k+1)A-k'B) & \text{if } l = i-1 \text{ and } l' \neq r-i, \\ \mathcal{L}|_C(-kA-(k'+1)B) & \text{if } l \neq i-1 \text{ and } l' = r-i, \\ \mathcal{L}|_C(-(k+1)A-(k'+1)B) & \text{if } l = i-1 \text{ and } l' = r-i, \end{cases} \tag{5.9}$$

and

$$\mathcal{L}_{i,j}|_{E_m} = \begin{cases} \mathcal{O}_{E_m}(k+k'+2) & \text{if } m=i \\ \mathcal{O}_{E_m}(-1) & \text{if } m=i-l-1 \text{ or } m=i+l'+1 \\ \mathcal{O}_{E_m} & \text{otherwise.} \end{cases} \tag{5.10}$$

It follows from (5.10) that

$$\mathrm{h}^0(\mathcal{L}_{i,j-1}|_{F_{i,j}}) = k+k'+1.$$

Also, setting $\widehat{F_{i,j}} = \overline{X - F_{i,j}}$, we have, by Equation (5.9) and Lemma 5.2.1, a map

$$V_{i,j-1}|_{\widehat{F_{i,j}}} \to V|_C(-D_{i,j-1}\cdot C)$$

that is an injection by Equation (5.10), and whose image is included in $\widetilde{V}(-kA-k'B)$; just match the cases of (5.10) with those of (5.9). Therefore,

$$\dim V_{i,j-1}|_{\widehat{F_{i,j}}}\left(-\sum_{P\in\widehat{F_{i,j}}\cap F_{i,j}} P\right) \leq \dim \widetilde{V}(-kA-k'B) = r-k-k',$$

the equality by the hypothesis of the lemma. However, by Exact Sequence (5.7), we have that

$$\dim V_{i,j-1}|_{F_{i,j}} + \dim V_{i,j-1}|_{\widehat{F_{i,j}}}\left(-\sum_{P\in\widehat{F_{i,j}}\cap F_{i,j}} P\right) = r+1,$$

and hence

$$\begin{aligned} \dim V_{i,j-1}|_{F_{i,j}} &= k+k'+1, \\ \dim V_{i,j-1}|_{\widehat{F_{i,j}}}\left(-\sum_{P\in\widehat{F_{i,j}}\cap F_{i,j}} P\right) &= r-k-k'. \end{aligned}$$

Now, just apply Equation (5.8), to see that

$$m_{E_i}(j-1) = m_{E_i}(j) + (r-k-k')$$

where $m_{E_i}(j)$ is the multiplicity of E_i in the degeneration scheme of $\mathcal{V}_{i,j}$, for $j =, 0, \ldots, i(r+i-1)-1$. Summing up, we get our result.

Lemma 5.2.4 *Let C be a semistable curve, $p : \mathcal{C} \to \Sigma$ a smoothing of C, $\mathcal{L}$ a line bundle over $\mathcal{C}$ and $\mathcal{V} = p_*(\mathcal{L})$. Let $r+1$ be the rank of $\mathcal{V}$. Write $C = E \bigcup F$, where E and F are subcurves of C without common components. Then the following two statements hold:*

1. *If*

$$\mathrm{h}^0(E, \mathcal{L}|_E) + \mathrm{h}^0(F, \mathcal{L}(-E)|_F) \leq r+1,$$

then $\mathrm{h}^0(C, \mathcal{L}|_C) = r+1$*, and we have equality above.*

2. *If* $\mathrm{h}^0(C, \mathcal{L}|_C) = r+1$ *then the image of the map*

$$\mathrm{H}^0(\mathcal{C}, \mathcal{L}) \to \mathrm{H}^0(E, \mathcal{L}|_E)$$

is equal to the image of the map

$$\mathrm{H}^0(C, \mathcal{L}|_C) \to \mathrm{H}^0(E, \mathcal{L}|_E).$$

In particular, if the inequality in item 1 is true, both maps are surjective, i.e., $V|_E = H^0(E, \mathcal{L}|_E)$*.*

Proof. From the exact sequence

$$0 \to \mathcal{L}(-E)|_F \to \mathcal{L}|_C \to \mathcal{L}|_E \to 0$$

we get a left-exact sequence

$$0 \to \mathrm{H}^0(\mathcal{L}(-E)|_F) \to \mathrm{H}^0(\mathcal{L}|_C) \to \mathrm{H}^0(\mathcal{L}|_E) \to 0;$$

hence, by hypothesis,

$$\mathrm{h}^0(\mathcal{L}|_C) \leq \mathrm{h}^0(\mathcal{L}(-E)|_F) + \mathrm{h}^0(\mathcal{L}|_E) \leq r+1.$$

But, by semicontinuity, $\mathrm{h}^0(\mathcal{L}|_C) \geq r+1$; therefore we have equality and the latter sequence is in fact right-exact as well.

To prove the second statement we just notice that the map

$$\mathrm{H}^0(\mathcal{L}) \to \mathrm{H}^0(\mathcal{L}|_C)$$

is surjective by the base-change theorem, because $\mathrm{h}^0(\mathcal{L}|_C) = r+1$.□

Lemma 5.2.5 *Let C be a smooth curve, V a linear system, and $V' := V(-P)$ for any point $P \in C$ such that $V' \neq V$. Assume that V and V' do not have special ramification points. Then V' and V have no common ramification points distinct from P.*

Proof. Let $r := \dim V - 1$. Let $Q \in C \backslash \{P\}$ be a ramification point of V'; since Q is nonspecial, the vanishing sequence at Q of V' is $0, 1, \ldots, r-2, r$, which means the vanishing sequence at Q of V cannot be $0, 1, \ldots, r-1, r+1$, the only possible sequence for a nonspecial ramification point.□

Lemma 5.2.6 *Let C be a smooth curve, L a line bundle on C and $V \subset H^0(L)$ a linear system with no special ramification points. Let a and b be positive integers such that $\dim V = a+b$. Let $p_1, p_2 : C \times C \to C$ be the projections, and define the locally free sheaves*

$$\begin{aligned} V_{p_1}(-a\Delta) &= V \otimes \mathcal{O}_C \cap p_{1*}(p_2^*(L)(-a\Delta)) \subset p_{1*}(p_2^*(L)(-a\Delta)), \\ V_{p_2}(-b\Delta) &:= V \otimes \mathcal{O}_C \cap p_{2*}(p_1^*(L)(-b\Delta)) \subset p_{2*}(p_1^*(L)(-b\Delta)). \end{aligned}$$

Then $V_{p_1}(-a\Delta)$ and $V_{p_2}(-b\Delta)$ are relative linear systems of ranks b and a, respectively, whose ramification divisors concide.

Proof. The ramification divisor W_{p_1} of $V_{p_1}(-a\Delta)$ is the degeneration scheme of the evaluation map

$$p_1^*(V_{p_1}(-a\Delta)) \to J^{b-1}_{p_1}(p_2^*(L)(-a\Delta)).$$

(Likewise for W_{p_2}.) But we have a map

$$J^{b-1}_{p_1}(p_2^*(L)(-a\Delta)) \to J^{b-1}_{p_1}(p_2^*(L))$$

which is an isomorphism outside Δ. Also $J^{b-1}_{p_1}(p_2^*(L)) = p_2^*(J^{b-1}_C(L))$, and we have an exact sequence

$$0 \to p_{2*}(p_1^*(L)(-b\Delta)) \to p_{2*}p_1^*(L) \to J^{b-1}_C(L)$$

because

$$J^{b-1}(L) = p_{2*}\left(\frac{p_1^*(L)}{p_1^*(L)(-b\Delta)}\right)$$

by definition.

Since $V \subset \mathrm{H}^0(L)$ and $\mathrm{H}^0(L) \otimes \mathcal{O}_C = p_{1*}p_2^*(L) = p_{2*}p_1^*(L)$, we have an induced exact sequence

$$0 \to p_2^*(V_{p_2}(-b\Delta)) \to p_2^*(V \otimes \mathcal{O}_C) \to J^{b-1}_{p_1}(p_2^*L) \to 0.$$

This sequence is right-exact because V has no special ramification point, and left-exact because of the definition of $V_{p_2}(-b\Delta)$, which proves that $V_{p_2}(-b\Delta)$ has rank a. Hence W_{p_1} (outside the diagonal) is the degeneration scheme of the map

$$p_1^*(V_{p_1}(-a\Delta)) \oplus p_2^*(V_{p_2}(-b\Delta)) \to p_2^*(V \otimes \mathcal{O}_C).$$

By the same argument, so is W_{p_2}. Since W_{p_1} and W_{p_2} are divisors that do not contain the diagonal, because they intersect the diagonal on the ramification points of V, they are the same.□

Lemma 5.2.7 *Let C be a smooth curve, and V a linear system on C of rank $r+1$ with no special ramification points. Let P and Q be distinct points of C, and a and b positive integers with $a+b=r+1$. Then the following two statements are equivalent:*

1. *Q is a special ramification point of the system $V(-aP)$ and P is a special ramification point of the system $V(-bQ)$.*
2. *The points P and Q satisfy one of the following four conditions:*
   - *(a)* $\dim V(-(a+1)P-(b+1)Q) \geq 1$
   - *(b)* $\dim V(-(a-1)P-(b-1)Q) \geq 3$
   - *(c)* $\dim V(-aP-(b+1)Q) \geq 1$ *and* $\dim V(-(a-1)P-bQ) \geq 2$
   - *(d)* $\dim V(-(a+1)P-bQ) \geq 1$ *and* $\dim V(-aP-(b-1)Q) \geq 2$

Proof. Notice first that, since V has no special ramification points and $a, b \leq r$, we have

$$\dim V(-aP) = r + 1 - a = b$$
$$\dim V(-bQ) = r + 1 - b = a.$$

We will prove first that (1) implies (2). The point Q being a special ramification point of $V(-aP)$ means that

$$\dim V(-aP - (b-1)Q) \geq 2 \quad \text{or} \quad \dim V(-aP - (b+1)Q) \geq 1. \tag{5.11}$$

Similarly, for P

$$\dim V(-bQ - (a-1)P) \geq 2 \quad \text{or} \quad \dim V(-bQ - (a+1)P) \geq 1 \tag{5.12}$$

If $\dim V(-aP - bQ) \geq 2$ then obviously P and Q satisfy (2c). Therefore, we can assume that $\dim V(-aP - bQ) = 1$.

If the second alternatives in (5.11) and (5.12) hold then P and Q are base points of $V(-aP - bQ)$, and thus

$$\dim V(-(a+1)P - (b+1)Q) \geq 1,$$

which is (2a). On the other hand, if the first alternatives in (5.11) and (5.12) hold and (2b) does not hold, that is,

$$\dim V(-(a-1)P - (b-1)Q) \leq 2,$$

then P and Q are base points of $V(-(a-1)P - (b-1)Q)$, which implies $\dim V(-aP - bQ) = 2$, a contradiction. The two other cases are trivial.

We will now prove that (2) implies (1). This is obvious if (2c) or (2d) holds. Also, (2a) implies that

$$\dim V(-(a+1)P - bQ) \geq 1 \quad \text{and} \quad \dim V(-aP - (b+1)Q) \geq 1,$$

and (2b) implies that

$$\dim V(-(a-1)P - bQ) \geq 2 \quad \text{and} \quad \dim V(-aP - (b-1)Q) \geq 2.$$

In any case, (5.11) and (5.12) hold.□

5.3 The general curve

Since the Thom–Porteous formula may be applied only if the degeneration scheme has the expected codimension, we devote a section to prove that the several loci that will appear have the expected codimension. Since we are interested in the Picard group, we are only concerned with proving that certain loci have the expected codimension 1 (i.e., they are in fact a divisor) and certain other loci have the expected codimension at least 2 (i.e., we can avoid these loci).

Let ρ be an open property of smooth curves, and let U_ρ be the open set in M_g corresponding to ρ and V_ρ the complement of U_ρ. To prove that the codimension of V_ρ is at least 1, it suffices to exhibit one curve C satifying ρ, since ρ is an open condition and M_g is irreducible. One way to do this is to degenerate, that is, to show that there exists a curve in the boundary $\Delta \subset \overline{M_g}$ which cannot be the limit of curves in M_g not satisfying ρ. This curve will in general be a nodal union of two general curves of lesser genus, for which we can apply induction.

Let's state first two already known facts.

Proposition 5.3.1 *The locus in M_g, $g \geq 4$, of curves possessing a weight-3 Weierstras point has codimension 2.*

Proof. See [16], Thm. 3.7.

Proposition 5.3.2 *Let (C, A, B) be a general genus-g smooth pointed curve, and a_0, b_0 positive integers. Then for all $0 < a \leq a_0$ and $0 < b \leq b_0$, the linear system $H^0(\omega_C(aA + bB))$ has at most simple ramification points, and A and B are not among them. Furthermore, the linear system $H^0(\omega_C(aA))$ has only simple ramification points distinct from A.*

Proof. See [4] Prop. 4.4 and [5] Prop. 3.1.

Proposition 5.3.3 *Let (C, A, B) be a two-pointed general smooth genus-g curve. Then, for all nonnegative integers a and b with $a + b \leq g - 1$, the complete linear system $H^0(\omega_C(-aA - bB))$ has at most simple ramification points.*

Proof. Since the condition is an open property, we need only to show that there exists a two-pointed general curve with the stated property. In fact, we need only prove the case where $b = 0$, as we can make the points A and B coalesce. Now, degenerate and do an induction: Pick a pointed curve (C, A), the nodal union of a general pointed genus-$(g-1)$ curve (X, Q) and a general two-pointed elliptic curve (Y, R, A), identifying Q and R.

Take now a smoothing $\pi : \mathcal{C} \to \Sigma$ of C. Choose a section σ_A of π such that $\sigma_A(0) = A$ and let Γ_A be the image of σ_A. If the statement is not true for the generic curve $(\mathcal{C}_\eta, \sigma_A(\eta))$, then there is a point P_η on $\mathcal{C}_\eta$ which is a special ramification point for the linear system $\mathrm{H}^0(\omega_{\mathcal{C}_\eta}(-a\sigma_A(\eta)))$. After a base change, i.e., a map $\Sigma \to \Sigma$ taking $t \to t^{n+1}$ for some n, we may assume that this point is rational, and thus induces a section σ_{P_η} of π, with $\sigma_{P_\eta}(0) =: P$ a smooth point of C. However, after this base change, the node of C becomes a singular point of $\mathcal{C}$, locally given by the equation $t^{n+1} - xy$. After desingularization, the node of C is replaced by a chain of rational smooth curves $(E_1, E_2, \ldots, E_n)$ with $E_1 \cap X = \{Q\}$ and $E_n \cap Y = \{R\}$. We will still denote by π the newmap and by C its special curve.

Let $\mathcal{L} := \omega_\pi(-a\Gamma_A)$ and $\mathcal{V} := p_*(\mathcal{L})$. Since (C, A) is general, by Propositions 5.3.2 and 5.3.1, $\mathcal{V}$ is a vector bundle of rank $g - a$. From Equation (5.6) we see that P must be a special ramification point of one of the limit linear systems V_X, V_Y and V_{E_i} with focus on X, Y and the E_i. Since C is of compact type, there exists a divisor D_X supported on C such that

$$\begin{aligned}
\mathcal{L}(D_X)|_X &= \omega_X(-(a-1)Q) \\
\mathcal{L}(D_X)|_{E_j} &= \mathcal{O}_{E_j} \quad \text{for } j = 1, \ldots, n-1 \\
\mathcal{L}(D_X)|_Y &= \mathcal{O}_Y(((a+1)R - aA)).
\end{aligned}$$

Hence, $\mathcal{L}(D_X)$ has focus on X. Furthermore, since $\mathrm{h}^0(\mathcal{L}(D_X)|_X) = g - a$, we have $V_X = \mathrm{H}^0(\omega_X(-(a-1)Q))$. By induction, V_X has at most simple ramification points.

Analogously, $V_Y = \mathrm{H}^0(\mathcal{O}_Y(gR - aA))$ and $V_{E_i} = \mathrm{H}^0(\mathcal{O}_{E_i}(g-1-a))$, neither of them having special ramification points.$\square$

For the next propositions we will be using a similar approach, but now, we will have two points varying. Pick the curve (C, R) defined as the union of two general pointed curves (X, A) and (Y, B, R), with genus $g_X := g-1$ and 1 respectively, joined by a chain of rational curves $E_1, E_2, ..., E_{n-1}$ with $A \in E_1$ and $B \in E_{n-1}$. Define $A_j := E_{j-1} \cap E_j$ and $B_j := E_j \cap E_{j+1}$; see Figure 5.3. (By convention, $E_0 = X$ and $E_n = Y$.) Now, pick a smoothing $\pi : \mathcal{C} \to \Sigma$ of C. Let $\sigma_R : \Sigma \to \mathcal{C}$ be a section through the smooth locus of this smoothing (we will call the image Γ_R) such that $\sigma_R(0) = R$. Let

$$\mathcal{L} := \omega_\pi((1+i)\Gamma_R) \tag{5.13}$$

for $i \geq 0$, and $\mathcal{V} = \pi_*(\mathcal{L})$. Note that $\mathrm{h}^0(\mathcal{C}_\eta, L|_{\mathcal{C}_\eta}) = g+i$, whence $\mathcal{V}$ is a vector bundle of rank $g+i$.

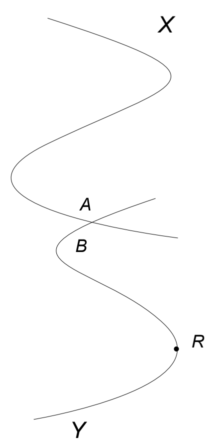


Figure 5.2: The curve (C, R).

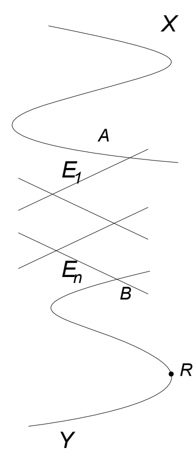


Figure 5.3: (C, R) after base change.

Since C is of compact type, there exists an effective divisor D_X supported on C but not containing X such that

$$\begin{aligned} \mathcal{L}(D_X)|_X &= \omega_X((2+i)A) \\ \mathcal{L}(D_X)|_{E_j} &= \mathcal{O}_{E_j} \quad \text{for } j = 1, \ldots, n-1 \\ \mathcal{L}(D_X)|_Y &= \mathcal{O}_Y((1+i)R - iB). \end{aligned} \tag{5.14}$$

Hence, by Lemma 5.2.4,

$$V_X := V(D_X)|_X = \mathrm{H}^0(\omega_X((2+i)A)) \tag{5.15}$$

is a limit linear system with focus on X.

Considering Y, first notice that $V|_Y = \mathrm{H}^0(\mathcal{O}_Y((1+i)R))$. Also, there exists an effective divisor D_Y supported on C but not containing Y, such that

$$\begin{array}{rclcl} \omega_\pi(D_Y)|_X & = & \mathcal{L}(D_Y)|_X & = & \omega_X((1-g_X)A) \\ \omega_\pi(D_Y)|_{E_j} & = & \mathcal{L}(D_Y)|_{E_j} & = & \mathcal{O}_{E_j} \quad \text{for } j = 1, \dots, n-1 \\ & & \omega_\pi(D_Y)|_Y & = & \mathcal{O}_Y((1+g_X)B) \\ & & \mathcal{L}(D_Y)|_Y & = & \mathcal{O}_Y((1+g_X)B + (1+i)R), \end{array} \tag{5.16}$$

which means that

$$V(D_Y - (1+i)\Gamma_R)|_Y = \mathrm{H}^0(\mathcal{O}_Y((1+g_X)B)),$$

because $\mathcal{L}(-(1+i)\Gamma_R) = \omega_\pi$. Hence, since $\mathcal{L}(D_Y)$ has focus on Y, by Lemma 5.2.1,

$$\begin{array}{rcl} \mathrm{H}^0(\mathcal{O}_Y((1+g_X)B)) & \subset & V_Y(-(1+i)R) \\ \mathrm{H}^0(\mathcal{O}_Y((1+i)R)) & \subset & V_Y(-(1+g_X)B), \end{array}$$

where $V_Y := V(D_Y)|_Y$. Therefore, by dimension considerations,

$$V_Y = V(D_Y)|_Y = \mathrm{H}^0(\mathcal{O}_Y((1+i)R)) + \mathrm{H}^0(\mathcal{O}_Y((1+g_X)B)), \tag{5.17}$$

viewed as a subspace of $\mathrm{H}^0(\mathcal{O}_Y((1+i)R + (1+g_X)B))$, is a limit linear system with focus on Y.

As for E_j, there are effective divisors $D_{j,1}, D_{j,2}$ supported on C but not containing E_j, such that

$$\begin{array}{rcl} \omega_\pi(D_{j,1})|_X & = & \omega_X((1-g_X)A) \\ \omega_\pi(D_{j,1})|_{E_l} & = & \mathcal{O}_{E_l} \quad \text{for all } l \neq j \\ \omega_\pi(D_{j,1})|_{E_j} & = & \omega_{E_j}((1+g_X)A_j + B_j) \\ \omega_\pi(D_{j,1})|_Y & = & \mathcal{O}_Y(B) \end{array} \tag{5.18}$$

and

$$\begin{array}{rcl} \mathcal{L}(D_{j,2})|_X & = & \omega_X(A) \\ \mathcal{L}(D_{j,2})|_{E_l} & = & \mathcal{O}_{E_l} \quad \text{for all } l \neq j \\ \mathcal{L}(D_{j,2})|_{E_j} & = & \omega_{E_j}(A_j + (2+i)B_j) \\ \mathcal{L}(D_{j,2})|_Y & = & \mathcal{O}_Y((1+i)R - iB). \end{array} \tag{5.19}$$

Hence, since B is a base point of $\mathrm{H}^0(\mathcal{O}_Y(B))$ and A is a base point of $\mathrm{H}^0(\omega_X(A))$,

$$V(D_{j,1} - (1+i)\Gamma_R)|_{E_j} = \mathrm{H}^0(\omega_{E_j}((1+g_X)A_j))$$

and

$$V(D_{j,2})|_{E_j} = \mathrm{H}^0(\omega_{E_j}((2+i)B_j)).$$

Now, since $\mathcal{L}(D_{j,1} + D_{j,2})$ has focus on E_j, $V_{E_j} := V(D_{j,1} + D_{j,2})|_{E_j}$ is a limit linear system with focus on E_j, and by Lemma 5.2.1,

$$\begin{array}{rcl} \mathrm{H}^0(\omega_{E_j}((1+g_X)A_j)) & \subset & V_{E_j}(-(1+i)B_j) \\ \mathrm{H}^0(\omega_{E_j}((2+i)B_j)) & \subset & V_{E_j}(-g_X A_j). \end{array}$$

Hence, by dimension considerations,

$$V_{E_j} = \mathrm{H}^0(\omega_{E_j}((1+g_X)A_j)) \oplus \mathrm{H}^0(\omega_{E_j}((2+i)B_j)) \tag{5.20}$$

as a subspace of $\mathrm{H}^0(\omega_{E_j}((1+g_X)A_j + (2+i)B_j))$.

Proposition 5.3.4 *Let i_0 be a fixed positive integer. Then for a general curve C of genus g and a general point $R \in C$,*

$$h^0\left(\omega_C((1+i)R - (a+1)P - (b+1)Q)\right) = 0 \tag{5.21}$$

for every $P, Q \in C$, every $i = 0, 1, ..., i_0$ and every nonnegative integers a and b with $a+b = g+i$.

Proof. We will do a double induction on g and i_0. For $g \leq 2$ we have that for every nonnegative integer i

$$\deg \omega_C((i+1)R - (a+1)P - (b+1)Q) = 2g - 2 + 1 + i - 1 - a - 1 - b = g - 3 < 0,$$

which means that the Proposition holds. For $g \geq 3$ all we have to do is to find a pointed curve with the required property. Since this property is open, the proposition follows. Our method to do this is to degenerate. Let (C, R) be as in Figure 5.2 and $\pi : \mathcal{C} \to \Sigma$ a smoothing of C, with a section $\sigma_R : \Sigma \to \mathcal{C}$ such that $\sigma_R(0) = R$. If for every pointed smooth curve there exist P and Q negating (5.21), then there exist P_η and Q_η points of the generic curve $\mathcal{C}_\eta$ with the same property. Hence, possibly after a base change, under which C is transformed into the curve depicted in Figure 5.3, we have sections σ_P and σ_Q of π with $\sigma_P(\eta) = P_\eta$ and $\sigma_Q(\eta) = Q_\eta$. Let $P := \sigma_P(0)$ and $Q = \sigma_Q(0)$. We may assume P and Q lie on the smooth locus of C. Also, let Γ_P and Γ_Q be the images of σ_P and σ_Q. Define $\mathcal{L}' := \mathcal{L}(-(a+1)\Gamma_P - (b+1)\Gamma_Q)$, where $\mathcal{L}$ is defined in (5.13), and $\mathcal{V}' := \pi_*(\mathcal{L}')$. Then $\mathcal{V}'$ is a vector bundle of rank at least 1.

Up to exchanging P and Q, there are 6 cases to consider:

Case 1: Assume P and Q lie on the E_i's. Since C is of compact type we can choose a divisor D of $\mathcal{C}$ supported on C such that

$$\begin{aligned} \mathcal{L}'(D)|_X &= \omega_X(-g_X A) \\ \mathcal{L}'(D)|_{E_i} &= \mathcal{O}_{E_i} \quad \text{for } i = 1, \ldots, n-1 \\ \mathcal{L}'(D)|_Y &= \mathcal{O}_Y(-(i+1)B + (1+i)R). \end{aligned}$$

Since A, B and R are general, we have

$$\mathrm{h}^0(\omega_X(-g_X A)) = \mathrm{h}^0(\mathcal{O}_Y(-(1+1)B + (i+1)R)) = 0,$$

a contradiction by Lemma 5.2.4.

Case 2: Assume $P \in E_i$ and $Q \in X$. Again, we can choose D such that

$$\begin{aligned} \mathcal{L}'(D)|_X &= \omega_X((1-g_X+b)A - (b+1)Q) \\ \mathcal{L}'(D)|_{E_i} &= \mathcal{O}_{E_i} \quad \text{for } i = 1, \ldots, n-1 \\ \mathcal{L}'(D)|_Y &= \mathcal{O}_Y(-(i+1)B + (i+1)R). \end{aligned}$$

Since A, B and R are general, we have

$$\mathrm{h}^0(\mathcal{O}_Y(-(i+1)B+(i+1)R))=0,$$

and, by Propositions 5.3.2 and 5.3.3,

$$\mathrm{h}^0(\omega_X((b-g_X)A-(b+1)Q))=0,$$

again a contradiction by Lemma 5.2.4

Case 3: Assume $P\in E_i$ and $Q\in Y$. Choose D such that

$$\begin{array}{rcl}\mathcal{L}'(D)|_X & = & \omega_X((1-g_X)A)\\ \mathcal{L}'(D)|_{E_i} & = & \mathcal{O}_{E_i}\quad \text{for } i=1,\ldots,n-1\\ \mathcal{L}'(D)|_Y & = & \mathcal{O}_Y((g-a-1)B+(i+1)R-(b+1)Q).\end{array}$$

Since

$$\mathrm{h}^0(\mathcal{O}_Y((g-a-1)B+(i+1)R-(b+1)Q))=0$$

and $\mathrm{h}^0(\omega_X(-g_XA))=0$, we have a contradiction by Lemma 5.2.4.

Case 4: Assume $P\in Y$ and $Q\in X$. First, suppose $b\leq g_X$; hence $a\geq i+1$. Recall Equation (5.17):

$$V_Y=V(D_Y)|_Y=\mathrm{H}^0(\mathcal{O}_Y((1+i)R))+\mathrm{H}^0(\mathcal{O}_Y((1+g_X)B)).$$

Then there exists an effective divisor D not containing Y such that $\mathcal{L}(D_Y-D)|_{E_i}=\mathcal{O}_{E_i}$ for all i, and such that

$$V(D_Y-D)|_Y\subset V_Y(-bB)=\mathrm{H}^0(\mathcal{O}_Y((1+i)R))+\mathrm{H}^0(\mathcal{O}_Y((1+g_X-b)B))$$

and

$$V(D_Y-D)|_X\subset \mathrm{H}^0(\omega_X((1-g_X+b)A)).$$

Hence, by Lemma 5.2.1 and Proposition A.2.1 applied to $V_Y(-bB)$,

$$V'(D_Y-D)|_Y\subset V_Y(-bB-(a+1)P)=0.$$

Also, by Proposition 5.3.3,

$$V'(D_Y-D)|_X(-A)\subset \mathrm{H}^0(\omega_X((-g_X+b)A-(b+1)Q))=0,$$

which, by Equation (5.7), implies that

$$V'(D_Y-D)|_C=0,$$

a contradiction.

If $b\geq g$ then there exists a divisor D such that

$$\begin{array}{rcl}\mathcal{L}'(D)|_X & = & \omega_X((1-g_X+b)A-(b+1)Q)\\ \mathcal{L}'(D)|_{E_j} & = & \mathcal{O}_{E_j}\quad \text{for } i=1,\ldots,n-1\\ \mathcal{L}'(D)|_Y & = & \mathcal{O}_Y((1+i)R+(1+g_X-b)B-(a+1)P).\end{array}$$

However, by Proposition 5.3.2,

$$\mathrm{h}^0(\mathcal{L}'(D)|_X) = 0.$$

Thus, since

$$\deg \mathcal{L}'(D)|_Y(-B) = -1,$$

it follows from Lemma 5.2.4 that

$$\mathrm{h}^0(\mathcal{L}'(D)|_C) = 0,$$

a contradiction.

Case 5: Assume $P, Q \in X$. Choose D such that

$$\begin{array}{rcl} \mathcal{L}'(D)|_X & = & \omega_X((2+i)A - (b+1)Q - (a+1)P) \\ \mathcal{L}'(D)|_{E_i} & = & \mathcal{O}_{E_i} \quad \text{for } i = 1, \ldots, n-1 \\ \mathcal{L}'(D)|_Y & = & \mathcal{O}_Y(-iB + (i+1)R). \end{array}$$

Since by induction

$$\mathrm{h}^0(\omega_X((2+i)A - (b+1)Q - (a+1)P)) = 0,$$

and since $\mathrm{h}^0(\mathcal{O}_Y(-(i+1)B + (i+1)R)) = 0$, it follows that $\mathrm{h}^0(\mathcal{L}'(D)|_C) = 0$, a contradiction.

Case 6: Assume $P, Q \in Y$. Choose D such that

$$\begin{array}{rcl} \mathcal{L}'(D)|_X & = & \omega_X((1-g_X)A) \\ \mathcal{L}'(D)|_{E_i} & = & \mathcal{O}_{E_i} \\ \mathcal{L}'(D)|_Y & = & \mathcal{O}_Y(gB + (i+1)R - (a+1)P - (b+1)Q). \end{array}$$

Since

$$\mathrm{h}^0(\mathcal{O}_Y(gB + (i+1)R - (a+1)P - (b+1)Q)) = 0$$

by degree considerations, and since $\mathrm{h}^0(\omega_X(-g_X A)) = 0$, it follows again from Lemma 5.2.4 that $\mathrm{h}^0(\mathcal{L}'(D)|_C) = 0$, a contradiction.$\square$

Proposition 5.3.5 *Let i_0 be a fixed positive integer. Then for a general curve C of genus g and a general point $R \in C$ there are no points $P, Q \in C \backslash \{R\}$ satisfying both:*

$$h^0\left(\omega_C((1+i)R - aP - (b-1)Q)\right) \geq 2$$

and

$$h^0\left(\omega_C((1+i)R - (a+1)P - bQ)\right) \geq 1$$

for any $i = 0, 1, ..., i_0$ and any nonnegative integers a and b with $a + b = g + i$.

Proof. We will do an induction on g and follow the path of the proof of Proposition 5.3.4. For $g \leq 1$, the statement follows easily because

$$\deg(\omega_C((1+i)R - (a+1)P - bQ)) = g - 2 < 0.$$

Keeping the notation and the initial constructions as in the proof of Proposition 5.3.4 and defining

$$\begin{aligned} \mathcal{L}_1 &:= \mathcal{L}(-a\Gamma_P - (b-1)\Gamma_Q) \quad \text{and} &(5.22)\\ \mathcal{L}_2 &:= \mathcal{L}(-(a+1)\Gamma_P - b\Gamma_Q), &(5.23)\end{aligned}$$

we see that $\mathcal{V}_1 := \pi_*(\mathcal{L}_1)$ and $\mathcal{V}_2 := \pi_*(\mathcal{L}_2)$ are vector bundles of rank at least 2 and 1, respectively. Reasoning by contradiction, we will prove that on the special fiber either $V_1(D_1)|_C$ has dimension at most 1 or $V_2(D_2)|_C$ has dimension 0 for certain divisors D_1 and D_2 of $\mathcal{C}$ supported on C.

Case 1: Assume P and Q lie on some of the E_i's. Since C is of compact type we can choose a divisor D_2 supported on C such that

$$\begin{aligned} \mathcal{L}_2(D_2)|_X &= \omega_X((1-g_X)A)\\ \mathcal{L}_2(D_2)|_{E_j} &= \mathcal{O}_{E_j} \quad \text{for } j = 1, \dots, n-1\\ \mathcal{L}_2(D_2)|_Y &= \mathcal{O}_Y((1+i)R - (1+i)B).\end{aligned}$$

Now, since

$$\mathrm{h}^0(\mathcal{O}_Y((1+i)R - (1+i)B)) = 0$$

and $\mathrm{h}^0(\omega_X(-g_X A)) = 0$, we get from Lemma 5.2.4 that $\mathrm{h}^0(\mathcal{L}_2(D_2)|_C) = 0$, a contradiction.

Case 2: Assume $P \in E_i$ for a certain i, and $Q \in X$. First assume $b \geq g$. Then we can choose D_1 such that

$$\begin{aligned} \mathcal{L}_1(D_1)|_{E_j} &= \mathcal{O}_{E_j} \quad \text{for } j = 1, \dots, n-1\\ \mathcal{L}_1(D_1)|_Y &= \mathcal{O}_Y((1+i)R - iB)\\ \mathcal{L}_1(D_1)|_X &= \omega((1-g_X+b)A - (b-1)Q).\end{aligned}$$

Since $\mathcal{L}_1(D_1)$ has focus on X and $\mathrm{h}^0(\omega_X((1-g_X+b)A - (b-1)Q)) = 1$ by Proposition 5.3.2, it follows that $\mathrm{h}^0(\mathcal{L}_1(D_1)|_C) \leq 1$, a contradiction.

If $b \leq g_X$, then, by Equation (5.20) and because C is of compact type, there exists an effective divisor D_1, which does not contain E_i, such that

$$V(D_{i,1} + D_{i,2} - D_1 - a\Gamma_P - (b-1)\Gamma_Q)|_{E_i} \subset V_{E_j}(-aP - bA_j) = 0,$$

where the last equality follows from Section A.1, and

$$\begin{aligned} \mathcal{L}_1(D_{i,1} + D_{i,2} - D_1)|_{E_j} &= \mathcal{O}_{E_j} \quad \text{for } j \neq i,\\ \mathcal{L}_1(D_{i,1} + D_{i,2} - D_1)|_X &= \omega_X((1-g_X+b)A - (b-1)Q).\end{aligned}$$

This means that

$$V_1(D_{i,1} + D_{i,2} - D_1)|_X(-A) \subset \mathrm{H}^0(\omega_X((-g_X+b)A - (b-1)Q)).$$

Since $\mathrm{h}^0(\omega_X((-g_X+b)A-(b-1)Q))=1$ by Proposition 5.3.3, if follows from Exact Sequence (5.7) that

$$\dim V_1(D_{i,1}+D_{i,2}-D_1)|_C \leq \dim V(D_{i,1}+D_{i,2}-D_1-a\Gamma_P-(b-1)\Gamma_Q)|_{E_i}+\dim V_1(D_{i,1}+D_{i,2}-D_1)|_X(-A),$$

thus bounded by 1, a contradiction.

Case 3: Assume $P, Q \in X$. We can choose D_1 and D_2 such that

$$\begin{array}{rclcl} \mathcal{L}_1(D_1)|_{E_i} & = & \mathcal{L}_2(D_2)|_{E_i} & = & \mathcal{O}_{E_i} \quad \text{for } i=1,\dots,n-1 \\ \mathcal{L}_1(D_1)|_Y & = & \mathcal{L}_2(D_2)|_Y & = & \mathcal{O}_Y((1+i)R-iB) \\ & & \mathcal{L}_1(D_1)|_X & = & \omega_X((2+i)A-aP-(b-1)Q) \\ & & \mathcal{L}_2(D_2)|_X & = & \omega_X((2+i)A-(a+1)P-bQ), \end{array}$$

and then argue by induction.

Case 4: Assume $P, Q \in Y$. By Equation (5.17) we just have to prove that the linear system

$$V_Y = \mathrm{H}^0(\mathcal{O}_Y(1+g_X)B)+\mathrm{H}^0(\mathcal{O}_Y((1+i)R)) \subset \mathrm{H}^0(\mathcal{O}_Y((1+g_X)B+(1+i)R))$$

does not admit points P, Q satisfying

$$\begin{array}{rcl} \dim(V_Y(-(a+1)P-bQ)) & \geq & 1 \quad \text{and} \\ \dim(V_Y(-aP-(b-1)Q)) & \geq & 2 \end{array} \tag{5.24}$$

But, when $P, Q \neq R$, this is just Proposition A.2.2.

Now, if $P=Q=R$ then (5.24) does not hold because

$$V_Y(-(a+b+1)R) \subset \mathrm{H}^0(\mathcal{O}_Y((1+g_X)B-(1+g_X)R))$$

and R and B are general. If $P=R$, $Q \neq R$ and $a \leq i$ then (5.24) does not hold because

$$V_Y(-aP) = \mathrm{H}^0(\mathcal{O}_Y(1+g_X)B)+\mathrm{H}^0(\mathcal{O}_Y((i+1-a)R)),$$

and by Proposition A.2.1 such a linear system does not admit special ramification points. So we are left with the case where $P=R$, $Q \neq R$ and $a \geq i+1$, which implies $b \leq g_X$. Blowing up $\mathcal{C}$ at R, we add an exceptional divisor, which we will call E, and let $P_E := \widetilde{\Gamma_P} \cap E$ and $R_E := \widetilde{\Gamma_R} \cap E$, where $\widetilde{\Gamma_P}$ and $\widetilde{\Gamma_R}$ are the strict transforms; see Figure 5.4. If $P_E = R_E$ we blow up again; however, there is no loss of generality in assuming that $P_E \neq R_E$. With the same reasoning that led to Equation (5.17) we can see that there exists D_E such that

$$\begin{array}{rcl} \mathcal{L}(D_E)|_X & = & \omega_X((1-g_X)A) \\ \mathcal{L}(D_E)|_{E_j} & = & \mathcal{O}_{E_j} \\ \mathcal{L}(D_E)|_Y & = & \mathcal{O}_Y((1+g_X)B-g_XR) \\ \mathcal{L}(D_E)|_E & = & \omega_E((2+g_X)R+(1+i)R_E) \end{array}$$

and

$$V_E := V(D_E)|_E = \mathrm{H}^0(\omega_E((2+g_X)R)) \oplus \mathrm{H}^0(\omega_E((1+i)R_E))$$

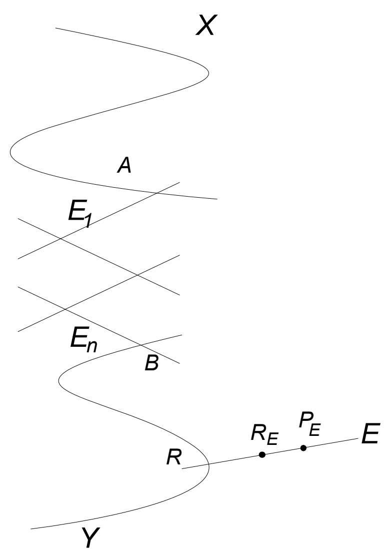


Figure 5.4: The blow up on R

is a limit linear system with focus on E. Since there exists an effective divisor D_2 not containing E such that

$$\begin{aligned}\mathcal{L}(D_E - D_2)|_Y &= \mathcal{O}_Y((1+g_X)B - (g_X - b + 1)R)\\ \mathcal{L}(D_E - D_2)|_E &= \omega_E((2 + g_X - (b-1))R + (1+i)R_E),\end{aligned}$$

and since $b \leq g_X$, by Section A.1,

$$V_2(D_E - D_2)|_E \subset V_E(-(b-1)R - (a+1)P_E) = 0.$$

Since

$$V_2(D_E - D_2)|_Y(-R) \subset \mathrm{H}^0(\mathcal{O}_Y((1+g_X)B - (g_X - b + 2)R - bQ)) = 0,$$

Exact Sequence (5.7) implies that $V_2(D_E - D_2)|_C = 0$, a contradiction.

Case 5: Assume $P \in E_j$ for a certain j, and $Q \in Y$. If $b - 1 \leq i + 1$ then there exists an effective divisor D_2 not containing E_j such that

$$\begin{aligned}\mathcal{L}_2(D_{j,1} + D_{j,2} - D_2)|_X &= \omega_X((1-g_X)A)\\ \mathcal{L}_2(D_{j,1} + D_{j,2} - D_2)|_{E_l} &= \mathcal{O}_{E_l} \quad \text{for } l \neq j\\ \mathcal{L}_2(D_{j,1} + D_{j,2} - D_2)|_{E_j} &= \omega_{E_j}((1+g_X)A_j + ((2+i) - (b-1))B_j - (a+1)P)\\ \mathcal{L}_2(D_{j,1} + D_{j,2} - D_2)|_Y &= \mathcal{O}_Y((1+i)R - (i - (b-1))B - bQ)),\end{aligned}$$

where $D_{j,1}$ and $D_{j,2}$ are those divisors that yielded Equations (5.18) and (5.19). Therefore, by Lemma 5.2.1 and Equations (5.20) and (5.23), and by Section A.1 as well, applied to $V_{E_j}(-(b-$

$1)B_j)$,

$$\begin{aligned} V_2(D_{j,1}+D_{j,2}-D_2)|_{E_j} &\subset V_{E_j}(-(a+1)P-(b-1)B_j)=0 \quad \text{and} \\ V_2(D_{j,1}+D_{j,2}-D_2)|_Y(-B) &\subset \mathrm{H}^0(\mathcal{O}_Y((1+i)R-(i+1-(b-1))B-bQ))=0. \end{aligned}$$

Hence

$$V_2(D_{j,1}+D_{j,2}-D_2)|_C=0,$$

a contradiction.

If $Q\neq R$ and $b-1\geq i+2$, then $a\leq g_X-2$. Following the steps above, there exists an effective divisor D_2, not containing Y, such that

$$\begin{aligned} \mathcal{L}_2(D_Y-D_2)|_X &= \omega_X((1-g_X)A) \\ \mathcal{L}_2(D_Y-D_2)|_{E_l} &= \mathcal{O}_{E_l} \quad \text{for } l\neq j \\ \mathcal{L}_2(D_Y-D_2)|_{E_j} &= \mathcal{O}_{E_j}((a+1)B_j-(a+1)P) \\ \mathcal{L}_2(D_Y-D_2)|_Y &= \mathcal{O}_Y((1+i)R+(1+g_X-(a+1))B-bQ), \end{aligned}$$

where D_Y is as in Equation (5.16). Therefore, by Lemma 5.2.1, Equations (5.17) and (5.23) and Proposition A.2.1,

$$\begin{aligned} V_2(D_Y-D_2)|_Y &\subset V_Y(-bQ-(a+1)B)=0 \\ V_2(D_Y-D_2)|_{E_j}(-B_j) &\subset \mathrm{H}^0(\mathcal{O}_{E_j}(aB_j-(a+1)P))=0. \end{aligned}$$

Hence, by Exact Sequence (5.7),

$$V_2(D_Y-D_2)|_C=0,$$

a contradiction.

The case $Q=R$ follows as in the case $Q=R$ in Case 4.

Case 6: Assume $P\in X$ and $Q\in Y$. If $a\geq g_X$ then we can choose D_1 and D_2 such that

$$\begin{aligned} \mathcal{L}_1(D_1)|_{E_i} = \mathcal{L}_2(D_2)|_{E_i} &= \mathcal{O}_{E_i} \quad \text{for } i=1,\dots,n-1 \\ \mathcal{L}_1(D_1)|_Y &= \mathcal{O}_Y((1+i)R-(b-1)Q+(b-i-1)B) \\ \mathcal{L}_2(D_2)|_Y &= \mathcal{O}_Y((1+i)R-bQ+(b-i)B) \\ \mathcal{L}_1(D_1)|_X &= \omega_X((2+a-g_X)A-aP) \\ \mathcal{L}_2(D_2)|_X &= \omega_X((1+a-g_X)A-(a+1)P). \end{aligned}$$

Since either

$$\begin{aligned} \mathrm{h}^0(\mathcal{O}_Y((1+i)R-(b-1)Q+(b-i-2)B)) &= 0 \quad \text{or} \\ \mathrm{h}^0(\mathcal{O}_Y((1+i)R-bQ+(b-i-1)B)) &= 0, \end{aligned}$$

and since P cannot be a special ramification point of either $\mathrm{H}^0(\omega_X((2+a-g_X)A))$ or $\mathrm{H}^0(\omega_X((1+a-g_X)A))$ by Proposition 5.3.2, Lemma 5.2.4 finishes the proof.

Suppose now that $a \leq g_X - 1$. Then $b \geq i + 2$. Suppose $Q \neq R$. Then there exist effective divisors D_1, D_2 not containing Y such that

$$\mathcal{L}_1(D_Y - D_1)|_{E_j} = \mathcal{L}(D_Y - D_2)|_{E_j} = \mathcal{O}_{E_j} \quad \text{for } j = 1, \ldots, n-1$$

and, using Lemma 5.2.1,

$$\begin{aligned} V_1(D_Y - D_1)|_Y &\subset V_Y(-(b-1)Q - (a+1)B) \\ V_2(D_Y - D_2)|_Y &\subset V_Y(-bQ - aB) \end{aligned}$$

and

$$V_1(D_Y - D_1)|_X(-A) \subset \mathrm{H}^0(\omega_X(-(g_X - a - 1)A - aP)) \tag{5.25}$$

$$V_2(D_Y - D_2)|_X(-A) \subset \mathrm{H}^0(\omega_X(-(g_X - a)A - (a+1)P)), \tag{5.26}$$

with D_Y as in Equation (5.16). But Proposition 5.3.3 says that the complete linear systems in (5.25) and (5.26) have the expected dimensions, namely 1 and 0. Hence we get from Exact Sequence (5.7), that both

$$\dim V_Y(-bQ - aB) \text{ and } \dim V_Y(-(b-1)Q - (a+1)B)$$

are positive, a contradiction by Proposition A.2.1 and Lemma 5.2.5.

The case $Q = R$ is handled similarly to the case $Q = R$ in Case 4.$\square$

Proposition 5.3.6 *Let i_0 be a fixed positive integer. Then, for a general curve C of genus g and a general point $R \in C$,*

$$\mathrm{h}^0\left(\omega_C((1+i)R - (a-1)P - (b-1)Q)\right) = 2$$

for every $P, Q \in C \backslash \{R\}$, every $i = 0, 1, ..., i_0$ and every positive integers a, b with $a + b = g + i$.

Proof. Since $\mathrm{H}^0(\omega_C((1+i)R))$ has no special ramification points, we may assume that $a > 1$ and $b > 1$. The condition

$$\mathrm{h}^0\left(\omega_C((1+i)R - (a-1)P - (b-1)Q)\right) \geq 3$$

is equivalent to

$$\mathrm{h}^0(\mathcal{O}_C((a-1)P + (b-1)Q - (1+i)R)) \geq 1$$

which is equivalent to the existence of a map $C \to \mathbb{P}^1$ of degree $a+b-2$ taking P and Q to the same point and having ramification degrees $a-2$ and $b-2$ on P and Q, respectively, and ramification degree at least i on R. If $i > 0$, considering the Hurwitz scheme $\mathcal{H}_i$ parametrizing covers of this type (see [7] and [15]), we see that its dimension is

$$2g - 2 + 2(a - 1 + b - 1) - (a-2) - (b-2) - i - 1 = 3g - 3.$$

Hence the map $\mathcal{H}_i \to M_{g,1}$ taking a covering $C \to \mathbb{P}^1$ to the curve (C, R) cannot be dominant.

If $i = 0$ the dimension of $\mathcal{H}_0$ will actually be

$$2g - 2 + 2(a - 1 + b - 1) - (a - 2) - (b - 2) - 2 = 3g - 4.$$

Hence the map $\mathcal{H}_0 \to M_g$ taking a covering $C \to \mathbb{P}^1$ to the curve C cannot be dominant.$\square$

Corollary 5.3.1 *Let (C, R) be a general genus-g pointed smooth curve, and a, b, i integers such that $i \geq 0$, $a, b \geq 1$ and $a + b = g + i$. Then there are no points $P, Q \in C\backslash\{R\}$ such that Q is a special ramification point of $H^0(\omega_C((1 + i)R - aP))$ and Q is a special ramification point of $H^0(\omega_C((1 + i)R - bQ))$.*

Proof. Just combine Lemma 5.2.7 with Propositions 5.3.4, 5.3.5 and 5.3.6.

Corollary 5.3.2 *Let (C, A, B) be a general genus-g two-pointed smooth curve, and a, b integers such that $a, b > 1$ and $a + b = g + 1$. Then there are no points $P, Q \in C\backslash\{A, B\}$ such that Q is a special ramification point of $H^0(\omega_C(A + B - aP))$ and P is a special ramification point of $H^0(\omega_C(A + B - bQ))$.*

Proof. We will just make the points A and B coalesce. Let X be the union of a general genus-g 1-pointed curve (C, R_1) with a 3-pointed rational curve $(\mathbb{P}^1, R_2, A, B)$, identifying R_1 with R_2. Let $\pi : \mathcal{C} \to \Sigma$ be a smoothing of X and σ_A and σ_B sections of π intersecting X at A and B. If every 2-pointed curve (C, A, B) admits points P and Q satisfying the conditions negated by the Corollary, then (possibly after a base change) there exist sections σ_P and σ_Q through the smooth locus of π intersecting the general fiber at those points. Let $P := \sigma_P(0)$ and $Q := \sigma_Q(0)$, denote by Γ_A, Γ_B, Γ_P and Γ_Q the images of σ_A, σ_B, σ_P and σ_Q, and set $\mathcal{L} := \omega_\pi(\Gamma_A + \Gamma_B)$ and $\mathcal{V} := \pi_*(\mathcal{L})$. It is straightforward to see that for every $i = 1, \ldots, g$ we have

$$\begin{aligned}
\mathcal{L}(-C)|_C &= \omega_C(2R_1) \\
\mathcal{L}(-C)|_{\mathbb{P}^1} &= \omega_{\mathbb{P}^1}(A + B) \\
\mathcal{L}(-i\mathbb{P}^1)|_C &= \omega_C((1 - i)R_1) \\
\mathcal{L}(-i\mathbb{P}^1)|_{\mathbb{P}^1} &= \omega_{\mathbb{P}^1}((1 + i)R_2 + A + B).
\end{aligned}$$

Thus, using Lemma 5.2.1,

$$\begin{array}{rcccl}
V_C &=& \mathcal{V}(-C)|_C &=& \mathrm{H}^0(\omega_C(2R_1)) \\
V_{\mathbb{P}^1} &=& \mathcal{V}(-g\mathbb{P}^1)|_{\mathbb{P}^1} &=& \mathrm{H}^0(\omega_{\mathbb{P}^1}(A + B)) + \mathrm{H}^0(\omega_{\mathbb{P}^1}((1 + g)R_2)) \\
&& \mathcal{V}(-i\mathbb{P}^1)|_{\mathbb{P}^1} &=& \mathrm{H}^0(\omega_{\mathbb{P}^1}(A + B)) + \mathrm{H}^0(\omega_{\mathbb{P}^1}((1 + i)R_2)).
\end{array} \tag{5.27}$$

We now have some cases to check.

Case 1: Assume $P, Q \in C$ (see Figure A.1). Then the result follows from Corollary 5.3.1.

Case 2: Assume $P \in C$ and $Q \in \mathbb{P}^1$ (see Figure A.2). If P is not a ramification point of $\mathrm{H}^0(\omega_C(-(g - a)R_1))$, then

$$\mathcal{V}(-a\Gamma_P)_{\mathbb{P}^1} = \mathcal{V}(-(g - a)\mathbb{P}^1 - a\Gamma_P)|_{\mathbb{P}^1} = \mathrm{H}^0(\omega_{\mathbb{P}^1}(A + B)) + \mathrm{H}^0(\omega_{\mathbb{P}^1}((1 + g - a)R_2)),$$

which, by Proposition A.1.4, does not have special ramification points. Thus, P must be ramification point of $\mathrm{H}^0(\omega_C(-(g-a)R_1))$, and hence, by Lemma 5.2.5 and Proposition 5.3.3 we see that P is not a ramification point of the linear systems $\mathrm{H}^0(\omega_C(-(g-a-1)R_1))$ and $\mathrm{H}^0(\omega_C(-(g-a+1)R_1))$.

However,

$$\begin{array}{rcl}\mathcal{L}(-(b-1)\mathbb{P}^1 - b\Gamma_Q)|_{\mathbb{P}^1} & = & \mathcal{O}_{\mathbb{P}^1} \quad \text{and} \\ \dim \mathcal{V}(-(b+1)\mathbb{P}^1)|_{\mathbb{P}^1}(-(b+1)Q) & = & 1\end{array}$$

by Proposition A.1.4. Therefore, by Exact Sequence 5.7 and dimension considerations, we get that

$$\mathrm{H}^0(\omega_C(-(g-a+1)R_1)) \subset \mathcal{V}(-b\Gamma_Q)_C \subset \mathrm{H}^0(\omega_C(-(g-a-1)R_1)).$$

Since P is a special ramification point of $\mathcal{V}(-b\Gamma_Q)_C$, P must be a ramification point of either

$$\mathrm{H}^0(\omega_C(-(g-a-1)R_1)) \quad \text{or} \quad \mathrm{H}^0(\omega_C(-(g-a+1)R_1)),$$

a contradiction.

Case 3: Assume $P, Q \in \mathbb{P}^1$ (see Figure A.3). Thus, by Lemma 5.2.7 and degree considerations, we have that

$$\dim V_{\mathbb{P}^1}(-(a-1)P-(b-1)Q) \geq 3,$$

which, by Proposition A.1.4, implies that $\{P, Q\} = \{A, B\}$.

Without loss of generality we can assume $P = A$ and $Q = B$. Now, we blow up the points A and B. Let $\mathbb{P}^1_A$ and $\mathbb{P}^1_B$ be the exceptional divisors. We will still denote by Γ_A, Γ_B, Γ_P and Γ_Q their strict transforms. Let $\tilde{A} := \Gamma_A \cap \mathbb{P}^1_A$ and $\tilde{B} := \Gamma_B \cap \mathbb{P}^1_B$ (see Figure A.7). Then there exists an effective divisor D on $\mathcal{C}$ supported on X such that

$$\begin{array}{rcl}\mathcal{L}(D)|_C & = & \omega_C((1-g)R_1) \\ \mathcal{L}(D)|_{\mathbb{P}^1} & = & \omega_{\mathbb{P}^1}((1+g)R_2 + (1-a)A + (1-b)B) = \mathcal{O}_{\mathbb{P}^1} \\ \mathcal{L}(D)|_{\mathbb{P}^1_A} & = & \omega_{\mathbb{P}^1_A}((1+a)A + \tilde{A}) \\ \mathcal{L}(D)|_{\mathbb{P}^1_B} & = & \omega_{\mathbb{P}^1_B}((1+b)B + \tilde{B}),\end{array}$$

and by Lemma 5.2.1 applied to $\mathcal{L}(D - \Gamma_B - \Gamma_A)$ and dimension considerations, we get that

$$\begin{array}{rcl}\mathrm{H}^0(\omega_{\mathbb{P}^1_A}((1+a)A)) & = & \mathcal{V}(D)|_{\mathbb{P}^1_A} \\ \mathrm{H}^0(\omega_{\mathbb{P}^1_B}((1+b)B)) & = & \mathcal{V}(D)|_{\mathbb{P}^1_B}.\end{array}$$

Therefore, it is easy to see that $\mathcal{V}(-a\Gamma_P)_{\mathbb{P}^1_B} = \mathcal{V}(D)|_{\mathbb{P}^1_B}$, which does not have ramification points, a contradiction.

Case 4: Assume that we have to do a base change, i.e., P goes to the node. Let F be the exceptional divisor and $P \in F$ (see Figure A.8). Note that there might appear other exceptional divisors, but they are irrelevant for the computations below. Then there exist divisors D_C, D_F and $D_{\mathbb{P}^1}$ on $\mathcal{C}$ supported on X such that

$$\begin{array}{rcl} \mathcal{L}(D_C)|_{\mathbb{P}^1} & = & \omega_{\mathbb{P}^1}(A+B) \\ \mathcal{L}(D_C)|_F & = & \omega_F(2R_2+aR_1) \\ \mathcal{L}(D_C)|_C & = & \omega_C((2-a)R_1); \\ \mathcal{L}(D_F)|_{\mathbb{P}^1} & = & \omega_{\mathbb{P}^1}(A+B) \\ \mathcal{L}(D_F)|_F & = & \omega_F(2R_2+(1+g)R_1) \\ \mathcal{L}(D_F)|_C & = & \omega_C((1-g)R_1) \quad \text{and} \\ \mathcal{L}(D_{\mathbb{P}^1})|_{\mathbb{P}^1} & = & \omega_{\mathbb{P}^1}((1+b)R_2+A+B) \\ \mathcal{L}(D_{\mathbb{P}^1})|_F & = & \omega_F((1-b)R_2+(1+g)R_1) \\ \mathcal{L}(D_{\mathbb{P}^1})|_C & = & \omega_C((1-g)R_1). \end{array}$$

Now, Lemma 5.2.1 implies that

$$\mathcal{V}(D_C)|_F = \mathrm{H}^0(\omega_F(2R_2)) + \mathrm{H}^0(\omega_F(aR_1)),$$

and then, by Section A.1,

$$\mathcal{V}(D_C)|_F(-aP) = 0;$$

therefore,

$$\mathcal{V}(-a\Gamma_P)_C = \mathcal{V}(D_C - a\Gamma_P)|_C = \mathrm{H}^0(\omega_C((1-a)R_1)),$$

which does not have special ramification points, whence $Q \notin C$.

In case $Q \in \mathbb{P}^1$, Lemma 5.2.1 implies that

$$\mathcal{V}(-a\Gamma_P)_{\mathbb{P}^1} = \mathcal{V}(D_{\mathbb{P}^1} - a\Gamma_P)|_{\mathbb{P}^1} = \mathrm{H}^0(\omega_{\mathbb{P}^1}(bR_2)) + \mathrm{H}^0(\omega_{\mathbb{P}^1}(A+B)),$$

which does not have special ramification points by Proposition A.1.4. Finally, in case $Q \in F$, the result follows from Proposition A.1.2, since

$$\mathcal{V}(-a\Gamma_P)_F = \mathcal{V}(D_F - a\Gamma_P) = (\mathrm{H}^0(\omega_F(2R_2)) + \mathrm{H}^0(\omega_F((1+g)R_1)))(-aP).$$

The case where P and Q go to diferent rational curves after the base change is handled in a similar manner.□

Looking closely at the proof of Proposition 5.3.6, we see that the locus in M_g where the fibers of the map $\mathcal{H}_0 \to M_g$ have dimension at least 1 has codimension at least 2; hence the locus of curves C such that there exist infinitely many pairs of points (P, Q) satisfying

$$\mathrm{h}^0(\mathcal{O}_C((a-1)P + (b-1)Q)) \geq 2$$

has codimension at least 2 in M_g.

However we do not know the existence of parameter spaces to prove similar statements in the setup of Propositions 5.3.4 and 5.3.5. The approach used here to prove these propositions, although it can possibly be extended, would lead to a very long reasoning, since now the limit curve C will be a nodal union of curves (X, A) and (Y, B) with X a general curve, but we must allow A to vary, and in fact we must consider other kinds of curves C, some not of compact type. Therefore, we just state:

Hypothesis 5.3.1 *For $g \geq 4$ and integers $a, b > 1$ with $a+b = g$, the locus in M_g of curves C having infintely many pairs of points (P, Q) such that P is a special ramification point of $H^0(\omega_C(-aQ))$ and Q is a special ramification point of $H^0(\omega_C(-bP))$ has codimension at least 2.*

For genus 4, and thus $a = b = 2$, it is easy to see that the hypothesis is true, since a nonhyperelliptic genus-4 curve is canonically embbedded in $\mathbb{P}^3$. Indeed, supose there are infinitely many pairs of points (P, Q) such that P is a special ramification point of $\mathrm{H}^0(\omega_C(-2Q))$ and Q is a special ramification point of $\mathrm{H}^0(\omega_C(-2P))$. The condition

$$\mathrm{h}^0(\omega_C(-2P - Q)) \geq 2$$

means that the tangent line at P intersects the curve again at Q, which is impossible for every point P. To see this, just note that C lies on a quadric Q, whence the tangent L line at P cannot intersect the quadric in another point, because $L \cdot Q = 2$, unless $L \subset Q$, but it is clear that for a smooth nonrational curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ not every point $P \in C$ has vertical or horizontal tangent line.

So we must have infinitely many pairs of points (P, Q) with

$$\mathrm{h}^0(\omega_C(-3P - 3Q)) \geq 1. \tag{5.28}$$

But for a general point P, $\mathrm{h}^0(\omega_C(-3P)) = 1$; then there is a unique point Q satisfying (5.28). This gives an automorphism of C, and the locus of curves with nontrivial automorphisms has codimension 2 in M_4. (In general, the locus of curves with nontrivial automorphisms has codimension $g-2$ in M_g; see [22]).

Chapter 6

The divisor

6.1 Introduction

Our aim in this section is to compute the class of the divisor $\overline{\mathcal{R}_g}$ in $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_g)$, for $g = 2n$ and $n \geq 2$, that is defined as the closure of the locus of smooth curves C with a pair of points (P, Q) satisfying that P is a special ramification point of the linear system $\mathrm{H}^0(\omega_C(-nQ))$ and Q is a special ramification point of $\mathrm{H}^0(\omega_C(-nP))$. In fact, $\overline{\mathcal{R}_g}$ is going to be defined more carefully later, but its support in M_g is the locus above.

But how do we compute this class? Let's first understand what happens on the smooth locus. Let $\pi \colon \mathcal{X} \to T$ be a family of smooth curves, and denote by X_t the fiber over a point t. The idea is to take the double product $\mathcal{Y} := \mathcal{X} \times_T \mathcal{X}$, viewed as a family of curves by the first projection $p_1 : \mathcal{Y} \to \mathcal{X}$, and consider the ramification divisor W of the line bundle $\omega_{p_1}(-n\Delta)$ with respect to p_1; this divisor is also the ramification divisor of $\omega_{p_2}(-n\Delta)$ with respect to p_2 by Lemma 5.2.6.

Following the steps in Section 5.1, the next part is to find the special ramification locus SW_{p_1} (as a set, this is $\{(t, P, Q),\ P, Q \in X_t$ such that Q is a special ramification point of $\mathrm{H}^0(\omega_{X_t}(-nP))\}$), which is given as the degeneracy scheme of the map $\mathcal{O}_{\mathcal{Y}}(W) \to J^1_{p_1}(\mathcal{O}_{\mathcal{Y}}(W))$, and intersect it with SW_{p_2}, which can be defined analogously. It is easy to see that this intersection is the degeneracy scheme SW_{p_1,p_2} of the map

$$\mathcal{O}_{\mathcal{Y}}(W) \to J^{1,1}_{p_1,p_2}(\mathcal{O}_{\mathcal{Y}}(W))$$

where $J^{l,k}_{p_1,p_2}(\mathcal{L})$ is the fibered sum:

$$\begin{array}{ccc} J^{l,k}_{p_1,p_2}(\mathcal{L}) & \to & J^l_{p_1}(\mathcal{L}) \\ \downarrow & & \downarrow \\ J^k_{p_2}(\mathcal{L}) & \to & \mathcal{L} \end{array}$$

All we have to do now is to compute $\pi_* p_{1*}([SW_{p_1,p_2}])$. To do this we need the class of W first, which is given by the Thom–Porteous Formula:

$$[W] = c_1(J^{n-1}_{p_1}(\omega_{p_1}(-n\Delta))) - c_1(p_1^* p_{1*}(\omega_{p_1}(-n\Delta))).$$

Using the truncation sequences to get

$$c_1(J^{n-1}_{p_1}(\omega_{p_1}(-n\Delta))) = \binom{n}{2} c_1(\omega_{p_1}) + nc_1(\omega_{p_1}(-n\Delta))$$

and Grothendieck–Riemann–Roch to see that

$$c_1(p_{1*}(\omega_{p_1}(-n\Delta))) = \pi^*\lambda - \binom{n+1}{2} K_\pi,$$

we have (cf. Equation (6.4))

$$[W] = \binom{n+1}{2}(K_{p_1} + K_{p_2}) - n^2\Delta - p_1^*\pi^*\lambda,$$

where $K_\pi := c_1(\omega_\pi)$ and $K_{p_i} := p_{3-i}^* K_\pi$ for $i = 1, 2$.

Now, $[SW_{p_1,p_2}]$ is given by the Thom–Porteous Formula:

$$[SW_{p_1,p_2}] = c_3(J^{1,1}_{p_1,p_2}(\mathcal{O}_{\mathcal{Y}}(W))) = (K_{p_1} + [W])(K_{p_2} + [W])[W].$$

And using formulas like $K^3_{p_1} = K^3_{p_2} = 0$ and $\pi_* p_{1*}(K_{p_1} K_{p_2} \Delta) = \kappa$ we get:

$$\pi_* p_{1*}([SW_{p_1,p_2}]) = (36n^7 - 18n^6 + 30n^5 - 26n^4 - 2n^3 - 30n^2 - 16n - 4)\lambda_\pi$$

The steps above suggest us how to define properly the divisor we are able to compute. Let $\widetilde{M^0_g}$ be the locus in M^0_g of curves C such that there is no point $P \in C$ with $\mathrm{h}^0(\omega_C(-nP)) \geq n + 1$ and there is at most a finite number of pairs (P, Q) such that P is a special ramification point of $\mathrm{H}^0(\omega_C(-nQ))$ and Q is a special ramification point of $\mathrm{H}^0(\omega_C(-nP))$. The complement of this locus has codimension at least 2 by Proposition 5.3.1 and by Hypothesis 5.3.1. Also, by Corollary 5.3.1, the image $\pi \circ p_1(SW_{p_1,p_2})$ has codimension 1 in $\widetilde{M^0_g}$.

Definition 6.1.1 *Let $\pi : \widetilde{\mathcal{C}^0} \to \widetilde{M^0_g}$ be the universal family. Define $\overline{\mathcal{R}_g} := \overline{\pi \circ p_1(SW_{p_1,p_2})}$ in $\overline{M_g}$.*

But how to generalize the steps taken above to stable curves? First of all, we can reduce our family to a general 1-parameter family, therefore avoiding unwanted singular curves. In fact we can assume that the singular curves we have in our family are of δ_i type, i.e., are general members of Δ_i having only one node. And furthermore, that these curves are not in $\overline{\mathcal{R}_g}$, which means that any points of SW_{p_1,p_2} found on the singular fibers are excess points. The natural idea from here on is to restrict the family to its smooth locus, and use limit linear series to understand the closure of W. However, if we have a δ_0 type curve X_0 with node P, one can expect that the closure of W will contain $\{P\} \times X_0$; therefore SW_{p_1,p_2} will also contain $\{P\} \times X_0$, hence will not have the expected codimension, invalidating the Thom–Porteous Formula.

To solve this, take a finite map $T' \to T$ of degree n fully ramified over the points t of T such that X_t is an irreducible singular fiber and unramified over the points where the fiber is reducible. Let $\mathcal{X}'$ be the desingularization of $\mathcal{X} \times T'$. Then our irreducible singular curves become the union of a genus-$(g-1)$ curve with a chain of $n-1$ rational smooth curves joining the points over the node.

With this configuration the closure of W will not contain all the nodes, it will still contain some nodes; but since we are differentianting in both directions, the nodes will not appear in SW_{p_1,p_2}.

But this new family introduces some other complications: we now have a lot of singularities in the double product, which are solved by blowing up specific codimension-1 subschemes. Moreover, after solving these singularities the sheaf $p_1^* p_{1*}(\omega_{p_1}(-n\widetilde{\Delta}))$ will not be a vector bundle, and again we cannot apply Thom–Porteous Formula. We solve this problem by twisting the line bundle $\omega_{p_1}(-n\widetilde{\Delta})$ with a divisor supported in the singular fibers, thus leaving unchanged the divisor W. Finally, the only concern that remains is whether the closure of W will be flat, but this is fortunately the case. After all the work we end up with our main theorem:

Theorem 6.1.1 *For each $n \geq 2$, the following formula holds in* $\mathrm{Pic}_{\mathrm{fun}}(\overline{M_{2n}}) \otimes \mathbb{Q}$*:*

$$\overline{\mathcal{R}_g} = a(n)\lambda - b_0(n)\delta_0 - \sum_{i=1}^{n} b_i(n)\delta_i$$

where

$$\begin{aligned} a(n) &:= 36n^7 - 18n^6 + 30n^5 - 26n^4 - 2n^3 - 30n^2 - 16n - 4, \\ b_0(n) &:= 4n^7 - n^6 + \frac{2}{3}n^5 - \frac{4}{3}n^4 - \frac{5}{3}n^3 - \frac{5}{3}n^2 - 2, \\ b_i(n) &:= 12n^7 i + 6n^6 i^2 - 24n^5 i^3 + 6n^4 i^4 + 12n^3 i^5 - 6n^2 i^6 - 6n^6 i + 42n^5 i^2 - 33n^4 i^3 - 30n^3 i^4 \\ &\quad +33n^2 i^5 - 6ni^6 + 6n^6 - 52n^5 i + 70n^4 i^2 + 8n^3 i^3 - 59n^2 i^4 + 29ni^5 - 4i^6 + 12n^5 - 43n^4 i \\ &\quad +24n^3 i^2 + 46n^2 i^3 - 51ni^4 + 14i^5 + 10n^4 - 36n^3 i - 4n^2 i^2 + 41ni^3 - 18i^4 + 4n^3 - 17ni^2 \\ &\quad +11i^3 - 4ni + 2i^2 - i. \end{aligned}$$

6.2 Setup

Let $\pi' : \mathcal{X}' \to T'$ be a general 1-parameter family of stable curves over a smooth curve T'. In particular, $\mathcal{X}'$ is smooth. Let $\tau : T \to T'$ be a finite map of degree n, fully ramified over the points t' of T' such that $\mathcal{X}'_{t'}$ is an irreducible singular fiber, and unramified over the points where the fiber is reducible. That such map exists can be seen in [10]. Let $\mathcal{X}$ be the desingularization of $\mathcal{X}' \times T$.

In the following calculations we will restrict ourselves to a neighborhood in T of some point t_0, such that the fiber of $\mathcal{X}$ over t_0 is singular, and such that this fiber is the only singular fiber. It will be clear how to "patch" these calculations later. We will still denote the family by $\pi : \mathcal{X} \to T$ and by τ the restricted map. We have then two cases to analyze, when the singular fiber is irreducible and when it's not.

6.2.1 The irreducible case

The family $\pi : \mathcal{X} \to T$ satifies $\mathcal{X}_t = \mathcal{X}'_{\tau(t)}$ if $t \neq t_0$. Let $t'_0 := \tau(t_0)$; then $X' := \mathcal{X}'_{t'_0}$ is a nodal irreducible curve with just one node, because the family is general. Let C be the normalization of X' and denote by A and B the points over the node. Then $X := \mathcal{X}_{t_0}$ is the union of C and a chain of $n-1$ rational smooth curves $E_1, E_2, \ldots, E_{n-1}$ connecting A and B (with $\{A\} = E_1 \cap C$ and $\{B\} = E_{n-1} \cap C$); see Figure 5.1. We extended the notation by setting $E_0 = E_n := C$, and denoting by A_{E_i} and B_{E_i}, for $i = 1, \ldots, n-1$ the intersection of E_{i-1} with E_i and the intersection of E_i with E_{i+1}, respectively.

Now we define $\mathcal{Y} := \mathcal{X} \times_T \mathcal{X}$ with projections p_1, p_2. Let $E'_{i,j} := E_i \times E_j$ for $i, j = 0, 1, 2, \ldots, n$. Since $\mathcal{Y}$ is singular at the points (P, Q) where P and Q are singular points of X, we desingularize it by blowing up in the following way: First, we blow up the diagonal; then we blow up the (strict transforms of the) $E'_{i,j}$ with $|i-j| = 0$, then the (strict transforms of the) $E'_{i,j}$ with $|i-j| = 1$ and so on. Let $b : \mathcal{B} \to \mathcal{Y}$ denote the desingularization. Let $\widetilde{\Delta}$ and $E_{i,j}$ denote the strict transforms of Δ and $E'_{i,j}$; also, let $\rho_i := p_i \circ b$.

In this desingularization, the preimage of a singular point of $\mathcal{Y}$ is a rational smooth curve. Moreover if $\{R\} = E'_{i,j} \cap E'_{i,j+1} \cap E'_{i+1,j} \cap E'_{i+1,j+1}$, then, R will be desingularized by the blowup of $E'_{i,j+1}$ or $E'_{i+1,j}$ if $i \neq j$, because in this case $|i-j| > \min\{|i-j-1|, |i-j+1|\}$, and by the blowup of the diagonal if $i = j$.

Let $P \in \mathcal{X}$. The family $\rho_1 : \mathcal{B} \to \mathcal{X}$ satisfies $\mathcal{B}_P = \mathcal{X}_{\pi(P)}$ if P isn't a node of $\mathcal{X}_{\pi(P)}$. In particular, if $P \in X$ and is not a node, then the curve $\mathcal{B}_P$ is the union of C_P with a chain of $n-1$ rational curves $E_{1,P}, E_{2,P}, \ldots, E_{n-1,P}$, where $E_{i,P} = \{P\} \times E_i$ and $C_P = \{P\} \times C$. We will often denote $E_i = E_{i,P}$ and $C = C_P$ when it is clear that we are on the fiber $\mathcal{B}_P$.

When P is a node of X, we have that $\mathcal{B}_P$ is the union of C_P with a chain of $2n-1$ rational curves $F_{1,P}, E_{1,P}, F_{2,P}, E_{2,P}, ..., E_{n-1,P}, F_{n,P}$ with $\{A\} = F_{1,P} \cap C$ and $\{B\} = F_{n,P} \cap C$, where $F_{i,P}$ is the rational smooth curve over the singularity $\{P\} \times (E_{i-1} \cap E_i)$ of $\mathcal{Y}$ for $i = 1, \ldots, n$. As for the E_i we will often denote $F_i := F_{i,P}$. See figures 6.1 and 6.2. We define A_{F_i} and B_{F_i} in a similar way to the A_{E_i} and B_{E_i}. It's easy to see that if $P \in E_i \cap E_{i+1}$ then

$$E_{i,j} \cap \mathcal{B}_P = F_j \cup E_j, \quad E_{i+1,j} \cap \mathcal{B}_P = E_j \cup F_{j+1} \quad \text{and} \quad \widetilde{\Delta} \cap \mathcal{B}_P = F_{i+1}$$

for $i = 0, 1, \ldots, n-1$.

Let

$$\mathcal{L} := \omega_{\rho_1}(-n\widetilde{\Delta} - E)$$

where

$$E := \sum_{i,j=1}^{n-1} a_{i,j} E_{i,j}$$

and $a_{i,j} := n \min\{i, j\} - ij$. Let also $\mathcal{E} := \rho_1^* \rho_{1*}(\mathcal{L})$ and $\mathcal{F} := J^{n-1}_{\rho_1}(\mathcal{L})$, and let $e : \mathcal{E} \to \mathcal{F}$ be the evaluation map.

Proposition 6.2.1 *$\mathcal{E}$ is a vector bundle of rank n.*

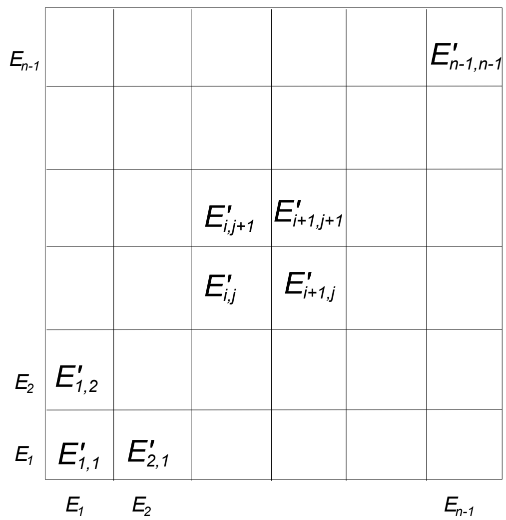


Figure 6.1: Before the blowup

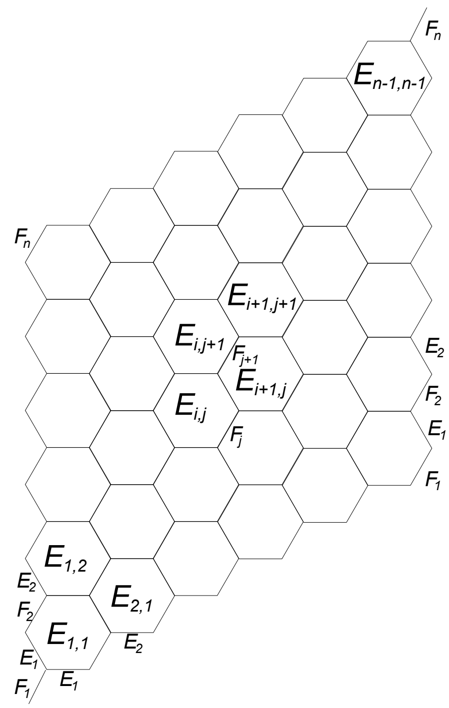


Figure 6.2: After the blowup

Proof. We just need to show that $h^0\left(\mathcal{L}|_{\mathcal{B}_P}\right) = n$ for every $P \in \mathcal{X}$. Thus we have 4 cases to check:

Case 1: P lies on a smooth fiber of π. Then

$$\mathrm{H}^0(\mathcal{L}|_{\mathcal{B}_P}) = \mathrm{H}^0(\omega_{\mathcal{X}_{\pi(P)}}(-nP))$$

which has dimension n by Proposition 5.3.1. (Indeed, since π is a general family, it avoids a codimension-2 locus in M_g.)

Case 2: P lies on $C \backslash \{A, B\}$. Then

$$\mathcal{B}_P = C \cup E_1 \cup \ldots \cup E_n.$$

Also, $\mathcal{L}|_C = \omega_C(A + B - nP)$ and $\mathcal{L}|_{E_i} = \mathcal{O}_{E_i}$ for $i = 1, \ldots, n-1$. So

$$h^0(\mathcal{L}|_{\mathcal{B}_P}) = h^0(\omega_C(A + B - nP)),$$

and since (C, A, B) is a general 2-pointed curve with genus $g-1$, by Proposition 5.3.2 this dimension is n.

Case 3: P lies on a certain E_i, with $0 < i < n$, and is not a node. Then

$$\begin{aligned}
\mathcal{L}|_C &= \omega_C((1-n+i)A + (1-i)B) \\
\mathcal{L}|_{E_j} &= \mathcal{O}_{E_j}(2a_{i,j} - a_{i,j-1} - a_{i,j+1}) \quad \text{for} \quad j \neq i \\
\mathcal{L}|_{E_i} &= \mathcal{O}_{E_i}(2a_{i,i} - a_{i,i-1} - a_{i,i+1} - n).
\end{aligned}$$

This means that $\mathcal{L}|_{E_j} = \mathcal{O}_{E_j}$ for $j = 1, \ldots, n-1$; then, by Lemma 5.2.4, $h^0(\mathcal{L}|_{\mathcal{B}_P}) = n$.

Case 4: $P \in E_i \cap E_{i+1}$ for $i = 0, 1, \ldots, n-1$. Then

$$\begin{aligned}
\mathcal{L}|_C &= \omega_C((1-n+i)A + (-i)B) \\
\mathcal{L}|_{E_j} &= \mathcal{O}_{E_j}(a_{i,j} + a_{i+1,j} - a_{i,j+1} - a_{i+1,j-1}) && \text{if } j \neq i, i+1 \\
\mathcal{L}|_{E_j} &= \mathcal{O}_{E_j}(a_{i,j} + a_{i+1,j} - a_{i,j+1} - a_{i+1,j-1} - n) && \text{if } j = i, i+1 \\
\mathcal{L}|_{F_j} &= \mathcal{O}_{F_j}(a_{i,j} + a_{i+1,j-1} - a_{i,j-1} - a_{i+1,j}) && \text{if } j \neq i+1 \\
\mathcal{L}|_{F_j} &= \mathcal{O}_{F_j}(a_{i,j} + a_{i+1,j-1} - a_{i,j-1} - a_{i+1,j} + n) && \text{if } j = i+1
\end{aligned}$$

which implies $\mathcal{L}|_{E_j} = \mathcal{O}_{E_j}(-1)$ and $\mathcal{L}|_{F_j} = \mathcal{O}_{F_j}(1)$ for $j = 1, \ldots, n$. Then, by Exact Sequence (5.7),

$$h^0(\mathcal{L}|_{\mathcal{B}_P}) = h^0(\omega_C((1-n+i)A - iB)),$$

which gives us $h^0(\mathcal{L}|_{\mathcal{B}_P}) = n$. □

Since $\mathcal{E}$ is a vector bundle we can define W' as the degeneration scheme of the evaluation map $e : \mathcal{E} \to \mathcal{F}$. In addition let $\widetilde{W} := W' \cap \mathcal{B}_{ns}$, where $\mathcal{B}_{ns}$ is the smooth locus of p_1, and let W be the ramification divisor, i.e., $W := \overline{\widetilde{W}}$.

Proposition 6.2.2

$$W = W' - \sum_{i=1}^{n-1} \frac{ni(n-i)}{2} \rho_2^* E_i$$

.

Proof. This equality is obvious off $\mathcal{B}_{t_0}$. To prove the equality on $\mathcal{B}_{t_0}$ we take slices Σ in $\mathcal{X}$ intersecting X transversally at a point $P \in X$, i.e., we will take a map $\mathrm{Spec}(\mathbb{C}[[t]]) \to \mathcal{X}$ taking the closed point to P and such that the image is transversal to X. (When P is a node, by "transversal" we mean transversal to each component of X.) Then we consider the fiber product $\mathcal{S}$:

$$\begin{array}{ccc}
\mathcal{S} & \xrightarrow{f} & \mathcal{B} \\
{\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\pi} \\
\Sigma & \xrightarrow[i]{} & \mathcal{X}
\end{array} \qquad (6.1)$$

and prove the equality in $\mathcal{S}$.

Above, σ is a family of genus-g curves with smooth general fiber. Since the formation of the degeneration scheme comutes with base change, we get that $f^*(W')$ is the degeneration scheme of the line bundle $\mathcal{L}_\Sigma := f^*(\mathcal{L})$.

Assume now that $P \in E_i$, with $0 < i < n$, and P is not a node of X; then, following the proof of Proposition 6.2.1, we get

$$\begin{aligned} \mathcal{L}_\Sigma|_C &= \omega_C((1-n+i)A + (1-i)B), \\ \mathcal{L}_\Sigma|_{E_j} &= \mathcal{O}_{E_j} \quad \text{for } j = 1, \ldots, n-1. \end{aligned}$$

We are done if we can apply Lemma 5.2.3. We see that, following the notation of that lemma,

$$\widetilde{V} = \mathrm{H}^0(\omega_C(-(n-i)A - iB)).$$

Hence, the hypotheses of the lemma become:

$$\begin{aligned} \mathrm{h}^0(\omega_C(-jA - ((g-1)-j-1)B)) &= 1, \\ \mathrm{h}^0(\omega_C(-(j+1)A - ((g-1)-j)B)) &= 0 \end{aligned}$$

for all $j = 0, \ldots, g-1$, which hold since (C, A, B) is a 2-pointed general curve of genus $g-1$.

On the other hand, if $P \in C \backslash \{A, B\}$ then

$$\begin{aligned} \mathcal{L}_\Sigma|_C &= \omega_C(A + B - nP), \\ \mathcal{L}_\Sigma|_{E_j} &= \mathcal{O}_{E_j} \quad \text{for } j = 1, \ldots, n-1. \end{aligned}$$

As before, we want to apply Lemma 5.2.3. Now, $\widetilde{V} = \mathrm{H}^0(\omega_C(-nP))$, which means that we need only check that

$$\begin{aligned} \mathrm{h}^0(\omega_C(-jA - (n-2-j)B - nP)) &= 1, \\ \mathrm{h}^0(\omega_C(-(j+1)A - (n-1-j)B - nP)) &= 0 \end{aligned}$$

for all $j = 0, \ldots, n-2$. But (C, A, B) is a 2-pointed general curve; hence the linear systems $\mathrm{H}^0(\omega_C(-jA - (n-2-j)B))$ and $\mathrm{H}^0(\omega_C(-(j+1)A - (n-1-j)B))$ do not have special ramification points by Proposition 5.3.3, whence the equalities above follow.

By Lemma 5.2.3, the diference $f^*(W') - \sum_{i=1}^{n-1} \frac{ni(n-i)}{2} E_i$ has no vertical components, and hence the proposition is proved.□

Proposition 6.2.3 *The divisor W is flat over $\mathcal{X}$. If P is a node of X then $W \cap \mathcal{B}_P$ is a reduced scheme. At any rate, $W \cap (E_{i,P} \backslash \{A_{E_i}, B_{E_i}\})$ is reduced for every $i = 1, \ldots, n-1$ and every $P \in X$.*

Proof. Let Σ be a slice through a point $P \in X$ and $\mathcal{S}$ as in Cartesian Diagram (6.1). Set $S_0 := \mathcal{B}_P$, the special fiber of σ. Assume first that $\{P\} = E_i \cap E_{i+1}$, for $i = 0, 1, \ldots, n-1$. Then

$$S_0 = C \cup F_1 \cup E_1 \cup \ldots \cup E_{n-1} \cup F_n.$$

.

Fix $l \in \{1, 2, \ldots, n\}$. Recall that $F_l \subset E_{i,l} \cap E_{i+1,l-1}$. Let

$$\mathcal{L}_{i,l} := \mathcal{L}(-\sum_{j=1}^{n-1}(a_{j,l} E_{i,j} + a_{j,l-1} E_{i+1,j})).$$

Then, over P, that is, on $\mathcal{B}_P$,

$$\mathcal{L}_{i,l}|_{F_l} = \mathcal{O}_{F_l}(1 + a_{l,l} + a_{l-1,l-1} - a_{l-1,l} - a_{l,l-1}) = \mathcal{O}_{F_l}(n),$$

whereas for $j < l$

$$\begin{array}{rcccl} \mathcal{L}_{i,l}|_{F_j} & = & \mathcal{O}_{F_j}(1 + a_{j,l} + a_{j-1,l-1} - a_{j-1,l} - a_{j,l-1}) & = & \mathcal{O}_{F_j}, \\ \mathcal{L}_{i,l}|_{E,j} & = & \mathcal{O}_{E_j}(-1 + a_{j,l} + a_{j,l-1} - a_{j-1,l-1} - a_{j+1,l}) & = & \mathcal{O}_{E_j}. \end{array}$$

Similarly, $\mathcal{L}_{i,l}|_{E_j}$ and $\mathcal{L}_{i,j}|_{F_j}$ are trivial ofr $j > l$ as well. As for C, we get

$$\mathcal{L}_{i,l}|_C = \omega_C((1 - 2n + i + l)A + (1 - i - l)B),$$

which means that $\mathrm{h}^0(\mathcal{L}_{i,l}|_{\mathcal{B}_P}) = n$ by Lemma 5.2.4. Therefore $\rho_{1*}(\mathcal{L}_{i,l})$ is a vector bundle in a neighborhood of P. Also, for every Σ through P, the degeneration scheme $W'_{\Sigma,i,l}$ of $\sigma_* f^*(\mathcal{L}_{i,l})$ does not contain F_l, because $\sigma_* f^*(\mathcal{L}_{i,l})$ has focus on F_l.

Because the formation of the degeneration scheme commutes with base change, we see that $W'_{\Sigma,i,l}$ is the pullback of the degeneration scheme of $\pi_*\mathcal{L}_{i,l}$. But this degeneration scheme contains W, and hence W does not contain F_l.

Moreover, we see that the limit linear system with focus on F_l is a subspace $V_{F_l} \subset \mathrm{H}^0(\mathcal{O}_{F_l}(n))$ such that $V(-A_{F_l}) = V(-B_{F_l}) = V(-A_{F_l} - B_{F_l})$, because $\mathrm{h}^0(\mathcal{L}_{i,l}|_C(-A)) = \mathrm{h}^0(\mathcal{L}_{i,l}|_C(-B)) = 0$. Hence

$$V(-nA_{F_l}) = V(-nA_{F_l} - B_{F_l}) \subset \mathrm{H}^0(\mathcal{O}_{F_l}(n))(-nA_{F_l} - B_{F_l}) = 0,$$

which means that A_{F_l} (analogously, B_{F_l}) is not a ramification point of V_{F_l}. Also, for $Q \in F_l \backslash \{A, B\}$

$$V_{F_l}(-(n+1)Q) \subset \mathrm{H}^0(\mathcal{O}_{F_l}(n))(-(n+1)Q) = 0.$$

Moreover, since

$$\mathrm{H}^0(\mathcal{O}_{F_l}(n))(-A_{F_l} - B_{F_l}) \subset V_{F_l}$$

and $\mathrm{H}^0(\mathcal{O}_{F_l}(n))(-A_{F_l} - B_{F_l})$ is a complete linear system, the vanishing sequence of V_{F_l} at Q starts with $0, 1, \ldots, n-2$, which means that $\dim(V_{F_l}(-(n-1)Q)) = 1$. Therefore, V_{F_l} has no special ramification point. By Plücker Formula, it has exactly n ramification points.

As for C, we know that $\mathcal{L}_\Sigma$ has focus on C and the limit linear system is $\mathrm{H}^0(\omega_C((1-n+i)A-iB))$. Since (C, A, B) is a general 2-pointed curve, by Proposition 5.3.3 this linear system has no special ramification points, and A and B are not ramification points. By Plücker Formula, this linear system has $n(n-1)(2n+1)$ points. Moreover, the proof of Proposition 6.2.2, shows that W is flat over $\mathcal{X}$ away from the nodes of X. Indeed, the proof of Proposition 6.2.2 shows that, for each $P \in X$ not a node and each slice Σ through P, f^*W does not have vertical components in $\mathcal{S}$.

Since $W \cap S_0$ contain the $n(n-1)(2n+1) + n^2 = n(2n^2-1)$ ramification points described above and W is flat of degree $n(2n^2-1)$ in a punctured analytic neighboorhood of P, we see that W does not contain any E_j and there are no more points in $W \cap S_0$ other than those described above.

Since W is flat, the intersection $W \cap \mathcal{B}_P$ is equal to the intersection of the ramification divisor of the slice with S_0. And this divisor is given by Equation (5.6). Therefore $W \cap S_0$ is reduced.

Assume now that $P \in E_i$, for $0 < i < n$, but P is not a node of X. Let Σ be a slice through P, and set

$$\mathcal{L}_l := \mathcal{L}_\Sigma \left(\sum_{j=1}^{n-1} a_{l,j} E_j \right). \tag{6.2}$$

Then

$$\begin{aligned} \mathcal{L}_l|_{E_j} &= \mathcal{O}_{E_j} \quad \text{for} \quad j \neq l, \\ \mathcal{L}_l|_{E_l} &= \mathcal{O}_{E_l}(n), \\ \mathcal{L}_l|_C &= \omega_C((1-2n+i+l)A + (1-i-l)B). \end{aligned}$$

Therefore, $\mathcal{L}_l$ has focus on E_l, and again the limit linear system V_{E_l} is a subspace of $\mathrm{H}^0(\mathcal{O}_{E_l}(n))$ satisfying

$$V_{E_l}(-A_{E_l}) = V_{E_l}(-B_{E_l}) = V_{E_l}(-A_{E_l} - B_{E_l}). \tag{6.3}$$

As before, V_{E_l} has no special ramification points, n ramification points and A_{E_l}, B_{E_l} are not ramification points.

If $P \in C \backslash \{A, B\}$, then we can still define $\mathcal{L}_l$ as in Equation (6.2), but now one restriction changes:

$$\mathcal{L}_l|_C = \omega_C((1-n+l)A + (1-l)B - nP).$$

Since (C, A, B) is a general 2-pointed curve, by Proposition 5.3.3, $\mathrm{h}^0(\mathcal{L}_l|_C) = 1$. There are two cases to consider: First, if

$$\mathrm{h}^0(\omega_C(-(n-l)A - (l-1)B - nP)) = \mathrm{h}^0(\omega_C(-(n-l-1)A - lB - nP)) = 0$$

then V_{E_l} satisfies (6.3) as well. On the other hand, if $\mathrm{h}^0(\omega_C(-(n-l)A-(l-1)B-nP)) = 1$ (and analogously for B) then the only section of $\mathrm{H}^0(\mathcal{L}_l|_C)$ vanishes on A, and hence A_{E_l} is going to be a base point of V_{E_l}; therefore there will be no other ramification points by Plücker Formula. This concludes the proof of the proposition. □

Corollary 6.2.1 $SW_{\rho_1,\rho_2} \cap \mathcal{B}_{t_0} = \emptyset$

Proof. From the proposition above and the symmetry of W and SW_{ρ_1,ρ_2}, we get that if $R \in SW_{\rho_1,\rho_2} \cap \mathcal{B}_{t_0}$, then $\rho_1(R), \rho_2(R) \in C \backslash \{A, B\}$. Denoting $P := \rho_1(R)$ and $Q := \rho_2(R)$, the fact that R is in the intersection above means that P is a special ramification point of $\mathrm{H}^0(\omega_C(A+B-nQ))$ and Q is a special ramification point of $\mathrm{H}^0(\omega_C(A+B-nP))$. But since (C, A, B) is general, by Corollary 5.3.2 there are no such pairs of points (P, Q).

Proposition 6.2.4 $R^1\rho_{1*}\mathcal{L} \cong \mathcal{O}_{\mathcal{X}}$.

Proof. First we see that $h^1(\mathcal{L}|_{\mathcal{B}_P}) = 1$ for every $P \in \mathcal{X}$, because of Proposition 6.2.1 and Riemann–Roch. This means that $\mathrm{R}^1\rho_{1*}\mathcal{L}$ is a line bundle.

Now, set $D = n\tilde{\Delta} + E$ and consider the exact sequence.

$$0 \to \mathcal{L} \to \omega_{\rho_1} \to \omega_{\rho_1}|_D \to 0$$

Take the pushforward by ρ_1, which gives us the long exact sequence:

$$\begin{array}{ccccccccc} 0 & \to & \rho_{1*}(\mathcal{L}) & \to & \rho_{1*}(\omega_{\rho_1}) & \to & \rho_{1*}(\omega_{\rho_1}|_D) & \to & \\ & & \mathrm{R}^1\rho_{1*}(\mathcal{L}) & \to & \mathrm{R}^1\rho_{1*}(\omega_{\rho_1}) & \to & \mathrm{R}^1\rho_{1*}(\omega_{\rho_1}|_D) & \to & 0. \end{array}$$

Since $\mathrm{R}^1\rho_{1*}(\omega_{\rho_1}) = \mathcal{O}_{\mathcal{X}}$, we have a map $\mathrm{R}^1\rho_{1*}(\mathcal{L}) \to \mathcal{O}_{\mathcal{X}}$. Since $\mathrm{R}^1\rho_{1*}(\mathcal{L})$ is a line bundle, if we prove that $\mathrm{R}^1\rho_{1*}(\omega_{\rho_1}|_D) = 0$ in codimension 2, then the map is an isomorphism. To prove this we take the exact sequence:

$$0 \to \omega_{\rho_1}(-E)|_{n\tilde{\Delta}} \to \omega_{\rho_1}|_D \to \omega_{\rho_1}|_E \to 0.$$

Taking the pushforward under ρ_1, and observing that $\mathrm{R}^1\rho_{1*}(\omega_{\rho_1}(-E)|_{n\tilde{\Delta}}) = 0$ in codimension 2, because it is supported on the nodes of X, we get an isomorphism $\mathrm{R}^1\rho_{1*}(\omega_{\rho_1}|_D)\tilde{\to}\mathrm{R}^1\rho_{1*}(\omega_{\rho_1}|_E)$ in codimension 2. Therefore we need only prove that $\mathrm{R}^1\rho_{1*}(\omega_{\rho_1}|_E) = 0$, or equivalently, $H^1(\mathcal{B}_P, (\omega_{\rho_1}|_E)|_{\mathcal{B}_P}) = 0$ for every $P \in \mathcal{X}$ away from a codimension-2 locus. If $P \notin \bigcup_{i=1}^{n-1} E_i$ then this is obvious. Suppose now that $P \in E_i$ and P is not a node of X. (The locus of the nodes has codimension 2 in $\mathcal{X}$.) Consider the exact sequence

$$0 \to \omega_{\rho_1}(-E) \to \omega_{\rho_1} \to \omega_{\rho_1}|_E \to 0.$$

Restricting to $\mathcal{B}_P$, we get a right-exact sequence:

$$\omega_{\rho_1}(-E)|_{\mathcal{B}_P} \to \omega_{\mathcal{B_P}} \to (\omega_{\rho_1}|_E)|_{\mathcal{B}_P} \to 0.$$

Now, writing $\mathcal{B}_P = C \cup (\bigcup E_i)$, we see that the image of the first map is $\omega_C(A+B-E\cdot C)$, because $E_i \subset E$ and thus the map is 0 over the E_i. Therefore, we get an exact sequence of the form:

$$0 \to \omega_C(A+B-E\cdot C) \to \omega_{\mathcal{B}_P} \to (\omega_{\rho_1}|_E)|_{\mathcal{B}_P} \to 0.$$

Taking the long exact sequence in cohomology,

$$\mathrm{H}^1(C, \omega_C(A+B-E\cdot C)) \to \mathrm{H}^1(\mathcal{B}_P, \omega_{\mathcal{B}_P}) \to \mathrm{H}^1(\mathcal{B}_P, (\omega_{\rho_1}|_E)|_{\mathcal{B}_P}) \to 0$$

is exact. Therefore, we just have to prove that the first map is a surjection, which is the same, by duality, as proving that the map $\mathrm{H}^0(\mathcal{B}_P, \mathcal{O}_{\mathcal{B}_p}) \to \mathrm{H}^0(C, \mathcal{O}_C((n-i-1)A + (i-1)B))$ is injective, which is obvious.$\square$

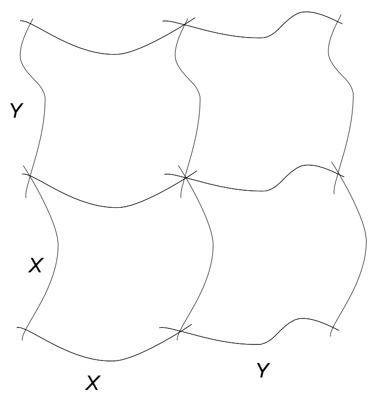


Figure 6.3:

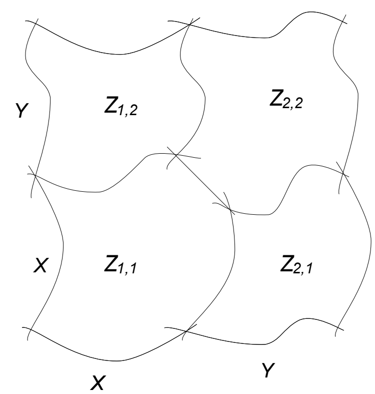


Figure 6.4:

6.2.2 The reducible case

Our special fiber now is the nodal union of a general genus-i pointed curve (X, A) with a general genus-$(2n-i)$ pointed curve (Y, B) identifying A with B, for $i \le n$. Then on $\mathcal{Y} := \mathcal{X} \times_T \mathcal{X}$ we have 4 subschemes $X \times X$, $X \times Y$, $Y \times X$ and $Y \times Y$, and we have a singularity on the commom point of those subschemes. To solve this singularity we blow-up $X \times X$, which gives us a $\mathbb{P}^1$ over the point; we will call $\mathcal{B}$ this blowup. We will denote the strict transform of $X \times X$, $X \times Y$, $Y \times X$ and $Y \times Y$ by $Z_{1,1}$, $Z_{1,2}$, $Z_{2,1}$ and $Z_{2,2}$ respectively. We will also call the strict transform of the diagonal by $\widetilde{\Delta}$. A local analysis shows that the new $\mathbb{P}^1$ is contained in $Z_{1,1}$ and $Z_{2,2}$, but not in $Z_{1,2}$ or $Z_{2,1}$, and intersects $\widetilde{\Delta}$ properly. See figures 6.3 and 6.4.Set $\mathcal{L} := \omega_\pi(-n\widetilde{\Delta} - nZ_{1,1})$, $\mathcal{E} := \rho_1^*\rho_{1*}(\mathcal{L})$ and $\mathcal{F} := J_{\rho_1}^{n-1}(\mathcal{L})$.

Proposition 6.2.5 *$\mathcal{E}$ is a vector bundle of rank n if $i < n$. If $i = n$ then $\mathcal{E}$ is a vector bundle outside a locus of codimension 2.*

Proof. As in Proposition 6.2.1, it is sufficient to prove that $h^0(\mathcal{L}|_{\mathcal{B}_P}) = n$ for every $P \in \mathcal{X}$ (for every $P \in \mathcal{X}$ outside a locus of codimension 2 if $i = n$). We have then 4 cases to check:

Case 1: P lies on a smooth fiber of $\mathcal{X}$. Then

$$H^0(\mathcal{L}|_{\mathcal{B}_P}) = H^0(\omega_{\mathcal{X}_{\pi(P)}}(-nP)),$$

which has dimension n because $\mathcal{X} \to T$ is a general family.

Case 2: P lies on $X \backslash \{A\}$. Then

$$\begin{aligned} \mathcal{L}|_X &= \omega_X((n+1)A - nP), \\ \mathcal{L}|_Y &= \omega_Y(-(n-1)B), \end{aligned}$$

and since (X, A) and (Y, B) are general pointed curves we have $h^0(\mathcal{L}|_{\mathcal{B}_P}) = n$.

Case 3: P lies on $Y \backslash \{B\}$. Then

$$\begin{aligned} \mathcal{L}|_X &= \omega_X(A), \\ \mathcal{L}|_Y &= \omega_Y(B - nP). \end{aligned}$$

Again, since (X, A) and (Y, B) are general pointed curves, we have that $h^0\left(\mathcal{L}|_{\mathcal{B}_P}\right) = n$ if $i < n$ and if $i = n$ or P is not a Weierstrass point of Y (thus, away form a codimension-2 locus in $\mathcal{X}$), by Lemma 5.2.4.

Case 4: P is the node. Then

$$\begin{aligned} \mathcal{L}|_X &= \omega_X(A), \\ \mathcal{L}|_{\mathbb{P}^1} &= \mathcal{O}_{\mathbb{P}^1}(0), \\ \mathcal{L}|_Y &= \omega_Y(-(n-1)B), \end{aligned}$$

which implies that $h^0\left(\mathcal{L}|_{\mathcal{B}_P}\right) = n$ by Lemma 5.2.4.□

Let $e : \mathcal{E} \to \mathcal{F}$ be the evaluation map, and define W' as the degeneration scheme of this map, if $i < n$. If $i = n$, we define W' as the closure of the degeneration scheme of the restriction of e to the open set where $\mathcal{E}$ is a vector bundle. Since the complement of this open set has codimension 2, the Chow groups of codimension-1 cycles are isomorphic, hence we will be able to apply the Thom–Porteous formula later. Let $W := \overline{W' \cap \mathcal{B}_{ns}}$; then $W' > W$.

Let P_j, for $j = 1, 2 \ldots, i(n^2 - 1)$, be the ramification points of $\mathrm{H}^0(\omega_X((1 + n - i)A))$ distinct from A; note that Proposition 5.3.2 states that these points are distinct. Also, let Q_l, for $l = 1, 2, \ldots, n(2n^2 - in - 1)$, be the ramification points of $\mathrm{H}^0(\omega_Y(-(n - i)B))$; note that Proposition 5.3.3 state that these points are also distinct.

Proposition 6.2.6

$$W = W' - \binom{n-i+1}{2}(Z_{1,1} + Z_{2,1}) - \binom{i}{2} Z_{1,2} - \binom{i+1}{2} Z_{2,2}.$$

Proof. As in the proof of Proposition 6.2.2 we take a slice Σ through a point $P \in \mathcal{X}_{t_0}$, let $\mathcal{S}$ denote the fibered product, and prove the equality on $\mathcal{S}$:

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{f} & \mathcal{B} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\rho_1} \\ \Sigma & \xrightarrow[i]{} & \mathcal{X} \end{array}$$

Here, $\sigma : \mathcal{S} \to \Sigma$ is a family of genus-g curves, with smooth generic fiber. The pullback $f^*(W')$ is the degeneration scheme of $f^*\mathcal{E} \to f^*\mathcal{F}$ because of the formation of the degeneration scheme commutes with base change. Since the sheaf of jets commutes with base change as well, we get

$$f^*\mathcal{F} = J_\sigma^{n-1}(\omega_\sigma(-nf^*(\tilde{\Delta}) - nf^*Z_{1,1})).$$

Then it is sufficient to prove that

$$f^*(W') - \binom{n-i+1}{2}(f^*(Z_{1,1}) + f^*(Z_{2,1})) - \binom{i}{2} f^*(Z_{1,2}) - \binom{i+1}{2} f^*(Z_{2,2})$$

is effective, and has no vertical components for P different from the P_j, the Q_l and the node. (Note that the Q_l are exactly the points of $\mathcal{X}$ over which $\mathcal{E}$ might not be locally free if $i = n$.)

If $P \in X\backslash\{A\}$ then

$$f^*(Z_{2,j}) = 0 \quad j = 1, 2;$$

thus we need only prove that the degeneration scheme contains X with multiplicity $\binom{n-i+1}{2}$ and Y with multiplicity $\binom{i}{2}$. Since $P \neq P_l$, the point A is not a ramification point of the complete linear system $V|_X = H^0(\omega_X((n+1)A - nP))$, and hence has total weight $\binom{i}{2}$, which, by lemma 5.2.2, implies that the multiplicity of Y is the desired one. Also, B is not a ramification point of the complete linear system $V|_Y = \mathrm{H}^0(\omega_Y(-(n-1)B))$, and hence has total weight $\binom{n-i+1}{2}$. For $P \in Y\backslash\{B\}$ the proof is similar.

To see that the hypotheses of Lemma 5.2.2 are satisfied, just notice that, for $P \neq P_j$,

$$V|_X(-(j+1)A) = 0$$

for every $j \geq i$, whereas for $j \leq i-1$ we have

$$V(-jY)|_Y(-B) \subset \mathrm{H}^0(\omega_Y(-(n-j)B)),$$

which implies $\dim V(-jY)|_Y(-B) \leq n - i + j$ because B is general and

$$V|_X(-jA) \subset \mathrm{H}^0(\omega_X((1+n-j)A - nP)),$$

which implies $\dim V|_X(-jA) \leq i - j$, by Proposition 5.3.2. The argument is similar for the other hypothesis.$\square$

Unlike the irreducible case, W is not flat in this case, this means that SW_{ρ_1,ρ_2} will contain excess points on $\mathcal{B}_{t_o}$. Fortunately, we are able to write the local equation of W at these points, and hence compute their weights in SW_{ρ_1,ρ_2}.

Proposition 6.2.7 *$W \cap \mathcal{Z}_{1,2}$ is the union of the $\{P_j\} \times Y$ and the $X \times \{Q_l\}$. And in the ring $\widehat{\mathcal{O}_{P_j \times Q_l}}$ the local equation of W is*

$$t + xy + tf(x, y, t) + [\text{degree} > 2]$$

with $f(0,0,0) = 0$, where t is a local parameter of T at $\pi(P_j)$, and x and y are local parameters of X at P_j and Y at Q_l respectively.

Proof. Let $P \in X$ and $P \neq P_j, A$. Note that, by Proposition 6.2.6 $W \cap \mathcal{B}_P$ is finite, and over a slice Σ through P, the limit linear systems will be $V_X = \mathrm{H}^0((\omega_X((1+2n-i)A - nP)))$ and $V_Y = \mathrm{H}^0(\omega_Y(-(n-i)B))$; the first has $n^2 i$ ramification points (none of which are A) and the second $n(2n^2 - ni - 1)$ (none of which are B). These add up to $n(2n^2 - 1)$, which is the total number of ramification points. Hence W does not contain the node, and intersects Y in a reduced subscheme because of Propostion 5.3.3.

Now, if $Q \in Y$, $Q \neq Q_l, B$, then over a slice through Q, the limit linear systems will be $V_X = \mathrm{H}^0(\omega_X((1+n-i)A))$ and $V_Y = \mathrm{H}^0(\omega_Y((1+i)B - nQ))$; the first has $n^2 i$ ramification points (now, A has weight i) and the second has $n(2n^2 - ni)$ ramification points (B has weight $n-i$). These add up to $n(2n^2-1)$ ramification points away from the nodes, which is the total number of ramification points; hence W does not contain the node and intersects X is a reduced subscheme by Proposition 5.3.2.

However, for every $P \in X$ with $P \neq P_j$, V_Y does not depend on the choice of P. This means that $W \cap Z_{1,2}$ contains $X \times \{Q_l\}$. With the same reasoning, we conclude that $W \cap Z_{2,1}$ contains $Y \times \{P_j\}$, which by the symmetry of W, implies that $W \cap Z_{1,2}$ contains $\{P_j\} \times Y$. Moreover, $W \cap Z_{1,2} = \bigcup(X \times \{Q_l\}) \cup \bigcup(\{P_j\} \times Y)$, because $W \cap (\{P\} \times Y)$ and $W \cap (X \times \{Q\})$ is reduced for every $P \neq P_j$ and $Q \neq Q_l$.

Now, let Σ be a slice through $P := P_j$. Let $L := f^*(\mathcal{L})((i-2)X)$, $L' := L(2X)$ and $L'' := L(3X)$. Then

$$\begin{array}{rcl} L'|_Y &=& \omega_Y(-(n-1-i)B) \\ L'|_X &=& \omega_X((1+n-i)A - nP) \\ L''|_Y &=& \omega_Y(-(n-i-2)B) \\ L''|_Y &=& \omega_X((n-i)A - nP). \end{array}$$

If $i < n$, the line bundle L' has focus on Y, because $\mathrm{h}^0(\omega_X((n-i)A - nP)) = 0$ by Proposition 5.3.2. Let $V_Y \subset \mathrm{H}^0(\omega_Y(-(n-1-i)B))$ the limit linear system. Since

$$\begin{array}{rcl} L|_Y &=& \omega_Y(-(n+1-i)B), \\ L|_X &=& \omega_X((3+n-i)A - nP) \end{array}$$

and $\mathrm{h}^0(\omega_X((2+n-i)A - nP)) = 1$ we get that $\mathrm{H}^0(\omega_Y(-(n+1-i)B)) \subset V_Y(-2B)$ by Lemmas 5.2.4 and 5.2.1. On the other hand, for $i = n$, the line bundle L'' has focus on Y, because $\mathrm{h}^0(\omega_X(-A - nP)) = 0$ by Proposition 5.3.3. In this case we have

$$\mathrm{H}^0(\omega_Y(-B)) \subset V_Y \subset \mathrm{H}^0(\omega_Y(2B))$$

Claim 1 *If $V_Y \neq H^0(\omega_Y(-(n-i)B))$ then Q_l is not a ramification point of V_Y.*

Proof. Assume $i < n$. Since Q_l is a ramification point of $\mathrm{H}^0(\omega_Y(-(n-i)B))$ then Q_l is not a ramification point of either $\mathrm{H}^0(\omega_Y(-(n-i-1)B))$ or $\mathrm{H}^0(\omega_Y(-(n-i+1)B))$ ($\mathrm{H}^0(\omega_Y(2B))$ if $i = n$), by Lemma 5.2.5 and because $\mathrm{H}^0(\omega_Y(-aB))$ does not admit special ramification points for any a by Proposition 5.3.3. Hence, the vanishing sequence at Q_l of V_Y starts with $0, 1, \ldots, n-2$ but the last order cannot be n, because if s is the only, up to multiples, section in $\mathrm{H}^0(\omega_Y(-(n-i-1)B))$ that vanishes with order n at Q_l, then $s \in \mathrm{H}^0(\omega_Y(-(n-i)B))$. But

$$V_Y \cap \mathrm{H}^0(\omega_Y(-(n-i)B)) = \mathrm{H}^0(\omega_Y(-(n-i+1)B)),$$

and $s \notin \mathrm{H}^0(\omega_Y(-(n-i+1)B))$ thus $s \notin V_Y$. This argument clearly works for $i = n$ as well. This finishes the proof of the claim.

Let $x = \alpha t$, for $\alpha \in \mathbb{C}$, be the local equation of Σ and V'_C the image of the map $\mathrm{H}^0(\mathcal{S}, L') \to \mathrm{H}^0(C, L'|_C)$. If $V_Y = \mathrm{H}^0(\omega_Y(-(n-i)B))$, then B is base point of V_Y, which means that every section of V_C vanishes on all X, because $L'|_X = \omega_X((1-n+i)A - nP)$. Therefore these sections will be sections of $L'(-X)$ that are in the image of

$$\mathrm{H}^0(\mathcal{S}, L'(-X)) \to \mathrm{H}^0(C, L'(-X)|_C),$$

and then $L'(-X)$ will have focus on Y. However, the image of the map

$$\mathrm{H}^0(\mathcal{S}, L'(-X)) \to \mathrm{H}^0(Y, L'(-X)|_Y)$$

is clearly contained in $\mathrm{H}^0(\omega_Y(-(n-i+1)B))$, because A is a base point of $\mathrm{H}^0(\omega_X((2+n-i)A-nP))$, a contradiction. Thus, $V_Y \neq \mathrm{H}^0(\omega_Y(-(n-i)B))$.

Write now a local equation for W at (P_j, Q_l):

$$a_1 t + a_2 x + a_3 y + b_1 tx + b_2 ty + b_3 xy + [\text{degree} \geq 3],$$

Then the local equation of f^*W is

$$t(a_1 + \alpha a_2) + a_3 y + t(\alpha b_1 t + b_2 y) + \alpha b_3 ty + [\text{degree} \geq 3].$$

But $f^*(W)$ contains Y and $f^*(W) - Y$ does not pass through Q_l. Thus the local equation is a multiple of t, whence $a_3 = 0$, and $a_1 + \alpha a_2 \neq 0$. Since this is valid for almost every α, we must have $a_2 = 0$ and $a_1 \neq 0$.

To compute b_3 we make $t = 0$ and invert x. Geometrically this means to consider $\mathcal{B}_{t_0} \backslash \mathcal{B}_{P_j}$. But, in a punctured neighborhood of P_j, the limit linear system on Y is given by $\mathrm{H}^0(\omega_Y(-(n-i)B))$, which has Q_l as a ramification point of weight 1, hence y appears with power 1. By symmetry, also x appears with power 1, hence $b_3 \neq 0$, which concludes the proof.$\square$

Proposition 6.2.8 *$SW_{\rho_1,\rho_2} \cap \mathcal{B}_{t_0}$ is the union of the set of points (P_j, Q_l) and (Q_l, P_j). Furthermore SW_{ρ_1,ρ_2} is reduced in a neighborhood of these points.*

Proof. By the proof of Proposition 6.2.7, for a point $(P, Q) \in X \times X$ $P \neq A \neq Q$, to be in SW_{ρ_1,ρ_2} it must satisfy that P is a special ramification point of $\mathrm{H}^0(\omega_X((1+2n-i)A - nQ))$ and Q is a special ramification point of $\mathrm{H}^0(\omega_X((1+2n-i)A-nP))$, which is not possible by Corollary 5.3.1. The same argument holds for a point of $Y \times Y$. For $(P, Q) \in X \times Y$, we use the fact that $W \cap \{P\} \times Y$ is reduced if $P \neq P_j$, and $W \cap X \times \{Q\}$ is also reduced if $Q \neq Q_l$. Then

$SW_{\rho_1,\rho_2} \cap \mathcal{B}_{t_0}$ is supported in the set of points $\{(P_j, Q_l), (Q_l, P_j)\}$. To see that SW_{ρ_1,ρ_2} is reduced, just notice that, by Proposition 6.2.7, SW_{ρ_1,ρ_2} is given locally by equations whose linear terms are $t, b_1t + y, b_2t + x$ with $b_1, b_2 \in \mathbb{C}$. Hence SW_{ρ_1,ρ_2} is reduced in a neighborhood of (P_j, Q_l).

Let now P be the node. Write $\rho_2^{-1}(P) = X_P \cup \mathbb{P}_1 \cup Y_P$. Since for a general $Q \in \mathcal{X}_{t_0}$ we proved that W does not contain the node of $\mathcal{B}_Q$, we have that W contains neither X_P nor Y_P. However, W does contain (P_j, P) and (Q_l, P) (because it contains $\{P_j\} \times Y$ and $\{Q_l\} \times X$). Hence, by the symmetry of W we see that $W \cap \mathcal{B}_P$ contains neither X nor Y but contains the points P_j, Q_l. But there are $n(2n^2 - 1) - i$ such points; thus we need to understand what happens on the $\mathbb{P}^1$.

Take now a slice Σ through the node. Define $L := f^*\mathcal{L}$ and $L' = L(iX + (n-i)Y + nf^*\tilde{\Delta})$, then

$$\begin{aligned} L'|_X &= \omega_X((1-i)A) \\ L'|_Y &= \omega_Y((1-2n+i)B) \\ L'|_{\mathbb{P}^1} &= \mathcal{O}_{\mathbb{P}^1}(2n). \end{aligned}$$

So L' has focus on $\mathbb{P}^1$ and $V'|_{\mathbb{P}^1} \subset \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(2n))$. By Lemma 5.2.1, we see that

$$V'(-iX)|_{\mathbb{P}^1} \subset V'|_{\mathbb{P}^1}(-iA) \subset \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(2n - iA)).$$

But $L'(-iX)|_X = \omega_X(A)$, and hence $V'(-iX)|_{\mathbb{P}^1}$ has A as a base point. Therefore

$$V'(-iX)|_{\mathbb{P}^1} \subset \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(2n - (i+1)A)).$$

But, by the Exact Sequence (5.7), we see that $\dim V'(-iX)|_{\mathbb{P}^1} = 2n - i$; hence we have an equality above. This gives us

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(2n - (i+1)A)) \subset V'|_{\mathbb{P}^1}.$$

Analogously,

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(2n - (2n - i + 1)B)) \subset V'|_{\mathbb{P}^1}.$$

But

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(2n - (i+1)A)) \bigcap \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(2n - (2n - i + 1)B)) = 0$$

as subspaces of $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(2n))$; then, by dimension considerations,

$$V'|_{\mathbb{P}^1} = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(2n - (i+1)A)) \oplus \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(2n - (2n - i + 1)B)).$$

Let $R := \mathbb{P}^1 \cap \tilde{\Delta}$. Using Lemma 5.2.1 again we get

$$V'(-nf^*\tilde{\Delta})|_{\mathbb{P}^1} \subset V'|_{\mathbb{P}^1}(-nR).$$

Since the only ramification points of $V'|_{\mathbb{P}^1}$ are A and B, we see that $\dim V'|_{\mathbb{P}^1}(-nR) = n$, and hence we have equality. Note that $V'(-nf^*\tilde{\Delta})|_{\mathbb{P}^1}$ is the limit linear system of $\mathcal{L}$ wiht focus on $\mathbb{P}^1$.

Since this limit linear system does not depend on the slice Σ we see that W does not contain $\mathbb{P}^1$, and by Proposition A.1.3 such linear system has only simple ramification points. This concludes the proof. $\square$

Proposition 6.2.9 $R^1\rho_{1*}(\mathcal{L}) = \mathcal{O}_{\mathcal{X}}$

Proof. As in the proof of Proposition 6.2.4 we just have to prove that $\mathrm{R}^1\rho_{1*}(\omega_{\rho_1}|_{nZ_{1,1}}) = 0$; so we will prove that the first cohomology of the restriction of $\omega_{\rho_1}|_{nZ_{1,1}}$ over each fiber of ρ_1 is 0. As in Proposition 6.2.4, we need only consider the fiber over $P \in X$. By the same arguments, we will have a right-exact sequence:

$$0 \to \omega_Y(-nZ_{1,1} \cdot Y) \to \omega_{\mathcal{B}_P} \to \left(\omega_{\rho_1}|_{nZ_{1,1}}\right)\big|_{\mathcal{B}_P} \to 0,$$

and again taking the long exact sequence in cohomology, will be left to prove that the induced map

$$\mathrm{H}^1(Y, \omega_Y(-n\mathcal{Z}_{1,1} \cdot Y)) \to \mathrm{H}^1(\mathcal{B}_P, \omega_{\mathcal{B}_P})$$

is surjective, which is the case if and only if $\mathrm{H}^0(\mathcal{B}_P, \mathcal{O}_{\mathcal{B}_P}) \to \mathrm{H}^0(Y, \mathcal{O}_Y(nB))$ is injective, which is obvious.

6.3 The calculation

6.3.1 The irreducible case

Lemma 6.3.1 *The map $\rho_1|_{\tilde{\Delta}} : \tilde{\Delta} \to \mathcal{X}$ is the blowup of $\mathcal{X}$ at the nodes of X and we have the following:*

$$\rho_{1*}\left(\tilde{\Delta}^2\right) = \omega_\pi^{-1}$$

$$\pi_*\rho_{1*}\left(\tilde{\Delta}^3\right) = \kappa - \delta$$

Proof. We have the commutative diagram:

$$\begin{array}{ccc} \tilde{\Delta} & \longrightarrow & \mathcal{B} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ \Delta & \longrightarrow & \mathcal{Y} \end{array}$$

where $\mathcal{Y} := \mathcal{X} \times_T \mathcal{X}$, and we have the projection $p_1 : \mathcal{Y} \to \mathcal{X}$, which becomes an isomorphism when restricted to Δ. Let P be a node of the fiber $\mathcal{X}_{t_0}$. Let t be a local equation of T at t_0 and x, y the local equations of $\mathcal{X}$ at P, such that $t = xy$. Then, we can write the (completion of the) local ring of $\mathcal{Y}$ at (P, P):

$$\frac{\mathbb{C}[[t, x_1, y_1, x_2, y_2]]}{(t - x_1y_1, x_1y_1 - x_2y_2)}$$

Since the ideal of the diagonal is generated by $x_1 - x_2, y_1 - y_2$, locally the blowup is given by

$$\frac{\mathbb{C}[[t, x_1, y_1, x_2, y_2, \lambda]]}{(t - x_1y_1, \lambda y_1 + x_2, (x_1 - x_2) - \lambda(y_1 - y_2))}$$

as we note that $x_1y_1 - x_2y_2 = (x_1 - x_2)y_1 + x_2(y_1 - y_2)$. The ideal of the exceptional divisor is generated by $y_1 - y_2$. Since $\widetilde{\Delta}$ is the exceptional divisor, its local ring is given by the quotient

$$\frac{\mathbb{C}[[t, x_1, y_1, x_2, y_2, \lambda]]}{(t - x_1y_1, y_1 - y_2, x_1 - x_2, \lambda y_1 + x_1)} \equiv \frac{\mathbb{C}[[t, x_1, y_1, \lambda]]}{(t - x_1y_1, \lambda y_1 + x_1)}.$$

Therefore, the map $\rho_1|_{\widetilde{\Delta}} : \widetilde{\Delta} \to \mathcal{X}$ is the blowup of $\mathcal{X}$ at the nodes of $\mathcal{X}_{t_0}$. We will call the exceptional divisors of $\tilde{\Delta}$ by F_j: the figure is pretty much the same we had before, but now the multiplicity of this F_j in the fiber over T is 2 instead of 1. Now, we consider the exact sequence:

$$0 \to \mathcal{I}_\Delta^2 \to \mathcal{I}_\Delta \to \frac{\mathcal{I}_\Delta}{\mathcal{I}_\Delta^2} \to 0.$$

By definition $\Omega_\pi^1 \cong \frac{\mathcal{I}_\Delta}{\mathcal{I}_\Delta^2}$ (identifying Δ and $\mathcal{X}$ by the natural isomorphism). Now we pull back the sequence by b, obtaining the right-exact sequence:

$$b^*\left(\mathcal{I}_\Delta^2\right) \longrightarrow b^*\left(\mathcal{I}_\Delta\right) \longrightarrow b^*\left(\frac{\mathcal{I}_\Delta}{\mathcal{I}_\Delta^2}\right) \longrightarrow 0.$$

But we have natural surjective maps $b^*(\mathcal{I}_\Delta^2) \to \mathcal{O}_\mathcal{B}(-2\tilde{\Delta})$ and $b^*(\mathcal{I}_\Delta) \to \mathcal{O}_\mathcal{B}(-\tilde{\Delta})$; thus we end up with a diagram:

$$\begin{array}{ccccccc} b^*\left(\mathcal{I}_\Delta^2\right) & \to & b^*\left(\mathcal{I}_\Delta\right) & \to & b^*\left(\frac{\mathcal{I}_\Delta}{\mathcal{I}_\Delta^2}\right) & \to & 0 \\ \downarrow & & \downarrow & & & & \\ \mathcal{O}_\mathcal{B}(-2\tilde{\Delta}) & \to & \mathcal{O}_\mathcal{B}(-\tilde{\Delta}) & \to & \mathcal{O}_\mathcal{B}(-\tilde{\Delta})|_{\tilde{\Delta}} & \to & 0 \end{array}$$

giving us a surjective map $f^*\left(\Omega_\pi^1\right) \to \mathcal{O}_\mathcal{B}(-\tilde{\Delta})|_{\tilde{\Delta}}$. However, $\Omega_\pi^1 = \omega_\pi \otimes \mathcal{I}$, where $\mathcal{I}$ is the ideal sheaf of the nodes, and since $f^*\mathcal{I}$ modulo torsion is $\mathcal{O}_{\tilde{\Delta}}(-\sum_{j=1}^n F_j)$ we conclude that $\mathcal{O}_\mathcal{B}(-\tilde{\Delta})|_{\tilde{\Delta}} = \omega_{\rho_1|_{\tilde{\Delta}}}(-\sum_{j=1}^n F_j)$. From this we easily have the stated equalities, because $F_j^2 = -1$.

Lemma 6.3.2 *We have the following equalities:*

1. $$\pi_*\left(\left(\sum_{i=1}^{n-1}\frac{ni(n-i)}{2}E_i\right)^2\right) = -\frac{(n-1)n^2(n+1)}{12}\delta_0$$

2. $$\rho_{2*}\left(E^2\right) = \rho_{1*}\left(E^2\right) = -\sum_{i=1}^{n-1} ni(n-i)E_i$$

3. $$\pi_*\rho_{1*}\left(E^3\right) = \frac{3}{2}(n-1)^2 n\delta_0$$

4. $$\rho_{1*}\left(\tilde{\Delta}\cdot E\right) = \sum_{i=1}^{n-1} i(n-i)E_i$$

5. $$\pi_*\rho_{1*}\left(\tilde{\Delta}^2\cdot E\right) = (n-1)\delta_0$$

6. $$\pi_*\rho_{1*}\left(\tilde{\Delta}\cdot E^2\right) = -\frac{2}{3}(n-1)(2n-1)\delta_0$$

where $E_1, \ldots, E_{n-1}$ and E are defined in the beginning of Section 6.2.1.

Proof. First of all, note that $\delta_0 = n[t_0]$, because we have n nodes over t_0.

1. We start with

$$\star := \left(\sum_{i=1}^{n-1} \frac{ni(n-i)}{2} E_i\right)^2 = \sum_{i=1}^{n-1} \frac{ni(n-i)}{2} \left(E_i \left(\sum_{j=1}^{n-1} \frac{nj(n-j)}{2} E_j\right)\right).$$

But

$$E_i \cdot E_j = \begin{cases} 1 & \text{if } |i-j| = 1 \\ -2 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\begin{aligned} \star &= \sum_{i=1}^{n-1} \frac{ni(n-i)}{2} \left(\frac{n(i-1)(n-i+1) + n(i+1)(n-i-1) - 2ni(n-i)}{2}\right) \\ &= \sum_{i=1}^{n-1} \frac{ni(n-i)}{2} \left(\frac{-2n}{2}\right) = -\sum_{i=1}^{n-1} \frac{n^2 i(n-i)}{2} = -\frac{(n-1)n^3(n+1)}{12}. \end{aligned}$$

2. The first equality is easy to see because of the symmetry of $\mathcal{B}$ and E. For the second equality, observe first that:

$$\rho_{1*}\left(E_{i,j} \cdot E_{i',j'}\right) = \begin{cases} E_i & \text{if } i = i' \text{ and } |j-j'| = 1 \\ -2E_i & \text{if } i = i' \text{ and } j = j' \\ 0 & \text{otherwise} \end{cases}$$

because $E_{i,j}^2 = -E_{i,j} \cdot (\mathcal{B}_{t_0} - E_{i,j})$. This implies:

$$\begin{aligned} \rho_{1*}\left(\left(\sum_{i,j=1}^{n-1} a_{i,j} E_{i,j}\right)^2\right) &= \rho_{1*}\left(\sum_{i,j=1}^{n-1} a_{i,j} E_{i,j} \left(\sum_{i',j'=1}^{n-1} a_{i',j'} E_{i',j'}\right)\right) \\ &= \sum_{i,j=1}^{n-1} a_{i,j}\left(a_{i,j-1} + a_{i,j+1} - 2a_{i,j}\right) E_i \\ &= \sum_{i=1}^{n-1} E_i \left(\sum_{j=1}^{n-1} a_{i,j}\left(a_{i,j-1} + a_{i,j+1} - 2a_{i,j}\right)\right) \\ &= -\sum_{i=1}^{n-1} n a_{i,i} E_i = -\sum_{i=1}^{n-1} ni(n-i) E_i \end{aligned}$$

because

$$a_{i,j-1} + a_{i,j+1} - 2a_{i,j} = 0$$

unless $i = j$, in which case

$$a_{i,i-1} + a_{i,i+1} - 2a_{i,i} = -n.$$

3. We begin with the following equalities:

Let

$$J := \{(i-1,j),(i-1,j+1),(i,j-1),(i,j+1),(i+1,j-1),(i+1,j)\}$$

for each i, j with $1 \leq i, j \leq n-1$. Then

$$E_{i,j} \cdot E_{i',j'} E_{i'',j''} = \begin{cases} 6 & \text{if } (i,j) = (i',j') = (i'',j'') \\ 1 & \text{if } \{(i',j'),(i'',j'')\} \subset J \text{ and } |i'-i''| + |j'-j''| = 1 \\ -1 & \text{if } (i',j') \in J \text{ and } (i'',j'') = (i,j) \text{ or } (i'',j'') = (i',j') \\ 0 & \text{otherwise} \end{cases}$$

Because $E_{i,j}^3 = (E_{i,j} \cdot (\mathcal{B}_{t_0} - E_{i,j}))^2$, where the square is taken in $E_{i,j}$. But $E_{i,j}$ can be identified with the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $((0:1),(0:1))$ and $((1:0),(1:0))$. Let L_1 and L_2 be the exceptional divisors, and let L_3, L_4, L_5 and L_6 be the strict transforms of $(0:1) \times \mathbb{P}^1$, $(1:0) \times \mathbb{P}^1$, $\mathbb{P}^1 \times (0:1)$ and $\mathbb{P}^1 \times (1:0)$. Then $E_{i,j} \cdot (\mathcal{B}_{t_0} - E_{i,j}) = L_1 + \ldots + L_6$ (just look at Figure 6.2), and clearly $(L_1 + \ldots + L_6)^2 = 6$. And $E_{i,j} \cdot E_{i',j'}^2 = L_k^2 = -1$ for some k if $(i',j') \in J$.

Then we can see that:

$$E^3 = \sum_{i,j=1}^{n-1} a_{i,j} b_{i,j}$$

where

$$\begin{aligned} b_{i,j} \;\; = \;\; & (6a_{i,j}^2 + 2(a_{i-1,j}a_{i,j-1} + a_{i,j-1}a_{i+1,j-1} + a_{i+1,j-1}a_{i+1,j} + a_{i+1,j}a_{i,j+1} + a_{i,j+1}a_{i-1,j+1} \\ & + a_{i-1,j+1}a_{i-1,j}) - a_{i,j}(a_{i-1,j} + a_{i,j-1} + a_{i+1,j-1} + a_{i+1,j} + a_{i,j+1} + a_{i-1,j+1}) - (a_{i-1,j}^2 \\ & + a_{i,j-1}^2 + a_{i+1,j-1}^2 + a_{i+1,j}^2 + a_{i,j+1}^2 + a_{i-1,j+1}^2)). \end{aligned}$$

But

$$b_{i,j} = \begin{cases} -2 & \text{if } j \neq i-1, i, i+1 \\ -n^2 + 2n - 2 & \text{if } j = i-1, i+1 \\ 2n^2 - 2 & \text{if } j = i \end{cases}$$

Thus we end up with

$$\begin{aligned} E^3 &= \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} -2a_{i,j} - (i-1)(n-i)(n^2-2n) + 2i(n-i)n^2 - i(n-i-1)(n^2-2n) \right) \\ &= \sum_{i=1}^{n-1} (-ni(n-i) - (i-1)(n-i)(n^2-2n) + 2i(n-i)n^2 - i(n-i-1)(n^2-2n)) \\ &= \sum_{i=1}^{n-1} -3i^2 n + 3in^2 + n^3 - 2n^2 \\ &= \frac{3}{2}(n-1)^2 n^2 \end{aligned}$$

4. For this one all we need is to observe that

$$\rho_{1*}\left(\tilde{\Delta}\cdot E_{i,j}\right)=\begin{cases} E_i & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}$$

5. First of all, since $\tilde{\Delta}\cap E_{i,j}$ is the diagonal in $E_{i,j}$ when $i=j$ and L_1 or L_2 when $|i-j|=1$,

$$\left(\tilde{\Delta}^2\cdot E_{i,j}\right)=\begin{cases} 2 & \text{if } i=j\\ -1 & \text{if } |i-j|=1\\ 0 & \text{otherwise}\end{cases}$$

which implies

$$\left(\tilde{\Delta}^2\cdot\sum_{i,j=1}^{n-1}a_{i,j}E_{i,j}\right)=\sum_{i=1}^{n-1}(2a_{i,i}-a_{i-1,i}-a_{i,i+1})=n(n-1)$$

6.

$$\tilde{\Delta}\cdot E_{i,j}\cdot E_{i',j'}=\begin{cases} 1 & \begin{array}{l}\text{if either } i=j \text{ and } (i',j')\in\{(i-1,i),(i,i-1),(i,i+1),(i+1,i)\}\\ \text{or } i'=j' \text{ and } (i,j)\in\{(i'-1,i'),(i',i'-1),(i',i'+1),(i'+1,i')\}\end{array}\\ \\ -1 & \text{if } \{(i,j),(i',j')\}\subseteq\{(l,l+1),(l+1,l)\} \text{ for some } l\\ \\ -4 & \text{if } (i,j)=(i',j')=(i,i)\\ \\ 0 & \text{otherwise}\end{cases}$$

Since $E_{i,i}^2=-(L_1+\ldots+L_6)$ and $\widetilde{\Delta}\cdot E_{i,i}$ is the diagonal in $E_{i,i}$, we have $\widetilde{\Delta}\cdot E_{i,i}^2=-4$. In addition, $E_{l,l+1}\cdot\widetilde{\Delta}=E_{l+1,l}\cdot\widetilde{\Delta}=F_l$ in $\widetilde{\Delta}$, and $F_l^2=-1$, as we saw in Lemma 6.3.1. The nonnegative intersection numbers are easy to obtain. This implies

$$\begin{aligned}\tilde{\Delta}\cdot E^2 &= 2\sum_{i=1}^{n-1}a_{i,i}(a_{i-1,i}+a_{i,i-1}+a_{i,i+1}+a_{i+1,i})-4\sum_{i=1}^{n-1}a_{i,i}^2-4\sum_{i=1}^{n-1}a_{i,i+1}^2\\ &= -4\sum_{i=1}^{n-1}\left(i(n-i)^2-i^2(n-i)+i^2\right)\\ &= -4\frac{(n-1)n(2n-1)}{6}\\ &= -\frac{2}{3}(n-1)n(2n-1)\end{aligned}$$

With the above lemma and the projection formula, we are able to compute all the intersections that will appear. Let $K_\pi:=c_1(\omega_\pi)$ and $K_{\rho_i}:=c_1(\omega_{\rho_i})$. Note that $\omega_{\rho_1}=\rho_2^*(\omega_\pi)$ and $K_\pi\cdot E_i=0$.

To calculate the class $[W]$, we calculate the class $[W']$ and apply Proposition 6.2.2. By Thom–Porteous Formula the class $[W']$ is given by $[W']=c_1(\mathcal{F}-\mathcal{E})=c_1(\mathcal{F})-c_1(\mathcal{E})$, where $\mathcal{E}=\rho_1^*\rho_{1*}(\mathcal{L})$

and $\mathcal{F} = J^{n-1}_{\rho_1}(\mathcal{L})$. Using the truncation sequences of jets we get that

$$c_1(\mathcal{F}) = \begin{pmatrix} n+1 \\ 2 \end{pmatrix} K_{\rho_1} - n^2\tilde{\Delta} - n\sum_{i,j=1}^{n-1} a_{i,j}E_{i,j}.$$

To compute $c_1(\mathcal{E})$ we apply the Grothendieck–Riemann–Roch Formula:

$$\begin{aligned}
\mathrm{ch}\left(\rho_{1!}\left(\mathcal{L}\right)\right) &= \rho_{1*}\left(\mathrm{ch}(\mathcal{L})\cdot \mathrm{td}\left(\mathcal{T}_{\mathcal{B}/\mathcal{X}}\right)\right) \\
&= \rho_{1*}\left(\left(1 + c_1(\mathcal{L}) + \frac{c_1(\mathcal{L})^2}{2} + ...\right)\cdot\left(1 - \frac{K_{\rho_1}}{2} + \mathrm{td}_2(\mathcal{T}_{\mathcal{B}/\mathcal{X}}) + \ldots\right)\right) \\
&= \rho_{1*}\left(1 + \left(c_1(\mathcal{L}) - \frac{K_{\rho_1}}{2}\right) + \left(\frac{c_1(\mathcal{L})^2}{2} - \frac{K_{\rho_1}c_1(\mathcal{L})}{2} + \mathrm{td}_2(\mathcal{T}_{\mathcal{B}/\mathcal{X}})\right) + \ldots\right) \\
&= (n-1) + \left(\pi^*\lambda - \begin{pmatrix} n+1 \\ 2 \end{pmatrix} K_\pi + \sum_{i=1}^{n-1}\frac{ni(n-i)}{2}E_i\right) + ...
\end{aligned}$$

The last equation came from Lemma 6.3.2 and Equation (4.6), and the fact that $c_1(\mathcal{L}) = K_{\rho_1} - n\widetilde{\Delta} - E$. Therefore, since $R^1\rho_{1*}(\mathcal{L}) = \mathcal{O}_{\mathcal{X}}$, we have

$$c_1(\rho_{1*}(\mathcal{L})) = \pi^*\lambda - \begin{pmatrix} n+1 \\ 2 \end{pmatrix} K_\pi + \sum_{i=1}^{n-1}\frac{ni(n-i)}{2}E_i,$$

and thus we end up with

$$[W] = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}(K_{\rho_1} + K_{\rho_2}) - \rho_1^*\pi_1^*\lambda - n^2\tilde{\Delta} - n\sum_{i,j=1}^{n-1} a_{i,j}E_{i,j} - \sum_{i=1}^{n-1}\frac{ni(n-i)}{2}\left(\rho_1^*E_i + \rho_2^*E_i\right). \quad (6.4)$$

Since

$$\pi_*\rho_{1*}([SW_{\rho_1,\rho_2}]) = \pi_*\rho_{1*}\left([W]\cdot([W] + K_{\rho_1})\cdot([W] + K_{\rho_2})\right),$$

it follows from Lemma 6.3.2 that

$$b_0(n) = 4n^7 - n^6 + \frac{2}{3}n^5 - \frac{4}{3}n^4 - \frac{5}{3}n^3 - \frac{5}{3}n^2 - 2n$$

6.3.2 The reducible case

Lemma 6.3.3 *The map $\rho_1|_{\tilde{\Delta}} : \tilde{\Delta} \to \mathcal{X}$ is an isomorphism and we have:*

$$\rho_{1*}\left(\tilde{\Delta}^2\right) = \omega_\pi^{-1}$$

$$\rho_{1*}\left(\tilde{\Delta}^3\right) = \kappa$$

Proof. Since the map $\mathcal{B} \to \mathcal{Y}$ is the blowup along $X \times X$, a local analysis, as in Lemma 6.3.1, shows that $\widetilde{\Delta} \to \mathcal{X}$ is an isomorphism, and we have the same diagram as in Lemma 6.3.1:

$$\begin{array}{ccccccc}
b^*\left(\mathcal{I}_\Delta^2\right) & \to & b^*\left(\mathcal{I}_\Delta\right) & \to & b^*\left(\frac{\mathcal{I}_\Delta}{\mathcal{I}_\Delta^2}\right) & \to & 0 \\
\downarrow & & \downarrow & & & & \\
\mathcal{O}_{\mathcal{B}}(-2\tilde{\Delta}) & \to & \mathcal{O}_{\mathcal{B}}(-\tilde{\Delta}) & \to & \mathcal{O}_{\mathcal{B}}(-\tilde{\Delta})|_{\tilde{\Delta}} & \to & 0
\end{array}$$

which gives the surjective map $f^*(\Omega^1_\pi) \to \mathcal{O}_\mathcal{B}(-\widetilde{\Delta})|_{\widetilde{\Delta}}$. But now, f is an isomorphism, hence the map factors through ω_π. This gives us $\mathcal{O}_\mathcal{B}(-\widetilde{\Delta})|_{\widetilde{\Delta}} = \omega_\pi$, and from this the result follows.

Lemma 6.3.4 *We have the following equalities:*

1. $$\pi_*\rho_{1*}\left(\tilde{\Delta}^2 \cdot Z_{1,1}\right) = -(2i-1)\delta_i$$

2. $$\pi_*\rho_{1*}\left(\tilde{\Delta}^2 \cdot Z_{2,2}\right) = -(4n-2i-1)\delta_i$$

3. $$\pi_*\rho_{1*}\left(\tilde{\Delta} \cdot Z^2_{1,1}\right) = \pi_*\rho_{1*}\left(\tilde{\Delta} \cdot Z^2_{2,2}\right) = -\delta_i$$

4. $$\pi_*(X^2) = \pi_*(Y^2) = -\delta_i$$

5. $$\pi_*\rho_{1*}\left(\tilde{\Delta} \cdot Z_{1,1} \cdot Z_{2,2}\right) = \delta_i$$

6. $$\pi_*\rho_{1*}\left(Z^3_{1,1}\right) = \pi_*\rho_{1*}\left(Z^3_{2,2}\right) = \delta_i$$

7. $$\rho_{1*}\left(Z^2_{1,1}\right) = \rho_{2*}\left(Z^2_{1,1}\right) = -X$$

8. $$\rho_{1*}\left(Z^2_{2,2}\right) = \rho_{2*}\left(Z^2_{2,2}\right) = -Y$$

9. $$\pi_*\rho_{1*}\left(Z^2_{1,1} \cdot Z_{2,2}\right) = \pi_*\rho_{1*}\left(Z_{1,1} \cdot Z^2_{2,2}\right) = -\delta_i$$

10. $$\rho_{1*}\left(Z_{1,1} \cdot Z_{2,2}\right) = \rho_{2*}\left(Z_{1,1} \cdot Z_{2,2}\right) = 0$$

.

Proof.

1. First we see that $\tilde{\Delta} \cdot Z_{1,1}$ is the strict transform of the diagonal in the blowup $Z_{1,1} \to X \times X$; thus the self-intersection of this transform is the self-intersection of the diagonal -1, because we are exploding $X \times X$ in a point belonging to the diagonal. Since the self-intersection of the diagonal is $-(2g_X - 2)$ we have the result.

2. Same as above.

3. Since $Z_{1,1} + Z_{1,2} + Z_{2,1} + Z_{2,2} = 0$, we get that $Z^2_{1,1}$ is $-(\{A\} \times X + X \times \{A\} + \mathbb{P}^1)$. Then, since the strict transform of the diagonal intersects only the $\mathbb{P}^1$, and transversally at just one point, we are done. The proof is the same for $Z_{2,2}$.

4. Obvious from the fact that $X + Y = 0$.

5. What we need is to observe that $Z_{1,1} \cdot Z_{2,2} = \mathbb{P}^1$ and, again, the diagonal intersects $\mathbb{P}^1$ transversally at just one point.

6. We need to compute $(\{A\} \times X + X \times \{A\} + \mathbb{P}^1)^2$ on $Z_{1,1}$, which is 1. The same holds for $Z_{2,2}$.

7. Again, since $Z_{1,1}^2$ is $-(\{A\} \times X + X \times \{A\} + \mathbb{P}^1)$, and both $\mathbb{P}^1$ and $\{A\} \times X$ are contracted by ρ_1, all that remains is $-X$. The same for ρ_2.

8. As above.

9. All we have to compute is $-(\{A\} \times X + X \times \{A\} + \mathbb{P}^1) \cdot \mathbb{P}^1$, which is -1.

10. The intersection $Z_{1,1} \cdot Z_{2,2}$ is $\mathbb{P}^1$, which is vertical with respect to both projections.$\square$

To compute the class $[W]$, we compute the class $[W']$ and use Proposition 6.2.6. Now, W' is given by $[W'] = c_1(\mathcal{F}) - c_1(\mathcal{E})$, where $\mathcal{E} = \rho_1^* \rho_{1*}(\mathcal{L})$ and $\mathcal{F} = J_{\rho_1}^{n-1}(\mathcal{L})$. Using the truncation sequences for jets, we get:

$$c_1(\mathcal{F}) = \binom{n+1}{2} K_{\rho_1} - n^2 \tilde{\Delta} - n^2 Z_{1,1}.$$

As for $c_1(\mathcal{E})$ we apply the Grothendieck–Riemann–Roch Formula, as in Section 6.3.1, and use Lemma 6.3.4 and Equation (4.6) together with the fact that $c_1(\mathcal{L}) = K_{\rho_1} - n\widetilde{\Delta} - nZ_{1,1}$, to get

$$c_1(\mathcal{E}) = \rho_1^* \pi^* \lambda - \binom{n+1}{2} K_{\rho_2} + \binom{n+1}{2} \rho_1^* X - ni\rho_1^* X.$$

Thus

$$\begin{aligned}
[W] &= \binom{n+1}{2} (K_{\rho_1} + K_{\rho_2}) - \rho_1^* \pi^* \lambda - n^2 \tilde{\Delta} - n^2 Z_{1,1} - \binom{n+1}{2} \rho_1^* X + ni\rho_1^* X \\
&\quad - \binom{n-i+1}{2} (Z_{1,1} + Z_{2,1}) - \binom{i}{2} Z_{1,2} - \binom{i+1}{2} Z_{2,2} \\
&= \binom{n+1}{2} (K_{\rho_1} + K_{\rho_2}) - \rho_1^* \pi^* \lambda - n^2 \tilde{\Delta} - \binom{2n-i+1}{2} Z_{1,1} \\
&\quad - \binom{n-i+1}{2} (Z_{1,2} + Z_{2,1}) - \binom{i+1}{2} Z_{2,2} \\
&= \binom{n+1}{2} (K_{\rho_1} + K_{\rho_2}) - \rho_1^* \pi^* \lambda - n^2 \tilde{\Delta} + \left(\binom{i}{2} - n^2 \right) Z_{1,1} \\
&\quad - \binom{n-i+1}{2} (\rho_1^* X + \rho_2^* X) - \binom{i+1}{2} Z_{2,2} \qquad (6.5)
\end{aligned}$$

Since

$$\pi_* \rho_{1*}([SW_{\rho_1,\rho_2}]) = \pi_* \rho_{1*} \left([W] \cdot ([W] + K_{\rho_1}) \cdot ([W] + K_{\rho_2}) \right)$$

and

$$\overline{\mathcal{R}_{g_\pi}} = \pi_* \rho_{1*}([SW_{\rho_1,\rho_2}]) - e(n,i)\delta_i$$

where $e(n,i)$ is the number of points of SW_{ρ_1,ρ_2} lying on $\mathcal{B}_{t_o}$, that is, the points (P_j, Q_l) and (Q_l, P_j). Remember that the points P_j, for $j = 1, \ldots, i(n^2-1)$, are the ramification points of $\mathrm{H}^0(\omega_X((1+n-i)A))$ distinct from A and the points Q_l, for $l = 1, \ldots, n(2n^2 - in - 1)$, are the ramification points of $\mathrm{H}^0(\omega_Y(-(n-i)B))$. Thus, we end up with

$$e(n,i) = 2i(n+1)(n-1)n(2n^2 - ni - 1).$$

Hence, using Lemma 6.3.4, we get

$$\begin{aligned} b_i(n) &= 12n^7 i + 6n^6 i^2 - 24n^5 i^3 + 6n^4 i^4 + 12n^3 i^5 - 6n^2 i^6 - 6n^6 i + 42n^5 i^2 - 33n^4 i^3 - 30n^3 i^4 \\ &\quad +33n^2 i^5 - 6ni^6 + 6n^6 - 52n^5 i + 70n^4 i^2 + 8n^3 i^3 - 59n^2 i^4 + 29ni^5 - 4i^6 + 12n^5 - 43n^4 i \\ &\quad +24n^3 i^2 + 46n^2 i^3 - 51ni^4 + 14i^5 + 10n^4 - 36n^3 i - 4n^2 i^2 + 41ni^3 - 18i^4 + 4n^3 - 17ni^2 \\ &\quad +11i^3 - 4ni + 2i^2 - i. \end{aligned}$$

This finishes the proof of Theorem 6.1.1.

6.4 Decomposition of the divisor

As seen in Lemma 5.2.7, the condition for a smooth genus-g curve C, where $g = 2n$, to have points P, Q such that P is a special ramification point of $\mathrm{H}^0(\omega_C(-nQ))$ and Q is a special ramification point of $\mathrm{H}^0(\omega_C(-nP))$ can be split in 4 cases:

1. C has a special Weierstrass point (which would be $P = Q$);
2. C has points P, Q with
$$\mathrm{h}^0(\omega_C(-(n+1)P - (n+1)Q)) \geq 1;$$
3. C has points P, Q with
$$\mathrm{h}^0(\omega_C(-(n-1)P - (n-1)Q)) \geq 3;$$
4. C has points P, Q with
$$\begin{aligned} \mathrm{h}^0(\omega_C(-(n-1)P - nQ)) &\geq 2 \\ \mathrm{h}^0(\omega_C(-nP - (n+1)Q)) &\geq 1. \end{aligned}$$

Each one of these conditions defines an effective divisor in M_g. The first defines the divisor SW_g studied in [4]. To define the others, let $\widetilde{M}_g^0$ be the locus in M_g^0 of curves C such that there is no point P with $\mathrm{h}^0(\omega_C(-(n+1)P)) \geq n$. Note that the complement of this locus has codimension at least 2 by Proposition 5.3.1 if $g \geq 6$. Let $\pi : \widetilde{\mathcal{C}^0} \to \widetilde{M}_g^0$ be the universal family and $p_1, p_2 : \widetilde{\mathcal{C}^0} \times \widetilde{\mathcal{C}^0} \to \widetilde{\mathcal{C}^0}$ the projections.

Definition 6.4.1 *Let*

$$e_+ : p_1^* p_{1*}(\omega_{p_1}(-(n+1)\Delta)) \to J^n_{p_1}(\omega_{p_1}(-(n+1)\Delta))$$

be the evaluation map. Set $\mathcal{B} := \overline{\pi_* p_{1*}([D_{n-2}(e_+)])}$ *in* $\overline{M}_g$*, where* $D_{n-2}(e_+)$ *is the locus where the map* e_+ *has rank at most* $n-2$.

Definition 6.4.2 *Let*

$$e_- : p_1^* p_{1*}(\omega_{p_1}(-(n-1)\Delta)) \to J^{n-2}_{p_1}(\omega_{p_1}(-(n-1)\Delta))$$

be the evaluation map. Set $\mathcal{T} := \overline{\pi_* p_{1*}([D_{n-2}(e_-)])}$ *in* $\overline{M_g}$.

Definition 6.4.3 *Let*

$$\begin{aligned} e_1 : p_1^* p_{1*}(\omega_{p_1}(-(n-1)\Delta)) &\to J^{n-1}_{p_1}(\omega_{p_1}(-(n-1)\Delta)) \\ e_2 : p_1^* p_{1*}(\omega_{p_1}(-n\Delta)) &\to J^{n}_{p_1}(\omega_{p_1}(-n\Delta)) \end{aligned}$$

be the evaluation maps. Set $\mathcal{U} := \overline{\pi_* p_{1*}([D_{n-1}(e_1) \cap D_{n-1}(e_2)])}$ *in* $\overline{M_g}$.

Note that $\mathcal{U}$ is in fact a divisor, since, outside the diagonal, $D_{n-1}(e_1) \cap D_{n-1}(e_2)$ can be viewed as the degeneration scheme of the flag

$$p_1^* p_{1*}(\omega_{p_1}(-n\Delta)) \to p_1^* p_{1*}(\omega_{p_1}(-(n-1)\Delta)) \to J^n_{p_1}(\omega_{p_1}(-(n-1)\Delta)) \to J^{n-1}_{p_1}(\omega_{p_1}(-(n-1)\Delta))$$

as defined in [13].

Further computations show that in $\mathrm{Pic}(M_g)$

$$\begin{aligned} \mathcal{B}|_{M_g} &= (n+1)^2(6n^5 + 13n^4 - 5n^3 - 28n^2 - 20n - 6)\lambda \\ \mathcal{T}|_{M_g} &= n^2(n-1)^3(n-2)(6n-1)\lambda \end{aligned}$$

and suggests, although this is not completely clear, that

$$\mathcal{U}|_{M_g} = n(n+1)(n-1)(12n^4 - 6n^3 - 16n^2 - 5n - 2)\lambda$$

and that one can expect that

$$\overline{\mathcal{R}_g} = E_{-1} + E_1 + \mathcal{B} + \mathcal{T} + 2\mathcal{U}$$

where E_{-1}, E_1 are the Cukierman and Diaz divisors, defined as the locus of curves possessing a special Weierstrass point of type $g+1$ and $g-1$, respectively.

To follow the steps in [4] and compute these divisors in $\mathrm{Pic}(\overline{M_g})$, we can try to use test curves. To use such curves we must be able to compute the number of triples of points (R, P, Q) on a general curve C satisfying the aforementioned properties (with $\omega_C((1+i)R)$ instead of just ω_C). But so far we were unable to do such calculations, specially because we are now over the triple product $C \times C \times C$, and then, when trying to apply the Thom–Porteous formula, we get excess in codimension 2, which is somewhat hard to deal with.

Appendix A

Linear systems on rational and elliptic curves

A.1 Linear systems on rational curves

Let A, B and P be distinct points on $\mathbb{P}^1$, and a, b and i positive integers with $a+b>i$. Set $j := a+b-i$. Define the $(a+b)$-dimensional linear system

$$V_{a,b} := \mathrm{H}^0(\omega_{\mathbb{P}^1}((1+a)A)) \oplus \mathrm{H}^0(\omega_{\mathbb{P}^1}((1+b)B)) \subset \mathrm{H}^0(\omega_{\mathbb{P}^1}((1+a)A+(1+b)B)).$$

Since we have an isomorphism $\omega_{\mathbb{P}^1}((1+a)A+(1+b)B) \cong \mathcal{O}_{\mathbb{P}^1}(a+b)$, and via this isomorphism we have identifications

$$\begin{aligned}
\mathrm{H}^0(\omega_{\mathbb{P}^1}((1+a)A)) &= \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(a+b-(1+b)B)) \\
\mathrm{H}^0(\omega_{\mathbb{P}^1}((1+b)B)) &= \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(a+b-(1+a)A)),
\end{aligned}$$

we can write

$$V_{a,b} = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(a+b-(1+a)A)) \oplus \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(a+b-(1+b)B)).$$

It is easy to see that the vanishing sequence of $V_{a,b}$ at A is $0,1,\ldots,a-1,a+1,\ldots,a+b$ and at B is $0,1,\ldots,b-1,b+1,\ldots,a+b$. Hence $\mathrm{wt}_{V_{a,b}}(A)=b$ and $\mathrm{wt}_{V_{a,b}}(B)=a$, which means, by Plücker Formula, that we have no other ramification points. So the following linear system is $(a+b-i)$-dimensional:

$$V^i_{a,b} := V_{a,b}(-iP). \tag{A.1}$$

To understand better $V^i_{a,b}$ let's assume $A=(0:1), B=(1:0), P=(-1:1)$. Then $V_{a,b}$ is the linear system of homogeneous polynomials $F(X,Y)$ of degree $a+b$ whose coefficient of the monomial X^aY^b is zero. Therefore $V^i_{a,b}$ is the system of homogeneous polynomials $G(X,Y)$ of degree $a+b-i$ such that $(X+Y)^iG(X,Y) \in V_{a,b}$.

Thus the point A is a ramification point of $V_{a,b}^i$ if $X^{a+b-i}(X+Y)^i$ has 0 for the coefficient of X^aY^b, which happens if and only if $a+b-i>a$, and in this case the vanishing sequence at A of $V_{a,b}^i$ is $0,1,\ldots,a-1,a+1,\ldots,a+b-i$, because this sequence is the unique subsequence of the vanishing sequence at A of $V_{a,b}$ with orders at most $a+b-i$, the degree of $V_{a,b}^i$. This means that the weight of A in $V_{a,b}^i$ is $w_A := \max\{b-i,0\}$. Analogously the weight of B is $w_B := \max\{a-i,0\}$.

Now, for a point $Q := (-\lambda : 1)$ to be a ramification point of $V_{a,b}^i$, it is necessary and sufficient that there exist $G \in V_{a,b}^i$ whose order of vanishing at Q is $a+b-i$, in other words, $G(X,Y) = (X+\lambda Y)^{a+b-i}$.

For each i and j positive integers, define the degree-n homogeneous polynomials $H_{i,j}^{[n]}(Z,W)$ as the coefficient of X^{i+j-n} in the expansion of $(X+Z)^i(X+W)^j$. Note that, in fact, $H_{i,j}^{[n]}$ has degree n only when $0 \le n \le i+j$, being 0 otherwise. These polynomials have the following properties

$$\begin{array}{rcl} H_{i,j}^{[n]} & = & H_{i-1,j}^{[n]} + ZH_{i-1,j}^{[n-1]} \\ H_{i,j}^{[n]} & = & H_{i,j-1}^{[n]} + WH_{i,j-1}^{[n-1]} \\ H_{0,j}^{[n]} & = & \binom{j}{n}W^n \\ H_{i,0}^{[n]} & = & \binom{i}{n}Z^n. \end{array} \tag{A.2}$$

And can be written explicitly as

$$H_{i,j}^{[n]} = \sum_{l=0}^{n} \binom{i}{l}\binom{j}{n-l} Z^l W^{n-l}.$$

Moreover, differentiating $H_{0,j}^{[n]}$ with respect to W and Z and doing induction on i and j, we get equalities:

$$\begin{array}{rcl} \frac{\partial}{\partial Z} H_{i,j}^{[n]} & = & iH_{i-1,j}^{[n-1]} \\ \frac{\partial}{\partial W} H_{i,j}^{[n]} & = & jH_{i,j-1}^{[n-1]}. \end{array} \tag{A.3}$$

Let $h_{i,j}^{[n]} := H_{i,j}^{[n]}(1,W)$.

With these definitions, we can see that Q is a ramification point of $V_{a,b}^i$ if and only if

$$h := h_{i,j}^{[i+j-a]}(\lambda) = 0.$$

Furthermore, so that $V_{a,b}^i$ does not have a special ramification point, h must not have double roots distinct from 0.

But, when $i \ge n$, $h_{i,j}^{[n]}$ is the hypergeometric function

$$F\left(\begin{array}{c} -j, -n \\ i+1-n \end{array} \middle| \, z \right) \binom{i}{n}$$

which means that $h_{i,j}^{[n]}$ satisfies the differential equation (see [18] Equation (5.108)):

$$W(1-W)(h_{i,j}^{[n]})'' + (i-n+1+W(j+n-1))(h_{i,j}^{[n]})' - jnh_{i,j}^{[n]} = 0.$$

Hence, if λ is a nonzero double root of $h_{i,j}^{[n]}$ then it is a double root of $(h_{i,j}^{[n]})'$ (because clearly $\lambda \neq 1$). Therefore, by induction on j we have no double roots. At any rate, the diferential equation above is valid for every i; indeed, it holds clearly when $i = 0$, and it is easy to do an induction on i. This proves the following propositions:

Proposition A.1.1 *The polynomial $h^{[n]}_{i,j}$ does not have double roots distinct from 0 and 1.*

Proposition A.1.2 *Let a and b be positive integers and $A, B \in \mathbb{P}^1$. Then $V^i_{a,b}$ does not have special ramification points for any positive integer i such that $i < a + b$.*

Let A, B and C be distinct points on $\mathbb{P}^1$, and a, b and i positive integers with $a + b \geq i$. Set $j = a + b + 2 - i$. Define the $(a + b + 1)$-dimensional linear system

$$V := \mathrm{H}^0(\omega_{\mathbb{P}^1}((1 + a)A)) \oplus \mathrm{H}^0(\omega_{\mathbb{P}^1}((1 + b)B)) \oplus \mathrm{H}^0(\omega_{\mathbb{P}^1}(2C)) \subset \mathrm{H}^0(\omega_{\mathbb{P}^1}((1 + a)A + (1 + b)B + 2C)).$$

Since $\omega_{\mathbb{P}^1}((1+a)A+(1+b)B+2C) \cong \mathcal{O}_{\mathbb{P}^1}(i+j)$, and via this isomorphim we have the identifications

$$\begin{aligned}
\mathrm{H}^0(\omega_{\mathbb{P}^1}((1 + a)A)) &= \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(i + j - (1 + b)B - 2C)) \\
\mathrm{H}^0(\omega_{\mathbb{P}^1}((1 + b)B)) &= \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(i + j - (1 + a)A - 2C)) \\
\mathrm{H}^0(\omega_{\mathbb{P}^1}((2C)) &= \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(i + j - (1 + b)B - (1 + a)A)),
\end{aligned}$$

we can thus write

$$V = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(i+j-(a+1)A-2C))\oplus\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(i+j-(b+1)B-2C))\oplus\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(i+j-(a+1)A-(b+1)B)).$$

As for the linear system $V_{a,b}$, we can easily see that the vanishing sequences of A, B and C are

$$\begin{gathered}
0, 1, \ldots, a - 1, a + 1, \ldots, a + b + 1 \\
0, 1, \ldots, b - 1, b + 1, \ldots, a + b + 1 \\
0, 2, 3, \ldots, a + b + 1.
\end{gathered}$$

Therefore, by Plücker Formula, V does not have other ramification points.

Proposition A.1.3 *There are no points P and Q in $\mathbb{P}^1\backslash\{A, B, C\}$ satisfying both*

$$\dim V(-iP - jQ) \geq 1 \quad \textit{and} \quad \dim V(-(i - 1)P - (j - 1)Q) \geq 2.$$

Moreover, if we set $P = A$ and only ask for $Q \neq C$, then there exists such point Q only if $i \geq a+1$.

Proof. Let's proceed by contradiction, and assume that P and Q satisfy the conditions above. Since the vanishing sequence of C starts with $0, 2, \ldots$, this happens if and only if

$$\begin{aligned}
\dim V(-2C - (i - 1)P - (j - 1)Q) &\geq 1 \\
\dim V(-iP - jQ) &\geq 1,
\end{aligned} \tag{A.4}$$

because clearly

$$\begin{aligned}
V(-2C - (i - 1)P - (j - 1)Q) + V(-iP - jQ) &\subset V(-(i - 1)P - (j - 1)Q) \quad \text{and} \\
V(-2C - (i - 1)P - (j - 1)Q) \cap V(-iP - jQ) &= 0.
\end{aligned}$$

Without loss of generality, we can assume $A = (0:1)$, $B = (1:0)$, $C = (1:1)$, $P = (-\lambda_1 : 1)$ and $Q = (-\lambda_2 : 1)$. Then (A.4) happens if and only if

$$(X+\lambda_1 Y)^i (X+\lambda_2 Y)^j = (X-Y)^2 J(X,Y) + cX^{a+1}Y^{b+1}, \tag{A.5}$$

with $J(X,Y)$ and $(X+\lambda_1 Y)^{i-1}(X+\lambda_2 Y)^{j-1}$ having zero as coefficient of X^aY^b. The latter implies that

$$H^{[i+j-2-a]}_{i-1,j-1}(\lambda_1,\lambda_2) = 0. \tag{A.6}$$

Setting $Y = 1$ in (A.5), and multiplying by $(1-X)^{-2}$ we get

$$\left(\sum_{n=0}^{i+j} H^{[n]}_{i,j}(\lambda_1,\lambda_2)X^{i+j-n}\right)(1+2X+3X^2+\ldots) = J(X,1) + cX^{a+1}(1+2X+3X^2+\ldots).$$

But, the coefficient of X^a on the right-hand side is 0, whence

$$\sum_{n=i+j-a}^{i+j} (a+n-i-j+1)H^{[n]}_{i,j}(\lambda_1,\lambda_2) = 0. \tag{A.7}$$

Now, dividing (A.5) (still setting $Y = 1$) by X^{a+1}, differentiating with respect to X and setting $X = 1$, we get

$$i(1+\lambda_1)^{i-1}(1+\lambda_2)^j + j(1+\lambda_1)^i(1+\lambda_2)^{j-1} - (a+1)(1+\lambda_1)^i(1+\lambda_2)^j = 0,$$

which implies

$$i+j-a-1 = \frac{i\lambda_1}{1+\lambda_1} + \frac{j\lambda_2}{1+\lambda_2}. \tag{A.8}$$

Repeating this step, but with $(X+\lambda_1)^i(X+\lambda_2)^j$ replaced by its expansion, we get

$$\sum_{n=0}^{i+j}(i+j-n-a-1)H^{[n]}_{i,j}(\lambda_1,\lambda_2) = 0,$$

which, together with (A.7), imply

$$\sum_{n=0}^{i+j-a-1} (i+j-n-a-1)H^{[n]}_{i,j}(\lambda_1,\lambda_2) = 0. \tag{A.9}$$

Define now

$$L^{[a]}_{i,j} := \sum_{n=0}^{i+j-a-1} H^{[n]}_{i,j}.$$

Then, from (A.2) we get that

$$\begin{array}{rclcl} L^{[a]}_{i,j} &=& (1+Z)L^{[a]}_{i-1,j} + H^{[i-1+j-a]}_{i-1,j} &=& L^{[a-1]}_{i-1,j} + ZL^{[a]}_{i-1,j} \\ L^{[a]}_{i,j} &=& (1+W)L^{[a]}_{i,j-1} + H^{[i+j-1-a]}_{i,j-1} &=& L^{[a-1]}_{i,j-1} + WL^{[a]}_{i,j-1}. \end{array} \tag{A.10}$$

From these and (A.6), we have that

$$\begin{array}{rclcl} 0 &=& H^{[i+j-2-a]}_{i-1,j-1}(\lambda_1,\lambda_2) &=& L^{[a]}_{i,j-1}(\lambda_1,\lambda_2) - (1+\lambda_1)L^{[a]}_{i-1,j-1}(\lambda_1,\lambda_2) \\ 0 &=& H^{[i+j-2-a]}_{i-1,j-1}(\lambda_1,\lambda_2) &=& L^{[a]}_{i-1,j}(\lambda_1,\lambda_2) - (1+\lambda_2)L^{[a]}_{i-1,j-1}(\lambda_1,\lambda_2), \end{array}$$

whence

$$\frac{L^{[a]}_{i,j-1}(\lambda_1,\lambda_2)}{1+\lambda_1} = \frac{L^{[a]}_{i-1,j}(\lambda_1,\lambda_2)}{1+\lambda_2}. \tag{A.11}$$

Going back to equation (A.9), we see that

$$0 = (i+j-a-1)L^{[a]}_{i,j}(\lambda_1,\lambda_2) - \sum_{n=0}^{i+j-a-1} nH^{[n]}_{i,j}(\lambda_1,\lambda_2). \tag{A.12}$$

But

$$\begin{aligned} nH^{[n]}_{i,j} &= W\frac{\partial}{\partial W}H^{[n]}_{i,j} + Z\frac{\partial}{\partial Z}H^{[n]}_{i,j} \\ &= jWH^{[n-1]}_{i,j-1} + iZH^{[n-1]}_{i-1,j}. \end{aligned}$$

Summing up,

$$\sum_{n=0}^{i+j-a-1} nH^{[n]}_{i,j} = jWL^{[a]}_{i,j-1} + iZL^{[a]}_{i-1,j}.$$

Substituting in (A.12), we get

$$(i+j-a-1)L^{[a]}_{i,j}(\lambda_1,\lambda_2) = j\lambda_2 L^{[a]}_{i,j-1}(\lambda_1,\lambda_2) + i\lambda_1 L^{[a]}_{i-1,j}(\lambda_1,\lambda_2). \tag{A.13}$$

Finally, equations (A.8), (A.11) and (A.13) imply that the vectors

$$\left(1, \frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}\right) \text{ and } \left(L^{[a]}_{i,j}(\lambda_1,\lambda_2), \lambda_1 L^{[a]}_{i-1,j}(\lambda_1,\lambda_2), \lambda_2 L^{[a]}_{i,j-1}(\lambda_1,\lambda_2)\right)$$

must be colinear, and hence that

$$L^{[a]}_{i,j}(\lambda_1,\lambda_2) = (1+\lambda_1)(L^{[a]}_{i-1,j}(\lambda_1,\lambda_2)),$$

and then, by (A.10),

$$H^{[i+j-1-a]}_{i-1,j}(\lambda_1,\lambda_2) = 0.$$

This, combined with (A.6), gives us

$$\begin{aligned} h^{[i+j-1-a]}_{i-1,j}\left(\frac{\lambda_2}{\lambda_1}\right) &= 0 \\ h^{[i+j-2-a]}_{i-1,j-1}\left(\frac{\lambda_2}{\lambda_1}\right) &= 0, \end{aligned}$$

but $(h^{[i+j-1-a]}_{i-1,j})' = jh^{[i+j-2-a]}_{i-1,j-1}$, which implies that λ_2/λ_1 is a double root of $h^{[i+j-1-a]}_{i-1,j}$. But Proposition A.1.1 says that this is only possible when $\lambda_1 = \lambda_2$, i.e., $P = Q$, but then P is ramification point of V, a contradiction.

To finish, if $P = A$, Equation (A.6) implies that $i \geq a+1$ or $Q = A, B$. But, if $Q = B$ then Equation (A.8) gives us $i = a+1$, whereas equation (A.5) proves that $Q \neq A$.$\square$

Let A, B and C be distinct points on $\mathbb{P}^1$ and a, i and j positive integers with $j+i=a-1$. Define the $(a+1)$-dimensional linear system inside $\mathrm{H}^0(\omega_{\mathbb{P}^1}((1+a)C+A+B))$

$$V := \mathrm{H}^0(\omega_{\mathbb{P}^1}((1+a)C)) \oplus \mathrm{H}^0(\omega_{\mathbb{P}^1}(A+B)).$$

Via an isomorphism $\omega_{\mathbb{P}^1}((1+a)C+A+B) \cong \mathcal{O}_{\mathbb{P}^1}(a+1)$ we can write

$$V = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(a+1-A-B)) \oplus \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(a+1-(a+1)C)).$$

Proposition A.1.4 *The linear system V does not have special ramification points. Furthermore, the only pairs of points (P,Q) of $\mathbb{P}^1$ satisfying*

$$\dim V(-iP-jQ) \geq 3,$$

are (A,B) and (B,A).

Proof. Without loss of generality, we can assume $A=(0:1)$, $B=(-1,1)$, $C=(1:0)$, $P=(-\lambda_1,1)$ and $Q=(-\lambda_2,1)$. Then V is the space of polynomials of degree $a+1$ of the form $cY^{a+1}+X(X+Y)F(X,Y)$. By degree considerations, V does not have a special ramification point of type $a+2$, and for P to be a special ramification point of type a, we must have that all polynomials of the form $(t_1X+t_2Y)(X+\lambda_1Y)^a$ are in V. However setting $t_1=1$ and $t_2=0$ we get that $\lambda_1=1$ and setting $t_1=t_2=1$ we get that $\lambda_1=0$, a contradiction.

As for the second statement, first note that $V(-iP-jQ)$ is the subspace of V of polynomials of the form $D(X,Y)(X+\lambda_1Y)^i(X+\lambda_2Y)^j$ with $\deg D=2$. But $XY(X+\lambda_1Y)^i(X+\lambda_2Y)^j$ is of the form $cY^{a+1}+X(X+Y)F(X,Y)$ only when $\lambda_1=1$ or $\lambda_2=1$; analogously, $(X+Y)(X+\lambda_1Y)^i(X+\lambda_2Y)^j$ is of the form $cY^{a+1}+X(X+Y)F(X,Y)$ only when $\lambda_1=0$ or $\lambda_2=0$. This concludes the proof.$\square$

A.2 Linear systems on elliptic curves

Fix a and b positive integers. Let A and B be general distinct points on a elliptic curve E. Let $L_{A,B}=\mathcal{O}_E((1+a)A+(1+b)B)$ and

$$V_{A,B} := \mathrm{H}^0(\mathcal{O}_E((1+a)A)) + \mathrm{H}^0(\mathcal{O}_E((1+b)B)) \subset \mathrm{H}^0(L_{A,B}).$$

Then $V_{A,B}$ is a linear system of dimension $a+b+1$. (Note that we do not need generality for this.) The vanishing sequence at A of $V_{A,B}$ is

$$0,1,\ldots,a-1,a+1,\ldots,a+b+1$$

and the vanishing sequence at B is

$$0,1,\ldots,b-1,b+1,\ldots,a+b+1$$

because both $(1+a)(A-B)$ and $(1+b)(A-B)$ are not equivalent to zero by the generality assumption. Hence $\mathrm{wt}_{V_{A,B}}(A) = b+1$ and $\mathrm{wt}_{V_{A,B}}(B) = a+1$.

Take now the double product, viewed as a family by the first projection $p_1 : E \times E \to E$, and consider the line bundle given by $\mathcal{L} := \mathcal{O}_{E\times E}((1+a)\Gamma + (1+b)\Delta)$, where $\Gamma := E \times \{A\}$ and Δ is the diagonal. Define the sheaf $\widetilde{\mathcal{V}}$ as the image of the map

$$p_{1*}(\mathcal{L}(-(1+a)\Gamma)) \oplus p_{1*}(\mathcal{L}(-(1+b)\Delta)) \longrightarrow p_{1*}(\mathcal{L})$$

Then for $R \in E\backslash\{A\}$ we have

$$\widetilde{\mathcal{V}}_R := \widetilde{\mathcal{V}}|_R = V_{A,R}.$$

This means that, restricted to $E\backslash\{A\}$, $\widetilde{\mathcal{V}}$ is a vector bundle, and if R is general we see that A has weight $b+1$ and R has weight $a+1$ in $\widetilde{\mathcal{V}}_R$.

Since E is a smooth curve, there exists a vector bundle $\mathcal{V}$ of rank $a+b+1$, such that $\mathcal{V} \subset p_{1*}(\mathcal{L})$ and

$$\mathcal{V}|_{E\backslash\{A\}} = \widetilde{\mathcal{V}}|_{E\backslash\{A\}}$$

Therefore $W_{\mathcal{V}} \geq (1+b)\Gamma + (1+a)\Delta$, and hence $R_{\mathcal{V}_A} = W_{\mathcal{V}} \cap p_1^{-1}(A) \geq (a+b+2)A$. However, $\mathcal{V}_A \subset \mathrm{H}^0(\mathcal{O}_E((2+a+b)A))$; therefore the vanishing sequence at A of $\mathcal{V}_A$ is a subsequence of $0, 1, \ldots, a+b, a+b+2$; thus A is a base point of $\mathcal{V}_A$, which means that $\mathcal{V}_A = \mathrm{H}^0(\mathcal{O}_E((1+a+b)A))$.

Let $W := W_{\mathcal{V}} - (1+b)\Gamma - (1+a)\Delta$, and let SW_{p_1} denote the zero scheme of the section $\mathcal{O}_{E\times E} \to J^1_{p_1}(\mathcal{O}(W))$ induced by W. Since $\mathcal{V}_A$ does not have special ramification points other than A, we get that $SW_{p_1} \cap p_1^{-1}(A) = \emptyset$, and hence that SW_{p_1} has codimension 2. Therefore, for a general point B, the linear system $V_{A,B}$ does not have special ramification points distinct from A and B.

Proposition A.2.1 *Let (E, A, B) be a general two-pointed elliptic curve and a and b positive integers. Then the linear system $\mathrm{H}^0(\mathcal{O}_E((1+a)A)) + \mathrm{H}^0(\mathcal{O}_E((1+b)B) \subset \mathrm{H}^0(\mathcal{O}_E((1+a)A+(1+b)B))$ does not have special ramification points other than A and B.*

Proof. Follows from the previous discussion.□

Following the same idea, we now want to prove:

Proposition A.2.2 *Let (E, A, B) be a general two-pointed elliptic curve, and a, b, i and j be positive integers such that $i + j = a + b + 2$. Then there do not exist points P and Q in $E\backslash\{A, B\}$ such that*

$$\begin{aligned} \dim V_{A,B}(-iP - jQ) &\geq 1 \quad \textit{and} \\ \dim V_{A,B}(-(i-1)P - (j-1)Q) &\geq 2. \end{aligned} \tag{A.14}$$

Proof. Note that by degree considerations we have equality in both equations if they are satisfied and if $j = 1$ or $i = 1$ the result follows from Proposition A.2.1. In order to prove the Proposition, we will repeat the same argument as above, but we will not allow the points P and Q to coincide with A or B.

Let X be the union of a general pointed elliptic curve (E, R_1) with a 3-pointed rational curve $(\mathbb{P}^1, R_2, A, B)$, identifying R_1 with R_2 (see Figure A.2). Let $\pi : \mathcal{C} \to \Sigma$ be a smoothing of X and σ_A and σ_B sections of π intersecting X at A and B. If every 2-pointed elliptic curve (E, A, B) admits P and Q as in Proposition A.2.2, then (possibly after a base change, which will introduce a chain of rational curves over the node) there exist sections σ_P and σ_Q of π through its smooth locus intersecting the general fiber at these points. Let $P := \sigma_P(0)$ and $Q := \sigma_Q(0)$. Note that, as seen in the proof of Proposition 5.3.4, we can disregard the rational curves that do not contain some of the special points. Also, let Γ_A, Γ_B, Γ_P and Γ_Q be the images of σ_A, σ_B, σ_P and σ_Q. Define $\mathcal{L} := \omega_\pi((1+a)\Gamma_A + (1+b)\Gamma_B)$ and $\widetilde{\mathcal{V}}$ as the image of the map

$$\pi_*(\mathcal{L}(-(1+a)\Gamma_A)) \oplus \pi_*(\mathcal{L}(-(1+b)\Gamma_B)) \to \pi_*(\mathcal{L}).$$

As before, there exists a vector bundle $\mathcal{V} \subset \pi_*(\mathcal{L})$ such that

$$\mathcal{V}|_\eta = \widetilde{\mathcal{V}}|_\eta,$$

where η is the generic point of Σ. Define $\mathcal{V}_1$ and $\mathcal{V}_2$ as follows:

$$\begin{aligned} \mathcal{V}_1 &:= \mathcal{V} \cap \pi_*(\mathcal{L}(-i\Gamma_P - j\Gamma_Q)) \\ \mathcal{V}_2 &:= \mathcal{V} \cap \pi_*(\mathcal{L}(-(i-1)\Gamma_P - (j-1)\Gamma_Q)). \end{aligned}$$

Since we have equality in (A.14), $\mathcal{V}_1$ and $\mathcal{V}_2$ are vector bundles of rank 1 and 2.

Also, for each positive integer $l < a+b+1$, define

$$\begin{aligned} \mathcal{V}(-l\Gamma_P) &:= \mathcal{V} \cap \pi_*(\mathcal{L}(-l\Gamma_P)) \quad \text{and} \\ \mathcal{V}(-l\Gamma_Q) &:= \mathcal{V} \cap \pi_*(\mathcal{L}(-l\Gamma_Q)). \end{aligned}$$

By Proposition A.2.1, both $\mathcal{V}(-l\Gamma_P)$ and $\mathcal{V}(-l\Gamma_Q)$ are vector bundles of rank $a+b+1-l$.

Note that

$$\begin{aligned} \mathcal{L}(-\mathbb{P}^1)|_{\mathbb{P}^1} &= \omega_{\mathbb{P}^1}((1+a)A + (1+b)B + 2R_2) \\ \mathcal{L}(-\mathbb{P}^1)|_E &= \mathcal{O}_E, \end{aligned}$$

whence $\mathcal{V}(-\mathbb{P}^1)$ has focus on $\mathbb{P}^1$. Applying now Lemma 5.2.1 to $\mathcal{V}(-\mathbb{P}^1 - (1+a)\Gamma_A - 3E)$, to $\mathcal{V}(-\mathbb{P}^1 - (1+b)\Gamma_B - 3E)$ and to $\mathcal{V}(-\mathbb{P}^1 - (1+a)\Gamma_A - (1+b)\Gamma_B)$, we get that

$$\begin{aligned} \mathcal{V}_{\mathbb{P}^1} = \mathcal{V}(-\mathbb{P}^1)|_{\mathbb{P}^1} &= \mathrm{H}^0(\omega_{\mathbb{P}^1}((1+a)A)) + \mathrm{H}^0(\omega_{\mathbb{P}^1}((1+b)B)) + \mathrm{H}^0(\omega_{\mathbb{P}^1}(2R_2)) \\ &\subset \mathrm{H}^0(\omega_{\mathbb{P}^1}((1+a)A + (1+b)B + 2R_2)) \end{aligned}$$

and then, since $\mathcal{V}_{\mathbb{P}^1}(-(a+b+2)R_2) = 0$ by Section A.1,

$$\mathcal{V}_E = \mathcal{V}(-(a+b+1)E) = \mathrm{H}^0(\mathcal{O}_E((a+b+1)R_1)).$$

Here $\mathcal{V}_{\mathbb{P}^1}$ and $\mathcal{V}_E$ are the limit linear systems over $\mathbb{P}^1$ and E. We have now some cases to check.

Case 1: Assume $P, Q \in E$ (see Figure A.1). Then $\mathcal{V}_1|_E = \mathrm{H}^0(\mathcal{O}_E((a+b+1)A))(-iP-jQ) = 0$, a contradiction.

Case 2: Assume $Q \in E$ and $P \in \mathbb{P}^1 \backslash \{A, B\}$ (see Figure A.2) . Then, by Lemma 5.2.1, we get that

$$\begin{aligned} V := \mathcal{V}|_{\mathbb{P}^1} &= \mathrm{H}^0(\omega_{\mathbb{P}^1}((1+a)A)) + \mathrm{H}^0(\omega_{\mathbb{P}^1}((1+b)B)) \quad \text{and} \\ \mathcal{V}|_E &= \mathrm{H}^0(\mathcal{O}_E(R_1)). \end{aligned}$$

Note that in fact we have $V \subset \mathrm{H}^0(\omega_{\mathbb{P}^1}((1+a)A + (1+b)B + R_2))$, with R_2 as a base point of V, but for simplicity we will assume $V \subset \mathrm{H}^0(\omega_{\mathbb{P}^1}((1+a)A + (1+b)B))$. Therefore, there exist D_1 and D_2 effective divisors on $\mathcal{C}$ supported on X not containing $\mathbb{P}^1$, in fact $D_1 = (j-1)E$ and $D_2 = (j-2)E$, such that

$$\begin{aligned} \mathcal{V}_1(-D_1)|_{\mathbb{P}^1} &= V(-(j-2)R_2 - iP) \\ \mathcal{V}_1(-D_1)|_E &= \mathrm{H}^0(\mathcal{O}_E(jR_1 - jQ)) \\ \mathcal{V}_2(-D_2)|_{\mathbb{P}^1} &= V(-(j-3)R_2 - (i-1)P) \\ \mathcal{V}_2(-D_2)|_E &= \mathrm{H}^0(\mathcal{O}_E((j-1)R_1 - (j-1)Q)). \end{aligned}$$

Since $\mathrm{h}^0(\mathcal{O}_E(jR_1 - jQ)) = 0$ or $\mathrm{h}^0(\mathcal{O}_E((j-1)R_1 - (j-1)Q)) = 0$, then

$$\begin{aligned} \dim V(-(j-1)R_2 - iP) &= 1 \quad \text{or} \\ \dim V(-(j-2)R_2 - (i-1)P) &= 2, \end{aligned}$$

which means that P must be a special ramification point of $V(-(j-1)R_2)$ or $V(-(j-2)R_2)$, a contradiction by Proposition A.1.2.

Case 3: Assume $P, Q \in \mathbb{P}^1 \backslash \{A, B\}$ (see Figure A.3). Then Proposition A.1.3 solves our problem.

Case 4: Assume $P = A$. If $Q \in \mathbb{P}^1$, then by Proposition A.1.3 we have that $i \geq a+1$. Now, we blow up $\mathcal{C}$ at A. Let $\mathbb{P}^1_A$ be the exceptional divisor. We will still denote by Γ_A and Γ_P their strict transforms; also, let $\tilde{A} = \Gamma_A \cap \mathbb{P}^1_A$. Assume first that $Q \in \mathbb{P}^1_A$ (see Figure A.5). Then, there exists an effective divisor D_A such that

$$\begin{aligned} \mathcal{L}(D_A - (1+a)\Gamma_A)|_{\mathbb{P}^1_A} &= \omega_{\mathbb{P}^1_A}(2A) = \mathcal{O}_{\mathbb{P}^1_A} \\ \mathcal{L}(D_A - (1+a)\Gamma_A)|_{\mathbb{P}^1} &= \omega_{\mathbb{P}^1}((1+b)B + 2R_2) \\ \mathcal{L}(D_A - (1+a)\Gamma_A)|_E &= \mathcal{O}_E. \end{aligned}$$

By Lemma 5.2.1, we get that

$$\mathcal{V}(-(1+a)\Gamma_A)_{\mathbb{P}^1} = \mathrm{H}^0(\omega_{\mathbb{P}^1}((1+b)B)) + \mathrm{H}^0(\omega_{\mathbb{P}^1}(2R_2)).$$

However, this $(b+1)$-dimensional linear system does not have A as a ramification point, whence

$$\mathcal{V}(-(1+a)\Gamma_A)_{\mathbb{P}^1_A} = \mathcal{V}(-(1+a)\Gamma_A - b\mathbb{P}^1_A) = \mathrm{H}^0(\omega_{\mathbb{P}^1_A}((2+b)A)).$$

This, together with the fact that $\mathcal{V}(-(1+b)\Gamma_B)|_{\mathbb{P}^1_A} = \mathrm{H}^0(\omega_{\mathbb{P}^1_A}((1+a)\tilde{A}))$, implies that

$$\mathcal{V}(-i\Gamma_P)_{\mathbb{P}^1_A} = (\mathrm{H}^0(\omega_{\mathbb{P}^1_A}((1+a)\tilde{A})) + \mathrm{H}^0(\omega_{\mathbb{P}^1_A}((2+b)A)))(-iP),$$

which does not have special ramification points by Proposition A.1.2, a contradiction.

Assume now that $Q \in \mathbb{P}^1$ (see Figure A.4). Since $i \geq a+1$, A is not a ramification point of $\mathcal{V}(-i\Gamma_P)_{\mathbb{P}^1_A}$, because the weight of A is $\max\{a-i, 0\}$ by Section A.1. Note that $\dim \mathcal{V}(-i\Gamma_P) = j-1$, which means that $\mathcal{V}(-i\Gamma_P)_{\mathbb{P}^1_A}(-(j-1)A) = 0$. Then there exists a divisor D such that

$$\begin{array}{rcl} \mathcal{L}(D-i\Gamma_P)|_{\mathbb{P}^1_A} & = & \omega_{\mathbb{P}^1_A}((1+a)\tilde{A} + (2+b)A - iP - (j-2)A) \\ \mathcal{L}(D-i\Gamma_P)|_{\mathbb{P}^1} & = & \mathcal{O}_{\mathbb{P}^1}(j-1) \\ \mathcal{L}(D-i\Gamma_P)|_E & = & \mathcal{O}_E. \end{array}$$

then, Lemma 5.2.1 and Exact Sequence (5.7) yield that

$$\mathcal{V}(-i\Gamma_P)_{\mathbb{P}^1} \subset \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(j-1)).$$

And, repeating the same argument for $D_1 = D - E$, we also get that

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(j-1-2R_2)) \subset \mathcal{V}(-i\Gamma_P)_{\mathbb{P}^1}.$$

Therefore, $\mathcal{V}(-i\Gamma_P)_{\mathbb{P}^1}$ does not have special ramification points away from R_2, a contradiction. Note that this argument includes the case where $Q = B$.

If $Q \in E$ (see Figure A.6), then there exists a divisor D_E such that

$$\begin{array}{rcl} \mathcal{L}(D_E-i\Gamma_P)|_{\mathbb{P}^1_A} & = & \omega_{\mathbb{P}^1_A}((1+a)\tilde{A} + (2+b)A - iP - (j-2)A) \\ \mathcal{L}(D_E-i\Gamma_P)|_{\mathbb{P}^1} & = & \mathcal{O}_{\mathbb{P}^1} \\ \mathcal{L}(D_E-i\Gamma_P)|_E & = & \mathcal{O}_E((j-1)R_2). \end{array}$$

Again, Lemma 5.2.1 and Exact Sequence (5.7) imply that $\mathcal{V}(-i\Gamma_P)_E = \mathrm{H}^0(\mathcal{O}_E((j-1)R))$, which does not have special ramification points, a contradiction.

The case where P or Q goes to the node, i.e., where a base change is necessary (for instance, see Figure A.8, although a more longer chain of rational curves may arise), as well as the more degenerated case where $P = \tilde{A}$, can be handled similarly.$\square$

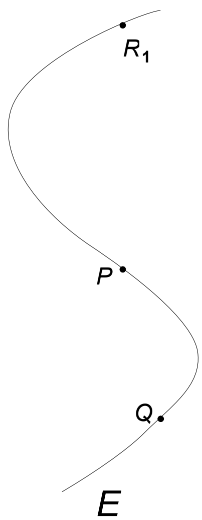


Figure A.1: $P, Q \in E$

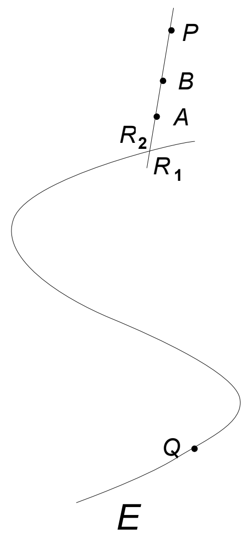


Figure A.2: $P \in \mathbb{P}^1$ and $Q \in E$

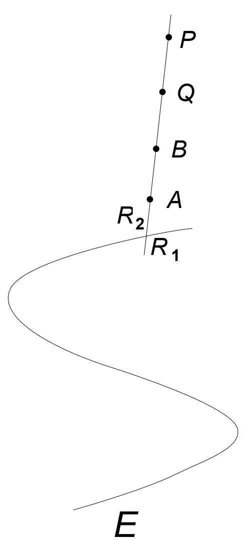


Figure A.3: $P, Q \in \mathbb{P}^1$

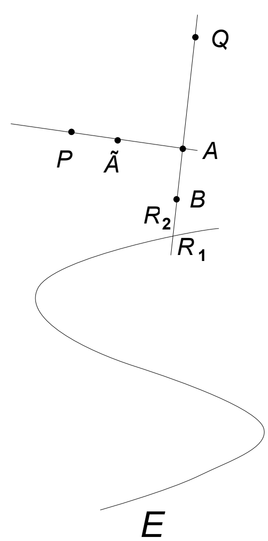


Figure A.4: $P = A$ and $Q \in \mathbb{P}^1$

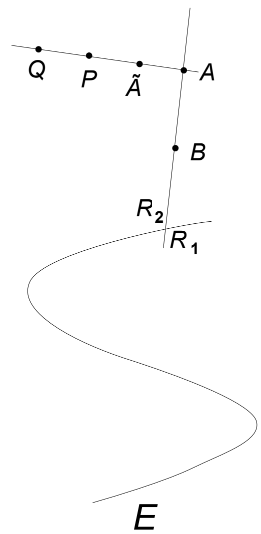


Figure A.5: $P = Q = A$

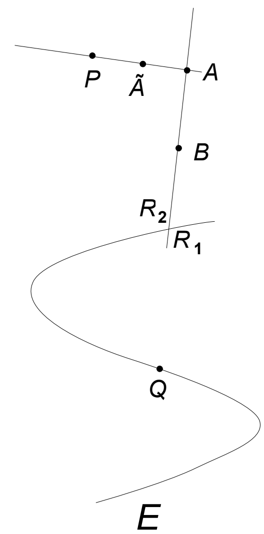


Figure A.6: $P = A$ and $Q \in E$

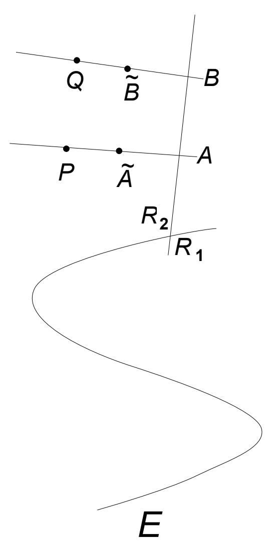


Figure A.7: $P = A$ and $Q = B$

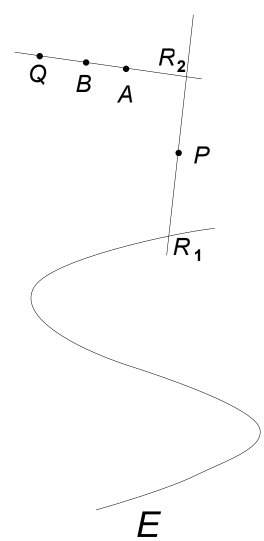


Figure A.8: P goes to the node

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